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THE GIST
OF
MATHEMATICS

PRENTICE-HALL MATHEMATICS SERIES

ALBERT A. BENNETT, EDITOR

THE GIST OF MATHEMATICS

BY

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To

CONSUELO, HENRY, JOHN, AND JIMMY

PREFACE

Many adults wish that they could remember the essentials of mathematics long since known and forgotten. In their earlier years the subject was not closely related to life, and when later experience indicates, even vaguely, the presence of mathematical relationships in everyday affairs, there comes a rather frustrated feeling that it is too late to learn. Many people are afraid of mathematics. In no subject, perhaps, is there such a widely prevailing timidity.

Accounting has now become a flourishing profession, reaching into nearly all the nooks and crannies of public and private business. Members of this profession do not go far in practice without running into a multitude of problems that cannot be readily solved by arithmetic. Economists, biologists, and scientific workers in all fields have constant need of the simple, elementary concepts that are elaborated in the present volume. These concepts in fact are the *gist of mathematics*. No matter how deeply one may penetrate the intricacies of such a field as actuarial principles, or the designing of television sets, or the theory of atomic structure, the fundamental tools of the human mind for exploring these new aspects of science and technology are none other than the ideas set forth in the following pages.

An idea can be expressed only by means of a symbol. Without symbols there can be no communication and no thought. The symbol may be a word or a number or a figure of some sort possessing a conventional meaning. This international language of mathematics has grown, from the first stammerings thousands of years ago, into a flexible and unbounded means of expression, capable of being fitted like a map to the observed facts of the universe and of revealing in those facts entirely unsuspected truths. Some modern industries involving hundreds of millions of dollars of capital were literally born from equations that could be written on one small sheet of paper.

To study mathematics is to learn a new language. Just as children learn their mother tongue almost without effort, so it is possible to set forth step by step an easy approach to mathematics.

J. H. M.
J. A. M.

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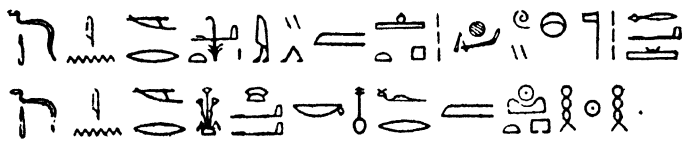
THE GIST
OF
MATHEMATICS

1 THE LANGUAGE OF MATHEMATICS



"Mert-shemā says: 'Come in peace, protector of the Great Company of the Gods,' and Mert-meht says: 'Thou risest [in] beauty in the Horizon of Eternity.'" (From E. A. Wallis Budge, *Osiris and the Egyptian Resurrection* [London, 1911], Vol. I, p. 45.)

Osiris of Philae.

If a Chinese laundryman should say: "Shee kin sham mee kin yat kock wu fan kung tsik liu kock," his words would be meaningless to most Americans. Should some able bystander tell us that they mean, "Four shirts at 15 cents apiece makes 60 cents," we should have an accurate notion of the numerical values intended. Ideas are expressed by means of symbols. These symbols may be sounds combined to make up words, or marks arranged upon paper. Numbers are ideas; the words or figures which represent them are only symbols. The figures "XV" and "15" are entirely different symbols, yet they represent the same numerical value. All branches of mathematics are concerned with recording by means of symbols some kind of numerical relationship.

The written language of mathematics does not possess an absolutely perfect set of symbols each of which conveys one unequivocal meaning. Written music is probably more universally understandable than the language of certain mathe-

matical fields where pioneers have invented symbols which have not yet come into universal usage. Beginners are puzzled by the multiplicity of ways of expressing one idea. Thus, "fifteen multiplied by five" may be written: (*a*) 15×5; (*b*) $15(5)$; or (*c*) $15 \cdot 5$. Even in accountancy no absolutely identical way of recording business data prevails in modern industrial nations. In the United States, debts are put on the right-hand side of a ledger; in England, the same items appear on the left-hand side.

These apparently trivial imperfections in the use of symbols are mentioned here to emphasize the fact that we can never know what another person is talking about unless we have a clear agreement as to the meaning of the sounds he makes or of the marks he puts upon paper. A professional mathematician should never feel disdain for the layman who is tripped up by unfamiliarity with conventional shorthand expressions that are so easily forgotten. For example, in an expression like

$$15 \div 5 + 4 \times 12 - 6,$$

one meets five quantities and four signs of operation. Nothing shows the order in which these four operations are to be performed. In fact, unless we know the convention that multiplication and division operations are to be performed first, in the order in which they appear from left to right, and addition and subtraction second, there would *not* be one and only one possible value for the above expression.

Words and grammar were devised, not upon scientific principles, but according to the give-and-take of thousands of persons trying to communicate their thoughts. It is not surprising, therefore, that words are not always precise and unequivocal in meaning. When we say that a person is *honest*, we may mean that he is *straightforward, genuine, frank, equitable, candid, trustworthy*, or *upright*. No one of these words has exactly the same meaning as *honest*. The synonyms overlap and in a way mutually delimit each other. The part which is common to all these terms conveys, more or less closely, the meaning of *honest*.

For practical purposes of computation, the language of mathematics needs to be far more exact and definite. A symbol should mean one thing and only one thing. For any kind of clear thinking, words and symbols must always be used with

a definite understanding of their meaning. Mathematical symbols have the advantage of not being emotive. The expression "isosceles triangle," unlike "communism," excites neither approbation nor dislike. A means of communication which does not set up overtones of emotion in the mind is better adapted to the clear, dispassionate expression of truth than is our everyday speech.

A peculiarity of the language of mathematics is its intentionally abstract character. When we say, "1 + 1 = 2," we are expressing a truth about numbers without any concrete connotations. If we say, "1 gallon + 1 gallon = 2 gallons," we are expressing another abstract idea, because the word *gallon* indicates a comparison, and not an object. Nobody ever saw or touched a gallon. If now we should say, "One gallon of water plus one gallon of alcohol makes two gallons of a mixed liquid," we should be stating a falsehood, for it is a curious fact that such a mixture would be slightly less than two gallons in volume. It is most important, as this illustration shows, not to fall into the error of assuming that what may be true of numerical relationships is necessarily true of the physical objects which are enumerated.

As we learn more about nature, our vocabulary necessarily grows richer in order to express our increasing wealth of ideas. For example, the words *radio*, *airplane*, *Diesel engine*, and *spark-plug* would have been meaningless to George Washington. Mathematics, too, has to grow if it is to fit the increasingly complex structure of the universe as revealed by science. The word *number* once denoted only positive integers (integers are often called *whole numbers*). But the word became more generalized and gradually included fractions, irrationals like $\sqrt{2}$, imaginaries like $\sqrt{-1}$, and transcendentals like π. Progress in mathematics has revealed unsuspected truths about physics and chemistry, and these studies in turn have brought about more progress in mathematics.

NUMBERS

The first to give number its abstract quality were the Greeks (about 500 B. C.). They took the tremendous step forward of expressing the concept *two*, whether applied to dogs or apples or to other concrete objects or abstract things. Yet they were thwarted by a lack of proper symbols to denote num-

bers. The disadvantage of computing by means of the Greek system, in which symbols did not have place value and in which no symbol for zero and no decimal point existed, is shown by contrasting the ancient with the modern method of multiplication:

Greek	*Translation*	*Modern Method*
$\rho\ \lambda\ \eta'$	138	138
$\rho\ \lambda\ \eta'$	138	138
$,\iota\quad ,\gamma\quad \omega'$	10,000 3,000 800	1104
$,\gamma\quad \pi'\quad \sigma\mu'$	3,000 900 240	414
$\omega'\ \sigma\mu'\quad \xi\delta'$	800 240 64	138
$,\iota\theta\mu\delta'$	19,044	19,044

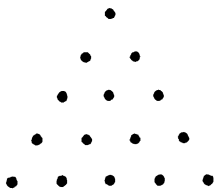


Fig. 1.

Without adequate symbols, there could be no arithmetic readily usable by people of only average intelligence. Problems which a normal child can solve today in a few minutes required in ancient times hours of hard work. In fact, a person quick in computation was thought to possess almost supernatural powers. This aura of mystery and awe reached its zenith during the fifth century B. C., when Pythagoras and his School attributed divine significance to numbers.

The successive addition of the natural numbers or integers formed the "triangular" numbers: 1, 3, 6, 10, 15, 21, and so on. Figure 1 shows clearly the reason for the name. Count the dots in the triangle formed by taking three rows; 6, the third number in the series, is the result. If four rows are taken, the dots add up to 10, the fourth number in the series. Curiosity regarding these magical triangular numbers led ultimately to the study of series—a broad and practical topic.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Fig. 2.

A number of integers arranged in the form of a square (Figure 2) so that the sum of the numbers in every row, in every column, and in each diagonal is the same, make up what is known as a *magic square.* The particular arrangement shown in this figure was drawn by Albert Dürer in the picture *Melan-*

choly, painted in 1514. These squares, engraved on metal, were worn as a charm against disease and malignant fever.

RATIONAL AND IRRATIONAL NUMBERS

Pythagoras' discovery that *in a right triangle, the square on the hypotenuse is equal to the sum of the squares of the other two sides* (see page 295), led his disciples to a very thorough study of the properties of right triangles. While working on the solution of the familiar case of a right triangle having two equal sides one unit long (Figure 3), they made the surprising discovery that the length of the hypotenuse, or longest side, is "incommensurable": that is, a line whose length is $\sqrt{2}$ cannot be exactly expressed by any whole number or by any ratio of whole numbers. Number was no longer divine! It could not even account for a very simple fact. This discovery caused so much dismay and anxiety that such an entity as $\sqrt{2}$ was called *alogon*, or "the unutterable."

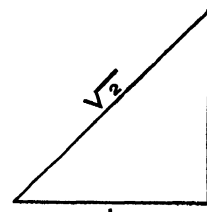


Fig. 3.

Whole numbers (integers) and ratios of whole numbers are called *rational numbers*. Numbers which cannot be expressed by a ratio of whole numbers are called *irrational numbers*.

The fact that a numerical value for an irrational number cannot be obtained by dividing one whole number by another need not worry us. An irrational number is expressed arithmetically by a *nonending, nonrepeating* decimal. And, for practical purposes, it is sufficient to note that an irrational number may be expressed to any degree of accuracy by a rational approximation. Thus, $\sqrt{2} = 1.41421\ldots$ may be taken *approximately* as $\frac{17}{12} = 1.4166\ldots$, or $\frac{707}{500} = 1.414$.*

SIGNED NUMBERS

The Hindu mathematicians, who had little interest in mathematical methods but were very much concerned with business procedure, introduced the concept of negative numbers and used it to distinguish between assets and debts.

* It is important to note that 1.4166 . . ., although a non-ending decimal, can be expressed by the ratio of two whole numbers, because the decimal repeats itself or is "periodic"; in fact, it is the rational number $\frac{17}{12}$. Moreover, the number 1.285714285714 . . . is rational for the same reasons, and is equal to $\frac{9}{7}$.

In everyday life, things are moving and changing. An airplane changes its elevation while traveling from one place to another. Temperature varies in the course of a day. Wholesale and retail prices, taxes and quotations of stock, continually fluctuate in value. Change of any kind is usually represented in terms of *more* or *less;* that is, *quantitatively.* Numbers represent quantities. Numbers, therefore, besides representing the change in a quantity, should indicate whether the change is up or down, east or west, profit or loss; that is, whether the change is an increase or a decrease in quantity. In other words, *a number must be able to contrast a quantity with another quantity of opposite character.*

Consider the following problem: "Mr. Rogers travels in his car 25 miles along a straight road. He goes from his home to town B, 15 miles away, and then travels along the same straight road from town B to town A, a distance of 10 miles. How far is he from home?" The problem, as stated, has no meaning unless we know whether or not the distance from B to A is in the same direction as the distance from Rogers' home to B.

But assume that a road map shows the location of towns A and B on the same straight road as in Figure 4. Since, in

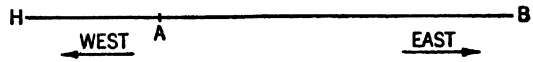


Fig. 4.

traveling from his home to B, Mr. Rogers goes directly east, while in traveling from B to A he goes directly west, the distance from B to A is thought of as having an opposite character to that of the distance from his home to B. The latter represents an increase in the distance between Rogers and his home, while the distance from B to A represents a decrease.

So that this contrast in the character of a number may be represented, *signed* numbers are introduced. The symbols $+$ and $-$ are used to distinguish quantities of opposite character. A number preceded by the symbol of addition, $+$, or with no symbol at all before it, is called a *positive* number. A number preceded by the symbol of subtraction, $-$, is called a *negative* number. *The sign must always be written before a negative number; if no sign appears before a number, the number is positive.*

Thus, if the distance traveled by Rogers from his home H to town B is distance traveled toward the east, and if it is considered positive, then the distance traveled from town B to town A, or distance toward the west, should be considered negative. A negative quantity is, therefore, exactly opposite in quality or condition to a positive quantity. The terms *positive* and *negative* are relative. Whether a given physical measure is represented by a positive or by a negative number is a matter of agreement.

Note that if Rogers had traveled west 15 miles from town B, he would have been back at his starting point, or at a distance of zero miles from his home. Thus, if the two distances are added numerically, taking into consideration their opposite qualities of direction, the result should be zero; that is, $(+15) + (-15) = 0$. In general, the addition of a positive number and a numerically equal negative number gives zero as the value of the answer.

The following examples are sufficient to explain the rules for adding signed numbers. In these examples, let profit be represented by a positive number and loss by a negative number.

(1) If a man makes a profit of \$10, followed by another profit of \$5, his total profit is \$15.

Thus,

$$(+10) + (+5) = +15.$$

(2) If a man sustains a loss of \$10, followed by another loss of \$5, his total loss is \$15.

Thus,

$$(-10) + (-5) = -15.$$

From these two illustrations the following rule may be inferred:

In adding two signed numbers having like signs, disregard the signs. Find the sum, and prefix to this sum the common sign.

(3) If a man makes a profit of \$10, followed by a loss of \$3, the resulting profit is \$7.

Thus,

$$(+10) + (-3) = +7.$$

(4) If a man loses \$10, and then gains \$2, his net loss is \$8.

Thus,

$$(-10) + (+2) = -8.$$

From the last two illustrations a second rule may be inferred:

In adding two signed numbers having unlike signs, disregard the signs. Subtract the smaller number from the larger and prefix to this difference the sign of the larger number.

Untrained persons believe subtraction possible only in case the number to be subtracted is the smaller. Consider, however, the following problem: "A man wishing to buy a radio worth \$35 has only \$15. He arranges to pay \$15 now, with the promise to pay \$20 as soon as possible." This \$20 is a debt, a claim on the earnings of the man; hence, it is thought of as having an opposite character to that of the money he earns. It is money to be subtracted from his earnings, and therefore should be represented as a negative quantity; that is, −\$20. The result of subtracting \$35 from \$15 is, therefore, a debt of \$20, or −\$20. Thus, $15 - 35 = -20$ implies that 15 of the 35 is subtracted at once, while the balance of 20 remains as a debt to be paid (or subtracted) as soon as possible. This method gives rise to the idea of credit in business transactions.

Fig. 5.

Figure 5 represents a Fahrenheit thermometer. Signed numbers are used to represent temperatures above or below zero. Thus, −8° means a temperature of 8 degrees below zero; and +8° (or simply 8°) means a temperature of 8 degrees above zero. In other words, direction upward is considered positive while direction downward is considered negative. Hence, an increase in temperature is indicated by a positive number, and a drop by a negative number. The subtraction of a signed number b from another signed number a does *not* mean to take b away from a, but *to find a signed number* c *which added to* b *gives* a. For example, the subtraction of +5 from +8 does not mean to take 5 away from 8, but to find a number which, added to +5, gives +8. The answer is obviously +3. This result may be illustrated by the thermometer's scale as follows: The subtraction of +5° from +8° means to count the number of degrees from +5° to +8°, which gives 3°. Since the counting from +5° to +8° is upward, that is, in the positive direction, the result is called "+3°."

Thus,

$$(+8) - (+5) = +3.$$

To change from 8° to −5°, the temperature must drop 13°. That is,

$$(-5) - (+8) = -13.$$

To change from −5° to +8°, it must rise 13°. That is,

$$(+8) - (-5) = +13.$$

To change from −6° to −10°, the temperature must drop 4°. That is,

$$(-10) - (-6) = -4.$$

A careful study of the following examples shows that subtraction of signed numbers may be changed into addition:

$$\begin{aligned}(+8) - (+5) &= +3 &&= (+8) + (-5).\\ (-5) - (+8) &= -13 &&= (-5) + (-8).\\ (+8) - (-5) &= +13 &&= (+8) + (+5).\\ (-10) - (-6) &= -4 &&= (-10) + (+6).\end{aligned}$$

Note that, in every case, the result may be obtained by merely changing the sign of the subtrahend and then proceeding as in ordinary addition. Therefore: *To subtract one number from another, change the sign of the subtrahend and add.*

If profit and future time are considered positive, while loss and time already past are considered negative, the following examples will illustrate the rule for multiplication of signed numbers:

(1) If a man makes a profit of $10 a day, his profit after 3 days is $30. That is, $(+10) \times (+3) = +30$, or

$$\text{plus} \times \text{plus} = \text{plus}.$$

(2) If a man expends $10 a day, 3 days from now his expenses will be $30. That is, $(-10) \times (+3) = -30$, or

$$\text{minus} \times \text{plus} = \text{minus}.$$

(3) If a man for some time has been depositing $10 a day with a trustee, then his account 3 days ago was $30 less than it is today. That is, $(+10) \times (-3) = -30$, or

$$\text{plus} \times \text{minus} = \text{minus}.$$

(4) If a man for some time has been losing \$10 a day, then 3 days ago he had \$30 more than he has today.

That is,

$$(-10) \times (-3) = +30,$$

or

$$\text{minus} \times \text{minus} = \text{plus}.$$

The results obtained above suggest the following rule: *To multiply one signed number by another, disregard the signs, and find their product. The product is positive if the two signed numbers have like signs. The product is negative if the two signed numbers have unlike signs.*

Since division is the inverse of multiplication, the rule for the division of signed numbers may be obtained directly from the rules of multiplication. That is, $18 \div 6 = 3$ because $18 = 6 \times 3$. Reasoning in this manner leads to the following conclusions:

(1) $\dfrac{+18}{+6} = +3$ because $+18 = (+6) \times (+3)$.

(2) $\dfrac{-18}{+6} = -3$ because $-18 = (+6) \times (-3)$.

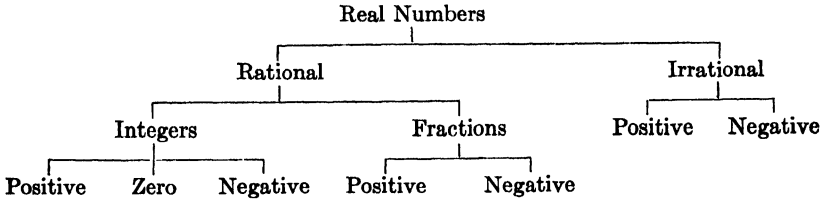
(3) $\dfrac{+18}{-6} = -3$ because $+18 = (-6) \times (-3)$.

(4) $\dfrac{-18}{-6} = +3$ because $-18 = (-6) \times (+3)$.

From these results the following rule is inferred: *The quotient of two numbers having like signs is positive. The quotient of two numbers having unlike signs is negative.*

The use of negative numbers makes always possible the operation of subtraction between any two numbers. This is not the case with division, *for division by zero is excluded.* To divide a number a by another number b means to find a number c such that $a = b \times c$. That is, as previously stated, 6 divided by 2 is said to be equal to 3 because $6 = 2 \times 3$. Thus, if $4 \div 0 = 0$, then $4 = 0 \times 0$, which is absurd; and if $4 \div 0 = 4$, then $4 = 0 \times 4$, which again is obviously nonsensical. In general, if $b = 0$, then a number c is to be found such that $a = 0 \times c$. If $a = 0$, then c may be any number; but if a is not zero, there is no number c.

Positive and negative rational and irrational numbers and zero constitute what is known as the "real number system":



Besides the real numbers, there are other kinds, which will be discussed later.

DEDUCTIVE REASONING

These introductory remarks about symbolism in mathematics must not obscure the fact that symbols, after all, are merely convenient tools used in thinking. The mental process known as "reasoning" consists in combining certain statements called *premises* in order to obtain a new statement called a *conclusion.* For example:

Major premise: All mammals are warm-blooded.
Minor premise: Whales are mammals.
Conclusion: Whales are warm-blooded.

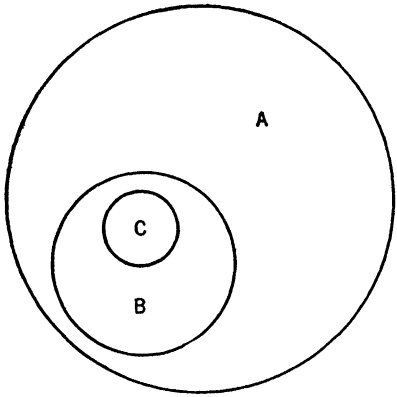
Major premise: The sum of the angles of a plane triangle is 180°.
Minor premise: In the triangle ABC (with angles A, B, C), the angle $C = 90°$.
Conclusion: The sum of the angles A and B is 90°.

Each of the above two sets of statements is called a *syllogism.* The type of reasoning just given is called *deductive* reasoning. Deductive reasoning is that form of reasoning in which the conclusion must necessarily follow if we accept the premises as true. In deduction, it is impossible for the premises to be true and the conclusion false. Naturally, if a premise is too broadly stated, too general, a false conclusion can be drawn from it. For example:

Major premise: Red-haired people are quick-tempered.
Minor premise: Mary has red hair.
Conclusion: Mary is quick-tempered.

If we accept as true the major premise about red-haired people, then, if Mary has red hair, it is a *valid* conclusion that

she is quick-tempered. If the premises are not true, however, the conclusion may not be true. In deduction, it is only necessary that the conclusion be obtained by a valid argument. In certain types of insanity, the reasoning is as logical as with sane people; the difference is that the insane person believes some general statement which sane people know to be false. Thus, one afflicted with the mania of persecution believes that people are conspiring to harm or kill him. He then may commit an act of violence under the logical conviction that he is acting in self-protection.



Note that the word *deduction* as here used does not have the everyday meaning of drawing any kind of conclusion, but rather the restricted meaning of drawing the special kind of conclusion validated by the premises. Formal logic contains discussions of the relationship between *general* and *particular* premises, and general and particular conclusions, but we cannot go into these matters in detail. The tests of validity in deductive reasoning are discussed in textbooks on logic.

A common and convenient method of indicating the inclusiveness, or the lack of it, with regard to premises and conclusions is the use of circles. Thus, in the syllogism of the whales above, the circle A can indicate the class of warm-blooded animals; the circle B, the class of all mammals; and C, the class of all whales. The major premise, "All mammals are warm-blooded," is indicated by the position of circle B in relation to circle A. Similarly, the minor premise is illustrated by the fact that circle C is entirely within circle B. Since, then, circle C is entirely within circle A, one infers from the diagram that "Whales are warm-blooded," which is the desired conclusion.

INDUCTIVE REASONING

Nondeductive reasoning needs to be distinguished from deductive reasoning. The most important type of the former is

known as *inductive* reasoning, or *induction*. This type of reasoning consists in concluding a general statement from a group of particular facts. For some years prior to 1930, it was discovered that the average of monthly wholesale prices of a large group of commodities showed fluctuations up and down somewhat in advance of similar fluctuations in the monthly average of various rates of interest. This relationship between the movements of two statistical series enabled certain banks to profit in their investment policy by being able to gauge in advance the trend in interest rates. After the depression got under way, however, the relationship ceased to prevail. Thus, from this and many other examples, it is clear that a conclusion reached by induction is not necessarily certain to follow from the premises, but is only more or less probable.

Since induction is a method of reaching a general conclusion from particular statements, it often is a means of discovering new truths. Science began with the use of induction. Reliance by the ancients (and, indeed, by the unthinking today) upon general statements unverified or unsupported by experience, led to gross falsehoods and superstition. The persistent belief that an atom* is an indivisible entity was not proved to be false until the present century. Mathematical history relates many views of number which are tinged with superstition.

INTUITION

Intuition is another type of reasoning. It has great potential creative value, but is beset by many pitfalls. Experience can sometimes develop intuition into what is virtually a sixth sense; but intuition, even if based upon experience, can lead the unwary into unjustifiable inferences. Some of the most brilliant mathematical discoveries were intuitive, even inspirational; on the other hand, there are very capable mathematicians who eschew intuition as something untrustworthy and dangerous. Agelong intuitions in regard to space are now demonstrably false.

Since intuition is peculiarly personal, it is desirable to base our knowledge of mathematics upon something more objective; namely, reasoning or logic, the rules of which have been under

* *a-tomos* means "something which cannot be cut up."

constant scrutiny for nearly 25 centuries. Inasmuch as non-deductive reasoning gives only probable conclusions even if the premises be accepted, while deductive reasoning from true premises gives conclusions which are certain, we strive to base as much of our knowledge as possible upon deductive reasoning.

MATHEMATICAL REASONING

In order to reason deductively, we must have some statements or premises which must be taken for granted in order to draw conclusions from them. Such statements may have been obtained as the conclusions of previous deductions, or they may be agreed upon and accepted without proof. These assumptions or statements are basic. They are in accord with experience and therefore seem to be true. Mathematical proofs are deductive and rest finally upon these statements.

It is the crowning glory of mathematics that a purely abstract subject, free of the trammels of sense or of matter, should have proved to be the best medium of recording and of clarifying the laws of nature underlying the rapidly changing phenomena of life and of material events.

NOTE TO READERS

So far, everything has been rather easy. But mathematics cannot be learned only through the eyes. Work is necessary; there is no royal road to learning. Thus, at the end of each chapter, we shall give a few problems designed as a self-imposed test of progress. Some problems are *easy;* some are *medium;* and some are *difficult.* Doubtless many readers will laugh at the designation of any such problems as difficult. Nonetheless, we urge that pencil and paper be used diligently to obtain an answer. Here and there a few little twists and surprises are included just to make things interesting. Wherever possible, practical problems are used in the hope that they will be more fascinating and thought-provoking than crossword puzzles.

1. Caesar was assassinated in 44 B. C. The sixth Roman emperor, Nero, met a violent death 111 years later. What was the date of Nero's death?

2. Absolute zero is defined as $-273°$ Centigrade. Water freezes at 0° C. and boils at 100° C. How many degrees are there between absolute zero and the boiling point of water?

3. A man is selling 30 articles at 3 for 1¢, and 30 other articles at 2 for 1¢. Would his total receipts be more, less, or the same, if he sold all 60 at 5 for 2¢?

4. A heavy cannon ball and a tennis ball are at the height of 6 feet from the ground and are dropped simultaneously. Which one hits the ground first? What kind of reasoning?

5. Acids change the color of litmus paper from blue to red. Salicylic acid does not thus change the color of litmus paper. What is your conclusion?

6. If a liter of water is added to a liter of sulphuric acid, how many liters of the mixture will there be?

7. If a pound of sodium is added to a pound of water, what will be the result?

8. A young man has 12 socks, forming 6 pairs, scattered about in a drawer. The socks are of solid brown, gray, white, red, black, and green. If the room is dark, what is the least number of socks that have to be withdrawn to be sure that there would be a pair of one color?

9. If $1.00 at 5 per cent compound interest doubles in 15 years, how long would it take for $5.00 to double itself?

10. Each bacterium in a certain culture doubles itself in one minute. If, starting with one bacterium, a certain container is completely filled in 1,000 minutes, how much time would be required to fill the same container if there were two bacteria at the outset?

11. Does it take an aviator the same time to travel 2 miles, (*a*) when he goes the first mile at 60 miles per hour and the second mile at 120 miles per hour, as (*b*) when he travels the 2 miles at 90 miles per hour?

12. Halley's comet travels round the sun in an elliptical orbit with the sun at one focus of the ellipse, the time of the orbit being about 75 years. The Great Comet of 1811, having a head larger than the sun, has not been seen since. Will this comet return again? What kind of reasoning will be used in answering this question?

13. After the planet Uranus was discovered by William Herschel in 1782, the known planets and their respective distances from the sun, assuming the distance of the earth to be one unit, were as follows:

Mercury.....	0.39	Jupiter.....	5.20
Venus.....	0.72	Saturn.....	9.54
Earth.....	1.00	Uranus.....	19.19
Mars.....	1.52		
.....	.....		

In 1772, Johann Ehlert Bode (1747-1826) had noticed that this sequence of numbers follows approximately a certain law. Write a series of 4's; to the second add 3; to the third add $3 \times 2 = 6$; to the fourth add $6 \times 2 = 12$; and so on, doubling the number each time. Then point off one decimal place. Thus:

4	4	4	4	4	4	4	4	4
	3	6	12	24	48	96	192	384
.4	.7	1.0	1.6	2.8	5.2	10.0	19.6	38.8

These numbers are seen to agree approximately with those of the preceding table, except for an apparent gap at 2.8. Bode suggested that some day another planet might be found at the distance 2.8.

(*a*) What kind of reasoning was this? (In 1801, a Sicilian astronomer located a small planet, which he called Ceres, 2.77 times as far away from the sun as the earth.)

(*b*) When Neptune was discovered in 1846 at a distance from the sun not 38.8 but 30.07 times the distance of the earth from the sun, did this fact invalidate Bode's inference?

14. Why does ice float? What kind of reasoning is involved in the answer?

15. Can we assume that natural laws are uniform everywhere in the universe?

16. A certain man with his wife did usually drink out a vessel of beer in 12 days, and the husband found by frequent experience that, his wife being absent, he drank it out in 20 days. The question is, In how many days could the wife alone drink it out?*

17. Through the following nine points draw four straight lines without removing the pencil from the paper and without retracing any part of any line.

* From *A Plain and Familiar Method for Attaining the Knowledge and Practice of Common Arithmetic*, by Edmund Wingate, published in London in 1751 A. D.

2

SYMBOL TECHNIQUE— ADDITION AND SUBTRACTION

If all the factories and equipment in this country were destroyed by some cataclysm, but our records and books were left intact, within a score of years everything could be rebuilt,

Courtesy of Librairie Ancienne, Leo S. Olschki, Florence

The woodcut is from *Gregorius Reisch Margarita Philosophica*, Basileae, 1517.

even better than before. But if the formulas, tables, and mathematical knowledge were eradicated, our civilization would perish and possibly never be reconstructed.

FORMULA

Man is the only animal who leaves records of his thoughts. We are spared the need of taking the same laborious steps which our forebears took in acquiring knowledge. Each generation has a more powerful springboard from which to dive into the ever-widening ocean of mystery that surrounds all science. There is literally no economy so great in the acquisition of new knowledge of the real world as the expression of known laws in mathematical form. Without such expression, we should be condemned to repeat over and over again a lot of wasted labor. For example, Mr. Babbage wants to know how many gallons of water will be contained in a cylindrical tank, 6 feet high and 4 feet in diameter, on the roof of his barn. One way to find out would be to empty successively several gallon cans of water into the tank, until it was filled, and keep careful count meanwhile. This rule-of-thumb method is laborious and stupid. One could ascertain in this way that the above-mentioned cylinder would hold 564 gallons of water. This empirical information, however, would be of little use in estimating the volume of another cylinder of different dimensions. But if we have a *general* statement giving the relationship between the dimensions of a cylinder and its volume, and if we know also the volume of a gallon of water, we can ascertain the cubic contents of any cylinder.

The expression in symbols of a rule of computation is called a *formula.*

Thus, the rule, *to find the volume of a circular cylinder, multiply the area of the base by the altitude,* can be expressed by letters as follows:

Let d = diameter of cylinder.
r = radius of cylinder.
h = height of cylinder, or altitude.
V = volume of cylinder.

Then, $V = \pi r^2 h = \frac{1}{4}\pi d^2 h.$

This is the formula for the volume of a circular cylinder (that is, *any* circular cylinder). For the meaning of π, see page 362.

The use of letters in mathematics to designate numbers has the following great advantages:

1. Letters *generalize* numerical relationships.

2. Letters enable us to distinguish different meanings and to preserve these distinctions throughout a discussion.

For example: "George gave Henry 5 apples. He paid him 5 cents." We assume that it was Henry who paid the money, but we may be wrong. If, however, we say:

G gave 5 apples to H, and
H gave 5 cents to G,

there is no possible confusion regarding the meaning of *he* and *him*.

3. Letters exhibit clearly and concisely the *form* of a proposition.

As will be explained later, *any* equation of the form $ax = by$, where a and b may be any real numbers, is called a *linear* equation and can be expressed graphically by a straight line.

4. Adequate symbols make clear just what is constant or invariable in a proposition and what is variable. If we tried to put into words the fundamental truths of electricity, we should require many pages, and the essential relations between the different factors involved would be hidden. In symbols, however, these fundamental laws can be stated in less than half a page.

5. A fifth advantage of symbols is their labor-saving and thought-saving function. With them much that otherwise would require concentrated attention can be performed almost mechanically. Very often a good symbolism may suggest conclusions which otherwise would escape the thinker completely. If Professor Pupin, for example, had been confined to the old Roman method of writing numbers, he never could have devised, mathematically at first and then with apparatus, a method of constructing the relay coils that enable us to telephone from New York to London.

EQUATION

An equation is a mathematical statement of equality between two quantities. That is, an equation is a sentence expressed in mathematical symbols. Just as a sentence is a means of stating a thought, so an equation is a means of expressing a mathematical relation. Thus, the sentence: "The cost of a concrete road was $750,000; the county paid twice as much as the state: the state paid three times as much as the

federal government," may be expressed in mathematical symbolism as follows:

Mathematically		*In English*
x	=	The amount paid by the federal government.
$3x$	=	The amount paid by the state.
$6x$	=	The amount paid by the county.
$10x$	=	\$750,000, the cost of the concrete road.

We are thus told succinctly that the total cost of the road is ten times the amount contributed by the federal government.

The expression $10x$ is the *left* side of the above equation, and \$750,000 is the *right* side.

In an equation (of the sort given above), an expression which, while not divided into parts by a + or a − sign, is separated from the other parts by such a sign, is called a *term*. Thus, $3x$, y, and 7 are terms of the equation $3x - y = 7$.

A *factor* is any one of two or more whole numbers which are multiplied together to form a product. Thus, 2 and 3 are factors of 6 because $2 \times 3 = 6$. The word *factor* has now been given an extension of meaning. For example, in the expression $3x$, 3 and x are called *factors*, even though x may not denote an integral number.

In a term such as $3x$, 3 is called the *numerical coefficient*, or simply the *coefficient* of x. Evidently, $1 \times y = y$. Hence, whenever a term appears without a coefficient, it is implied that 1 is its coefficient. Thus, xy implies $1xy$ and means $1 \times x \times y$.

The notation $5 \times 5 \times 5$ is somewhat clumsy. A short way of writing $5 \times 5 \times 5$ is 5^3. The small number 3 written above and to the right of 5 is called an *exponent*. An exponent is a number written to the right and above another number, called the *base*, to indicate how many times the base is to be taken as a factor. It is evident that $5^1 = 5$. In general, whenever a number or quantity appears without an exponent, that number or quantity, by implication, is raised to the first power.

Thus,

$$a^1 = a, \text{ and } (1.06)^1 = 1.06.$$

Thus,

$$3b^3 = 3 \times b \times b. \quad \text{And if } b = 5,$$

then

$$3b^2 = 3 \times 5 \times 5 = 75.$$

The *degree of a term* is the sum of the exponents of its literal (or letter) factors. Thus, $3x^2$ is a term of the second degree; $115x^2y$ is a term of the third degree; $2x^3yz$ is a term of the fifth degree.

The *degree of an equation* is (by definition) the degree of the term of highest degree in the expression. Thus, $5x^3 + 7x - 2 = 0$ is an equation of the third degree.

Unfortunately, the degree of an equation containing more than one letter may be spoken of in different ways.

Thus, $x^2y^2 = 6x^3 - 2xy + 3$ is an equation of the third degree in x. It is an equation of the second degree in y. It is an equation of the fourth degree in x and y together.

The equation $xy = 5$ is an equation of the first degree in x. It is an equation of the first degree in y. It is an equation of the second degree in x and y together.

In general, in any equation where more than one variable occurs, the degree of the equation is usually taken to mean the degree of the equation in x AND y, and NOT the degree of the equation in x OR y. We shall be careful to avoid ambiguity on this point in subsequent pages.

ADDITION OF SYMBOLS

It is assumed that the reader knows how to perform in arithmetic the four fundamental operations: addition, subtraction, multiplication, and division.

It is a commonplace to say that addition of unlike objects is impossible. Thus, if we put in a basket 5 apples and 3 bananas, we cannot say that it contains 8 apples or 8 bananas. If, however, each object in the basket is called a gift, we can say that the basket contains 8 gifts. Thus, in symbols we can write $5g + 3g = 8g$. But if the apples are designated by the symbol a and the bananas by b, the putting together of these two kinds of fruit, in the amounts given, can only be indicated as $5a + 3b$.

Terms which contain the same literal factors raised to the same powers are called *like* (or *similar*) *terms*. That is, $5d$ and $3d$ are similar terms. Moreover, $5d$ and $-3d$ are like (or similar) terms. Also, $7a^3b$ and $4a^3b$. Likewise, $-7a^3b$ and

$4a^3b$. Also $-7a^3b$ and $-4a^3b$. In other words, *like* (or *similar*), in this case, does not refer to the sign of the coefficient of a term.

The addition of like (or similar) terms is effected by adding the numerical coefficients, taking account of the signs, and prefixing this sum to the literal factors.

Thus,

$$-3e^2k + 7e^2k = 4e^2k,$$

and

$$2m^3n^4 + 8m^3n^4 = 10m^3n^4.$$

The addition of unlike or dissimilar terms can only be indicated. Thus, $2xy^2 + 6xy^2 = 8xy^2$. But, if $9xy^2$ is to be added to $9xy$, the operation can only be indicated and written $9xy^2 + 9xy$.

As already stated, to add symbols or letters means to find the sum, taking account of the signs. The coefficients of the terms may be either positive or negative numbers; hence, the addition of these terms follows the rules of addition for signed numbers. Thus, just as -3 added to $+5$ means $5 + (-3) = 5 - 3 = 2$, so $9d + (-3d) = 9d - 3d = 6d$.

Hence, if a series of terms is to be added, arrange the terms in columns, placing similar terms in the same column. Then add the numerical coefficients of each column and prefix this sum to the literal factors of the terms in the column.

Thus, to add the series of terms $5a^2b + 2ab - 3ab^2 - 4ab + 7a^2b + ab^2 - 6a^2b - 8ab^2 + 9ab$, arrange them as follows:

$$\begin{array}{rr} & 5a^2b + 2ab - 3ab^2 \\ & 7a^2b - 4ab + ab^2 \\ & -6a^2b + 9ab - 8ab^2 \\ \hline \text{Sum,} & 6a^2b + 7ab - 10ab^2. \end{array}$$

To check this result, let a be any number, say 2, and b any number, say 1. Then find the numerical value of each row, and compare the sum of these results with the numerical value of the expression representing the sum.

SUBTRACTION OF SYMBOLS

The rules for subtraction are fundamentally the same as those for addition. Subtraction of like (or similar) terms is effected by subtracting the numerical coefficients, taking

account of the signs, and prefixing this difference to the literal factors. Thus, since -6 subtracted from -2 means $-2 - (-6) = -2 + 6 = 4$, then $-2d - (-6d) = -2d + 6d = 4d$.

The subtraction of unlike (or dissimilar) terms can only be indicated. Thus, if $5a^3b$ is to be subtracted from $6ab^3$, the operation can only be indicated and written $6ab^3 - 5a^3b$.

Hence, to subtract one series of terms from another, arrange the terms in columns, placing similar terms in the same column. Change the signs of *all* the terms in the subtrahend. Then add the numerical coefficients of each column and prefix this sum to the literal factors of the terms in the column. Thus, to subtract $2x^3y + 5xy^2 - 3y - x + 15xy + 9$ from $10 + 7x + 6xy^2 - y + 4x^3y - 15xy$, rearrange the terms as follows, change the signs of all the terms of the subtrahend, and then add:

$$\begin{array}{r} 4x^3y + 6xy^2 - y + 7x - 15xy + 10 \\ -2x^3y - 5xy^2 + 3y + x - 15xy - 9 \\ \hline 2x^3y + xy^2 + 2y + 8x - 30xy + 1. \end{array}$$

The result may be checked by substituting any numbers for x and y and comparing the result of the difference of the numerical value of the rows with the numerical value of the literal expression representing the difference.

PARENTHESES

Computations may be simplified by indicating all the arithmetic operations before performing them. In this manner, operations which nullify or cancel each other are readily recognized. As already stated, in a sequence indicating arithmetical operations, common practice demands that before any additions or subtractions are effected, *all multiplications and divisions must be performed in the order in which they appear from left to right.*

Thus

$$11 + 15 \div 5 - 6 \times 3 + 12 \div 2$$

is equal to

$$11 + 3 - 18 + 6 = 2.$$

To avoid any doubt in the grouping of terms, the parenthesis is used. Terms which are to be treated as a group are put

together within parentheses. *Brackets* [ ] and *braces* { } and a bar (called a *vinculum*) over an expression are used in the same manner. Thus, the above sequence becomes much clearer when written:

$$11 + (15 \div 5) - (6 \times 3) + (12 \div 2) = 2.$$

The insertion of parentheses in different ways changes the meaning of an indicated operation. Thus:

$$18 + (12 \div 2) - (3 \times 2) + (20 \div 5) = 22,$$

while

$$(18 + 12) \div 2 - 3(2 + 20) \div 5 = 1\tfrac{4}{5}.$$

The latter sequence could have been written as follows without changing its meaning:

$$(18 + 12) \div 2 - [3 \times (2 + 20) \div 5] = 1\tfrac{4}{5}.$$

The expression $(5x - 7) + (3x - 9)$ means that $(3x - 9)$ is to be added to $(5x - 7)$. As already explained, the addition is easy to perform if we keep in mind that the letter (in this case, x) is used to represent some quantity. That is, x may represent dollars, miles per hour, or feet. The case is exactly the same as that of adding numbers representing concrete quantities. Thus, to add 2 hours and 10 minutes to 3 hours and 15 minutes, the similar terms are grouped together and then added. Hours are added to hours and minutes to minutes, as follows:

3 hours	15 minutes
2 hours	10 minutes
5 hours	25 minutes.

In a similar manner, we add:

$$\begin{array}{r} 5x - 7 \\ 3x - 9 \\ \hline 8x - 16. \end{array}$$

Note that the same result is obtained if the parentheses are removed and the indicated operations are performed.

That is,

$$\begin{aligned}(5x - 7) + (3x - 9) &= 5x - 7 + 3x - 9 \\ &= 5x + 3x - 7 - 9 = 8x - 16.\end{aligned}$$

The expression $(5x - 7) - (3x - 9)$ indicates that $(3x - 9)$ is to be subtracted from $(5x - 7)$. By the rule for the subtraction of signed numbers, the operation may be changed to addition by merely changing the signs of the subtrahend. So that:

$$\begin{aligned}(5x - 7) - (3x - 9) &= (5x - 7) + (-3x + 9)\\ &= 5x - 7 - 3x + 9 = 5x - 3x - 7 + 9\\ &= 2x + 2.\end{aligned}$$

These illustrations suggest the following rules: *When an expression within parentheses is preceded by the sign +, the parentheses may be removed without making any change in the expression.*

Conversely, *any part of an expression may be enclosed within parentheses and the sign + prefixed, the sign of every term within the parentheses remaining unaltered.*

When an expression within parentheses is preceded by the sign −, the parentheses may be removed if the sign of every term within the parentheses is changed.

Conversely, *any part of an expression may be enclosed within parentheses and the sign − prefixed, provided that the sign of every term within the parentheses is changed.*

Where there are parentheses and brackets inside braces, or vice versa, and it is desired to simplify the expression, it is usually best to begin with the inside pair, remove it from the parentheses, and deal with each pair in succession, following the two rules just given above.

Thus, to simplify the expression:

$$\begin{aligned}&-[-2x - \{6y - (3x - 3y) + (3x - 2y)\} + 2x]\\ &= -[-2x - \{6y - 3x + 3y + 3x - 2y\} + 2x]\\ &= -[-2x - 6y + 3x - 3y - 3x + 2y + 2x]\\ &= 2x + 6y - 3x + 3y + 3x - 2y - 2x\\ &= 7y. \quad (Answer)\end{aligned}$$

So far, there has been nothing puzzling about the use of symbols. They can be added and subtracted in just the same way as the numbers which they represent. What does puzzle a beginner, however, is to express the conditions of a question or problem by means of symbols. For example, some readers might be uncertain how to carry out the following: "Find a number greater than y by a." If the question, however, were,

"Find a number greater than 49 by 6," the answer would be at once seen to mean addition, namely, $49 + 6 = 55$. Thus, the number greater than y by a is $y + a$.

Another example: If x is one part of 97, what is the other part? Evidently, $97 - x$.

By how much does z exceed 17? If this question at first seems vague, take a similar one with a number instead of the letter: "By how much does 29 exceed 17?" The answer obviously is 12, which is equal to $29 - 17$. Hence, the excess of z over 17 is $z - 17$.

Readers are urged to practise on the following queries. Most of them are easy. There is one rather difficult one. Which is it? Why is it more difficult than the others?

1. What must be added to x to make y?

2. If a is one factor of b, what is the other?

3. If the difference of two numbers be 11, and if the smaller be y, what is the greater?

4. In a government bureau, x employees are working at corporation tax reports, y at personal tax reports, and the rest are idle. What is the excess of workers over idlers?

5. An airplane travels x miles per hour. How long does it take for it to go from New York to San Francisco, a distance of 3,000 miles?

6. In x years a building will be 50 years old. How old is it now?

7. If x men take 5 days to paint a wall, how long will one man take?

8. How far can I drive in x hours at the rate of p miles per hour?

9. By how much does $3x - 7$ exceed $x + 1$?

10. What number must be taken from $a - 2b$ to leave $a - 3b$?

11. If a debt is shared equally among x persons, and each pays \$10.75, how many cents does the debt amount to?

12. In how many weeks will x horses eat 150 bushels of oats, if one horse eats y bushels a week?

13. By what number must 657 be divided to give a quotient of 13 and a remainder of 20?

14. If the divisor is x, the quotient y, and the remainder z, what is the dividend?

15. What is the length of an edge of a cube which contains as many cubic inches as there are square inches in its surface?

16. The total height of the Congressional Library is 64 feet. The second story is 7 feet less than twice the height of the first story. The third story is 1 foot in excess of the height of the first story. What is the height of each story?

17. There are two consecutive odd integers whose sum is 188. What are they?

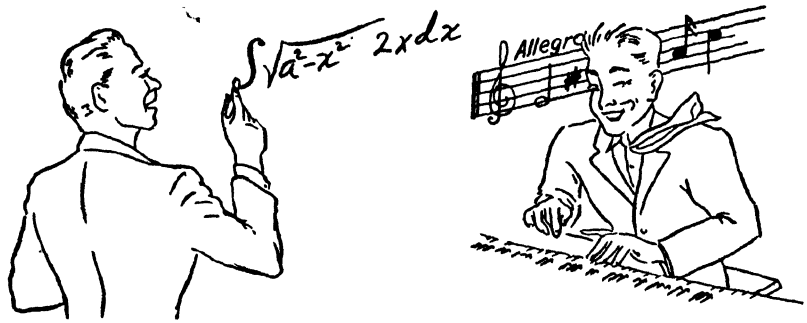
18. The sum of certain three consecutive integers is 216. What are they?

19. A chemist has 77.6 grams of silver nitrate solution. He desires to separate the liquid into two parts such that the larger exceeds the smaller by 19.4 grams. How many grams must each part contain?

20. A board 19 feet long is to be cut into two pieces such that one will be $1\frac{3}{4}$ feet longer than the other. What is the length of each piece?

21. A brass pipe 126 inches long is to be sawed into three lengths, such that the second is 14 inches less than the first and the third exceeds twice the first by 8 inches. What are the three lengths?

3 SYMBOL TECHNIQUE CONTINUED—MULTIPLICATION AND DIVISION



Long practice at scales and arpeggios is required before the player of a musical instrument can learn the best technique for rendering some beautiful fugue or sonata. Similarly in mathematics many conventions, signs, and symbols must be learned and considerable practice in manipulating them must be had before exploring the many mansions of this subject.

From the definition of *exponent* on page 20, it is easily seen that any number, a, taken as a factor four times is written a^4. And, in general, any number, a, taken as a factor any number, n, times, where n is a positive integer, is written a^n.

The expression a^n indicates the result of an operation, namely, that a has been taken as a factor n times. The result of this operation is called the nth power of a. Thus, if $a^n = b$, then b is the nth power of a. However, it is common practice to speak of the expression a^n also as the "nth power of a."

The expression $5^3 \times 5^2$ means: "5 taken as a factor three times multiplied by 5 taken as a factor two times." That is, $5^3 \times 5^2 = (5 \times 5 \times 5) \times (5 \times 5) = 5 \times 5 \times 5 \times 5 \times 5$.

The figures just given indicate that 5 taken as a factor five times may be written 5^5; therefore, $5^3 \times 5^2 = 5^5$.

Notice, in this case, that if the exponents 3 and 2 are added, the result, 5, is the new exponent.

Similarly, $a^3 \times a^2$, where a represents any number, is equal to $(a \times a \times a) \cdot (a \times a) = a \times a \times a \times a \times a$, and may be written a^5. Hence, $a^3 \times a^2 = a^5$. Again, notice that if the exponents 3 and 2 are added, the result, 5, is the new exponent. The numerical value of the base (see page 20), therefore, plays no part in the number of times it is taken as a factor.

In general, $a^m \times a^n$, when a represents any number and m and n any positive whole numbers whatever, may be written

$$a^m \times a^n = \underbrace{(a \times a \times a \times \ldots\ldots \times a)}_{m \text{ times}} \times \underbrace{(a \times a \times \ldots\ldots \times a)}_{n \text{ times}}.$$

This notation indicates that a is taken as a factor $m + n$ times and may be written a^{m+n}; hence, $a^m \times a^n = a^{m+n}$. Therefore, the first law of exponents is:

*The exponent of the product of two powers whose exponents are positive whole numbers and whose bases are the same is the sum of the original exponents.**

This rule may be extended to apply to the product of any number of powers, for:

$$a^5 \times a^3 \times a^6 \times a^2 \times a^7 = (a^5 \times a^3)(a^6 \times a^2)\, a^7$$
$$= a^8 \times a^8 \times a^7 = (a^8 \times a^8)a^7 = a^{16} \times a^7 = a^{23} = a^{5+3+6+2+7}.$$

The expression $(4 \times 3)^3$ means that the product (4×3) is taken as a factor three times. It is readily seen that:

$$(4 \times 3)^3 = (4 \times 3)(4 \times 3)(4 \times 3) = (4 \times 4 \times 4)(3 \times 3 \times 3) = 4^3 \times 3^3.$$

In general, if a and b are any numbers and m any positive whole number,

$$(a \times b)^m = \underbrace{(a \times b)(a \times b) \ldots\ldots (a \times b)}_{m \text{ times}}.$$

The above expression may be written as on the next page.

* Note that, in applying this or any other rule of exponents, *the bases must be the same.* For it is obvious, if the meaning of an exponent is kept in mind, that $4^3 \times 5^3$ does not equal 4^5 or 5^5, because the real situation is as follows:

$$4^5 = 4 \times 4 \times 4 \times 4 \times 4 = 1024$$
$$5^5 = 5 \times 5 \times 5 \times 5 \times 5 = 3125$$
$$4^3 \times 5^3 = 4 \times 4 \times 4 \times 5 \times 5 = 1600.$$

$$(a \times b)^m = \underbrace{(a \times a \times a \times \ldots\ldots \times a)}_{m \text{ times}} \underbrace{(b \times b \times b \times \ldots\ldots \times b)}_{m \text{ times}}.$$

That is, a appears as a factor m times and b appears as a factor m times also, so that the results of these operations may be indicated by a^m and b^m. Hence, $(a \times b)^m = a^m \cdot b^m$. This second law of exponents may be stated as follows:

To raise a product to a positive integral power, raise each factor to that power.

The expression $(4^3)^2$ is read: "4 is taken as a factor three times, and the result is then taken as a factor two times." Thus:

$$(4^3)^2 = (4 \times 4 \times 4)^2 = (4 \times 4 \times 4)(4 \times 4 \times 4)$$
$$= 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6.$$

If the exponents are multiplied together, namely, 3×2, the result, 6, is the new exponent.

Similarly, $(a^m)^n$, where a is any number and m and n are any positive whole numbers whatever, may be written:

$$(a^m)^n = \underbrace{(a \times a \times \ldots\ldots \times a)^n}_{m \text{ times}}$$

$$= \overbrace{\underbrace{(a \times a \times \ldots\ldots \times a)}_{m \text{ times}} \ldots\ldots \underbrace{(a \times a \times \ldots \times a)}_{m \text{ times}}}^{n \text{ factors}}.$$

The right-hand side of the above notation indicates that there are n factors with m number of a's in each factor, so that there are mn number of a's in the whole expression. That is, a is taken as a factor mn times, and hence may be written a^{mn}. Therefore, $(a^m)^n = a^{mn}$, and the third law of exponents is:

The exponent of the result of raising to a positive integral power a positive integral power of a quantity is the product of the exponents.

The expression $4^5 \div 4^3$ could be read aloud as follows: "4 taken as a factor five times is divided by 4 taken as a factor three times." The same expression could be written as:

$$\frac{4^5}{4^3} = \frac{4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4}.$$

Cancelling the 4's in both numerator and denominator gives

$$\frac{\overset{1}{4}\times\overset{1}{4}\times\overset{1}{4}\times\overset{1}{4}\times\overset{1}{4}}{\underset{1}{4}\times\underset{1}{4}\times\underset{1}{4}} = \frac{4\times 4}{1} = 4^2.$$

If, however, instead of performing this unnecessary labor, we subtract the exponent of the denominator from the exponent of the numerator, we have $4^{5-3} = 4^2$, obtaining the same result more quickly.

By similar reasoning, $a^m \div a^n = a^{m-n}$, where a is any number, m and n are any positive whole numbers, and m is greater than n (in symbols, $m > n$).

So that a fourth law of exponents can now be stated as follows:

If the ratio of two powers having positive integral exponents and the same bases is such that the exponent of the numerator is greater than the exponent of the denominator, the exponent of the quotient is found by subtracting the exponent of the divisor from that of the dividend.

The operation $4 \div 4$ can be written $\frac{4}{4}$ or $\frac{4^1}{4^1}$. Applying the fourth law of exponents, just stated,

$$\frac{4^1}{4^1} = 4^{1-1} = 4^0.$$

But

$$\frac{4^1}{4^1} = \frac{4}{4} = 1.$$

Therefore, we define 4^0 as equal to 1. Similarly, $a^m \div a^m$ can be written $\frac{a^m}{a^m} = a^{m-m} = a^0$. But evidently, $\frac{a^m}{a^m} = 1$. Therefore, a^0 is defined as equal to 1.

That is, *any* quantity raised to the zero power equals 1.* For example: $(96 \div 6)^0 = 1$, for if the indicated division be performed, we have $16^0 = 16^{1-1} = \frac{16}{16} = 1$, and similarly for any quantity, however complex the expression.

* The previous discussion does not apply when $a = 0$, that is, for $0^m \div 0^m$, since division by zero is excluded. However, many mathematicians define 0^0 as 1, since this convention introduces more simplifications than difficulties.

The fourth law of exponents, that is, $a^m \div a^n = a^{m-n}$, where m is greater than n (that is, $m > n$), may be extended as follows:

Consider m smaller than n (that is, $m < n$), as in $4^3 \div 4^5$. Then:

$$\frac{4^3}{4^5} = \frac{4 \times 4 \times 4}{4 \times 4 \times 4 \times 4 \times 4} = \frac{1}{4 \times 4} = \frac{1}{4^2}.$$

But applying the fourth law of exponents,

$$4^3 - 4^5 = 4^{3-5} = 4^{-2}.$$

That is,

$$4^{-2} = \frac{1}{4^2}.$$

In general,

$$a^{-m} = \frac{1}{a^m}.$$

So that *any factor may be changed from the numerator to the denominator, or vice versa, by changing the sign of its exponent.* In other words, any number raised to a negative power is equal to the reciprocal* of that number to the corresponding positive power.

Thus,

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25}, \text{ and } \frac{1}{(P+i)^{-6}} = (P+i)^6.$$

If the fraction $\frac{2}{3}$ is to be raised to the third power, the operation would be indicated as follows: $(\frac{2}{3})^3$. But

$$\left(\frac{2}{3}\right)^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2}{3 \times 3 \times 3} = \frac{2^3}{3^3}.$$

Similarly, $\left(\frac{a}{b}\right)^m$, where a and b are any numbers and m is any positive whole number, can be shown to be equal to $\frac{a^m}{b^m}$. So that a fifth law of exponents may be stated as follows:

* The reciprocal of a number (except zero) is the quantity obtained by dividing 1 by that number. Thus, the reciprocal of 4 is $\frac{1}{4}$.

The positive integral power of a quotient is equal to the quotient of the powers.

A *monomial* is an expression consisting of only one term. Thus, $-3mpq$ and $5(a)^{-n}$ are two monomials.

In multiplying monomials, the numerical coefficient of the product is equal to the product of the numerical coefficients of the factors. The letters in the product are the same as the letters in the factors, each letter in the product having for its exponent the sum of its exponents in the various factors.

Thus: $-2b^2fg \times 4f^2g^3s^2 = -8b^2f^3g^4s^2$.

Etymology justifies the notion that a *polynomial** is an expression of two or more terms, such as:

$$3a^2b + 5ab - 4ab^2$$

or

$$6x^3 - 7x^2 + 3x - 5.$$

To multiply a polynomial by a monomial, multiply successively each term of the polynomial by the monomial, following the rules just given.

ILLUSTRATION:

$$\begin{array}{r} 3a^3b^2 + 5ab - 2b^2c - 4c^2 \\ 2ab^2 \\ \hline 6a^4b^4 + 10a^2b^3 - 4ab^4c - 8ab^2c^2 \end{array}$$

Before discussing the multiplication of one polynomial by another, consider how multiplication of *numbers* is tremendously simplified by the fact that digits have place value. If this principle of place value were revoked and the present-day symbols still used, the number 57 might then be written $50 + 7$. Suppose now that, under this condition, the multiplication of 57 by 14 were to be performed. A different technique would then have to be introduced. The two methods can be compared as on the next page.

* Modern mathematicians have found that restricted definitions excluding important special cases become a source of trouble. For this reason a polynomial in x of degree n is defined as of the form:

$$c_1x^n + c_2x^{n-1} + \ldots + c_{n-1}x + c_n,$$

where the c's are any constants and n is any positive integer or zero. Note that a polynomial of 2 terms is called a *binomial*, and that a polynomial of 3 terms is called a *trinomial*.

USUAL METHOD:

$$\begin{array}{l} 57 \\ \underline{14} \\ 228 \text{ (multiplication of 57 by 4)} \\ \underline{57} \text{ (multiplication of 57 by 1)} \\ 798 \text{ (sum of products)} \end{array}$$

SECOND METHOD:

$$\begin{array}{l} 50 + 7 \\ \underline{10 + 4} \\ 200 + 20 + 8 \text{ (multiplication of } 50 + 7 \text{ by 4)} \\ \underline{500 + 70} \text{ (multiplication of } 50 + 7 \text{ by 10)} \\ 700 + 90 + 8 = 798 \text{ (sum of products)} \end{array}$$

When letters are used to designate numbers, the short, usual method of arithmetic cannot be used. Letters do not have place value. Thus,

$$\begin{array}{l} \qquad 6x - 2y \\ \qquad \underline{3x + 5y} \\ \qquad 30xy - 10y^2 \text{ (multiplication of } 6x - 2y \text{ by } 5y) \\ \underline{18x^2 - 6xy} \text{ (multiplication of } 6x - 2y \text{ by } 3x) \\ 18x^2 + 24xy - 10y^2 \text{ (sum of the products)} \end{array}$$

Thus, to multiply a polynomial by a polynomial, multiply the multiplicand successively by each term of the multiplier, and add the partial products.

It is customary (but not imperative) to follow the neat practice of arranging a polynomial according to the powers of some letter, so that the exponents of that letter either descend or ascend in order of magnitude.

Note that the product of several quantities is not altered by changing the order of the factors. That is, we can start multiplying at the right or at the left.

ILLUSTRATION:

$$\begin{array}{l} 2x^2 - 5xy + 3y^2 \\ \underline{6x^2 + 7xy - 4y^2} \\ 12x^4 - 30x^3y + 18x^2y^2 \\ \qquad 14x^3y - 35x^2y^2 + 21xy^3 \\ \qquad\qquad - 8x^2y^2 + 20xy^3 - 12y^4 \\ \hline 12x^4 - 16x^3y - 25x^2y^2 + 41xy^3 - 12y^4 \end{array}$$

There are three special cases of multiplication of one binomial by another which are so frequently found in practical problems that they deserve mention here.

I. $$(a + b)(a + b) = (a + b)^2 = a^2 + 2ab + b^2.$$

That is, *the square of the sum of two terms is equal to the square of the first term, plus twice the product of the two terms, plus the square of the second term.*

ILLUSTRATION: $(x + 5)^2 = x^2 + 2(x)(5) + 25 = x^2 + 10x + 25.$

This fact can be used in mental arithmetic. Thus, if we wish to square 57 mentally, we can picture the operation as follows:

$$\begin{aligned}(57)^2 = (50 + 7)^2 &= (50)^2 + 2 \times 7 \times 50 + (7)^2 \\ &= 2500 + \quad 700 \quad + 49 \\ &= 3249.\end{aligned}$$

II. $$(a - b)(a - b) = (a - b)^2 = a^2 - 2ab + b^2.$$

That is, *the square of the difference of two terms is equal to the square of the first, minus twice the product of the two terms, plus the square of the second term.*

ILLUSTRATION: $(x - 3)^2 = x^2 - (2)(x)(3) + 9 = x^2 - 6x + 9.$

Note that in reality this second rule is really unnecessary, for it is covered by the first rule. Since a and b are symbols of any real numbers, either a or b or both of them might be negative. Rule 2, therefore, is stated merely for the convenience of the reader.

III. $$(a + b)(a - b) = a^2 - b^2.$$

That is, *the product of the sum of two given terms by their difference is equal to the square of the first term minus the square of the second term.*

ILLUSTRATIONS: $(7 + 2)(7 - 2) = 7^2 - 2^2 = 49 - 4 = 45.$
$(x + 4)(x - 4) = x^2 - 16.$

Division, as well as multiplication, of *numbers* is greatly simplified by the fact that digits have place value.

If the principle of place value were revoked, division would have to be performed by a more cumbersome method. Consider $378 \div 13$.

Usual Method	Second Method
29	$30 - 2 + 1$
$13\overline{)378}$	$10 + 3\overline{)300 + 70 + 8}$
26	$300 + 90$
118	$-20 + 8$
117	$-20 - 6$
1	$10 + 4$
	$10 + 3$
	1

The second method would be explained in words as follows:

The first number of the dividend 300 is divided by 10, the first term of the divisor, giving 30, the first number in the quotient.

The divisor, $10 + 3$, is then multiplied by 30, the first number of the quotient, giving the product $300 + 90$. This product is placed under the corresponding numbers of the dividend and subtracted. The difference, -20, and the next number of the dividend, 8, which is brought down, form the new dividend. The process is then repeated until the remainder is less than the first number of the divisor or until there is no remainder.

The quotient is $30 - 2 + 1 = 29$, and the remainder is $4 - 3 = 1$.

Since, as stated before, letters representing quantities have no place value, the division of two literal expressions must be performed by the second method just described.

Consider the operation: $(8x^3 - 6x^2 + 3x + 5) \div (4x + 3)$.

$$
\begin{array}{r}
2x^2 - 3x + 3 \\
\hline
4x + 3\overline{)8x^3 - 6x^2 + 3x + 5} \\
8x^3 + 6x^2 \\
\hline
-12x^2 + 3x \\
-12x^2 - 9x \\
\hline
12x + 5 \\
12x + 9 \\
\hline
-4
\end{array}
$$

The above division operation would be explained in words as follows:

Divide the first term of the dividend, $8x^3$, by the first term of the divisor, $4x$. The result, $2x^2$, is the first term of the quotient.

The divisor, $4x + 3$, is then multiplied by $2x^2$, the first term of the quotient, giving the product $8x^3 + 6x^2$. This product is placed under the corresponding terms of the dividend and subtracted.

The difference, $-12x^2$, and the next term of the dividend, $3x$, which is brought down, form the new dividend. The process is then repeated until the remainder is less in degree than the first term of the divisor or there is no remainder.

The quotient is $2x^2 - 3x + 3$, and the remainder is -4.

The manipulation of symbols, as they have been described thus far, is a very simple matter if one is meticulously careful to follow the rules. What does test one's ingenuity and inventiveness sometimes is to apply symbols to situations in real life where some practical solution is called for. Truly, "mathematics maketh an ingenious man."

Like chess, mathematics should develop accuracy of observation and thought. What is important always is the logical meaning behind symbols. Occasionally a somewhat absurd problem will be given in this book just for the purpose of reminding the reader that symbolism does not justify a failure to use common sense.

Symbol Practice

1. Express x yards, y feet, and 8 inches in feet.

2. An accountant earns \$68 a week. His living expenses average E dollars a day. How much can he save in 4 weeks?

3. A servant buys k pounds of butter at n cents a pound. How many cents change does she get from a five-dollar bill?

4. A reporter travelled r miles by auto, s miles by boat, t miles by airplane, and the remainder by train. How many miles did he travel by train, if the whole journey was 895 miles?

5. Express 7 consecutive even numbers, if the middle one is $2n$.

6. A square piece of copper has an edge y inches long. From each corner a little square 2 inches wide is cut out. Find the area of the remaining copper.

7. A rectangle is 17 feet long and 8 feet wide. If its length is increased by x feet and its width by y feet, what is its new area in square feet?

8. The four sides of a square plot total s linear feet. What is the area of the square?

9. If $A = \pi r^2$, and $\pi = 3.14159$ and $r = 6.2$, what is the value of A?

10. If $H = a + vt - 16t^2$, find H when $a = 1{,}000$, $v = 50$, and $t = 7$.

11. If $s = \frac{1}{2}gt^2$, find s, when $g = 32$ and $t = 1.5$.

12. Perform the following multiplications and check the correctness of your answers:

(*a*) $x^3 - 3x^2 + 5x - 1$ by $x - 2$.
(*b*) $25 + x^4 - 6x^2$ by $x^2 + 5$.
(*c*) Simplify: $(x + y)(x^2 - y^2) - xy\,(x - y)$.
(*d*) $(x + 3)(x - 4)(2x - 3)$.
(*e*) $a^4 + 2a^2b^2 - b^4$ by $a^4 - 2a^2b^2 - b^4$. Check by letting $a = 1$ and $b = 2$.
(*f*) Prove that $a - 3$ is a factor of $a^3 - 3a - 18$.

13. Perform the following divisions and check the correctness of your answers:

(*a*) $(a^3 + 3a^2x + 3ax^2 + 4x^3) \div (a^2 + x^2 + 2ax)$.
(*b*) $(9x^2 + 9x - 40) \div (3x - 5)$.
(*c*) $(6m^2 + 19m - 7) \div (3m - 1)$.
(*d*) $(r^3 + 8) \div (r + 2)$.
(*e*) Find the dividend if the divisor is $y - 2$ and the quotient $15y - 7$.
(*f*) What is the divisor if the quotient is $s + 4$, the dividend $s^2 + 6s + 9$, and the remainder 1?
(*g*) What is the value of z which will make $d - 5$ an exact divisor of $3d^2 - 12d - z$?

14. Give instantly the *numerical* value of the following expression:

$$\left[\left(1 - \frac{j}{m}\right)^{\frac{m}{p}}\right]^0.$$

15. Which is greater: a^{r-n} or $\dfrac{a^r}{a^n}$?

16. How much dirt is there in a cylindrical hole two feet in radius and two feet deep?

17. Logic and common sense: A jeweler has six pieces of gold chain, each piece consisting of four links. Assuming that it will cost 5 cents to cut a link and 5 cents to weld a link, what is the lowest cost of making the above 6 pieces into one straight chain?

18. Find the value of the following expression: 2^{2^2}.

19. You ask a friend for a loan. He says that he will lend the following number of dollar bills:

$$(2 + 2 \times 2 - 2 \times 2)^{2-2} - 2 \div 2.$$

What is the size of the loan?

20. A telephone girl, when asked the time of day, replied: "It is between 4 and 5 o'clock, and the hour and minute hands are together." What was the time?

4

BALANCING IDEAS— EQUATIONS AND FACTORING

A businessman once sent the authors a letter containing the following problem, which, he claimed, was impossible of solution:

The general manager of a firm is to receive a bonus of 4 per cent of the net profit, after deducting from this net profit the income taxes of $25,000 and his bonus. The net profit before deducting the bonus or income taxes is $265,000. What is his bonus?

By the use of one symbol, x, this problem can be solved very easily. Let the reader get out his pencil and wrestle with the task. It is more interesting and useful than guessing which horse will win the next race. If the reader solves it, let him see next how awkward the solution would be without using some letter such as x to denote the unknown quantity. The point to be emphasized here is that arithmetic alone is not enough for practical, everyday businessmen, not to speak of engineers, bankers, accountants, and social workers. We are surrounded by mathematical problems every day, but often without realizing the fact. A large bank was equipping its offices with modern lighting fixtures suspended from the ceiling. The man in charge was asked how he had found out the correct distance that the lights should be beneath the ceiling, and answered:

"It makes no difference, as long as they look all right."

However, for a given strength of current and type of electric bulb, there is *one* distance *and only* one distance at which the fixtures would furnish a *maximum* of illumination. Any other distance would entail much waste of money, because the bank would be paying for illumination which it did not receive. It is easy enough to ascertain the one correct distance in this problem—but not by arithmetic. Arithmetic must be *generalized* by symbols in order to pave the way for the solution of a wide range of problems of practical importance.

For example, the symbol r may be used to designate the resistance of a wire to the passage of an electric current. The value of r in a particular case depends, among other things, upon the length of the diameter of the wire. Since there are wires of many sizes, the value of r may vary in different situations. A quantity to which an unlimited number of values can be assigned, in an investigation, is called a *variable.*

TYPES OF EQUATIONS

An equation, as already stated, is a mathematical sentence. Obviously, a sentence should always have a meaning. Hence, it is necessary to be thoroughly versed in the art of translating ordinary English into mathematical shorthand, and also in translating this shorthand into English. The backbone of symbol technique in mathematics is, beyond any doubt, the equation and its applications.

It is important to note that an equation may be either an equation of identity or an equation of condition.

An equation of identity states a fact which is true of an unlimited number of values of the variables in the equation. For example, in the following equation:

$$(x - 2)(x - 3) = x^2 - 5x + 6,$$

the value of x might be 1, 2, or 3, or any other real number.

An equation of condition states that, for some particular value or values of the variables in the equation, the equality expressed is true, but for all other values of the variables, the equality is not true. Thus,

$$x^2 - 5x + 6 = 0$$

raises the question: "What are the values of x such that the square of the number less five times the number plus 6 is zero?" By methods which will later be explained, the particular values of x for which this equality is true are found to be 2 and 3. Thus,

$$(2)^2 - 5(2) + 6 = 0, \text{ and } (3)^2 - 5(3) + 6 = 0.$$

SOLUTION OF AN EQUATION

The expressions on the left and on the right of the equality sign ($=$) are called, respectively, the *left side* and the *right side* of the equation. An equation may be compared to a balance.

The two sides of the equation may be considered as the equal weights, which, placed on the scale pans, balance each other. Obviously, if one of these weights is changed, a corresponding change must be made in the other weight to keep the scale balanced. Moreover, if the fact that an equation is a statement of the equality of the two expressions be kept in mind, it is readily seen that the equality is not changed so long as the quantities added to or subtracted from one side are equal to the quantities added to or subtracted from the other side.

Since this fact apparently holds good in all cases which might be observed, we make the general assumption that:

If equals are added to equals, the sums are equal.

If equals are subtracted from equals, the remainders are equal.

By similar reasoning, the following assumptions are held to apply to quantities in general:

If equals are multiplied by equals, the products are equal.

If equals are divided by equals, the quotients are equal, division by zero being always excluded.

If equals are raised to the same power, the results are equal.

If the square root (see page 75) *or any other root be taken of both sides of an equation, the results are equal.*

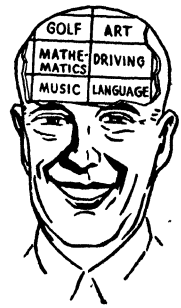
That is, one side of an equation may be multiplied or divided by any number, if the other side is multiplied or divided by the same number (provided that we do not try to divide by zero).* And each side of an equation may be raised to the same power without disturbing the balance, or the same root (see page 75) may be taken of each side, and the equality still remain.

SOLUTIONS OF PROBLEMS

The solution of problems should follow a process of logical reasoning. The bad habit of solving problems by pigeonholing in the mind different types of problems and memorizing a method of solution is unfortunately easily acquired. The result of this habit is the complete loss of the meaning of the symbols, and therefore the problem becomes a jigsaw puzzle in

* Note that in multiplying both sides of an equation by a number, *every term* in the equation must be multiplied by the number. That is, if Smith and Burns together have the same amount of money that Allen and Crow have, then if the amount Smith has is multiplied by 20, it is evident that the amounts owned by each of the other three men must also be multiplied by 20 if the first statement is to remain true.

which different symbols with no meaning whatever are to be shifted about or put together in an effort to get a solution. Hence, although the ability to translate into equations specific conditions stated in words is of the utmost importance, the habit of constantly keeping in mind what the equation stands for is no less important. That is, *the meaning of the symbols should be constantly kept before the mind.*



In the solution of simple elementary problems, the following procedure is suggested:

1. Read the problem carefully until you understand it clearly. Determine what is given and what is sought.
2. Express the quantity to be found by some letter (possibly by the initial letter of the word).
3. Next, express all the other quantities in terms of the same letter.
4. Look for any words that indicate a relation of equality. Express the relation of equality in the form of an equation.
5. Solve the equation.
6. Check the result by using the original statements of the problem.

For example:

The Windsor Department Store has a stock of 735 pairs of men's shoes which sell for \$2.25 a pair. After the sale of a certain number, the price is reduced to \$1.90 a pair, and the remainder is sold. The total sales from these shoes amount to \$1,589. It is desired to know how many pairs of shoes are sold at each price.

After reading the problem carefully and determining that what is sought is the number of pairs of shoes sold at each price, let

$$p = \text{number of pairs of shoes sold at \$2.25 a pair.}$$

Go back to the beginning of the problem and read it again, putting each English statement into mathematical shorthand.

"The Windsor Department Store has a stock of 735 pairs of men's shoes."

Since p is the number of pairs of shoes sold at \$2.25 a pair, then obviously,

$735 - p$ = pairs of men's shoes sold at \$1.90.

"Which sell for \$2.25 a pair."

If p pairs of shoes were sold at \$2.25 a pair, then

$2.25p$ = amount received from the sale of the p pairs of shoes sold at \$2.25 a pair.

"After selling a certain number, the price is reduced to \$1.90 a pair and the remainder is sold."

The remainder, $735 - p$ pairs of shoes, is sold at \$1.90 a pair. Hence,

$1.90(735 - p)$ = amount received from the sale of the $(735 - p)$ pairs sold at \$1.90 a pair.

"The total sales from these shoes amounted to \$1,589."

It is now easy to see that the amount received from the sale of the p pairs of shoes sold at \$2.25 a pair, added to the amount received from the sale of the $(735 - p)$ pairs sold at \$1.90 a pair, is the total sales, or \$1,589. Therefore,

$$2.25p + 1.90(735 - p) = 1{,}589.$$

Performing the operation indicated by the parenthesis, namely, multiplying $735 - p$ by 1.90, gives:

$$2.25p + 1{,}396.50 - 1.90p = 1{,}589.$$

Subtracting 1,396.50 from both sides of the equation evidently leaves the equality unchanged:

$$2.25p + 1{,}396.50 - 1{,}396.50 - 1.90p = 1{,}589.00 - 1{,}396.50.$$

The two quantities $+1{,}396.50$ and $-1{,}396.50$ on the left-hand side of the equation cancel each other. So that:

$$2.25p - 1.90p = 1{,}589.00 - 1{,}396.50. \qquad (a)$$

Notice that the operation of subtracting 1,396.50 from both sides of the equation merely placed 1,396.50 on the other or right side of the equation, changing its sign from $+$ to $-$. *That is, a term may be moved from one side of an equation to the other if the sign before the term is changed.* This operation is called *transposition.*

For example, in the expression:

$$5x - 20 = x,$$

the term x can be transposed to the left side of the equation and the term -20 to the right side, the signs being changed. So that:

$$5x - x = 20.$$

However, the fact that in each case this is really a short method of adding or subtracting the same quantity to both sides of the equation should always be kept in mind.

Returning to the solution of the Windsor Department Store problem, and carrying out the indicated operations in equation (a), we obtain:

$$0.35p = 192.50.$$

Finally, dividing both sides by 0.35 gives:

$$p = \frac{192.50}{0.35} = \frac{19250}{35} = 550.$$

That is, 550 pairs of shoes were sold at \$2.25 a pair, and, therefore, $735 - 550 = 185$ pairs of shoes were sold at \$1.90.

A problem must always be checked in the statements of the problem itself, and not by direct substitution in the equation. It is easy to make a mistake in the translation of the English sentences into mathematical shorthand and write an equation which does not depict the facts stated in the problem. This equation can then be solved correctly, and an answer which is the correct answer for the false equation may be obtained. However, since in that case the equation does not represent the facts of the problem correctly, the answer does not fit the problem. To check the problem, therefore, reason as follows: 550 pairs were sold at \$2.25 a pair. So that $550 \times 2.25 =$ \$1,237.50, the amount received from this sale. The 185 pairs sold at \$1.90 a pair amount to $185 \times 1.90 =$ \$351.50. The total sales are, therefore, \$1,237.50 + \$351.50 = \$1,589.00, which checks the statement of the problem.

FACTORING

To acquire greater skill and speed in operating with equations, it is advantageous to learn some methods or techniques which can be used to save time and effort. One of these is *factoring*. The practical purpose of studying factoring is to be able to handle equations and simplify them before proceeding to a solution of the problem involved.

A factor of a whole number is defined as any one of two or more whole numbers which are multiplied together to form a product. Thus, since $15 = 3 \times 5$, then 3 is a factor of 15, and so also is 5. Consequently, every factor of a product is an exact divisor of that product. Similarly, when dealing with symbols, a factor is defined as any one of two or more expressions* which are multiplied together to form a product. Thus, since $a \cdot b = a \times b$, then a is a factor of $a \cdot b$, and so also is b. Moreover, since $2x^2 - 7x - 15 = (2x + 3)(x - 5)$, then $2x + 3$ and $x - 5$ are factors of $2x^2 - 7x - 15$.

Factoring a mathematical expression is, therefore, the process of finding two or more expressions which, when multiplied together, will give the original expression.

A mathematical expression in which the exponents of all the letters are integers is a *rational expression.* Thus, $x^2 + 2xy + y^2$ is a rational expression, while $x^2 + 2\sqrt{xy} + y^2$, which also may be written $x^2 + 2(xy)^{\frac{1}{2}} + y^2$, is not.

Frequently one or more factors of a mathematical expression can be factored again. Thus, the expression $x^3 + 3x^2 - 25x - 75 = (x + 3)(x^2 - 25)$. So that $(x + 3)$ and $(x^2 - 25)$ are factors of the expression. However, the factor $(x^2 - 25)$ can be factored again, because $x^2 - 25 = (x + 5)(x - 5)$. As a rule, when factoring a mathematical expression, one should factor it *completely.* That is, all polynomials which occur in the process of factoring should be factored again, *whenever possible,* by any of the methods discussed in this book.

The easiest case of factoring occurs in an expression whose terms contain a common factor. The common factor is easily found by examining the terms of the expression.

For example, by examining the expression:

$$8x^2 - 5xy,$$

it is easily seen that x is a common factor of the two terms. When the expression is divided by the common factor x, the other factor is found to be $8x - 5y$. Hence,

$$8x^2 - 5xy = x(8x - 5y).$$

Similarly, in the expression:

$$7axy + 21aby + 14a^2b^2y,$$

* In these pages, attention is confined to monomials and polynomials.

the numerical coefficients have 7 as a factor, and the literal part has ay as a factor; hence, $7ay$ is a factor of every term. Dividing by this common factor gives $x + 3b + 2ab^2$. Hence,

$$7axy + 21aby + 14a^2b^2y = 7ay(x + 3b + 2ab^2).$$

The last two terms of the factor $x + 3b + 2ab^2$ contain b as a factor, so that the factor

$$x + 3b + 2ab^2 = x + b(3 + 2ab),$$

and hence the original expression may be written:

$$7axy + 21aby + 14a^2b^2y = 7ay[x + b(3 + 2ab)].$$

An example of the application of this simple case of factoring to the solution of an equation follows.

SIMPLE INTEREST

Interest is payment for the use of borrowed money or credit.* The interest payment agreed upon for a certain borrowed sum depends upon the size of the sum borrowed, the duration of the indebtedness, and the rate per cent charged. The sum loaned, which may also be looked upon as the capital invested, is called the *principal.* By convention the letter P is used to denote the principal, n to denote the number of years, i to denote the rate of interest, and I the interest. Simple interest is calculated on the original principal, which remains always unchanged, and is the product of principal, time, and rate. So that, applying the conventional notation just described:

$$I = Pni.$$

The sum obtained by adding the interest to the principal is called the *amount,* and is usually denoted by S. Hence,

$$S = P + Pni.$$

Now, suppose it is desired to find an expression for P in terms of the other quantities involved. It is seen at once that P is a factor of the two terms on the right-hand side of the above expression, so that:

$$S = P(1 + ni).$$

* For a further discussion of interest, and many practical applications, see Moore and Mira, *Practical Business Mathematics,* New York, Longmans, Green and Company, 1938.

Dividing both sides of this expression by $1 + ni$ gives the value of P immediately:

$$\frac{S}{1 + ni} = P.$$

The application to a concrete case is, of course, simply a matter of substitution. Assume that it is desired to know how much money must be invested now at simple interest at 6 per cent per annum in order to receive \$5,000 at the end of 10 years. Here $S = \$5{,}000$, $n = 10$, $i = 0.06$, and P is the unknown. Substituting these quantities in the last expression obtained:

$$\frac{5{,}000}{1 + 10(.06)} = P = \frac{5{,}000}{1.6} = \$3{,}125 \quad (\textit{Answer}).$$

SPECIAL PRODUCTS

Several simple types of expressions occur very frequently in mathematical operations. Some of the expressions are treated in the following discussion, while others are dealt with in the investigation of quadratic equations in Chapter 16.

Consider the expression:

$$mx + my + nx + ny.$$

The terms of this expression can be grouped in pairs to show a common factor as follows. Factor the first two and the last two terms:

$$mx + my + nx + ny = m(x + y) + n(x + y).$$

Observe that when the terms are properly grouped and then factored, $(x + y)$ is found to be a factor of the two terms on the right-hand side of the expression. Thus,

$$mx + my + nx + ny = m(x + y) + n(x + y) = (x + y)(m + n).$$

Similarly, observe that in the expression:

$$3xy^2 + 3my - 9my^2 - xy,$$

the first and third, and the second and fourth terms may be grouped in pairs to show a common factor. Thus:

$$3xy^2 - 9my^2 - xy + 3my = 3y^2(x - 3m) - y(x - 3m).$$

The two terms on the right-hand side contain $x - 3m$ as a common factor; therefore,

$$3y^2(x - 3m) - y(x - 3m) = (x - 3m)(3y^2 - y).$$

But the factor $3y^2 - y$ contains y as a common factor of its two terms, so that $3y^2 - y = y(3y - 1)$. Hence,

$$3xy^2 - 9my^2 - xy + 3my = (x - 3m)(3y - 1)y.$$

Another simple type, extremely important because of its usefulness, is the product $(a + b)(a - b)$; that is, the product of the sum of two quantities by their difference.

If the indicated multiplication is carried out, the product is:

$$(a + b)(a - b) = a^2 - b^2.$$

Therefore, the difference of the squares of two quantities, a and b, is equal to the product of the sum, $(a + b)$, of the two quantities and the differences, $(a - b)$, of the two quantities. Thus,

$$4x^2 - 16y^2 = (2x + 4y)(2x - 4y),$$

and

$$9m^2 - 25 = (3m + 5)(3m - 5).$$

The sum or difference of two cubes occurs often enough to make their factors important to remember.

The sum of the cubes of two quantities is equal to the sum of the quantities multiplied by an expression formed by the sum of the squares of the quantities less their product. Thus,

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

The difference of the cubes of two quantities is equal to the difference of the quantities multiplied by an expression formed by the sum of the squares of the quantities plus their product. Thus,

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

The more general case of the sum or difference of the powers of two quantities, although not so important as the two cases first mentioned, are given here for reference. These cases may be subdivided as follows:

(1) The *sum* of the *odd* powers of two quantities in x and y, for example, $x^5 + y^5$, has $(x + y)$ as a factor. Thus,

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$$

(2) The *sum* of the *even* powers of two quantities in x and y, for example, $x^4 + y^4$, does *not* have $(x + y)$ or $(x - y)$ as a factor. One may, however, factor the special case when the expression can be put in the form of the preceding instance. For example:

$$x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4).$$

(3) The *difference* of the *odd* powers of two quantities in x and y, for example, $x^5 - y^5$, has $(x - y)$ as a factor:

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4).$$

(4) The *difference* of the *even* powers of two quantities in x and y, for example, $x^4 - y^4$, has $(x - y)$ and $(x + y)$ as factors:

$$x^6 - y^6 = (x - y)(x + y)(x^2 - xy + y^2)(x^2 + xy + y^2).$$

The following table is a symbolic summary of these cases which may be used as a set of reference, the symbol n standing for a positive integral number.

(1) $x^n + y^n$ has the factor $(x + y)$ if n is odd.
(2) $x^n - y^n$ has the factor $(x - y)$ if n is odd.
(3) $x^n - y^n$ has the factors $(x + y)$ and $(x - y)$ if n is even.
(4) $x^n + y^n$ cannot be factored into rational factors if n is even. For example, given the expression $x^2 + y^2$. If $2xy$ is added and subtracted, the value of the expression is not changed, and we have $x^2 + 2xy + y^2 - 2xy$. The first three terms of this expression, namely, $x^2 + 2xy + y^2$ equal $(x + y)^2$. So that the given expression may be written $(x + y)^2 - 2xy$. The last term, namely, $2xy$, can be considered as the square of $\sqrt{2xy}$, because $\sqrt{2xy} \cdot \sqrt{2xy} = 2xy$. In its last form, that is, the form $(x + y)^2 - 2xy$, the expression may be factored as the difference of two squares. Thus, $x^2 + y^2 = (x + y)^2 - 2xy = (x + y + \sqrt{2xy})(x + y - \sqrt{2xy})$. But these factors are not rational, for they contain the irrational term $\sqrt{2xy}$.

A little familiarity with these expressions will show that in the first three cases the second factor, when arranged in descending powers of x, is such that:

(a) The first term is positive and contains the variable x only, raised to a power which is one less than that of x in the original expression.

(b) If the original expression has $(x + y)$ as a factor, the signs of the terms of the other factor alternate; while if $(x - y)$ is a factor, the signs of the terms of the other factor are all positive.

(c) The exponent of x decreases by one in each succeeding term, and the exponent of y increases by one in each succeeding term.

(d) The last term contains the variable y only raised to a power which is one less than that of y in the original expression.

The necessity of constantly keeping in mind the meaning of the symbols may be illustrated by the well-known mathematical "trick" which seems to "prove" that twice a number is equal to the number itself:

Assume that a and b represent any two numbers, and let

$$a = b.$$

Multiplying both sides by a,

$$a^2 = a \cdot b.$$

Subtracting b^2 from both sides,

$$a^2 - b^2 = a \cdot b - b^2$$

The left-hand side of the equation may be factored by observing that it falls under type (3), so that it contains $(a - b)$ and $(a + b)$ as factors, that is, $a^2 - b^2 = (a + b)(a - b)$. The terms of the right-hand side of the equation obviously contain b as a factor. Hence, factoring both sides gives:

$$(a + b)(a - b) = b(a - b).$$

Dividing both sides by $(a - b)$,

$$a + b = b.$$

Now, since $a = b$, then:

$$b + b = b$$

and

$$2b = b.$$

That is, twice any number b is equal to the number itself.

The trick, of course, is effected by deliberately throwing sand into the reader's eyes in choosing two different symbols to represent the same thing, namely, $a = b$. The fallacy consists in the fact that, since $a = b$, then division by $a - b$, which is evidently $b - b = 0$, is excluded (see page 10). Therefore, the apparent proof is no proof at all.

Problems

1. A salesman started on a trip with a certain sum of money. He spent $\frac{1}{4}$ of it for railroad fares and $\frac{1}{2}$ of it for hotel expenses. On returning, he had \$96.00 left. How much did he have when he started?

2. The sum of the three sides of a triangle is 68 inches. If the second side is 8 inches longer than the first and 7 inches shorter than the third, find the length of each side.

3. If 10 be subtracted from a certain number, three-fourths of the remainder is 9. What is the number?

4. A tank can be filled by one pipe in 9 hours, or by another pipe in 18 hours. How long will it take to fill the tank if both pipes are open?

5. A can do a piece of work in 4 days, B in 6 days, and C in 8 days. How long will it take them working together?

6. A 12-gallon mixture of alcohol and water contains 80 per cent alcohol. How much water must be added to make it contain only 60 per cent alcohol?

7. How much water must be added to 64 pounds of a 10 per cent salt solution to reduce it to an 8 per cent solution?

8. Coffee is bought at 25 cents a pound and chicory at 6 cents a pound. In what proportion must they be mixed in order for the merchant, by selling the mixture at 22 cents a pound, to make a profit of 10 per cent?

9. A basket of lemons is emptied by one person taking half of them and 1 more, a second person taking half of the remainder and 1 more, and a third person taking half of the remainder and 6 more. How many did the basket contain at first?

10. A man receives a certain amount of dollars and cents. He spends half as much as he was given and has left as many cents as he had dollars and half as many dollars as he had cents. How much was he given?

5 THE BUGBEAR—FRACTIONS

For some unknown reason, fractions mystify millions of adults. It may be safely asserted that more mistakes arise in daily business computations as a result of carelessness in handling fractions than from any other one cause. A little patience in reviewing fundamentals will save hours of vexation and confusion. Even college students of higher mathematics often go astray because of unfamiliarity with this humble subject. "So that now, Arithmetic in whole Numbers is plainly and fully handled before any entrance be made into the craggy paths of fractions, at the sight of which some learners are so discouraged, that they make a stand, and cry out, *Non plus ultra*, there's no progress farther."*

A numerical fraction is one or more equal integral parts of a unit or whole. It is expressed by two whole numbers separated by a horizontal or slanting line. Thus, $\frac{2}{5}$ or 2/5 is a fraction which denotes that, after dividing the unit or whole into five equal parts, two of these five equal parts are to be taken.

The quantity above the line is called the *numerator* and the quantity below the line is called the *denominator*. The numerator indicates how many equal integral parts of the whole are taken. The denominator indicates the number of equal integral parts into which the unit or whole has been divided.

A fraction, therefore, implies the operation of division.

On the basis of this fact, the arithmetical notion of a fraction is extended so that, when dealing with a mathematical expression of an unexecuted division, the indicated quotient is called a *fraction*. For example, $\frac{a}{b}$, where a and b represent any numbers whatever, except for those values which render b equal to zero, is a fraction.

Consequently, $\frac{-a}{b}$ may represent an expression like $\frac{3}{5\frac{2}{7}}$. Since

* Edmund Wingate, *A Plain and Familiar Method for Attaining the Knowledge and Practice of Common Arithmetic*, London, 1751.

a thing cannot be divided into $5\frac{4}{7}$ equal parts, all fractions are not accurately described by the definition commonly given in arithmetic. But an expression like $\frac{35}{7}$, regarded as 35 sevenths, is equivalent to 5, or $35 \div 7$, so that the numerator of a fraction may be regarded as a dividend, and the denominator as its divisor. This interpretation of a fraction is broad enough to include the fraction $\frac{a}{b}$ when a and b represent any numbers whatever, except that b cannot be equal to zero, since division by zero is impossible. Hence, the expression of an unexecuted division, in which the dividend is the numerator and the divisor the denominator, is a fraction. Thus, in the fractions $\frac{2}{5}$, $\frac{a}{b}$, and $\frac{x-25}{x^2+x-6}$, the respective numerators are 2, a, and $x - 25$. The respective denominators are 5, b, and $x^2 + x - 6$.

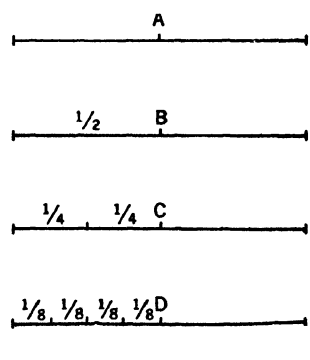
FUNDAMENTAL PRINCIPLE—CANCELLING

The reader should bear in mind that a fraction is *not defined* for those values of the quantities which render the denominator zero. Thus, the fraction $\frac{38}{x-a}$ is not defined for the value $x = a$. The expression $\frac{38}{0}$ has no meaning (see page 10).

In operations with fractions, it is necessary to depend constantly on the following fundamental principle:

Multiplying or dividing the numerator and the denominator of a fraction by the same number (except zero) does not change the value of the fraction.

This principle may be illustrated as follows:



The length of a line may be divided into any number of equal parts. In the figure the lines A, B, C, and D are all of the same length. Lines B, C, and D are divided into 2, 4, and 8 equal parts, respectively. From the figure it is easily seen that:

$$\tfrac{1}{2} = \tfrac{2}{4} = \tfrac{4}{8}.$$

If in the fraction $\frac{4}{8}$ both the numerator and denominator are divided by 4, the result is:

$$\frac{4 \div 4}{8 \div 4} = \frac{1}{2}.$$

But it has just been shown that $\frac{4}{8} = \frac{1}{2}$. So that the division of the numerator and denominator by the same quantity has not changed the value of the fraction.

Similarly, if in the fraction $\frac{2}{4}$ both the numerator and the denominator are multiplied by 2, the result is:

$$\frac{2 \times 2}{4 \times 2} = \frac{4}{8}.$$

Again, the value of the fraction remains unchanged, since

$$\tfrac{2}{4} = \tfrac{4}{8}.$$

That is, in all these cases the *form* of the fractions has been changed but not their *value.*

Since in mathematical operations letters are used to represent numbers, this principle is extended to include such expressions.

When the numerator and denominator contain one or more common factors, the fraction may be reduced to a fraction of equivalent value, expressed in its simplest form, by dividing both numerator and denominator by the common factor or factors.

For example: $\frac{105}{140}$ can be factored and written $\frac{7 \times 3 \times 5}{7 \times 4 \times 5}$. If both the numerator and the denominator of this fraction are divided by the common factors 7 and 5, the result is $\frac{3}{4}$. Hence, $\frac{105}{140} = \frac{3}{4}$.

This elimination of equal factors is called *cancelling.*

Similarly, in the expression:

$$\frac{3mx + 6my - nx - 2ny}{5mx + 10my - nx - 2ny}$$

by factoring we obtain:

$$\frac{3mx + 6my - nx - 2ny}{5mx + 10my - nx - 2ny} = \frac{3m(x + 2y) - n(x + 2y)}{5m(x + 2y) - n(x + 2y)} = \frac{(3m - n)(x + 2y)}{(5m - n)(x + 2y)}.$$

Dividing both numerator and denominator by the common factor $(x + 2y)$ gives:

$$\frac{3m - n}{5m - n}.$$

In applying this principle, *always cancel factors and not terms.* It is of the utmost importance to guard against this very common error. Whenever in doubt, substitute numbers for the letters involved. For example: the expression $\frac{a+3}{a+4}$ is *not* equal to $\frac{\cancel{a}+3}{\cancel{a}+4}$ or $\frac{3}{4}$. For, let a equal some number, say 5, then $\frac{a+3}{a+4} = \frac{5+3}{5+4} = \frac{8}{9}$, and this is obviously not equal to $\frac{\cancel{5}+3}{\cancel{5}+4}$ or $\frac{3}{4}$. Moreover, $\frac{2m+n}{am}$ is *not* equal to $\frac{2\cancel{m}+n}{a\cancel{m}}$ or $\frac{2+n}{a}$ because m is not a factor of *all* the terms in the numerator, that is, of the entire numerator, but only of *one* term of the numerator.

Again the mistake here is easily prevented if numbers are substituted for the letters. Let $m = 3$, $n = 5$, and $a = 4$; then:

$$\frac{2m + n}{am} = \frac{2 \times 3 + 5}{4 \times 3} = \frac{11}{12},$$

and this evidently is not equal to:

$$\frac{2 \times \cancel{3} + 5}{4 \times \cancel{3}} \text{ or } \frac{7}{4}.$$

TYPES OF FRACTIONS

A numerical fraction whose numerator is less than its denominator is called a *proper fraction.* Thus, $\frac{2}{5}$ and $\frac{3}{4}$ are proper fractions.

Similarly, a literal fraction in which the *degree* of the numerator is less than the degree of the denominator, such as $\frac{2x+5}{x^2+x-6}$, is also called a proper fraction.

If there is a number of equal units or whole things, each of

the units or whole things may be divided into equal parts. For example, there may be three equal whole things, each one of which may have been divided into four equal parts. The result of this partition would be a total of twelve equal parts, each one of which is one-fourth of one whole thing.

Evidently, seven of these equal parts are seven-fourths of a unit or whole; that is, $\frac{7}{4}$. Or, it might be said that seven of these equal parts constitute one whole unit and three-fourths of another unit; that is, $1\frac{3}{4}$.

Hence, a numerical fraction may have a numerator larger than the denominator, and in that case the quantity may be expressed as a whole number and a fraction.

A numerical fraction whose numerator is greater than its denominator is called an *improper fraction*. Thus, $\frac{7}{4}$ and $\frac{5}{3}$ are improper fractions.

An improper fraction expressed as a whole number and a fraction is called a *mixed number*. Thus, $1\frac{3}{4}$ and $1\frac{2}{3}$ are mixed numbers. Note that the juxtaposition of 1 and $\frac{3}{4}$ or of 1 and $\frac{2}{3}$ indicates that the two combinations of quantities are to be ADDED. Thus, a mixed number such as $1\frac{3}{4}$ means $1 + \frac{3}{4}$. If letters are used, this is NOT the case.

Denoting numbers by means of letters does not change the fundamental ideas involved. An expression in which the degree of the numerator equals or exceeds the degree of the denominator is also called an improper fraction. That is, $\frac{x^2 - 6}{x + 5}$ and $\frac{x - 3}{x}$ are improper fractions.

Since a fraction is an indicated division, an improper fraction may be expressed as a mixed expression by dividing the numerator by the denominator. If there is a remainder, this is written over the denominator, annexing the result to the quotient with the proper sign. For example, to express $\frac{x^2 - 6}{x + 5}$ as a mixed expression, divide $x^2 - 6$ by $x + 5$. Thus:

$$
\begin{array}{r}
x - 5 \\
x + 5 \overline{)\, x^2 \qquad - 6} \\
\underline{x^2 + 5x} \\
\quad - 5x - 6 \\
\quad \underline{- 5x - 25} \\
\qquad\qquad 19
\end{array}
$$

Therefore,

$$\frac{x^2 - 6}{x + 5} = x - 5 + \frac{19}{x + 5}.$$

Similarly,

$$\frac{x - 3}{x} = 1 - \frac{3}{x}.$$

If a mixed expression is to be changed to an improper fraction, the process is simply reversed. That is, the integral part of the expression is multiplied by the denominator of the fractional part. To this product, the numerator of the fractional part is then added, and the result is written above the denominator. For example: to write $x - 5 + \frac{19}{x + 5}$ as an improper fraction, multiply $(x - 5)$ by $(x + 5)$; then, to the product, which is $x^2 - 25$, add 19. Write the result $x^2 - 6$ over the denominator $x + 5$, and obtain the required improper fraction. That is,

$$x - 5 + \frac{19}{x + 5} = \frac{(x - 5)(x + 5) + 19}{x + 5} = \frac{x^2 - 6}{x + 5}.$$

Similarly,

$$1 - \frac{3}{x} = \frac{(1)(x) - 3}{x} = \frac{x - 3}{x}.$$

SIGNS OF A FRACTION

To determine whether a fraction is positive or negative, *three* signs in reality must be considered; namely, the sign of the numerator, the sign of the denominator, and the sign before the dividing line of the fraction. Because of the now often repeated fact that a fraction is an indicated quotient, the laws of signs for division are applied (see page 10).

The value of the fraction is not changed if *any two* of the three signs of a fraction are changed. In other words, both the sign of the numerator and the sign of the denominator or both the sign before the fraction and the sign of *either* the numerator or the denominator may be changed without changing the value of the fraction. That is,

$$+\frac{+a}{+b} = +\frac{-a}{-b} = -\frac{-a}{+b} = -\frac{+a}{-b}.$$

The reason is that these operations are equivalent to multiplying both numerator and denominator by -1 or to multiplying the fraction twice by -1. In either case, the value of the fraction is not changed, since multiplying both terms of a fraction by the same number does not change its value, and $(-1)(-1) = 1$.

It is important to remember that in changing the sign of a numerator or denominator which contains more than one term, the signs of *all* the terms of the numerator or of the denominator must be changed. For example:

$$+\frac{3x^2 - 6x + 7}{x^3 - 7x^2 + 3} = +\frac{-3x^2 + 6x - 7}{-x^3 + 7x^2 - 3}$$

$$= -\frac{-3x^2 + 6x - 7}{x^3 - 7x^2 + 3} = -\frac{3x^2 - 6x + 7}{-x^3 + 7x^2 - 3}.$$

ADDITION AND SUBTRACTION OF FRACTIONS

The difficulties experienced in operating with fractions are in most cases easily overcome. The source of all these difficulties is eliminated if it is kept in mind that an integral numerical denominator, by indicating the number of equal parts into which a unit or whole has been divided, actually names the things we are dealing with.

Nobody has any trouble in realizing that one five-ounce bottle of ink added to two five-ounce bottles of the same ink gives as a result three five-ounce bottles of ink. Nor that one three-ounce bottle of ink added to two five-ounce bottles of ink does *not* give as a result three three-ounce bottles or three five-ounce bottles. Everyone realizes that before adding these bottles, it is necessary to express them on the basis of a common denomination. That is, either the three-ounce bottle must be expressed in terms of the five-ounce bottle or the five-ounce bottle must be expressed in terms of the three-ounce.

It is just as easy to realize that the fraction *one-fifth* can be added to the fraction *two-fifths*, giving as a result *three-fifths*. But in order to add the fraction *one-third* to the fraction *two-fifths*, it is necessary to express these fractions on the basis of a common denomination. This is accomplished by changing the *form* of the fractions, but not their *values*, so that both will have the same or a *common denominator*. This common

denominator must be a common multiple of the given denominators.

Thus, to add $\frac{1}{3}$ and $\frac{2}{5}$, the least common multiple of the denominators should first be found. This least common denominator is evidently, in this case, 15. To express the fraction $\frac{1}{3}$ in terms of 15, the denominator 3 must be multiplied by 5. In order to change the form but not the value of the fraction $\frac{1}{3}$, both the numerator and the denominator must be multiplied by 5; so that $\frac{1}{3} = \frac{1 \times 5}{3 \times 5} = \frac{5}{15}$. Similarly, in order to express the fraction $\frac{2}{5}$ in terms of 15, both the numerator and the denominator must be multiplied by 3; that is,

$$\frac{2}{3} = \frac{2 \times 3}{5 \times 3} = \frac{6}{15}.$$

Hence,

$$\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15} = \frac{5+6}{15} = \frac{11}{15}.$$

By the same reasoning, it is seen at once that any fractions with a common denominator are subtracted by subtracting the numerators and writing the remainder over the common denominator; but that, in order to subtract fractions which do not have a common denominator, these fractions must first be expressed with the same, or a common, denominator.

In general, to add or subtract numerical fractions, first, if necessary, express all the fractions as equivalent fractions having a *lowest* common denominator. Then add or subtract the numerators, writing this sum or difference over the lowest common denominator. Finally, cancel any common factors, so that the resulting fraction is expressed in its simplest form.

These conclusions, of course, must hold when dealing with fractions containing letters. For example, to add:

$$\frac{2x-3}{x^2-25} + \frac{x-6}{x^2+2x-15} - \frac{3}{x-5},$$

the fractions should first be expressed as equivalent fractions having a lowest common denominator. The lowest common multiple of $x^2 - 25$, $x^2 + 2x - 15$, and $x - 5$ is the expression of lowest degree which can be exactly divided by each of the expressions. To find it, factor each of the denominators com-

pletely, and then form a product by multiplying together *all* the different factors of the several denominators, *taking each factor the greatest number of times it occurs in any of the denominators.* The result is the lowest common multiple of the expressions, namely, the *lowest common denominator.* Thus,

$$\begin{aligned} x^2 - 25 &= (x + 5)(x - 5), \\ x^2 + 2x - 15 &= (x - 3)(x + 5), \text{ and} \\ x - 5 &= x - 5. \end{aligned}$$

The different factors are $(x + 5)$, $(x - 5)$, and $(x - 3)$. The greatest number of times each occurs in any of the expressions is one. Hence,

$$(x + 5)(x - 5)(x - 3) = (x^2 - 25)(x - 3) = x^3 - 3x^2 - 25x + 75;$$

the latter expression is the least common denominator.

To express $\dfrac{2x - 3}{x^2 - 25}$ in terms of $x^3 - 3x^2 - 25x + 75$, the denominator $(x^2 - 25)$ must be multiplied by $(x - 3)$. In order to change the form, but not the value, of the fraction, both the numerator and the denominator must be multiplied by $(x - 3)$. So that:

$$\frac{2x - 3}{x^2 - 25} = \frac{(2x - 3)(x - 3)}{(x^2 - 25)(x - 3)} = \frac{(2x - 3)(x - 3)}{x^3 - 3x^2 - 25x + 75}.$$

Similarly, since $x^2 + 2x - 15$ must be multiplied by $(x - 5)$ to give the same least common denominator, then the numerator $(x - 6)$ must also be multiplied by $(x - 5)$. Finally, the numerator 3 must be multiplied by $(x + 5)(x - 3)$, for the denominator $(x - 5)$ must be multiplied by this expression to change the fraction to the proper form. That is,

$$\frac{2x - 3}{x^2 - 25} + \frac{x - 6}{x^2 + 2x - 15} - \frac{3}{x - 5}$$

$$= \frac{(2x - 3)(x - 3) + (x - 6)(x - 5) - 3(x + 5)(x - 3)}{x^3 - 3x^2 - 25x + 75}$$

$$= \frac{2x^2 - 9x + 9 + x^2 - 11x + 30 - 3x^2 - 6x + 45}{x^3 - 3x^2 - 25x + 75}$$

$$= \frac{-26x + 84}{x^3 - 3x^2 - 25x + 75} = -\frac{26x - 84}{x^3 - 3x^2 - 25x + 75}$$

$$= -\frac{2(13x - 42)}{x^3 - 3x^2 - 25x + 75} \quad (\textit{Answer}).$$

Various short cuts are often possible in adding fractions. For example, in adding a series of fractions, if two or more of them have the same denominator, these particular fractions can be added first, and then the result is added to the remaining fractions of the series. Thus, in the following series:

$$\frac{1}{x-1} + \frac{1}{x^2-1} + \frac{4}{x-1} + \frac{3}{x^2+2} + \frac{5}{x-1},$$

the three fractions having $x - 1$ as the denominator should be added first, obtaining:

$$\frac{10}{x-1} + \frac{1}{x^2-1} + \frac{3}{x^2+2}. \tag{1}$$

Furthermore, in this series of fractions, the indicated operations may be facilitated by applying the facts already set forth in Chapter 4 regarding factors. Thus, it is easily seen that, since $x^2 - 1 = (x + 1)(x - 1)$, the common denominator of the first two fractions in (1) is $x^2 - 1$. Hence, adding the first two fractions, we obtain:

$$\frac{10}{x-1} + \frac{1}{x^2-1} + \frac{3}{x^2+2} = \frac{10(x+1)+1}{x^2-1} + \frac{3}{x^2+2}$$

$$= \frac{10x + 11}{x^2 - 1} + \frac{3}{x^2 + 2}.$$

The addition of the two resulting fractions may then be carried out according to the rule already stated.

The addition or subtraction of an integral expression to a fraction conforms, of course, to the general rule. The integral expression is indicated in terms of the denominator of the fraction, the numerators are then added or subtracted, as the case may be, and the result is placed over the common denominator.

The addition $3 + \frac{2}{5}$ can be simply written $3\frac{2}{5}$, or the integral expression 3 might be indicated in terms of 5, namely, $3 = \frac{15}{5}$,

and then added to $\frac{3}{5}$. Thus, $3 + \frac{3}{5} = \frac{15}{5} + \frac{3}{5} = \frac{15+3}{5} = \frac{18}{5}$.

Moreover, the addition $m + \frac{a}{b}$ can be expressed simply as it is, or for convenience the integral expression m can be indicated in terms of b, namely, $m = \frac{mb}{b}$, and then added to $\frac{a}{b}$. Thus,

$$m + \frac{a}{b} = \frac{mb}{b} + \frac{a}{b} = \frac{mb + a}{b}.$$

Note that this process is merely a repetition of the method given to change a mixed expression to an improper fraction, as it should be, for a mixed expression is merely an indicated addition or subtraction.

When dealing with two fractions whose numerators are *one*, such as $\frac{1}{a}$ and $\frac{1}{b}$, it is convenient to remember that their sum or difference is:

$$\frac{1}{a} \pm \frac{1}{b} = \frac{b \pm a}{ab}.$$

Thus,

$$\frac{1}{3} + \frac{1}{5} = \frac{5+3}{15} = \frac{8}{15},$$

and

$$\frac{1}{3x} - \frac{1}{2m} = \frac{2m - 3x}{6mx}.$$

As already explained, short cuts should be used whenever possible. For example, in the following series of indicated operations,

$$8x + 1 + \frac{2x}{x-1} + 4x^2 - 5x - 2 + \frac{2}{x-1},$$

the similar terms should be added first, and the fractions having the same common denominator should then be added, giving:

$$4x^2 + 3x - 1 + \frac{2x+2}{x-1}.$$

If it is desired to complete the indicated operations, the rule just stated is now applied. Thus,

$$4x^2 + 3x - 1 + \frac{2x+2}{x-1} = \frac{4x^2(x-1) + 3x(x-1) - (x-1) + 2x + 2}{x-1}$$

$$= \frac{4x^3 - x^2 - 2x + 3}{x-1}.$$

MULTIPLICATION OF FRACTIONS

The operation of the multiplication of a fraction by a positive whole number is fundamentally, like the multiplication of integers, a short method of repeated addition. To multiply $\frac{2}{3}$ by 4 means to take* $\frac{2}{3}$ four times. Since $\frac{2}{3}$ taken four times is $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{8}{3}$, it is evident that the operation is the repeated addition of the same fraction. Obviously, since it is the same fraction, the denominators are all equal. To add fractions with the same denominator, the numerators are added (page 60). But in this case, the numerators are also equal. To add a series of numbers which are all equal, this common number is multiplied by the number of times which it occurs. Therefore, to multiply a fraction by a positive whole number, the numerator of the fraction is multiplied by the whole number, and this product is written above the denominator of the fraction.

Thus,

$$\frac{2}{3} \times 4 = \frac{2 \times 4}{3} = \frac{8}{3} \text{ or } 2\tfrac{2}{3}.$$

Moreover, $\frac{2}{3} \times 4$ means two-thirds of four. Now, to obtain two-thirds of any quantity, the quantity must first be divided into three equal parts and then two of these three equal parts must be taken. That is, the multiplication of a whole number by a fraction implies first an operation of division, and then an operation of multiplication, although these operations may be performed in the other order.

It is very important to remember that *when using numerals*

* The word *take* means to multiply. Thus, "take 3 times 5" or "take $\frac{3}{5}$ of $\frac{2}{7}$" mean, respectively, "multiply 3 by 5" and "multiply $\frac{3}{5}$ by $\frac{2}{7}$." *Take* is not to be confused with *take away*. The latter indicates subtraction.

only, $1\frac{2}{3}$ means $1 + \frac{2}{3}$ and *not* $1 \times \frac{2}{3}$. Thus, $1\frac{2}{3} = 1 + \frac{2}{3} = \frac{3+2}{3} = \frac{5}{3}$.

When using letters, however, $m\frac{a}{b}$ means $m \times \frac{a}{b}$ and not $m + \frac{a}{b}$. Hence, $m\frac{a}{b} = m \times \frac{a}{b} = \frac{ma}{b}$. Again, $3\frac{2}{x-5}$ means $3 \times \frac{2}{x-5} = \frac{3 \times 2}{x-5} = \frac{6}{x-5}$. This variation in symbolism is exceedingly unfortunate, but it is so firmly established by usage that we must conform.

Consider next the multiplication of a fraction by another fraction. What is $\frac{2}{5} \times \frac{4}{7}$? To take $\frac{2}{5}$ of $\frac{4}{7}$ implies that the quantity $\frac{4}{7}$ is to be divided into five equal parts and two of these five equal parts are to be taken. This operation might be done as follows: one-fifth of $\frac{1}{7}$ is one of the five equal parts into which $\frac{1}{7}$ may be divided. If each seventh is divided into five equal parts, $\frac{1}{7}$ or the unit will be divided into 7×5 or 35 equal parts, and each part will be $\frac{1}{35}$. Hence, $\frac{1}{5}$ of $\frac{1}{7}$ is $\frac{1}{35}$. So $\frac{1}{5}$ of $\frac{4}{7}$ is $4 \times \frac{1}{35}$, namely, $\frac{4}{35}$; and $\frac{2}{5}$ of $\frac{4}{7}$ is $2 \times \frac{4}{35}$, that is, $\frac{8}{35}$. But $\frac{2 \times 4}{5 \times 7} = \frac{8}{35}$. Therefore,

To multiply one fraction by another, multiply the numerators to form the numerator of the product; multiply the denominators to form the denominator of the product; reduce the product to its simplest form.

For example:

$$\frac{x-5}{x+3} \cdot \frac{x^2+8x+15}{x^2-25} = \frac{(x-5)(x^2+8x+15)}{(x+3)(x^2-25)} = 1.$$

Of course, as more practice is acquired, some of the steps just set forth may be eliminated. Thus, an expert would immediately note that $x - 5$ is a factor of $x^2 - 25$, and hence would first cancel $x - 5$, obtaining:

$$\frac{1}{\cancel{x-5}} \cdot \frac{x^2+8x+15}{\cancel{x^2-25}}; \qquad x+5$$

then, noticing that $x^2 + 8x + 15 = (x + 3)(x + 5)$, he would immediately arrive at the answer, namely, 1.

DIVISION OF FRACTIONS

Suppose that a cafeteria divides an apple pie into eight equal parts and sells each part or one-eighth of a pie as a portion. In two pies, there will be 16 portions, namely, 16 eighths. That is, a portion or one-eighth of a pie is contained in 2 pies 16 times. Hence, $2 \div \frac{1}{8} = 16$.

Now suppose that 5 is to be divided by $\frac{3}{4}$. The fraction $\frac{1}{4}$ is contained 4 times in 1. Therefore, $\frac{3}{4}$, or 3 times $\frac{1}{4}$, is contained in 1 one-third part of 4 times, or $\frac{4}{3}$ times. Since $\frac{3}{4}$ is contained $\frac{4}{3}$ times in 1, it will be contained 5 times $\frac{4}{3}$ in 5, or $\frac{20}{3}$.

In general, consider that any whole number c is to be divided by the fraction $\frac{a}{b}$, where neither a nor b is equal to zero. The fraction $\frac{1}{b}$ is contained b times in 1. Therefore, $\frac{a}{b}$, or a times $\frac{1}{b}$, is contained in 1 the $\frac{1}{a}$ part of b times, or $\frac{b}{a}$ times. Since $\frac{a}{b}$ is contained $\frac{b}{a}$ times in 1, it will be contained c times $\frac{b}{a}$, or $\frac{cb}{a}$, in c.

The problems just considered involved the division of a whole number by a fraction.

Consider next the division of a fraction by a whole number. What is $\frac{2}{3} \div 2$? If each third is divided into two equal parts, $\frac{3}{3}$ or the unit will be divided into 3×2 or 6 equal parts, and each part will be $\frac{1}{6}$. Hence, $\frac{1}{3} \div 2 = \frac{1}{6}$; and, therefore, $\frac{2}{3} \div 2 = \frac{2}{6}$, or $\frac{1}{3}$.

In general, suppose that any fraction $\frac{a}{b}$ is to be divided by any whole number c, but that neither b nor c is equal to zero. The whole number c is contained $\frac{1}{c}$ times in 1. Therefore, it will be contained $\frac{a}{b}$ times $\frac{1}{c}$ in $\frac{a}{b}$, or $\frac{a}{b \cdot c}$.

Therefore, the result in both of these divisions can be obtained by *multiplying the dividend by the divisor inverted.* That

is, in the first case, $2 \div \frac{1}{8} = 16$; the number 2 or dividend multiplied by the divisor $\frac{1}{8}$ inverted gives $2 \times \frac{8}{1} = \frac{2 \times 8}{1} = 16$. In the second case, the divisor 2 can be written as $\frac{2}{1}$, and $\frac{2}{3} \div \frac{2}{1} = \frac{2}{3} \times \frac{1}{2} = \frac{2}{6}$.

Similarly:

$$(2x - 3) \div \frac{5}{x - 2} = (2x - 3) \times \frac{x - 2}{5}$$

$$= \frac{(2x - 3)(x - 2)}{5} = \frac{2}{5}x^2 - \frac{7}{5}x + \frac{6}{5};$$

and

$$\frac{6}{x - 5} \div (2x - 3) = \frac{6}{x - 5} \div \frac{2x - 3}{1} = \frac{6}{x - 5} \times \frac{1}{2x - 3}$$

$$= \frac{6}{2x^2 - 13x + 15}.$$

Finally, the case of the division of a fraction by another fraction can be analyzed as follows: What is $\frac{2}{3} \div \frac{3}{5}$? If one-third is divided into five equal parts, $\frac{3}{3}$ or the unit is divided into $3 \times 5 = 15$ equal parts, and each part is $\frac{1}{15}$. So that $\frac{1}{3}$ is $\frac{5}{15}$, and hence $\frac{2}{3}$ is $\frac{10}{15}$. Again, if one-fifth is divided into three equal parts, $\frac{5}{5}$ or the unit is divided into 5×3 or 15 equal parts, and each part is $\frac{1}{15}$. So that $\frac{1}{5}$ is $\frac{3}{15}$, and $\frac{3}{5}$ is $\frac{9}{15}$. Now, it is easily seen that $\frac{9}{15}$ is contained in $\frac{10}{15}$ as many times as 9 is contained in 10. That is, $\frac{2}{3} \div \frac{3}{5} = \frac{10}{15} \div \frac{9}{15} = 10 \div 9 = \frac{10}{9} = 1\frac{1}{9}$.

But notice this: If the divisor fraction is inverted and the fractions are then multiplied, the same result is obtained. Thus,

$$\frac{2}{3} \div \frac{3}{5} = \frac{10}{9} = \frac{2}{3} \times \frac{5}{3}.$$

In general, consider that the fraction $\frac{a}{b}$ is to be divided by $\frac{c}{d}$, where b, c, and d are not equal to zero. The fraction $\frac{1}{d}$ is contained d times in 1. Therefore, $\frac{c}{d}$, or c times $\frac{1}{d}$, is contained in 1 the $\frac{1}{c}$ part of d times, or $\frac{d}{c}$ times. Since $\frac{c}{d}$ is contained $\frac{d}{c}$

times in 1, it will be contained $\frac{a}{b}$ times $\frac{d}{c}$, or $\frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$ times, in $\frac{a}{b}$.

Therefore, *to divide fractions, multiply the dividend fraction by the inverted divisor fraction.*

For example:

$$\frac{a-b}{mn^2} \div \frac{a-b}{m^3n} = \frac{a-b}{mn^2} \cdot \frac{m^3n}{a-b} = \frac{m^2}{n}.$$

The division of a fraction by another fraction can be indicated as it has been done so far; namely, horizontally or vertically. That is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

A fractional expression that has one or more fractions in its numerator or its denominator or in both is called a *complex fraction.*

A complex fraction is simplified by carrying out the indicated operations in the usual way. For example:

$$\frac{\frac{a}{m}}{\frac{n}{}} = a \cdot \frac{n}{m} = \frac{an}{m};$$

$$\frac{m + \frac{x}{y}}{n - \frac{x}{y}} = \frac{\frac{my + x}{y}}{\frac{ny - x}{y}} = \frac{my + x}{y} \cdot \frac{y}{ny - x} = \frac{my + x}{ny - x}.$$

FRACTIONAL EQUATIONS

The technique just set forth for the manipulation of fractions has its practical application in the solution of problems. Consider the following case: "A merchant invests a certain amount of money in a business. His living expenses amount to $5,000 yearly, and he increases the remainder by $33\frac{1}{3}\%$. After three years, he finds that his capital is twice as large as at first. Find the amount of money invested."

If S = amount of money invested,

$S - 5000$ = remainder at end of first year,

$\frac{S - 5000}{3}$ = increase at end of first year,

and $S - 5000 + \frac{S - 5000}{3}$ = capital at the beginning of the second year.

Adding these expressions,

$$S - 5000 + \frac{S - 5000}{3} = \frac{3S - 15000 + S - 5000}{3} = \frac{4S - 20000}{3}.$$

Since he spends \$5,000 every year,

$\frac{4S - 20000}{3} - 5000$ = remainder at end of second year.

$$\frac{4S - 20000}{3} - 5000 = \frac{4S - 20000 - 15000}{3} = \frac{4S - 35000}{3}.$$

$\frac{4S - 35000}{3} \div 3$ = increase at end of second year.

$$\frac{4S - 35000}{3} \div 3 = \frac{4S - 35000}{3} \times \frac{1}{3} = \frac{4S - 35000}{9}.$$

$\frac{4S - 35000}{3} + \frac{4S - 35000}{9}$ = capital at the beginning of the third year.

$$\frac{4S - 35000}{3} + \frac{4S - 35000}{9} = \frac{12S - 105000 + 4S - 35000}{9}$$

$$= \frac{16S - 140000}{9}.$$

He again spends \$5,000 during the third year.

$\frac{16S - 140000}{9} - 5000$ = remainder at end of third year.

$$\frac{16S - 140000}{9} - 5000 = \frac{16S - 140000 - 45000}{9} = \frac{16S - 185000}{9}.$$

$\frac{16S - 185000}{9} \div 3$ = increase at end of third year.

$$\frac{16S - 185000}{9} \div 3 = \frac{16S - 185000}{9} \times \frac{1}{3} = \frac{16S - 185000}{27}.$$

$$\frac{16S - 185000}{9} + \frac{16S - 185000}{27} = \text{capital at end of the third year.}$$

$$\frac{16S - 185000}{9} + \frac{16S - 185000}{27} = \frac{48S - 555000 + 16S - 185000}{27}$$

$$= \frac{64S - 740000}{27}.$$

Since at the end of three years his capital is twice as large as at first,

$$\frac{64S - 740000}{27} = 2S$$

$$64S - 740000 = 54S$$

$$10S = 740000$$

$$S = \$74{,}000 \quad (\textit{Answer}).$$

Observe now another case: "The pilot of an airplane flies 250 miles with the wind in the same time that he flies 150 miles against the wind. The plane can fly at the rate of 100 miles per hour in calm weather. Find the velocity of the wind."

Let x = velocity of the wind.

Then,

$100 + x$ = velocity of the plane with the wind, and
$100 - x$ = velocity of the plane against the wind.

Since distance divided by the velocity is equal to the time necessary to cover the given distance,

$\frac{250}{100 + x}$ = time taken to fly 250 miles with the wind, and

$\frac{150}{100 - x}$ = time taken to fly 150 miles against the wind.

These times are equal; therefore,

$$\frac{250}{100 + x} = \frac{150}{100 - x}.$$

Multiplying both sides by $100 + x$,

$$\frac{250(100 + x)}{100 + x} = \frac{150(100 + x)}{100 - x}$$

or

$$250 = \frac{150(100 + x)}{100 - x}$$

Multiplying both sides by $100 - x$,

$$250(100 - x) = \frac{150(100 + x)(100 - x)}{100 - x}$$

or $$250(100 - x) = 150(100 + x);$$

that is, $$25000 - 250x = 15000 + 150x.$$

Transposing $-250x$ to the right side of the equation and 15000 to the left side gives:

$$25000 - 15000 = 150x + 250x$$

so that $$10000 = 400x$$

$$\frac{10000}{400} = x,$$

and $$x = 25 \quad (\textit{Answer}).$$

Therefore, the velocity of the wind is 25 miles per hour.

Problems

1. In designing a new automobile, it is found that, according to the tentative plan, the front axle is carrying a load of 400 pounds and the rear axle a load of 500 pounds. How many pounds must be taken from the load of the front axle and added to that of the rear axle so that the former will have three-fifths as much to carry as the latter?

2. In a certain factory, one machine can turn out a certain number of articles in 4 hours, a second can turn out the same number in 8 hours, and a third in 10 hours. How long will it take to fill a rush order for this number of articles, if all three machines are worked simultaneously?

3. It is desired to separate 6200 ball bearings into two parts such that four-ninths of the greater shall be 300 more than one-half of the smaller. How many ball bearings should be in each group?

4. A post is a third of its length in ground under water, a fourth of its length is in water, and it extends 25 feet above water. What is the length of the post?

5. A broker left $49,000 to his wife, three sons, and two daughters. Each son received $\frac{4}{5}$ and each daughter $\frac{3}{4}$ as much as the wife's share. What was the share of each?

6. A battle cruiser travelling 28 miles an hour starts $\frac{3}{4}$ of an hour before a second cruiser which travels 32 miles an hour. In how many hours will the first cruiser be overtaken by the second?

7. After expending $\frac{2}{5}$ of his money and \$16, a sailor had left $\frac{3}{7}$ of his money and \$20. How much money had he at first?

8. Divide 152 into two parts such that, if the greater is divided by the smaller, the quotient is $\frac{5}{3}$.

9. An ounce of gold when weighed in water loses $\frac{1}{19}$ of an ounce, and an ounce of silver $\frac{2}{21}$ of an ounce. How many ounces of gold and silver are there in a mixed mass weighing 150 ounces in air and losing 10 ounces when weighed in water?

10. Two camel caravans start at the same hour from 2 oases 160 miles apart, the first travelling $3\frac{1}{2}$ miles an hour and the second $4\frac{1}{2}$ miles per hour. After how many hours do they meet, and how many miles does each travel?

6

MODERN WIZARDRY— FRACTIONAL EXPONENTS

The concept of a fraction, although it originated in ancient times, probably arose much later than the concept of a whole number. The Egyptians and Babylonians invented symbols for fractions and calculated with these symbols in much the same manner as we do today. However, not even the Greeks, who were the first abstract thinkers of real ability, ever dreamed of using a fraction as an exponent.*

An exponent, as explained in Chapter 3, indicates the number of times another number is to be taken as a factor. Thus, $6^3 = 6 \times 6 \times 6 = 216$, $a^3 = a \times a \times a$, and a^n means that any number a is to be taken as a factor any number, n, of times.

An arithmetic fraction, as already explained, is one or more equal integral parts of a unit or whole. The denominator indicates the number of equal integral parts into which the unit or whole has been divided, and the numerator the number of equal integral parts of the whole that are taken.

FRACTIONAL EXPONENTS

If these definitions are to remain applicable, what does $(16)^{\frac{1}{2}}$ mean? Here is a fraction written in the place corresponding to an exponent. It must, therefore, refer to factors. It does. It is a fraction; hence, it must indicate the number of equal integral parts into which the quantity has been divided and how many of these parts are to be taken. It does just that. The fraction $\frac{1}{2}$ written as an exponent indicates that the quantity, in this case 16, is to be divided into two equal factors, and that only one of these factors is to be taken. The two equal positive factors which will produce 16 are 4 and 4; if only one

* In this chapter, attention is confined to positive fractional exponents.

is taken, the answer is 4. In other words, $(16)^{\frac{1}{2}} = (4 \cdot 4)^{\frac{1}{2}} = 4$. The number 4 is then said to be the *square root* of 16. Thus, *the square root of a number is one of the two equal factors of that number.*

The notation $(27)^{\frac{1}{3}}$ means, therefore, that one of the three equal factors of 27 is to be taken. The three equal factors of 27 are $3 \times 3 \times 3$. If one of these equal factors is taken, the answer is 3. That is, $(27)^{\frac{1}{3}} = (3 \times 3 \times 3)^{\frac{1}{3}} = 3$. The number 3 is then said to be the cube root of 27. Thus, *the cube root of a number is one of the three equal factors of that number.*

Finally, $(a)^{\frac{1}{n}}$ means that one of the n equal factors of a is to be taken. Thus, the nth root of a number is one of the n equal factors of the number.

The symbol $\sqrt{\ }$, called the *radical sign*, is also used to denote the operation of finding a root. It is placed before the number whose root is to be found. The number whose root is to be found is called the *radicand.* The small number written outside the radical sign and slightly to the left and above it, and indicating the root to be found, is called the *index.*

Since the first root of a number is the number itself, the expression $\sqrt[1]{4} = 4$, and $\sqrt[1]{a} = a$.

The convention has been generally adopted that if there is no index before the radical sign, the latter symbol denotes the square root. Indeed, for centuries this was the usual meaning of the radical sign.

Thus, $(16)^{\frac{1}{2}}$, or $\sqrt{16}$, means that the square root of 16 is to be found. It is no crime to write $\sqrt[2]{16}$, but the usual custom is to omit the index 2 when a square root is indicated. Thus, $\sqrt[2]{16} = \sqrt{16}$.

$(27)^{\frac{1}{3}}$, or $\sqrt[3]{27}$, means that the cube root of 27 is to be found.

$(a)^{\frac{1}{n}}$, or $\sqrt[n]{a}$, means that the nth root of a is to be found.

EXTRACTION OF ROOTS

The *extraction of roots* now requires a brief discussion. When a number is composed of equal integral factors, two, three, or four in number, it is said to be a perfect square, perfect cube, or perfect fourth power, respectively. Thus, for the first ten integers we have:

Number.....	1	2	3	4	5	6	7	8	9	10
Perfect Square....	1	4	9	16	25	36	49	64	81	100
Perfect Cube.....	1	8	27	64	125	216	343	512	729	1,000
Perfect 4th Power.	1	16	81	256	625	1,296	2,401	4,096	6,561	10,000

Between the first three perfect squares (1, 4, and 9) lie other numbers, namely, 2 and 3 and 5, 6, 7, and 8. The latter numbers are each, respectively, the square of some number. Thus, 2 is approximately equal to 1.414×1.414. In other words, $\sqrt{2} = 1.414$ (approximately).

USE OF THE TABLE OF SQUARE ROOTS

The square root of any positive number, whether it be an integer or a decimal, can be obtained, approximately, by arithmetic, but it can be found much more quickly and easily from tables. Similarly, cube roots can be obtained by arithmetic, but the process is laborious and will not be discussed. Tables of cube roots can be used readily. Fourth, fifth, sixth, and other roots can be obtained by logarithms (see page 445).

Two pages of square roots and cube roots are here given (page 78) for numbers from 10 to 100. Particular attention is called to the footnotes showing how the square or cube root of numbers from 1 to 10, or of multiples of 10 to 100, can be obtained. For ordinary working purposes, this table is sufficient.

For review purposes, the arithmetical method of extracting a square root is given in Appendix A, on page 693. Square roots of perfect squares can sometimes be obtained by inspection or by factoring the numbers. Thus, $\sqrt{225}$ is seen at once to be 15, and $\sqrt{400}$ to be 20. But the value of $\sqrt{441}$ may not be immediately apparent. Factoring gives $\sqrt{441} = \sqrt{7 \times 7 \times 3 \times 3}$, or $\sqrt{21 \times 21}$; thus, the value of $\sqrt{441} = 21$.

From the small table on the preceding page, it is seen that the square root of a number consisting of one or two figures is a number of one figure.

The squares of:	10,	11,	12,	13	98,	99
are	100,	121,	144,	169	9,604,	9,801.

Hence, the square root of a number consisting of three or four figures is a number having two figures.

Similarly, it can be shown that the square root of a number of five or six figures which is a perfect square is a number of three figures. Hence, mark off the figures of the number in groups of two figures each, beginning at the right, and counting any single figure remaining at the extreme left as a group. The number of figures in the square root is the same as the number of groups.

In other words, *moving the decimal point one place in a number will move it two places in the square of the number.*

To change 12 into 0.12, the decimal point must be moved two places to the left; therefore, it must be moved four places to the left in the power. Thus, $(0.12)^2 = 0.0144$, for the square of 12 is 144. Moreover, $\sqrt{1.69} = 1.3$ because $\sqrt{169} = 13$. To change 169 into 1.69, the decimal point must be moved two places to the left; hence, it must be moved only one place to the left in the root.

The rule that moving the decimal point one place in a number will move it two places in the square is very useful in using tables of square roots. Suppose that the square of 120 is to be found from a table that gives only squares of numbers up to 100. The square of 12 is seen to be 144. If one zero is added to the root 12, the number becomes 120, and the decimal point has been moved one place to the right. Therefore, it must be moved two places in the square; that is, two zeros must be added to 144, making the number 14,400, so that $120^2 = 14{,}400$. In the same manner, if the square root of 16,900 is desired, the rule is applied as follows: The square root of 169 is 13; to write 169 as 16,900, the decimal point must be moved two places to the right; therefore, the decimal point must be moved only one place to the right in the root, so that $\sqrt{16{,}900} = 130$.

As may be seen in the Appendix, page 693, the first step in obtaining the square root is to separate the number into groups of two digits each. Groups are indicated from the decimal point to the left for the integral part of the root, and from the decimal point to the right for the decimal part of the root. Thus, for the square root of 171.61, groups are indicated by writing this number 1′71′.61′; its square root is 13.1. For the square root of 17.161, groups are indicated by writing this num-

SQUARE ROOTS and CUBE ROOTS

	Square Roots		Cube Roots		
n	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
10	3.16228	10.0000	2.15443	4.64159	10.0000
11	3.31662	10.4881	2.22398	4.79142	10.3228
12	3.46410	10.9545	2.28943	4.93242	10.6266
13	3.60555	11.4018	2.35133	5.06580	10.9139
14	3.74166	11.8322	2.41014	5.19249	11.1869
15	3.87298	12.2474	2.46621	5.31329	11.4471
16	4.00000	12.6491	2.51984	5.42884	11.6961
17	4.12311	13.0384	2.57128	5.53966	11.9348
18	4.24264	13.4164	2.62074	5.64622	12.1644
19	4.35890	13.7840	2.66840	5.74890	12.3856
20	4.47214	14.1421	2.71442	5.84804	12.5992
21	4.58258	14.4914	2.75892	5.94392	12.8058
22	4.69042	14.8324	2.80204	6.03681	13.0059
23	4.79583	15.1658	2.84387	6.12693	13.2001
24	4.89898	15.4919	2.88450	6.21447	13.3887
25	5.00000	15.8114	2.92402	6.29961	13.5721
26	5.09902	16.1245	2.96250	6.38250	13.7507
27	5.19615	16.4317	3.00000	6.46330	13.9248
28	5.29150	16.7332	3.03659	6.54213	14.0946
29	5.38516	17.0294	3.07232	6.61911	14.2604
30	5.47723	17.3205	3.10723	6.69433	14.4225
31	5.56776	17.6068	3.14138	6.76790	14.5810
32	5.65685	17.8885	3.17480	6.83990	14.7361
33	5.74456	18.1659	3.20753	6.91042	14.8881
34	5.83095	18.4391	3.23961	6.97953	15.0369
35	5.91608	18.7083	3.27107	7.04730	15.1829
36	6.00000	18.9737	3.30193	7.11379	15.3262
37	6.08276	19.2354	3.33222	7.17905	15.4668
38	6.16441	19.4936	3.36198	7.24316	15.6049
39	6.24500	19.7484	3.39121	7.30614	15.7406
40	6.32456	20.0000	3.41995	7.36806	15.8740
41	6.40312	20.2485	3.44822	7.42896	16.0052
42	6.48074	20.4939	3.47603	7.48887	16.1343
43	6.55744	20.7364	3.50340	7.54784	16.2613
44	6.63325	20.9762	3.53035	7.60590	16.3864
45	6.70820	21.2132	3.55689	7.66309	16.5096
46	6.78233	21.4476	3.58305	7.71944	16.6310
47	6.85565	21.6795	3.60883	7.77498	16.7507
48	6.92820	21.9089	3.63424	7.82974	16.8687
49	7.00000	22.1359	3.65931	7.88374	16.9850
50	7.07107	22.3607	3.68403	7.93701	17.0998
n	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$

Square root. Mark the number off into sets of 2 figures beginning at the decimal point. Disregarding the decimal point, if the first set contains 2 figures, use column $\sqrt{n}$; if it contains 1 figure, take the first two sets and use column $\sqrt{10n}$.

$$\sqrt{.0034} = \sqrt{34 \times .0001} = .01\sqrt{34} = .01 \times 5.83095 = .0583095$$
$$\sqrt{34{,}000} = \sqrt{340 \times 100} = 10\sqrt{340} = 10 \times 18.4391 = 184.391$$

Cube root. Mark the number off into sets of 3 figures beginning at the decimal point. Disregarding the decimal point, if the first set contains 2 figures, use column $\sqrt[3]{n}$; if it contains 3 figures, use column $\sqrt[3]{10n}$; if it contains 1 figure, take the first two sets and use column $\sqrt[3]{100n}$.

SQUARE ROOTS and CUBE ROOTS (Continued)

	Square Roots		Cube Roots		
n	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
51	7.14143	22.5832	3.70843	7.98957	17.2130
52	7.21110	22.8035	3.73251	8.04145	17.3248
53	7.28011	23.0217	3.75629	8.09267	17.4351
54	7.34847	23.2379	3.77976	8.14325	17.5441
55	7.41620	23.4521	3.80295	8.19321	17.6517
56	7.48331	23.6643	3.82586	8.24257	17.7582
57	7.54983	23.8747	3.84850	8.29134	17.8632
58	7.61577	24.0832	3.87088	8.33955	17.9670
59	7.68115	24.2899	3.89300	8.38721	18.0697
60	7.74597	24.4949	3.91487	8.43433	18.1712
61	7.81025	24.6982	3.93650	8.48093	18.2716
62	7.87401	24.8998	3.95789	8.52702	18.3709
63	7.93725	25.0998	3.97906	8.57262	18.4691
64	8.00000	25.2982	4.00000	8.61774	18.5664
65	8.06226	25.4951	4.02073	8.66239	18.6626
66	8.12404	25.6905	4.04124	8.70659	18.7578
67	8.18535	25.8844	4.06155	8.75034	18.8520
68	8.24621	26.0768	4.08166	8.79366	18.9454
69	8.30662	26.2679	4.10157	8.83656	19.0378
70	8.36660	26.4575	4.12129	8.87904	19.1293
71	8.42615	26.6458	4.14082	8.92112	19.2200
72	8.48528	26.8328	4.16017	8.96281	19.3098
73	8.54400	27.0185	4.17934	9.00411	19.3988
74	8.60233	27.2029	4.19834	9.04504	19.4870
75	8.66025	27.3861	4.21716	9.08560	19.5743
76	8.71780	27.5681	4.23582	9.12581	19.6610
77	8.77496	27.7489	4.25432	9.16566	19.7468
78	8.83176	27.9285	4.27266	9.20516	19.8319
79	8.88819	28.1069	4.29084	9.24434	19.9163
80	8.94427	28.2843	4.30887	9.28318	20.0000
81	9.00000	28.4605	4.32675	9.32170	20.0830
82	9.05539	28.6356	4.34448	9.35990	20.1653
83	9.11043	28.8097	4.36207	9.39780	20.2469
84	9.16515	28.9828	4.37952	9.43539	20.3279
85	9.21954	29.1548	4.39683	9.47268	20.4083
86	9.27362	29.3258	4.41400	9.50969	20.4880
87	9.32738	29.4958	4.43105	9.54640	20.5671
88	9.38083	29.6648	4.44796	9.58284	20.6456
89	9.43398	29.8329	4.46475	9.61900	20.7235
90	9.48683	30.0000	4.48140	9.65489	20.8008
91	9.53939	30.1662	4.49794	9.69052	20.8776
92	9.59166	30.3315	4.51436	9.72589	20.9538
93	9.64365	30.4959	4.53065	9.76100	21.0294
94	9.69536	30.6594	4.54684	9.79586	21.1045
95	9.74679	30.8221	4.56290	9.85048	21.1791
96	9.79796	30.9839	4.57886	9.86485	21.2532
97	9.84886	31.1448	4.59470	9.89898	21.3267
98	9.89949	31.3050	4.61044	9.93288	21.3997
99	9.94987	31.4643	4.62606	9.96655	21.4723
100	10.00000	31.6228	4.64159	10.00000	21.5443
n	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$

$\sqrt[3]{34{,}000} = \sqrt{34 \times 1{,}000} = 10\sqrt[3]{34} = 10 \times 3.23961 = 32.3961$

$\sqrt[3]{0.34} = \sqrt{340 \times .001} = .1\sqrt[3]{340} = .1 \times 6.97953 = .697953$

$\sqrt[3]{3.4} = \sqrt{3400 \times .001} = .1\sqrt[3]{3400} = .1 \times 15.0369 = 1.50369$

ber '17'.16'10'; its square root is 4.104. Notice that the sequence of digits in the square root is not independent of the position of the decimal point. Therefore, it is suggested that, in finding the square root of any number, the first digit in the root be ascertained by considering the first group of digits in the square.

To find the square root of a common fraction, if the terms of the fraction are perfect squares, extract the square roots of the numerator and denominator separately. If the terms of the fraction are not perfect squares, reduce the fraction to a decimal, and then extract the square root.

ILLUSTRATION: Find the value of $\sqrt{\frac{49}{289}}$.

SOLUTION: Here it is seen that $49 = 7^2$ and $289 = 17^2$.

$$\sqrt{\frac{49}{289}} = \frac{\sqrt{49}}{\sqrt{289}} = \frac{7}{17}.$$

As for *roots of order higher than 3,* from an earlier paragraph it is clear that the square root of a square root of any number is the same as the fourth root of that number. For example,

$$\sqrt{\sqrt{16}} = (16^{1/2})^{1/2} = 16^{1/4} = \sqrt[4]{16} = 2.$$

Similarly, the cube root of a cube root of any number is the same as the ninth root of that number:

$$\sqrt[3]{\sqrt[3]{512}} = (512^{1/3})^{1/3} = (512)^{1/9} = \sqrt[9]{512} = 2.$$

In actual business practice, such as in bond, annuity, and sinking fund computations, where quantities often run to many decimal places, roots and powers of all kinds are easily and quickly obtained by means of logarithms, as will be explained in Chapter 25.

MEANING OF FRACTIONAL EXPONENTS

The symbol $(64)^{2/3}$ states that two of the three equal factors of 64 are to be taken. The three equal factors of 64 are $4 \times 4 \times 4$. If two of these equal factors are taken, the answer is $4 \times 4 = 16$. That is, $(64)^{2/3} = (4 \times 4 \times 4)^{2/3} = 4 \times 4 = 16$.

The operation just performed can be described by stating that (*a*) the cube root of 64 was found, and (*b*) that this cube root, namely, 4, was then raised to the second power.

So that, using the radical sign, this operation may be symbolized as follows: $(\sqrt[3]{64})^2$. That is, $(64)^{\frac{2}{3}} = (\sqrt[3]{64})^2$.

In general, *in a fractional exponent, the denominator indicates the required root, while the numerator denotes the power to which the root is to be raised.*

Thus, if a, m, and n represent any numbers,

$$(\sqrt[n]{a})^m = a^{\frac{m}{n}}.$$

Or, in other words, the nth root of a is to be found, and this root is to be raised to the mth power.

Since the expression $a^{\frac{1}{n}}$ means $\sqrt[n]{a}$, then the nth root of a raised to the mth power may also be written $[a^{\frac{1}{n}}]^m$. Therefore,

$$[a^{\frac{1}{n}}]^m = a^{\frac{m}{n}}.$$

This result could be obtained by applying to the expression on the left-hand side of this equality the third law of exponents (see page 30).

So that, from the meaning given to fractional exponents in these discussions, it follows that the laws of exponents hold whether the exponents are integral or fractional, positive or negative.

The order in which these operations are performed is immaterial. For example, $(\sqrt[3]{64})^2 = \sqrt[3]{(64)^2}$, since $\sqrt[3]{(64)^2} = \sqrt[3]{64 \times 64} = \sqrt[3]{4{,}096} = (16 \times 16 \times 16)^{\frac{1}{3}} = 16 = (\sqrt[3]{64})^2$.

In general,

$$[a^{\frac{1}{n}}]^m = [a^m]^{\frac{1}{n}}.$$

Applying the second law of exponents,

$$(a \cdot b)^{\frac{1}{n}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}.$$

This notation may be expressed in its equivalent form by using the radical sign:

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

Translating this symbolism into English: If the signs of the roots of the product of two or more numbers are not taken into account, the root of the product is equal to the product of the roots of the numbers.

If the signs of the roots are taken into account, it is always possible by a proper choice of the sign to extend the statement generally. Thus, $\sqrt{4 \times 9} = \sqrt{36} = \pm 6$. But $\sqrt{4 \times 9} = \sqrt{4}\sqrt{9} = (\pm 2)(\pm 3)$. To obtain 6 as the chosen root, we take either $(+2) \times (+3) = 6$ or $(-2) \times (-3) = 6$. To obtain -6 as the chosen root, we take either $(-2) \times (+3) = -6$ or $(+2) \times (-3) = -6$.

In practice, the sign $\sqrt{}$ over a positive quantity means, unless otherwise indicated, the positive square root. If a is positive, $a^{\frac{1}{2}}$, that is, $\sqrt{a}$, means not merely some number whose square is a, but the positive number whose square is a. Thus, $\sqrt{4 \times 9} = \sqrt{36} = 6$.

SIMPLIFICATION OF RADICALS

By means of the principles just set forth, we can simplify a quantity under a radical sign. For example:

$$\sqrt[3]{16} = \sqrt[3]{8 \times 2} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}.$$

Moreover, $3\sqrt{48m^5x^3y^2} = 3\sqrt{16m^4x^2y^2 \cdot 3mx} = 3\sqrt{16m^4x^2y^2}\sqrt{3mx} = 3(4m^2xy)\sqrt{3mx} = 12m^2xy\sqrt{3mx}$.

If the quantity under the radical sign is a fraction, the expression can be simplified by multiplying both the numerator and the denominator of the fraction by a quantity such that the new denominator thus obtained can be expressed as a perfect power having an exponent equal to the index of the radical.

For example, to simplify $\sqrt[3]{\frac{16}{25}}$, multiply both terms of the fraction by a number which will make the denominator a perfect cube. Since 25 is equal to 5^2, the number sought is 5. That is, $5 \times 5 \times 5 = 125 = 5^3$.

Hence,

$$\sqrt[3]{\tfrac{16}{25}} = \sqrt[3]{\tfrac{16}{25} \cdot \tfrac{5}{5}} = \sqrt[3]{\tfrac{80}{125}} = \sqrt[3]{\tfrac{8}{125} \cdot 10} = \tfrac{2}{5}\sqrt[3]{10}.$$

Similarly,

$$\sqrt{\frac{2mx}{3y}} = \sqrt{\frac{2mx}{3y} \cdot \frac{3y}{3y}} = \sqrt{\frac{6mxy}{9y^2}} = \frac{\sqrt{6mxy}}{3y}.$$

A radical in its simplest form must *not* contain under the radical sign:

(*a*) factors of which the root indicated may be taken, or (*b*) fractions.

SIMILAR RADICALS

Two or more radicals having the same indices and the same radicand are said to be *similar*. When various radicals have each been reduced to the simplest form, it is then much easier to ascertain whether or not they are similar.

For example, it is difficult to determine whether $5\sqrt[3]{32}$ and $2\sqrt[3]{108}$ are similar. However, after reducing these radicals to their simplest forms,

$$5\sqrt[3]{32} = 5\sqrt[3]{8 \cdot 4} = 5\sqrt[3]{8} \cdot \sqrt[3]{4} = 10\sqrt[3]{4}, \text{ and}$$

$$2\sqrt[3]{108} = 2\sqrt[3]{27 \cdot 4} = 2\sqrt[3]{27} \cdot \sqrt[3]{4} = 6\sqrt[3]{4},$$

it is easily seen that the given radicals are similar.

Now,

$$(m - n)\sqrt{\frac{m + n}{m - n}} \text{ and } (m + n)\sqrt{\frac{m - n}{m + n}}$$

appear to be dissimilar in their present form. When the expressions are reduced to a simpler form, the following result is obtained:

$$(m - n)\sqrt{\frac{m + n}{m - n}} = (m - n)\sqrt{\frac{m + n}{m - n} \cdot \frac{m - n}{m - n}} = (m - n)\sqrt{\frac{m^2 - n^2}{(m-n)^2}}$$

$$= \frac{m - n}{m - n}\sqrt{m^2 - n^2} = \sqrt{m^2 - n^2}.$$

$$(m + n)\sqrt{\frac{m - n}{m + n}} = (m + n)\sqrt{\frac{m - n}{m + n} \cdot \frac{m + n}{m + n}} = (m + n)\sqrt{\frac{m^2 - n^2}{(m+n)^2}}$$

$$= \frac{m + n}{m + n}\sqrt{m^2 - n^2} = \sqrt{m^2 - n^2}.$$

That is, the given expressions are not only similar: they are equal.

OPERATIONS WITH EXPONENTS

It is sometimes convenient to express radicals of different degrees by means of the same index. For example, $\sqrt[3]{7}$ and $\sqrt{3}$ can be written with the same index, as follows: The $\sqrt[3]{7} = (7)^{\frac{1}{3}}$, and $\sqrt{3} = (3)^{\frac{1}{2}}$. The fractional exponents $\frac{1}{3}$ and $\frac{1}{2}$

can be expressed with the common denominator 6 by writing $\frac{1}{3} = \frac{2}{6}$ and $\frac{1}{2} = \frac{3}{6}$. So that $\sqrt[3]{7} = (7)^{\frac{1}{3}} = (7)^{2/6} = \sqrt[6]{(7)^2}$; and $\sqrt{3} = (3)^{\frac{1}{2}} = (3)^{3/6} = \sqrt[6]{(3)^3}$.

Only similar radicals can be united by addition or subtraction. The addition or subtraction of dissimilar terms can only be indicated.

Thus, the operation $\sqrt{72} - \sqrt{50} + \sqrt{18}$ is performed by first reducing the radicals to their simplest form and then carrying out the indicated operations:

$$\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$$

$$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$$

$$\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$$

$$\sqrt{72} - \sqrt{50} + \sqrt{18} = 6\sqrt{2} - 5\sqrt{2} + 3\sqrt{2} = 4\sqrt{2} \quad (\textit{Answer}).$$

Similarly,

$$4\sqrt{\frac{x}{2}} - x\sqrt{\frac{2}{x}} + 3\sqrt{2x} = 4\sqrt{\frac{x}{2} \cdot \frac{2}{2}} - x\sqrt{\frac{2}{x} \cdot \frac{x}{x}} + 3\sqrt{2x}$$

$$= 4\sqrt{\frac{2x}{4}} - x\sqrt{\frac{2x}{x^2}} + 3\sqrt{2x} = 2\sqrt{2x} - \sqrt{2x} + 3\sqrt{2x} = 4\sqrt{2x}$$

(*Answer*).

From the definition of exponents and roots, it is evident that:

$$(\sqrt{a})^2 = \sqrt{a} \cdot \sqrt{a} = a,$$

for

$$(a^{1/2})^2 = a^{2/2} = a^1 = a.$$

As previously explained on page 81,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b};$$

and since

$$\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}},$$

then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Therefore, *to multiply or divide radicals, reduce them, if neces-*

sary, to the same index. Multiply or divide the coefficients together for a new coefficient. Multiply or divide the quantities under the radical sign together for a new quantity under the radical sign. Simplify the result.

Thus,

$$3\sqrt{27} \times 5\sqrt{6} = 15\sqrt{27 \times 6} = 15\sqrt{162} = 15\sqrt{81 \cdot 2} = 135\sqrt{2};$$

and

$$\frac{3\sqrt{27}}{5\sqrt{6}} = \frac{3}{5}\sqrt{\frac{27}{6}} = \frac{3}{5}\sqrt{\frac{27}{6} \cdot \frac{6}{6}} = \frac{3}{5 \times 6}\sqrt{162} = \frac{1}{10}\sqrt{81 \cdot 2} = \frac{9}{10}\sqrt{2}.$$

Similarly,

$$\sqrt{\frac{3mx^3}{2b^2y}} \cdot \sqrt{\frac{4b^4y}{9mx^4}} = \sqrt{\frac{12mx^3b^4y}{18b^2ymx^4}} = \sqrt{\frac{2b^2}{3x}} = \sqrt{\frac{2b^2}{3x} \cdot \frac{3x}{3x}} = \frac{b\sqrt{6x}}{3x};$$

and

$$\frac{\sqrt{8x^3y^5}}{\sqrt{2xy^2}} = \sqrt{\frac{8x^3y^5}{2xy^2}} = \sqrt{4x^2y^3} = 2xy\sqrt{y}.$$

Expressions containing one or more radical terms are multiplied by other such expressions in the same manner as any mathematical expression containing two or more different terms (see page 34). Thus, $(\sqrt{6} + \sqrt{8})$ is multiplied by $(\sqrt{2} + \sqrt{8})$ as follows:

$$\begin{array}{l} \sqrt{6} + \sqrt{8} \\ \sqrt{2} + \sqrt{8} \\ \hline \sqrt{12} + \sqrt{16} + \sqrt{48} + 8 = 2\sqrt{3} + 4 + 4\sqrt{3} + 8, \end{array}$$

or $12 + 6\sqrt{3}$.

Similarly,

$$\begin{array}{l} \sqrt{2x} + \sqrt{3y} \\ \sqrt{x} + \sqrt{y} \\ \hline x\sqrt{2} + \sqrt{3xy} + \sqrt{2xy} + y\sqrt{3}, \text{ or } \sqrt{2}x + (\sqrt{3} + \sqrt{2})\sqrt{xy} + \sqrt{3}y. \end{array}$$

The indicated quotient of two radical expressions containing one or more terms may be simplified by expressing the denominator free of radicals. This is accomplished by multiplying both terms of the fraction by a properly selected radical ex-

pression. This radical expression is selected from the special products discussed in Chapter 5 and repeated here for convenience:

$$\text{I.} \quad (a+b)(a-b) = a^2 - b^2.$$
$$\text{II.} \quad (a+b)(a^2 - ab + b^2) = a^3 + b^3.$$
$$\text{III.} \quad (a-b)(a^2 + ab + b^2) = a^3 - b^3.$$

Thus,

$$\frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{4}} = \frac{3}{2}\sqrt{2},$$

and

$$\frac{3\sqrt{b}}{\sqrt{a}} = \frac{3\sqrt{b}}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{3\sqrt{ab}}{\sqrt{a^2}} = \frac{3}{a}\sqrt{ab}.$$

Moreover,

$$\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \cdot \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} - \sqrt{y}} = \frac{x - 2\sqrt{xy} + y}{x - y}$$

and

$$\frac{\sqrt{a} + \sqrt{b}}{3\sqrt{a} - \sqrt{b}} = \frac{\sqrt{a} + \sqrt{b}}{3\sqrt{a} - \sqrt{b}} \cdot \frac{3\sqrt{a} + \sqrt{b}}{3\sqrt{a} + \sqrt{b}} = \frac{3a + 4\sqrt{ab} + b}{9a - b}$$

and

$$\frac{\sqrt{x} + \sqrt{y}}{\sqrt[3]{x} - \sqrt[3]{y}} = \frac{\sqrt{x} + \sqrt{y}}{\sqrt[3]{x} - \sqrt[3]{y}} \cdot \frac{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}$$
$$= \frac{x + 2\sqrt[3]{x^2y} + 2\sqrt[3]{xy^2} + y}{x - y}.$$

SOLUTION OF EQUATIONS INVOLVING RADICALS

The necessity of thoroughly covering the most important cases that occur in the manipulation of radicals arises from the fact that situations involving radicals are often encountered in the solution of practical problems.

Assume that the equation

$$\sqrt{x + 10} - \sqrt{x - 2} = 2$$

arises from the solution of a problem involving a moving object. (A similar situation will be encountered later in the develop-

ment of the equations of two well-known curves, the ellipse and the hyperbola.)

Transposing the term $\sqrt{x-2}$, so that one radical shall be alone on one side of the equation,

$$\sqrt{x+10} = 2 + \sqrt{x-2}.$$

Squaring both sides,

$$x + 10 = (2 + \sqrt{x-2})^2 = (2 + \sqrt{x-2})(2 + \sqrt{x-2})$$

or

$$x + 10 = 4 + 4\sqrt{x-2} + x - 2.$$

Collecting and transposing terms, so that the term containing the radical shall be alone on one side of the equation,

$$8 = 4\sqrt{x-2}.$$

Dividing both sides by 4,

$$2 = \sqrt{x-2}.$$

Squaring both sides,

$$4 = x - 2.$$

Hence,

$$6 = x.$$

Substituting this solution in the original equation to check the problem,

$$\sqrt{6+10} - \sqrt{6-2} = 2,$$

$$\sqrt{16} - \sqrt{4} = 4 - 2 = 2.$$

It is very important to note that *squaring each of the two members of an equation may introduce solutions which the original equation did not possess.*

These solutions, introduced in the process of solving an equation, but which the original equation does not possess, are called *extraneous solutions.*

In some cases, all the solutions found after squaring each of the two members of the given equation are extraneous. Under such circumstances, the given equation does not represent an equality which is true for some value of the unknown.

Thus, to solve:

$$\sqrt{x} - \sqrt{x+2} = 2,$$

transpose, so that one radical shall be alone on one side of the equation:

$$\sqrt{x} - 2 = \sqrt{x + 2}.$$

Square both sides:

$$x - 4\sqrt{x} + 4 = x + 2.$$

Collect and transpose terms so that again one radical shall be alone on one side of the equation:

$$2 = 4\sqrt{x}.$$

Divide by 4:

$$\tfrac{1}{2} = \sqrt{x}.$$

Square both sides:

$$\tfrac{1}{4} = x.$$

But on substituting this value in the original equation,

$$\sqrt{\tfrac{1}{4}} - \sqrt{\tfrac{1}{4} + 2} = \tfrac{1}{2} - \sqrt{\tfrac{9}{4}} = \tfrac{1}{2} - \tfrac{3}{2} = -\tfrac{2}{2} = -1 \neq 2.*$$

Therefore, the original equation is not true, because it does not represent an equality.

Problems

Note: The various formulas introduced in the following problems will be explained later. They are given here merely for the sake of practice with evolution and involution as the latter occur in practical subjects.

1. Formulas for the area of a circle are:

$$A = \pi r^2.$$
$$A = \tfrac{1}{4}\pi d^2.$$

A = area; $\pi = 3.1416$; r = radius; and d = diameter. Find the area of a circular sheet of iron whose diameter is 8 inches

2. What is the area of one side of a flat iron ring whose inside diameter is 14 inches and whose outside diameter is 18 inches?

3. The glass mosaics to go into the design of a circular church window have an aggregate area of 314.16 square feet. What diameter must be allowed for the window?

4. The total surface (S) of a right circular cylinder, exemplified

* The symbol $\neq$ is read "is not equal to."

by an ordinary can of condensed milk, is equal to the curved surface plus the area of the two ends. The curved surface is obtained by multiplying the circumference by the altitude.

Thus, $$S = \pi dh + 2\frac{\pi d^2}{4},$$

or $$S = 2\pi rh + 2\pi r^2.$$

Find the total surface of a cast steel piston barrel, the dimensions of which are: diameter, 4 inches; length, 14 inches.

5. The total surface of a right circular cylinder 2 feet high is to be 416.26 square inches. How large must the diameter be?

6. A cube of gold containing 343 cubic inches is wanted. What will have to be the length of an edge?

7. [The volume of a cylinder is equal to the area of a base times the height. A gallon of water (231 cubic inches) weighs 8.345 pounds. A cubic foot of water weighs 62.428 pounds.] How many gallons of water will be contained in a cylinder whose inside diameter is 4 feet and whose length is 16 feet? What is the weight of this water?

8. A gallon of a certain kind of paint will cover 700 square feet of metal (with one coating). How many gallons will be needed to paint one mile of cable 2 inches in diameter?

9. A cylinder, to be 10 feet long, is to contain 528.8 gallons of water. What will be its diameter?

10. (The volume of a sphere is $\frac{4\pi}{3}$ times the cube of the radius. That is, $V = \frac{4\pi r^3}{3}$.)

(*a*) A sphere is to contain 2,144.7 cubic inches. What must be its diameter?

(*b*) A spherical copper retort is to contain 4,189 gallons of water. What will the diameter of the retort need to be?

7 SOME ELEMENTARY CONCEPTS OF SPACE

Through our senses, directly or indirectly, we observe differences between objects compared. These differences may be in shape, size, weight, color, time, or various other characteristics. The process of measuring, as of counting, consists fundamentally of comparing one thing with another.

Courtesy of Librairie Ancienne, Leo S. Olschki, Florence

The woodcut is from *Annali letterarj*, Modena, Zatta, 1762-64. Notice the palm, width of the human hand, the foot width of four hands, the simple step, and the geometric step.

Primitive measurements were made by comparing an object with various parts of the human body. Thus, originally an *inch* was the length of the second bone of an adult's thumb, a *palm* was the width of a hand, an *ell* the distance from the elbow to the end of the middle finger, and a *foot* the length of a human foot. Since human beings are not all of the same size, the little man, in selling goods, had an advantage over his customers. We can imagine the possibilities of haggling and bickering when measurements were so variable.

The need for some unchanging standard, independent of personal characteristics or mental experiences, whereby differences between objects might be measured, was gradually supplied. Agreement upon standards is a most important basis of modern civilization.

STANDARDS OF LENGTHS

The fundamental units of our national system of weights and measures are defined in terms of the metric system. Our common units of length, such as the inch, foot, yard, rod, and mile, were derived from British standards. In some cases, they have been altered since the original United States legislation. American measures of length are given in the following table:

LINEAR MEASURE		SURVEYORS' LINEAR MEASURE	
12 inches (in.)	= 1 foot (ft.)	7.92 inches	= 1 link
3 feet	= 1 yard (yd.)	25 links	= 1 rod
5½ yards or 16½ feet	= 1 rod (rd.)	4 rods	= 1 chain
320 rods	= 1 mile (mi.)	80 chains	= 1 mile
1 mile (1,760 yds.)	= 5,280 ft.	1 mile = 80 chains	= 5,280 ft.

MARINERS' LINEAR MEASURE

9 inches	= 1 span (sp.)
8 spans (6 ft.)	= 1 fathom (fath.)
120 fathoms	= 1 cable length
7½ cable lengths	= 1 knot or nautical mile
3 miles	= 1 league
1 degree (at Equator)	= 68.7 statute miles

LENGTH—METRIC SYSTEM

The base of the metric system is the *meter*, which, by definition, is the length, under standard conditions, of a bar made of

platinum and iridium. This bar, called the International Prototype Meter, is kept in France. Two copies of this bar are in the possession of the United States Bureau of Weights and Measures in Washington, D. C.*

Multiples and subdivisions are obtained by multiplying or dividing by 10. The great practical advantage of this system is that measures of the same kind are converted from one unit to another simply by shifting the decimal point. To indicate multiples, Greek prefixes are used; Latin prefixes are used to indicate subdivisions.

Myria signifies ten thousand.	*Meter* is the unit of measurement.
Kilo signifies one thousand.	*Deci* signifies one-tenth.
Hecto signifies one hundred.	*Centi* signifies one-hundredth.
Deca signifies ten.	*Milli* signifies one-thousandth.

The measures of length are, therefore:

		Abbreviation
Myriameter	= 10,000 meters	Mm.
Kilometer	= 1,000 "	Km.
Hectometer	= 100 "	Hm.
Decameter	= 10 "	Dm.
Meter	= the unit of length	m.
Decimeter	= 0.1 of a meter	dm.
Centimeter	= 0.01 of a meter	cm.
Millimeter	= 0.001 of a meter	mm.

The following table gives the equivalences of American and metric measures of length:

POINT AND LINE

The ancient Egyptians developed a system of land measurement to redistribute farm land the boundaries of which, every year, were eradicated by the annual floods of the Nile. This system consisted simply of a body of facts applied by rule-of-thumb methods. These rules seem to have been obtained not by deductive reasoning, but by long experience. The selection of definite standards of measurement and agreement as to

* This bar was intended to be equal in length to one ten-millionth of the earth's quadrant. Later, by more careful investigation, scientists ascertained that a quadrant of a meridian is 10,002,208 meters, when this bar is used as a unit of measurement.

The standard remains, however, as first computed. Therefore, the present standard meter is a little less than $\frac{1}{10,000,000}$ of a quadrant of the terrestrial meridian. The International Prototype Meter is 1,553,164.13 times the wave length of the red cadmium line in air, at 760 mm. pressure, 15° Centigrade.

AMERICAN EQUIVALENTS OF METRIC SYSTEM		METRIC EQUIVALENTS OF AMERICAN SYSTEM	
1 millimeter	= 0.03937 of an inch	1 inch	= 2.54 centimeters
1 centimeter	= 0.3937 of an inch	1 foot or 12 inches	= 30.48 centimeters
1 decimeter	= 3.937 inches or 0.3281 of a foot	1 yard or 3 feet	= 91.44 centimeters or 0.9144 of a meter
1 meter	= 39.37 inches or 3.281 feet or 1.0936 yards	1 rod or 16.5 feet	= 5.0292 meters
1 decameter	= 10.936 yards or 1.98838 rods	1 furlong or 220 yards	= 201.168 meters
1 hectometer	= 19.8838 rods	1 mile or 5,280 feet	= 1.60935 kilometers
1 kilometer	= 0.62137 of a mile	1 degree (at Equator)	= 110.56 kilometers

certain rules of computation were the first steps toward the creation of a mathematical science. To the Egyptians, a point, a straight line, a circle, or a plane was a physical, concrete thing. For the further progress of mathematics and, therefore, of civilization, it was necessary to form abstract ideas of these matters.

The shape and size of objects and the motion of objects cannot be conveyed by human language unless there be clear agreement in regard to certain elementary concepts regarding space. It was stated on page 1 that numbers are ideas, and that the words or figures which represent them are merely symbols. Similarly, when dots, lines, curves, and so on, are used to denote something occurring in space, it is highly important to remember always that what we draw on paper are symbols and *nothing else*. Thus, a point is not the dot or period at the end of this sentence. *A dot symbolizes a point.* A point exists only in the mind, because it means something which has neither length, breadth, nor thickness. A point indicates location only. But a dot on paper must have some length, some breadth, and some thickness, for otherwise we could not see it. Speaking precisely, nobody has ever seen a separate point.

A straight line may be described as the path traced by a point moving in a constant direction. Therefore, a line has only one

dimension, length. Nobody can possibly draw a straight line, because, first, what one draws, as already stated, has three dimensions, and second, no matter how carefully the line be constructed, a microscope will reveal a multitude of imperfections. The keenest razor blade, if greatly magnified, looks like a saw. Unfortunately, because of the limitations of our senses and of our human speech, conceived in ignorance and born with inexactitude, we must in our daily vocabulary call this ——— a straight line, although strictly it is only the *symbol* of a straight line (or portion of one).

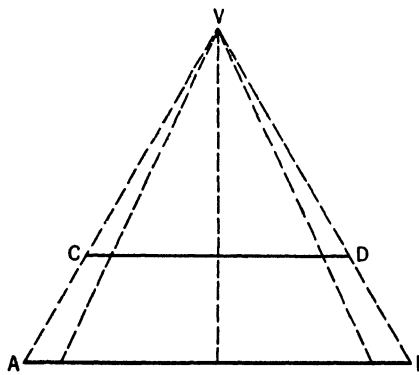


Fig. 6.

The erroneous idea that a straight line is built by a very large (but finite) number of points, placed one next to the other, leads to an obvious contradiction. Assume this definition of a straight line to be true. Then it is easy to show that the two lines AB and CD shown in Figure 6, although evidently different in length, are nevertheless equal in the sense of being composed of the same number of points. Draw two lines passing through the points AC and BD and meeting at the point V. From every point on the line AB, a line may be drawn to V. Every one of these lines will cut the line CD at a point. This is easy to visualize as follows: Hold a thin piece of thread so that one end is at V and the other end is placed at B. Then, keeping the end at V fixed at this point, and holding the thread taut, move the thread at B along the line AB toward A. Since every one of these lines cuts CD, for every point on the line AB there is a corresponding point on the line CD. Hence, the line CD has exactly the same number of points as AB, and therefore is equal to it!

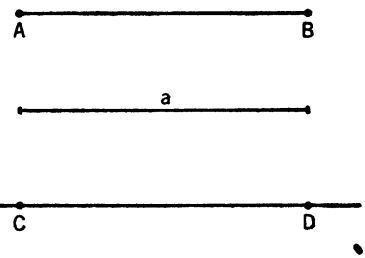
The fallacy here is not in the reasoning, but in the assumption. In assuming that a line is made up of a very large finite number of adjacent points, we contradict the definition that a point indicates location only and has neither length, breadth, nor thickness. That is, we confuse the symbol (·) used to denote a point with the abstract idea of a point. For it is evident

that a finite number of these dots (·) used to symbolize a point when put together will make the symbol ———— used to denote a line; but it is also easy to see that since a point has no dimensions, no matter what finite number of them are put together, we still cannot obtain something that does possess a dimension.

Consequently, the definitions that follow should be read carefully and the distinction between the idea and the symbol used to denote it kept clearly and distinctly in mind.

STRAIGHT LINE: DEFINITIONS

A straight line is regarded as extending indefinitely in each direction, that is, as infinite in length. That portion of a straight line which measures the distance between two particular points is called a *line segment* or, more briefly, a *segment*, of which the two given points are end-points.



ASSUMPTION: *A line segment may be extended to any required length.*

It is quite common to call a segment simply a straight line. This is one of the many little ambiguities in mathematics which may puzzle the beginner. However, to avoid misunderstanding, a particular straight line or line segment is identified by naming two of its points: as the segment AB (a line segment with end-points A and B), or by using some letter such as a to designate the line segment; or by calling the line CD, where C and D are distinct points chosen at random on the line, indefinite in length.

Usually, in elementary mathematical work, we are interested only in the length of a segment. But with a segment the idea of *direction* may also be associated. Thus, if the segment AB is regarded as extending from A to B, we would call it the *directed segment* AB. If, however, this segment is regarded as extending from B to A, it would be called the *directed segment* BA. The magnitudes of the directed segments AB and BA are the same, but their directions are opposite. This difference in direction is analogous to the difference in signed num-

bers. Thus, the signs + and − may also be used to indicate the difference in direction of a segment. Note that a directed line segment is designated by naming the initial point first. If one direction is chosen as positive, the opposite direction is negative. Thus, if AB is positive, then BA is negative, and $AB = -BA$. In the following pages by the words *line segment* (or *segment*) we shall mean, unless otherwise stated, an *undirected segment* only.

CURVE AND PLANE

The word *curve* is described as the path of a moving point. As already stated, if the motion is in a constant direction, a straight line is formed by its path, so that a straight line is really a special case included in the concept "curve." In ordinary as well as in mathematical language, the word *curve* is also used with the special meaning of the path of a moving point which changes continuously in direction.

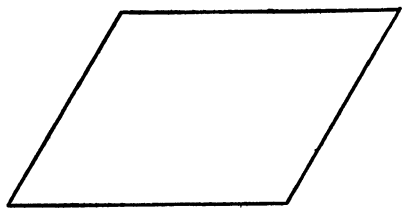
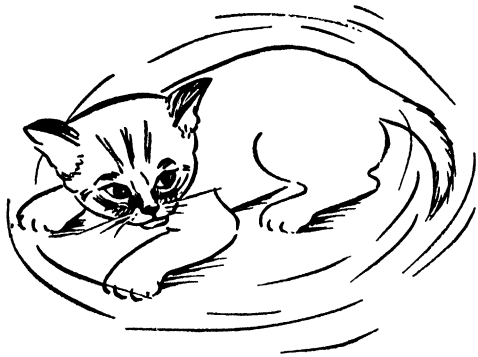


Fig. 7.

Since a straight line is always considered as potentially extending indefinitely in each direction, that is, as infinite in length, then obviously, if the line is not allowed to rotate about any one of its points, it can be thought of as moving only laterally. A straight line conceived as moving in a constant direction generates a *plane surface.* A plane surface, when considered as extending indefinitely in each of its directions, is called a *plane.* A plane is ideally and perfectly flat. Of course, in the physical world, no surface, however smooth, is perfectly flat, not even a glass table top. *Plane* is a word which symbolizes an idea, just as Figure 7 symbolizes a plane.

Readers will notice that certain words, like *direction, continuous, smooth,* and others, have been used without being strictly defined. A thing, to be defined, should be described in terms simpler than the thing itself. Definitions of ordinary terms like these accomplish little, because our understanding of their meaning may remain at best intuitive. That is, there are terms so simple and so well understood that it is not only im-

possible but unnecessary to define them. Definitions of simple terms suggest a kitten chasing its tail: we move rapidly from



one word to another, and then on to a third, rounding out our vocabulary but accomplishing little else.

FUNDAMENTAL ASSUMPTIONS

From the preceding discussion and definitions, the following facts about straight line segments may be deduced:

Through two distinct points, one and only one straight line can be drawn; that is, two distinct points determine a straight line.

In a plane, a straight line segment is the shortest path between two points.

Two straight lines in the same plane either intersect or do not meet or coincide.

If two straight line segments intersect, they can do so in only one point.

If two straight line segments in the same plane do not meet, no matter how far they are extended, they are said to be *parallel.*

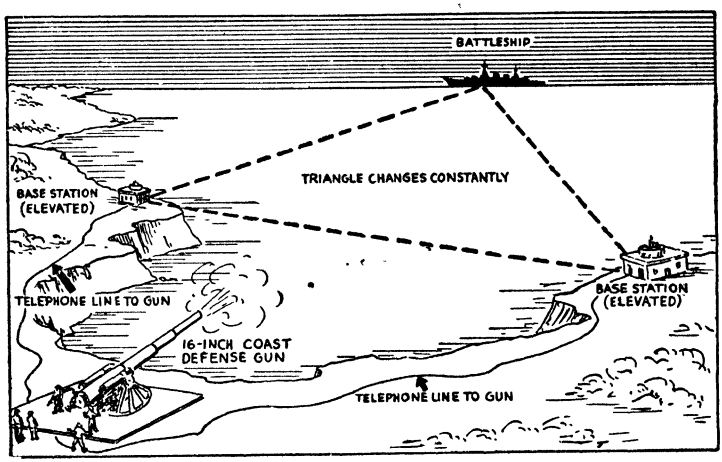
If two line segments have two distinct points in common, they coincide.

And, finally, a straight line segment can be divided into two equal parts at but one point.

ASSUMPTION: From the foregoing statements, it is assumed that *through a point outside the line one and only one line parallel to this line can be drawn.*

Under this assumption, *two intersecting lines cannot both be parallel to a third straight line.*

An aviator who can enter an airplane, "fly blind" to some given destination, and land his craft there in safety owes this achievement not only to many marvelous modern instruments and discoveries, but also indirectly to several famous Greeks who, more than two thousand years ago, studied triangles and circles and explored the ways in which these can be used in taking measurements. The properties of a triangle and a circle make possible the art of surveying and navigation. The



deadly accuracy of modern cannon is due, among other things, to the knowledge of the relationship between the sides and angles of a triangle.

RAY AND CIRCLE

We now need another concept relating to straight lines. A part of a straight line which starts at a given point and extends indefinitely in one direction is called a *ray*. The point from which a ray starts is called the *origin* of the ray. Thus, in Figure 8, ON is a ray whose origin is the point O.

Consider two rays OM and ON in Figure 8, lying in the same plane, in such a position that the ray ON lies right on OM, that is, coincides with OM. Now assume these two rays to be hinged at the point O like the hands of a watch. If we now

rotate the ray ON upward toward G, that is, in a counterclockwise sense, it will assume all possible positions on its journey of rotation until it again coincides with the ray OM. If three points, such as A, B, and C, are marked on ON, these points will each generate a closed curve, called a *circle*, in one complete revolution. That is, *a circle is generated by a point which moves*

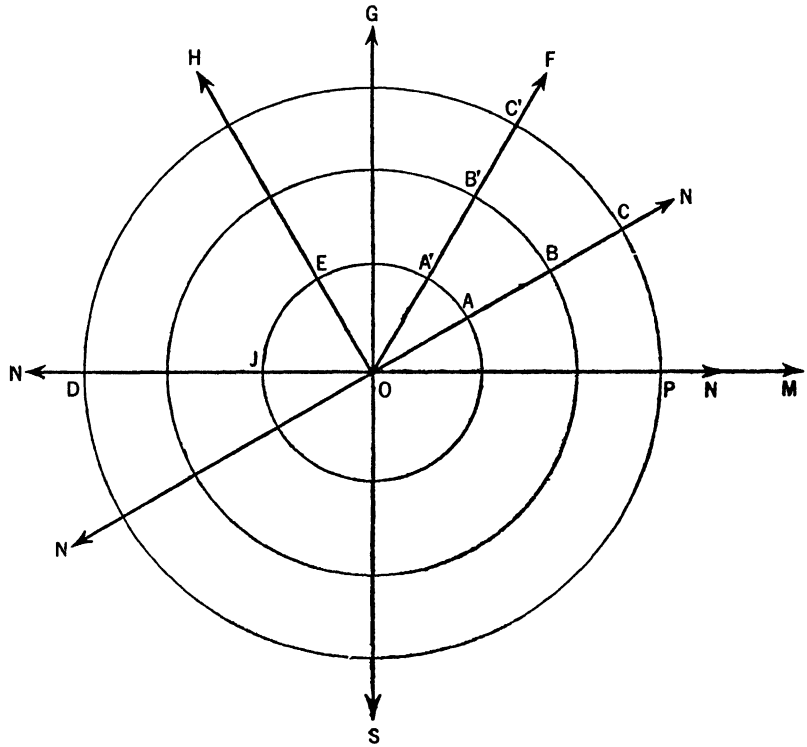


Fig. 8.

so that it is always at the same distance from a fixed point. A circle is a closed curve all points of which lie in the same flat surface or plane and are equidistant from a fixed point called the *center*.

Any portion of a circle, such as BB' or EJ, is called an *arc*. A *radius* of a circle is a line segment drawn from the center to any point on the circle; OC is a radius of the outside circle, and OA and OJ are radii of the inner circle. From the definition of a circle it is obvious that *all radii of the same circle, or of equal circles, are equal.*

ASSUMPTION: *A circle may be drawn from any point as center and with any length as radius.*

A line segment drawn through the center of a circle and terminated by two points on the circle is a *diameter.*

PLANE ANGLE

The figure formed by two rays which have the same origin is called an *angle.*

The *sides* of an angle are the rays which form the angle.

The origin of the rays which form an angle is called the *vertex* of that angle.

A precise way of naming an angle is to use three letters: one for a point in each side of the angle and a third letter for the vertex between these two. For example, the angle formed by the two line segments OC' and OC (Figure 8) is the angle COC'. The sides of this angle are OC and OC', and the vertex is at O.

An angle can also be considered as formed by the rotation of a ray on a plane about its origin. The angle POC (Figure 8), formed by the lines OP and OC, may be generated by a moving ray which rotates about the point O from the position OP to the position OC in a counterclockwise sense. This same angle could also be generated by a moving ray which rotates about the point O from the position OC to the position OP in a clockwise sense. By convention, if an angle is generated by rotation in a counterclockwise sense, the angle is considered *positive;* if generated by rotation in a clockwise sense, the angle is considered *negative.* It is frequently advantageous to regard the positive angle POC as the angle formed by the rotation of the complete line DP about O in the counterclockwise sense, which first brings it into coincidence with the complete line OC.

MAGNITUDE OF AN ANGLE

The portion of the plane swept by the rotating ray is the magnitude of the angle. This magnitude is the same whether the rotation be in the counterclockwise or in the clockwise sense.

The magnitude of the angle POC can be measured by the arc AA', BB', or CC'.

To take advantage of this fact, the length of any circle, called its *circumference,* is considered, by convention, to con-

tain 360 equal parts. That is, the outside circle, the middle circle, and the inner circle of Figure 8 are each considered to be divided into 360 equal parts. One of the equal parts is represented by the symbol 1°, called one *degree*. Naturally, the arc AA' contains the same number of degrees of the inner circle as the arc BB' contains degrees of the middle circle, or, in fact, as any arc cut off by the sides OC and OC' contains degrees of any circle drawn with O (the vertex of the angle) as its center, because the portion of the circumference of a circle drawn in this manner, and cut off by the two sides of the angle is the same proportionate part of the circumference no matter what the size of the circle may be. Hence, an angle is measured by the number of degrees contained in an arc of a circle intersected by the two sides of the angle, if the center of the circle is the vertex of the angle.

A degree is considered to contain 60 equal parts. One of these equal parts is represented by the symbol 1′, called one *minute*. The minute is then subdivided into 60 equal parts. One of these equal parts is represented by the symbol 1″, read "one *second*." Thus, 30° 16′ 12″ is read "thirty degrees sixteen minutes and twelve seconds."

CLASSIFICATION OF ANGLES

Before the ray ON, in Figure 8, starts to rotate round the point O, it coincides with the ray OM. There is then no opening between the rays, and therefore the angle formed by the two rays measures 0°. As we rotate the ray ON about the point O in a counterclockwise sense, and before it assumes the position OG, that is, before it cuts off $\frac{1}{4}$ of the circumference of the circle or 90°, the ray ON will form with the ray OM a multitude of angles. These angles, which are greater than 0° but less than 90°, are called *acute angles*.

When the ray ON has reached the position OG, cutting off $\frac{1}{4}$ of the circumference of the circle, or 90°, it forms with the ray OM a *right angle*. In this case, that is, when ON makes a right angle with OM, the ray ON is said to be *perpendicular* to OM, or vice versa.

Angles which have a common vertex and a common side which separates them are called *adjacent angles*. Thus, in Figure 8, the angles POC and COC' are adjacent. If two angles are adjacent, another angle is formed by the two ex-

terior sides. For example, the adjacent angles POC and COC' form angle POC'. This other angle formed in this way could also be regarded as formed by the rotation of the exterior side of the first angle about the vertex until it coincides with the exterior side of the second angle. Thus, angle POC' may be regarded as formed by the rotation of the ray ON in a counterclockwise sense about O until it coincides with OC'. The space swept by this rotation comprises only the two parts represented by the first and second angles. This third angle is the *sum* of the two adjacent angles. Thus, angle POC' = angle POC + angle COC'.

If the sum of two angles equals a right angle, the angles are said to be *complementary*. Hence, the *complement of an angle is the difference between that angle and one right angle.* Angle POC' and angle $C'OG$ are complementary angles.

One of the fundamental assumptions in regard to space is that *any figure, such as an angle, can be moved without altering its shape or size.*

Two angles are said to be *equal* if one can be superimposed upon the other so that the two angles coincide.

Since by definition a right angle has 90°, we infer that *all right angles are equal,* because all of them can be made to coincide.

Since halves of the same thing are equal, then *a given angle can be divided into two equal angles by but one ray.*

Complements of the same angle or of two equal angles are equal.

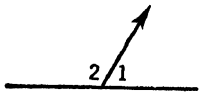
As we continue rotating the ray around the point O from the position OG, and before it assumes the position OD, cutting off $\frac{1}{2}$ of the circumference of the circle, or 180°, the ray ON assumes all intermediate positions. The angles thus formed, greater than 90° but less than 180°, are called *obtuse angles.* The angle POH, for example, is obtuse. When the ray ON assumes the position OD, cutting $\frac{1}{2}$ of the circumference, or 180°, it is said to form a *straight angle.* That is, a straight angle is an angle whose sides lie in the same straight line and extend in opposite directions from the vertex, such as the angle POD.

Two angles whose sum equals a straight angle are said to be *supplementary* angles. Hence, the *supplement* of an angle is the difference between that angle and a straight angle. Angle

POC and angle *COD*, for example, or angle *MOF* and angle *FOD*, are supplementary angles.

Supplements of the same angle or of two equal angles are equal.

It follows that *all straight angles are equal;* that since a given angle can be divided into two equal angles by but one ray, then *at a given point in a straight line there can be one and only one perpendicular to that line.*



The following properties of adjacent angles can now be inferred:

If a ray has as its origin any point on a straight line segment, the sum of the two adjacent angles thus formed is a straight angle, for the angles thus formed are supplementary.

If two adjacent angles are together equal to a straight angle (or two right angles), the sides which they do not have in common, namely, their exterior sides, form a straight line, for their exterior sides form a straight angle and therefore must lie in a straight line.

If the rotation of the ray *ON* is continued around the point *O* (Figure 8), it will form angles greater than 180° but less than 360°, called *reflex angles*. When the ray *ON* assumes the position *OS*, or makes an angle of 270° with *OM*, it becomes simply a continuation of the ray *OG*, and hence it is again perpendicular to *OM*. When the ray *ON* again coincides with *OM* after making one complete revolution, we say, since the point *N* has traveled the whole length of the circumference, that *ON* makes an angle of 360° with *OM*.

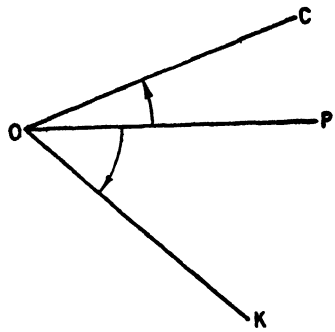


Fig. 9.

There is no reason whatever why a line cannot be rotated in a clockwise instead of a counterclockwise direction. To avoid confusion, it has been agreed, as already stated, that angles formed by counterclockwise rotation are positive, while angles formed by clockwise rotation are negative. Thus, in Figure 9 the angle *POK* is negative, while the angle *POC* is positive.

THE PROTRACTOR

A protractor is a convenient instrument for measuring angles. The most common type consists of a half-circle divided into 180 equal parts. Each of these parts is one degree. The center of the half-circle is indicated by a dot or dash (point B in

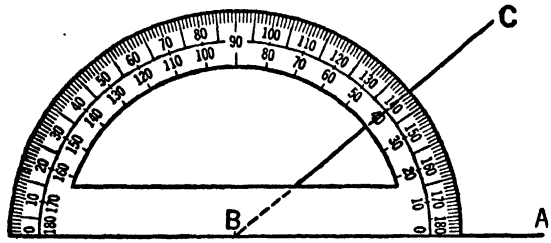


Fig. 10.

Figure 10). To measure an angle, such as ABC, place the dot or dash of the protractor at the vertex of the angle, and the base edge of the protractor along one side of the angle. The point where the other side of the angle (extended, if necessary) crosses the protractor scale indicates the number of degrees in the angle. Thus, in the figure, the angle $ABC = 40°$.

ANGLES FURTHER CLASSIFIED

Acute, obtuse, and reflex angles are, in general, called *oblique angles*. That is, an angle which is neither a right angle nor a straight angle is an oblique angle. An *oblique line* is a line which makes an oblique angle with another given line. The lines OC, OH, OI, and OJ (Figure 11) are all oblique to OM.

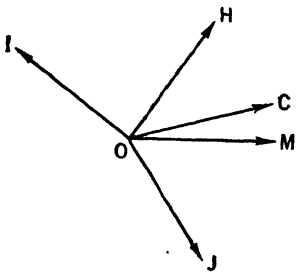


Fig. 11.

Straight line segments (in the same plane) which have different directions are said to be "at an angle" with each other, even though they do not touch. Thus, AB and CD in Figure 12 are "at an angle"; this means that if prolonged upward until they met, the two segments would form an angle.

Lines which pass through the same point are said to be *concurrent*. If a number of rays have the same origin, they are concurrent. These rays will form a number of successive adja-

cent angles, such as angles MOC, COH, HOI, IOJ, and JOM in Figure 11. If the common origin of these rays be taken as the center of a circle, the circle thus drawn will intersect each ray. This circle will have 360°. But the sum of the successive adjacent angles formed by rays is equal to the angle formed by one complete rotation of a ray about its origin. Hence,

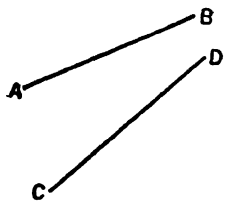


Fig. 12.

If a number of rays have a common origin, the sum of the successive adjacent angles about this point equals two straight angles or 360°.

Similarly, *if a number of rays start from a point in a straight line and extend on the same side of that line, the sum of the successive adjacent angles formed equals a straight angle or 180°.*

If the sides of an angle are extended through the vertex, another angle is formed whose sides are the extensions of the sides of the given angle and having the same vertex. These two angles are called *vertical* (or opposite) angles. For example, angles AOC and DOB or angles AOD and COB in Figure 13.

An angle may be designated by the use of some letter, inserted near the vertex between the sides. Thus, in Figure 13, angle DOB can be called angle a, and the other angles can be called by any other letters. It is common practice to use Greek letters to designate angles. The Greek alphabet is given on page 691.

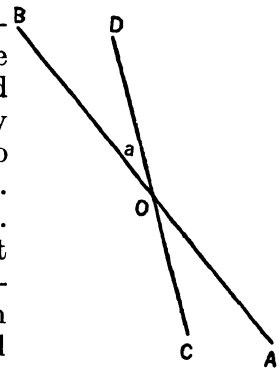


Fig. 13.

In Figure 13, angle a is the supplement of angle AOD; angle a is also the supplement of angle COB. Since things which are equal to the same thing, or to equal things, are equal to each other, it follows that the angles AOD and COB are equal. That is,

Fact 1: If two straight lines intersect each other, the vertical angles are equal.

POLYGON

A point moving in a plane and occasionally changing direction abruptly but not continually, generates a *broken line*

(Figure 14). Hence, a broken line can be considered as made up of different straight line segments.

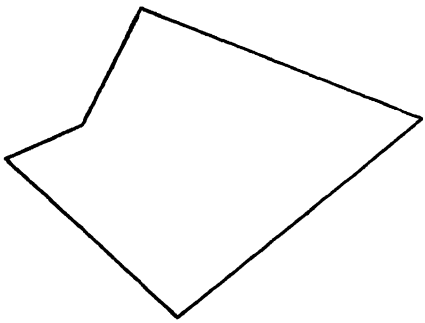
A *polygon* is a single finite portion of a plane bounded by straight line segments no two of which intersect.

Fig. 14.

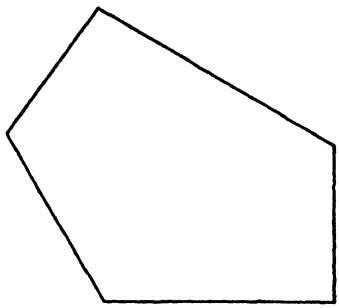
The bounding line segments are the *sides*. The angles included by the adjacent sides are the *angles* of the polygon, and the vertices of these angles are the *vertices* of the polygon. The number of sides of a polygon is equal to the number of its angles.

A *convex* polygon is a polygon of which no side, when produced, will enter the polygon.

A *concave* polygon is a polygon of which two or more sides, when produced, will enter the polygon.



CONCAVE



CONVEX

TRIANGLE

A *triangle* is a polygon of three sides. It is the simplest and one of the most useful polygons. Because of its simplicity, most of the other plane figures are dissected into triangles before being measured, and because the shape of triangular objects cannot be altered without a change in the length of their sides, the triangle is extremely useful in measuring areas.

In Figure 15, the triangular frame is seen to be the only one whose shape cannot be changed by pressing upon opposite sides or vertices. This property of rigidity is characteristic of all triangular frames. Consequently, the struts of bridges,

radio towers, and other structures are frequently triangular in shape.

A triangle has three angles, usually named by the capital letters placed at the vertices (such as the angles A, B, and C in Figure 16), and three sides, which may be denoted by the two end-points of each line segment or by the small letters corresponding to the opposite angles: AB or c; BC or a; and AC or

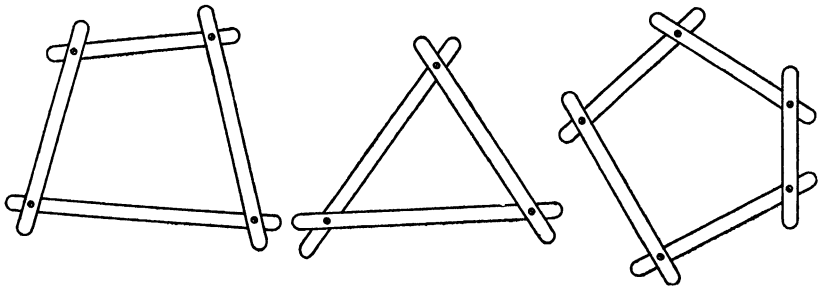


Fig. 15.

b. The vertices of the angles are also called the vertices of the triangle.

A triangle is called *right* or *obtuse* (see Figure 17) according to whether one of its angles is right or obtuse. A triangle is *acute* if all of its angles are acute.

If a triangle has all of its sides equal, it is said to be *equilateral*. If all its angles are equal, it is *equiangular*. If two of its sides are equal, it is *isosceles*.

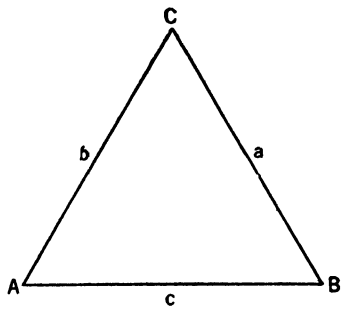


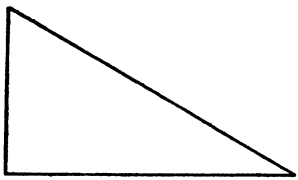
Fig. 16.

Any side of a triangle may be called its *base*. Usually, one side is drawn horizontal, and then this side is called the base of the triangle. The vertex opposite the base is called the vertex of the triangle. The altitude of a triangle is the perpendicular distance from the vertex to the base, or to the base extended: h in any one of the triangles in Figure 17.

From the definition of a triangle and its characteristics, just discussed, it can be seen that, since a line segment is the shortest path joining two points in a plane, then *the sum of any two sides of a triangle is greater than the third side.*

The sum of all the sides of a triangle (or of any polygon) is its *perimeter*.

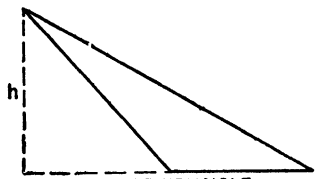
As already stated, it is assumed that a figure may be freely moved in space without any change in form or size. This is a most important assumption. (In Einstein's theory of relativity, this assumption is not made.)



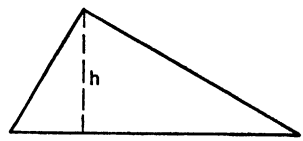
RIGHT TRIANGLE

Any two figures that can be made to coincide are called *congruent figures*.

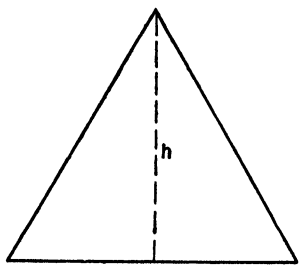
When two congruent figures are superimposed and coincide, the sides which coincide are called the *corresponding sides*. Furthermore, the angles which coincide are called the *corresponding angles*.



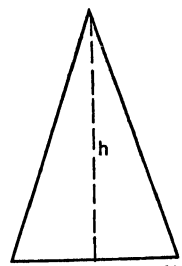
OBTUSE TRIANGLE



ACUTE TRIANGLE



EQUILATERAL AND EQUIANGULAR TRIANGLE



ISOSCELES TRIANGLE

Fig. 17.

Problems

1. Some soldiers wish to enclose a field with wire. The field is oblong, 136 rods by 192 rods. If wire netting costs 30 cents a linear meter, what will be the entire cost? What will be the cost of posts at 50 cents apiece if they are set 12 feet apart?

2. What is the diameter, in inches, of a 75-mm. gun?

3. An American tourist abroad is arrested for speeding. He proves that his car was going 50 miles per hour. If the permitted speed is 80 kilometers, can he be held guilty?

4. An American artillery officer in a foreign field is told that a certain landmark in enemy territory is 15 kilometers away. For what distance in yards shall he sight his cannon?

5. What is the perimeter in inches of a triangle whose sides are, respectively, 26, 34, and 79 cm.?

6. A British steamship takes soundings. The water is 21 fathoms deep. A Swiss lady inquires how many meters are the equivalent of this depth.

7. Assume the distance from New York to San Francisco to be 3,000 miles. What is the distance in feet on a map drawn to a scale of 1 to 5,000,000?

8. How many degrees are there in an angle of an equilateral triangle?

9. Three-fourths of a certain angle equals three-fifths of the complement of the angle. What is the angle?

10. One-tenth of the supplement of a certain angle exceeds one-half the complement of the angle by 6 degrees. Find the angle.

11. Two-fifths of the supplement of a certain angle exceeds six-sevenths of the complement of that angle by 20°. Find the angle.

12. The perimeter of a triangular plot of ground is 490 feet. The second side is $\frac{5}{4}$ of "10 less than the first side"; the third side is $\frac{2}{3}$ of "10 more than the first side." What are the lengths of the sides of this plot?

8 MEASURING SPACE BY TRIANGLES

Certain facts regarding triangles are of fundamental importance in reckoning distances. The average boy finds it hard to believe that certain distances can be measured far more accurately by mental computation than by using a measuring tape.

As will be seen later, the same kind of calculations used in astronomy to foretell the date of eclipses of the sun and moon have been discovered to be useful in investigating the electrical charges within the atom. Although our knowledge of this subject is as yet only fragmentary, the invention of the photo-electric cell, which now opens doors when a person approaches, counts and classifies cigars, and does many other daily jobs with silent competence, came about indirectly from previous study of the triangle. Triangles and circles are the keys which have unlocked secrets of the radio, the X-ray, television, and many more inventions. The same keys will admit us to countless other discoveries in the future.

CONGRUENT TRIANGLES

In the following pages, unless otherwise stated, all figures and lines are considered as lying in the same plane.

In the preceding chapter certain statements and definitions were given in regard to congruent triangles. This idea of congruence is perhaps the most important and most far-reaching of all. To illustrate the practical usefulness of congruence, let us consider the following problem:

Find the distance between two points A and B on opposite sides of an impassable marsh. Taking a convenient position C (see Figure 18), measure BC and produce BC to D so that CD is equal to BC. Then, measuring AC, produce AC to E so that CE is equal to AC. If the triangle CDE, constructed in this manner, can be shown to be congruent to the triangle ABC, then the distance AB can be found by simply measuring ED.

Angles ACB and ECD are vertical angles, and therefore, by Fact 1, they are equal. The side AC has been constructed equal to CE, and the sides BC and CD are also equal for the same reason.

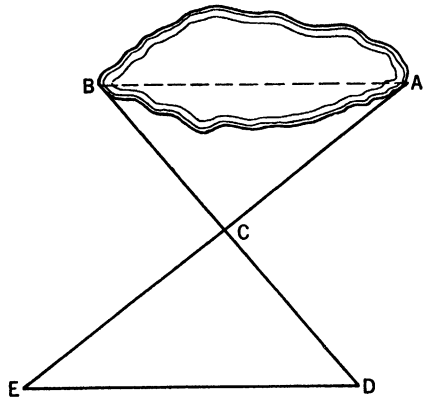


Fig. 18.

Since it is assumed that a figure may be moved in space without changing its form or size, place the triangle ABC on triangle CDE so that BC coincides with its equal CD, and AC falls on the same side of ED as CE. Since angle ACB is equal to angle ECD, then AC will fall on CE and point A will coincide with point E, for AC and CE are equal. Points B and D will coincide also, because the side BC was placed so that it would coincide with its equal CD. Hence, the side AB will coincide with the side ED, because through two points one and only one straight line can be drawn. That is, the two triangles ABC and CDE will coincide and therefore are congruent.

In other words:

Fact 2: If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, the triangles are congruent.

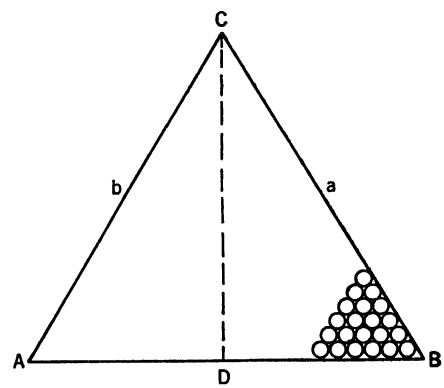


Fig. 19.

The cross section of the attic of a barn is represented in Figure 19. The rafters a and b are known to be equal in length; that is, the triangle ABC is an isosceles triangle. Suppose that the angles A and B have to be measured. Although the size of angle A can be easily ascertained, in order to measure angle B a pile of heavy barrels would have to be moved. However, since a is given equal to

b, if we can prove that angle A must then be equal to angle B, this work of moving the barrels would be unnecessary.

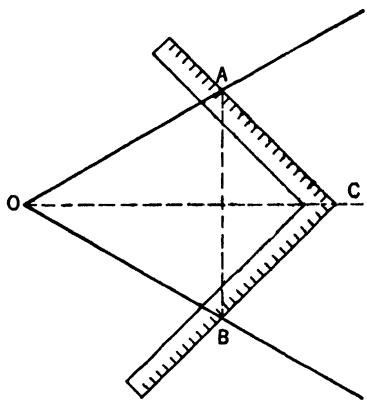
Draw the line CD so that it bisects angle C and meets AB at D. This line divides the original triangle into the two triangles ACD and BCD. In these triangles, the side a was given equal to side b. Since the line CD bisects angle C, then angle ACD is equal to angle BCD. Finally, the side CD is common to both triangles. That is, the sides b and CD and the included angle ACD of triangle ACD are equal to the two sides a and CD and the included angle BCD of triangle BCD. Therefore, by Fact 2, these triangles are congruent; and since corresponding angles of congruent triangles are equal, it follows that angle A is equal to angle B. So that,

Fact 3: If two sides of a triangle are equal, the angles opposite these sides are equal.

From this follow:

Fact 4: The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base; and

Fact 5: An equilateral triangle is equiangular.



To bisect an angle, the following method is frequently used. A carpenter's square is placed so that the edges pass through two points A and B on the sides of the angle, at the same distance from O, and so that AC is equal to BC. Then OC is drawn. In order to prove that OC bisects angle AOB, we must show that the triangles AOC and BOC are congruent.

The two triangles AOC and BOC were constructed so that the sides OA and OB, and AC and BC are respectively equal. The side OC is common to both triangles. Hence, we have two triangles whose three sides are respectively equal.

Draw the line AB. Since OA was given equal to OB, the triangle AOB is isosceles. It follows, by Fact 3, that angle OAB is equal to angle OBA.

Similarly, in triangle ACB, sides AC and BC were con-

structed equal, and therefore angles CAB and CBA are also equal.

If equals are added to equals, the sums are equal. Hence, angle OAB + angle CAB = angle OBA + angle CBA. But angle OAB + angle CAB = angle OAC, and angle OBA + angle CBA = angle OBC. The whole is equal to the sum of its parts. Therefore, angle OAC is equal to angle OBC. That is, the triangle OAC has two sides OA and AC and the included angle OAC equal respectively to the two sides OB and BC and the included angle OBC of triangle OBC. Hence, angle AOC is equal to angle BOC. In other words:

Fact 6: Two triangles are congruent if the three sides of one are equal respectively to the three sides of the other.

EXTERIOR ANGLES OF A TRIANGLE

If one side of a triangle is extended, an angle outside of the triangle is formed. Thus, by prolonging the segment AB, in Figure 20, the angle FBC is formed. Such an angle is called an *exterior angle* of the triangle.

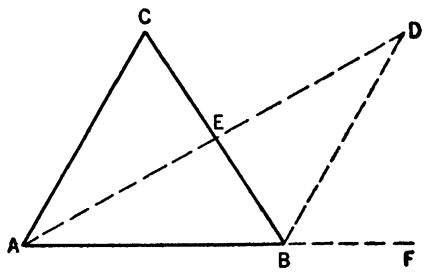


Fig. 20.

It can be shown that the angle FBC is greater than the angle ACB or the angle BAC.

Let E be the mid-point of line CB, and draw the line segment AD passing through E, and let AD be of such length that AE is equal to ED. Draw BD.

In the two triangles ACE and BDE, the sides CE and EB are equal, because E is the mid-point of the line CB. By construction, AE is equal to ED. Since, by Fact 1, vertical angles are equal, angles CEA and DEB are also equal. That is, we again have two sides and the included angle of one triangle, namely, triangle ACE, equal to two sides and the included angle of another triangle, that is, BDE. Therefore, by Fact 2, the triangles are congruent.

It follows that the angle DBC is equal to the angle ACB, for they are corresponding angles of congruent triangles. Since the whole is greater than any of its parts, angle FBC is greater

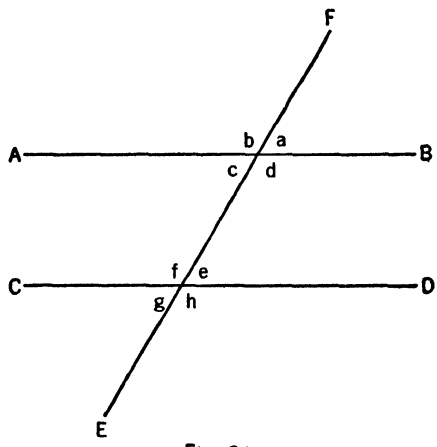


Fig. 21.

than angle DBC. But angle DBC is equal to angle ACB, so that angle FBC is greater than angle ACB.

The reader may prove angle FBC greater than angle BAC by taking the mid-point of AB, drawing a line to it from C, as before, and proceeding with the same argument as above. Therefore:

Fact 7: An exterior angle of a triangle is greater than either opposite interior angle.

PARALLELS AND TRANSVERSALS

A line that intersects two or more other lines is called a *transversal.* Thus, EF, in Figure 21, is called a transversal. The angles formed by the intersection of the transversal with two straight lines are of particular importance in various proofs; hence, it is convenient to give these angles special names.

In Figure 21, a, b, g, and h are *exterior* angles; c, d, e, and f are *interior* angles; c and e and d and f form pairs of *alternate interior* angles; a and e, b and f, c and g, and d and h form pairs of *corresponding* angles.

Assume that, at a certain point, A, on the bank of a river, the width AB of the river is to be found. This width may be ascertained as follows:

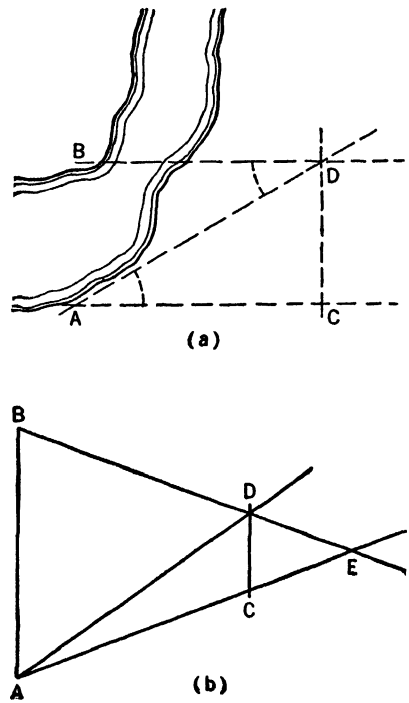


Fig. 22.

From B, sight on A, and locate BD perpendicular to AB. At any convenient point D, along BD, sight on B and lay out line DC so that angle BDC is a right angle. Measure angle BDA. From A, sight on D and locate line AC so that angle DAC is equal to angle BDA.

If we can show that in that case the lines BD and AC are parallel, then, since parallel lines are everywhere equidistant (see Fact 39, page 159), the width of the river at point A, namely AB, may be ascertained simply by measuring DC.

The lines BD and AC are cut by the transversal AD so that angle BDA is equal to angle DAC. It is to be proved that line AC must then be parallel to BD.

There are two possibilities: either AC is parallel to BD or it is not.

Assume AC is not parallel to BD. If AC is not parallel to BD, then it must intersect BD at some point such as E (see Figure 22), for by definition two lines are parallel only if they do not meet.

Now, if AC intersects BD at point E, angle BDA is an exterior angle of the triangle ADE, and therefore angle BDA is greater than angle DAC. For we have just shown by Fact 7 that an exterior angle of a triangle is greater than either opposite interior angle. But angle BDA is given equal to angle DAC; hence, AC cannot meet BD. Therefore, AC is parallel to BD. If the point E were on the left side of AB, instead of on the right side, as in Figure 22, precisely the same line of reasoning would apply, and thus need not be repeated. That is:

Fact 8: When two lines are cut by a transversal, if a pair of alternate interior angles are equal, the two lines are parallel.

It follows that:

Fact 9: When two lines are cut by a transversal, if the alternate interior angles are unequal, the two lines are not parallel.

By assuming, in Figure 21, that two corresponding angles are equal, or that two interior angles on the same side of the transversal are supplementary, the reader may show by first applying Fact 1 and then Fact 8:

Fact 10: When two lines are cut by a transversal, if two corresponding angles are equal, or if two interior angles on the same side of the transversal are supplementary, the lines are parallel.

Since, by the above fact, if two corresponding angles are equal the lines are parallel, then

Fact 11: Two distinct lines in the same plane, perpendicular to the same line, are parallel.

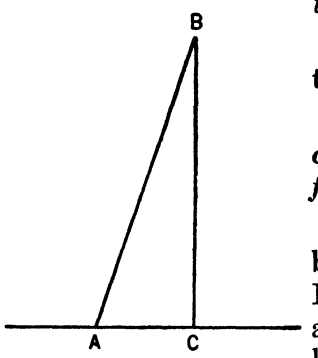


Fig. 23.

Moreover, from Fact 11 it follows that:

Fact 12: Not more than one perpendicular can be drawn to a given line from a given external point.

For if AB and BC in Figure 23 were both perpendicular to AC, then by Fact 11 they would have to be parallel and we would have two parallel lines meeting at a point B, which is contrary to the assumption on page 97.

DIFFERENT FORMS OF DEDUCTIVE REASONING

In reading the pages of this chapter, the attentive reader must have noted that from certain fundamental assumptions, which may be called the seeds of thought, there grew and flowered a series of propositions related to each other with an inevitableness that seems like fatalism. This unfolding of abstract truth is the glory of logical thought. Since all mathematics originates and develops in this manner, it will now be interesting and worth while to go back and analyze the different forms of deductive reasoning used thus far in demonstrating the preceding facts.

The proofs just given of Facts 8 and 12 consisted of demonstrations that if the particular fact were assumed to be false, it would entail the contradiction of a given condition and would therefore lead to an impossible result. Consequently, we concluded that Fact 8 and Fact 12 could not be false and therefore must be true. A demonstration of this sort is called an *indirect* proof.

The demonstration of Fact 2 consisted in proving two figures equal by placing one figure upon the other and showing that the two figures must then coincide throughout. A demonstration of this kind is called a proof by *superposition.*

The third and most important kind of proof, because most

logical and hence most satisfactory, consists in showing that the fact is true by singling out this possibility and by using the assumptions already stated and the facts already established, thus proving it directly with no reference to other possibilities. This kind of proof, used with Facts 3 and 7, is called a *direct proof*.

CONCERNING ANGLES FORMED BY A TRANSVERSAL

Fact 13: If two parallel lines are cut by a transversal, the alternate interior angles are equal.

To demonstrate this fact, consider the following interesting problem: The most direct method of testing whether or not the

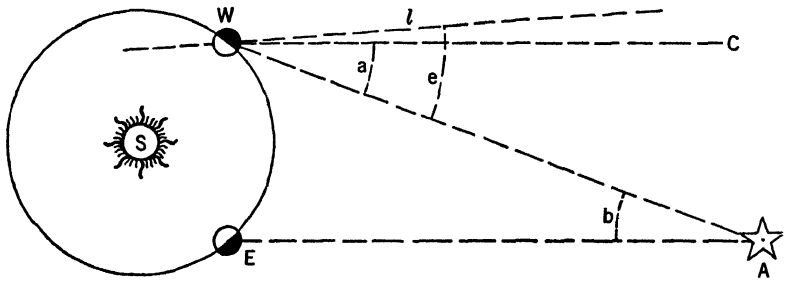


Fig. 24.

earth moves is to find whether the direction of a star, regarded as fixed, is the same when observed at different times of the year.

The points E and W, in Figure 24, represent two positions of the earth on its path around the sun. By making observations at these different points where the lines WC and EA are known to be parallel, it is possible to determine the angle CWA. If we can show that the angle a must then be equal to angle b, the fact that the direction of the star has changed proves that the earth has moved.

The parallel lines WC and EA are cut by the transversal AW. We have to show that angle a is equal to angle b.

Through the point W draw the line l such that angle e is equal to angle b. Then the lines l and EA must be parallel, for these two lines are cut by the transversal AW so that the alternate interior angles e and b are equal (Fact 8). But since it was assumed that through a point outside the line EA (point

W) only one parallel to a line (EA) can be drawn, then the lines l and WC must coincide instead of being different lines as in the figure. Hence, angle e coincides with its equal angle a. But angle e was drawn equal to angle b; therefore, angle a is equal to angle b.

From the preceding proof, the following facts can at once be deduced:

Fact 14: If two parallel lines are cut by a transversal, the corresponding angles are equal.

Fact 15: If two parallel lines are cut by a third straight line, the sum of the two interior angles on the same side of the transversal is equal to two right angles.

Fact 16: If a straight line is perpendicular to one of two parallel lines, it is also perpendicular to the other.

Fact 17: If two different lines are parallel to the same line, they are parallel to each other.

Distinct lines parallel to parallel lines are parallel.

Distinct lines perpendicular to parallel lines are parallel.

PERTAINING TO ANGLES OF A TRIANGLE

Let us return to that very important figure, the triangle, and consider again the problem, stated on page 111, of finding the angles made by the rafters of the attic of a barn. By the previous discussion we found that, since the rafters AC and CB are equal (Figure 25), by simply measuring angle A, the size of angle B is immediately known, for the two angles must be equal. If we can show that the sum of the interior angles of the triangle is a straight angle, then angle C can be found by simply subtracting the sum of the angles A and B from 180°. In this way, we shall be spared the difficult job of climbing up a ladder and actually measuring C.

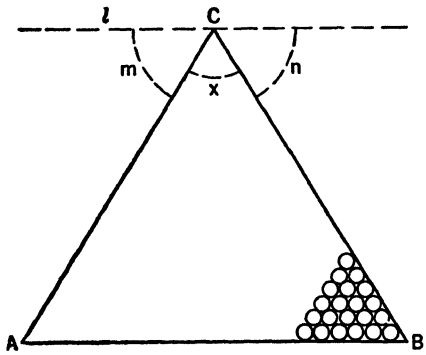


Fig. 25.

This fact is easily proved as follows: Through C draw the line l parallel to AB. The two parallel lines l and AB are cut by the transversals AC and BC, so that, by Fact 13, the alter-

nate interior angles are equal. That is, angle m is equal to angle A, and angle n is equal to angle B.

Since by definition angle m + angle C (which in Figure 25 is called x) + angle n = a straight angle, then substituting angle A for its equal angle m and angle B for its equal angle n, we have:

angle A + angle C + angle B = a straight angle, or 180°.

That is,

Fact 18: The sum of the interior angles of a triangle is a straight angle.

It is interesting to note that to give this obviously very simple proof, it has been necessary to establish the truth of the entire chain of preceding facts.

From Fact 18 the following series of deductions can be made:

Fact 19: An exterior angle of a triangle is equal to the sum of the two opposite interior angles.

Fact 20: In a right triangle, the sum of the two acute angles equals one right angle. Therefore,

Fact 21: If an acute angle of one right triangle equals an acute angle of another right triangle, the remaining acute angles of the two triangles are equal.

Fact 22: If two angles of one triangle are equal respectively to two angles of a second triangle, the third angles are equal.

Fact 23: If two angles and a side of one triangle are equal, respectively, to two angles and the corresponding side of another triangle, the triangles are congruent.

Fact 24: Two right triangles are congruent if the hypotenuse and an acute angle of one are equal, respectively, to the hypotenuse and an acute angle of the other.

Fact 25: Each angle of an equilateral triangle contains 60°.

The distance of a ship off shore may be found by the following method. Taking a large wooden isosceles triangle, sight along a side of the angle included between the two equal sides to the ship S, Figure 26. Then, sighting along the other equal side, lay the line AB. Walk along AB until a point, B, is found such that the point A is sighted along an equal side of the triangle while the ship S is sighted along the base. The triangle BSA then has angles S and B equal. If we can show that the side AS must then be equal to AB, then the distance

the ship is from A, that is, AS, can be found by simply measuring AB.

Since a given angle can be divided into two equal parts by only one ray, draw the ray AC so that it bisects the angle A

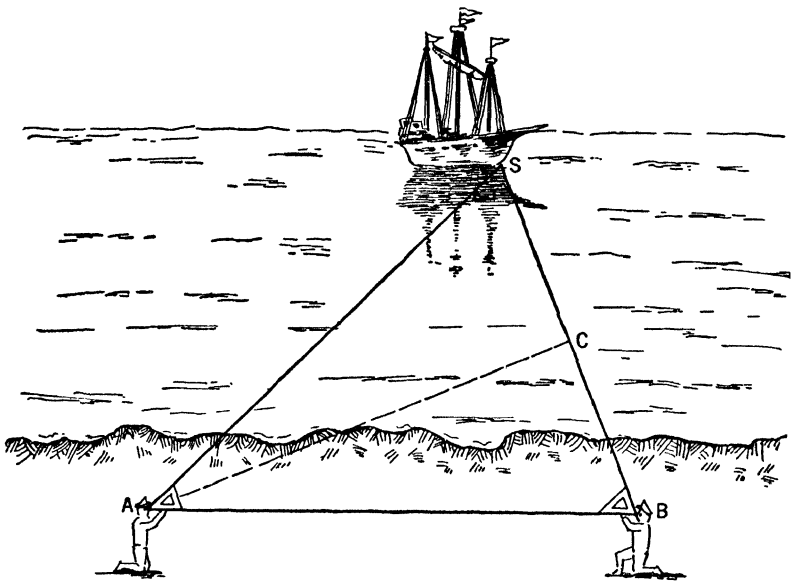


Fig. 26.

and meets the line SB at C. The triangle BAS is then divided into the two triangles BAC and SAC. In these triangles, angle B was constructed equal to angle S. The side AC is common to both. Thus, the triangle BAC has the two angles B and CAB and the side AC equal respectively to the two angles S and CAS and the side AC of the triangle CAS. Therefore, by Fact 23, the triangles are congruent. Since corresponding sides of congruent triangles are equal, it follows that AS is equal to AB. That is,

Fact 26: If two angles of a triangle are equal, the sides opposite these angles are equal.

Consider, once more, the problem of the cross section of an attic, and assume that a partition CD, Figure 27, divides the attic. The rafters a and b, as before, are known to be equal. Suppose that the distances AD and DB are to be measured. Although the distance AD can be easily ascertained, the pile

of barrels would have to be moved so that DB might be measured. However, if we can show that the right triangles ACD and BCD are congruent, then it follows that the distances AD and DB are equal, and hence only AD would have to be measured.

The right triangles ACD and BCD have their respective hypotenuses b and a equal, and the side CD is common to both. It is to be shown that the triangles are then congruent.

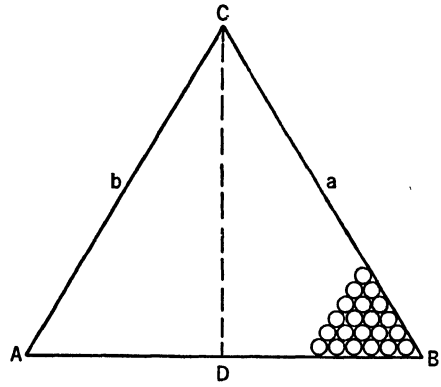


Fig. 27.

The triangle ABC has the two sides a and b equal; hence, by Fact 3, the angles opposite these sides are equal. That is, angle A is equal to angle B. Therefore, triangle ACD has the hypotenuse b and the acute angle A equal respectively to the hypotenuse a and the acute angle B of triangle BCD. Therefore, by Fact 24, the triangles are congruent. That is,

Fact 27: Two right triangles are congruent if the hypotenuse and a leg of one are equal, respectively, to the hypotenuse and a leg of the other.

Problems

1. How many centimeters are there in the perimeter of an equilateral triangle one side of which is 9 millimeters?

2. Find the length in decimeters of one side of an equilateral triangle whose perimeter is 57 centimeters.

3. Each of the equal sides of an isosceles triangle is double the base, and the perimeter is 55 centimeters. Find each side.

4. The exterior angle formed by prolonging the base of an isosceles triangle contains 3 times as many degrees as the interior base angle. (*a*) How many degrees are there in the sum of the base angles? (*b*) How many in their difference?

5. The exterior angle formed by prolonging one of the legs of an isosceles triangle is 26° larger than a base angle. How many degrees are there in each base angle?

6. How many degrees are there in the angle x formed by prolonging a side of an equilateral triangle?

7. The perimeter of a scalene* triangle is 65 inches. One side is 14 inches, and another $1\frac{1}{2}$ times as long. Find the third side.

8. How many degrees are there in each angle of an isosceles triangle whose vertical angle is equal to the sum of the base angles?

9. An isosceles triangle ABC has a vertical angle at B of 38°. AE bisects one base angle. DC bisects the other. The two bisectors cross at F. Angles of how many degrees are formed at F?

10. If three rods of different lengths are put together, their ends touching, can a triangle always be formed? Can more than one triangle ever be formed?

11. Prove that the medians to the sides of an isosceles triangle are equal. (A median of a triangle is a line segment from any vertex to the mid-point of the opposite side.)

12. Prove that if the base of an isosceles triangle is trisected and from each point of division perpendiculars are drawn upon the nearest side, these perpendiculars are equal.

* A *scalene* triangle is one having no two sides of equal length.

9 FUNDAMENTAL CONSTRUCTIONS

To be able to draw the figures involved in the solution of a problem is a great advantage. A drawing helps one to visualize the object and to discover relations which otherwise would be virtually impossible to see. Obviously, the more exactly a figure is drawn, the easier it is to perceive these relations. However, even when figures are drawn very accurately, it is easy to be deceived by appearances.

For example, the drawings shown in Figure 28 are not what they seem to be. The two lines *AB* and *CD* appear to be different in length. A careful check with the compasses or an accurate measurement with a ruler will show that both are of the same length. Furthermore, lines *EF* and *MN* are parallel, although they do not seem to be. The line *m* appears to be a prolongation of *l*, but is not.

From these examples, it is evident that we should not trust too much to our sense of sight; optical illusions are of frequent occurrence. Therefore, not only must methods of drawing

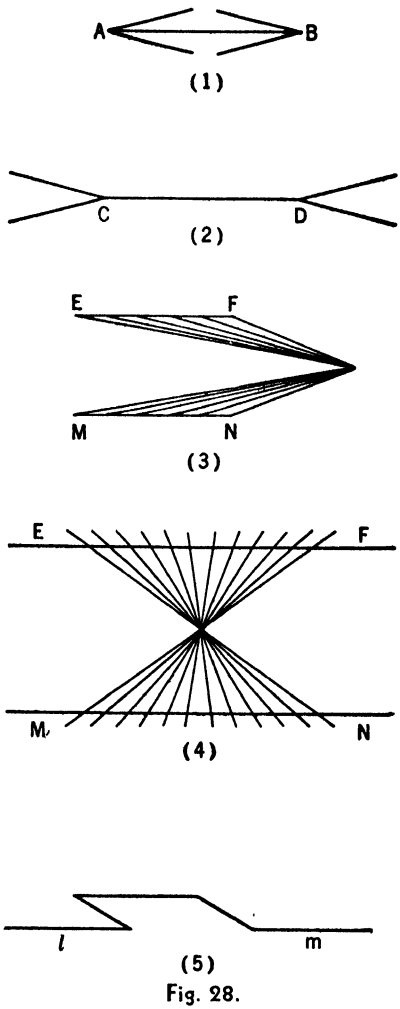


Fig. 28.

figures be developed, but the correctness of the methods must also be demonstrated.

EQUAL ANGLES

To construct an angle equal to a given angle, through a given point on a straight line, simply use a protractor (see page 104). However, if a protractor is not at hand, the angle may be drawn in a more accurate manner by means of a ruler and compasses.

The direction of the rays of light issuing from the headlights of an automobile depends on the position of the bulb. It is shown in physics, by experiment, that if a bulb, regarded as a point-source, be placed at a certain point, all the rays flowing from the source of light will be reflected in the same direction. That is, a ray of light striking the surface of the headlight is reflected in such a way that angles a and e, in Figure 29, are equal. This principle is used in manufacturing searchlights.

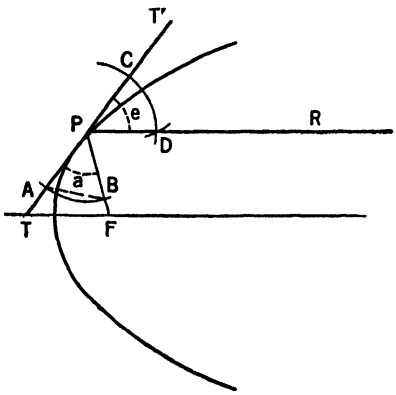


Fig. 29.

Assume the source of light to be at the point F. In order to find the direction in which the ray of light will be reflected, it is necessary to draw a line, PR, making an angle, e, with the tangent line (see page 322), TT', equal to angle a.

With P as a center, and any convenient radius, draw an arc cutting TT' at A and PF at B*. With the same radius and P as center, describe an arc CD cutting TT' at C. Draw the line segment AB; and with C as center and AB as radius, describe an arc cutting the arc CD at D. Draw the line segment PD. Then angles a and e are equal. The reader may prove the last statement by drawing a straight line from C to D, and showing that triangles PAB and CPD are congruent.

* The reader will please keep in mind the assumptions made in Chapter 7, namely: (1) A circle may be drawn about any given point as a center, and with any given radius. (2) A straight line may be drawn through or connecting any two points. (3) Finally, a line segment may be extended indefinitely. It seems unnecessary to keep repeating these assumptions once they have been stated and accepted.

BISECTION OF ANGLES

A method of bisecting an angle by means of a carpenter's square was discussed on page 112. This method, although very useful and convenient in carpentry, is of little value in drawing or in metal work, where great accuracy is required.

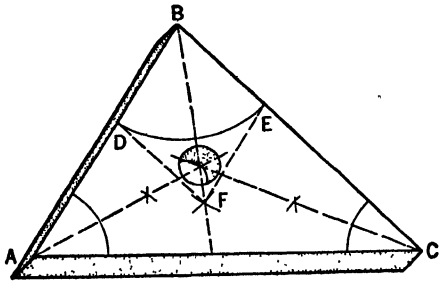


Fig. 30.

Assume that a circular hole is to be drilled on a triangular section of steel. This hole, Figure 30, must be at the intersection of the bisectors of the angles.

To bisect angle *ABC*, draw, with *B* as a center and any radius, *BD*, an arc cutting *AB* at *D* and *BC* at *E*. With *D* and *E* as centers and any convenient radius greater than one-half the distance *DE*, describe arcs intersecting at *F*. Draw *BF*. This line *BF* may be proved the bisector of angle *ABC* as follows:

Draw the line segments *DF* and *FE*, forming the triangles *BDF* and *BEF*. The two line segments *BD* and *BE* were constructed equal. The line segment *BF* is common to both triangles, and *DF* and *EF* were also constructed equal. That is, triangle *BDF* has its three sides, *BD*, *DF*, and *BF* equal, respectively, to the three sides *BE*, *EF*, and *BF* of triangle *BEF*. Therefore, by Fact 6, the triangles are congruent. Since corresponding angles of congruent triangles are equal, it follows that angle *DBF* is equal to angle *EBF*.

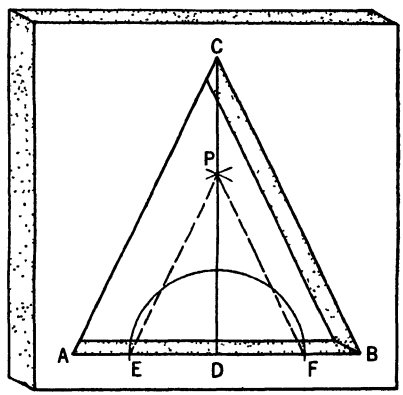


Fig. 31.

Through bisecting angles *BAC* and *BCA*, in the same manner, the center of the circular hole is found.

A die for punching pieces is to be in the form of a triangle. The point *C*, Figure 31, must lie on a perpendicular line erected at point *D*, midway between *A* and *B*.

The point D divides the straight line segment AB into two segments, AD and DB. These two segments can be considered as forming a straight angle. To erect a perpendicular at a given point, D, in a straight line is merely to bisect the straight angle, namely, ADB; for it is easily seen that if the straight angle is bisected, the bisecting line will form two right angles with AB, and therefore will be perpendicular to it. Hence, the process of construction and proof is exactly the same as the one just given for bisecting an angle. The reader can easily duplicate it from the drawing in Figure 31.

Instead of drawing by means of compasses the line CD, in Figure 31, perpendicular to AB at D, where D is the mid-point of EF, suppose a protractor had been used. Then it is easy to show that PE and PF must be equal.

In the triangles EPD and FPD, segments ED and FD are constructed equal, and segment PD is common to both. Segment PD is perpendicular to AB; hence, angles PDE and PDF are right angles. Since all right angles are equal, angles PDE and PDF are equal. That is, triangle PDE has sides ED and PD and the included angle PDE equal, respectively, to the two sides FD and PD and the included angle PDF of triangle FPD. Therefore, by Fact 2, the triangles are congruent. Since corresponding sides of congruent triangles are equal, it follows that PE is equal to PF. That is,

Fact 28: If two straight line segments drawn from a point in a perpendicular to a given line cut off on the given line equal segments from the foot of the perpendicular, they are equal.*

BISECTING A LINE SEGMENT

Three circular holes, A, B, and C, Figure 32, must be drilled in a rectangular steel plate 12 inches wide by 16 inches long, the three holes to be in a straight line 6 inches from the smaller side and such that A and B are two inches from the longer side and C is exactly in the middle between A and B. In order to get the location of C as accurately as possible, it is necessary to bisect the line segment AB.

With A and B as centers, and with any convenient radius greater than one-half the distance AB, construct two arcs in-

* The point in which a perpendicular line meets another line is called the *foot* of the perpendicular.

tersecting at D and E. Draw the line DE cutting AB at C. Then DE bisects AB at C.

In order to prove C equidistant from A and B, it is sufficient to prove AC equal to BC.

Draw the line segments AE, AD, BE, and BD. The triangle ADE has, by construction, its three sides AD, DE, and AE equal, respectively, to the three sides BD, DE, and BE of triangle BDE. Therefore, by Fact 6, the triangles are congruent. Corresponding angles of congruent triangles are equal; hence, angle ADE is equal to angle BDE.

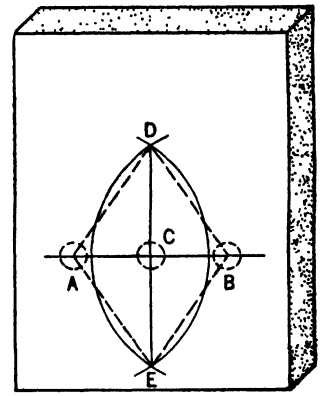


Fig. 32.

Now consider the triangles ADC and BDC. Line segment AD was constructed equal to BD, side DC is common to both, and angle ADC has just been proved equal to angle BDC. That is, triangle ADC has two sides, AD and DC, and the included angle ADC equal, respectively, to the two sides BD and DC and the included angle BDC of triangle BDC. Therefore, by Fact 2, the triangles are congruent. Since corresponding sides of congruent triangles are equal, it follows that AC is equal to BC; that is, C is the mid-point of AB.

PERPENDICULAR TO A LINE SEGMENT

As seen from a point A, Figure 33, on the shore of a lake, an islet, C, is 20 degrees north of east. Observed from another point, B, on the shore of the lake and two miles directly east of A, the islet is found to be 30 degrees north of west. The distance which the islet, C, is directly east of A may be found by making a drawing with any convenient scale, and then simply dropping a perpendicular from C to AB and measuring the distance from A to the foot of the perpendicular.

To drop a perpendicular from a point C outside the given line segment AB to this line segment AB, draw, with C as a center and a radius sufficiently long, an arc cutting AB at M and N. Then, with M and N as centers and any radius greater than one-half MN, construct two arcs intersecting at P below

AB. Draw *CP* intersecting *AB* at *D*. Then *CD* is perpendicular to *AB*.

The proof follows exactly the same reasoning used in the preceding construction. The line segments *CM*, *MP*, *CN*, and *NP* are drawn. The two triangles *CMP* and *CNP* are congruent because, by construction, *CM* and *MP* are respectively equal to *CN* and *NP*, and *CP* is common to both triangles. Therefore, Fact 6 can be applied. Hence, angles *MCP* and *NCP* are equal. Then, considering the triangles *MCD* and *NCD*, it is easily seen that *CM* and *CN* were constructed equal, *CD* is common to both triangles, and the included angles, *MCD* and *NCD*, have just been proved equal. Therefore, by Fact 2, the triangles are congruent and angles *CDM* and *CDN* are equal. Since angle *MDN* is a straight angle, it is evident that angles *CDM* and *CDN* are right angles. Therefore, *CD* is perpendicular to *AB*.

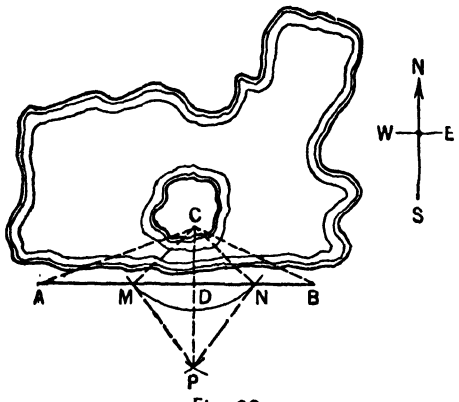


Fig. 33.

Finally, measuring *AD*, the islet *C* is found to be approximately $1\frac{4}{21}$ miles directly east of *A*.

PARALLEL LINE CONSTRUCTION

Assume that, in the rectangular steel plate shown in Figure 32, three more circular holes are to be drilled on a line parallel to the center of the holes already drilled and passing through a point three inches above this line.

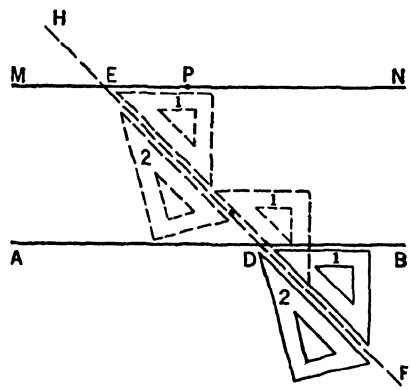


Fig. 34.

A line parallel to another line, passing through a given point, may be drawn by using two "draftsman's triangles," as fol-

lows: Set triangle 1, Figure 34, so that an edge coincides with the line AB. Place triangle 2 so that one of its edges lies flat against triangle 1. Holding triangle 2 motionless, slide triangle 1 along the edge of triangle 2. Repeat this operation as many times as necessary, holding one of the triangles motionless and sliding the other along the edge of the one held fixed until point P is reached. Draw the line MN.

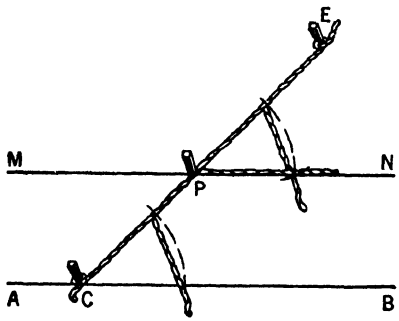


Fig. 35.

The proof that the line MN is parallel to AB is very simple. Angles NED and BDF are equal by construction. That is, the two lines MN and AB are cut by the transversal HF in such a way that the corresponding angles, NED and BDF, are equal. Therefore, by Fact 10, the two lines are parallel.

In laying out a tennis court it is necessary to run a line parallel to AB through a point, P, Figure 35. The following convenient method might be used: Choosing any point, C, on the line AB, run the line CE passing through P. Then, with a rope of any convenient length as radius and using the method already shown on page 124, construct angle EPN equal to angle PCB. The two lines MN and AB are then cut by the transversal, PC, so that the corresponding angles, EPN and PCB, are equal. Therefore, by Fact 10, the lines MN and AB are parallel.

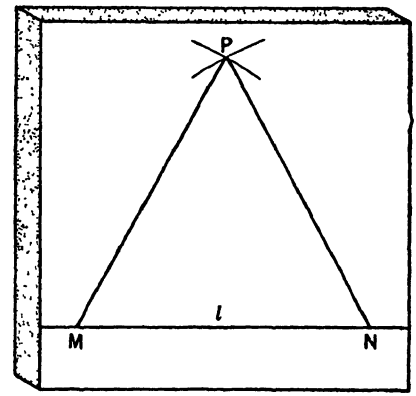


Fig. 36.

DRAWING EQUAL TRIANGLES

A triangular die exactly the same as the one shown in Figure 31 is to be made. To draw a triangle of exactly the same size on another steel plate, the following method can be used: Locate point M, Figure 36, at the same distance from both

edges, on the new plate, as point A in Figure 31. Through M draw the line l parallel to the edge of the steel plate. With M as center, mark off on l, with the compasses, a line segment MN equal to AB in Figure 31. Using the length AC, Figure 31, as radius and with M as center, draw an arc. Then, with N as center and a radius equal to BC, Figure 31, draw an arc cutting the other arc at P. Draw MP and NP. Then MNP is the required triangle, for by construction MP, NP, and MN are equal respectively to AC, BC, and AB.

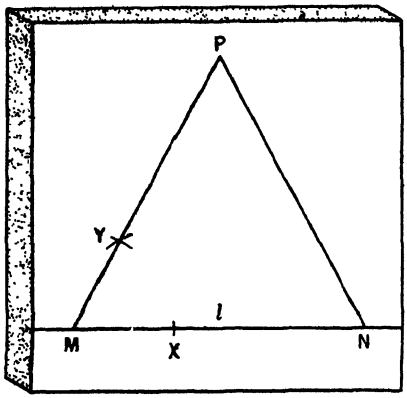


Fig. 37.

The procedure just followed is not, of course, the only method of duplicating the triangular die on page 125. Suppose that instead of measuring the three sides, only two sides and the included angle had been determined. The problem then is to construct a triangle when two sides and the included angle are given.

As before, locate point M; draw the line l parallel to the edge of the steel plate and mark off the line segment MN. Using the method shown on page 124, construct an angle XMY, Figure 37, equal to angle BAC of Figure 31. On MY mark off the line segment MP. Draw NP. Then the triangles ABC of Figure 31 and MNP of Figure 37 have by construction two sides and the included angle of one equal, respectively, to two sides and the included angle of the other. Therefore, by Fact 2, the triangles are congruent.

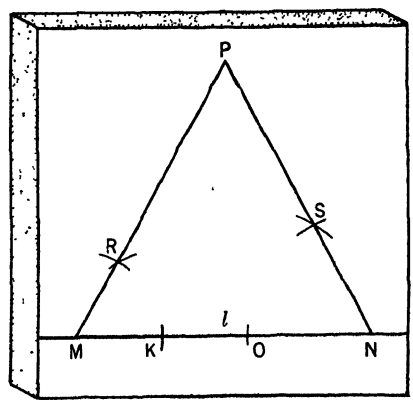


Fig. 38.

The triangular die can be duplicated by still another method. This time ascertain the length of AB and the size of angles BAC and ABC of Figure 31. The problem then is to construct a triangle when two angles and a side are given.

As in the two preceding cases, locate point M, draw l parallel to the edge of the steel plate, and mark off MN. Construct angles KMR and ONS, Figure 38, equal respectively to angles BAC and ABC of Figure 31, by using the method shown on page 124. Extend MR and NS until they meet at P. The triangles MNP of Figure 38 and ABC of Figure 31 have two angles and a side of one equal, respectively, to two angles and a side of the other. Therefore, by Fact 23, the triangles are congruent.

If the side chosen is not included by the angles, such as angles BAC and ABC and side BC of Figure 31, then simply find the third angle by using Fact 18—that is, the sum of the three angles of a triangle is a straight angle—and proceed as before.

INEQUALITIES

Conditions have been discussed under which an object or quantity is *equal* to another. It is just as important to examine the conditions under which an object or quantity is *unequal* to another.

In this new topic, as always in mathematics, there first must be laid a foundation upon which a logical structure can be erected. The foundation consists of definitions and assumptions. From them, by deduction, can be built up a useful body of facts.

A statement indicating that one of two quantities is greater or less than the other is called an *inequality*. The symbol "$a > b$" is read "a is greater than b"; "a is less than b" is written: $a < b$.

A number* is greater than another if the remainder obtained by subtracting the second from the first is *positive*. Thus, if $a - b$ is a positive number, a is greater than b; $(a > b)$. For example: $7 - 5 = 2$. Since 2 is positive, $7 > 5$. Moreover, $(-3) - (-4) = -3 + 4 = 1$; hence, $-3 > -4$.

If the remainder obtained by subtracting a number from another is *negative*, the first is less than the second. Thus, if

* It is assumed in this discussion that we are dealing with real numbers. The reader will recall, from the discussion on numbers in Chapter 1, that of two negative numbers, the larger numerically is the less. That is, -2 is less than -1, -3 is less than -2, -4 is less than -3, and so forth, and each is less than zero.

$a - b$ is negative, number a is less than b, that is, $a < b$. For example: $5 - 7 = -2$. Since -2 is negative, $5 < 7$. Moreover, $(-5) - (-2) = -5 + 2 = -3$. Therefore, $-5 < -2$.

The statement "a is greater than or equal to b" is written: $a \geqq b$, and the symbol "$a \leqq b$" is read, "a is less than or equal to b."

The expression on the left of an inequality is called the *first member*, and the expression on the right is the *second member*. Thus, in the inequality $a + b > a - b$, $a + b$ is the first member and $a - b$ is the second member.

INEQUALITIES CLASSIFIED

As in the case of equations of identity and condition (see Chapter 4, page 41), inequalities are *absolute* or *conditional*. An inequality is *absolute* if it is true for all values of any letters involved. For example: $a^2 + b^2 > -1$ is an absolute inequality; for whether a and b are positive, zero, or negative, their squares are non-negative and therefore the sum of the squares is non-negative and greater than -1. If an inequality is true for only certain values of the letters involved, it is called a *conditional* inequality. Thus, $4x - 8 > 0$ is conditional; for it is true only for values of x greater than 2.

Two inequalities exist in the *same sense* if the first member, in both expressions, is greater (or less) than the second member. Thus, the inequalities $7 > 5$ and $4 > 3$ exist in the same sense. When the first member is greater in one inequality and less in another, the inequalities exist in *opposite senses*. For example: $7 > 5$ and $-4 < 2$ exist in opposite senses.

The following assumption, universally accepted, is the foundation upon which much of the work in mathematics rests:

A given magnitude, when compared with another magnitude, must be either equal to, less than, or greater than the second magnitude.

OPERATIONS WITH INEQUALITIES

The following important properties of inequalities are easily verified:

If equals are added to or subtracted from both members of an inequality, the inequality continues to exist in the same sense. That is, the results are unequal in the same order.

Either one of the operations increases or decreases both members by the same amount. Therefore, that member which was greater before performing either operation is also greater afterwards.

For readers who prefer a demonstration involving the use of letters, the above statement can be shown to be true as follows: Let a, b, and c represent any numbers such that a is greater than b (that is, $a > b$). We want to show that $a + c > b + c$. It is easily seen that $(a + c) - (b + c) = a - b$. Since a is greater than b, the difference, $a - b$, is positive. It follows that $(a + c) - (b + c)$ must also be positive. Therefore, $a + c$ is greater than $b + c$.

For example: Given $7 > 5$. Adding 3 to both members, we have $(7 + 3) > (5 + 3)$; that is, $10 > 8$. Subtracting 3 from both members, we have $(7 - 3) > (5 - 3)$; that is, $4 > 2$.

If both members of an inequality are multiplied or divided by equal positive numbers, the inequality continues to exist in the same sense. In other words, *the results are unequal in the same order.*

The very simple demonstration of this truth follows: Let x and y be any numbers such that x is greater than y, that is, $x > y$; and let c be any positive number. We want to show that xc is greater than yc, that is, $xc > yc$. It is easily seen that $xc - yc = c(x - y)$. The number x is greater than y, so that the difference, $x - y$, is positive; and since c is positive, then $c(x - y)$ is positive. It follows that the difference $xc - yc$ is also positive. Therefore, xc is greater than yc.

In a similar manner, the reader may show that if both members of the inequality are divided by an equal positive number, the inequality continues to exist in the same sense.

The following illustrations clarify the demonstration just given: If $7 > 5$, then $7 \times 3 > 5 \times 3$, that is, $21 > 15$. Moreover, if $8 > 2$, then $8 \div 2 > 2 \div 2$; that is, $4 > 1$.

The reader may show, as an easy and interesting exercise, that: *If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.**

Thus, if $9 > 7$ and $7 > 5$, then $9 > 5$.

---

* *Hint:* Let a, b, and c be any three numbers such that $a > b$ and $b > c$. Then $(a - b) + (b - c) = a - c$. Show that $a - c$ must be positive.

If unequals are added to unequals in the same sense, the inequality continues to exist in the same sense. That is, the sums are unequal in the same order.

For, let a, b, c, and d be any numbers such that a is greater than b (that is, $a > b$) and c is greater than d (that is, $c > d$). We want to show that $a + c$ is greater than $b + d$.

Since a is greater than b, then $a + d$ is greater than $b + d$; for if equals are added to both members of an inequality, the results are unequal in the same order (page 132). Moreover, since c is greater than d, then $a + c$ is greater than $a + d$; for again we have added an equal number, a, to both members of the inequality, $c > d$, that is, $a + c > a + d > b + d$.

But the reader has just shown that if the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third. Therefore, $a + c$ is greater than $b + d$.

For example: If $9 > 7$ and $6 > 5$, then $9 + 6 > 7 + 5$; that is, $15 > 12$.

If both members of an inequality are multiplied or divided by a given negative number, the inequality will then exist in the opposite sense. That is, *the results are unequal in the reverse order.*

For, let m and n be any numbers such that m is greater than n (that is, $m > n$); and let c be a negative number. We want to show that mc is less than nc (that is, $mc < nc$). It is easily seen that $mc - nc = c(m - n)$. Since m is greater than n, then $m - n$ is positive. The product $c(m - n)$ however, is negative, since a negative quantity multiplied by a positive gives a negative quantity. It follows that the difference $mc - nc$ is also negative. Therefore, mc is less than nc; that is, $mc < nc$.

Similarly, it can be shown that if both members of an inequality are divided by a negative number, the results are unequal in the reverse order.

Thus, given $9 > 6$. Dividing both members by -3, we have $(9) \div (-3) < (6) \div (-3)$; that is $-3 < -2$. Multiplying both members by -3, we have $(9) \times (-3) < (6) \times (-3)$; that is, $-27 < -18$.

If each member of an inequality is subtracted from the corresponding member of an equation, the resulting inequality will exist in the opposite sense. In other words, *if unequals are subtracted from equals, the remainders are unequal in reverse order.*

Let a, b, and c be any three numbers such that a is greater than b (that is, $a > b$). Then the difference $a - b$ is positive. Obviously, if a positive number is subtracted from any number, the latter will be diminished. So that $c - (a - b) < c$, that is, $c - a + b < c$. Subtracting the same number, b, from both members of this inequality leaves the results unequal in the same order. Hence, $c - a < c - b$. That is, if each member of the inequality $a > b$ is subtracted from the corresponding member of the equation $c = c$, the result is an inequality existing in the opposite sense.

RELATIONS OF SIDES AND ANGLES OF A TRIANGLE

A force producing motion very often must be applied obliquely. A force applied obliquely, such as the force OV exerted by a boy pulling a sled (Figure 39), evidently produces

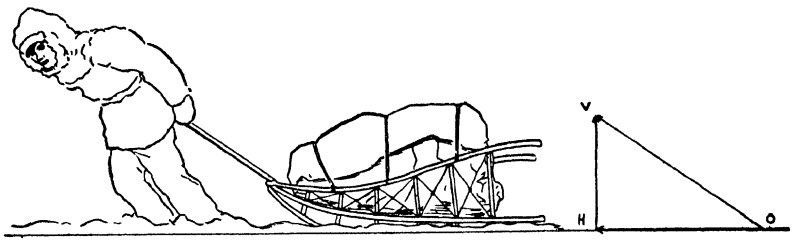


Fig. 39.

two effects: it moves the sled forward along the ground and it tends to lift it. Forces have not only magnitude, but also direction. Quantities which require a specification of direction as well as magnitude are called *vector quantities*. Forces, therefore, are vector quantities. They are represented by arrows, as many units long as there are units of force, to indicate direction and magnitude. The arrowhead indicates the direction in which the force is acting. The two effects produced by a force acting obliquely are called its *components*. The force VH is called the *vertical component* and the force OH, the *horizontal component*. It is shown, in physics, that if the force OV is represented by a vector drawn to scale, its vertical and horizontal components are the legs of the right triangle formed by drawing a horizontal line through O and dropping a perpendicular to this horizontal line from the point V. That is, OH is the horizontal component and represents the horizontal force

pulling the sled along the ground, while VH is the vertical component and represents the vertical force tending to lift the sled. These two forces are, therefore, together the equivalent of the force OV. If we can show that if in a triangle one side is greater than another, the angle opposite the greater side is the greater angle, then it is easily seen that if the horizontal component of the force is larger than the vertical component, angle OVH will be greater than angle VOH. That is, the force OV will be nearer to the horizontal position.

Assume, then, that any triangle ABC (Figure 40), is given such that AC is greater than CB, that is, $AC > BC$. We want to show that angle CBA is greater than angle BAC.

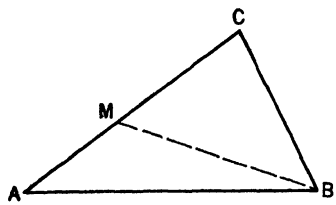


Fig. 40.

Since AC is greater than CB, on AC lay off CM equal to CB. Draw MB. Angle CBA is greater than angle CBM, for the whole is greater than any of its parts. Angle CMB is an exterior angle of triangle AMB; hence, by Fact 7, angle CMB is greater than the opposite interior angle BAC. Now, if angle CMB can be shown to be equal to angle CBM, we can prove angle CBA greater than angle BAC by using one of the properties of inequalities just discussed in the preceding pages. The two sides CM and CB of triangle CMB were constructed equal; hence, by Fact 3, the angles CBM and CMB opposite these equal sides are equal. That is, we have angle $CBA >$ angle CBM, and angle $CMB >$ angle BAC, and angles CBM and CMB equal. Substituting angle CBM for its equal angle CMB, then angle $CBA >$ angle CBM and angle $CBM >$ angle BAC. That is, the first of three quantities (angle CBA) is greater than the second (angle CBM), and the second (angle CBM) is greater than the third (angle BAC). Therefore, by the property of an inequality, shown on page 133, the first is greater than the third. So that angle CBA is greater than angle BAC. In other words:

Fact 29: If one side of a triangle is greater than a second side, the angle opposite the first side is greater than the angle opposite the second side.

In pulling an object such as a sled, we have undoubtedly noticed that the nearer to the ground the force is applied, the

easier it is to move the object. The explanation of this fact is easily seen if we can show that if two angles of a triangle are unequal, the side opposite the greater angle is the greater. For obviously, as the angle VOH, Figure 39, is made smaller, since OHV is a right triangle, angle OVH will be greater. So that the horizontal component of the force applied through the rope pulling the sled will be greater.

Assume, then, that any triangle, ABC, Figure 40, has angle B greater than angle A. We want to show that AC is greater than CB. On page 132 it was assumed that a given quantity, when compared with another quantity of the same kind, must be either equal to, less than, or greater than the other quantity. So that AC is either equal to CB, less than CB, or greater than CB. If AC is equal to CB, triangle ABC has two sides equal; hence, by Fact 3, the angles, A and B, opposite these sides, are equal. But this conclusion is contrary to the fact that angle B is greater than angle A. If AC is less than CB, that is, CB is greater than AC, then by Fact 29, just proved, angle A opposite the greater side CB would be greater than angle B. This is also impossible, for it contradicts the fact that angle B is greater than angle A. That is, AC cannot be equal to or less than CB. Therefore, AC is greater than CB. In other words,

Fact 30: If one angle of a triangle is greater than a second angle, the side opposite the first angle is greater than the side opposite the second angle.

SHORTEST LINE SEGMENT

The reader has probably surmised that most of the demonstrations of the properties of inequalities seem to be superfluous. These properties appear to be so self-evident! However, at this point it is well to return to page 123 of this chapter and inspect again the figures there. This reminder of the tricks that our eyes can play on us will unquestionably renew the conviction already formed, of the necessity of proving, as far as possible, even what seem to be the most obvious facts. For example, it seems evident that, in a plane,

Fact 31: The perpendicular is the shortest line segment that can be drawn from a given point not on a given straight line, to this line.

However, by this time, the reader is justly sceptical about

concurring, in general, with what his senses seem to indicate are *obvious* facts. Hence proof is wanted to show that, given a point, P, Figure 41, and a line segment, MN, then a perpendicular is the shortest line segment that can be drawn from P to MN.

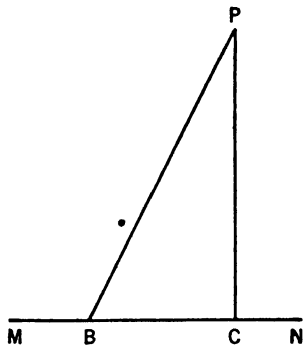


Fig. 41.

Draw a perpendicular from P to MN, meeting MN at C, and any other line segment PB. It is desired to show that PC is less than PB, that is, $PC < PB$. This fact will be proved if we can show that angle PBC is less than angle PCB, for in that case a simple application of Fact 30 would suffice.

Since PC is perpendicular to MN, then angle PCB is a right angle and evidently the figure PCB is a right triangle. Hence, by Fact 20, the sum of the other two angles of the triangle, namely, PBC and BPC, is equal to one right angle. So that angle PBC is, thus, less than a right angle. That is, angle $PBC < PCB$. Therefore, by Fact 30, the side PB, opposite the greater angle, PCB, is greater than the side PC. In other words, the perpendicular PC is the shortest line segment that can be drawn from the point P to the straight line MN.

The converse of this fact follows immediately. That is, if PC is the shortest line segment from P to MN, it must be perpendicular to MN. For if PC were not perpendicular to MN, we could have, by drawing a perpendicular and applying Fact 31, just proved, a line segment from P to MN shorter than PC. But this conclusion is contrary to the fact that PC is the shortest line segment from P to MN. Therefore,

Fact 32: If a line segment is the shortest that can be drawn from a given point to a given line, it is the perpendicular from the point to the given line.

RELATION OF SIDES AND ANGLES OF TRIANGLES

A trough is formed by nailing together, edge to edge, two boards, so that a cross section is a triangle (ABC, Figure 42). Usually, troughs are made so that the boards which make up

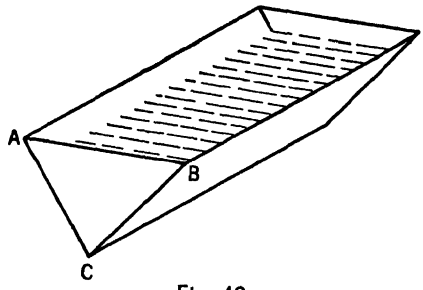


Fig. 42.

the sides are of the same width, that is, AC in Figure 42 is equal to BC. In order to insure the greatest carrying capacity, the vertex angle ACB, formed by the sides, is made a right angle. Suppose two troughs are constructed with boards of the same width. The fact that of the two troughs the one made with the greater vertex angle will have the greater top width seems evident. A logical demonstration of this fact may be given as follows:

Assume that triangles ABC and QRP have sides b equal to r and a equal to q, but angle C greater than angle P. It is to be shown that side c is greater than side p. Place triangle QRP so that r coincides with its equal b and R lies on the same side of AC as B. Then, since angle P is less than angle C, side q falls within angle ACB. Construct CM so that it bisects angle RCB and draw MR.

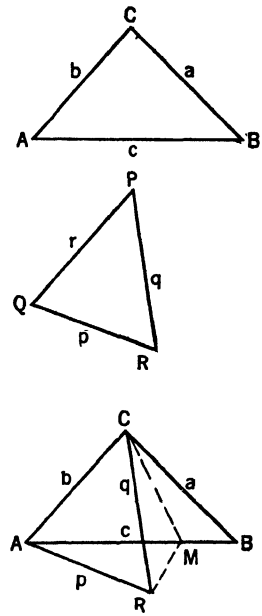


Fig. 43.

If triangles CMR and CMB can be shown congruent, then p can be easily proved less than c. In these triangles, a is equal to q by construction, CM is common to both triangles and bisects angle RCB, so that angles RCM and MCB are equal. That is, two sides and the included angle of triangle CMR are equal, respectively, to two sides and the included angle of triangle CMB. So that, by Fact 2, the triangles are congruent. It follows that MR is equal to MB. Since, in a plane, a straight line segment is the shortest distance between two points, $AM + MR$ is greater than p. Or, substituting MB for its equal, MR, $AM + MB$ is greater than p. But $AM + MB$ is equal to line c; therefore, c is greater than p. In general, then,

Fact 33: If two triangles have two sides of one equal, respectively,

to two sides of the other, and the included angles unequal, the triangle which has the greater included angle has the greater third side.

It is now easy to show that if the triangles ABC and QRP of Figure 43 have, as before, sides b and r, and a and q respectively equal, but this time side c is known to be greater than p, then angle C must be greater than angle P.

The demonstration is of the same type as that given for Fact 30. Angle C is either equal to, less than, or greater than angle P. If angle C is equal to angle P, the triangle ABC has the two sides a and b and the included angle C equal, respectively, to the two sides r and q and the included angle P of triangle QRP. Hence, by Fact 2 the triangles are congruent and c is equal to p. This conclusion contradicts the fact that c is given greater than p, so it is impossible.

If angle C is less than angle P, then by Fact 33, just proved, c is less than p. This is also impossible, because it again contradicts the fact that c is given greater than p.

That is, neither of these two conclusions can be true; therefore, by the assumption stated on page 132, the only remaining possibility must be true, and angle C is greater than angle P. In other words,

Fact 34: If two triangles have two sides of one equal, respectively, to two sides of the other and the third sides unequal, the triangle which has the greater third side has the greater angle opposite the third side.

Before the year 1675, the velocity of light was thought to be instantaneous. However, in that year Olaf Roemer, a Danish astronomer, made the following observations of one of the moons of the planet Jupiter. A moon of Jupiter, M, in Figure 44, in its path around the planet, passes into Jupiter's shadow once in $42\frac{1}{2}$ hours, approximately. In passing through Jupiter's shadow, the moon, M, is darkened and it is then said to be "eclipsed." Roemer accurately observed, when the earth was at E, the time at which Jupiter's moon, M, was eclipsed. He then computed the exact time at which an eclipse should occur six months later when the earth was at E'. However, he noticed that the eclipse occurred about sixteen and a half minutes later.

If we can show that the distance ME' is greater than ME, then we can conclude, as Roemer did, that the difference be-

tween his computation and the actual instant in which the eclipse occurred is due to the fact that it takes light about 16.6 minutes to cross the orbit of the earth, a distance of about 185,800,000 miles. This gives the velocity of light as about 186,000 miles per second.

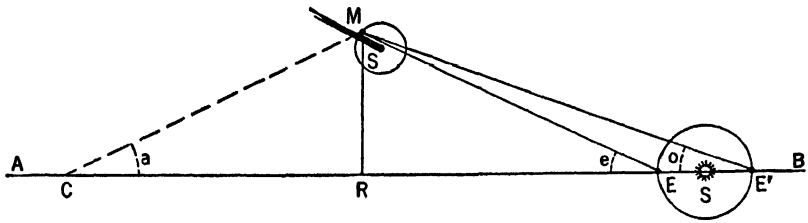


Fig. 44.

It is interesting to note that although this speed seems fabulous to us, it is so small in comparison with the distance between stars that the light reaching us now from the pole star started about $54\frac{1}{2}$ years ago. That is, if an observer on the pole star had a telescope powerful enough to be able to see events on the earth, he would now be seeing the closing years of the reign of Queen Victoria.

Through E and E', in Figure 44, draw the line AB. From the point M drop the perpendicular, MR, to AB. Draw the line segment MC so that RC is equal to RE. Then, since these line segments MC and ME drawn from the perpendicular, MR, cut off equal segments, RC and RE, from the foot of the perpendicular, by Fact 28, the line segments MC and ME are equal.

Now, ME' is greater than ME if ME' is greater than MC, which is equal to ME. The line segment ME' can be shown to be greater than MC if we can show angle a greater than angle o. Angle a is greater than angle o if angle e can be shown equal to angle a and greater than angle o. These different steps can be demonstrated as follows:

In the triangle CME, the two sides MC and ME have just been proved equal; hence, by Fact 3, the angles e and a opposite these sides are equal.

Angle e is an exterior angle of triangle EME', so that by Fact 7 it is greater than the opposite interior angle o. Substituting angle a for its equal, angle e, we have angle a greater than angle o.

Then, in triangle CME', since angle a has just been shown greater than angle o, then, by Fact 30, the side ME' opposite the greater angle a is greater than the side MC opposite angle o. Since MC is equal to ME, then ME' is greater than ME. That is:

Fact 35: If two straight line segments drawn from a point in a perpendicular to a given line, cut off on the given line unequal segments from the foot of the perpendicular, the more remote of the two given straight line segments is the greater.

SOLUTION OF INEQUALITIES

A conditional inequality in one variable such that the unknown appears to the first degree only is called a *linear inequality*. The solution of a conditional inequality in one variable consists in finding all values of the variable for which the conditional inequality is true. An inequality, however, unlike an equation, will generally be satisfied by a certain range of values of the variable rather than by one or a few particular values.

From the preceding discussions of the properties of inequalities, it is easily seen that certain operations which are used in handling equations can be applied to the solution of inequalities. Hence, in solving an inequality, we treat it in much the same way as an equation. For example: To solve the inequality $3x - 10 > 11$, we transpose -10 to the right-hand side or second member (by using the principle stated on page 132), and obtain $3x > 21$. If we now divide both members by 3 (using the principle demonstrated on page 133), we then get $x > 7$.

Two inequalities in one variable impose two conditions on the values for which the inequalities are true. Thus, if the inequality of the preceding example is qualified by the additional inequality $2x - 15 < 5$, the solution consists of finding the value of x for which both the inequalities $3x - 10 > 11$ and $2x - 15 < 5$ are true.

From the preceding example, we know the first inequality is true for all values of x greater than 7. Solving the second inequality, we first transpose -15, obtaining $2x < 20$. Then, dividing both members by 2, we get $x < 10$. Thus, the second inequality is true for all values of x less than 10. Since the first inequality imposes the condition that x be greater than 7, while the second inequality imposes the condition that x be

less than 10, then, any number greater than 7 but less than 10 will satisfy both inequalities, while no other number will.

Problems

1. The treasurer of a small corporation wishes to make a purchase of machinery for \$2,800 but has not enough money. If he borrows one-third as much money as he now has, he will be able to make the purchase and have more money left than he now lacks. How much money has he now in the treasury?

2. The daily payroll for plumbers and electricians of a contractor who pays electricians \$7.50 a day and plumbers \$6.50 a day is between \$250 and \$260. There are 4 more electricians than plumbers. Find the number of each.

3. Two insured properties are, respectively, 12 and 8 years old. What is the shortest time after which the age of the older property will be less than $1\frac{3}{10}$ times the age of the younger?

4. If only 5 people be seated on each bench in a courtroom containing fewer than 6 benches, 4 people will be without seats; but if 6 people be seated on each bench except the last, not all of the seats on the latter will be occupied. Find the number of benches.

5. A submarine is 50 miles east of a cruiser, a passenger steamer is 40 miles south of the submarine, and a freighter is 40 miles northwest of the submarine. Show that the cruiser is at a greater distance from the passenger steamer than from the freighter.

6. Prove that the median of a triangle is less than half the sum of the two adjacent sides.

7. A factory manager in comparing two new methods of computing the cost of disposing of certain waste products symbolizes the methods by the two expressions $a^3 + b^3$ and $a^2b + ab^2$. Which is greater?

8. A realtor is making tentative measurements for a triangular space between some plots on which dwellings are to be erected. Two sides of the triangle are to be 30 and 80 feet. Between what numerical limits will the length of the third side lie?

9. Within the triangular course of a yacht race, ABC being the triangle, a launch is anchored at the point P, the distance CP being equal to CB. Prove that $AB > AP$.

10. An air pilot flying at a constant altitude of 10,000 feet and in a straight line in the direction ABC observes, from A, a beacon at Q and finds angle BAQ to be 35°. After flying in the same direction 7 miles to B, he observes angle CBQ to be 70°. How far is the plane now from the beacon?

10

COMPARISON OF NUMERICAL QUANTITIES

The idea of ratio, implied by the very common word *rate*, is one of the most important cornerstones of mathematics. Quantities, in everyday life, are continuously being compared

Courtesy of Librairie Ancienne, Leo S. Olschki, Florence

Woodcut from Philippus Calandri, *Arithmetica*, Firenze, 1491.

in manifold ways and their relations measured by important ratios.

Ratio is, probably, the most widely used arithmetical device in business. The relationship between current assets and current liabilities, called "the current ratio," is one item to which bankers give keen attention when considering applications of business firms for credit.

Some of the most important definitions in science are given in terms of ratios. Thus, the specific gravity of a body is defined as the ratio of its weight to the weight of an equal volume of some other body assumed as a standard.

Ratio originates in the comparison of numbers. In developing the idea of number, we begin by numbering concrete objects, that is, with *denominate numbers*. Denominate numbers are concrete in the sense that the number is associated with some particular unit. Thus, 12 cases of merchandise, 5 men, and 2 shoes are denominate or concrete numbers. Much later the numerical idea is met dissociated from the objects, and *pure* or *abstract numbers* are obtained. Thus, 12, 5, and 2 are abstract numbers.

RATIO

The ratio between two quantities is the *numerical measure* of their relation. A relation of 12 to 2 is that "12 is six times 2," but the ratio of 12 to 2 is *six;* and this ratio, six, is the number which *measures* this relation of 12 compared to 2. The word *ratio* is, therefore, more precise than *relation.*

Since a ratio measures the numerical relation of two quantities, it is always an abstract number. For example, if the ratio of 12 cases of perfect-quality merchandise to 2 cases of spoiled merchandise should be under consideration in adjusting a claim for damages, the ratio is the measure of the relationship of 12 to 2, that is, six, a pure or abstract number.

The ratio of one number to another is the result obtained by dividing the first number by the other. Thus, the ratio of 3 to 5 is $\frac{3}{5} = 0.6$.

Here 3 is the dividend, 5 is the divisor, and 0.6 is the quotient. *Ratio* and *quotient* have almost but not exactly the same meaning. *Ratio* states a relationship; *quotient* gives the numerical value of that relationship. If we write $\frac{3}{5}$, we may wish to indicate that the operation of division is to be performed; if we

write 0.6, we have the quotient or result obtained by division.

Ratios are expressed either in the form of a fraction or by the symbol : placed between the two numbers. Thus, 4 : 7 and $\frac{4}{7}$ mean the same thing. In general, the ratio of a to b is written either $a : b$ or $\frac{a}{b}$.

Since a ratio may be expressed as a fraction, obviously the rules of operations with fractions also hold with ratios. The value of a fraction is not changed if both numerator and denominator are either multiplied or divided by the same number. Thus, the ratio of 2 to 8 and the ratio of 1 to 4 are equal, because $\frac{2}{8} = \frac{1}{4}$.

The magnitudes of a ratio must be expressed in the same unit. For example, what is the ratio of 1 gallon to 3 quarts? Evidently *not* 1 : 3, but rather 4 : 3.

PROPORTION

The statement of the equality of two ratios is a *proportion.* Thus, $\frac{2}{8} = \frac{1}{4}$ or $2 : 8 = 1 : 4$ is a proportion. In general, $\frac{a}{b} = \frac{c}{d}$ or $a : b = c : d$ is read "a is to b as c is to d." The first and last terms, that is, 2 and 4, or a and d, are called the *extremes,* or outer terms. The second and third terms, that is, 8 and 1, or b and c, are called the *means,* or inner terms. The first important relation existing between two terms of a proportion is easily deduced. Given $\frac{2}{8} = \frac{1}{4}$, observe that $2 \times 4 = 8 \times 1$, that is, the "cross products" are equal.

In general, suppose that $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$, and let r equal the ratio, so that $\frac{a}{b} = r$ and $\frac{c}{d} = r$. Then, since $\frac{a}{b} = r$, it follows that $a = br$. Similarly, since $\frac{c}{d} = r$, then $c = dr$. Hence, the proportion $a : b = c : d$ may be written $br : b = dr : d$. It is now easily seen that the product of the means is $b \times dr$ and the product of the extremes is also $br \times d$; hence, the two products are equal. So that, if $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$, then,

$$ad = bc. \tag{1}$$

That is: *The product of the means equals the product of the extremes.*

This fundamental law of proportion is often called the "*rule of three.*"

Moreover, if $\frac{2}{8} = \frac{1}{4}$, then $\frac{2 \times 4}{8} = 1$; and if $\frac{3}{8} = \frac{6}{16}$, then $16 = \frac{6 \times 8}{3}$.

In general, if $\frac{a}{b} = \frac{c}{d}$, the following four equations can be derived by multiplication and division:

$$a = \frac{b \times c}{d}; b = \frac{a \times d}{c}; c = \frac{a \times d}{b}; d = \frac{b \times c}{a}.$$

Hence, if any three terms of a proportion are given, the fourth may be found.

Thus:

$$3 : 5 = 6 : \text{What number?}$$

or

$$\frac{3}{5} = \frac{6}{d}.$$

Then,

$$d = \frac{5 \times 6}{3} = 10.$$

So that

$$\frac{3}{5} = \frac{6}{10}.$$

SIMPLE PROPORTION

By applying one of the solutions just given, a multitude of practical problems can be solved. Any particular problem should be studied until the conditions are thoroughly understood. The forming of the proportion should then offer little difficulty. Thus, suppose the problem to be, "If an automobile consumed 5 gallons of gasoline in going 90 miles, how many gallons of gasoline would it require for a trip of 2700 miles?"

Here it is easily seen that, other things being equal, the number of gallons of gasoline required for a trip of 2700 miles bears the same relation to the number of gallons (5) of gasoline

required for a trip of 90 miles that 2700 miles bears to 90 miles. Hence, we have the proportion, number of gallons for a trip of 2700 miles : 5 = 2700 : 90. So that, if we let a represent the number of gallons required for a trip of 2700 miles, then $a : 5 = 2700 : 90$, or $\frac{a}{5} = \frac{2700}{90}$. Hence, $a = \frac{2700 \times 5}{90} = 150$ gallons.

If the statement of a proportion should offer some difficulty to the reader, the problem may be solved by analysis as follows: If the automobile consumed 5 gallons of gasoline in going 90 miles, it will consume $\frac{5}{90} = \frac{1}{18}$ of a gallon to go one mile, and $2700 \times \frac{1}{18} = 150$ gallons for a trip of 2700 miles.

Although the latter method may seem much easier, the comparison of numbers by proportion is necessary in the solution of problems which involve comparison of magnitudes and also in higher mathematics. Readers should therefore master the method.

After the process of solving a proportion is well known and understood, the following mechanical method of solution may be used. Make the required term the first term. The term which is of the same kind as the first term is made the second, so that the third and fourth are of the same kind and denomination. To ascertain which one should be the third or fourth term, determine from the problem whether the answer should be greater or less than the second term.

If the answer should be greater than the second term, then make the greater of the other two terms the third. If the answer should be less than the second term, make the lesser of the other two terms the third.

Thus, if an automobile consumed 5 gallons of gasoline in going 90 miles, find the number of gallons of gasoline required for a trip of 2700 miles.

The answer is to be in gallons; hence, make gallons the second term. The answer is to be greater than 5 gallons; hence, make 2700 the third term. So that $a : 5 = 2700 : 90$, or $\frac{a}{5} = \frac{2700}{90}$. Hence, $a = \frac{2700 \times 5}{90} = 150$ gallons.

It is important to note that "the rule of three" works only in cases where the quantities are in proportion, that is, when the ratios of the quantities to each other are equal and *remain constant.* This rule does not work in any other case. For

example: A stone dropped from the Empire State building, in New York City, 1248 feet high, will hit the ground 8.9 seconds afterward. How high is the R.C.A. building at Rockefeller Center if a stone hits the ground 7.5 seconds after being dropped? The answer is *not* given by the proportion $3.9 : 7.5 = 1248 : d$, for in this case d would be equal to $\frac{1248 \times 7.5}{8.9} = 1051.79$ feet, while the correct answer is 850 feet. The rule of three does not work in this case simply because the distance fallen is not the same for the first second as for every succeeding second.

PROPERTIES OF PROPORTION

From the rule of three some basic properties of proportion may be deduced.

Assume that $a : x = a : y$, that is, $\frac{a}{x} = \frac{a}{y}$. Then, by the fundamental law of proportion [see (1) on page 146],

$$ay = ax,$$

and, dividing both sides of this equality by a, we obtain $y = x$. So that, if the first and third terms of a proportion are equal, the second and fourth terms are equal.

Thus, if $5 : 3 = 5 : d$, it follows that $d = 3$.

The reader can easily show that, if $a : b = c : d$ and $a : b = c : x$, then $x = d$. That is, if three terms of one proportion are equal to the corresponding three terms of another proportion, the fourth terms of the two proportions are equal.

So that, if $5 : 3 = 10 : 6$ and $5 : 3 = 10 : x$, then $x = 6$.

It is easy to see that, when four numbers are in proportion, the ratio between any two of them (if the two chosen are not the means or the extremes) is the same as the ratio between the other two. Thus, in the proportion $2 : 8 = 1 : 4$, or $\frac{2}{8} = \frac{1}{4}$, by (1) $2 \times 4 = 8 \times 1$, $2 = \frac{8 \times 1}{4}$, and $\frac{2}{1} = \frac{8}{4}$, that is, $2 : 1 = 8 : 4$.

Following exactly this procedure, the reader may show that, if

$$a : b = c : d,$$

then

$$a : c = b : d \tag{2}$$

and

$$b : a = d : c. \tag{3}$$

If four numbers are in proportion, then (2) shows that they are in proportion by *alternation;* that is, the first term is to the third as the second is to the fourth. Moreover, (3) shows that the numbers are also in proportion by *inversion;* that is, the second term is to the first as the fourth is to the third. So that, if $5 : 3 = 10 : 6$, then by (2), $5 : 10 = 3 : 6$; and by (3), $3 : 5 = 6 : 10$.

In the case $a : b = b : c$, the term b is called *the mean proportional* between a and c. There is only one positive mean proportional between two numbers, and hence we speak of *the* mean proportional.

If $a : b = b : c$, or $\frac{a}{b} = \frac{b}{c}$, then $ac = b^2$ and $b = \sqrt{ac}$. Hence, *to find the mean proportional between two numbers, take the square root of their product.*

Thus, the mean proportional between 2 and 32 is equal to $\sqrt{32 \times 2} = \sqrt{64} = 8$.

If in the proportion $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$, 1 is added to each side, the equality remains unchanged. Hence, if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + 1 = \frac{c}{d} + 1$, or $\frac{a+b}{b} = \frac{c+d}{d}$; that is,

$$(a + b) : b = (c + d) : d. \tag{4}$$

So that, (4), if four numbers are in proportion, they are in proportion by *addition;* that is, the sum of the first two terms is to the second as the sum of the last two terms is to the fourth. Thus, if $2 : 3 = 4 : 6$, then $(2 + 3) : 3 = (4 + 6) : 6$, or $5 : 3 = 10 : 6$.

Similarly, it can be easily shown that:

$$(a + b) : a = (c + d) : c. \tag{5}$$

Hence, if $2 : 3 = 4 : 6$, then $(2 + 3) : 2 = (4 + 6) : 4$, or $5 : 2 = 10 : 4$.

Now consider the series of equal ratios $a : b = c : d = m : n$, or $\frac{a}{b} = \frac{c}{d} = \frac{m}{n}$.

Since $\frac{a}{b} = \frac{m}{n}$, then by (1) $bm = an$. Similarly, since $\frac{a}{b} = \frac{c}{d}$, then by (1) $bc = ad$, and obviously $ab = ab$.

If equals are added to equals, the sums are equal. Hence, $ab + bc + bm = ab + ad + an$, or $b(a + c + m) = a(b + d + n)$. Dividing both sides by $b(b + d + n)$, we obtain $\frac{a + c + m}{b + d + n} = \frac{a}{b}$; that is,

$$(a + c + m) : (b + d + n) = a : b.$$

Hence, in a series of equal ratios, the sum of all the first terms is to the sum of all the second terms as the first term in any of the given ratios is to its second term.

For example: if $2 : 3 = 4 : 6 = 8 : 12$, it follows that $(2 + 4 + 8) : (3 + 6 + 12) = 2 : 3$; that is, $14 : 21 = 2 : 3$.

Assume now that $a : b = c : d$, $m : n = h : l$, and $r : s = v : w$; that is, $\frac{a}{b} = \frac{c}{d}$, $\frac{m}{n} = \frac{h}{l}$, and $\frac{r}{s} = \frac{v}{w}$.

Since $\frac{m}{n} = \frac{h}{l}$, we may multiply $\frac{a}{b}$ by $\frac{m}{n}$ and $\frac{c}{d}$ by $\frac{h}{l}$ without changing the equality $\frac{a}{b} = \frac{c}{d}$. For, if equals are multiplied by equals, the products are equal. Hence,

$$\frac{am}{bn} = \frac{ch}{dl}.$$

Moreover, since $\frac{r}{s} = \frac{v}{w}$, the expression $\frac{am}{bn}$ may be multiplied by $\frac{r}{s}$, and $\frac{ch}{dl}$ by $\frac{v}{w}$, without changing the equality $\frac{am}{bn} = \frac{ch}{dl}$. So that

$$\frac{amr}{bns} = \frac{chv}{dlw}.$$

By using the fact that if equals are divided by equals, the

quotients are equal, the reader may easily show that $\frac{a}{b} \div \frac{m}{n} \div \frac{r}{s} = \frac{c}{d} \div \frac{h}{l} \div \frac{v}{w}$; that is,

$$\frac{ans}{bmr} = \frac{clw}{dhv}.$$

Hence, *the products or the quotients of the corresponding terms of two or more proportions are proportional.*

Thus, if $2 : 3 = 4 : 6$, $3 : 5 = 6 : 10$, and $4 : 7 = 8 : 14$, then $\frac{2 \times 3 \times 4}{3 \times 5 \times 7} = \frac{4 \times 6 \times 8}{6 \times 10 \times 14}$; that is, $\frac{24}{105} = \frac{192}{840}$, or $24 : 105 = 192 : 840$.

Moreover,

$$\frac{2 \times 5 \times 7}{3 \times 3 \times 4} = \frac{4 \times 10 \times 14}{6 \times 6 \times 8}; \text{ that is, } \frac{70}{36} = \frac{560}{288}.$$

INVERSE PROPORTION

A certain type of problem gives rise to a proportion whose ratios do not vary in the same order. In this class of problems the magnitudes of the same kind are to each other, not directly as the other two magnitudes, but rather inversely as those magnitudes. This type of proportion is called an *Inverse Proportion.* Inverse proportions are calculated in the same manner as direct proportions, except that, in writing the proportion, the third and fourth terms are reversed. Thus: "Two driving wheels, whose diameters are, respectively, 9 and 5 inches, are connected by a belt. If the larger wheel revolves 120 times per minute, how many will be the revolutions of the smaller wheel?"

In this case, the number of revolutions of the smaller wheel is to 120, not as 5 inches is to 9 inches, but as 9 inches is to 5 inches—that is, inversely as the order indicated by the terms of the first ratio. So that, if a is the number of revolutions of the smaller wheel, $a : 120 = 9 : 5$, or $5a = 120 \times 9$; and $a = \frac{120 \times 9}{5} = 216$ revolutions.

COMPOUND PROPORTION

A *compound ratio* is the product of two or more simple ratios It may be expressed either as the product of two or more frac-

tions, such as $\frac{2}{3} \times \frac{4}{5}$, or by the expression: $\left\{\begin{matrix} 2:3 \\ 4:5 \end{matrix}\right\}$. If a compound ratio is compared to an equal single ratio, or if two compound ratios are compared with each other, we obtain a *compound proportion.* Thus, $8:15 = \left\{\begin{matrix} 2:3 \\ 4:5 \end{matrix}\right\}$, that is, $\frac{8}{15} = \frac{2}{3} \times \frac{4}{5}$, and $\left\{\begin{matrix} 4:6 \\ 8:10 \end{matrix}\right\} = \left\{\begin{matrix} 2:3 \\ 4:5 \end{matrix}\right\}$, that is, $\frac{4}{6} \times \frac{8}{10} = \frac{2}{3} \times \frac{4}{5}$, are compound proportions. Hence, a proportion in which one or both ratios are compound is called a compound proportion.

Now consider the compound proportion

$$\left\{\begin{matrix} a:b \\ c:d \end{matrix}\right\} = \left\{\begin{matrix} m:n \\ v:w \end{matrix}\right\}, \text{ or } \frac{a}{b} \times \frac{c}{d} = \frac{m}{n} \times \frac{v}{w}.$$

This may be written $\frac{ac}{bd} = \frac{mv}{nw}$; and by (1), $acnw = mvbd$, or:

$$a \times c \times n \times w = m \times v \times b \times d. \tag{6}$$

Hence, (6) *in a compound proportion the product of all the terms in the means equals the product of all the terms in the extremes.*

Thus, if $\left\{\begin{matrix} 3:5 \\ 2:3 \end{matrix}\right\} = \left\{\begin{matrix} 6:10 \\ 4:6 \end{matrix}\right\}$, or $\frac{3}{5} \times \frac{2}{3} = \frac{6}{10} \times \frac{4}{6}$, then,

$$3 \times 2 \times 10 \times 6 = 5 \times 3 \times 6 \times 4.$$

Furthermore, if $a : 4 = \left\{\begin{matrix} 5:2 \\ 3:5 \\ 2:3 \end{matrix}\right\}$, it follows that:

$$a \times 2 \times 5 \times 3 = 4 \times 5 \times 3 \times 2;$$

that is,

$$30a = 120, \text{ and } a = \frac{120}{30} = 4.$$

This fact, as the reader probably has already noticed, is simply an extension of the "rule of three," that is, of the fundamental law of proportion stated on page 147.

In solving a problem in which more than two quantities are considered, the method of attack is, nevertheless, the same as in a simple proportion. After a careful study of the conditions involved in the problem, the proportion should be formed.

Thus, suppose the problem to be, "If 15 men can dig an irrigation ditch 100 yards long in 7 days, how many days will 10 men require to dig a ditch 150 yards long?"

Here it is easy to see that the number of days required is in proportion inversely to the number of men and directly to the length of the ditch. Hence, the number of days required for 10 men to dig the 150-yard ditch is to 7 days as 15 men to 10 men and also as 150 yards to 100 yards. That is,

$$\text{No. of days} : 7 = \left\{ \begin{matrix} 15 : 10 \\ 150 : 100 \end{matrix} \right\}.$$

So that, if we let a equal the number of days, then $a : 7 = \left\{ \begin{matrix} 15 : 10 \\ 150 : 100 \end{matrix} \right\}$; and it follows by (6) that $a \times 10 \times 100 = 7 \times 15 \times 150$, or $1000a = 15750$ and $a = \dfrac{15750}{1000} = 15\frac{3}{4}$ days.

Once more, as in the simple proportion case, this problem may be solved by analysis as follows: If 100 yards are dug in 7 days by 15 men, then one yard will be dug in $\frac{1}{100}$ of 7 days, and 150 yards will be dug in 150 times $\frac{1}{100}$ of 7, or $\dfrac{150}{100} \times 7$ days. If 150 yards can be dug in $\dfrac{150}{100} \times 7$ days by 15 men, then one man will take 15 times $\dfrac{150}{100} \times 7$ days to dig 150 yards, and 10 men will take $\frac{1}{10}$ of 15 times $\dfrac{150}{100} \times 7$ days. That is, $\dfrac{15 \times 150 \times 7}{10 \times 100} = 15\frac{3}{4}$ days.

This time the solution by analysis is not so simple as before, and naturally, the greater the number of terms involved, the more difficult becomes a solution by this method. Hence, it is advantageous to master both methods, so as to be able to apply one or the other according to the nature of the problem.

As before, the following mechanical method of solving a compound proportion is given, but again the reader is warned not to use this method until the logical process of solution is well understood.

After a careful study of the conditions involved in the problem, make the required quantity the first term, and the quan-

tity which is of the same kind as the required term, the second. Usually there are two or more third and fourth terms. Arrange each pair, of the same kind, of these third and fourth terms separately, by the same reasoning followed in a single proportion. Write down all the third terms in a column, and all the fourth terms in another column, forming the compound proportion. Apply principle (6) for compound proportions and solve for the unknown:

Thus, assume the problem to be, "If 15 men can dig an irrigation ditch 100 yards long and 3 yards wide in 7 days, how many days will 10 men require to dig a ditch 150 yards long and 2 yards wide?"

The required quantity is days, so make days the first term. The term of the same denomination is 7 days; hence, 7 is the second term. There are three pairs of third and fourth terms, as follows:

$$\text{men } \begin{cases} 15 \\ 10 \end{cases} \qquad \text{length } \begin{cases} 100 \\ 150 \end{cases} \qquad \text{width } \begin{cases} 3 \\ 2 \end{cases}$$

Considering only the ratio of the first pair, that is, men, to the first and second terms, the answer should be greater. Hence, make 15 a third term and 10 a fourth. The same condition exists when the proportion formed by the second pair with the first and second terms is considered separately. So that 150 is a third term and 100 is a fourth. But considering only the proportion formed by the third pair and the first and second terms, the answer, obviously, should be less. Hence, make 2 a third term and 3 a fourth. Thus:

$$a : 7 = \begin{array}{c} \textit{Third Terms} \\ \left\{ \begin{array}{rcr} 15 & : & 10 \\ 150 & : & 100 \\ 2 & : & 3 \end{array} \right. \end{array} \begin{array}{c} \\ \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \end{array} \textit{Fourth Terms}$$

Applying principle (6), we obtain:

$$a \times 10 \times 100 \times 3 = 7 \times 15 \times 150 \times 2$$

or

$$3000a = 31{,}500; \text{ that is, } a = \frac{31{,}500}{3000} = 10\tfrac{1}{2} \text{ days.}$$

The operation may be made simpler by cancellation, writing the above calculation as follows:

$$a = \frac{\overset{1}{\cancel{7}} \times \overset{3}{\cancel{15}} \times \overset{1}{\cancel{150}} \times \overset{1}{\cancel{2}}}{\underset{1}{\cancel{10}} \times \underset{2}{\cancel{100}} \times \underset{1}{\cancel{3}}} = \frac{21}{2} = 10\tfrac{1}{2} \text{ days.}$$

Problems

1. If 15 pounds of coffee cost \$6.75, how much would 33 pounds cost?

2. A tree 15 feet high casts a shadow 9 feet long. How high is a building which, at the same time, casts a shadow 36 feet long?

3. Two gears have a speed ratio of 2 : 3. If the larger gear has 45 teeth, how many teeth has the smaller gear?

4. For a house assessed for \$12,000, one must pay \$264 in taxes. How much should one pay in taxes on a house assessed for \$7,200?

5. A farmer raised wheat and barley in the ratio of 3 : 2. His neighbor raised 30 bushels more of each, and his crop of wheat was to his crop of barley as 5 : 4. How many bushels of each grain did the first farmer raise?

6. Two barrels, partly full, contain, respectively, 30 and 74 gallons. How many gallons must be transferred from one to the other so that the contents may be in the ratio 3 : 5?

7. Six masons, working 8 hours a day, finish a wall in 9 days. How long would it take 10 masons, working 6 hours a day, to finish a similar wall?

8. If a building 40 feet long, 30 feet wide, and 24 feet high costs \$25,000, how much would a building 45 feet long, 25 feet wide, and 30 feet high cost?

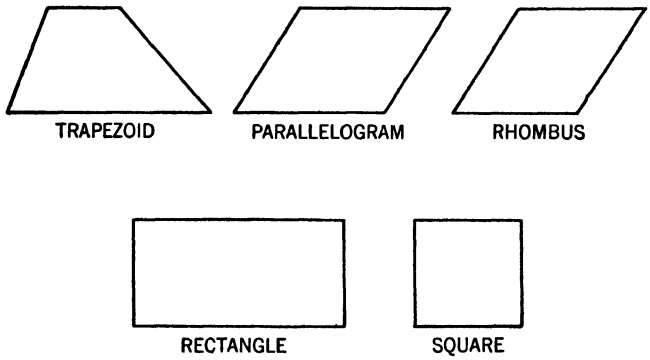
9. The manager of a cafeteria estimates the food consumed by 3,500 persons, eating 3 meals a day for 7 days, costs \$1,530. How much will be the cost of food for 2,000 additional customers, eating 2 meals a day, and the same quantity of food per meal, for 15 days?

10. The freight charges for shipping 6,000 pounds of sugar a distance of 1,350 miles is \$324. How much will it cost to ship 1,375 pounds of sugar a distance of 3,600 miles?

11. Messrs. Young, Rogers, and Morris engaged in a temporary partnership with a joint capital of \$74,000. Young's capital was invested for 3 months, Rogers' for 2 months, and Morris' for 5 months. Young's share of the gain was \$781.25, Rogers' \$625, and Morris' \$989.58. What was the capital of each?

11 COMPARISON OF MAGNITUDES

At a very early date, surveyors used to measure land with ropes and to record the results. These ancient surveyors encountered great difficulties because they had at their disposal only direct means of measurement. The outstanding accomplishment of the comparison of magnitudes, from the practical point of view, is, perhaps, the development of useful formulas for measuring *indirectly* lengths, areas, and volumes. Measurements which by direct methods could be made only with the greatest difficulty or not at all can be determined easily and



accurately by indirect methods. Thus, we may measure indirectly the width of a lake or river, the height of the tallest building, and even the distance and size of a remote planet.

However, before this method can be used, a few definitions are necessary and some facts must be proved.

QUADRILATERALS

A closed plane figure, formed by four straight line segments, bounding a single portion of the plane, is called a *quadrilateral.**

* Unless otherwise stated, by "quadrilateral," in fact by "polygon," we shall mean a convex polygon.

These given line segments are the *sides* of the quadrilateral. A *diagonal* of a quadrilateral is a straight line segment joining two non-consecutive vertices.

A quadrilateral is called a *trapezoid* if it has only two of its sides parallel. If the nonparallel sides of a trapezoid are equal, the trapezoid is said to be *isosceles*.

PARALLELOGRAMS

A *parallelogram* is a quadrilateral whose opposite sides are parallel. A parallelogram is called a *rectangle* if all its angles are right angles; a *rhombus* if all its sides are equal. A rectangle is called a *square* if all its sides are equal.

That one of two parallel sides upon which the parallelogram or trapezoid seems to stand is called its *lower base*. The opposite side is called the *upper base*.

The *altitude* or height of a parallelogram or a trapezoid is the length of the perpendicular between the lines of the bases.

PROPERTIES OF PARALLELOGRAMS

Assume that the distance between two points, A and B in Figure 45, on opposite sides of a lagoon is to be measured. This distance may be determined as follows: From B, sighting on A, extend the line segment AB. Lay off the line AD as shown in Figure 45. From any convenient point D on AD, set the line DC parallel to AB by making angle x equal to angle o, as shown on page 124, Chapter 9. Locate the line BC parallel to AD by making angles x and w equal. The figure formed, $ADCB$, is by definition a parallelogram. If we can show that DC is equal to AB, the distance AB can be ascertained by simply measuring DC.

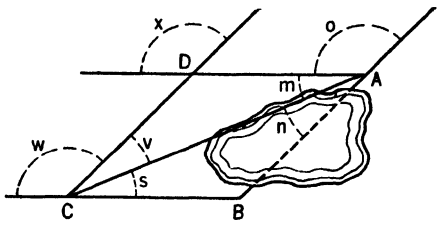
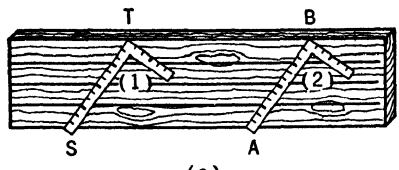


Fig. 45.

Draw the diagonal AC, dividing the parallelogram into two triangles ACD and ACB. Then, in the triangles ACD and ACB, since line segments AD and BC, cut by the transversal AC, were constructed parallel, by Fact 13 the alternate interior angles s and m are equal. Similarly, line segments AB and

DC, cut by the transversal AC, are also constructed parallel; hence, applying Fact 13 again, we know that angle v is equal to angle n. Since AC is common to both triangles, it follows that triangle ACD has two angles m and v and a side AC equal, respectively, to two angles n and s and a side AC of triangle ACB. Therefore, by Fact 23, the triangles are congruent. So that AD is equal to BC, AB is equal to DC, and angle D is equal to angle B, for corresponding parts of congruent triangles are equal. Moreover, angles s and m are equal, and angle v is equal to angle n; hence, angle s + angle v is equal to angle m + angle n, for if equals are added to equals, the sums are equal. That is, angle C is equal to angle A. Hence,



(a)
Fig. 46.

Fact 36: The opposite sides of a parallelogram are equal and the opposite angles are equal.

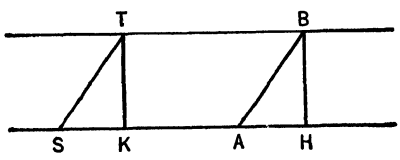
The following facts may now be inferred:

Fact 37: A diagonal divides a parallelogram into two congruent triangles.

Fact 38: Segments of parallel lines cut off by parallel lines are equal.

Fact 39: Two parallel lines are everywhere equidistant.

A carpenter wishing to saw a plank into four strips of equal width places the square as in (1), Figure 46 (*a*), and makes marks at 3″, 6″, and 9″. He then repeats the operation with the square similarly placed at some other convenient point such as (2). In order to justify his method, the carpenter must prove that the lines joining the corresponding marks are parallel. The carpenter in drawing these lines knows that the line segments TB and SA representing the edges of the planks are parallel, and that the line segment TS is equal to AB.

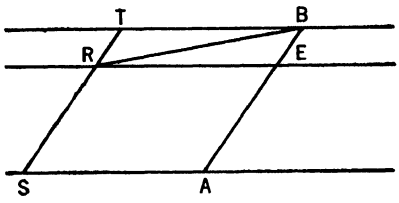


(b)
Fig. 46

Draw TK and BH perpendicular to SA, Figure 46 (*b*), forming the right triangles STK and ABH. In these triangles ST by construction is equal to AB. By Fact 39, two parallel lines

are everywhere equidistant, so that TK is equal to BH. That is, the right triangle STK has the hypotenuse ST and a leg TK equal, respectively, to the hypotenuse AB and a leg BH of triangle ABH. Therefore, by Fact 27, the triangles are congruent. Hence, angles S and A are equal, for they are corresponding angles of congruent triangles. That is, the line segments ST and AB cut by the transversal SH have the corresponding angles S and A equal. Therefore, by Fact 10, the line segments ST and AB are parallel.

Now consider Figure 46 (*c*), in which RE is one of the line segments drawn by the carpenter to cut the plank into four equal strips. In this case we know that TR is parallel and equal to BE, for they are part of the lines ST and AB which have just been proved parallel. We want to prove that RE is parallel to TB.



(c)
Fig. 46.

Draw the diagonal RB. In the triangles BRT and BRE, the lines TR and BE are equal by construction. The two parallel line segments TR and BE are cut by the transversal RB; hence, by Fact 13 the alternate interior angles BRT and RBE are equal; and the line segment RB is common to both triangles. That is, the triangle BRT has the two sides TR and RB and the included angle BRT equal, respectively, to the two sides BE and RB and the included angle RBE of triangle BRE. Therefore, by Fact 2, the triangles are congruent. Hence, angles TBR and BRE are equal, for they are corresponding angles of congruent triangles. That is, the line segments TB and RE cut by the transversal RB have the alternate interior angles TBR and BRE equal. So that, by Fact 8, the line segments TB and RE are parallel.

It is obvious that, in the same manner, any of the other line segments can be proved parallel to TB. Since, by Fact 17, lines parallel to the same line are parallel to each other, it follows that all lines drawn by the carpenter in this manner are parallel to each other.

Incidentally, in proving RE parallel to TB, having been given TR parallel and equal to BE [Figure 46, (c)], we have shown that the figure $TREB$ is a parallelogram. Hence

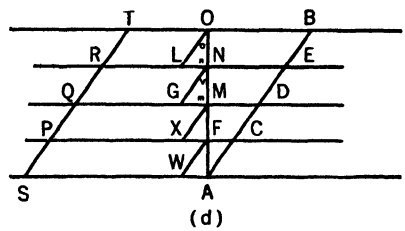
Fact 40: If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

PARALLELS ON TRANSVERSALS

Having proved that the lines drawn by the carpenter are parallel, it remains to be shown that the strips are of equal width.

Consider, then, Figure 46 (*d*), where *BT* and *SA* represent, as before, the edges of the plank, *ST* and *AB* the different positions of the carpenter's square, and *RE*, *QD*, and *PC* the line segments drawn by the carpenter to saw the plank.

From the previous discussion, we know that *BT*, *RE*, *QD*, *PC*, and *SA* are parallel line segments intercepting the equal parts *AC*, *CD*, *DE*, and *EB* on the transversal *AB*.



(d)

Fig. 46.

Draw the line segment *AO* perpendicular to *BT*. Then *AO* is also a transversal intercepting the parallel lines. To show that the strips are of equal width, we have to prove that the segments *AF*, *FM*, *MN*, and *NO* cut off by the parallel lines *BT*, *RE*, *QD*, *PC*, and *SA* on the transversal *AO* are equal.

Through *O*, *N*, *M*, and *F* draw *OL*, *NG*, *MX*, and *FW* parallel to *AB*. The parallel segments *OL* and *EB* are cut off by the parallel lines *BT* and *RE;* hence, by Fact 38, they are equal. Similarly, *NG* is equal to *DE*, *MX* is equal to *CD*, and *FW* is equal to *AC*. Since the line segments *AC*, *CD*, *DE*, and *EB* are equal, it follows that *OL*, *NG*, *MX*, and *FW* are equal, for things which are equal to equal things are equal to each other.

Now consider the triangles *OLN* and *NGM*. The parallel line segments *OL* and *NG*, just proved equal, are cut by the transversal *AO;* hence, by Fact 14, the corresponding angles *o* and *v* are equal. Similarly, the parallel lines *RE* and *QD* are cut by the transversal *AO;* hence, again by Fact 14, the corresponding angles *n* and *m* are equal. That is, the triangle *OLN* has two angles *n* and *o* and a side *OL* equal, respectively, to two angles *m* and *v* and the corresponding side *NG* of triangle *NGM*. Therefore, by Fact 23, the triangles are congruent.

Hence, the corresponding sides NO and MN, of the congruent triangles, are equal.

Similarly, the reader may prove FM equal to MN, and AF equal to FM. The strips are, therefore, of equal width.

In proving these strips to be of equal width we have shown that:

*Fact 41: If three or more parallels cut off equal segments on one transversal, they cut off equal segments on any transversal.**

It may now be deduced that:

Fact 42: A line parallel to one side of a triangle bisecting a second side, bisects the third side.

SEGMENTS IN PROPORTION

The *numerical measure* of a magnitude is the number of times the magnitude contains a unit of the same kind. Thus, the numerical measure, or simply the measure of the length, of a desk in inches is the number of times the length contains the unit of length 1 inch.

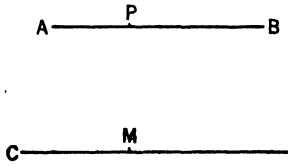


Fig. 47.

As in the case of numerical quantities, the ratio of two magnitudes is a measure of the relation of the numerical measures of the magnitudes expressed in terms of a common unit. Thus, if a desk is 4 feet 3 inches long by 2 feet 3 inches wide, the ratio of the length to the width, expressed in inches, is 51 : 27, or $\frac{51}{27}$, that is, $\frac{17}{9}$.

If two line segments are divided so that the corresponding segments form a proportion, the lines are said to be *divided proportionally*. For example, if the two line segments AB and CD in Figure 47 are divided so that AP is one-third of AB, and CM is also one-third of CD, then $AP : CM = AB : CD$. The line segments AB and CD are then said to be divided proportionally.

Now, suppose that the distance across a pond, AD in Figure 48, is to be measured. To determine this distance, stakes may be set up at A and D. The line segment AD is then produced by driving a stake at B in line with A and D. Lay off the line

* The reader will note that whether the line AO is drawn perpendicular to BT or not is immaterial to the proof of this fact.

segment AC skirting the shore of the pond. From D locate E on the line segment AC so that DE is parallel to BC.

The distances DB, EC, and AE can be easily measured. Hence, if we can show that $EC : AE = BD : AD$, that is, $\frac{EC}{AE} = \frac{BD}{AD}$, the distance AD is then easily determined.

Assume that a common unit of measure, such as BM, can be found that will be contained in EC 2 times and in AE 5 times. Then, since the ratio of two segments is the ratio of their numerical measure, $\frac{EC}{AE} = \frac{2(BM)}{5(BM)} = \frac{2}{5}$.

At the points of division of EC and AE, construct line segments parallel to BC. These parallel line segments cut off equal segments on the transversal AC; hence, by Fact 41, they cut off equal segments on the transversal AB. That is, the line segment AB will then be divided into 7 parts, of which BD contains 2 parts and AD contains 5 parts. The ratio of the two segments BD and AD is the ratio of their numerical measures; hence, $\frac{BD}{AD} = \frac{2(BM)}{5(BM)} = \frac{2}{5}$.

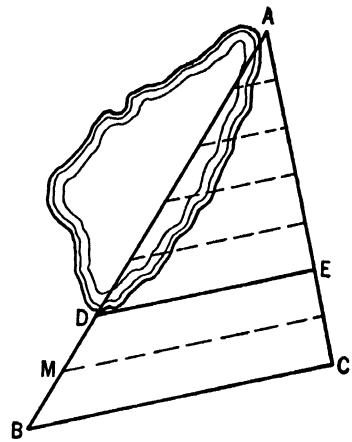


Fig. 48.

Since things equal to the same thing are equal to each other, $\frac{EC}{AE} = \frac{BD}{AD}$.

If no common unit of measure can be found such that the lengths of EC and AE can be exactly expressed by any whole number or fraction, that is, if the line segments are incommensurable, this fact can still be proved. However, the proof is beyond the scope of this book, and hence will be taken for granted.

Therefore, it may be concluded that:

Fact 43: A line parallel to one side of a triangle and intersecting the other two sides divides these two sides proportionally.

PROPORTION BY ADDITION

Two additional properties may be derived very easily.

It has just been shown that in Figure 48, $\frac{EC}{AE} = \frac{BD}{AD}$. Since adding 1 to each member of this proportion leaves the equality unchanged, we may write $\frac{EC}{AE} + 1 = \frac{BD}{AD} + 1$, or $\frac{EC + AE}{AE} = \frac{BD + AD}{AD}$. But $EC + AE = AC$, and $BD + AD = AB$; hence, $\frac{AC}{AE} = \frac{AB}{AD}$.

Moreover, by the principle of proportion (3), shown on page 150, Chapter 10, if $\frac{EC}{AE} = \frac{BD}{AD}$, then $\frac{AE}{EC} = \frac{AD}{BD}$.

By beginning with this proportion and following exactly the same procedure, it may be easily shown that $\frac{AC}{EC} = \frac{AB}{BD}$. So that,

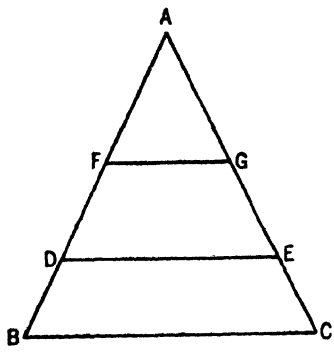


Fig. 49.

Fact 44: A line parallel to the base of a triangle cuts the other two sides so that one side is to a segment of that side as the other side is to the corresponding segment of the second side.

Assuming that FG and DE, Figure 49, are parallel to the base BC of triangle ABC, then by Fact 43 these lines divide the sides AB and AC proportionally. So that $\frac{DB}{AD} = \frac{EC}{AE}$. Or, applying the principle of proportion (2), shown on page 150, Chapter 10, $\frac{DB}{EC} = \frac{AD}{AE}$.

Similarly, considering the triangle ADE and the line FG parallel to its base, DE, by Fact 43, $\frac{FD}{AD} = \frac{GE}{AE}$. And by principle (2), page 150, Chapter 10, $\frac{FD}{GE} = \frac{AD}{AE}$.

Since things equal to equal things are equal to each other, $\frac{DB}{EC} = \frac{FD}{GE}$. That is,

Fact 45: Three or more parallel lines cut off proportional segments on any two transversals.

PROBLEMS OF CONSTRUCTION

The application of facts to the drawing of figures is, as was shown in Chapter 9, of fundamental importance.

Suppose that in determining the distance across the pond, on page 163 of this chapter, the lengths of the three line segments DB, EC, and AE, Figure 48, were, respectively, m, n, and p. The length of the line segment AD may then be ascertained without recourse to computation.

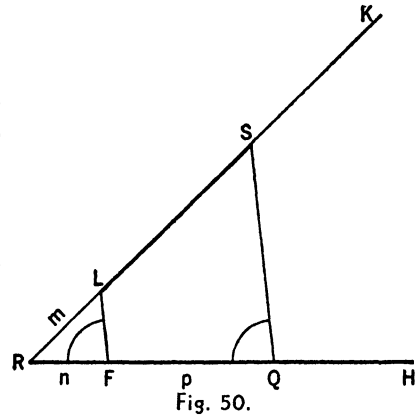


Fig. 50.

Since the lengths of the line segments AD, DB, EC, and AE are known to be in proportion, the problem is then to construct the fourth proportional to three given line segments.

Draw the lines RH and RK, Figure 50, making any convenient angle HRK. On RH, using compasses, mark off RF equal to n and FQ equal to p. Then on RK mark off RL equal to m. Draw LF. At Q, using the method shown on page 128, Chapter 9, draw a line segment parallel to LF meeting RK at S.

Then the line segment LF intersecting sides RS and RQ of triangle RSQ is parallel to the side SQ. Hence, by Fact 43, it divides these two sides, RS and RQ, proportionally. So that $\frac{RF}{FQ} = \frac{RL}{LS}$; and, substituting m, n, and p for their equals RL, RF, and FQ, we have $\frac{n}{p} = \frac{m}{LS}$.

Therefore, LS is the required fourth proportional, and by the principle shown on page 149, Chapter 10, it is equal to AD.

While any line segment may be divided approximately into

a given number of equal parts by measuring its length with a ruler, a more accurate method is often necessary. This greater accuracy can be attained as follows:

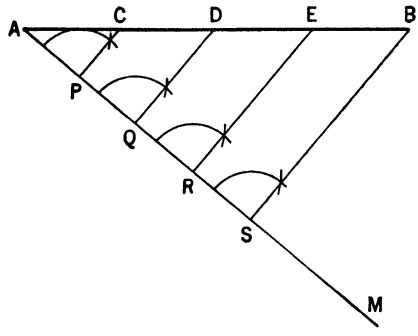


Fig. 51.

Assume that the line segment AB, Figure 51, is to be divided into a given number of equal parts.

From A draw the line AM making any convenient angle with AB. Take any convenient length, such as AP, and, with the compasses, apply it to AM as many times as the number of parts, say four, into which AB is to be divided. From S, the last point marked off in this manner, draw SB. From the other points of division P, Q, and R on AM, construct the line segments RE, QD, and PC parallel to SB.

Since the parallel line segments PC, QD, RE, and SB were so constructed that they cut off equal segments on the transversal AS, then by Fact 41 they cut off equal segments, AC, CD, DE, and EB, on the transversal AB. Therefore, the line segment AB is divided into four equal parts.

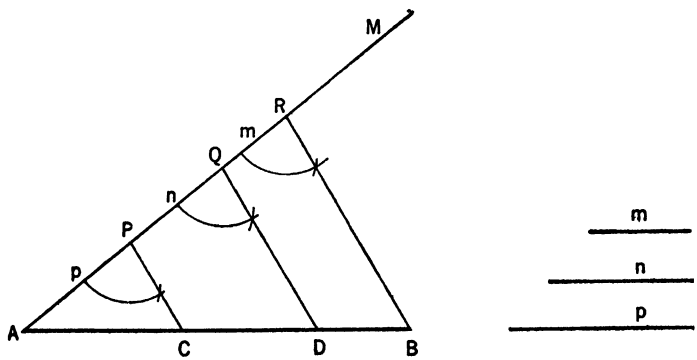


Fig. 52.

In connection with the division of line segments, there is still another useful method which should be used in cases where greater accuracy is desired.

Consider the line segments AB, m, n, and p, Figure 52, and

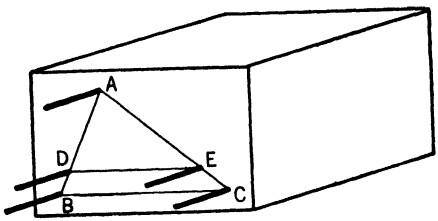
suppose the line segment AB is to be divided into parts proportional to m, n, and p.

Draw the line AM making any convenient angle with AB. On AM, with the compasses, mark off AP equal to p, PQ equal to n, and QR equal to m. Draw RB. From Q and P construct the line segments QD and PC parallel to RB. Then AC, CD, and DB are the required proportional parts.

The reader may take as an exercise the proof of the validity of this construction by showing that the corresponding parts are proportional.

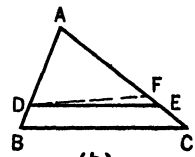
SEGMENTS ON TRANSVERSALS

A small steel safe is to be attached to a wall by five bolts as shown in Figure 53 (a). The mason, in checking the measurements to attach the safe to the wall, finds that AD, DB, AE, and EC are, respectively, 8, 2, 12, and 3 inches long. Since $8 : 2 = 12 : 3$, the lines are in the proportion $AD : DB = AE : EC$. He concludes that since the bolts D and E are in a line DE, which divides the two sides, AB and AC, of triangle ABC proportionally, then DE must be parallel to BC.



(a)
Fig. 53.

In order to justify his conclusion, consider the triangle ABC, Figure 53 (b), with DE intersecting AB and AC so that $DB : AD = EC : AE$. It is to be proved that DE is parallel to BC. This conclusion is demonstrated by the indirect method.



(b)
Fig. 53.

Suppose DE is not parallel to BC. Then from D draw a line DF parallel to BC. Since DF is parallel to the base BC, then by Fact 44 it cuts the other two sides AB and AC so that one side is to a segment of that side as the other side is to the corresponding segment of the second side. That is, $\frac{AB}{AD} = \frac{AC}{AF}$. But by construction, $\frac{DB}{AD} = \frac{EC}{AE}$. Hence, applying the princi-

ple of proportion (4), page 150, $\frac{DB + AD}{AD} = \frac{EC + AE}{AE}$. But $DB + AD$ is AB, and $EC + AE$ is AC. Hence, $\frac{AB}{AD} = \frac{AC}{AE}$. Since $\frac{AB}{AD} = \frac{AC}{AF}$, and things equal to equal things are equal to each other, it follows that $\frac{AC}{AE} = \frac{AC}{AF}$. So that, applying the principle of proportion, page 149, $AE = AF$.

Hence, the point F falls on E, and the line segments DF and DE must coincide, for in a plane two points determine a straight line. But DF was drawn parallel to BC. Therefore, DE, which coincides with DF, must be parallel to BC, for we assumed (page 97) that through a given point, not on a given line, only one line can be drawn parallel to a given line. That is,

Fact 46: If a line divides two sides of a triangle proportionally, it is parallel to the third side.

SIMILAR FIGURES

It is exceedingly important to distinguish sharply between the words *congruent* and *similar*. Congruent figures, as already explained on page 108, Chapter 7, are those figures which can be shifted in space without change of shape and size so that one can be exactly superimposed upon the other. Similar figures are those alike in shape but not necessarily in size. A map of a flat region, for example, is similar to the territory which it represents. A photograph can be enlarged, and the enlargement is similar to the original. Mathematically, however, we must be more precise in defining the similarity of two closed plane figures or polygons. Two corresponding polygons are said to be similar if:

(1) The corresponding angles are equal, and
(2) The corresponding sides are proportional.

That is, similarity means not only equality of angles, but also equality between the ratios of certain parts of one figure and the corresponding parts of the other. For example, the rectangle R and the parallelogram P, Figure 54, have the corresponding sides proportional but the corresponding angles are not equal. Hence, they are *not* similar. While, although

the rectangle Q and the square S have the corresponding angles equal, the corresponding sides are not proportional. So that

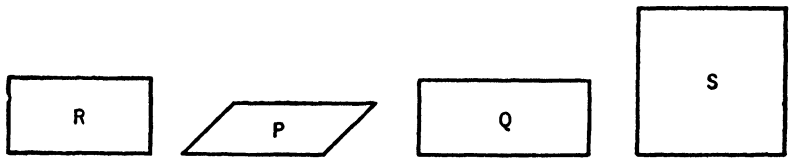


Fig. 54.

again the figures are not similar. The triangle is the only polygon such that if one of the two conditions is true, the other must be true also.

SIMILAR TRIANGLES

The indirect method of measurement was first used in an attempt to determine the comparative distances of two heavenly bodies by Aristarchus of Samos (310–230 B. C.), called "the mathematician." He based his calculations on an observation of the moon when half full. However, what appears to the naked eye to be a well-defined line is, owing to irregularities in the surface of the moon, a broken line. Consequently, he obtained very poor results.

Ptolemy of Alexandria, another famous Greek astronomer and mathematician who lived about the middle of the second century A.D., obtained a better approximation of the distance to the moon by a method virtually the same as that still in use. At each of two points A and B, Figure 55, on the same meridian* and as far apart as possible, the angles MAZ and MBZ' are measured. These angles are made by the imaginary lines CZ and CZ', passing from the center of the earth, C, upward through the observers at A and B, and the line segments, AM and BM, joining the center of the moon, M, to the positions of the observers.

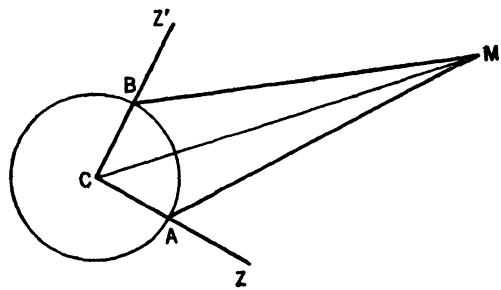
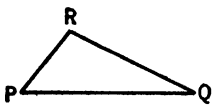


Fig. 55.

* A *meridian* is an imaginary circle on the earth passing through the poles and through the position of an observer on the earth.

Angles CAM and CBM, being supplementary angles to MAZ and MBZ', are then determined. Furthermore, angle BCM is also easily determined from the position of the observer at B; and CB, being the radius of the earth, is also known. That is, in the triangle CBM, we know two angles CBM and BCM and the side BC. Since by Fact 18 the sum of the angles of a triangle is a straight angle, then angle BMC, called the *parallax*, is also known.

Now, we can easily draw a triangle having exactly the same angles as those of the imaginary triangle CBM, and, by actually measuring its sides, determine the ratios of these sides to each other. Hence, if we can show that two mutually equiangular triangles are similar, then the ratios of the corresponding sides of the imaginary triangle and of the actual triangle drawn are proportional, so that the distance to the moon is then obtained by a simple proportion.



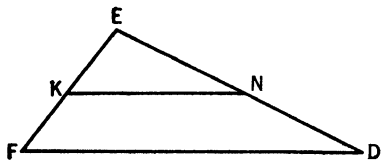


Fig. 56.

Consider, then, the triangles FDE and PQR, Figure 56, with angles F, D, and E equal, respectively, to angles P, Q, and R.

Place triangle PQR upon triangle FDE so that angle R coincides with its equal angle E, and PQ takes the position KN. By construction, angle EKN, equal to angle P, is equal to angle F. Then the two line segments KN and FD are cut by the transversal FE, so that the corresponding angles EKN and F are equal. Hence, by Fact 10, the line segments KN and FD are parallel. That is, the line segment KN parallel to the base FD of triangle FDE cuts the sides FE and DE. So that, by Fact 44, a side is to a segment of that side as the other side is to the corresponding segment of the second side. That is, $\frac{FE}{KE} = \frac{DE}{NE}$; or, since KE is equal to PR, and NE is equal to RQ, $\frac{FE}{PR} = \frac{DE}{RQ}$.

Similarly, by placing triangle PRQ upon triangle FDE so

that angle P coincides with its equal angle F, we can show that $\frac{FE}{PR} = \frac{FD}{PQ}$.

Hence, $\frac{FE}{PR} = \frac{DE}{RQ} = \frac{FD}{PQ}$. So that by definition the triangles are similar. That is:

Fact 47: If two triangles have the three angles of one equal, respectively, to the three angles of the other, the triangles are similar.

Returning to the problem of finding the distance to the moon, solved by Ptolemy, assume that angle CBM, Figure 55,

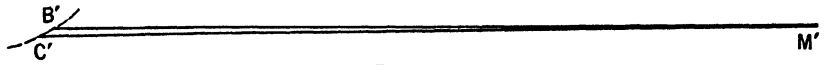


Fig. 57.

is 164°45′ and angle BCM is 15°. Then angle BMC is 180° − (164°45′ + 15°) = 15′. With a protractor, construct a triangle $C'B'M'$, Figure 57, so that angle $C'B'M'$ is 164°45′, angle $B'M'C'$ is 15′, and angle $B'C'M'$ is 15°. It will be found that $C'M'$ is approximately 60 times $C'B'$. That is, $C'M' : C'B' = 60 : 1$. Since triangle $C'B'M'$ is equiangular to the imaginary triangle CBM of Figure 55, then by Fact 47 they are similar. Therefore, $CM : CB = C'M' : C'B'$. That is, the distance from the center of the earth to the center of the moon : the earth's radius = 60 : 1. The earth's radius is approximately 3959 miles, so that $CM : 3959 = 60 : 1$; that is, $\frac{CM}{3959} = \frac{60}{1}$, or $CM = 60 \times 3959 = 237{,}540$ miles. The moon, then, is approximately 237,540 miles from the earth.

The sum of the angles in any triangle is, by Fact 18, equal to a straight angle. Hence, if two angles of one triangle are given equal to two angles of another triangle, the third angles are equal. That is, the triangles are mutually equiangular. Therefore, by Fact 47, the triangles are similar. So that:

Fact 48: If two triangles have two angles of one equal, respectively, to two angles of the other, the triangles are similar.

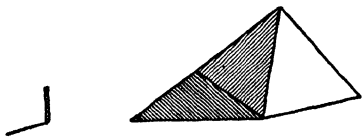
Moreover, it is easily seen that:

Fact 49: If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar.

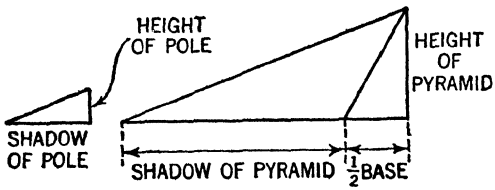
Although, probably, many men previously had used the in-

direct method of measurement, the first recorded person to use it was the Greek mathematician, Thales (624 to 547 B. C.). Whether Thales dumfounded the Egyptians by measuring the height of the pyramids or the Egyptians amazed Thales by teaching him the method is a matter for speculation. The Greeks, however, left a record of this event in writing, while the Egyptians did not.

Thales, besides being the first of the great Greek mathematicians, was also an accomplished astronomer. He was the first man to predict an eclipse. In fact, he was so interested in this branch of science that one night while star-gazing he fell into a well and was rallied by "a clever and pretty girl for being so eager to know what goes on in the heavens that he could not see what was straight in front of him, nay, at his very feet."*



According to Sir Thomas Heath's monumental work, *A History of Greek Mathematics*, the earliest and simplest version of Thales' method of measuring the heights of the pyramids is the following: Thales probably placed a vertical stick on the ground, waited until the length of the shadow cast by the stick and the length of the stick were equal, and at that moment measured the length of the shadow of the pyramid. At that moment an edge of the square base was at right angles with the sun's rays; the triangle formed by the height of the pyramid, the length of its shadow plus one-half its base, and a ray of sunlight passing from the



* Sir Thomas Heath, *A History of Greek Mathematics*, New York, Oxford University Press.

vertex or peak of the pyramid to the tip of the shadow is an isosceles right triangle. Therefore, Thales simply added the length of the shadow to half the length of the base of the pyramid.

There is another version, however, that gives Thales credit for using Fact 49. According to this version, Thales did not wait until the length of the shadow cast by the stick and the length of the stick were equal. Realizing (this is very doubtful) that he had two similar right triangles, he measured the stick, its shadow, half the base of the pyramid, and the length of its shadow. Then forming the proportion, height of pyramid : length of its shadow + half its base = height of stick : length of stick's shadow, he solved the problem.

On page 169 of this chapter, the statement was made that the triangle is the only polygon such that if one of the two conditions for similarity (see page 168) is true, the other must be true also. So far we have shown by Fact 47 that if the first condition is true, the second must be true. To prove that if the second condition is true, the first must be true also, we have to show that if two triangles exist so that their sides are in proportion, the triangles are similar and therefore equiangular.

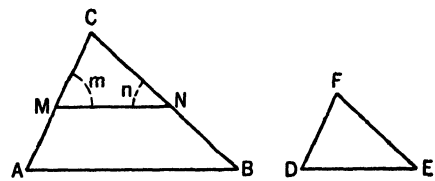


Fig. 58.

Assume that the triangles ABC and DEF, Figure 58, have their sides respectively proportional. That is,

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{CB}{FE}.$$

On AC mark off CM equal to DF, and on CB take CN equal to FE. Draw line segment MN. Then, by construction $\frac{AC}{CM} = \frac{CB}{CN}$. That is, the line segment MN divides two sides AC and CB of triangle ACB proportionally. Therefore, by Fact 46, line segment MN is parallel to AB. So that, by Fact 14 angles m and n are equal, respectively, to their corresponding angles A and B. That is, the triangle ABC has two angles A and B equal, respectively, to the two angles m and n of triangle MCN. Therefore, by Fact 48, the triangles are similar.

So that by definition $\frac{AC}{CM} = \frac{AB}{MN}$, that is, $\frac{AC}{DF} = \frac{AB}{MN}$. But by construction $\frac{AC}{DF} = \frac{AB}{DE}$; hence, $MN = DE$. Therefore, the triangles MCN and DEF are congruent (Fact 6). Since triangles ABC and MCN were proved similar, it follows that triangles ABC and DEF are similar and, of course, mutually equiangular. That is,

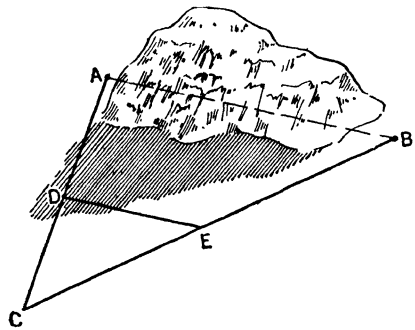


Fig. 59.

Fact 50: If two triangles have their sides respectively proportional, the triangles are similar.

A water line is to be carried through a mound from A to B, Figure 59. It is necessary to order in advance the exact length of pipe needed. The distance AB may be determined as follows:

Choose any convenient point C. On the line segments AC and BC mark off points D and E so that $\frac{AC}{DC} = \frac{BC}{CE}$. For example, if AC and BC were 60 and 80 feet, respectively, DC could be 15 feet and CE 20 feet. The length of AB can then be easily determined if the triangles ABC and CDE are shown to be similar.

By construction the triangles ABC and CDE have an angle equal, for angle C is common to both triangles, and the sides AC, BC, CD, and CE including this angle, C, proportional. That is, $\frac{AC}{DC} = \frac{BC}{CE}$. In other words, the line segment DE divides the two sides AC and BC of triangle ABC proportionally. Therefore, by Fact 46, DE is parallel to AB.

Since the parallel lines DE and AB are cut by the transversal AC, it follows, by Fact 14, that the corresponding angles BAC and CDE are equal. Similarly, the transversal BC cuts the parallel lines DE and AB, and, therefore, again by Fact 14, the corresponding angles ABC and DEC are equal.

That is, triangles ABC and CDE have their angles respectively equal. So that, by Fact 47, the triangles are similar. In general,

Fact 51: If two triangles have an angle of one equal to an angle of the other and the sides including these angles proportional, the triangles are similar.

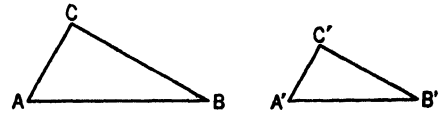


Fig. 60.

This conclusion applies to any two triangles meeting the stated conditions. For, suppose triangles ABC and $A'B'C'$, Figure 60, so constructed that angles C and C' are equal and $\frac{AC}{A'C'} = \frac{BC}{B'C'}$. It is easily seen that, by placing triangle $A'B'C'$ upon triangle ABC so that angle C' coincides with its equal angle C and $C'B'$ falls along CB, we have the case just discussed.

Assume that, as suggested on page 174, AC were 60 feet, BC 80 feet, DC 15 feet, and CE 20 feet. To find AB, simply measure DE. If DE were found to be 16 feet, then, since the triangles are similar, $\frac{60}{15} = \frac{AB}{16}$; that is, $\frac{60 \times 16}{15} = AB$, or $AB = 64$ feet.

Problems

1. A map is drawn to the scale of 1″ equals 100 miles. If Chicago is $4\frac{3}{4}$ inches from another city on the map, what is the actual distance between them?

2. A snapshot is $2\frac{1}{2}'' \times 3\frac{1}{2}''$. (*a*) If an enlargement is made so that the $2\frac{1}{2}''$ dimension becomes $7\frac{1}{2}''$, what is the other dimension?
(*b*) If a child is 2″ tall in the smaller picture, what will be its height in the enlargement?

3. (*a*) What is the third proportional* to 3 and 9?
(*b*) What is the mean proportional between 2 and 18?

4. The angle of elevation of the sun is 45°. At a point 200 feet from a radio tower, the sun is observed to be in line with the top of the tower. What is the height of the radio tower above the eye of the observer?

5. For every degree change in the vertex angle of an isosceles triangle, what change takes place in a base angle?

6. The sides of a triangle are 3, 5, and 7. Find the sides of a similar triangle if the side corresponding to 3 is 12.

* In the proportion $a : b = b : c$, b is called a *mean proportional* between a and c, and c is called a *third proportional* to a and b.

7. If $ab = 6fy$, write a proportion whose first term is $2y$.

8. A tower casts a shadow 70 feet long on level ground when a vertical pole 14 feet high casts a shadow of 10 feet. How high is the tower?

9. Divide the given line segment AB into segments which have the ratio of 2 : 3.

10. Prove that a line drawn perpendicular to either leg of a right triangle at its middle point passes through the middle point of the hypotenuse.

11. Mr. Chandon dies, leaving a triangular garden measuring 80′ by 170′ by 150′. His will provides that this land shall be divided unequally between two nephews, Frank and Henry, by a line 100 feet long parallel to the longest side. Frank has the smaller patch and Henry the larger. Each one buys fencing to enclose his land from the outside, but no fencing is set up on the line between the two nephews. How much fencing does each have to buy?

12. A deep, rushing stream separates two hostile forces. To use trench mortars from Q, a point on one side of the stream against a machine gun nest at P, a point on the other side of the stream, and not to waste ammunition, it is desired to know accurately the distance PQ. Show how this distance PQ may be obtained.

12

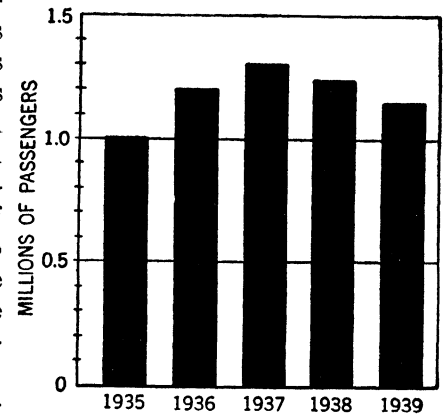
GRAPHS

Advertising men know that, to attract attention, a clever cartoon is far more effective than words. Similarly, a graph is more arresting and easier to understand than the numbers which it represents. Relationships between numbers often are more readily conveyed by graphs than by the numbers themselves. Thus, in business and in the sciences, graphs are constantly used to save time and energy in presenting significant information.

The most common types of graphs are *bar diagrams, circular* or *pie diagrams, pictograms,* and *line* or *curve diagrams.*

BAR CHARTS

If the quantities to be represented are such that each item is a total in itself, the most accurate visual impression is given by a bar chart. This and other types of graphs are most readily made on graph paper which is sold by stationers and which commonly contains a network of squares each subdivided into very small squares. The sides of the squares can be chosen to represent any desired unit of measurement. If graph paper is not available, it can be replaced by a sheet ruled with penciled lines at right angles to each other, each line being marked to represent some convenient scale. The choice of the scale will naturally be determined by the data which are to be depicted by the bar chart. The accompanying figure portrays a comparison of aerial passenger traffic

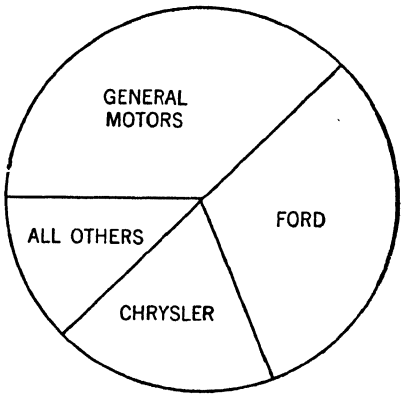


Passengers Carried for Hire by Civil Aviation in U. S. (*Data from The World Almanac.*)

for the years indicated. It is good practice to separate the bars by spaces about $\frac{3}{4}$ as wide as the bars themselves. The bars, of course, are of uniform width, since differences in the annual passenger totals are indicated only by the respective heights of the bars. All graphs should be properly labeled and should have adequate descriptive captions.

PIE DIAGRAM

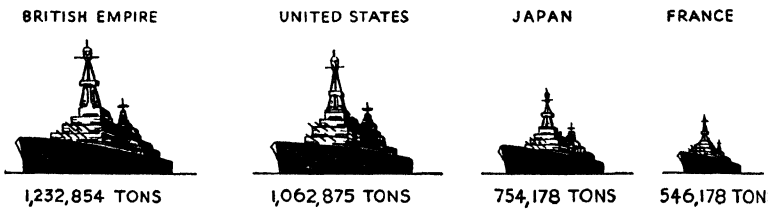
A circular graph or pie diagram is used to show the relation between the parts that go to make up a total on a percentage basis. To draw a pie diagram, find the per cent which each particular item is of the total. The whole circle represents 100 per cent. Since every circle contains 360°, the proper number of degrees to plot for each section is obtained by multiplying the number of per cent by 3.6. Using a protractor, lay off the number of degrees for each item, labeling each section. Thus, in the figure, General Motors' slice of the pie measures approximately 140°, thereby showing that it sold about 39 per cent of the cars marketed in the United States in 1935.



Division of the American Market Among Automobile Producers in 1935. (*Data from Young Management Corporation.*)

PICTOGRAMS

Pictures are sometimes used to represent the comparative sizes of the items. The basis of comparison between pictorial devices is usually that of length only. This method of pres-



Navies of the World, as of October 1936. (*Data from Office of Chief of Naval Operations, Washington.*)

entation is used where great accuracy is not important. It is open to one serious objection, namely, that the public may think of the surface or weight of the objects pictured, rather than only the *length*.

FUNCTIONS; VARIABLES

Comparisons of quantities form the essence of scientific study in any field. Factual data and their mutual relationships, indeed, are the fundamentals of knowledge. Without numerical information, progress would be terribly slow and uncertain. Moreover, the most significant kind of scientific information is that relating to *changes* in one quantity along with *changes* in another quantity, if there be some sort of cause-and-effect relationship between the two varying quantities.

For example, under normal conditions, the nearer one approaches a hot stove, the warmer becomes the air; that is, temperature changes with distance. Whenever two quantities are so related that the value of one is determined by the value of the other, the first quantity is said to be a *function* of the second. Thus, in the preceding example, other things being constant, the temperature is said to be a function of the distance.

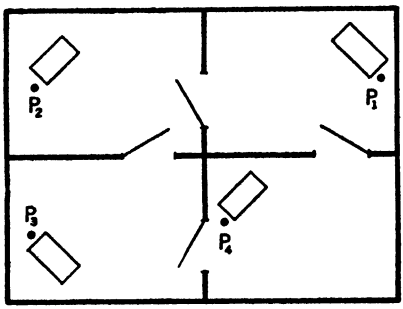
Any quantity that changes in value or magnitude is called a *variable*. The variable whose value depends on that of another is called the *dependent* variable. The variable whose value may be changed at will is the *independent* variable. Thus, one may change his distance from a hot stove at will, but the temperature takes on definite corresponding values with variations in distance; hence, temperature is a function of distance, since the former depends on the latter. Here both distance and temperature are variables; distance is the independent and temperature is the dependent variable.

From the very familiar and simple idea of referring the measurement of objects to a common line, Descartes (1596–1650), often described as the first modern mathematician, developed and applied this idea in mathematics, giving us one of the most powerful and effective tools in the study of relationships.

FRAMES OF REFERENCE

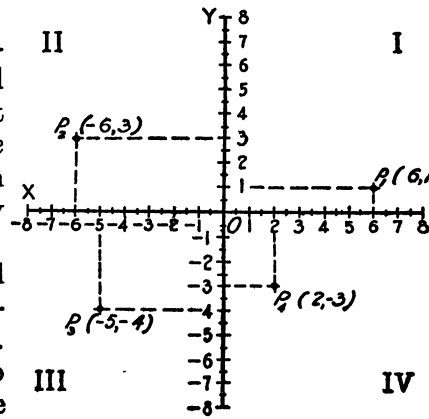
Assume, for example, that Riggs & Company decide to enlarge their office space by taking a suite of four rooms. In planning their new office, they give instructions to the contractor to place four electric plugs in the floor to accommodate con-

nections for the four desks indicated in the sketch. The four points decided upon are indicated by P_1, P_2, P_3, and P_4. To give clear instructions to the contractor, there must be agreement as to: (1) the unit of measurement (in the United States, feet and inches would be used; in France, meters and centimeters) and (2) the place from which the measurements are to be taken.



The walls of the rooms afford what is called technically a "frame of reference." In particular, the two inside walls dividing the suite into four rooms would give the best frame of reference, for all the plugs or points can be referred to them. A point in a plane can be fully determined if we know its distances from two straight lines in the same plane and (not necessarily, but preferably) at right angles to each other. Incidentally, the distance of a point from a line is measured always by the perpendicular drawn from the point to that line.

Thus, the points, indicating the electric plugs, would be measured from two straight lines, known as the *coordinate axes*, at right angles to each other, like the lines made by the two inside walls.



The vertical axis is called the *Y-axis* and the horizontal axis is called the *X-axis*. The point where the two axes intersect is called the *origin* and is usually designated by the letter *O*.

A convenient unit of length is selected and each of the coordinate axes is divided into segments of unit length, beginning at the origin. The number 0 is assigned to the origin; 1, 2, 3, 4, . . . and so forth, to the points of division in succession to the *right* of the origin on the *X*-axis, and -1, -2, -3, -4, . . .

and so forth, to the points of division to the *left* of the origin. In the same way, positive integers are assigned to the points of division *upward* from the origin on the Y-axis, and negative integers to the points of division *downward* from the origin. This selection of positive or negative integers was historically entirely arbitrary but is now the convention universally observed.

Any point in a plane is expressed in terms of its distances from the X- and Y-axes, respectively. These distances are always measured along a perpendicular line from the point to the coordinate axes. Thus, the point P_1 is defined by its distances from the X- and Y-axes, called the *coordinates* of the point, and is expressed in terms of its coordinates by the notation P_1 (6, 1) or simply (6, 1), *always writing the x-coordinate first.* That is, the x-coordinate, or *abscissa*, of the point P_1 is 6, and the y-coordinate, or *ordinate*, is 1. In other words, although a plane has an infinite number of points, if we say $x = 6$ and $y = 1$, we can immediately and infallibly locate the particular point which is here designated by the symbol P_1. Similarly, P_2, P_3, and P_4 are, respectively, $(-6, 3)$, $(-5, -4)$, and $(2, -3)$.

The coordinate axes divide a plane into four *quadrants*, numbered counterclockwise I, II, III, and IV. By convention, horizontal distances, or abscissas, to the right of the Y-axis are considered *positive*, while those to the left are *negative*. Vertical distances, or ordinates, upward from the X-axis are *positive* while those downward are *negative*. That is, the *abscissa* of a point will be positive in quadrants I and IV, and negative in quadrants II and III.

GRAPHS: APPLICATION TO BUSINESS

A practical business application of the graphical measurement conventions just discussed is shown by the following method of analyzing manufacturing costs and fluctuations in the volume of production, in prices, and in profits. Assume that Riggs and Company, manufacturers of a stamped metal article, know by experience that the following figures are true.

Observe how much clearer become the mutual relationships between these sets of variables when the values are plotted in graphical form.

Let the horizontal scale represent the number of articles manufactured, and the vertical scale be in dollars.

Number of Articles	Total Cost	Selling Price at \$0.15 each	Total Profit
20	\$ 5.00	\$ 3.00	−\$2.00
30	5.10	4.50	− 0.60
40	5.25	6.00	0.75
50	5.50	7.50	2.00
60	6.75	9.00	2.25
70	8.35	10.50	2.15
80	10.00	12.00	2.00
90	11.70	13.50	1.80
100	13.25	15.00	1.75

The points are then located or "plotted." Thus, the point A (Figure 61) in the sales graph corresponds to the values (20, 3); that is, 20 articles are sold for \$3.00. In a similar fashion the other points in the sales graph are plotted. In case all the points fall in a straight line, as in the sales graph, simply connect them by drawing a straight line through them.

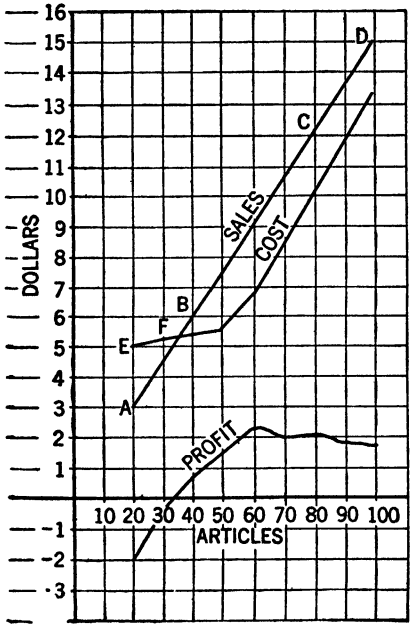


Fig. 61.

The cost graph is then drawn by: (1) Plotting the points so that E corresponds to (20, 5), F to (30, 5.10), and so forth, and (2) drawing a "smooth curve" through the points. The profit graph is drawn similarly.

Figure 61 readily reveals the following facts:

(*a*) Since the sales graph is a straight line, the total sales increase uniformly.

(*b*) Since the cost increases

with the number of articles manufactured but does not increase uniformly, the cost graph is a curve which rises as the number of articles increases.

(*c*) The profit line rises up to a certain point and then falls, so that the profit at first increases and then decreases.

From these facts, the following conclusions may be drawn:

(1) Graphs may be used to make *approximate* calculations rapidly. Thus, the selling price of 42 articles appears, from the graph, to be \$6.25. Actually, it is \$6.30.

(2) For 35 articles, the cost and the selling price are the same, for the two graphs intersect at that point. This fact is checked from the profit graph, since at 35 articles the graph crosses the horizontal axis, showing neither profit nor loss.

(3) The most profitable output for the machine is 62 articles, since this number gives the highest point in the profit graph or, in other words, the maximum profit.

FUNDAMENTAL PRINCIPLE

Generalizing from the discussion on previous pages, it is now evident that, in a given plane, and for a given choice of origin and scales:

(*a*) To every pair of real numbers there corresponds one and only one point on a plane; and

(*b*) To every point on a plane there corresponds one and only one pair of real numbers or coordinates.

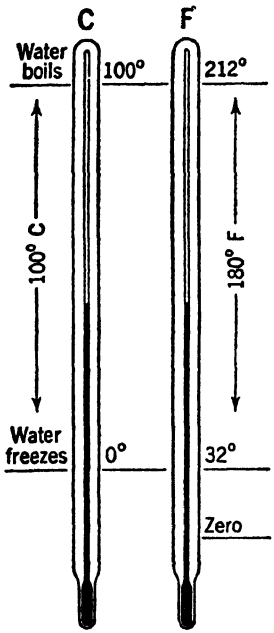
It follows that to have the position of a point given is equivalent to having the coordinates of the point given; conversely, having the coordinates of a point given is equivalent to having the position of the point given.

This simple statement links numbers with lines and curves; if observed facts can be stated numerically, they can be portrayed by graphs; if they can be portrayed by graphs, various relationships and, in particular, comparative rates of change, can be made visually apparent; if the variables be functions of each other, the presence of some natural law connecting them may be revealed; and, finally, such a law can be stated in mathematical form by means of an equation. For the present we are discussing only straight lines, reserving the various types of curves for treatment in subsequent chapters. It is necessary to discuss many mathematical aspects of the straight line

in considerable detail to give a firm foundation for the broader generalizations to follow.

THERMOMETERS

"Empirical" means something ascertained by direct experience or observation. The data of sales, costs, and profits of Riggs and Company on page 182 were empirical. They were just as objective as the fact that a stone falls a certain distance in a given time, or that light striking a burnished surface is reflected at a certain angle. An easy illustration of the method of preparing a chart from empirical data (where the graph will be a straight line) is furnished by the relationship between the Centigrade and Fahrenheit scales for measuring temperature.



In the attached picture, two thermometers are shown—a Centigrade on the left and a Fahrenheit on the right. The boiling point of water, at sea level and under ordinary atmospheric conditions, is 212 degrees Fahrenheit (written 212° F) and 100 degrees Centigrade (written 100° C). The freezing point of water, under the same circumstances, is 32° F and 0° C, respectively. Between these two points there are 180 degrees on the Fahrenheit scale and 100 degrees on the Centigrade scale. Hence, the former are $\frac{100}{180}$ or $\frac{5}{9}$ as large as the latter. But since zero on the Centigrade scale* corresponds to 32° on the Fahrenheit scale, 32 must be subtracted from a Fahrenheit reading to express degrees Fahrenheit

* *Conversion of Centigrade into Fahrenheit.* Since 100° C = 180° F, a degree Centigrade is equal to $\frac{180}{100}$ or $\frac{9}{5}$ of a degree Fahrenheit. But since 0° C = 32° F, to convert degrees Centigrade into degrees Fahrenheit, the former must be multiplied by $\frac{9}{5}$, and to the product 32° must be added. That is,

$$F = \tfrac{9}{5}C + 32.$$

For example, if it be desired to know the Fahrenheit equivalent of 30° C, the calculation would be as follows:

$$F = \tfrac{9}{5}(30) + 32 = \tfrac{270}{5} + 32 = 54 + 32 = 86° \quad (\textit{Answer}).$$

in degrees Centigrade, and one must multiply the remainder by $\frac{5}{9}$. That is,

$$C = \tfrac{5}{9}(F - 32).$$

For example, if 86°F is to be represented on the Centigrade scale:

$$C = \frac{5}{9}(86 - 32) = \frac{5}{9}(54) = \frac{5 \times 54}{9} = 30° \quad (\textit{Answer}).$$

Preparatory to making a graph, a small table of equivalent values might be made up as follows:

C	F
−35.5	−32
−17.8	0
0	32
30	86
100	212

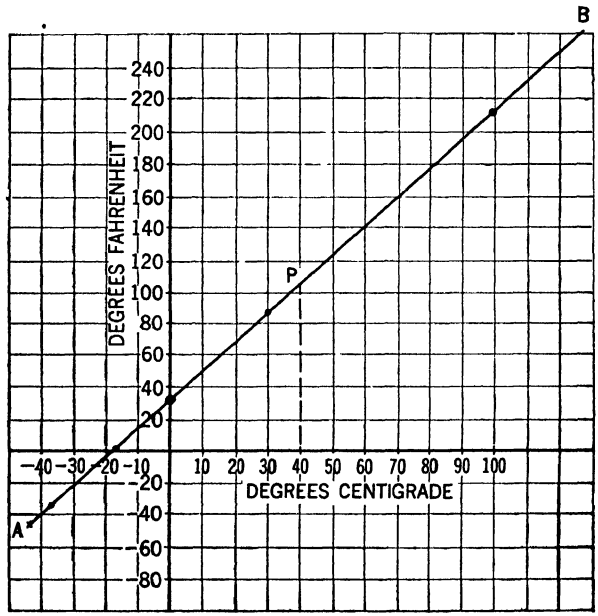
These values are then plotted, as shown by the points encircled on the accompanying chart, and the points are then connected by the straight line AB. By means of this graph, if it be carefully enough constructed, equivalences can be read off at a glance. For example, if a child be ill with a fever of 104° F, it will show a temperature on a Centigrade thermometer of 40° C. [Find the point P where a perpendicular to the vertical axis at the point 104° F intersects line AB, and drop a perpendicular to the horizontal axis.]

Note that in statistical work, from a few empirical data which *seem* to lie along a straight line, it should not be inferred that other data, much smaller or larger in value, will also lie along the straight line. Some types of curves along part of their path are nearly rectilinear, but in other parts are bent sharply.

APPLICATION TO PROBLEMS

Graphs not only can present visually the relationship between two or more variables, but they can also be used for the

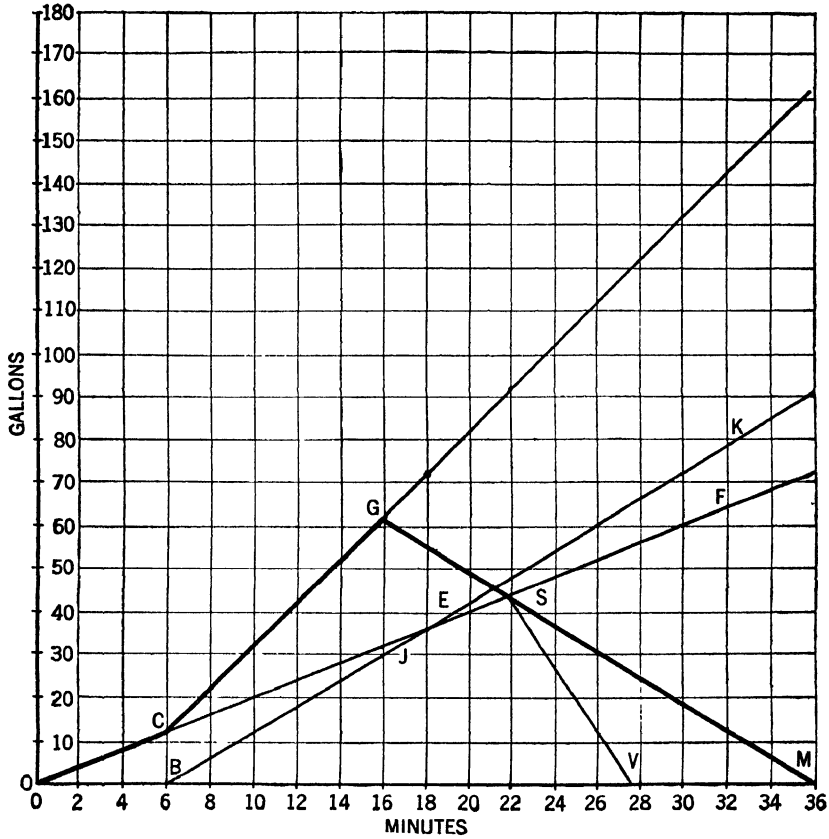
very practical purpose of ascertaining approximately some intermediate value not included in the data upon which the graph was based. Thus, as just seen, the child with a fever of 104° F would be described as having a fever of 40° C. Travellers in tropical countries, using the local Centigrade thermometers, might, indeed, be in trouble in taking their own temperature if there were any doubt about the equivalences already described. But there are other situations in business or indus-



try where problems can be solved by graphs instead of by computation. This truth will be emphasized in subsequent chapters.

For example, consider an empty tank which is being filled by a faucet supplying 2 gallons of water per minute. After this faucet has been running 6 minutes, a second faucet is turned on which independently supplies 3 gallons of water per minute. When the two faucets have been running together for 10 minutes, an outlet is opened, but both faucets continue to run. If the tank is empty at the end of 36 minutes from the start, draw a graph representing the amount of water in the tank at any instant.

Draw a scale in which the horizontal line represents minutes and in which the vertical line represents gallons of water in the tank. The inflow of water per minute from each faucet is constant, and the outflow per minute when the outlet is opened is also constant.



When the time, t, is 0, the amount of water, w, in the tank is also 0. The increase of water in the tank due to the first faucet is represented by the line OF. At the point C on this line, at 6 minutes on the horizontal scale, the amount on the vertical scale is 12 (that is, 2×6). The increase of water in the tank due to the second faucet is represented by the line BK. At the point J on this line, at 16 minutes (that is, $6 + 10$)

on the horizontal scale, the amount on the vertical scale is 30 (that is, 3×10).

The two lines cross at the point E, which represents 18 minutes from the start and a total of 36 gallons for each faucet or a combined total of 72 gallons. It can be ascertained by inspecting the graph or by solving the following equation:

Let t = time,
w_1 = water from 1st faucet,
w_2 = water from 2nd faucet.

$$2t = 3(t - 6)$$
$$2t = 3t - 18$$
$$-t = -18$$
$$t = 18$$

$$w_1 = 2t = 2 \times 18 = 36,$$
$$w_2 = 3(t - 6) = 3(18 - 6) = 36.$$

In other words, the time taken by each of the two faucets to pour the *same* amount of water into the tank is 18 minutes for the first and 12 minutes for the second.

In the problem as given, from the instant $t = 6$, the two faucets are running simultaneously. The sum of the two functions is, thus, represented by the line segment CG, where $C = (6, 12)$ and $G = (16, 30 + 32) = (16, 62)$. That is, G shows that at 16 minutes from the start, the tank contains 62 gallons.

When $t = 16$, a new factor enters, namely, an outlet is opened, causing the amount of water in the tank to go down to zero when $t = 36$. This is shown by the line GM. Thus, the amount of water in the tank at any time during the entire period of 36 minutes is shown by the broken line $OCGM$. That is, any point on GM indicates the amount of water in the tank at a particular instant during the last 20 minutes of the 36-minute period of the problem. Thus, 4 minutes after the outlet is opened (or 20 minutes from the start), the graph shows that the tank contains approximately 50 gallons.

Consider next a new problem: Assume that the two faucets were turned off when $t = 22$, the outlet having been opened, as before, 16 minutes from the start, and remaining open. When would the tank be empty? Draw a line showing the amount of water in the tank from the twenty-second minute onward. At the end of the 22nd minute, the amount of water in the tank is indicated by the point S on the line GM, or approximately 43 gallons.

The downward direction of GM indicates that the capacity of the outlet is greater than the combined inflow indicated by

the line CG plus the accumulation of 62 gallons in the tank, indicated by the point G, at the time the outlet was opened.

The outflow per minute, while the two faucets were open, was $5 - (-3.1)$, or 8.1 gallons. At the end of the 22nd minute, there were approximately 43 gallons in the tank, and no further addition of water occurred thereafter. The time for the outlet to empty the tank is, therefore, $43 \div 8.1$, or approximately 5.3 minutes. The situation is shown graphically by the line SV, which naturally is steeper than that of GM.* The next chapter will discuss how to measure steepness and will look further into the important subject of rate of change.

Problems

1. Of a family's income, 20% is spent for food; 20% for rent, light, and insurance; 15% for clothing; 10% for maintenance; 10% for recreation; 15% for miscellaneous expenses. The remainder is saved. In making a circle graph ("pie chart") to show the distribution of income, how many degrees should there be in the arc of the sector representing the family savings?

2. Using a scale of one-quarter of an inch to the foot, a draftsman should represent a rectangular room 17′ 6″ by 12′ 9″ by a rectangle of what proportions?

3. Draw a graph showing the relationship between pounds and kilograms. (1 kg. = 2.2 lbs., approximately; see page 488.) Let the vertical scale be 0–24 pounds and the horizontal scale 0–10 kilograms. (*a*) From the scale, find the approximate equivalence of 7.5 kilograms. (*b*) The equivalence of 20 pounds.

4. Draw a graph showing the relationship between acres and hectares. (1 hectare = 2.471 acres; see p. 255.) Let the vertical scale be 0–1,000 acres in intervals of 100, and the horizontal scale 0–500 hectares.

(*a*) From the scale find the approximate equivalence of 260 hectares.

(*b*) The approximate equivalence of 900 acres.

5. Draw a graph showing the lengths of the circumferences of circles in terms of their radii. (Circumference = $2\pi \times$ radius; see

* In the first problem, if the line GM were prolonged to meet the Y-axis, it would meet it at the point corresponding to 112 gallons, as can be easily verified by laying a straightedge along GM. This figure, 112 gallons, has no *meaning or significance.* In general, mathematical symbolism should not be used incautiously in situations where the symbolism has been applied to real or concrete facts, since absurd or fantastic deductions might thus seemingly be justified.

p. 362.) Let the vertical scale be 0–40 inches, and the horizontal scale 0–250 inches.

(*a*) From the scale, find the approximate length of the circumference when the radius is 10 inches.

(*b*) The approximate length of the radius when the circumference is 200 centimeters.

6. Duplicate the vertical scale used in the preceding problem but multiply the horizontal scale by 20 and let it represent the area in terms of square units. (Area = πr^2; see page 362.) Use the following values to plot points, and draw a smooth curve through them.

Radius	*Area*
10	314.6
15	706.9
20	1,256.6
25	1,963.5
30	2,827.4
35	3,848.5
40	5,026.5

(*a*) From the scale, find the approximate area of a circle when the radius is 28 inches.

(*b*) The approximate length of the radius when the area is 4,500 square centimeters.

7. Plot the equation $x + 2y = 4$. Also, on the same grid, $3x - y = 6$. (*Hint:* In each case, to find where the line crosses the coordinate axes, put $x = 0$ and find y, and then $y = 0$ and find x.) At what point do the two lines cross?

8. Represent graphically the following data of the population of the United States. (*Hint:* Plot time on the horizontal axis.) Is the growth of population constant or otherwise?

Year	*Population*	*Year*	*Population*
1790	3,929,000	1870	38,655,000
1800	5,308,000	1880	50,262,000
1810	7,230,000	1890	63,056,000
1820	9,664,000	1900	76,129,000
1830	12,806,000	1910	92,267,000
1840	17,069,000	1920	106,543,000
1850	23,261,000	1930	123,091,000
1860	31,503,000	1940*	132,000,000

*Estimated.

13 SLOPES, RATES OF CHANGE, AND TANGENTS

The fundamental concepts necessary to the study of relationships existing between quantities have just been discussed. The usefulness of coordinates in mapping data gathered by experiment has also been seen. We shall now see how coordinates are useful not only in this respect, but also in much more important fields: the study of the motion, shape, and size of objects.

LOCUS OF A POINT

Suppose that an airplane starts from Newark, 9 miles from New York, on its way to Washington and travels toward the latter city at the rate of 112 miles per hour. In x hours, the airplane travels $112x$ miles. If the distance from New York is denoted by y, then $y = 112x + 9$ is the equation that gives the relationship between distance and time for this airplane.

As already suggested, the graph of an equation will help in the study of the relationship between the quantities involved. The equation $y = 112x + 9$ defines distance as a function of time. Obviously, before the plane starts from Newark, it is 9 miles away from New York City. Hence, if the horizontal axis denotes time and the vertical axis denotes distance, then the point P, or (0, 9), in the graph would mark the point of departure.

The point P has been defined by the pair of values (0, 9). If these values are allowed to vary, the point P will assume different positions; in other words, as it moves on the plane, it will trace a definite line or curve. The line or curve obtained in this manner is called the *locus* of the point. In general, the path of a point which moves according to certain conditions is called the locus of the point or of the *equation of its path.*

Choosing values arbitrarily for the time x, and substituting

these in the airplane equation, we obtain values of the distance y. These pairs of values are shown in the following table:

x	y
0	9
$\frac{1}{2}$	65
1	121
$1\frac{1}{2}$	177
2	233

When $x = 0$, $y = 112(0) + 9 = 9$.

When $x = \frac{1}{2}$, $y = 112(\frac{1}{2}) + 9 = 65$.

When $x = 1$, $y = 112(1) + 9 = 121$.

When $x = 1\frac{1}{2}$, $y = 112(\frac{3}{2}) + 9 = 177$.

When $x = 2$, $y = 112(2) + 9 = 233$.

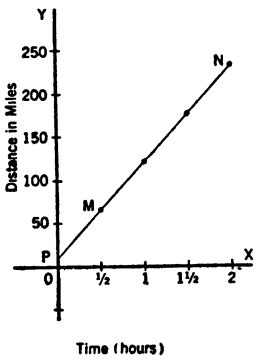


Fig. 62.

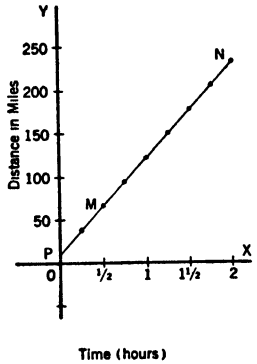


Fig. 63.

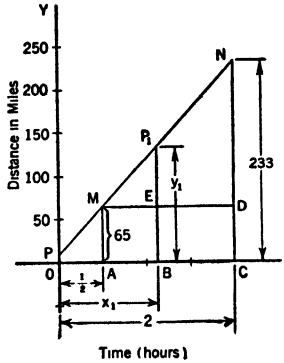


Fig. 64.

These points, when plotted (see Figure 62), appear to lie in a straight line. If *intermediate* values of the time, x, are chosen, the following values of the distance, y, are obtained:

x	y
$\frac{1}{4}$	37
$\frac{3}{4}$	93
$1\frac{1}{4}$	149
$1\frac{3}{4}$	205

When $x = \frac{1}{4}$, $y = 112(\frac{1}{4}) + 9 = 37$.

When $x = \frac{3}{4}$, $y = 112(\frac{3}{4}) + 9 = 93$.

When $x = 1\frac{1}{4}$, $y = 112(\frac{5}{4}) + 9 = 149$.

When $x = 1\frac{3}{4}$, $y = 112(\frac{7}{4}) + 9 = 205$.

Plotting these additional points (see Figure 63), we find that they appear to lie in the same straight line with the others.

A pair of values such as (2, 233) is said to *satisfy* the equation $y = 112x + 9$ if, when 2 is substituted for x, and 233 is substituted for y, the equation is true. Thus, $233 = 112(2) + 9$.

The fact that all the points appear to lie in the same straight line suggests that the points obtained by giving x *any* values and finding the corresponding values of y will lie in the same straight line. To show that this is the case, let x_1 (read "x sub-one") be *any particular value* of the time, and y_1 (read "y sub-one") *its corresponding* distance; that is, let the pair of values (x_1, y_1) satisfy the equation, so that:

$$y_1 = 112x_1 + 9.*$$

Then, (x_1, y_1) define a point P_1, which can be shown to lie in the same straight line.

Draw the right triangles MEP_1 and MND by drawing the line segments MA, P_1B, and NC perpendicular to the X-axis, and the line segment MD parallel to the X-axis. (See Figure 64.) The line segments MA, P_1B, and NC are parallel. (Fact 11.)

If P_1 is on the line PN, then the line segment P_1E parallel to the base ND of triangle MND cuts off a triangle MP_1E similar to the triangle MND. (Facts 44 and 50.)

Since triangle MP_1E is similar to triangle MND,

$$\frac{P_1E}{ND} = \frac{ME}{MD}.$$

Now, the parallel line segments MA and EB, cut off by parallel line segments, ME and AB, are equal. (Fact 38.)

That is,

$$P_1E = P_1B - EB = P_1B - MA = y_1 - 65.$$

Similarly,

$$ND = NC - DC = NC - MA = 233 - 65 = 168,$$
$$ME = AB = OB - OA = x_1 - \tfrac{1}{2},$$

and

$$MD = AC = OC - OA = 2 - \tfrac{1}{2} = 1\tfrac{1}{2}.$$

* Note that x_1 does not mean $x = 1$, or "one x," but, as stated, merely indicates a particular value of x, and is read "x sub-one."

So that, expressed in terms of the coordinates of the points M, P_1, and N, the proportion becomes:

$$\frac{y_1 - 65}{168} = \frac{x_1 - \frac{1}{2}}{\frac{3}{2}}.$$

Multiplying both sides by 168 gives:

$$y_1 - 65 = 112x_1 - 56;$$

whence,

$$y_1 = 112x_1 + 9.$$

Since it was assumed that (x_1, y_1) satisfied the equation, it has been shown that:

Every pair of values that satisfies the equation defines a point on the straight line. Moreover,

Every point on the straight line must satisfy the equation because, if the point is on the straight line, then the triangles MEP_1 and MND in Figure 64 are similar, and hence:

$$\frac{y_1 - 65}{168} = \frac{x_1 - \frac{1}{2}}{\frac{3}{2}},$$

or

$$y_1 = 112x_1 + 9.$$

AVERAGE RATE OF CHANGE

From Figure 65 and the preceding discussion, it is easy to see that if a straight line is given, then *all* right triangles formed by drawing lines parallel to the axes, using part of the given line for their longest side or *hypotenuse*, are similar; for the point P_1 or (x_1, y_1) was assumed to be *any* point in the line, and hence its actual position in the line is immaterial. Thus, no matter what two points in the line are chosen to form the right triangle, the ratio of the vertical side to the horizontal side is always constant. In Figure 65, the reader may prove, by applying Fact 14 and Fact 49, that:

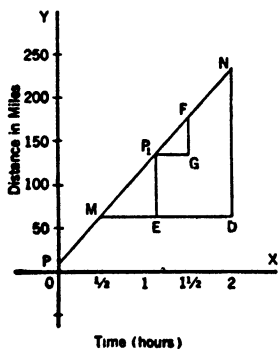


Fig. 65.

$$\frac{ND}{MD} = \frac{FG}{P_1G} = \frac{P_1E}{ME}.$$

From the preceding discussion, it is easy to see that whether a point moving on a plane traces a certain straight line or curve depends on how the point changes its direction and on what this direction is. The changes of direction and the direction of the point are defined by the relationship of dependence between the quantities which set the path of the point in the plane. Thus, the relationship of dependence of distance to time given by the equation $y = 112x + 9$ sets the direction and the changes of direction of the point tracing the straight line which is the locus of this equation. Hence, in studying a function, it is very convenient to know how fast it is increasing or decreasing; that is, the *rate* at which it is changing.

The *average rate of change* of a function is found by dividing the change in the function by the proportionate amount of change in the independent variable. Thus, $y = 112x + 9$ defines the distance y as a function of the time x. From Figure 65, or from the tables on page 192, it is seen that a half-hour after the plane had left Newark, it was 65 miles away from New York City (point M in the graph, Figure 65); and 2 hours after it had left Newark, it was 233 miles away from New York City (point N in the graph). The change in the function (in this case, the distance) is the difference between 233 miles and 65 miles; that is, the plane traveled $233 - 65 = 168$ miles during the interval under consideration. The proportionate amount of change in the other quantity is the change in the time corresponding to the period during which the plane traveled the 168 miles: in this case, the difference between 2 hours and $\frac{1}{2}$ hour; that is, it took the plane $2 - \frac{1}{2} = 1\frac{1}{2}$ hours to travel the 168 miles. Hence, the average rate of change of the function is $168 \div 1\frac{1}{2} = 112$ miles per hour. Thus, the distance changed by 112 miles every hour during this interval.

In general, then, the average rate is the change in the function per unit-change in the independent variable.

SLOPE

Graphically, the operation just performed is executed simply by finding the length of ND (or, $233 - 65 = 168$), and dividing it by the length of MD (or, $2 - \frac{1}{2} = 1\frac{1}{2}$). That is, the average rate is:

$$\frac{ND}{MD} = \frac{168}{\frac{3}{2}} = 112.$$

But it has been shown (page 194) that this ratio of the vertical side of the right triangle, formed by using part of the given line as the hypotenuse, to the horizontal side is always constant. Therefore, *the ordinates of any point moving in a straight line have a constant rate of change with respect to the abscissas.*

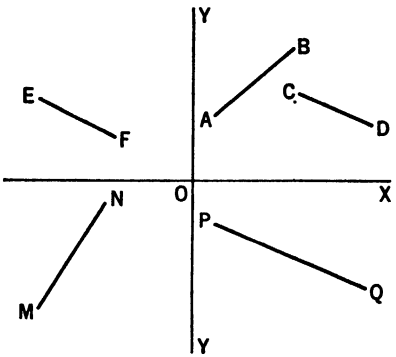


Fig. 66.

The rate of change of the ordinates of a point moving in a straight line, with respect to the abscissas, is called the *slope.* So that, *the slope of a straight line is the rise or fall of the line per horizontal unit.*

To avoid any confusion which might arise as to whether a line is going up or down, it is accepted that *the direction of the line is always considered from left to right.*

Thus, in Figure 66, the lines AB and MN are going up, while the lines CD, EF, and PQ are going down. If the line goes *up* from *left to right*, as MN and AB, the slope is said to be *positive.* If the line goes *down* from *left to right*, as CD, EF, and PQ, the slope is said to be *negative.*

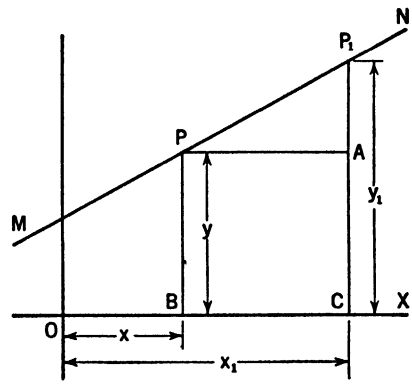


Fig. 67.

Now assume that a straight line, such as MN in Figure 67, is given. Take any point P, or (x, y), on the line, and any other point P_1 or (x_1, y_1), also on the line. Draw the horizontal line segment PA and the vertical line segments PB and P_1C. Then, by definition, the slope of the line, usually denoted by m, is:

$$\frac{P_1A}{PA}.$$

But

$$P_1A = P_1C - AC = P_1C - PB = y_1 - y$$

and

$$PA = BC = OC - OB = x_1 - x;$$

hence,

$$m = \frac{y_1 - y}{x_1 - x}.$$

That is, *the slope of a line is equal to the change in y divided by the corresponding change in x.*

This ratio of the change in the function y to the corresponding change in x, which expresses the already familiar proportionality of corresponding sides of similar triangles, is, as stated before, of the greatest importance.

To avoid writing this somewhat clumsy fraction, the Greek letter Δ (delta) is introduced. This letter Δ, translated into mathematical language, means "the change in." Thus, Δy is read "the change in y," and Δx is read "the change in x." Note that the symbol Δy does *not* mean some quantity Δ times the quantity y, but simply, as just stated, the change in y.

Using this new notation, the ratio giving the slope of a line in terms of the coordinates of two of its points is written:

$$m = \frac{\Delta y}{\Delta x}$$

and read aloud, "The slope is equal to the ratio of the change in y to the corresponding change in x."

For example, to find the slope of the straight line connecting points $(-2, 4)$ and $(3, 2)$, choose either of the two points as P and the other as P_1. Thus, let P be the point $(-2, 4)$ and P_1 the point $(3, 2)$. Then, $\Delta y = y_1 - y = 2 - 4 = -2$, and $\Delta x = x_1 - x = 3 - (-2) = 5$. Hence,* $m = \frac{\Delta y}{\Delta x} = -\frac{2}{5}$.

EQUATION OF THE STRAIGHT LINE

In the particular problem considered at the beginning of this chapter, the equation $y = 112x + 9$ was shown to be repre-

* Note that the same result is obtained if $(3, 2)$ is taken as P and $(-2, 4)$ as P_1.

sented graphically by a straight line. It will be very useful to find out if all straight lines in a plane are represented by equations of the first degree in x and y, such as the one just mentioned, and if all equations of the first degree in two variables have a straight-line graph.

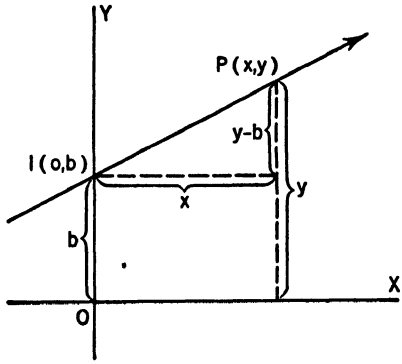


Fig. 68.

Consider, then, any point I in a plane (Figure 68). Through point I draw a vertical line to represent the Y-axis. At any distance, b, from point I draw a horizontal line to indicate the X-axis. Point I is on the Y-axis; therefore, its distance from it, that is, its abscissa, is zero. The ordinate of point I is b, for the X-axis was drawn b units from the point. Hence, the coordinates of I are $(0, b)$.

If point I is moved in a constant direction, it will generate a straight line. To find an expression for the slope of this line, any other point, P, on the line may be chosen; for it has been shown, on page 194, that a straight line has a constant slope. Denote the coordinates of this other point P by (x, y). Then, applying the formula developed on page 197 for the slope of a line,

$$m = \frac{y - b}{x - 0}, \text{ that is, } m = \frac{y - b}{x}.$$

Multiply both sides of this equation by x, and obtain:

$$mx = y - b.$$

Then, transposing b, we get:

$$mx + b = y, \text{ or } y = mx + b.$$

The equation $y = mx + b$ represents any straight line which intersects the Y-axis, where the coefficient of x, that is, m, is the slope of the line, and the constant term b is the distance from the X-axis at which the line crosses the Y-axis. A line parallel to the Y-axis has, by definition, all its points equidistant from this axis. That is, the abscissas of all points on

such a line are equal. Hence, the equation of any line parallel to the Y-axis has the form $x = a$, where a is constant.

A horizontal line does not rise or fall; that is, its slope is zero. Therefore, the equation of a line parallel to the X-axis is $y = 0x + b$, or simply $y = b$, where b is the Y-intercept. Thus, the equation of a horizontal line 3 units below the X-axis is $y = -3$.

The two forms $y = mx + b$ and $x = a$ will, then, represent all straight lines in the plane. Since these equations are of the first degree in x and y, we conclude that *a straight line is represented by an equation of the first degree in two variables.*

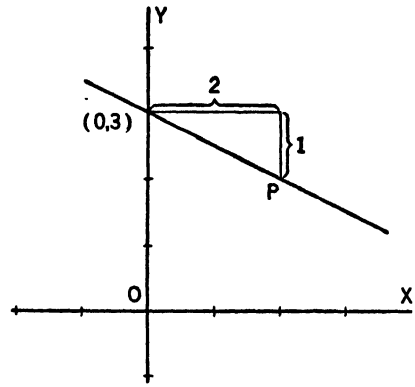


Fig. 69.

Thus, the equation of a straight line whose slope is 2 and which crosses the Y-axis 5 units above the X-axis is $y = 2x + 5$. The equation of a line parallel to the Y-axis and 4 units to the left of it is $x = -4$.

Moreover, the equation $y = -\frac{1}{2}x + 3$ is evidently of the form $y = mx + b$, where the coefficient of x, that is, m, is $-\frac{1}{2}$, and the constant term, or Y-intercept, b, is 3. Hence, the equation $y = -\frac{1}{2}x + 3$ is represented graphically by a straight line crossing the Y-axis 3 units above the X-axis and having a slope of $-\frac{1}{2}$. To plot the graph of this equation, mark off the Y-intercept (0, 3), as in Figure 69. Since the slope is negative and equal to $\frac{1}{2}$, the line goes down $\frac{1}{2}$ unit for every unit along the horizontal, or one unit for every two along the horizontal. Therefore, from (0, 3) count two units to the right and one unit down, and plot point P, as in Figure 69. The straight line representing the equation must then pass through the points (0, 3) and P.

GENERAL EQUATION OF THE FIRST DEGREE

An equation of the first degree in two variables, say x and y, can contain at most a term in x, a term in y, and a constant term. The expression $Ax + By + C = 0$, where A, B, and C

are any real numbers, and A and B are not both zero, contains all these terms and no more. Therefore, it is said to be the *general equation of the first degree in* x *and* y.

The expression:

$$y = mx + b \tag{1}$$

represents any straight line which intersects the Y-axis. It is easy to show that the general equation of the first degree, that is, $Ax + By + C = 0$, where A and B are not both zero, can always be expressed in form (1) or in the form $x = a$. Transpose the term in x and the constant term, obtaining $By = -Ax - C$. If B is not equal to zero, dividing both sides of the equation by B gives:

$$y = -\frac{A}{B}x - \frac{C}{B}, \tag{2}$$

which is evidently of the form $y = mx + b$. But if B is equal to zero, the reasoning fails, for we cannot divide by zero (Chapter 1, page 10). However, if $B = 0$, and hence, by hypothesis, $A \neq 0$, the general equation becomes:

$$Ax + 0y + C = 0,$$

or simply

$$Ax + C = 0.$$

Transposing C and dividing both sides by A, we may write:

$$x = -\frac{C}{A}.$$

This equation is obviously of the form $x = a$, and therefore represents a line parallel to the Y-axis. Hence, we have shown that all equations of the first degree in two variables have a straight-line graph.

Comparison of the expressions

$$y = mx + b \tag{1}$$

and

$$y = -\frac{A}{B}x - \frac{C}{B} \tag{2}$$

shows that, since m is the slope of the straight line represented by (1), then $-\frac{A}{B}$ is the slope of the straight line represented

by (2). Therefore, in equation (2), $m = -\frac{A}{B}$. Similarly, since b is the Y-intercept of the straight line represented by (1), then $-\frac{C}{B}$ is the Y-intercept of the straight line represented by (2). In other words, if an equation is given in the form $Ax + By + C = 0$, to find the slope, divide the coefficient of x by the coefficient of y and change the sign of the resulting expression. To find the Y-intercept, divide the constant term by the coefficient of y and change the sign of the resulting expression.

For example, let it be required to find the slope and the Y-intercept of the line represented by the equation $2x - 3y + 5 = 0$. The slope $m = -\frac{A}{B} = -\frac{2}{-3} = +\frac{2}{3}$. The Y-intercept $b = -\frac{C}{B} = -\frac{5}{-3} = +\frac{5}{3}$. The line may then be plotted, as in the example shown on page 199. The graph of $2x - 3y + 5 = 0$ is shown in Figure 70.*

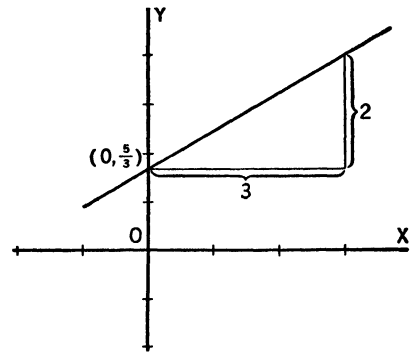


Fig. 70.

There are literally thousands of practical applications for the ratio $\frac{\Delta y}{\Delta x}$ whereby, with great ease and speed, solutions of problems can be effected without the need of tiresome arithmetical calculations. But first we must discuss further the use of ratios in measuring angles.

* Note that in the expression $Ax + By + C = 0$, all terms appear on one side of the equation. Hence, before the formulas $m = -\frac{A}{B}$ and $b = -\frac{C}{B}$ are applied, all terms must be placed on the same side of the equation. For example, the slope of $5x = 7y + 6$ is not $-\frac{5}{7}$. Before the slope and Y-intercept are found, the equation should be written $5x - 7y - 6 = 0$. The slope is $-\frac{5}{-7} = +\frac{5}{7}$, and the Y-intercept is $-\frac{-6}{-7} = -\frac{6}{7}$.

TANGENT OF AN ANGLE

The necessity for measuring distances between inaccessible points and for establishing boundary lines led the "rope stretchers" of ancient Egypt and Greece to the study of similar figures—in particular, to the investigation of similar triangles and the relationship of their sides (Chapter 11). These ancient surveyors developed to a high degree the art of measuring distances by constructing lines on the earth's surface. Thales' measurement of the height of the pyramids is an outstanding example (see Chapter 11, page 172). The earth's surface, however, is usually very rough and for geographical distances perceptibly curved; hence, the construction of these lines contained slight errors, which in the measurement of large distances accumulated and became great discrepancies.

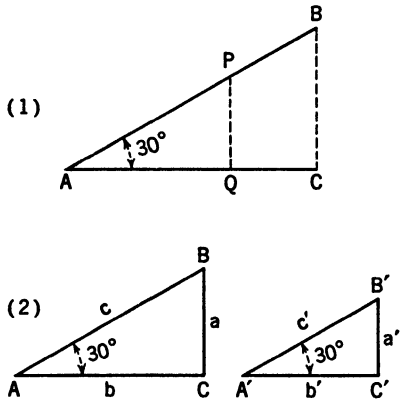


Fig. 71.

Angles, however, can be measured much more accurately. It has already been seen that an angle can be measured by the number of degrees contained in the arc of a circle intercepted by the two sides of the angle, if the center of the circle is the vertex of the angle (see Chapter 7, page 100). But there are many instances where the use of degrees is not convenient. It is then that the relationship of the sides of a right triangle to each other are of great value.

Consider any angle equal to 30°, such as BAC, Figure 71 (1). From any point P, on line segment AB, drop PQ perpendicular to AC. Through B, also on AB, draw BC perpendicular to AC. The right triangles APQ and ABC have the acute angle A common to both. Hence, by Fact 49, the triangles are similar. Therefore, $\frac{PQ}{AQ} = \frac{BC}{AC}$. That is, the ratio:

$$\frac{\text{length of side opposite to 30° angle}}{\text{length of side adjacent to 30° angle}} \tag{3}$$

is equal in both triangles.

Moreover, since if two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar (Fact 49), the above ratio would be the same for any right triangle having an angle equal to 30°. Thus, in Figure 71 (2), angles A and A' are equal to 30°. Therefore, the triangles are similar and $\frac{a}{b} = \frac{a'}{b'}$. Thus, for all right triangles containing an angle equal to 30°, ratio (3) has the same value, no matter what the lengths of the sides of the triangles are. This ratio (3) is called the *tangent* of 30°. It follows that angle A can be measured either by stating that it is 30° or by giving the value of the tangent of 30°.

In general, the abbreviation *tan* is used for tangent; thus, if A is an acute angle of a right triangle,

$$\tan A = \frac{\text{length of side opposite to angle } A}{\text{length of side adjacent to angle } A}.$$

Note that tan A is nothing more than the slope of line AB if the adjacent side is drawn horizontally toward the right and the opposite side vertically upward. Therefore, if rectangular coordinates are used, tan A may be written equal to the ratio $\frac{\Delta y}{\Delta x}$.

Obviously, the value of this ratio changes with the size of the angle. It is very important, then, for practical purposes, to ascertain the value of the tangent for different angles.

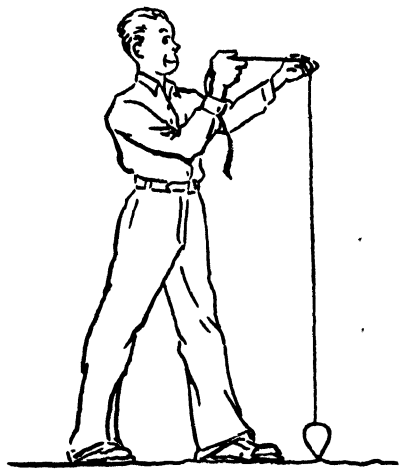


Fig. 72.

TABLE OF TANGENTS

In Figure 71 (2) the ratio of side a to side b is approximately 0.6. In the same figure, the ratio of a' to b' is approximately 0.565. The tangent of 30° may, then, be said to be approximately 0.58. The result is approximately correct, but obviously not good enough if great accuracy is desired. Mathematicians, by means of

methods more exact than visual measurements, have calculated the values of the tangents of different angles. These values, to three decimal places, are found in the table below.

TABLE OF TANGENTS OF ANGLES FROM 0° TO 89°

Angle	*Tangent*	*Angle*	*Tangent*	*Angle*	*Tangent*
0°	.000	30°	.577	60°	1.732
1	.017	31	.601	61	1.804
2	.035	32	.625	62	1.881
3	.052	33	.649	63	1.963
4	.070	34	.675	64	2.050
5	.087	35	.700	65	2.145
6	.105	36	.727	66	2.246
7	.123	37	.754	67	2.356
8	.141	38	.781	68	2.475
9	.158	39	.810	69	2.605
10	.176	40	.839	70	2.747
11	.194	41	.869	71	2.904
12	.213	42	.900	72	3.078
13	.231	43	.933	73	3.271
14	.249	44	.966	74	3.487
15	.268	45	1.000	75	3.732
16	.287	46	1.036	76	4.011
17	.306	47	1.072	77	4.331
18	.325	48	1.111	78	4.705
19	.344	49	1.150	79	5.145
20	.364	50	1.192	80	5.671
21	.384	51	1.235	81	6.314
22	.404	52	1.280	82	7.115
23	.424	53	1.327	83	8.144
24	.445	54	1.376	84	9.514
25	.466	55	1.428	85	11.430
26	.488	56	1.483	86	14.301
27	.510	57	1.540	87	19.081
28	.532	58	1.600	88	28.636
29	.554	59	1.664	89	57.290

Thus, for any acute angle of any size, there is a unique number called the *tangent* which measures its size. These numbers, computed to several decimal places, afford a notable saving of time and effort in making calculations, and make possible the accurate measurement of inaccessible distances.

APPLICATION OF TANGENTS

Before applying the facts just learned, it is convenient to give the meaning of a few terms.

We are all familiar with the little weight of lead or brass attached to a line and used by builders to determine whether a wall is standing up straight. (See Figure 72.) This cord with a weight attached is called a *plumb line* (Latin *plumbum* = lead). The line determined by the direction of the plumb line is called a *vertical line.* A vertical line is, therefore, perpendicular to the "plane of the horizon." A straight line lying wholly in the "plane of the horizon" is a *horizontal* line.

The angle formed by any line with a horizontal line in the same vertical plane is called the *angle of inclination.* Thus, in Figure 73, line AB makes the angle e with horizontal line AC, and angle d with a horizontal line drawn through B. Since all horizontal lines in the same vertical plane are parallel, angles d and e are equal (Fact 8). To avoid any confusion, these two equal angles are given different names.

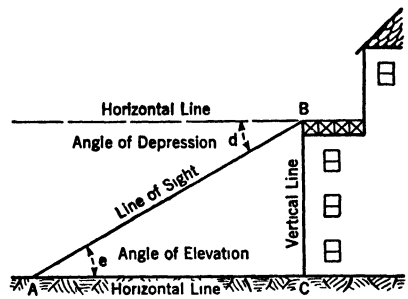


Fig. 73.

When an observer is standing on the ground, that is, at A in Figure 73, and looking up to B, his "line of sight" makes an angle in a vertical plane with the horizontal line AC passing through his eyes. This angle, e, is the *angle of elevation.*

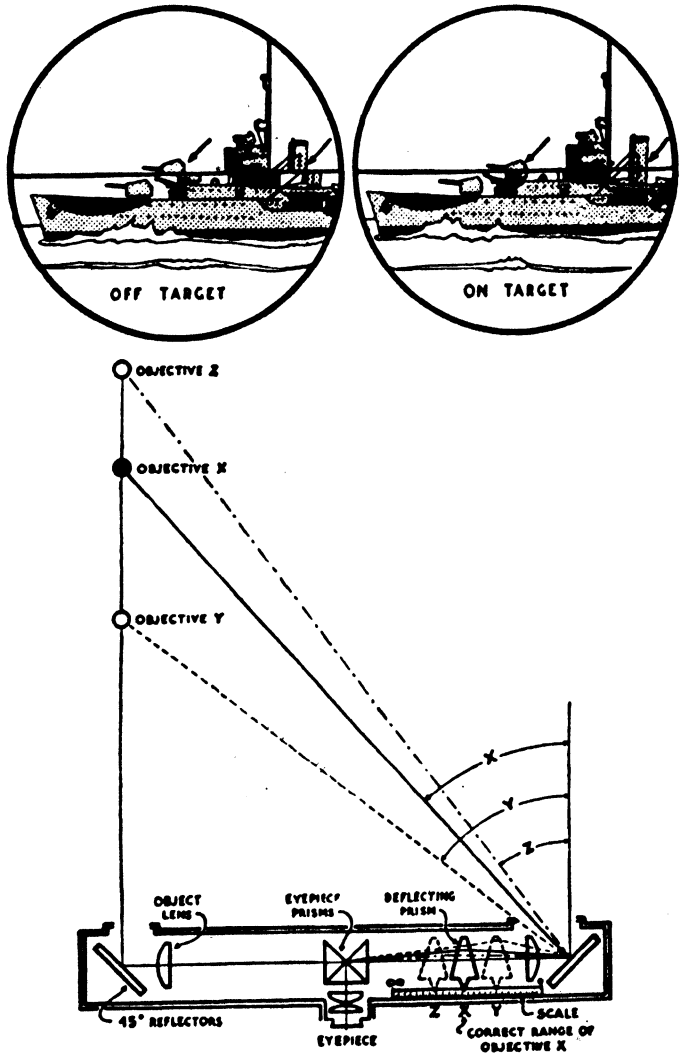
If, however, the observer is on the terrace of the building, that is, at B, and looks down to A, the angle, d, which his "line of sight" makes with the horizontal line through B is called the *angle of depression.*

The height of the balcony of the building shown in Figure 73 is now easily determined.

Suppose that an observer at A finds that the angle of elevation of the balcony is 30° and that AC is 62 feet. Then, $\tan e = \frac{BC}{AC}$, so that $\tan 30° = \frac{BC}{62}$, and $62 \times \tan 30° = BC$. From the table, $\tan 30°$ is found to be 0.577. Hence, $BC = 62 \times 0.577 = 35.8$ feet.

In the range-finder shown on the next page, manufactured by Bausch and Lomb, there are two reflectors. The one on the left is fixed at an angle of 45 degrees; consequently, a ray of light coming from the target and entering the aperture and striking the mirror is reflected at an angle of 90 degrees toward the second mirror at the right.

As seen in the diagram (taken, by courtesy of the publishers, from *Fortune Magazine*, the issue of October, 1940), a ray of light from the target striking the mirror on the right is reflected to the left through a movable deflecting prism.



Three possible positions of this prism are marked X, Y, and Z in the diagram.

The rays from the two mirrors reach the observer's eye after passing through the eye-piece prisms.

The movable deflecting prism can be adjusted by the observer so that the image is in focus. When it is in focus, the triangle formed (*a*) by the ray of

light from the target to the left-hand mirror, (*b*) the ray of light from the left-hand to the right-hand mirror, and (*c*) the ray of light from the target to the right-hand mirror is a right triangle.

Calling the angle of reflection (at the right-hand mirror) A, the distance between the two mirrors D, and the distance from the target to the left-hand mirror R, we have:

$$\tan A = \frac{R}{D}$$

or

$$R = D \tan A.$$

The values of $D \tan A$ are read on the scale, by means of a pointer attached to the movable deflecting prism. (The artist making this sketch is Mr. Rolf Klep.)

Problems

1. In Figure 74, ascertain the equations which are graphically represented by the lines A, B, C, and D.

2. A rectangular field is 100 feet long and 60 feet wide. Draw the rectangle. Extend its base and let this line represent the X-axis. Through one of the two vertical sides (choose the left one for convenience) draw the Y-axis. Find the slopes and inclination of the diagonals.

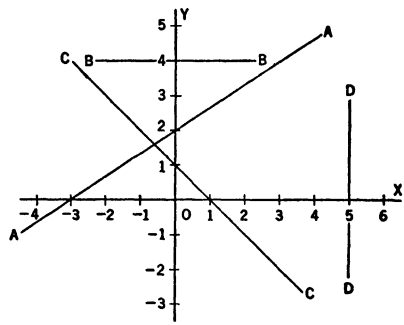


Fig. 74.

3. If the length of the shadow (on level ground) of a tree is twice the height of the tree, what is the angle of elevation of the sun?

4. It is desired to find the height of a certain chimney. At a point 150 feet from the foot, the angle of elevation of the top of the chimney is found to be 40°. How high is the chimney? (*Hint:* Use table of tangents on page 204.)

5. The vertex angle of an isosceles triangle is 30°. The altitude is 12 feet. Find the base of the triangle.

6. The legs of a right triangle are 12 and 16. Find the two acute angles.

7. An autoist drove from San Francisco to Los Angeles, a distance of 425 miles, in 17 hours. He drove back to San Francisco in 10 hours. What was his average speed for the round trip?

8. The sides of a triangle are in the ratio of 3, 4, and 5, and the perimeter is 960 yards. Find the sides. Show that the triangle is a right triangle.

9. The ratio of a father's age to his son's is 7 : 2, and the father is 40 years older than the son. Find the age of each.

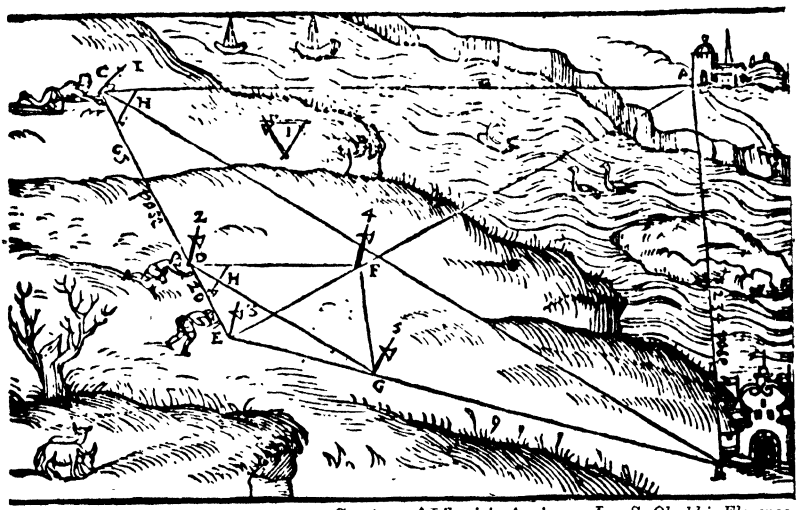
10. A road bed rises $7\frac{1}{2}$ feet in 600 feet; what is the slope? In how many feet will it rise 6 inches?

11. In a college boat race, crew A pulls 17 strokes to 16 strokes of crew B; but 28 strokes of crew B are as effective as 34 of crew A. Which is the faster crew and in what ratio?

12. If 10 equal gold coins and 40 equal silver coins are worth as much as 7 gold coins of the first value and 88 silver coins of the second value, what is the ratio of their values? Solve graphically.

14

TRIANGLE MEASURING (TRIGONOMETRY)



Courtesy of Librairie Ancienne, Leo S. Olschki, Florence

Woodcut from Thomas Digges, *A Geometrical Practise and Pantometria*, London, 1591.

The reader undoubtedly realizes the great possibilities in the new method of indirect measurement introduced in the preceding chapter. There is hardly a branch of science in the development of which the method of triangle measuring has not played an important part. Its greatest achievement, from a practical point of view, is, perhaps, its applications to surveying, navigation, astronomy, and mechanics.

TRIGONOMETRIC FUNCTIONS

An angle may be generated by the rotation of one of two rays assumed to be hinged like the hands of a clock. (See Chapter 7.)

If the vertex of the angle is placed at the origin of the coordi-

nate axes, the rotating ray will sweep the four quadrants in its journey around the origin. Thus, in Figure 75, a ray coinciding with the positive side of the X-axis will, if rotated counterclockwise, eventually assume the positions OA, OB, OC, and OD.

Every angle can be associated with a right triangle. In Chapter 13, page 202, angle BAC, equal to 30° (Figure 71), was associated with four right triangles of different size. It was also clearly shown in Chapter 11 that, since all right triangles having one acute angle equal are similar (Fact 49), the ratios of any two sides remain constant regardless of the length of the sides, as long as the angle remains unchanged.

Therefore, when the rotating ray assumes the position OA [Figure 75 (1)], angle AOX can be measured by dropping a perpendicular from any point on OA to the X-axis, for all the right triangles formed in this manner will be similar (Fact 49), and hence the ratio of any two sides will be constant. Therefore, we may choose, without loss of generality, a particular point in the rotating ray. So that the distance v, measured along the ray from the origin to this particular point, has the same fixed value no matter what position the rotating ray assumes.

As a ray is rotated around point O, and before it coincides with the Y-axis, the ray will form with the positive side of the X-axis all possible acute angles. That is, for all angles between 0° and 90°, the rotating ray will be in the first quadrant. For example, Figure 75 (1). If the angle is obtuse, such as XOB, Figure 75 (2), the rotating ray is in the second quadrant. For angles between 180° and 270°, for example, angle XOC, Figure 75 (3), the rotating ray is in the third quadrant; and finally, for angles between 270° and 360° [see XOD, Figure 75 (4)], the rotating ray is in the fourth quadrant.

Suppose that the rotating ray assumes any position in the first quadrant, say OA. From point A, on OA, and so that $OA = v$, drop a perpendicular, AP, to the X-axis. Then AP is the ordinate of point A, and OP is its abscissa, so that OP and AP may be denoted by x and y, respectively.

There are six fundamental ratios of the sides of a right triangle for each of its acute angles. Since these ratios are used very frequently, it is necessary to give them names. Hence, we define:

$$\text{Sine of angle } AOX = \frac{\text{ordinate of } A}{\text{distance of } A \text{ from origin}} = \frac{y}{v}$$

$$\text{Cosine of angle } AOX = \frac{\text{abscissa of } A}{\text{distance of } A \text{ from origin}} = \frac{x}{v}$$

$$\text{Tangent of angle } AOX = \frac{\text{ordinate of } A}{\text{abscissa of } A} = \frac{y}{x}$$

$$\text{Cotangent of angle } AOX = \frac{\text{abscissa of } A}{\text{ordinate of } A} = \frac{x}{y}$$

$$\text{Secant of angle } AOX = \frac{\text{distance of } A \text{ from origin}}{\text{abscissa of } A} = \frac{v}{x}$$

$$\text{Cosecant of angle } AOX = \frac{\text{distance of } A \text{ from origin}}{\text{ordinate of } A} = \frac{v}{y}$$

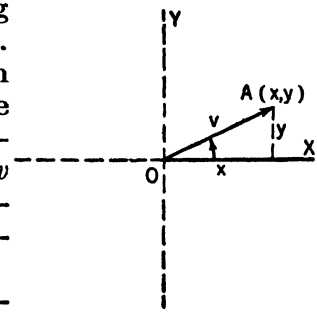
Since these ratios do not depend upon the lengths of the sides of the right triangle, but depend only upon the size of the angle, they are said to be *functions* of the angle. These functions, called the *trigonometric functions*, are usually written in an abbreviated form, as follows:

$$\sin AOX = \frac{y}{v}, \quad \tan AOX = \frac{y}{x}, \quad \sec AOX = \frac{v}{x}$$

$$\cos AOX = \frac{x}{v}, \quad \cot AOX = \frac{x}{y}, \quad \csc AOX = \frac{v}{y}$$

SIGNS OF THE TRIGONOMETRIC FUNCTIONS

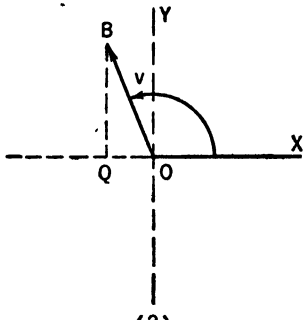
The values of the coordinates of any point in the rotating ray are positive or negative according to the quadrant in which this ray lies. The point in the rotating ray has been chosen so that its distance from the origin, measured along the ray, is constant and equal to v. This distance v is taken, for convenience, *always positive* no matter what position the rotating ray assumes.



(1)
Fig. 75.

As already stated, for all angles between 0° and 90°, the rotating ray is in the first quadrant. Therefore, for all acute angles, the coordinates of any point in the rotating ray will be positive. That is, in Figure 75 (1), if AOX is any acute

angle and A any point on OA such that $OA = v$, the coordinates of A, namely, x and y, are positive. Since v is always considered positive, it is easily seen that all the trigonometric functions of an acute angle are positive.



(2)
Fig. 75.

If the rotating ray is in the second quadrant, as in Figure 75 (2), the trigonometric functions of the obtuse angle are obtained (by definition) by dropping a perpendicular from any point, B, on the rotating ray to the X-axis. Thus, $\sin XOB = \frac{BQ}{OB}$, $\cos XOB = \frac{OQ}{OB}$, $\tan XOB = \frac{BQ}{OQ}$, and so forth.

If the right triangle OBQ is considered, it is easily seen that the trigonometric functions of obtuse angle XOB and acute angle BOQ are numerically the same. Angle BOQ is the *supplement* of angle XOB, so that *the trigonometric functions of an obtuse angle are equal numerically to the trigonometric functions*

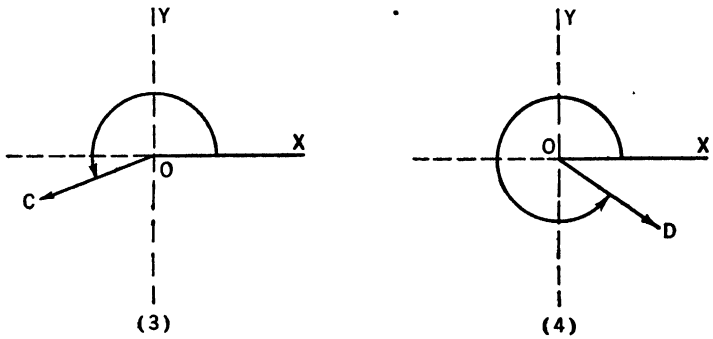


Fig. 75.

of its supplement. Thus, the trigonometric functions of 135° and 45° have the same numerical value; so that, since cos 45° = 0.707, then cos 135° has also the numerical value 0.707. However, if the rotating ray is in the second quadrant, the ordinate of any point in the rotating ray and the length $OB = v$ are positive, while the abscissa of the point is negative. That is, for any angle between 90° and 180°, such as angle XOB,

Figure 75 (2), the ordinate BQ and the length of OB, namely, v, are positive, while the abscissa OQ is negative. Hence,

$$\sin XOB = \frac{BQ}{OB} = \frac{+}{+} = + \qquad \cot XOB = \frac{OQ}{BQ} = \frac{-}{+} = -$$

$$\cos XOB = \frac{OQ}{OB} = \frac{-}{+} = - \qquad \sec XOB = \frac{OB}{OQ} = \frac{+}{-} = -$$

$$\tan XOB = \frac{BQ}{OQ} = \frac{+}{-} = - \qquad \csc XOB = \frac{OB}{BQ} = \frac{+}{+} = +$$

Therefore, although the cos 45° and cos 135° have the same numerical value, namely, 0.707, cos 45° = 0.707, while cos 135° = −0.707.

The following table may be verified by checking the signs of the trigonometric functions of angles in the third and fourth quadrants.

Quadrant	Sin.	Cos.	Tan.	Cot.	Sec.	Csc.
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

This table may be summarized by the following statements:

I Quadrant: All functions positive.

II Quadrant: Sine and cosecant positive, all other functions negative.

III Quadrant: Tangent and cotangent positive, all other functions negative.

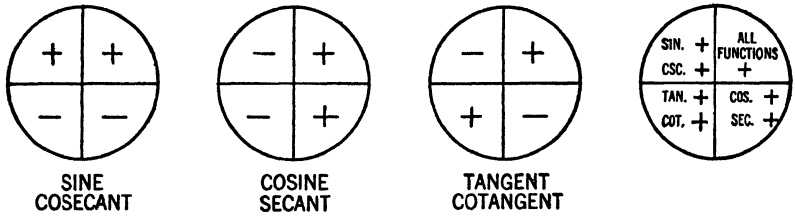
IV Quadrant: Cosine and secant positive, all other functions negative.

CO-FUNCTIONS

The reader may have been puzzled, when looking at the table of trigonometric functions on page 215, to see that the numbers indicating the size of the angle in the left-hand column run downward from 0° to 45°, while those in the right-

hand column run upward from 45° to 90°. The explanation is simple and necessary for the proper use of the table.

Consider any right triangle such as ABC in Figure 76. The



sum of the two acute angles, by Fact 20, equals one right angle. So that angles A and B are *complementary*.

By definition:

$$\sin A = \frac{a}{c} = \cos B; \quad \cot A = \frac{b}{a} = \tan B$$

$$\cos A = \frac{b}{c} = \sin B; \quad \sec A = \frac{c}{b} = \csc B$$

$$\tan A = \frac{a}{b} = \cot B; \quad \csc A = \frac{c}{a} = \sec B$$

The prefix *co* in *cosine, cotangent,* and *cosecant* is now seen to indicate a complementary relationship. In fact, it is because of this relationship that these functions received their names. They are the sine, tangent, and secant of the complementary angle. Thus, sin 25° is equal to the cosine of the complement of 25°, namely, 65°. So that sin 25° = cos 65°. The tan 16° = cot 74°, and so forth.

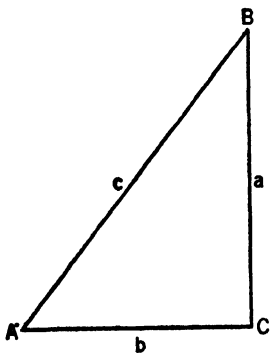


Fig. 76.

It now becomes clear why in the table on page 215 the size of the angles runs down on the left from 0° to 45° and up on the right from 45° to 90°; while each of the columns is "headed" by the name of the function and "tailed" by the name of the co-function.

To find, let us say, sin 25°, go down the left-hand column to 25° and then across to the column headed "sin." Sin 25° =

TRIGONOMETRIC FUNCTIONS TO THREE DECIMAL PLACES

α	sin α	cos α	tan α	cot α	sec α	csc α	
0°	.000	1.000	.000	——	1.000	——	**90°**
1°	.017	1.000	.017	57.290	1.000	57.299	89°
2°	.035	.999	.035	28.636	1.001	28.654	88°
3°	.052	.999	.052	19.081	1.001	19.107	87°
4°	.070	.998	.070	14.301	1.002	14.336	86°
5°	.087	.996	.087	11.430	1.004	11.474	**85°**
6°	.105	.995	.105	9.514	1.006	9.567	84°
7°	.122	.993	.123	8.144	1.008	8.206	83°
8°	.139	.990	.141	7.115	1.010	7.185	82°
9°	.156	.988	.158	6.314	1.012	6.392	81°
10°	.174	.985	.176	5.671	1.015	5.759	**80°**
11°	.191	.982	.194	5.145	1.019	5.241	79°
12°	.208	.978	.213	4.705	1.022	4.810	78°
13°	.225	.974	.231	4.331	1.026	4.445	77°
14°	.242	.970	.249	4.011	1.031	4.134	76°
15°	.259	.966	.268	3.732	1.035	3.864	**75°**
16°	.276	.961	.287	3.487	1.040	3.628	74°
17°	.292	.956	.306	3.271	1.046	3.420	73°
18°	.309	.951	.325	3.078	1.051	3.236	72°
19°	.326	.946	.344	2.904	1.058	3.072	71°
20°	.342	.940	.364	2.747	1.064	2.924	**70°**
21°	.358	.934	.384	2.605	1.071	2.790	69°
22°	.375	.927	.404	2.475	1.079	2.669	68°
23°	.391	.921	.424	2.356	1.086	2.559	67°
24°	.407	.914	.445	2.246	1.095	2.459	66°
25°	.423	.906	.466	2.145	1.103	2.366	**65°**
26°	.438	.899	.488	2.050	1.113	2.281	64°
27°	.454	.891	.510	1.963	1.122	2.203	63°
28°	.469	.883	.532	1.881	1.133	2.130	62°
29°	.485	.875	.554	1.804	1.143	2.063	61°
30°	.500	.866	.577	1.732	1.155	2.000	**60°**
31°	.515	.857	.601	1.664	1.167	1.942	59°
32°	.530	.848	.625	1.600	1.179	1.887	58°
33°	.545	.839	.649	1.540	1.192	1.836	57°
34°	.559	.829	.675	1.483	1.206	1.788	56°
35°	.574	.819	.700	1.428	1.221	1.743	**55°**
36°	.588	.809	.727	1.376	1.236	1.701	54°
37°	.602	.799	.754	1.327	1.252	1.662	53°
38°	.616	.788	.781	1.280	1.269	1.624	52°
39°	.629	.777	.810	1.235	1.287	1.589	51°
40°	.643	.766	.839	1.192	1.305	1.556	**50°**
41°	.656	.755	.869	1.150	1.325	1.524	49°
42°	.669	.743	.900	1.111	1.346	1.494	48°
43°	.682	.731	.933	1.072	1.367	1.466	47°
44°	.695	.719	.966	1.036	1.390	1.440	46°
45°	.707	.707	1.000	1.000	1.414	1.414	**45°**
	cos α	sin α	cot α	tan α	csc α	sec α	α

0.423. Note that 0.423 corresponds to the column tailed "cos" and across to 65° in the right-hand column. That is, cos 65° = 0.423 = sin 25°.

To find cot 56°, go up the right-hand column to 56° and then across to the column tailed "cot." Cot 56° = 0.675. Thus, it will be seen that cot 56° corresponds to the tangent of its complement, that is, cot 56° = 0.675 = tan 34°.

FUNCTIONS OF OBTUSE AND REFLEX ANGLES

The trigonometric functions of an obtuse angle are, as stated before, numerically equal to the trigonometric functions of its supplement. So that, *to find the functions of an angle between 90° and 180°, subtract the given angle from 180° and look up the functions of the supplement, prefixing the proper sign.* For example, to find tan 130°, subtract 130° from 180°, 180° − 130° = 50°. The tan 50° = 1.192; hence, tan 130° = −1.192.

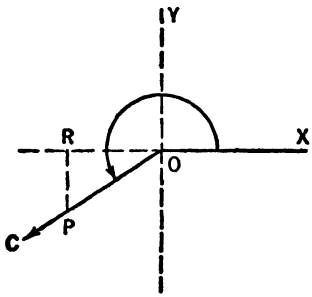


Fig. 77.

To obtain the trigonometric functions of an angle between 180° and 270°, such as angle *XOC*, Figure 77, a perpendicular, *PR*, must be drawn from any point on *OC* to the *X*-axis. So that the trigonometric functions of angles *XOC* and *ROP* are numerically equal. Hence, *to find the functions of an angle between 180° and 270°, subtract 180° from the given angle and obtain from the table the functions corresponding to the difference, prefixing the proper sign.* Thus, sin 205° is numerically equal to the sine of 205° − 180° = 25°. The sin 25° = 0.423; therefore, sin 205° = −0.423.

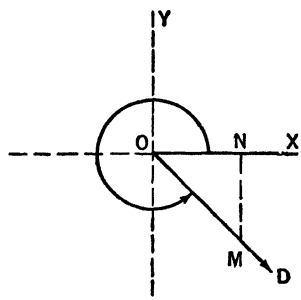


Fig. 78.

Similarly, the functions of an angle between 270° and 360°, such as angle *XOD*, Figure 78, are obtained by drawing a perpendicular, *MN*, from any point on *OD* to the *X*-axis. So that the trigonometric functions of angle *XOD* are numerically equal to the functions of angle *NOM*. Hence, *to find the functions of an angle between 270° and 360°, subtract the given angle from 360°, obtain from the table the functions of the remainder,*

and prefix the proper sign. For example, cot 295° is numerically equal to the cotangent of 360° − 295° = 65°. The cot 65° = 0.466; therefore, cot 295° = −0.466.

USE OF TABLES

If the angle involves minutes, a larger table, such as the one in the booklet on page 6, would allow us to simply read the corresponding function. However, since there are an infinite number of angles, it is impossible to print a complete table. The following method of finding the function of an angle not given in the table is therefore of prime importance.

Suppose sin 31° 15′ is to be determined. In order to obtain this function, *we assume that the change in the angle makes an approximately corresponding change in the function.*

Since 31° 15′ is between 31° and 32°, its sine must be between the sines of 31° and 32°.

Angle		*Sine*	
32°		0.530	
60′ { 31° 15′ }	15′	x { s }	0.015
31°		0.515	

Since 31° 15′ is $\frac{15}{60}$, that is, $\frac{1}{4}$, of the way from 31° to 32°, its sine is taken as $\frac{1}{4}$ of the way from 0.515 to 0.530. That is, $x = \frac{1}{4}$ of 0.015 = 0.0037, approximately. Hence, sin 31° 15′ = 0.515 + 0.0037 = 0.5187.

If the angle whose cotangent is 2.703 is to be found, the same method may be used. From the cotangent column in the table, it is seen that 2.703 is between the cotangents 2.605 and 2.747. It follows that the angle whose cotangent is 2.703 must be between 21° and 20°. Arranging the work as before, we have:

Angle		*Cotangent*	
21°		2.605	
60′ { A }	x	0.044 { 2.703 }	0.142
20°		2.747	

Since 2.703 is $\frac{0.044}{0.142} = \frac{44}{142} = \frac{22}{71}$ of the way from 2.747 to 2.605, angle A is taken as $\frac{22}{71}$ of the way from 20° to 21°,

namely, 60′. That is, $x = \frac{22}{71}$ of 60′ = 18.5′, or approximately 19′. Hence, angle $A = 20° + 19′ = 20°\ 19′$.

TABLE OF ANGLE FUNCTIONS IN FRACTIONAL FORM

Angle	sin	cos	tan	cot	sec	csc
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
90°	1	0	∞	0	∞	1
120°	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
180°	0	-1	0	∞	-1	∞
210°	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
270°	-1	0	∞	0	∞	-1
300°	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
360°	0	1	0	∞	1	∞

APPLICATIONS TO SURVEYING

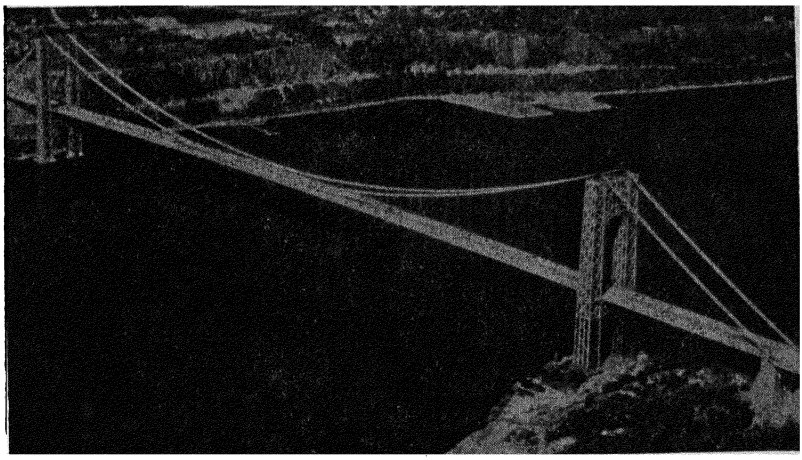
We are now in possession of enough facts concerning the ratios of the sides of a right triangle to show their applications to practical problems. Henceforth we shall use the four-place table on pages 6–14 of the booklet inside the back cover.

In the preliminary survey to build the George Washington Bridge over the Hudson River, the engineers had to ascertain the width of the river. The following method may have been used.

An engineer at point C, Figure 79, on the New York side of

the river sights to point B directly across the Hudson on the New Jersey side. He then lays out line DC so that angle BCD is 90°. Taking any convenient length on DC, say 1,000 feet, he marks point A. From A, angle CAB is found to be 74° 3′.

Triangle ABC is by construction a right triangle with angle



The George Washington Bridge over the Hudson River.

$CAB = 74° 3'$ and $AC = 1{,}000$ feet. Since we have given angle CAB and its adjacent side AC and we want to find the opposite side BC, it is obvious that the tangent ratio should be used. So that $\tan A = \frac{BC}{AC}$, that is, $\tan 74° 3' = \frac{BC}{1{,}000}$, and $1{,}000 (\tan 74° 3') = BC$. From the table, using the method shown to find the function of an angle not given in the table (page 217), $\tan 74° 3' = 3.499$. Therefore, $BC = 1{,}000 \times 3.499 = 3{,}499$ feet.

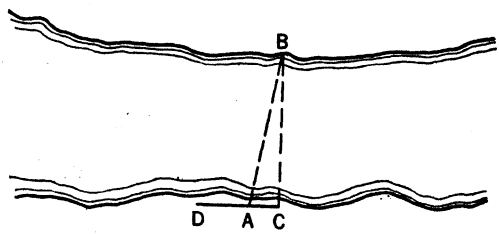


Fig. 79.

Another interesting problem occurs when the height of an object difficult to approach has to be found.

Suppose that the height of a hill is to be determined. At the foot of the hill, a forest makes impracticable a convenient

approach. To find its height, the following method can be used.

Two convenient points A and B, Figure 80, in line with the base of the hill are chosen so that the distance AB can be easily measured. The angles of elevation at A and B are then determined.

Assume that the distance AB is found to be 2,000 feet, while the angles of elevation at A and B are, respectively, 20° and 27°.

In right triangle ACD, $\cot 20° = \dfrac{2{,}000 + BD}{CD}$, that is,

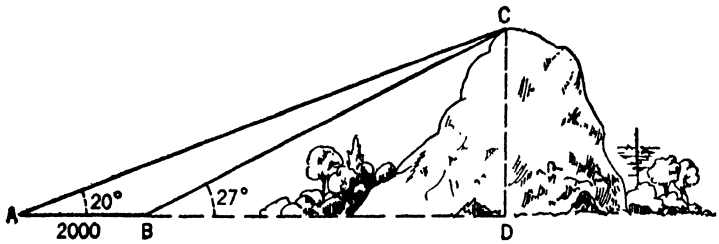


Fig. 80.

$CD\,(\cot 20°) = 2{,}000 + BD$, and $CD\,(\cot 20°) - 2{,}000 = BD$ (I).

In right triangle BCD, $\cot 27° = \dfrac{BD}{CD}$, that is, $CD\,(\cot 27°) = BD$ (II).

The left-hand sides of equations (I) and (II) are both equal to BD. Therefore, they are equal to each other. That is,

$$\begin{aligned} CD\,(\cot 20°) - 2{,}000 &= CD\,(\cot 27°); \text{ so that} \\ CD\,(\cot 20°) - CD\,(\cot 27°) &= 2{,}000 \\ 2.7475CD - 1.9626CD &= 2{,}000 \\ 0.7849CD &= 2{,}000 \\ CD &= \frac{2{,}000}{0.7849} = 2{,}548 \text{ feet.} \end{aligned}$$

APPLICATIONS TO PHYSICS

The effects of a force applied to an object were discussed in Chapter 9, page 135. We may recall the following important facts: Since a force has both direction and magnitude, it is a vector quantity. A force applied at an oblique angle, that is, neither vertically nor horizontally, produces two effects, one in each of these directions, and may be regarded as composed of

two forces, called its *horizontal and vertical components.* If a force is represented by a vector drawn to scale, F in Figure 81, its horizontal and vertical components are the legs H and V of the right triangle formed by dropping a perpendicular from F to the horizontal line. The direction of the force is given by the angle it makes with the horizontal, that is, by angle D.

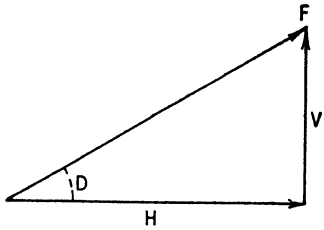
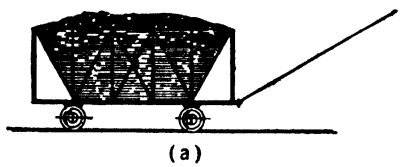


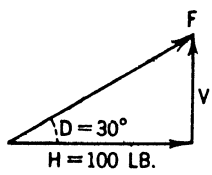
Fig. 81.

It is easily seen that, since the triangle formed is a right triangle, $\cos D = \frac{H}{F}$, so that $H = F \cos D$. That is, *the horizontal component of a force is equal to the force multiplied by the cosine of its direction angle.*

Moreover, $\sin D = \frac{V}{F}$, so that $V = F \sin D$. In other words, *the vertical component of a force is equal to the force multiplied by the sine of its direction angle.*

Assume that a coal car can be moved along level tracks by a horizontal force of 100 lbs. If the cable used to pull the car is inclined 30° from the horizontal, the force that must be exerted along the cable can now be easily found. It is important to note that Figure 82 (*a*) gives us a pictorial account of the problem, while Figure 82 (*b*) is a diagram representing forces. So that the unit used in drawing (*a*) bears no connection with that used in (*b*).


(a)



(b)

Fig. 82.

In Figure 82 (*b*), $\cos D = \frac{H}{F}$, that is, $F = \frac{H}{\cos D}$. Since $H = 100$ lbs. and $D = 30°$, then

$$F = \frac{100}{\cos 30°} = \frac{100}{0.8660} = 115.48 \text{ lbs.}$$

Forces are considered at rest, and therefore in equilibrium. To have equilibrium in any system of forces, there must be neither translation nor rotation. In order to balance each other, forces in the same plane must be such that the sum of their horizontal components equals zero, and the sum of their vertical components also equals zero.

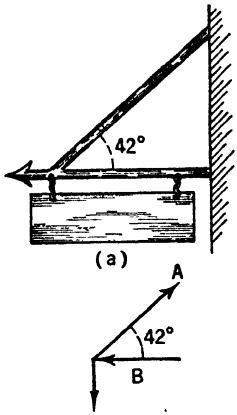


Fig. 83.

For example, suppose that a heavy sign weighing 200 pounds is supported by two rods hinged to a wall, as shown in Figure 83 (*a*), the inclined rod making an angle of 42° with the horizontal. The horizontal and vertical components of all the forces may be computed from Figure 83 (*b*), as follows: Since the weight of the sign pulls directly downward, its vertical component is the weight itself, that is, 200 lbs. Therefore, its horizontal component is zero. The horizontal and vertical components of force A are $A \cos 42°$ and $A \sin 42°$, respectively. Force B acts along the horizontal rod. Hence, its horizontal component is B itself, while its vertical component is zero.

In order to balance each other, the vertical components of the forces must balance, so that one exactly offsets the other; thus, $200 = A \sin 42°$ or $A = \dfrac{200}{\sin 42°} = \dfrac{200}{0.6691} = 298.9$ lbs.

If the horizontal components are to balance each other, $B = A \cos 42°$. Since A has been found equal to 298.9 lbs., then $B = 298.9 \times \cos 42° = 298.9 \times 0.7431 = 222.12$ lbs.

APPLICATIONS TO NAVIGATION

The location of a point in a plane is accomplished, as was shown in Chapter 12, by linking the point to a "frame of reference." This frame of reference consists of two perpendicular lines known as the coordinate axes. Any point in the plane is expressed in terms of its distances to the X- and Y-axes, that is, in terms of its abscissa and ordinate.

The earth is a huge ball which, rotating on its axis, rushes through space in its path around the sun. An object moving a considerable distance on the earth's surface must, therefore,

travel along a curved path. Hence, although a system of plane rectangular coordinates is very useful when a relatively small section of the earth is under consideration, a similar system of coordinates such that the position of a ship at sea could be accurately determined had to be developed before extensive navigation was made possible.

As is well known to those of us who live in the northern hemisphere, if we look up on a clear night and watch the stars for only a few hours, we notice that their position in the sky is slowly changing. An interested person would, if he continued to watch them all night, find that all the stars appear to move in circles around a point near the pole star. A person in the southern hemisphere would find that the stars appear to move in the same manner, that is, around a point in the sky. So that, to an observer on the earth, the stars appear to move as if they were fastened to the inside of an enormous sphere rotating on an axis passing through two fixed points in the sky. These two points in the sky around which the stars appear to move are called the *celestial poles*.

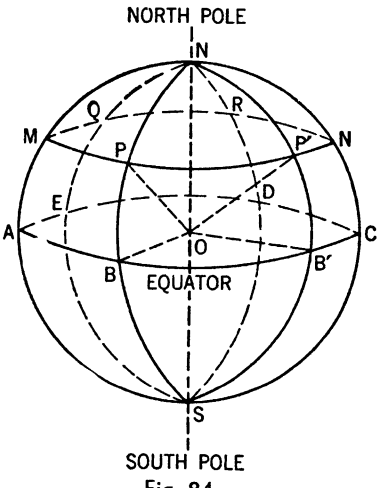


Fig. 84.

The apparent revolution of the stars is due, however, to the actual turning of the earth on its axis, one rotation every twenty-four hours. Therefore, an imaginary line (*NOS* in Figure 84) joining the celestial poles is the earth's axis. This axis pierces the earth's surface at two points (*N* and *S* in Figure 84), known as the *north* and *south poles*.

A plane perpendicular to the earth's axis and halfway between its poles cuts out on the earth's surface a circle (*ABB' CDE*, Figure 84). This circle, every point of which is at the same distance from the north and south poles, is called the *equator*. The position of an object, such as *P*, north or south of the equator (but not at a pole) is defined as follows: A circle on the earth's surface passing through *P* and the north and south poles intersects the equator at *B*. The two lines *OP* and

OB drawn from P and B to the center of the earth form an angle POB. This angle POB is called the *latitude* of P.

It is easily seen that, as in the case of the location of a point in a plane, this measurement of latitude is not sufficient to determine the exact position of P. For if a plane is passed through P parallel to the plane of the equator, it will cut out on the earth's surface a circle, $MPP'NRQ$, all points of which are at the same distance from the equator, so that angles POB and $P'OB'$ are equal. That is, all points on the circle $MPP'NRQ$ have the same latitude. It is necessary, therefore, to link P to another line of reference.

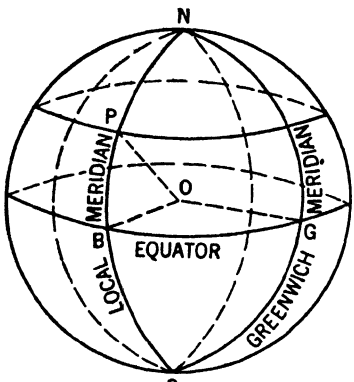


Fig. 85.

A circle on the earth's surface passing through the position of the object and the north and south poles is said to be the *meridian* of that object. Thus, circle $NPBSDR$ in Figure 84 is the meridian of P. The great observatory at Greenwich, England, was erected to furnish the observations necessary for the accurate determination of the position of a ship at sea. For this reason, nearly all nations have adopted the meridian passing through Greenwich as the standard meridian.

By this agreement, the position of an object, such as P in Figure 85, on its circle of latitude can now be determined. The directed angle, GOB in Figure 85, measured by the arc of the equator intercepted between the meridian at Greenwich and that of the object, is called the *longitude* of P.

In order to give the exact location of an object on the earth's surface, it is sufficient to state its latitude and longitude. These measurements are the coordinates of the object. For example: the latitude and longitude of Chicago, Ill., are, respectively, 41° 50′ N and 87° 36½′ W. That is, Chicago is 41° 50′ north of the equator and 87° 36½′ west of the meridian through Greenwich. Thus, the latitude and longitude of Chicago, namely, 41° 50′ N and 87° 36½′ W, locate uniquely this city on the surface of the earth.

The huge ball on which we live is slightly flattened at the

poles. The equatorial radius is 3,963.31 miles, while the polar radius is 3,949.87 miles. That is, the pole is 13.44 miles nearer to the center of the earth than a point on the equatorial circle. However, when its size is considered, the earth is as smooth and round as most billiard balls. Therefore, for practical purposes, the equatorial and meridian circles are considered perfect circles. The radius of these circles is the radius of the earth, so that the equatorial and meridian circles are all of the same size. The circles of latitude, however, vary in size, getting smaller as they get farther away from the equator until at the pole the circle of latitude is just a point.

It is easily seen that an object moving the same number of degrees along any meridian or along the equator covers the same distance, for these circles are equal. Thus, an object going from A to C (Figure 86) covers the same distance as an object going from B to D, for angles AOC and BOD are equal. Hence, two objects traveling along any meridian and changing their latitude by the same number of degrees cover the same distance.

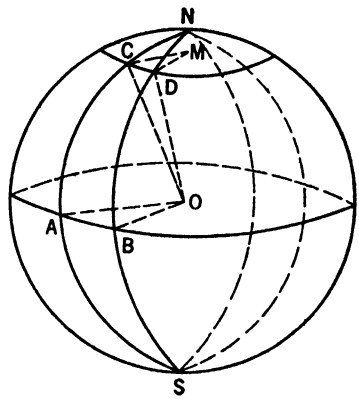


Fig. 86.

However, an object moving along a circle of latitude near the pole the same number of degrees that another object moves along the equator covers a much smaller distance, for the circle of latitude near the pole is smaller. Thus, in Figure 86, angles CMD and AOB are equal, so that arcs CD and AB contain the same number of degrees. However, a degree of arc CD is much smaller in length than a degree of arc AB. Hence, an object in going from C to D covers a much smaller distance than in going from A to B.

A degree of arc at the equator is 68.704 miles, so that one minute of arc at the equator is $68.704 \div 60 = 1.145$ miles. It may be shown by applying the formula for the length of any arc, given in Chapter 22, page 367, that one degree of arc of any circle of latitude is equal in length to the product of the length of one degree along the equator and the cosine of the latitude. Thus, if angle AOC is assumed to be 60°, and arc AB is 3°, then

the length of arc CD = length of arc $AB \times \cos AOC$. Since the length of arc $AB = 3 \times 68.704 = 206.11$ miles, then arc $CD = 206.11 \times \cos 60° = 206.11 \times 0.5 = 103.05$ miles.

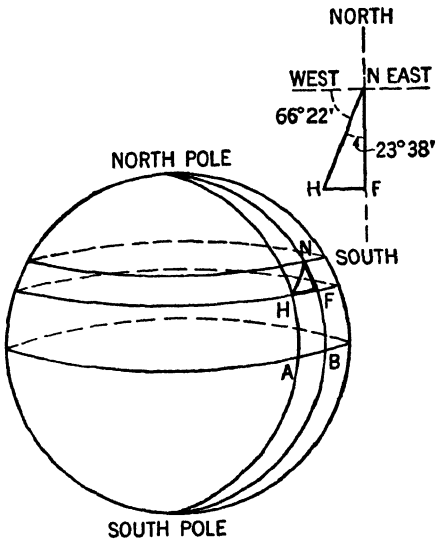


Fig. 87.

A plane flying from New York City to Havana, Cuba, must set its course in a certain direction. The latitude and longitude of New York are, respectively, 40° 48′ N and 73° 57′ W, while those of Havana are 23° 9′ N and 82° 21′ W. The direction of the course may be found as follows:

The difference in latitude is 40° 48′ − 23° 9′ = 17° 39′, or 17.65 degrees. Since a degree of arc of any meridian is equal to a degree of arc at the equator, the distance NF (Figure 87) is equal to $68.704 \times 17.65 = 1{,}212.63$ miles.

The difference in longitude is 82° 21′ − 73° 57′ = 8° 24′, or 8.4 degrees. So that the length of the arc on the equator corresponding to HF, that is, AB, $= 68.704 \times 8.4 = 577.11$ miles. Hence, $HF = 577.11 \times \cos 23° 9'$. The cos 23° 9′ is found from the table, and by interpolating, to be 0.9196. Therefore, $HF = 577.11 \times 0.9196 = 530.71$ miles.

In order to find the direction of the plane, it is necessary to find angle HNF. Triangle HFN can be considered, for practical purposes, as a plane right triangle having HN as its hypotenuse. In the right triangle HNF, we know the lengths of HF and NF, that is, of the sides opposite and adjacent to angle HNF. So that, using the tangent ratio $\tan HNF = \dfrac{HF}{NF}$ $= \dfrac{530.71}{1{,}212.63} = 0.4376$. The angle whose tangent is 0.4376 is found, from the table, to be between 23° and 24°. Again using the method shown on page 217, angle HNF is determined to be 23° 38′.

The direction to be followed by a plane flying from New York

to Havana may then be given as 23° 38′ west of south; or, subtracting 23° 38′ from 90°, as 66° 22′ south of west. A good approximation of the flying distance from New York to Havana can now be easily found. The sin $FNH = \frac{HF}{HN}$; that is, sin $23° 38' = \frac{530.71}{HN}$, or $HN \times \sin 23° 38' = 530.71$, and $HN = \frac{530.71}{\sin 23° 38'}$. Since sin $23° 38' = 0.4008$, then $HN = \frac{530.71}{0.4008}$ = 1,324.1 miles.

APPLICATIONS TO ASTRONOMY

The first and most fundamental problem of practical astronomy is to determine the distance of a heavenly body. No real conception of its actual size or trajectory can be obtained until its distance from the earth is determined. A simple method of finding the distance to the moon is already familiar to the reader. In Chapter 11, page 169, the "parallax" method, introduced by Ptolemy of Alexandria, was discussed and its application by means of a scale drawing suggested. This method is simplified and a greater accuracy obtained by use of trigonometric functions.

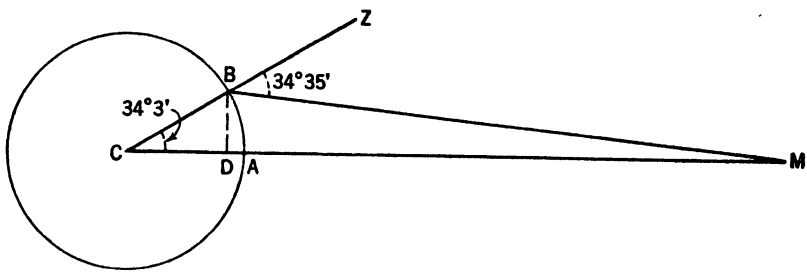


Fig. 88.

Assume that A and B in Figure 88 are two stations on the same meridian, so that A is on the equator and B's latitude is 34° 3′ N. At the exact time when the moon is directly overhead at A, the observer at B ascertains that angle ZBM is 34° 35′.

From B drop a perpendicular, BD, to CM. The triangle CBM is then divided into two right triangles, BDC and BDM.

The distance from the center of the earth to the moon, that is, CM, may be determined by finding CD and DM.

In the right triangle BDC, we know angle BCD is 34° 3′; for this angle measured B's latitude, and A was assumed to be on the equator. The line CB is the radius of the earth, which is known to be 3,960 miles. Hence, BD and CD can be easily found as follows: $\sin 34° 3' = \frac{BD}{3{,}960'}$, so that $(\sin 34° 3')(3{,}960) = (0.5599)(3{,}960) = BD$. That is, $BD = 2{,}217.20$ miles. Then $\cos 34° 3' = \frac{CD}{3{,}960'}$, so that $(\cos 34° 3')(3{,}960) = (0.8285)(3{,}960) = CD$. Hence, $CD = 3{,}280.86$ miles, or 3,281 miles, approximately.

Angles ZBM, MBD, and DBC together make a straight angle, or 180°. Angle ZBM is 34° 35′. Angle DBC is complementary to angle BCD, so that it is equal to 90° − 34° 3′, that is, 55° 57′. Therefore, 34° 35′ + angle MBD + 55° 57′ = 180°, so that angle MBD = 180° − 90° 32′ = 89° 28′.

In right triangle BDM, $\tan MBD = \frac{DM}{BD}$, so that $\tan 89° 28' = \frac{DM}{2{,}217.20'}$ and $DM = \tan 89° 28' \times 2{,}217.20 = 108.86 \times 2{,}217.20 = 241{,}360$ miles, approximately.

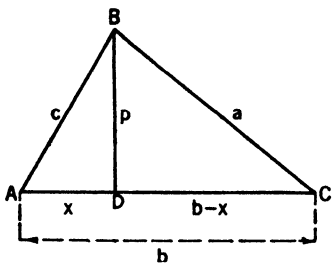


Fig. 89.

Therefore, the approximate distance from the center of the earth to the moon is $CD + DM$, or 3,281 + 241,360 = 244,641 miles.

The solution of oblique triangles, such as triangle CBM, in Figure 88, is frequently necessary in practical problems. Study of the general case involving this type of problem will undoubtedly prove useful. Consider, therefore, any oblique triangle ABC, Figure 89. From B drop a perpendicular, BD, to AC. The triangle ABC is then divided into two right triangles, BDA and BDC.

In the right triangle BDA, $\sin A = \frac{p}{c}$; so that

$$c \sin A = p.$$

In the right triangle BDC, $\sin C = \frac{p}{a}$; hence,

$$a \sin C = p.$$

Therefore,

$$c \sin A = a \sin C,$$

or

$$\frac{c}{\sin C} = \frac{a}{\sin A}.$$

By dropping a perpendicular from C to AB, the reader may easily show that:

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

So that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

That is, *the three sides of a triangle are proportional to the sines of the opposite angles.*

This relationship between the sides of a triangle and the sines of the opposite angles is known as the *Law of Sines.*

The distance from the earth's center to the moon, that is, CM in Figure 88, may now be easily determined as follows. In triangle CBM, angle BCM is 34° 3′; line CB, the radius of the earth, is 3,960 miles; and angle CBM = 180° − angle ZBM = 180° − 34° 35′ = 145° 25′.

In the given problem, side CB is known, while it is desired to obtain CM. Applying the sine law, we get:

$$\frac{CB}{\sin BMC} = \frac{CM}{\sin CBM}.$$

Since angles BCM and CBM are known, angle BMC is, by Fact 18, equal to 180° − (angle BCM + angle CBM), or 180° − (34° 3′ + 145° 25′), that is, 180° − 179° 28′ = 32′. Substituting known values gives:

$$\frac{3960}{\sin 32'} = \frac{CM}{\sin 145° 25'},$$

or

$$\frac{3960}{0.0093} = \frac{CM}{0.5676}$$

so that

$$CM = 241{,}688 \text{ miles.}^*$$

In any oblique triangle, if *a side and two angles* or *two sides and the angle opposite one of them* are known, the other angles and sides may be easily found by using the law of sines. However, if *three sides* or *two sides and their included angle* are known, the sine law will not solve the triangle.† A formula for the solution of a triangle in which these combinations of parts are given can be obtained by applying the very familiar theorem of Pythagoras, namely, "The square on the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides." (The proof of this theorem, that is, Fact 70, is given in Chapter 19, page 295.)

Consider the triangle ABC, Figure 89. The square on the hypotenuse, AB, of right triangle BDA, is, by Fact 70, equal to the sum of the squares of AD and BD. That is,

$$c^2 = x^2 + p^2,$$

or

$$c^2 - x^2 = p^2. \tag{1}$$

Similarly, in right triangle BDC,

$$a^2 = (b - x)^2 + p^2,$$

or

$$a^2 - (b - x)^2 = p^2. \tag{2}$$

Since the left-hand sides of (1) and (2) are both equal to p^2, they are equal to each other.

Hence,

$$a^2 - (b - x)^2 = c^2 - x^2,$$

or

$$a^2 - b^2 + 2bx - x^2 = c^2 - x^2,$$

so that,

$$a^2 = b^2 + c^2 - 2bx.$$

* The discrepancy between the answer previously obtained (see page 228), namely, 241,360 miles, and this answer of 241,688 miles is due to the fact that a four-place table has been used.

† If only three angles of a triangle are known, the size of the triangle is not fixed. Therefore, this combination of parts is not sufficient to obtain a solution.

In order to express x in terms of the parts of the given triangle, note that in the right triangle BDA,

$$\cos A = \frac{x}{c},$$

or

$$c \cos A = x.$$

Substituting this value for x in the above expression, we obtain:

$$a^2 = b^2 + c^2 - 2bc \cos A. \tag{3}$$

That is, *the square on one side of a triangle is equal to the sum of the squares of the other two sides, minus twice the product of those sides by the cosine of their included angle.*

This relationship is called the *Law of Cosines.* In a similar manner, the reader may easily show that:

$$b^2 = a^2 + c^2 - 2ac \cos B, \tag{4}$$

and

$$c^2 = a^2 + b^2 - 2ab \cos C. \tag{5}$$

For example, suppose that in triangle ABC, Figure 89, $b = 60$, $c = 80$, and angle $A = 61°$. Applying form (3) of the Law of Cosines, we obtain:

$$a^2 = (60)^2 + (80)^2 - 2(60)(80) \cos 61°,$$

or

$$a^2 = 10{,}000 - (9600)(0.4848) = 5345.92,$$

and

$$a = 73.11.$$

To find angle C, apply form (5) of the Law of Cosines, obtaining:

$$(80)^2 = (73.11)^2 + (60)^2 - 2(73.11)(60) \cos C$$
$$8773.20 \cos C = 2545.92$$
$$\cos C = \frac{2545.92}{8773.20} = 0.2902.$$

Hence,

$$\text{angle } C = 16° 53'.$$

Angle B may now be found by Fact 18. Thus, angle $B = 180° - (61° + 16° 53') = 102° 7'$.

The proof of these two very useful and important formulas has been given for an oblique triangle because by this assumption no loss of generality has been incurred. The reader may easily prove these two laws for an obtuse-angled triangle such as triangle ABC in Figure 80.

FUNCTIONS OF 0°, 90°, 180°, 270°, AND 360°

The functions of acute, obtuse, and reflex angles have been defined. To form a complete system, it is necessary to determine the functions of 0°, 90°, 180°, and their multiples.

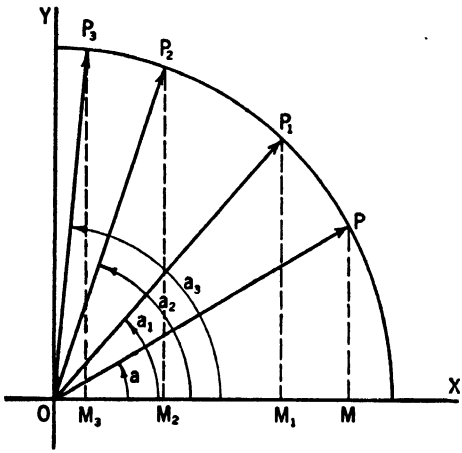


Fig. 90.

Consider the ray OP, Figure 90, forming the angle a with the X-axis. From P drop a perpendicular PM to the X-axis so that PM and OM are, respectively, the ordinate and abscissa of P. If the ray OP is rotated around point O, it assumes the positions OP_1, OP_2, OP_3, and so forth, before it coincides with the Y-axis, while angle a increases in value approaching 90°. Now, as angle a approaches 90°, the ordinate PM assumes the positions P_1M_1, P_2M_2, P_3M_3, and so forth. So that, since the ray OP remains of a fixed length, the ordinates PM, P_1M_1, P_2M_2, P_3M_3, and so forth, of the successive points P, P_1, P_2, P_3, and so forth, become larger and larger approaching the length of OP. Similarly, the abscissas OM, OM_1, OM_2, OM_3, and so forth, become smaller and smaller approaching the value zero. That is,

Angle a approaches 90°,	OP remains of a fixed length.
PM approaches OP,	OM approaches zero.

Since $\sin a = \frac{PM}{OP}$, then $\sin 90° = \frac{OP}{OP} = 1.$

" $\cos a = \frac{OM}{OP}$, " $\cos 90° = \frac{0}{OP} = 0.$

" $\cot a = \frac{OM}{PM}$, " $\cot 90° = \frac{0}{OP} = 0.$

" $\csc a = \frac{OP}{PM}$, " $\csc 90° = \frac{OP}{OP} = 1.$

Now, $\tan a = \frac{PM}{OM}$. When $a = 90°$, this ratio assumes the form $\frac{OP}{0}$, which, as stated in Chapter 1, page 10, is meaningless. However, it is easily seen that, as the denominator OM becomes smaller and smaller through positive values, the value of the fraction becomes larger and larger. Since the value of the fraction can be made as large as we please by making the denominator sufficiently small, we then say that tan a becomes infinite as angle a approaches 90° from the positive side of the X-axis in a counterclockwise sense. But if the angle a approaches 90° from the negative side of the X-axis in a clockwise sense, the corresponding fraction is throughout negative, so that tan a becomes negatively infinite in this case. These facts expressed symbolically are written $\tan 90° = \pm\infty$, and read "the tangent of any angle a increases without bound as angle a approaches 90° in a counterclockwise sense and decreases without bound (through negative values) as angle a approaches 90° in a clockwise sense."

It is important to note that the symbol ∞, read "infinity," does not represent a very large definite number, but simply denotes a quantity increasing without bound. A similar discussion will show that $\sec a = \frac{OP}{OM}$ increases without bound as angle a approaches 90° in a counterclockwise sense, and decreases without bound (through negative values) as angle a approaches 90° in a clockwise sense. Hence, we may write $\sec 90° = \pm\infty$.

To obtain the trigonometric functions of 0°, rotate the ray OP in a clockwise sense toward the X-axis, as in Figure 91. It is easily seen that,

Angle a approaches 0°,	OP remains of a fixed length.
PM approaches 0,	OM approaches OP.

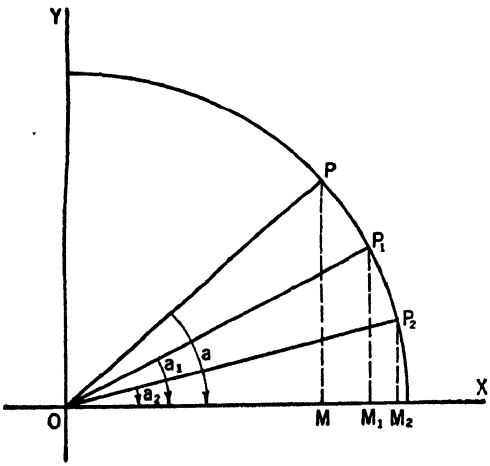


Fig. 91.

So that, following the same process of reasoning, we obtain:

$$\sin 0° = 0$$
$$\cos 0° = 1$$
$$\tan 0° = 0$$
$$\cot 0° = \pm \infty$$
$$\sec 0° = 1$$
$$\csc 0° = \pm \infty$$

The following table showing the values of the trigonometric functions of various angles may be easily verified.

Angle	0°	90°	180°	270°	360°
sin	0	1	0	−1	0
cos	1	0	−1	0	1
tan	0	±∞	0	±∞	0
cot	±∞	0	±∞	0	±∞
sec	1	±∞	−1	±∞	1
csc	±∞	1	±∞	−1	±∞

The variations of the trigonometric functions through the four quadrants are shown in the following table. The verification of this table is left as an exercise for the reader.

Function	*Quadrant I*	*Quadrant II*	*Quadrant III*	*Quadrant IV*
sin	0 to +1	+1 to 0	0 to −1	−1 to 0
cos	+1 to 0	0 to −1	−1 to 0	0 to +1
tan	0 to +∞	−∞ to 0	0 to +∞	−∞ to 0
cot	+∞ to 0	0 to −∞	+∞ to 0	0 to −∞
sec	+1 to +∞	−∞ to −1	−1 to −∞	+∞ to +1
csc	+∞ to +1	+1 to +∞	−∞ to −1	−1 to −∞

The following table showing the functions of complementary, obtuse, and reflex angles in terms of an acute angle may be easily verified by working out Problems 5 and 6 on pages 623-624.

$\sin(90° - \theta) = \cos\theta$ $\cos(90° - \theta) = \sin\theta$ $\tan(90° - \theta) = \cot\theta$ $\cot(90° - \theta) = \tan\theta$ $\sec(90° - \theta) = \csc\theta$ $\csc(90° - \theta) = \sec\theta$	$\sin(90° + \theta) = \cos\theta$ $\cos(90° + \theta) = -\sin\theta$ $\tan(90° + \theta) = -\cot\theta$ $\cot(90° + \theta) = -\tan\theta$ $\sec(90° + \theta) = -\csc\theta$ $\csc(90° + \theta) = \sec\theta$
$\sin(180° - \theta) = \sin\theta$ $\cos(180° - \theta) = -\cos\theta$ $\tan(180° - \theta) = -\tan\theta$ $\cot(180° - \theta) = -\cot\theta$ $\sec(180° - \theta) = -\sec\theta$ $\csc(180° - \theta) = \csc\theta$	$\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$ $\tan(180° + \theta) = \tan\theta$ $\cot(180° + \theta) = \cot\theta$ $\sec(180° + \theta) = -\sec\theta$ $\csc(180° + \theta) = -\csc\theta$
$\sin(270° - \theta) = -\cos\theta$ $\cos(270° - \theta) = -\sin\theta$ $\tan(270° - \theta) = \cot\theta$ $\cot(270° - \theta) = \tan\theta$ $\sec(270° - \theta) = -\csc\theta$ $\csc(270° - \theta) = -\sec\theta$	$\sin(270° + \theta) = -\cos\theta$ $\cos(270° + \theta) = \sin\theta$ $\tan(270° + \theta) = -\cot\theta$ $\cot(270° + \theta) = -\tan\theta$ $\sec(270° + \theta) = \csc\theta$ $\csc(270° + \theta) = -\sec\theta$
$\sin(360° - \theta) = -\sin\theta$ $\cos(360° - \theta) = \cos\theta$ $\tan(360° - \theta) = -\tan\theta$ $\cot(360° - \theta) = -\cot\theta$ $\sec(360° - \theta) = \sec\theta$ $\csc(360° - \theta) = -\csc\theta$	$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$ $\sec(-\theta) = \sec\theta$ $\csc(-\theta) = -\csc\theta$

Problems

1. Construct the angles, having given the following values of the functions:

(*a*)	$\sin \theta = \frac{2}{3}$	(*d*)	$\cot y = 3$
(*b*)	$\cos \phi = \frac{3}{5}$	(*e*)	$\sec z = \frac{5}{2}$
(*c*)	$\tan x = \frac{1}{2}$	(*f*)	$\csc r = \frac{4}{3}$

2. By using the relations given in the table on page 235, find the functions of the following angles:

(*a*)	150°	(*d*)	−120°
(*b*)	240°	(*e*)	315°
(*c*)	−45°	(*f*)	−330°

3. An observation balloon is known to be directly above a point 5,000 feet away from an artillery post. The artillery officer in command finds the angle of elevation of the balloon to be 40°. How high is the balloon?

4. A wire for a radio antenna is to be run from a pole 25 feet high and 20 feet away from a window. Find the necessary length of wire.

5. Wishing to determine the width of a river, a colonel observed a tree standing directly across on the opposite bank. The angle of elevation of the top of the tree was 30°. At 150 feet back from this point of observation and in the same direction from the tree, the angle of elevation of the top of the tree was 18°. Find the width of the river.

6. In surveying a coal mine, an engineer measures a length $AB = 250$ feet due east with a dip of 7°; then a length $BC = 400$ feet due south with a dip of 11°. How much lower is C than A?

7. In the side of a hill which slopes upward at an angle of 32°, a tunnel is bored sloping downward at an angle of 12° with the horizontal. How far below the surface of the hill is a point 125 feet down the tunnel?

8. From the summit of Rigi, 1,200 meters above the surface of Lake Lucerne, two boats are seen in the same vertical plane. The angle of depression of the first is 35° and of the second 24°. What is the distance between the boats?

9. Why does a man lead a horse, pulling a heavy load, from one side of the road to the other when going up a hill?

10. Find the force required to pull a body weighing 100 lbs. up a smooth plane having an inclination of 30°.

11. A man can exert a pull of 120 lbs. He wishes to raise a body weighing 180 lbs., and decides to use a smooth inclined plane. What is the greatest possible inclination of the plane?

12. A 26-foot ladder leans against a smooth wall with its foot 10 feet from the wall. If a 180-pound man climbs to its very top, what thrust is in the ladder and what pressure against the wall?

15 CONCURRING BALANCES, OR SIMULTANEOUS LINEAR EQUATIONS

In the solution of problems it has been sufficient so far to consider one unknown quantity. Those cases in which two unknowns occurred did not call for a particular solution. For example, the analysis of the airplane problem of Chapter 13 did not lead to one particular solution, but rather to a general idea of the problem as indicated by the graph of the equation. Moreover, instead of having only one value as the solution of this problem, very many values, in fact, every pair of corresponding values of the unknown quantities, satisfied the equation.

METHODS OF SOLUTION

Consider, however, an extension of the airplane problem as follows: A train leaves Pennsylvania Station at New York City for Washington 2 hours before an airplane leaves the Newark airport 9 miles from New York City on the road to Washington. The train and the airplane travel at the average rates of 50 miles an hour and 112 miles an hour, respectively. At what precise distance from New York City will the airplane pass the train?

There will be one and only one solution or answer. The facts stated in the problem are put in mathematical language as follows: From Chapter 13, the equation $y = 112x + 9$ defines y, or the distance the plane is away from New York City, as a function of x, the time elapsed since the airplane left Newark airport. Inasmuch as the train left two hours before the plane started, then $x + 2$ denotes the time elapsed since the train departed from Pennsylvania Station. The train travels at 50 miles an hour; hence, $50(x + 2)$ represents the distance the train is away from New York City and, therefore,

may be denoted by y. That is, $y = 50(x + 2)$, or $y = 50x + 100$.

Two equations have now been obtained, representing mathematically the conditions stated in the problem. These are:

$$y = 50x + 100 \tag{1}$$

and

$$y = 112x + 9. \tag{2}$$

The question naturally arises: How does one proceed to solve these equations? Or, in other words: Which are the values of x and y that satisfy both equations?

The solution can be obtained by any one of five different methods:

A. By addition and subtraction.
B. By substitution.
C. By comparison.
D. By graphs.
E. By determinants.

A. ADDITION AND SUBTRACTION METHOD

Rewriting equations (1) and (2) so that the constant term appears alone on one side of the equation, the first method may be applied as follows:

$$y - 50x = 100 \tag{1}$$
$$y - 112x = 9. \tag{2}$$

It is easily seen that if equation (2) is subtracted from equation (1), the y's are eliminated and an equation involving one unknown only is obtained. Thus:

$$\begin{array}{r} y - 50x = 100 \\ y - 112x = 9 \\ \hline 62x = 91 \end{array}$$

or

$$x = \frac{91}{62} = 1.468 \text{ hours.}$$

That is, the plane overtakes the train 1.468 hours after leaving Newark airport. In view of the fact that when the plane overtakes the train, both conveyances are at the same distance

from New York City, then the value of y may be obtained by substituting this value of x in either of the two equations.

Substituting in equation (1) gives:

$$y = 50(1.468) + 100$$

or

$$y = 173.4 \text{ miles.}$$

The same result could have been obtained immediately if, instead of eliminating the y's from the two equations, the x's were made to vanish. The operation would then be as follows:

If each side of equation (1) is multiplied by 112, or the coefficient of x in equation (2), and each side of equation (2) is multiplied by 50, or the coefficient of x in equation (1), the equality in the respective equations is not changed. Thus,

$$y - 50x = 100 \qquad (1)$$

and

$$y - 112x = 9 \qquad (2)$$

become

$$112y - 5{,}600x = 11{,}200$$

and

$$50y - 5{,}600x = 450.$$

Notice that the coefficients of x in the two equations are now equal and that therefore if equation (2) in its new form is subtracted from equation (1), the terms in x vanish. Thus,

$$\begin{array}{rr} 112y - 5{,}600x = & 11{,}200 \\ 50y - 5{,}600x = & 450 \\ \hline 62y = & 10{,}750 \end{array}$$

or

$$y = \frac{10{,}750}{62} = 173.4 \text{ miles.}$$

B. SUBSTITUTION METHOD

The second method or solution by substitution is carried out in the following manner. Solve *one* of the equations for *one* of the variables as an expression in terms of the other and then substitute this expression in the second equation. This gives an equation in one unknown which is solved by the methods already explained.

Thus: solve equation (1):

$$\begin{aligned} y - 50x &= 100 \qquad \text{for } y, \text{ obtaining:} \\ y &= 50x + 100. \end{aligned}$$

Substitute this expression for y in equation (2):

$$y - 112x = 9,$$

so that

$$(50x + 100) - 112x = 9.$$

Then, solving for x,

$$\begin{aligned} 100 - 9 &= 112x - 50x \\ 91 &= 62x \end{aligned}$$

and

$$\frac{91}{62} = x = 1.468 \text{ hours.}$$

Substituting, as before, this value of x in either of the two equations, the value of $y = 173.4$ miles is obtained.

C. COMPARISON METHOD

If the third method is to be used, solve both equations for the same unknown in terms of the other, and then set these expressions equal to each other. This again gives an equation in one unknown which is solved as before.

For example, if the value of y is desired at once, solve equations (1) and (2) for x. Thus:

$$\begin{aligned} y - 50x &= 100 \\ y - 100 &= 50x \\ \frac{y - 100}{50} &= x, \end{aligned} \qquad (1)$$

and

$$\begin{aligned} y - 112x &= 9 \\ y - 9 &= 112x \\ \frac{y - 9}{112} &= x. \end{aligned} \qquad (2)$$

Equating these expressions for x gives:

$$\frac{y - 100}{50} = \frac{y - 9}{112}.$$

Then solving for y:

$$\begin{aligned} 112(y - 100) &= 50(y - 9) \\ 112y - 11{,}200 &= 50y - 450 \\ 112y - 50y &= 11{,}200 - 450 \\ 62y &= 10{,}750 \\ y &= \frac{10{,}750}{62} = 173.4 \text{ miles.} \end{aligned}$$

D. GRAPHIC METHOD

In Chapter 13 it was shown that an equation of the first degree in two variables is represented graphically by a straight line. It was also pointed out that a straight line is the graph of an equation when and only when the coordinates of any point on the line satisfy the given equation.

The solution of a pair of simultaneous equations calls for a pair of values, one for x and the corresponding one for y, such that if these values are substituted in the equations, both are satisfied. Therefore, this particular pair of values must define a point on the plane which is common to the two lines. That is, the solution of two simultaneous linear equations may be accomplished graphically by finding the point which the two lines have in common.

Consider again the two equations:

$$y = 50x + 100 \tag{1}$$

and

$$y = 112x + 9. \tag{2}$$

A straight line on the plane is represented by an equation of the type

$$y = mx + b, \quad \text{(See Chapter 13, page 198)}$$

where m is the slope and b the distance from the X-axis at which the line crosses the Y-axis. The coefficient of x in equation (1) is 50, so that $m = 50$. The constant term is 100, that is, $b = 100$. Hence, equation (1) crosses the Y-axis 100 units above the X-axis and has a slope of 50. Its graph is the straight line (1) shown in Figure 92.

The line which represents equation (2) has a slope of 112 and crosses the Y-axis 9 units above the X-axis. So that its graph is the straight line (2) shown also in Figure 92. These two

lines have only the point S in common, therefore they have only one solution. The coordinates of S are approximately (1.45, 173). Therefore the plane overtakes the train 1.45 hours after leaving Newark, at a distance of 173 miles from New York City.

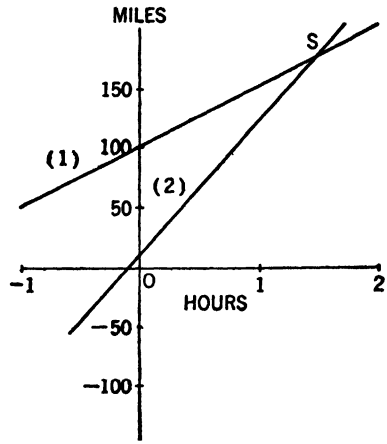


Fig. 92.

E. DETERMINANTS

The present discussion of determinants, from here to page 250, is merely introductory, inserted for the sake of completeness. Rarely, if ever, would determinants be used in solving *two* simultaneous linear equations, because the other methods of solutions just discussed are easy and effective. When a problem contains three or more equations, with three or more unknown quantities, the method of determinants becomes valuable. Readers who wish to pursue the subject a little further may consult some standard text on higher algebra for a complete exposition.

The solution of two linear equations may be effected by *the use of determinants*. This new method is a very valuable tool in the analysis of simultaneous equations and in more advanced work in mathematics and physics.

In general, to solve any pair of linear equations, such as

$$a_1x + b_1y = c_1 \tag{1}$$

and

$$a_2x + b_2y = c_2, \tag{2}$$

the first method can be applied.

The variable x may be eliminated by multiplying equation (1) by $-a_2$, the negative of the coefficient of x in (2), and equation (2) by a, the coefficient of x in (1). Thus,

$$\begin{aligned} -a_1a_2x - a_2b_1y &= -a_2c_1 \\ a_1a_2x + a_1b_2y &= a_1c_2 \end{aligned}.$$

Adding,

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1$$

or

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Similarly, by multiplying equation (1) by b_2 and equation (2) by $-b_1$ and adding the equations, y is eliminated. Thus:

$$\begin{array}{rrr} a_1b_2x + b_1b_2y = & c_1b_2 & (1) \\ -a_2b_1x - b_1b_2y = & -c_2b_1 & (2) \\ \hline (a_1b_2 - a_2b_1)x = & c_1b_2 - c_2b_1 & \end{array}$$

or

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}.$$

Note that the denominator in each of the two formulas giving the values of x and y is the same. This expression is so important and occurs so frequently in the discussion of simultaneous linear equations that a new symbol is introduced to represent it. Thus:

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

The symbol on the right is called a *determinant.* Note that this particular determinant is formed by the square array of the coefficients of x and y in equations (1) and (2), bounded by two vertical lines. For brevity, the determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

may be written simply $|a\ b|$. That is,

$$|\ a\ b\ | = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

The order of the determinant is fixed by the number of horizontal rows and vertical columns. A determinant having n rows and n columns is called a *determinant of the nth order.* The particular determinant just given is, therefore, of the *second order* because it contains two rows and two columns. The

symbols a_1, a_2, b_1, and b_2 are called *elements* of the determinant. The elements a_1 and b_2 form its *principal diagonal.*

Using this notation of determinants, the results obtained for x and y may be written:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{|c\ b|}{|a\ b|},$$

and

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{|a\ c|}{|a\ b|}.$$

Observe that the determinant which forms the numerator of the expression for x is formed from the denominator or $|a\ b|$ by replacing the column which contains the coefficients of x (namely, a_1 and a_2) by the column of constant terms, c_1 and c_2.

Moreover, the numerator of the expression for y is the determinant formed by replacing the column which contains the coefficients of y (namely, b_1 and b_2) by the column of constant terms, c_1 and c_2.

Consider the pair of linear equations obtained in the solution of the problem proposed at the beginning of this chapter:

$$-50x + y = 100 \tag{1}$$

and

$$-112x + y = 9. \tag{2}$$

Then:

$$|a\ b| = \begin{vmatrix} -50 & 1 \\ -112 & 1 \end{vmatrix} = (-50)(1) - (-112)(1) = -50 + 112 = 62,$$

$$|c\ b| = \begin{vmatrix} 100 & 1 \\ 9 & 1 \end{vmatrix} = (100)(1) - (9)(1) = 100 - 9 = 91,$$

and

$$|a\ c| = \begin{vmatrix} -50 & 100 \\ -112 & 9 \end{vmatrix} = (-50)(9) - (-112)(100) = -450 + 11{,}200 = 10{,}750.$$

Therefore,

$$x = \frac{\begin{vmatrix} c & b \end{vmatrix}}{\begin{vmatrix} a & b \end{vmatrix}} = \frac{91}{62} = 1.468$$

and

$$y = \frac{\begin{vmatrix} a & c \end{vmatrix}}{\begin{vmatrix} a & b \end{vmatrix}} = \frac{10{,}750}{62} = 173.4.$$

ANALYSIS OF EQUATIONS

The method of determinants, as just illustrated, may seem cumbersome and laborious by comparison with the other four methods. But, as already indicated, it is an effective tool in the analysis of these equations and a valuable labor- and time-saver when applied to equations involving three or more variables.

Consider again the simultaneous equations:

$$a_1x + b_1y = c_1 \quad (1)$$
$$a_2x + b_2y = c_2 \quad (2)$$

and their solution:

$$x = \frac{\begin{vmatrix} c & b \end{vmatrix}}{\begin{vmatrix} a & b \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & c \end{vmatrix}}{\begin{vmatrix} a & b \end{vmatrix}}.$$

If equations (1) and (2) are solved for y, they may be written as follows (supposing that b_1 and b_2 are both different from zero):

$$y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1}, \quad (1)$$

$$y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2}. \quad (2)$$

Comparing these equations with the type of equation

$$y = mx - b,$$

and letting m_1 and m_2 represent the slopes of (1) and (2), respectively, it is readily seen that

$$m_1 = -\frac{a_1}{b_1} \text{ and } m_2 = -\frac{a_2}{b_2}.$$

Now assume that in the solution of the simultaneous equations (1) and (2), the $\begin{vmatrix} a & b \end{vmatrix}$ is not equal to zero. That is,

$$a_1b_2 - a_2b_1 \neq 0.$$

Then obviously,

$$a_1b_2 \neq a_2b_1$$

and

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}.$$

So that, multiplying both sides by -1,

$$-\frac{a_1}{b_1} \neq -\frac{a_2}{b_2},$$

or

$$m_1 \neq m_2.$$

That is, if $| a\ b |$ is not equal to zero, the lines *have different slopes and, therefore, must intersect.* Two distinct straight lines can intersect in only one point. Therefore, equations (1) and (2) have one and only one solution. It is to be concluded, then, that *a pair of linear equations in two variables has a unique solution if the determinant* $| \boldsymbol{a\ b} |$ *is not equal to zero.*

Suppose, now, that $| a\ b | = 0$. In this case, the formal solution

$$x = \frac{| c\ b |}{| a\ b |} \quad \text{and} \quad y = \frac{| a\ c |}{| a\ b |}$$

becomes meaningless because division by zero is impossible. However, if $| a\ b | = 0$,

$$a_1b_2 - a_2b_1 = 0.$$

This can be possible (supposing that both b_1 and b_2 are different from zero) only if

$$a_1b_2 = a_2b_1$$

or

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

As was previously pointed out, the slope of equation (1), or m_1, is $-\frac{a_1}{b_1}$, and the slope of equation (2), or m_2, is $-\frac{a_2}{b_2}$.

Hence,

$$m_1 = m_2.$$

That is, *if the determinant* $| \boldsymbol{a\ b} | = 0$, *the slopes of the two lines are equal.*

Now, if the slopes of the two lines are equal, either the lines are parallel or they coincide.

Let us assume, then, that besides having $| a\ b | = 0$, the determinant $| c\ b | = 0$ also.

If $| c\ b | = 0$,

$$c_1b_2 - c_2b_1 = 0,$$
$$c_1b_2 = c_2b_1,$$

and

$$\frac{c_1}{b_1} = \frac{c_2}{b_2}.$$

So that

$$-\frac{c_1}{b_1} = -\frac{c_2}{b_2}.$$

Comparing equations (1) and (2) with the type equation of a straight line (page 198), it is easily seen that the graph of equation (1) crosses the Y-axis $-\frac{c_1}{b_1}$ units above the X-axis, for the constant term b of the type equation is $-\frac{c_1}{b_1}$ in equation (1). Similarly, equation (2) crosses the Y-axis at $-\frac{c_2}{b_2}$ units from the X-axis.

The above equality shows, then, that the two lines cross the Y-axis at the same point. The fact that $| a\ b | = 0$ shows that they have the same slope. But two lines that have a point in common and have the same slope must have all points in common. Therefore,

If a pair of linear equations in two variables has $| a\ b | = 0$ *and* $| c\ b | = 0$, *the two lines coincide and every solution of either is a solution of the other. That is, the equations are equivalent.*

Finally, if $| a\ b | = 0$ and $| c\ b | \neq 0$, it can be concluded from the preceding discussion that the two lines have the same slope but cross the Y-axis at different points. It follows, then, that the two lines are parallel and have no point in common. So that,

If a pair of linear equations in two variables has $| a\ b | = 0$ *and* $| c\ b | \neq 0$, *the lines are parallel and have no common solution.* For example:

I. The pair of linear equations:

$$2x + 3y = 5$$
$$3x - 5y = 2$$

has

$$|\,a\ b\,| = \begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix} = (2)(-5) - (3)(3) = -10 - 9 = -19.$$

Therefore, the equations have a unique solution, which can be found as follows:

$$|\,c\ b\,| = \begin{vmatrix} 5 & 3 \\ 2 & -5 \end{vmatrix} = -25 - 6 = -31,$$

$$|\,a\ c\,| = \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} = 4 - 15 = -11.$$

So that

$$x = \frac{|\,c\ b\,|}{|\,a\ b\,|} = \frac{-31}{-19} = 1\tfrac{12}{19} \text{ and } y = \frac{|\,a\ c\,|}{|\,a\ b\,|} = \frac{-11}{-19} = \tfrac{11}{19}.$$

II. The pair of linear equations:

$$3x + 4y = 8$$
$$6x + 8y = 16$$

has

$$|\,a\ b\,| = \begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} = 24 - 24 = 0$$

and

$$|\,c\ b\,| = \begin{vmatrix} 8 & 4 \\ 16 & 8 \end{vmatrix} = 64 - 64 = 0.$$

Therefore, the lines coincide and the equations are equivalent.

III. Finally, the pair of linear equations

$$2x + 7y = 3$$
$$4x + 14y = 2$$

has

$$|\,a\ b\,| = \begin{vmatrix} 2 & 7 \\ 4 & 14 \end{vmatrix} = 28 - 28 = 0,$$

but

$$| c \quad b | = \begin{vmatrix} 3 & 7 \\ 2 & 14 \end{vmatrix} = 42 - 14 = 28.$$

Therefore, the lines are parallel and have no common solution.

THREE UNKNOWNS

The solution of problems involving three unknowns can be accomplished by applying the methods already explained for a pair of simultaneous equations. Consider the following problem:

In the successive years 1936, 1937, and 1938, the Marsh Motors Co. manufactured automobiles with profits as follows: In the year 1936, it manufactured 600 two-door sedans, 200 four-door sedans, and 300 master sedans with a profit of $101,000; in 1937, 800 two-door sedans, 500 four-door sedans, and 300 master sedans with a profit of $142,500; in 1938, 500 two-door sedans, 300 four-door sedans, and 200 master sedans with a profit of $89,500. What were the average profits per car for each type of automobile?

Let x represent the average profit per two-door sedan, y the average profit per four-door sedan, and z the average profit per master sedan. Then,

$$\begin{aligned} 600x + 200y + 300z &= 101{,}000, && (1)\\ 800x + 500y + 300z &= 142{,}500, && (2)\\ 500x + 300y + 200z &= 89{,}500. && (3) \end{aligned}$$

Or, simplifying,

$$\begin{aligned} 6x + 2y + 3z &= 1{,}010, && (1)\\ 8x + 5y + 3z &= 1{,}425, && (2)\\ 5x + 3y + 2z &= 895. && (3) \end{aligned}$$

These equations may be solved by first eliminating one of the unknown quantities from one pair of equations. Thus, choosing the pair of equations (1) and (2) and subtracting (1) from (2), the variable z is made to vanish:

$$\begin{aligned} 8x + 5y + 3z &= 1{,}425 && (2)\\ 6x + 2y + 3z &= 1{,}010 && (1)\\ \hline 2x + 3y &= 415 && (4) \end{aligned}$$

The same unknown quantity is then eliminated from any other pair of the given equations. Choosing the pair (1) and (3), the z's are eliminated by multiplying (1) by -2 and (3) by 3 and adding:

$$\begin{aligned} -12x - 4y - 6z &= -2{,}020 \quad (1) \\ 15x + 9y + 6z &= 2{,}685 \quad (3) \\ \hline 3x + 5y &= 665 \quad (5) \end{aligned}$$

The result is two equations with two unknown quantities. These two equations, (4) and (5), are then solved as a pair of simultaneous equations in two variables. Multiplying (4) by -3 and (5) by 2 and adding, the x's are eliminated.

$$\begin{aligned} -6x - 9y &= -1{,}245 \quad (4) \\ 6x + 10y &= 1{,}330 \quad (5) \\ \hline y &= 85 \end{aligned}$$

Substituting the value of y in (4) or (5), the value of x is obtained. Choosing (4):

$$\begin{aligned} 2x + 3(85) &= 415 \\ 2x &= 415 - 255 \\ 2x &= 160 \\ x &= 80. \end{aligned}$$

The values obtained for the two unknowns x and y may now be substituted in any one of the three original equations to find the value of the third unknown. Thus, choosing (3):

$$\begin{aligned} 5(80) + 3(85) + 2z &= 895 \\ 2z &= 895 - 655 \\ 2z &= 240 \\ z &= 120. \end{aligned}$$

Problems

1. A savings bank has \$500,000 invested, partly in mortgages averaging 5 per cent interest, and partly in bonds averaging 3 per cent. Income from the two sources aggregates \$18,700 per year. How much money is in each of these two kinds of investment?

2. The perimeter of a certain rectangle is 12 feet less than 3 times the base. If the base is 5 times the altitude, what are the lengths of the base and altitude?

3. The receipts from the sale of 500 tickets for an art lecture were \$335.50. Adults paid 75 cents and children 25 cents each. How many tickets of each kind were sold?

4. A grocery chain wishes to make 100,000 pounds of coffee worth 32 cents a pound by mixing two other grades, one of which is worth 35 cents a pound and the other 20 cents a pound. How much of each must be used?

5. A messenger went to the bank to deposit some bills, some of them being \$1 bills and the rest \$5 bills. If there were 320 bills in all and their combined value was \$524, how many of each were there?

6. Two ports are 4,080 miles apart. To travel the distance between them by passenger steamer takes 136 hours less time than by freight steamer. But if the freight steamer has a start of 1,632 miles, each arrives at the same time. What is the speed of each steamer?

7. A man can row 18 miles down a stream and back in 8 hours. To row back takes him three times as long as to row down. What is the speed of the current, and of the rower in still water?

8. Two passengers in an airplane together have 154 lbs. of baggage. One pays \$1.90 and the other \$3.50 for excess above the weight allowed free. Had the baggage belonged to one person, he would have had to pay \$10.40. How much baggage is allowed free to a single passenger?

9. One vat contains 36 gallons of nitric acid and 24 gallons of hydrochloric acid; another vat contains 8 gallons of nitric and 24 gallons of hydrochloric acid. How many gallons must be taken from each vat so that when mixed in a third vat there may be 42 gallons, half nitric and half hydrochloric?

10. A factory has at its disposal three machines, A, B, and C. A contract may be filled by A and B together in 10 days, by A and C in 9 days, and by B and C in 8 days. How long will it take each machine alone to do the work?

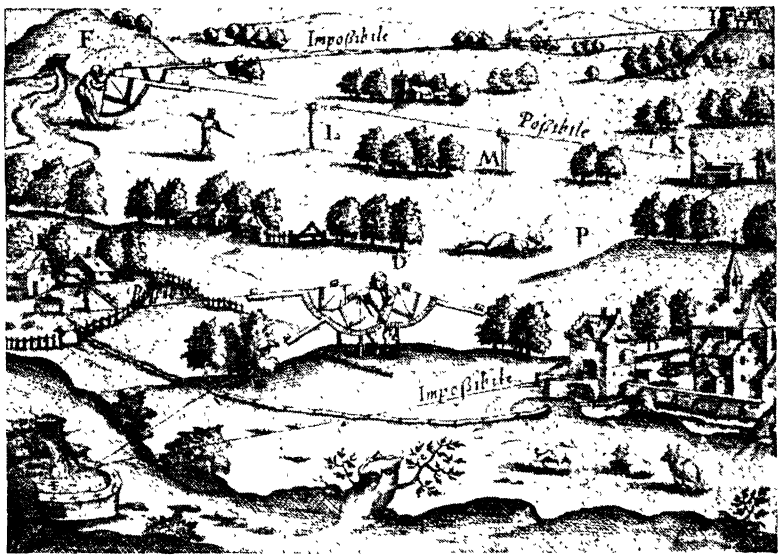
11. A purse contains 19 coins in quarters, dimes, and nickels, amounting to \$2.25. After spending 1 nickel for carfare, there are two more nickels than quarters. How many coins of each were there in the purse?

12. A swimming pool may be filled by each of three pipes. The first and second, running together, can fill it in 14 hours, the second and third can fill it in 12 hours, and the first and third in 10 hours. How long will it take each pipe, running alone, to fill the pool?

16

INTRODUCTION TO AREAS; QUADRATICS

Ancient Babylon and Egypt first recorded the periodicity of the seasons, invented units and rules of measurement, and formulated simple arithmetic. Improvements came slowly. Little by little, progress has been made in establishing conventions of measurement acceptable and accepted internationally,



Courtesy of Librairie Ancienne, Leo S. Olschki, Florence

Woodcut from Comelius de Judaeis, *De Quadrante Geometrica*, 1594.

although much still remains to be done. It was not until 1801 that France established the metric system which is now obligatory in most American and many European countries. But not only wide agreement as to the acceptance of conventional standards is necessary; equally important is the need of finding some common appeal to logical human thinking whereby sound-

minded men, reasoning along the same lines, would arrive at the same logical conclusions. The chaos into which this already confused world would otherwise be thrown can readily be imagined.

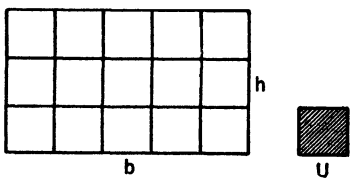
UNIT OF AREA

The area of a closed plane figure is the amount of surface enclosed within its boundaries. To measure the amount of surface, a unit of area must be employed. This unit of area is ordinarily a square, each side of which contains one linear unit. Thus, if the unit of length is the inch, the corresponding unit of area is a square each side of which is one inch long. Such a square is called a *square inch*. Similarly, if the unit of length is a foot, then the corresponding unit of area is a square foot.

American and metric measures of area are given in the following tables:

SQUARE MEASURE		SURVEYORS' SQUARE MEASURE	
144 sq. inches	= 1 sq. foot	625 sq. links	= 1 sq. rod
9 sq. feet	= 1 sq. yard	16 sq. rods	= 1 sq. chain
$30\frac{1}{4}$ sq. yards	= 1 sq. rod	10 sq. chains	= 1 acre
160 sq. rods	= 1 acre	640 acres	= 1 sq. mile
640 acres	= 1 sq. mile, 1 section	1 sq. mile	= 1 section
36 sections	= 1 township	36 sections	= 1 township

AREA OF A RECTANGLE



If a rectangle, such as the one in the figure, is 5 units long and 3 units wide, its area is easily found. The small square U is the unit of area. It is evident that there are 5 unit squares in each row and that there are 3 rows; hence, there are $5 \times 3 = 15$ unit squares in the whole rectangle. In general, if the sides have a common unit of measure, that is, are commensurable, then

Fact 52: The area of a rectangle is the product of its base and altitude.

For, if the base contains b units and the height h units, then there are b unit squares in each row and there are h rows;

AMERICAN EQUIVALENTS OF THE METRIC SYSTEM

1 sq. millimeter	=	0.00155 sq. in.
1 sq. centimeter	=	0.15499 sq. in.
1 sq. decimeter	=	15.4999 sq. in.
1 sq. meter (centare)	=	10.76387 sq. ft. or 0.0395 sq. rod
1 sq. decameter (are)	=	0.02471 acre
1 sq. hectometer (hectare)	=	2.471 acres
1 sq. kilometer	=	247.1 acres 0.3861 sq. mile

METRIC EQUIVALENTS OF THE AMERICAN SYSTEM

1 sq. inch	=	645.16 sq. millimeters 6.4516 sq. centimeters
1 sq. foot or 144 sq. in.	=	929.034 sq. centimeters or 0.0929 sq. meter
1 sq. yard or 9 sq. feet	=	8,361.31 sq. centimeters or 0.83613 sq. meter
272.25 sq. ft. or 1 sq. rod or 0.00625 acre	=	25.293 sq. meters (centares)
4,840 sq. yards or 160 sq. rods or 1 acre	=	4,046.873 sq. meters or (centares) 0.4047 hectare
640 acres or 1 sq. mile	=	258.999 hectares or 2.5899 sq. kilometers

hence, there are $b \times h$ unit squares in the whole rectangle. Or

$$A = b \cdot h.$$

If the sides of the rectangle have no common unit of measure, that is, are incommensurable, it is easily seen from the definition of irrational numbers given on page 5, that the length of the base and altitude can be approximated to any desired degree of accuracy by choosing, in each case, an appropriate rational number. For, suppose the lengths of the base and altitude to be each an irrational number, such as $\sqrt{2}$, this

length can be approximated to any degree of accuracy desired by simply applying the well-known method of extracting the square root of a number (see page 693). Performing this operation, the successive approximations 1.4, 1.41, 1.414, 1.4142, and so forth, are obtained. So that, as the operation is continued, the number varies, getting closer and closer to the actual value of the $\sqrt{2}$. It is then said that the variable decimal 1.4, 1.41, 1.414, 1.4142, and so forth, approaches the $\sqrt{2}$ as its limit. In fact, for practical purposes, we write $\sqrt{2} = 1.41421\ldots$.

Moreover, if the base and altitude are incommensurable, there is no unit which will give simultaneously a whole number of squares in the row along the base and a whole number of rows along the height. However, the part left over on each line, no matter what unit is used, will be less than the unit length. So that, the smaller the unit square used, the less is the discrepancy in treating the base as though it contained a whole number of squares, and the height a whole number of rows. Therefore, the more nearly will the product of these numbers equal the area of the rectangle. Hence, Fact 52 holds good whether the quantities involved are commensurable or incommensurable.

Since a square is a rectangle whose sides are equal, it follows that:

Fact 53: The area of a square is equal to the square of one of its sides.

QUADRATIC EQUATIONS

Hence, if the side of a square is denoted by x, then its area, A, is equal to x^2. That is,

$$A = x^2.$$

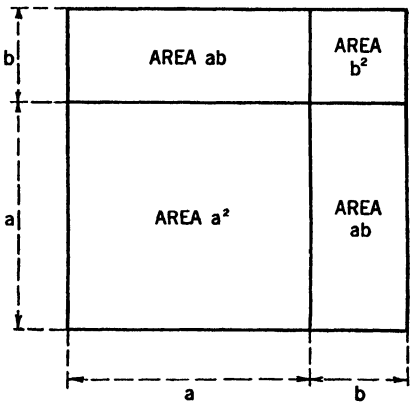
This equation is of the second degree.

An equation in one variable, containing a term of the second degree in the variable, but none of a higher degree, is called a *quadratic equation.* The word *quadratic* means "square." Thus, the equations $x^2 + 3x - 5 = 0$, $2x - 6 = 5x^2$, and $16x^2 - 25 = 0$ are examples of quadratic equations.

Any quadratic equation can be expressed in the form:

$$ax^2 + bx + c = 0,$$

where a, b, and c represent any numbers and a is not equal to zero. Thus, $5x^2 + 8x - 3 = 3x^2 + 2x - 4$ reduces to $2x^2 + 6x + 1 = 0$ by subtracting $3x^2 + 2x - 4$ from both sides of the equation. In the reduced equation $2x^2 + 6x + 1 = 0$, the coefficient of x^2 is 2, so that $a = 2$. The coefficient of x is 6; hence, $b = 6$. Finally, $c = 1$. The coefficient of x^2, namely, a, cannot take the value zero. For, if $a = 0$, the term of the second degree vanishes and, in that case, the equation is no longer a quadratic.



If the sides of a square are expressed as the sum of two numbers, such as $a + b$, the area of the square is then:

$$(a + b)(a + b) = (a + b)^2 = a^2 + 2ba + b^2.$$

The truth of this statement is made evident in the accompanying figure.

The square whose sides are $a + b$ is made up of one square whose sides are a, another square whose sides are b, and two equal rectangles whose sides are a and b. The sum of the areas of these figures is the area of the square whose sides are $(a + b)$. Since the area of the square whose sides are $a + b$ is $(a + b)(a + b)$ or $(a + b)^2$, then $(a + b)^2 = a^2 + 2ba + b^2$.

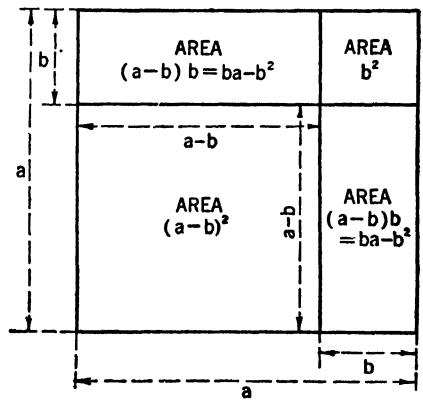


Fig. 93.

Similarly, in Figure 93, the area of the square whose sides are $(a - b)$ is equal to the area of the square whose sides are a, minus the area of the square whose sides are b and the area of the two equal rectangles whose sides are $(a - b)$ and b. That is,

$(a - b)^2 = a^2 - [b^2 + 2(ba - b^2)]$, or
$(a - b)^2 = a^2 - (b^2 + 2ba - 2b^2) = a^2 - (2ba - b^2)$, so that
$(a - b)^2 = a^2 - 2ba + b^2$.

PERFECT SQUARES

The simple ideas of measurement just discussed can be used, as will be seen presently, to solve a great multitude of practical problems wherein numbers only may be involved and not areas. But first we must generalize the two equations

$$(a + b)^2 = a^2 + 2ba + b^2 \quad \text{and}$$
$$(a - b)^2 = a^2 - 2ba + b^2.$$

The first letters of the alphabet, namely, a, b, c, d, and so forth, are usually used to represent constants, that is, quantities whose values are fixed in any investigation. Constants to which numerical values may be assigned, but that retain these assigned values throughout the investigation, are *arbitrary constants*. The last letters, v, w, x, y, z, are used to denote variables. If a is considered the variable in the above two equations, and the symbol x is substituted for it, we have,

$$(x + b)^2 = x^2 + 2bx + b^2 \quad \text{and} \qquad (1)$$
$$(x - b)^2 = x^2 - 2bx + b^2. \qquad (2)$$

The arbitrary constant b can be any number. Assume that it is 11. Then we have:

$$(x + 11)^2 = x^2 + 2(11)x + (11)^2 = x^2 + 22x + 121 \quad \text{and}$$
$$(x - 11)^2 = x^2 - 2(11)x + (11)^2 = x^2 - 22x + 121.$$

Whenever a quadratic equation is in either one of the two forms (1) or (2), whatever be the value of b, it is called a *perfect square*.

COMPLETING THE SQUARE

The coefficients of the middle term in equations (1) and (2), that is, the coefficients of the terms which contain the first power of x, are $2b$ and $-2b$, respectively. The last, or constant, term of the perfect square is in both cases equal to one-half the coefficient of the middle term, raised to the second power. That is, $\left(\frac{2b}{2}\right)^2 = b^2$ and $\left(\frac{-2b}{2}\right)^2 = b^2$.

Suppose we had the expression $x^2 + 22x$ and wanted to make

a perfect square by adding a constant. The missing constant term must be equal to one-half the coefficient of x, raised to the second power. Hence, simply divide 22 by 2 and square the result, as follows: $\left(\frac{22}{2}\right)^2 = (11)^2 = 121$. So that, $x^2 + 22x + 121$ is a perfect square.

Similarly, the expression $x^2 - 8x$ can be made into a perfect square by taking one-half of -8, namely, -4, squaring it, obtaining $+16$, and adding it to the expression. Thus, $x^2 - 8x + 16$ is a perfect square and, therefore, $(x - 4)^2 = x^2 - 8x + 16$.

As will presently be seen, this process of "completing the square" is one of the methods used in ascertaining the value of the unknown in quadratic equations.

SOLUTION OF A QUADRATIC

To *solve* a quadratic equation is to find the values of the variable which, when substituted in the given equation, make the equation true. That is, the solutions of the quadratic equation $2x^2 + x - 15 = 0$ are -3 and $2\frac{1}{2}$, because if either of these two values is substituted in place of the variable x in the equation, the resulting value of the equation is zero. Thus, $2(-3)^2 + (-3) - 15 = 0$. These values are then said to *satisfy* the equation.

The values obtained by the solution of an equation in one unknown are so important that a special name is given to them. These values are called *roots* of the equation. In other words, the roots of an equation in one variable are those numbers which, when substituted for the variable, satisfy the equation. Thus, since the numbers -3 and $2\frac{1}{2}$ satisfy the above equation, they are its roots.

Quadratic equations can be solved by any one of four methods:

(1) Completing the square.
(2) Factoring.
(3) Using the formula for the quadratic equation.
(4) Graphs.

Since the solution of problems is an art as well as a science, it is important not only to understand these four methods, but also to know which one in a given situation can be used with the least expenditure of time and energy. The first two meth-

ods are explained in this chapter, and the second two in the following chapter.

SOLUTION BY COMPLETING THE SQUARE

The solution of a quadratic equation by the first method, namely, by completing the square, is illustrated and explained as follows:

A broker bought a certain number of shares of stock for \$1,960. When the price had advanced \$10 per share, he sold all but one share, receiving \$2,910. How many shares did he buy?

Let n = number of shares bought.

The price of the shares was \$1,960. Since n shares were bought, then $\frac{1,960}{n}$ = price paid for one share.

The broker sold after the price had advanced \$10 per share, so that, $\frac{1,960}{n} + 10$ = selling price of one share.

He sold all but 1 share; hence, $n - 1$ = number of shares sold.

Since he sold $(n - 1)$ shares for $\left(\frac{1,960}{n} + 10\right)$ per share, then $(n - 1)\left(\frac{1,960}{n} + 10\right)$ = total selling price for the $(n - 1)$ shares.

He received \$2,910 for the $(n - 1)$ shares; therefore,

$$(n - 1)\left(\frac{1,960}{n} + 10\right) = 2,910.$$

Performing the indicated multiplication on the left side gives:

$$\frac{1,960(n - 1)}{n} + 10(n - 1) = 2,910,$$

or

$$\frac{1,960n - 1,960}{n} + 10n - 10 = 2,910.$$

Transposing -10 gives:

$$\frac{1,960n - 1,960}{n} + 10n = 2,920.$$

Multiplying both sides by n gives:

$$1{,}960n - 1{,}960 + 10n^2 = 2{,}920n.$$

Transposing $2{,}920n$ and arranging terms, we have:

$$10n^2 - 960n - 1{,}960 = 0.$$

Dividing both sides of the equation by 10 gives:

$$n^2 - 96n - 196 = 0.$$

Since this quadratic equation is not a perfect square, transpose the constant term and rewrite the equation as follows:

$$n^2 - 96n = 196.$$

To make the left-hand side of the equation a perfect square, a number which is one-half of -96 (the coefficient of the middle term) squared must be added. Now, $-96 \div 2 = -48$, and $(-48)^2 = 2{,}304$. Since adding the same quantity to both sides of the equation does not change the equality, we may write:

$$n^2 - 96n + 2{,}304 = 196 + 2{,}304 = 2{,}500.$$

Since the left-hand side of the above equation, namely, $n^2 - 96n + 2{,}304$, is a perfect square, it is equal to $(n - 48)^2$. [See (2), page 258.] Hence, the equation in its new form may now be written:

$$(n - 48)^2 = 2{,}500.$$

Taking the square root of both sides of the equation again leaves the equality unchanged; hence,

$$n - 48 = \pm 50.$$

Choosing the positive value of the square root of 2,500, that is, 50, a solution is obtained thus:

$$n - 48 = 50;$$

then,

$$n = 50 + 48$$

and

$$n = 98.$$

Then choosing the negative value of the square root of 2,500, namely, -50, another solution is obtained as follows:

$$n - 48 = -50$$

or

$$n = -50 + 48;$$

that is,

$$n = -2.$$

Therefore, the roots of the quadratic equation $n^2 - 96n - 196 = 0$ are 98 and -2, because if these values are substituted in the given equation in place of the variable n, the equation is satisfied.

It is evident that the solution is 98 shares, because the value -2 has no meaning as a solution of this problem.

Problems involving quadratics always have two answers, but frequently the conditions of a practical problem exclude one of the answers as inapplicable. The two values of the unknown should always be given careful consideration before deciding to eliminate one of them as irrelevant.

So far, this method of solution has been applied to equations in which the coefficient of the term containing the second power has been 1. A consideration of the following problem will show the ease with which this method can be applied in other cases.

To ascertain the roots of the equation $2x^2 - x - 3 = 0$, first divide both sides of the equation by 2, obtaining:

$$x^2 - \tfrac{1}{2}x - \tfrac{3}{2} = 0.$$

Since the equation now has 1 for the coefficient of x^2, we proceed as before.

Transposing $-\frac{3}{2}$,

$$x^2 - \tfrac{1}{2}x = \tfrac{3}{2}.$$

Taking one-half of $-\frac{1}{2}$ gives $-\frac{1}{4}$. The square of $-\frac{1}{4}$ is $\frac{1}{16}$. So that

$$x^2 - \tfrac{1}{2}x + \tfrac{1}{16} = \tfrac{3}{2} + \tfrac{1}{16} = \tfrac{25}{16}.$$

Since the left-hand side is now a perfect square, the equation can be written:

$$(x - \tfrac{1}{4})^2 = \tfrac{25}{16},$$

or

$$x - \tfrac{1}{4} = \pm\tfrac{5}{4}.$$

Transposing $-\frac{1}{4}$,

$$x = \pm\tfrac{5}{4} + \tfrac{1}{4}.$$

Choosing the positive value first,

$$x = \tfrac{5}{4} + \tfrac{1}{4} = \tfrac{6}{4} = 1\tfrac{1}{2}.$$

Then choosing the negative value,

$$x = -\tfrac{5}{4} + \tfrac{1}{4} = -\tfrac{4}{4} = -1.$$

Therefore, the roots of the equation $2x^2 - x - 3 = 0$ are $1\frac{1}{2}$ and -1.

SOLUTION BY FACTORING

The second method of solving a quadratic equation, namely, the method of *factoring*, should be used only in the case where the factors of the equation can be easily seen. It is, therefore, very useful to remember the already familiar identities:

$$(x + b)^2 = x^2 + 2bx + b^2. \qquad \text{(Type A)}$$
$$(x - b)^2 = x^2 - 2bx + b^2. \qquad \text{(Type B)}$$
$$(x + b)(x - b) = x^2 - b^2. \qquad \text{(Type C)}$$

If the quadratic equation is of Type A or Type B, it is easily recognized by the fact, already stated, that the constant term is the square of one-half of the coefficient of the middle term.

Thus, $x^2 + 12x + 36 = 0$ is a perfect square, since $36 = (\frac{12}{2})^2$. Therefore, $x^2 + 12x + 36 = (x + 6)(x + 6) = 0$.

The product of two numbers can equal zero only if at least one of the numbers is zero. That is, $a \times 4 = 0$ only if a is zero; and if $4 \times b = 0$, then b must be zero. In other words, if the product of any two numbers, a and b, equals 0, then either a is zero or b must be zero. Now, since $x^2 + 12x + 36 = (x + 6)(x + 6) = 0$, and the two factors are equal, then obviously $x + 6 = 0$, and therefore $x = -6$.

In this case, since the two factors are equal, we get the same answer whether we equate the first factor to zero or whether the second factor is taken equal to zero. The quadratic equation is still said to have two roots, but the two roots are identical and equal to -6.

The same reasoning evidently applies to Type B. Thus, the equation $x^2 - 14x + 49 = 0$ is a perfect square, so that

$x^2 - 14x + 49 = (x - 7)(x - 7) = 0$. Therefore, the roots of the equation are identical and equal to $+7$.

If the equation consists of the difference of two squares, that is, is of Type C, then it may be factored and solved. For example: $4x^2 - 9 = 0$ is of Type C; hence, $4x^2 - 9 = (2x - 3)(2x + 3) = 0$. Now, $(2x - 3)(2x + 3)$ can equal zero only if $2x - 3 = 0$ or if $2x + 3 = 0$. If $2x - 3 = 0$, then $2x = 3$ and $x = \frac{3}{2}$, or $1\frac{1}{2}$; and if $2x + 3 = 0$, then $2x = -3$ and $x = -\frac{3}{2}$, or $-1\frac{1}{2}$. Therefore, the roots of the equation $4x^2 - 9 = 0$ are $1\frac{1}{2}$ and $-1\frac{1}{2}$.

By observing the product of other types of factors, solution by factoring may be easily accomplished. The product $(x + a)(x + b) = x^2 + (a + b)x + a \cdot b$. Thus, if an expression

$$x^2 + mx + n = 0 \qquad \text{(Type D)}$$

is such that the coefficient of x is equal to the sum of two numbers a and b, that is, $m = a + b$, and the constant term is the product of the same two numbers, that is, if $n = a \cdot b$, the factors of the equation are $(x + a)$ and $(x + b)$. In other words, in order to factor an equation of Type D, two numbers whose sum is the coefficient of the middle term, that is, m, and whose product is the constant term or n, must be found.

For example: the equation $x^2 + 9x - 36 = 0$ is of Type D. The numbers whose product is -36 and whose sum is $+9$ are $+12$ and -3. Therefore, the factors are $(x + 12)(x - 3) = 0$. Hence, the roots are -12 and $+3$, obtained by equating $x + 12$ to zero and then equating $x - 3$ to zero.

If the coefficient of x^2 is the product of two numbers a and c, and the constant term is the product of another pair of numbers b and d, the product $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$. Thus, if an expression,

$$lx^2 + mx + n = 0 \qquad \text{(Type E)}$$

is such that the coefficient of x^2 is equal to the product of two numbers a and c, the constant term is also equal to the product of another pair of numbers b and d, that is, $n = bd$, and the coefficient of the middle term, m, is the sum of the products $ad + bc$, then the factors of the equation are $ax + b$ and $cx + d$.

Therefore, to factor an equation of Type E, a pair of numbers whose product is l, and another pair whose product is n,

must be found. These pairs must then be arranged so that the sum of their products is equal to the coefficient of the middle term. Thus, the equation $6x^2 - x - 12 = 0$ is of Type E. The factors of the coefficient of x^2, or 6, are: 1 and 6, -1 and -6, 2 and 3, and -2 and -3. The factors of the constant term -12 are: -1 and 12, 1 and -12, -2 and 6, 6 and -2, 3 and -4, and -3 and 4. If we are to succeed in factoring by this method, the arrangement of these numbers must be such that $ad + bc$ will be equal to -1, the coefficient of the middle term. This arrangement might be arrived at by using the following method:

Note that if the factors are written:

$$\begin{array}{ccc} ax + b & & \\ & \times & \\ cx + d & & \end{array}$$

the sum of the cross products $(ax \cdot d) + (cx \cdot b) = adx + bcx = (ad + bc)x$, which is the middle term.

Choosing the pairs 2 and 3 and -3 and 4, write:

$$\begin{array}{ccc} 2x - 3 & & \\ & \times & \\ 3x + 4 & & \end{array}$$

the sum of the cross products $[(2x)(4)] + [(3x)(-3)] = 8x - 9x = -x$, which is the middle term. Therefore, the factors of the equation $6x^2 - x - 12$ are $(2x - 3)$ and $(3x + 4)$. So that $6x^2 - x - 12 = (2x - 3)(3x + 4) = 0$. Letting $2x - 3 = 0$ gives $2x = 3$ and $x = \frac{3}{2}$, or $1\frac{1}{2}$. Then, letting $3x + 4 = 0$, $3x = -4$ and $x = -\frac{4}{3}$ or $-1\frac{1}{3}$. Hence, the roots of the given equation are $1\frac{1}{2}$ and $-1\frac{1}{3}$.

Obviously, the ability to choose the correct arrangement requires practice. Moreover, since not every expression of Type E can be so factored that the pairs of numbers sought are whole numbers, equations should be solved by this method only when the arrangement of the numbers is promptly and easily seen. In other cases, the solution should be performed by means of the general formula.

It is very important, therefore, to derive a general formula by means of which any quadratic equation can be solved. This is the third method of solving a quadratic; it will be discussed in the following chapter.

EQUATIONS IN THE QUADRATIC FORM

An equation not of the second degree but containing at most only two powers of a variable may sometimes be expressed as a quadratic equation by a simple substitution. For example: if in the equation $x^4 - 13x^2 + 36 = 0$, we let $y = x^2$; then $y^2 = x^4$, and the equation may be written $y^2 - 13y + 36 = 0$. The latter equation is in quadratic form and may be solved as a quadratic equation. Thus, $y^2 - 13y + 36 = (y - 9)(y - 4) = 0$, so that $y = 9$ and $y = 4$. Since $y = x^2$, $x^2 = 9$ and $x^2 = 4$. Therefore, $x = \pm 3$ and $x = \pm 2$.*

In general, any equation which can be expressed in the form $ax^{2n} + bx^n + c = 0$, where n is any real number, may be reduced to a quadratic equation by substituting y for x^n, and the resulting equation may then be solved.

Thus, the equation $x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0$ is easily seen to be of the form $ax^{2n} + bx^n + c = 0$. If we let $y = x^{\frac{1}{4}}$, then $y^2 = x^{\frac{1}{2}}$, and the equation may be written $y^2 - y - 6 = 0$, which is in quadratic form and may be solved as a quadratic equation. Thus, $y^2 - y - 6 = (y - 3)(y + 2) = 0$, so that $y = 3$ and $y = -2$. Since $y = x^{\frac{1}{4}}$, $x^{\frac{1}{4}} = 3$ and $x^{\frac{1}{4}} = -2$. Therefore, $x = 81$ and $x = 16$.

Problems

1. The total area of two square plots of ground in an oil field is 43,300 square yards. The side of one exceeds the side of the other by 50 yards. How long is the side of each?

2. The area of the oblong main waiting room of the railroad station in Washington is 28,600 square feet. Its perimeter is 700 feet. What are its dimensions?

3. In a city park, a flower bed was spaded to the dimensions 40 feet by 20 feet. The park commissioner then decided to double the area by adding a border of uniform width throughout. How wide shall the border be?

4. A tank car is filled by two pipes together in 45 minutes. The larger pipe used alone will fill it in two hours' less time than the smaller one alone. In how many minutes will the smaller pipe alone fill the tank car?

5. To pave a Florida patio with square tiles of a certain size,

* It is important to note that all roots obtained in this manner must be checked in the original equation before acceptance.

31,104 tiles are needed. If each tile were one inch longer and one inch wider, 21,600 tiles would be needed. What are the dimensions of the two kinds of tiles?

6. A reconnaissance airplane traveling a distance of 2,300 kilometers would arrive 24 minutes earlier if the speed per hour were increased by 40 kilometers. Find the airplane's rate of speed.

7. A line segment 150 feet long is to be divided into two parts AC and CB, so that AC is the mean proportional between CB and AB. How long is AC?

8. A number of soldiers on furlough bought a boat, each paying as many dollars as there were soldiers. Had there been 16 soldiers more, each would have paid \$8 less. What was the price of the boat?

9. Find two consecutive numbers the sum of whose squares is 1,625.

10. Two angles are supplementary. The square of the number of degrees in the larger exceeds 500 times the number of degrees in the smaller by 3,525°. What are the angles?

11. A chemical factory was planned whose length exceeded its width by 25 feet. Owing to war business, it was decided to increase both dimensions by 45 feet. This change increased the floor area of the factory by 9,900 square feet. What were the original and the new dimensions?

12. A pencil manufacturer discovers that by lowering the price of grade A drawing pencils and selling them 10 cents a dozen cheaper, he can sell 180 more pencils than he used to sell for \$45. What was the original price per dozen?

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AREAS AND QUADRATICS, CONTINUED

Two methods of solving quadratics have been presented, namely, (1) the method of completing the square, and (2) the method of factoring. It now remains to discuss:

(3) The quadratic formula.

(4) Graphical solution.

THE QUADRATIC FORMULA

Consider now the general form of the quadratic equation, introduced on page 256,

$$ax^2 + bx + c = 0,$$

where a, b, and c are real rational numbers and a is not zero.

This general equation is now solved by applying the method of completing the square as follows:

Transpose the constant term:

$$ax^2 + bx = -c.$$

Divide both sides of the equation by a, the coefficient of x^2:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

To make the left-hand side of the equation a perfect square, a number which is the square of one-half of $\frac{b}{a}$ (the coefficient of the middle term) must be added. Thus, $\frac{1}{2}$ of $\frac{b}{a}$ is $\frac{b}{2a}$, and $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$. Adding this quantity to both sides of the equation, we obtain:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

The left-hand side of the equation is now a perfect square, of Type A, page 263. Hence,

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$

The common denominator of the right-hand side of the equation is $4a^2$ (for a is contained in $4a^2$). Therefore, the expression $\frac{b^2}{4a^2} - \frac{c}{a}$ can be written: $\frac{b^2}{4a^2} - \frac{4ac}{4a^2}$, or $\frac{b^2 - 4ac}{4a^2}$. That is,

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of both sides of this equation gives:

$$x - \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

Transposing $\frac{b}{2a}$,

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Or, since the expression on the right has the common denominator $2a$, it can be written:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

That is, the solutions of any quadratic equation are obtained by taking the coefficient of x with the sign changed plus or minus the square root of the difference of the square of the coefficient of x and four times the product of the coefficient of x^2 and the constant term, and dividing this result by twice the coefficient of x^2. Thus, the equation $6x^2 - x - 12 = 0$, already solved by factoring, can now be solved by the simple application of the formula as follows: Note that $a = 6$, $b = -1$, and $c = -12$. Hence,

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(6)(-12)}}{2(6)}$$

or

$$x = \frac{1 \pm \sqrt{1 + 288}}{12} = \frac{1 \pm \sqrt{289}}{12} = \frac{1 \pm 17}{12}.$$

Choosing $+17$ gives:

$$x = \frac{1 + 17}{12} = \frac{18}{12} = 1\tfrac{1}{2}.$$

Then, choosing -17, we obtain:

$$x = \frac{1 - 17}{12} = \frac{-16}{12} = -1\tfrac{1}{3}.$$

Thus, as before, the two roots of the equation are $1\frac{1}{2}$ and $-1\frac{1}{3}$.

NATURE OF THE ROOTS OF A QUADRATIC

The general formula for the solution of a quadratic not only affords a quick and easy way of finding the roots, but in addition it allows us to predict the nature of these roots without the necessity of solving the equation.

Before starting this discussion on the nature of the roots of a quadratic, it may be advisable, for those who possibly have forgotten, to consult Chapter 1 (page 11) for the definition of real numbers and the distinction between rational and irrational numbers.*

If the general formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is examined carefully, it may be seen that the value of the part under the radical sign, namely, $b^2 - 4ac$, is of prime importance in discovering the type of numbers the roots will be. In fact, an inspection of the value of $b^2 - 4ac$ is sufficient to determine the nature of the roots. It is for this reason that $b^2 - 4ac$ is called the *discriminant* of the quadratic equation.

Consider the three cases which might occur:

(1) $b^2 - 4ac$ might be positive,
(2) $b^2 - 4ac$ might be equal to zero, and
(3) $b^2 - 4ac$ might be negative.

If $b^2 - 4ac$ is positive, its square root can be found by the usual method. There are two real square roots of a positive number, one positive and one negative. For example: $\sqrt{16}$ is $+4$ or -4, since $(+4)^2 = 16 = (-4)^2$. Hence, if the positive

* Imaginary numbers are discussed in Chapter 30.

value of the square root is used, a solution $\frac{-b+\sqrt{b^2-4ac}}{2a}$ is obtained. If the negative value of the square root is used, another solution $\frac{-b-\sqrt{b^2-4ac}}{2a}$ is obtained. Therefore, the roots of the equation are real and unequal. Whether the roots of the equation are rational or irrational depends upon the kind of number obtained in extracting the square root of the discriminant. If the number thus obtained can be expressed as an arithmetic fraction, the roots of the equation are rational. If on extracting the square root of $b^2 - 4ac$ an unending and non-repeating decimal is obtained, the roots of the equation are real, distinct, and irrational.

For example: In the equation $3x^2 - 9x + 6 = 0$, the discriminant $b^2 - 4ac = (-9)^2 - (4)(3)(6) = 81 - 72 = 9$. Since the result is positive, and the $\sqrt{9}$ is equal to the integers ± 3, the roots are real, distinct, and rational.

If $b^2 - 4ac$ is equal to zero, the general formula reduces to $x = \frac{-b}{2a}$, and it is evident that the two values of x are identical. Therefore, the two roots of the equation are real, equal, and rational.

Thus, in the equation $9x^2 + 12x + 4 = 0$, the discriminant $b^2 - 4ac = (12)^2 - (4)(9)(4) = 144 - 144 = 0$. Since the result is zero, the roots are real, equal, and rational.

Finally, if $b^2 - 4ac$ is negative, we find ourselves confronted with a new kind of number, namely, the square root of a negative number. Since, (page 10), the square of a positive real number is positive and the square of a negative real number is positive also, there is no real number whose square is negative. That is, $\sqrt{-4}$ is not equal to -2, because $(-2)^2 = +4$, and $\sqrt{-4}$ is not equal to $+2$, because $(+2)^2 = +4$. Therefore, the equation is not solvable if only real numbers can be used. In order to be able to find all the roots of the equation, it is necessary that an expression like $\sqrt{-1}$ have a definite meaning. The expression $\sqrt{-1}$ is called the *imaginary unit* and is denoted by the letter i. The square root of a negative number is called an *imaginary number*. These so-called imaginary numbers have just as much actual existence as the real numbers, and are discussed in Chapter 30.

It is sufficient for the present to know that, if $b^2 - 4ac$ is negative, the roots of the equation are imaginary.

For example: In the equation $5x^2 + 3x + 4 = 0$, the discriminant $b^2 - 4ac = (3)^2 - (4)(5)(4) = 9 - 80 = -71$. Since the result is negative, the roots of the equation are imaginary.

The preceding discussion may be summarized as follows: The general quadratic equation $ax^2 + bx + x = 0$ has two solutions, namely,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the values of a, b, and c from any particular quadratic equation are substituted in the above formula, the roots of the equation are obtained. The nature of these roots may be predicted by means of the discriminant as follows:

I. If $b^2 - 4ac$ is positive, the roots of the equation are real, unequal, and rational or irrational.

II. If $b^2 - 4ac = 0$, the roots are real, rational, and equal.

III. If $b^2 - 4ac$ is negative, the roots are imaginary.

GRAPHICAL SOLUTION

The advantage of drawing a graph has been stressed so many times already that its value in presenting the relation between two quantities clearly and vividly to the eye ought to be evident now. In the preceding discussion, only the quadratic equation $ax^2 + bx + c = 0$ has been considered. Since there are only two values of x which satisfy a quadratic equation, if the equation as it stands were to be plotted, the graph would consist of two points only. However, if only the expression $ax^2 + bx + c$ is considered, it is easily seen that the value of this expression depends on any arbitrary value that may be assigned to x. That is, the expression $ax^2 + bx + c$ is a function of x, and hence it is called a quadratic function. If, for the sake of simplicity, we let y stand for the function of x, we may write the quadratic function:

$$y = ax^2 + bx + c.$$

Although only two values satisfy the quadratic equation $ax^2 + bx + c = 0$, any number of pairs of values of x and y may be found satisfying the quadratic function $y = ax^2 + bx$

$+ c$, because any value of x when substituted in this equation will determine a corresponding value of y. The graph of the quadratic function will, therefore, consist of some curve in which the real values satisfying the quadratic equation $ax^2 + bx + c = 0$ will be included, since the real roots of the quadratic equation are simply those particular values of x which determine the value zero for y in the quadratic function.

Consider, for example, the following interesting problem:

The Woolworth Building in New York City is 792 feet high. Assume that an antiaircraft gun is placed on its roof and fires directly upward with a muzzle velocity of 800 feet per second. If the air resistance, which for high velocities is considerable, be disregarded, it can be shown that the height of the projectile (h ft.) above the ground at any given time (t sec.) is given by the formula:

$$h = -16t^2 + 800t + 792.$$

Here h is a quadratic function of t. It is obvious that the graph of this function will be of great help in studying the variations of the height of the projectile in its path through the air, that is, how far it can go, how long it will take to go that distance, and other useful information.

Before plotting this function, it is well to recall that any pair of real values of t and h which satisfy this equation are the coordinates of a point on the graph of the equation.

In order to draw the graph of this function, a table of values is made up as follows:

$$\text{At } t = 0,\ h = -16(0)^2 + 800(0) + 792 = 792.$$
$$\text{At } t = 10,\ h = -16(10)^2 + 800(10) + 792 = 7192.$$

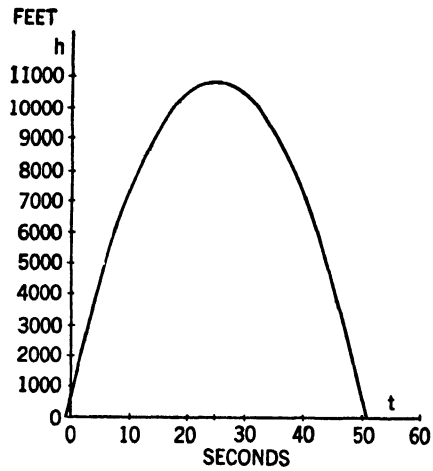
Continuing in this manner, the following table is obtained:

h	792	7192	10,392	10,392	7192	792
t	0	10	20	30	40	50

Plotting the corresponding values of t and h given in the table, a series of points is obtained. If straight line segments were drawn connecting these points, the segments would indicate, from the discussion of rate of change on page 196, that the projectile would travel at the same velocity between, let us say,

the tenth and the twentieth second, changing suddenly in velocity on reaching the twentieth second. This, obviously, is not the case, for it is evident that the velocity of the projectile is changing continuously. Therefore, a smooth curve is drawn through the points. The curve is the graph of this quadratic function.

It is important to note that the graph shows only the relation between the two quantities t and h and is not a representation of the path of the projectile through the air.



A careful examination of the graph reveals several important truths:

(1) The distance from any point on the curve to the horizontal axis represents the height of the projectile at that instant. Hence, the highest point of the curve, approximately 10,792 feet, is the highest point reached by the projectile. This highest point of the curve is called the *maximum value* of the function.

(2) Since the value 10,792 for h corresponds to the value 25 for t, the projectile reached its maximum height approximately 25 seconds after being fired.

(3) The curve crosses the horizontal axis approximately at 51. At this point the value of h is zero. That is, the coordinates of the point at which the curve crosses the horizontal axis are (51, 0). Therefore, the projectile struck the ground approximately 51 seconds after being fired.

For the sake of emphasis, let us consider another quadratic function:

$$y = x^2 - 3x - 4.$$

As usual, distances along the horizontal axis represent the values of the independent variable x, while the corresponding values of the function y are measured along the vertical axis.

Taking arbitrary values of x (the independent variable) and

substituting these values in the quadratic function, corresponding values of y are obtained. These values are given in the following table:

y	6	0	-4	-6	-6	-4	0	6
x	-2	-1	0	1	2	3	4	5

Plotting the corresponding values of x and y, and drawing as before a smooth curve through these points, the graph of the quadratic function is obtained. (See Figure 94.)

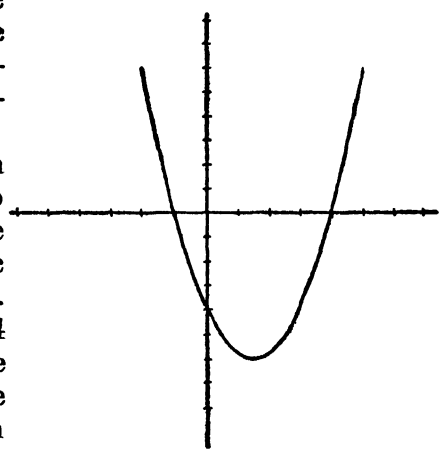


Fig. 94.

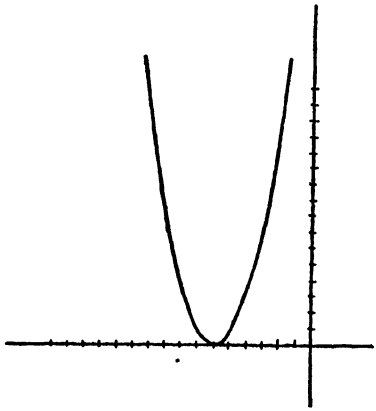
The values of x which make the function equal to zero are obviously the same as the roots of the quadratic equation $x^2 - 3x - 4 = 0$. For example, the value 4 when substituted for x in the quadratic function makes the value of y zero, and it is seen at once that 4 is also a root of the quadratic equation $x^2 - 3x - 4 = 0$. These values are shown in the graph at the points where the curve crosses the X-axis. The fact is that since these points on the curve are on the X-axis, they are at no distance from it, so that their ordinates, that is, their y values, are zero. The roots of the quadratic equation $x^2 - 3x - 4 = 0$ are, therefore, 4 and -1, since these are the values of the abscissas of the points on the curve which are on the X-axis.

The lowest point of the curve is $(1\frac{1}{2}, -6\frac{1}{4})$. The least or *minimum value* of the function is, therefore, the value of the ordinate of this point, that is, $-6\frac{1}{4}$.

The curves of the two quadratic functions just discussed are evidently of the same type. This type of curve is called a *parabola*. It will be shown in Chapter 23 that the graph of *any* quadratic function is a parabola.

NATURE OF ROOTS DETERMINED BY GRAPHS

The points at which the curve crosses the X-axis have been just shown to be the roots of the equation. Since the graph of a quadratic function is a parabola, the curve may cut the X-axis at two points; it may touch the X-axis at only one point; or it may not touch the X-axis at all. Therefore, the nature of the roots of a quadratic equation can be determined by the graph of the quadratic function, for it is easily seen that:



(*a*) If the graph cuts the X-axis at two points, the roots are real and unequal.

(*b*) If the graph touches the X-axis at only one point, the roots are real and equal.

(*c*) If the graph does not touch the X-axis at all, the roots are imaginary.

The first case is illustrated by the preceding example, where it was shown that, since the quadratic function $y = x^2 - 3x - 4$ cuts the X-axis at the two points $(4, 0)$ and $(-1, 0)$, the roots of the quadratic equation $x^2 - 3x - 4 = 0$ are real and unequal, namely, 4 and -1.

The second case may be illustrated by considering the quadratic equation $x^2 + 12x + 36 = 0$ already solved on page 263. The graph of the quadratic function $y = x^2 + 12x + 36$ is shown in the above figure.

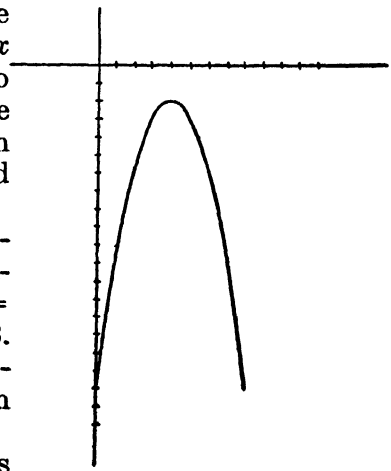


Fig. 95.

The curve touches the X-axis at only one point, namely, $(-6, 0)$. Therefore, the roots of the equation are identical, and are equal to -6.

Finally, the third case can be illustrated by the quadratic equation $-x^2 + 8x - 18 = 0$. The graph of the quadratic

function $y = -x^2 + 8x - 18$ is shown in Figure 95. The curve will not touch the X-axis no matter how far it is extended. Therefore, the roots are imaginary.

SOLUTION OF SYSTEMS INVOLVING EQUATIONS OF THE SECOND DEGREE

Many systems involving equations of the second degree may be solved by methods similar to those used for the solution of simultaneous linear equations using the technique explained in this chapter and in the preceding chapter. The most frequent cases involved are:

I. Systems involving an equation of the second degree and a linear equation.

II. Systems involving two equations of the second degree.

In case I, that is, in systems involving an equation of the second degree and a linear equation, the solution may be accomplished by the method of substitution. For example, to solve the equations

$$2x^2 + y^2 = 41 \quad (1)$$

and

$$x - y = 7, \quad (2)$$

solve (2) for y:

$$y = x - 7.$$

Then, substituting in (1), $2x^2 + (x - 7)^2 = 41$

or

$$3x^2 - 14x + 8 = 0.$$

Solving,

$$x = 4 \text{ or } \tfrac{2}{3}.$$

Substituting 4 in (2),

$$y = -3.$$

Substituting $\frac{2}{3}$ in (2),

$$y = -6\tfrac{1}{3}.$$

That is, x and y each have two values:

When $x = 4$, $y = -3$.
When $x = \frac{2}{3}$, $y = -6\frac{1}{3}$.

Case II, involving two equations of the second degree, may be solved by again using the method of substitution or by the method of addition or subtraction. For example, to solve the equations:

$$x^2 + y^2 = 13 \quad (1)$$

and

$$xy = 6, \quad (2)$$

solve (2) for y:

$$y = \frac{6.}{x} \quad (3)$$

Then, substituting in (1), $x^2 + \frac{36}{x^2} = 13$

or $x^4 - 13x^2 + 36 = 0.$

Solving: $x = \pm 2$ or $\pm 3.$

Substituting ± 2 in (3), $y = \pm 3.$

Substituting ± 3 in (3), $y = \pm 2.$

That is, x and y each have four values:

When $x = 2, y = 3.$
When $x = -2, y = -3.$
When $x = 3, y = 2.$
When $x = -3, y = -2.$

Another example. Solve the equations:

$$2x^2 + 4y^2 = 48 \quad (1)$$

and $$x^2 - y^2 = 12. \quad (2)$$

Rewriting (1), $2x^2 + 4y^2 = 48.$ (1)

Multiplying (2) by 4: $4x^2 - 4y^2 = 48.$ (3)

Adding (1) and (3): $6x^2 = 96.$

Solving: $x = \pm 4.$

Substituting 4 in (2): $y = \pm 2.$

Substituting -4 in (2): $y = \pm 2.$

That is, the four solutions obtained are:

When $x = 4, y = 2.$
When $x = 4, y = -2.$
When $x = -4, y = 2.$
When $x = -4, y = -2.$

Problems

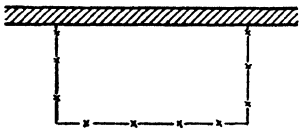
1. A keg contains 10 gallons of pure wine. A certain amount is drawn out and replaced by pure water. This operation is then repeated a second time, after which the keg contains a mixture 36 per cent of which is water. Find the amount drawn off each time.

2. Sixteen persons contributed \$60 to pay the expenses of a fishing trip. One-half of the amount was contributed by men and the other half by boys, each man paying \$2 more than each boy. What did each man and what did each boy contribute?

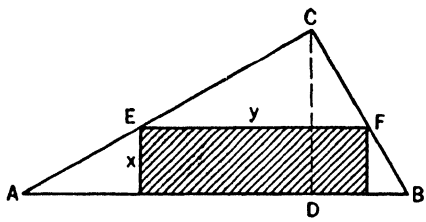
3. A cistern whose outlet was left open was filled in 6 hours by two pipes. The larger pipe alone, with the outlet closed, can fill it twice as fast as the smaller. The smaller pipe alone can fill the cistern in 20 minutes' less time than it takes the outlet to empty it.

Find the times required by each pipe alone to fill the cistern if the outlet is closed, and the time required for the outlet to empty it after it has been filled and the pipes have been shut off.

4. A rectangular sheep pen is to be fenced in, and a straight fence already built can be used as one side of the rectangle. There are only 60 feet of wire fencing available. Find the dimensions of the rectangle that will enclose the greatest area.



5. A rectangular building is to be constructed on a triangular lot whose frontage, AB, is 200 feet. The altitude of the triangle, CD, is 80 feet. Find the dimensions of the building that will give the maximum floor area.



6. If a projectile is fired directly upward with a muzzle velocity of v feet per second from a height of d feet, its height (h feet) above the ground at any given time (t seconds) is given by the quadratic function:

$$h = -16t^2 + vt + d.$$

(a) Find the maximum height attained by a projectile fired directly upward with a muzzle velocity of 320 feet per second if the gun was fired from the roof of a building 500 feet high.

(b) Find when the projectile struck the ground. Check the accuracy of this approximate answer by solving the proper quadratic equation.

7. A trough is to be made of a long, rectangular piece of tin by bending up two sides so as to give a rectangular cross section. If the width of the piece is 16 inches, how deep should the trough be made so that the area of the cross section is (a) 30 square inches? (b) 20 square inches?

8. A portrait measuring $18'' \times 24''$ is to be framed, the sides of the frame to be equal all around and the frame to be equal in area to the area of the portrait. What must be the width of the framing?

9. If it takes 26 rods of fence to enclose a rectangular garden containing $\frac{1}{4}$ of an acre, what are the dimensions of the garden? (For measures, see pages 91 and 254.)

10. To pave a patio with square tiles of a certain size, 2,000 tiles are needed. If each tile were one inch longer and one inch wider, 1,280 tiles would be needed. Find the dimensions of the tiles.

18

BORDERING SPACE— AREAS OF POLYGONS

Countless businessmen, not to mention carpenters, electricians, and engineers, have occasion daily to solve problems involving surfaces. In everyday language, by *surface* is meant the exterior part of anything. Mathematically, the surface of a solid is the boundary of the solid, or simply that which separates it from surrounding space. The surface is not considered as part of a solid; hence, it has no thickness. A plane surface is a portion of a plane enclosed by boundary lines. These boundary lines give a plane surface its shape and size. The

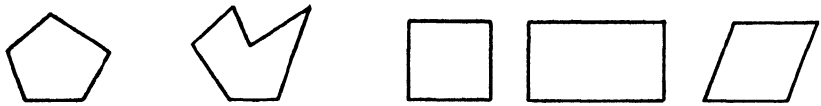


Fig. 96.

simplest kind of closed plane surface is the polygon. A *polygon* was defined, in Chapter 7, page 106, as a single finite portion of a plane bounded by straight line segments no two of which intersect.

The sides of a polygon are the straight line segments which bound it. The angles included by the adjacent sides of a polygon are its angles; and the vertices of these angles are the vertices of the polygon.

POLYGONS CLASSIFIED

A polygon is said to be *equilateral* if all its sides are equal (*a* and *c*, Figure 96).

A *convex* polygon, as already stated, is a polygon of which no side, when produced, will enter the polygon. A *concave* polygon is a polygon of which two or more sides, when pro-

duced, will enter the polygon. Unless otherwise stated, by *polygon* we shall mean a convex polygon.

Polygons are named according to the number of sides. We are already familiar with the simplest, that is, the *triangle*, or polygon of three sides. The polygon of four sides, namely, the *quadrilateral*, has already been defined and some of its properties demonstrated in Chapter 11. The next few, in order according to the number of sides, are: *pentagon* (five sides), *hexagon* (six sides), *heptagon* (seven sides), *octagon* (eight sides), *nonagon* (nine sides), and *decagon* (ten sides.)

A polygon which is both equilateral and equiangular is called a *regular* polygon. For example, (*a*) in Figure 96.

Two polygons are said to be mutually equiangular if their corresponding angles are equal (*a* and *b*, Figure 96), and mutually equilateral if their corresponding sides are equal (*a* and *c*, Figure 96). It is easily seen that two polygons may be mutually equiangular without being mutually equilateral (*a* and *b*, Figure 96), and mutually equilateral without being mutually equiangular (*a* and *c*, Figure 96).

AREA OF A PARALLELOGRAM

From the definition of congruent figures given in Chapter 7, page 108, it is evident that polygons that are mutually equiangular and mutually equilateral are congruent, for they may be made to coincide.

A lot in the form of a parallelogram is bordered on two sides by straight roads intersecting, as in Figure 97. The price of this land is \$7 per square foot. In order to ascertain the total price, a prospective buyer must determine the area of the lot. The rule of measurement by which the area of any parallelogram is found may be easily developed as follows: Consider the lot *ABCD* shaped in the form of a parallelogram. Denote the base, *AB*, by *b* and the altitude by *h*. From *D* and *C* drop perpendicular lines to *AB* (extended, if necessary), meeting *AB* at *E* and *F*. Then the lines *DE* and *CF*, perpendicular

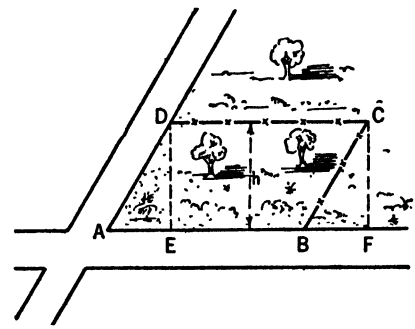


Fig. 97.

to the parallel lines AB and DC, are parallel (Fact 17). Therefore, the figure $EFCD$ is a rectangle.

By construction, DC is equal to AB. Because $EFCD$ is a rectangle, DC is equal to EF. Hence, EF is equal to AB, for things equal to the same thing are equal to each other. That is, the base of the rectangle, namely, EF, is b, and of course its altitude is h.

Since the rectangle $EFCD$ and the parallelogram $ABCD$ have their bases equal and the same altitude, if we can show the rectangle equal to the parallelogram, then the area of the latter will be determined. The parallel line segments AD and BC, also DE and CF, cut by the parallel lines AB and DC, are equal (Fact 38). That is, AD is equal to BC and DE is equal to CF. So that the right triangle ADE has its hypotenuse, AD, and one leg DE equal, respectively, to the hypotenuse, BC, and one leg, CF, of the right triangle BCF. Therefore, by Fact 27, the triangles are congruent.

If the right triangle ADE is taken away from the quadrilateral $AFCD$, there remains the rectangle $EFCD$. If the right triangle BCF is taken away from the same figure, $AFCD$, there remains the parallelogram $ABCD$. Since the triangles ADE and BCF have just been shown to be congruent, the same quantity is being subtracted from the quadrilateral $AFCD$. Therefore, the remainder in either case is the same. That is, the rectangle $EFCD$ is equal to the parallelogram $ABCD$. But the area of the rectangle $EFCD$ is, by Fact 52, the product of its base and altitude, that is, $b \cdot h$. Therefore, the area of the parallelogram $ABCD$ is also $b \cdot h$. That is,

Fact 54: The area of a parallelogram is equal to the product of its base and altitude.

Thus, if the prospective buyer of the lot ascertained that AB is 40 feet and determined the perpendicular height h as 20 feet, then the area of the lot $ABCD$ is $40 \times 20 = 800$ square feet. Hence, at \$7 per square foot, the lot is worth $800 \times 7 = \$5{,}600$.

From Fact 54, it immediately follows that:

Fact 55: Two parallelograms having equal bases and equal altitudes are equal in area.

Now, suppose two parallelograms have equal bases. Denote the equal bases by b. Let the area of one parallelogram be $A = b \times h$ and the area of the other be $A_1 = b \times h_1$. Since

if equals are divided by equals the quotients are equal, $\frac{A}{A_1} = \frac{b \times h}{b \times h_1}$.

Cancelling b from the right side of this equation gives $\frac{A}{A_1} = \frac{h}{h_1}$; that is,

Fact 56: The ratio of the areas of two parallelograms, having equal bases, is equal to the ratio of their respective altitudes.

Similarly, the reader may show that,

Fact 57: The ratio of the areas of two parallelograms, having equal altitudes, is equal to the ratio of their respective bases; and

Fact 58: The ratio of the areas of any two parallelograms is equal to the ratio of the respective products of their bases and altitudes.

AREA OF A TRIANGLE

A building is to be erected in a triangular plot of ground (Figure 98). To ascertain the amount of floor space available, the area of triangle ABD must be determined. Denote the base AB by b and let h be its altitude. From B draw BC parallel to AD. Through D draw CD parallel to AB, meeting BC at C. Then, by construction, the figure $ABCD$ is a parallelogram. The line BD is now a diagonal of the parallelogram $ABCD$ and, therefore, by Fact 37, divides the parallelogram into the two congruent triangles ABD and DBC. So that the triangle ABD is one-half of the parallelogram $ABCD$. But, by Fact 54, the area of parallelogram $ABCD$ is $b \times h$. Hence, the area of triangle ABD is one-half of $b \times h$, or $\frac{1}{2}\, b \cdot h$. That is,

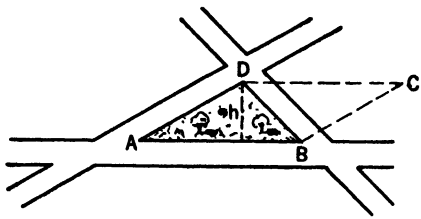


Fig. 98.

Fact 59: The area of a triangle is one-half the product of its base and altitude.

Thus, if b is 35 feet and h is 30 feet, the area of the triangular plot is $\frac{1}{2} \times 35 \times 30 = 525$ square feet.

From Fact 59 it is evident that:

Fact 60: Two triangles having equal bases and equal altitudes are equal in area.

By a process of reasoning similar to the one used in proving Fact 56, the reader may easily show that:

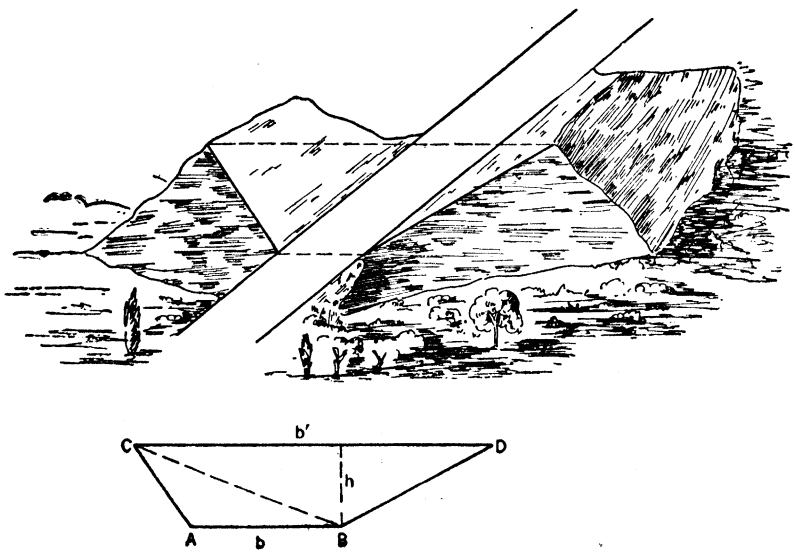
Fact 61: The ratio of the areas of two triangles having equal bases is equal to the ratio of their respective altitudes;

The ratio of the areas of two triangles having equal altitudes is equal to the ratio of their respective bases; and

The ratio of the areas of any two triangles is equal to the ratio of the respective products of their bases and altitudes.

AREA OF A TRAPEZOID

A road is to be cut through a mound. The cross section of the cut is made in the form of a trapezoid. To determine



the approximate amount of cubic feet of earth that must be excavated, it is necessary first to determine the area of the trapezoid forming the cross section. Assume that the trapezoid $ABCD$ is the cross section of the cut. Denote the bases AB and CD by b and b', and let h be the altitude. Draw the diagonal CB, dividing the trapezoid into the two triangles ABC

and CBD. Triangle ABC has base b and altitude h. Triangle CBD has base b' and the same altitude, h.

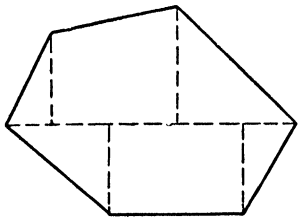


Fig. 99.

The area of a triangle is, by Fact 59, equal to one-half the product of its base and altitude. Hence, the areas of triangles ABC and CBD are $\frac{1}{2}\, b \cdot h$ and $\frac{1}{2}\, b' \cdot h$, respectively. But the trapezoid $ABCD$ consists of triangles ABC and CBD. Therefore, the area of the trapezoid is equal to the sum of the areas of the two triangles. That is, $A = \frac{1}{2}\, bh + \frac{1}{2}\, b'h$, which through factoring reduces to $A = \frac{1}{2}\, h(b + b')$. In other words:

Fact 62: The area of a trapezoid is equal to one-half the product of its altitude and the sum of its bases.

AREA OF AN IRREGULAR POLYGON

The area of an irregular polygon of four or more sides can usually be found by dividing the figure into more familiar types. The illustration (Figure 99) shows an irregular polygon which has been divided into triangles, rectangles, and trapezoids. The sum of the areas of the figures which make up the irregular polygon is its area.

AREAS OF SIMILAR TRIANGLES

A triangular die is to be constructed as shown in Figure 100. To determine the weight of the die, it is necessary first to compute the area of the face ABC, less the area of the hole DEF.

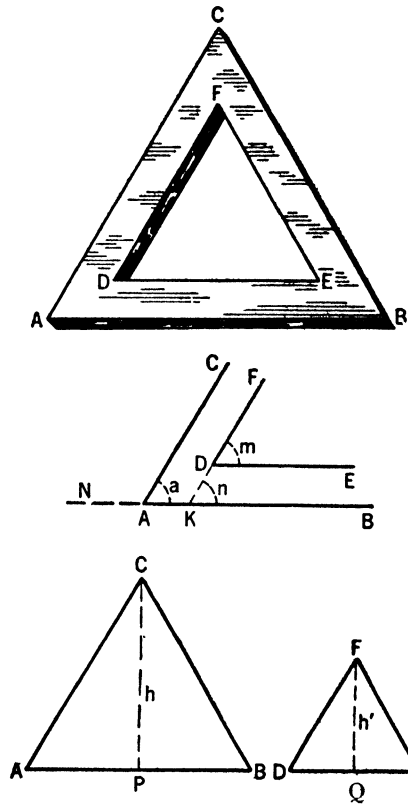


Fig. 100.

The triangular hole DEF is constructed so that its edges are parallel to the sides of triangle ABC. The computation can be considerably shortened if we can show that (I) the triangles ABC and DEF are similar, and (II) that in that case the areas of the triangles are to each other as the squares of any two corresponding sides.

The triangles may be proved similar by showing that the angles are equal.* By construction, DF is parallel to AC, and DE is parallel to AB. Extend the line DF to intersect AB at K, Figure 100 (2). The parallel lines DE and AB are now cut by the transversal KF, so that, by Fact 14, the corresponding angles n and m are equal. Similarly, the parallel lines KF and AC are cut by the transversal AB; so that, by Fact 14, the corresponding angles CAB and n are equal. Therefore, angle CAB is equal to angle FDE.

Similarly, angles E and F are equal, respectively, to angles B and C. Therefore, the triangles ABC and DFE are similar. If the line AB is extended to N, angle NAC is formed. This angle NAC has its sides AC and NA also parallel to the sides of angle FDE. Since angle NAC is the supplement of angle a, and angle a is equal to angle FDE, then angle NAC is the supplement of angle FDE. That is,

Fact 63: If two angles have their sides respectively parallel, they are either equal or supplementary.

The reader may show as an interesting exercise that,

Fact 64: If two angles have their sides respectively perpendicular, they are either equal or supplementary.

Let h and h' denote the corresponding altitudes of triangles ABC and DEF, Figure 100 (3). Indicate the corresponding areas of triangles ABC and DEF by A and A′. Then $A = \frac{1}{2} \times AB \times h$, and $A' = \frac{1}{2} \times DE \times h'$ (Fact 59). If equals are divided by equals, the quotients are equal; hence,

$$\frac{A}{A'} = \frac{\frac{1}{2} \times AB \times h}{\frac{1}{2} \times DE \times h'},$$

or, cancelling $\frac{1}{2}$,

$$\frac{A}{A'} = \frac{AB}{DE} \times \frac{h}{h'}. \qquad \text{(a)}$$

In the right triangles ACP and DFQ, angles A and D have

* By "equal" is meant having the same numerical value, without regard to sign.

just been proved equal. That is, right triangle ACP has acute angle A equal to acute angle D of right triangle DFQ. Therefore, by Fact 49, the triangles are similar, so that $\frac{AC}{DF} = \frac{h}{h'}$. Since the triangles ACB and DEF are similar, $\frac{AC}{DF} = \frac{AB}{DE}$. But things equal to the same thing are equal to each other; hence, $\frac{AB}{DE} = \frac{h}{h'}$.

Substituting $\frac{AB}{DE}$ for its equal $\frac{h}{h'}$ in (a), we have:

$$\frac{A}{A'} = \frac{AB}{DE} \times \frac{AB}{DE}, \quad \text{or} \quad \frac{A}{A'} = \frac{(AB)^2}{(DE)^2}. \quad \text{That is,}$$

Fact 65: The areas of two similar triangles are to each other as the squares of any two corresponding sides.

Thus, if the die of Figure 100 is such that AB is 6 inches, h is 5 inches, and DE is $1\frac{1}{2}$ inches, then the area of triangle ABC is $\frac{1}{2}(6)(5) = 15$ square inches; and the area of triangle DEF may be found by the proportion $\frac{15}{A'} = \frac{(6)^2}{(\frac{3}{2})^2}$, or $\frac{(15)(\frac{9}{4})}{36} = A' = 0.93$ square inches.

AREAS OF SIMILAR POLYGONS

By the already well-known process of dissecting a figure into more familiar and simpler types, the above fact may be extended to include any two similar polygons.

Assume that the two polygons in Figure 101 are similar, and draw all the diagonals from any two corresponding vertices (m, n, m', and n'). These diagonals divide the polygons into triangles. To show that Fact 65 holds for any two similar polygons, the triangles formed by the diagonals of the polygons must first be proved similar.

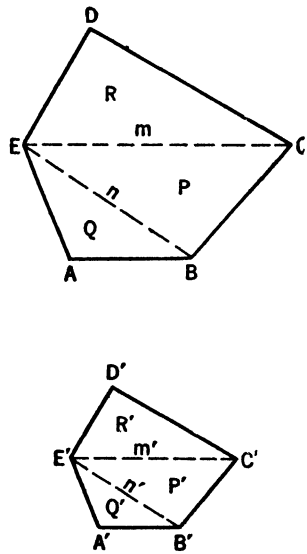


Fig. 101.

Consider the triangles EDC and $E'D'C'$. Since the polygons are similar, angles D and D' are equal, and $\frac{ED}{E'D'} = \frac{DC}{D'C'}$. That is, triangles EDC and $E'D'C'$ have angles D and D' equal and the sides including these angles proportional. Therefore, by Fact 51, the triangles are similar. So that the corresponding angles DCE and $D'C'E'$ are equal.

By construction, angle DCB is equal to angle $D'C'B'$. But angle DCB = angle DCE + angle ECB, and angle $D'C'B'$ = angle $D'C'E'$ + angle $E'C'B'$. That is, angle DCB − angle DCE = angle ECB; and angle $D'C'B'$ − angle $D'C'E'$ = angle $E'C'B'$. Since the left sides of these two equations are equal, the right sides must be equal. Hence, angle ECB is equal to angle $E'C'B'$.

Moreover, $\frac{DC}{D'C'} = \frac{CB}{C'B'}$, for the polygons are similar. Also, $\frac{DC}{D'C'} = \frac{EC}{E'C'}$, for the triangles EDC and $E'D'C'$ have been proved similar. But things equal to the same thing are equal to each other. Therefore, $\frac{CB}{C'B'} = \frac{EC}{E'C'}$. Again Fact 51 may be applied. Hence, the triangles ECB and $E'C'B'$ are similar. In like manner, triangles EBA and $E'B'A'$ may be proved similar. Therefore,

Fact 66: If two polygons are similar, they can be divided into the same number of triangles similar each to each and similarly placed.

The reader may prove the converse of this fact, that is,

Fact 67: If two polygons can be divided into the same number of triangles, similar each to each and similarly placed, the polygons are similar.

It is now easily shown that Fact 65 holds for two similar polygons. Denote the areas of the polygons in Figure 101 by S and S', respectively, and let R, P, Q, R', P', and Q' represent the areas of the respective triangles as shown in the figure. Since the triangles are similar, their areas are to each other as the squares of any two corresponding sides (Fact 65). That is, $\frac{R}{R'} = \frac{(m)^2}{(m')^2}$, and $\frac{P}{P'} = \frac{(m)^2}{(m')^2}$. Hence, $\frac{R}{R'} = \frac{P}{P'}$. Similarly,

$\frac{P}{P'} = \frac{(n)^2}{(n')^2}$ and $\frac{Q}{Q'} = \frac{(n)^2}{(n')^2}$. So that $\frac{P}{P'} = \frac{Q}{Q'}$. Therefore, $\frac{R}{R'} = \frac{P}{P'} = \frac{Q}{Q'}$.

In Chapter 10, page 151, it was shown that in a series of equal ratios, the sum of all the first terms is to the sum of all the second terms as the first term in any of the given ratios is to its second term. Hence, $\frac{R + P + Q}{R' + P' + Q'} = \frac{R}{R'}$. But by Fact 65, $\frac{R}{R'} = \frac{(ED)^2}{(E'D')^2}$, and obviously $R + P + Q$ and $R' + P' + Q'$ are, respectively, S and S', the areas of the given polygons. Substituting gives:

$$\frac{S}{S'} = \frac{(ED)^2}{(E'D')^2}.$$ That is,

Fact 68: The areas of two similar polygons are to each other as the squares of any two corresponding sides.

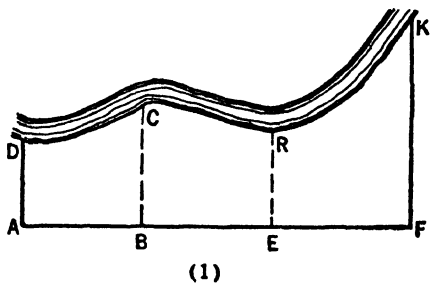
Further facts about polygons will be discussed later in analyzing the properties of a circle. The small table below sometimes saves time in computations. A moment's reflection shows that a line of any unit length may be the side of any number of regular polygons, having any number of sides, from 3 (when the polygon is an equilateral triangle) to any number however large. If the side be of some constant length, the greater the number of sides, the greater will be the area. Thus, in the following table, whatever be the length of its side, the area of a square is taken as unity, and the areas of other polygons, having sides of the same length as that of the square, are shown in sequence. Thus, by regarding these areas as constants, the area of any regular polygon of the class listed below may be found by multiplying the square of one of its sides by the corresponding number in the table. For example: Find the area of a regular pentagon whose sides are 7 inches in length. Solution: $(7)^2 \times 1.720477 = 84.303373$ sq. in.

TABLE OF AREAS OF REGULAR POLYGONS HAVING A SIDE EQUAL TO 1

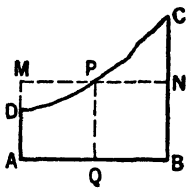
Name	*Area*	*Name*	*Area*
Triangle (equilat.)	0.433013	Octagon	4.828427
Square	1.000000	Nonagon	6.181824
Pentagon	1.720477	Decagon	7.694209
Hexagon	2.598076	Undecagon	9.365640
Heptagon	3.633912	Dodecagon	11.196152

AREA BOUNDED BY A CURVE

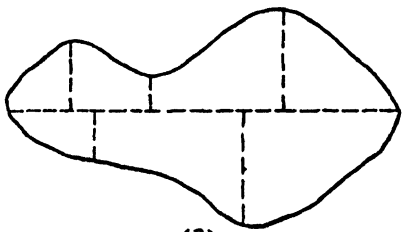
In actual practice, a large number of surfaces are enclosed by curves, for example, the field *AFKD*, Figure 102 (1), bounded on one side by a winding river. A surveyor wishing to find its area may first subdivide the field into convenient parcels by the lines *CB* and *RE*.



(1)



(2)



(3)

Fig. 102.

The area of a plane figure bounded by a curve, such as *ABCD*, Figure 102 (2), can be approximated from an accurate drawing of the figure by finding the average height of the curve and multiplying this height by the base. By the *average height* of the curve is meant the height of a rectangle equal in area to the area of the plane figure and having the same base. Thus, *PQ* in Figure 102 (2) is the average height. This average height may be approximated by drawing the horizontal line *MPN* so that the area *DMP*, which is added to form the rectangle *ABNM*, is as closely as possible equal to the area *CNP* which is cut off to form the rectangle. It is easily seen that, since the area added is approximately equal to the area cut off, the areas of the rectangle *ABNM* and of the plane figure *ABCPD* can be considered equal for practical purposes.

SIMPSON'S RULE

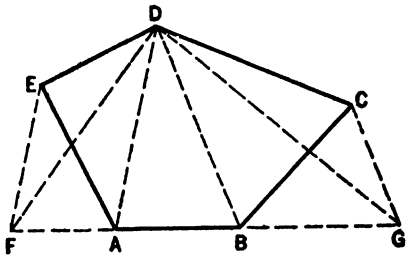
The average height may also be found or approximated by adding the heights *AD* and *CB*, Figure 102 (2), to four times the height at the middle and dividing the sum by 6. This

method is called Simpson's Rule. Thus, in Figure 102 (2), AD is 4, BC is 10, and the height at the middle is 5. Hence, the average height is $\frac{4 + 10 + 4(5)}{6} = \frac{34}{6} = 5\frac{2}{3}$. The area of figure $ABCPD$ is, then, $5\frac{2}{3}$ times AB. Since AB is 10, area $ABCPD$ is $10 \times 5\frac{2}{3} = 56\frac{2}{3}$ square units.

This method may be applied to any plane figure bounded by a curve by simply subdividing the figure into convenient portions, as shown in Figure 102 (3).

CONSTRUCTING A TRIANGLE EQUAL IN AREA TO A GIVEN POLYGON

It is sometimes convenient to construct a triangle equal in area to a given polygon. Suppose the polygon $ABCDE$ is given, and that a triangle equal in area is to be constructed. Draw the line AD, and from E draw a line parallel to AD meeting AB extended at F. Draw line DF.



Since EF is drawn parallel to AD, the perpendicular distance from E to AD is equal to the perpendicular distance from F to AD, for parallel lines are everywhere equidistant (Fact 39). Hence, the altitudes of the triangles AFD and AED are equal if the side AD, common to both triangles, is considered as their common base. It follows that the triangles AFD and AED, having equal bases and equal altitudes, are equal in area (Fact 60). Therefore, the polygon $ABCDE$ is equal in area to the polygon $FBCD$, which has one side less.

Similarly, take the three consecutive vertices B, C, and D, and draw BD. Then from C draw a line parallel to BD, meeting AB extended at G. Draw DG. Triangles BDG and BCD have a common base, DB, and the same altitude, since C and G are two points on a line parallel to DB (Fact 39). Therefore, by Fact 60, they are equal in area. It follows, then, that polygon $FBCD$ is equal in area to triangle FGD. Since $ABCDE$ is equal to $FBCD$, and $FBCD$ is equal to FGD, then $ABCDE$ is equal in area to triangle FGD, for things equal to the same

thing are equal to each other. Therefore, the triangle FGD is the required figure.

Problems

1. A canal is 15 feet deep, 62 feet wide at the top, and 45 feet wide at the bottom. Find the area of a cross section.

2. A house is 75 feet long, 50 feet high to the eaves, 68 feet high to the ridge pole, and 50 feet wide. Find the entire surface area of walls but not of the roof.

3. A hotel has a lobby in the form of a regular hexagon, each side of the hexagon measuring 25 feet. A certain kind of mosaic costs \$3 per square foot. What is the cost of the mosaic required to pave the lobby?

4. If the side AB of a triangle ABC be tripled, and the altitude on AB remain unchanged, what change takes place in the area of the triangle?

5. In a government office building, there are two square rooms, one of which has just double the perimeter of the other. Compare the areas.

6. A boat house measures 20 feet by 12 feet. A rowing club desires to construct another boat house on their property, similar in shape to the first but with double the linear dimensions. What is the area of the new building?

7. A swimming pool is 60 feet long, 20 feet wide, 3 feet deep at one end, and 10 feet deep at the other. It is planned to resurface the sides and bottom of the pool with green enamel tiles six inches square, which cost twenty cents apiece. What is the total cost of the tiles?

8. In a tulip garden there are two equilateral triangular beds. The side of one is equal to the altitude of the other. Find the ratio of their areas.

9. What is the area of a regular octagon whose sides are 10 feet long?

10. In two similar polygons, each having 17 sides, two corresponding sides are 3 and 5. The area of the first polygon is 100. What is the area of the second?

11. The base of a triangle is 16 meters and the altitude is 25 meters. What is the area of a triangle cut off by a line parallel to the base and 10 feet away from it?

12. The area of a regular pentagon is 27.527632 sq. centimeters. What is the length of its side?

19 LOCI OR THE PATHS OF A POINT

The ancient Greeks, thirsting for knowledge and curious about the nature of things, became masters in art and science. They created beautiful statues, great monuments, and imposing buildings. This high state of civilization could not have been achieved without an underlying knowledge of mathematics. As already indicated, Pythagoras is reputed to have proved one of the most useful and important mathematical truths demonstrated by the Greek mathematicians. Thus, the fact that the square on the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides is called the "theorem of Pythagoras." The applications of this fact are legion. One simple case will suffice to illustrate the demonstration.

PYTHAGORAS' THEOREM

A carpenter is to construct a timber roof truss, as shown in Figure 103. The span, AB, is to be 30 feet, and the slope of the roof $\frac{1}{2}$. The king rod, CD, is placed through point D, bisecting AB and perpendicular to it. In order to begin construction, the carpenter must know the lengths of the rafters AC and CB and of the king rod CD. Since the slope is to be $\frac{1}{2}$ and AD is 15, then $\frac{CD}{15} = \frac{1}{2}$ and $CD = \frac{15}{2} = 7\frac{1}{2}$ feet. Triangle ACD is by construction a right triangle whose sides AD and CD are, respectively, 15 feet and $7\frac{1}{2}$ feet. If we can prove the theorem of Pythagoras, the length of the rafter AC can then be easily determined.

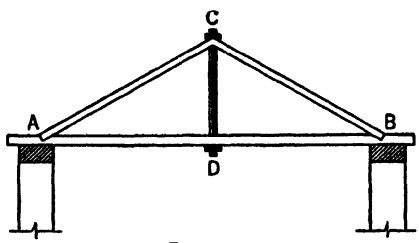


Fig. 103.

Consider any right triangle ABC, Figure 104, in which angle

C is the right angle; denote the sides by small letters corresponding to the opposite angles. From C drop a perpendicular to AB, meeting AB at D and dividing line AB into the segments x and y. The figure consists now of three right triangles, namely, the original right triangle ACB, and two right triangles ACD and CDB formed by dropping the perpendicular CD. The two right triangles ACB and ACD are similar, because the acute angle A is common to both (Fact 49). Likewise, triangles ACB and CDB are similar, for they have acute angle B in common (Fact 49).

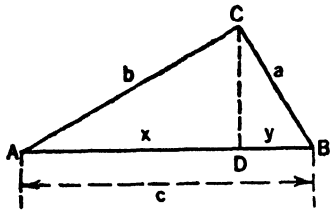


Fig. 104.

From the definition of similar triangles, it follows that triangle ACB is equiangular to triangles ACD and CDB. Therefore, triangles ACD and CDB are equiangular, and, hence, by Fact 47, similar. That is, the three right triangles are similar. Since triangles ACD and CDB are similar, their corresponding sides are in proportion. So that $\frac{x}{CD} = \frac{CD}{y}$ or $x : CD = CD : y$. The term CD, used twice in the proportion as a mean between the two given quantities x and y, is, as already stated, called a *mean proportional.* We have shown, then, that

Fact 69: If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse,

(a) The two triangles thus formed are similar to the given triangle and similar to each other;

(b) The perpendicular is the mean proportional between the segments of the hypotenuse.

Triangles ACB and ACD are similar. Hence, $\frac{b}{x} = \frac{c}{b}$; or, multiplying both sides by $b \cdot x$, we obtain $b^2 = cx$. (I)

In the similar triangles ACB and CDB, $\frac{a}{y} = \frac{c}{a}$. So that, multiplying both sides by $a \cdot y$, we obtain $a^2 = cy$. (II)

Equations (I) and (II) may be added without changing the equality, for if equals are added to equals, the results are equal. Hence,

$$a^2 + b^2 = cx + cy, \quad \text{or}$$

$$a^2 + b^2 = c(x + y). \tag{III}$$

But $x + y = c$. So that substituting c for $x + y$ in equation (III) gives $a^2 + b^2 = c \cdot c$, that is, $a^2 + b^2 = c^2$. In other words,

Fact 70: The square on the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

The length of rafter AC can now be determined. Since AC, Figure 103, is the hypotenuse of right triangle ACD, and AD and CD are, respectively, 15 and $7\frac{1}{2}$ feet, then $(AC)^2 = (15)^2 + (7\frac{1}{2})^2 = 225 + \frac{225}{4} = \frac{1,125}{4}$, and $AC = \sqrt{\frac{1,125}{4}} = 16.77$ feet. Obviously, the carpenter does not need to compute the length of CB, for both AC and CB cut off the equal segments AD and DB on line AB, so that, by Fact 28, they are equal.

FUNCTIONS OF 30°, 45°, AND 60°

Angles of 30°, 45°, and 60° occur very frequently in practical problems. The trigonometric functions of these angles can be readily obtained by means of Pythagoras' theorem. Since these functions are, fortunately, very easily determined, it is convenient to remember their values.*

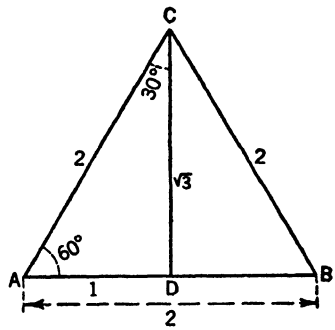


Fig. 105.

Consider the equiangular triangle ABC, Figure 105, with side AC equal to 2. Each angle of the triangle is, by Fact 25, equal to 60°, and each side is, by Fact 5, equal to 2. From C draw CD bisecting angle ACB. Then, by Fact 4, CD bisects the base AB and is perpendicular to AB. So that triangle ACD is a right triangle whose hypotenuse is 2 and whose side AD is equal to 1. Applying Fact 70, $(AC)^2 = (AD)^2 + (CD)^2$. That is, $(2)^2 = (1)^2 + (CD)^2$, or $4 - 1 = (CD)^2$, and $CD = \sqrt{3}$. Hence, by definition,

$$\sin 30° = \tfrac{1}{2} = \cos 60°; \qquad \cot 30° = \sqrt{3} = \tan 60°;$$

$$\cos 30° = \frac{\sqrt{3}}{2} = \sin 60°; \qquad \sec 30° = \frac{2}{\sqrt{3}} = \csc 60°;$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \cot 60°; \qquad \csc 30° = 2 = \sec 60°.$$

* See Chapter 14, page 218, for a table of these values.

Now consider an isosceles right triangle, ABC, Figure 106, with $AC = BC = 1$. The acute angles CAB and ABC are, by Fact 3, equal. So that, by Fact 20, angle CAB = angle $ABC = 45°$. Applying Fact 70, $(AB)^2 = (AC)^2 + (CB)^2$, or $(AB)^2 = (1)^2 + (1)^2 = 2$, and $AB = \sqrt{2}$. Hence, by definition,

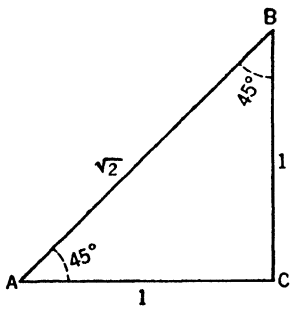


Fig. 106.

$$\sin 45° = \frac{1}{\sqrt{2}} = \cos 45°;$$

$$\tan 45° = 1 = \cot 45°;$$

$$\sec 45° = \sqrt{2} = \csc 45°.$$

DISTANCE BETWEEN TWO POINTS IN A PLANE

The application of Pythagoras' theorem also leads to one of the fundamental formulas in the scientific study of plane curves.

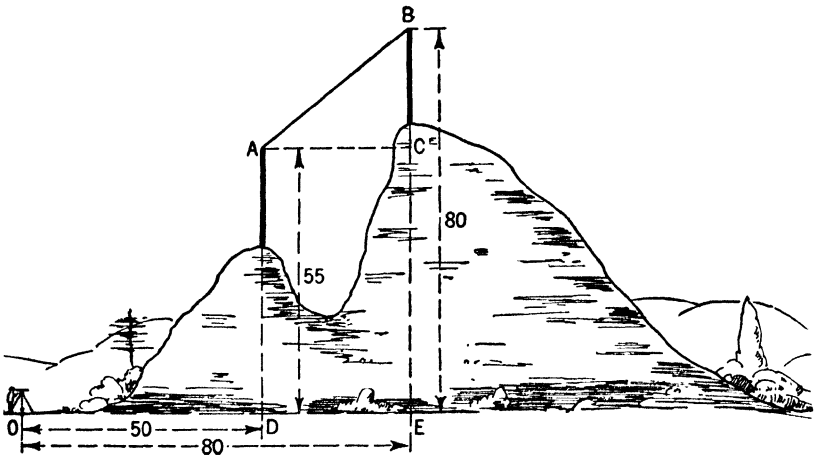


Fig. 107.

Assume that, in the extension of a telephone line, poles of a uniform height of 20 feet are used. Engineers making a preliminary survey find that the top of a mound is 35 feet high and 50 feet away from point O in Figure 107, while the top of a second hill is 60 feet high and 80 feet away from the same point. The length of cable necessary to connect the two poles may be determined as follows:

Through the top of the first pole, that is, through A, draw the horizontal line AC. Since the poles are assumed to be vertical, triangle ACB is a right triangle. In order to find AB, it is first necessary to determine AC and BC. By construction, $ADEC$ is a rectangle; hence, by Fact 36, DE is equal to AC. So that, $AC = DE = EO - OD = 80 - 50 = 30$.

Since the poles are 20 feet high, $AD = 55$ and $BE = 80$. In the rectangle $ADEC$, $AD = EC$ (Fact 36). So that, $BC = EB - EC = EB - AD = 80 - 55 = 25$. Applying Pythagoras' theorem gives $(AB)^2 = (AC)^2 + (BC)^2 = (30)^2 + (25)^2 = 1{,}525$. Therefore, $AB = \sqrt{1{,}525} = 39.06$ feet.

Consider next the general case. The horizontal line OE may be considered as the X-axis, point O as the origin, and a line perpendicular to the X-axis through O as the Y-axis. Now assume that A and B become *any two* points in the plane, not equidistant from the Y-axis, such as P and P_1 in Figure 108. To find the distance between any two points, P and P_1, in a plane, we simply follow the procedure just used to find the length of the cable connecting the two poles.

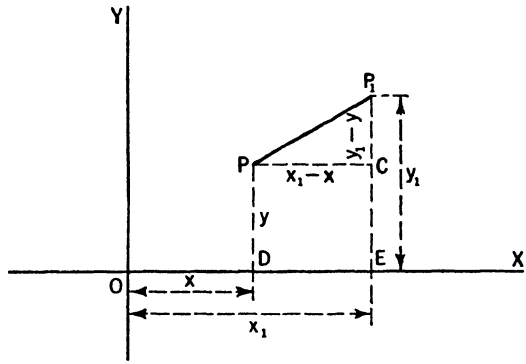


Fig. 108.

Let the coordinates of P be represented by x and y, and those of P_1 by x_1 and y_1. That is, P (x, y) is *any point* in the plane, and P_1 (x_1, y_1) is *any other point* not at the same distance from the Y-axis and in the same plane.

Draw line segments PD and P_1E perpendicular to the X-axis. Then PD is y, the ordinate of P, and P_1E is y_1, the ordinate of P_1. The line segment OE is equal to the perpendicular distance from P_1 to the Y-axis. Hence, it may be denoted by x_1, the abscissa of P_1. Line segment OD is equal to the perpendicular distance from P to the Y-axis, so that it can be denoted by x, the abscissa of P.

Draw line segment PC parallel to the X-axis. Since line segments PD and P_1E are perpendicular to DE, they are also

perpendicular to PC (Fact 16). Therefore, $PDEC$ is a rectangle, and, by Fact 36, $PD = CE$ and $PC = DE$. Triangle PP_1C is a right triangle having angle C as the right angle and PP_1 as its hypotenuse. Therefore, by Fact 70,

$$(PP_1)^2 = (PC)^2 + (P_1C)^2. \qquad (a)$$

As in the preceding example, it is easily seen that $PC = DE = OE - OD$. Since $OE = x_1$ and $OD = x$, then $PC = x_1 - x$. Moreover, $P_1C = P_1E - EC = P_1E - PD$; since $P_1E = y_1$ and $PD = y$, then $P_1C = y_1 - y$. Substituting these values of PC and P_1C in (a), we obtain:

$$(PP_1)^2 = (x_1 - x)^2 + (y_1 - y)^2, \text{ and}$$

$$PP_1 = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}.$$

The line segment PP_1 measures the distance between the two points P and P_1. These points are any two points in the plane, not equidistant from the Y-axis. Hence, we may write:

$$d = \sqrt{(x_1 - x)^2 + (y_1 - y)^2},$$

where d represents the distance between *any two points* $P\ (x, y)$ and $P_1\ (x_1, y_1)$ in the plane.

In the special case when P and P_1 are equidistant from the Y-axis, the same formula holds good, as the reader can very easily prove.

The Greek letter Δ (delta) was defined, in Chapter 13, page 197, as "the change in." In the formula just derived, $x_1 - x$ is the change in the value of the abscissas, while $y_1 - y$ is the change in the value of the ordinates. The preceding formula may, therefore, be written $d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

Thus, if the distance between the points $(3, -5)$ and $(-4, 6)$ is to be found, choose either of the two points as P and the other as P_1. For example, let P be the point $(3, -5)$ and P_1 the point $(-4, 6)$. Then, $\Delta x = x_1 - x = (-4) - (+3) = -7$, and $\Delta y = (+6) - (-5) = 11$. Hence, $d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(-7)^2 + (11)^2} = \sqrt{49 + 121} = \sqrt{170} = 13.038$.*

PARALLEL AND PERPENDICULAR LINES

The use of coordinates is a unifying element in the element-

* Note that the same result is obtained if $(-4, 6)$ is taken as P and $(3, -5)$ as P_1.

ary branches of mathematics. Accordingly, it is a most powerful tool in simplifying and clarifying the solution of mathematical problems. Obviously, the initial procedure will be to develop other standard formulas which may be applied for a quick solution of these problems.

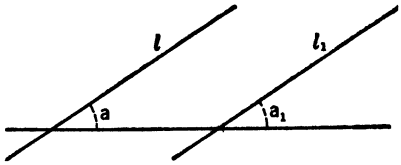


Fig. 109.

Suppose that lines l and l_1, Figure 109, are parallel. Draw a horizontal line intersecting the two parallel lines. Then, by Fact 14, the corresponding angles a and a_1 are equal. So that $\tan a = \tan a_1$. Since, as was shown in Chapter 13, page 203, the tangent ratios of a and a_1 measure the slopes of lines l and l_1, it follows that *if two lines are parallel, they have the same slope.*

Moreover, if l and l_1 have the same slope, then $\tan a = \tan a_1$, and therefore angles a and a_1 are equal. But if angles a and a_1 are equal, then, by Fact 10, the lines are parallel. Therefore, *if two lines have the same slope, the lines are parallel or they coincide.*

Now assume that lines l and l_1, Figure 110, are perpendicular to each other. The slopes of l and l_1 are measured by $\tan a$ and $\tan a_1$, respectively. Since l and l_1 are perpendicular to each other, angles a and a_1 are complementary. That is, if angle a is assumed to be 30°, then a_1 must be equal to 90° − 30°, that is, 60°.

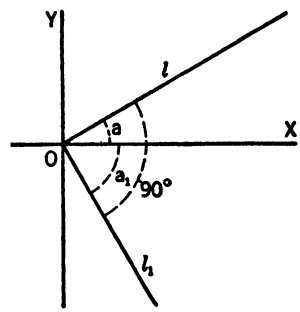


Fig. 110.

In Chapter 14, page 214, it was shown that if angles a and a_1 are complementary and in the same triangle, $\tan a$ is equal numerically to $\cot a_1$. However, if lines l and l_1 are assumed to be perpendicular to each other, then if l is in the first quadrant, l_1 will fall either in the second or the fourth quadrant, depending upon whether the right angle made by the two lines is measured in a counterclockwise or clockwise direction. In either case, the tangent and cotangent of a will have the sign opposite to that of the tangent and cotangent of a_1. The reader may easily ascertain, by referring back to Chapter 14, page 211, that this fact holds true regardless of the quadrant

in which line l is placed. For example: if l is placed in the second quadrant, l_1 will then fall in either the first or the third quadrant, and the tangent and the cotangent of the angle of inclination of l will have the sign opposite to that of the tangent and cotangent of the angle of inclination of l_1. Therefore, in general, if l and l_1 are perpendicular and a and a_1 are, respectively, their angles of inclination, $\tan a = -\cot a_1$.

From the definition of the trigonometric functions, given in Chapter 14, page 211, the reader can easily notice that these functions are related to each other. Thus, if A is any angle, it follows that:

$$\sin A = \frac{\text{ordinate of } A}{\text{distance of } A \text{ from origin}} = \frac{y}{v} = \frac{1}{\frac{v}{y}}$$

$$= \frac{1}{\frac{\text{distance of } A \text{ from origin}}{\text{ordinate of } A}} = \frac{1}{\csc A};$$

$$\cos A = \frac{\text{abscissa of } A}{\text{distance of } A \text{ from origin}} = \frac{x}{v} = \frac{1}{\frac{v}{x}}$$

$$= \frac{1}{\frac{\text{distance of } A \text{ from origin}}{\text{abscissa of } A}} = \frac{1}{\sec A};$$

$$\tan A = \frac{\text{ordinate of } A}{\text{abscissa of } A} = \frac{y}{x} = \frac{1}{\frac{x}{y}}$$

$$= \frac{1}{\frac{\text{abscissa of } A}{\text{ordinate of } A}} = \frac{1}{\cot A}.$$

That is, the sine and cosecant, the cosine and secant, the tangent and cotangent, respectively, of the same angle are reciprocals of each other.

We have just shown that, if l and l_1 are perpendicular and a and a_1 are, respectively, the angles of inclination of the two lines, $\tan a = -\cot a_1$. But from the preceding discussion, $\cot a_1 = \frac{1}{\tan a_1}$; hence, $\tan a = -\frac{1}{\tan a_1}$. Now, if m and m_1

denote, respectively, the slopes of l and l_1, then $m = \tan a$ and $m_1 = \tan a_1$. Hence, $m = -\frac{1}{m_1}$. In other words, *if two lines are perpendicular to each other, the slope of one is the negative reciprocal of the slope of the other.*

Thus, if one of two perpendicular lines has a slope of 4, the other must have a slope of $-\frac{1}{4}$. Or if one of two perpendicular lines has a slope of $-\frac{5}{6}$, the other must have a slope of $-\frac{1}{-\frac{5}{6}} = \frac{6}{5}$.

The reader may, by retracing the steps of the preceding proof, easily show that *if one of two lines has a slope equal to the negative reciprocal of the slope of the other, the lines are perpendicular.*

MID-POINT FORMULA

Another very important fundamental relation may be easily found as follows: Assume that the two distinct points P_1 (x_1, y_1) and P_2 (x_2, y_2), Figure 111, are the end points of some line segment which is not parallel to either axis; and denote the coordinates of the mid-point P by $(\bar{x}, \bar{y})$. From points P_1, P, and P_2, drop perpendicular line segments to the X-axis. Through P_1 and P draw the line segments P_1A and PB parallel to the X-axis. Then, since P_1A and PB are parallel to the same line, they are parallel to each other (Fact 17). Since PN and P_2R are perpendicular to the X-axis, then, by Fact 16, they are perpendicular to P_1A and PB, respectively. Hence, angles P_1AP and PBP_2 are right angles.

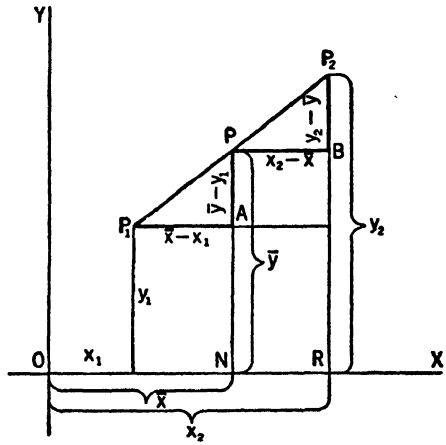


Fig. 111.

The parallel line segments P_1A and PB are cut by the transversal P_1P_2. So that, by Fact 14, the corresponding angles PP_1A and P_2PB are equal. The hypotenuse P_1P of right tri-

angle P_1PA is, by construction, equal to the hypotenuse PP_2 of right triangle PP_2B. That is, right triangle P_1PA has its hypotenuse P_1P and an acute angle PP_1A equal, respectively, to the hypotenuse PP_2 and an acute angle P_2PB of right triangle PP_2B. Therefore, by Fact 24, the triangles are congruent.

Corresponding sides of congruent triangles are equal. Hence, $P_1A = PB$. But $P_1A = \bar{x} - x_1$, and $PB = x_2 - \bar{x}$. Hence, $\bar{x} - x_1 = x_2 - \bar{x}$. Transposing, $2\bar{x} = x_2 + x_1$; and dividing both sides by 2, we obtain $\bar{x} = \frac{1}{2}(x_2 + x_1)$.

Similarly, since corresponding sides PA and P_2B are equal, $\bar{y} - y_1 = y_2 - \bar{y}$. So that, $2\bar{y} = y_2 + y_1$, and $\bar{y} = \frac{1}{2}(y_2 + y_1)$.

In the special cases where the line segment is parallel to either axis, the same formula holds good, as the reader can easily demonstrate.

In other words, *the abscissa* (or *ordinate*) *of the point midway between the two end points of any line segment is equal to one half the sum of the abscissas* (or *ordinates*) *of the end points.* Thus, if the end points of a line segment are $(-2, 6)$ and $(3, 4)$, the coordinates of the mid-point are $\bar{x} = \frac{1}{2}(-2 + 3) = \frac{1}{2}$ and $\bar{y} = \frac{1}{2}(6 + 4) = 5$.

THE EQUATION OF A LOCUS

The relationship between quantities is, in everyday life, very seldom given by a mathematical equation. Rather, this relationship is given by a definition of some kind. Very often the definition can be interpreted in terms of the *locus* of a point, that is, in terms of the curve traced by a point which moves according to a given law. The equation of the locus may then be obtained by (*a*) denoting the coordinates of the moving point P by (x, y); (*b*) by expressing the given condition as applied to P; and finally (*c*) by writing this condition in terms of the coordinates (x, y). The simplified expression in x and y is the equation of the locus.

Thus, assume that a point moves so that it is always equidistant from the points $(-1, 2)$ and $(3, 1)$. The equation of its locus may be found as follows:

Plot the given points as in Figure 112. Place the moving point P so as to be approximately equally distant from A $(-1, 2)$ and B $(3, 1)$, not in a particular place, such as the

axes, but rather in a general position conforming to the given condition, as shown in the figure. The statement of the problem imposes the condition that the distance from the moving point P to point A must remain equal to the distance from P to B. That is, $PA = PB$. To write this condition in terms of the coordinates (x, y) of point P, we apply the distance formula $d = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}$ (page 298). So that,

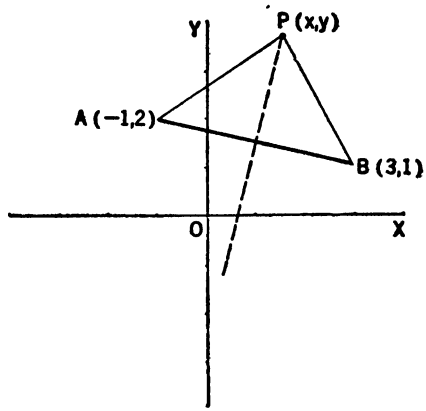


Fig. 112.

$$PA = \sqrt{(x + 1)^2 + (y - 2)^2}, \text{ and}$$

$$PB = \sqrt{(x - 3)^2 + (y - 1)^2}.$$

Since $PA = PB$, then,

$$\sqrt{(x + 1)^2 + (y - 2)^2} = \sqrt{(x - 3)^2 + (y - 1)^2}.$$

This expression is the equation of the locus, for it expresses the given law of motion. In its present form, however, the equation is not elegant and is awkward to manipulate. Hence, we simplify it as follows: Squaring both sides,

$$(x + 1)^2 + (y - 2)^2 = (x - 3)^2 + (y - 1)^2.$$

Expanding, we get:

$$x^2 + 2x + 1 + y^2 - 4y + 4 = x^2 - 6x + 9 + y^2 - 2y + 1.$$

Next, collecting terms, we obtain:

$$8x - 2y - 5 = 0.$$

This is an equation of the first degree in x and y, and hence shows that the point must move in a straight line.

The relative positions of line AB, connecting the two given points, and of the line traced by the moving point P might arouse the curiosity of a critical reader. The line traced by P seems to be the perpendicular bisector of line AB. Let us then investigate.

The slope of line AB may be easily found by applying the

formula $m = \frac{y_1 - y}{x_1 - x}$, discovered in Chapter 13, page 197. Choosing point A $(-1, 2)$ as P (x, y), and point B $(3, 1)$ as P_1 (x_1, y_1), the slope $m_{AB} = \frac{1-2}{3-(-1)} = \frac{-1}{4} = -\frac{1}{4}$. The line traced by point P is represented, as shown in the preceding discussion, by the equation $8x - 2y - 5 = 0$.

Hence, the slope $m_P = -\frac{A}{B} = -\frac{8}{-2} = +4$ (see page 201).

Evidently the negative reciprocal of m_P, namely, 4, is $-\frac{1}{4}$, which is the slope of AB. Therefore, the line traced by point P moving so that it is always equidistant from the points $(-1, 2)$ and $(3, 1)$ is perpendicular to line AB connecting these two points.

The fact that the line traced by P bisects line AB can be shown by proving that the coordinates of the mid-point of line AB lie also on the line traced by P. The coordinates of the mid-point of AB are:

$$\bar{x} = \tfrac{1}{2}(x_2 + x_1) = \tfrac{1}{2}[3 + (-1)] = 1,$$

and

$$\bar{y} = \tfrac{1}{2}(y_2 + y_1) = \tfrac{1}{2}(1 + 2) = \tfrac{3}{2}.$$

Every point on the line traced by P must satisfy its equation, namely, $8x - 2y - 5 = 0$. But substituting the coordinates of the mid-point of AB, that is, $(1, \frac{3}{2})$, in this equation gives $8(1) - 2(\frac{3}{2}) - 5 = 0$. Thus, the line traced by P passes through the mid-point of AB and therefore bisects AB.

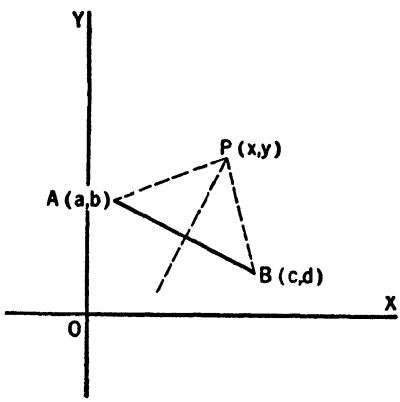


Fig. 113.

BISECTORS

This particular case leads us to suspect that any point moving so that it is always equidistant from two fixed points is the perpendicular bisector of the line joining them. The proof of this fact follows the same pattern as the particular case just discussed. Assume any two fixed points A and B, Figure 113, and let (a, b) and (c, d), respectively, represent the coordinates

of these points. Plot the moving point P so as to be approximately equally distant from A and B. As in the particular case, the law of motion states that the distance from P to A must remain equal to the distance from P to B. That is, $PA = PB$.

By the distance formula,

$$d = \sqrt{(x_1 - x)^2 + (y_1 - y)^2};$$

$$PA = \sqrt{(x - a)^2 + (y - b)^2}$$

and $$PB = \sqrt{(x - c)^2 + (y - d)^2}.$$

Since $PA = PB$,

$$\sqrt{(x - a)^2 + (y - b)^2} = \sqrt{(x - c)^2 + (y - d)^2}.$$

Squaring both sides and expanding gives:

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = x^2 - 2cx + c^2 + y^2 - 2dy + d^2.$$

Collecting terms, we have:

$$2cx - 2ax + 2dy - 2by + a^2 + b^2 - c^2 - d^2 = 0, \text{ or}$$
$$2(c - a)x + 2(d - b)y + a^2 + b^2 - c^2 - d^2 = 0. \qquad \text{(I)}$$

Since a, b, c, and d are any real numbers where $a \neq c$ and $b \neq d$, equation (I) is of the first degree in x and y, where $2(c - a)$ is the coefficient of x, $2(d - b)$ is the coefficient of y, and $a^2 + b^2 - c^2 - d^2$ is the constant term. Therefore, point P moves in a straight line.

The slope of line AB is $m_{AB} = \dfrac{\Delta y}{\Delta x} = \dfrac{d - b}{c - a}$. The slope of the line traced by P is, from equation (I), $m_P = -\dfrac{2(c - a)}{2(d - b)}$ $= -\dfrac{c - a}{d - b}$. But the negative reciprocal of m_{AB}, namely, $-\dfrac{1}{\dfrac{d - b}{c - a}} = (-1) \cdot \dfrac{c - a}{d - b}$ is m_P. That is, $m_P = -\dfrac{1}{m_{AB}}$. Therefore, the line traced by P is perpendicular to AB.

Moreover, the coordinates of the mid-point of AB are $\bar{x} = \frac{1}{2}(c + a)$ and $\bar{y} = \frac{1}{2}(d + b)$. Every point on the line traced by P must satisfy equation (I). But, substituting the coordinates of the mid-point of AB, namely, $\frac{1}{2}(c + a)$ and $\frac{1}{2}(d + b)$ in (I), we obtain:

$2(c - a)[\frac{1}{2}(c + a)] + 2(d - b)[\frac{1}{2}(d + b)] + a^2 + b^2 - c^2 - d^2 = 0$, or

$(c - a)(c + a) + (d - b)(d + b) + a^2 + b^2 - c^2 - d^2 = 0$, so that

$c^2 - a^2 + d^2 - b^2 + a^2 + b^2 - c^2 - d^2 = 0$, that is, $0 = 0$.

In other words, equation (I) is satisfied by the coordinates of the mid-point of AB. Hence, the line traced by P must bisect AB. Therefore,

*Fact 71: The locus of points equidistant from two given points is the perpendicular bisector of the line joining them.**

As a contrast to the analytic method of proof used in the above demonstration, and to recall the purely deductive reasoning used in preceding proofs, consider the following problem:

Fact 72: The locus of points within an angle equidistant from the sides is the ray that bisects the angle.

The solution of this problem requires that we show:

(1) Every point in the bisector of an angle equidistant from the sides of the angle; and

(2) Every point equidistant from the sides of an angle lying in the bisector of the angle.

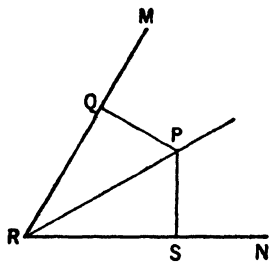


Fig. 114.

To prove the first part, it is necessary to demonstrate that, given any angle such as MRN in Figure 114 and its bisector RP, then any point in the bisector, say P, is equidistant from MR and NR.

Since the distance from a point to a line is always measured along the perpendicular from the point to the line, draw from P a perpendicular, PQ, to MR and PS perpendicular to RN. The triangles PQR and PSR formed in this manner are right triangles. Side PR is common to both triangles. Since PR is the bisector of angle MRN, angles QRP and PRS are equal. So that right triangle PQR has its hypotenuse PR and an acute

* The definition of the "locus of an equation" requires the proof that (1) the coordinates of every point of the locus satisfy the equation; and that (2) every point whose coordinates satisfy the equation lies on the locus. A complete proof of this type is given at the beginning of Chapter 13. The method used above demonstrates only the first part of the proof. However, since in the majority of cases the second part requires merely reversing the steps of the first part, it is usually left to the reader.

angle QRP equal, respectively, to the hypotenuse PR and an acute angle PRS of right triangle PSR. Therefore, by Fact 24, the triangles are congruent. It follows that the corresponding sides PQ and PS are equal. Hence, any point P in the bisector of angle MRN is equidistant from its sides.

The second part requires us to show that if any point P is equidistant from the sides of a given angle MRN, then PR is the bisector of the given angle.

From the statement of the problem, PQ is perpendicular to MR; PS is perpendicular to NR, and $PQ = PS$, for the distance from a point to a line is measured along the perpendicular from the point to the line, and P is given equidistant from MR and NR. To prove that PR is the bisector of angle MRN, it is sufficient to show that angles QRP and PRS are equal.

The triangles PQR and PSR are, by construction, right triangles. Side PR is common to both triangles, and PQ is given equal to PS. That is, right triangle PQR has its hypotenuse PR and a leg PQ equal, respectively, to the hypotenuse PR and a leg PS of right triangle PSR. Therefore, by Fact 27, the triangles are congruent, so that the corresponding angles QRP and PRS are equal. Hence, a point P equidistant from the sides of angle MRN lies in the bisector of the angle.

DEMONSTRATIONS BY ANALYSIS

Many more important facts may be demonstrated by analysis. However, if this method is used, the following important points should be noted. The property to be proved is independent of the position of the figure. Therefore, there is no loss of generality in assuming the figure in the most convenient position. That is, the figure may be drawn first, and then a careful choice of the position of the axes, with respect to the figure, can be made. However, the figure itself must not be made special in any case. For example, if a proof applying to triangles in general is being demonstrated, a right triangle or any other special type of triangle should not be used. The notation must be such that the coordinates themselves express the conditions imposed by the given problem. But again the conditions imposed must remain general, that is, numerical coordinates should not be assigned to the points.

For example: Consider any parallelogram. It is to be shown that:

Fact 73: The diagonals of a parallelogram bisect each other.

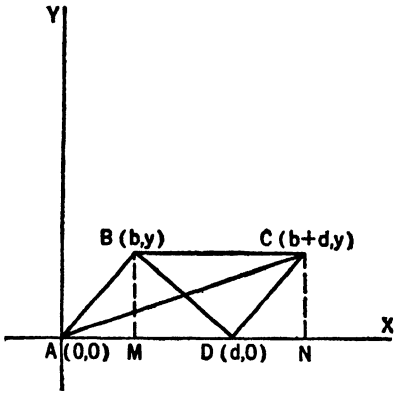


Fig. 115.

Draw the X-axis along one of the bases of the parallelogram, and the Y-axis through a vertex of this base, as shown in Figure 115.

The opposite sides of a parallelogram are equal (Fact 36). Hence, $BC = AD$. If the perpendicular lines BM and CN are drawn, the figure formed, $BMNC$, is a rectangle. So that, again by Fact 36, $BC = MN$. Therefore, $AD = MN$. It is easily seen from the figure that $AN = AD + DN$, or $AN - AD = DN$; and since $AD = MN$, then $AN - MN = DN$. Similarly, $AN = AM + MN$, or $AN - MN = AM$. Therefore, $DN = AM$.

The coordinates of A are (0, 0). Those of B may be chosen as (b, y), and those of D as $(d, 0)$. The coordinates of C are then determined as follows: The distance from C to the Y-axis is measured by AN. Line $AN = AD + DN$; or, since $DN = AM$, we may write $AN = AD + AM$. But AD is the distance from D to the Y-axis, and AM is the distance from B to the Y-axis. Hence, AD and AM may be, respectively, denoted by d and b. So that $AN = AM + AD = b + d$. That is, the abscissa of C is $b + d$. Moreover, since by construction BC is parallel to AD, points C and B are at the same distance from the X-axis (Fact 39). That is, the ordinates of the two points are equal in value. But the ordinate of B is y; hence, the ordinate of C may also be denoted by y. So that the coordinates of C are $(b + d, y)$.

The coordinates of the mid-point of BD are

$\bar{x} = \frac{1}{2}(x_2 + x_1) = \frac{1}{2}(b + d)$ and

$\bar{y} = \frac{1}{2}(y_2 + y_1) = \frac{1}{2}(y + 0) = \frac{1}{2}y$ (see page 302). Similarly, the coordinates of the mid-point of AC are

$\bar{x} = \frac{1}{2}[(b + d) + 0] = \frac{1}{2}(b + d)$ and

$\bar{y} = \frac{1}{2}(y + 0) = \frac{1}{2}y$.

That is, the mid-points of the diagonals AC and BD coincide. Therefore, the diagonals bisect each other.

Problems

1. The span of a roof truss, that is, the distance between the supporting walls, is 60 feet. If the height of the roof truss is 15 feet, find the length of a rafter.

2. A military map gives the location of two fortified positions with reference to a north-south line and an east-west line as follows: fort A is $(-3, 2)$ and fort B is $(5, -4)$. Find the distance between the two forts.

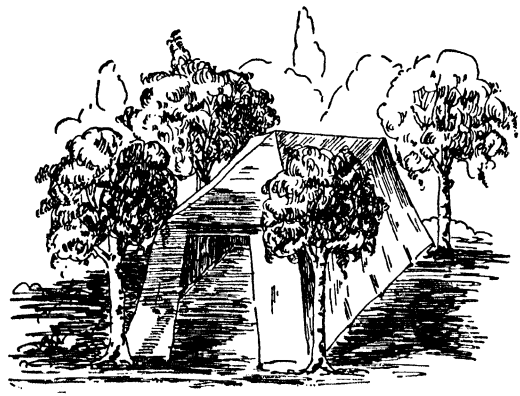
3. What must be the diameter of a round steel bar so that a square bolt head $1\frac{3}{4}$ inches on a side may be milled from it?

4. A military observer's balloon ascends from the ground along a line inclined 60° to the horizontal at a velocity of 3 miles an hour. What is the altitude when 10 minutes have elapsed?

5. In a mined harbor, the danger area is a triangle determined by the points A $(-5, 3)$, B $(10, 10)$, and C $(6, -7)$. If the unit of measurement is 100 yards, what is the area of the triangle?

6. A smokestack is held in position by 3 guy wires fastened to the stack 75 feet above the ground. If the guy wires reach the ground 50 feet from the foot of the stack, find the length of a guy wire.

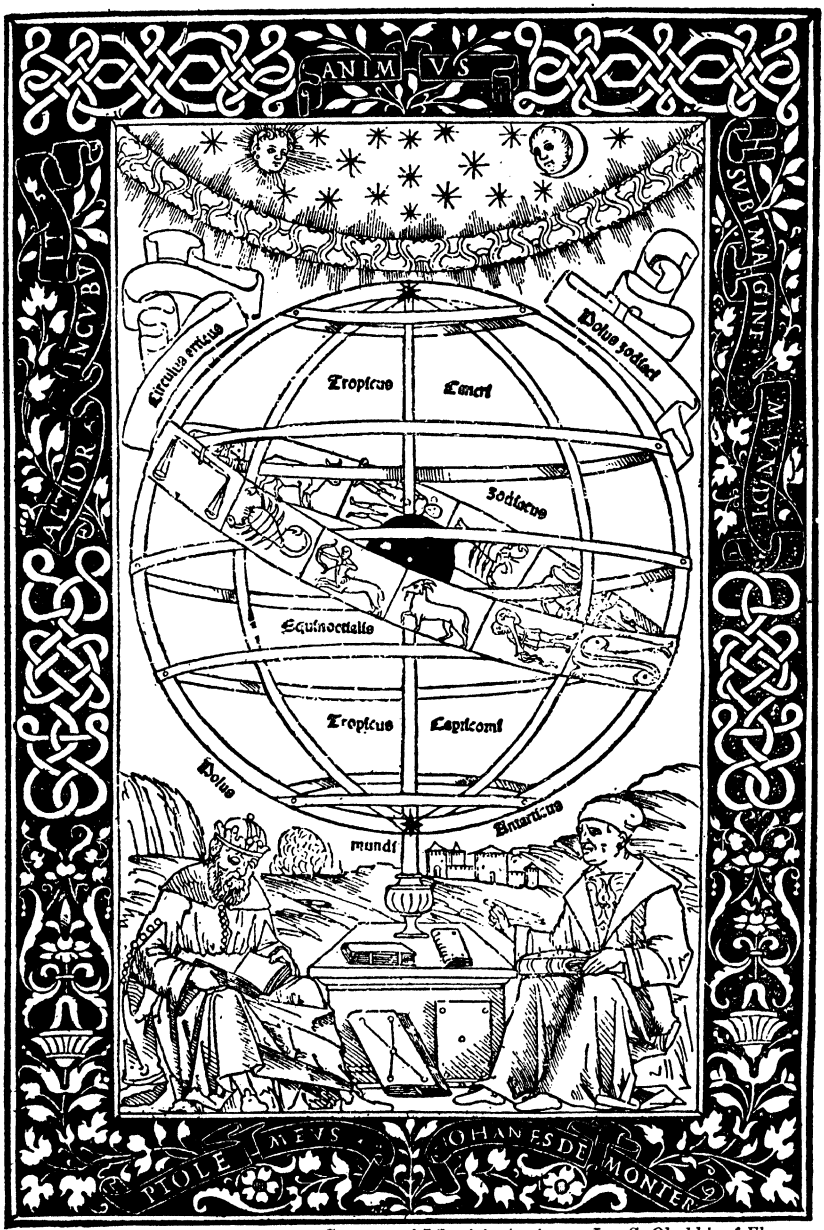
7. Four trees, A, B, C, and D, form a square. A tourist pitches a square tent among the trees as in the figure. After his departure another tourist comes and pitches a square tent of twice the floor space of the first tent amongst the same trees. How was it done?



8. A point moves so that its distance from $(5, 0)$ is always equal to its distance from the Y-axis. Find the equation of its locus.

9. Show that the points $(-3, -2)$, $(1, 2)$, and $(-5, 0)$ form a right triangle.

10. Show that the mid-point of the hypotenuse of a right triangle is equidistant from the vertices.



Courtesy of Librairie Ancienne, Leo S. Olschki, of Florence

From Johannes Regiomontanus, *Epitoma in Almagestum Ptolemaei*, 1496.

20 ENCIRCLING SPACE

Circles are as old as the history of mankind. Considered as the perfect curve, the circle was chosen by ancient astronomers to represent the path of a heavenly body. The circle was first defined mathematically by the Greeks, who took the practical but unconnected body of facts of the Egyptians and transformed them into a logically connected abstract science. To effect this change, the Greeks formulated a set of definitions and fundamental notions, or in other words, a set of rules. These rules, or postulates, limited them to the use of only two instruments, a straight unmarked ruler for drawing straight lines and the compasses for drawing circles. As a result of this limitation, there arose three famous problems, namely: squaring the circle, duplicating the cube, and the trisection of an arbitrary angle. Mathematicians struggled through centuries for their solution. It was not until the nineteenth century, when Galois, Cauchy, Jordan, and others developed the theory of groups that the impossibility of solving these problems, under the rules postulated by the Greeks, was demonstrated. Modern mathematics is not bound by these restrictions, and exact solutions to these problems have been obtained. In fact, as will be shown later, we can find to any degree of accuracy a square equal in area to a given circle.

EQUATION OF THE CIRCLE

A circle was defined, in Chapter 7, page 99, as the path in a plane of a point which moves so that it is always at the same distance from a fixed point.

To find the equation of a circle from this definition, draw the X- and Y-axes. Assume the fixed point C anywhere in the plane, as in Figure 116, and let the coordinates of C be represented by (h, k). Plot the moving point P at any distance, r, from C, where $r \neq 0$, and denote its coordinates by (x, y).

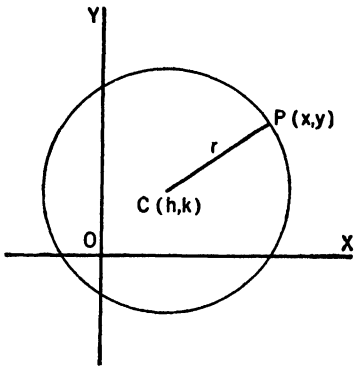


Fig. 116.

The law of motion states that the distance from the moving point $P(x, y)$ to the fixed point $C(h, k)$ must remain always equal to r. That is, $PC = r$.

Applying the distance formula, $d = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}$, we obtain:

$$PC = \sqrt{(x - h)^2 + (y - k)^2}.$$

Since $PC = r$, we may write:

$$\sqrt{(x - h)^2 + (y - k)^2} = r;$$

and squaring both sides,

$$(x - h)^2 + (y - k)^2 = r^2. \tag{1}$$

This expression is the equation of a circle whose center is (h, k), and whose radius is r, (in which r is not equal to zero).

If the center of the circle is at the origin, the coordinates of C, namely, (h, k), become $(0, 0)$, and equation (1) reduces to the form:

$$x^2 + y^2 = r^2. \tag{2}$$

That is, this expression is the equation of a circle whose center is at the origin and whose radius is r.

Thus, the equation of a circle whose center is at $(2, -3)$ and whose radius is equal to 5 is $(x - 2)^2 + (y + 3)^2 = 25$. Similarly, the equation of a circle whose center is at the origin and whose radius is equal to 3 is $x^2 + y^2 = 9$.

If equation (1) is expanded, and r^2 is transposed to its left-hand side, we obtain:

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 - r^2 = 0,$$

which may be written:

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

Since the letters h, k, and r represent any real numbers, and $r \neq 0$, we may conclude that *a circle is represented by an equation of the second degree in two variables, such as* x *and* y, *in which no* xy *term appears, and in which the terms of the second degree have equal coefficients and are the sum of the squares of* x *and* y.

Can we conclude that every equation of the second degree fulfilling the conditions just described represents a circle? To decide this question, let us first consider a special case, such as $x^2 + y^2 - 2x + 6y - 15 = 0$. If this equation can be expressed in form (1), it will represent a circle and its graph may then be easily plotted. The reader can easily note that form (1), namely, $(x - h)^2 + (y - k)^2 = r^2$, contains two perfect squares on its left-hand side. Therefore, the given equation may be expressed in form (1) by completing the square of the terms in x and y. As shown in Chapter 16, page 258, to complete the square, transpose the constant term and rewrite the given equation, thus, $x^2 - 2x + y^2 + 6y = 15$.

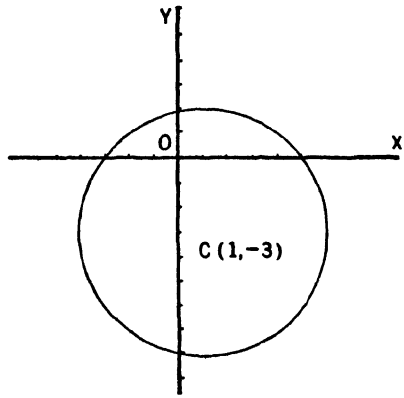


Fig. 117.

Consider first the terms in x, namely, $x^2 - 2x$. The missing constant term must be equal to the square of one-half the coefficient of x. Hence, taking one-half of -2, namely, -1, and squaring it, we obtain $+1$, which is the constant term. So that $x^2 - 2x + 1 = (x - 1)^2$. Similarly, $y^2 + 6y$ is made into a perfect square by taking one-half of $+6$, that is, $+3$, squaring it, thus obtaining $+9$, and adding it to the expression. Thus, $y^2 + 6y + 9 = (y + 3)^2$. In order to obtain the perfect squares, $+1$ and $+9$ were added to the left-hand side of the equation. Hence, the same quantities must be added to the other side. That is, $x^2 - 2x + 1 + y^2 + 6y + 9 = 15 + 1 + 9$, or simply $(x - 1)^2 + (y + 3)^2 = 25$. The equation is now of form (1), and therefore represents a circle whose center is $(1, -3)$ and whose radius is 5. The graph is shown in Figure 117.

GENERAL FORM OF THE EQUATION OF A CIRCLE

Although the solution of this particular case does not answer the question set up at the beginning of the problem, it nevertheless points the way to the method to be followed in the general case.

Equation (1), when expanded and rearranged, may be written, as previously shown, in the form:

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

Since h, k, and r are any constants and $r \neq 0$, this equation may be expressed in the form:

$$x^2 + y^2 + Dx + Ey + F = 0, \tag{3}$$

where D, E, and F are real numbers.

Proceeding as in the particular case just discussed, we complete the squares as follows:

Rewrite the equation thus,

$$x^2 + Dx + y^2 + Ey = -F.$$

Considering the terms in x, namely, $x^2 + Dx$, take one-half of D, that is, $\frac{D}{2}$, square it, obtaining $\frac{D^2}{4}$, and add it to the expression. So that $x^2 + Dx + \frac{D^2}{4} = \left(x + \frac{D}{2}\right)^2$. Similarly, $y^2 + Ey$ is made into a perfect square by taking one-half of E, namely, $\frac{E}{2}$, squaring it, obtaining $\frac{E^2}{4}$, and adding it to the expression. Thus, $y^2 + Ey + \frac{E^2}{4} = \left(y + \frac{E}{2}\right)^2$. In order to obtain perfect squares, $\frac{D^2}{4}$ and $\frac{E^2}{4}$ have to be added to the left-hand side of (3). Hence, adding the same quantities to the other side, we obtain:

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} = \frac{D^2}{4} + \frac{E^2}{4} - F$$

or

$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{1}{4}(D^2 + E^2 - 4F).$$

This equation is of form (1). Therefore, it represents a circle whose center is $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ and whose radius, r, is equal to $\frac{1}{2}\sqrt{D^2 + E^2 - 4F}$.

Since the radius is $\frac{1}{2}\sqrt{D^2 + E^2 - 4F}$, the graph of the circle depends on the value of $D^2 + E^2 - 4F$.

If $D^2 + E^2 - 4F$ is positive, the radius is a real number greater than zero, so that the locus of (3) is a circle.

If $D^2 + E^2 - 4F$ is equal to zero, the only real values satisfying (3) are $\left(-\frac{D}{2}, -\frac{E}{2}\right)$. Therefore, the graph of the given equation is the single point $\left(-\frac{D}{2}, -\frac{E}{2}\right)$. In this case, the locus is said to be a *point circle* or a circle whose radius is zero.

Finally, if $D^2 + E^2 - 4F$ is negative, there is no locus, for no real values satisfy (3).

Therefore, *the locus of an equation of the second degree in* x *and* y *in which no* xy *term appears, and in which the terms of the second degree have equal coefficients and are the sum of the squares of* x *and* y, *is a circle* or *a point circle, or there is no locus.*

Thus, the locus of $2x^2 + 2y^2 + 8x - 4y - 8 = 0$ is a circle or a point circle, or possibly the equation has no locus. To express this equation in form (3), divide both sides by 2, obtaining $x^2 + y^2 + 4x - 2y - 4 = 0$. A comparison with form (3) shows $D = 4$, $E = -2$, and $F = -4$. So that $D^2 + E^2 - 4F = 16 + 4 + 16 = 36$. Since the result obtained is positive, the locus of the given equation is a circle. The radius of the circle is $\frac{1}{2}\sqrt{36} = \frac{1}{2}(6) = 3$, and the coordinates of the center are $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ or $\left(-\frac{4}{2}, -\frac{-2}{2}\right)$, that is, $(-2, 1)$. The graph of the given equation is shown in Figure 118.

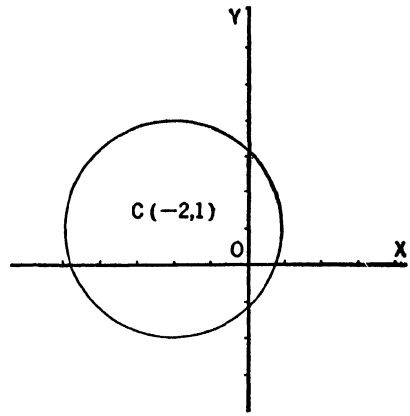


Fig. 118.

CIRCLE DETERMINED BY THREE POINTS

The *general form* of the equation of a circle, namely, form (3), contains three arbitrary constants, D, E, and F. Similarly, the *standard form*, that is, form (1), also contains three arbitrary constants, h, k, and r, where $r \neq 0$. If the equation of a circle is to be found, enough information about the circle must be given so that three equations can be set up to determine these con-

stants. So that, in general, a circle is determined by three conditions. In particular, if the coordinates of three points in turn are substituted in one of the two forms, we obtain three simultaneous equations which can be solved (see page 250) for D, E, and F or h, k, and r, as the case might be. Therefore, *a circle is determined by three points, provided that these are distinct and not in the same straight line.*

For example, find the equation of the circle passing through the points (1, −2), (4, 1), and (−1, −1).

Using form (3) and substituting the coordinates of these points in turn, we obtain:

$$\begin{aligned} D - 2E + F &= -5; \\ 4D + E + F &= -17; \\ -D - E + F &= -2. \end{aligned}$$

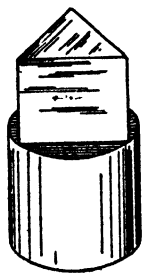
Solving these equations, as shown on page 250, $D = -\frac{21}{9}$, $E = -\frac{15}{9}$, and $F = -6$. Therefore, the required equation is:

$$x^2 + y^2 - \frac{21}{9}x - \frac{15}{9}y - 6 = 0,$$

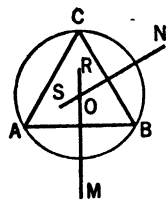
or

$$9x^2 + 9y^2 - 21x - 15y - 54 = 0.$$

The analytic proof just given, although brief, has the disadvantage of not showing the method of construction of a circle passing through three points.



(a)



(b)

Fig. 119.

Assume that a triangular punch with a cylindrical shank, as shown in Figure 119 (*a*), is to be made. The mechanic wishes to ascertain the shank of smallest diameter that he can use so that the punch will have the required triangular section.

The problem may be solved as follows: Draw the required triangular section ABC in Figure 119 (*b*). The diameter of the circle passing through points A, B, and C will determine the cylindrical shank to be used.

Construct, as shown in Chapter 9, page 127, RM, the per-

pendicular bisector of the line joining points A and B. Similarly, draw SN perpendicular to and bisecting the line joining points B and C. If RM and SN did not intersect, they would be parallel. But if RM and SN were parallel, AB and CB, by Fact 17, would have to be parallel. Since AB and CB are not parallel, RM and SN intercept at some point O.

Point O is on the perpendicular bisector of AB. Hence, it is equidistant from A and B (Fact 71). Similarly, point O is equidistant from C and B. Therefore, O is equidistant from A, B, and C. So that, if a circle with radius OA and with O as a center is drawn, it will pass through A, B, and C.

Since RM and SN can intersect at only one point, and only one straight line can be drawn from O to B, only one circle can be drawn through the three given points. That is,

Fact 74: Through any three given points not lying in a straight line, one circle and only one can be drawn.

The diameter of the circle passing through A, B, and C determines, therefore, the shank of smallest diameter which can be used to construct the required punch.

From the proof of Fact 74, just given, it is easily seen that:

Fact 75: The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the three vertices.

DEFINITIONS AND FUNDAMENTAL STATEMENTS

Before proceeding with the study of circles, it is convenient to define certain terms frequently used. Furthermore, as was pointed out in the first chapter, in order to reason deductively, we must have some fundamental statements which are taken for granted. Hence, a few of these intrinsic properties will be stated.

An arc that is half of a circle is called a *semicircle.*

Any straight line segment joining the extremities of an arc, such as CD in Figure 120, is called a *chord.* Every arc has one and only one chord. But evidently a chord determines or *subtends* two arcs of a circle—a *minor arc,* that is, an arc less than the semicircle, and a *major arc* larger than the semicircle, unless the chord is a diameter. Thus, arc CBD has the chord CD. But the chord CD subtends the minor arc CBD and the major arc $CMND$. Unless otherwise stated, by the arc of a chord we shall mean the minor arc.

If the vertex of an angle is at the center of a circle, and the sides are radii of the circle, the angle is called a *central angle*, such as MON in Figure 120. The minor arc MN cut off by the sides of the angle is its *intercepted arc*.

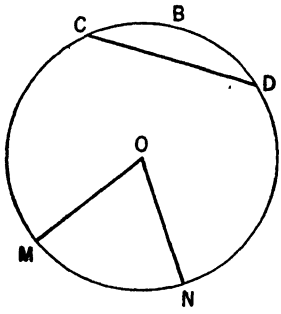


Fig. 120.

Every diameter bisects a circle, for by folding the circle on a diameter, the two arcs thus formed are seen to coincide.

A straight line cannot intersect a circle in more than two points, for if the line intersected the circle in more than two points, it would be possible to draw three or more radii from the center to the straight line. But, since all radii of the same circle are equal, this is impossible (Fact 35).

Fact 76: If in the same circle, or in equal circles, two central angles are equal, the arcs which they intercept are equal.

Since the central angles are equal, if the center and a side of the central angle of one circle are placed so that they coincide with the center and the corresponding side of the central angle of the other circle, the other side of the central angles will coincide. Therefore, the arcs will coincide, for all points on a circle are equidistant from the center.

Moreover, draw two equal circles and construct on them two equal arcs. Draw the sides of the central angles determined by these arcs. Superimpose one circle upon the other so that the centers and the arcs, respectively, coincide. Since now the corresponding sides of the central angles of both circles each have two points in common, the respective sides coincide. Therefore, the central angles are equal. Thus,

Fact 77: If in the same circle, or in equal circles, two arcs are equal, the central angles which they determine are equal.

ARCS AND CHORDS

The specifications of a stone bridge call for arches which are to be arcs of equal circles. After the arches are constructed, an engineer desires to ascertain whether they are in fact arcs of equal circles. He first measures the chord determined by each arc, and finds the chords to be equal in length. He then

concludes that the arcs are equal, basing this conclusion upon

Fact 78: If in the same circle, or in equal circles, two chords are equal, the arcs which they determine are equal.

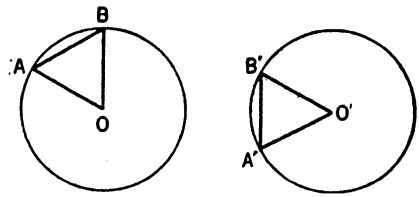


Fig. 121.

The reader may easily prove this fact by showing that, if the two chords AB and $A'B'$ in Figure 121 are equal, triangles AOB and $A'O'B'$ are congruent, so that the central angles are equal and therefore the arcs are equal.

Brown Brothers

Ponte Vecchio and the River Arno, Florence, Italy.

By following the same reasoning, the reader can easily show that

Fact 79: If in the same circle, or in equal circles, two arcs are equal, their chords are equal.

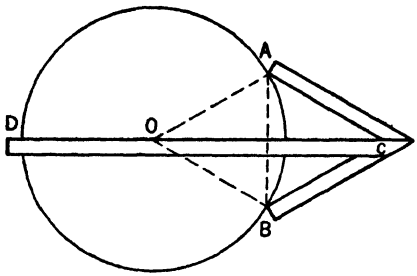


Fig. 122.

A mechanic wishing to locate the center of a cylindrical shaft or of a circular disk frequently uses the instrument shown in Figure 122. Here AC and CB are equal, and the edge DC bisects angle ACB. When points A and B touch the circle, edge CD is perpendicular to and bisects the chord AB, for the triangle ABC is isosceles ($AC = BC$), and the bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base (Fact 4). So that, in order to show that DC passes through the center, we must now prove that the perpendicular bisector of a chord passes through the center of the circle.

Consider a circle O, Figure 123, with a diameter CD perpendicular to a chord AB at M. Draw the radii OA and OB. Since all radii of the same circle are equal, $OA = OB$, so that right triangle OMA has its hypotenuse OA and a leg OM equal to the hypotenuse OB and a leg OM of right triangle OMB. Therefore, by Fact 27, the triangles are congruent.

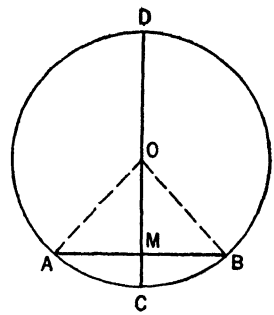


Fig. 123.

Since corresponding parts of congruent figures are equal, $AM = MB$ and angle AOM is equal to angle MOB. But if central angles AOM and MOB are equal, arcs AC and CB intercepted by them are, by Fact 76, equal. Moreover, angles DOA and DOB are also equal, for they are, respectively, the supplements of the equal angles AOM and MOB. So that, again by Fact 76, arc AD is equal to arc BD. Therefore,

Fact 80: The diameter perpendicular to a chord of a circle bisects the chord and the arcs determined by the chord.

Since the center of the circle is equidistant from A and B (Figure 123), it is on the perpendicular bisector of the chord, that is, on CD (Fact 71). Therefore, the perpendicular bisector of the chord must coincide with the diameter. For, if CD does not coincide with the diameter, we have, by construction, CD perpendicular to AB, and, by Fact 80, the diameter is also perpendicular to AB. Therefore, CD and the diameter would have to be parallel (Fact 11). So that we would have two perpendiculars, namely, CD and the diameter, drawn to line segment AB from an external point O, which, by Fact 12, is impossible. Therefore:

Fact 81: The perpendicular bisector of a chord of a circle passes through the center of the circle.

The mechanic wishing to locate the center of a circular disk applies the instrument shown in Figure 122 in two different positions, determining thereby two diameters. The intersection of the two diameters is the center of the disk.

EQUAL CHORDS

Consider next a number of shafts which have been grooved as shown in Figure 124. The foreman of the shop checked the width and depth of the grooves by measuring the distance from the bottom of the groove to the opposite side of the circular end. The line segment representing the bottom of the groove, if extended at both ends until it meets the circle, would then be a chord of the circle. In order to justify this method of checking, it must be shown that:

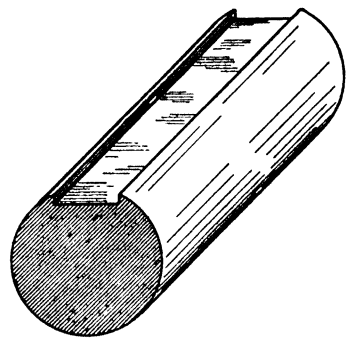


Fig. 124.

Fact 82: If in the same circle, or in equal circles, two chords are equidistant from the center, they are equal.

Consider the circle O, Figure 125, in which chords AB and CD are equidistant from the center, that is, $OE = OF$, and OE and OF are respectively perpendicular to AB and CD.

Draw the radii OA and OC. Since all radii of the same circle are equal, $AO = OC$; so that the right triangle AOE has its hypotenuse AO and a leg OE equal, respectively, to the

hypotenuse OC and a leg OF of right triangle COF. Therefore, by Fact 27, the triangles are congruent. It follows that the corresponding sides AE and CF are equal. But a diameter perpendicular to a chord bisects the chord (Fact 80). Therefore, chord AB is equal to chord CD.

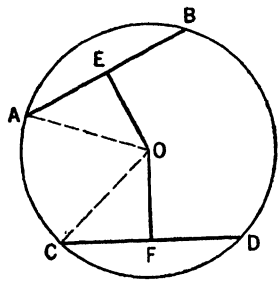


Fig. 125.

The reader may easily show that:

Fact 83: If in the same circle, or in equal circles, two chords are equal, they are equidistant from the center.

For in this case, CF is given equal to AE (Figure 125). By drawing the radii and applying again Fact 27, the right triangles may be proved congruent. Then $OE = OF$, and, therefore, the chords are equidistant from the center.

TANGENTS

A *tangent line* to a circle is defined as a straight line which touches the circle at only one point. This highly intuitive definition, although not applicable to all curves, is nevertheless adequate for the present.

The latitude of a point was defined in Chapter 14, page 224, as the angle made by a line segment (OP in Figure 126) drawn from the position of the observer on the earth's surface to the center of the earth with a line segment, OE, drawn from the point of intersection of the observer's meridian and the equator to the center of the earth. That is, the latitude of P is measured by angle EOP. Inquiring readers undoubtedly must have wondered how such an angle could be measured.

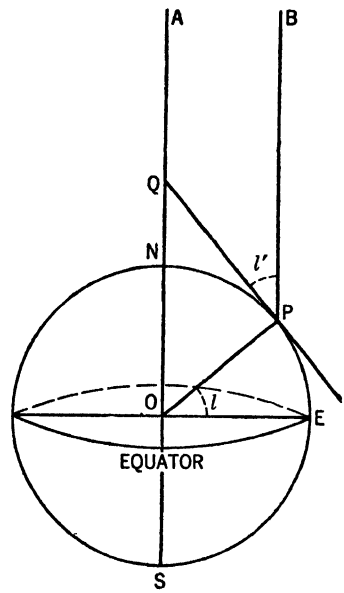


Fig. 126.

The question may be settled as follows: The distance from

the earth to the pole star is so great that, for all practical purposes, the earth's axis SOA and the line of observation to the pole star, namely, PB, may be considered parallel. A plane tangent to the earth at P, represented by tangent line PQ in the figure, determines the horizon at that point. If we can show that tangent line PQ is perpendicular to radius PO, angles l and l' may easily be proved equal.

Consider, then, any circle O, Figure 127, with MN tangent to the circle at P. From any other point on MN draw OA. Since MN is tangent to the circle, it touches the circle at point P only. Therefore, point A is outside the circle and line segment OA will intersect it at some point B. Segment OB is, then, a part of line segment OA. Hence, OA is greater than OB, for the whole is greater than any of its parts. Moreover, OB and OP, being radii of the same circle, are equal. Thus, OA is greater than OP. That is, OP is the shortest line segment that can be drawn from O to MN. Therefore, by Fact 32, it is the perpendicular from O to MN. In other words:

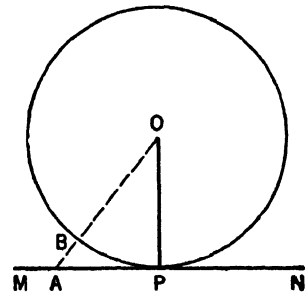


Fig. 127.

Fact 84: A tangent to a circle is perpendicular to the radius drawn to the point of contact.

Returning now to Figure 126, it is easily seen that, by the fact just proved, line PQ tangent to circle NES is perpendicular to the radius OP, so that angle OPQ is a right angle. Furthermore, angle EOS is, by construction, also a right angle.

As previously stated, due to the immense distance from the earth to the pole star, we have assumed, for practical purposes, lines SA and PB to be parallel. These two lines are cut by the transversal OP, so that, by Fact 13, the alternate interior angles POS and OPB are equal.

Angle POS = right angle EOS + angle l. Angle OPB = right angle OPQ + angle l'. Hence, right angle EOS + angle l = right angle OPQ + angle l'. But right angles EOS and OPQ are equal, for all right angles are equal. Therefore, angles l and l' are equal.

An observer at the earth's surface may, therefore, determine

his latitude by measuring the angle which the line of the horizon makes with the line of observation to the pole star.

A very inexpensive instrument to ascertain the approximate latitude of a point may be constructed as follows: On a carpenter's level, fix a protractor as shown in Figure 128. To determine the latitude, point a rotating stick, placed as in the figure, to the north star while maintaining the carpenter's level so that the bubble is in the center of the tube. The angle intercepted by the rotating stick is the observer's latitude, assuming, of course, that the rotating stick and the carpenter's level remain in a vertical plane.

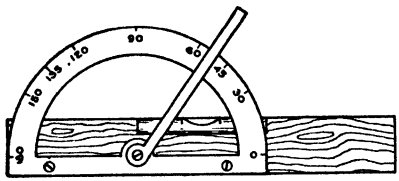


Fig. 128.

From the proof of Fact 84 it may be easily deduced that:

Fact 85: A line perpendicular to a radius at its extremity on the circle is tangent to the circle.

For, since every point on MN, Figure 127, except P is outside the circle, then by definition MN is tangent to the circle.

TWO TANGENTS

Suppose that two straight railroad tracks intersecting at a point M, Figure 129, are to be connected by a circular track. The circular track is to be placed tangent to tracks (1) and (2) and 1,000 ft. from point M measured along track (1). The point at which the circular track will come in contact with track (2) is to be marked off. If we can show that the distance along track (2) will be equal to the distance along track (1), it will then be a simple matter to locate the point of contact of track (2) with the circular track.

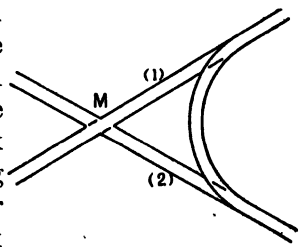
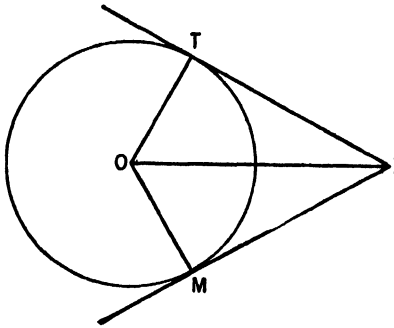


Fig. 129.

Consider, then, any circle O, with PT and PM tangent to O at T and M, respectively. Draw line segment OP and radii OT and OM. Then, by Fact 84, tangents PT and PM are, respectively, perpendicular to radii OT and OM drawn to the points of contact. So that triangles OTP and OMP are right triangles. Line segment OP is common to both triangles, and

since all radii of the same circle are equal, line segment OT is equal to line segment OM. That is, right triangle OTP has its hypotenuse OP and a leg OT equal, respectively, to the hypotenuse OP and a leg OM of right triangle OMP. Therefore, by Fact 27, the triangles are congruent. Since corresponding sides of congruent triangles are equal, it follows that PT and PM are equal. That is,



Fact 86: Tangents drawn to a circle from an external point are equal.

To locate the point of contact of the circular track along track (2) (Figure 129) with track (2), simply measure from point M the same distance at which the circular track is to contact track (1), namely, 1,000 ft.

PARALLEL CHORDS

A wooden pattern to be used in making a mold for a cast iron piece is to have the shape shown in Figure 130. It is important that the two concave sides have exactly the same curvature. To construct the pattern, a cylindrical piece of wood is first lathed to the proper diameter. The patternmaker then draws two parallel line segments at the proper distances from the center, on one of the flat circular ends. He concludes that the two arcs intercepted by the parallel line segments are equal, and therefore has the cylindrical piece of wood planed to meet these lines. This conclusion may be justified as follows:

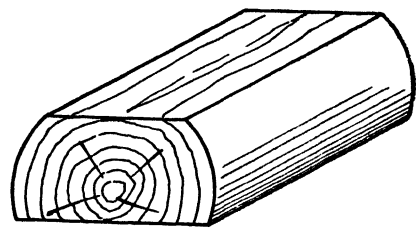


Fig. 130.

Consider the circle O, Figure 131, with AB and CD, two parallel lines, intersecting the circle at A, B, C, and D, respectively.

Draw the diameter MN perpendicular to CD. Then MN is also perpendicular to AB. For, by Fact 16, if a line is perpendicular to one of two parallel lines it is perpendicular to the other also. Since the diameter MN is perpendicular to chord CD, it bisects CD and arc CMD (Fact 80). So that arc CM is equal to arc MD. Similarly, since MN is perpendicular to AB, it bisects arc $ACMDB$ (Fact 80). That is, arc AM is equal to arc MB. But if from the two equal arcs AM and MB the equal arcs CM and MD are, respectively, taken away, the two remaining arcs AC and BD must be equal. For if equals are subtracted from equals, the remainders are equal. Therefore,

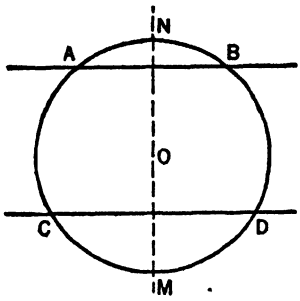


Fig. 131.

Fact 87: Two parallel chords intercept equal arcs on a circle.

Although this proof suffices to justify the conclusion of the patternmaker, it does not cover all possible cases. The parallel lines may be such that (*a*) one line is tangent to the circle while the other intersects the circle at two points or (*b*) the two lines may be tangent to the circle. The reader, by following the same course of reasoning as in the proof just given, can, however, easily prove these other two possibilities.

Problems

1. Lay together in a straight line a dime, a nickel, and a quarter. Draw a circle outside the three coins, tangent to the two terminal coins. Prove that the circumference of this circle is equal to the combined lengths of the circumferences of the three coins.

2. Two gears, one of $1\frac{1}{2}$-inch pitch diameter (twice the distance from the center to half the height of the teeth) and the other of 3-inch pitch diameter intermesh. What is the distance between the gear centers?

3. A round shaft is 6 inches in diameter. Find the maximum length of a side of a triangular end (an equilateral triangle) which can be made on the shaft.

4. Find the maximum length of (*a*) a square end in the above shaft, (*b*) a hexagonal end, (*c*) an octagonal end.

5. Two railroad tracks cross at an obtuse angle as shown in Figure 132. It was desired to lay a curved connecting track. The

curved track was to meet the crossing track at points 200 feet below the crossing point. The distance AC is 300 feet. What is the radius of the circular arc of the connecting tracks?

6. In laying out an airport, a cement rectangle is to be constructed whose altitude is 300 yards and whose area is 180,000 sq. yards. This rectangle is to have a circle circumscribed about it. What is the diameter of the circle?

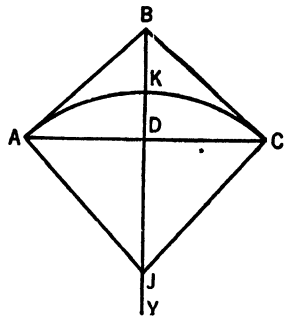


Fig. 132.

7. An advertisement has a red equilateral triangle inscribed in a blue circle. If the sides of the triangle are doubled but its angles remain the same, what change occurs in the radii of the circumscribed circle?

8. CA and CB are tangents to a circular tank at the points A and B and include an angle of 60°. If $CA = 50$ feet, what is the distance from A to B?

9. A portion of a spherical floating mine which was blown up by a cruiser was recovered. The chord was 2 feet across, and the perpendicular distance from the inside circular arc to the chord at its center was 3 inches. What was the inside diameter of this mine?

10. In mapping a volcanic region, it is desired to find the equation of a circle passing through the points $(-2, 1)$, $(3, 2)$, and $(2, -5)$.

21

ANGLE RECKONING

Measurement of angles has already been discussed to a considerable extent. The fact that angles can be measured much more accurately than distances was pointed out in Chapter 13, page 202. Since angles facilitate precise measurement, the study of them is not only highly useful but even fundamental to the development of more advanced work.

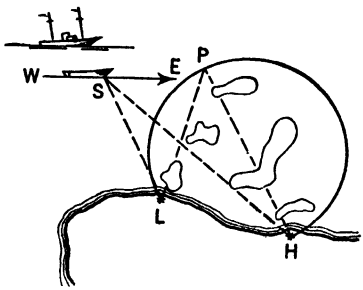


Fig. 133.

A navigator, whether on a steamer or in an airplane, steering the course of his ship, faces many occasions in which he must apply the technique of measuring angles. For example: the captain of a ship, *S* in Figure 133, sailing along a course W E, knows from the map the location of certain reefs situated off the coast near two lighthouses *L* and *H*. The chart also directs his attention to angle *LPH* inscribed* in a circle passing through *L* and *H* and surrounding the reefs. This angle *LPH* is known as the *Horizontal Danger Angle.*

The captain, to safeguard his ship from the rocks, must not continue in the direction of the line W E. He must veer to his left in order to avoid the circle. If he should continue in the direction W E, the angle *LSH* would soon be equal to the danger angle *LPH*, and his ship would then be inside the circle.

In selecting his course, the captain makes use of two facts, which are now to be proved, and which assure him that as long as angle *LSH* is less than angle *LPH*, his ship will be outside of the imaginary circle surrounding the reefs.

INSCRIBED ANGLE

Consider a circle *O*, Figure 134, with angle *LPH* intercept-

* An *inscribed angle* is an angle whose vertex is on the circle and whose sides are chords.

ing arc LH and such that the center of the circle lies inside of angle LPH. Draw the diameter PR dividing angle LPH into the two angles LPR and RPH. Thus, angle LPH = angle LPR + angle RPH, or angle n + angle y. Draw the radius OL. Then, since OL is equal to OP (radii of the same circle are equal), triangle POL is isosceles. Hence, by Fact 3, angles m and n, opposite the equal sides OL and OP, are equal. The exterior angle x is, by Fact 19, equal to the sum of the two opposite interior angles m and n. That is, $x = m + n$. So that, since $m = n$, we may write $x = 2n$ or $\frac{1}{2}x = n$. Angle x is measured by arc LR; hence, $\frac{1}{2}x$ is measured by $\frac{1}{2}$ arc LR. Therefore, n, which is equal to $\frac{1}{2}x$, is also measured by $\frac{1}{2}$ arc LR.

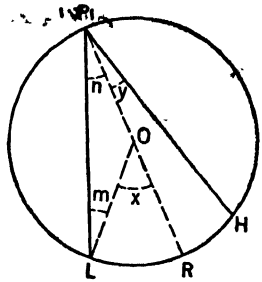


Fig. 134.

Similarly, angle y is measured by $\frac{1}{2}$ arc RH. Therefore, angle $(n + y)$ is measured by $\frac{1}{2}$ (arc LR + arc RH). But since angle $(n + y)$ is angle LPH, and $\frac{1}{2}$ (arc LR + arc RH) is equal to $\frac{1}{2}$ arc LH, then angle LPH is measured by $\frac{1}{2}$ arc LH. In other words,

*Fact 88:** *An angle inscribed in a circle is measured by one-half its intercepted arc.*

ANGLE FORMED BY TWO SECANTS

Before proceeding with the proof of the second part necessary to justify the captain's action in selecting his course, it is necessary to name a line segment. A straight line segment which, if extended far enough, intersects a circle in two points is called a *secant*. For example: SH in Figure 135.

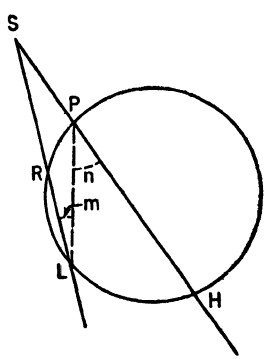


Fig. 135.

Now consider two secants, SL and SH in Figure 135, drawn from an external point S and intercepting two arcs RP and LH such that LH is greater than RP. Draw PL. Angle n, exterior to triangle SPL, is, by Fact 19, equal to the sum of the

* If the center of the circle lies on a side of the inscribed angle or outside the inscribed angle, the same fact holds good, as may be easily proved by the reader.

two opposite interior angles S and m. That is, $n = S + m$. Hence, $n - m = S$. But the inscribed angles n and m are, by Fact 88, respectively measured by $\frac{1}{2}$ arc LH and $\frac{1}{2}$ arc RP. So that angle $(n - m)$ is measured by $\frac{1}{2}$(arc LH − arc RP). Therefore, angle S, which is equal to $n - m$, is also measured by $\frac{1}{2}$(arc LH − arc RP). That is,

*Fact 89(a): An angle formed by two secants intersecting outside the circle is measured by one-half the difference of the intercepted arcs.**

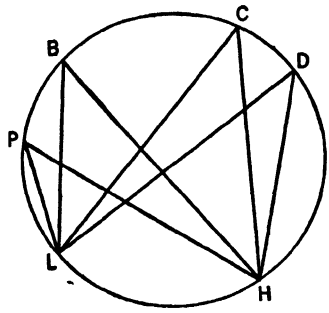


Fig. 136.

The captain's action is now easily justified as follows: Since, by Fact 88, an angle inscribed in a circle is measured by one-half its intercepted arc, then *all angles inscribed in the same segment† are equal.* For it is evident that angles P, B, C, and D, Figure 136, are all measured by one-half of arc LH (Fact 88). Therefore, they are equal. So that the horizontal danger angle, LPH, in Figure 133, is the same no matter where point P is, as long as P is on the circle, and is measured by $\frac{1}{2}$ of arc LH. It is also easily seen that an angle formed by two secants intersecting outside of the circle and intercepting the same arc as an inscribed angle must be smaller than the latter. For consider angles n and S in Figure 135. Angle n is measured by $\frac{1}{2}$ arc LH (Fact 88), while angle S is measured by $\frac{1}{2}$ (arc LH − arc RP) [Fact 89 (*a*)]. Evidently, $\frac{1}{2}$(arc LH − arc RP) is less than $\frac{1}{2}$ arc LH. Therefore, S is less than n. That is, the captain, in choosing his course so that angle LSH in Figure 133 is less than angle LPH, is simply making sure that his ship stays outside of the danger circle surrounding the rocks.

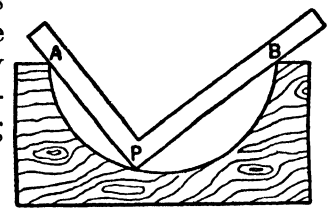


Fig. 137.

ANGLE INSCRIBED IN SEMICIRCLE

To determine whether or not a curve in a pattern is a semicircle, the patternmaker may use

* For a more generalized statement of this fact, see Fact 89 on page 337.
† A *segment* of a circle is the portion of a plane bounded by an arc of a circle

a carpenter's square. If, no matter where point P is placed along the cut (Figure 137), the outside edges of the square rest evenly on points A and B, the curve is a semicircle. The carpenter's square is so constructed that its two edges form a right angle. Therefore, to ascertain that the cut is a semicircle, we must show that if a point moves so that the straight lines connecting it to two fixed points always form a right angle, its locus is a circle.

Consider two fixed points A and B, Figure 138, and let the straight line connecting these two points represent the X-axis. Draw the Y-axis perpendicular to and bisecting the line segment AB. If the distance from B to the Y-axis is represented by a, the coordinates of B are $(a, 0)$ and the coordinates of A must then be $(-a, 0)$. Choose point P in a general position so that AP is perpendicular to BP, and denote the coordinates of P by (x, y). The problem imposes the condition that as point P moves in the plane, line AP must remain perpendicular to BP. Since if two lines are perpendicular to each other the slope of one is the negative reciprocal of the slope of the other (Chapter 19, page 301), then $m_{AP} = -\dfrac{1}{m_{BP}}$. The slope of line AP is $\dfrac{y-0}{x+a}$ (Chapter 13, page 197). Hence, we may write $m_{AP} = \dfrac{y}{x+a}$. Similarly, $m_{BP} = \dfrac{y}{x-a}$. Hence, $\dfrac{y}{x+a} = -\dfrac{1}{\dfrac{y}{x+a}}$ or $\dfrac{y}{x+a} = \dfrac{x-a}{-y}$. That is, $-y^2 = (x+a)(x-a)$, or $-y^2 = x^2 - a^2$. So that, through transposing, we obtain $a^2 = x^2 + y^2$. This is the equation of a circle whose center is at the origin and whose radius is a (Chapter 20, page 312).

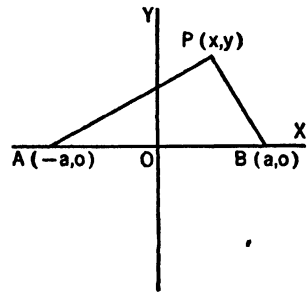


Fig. 138.

From this conclusion, and, indeed, as a deduction from Fact 88, it follows that:

Fact 90: An angle inscribed in a semicircle is a right angle.

and its chord. An angle is *inscribed* in a segment of a circle if its vertex is in the arc of the segment and its sides pass through the extremities of the arc.

ANGLE FORMED BY A TANGENT AND A CHORD

Engineers in laying out a curve for a railroad or highway sometimes use the method of "deflection angles." A *deflection*

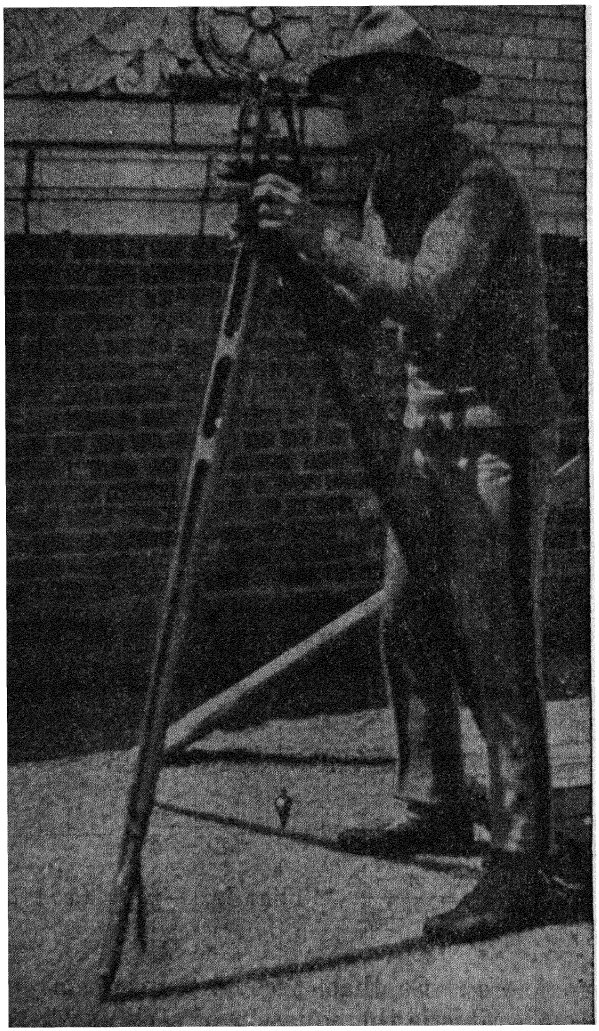


Fig. 139.

angle is the angle measured from the tangent to any given point along the curve, such as angle VCA in Figure 140. By the *degree of curve* is meant the central angle subtended by a chord of 100 feet. Clearly, the central angle in different-sized

circles, each having a chord of 100 feet, will be of different sizes. When the chord is of the fixed length 100 feet, the larger the central angle subtending this chord, the shorter will be the radius and, of course, the smaller the circle. Conversely, if the chord remains of the fixed length 100 feet, the smaller the central angle subtending this chord, the longer will be the radius and hence the larger the circle.

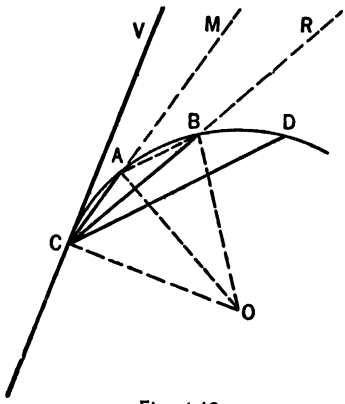


Fig. 140.

Assume that, having been given the location of point C, Figure 140, an engineer in the field is to lay cut a curve tangent to line VC. The curve to be laid out is an arc of a circle. The engineer is told by his chief in the home office the degree of the curve; that is, he is informed as to the size of the central angle subtended by a chord of 100 feet. Since point C is on the curve, a 100-foot chord can be measured from it. This 100-foot chord of the, as yet, unlocated arc of the circle will make some angle with the tangent VC. In order to carry out the work, the engineer must know the relationship between the angle made by the tangent and the chord, namely, the relationship between the deflection angle VCA and the central angle.

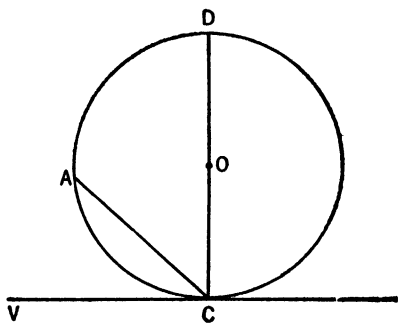


Fig. 141.

Assume circle O, Figure 141, with VC tangent to the circle at C and any chord AC. Draw the diameter DC. Since a diameter bisects a circle (Chapter 20, page 318), arc DAC is a semicircle or 180°, and hence $\frac{1}{2}$ arc DAC is 90°. Moreover, a tangent to a circle is, by Fact 84, perpendicular to the radius drawn to the point of contact. So that VC is perpendicular to DC, and hence angle DCV is a right angle, or 90°. Therefore, angle DCV is measured by $\frac{1}{2}$ arc DAC, for they are both equal to 90°. But, by Fact 88, angle ACD is measured by

$\frac{1}{2}$ arc AD. Therefore, angle VCA, which is equal to angle DCV − angle ACD, is measured by $\frac{1}{2}$(arc DAC − arc AD), which is $\frac{1}{2}$ arc AC. In other words,

Fact 91: An angle formed by a tangent and a chord drawn from the point of contact is measured by one-half the intercepted arc.

By this conclusion we know that the deflection angle VCA in Figure 140 is measured by $\frac{1}{2}$ arc CA. If chord CA is chosen 100 feet long, the degree of curve is the central angle COA, which is measured by arc CA. Therefore, angle VCA, which is measured by $\frac{1}{2}$ arc CA, is $\frac{1}{2}$ central angle COA, or $\frac{1}{2}$ the degree of curve.

How does the engineer proceed upon his task? He sets his transit (Figure 139, page 332) at point C, Figure 140, so that the plumb-bob is directly above the point, and sights along line CV. Then, by means of a small scale on the horizontal plate of the transit, he turns the telescope to the right until his line of sight, in the direction CM, forms an angle with CV equal to $\frac{1}{2}$ the central angle COA, namely, $\frac{1}{2}$ the degree of curve. One of the chainmen will place one end of the steel tape (sometimes called a *chain*) on the ground at point C. The other chainman stretches the 100-foot tape taut and moves to and fro until the engineer signals him that he is in the line of sight, CM. This second assistant, immediately upon receiving the signal, drives a stake in the ground at point A.

How next does the engineer locate point B? With the transit still at point C, he turns the telescope to the right until his new line of sight, CR, makes with line CA an angle, ACR, equal to angle VCA. A chainman then places one end of the tape at point A. The other chainman, keeping the 100-foot tape taut, moves the other end until the engineer signals him that he is in the line of sight CR. On receiving this signal, he drives a stake in the ground at point B.

Point B may be shown to lie in the arc of the circle as follows. Chord AB is 100 feet. So that central angle AOB, which measures arc AB, is equal to the degree of curve. Since angle ACB, inscribed in the circle, is, by Fact 88, measured by one-half arc AB, it is equal to $\frac{1}{2}$ the degree of curve. Therefore angle VCA, also equal to $\frac{1}{2}$ the degree of curve, is equal to angle ACB.

It is now easily seen that as long as the chords in the arc

of the circle are all chosen equal to 100 feet, the successive angles *VCA*, *ACB*, *BCD*, and so forth, are all equal to $\frac{1}{2}$ the degree of curve. Thus, all points along the arc may be located from point *C*.

ANGLE FORMED BY TWO CHORDS

The mathematical skill and reasoning power of the ancient Greeks are strikingly shown in their interpretation of astronomical phenomena. These ancient astronomers noticed the well-known fact that the sun appears to move more slowly during the spring than in any other season. To account for this event, Hipparchus (150 B. C.) developed a very ingenious device which, although not representing the actual movement of the sun, was nevertheless sufficient to give him a fairly accurate idea of the position of the sun for any day in the year. Hipparchus held the theory that the sun revolved around the earth. In fact, this theory was generally accepted until Copernicus, a quiet and unassuming Catholic priest who lived in the sixteenth century, adopted the views of another famous Greek astronomer, Aristarchus of Samos (third century B. C.), and developed mathematically the thesis that the planets revolve around the sun. The circle had been used by astronomers to represent the path of a planet until John Kepler published in 1609 A. D. his famous laws of planetary motion. Thus, naturally Hipparchus used a circle to represent the path of the sun.

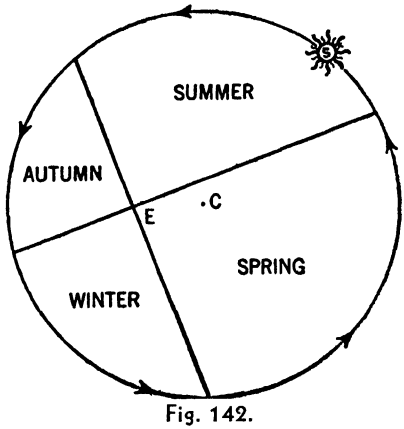


Fig. 142.

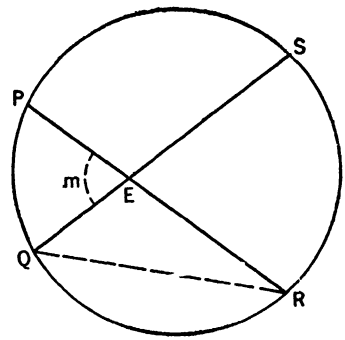


Fig. 143.

The device consisted of a circle whose center does not coincide with the position of the earth. Thus, the sun, *S* in Figure 142, moves along the circle so that it passes over equal arcs in

equal intervals of time. However, to the observer on the earth, E, the sun, being farther away, will seem to move more slowly during the spring. In determining the position of the sun, it was important to know the measure of an angle formed by two intersecting chords.

Consider a circle O, Figure 143, with chords PR and QS forming angle m. Draw QR. Angle m, exterior to triangle EQR, is, by Fact 19, equal to the sum of the two opposite interior angles R and Q. But the inscribed angle R is, by Fact 88, measured by $\frac{1}{2}$ arc PQ; and, similarly, angle Q is measured by $\frac{1}{2}$ arc SR. Therefore, angle m, which is equal

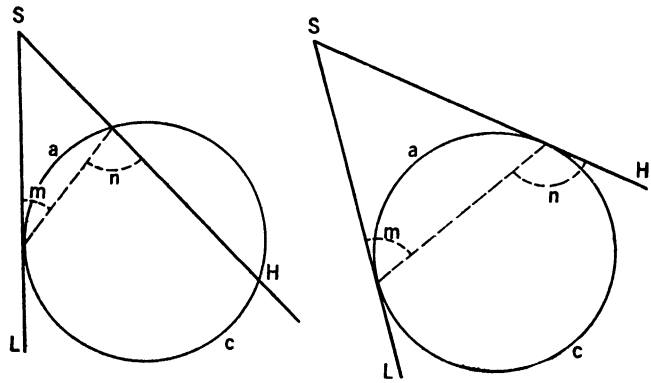


Fig. 144.

to angle $(R + Q)$, is measured by $\frac{1}{2}$(arc PQ + arc SR). That is,

Fact 92: An angle formed by two chords intersecting within a circle is measured by one-half the sum of the intercepted arcs.

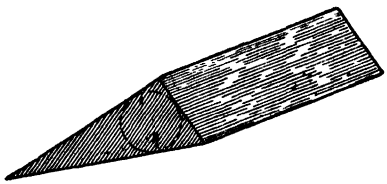
ANGLE FORMED BY TWO SECANTS

Following the same line of reasoning used in these proofs, the reader may easily generalize Fact 89 (a). For if the secant SL in Figure 135 moves to the position of tangency assumed in Figure 144 (a), or both secants move to the position of tangency as in Figure 144 (b), exterior angle n is, by Fact 19, equal to the sum of the two opposite interior angles S and m. So that, by applying Facts 88 and 91, angle S can be shown to be measured by $\frac{1}{2}$(arc c − arc a). Therefore, we may conclude that:

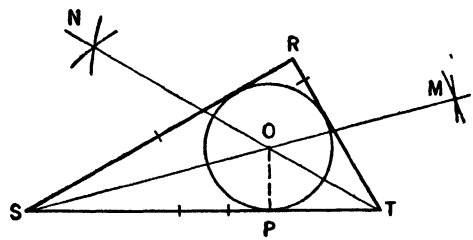
Fact 89: An angle formed by two secants, by a secant and a tangent, or by two tangents, intersecting outside a circle, is measured by one-half the difference of the intercepted arcs.

CIRCLE INSCRIBED IN A TRIANGLE

The foreman of a mill has a triangular piece of steel which he desires to machine into the form of a cylindrical shaft. To ascertain whether or not the triangular piece will yield a shaft having the desired diameter, the foreman must determine the diameter of the circle which can be inscribed in the triangular section.



Assume triangle *RST* is the triangular section of the steel piece. Using the method shown in Chapter 9, page 125, bisect angle *RST* by line *SM* and angle *RTS* by line *TN*. If lines *SM* and *TN* were parallel, the sum of the two interior angles *NTS* and *TSM* on the same side of the transversal *ST* would have to be equal to a straight angle (Fact 15). However, the sum of the interior angles of a triangle is, by Fact 18, a straight angle. So that the sum of angles *RST* and *RTS* must be less than 180°. And since angles *NTS* and *TSM* are, respectively, one-half of angles *RTS* and *RST*, it is evident that their sum is also less than a straight angle. Hence, lines *SM* and *TN* are not parallel and therefore intersect in some point *O*. From *O* construct, as shown in Chapter 9, page 127, *OP* perpendicular to *ST*. Since point *O* is in the bisector of angle *RST*, it is, by Fact 72, equidistant from the sides of the angle, that is, from *RS* and *ST*. Similarly, *O* is also equidistant from *RT* and *ST*, for it is in the bisector of angle *RTS*. Therefore, *O* is equidistant from all three sides of triangle *RST*. Hence, a circle described with *O* as its center and *OP* as its radius will touch all the sides of triangle *RST* and is therefore inscribed in it.



CONSTRUCTION OF TANGENTS

Two dead-end streets are connected by a circular road, as shown in Figure 145. The street on the left is to be extended tangent to the circular arc. To construct the tangent AT, a surveyor must first locate the center of the circular arc. After determining chord AB, the surveyor lays out OM perpendicular to and bisecting AB. The center O must be somewhere along this line OM, for, by Fact 81, the perpendicular bisector of a chord of a circle passes through the center of the circle. Similarly, ON is constructed perpendicular to and bisecting chord AM. So that again center O must also lie in line ON. Since two straight lines can intersect at only one point, O must be the center of the circular arc. Taking any convenient point C, outside line segment AO, as the center, and AC as the radius, a circle is described intersecting AO at P. Through P and C draw a straight line segment meeting the circle at T. Draw AT. Then, by construction, angle TAP is inscribed in a semicircle, so that, by Fact 90, it is a right angle. That is, AT is perpendicular to the radius AO at its extremity on the circle. Therefore, by Fact 85, it is tangent to the circle.

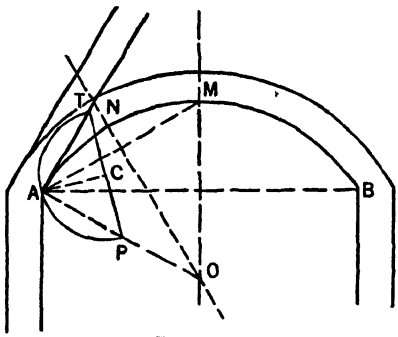


Fig. 145.

The darkening of a heavenly body, called an *eclipse*, was for many centuries a source of great superstitious terror. To show pictorially this phenomenon and to calculate the length of the earth's shadow, two methods of construction must be shown. That is, through a point outside a circle tangents to the circle must be drawn, and a common external tangent to two given circles has to be determined.

Assume that the circle with center S and with radius R, Figure 146, represents the sun, and let the circle with center E and with radius r denote the earth. With S as a center and with radius SA, equal to $R - r$, draw a circle. Before a common tangent from the sun's surface to the earth can be drawn to determine the earth's shadow, two tangents from E to the inner circle, whose radius is SA, must be drawn. Draw the line SE. Then, using the method shown in Chapter 9, page

126, bisect SE at P. With P as center and PS as radius, describe a circle intersecting the inner circle at A and B. Draw EA, EB, SA, and SB. Angle EAS is, by construction, in-

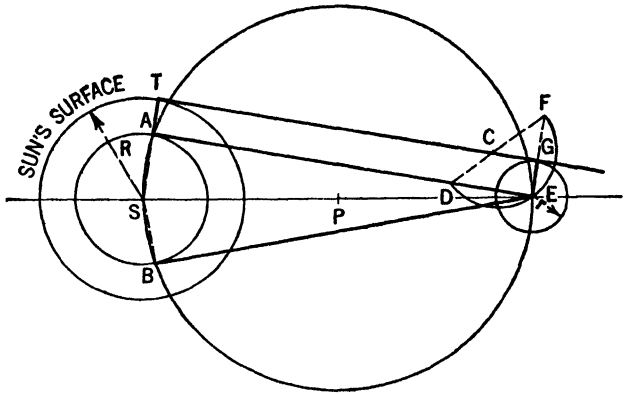


Fig. 146.

scribed in a semicircle. Therefore, by Fact 90, angle EAS is a right angle. So that the line segment EA perpendicular to the radius SA at its extremity on the circle is, by Fact 85, tangent to the circle. Similarly, it may be shown that EB is tangent to the inner circle.

To draw a common external tangent to the two circles, we proceed as follows. Extend line segment SA to meet circle S, representing the sun, at T. Draw EG perpendicular to AE, as shown in this chapter on page 338. That is, from any convenient point C, outside of line segment AE, and with CE as radius, describe a circle intersecting AE at D. Through D and C draw a straight line segment meeting the circle at F. Draw EF intersecting circle E at G. Then, by construction, angle FED is inscribed in a semicircle. Hence, by Fact 90, it is a right angle. Thus, EG is perpendicular to AE.

Since ST is also perpendicular to AE, it is, by Fact 11, parallel to EG. Moreover, by construction, AT is equal to EG. Hence, the figure $AEGT$ is, by Fact 40, a parallelogram. Hence, TG is parallel to AE. But since AT is perpendicular to AE, then AT is, by Fact 16, perpendicular to TG also. Similarly, EG is perpendicular to TG. That is, line TG is perpendicular to the radii ST and EG at their extremities on the circles. Therefore, by Fact 85, it is tangent to the two circles S and E.

The explanation of an eclipse of the moon may now be easily followed. The large circle S, in Figure 147, represents the sun.

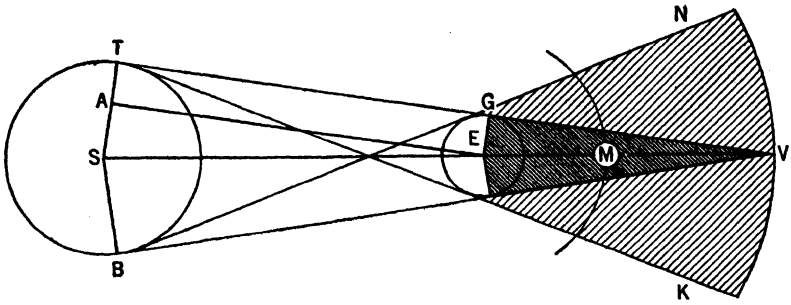


Fig. 147.

Circles E and M, respectively, represent the earth and moon. The common external tangents meeting at V represent the rays of the sun. Since the earth and sun are approximately spherical, the shadow of the earth, shown by the heavily dark triangle in the figure, has actually the shape of a cone. All sunlight should be excluded from this cone. However, the atmosphere of the earth bends some of the sun's rays into this cone, making it less dark and increasing it slightly where the moon crosses it. The lines BN and TK determine the penumbra within which a part, but not the whole, of the sunlight is cut off.

It is evident that if the path of the moon around the earth were in the same plane as the earth's path around the sun, as seems to be the case in the figure, the moon would pass directly behind the earth through the shadow cone at every revolution. But the plane of the moon's path is inclined about $5° 8'$ to the plane of the path of the earth. As a result, the moon is often above or below the earth's shadow. Lunar eclipses are, for this reason, rather infrequent, seldom occurring more than twice a year.

The length of the earth's shadow is easily found. Consider the triangles AES and GVE in Figure 147. Since, by construction, SA is perpendicular to AE, and EG is perpendicular to VG, triangles AES and GVE are right triangles. Line segments AE and VT are, by construction, parallel. These line segments are cut by the transversal VS. So that, by Fact 14, the corresponding angles AES and GVE are equal. That is, right triangle AES has an acute angle AES equal to the acute

angle GVE of triangle GVE. Therefore, by Fact 49, the triangles are similar, and we have $\frac{EG}{SA} = \frac{VE}{SE}$. In this proportion, EG represents the radius of the earth, known to be about 3,960 miles. Line segment SA is by construction equal to the difference between the radii of the sun and earth. Since the sun's radius is about 433,250 miles, $SA = 433{,}250 - 3{,}960 = 429{,}290$ miles. The distance from the sun to the earth, namely, SE, is approximately 92,900,000 miles. Thus, the length of the earth's shadow $VE = \frac{EG}{SA} \times SE = \frac{3{,}960}{429{,}290} \times 92{,}900{,}000 =$ 857,000 miles, approximately.

INEQUALITY RELATIONSHIPS

The three following facts are here merely stated, without proof, with the hope that those readers who enjoy mental exercise and like to try out their reasoning powers will attempt the proofs.*

Fact 93: If in the same circle, or in equal circles, two minor arcs are unequal, the greater arc determines the greater chord; and Conversely, *if two chords are unequal, the greater chord determines the greater minor arc.*

Fact 94: If in the same circle, or in equal circles, two chords are unequal, the longer chord is nearer the center.

Fact 95: If in the same circle, or in equal circles, two chords are unequally distant from the center, the chord nearer the center is the longer.

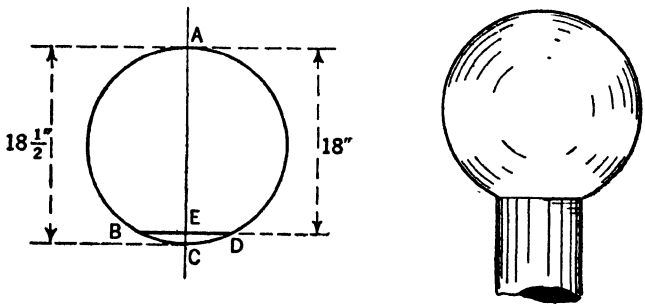


Fig. 148.

INTERSECTING CHORDS

The entrance post of a gate is to carry a glass sphere, as

* The proofs are given on pages 343-345 at the end of the present chapter.

shown in Figure 148. The glass sphere is to be $18\frac{1}{2}$ inches in diameter and is to have a vertical height of 18 inches. To order the post, it is necessary to compute its diameter so that it shall exactly match the flattened bottom of the sphere.

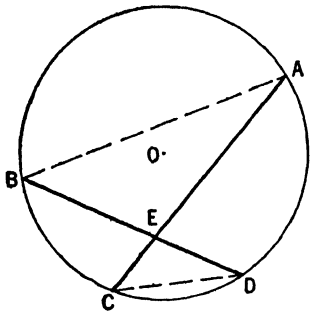


Fig. 149.

Consider, then, any circle O, Figure 149, with chords AC and BD intersecting at E. Draw AB and CD. Angle A is equal to angle D, for, by Fact 88, they are both measured by one-half of the intercepted arc BC. Similarly, angles C and B are both measured by one-half of the intercepted arc AD. Therefore, again by Fact 88, they are equal. That is, triangle AEB has two angles A and B equal to the two angles D and C of triangle ECD. Hence, by Fact 48, the triangles are similar. Thus, $\frac{EB}{EC} = \frac{EA}{ED}$. Multiplying both sides of this equality by $EC \times ED$, we obtain $EB \times ED = EA \times EC$. That is,

Fact 96: If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.

The diameter of the post supporting the glass sphere may now be found by the simple application of this fact. Considering the diameter AC and the chord BD, in Figure 148, we have $EB \times ED = AE \times EC$. Since, by construction, AC is a diameter perpendicular to chord BD, then by Fact 80, it bisects BD. So that $EB = ED$, and we may write $ED \times ED = AE \times EC$. That is, $(ED)^2 = AE \times EC$. Substituting the given values, $(ED)^2 = 18 \times \frac{1}{2} = 9$, or $ED = \sqrt{9} = 3$. Therefore, $BD = EB + ED = 2ED = 2(3) = 6$. The diameter of the post must therefore be 6 inches.

TANGENT AND SECANT

In a railroad location survey, it is desired to ascertain the distance from a track crossing to a circular curve, tangent to and connecting the two tracks. That is, the distance VM in Figure 150. Because of a large marsh, a direct measurement

of this distance is impossible. However, the tangent VC may be easily measured, and the radius OM of the curve is known. If a relation between these quantities and VM can be found, the problem is greatly simplified.

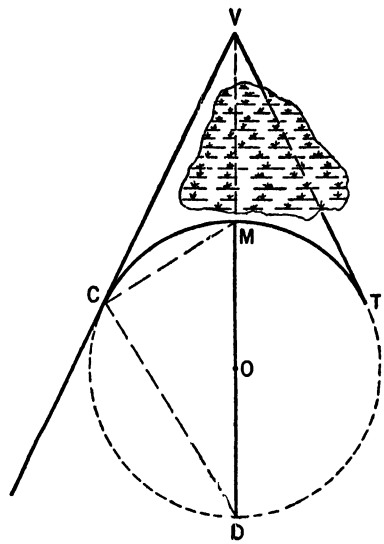


Fig. 150.

Complete the circle O, whose arc CT is the circular curve connecting the two tracks. Draw the secant VD and line segments CD and CM. Angle VCM, formed by the tangent VC and chord CM, is, by Fact 91, measured by one-half arc CM. Angle CDM inscribed in the circle is, by Fact 88, measured by one-half arc CM also. Therefore, angles VCM and CDM are equal. So that triangle VCD has two angles V and CDM equal to the two angles V and VCM of triangle VCM. Therefore, by Fact 48, the triangles are similar. Hence, $\frac{VD}{VC} = \frac{VC}{VM}$, or $VD : VC = VC : VM$. In other words:

Fact 97: If from a point outside a circle a tangent and a secant are drawn to the circle, the tangent is the mean proportional between the secant and its external segment.

Assume that the tangent VC is found to be 1,000 feet, and the radius OM of the circular curve is 500 feet. The diameter MD, equal to twice the radius, is then $2 \times 500 = 1{,}000$ feet. Applying the fact just proved, $\frac{VD}{VC} = \frac{VC}{VM}$. Multiplying both sides by $VC \times VM$, we obtain $VD \times VM = (VC)^2$. Since $VD = VM + MD$, we may write $(VM + MD)VM = (VC)^2$. Substituting known values, $(VM + 1{,}000)VM = (1{,}000)^2$; that is, $(VM)^2 + 1{,}000VM = 1{,}000{,}000$, or $(VM)^2 + 1{,}000VM - 1{,}000{,}000 = 0$. This expression is a quadratic equation in VM. Hence, applying the formula developed in Chapter 17, page 269, we obtain:

$$VM = \frac{-1{,}000 \pm \sqrt{(1{,}000)^2 + 4(1{,}000{,}000)}}{2}$$

$$= \frac{-1{,}000 \pm \sqrt{5{,}000{,}000}}{2},$$

or $VM = \dfrac{-1{,}000 \pm 2236.10}{2}$; that is, $VM = 618.05$ and -1618.05.

The negative answer does not apply in this case. Hence, the distance VM is 618.05 feet.

* * *

Proof of Fact 93

Consider circle O, Figure 151, with arc AB greater than arc CD. Draw the radii OA, OB, OC, and OD. Since all radii of the same circle are equal, $OA = OC$, and $OB = OD$. Arcs AB and CD are, respectively, measured by the central angles AOB and COD. So that, since, by construction, arc AB is greater than arc CD, the angle AOB is greater than angle COD. That is, triangle AOB has two sides AO and OB, respectively, equal to the two sides OC and OD of triangle COD, but the included angle AOB of the first triangle is greater than the included angle COD of the second triangle. Therefore, by Fact 33, side AB is greater than side CD.

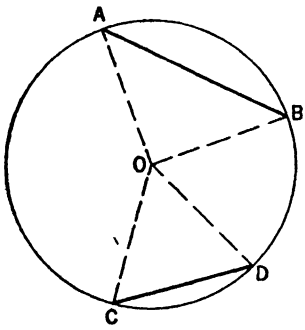


Fig. 151.

To prove the converse, consider circle O in the same figure with chord AB greater than chord CD. Show, by applying Fact 34, that angle AOB is greater than angle COD. Since central angles measure the arcs which they subtend, it follows that arc AB is greater than arc CD.

Proof of Fact 94

Consider circle O, Figure 152, with chord AB greater than chord CD. Draw OM and ON respectively perpendicular to CD and AB. From A mark off arc AR equal to arc CD.

Draw chord AR, and OS perpendicular to AR and intersecting AB at Q. Then, by Fact 79, $CD = AR$. Since AB is by construction greater than CD, then, by Fact 93, arc AB is greater than arc CD. Moreover, by Fact 83, $OM = OS$. By Fact 31, OQ is greater than ON. But OS is greater than OQ, for the whole is greater than any of its parts. Hence, OS is greater than ON. Since $OS = OM$, then OM is greater than ON, and therefore CD is more remote from O than AB.

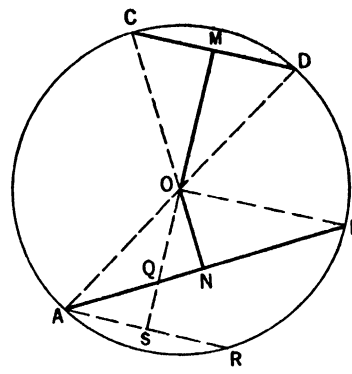


Fig. 152.

Proof of Fact 95

Consider circle O in the same Figure 152 with chords AB and CD such that CD is more remote from O than AB. Chord CD must either be (*a*) greater than AB, (*b*) equal to AB, or (*c*) less than AB. If CD is greater than AB, then, by Fact 94, AB is more remote than CD. This is contrary to the given fact that CD is more remote than AB. Therefore, it is impossible. If CD is equal to AB, then, by Fact 83, they are equidistant from the center. This again contradicts the statement that CD is more remote than AB. Therefore, it is also impossible. It follows that the only remaining possibility must be true. Therefore, CD is less than AB.

Problems

1. A tube is to enclose three smaller equal tubes which exactly fit within it. If the inside diameter of the large tube is 8 inches, what must be the outside diameters of the smaller tubes?

2. The shortest chord that can be drawn through a point 8 inches from its major arc is 2 inches. Find the diameter of the circle.

3. Part of the design of a colored glass window in a museum contains two intersecting circles. Their common chord is 24″ long. Find the distance between the centers of the circles if the radii are 36″ and 18″, respectively.

4. In an ornamental iron gate, a circle one foot in diameter is to enclose four smaller circles, all of the same size, each tangent to two

others and to the outer circle. What must be the diameter of the enclosed circles?

5. The driving bar of a wheel in a small machine is attached at one end to the rim of the wheel. The other end of the driving bar is attached to the head of a piston inside of the piston cylinder. As the piston moves to and fro, the portion of the driving bar outside of the piston cylinder varies in length. The driving bar of the wheel is 16 inches long when tangent to the wheel, measuring the distance from the point of tangency to the point of entrance into the piston cylinder. When the wheel has revolved a little, so that the driving bar is now a secant, touching the rim of the wheel at the point furthest from the point of entrance into the piston cylinder, the entire secant is four times its external segment. What is the length of the secant?

6. In the preceding problem, through how many degrees did the outer end of the driving bar move, from the first position to the second position?

7. A garret has a sloping roof making an angle of 30° with the floor of the garret. A keg 1 foot in diameter is rolled lengthwise into the angle as far as it will go. What is the diameter of a larger keg which will just touch the smaller keg when rolled against the sloping roof?

8. A tunnel bore is 10 feet in diameter. A steel tube, the outer diameter of which is 2 feet, lies lengthwise at the bottom of the tunnel bore, so that the line of centers of the two circles is vertical. Find the width of a wooden plank resting on top of the steel tube and touching with both ends the walls of the round tunnel.

9. An artist in designing a postage stamp draws a circle and two chords within it which form an angle of 72°. If one of the intercepted arcs is 102°, ascertain the number of degrees in the other arc.

10. In preparing a sketch of a trademark for a cheese company, a draftsman inscribes in a circle a triangle, two angles of which measure 56° and 64°, respectively. Find the number of degrees in each angle formed by a side of the triangle and the tangent drawn at a vertex of the triangle.

22 THE GIRDLED FIELD

The length of a curve is a deceptively simple idea, as one soon discovers in trying to frame a satisfactory definition. Intuitively we feel that the length of a curve is the amount of its extension. But in saying this, we have merely substituted some new words in place of the first ones and got nowhere. The area of a circle may be defined in terms of the length of its circumference, for the area is a function of that length.

A straight line is measured by comparing it to a certain unit segment, such as an inch or a foot. This unit segment, when repeatedly applied to a given line, may be contained an integral number of times in that straight line. In other cases, the unit segment is not contained an integral number of times in the given line. In fact, it may be impossible to obtain, as explained in Chapter 1, page 5, the exact numerical value of the length of a given line, although it may be approximated to any degree of accuracy. The impossibility of defining the length of *any* curve in terms of a curved unit may be easily grasped by simply visualizing the fact that a curved unit suitable to measure a circle of radius 5 could not be used for a circle of radius 6, for the simple reason that it would not fit the larger circle.

REGULAR POLYGONS AND CIRCLES

The length of a curve is, therefore, given *in terms of straight line units*. A satisfactory definition of the meaning and exact value of the length of any curve is beyond the scope of this book. Consequently, the present chapter will be confined to a discussion of the circle. For practical purposes, where great accuracy is not desired, the length of a circle may be easily approximated. A flexible tape can be made to coincide almost exactly with the curve and its length can then be read. To obtain a general formula, however, for the length of *any* circle,

it is first necessary to study certain relationships between polygons and circles.

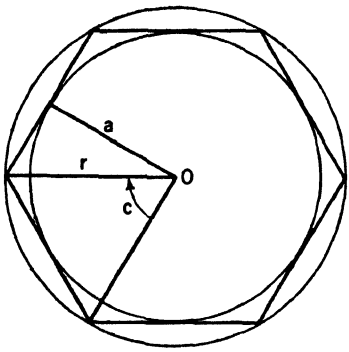
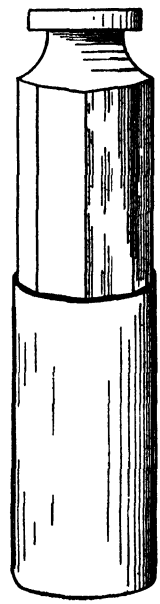


Fig. 153.

A polygon which is both equiangular and equilateral, as already stated, is said to be a *regular polygon*. If the sides of a polygon are all chords of a circle, the polygon is said to be *inscribed* in the circle. But if the sides of the polygon are all tangents to a circle, the polygon is *circumscribed* about the circle. In the first case, the circle is said to be *circumscribed* about the inscribed polygon. In the latter case, the circle is *inscribed* in the circumscribed polygon. Thus, the regular polygon shown in Figure 153 is circumscribed about the inner circle and inscribed in the outer circle. The inner circle is inscribed in the regular polygon, but the outer circle is circumscribed about the polygon.



The radius of the circle circumscribed about the regular polygon, namely, r in Figure 153, is the *radius of the polygon*. The radius of the circle inscribed in the regular polygon, that is, a in Figure 153, is the *apothem* of the polygon. Evidently, the apothem is, by Fact 84, perpendicular to the side of a regular polygon. The center of the circumscribed or inscribed circle is called the *center of the polygon*. Angle c, in Figure 153, formed by the radii drawn to the extremities of any side of the regular polygon, is an *angle at the center* of the polygon.

A perfume manufacturer bottles a particular brand in a container the flat end of which is in the form of a regular octagon. He desires to market this perfume enclosed in a cylindrical box as shown in the figure. To avoid breakage, the bottle must fit tightly in the cylindrical box. It is necessary, therefore, to find the diameter of the smallest circle that may contain the regular octagon. The problem, then, is to show that,

Fact 98: A circle can be circumscribed about any regular polygon.

Consider, then, the regular polygon *ABCDEFGH*, Figure 154. By construction, the three vertices *A*, *B*, and *C* cannot lie in a straight line. Therefore, by Fact 74, one circle and only one can be drawn through these three given points. Following the method shown in Chapter 20, page 316, construct a circle through the vertices *A*, *B*, and *C*, and let *O* be its center. Draw *OA*, *OB*, *OC*, and *OD*.

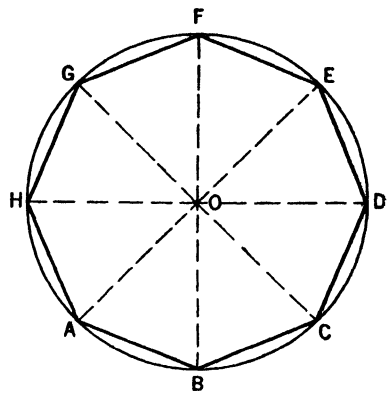


Fig. 154.

Since radii of the same circle are equal, *OB* is equal to *OC*. By construction (the given polygon is regular), *AB* is equal to *CD*, and angles *ABC* and *BCD* are equal. Moreover, since *OB* is equal to *OC*, triangle *OBC* is isosceles. Hence, by Fact 3, angles *CBO* and *BCO*, opposite the equal sides *OB* and *OC*, are equal. But if from the equal angles *ABC* and *BCD*, equal angles *CBO* and *BCO* are subtracted, the remainders, namely, angles *OBA* and *OCD*, are equal. That is, triangle *OAB* has the two sides *AB* and *OB* and the included angle *OBA* equal, respectively, to the two sides *CD* and *OC* and the included angle *OCD* of triangle *ODC*. Therefore, by Fact 2, the triangles are congruent, and hence *OA* is equal to *OD*.

By the definition of a circle, it is then evident that the circle through *A*, *B*, and *C* passes through *D*. Similarly, it can be shown that the circle through *B*, *C*, and *D* passes through *E*, and so on. Therefore, the circle constructed with *O* as a center and *OA* as a radius is circumscribed about the polygon.

Now suppose that a sheet metal worker is to cut a square metal sheet so as to form the largest possible circular disk. It is necessary to show that:

Fact 99: A circle can be inscribed in any regular polygon.

Consider then the square *ABCD*, Figure 155. Construct, using the method shown in the proof of Fact 98, a circle circumscribed about the given polygon *ABCD*, and let *O* be its center. Since the sides of *ABCD* are, by definition, page 348,

equal chords of the circumscribed circle, they are, by Fact 83, equidistant from the center O.

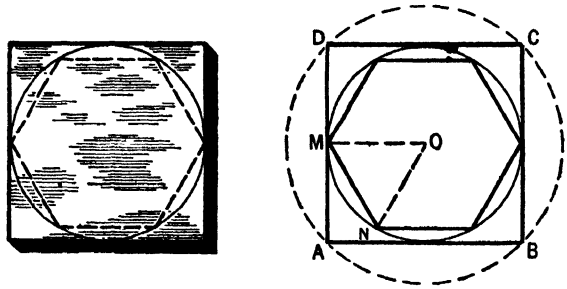


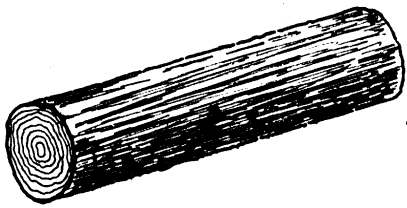
Fig. 155.

Draw OM perpendicular to the side AD. With O as a center and OM as radius, construct the circle passing through M and N. Then AD, perpendicular to the radius OM drawn to the point of contact, is, by Fact 85, tangent to the circle. Hence, by definition, page 348, the circle drawn with O as a center and with the perpendicular OM as radius is inscribed in the polygon. The reader can easily show as a deduction from this fact that:

Fact 100: An equilateral polygon inscribed in a circle is regular.

Having cut the circular disk, our metal worker is told to construct, out of this circular disk, the largest possible regular hexagonal metal sheet. The problem is to inscribe a regular hexagon in a circle. With M, Figure 155, as a center and a radius equal to OM, construct an arc intersecting the circle at N. Draw MN and ON. Then triangle MON is equilateral. So that, by Fact 25, angle MON is equal to 60° or $\frac{1}{6}$ of two straight angles, that is, $\frac{1}{6}$ of 360°. Therefore, chord MN, measured by angle MON, is, by Fact 100, a side of a regular inscribed hexagon. Hence, the required hexagon is inscribed by applying MN six times as a chord.

In manufacturing wooden columns, it is frequently necessary to cut logs so that any cross section, parallel to an end of the log, will have the shape of some regular polygon. A carpenter, to ascertain that the cross section of the column will have the shape of a regular polygon, divides the circular end of the log into as many equal parts as desired. To justify this carpenter's action, it is now necessary to show that:



Fact 101: If a circle is divided into any number of equal parts, the chords of these arcs form a regular inscribed polygon; and the tangents drawn at the points of division form a regular circumscribed polygon.

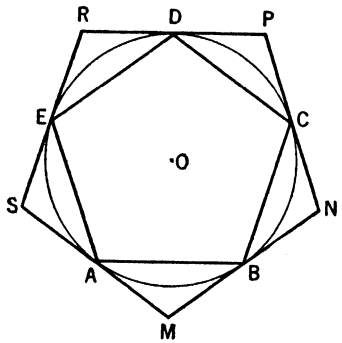


Fig. 156.

Consider the circle O, Figure 156, divided into equal arcs AB, BC, CD, DE, and EA. Since these arcs are equal, then, by Fact 79, their chords AB, BC, CD, DE, and EA are equal. Moreover, by definition, page 348, the figure $ABCDE$ is inscribed in circle O. Therefore, by Fact 100, $ABCDE$ is a regular inscribed polygon.

Consider again the circle O, in Figure 156, divided into equal arcs AB, BC, CD, DE, and EA. At the points A, B, C, D, and E, draw tangents to the circle. These intersecting tangents form the polygon $MNPRS$. The reader may easily show that $MNPRS$ is a regular circumscribed polygon by first applying Fact 89 and then, by joining consecutive points of contact of the sides, showing the isosceles triangles formed to be congruent. The polygon has then been shown to be equiangular and equilateral, and therefore it is regular.

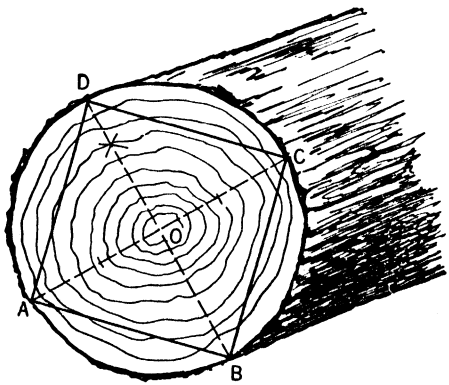
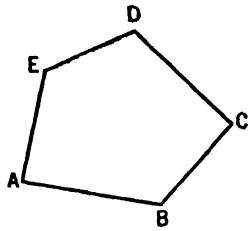


Fig. 157.

Observe next the special case in which the largest square column possible is to be cut from a log of given diameter. Evidently, the problem consists of *inscribing a square in a given*

circle. Consider the circle O, Figure 157, and draw any diameter AC. At O construct, as shown in Chapter 9, page 000, a diameter DB perpendicular to AC. Draw AB, BC, CD, and DA. Since angles ABC, BCD, CDA, and DAB are inscribed in semicircles, they are, by Fact 90, right angles. Moreover, the central angles are, by construction, right angles. So that the arcs AB, BC, CD, and DA, which the sides of these angles intercept, are, by Fact 76, equal. Hence, by Fact 79, their chords AB, BC, CD, and DA are equal; and the quadrilateral $ABCD$ is then, by definition, a square.



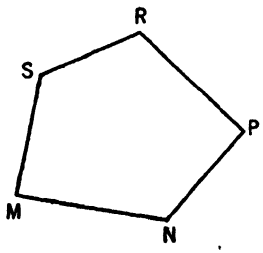


Fig. 158.

SIMILAR POLYGONS

An irregular field, enclosed by a fence, has the shape of a pentagon whose sides measure 75, 55, 70, 50, and 60 rods, respectively. It is decided to erect another fence surrounding the first and having the same shape, that is, forming a similar pentagon, but such that the longest side is 1 rod longer. The necessary amount of fencing may be found without measuring if we show that:

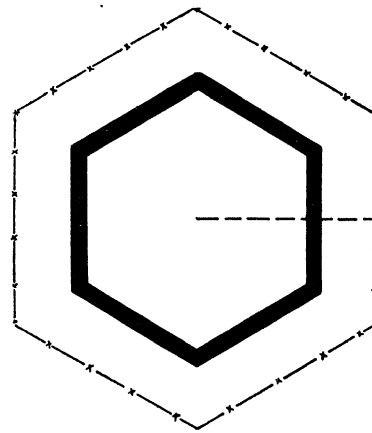
Fact 102: The perimeters of two similar polygons have the same ratio as any two corresponding sides.

Consider, then, the two similar polygons $ABCDE$ and $MNPRS$, Figure 158, with perimeters p and p', respectively. Since the polygons are similar, then, by definition, their corresponding sides are in proportion, that is, $\frac{AB}{MN} = \frac{BC}{NP} = \frac{CD}{PR} = \frac{DE}{RS} = \frac{EA}{SM}$. But in a series of equal ratios, the sum of all the first terms is to the sum of all the second terms as the first term in any of the given ratios is to its second term (see Chapter 10, page 151). Hence, $\frac{AB + BC + CD + DE + EA}{MN + NP + PR + RS + SM} = \frac{AB}{MN}$.

Since the numerator and denominator of the left-hand side of this equality are, respectively, p and p', then $\frac{p}{p'} = \frac{AB}{MN}$.

The amount of fencing necessary to surround the irregular field may now be easily calculated. The perimeter of the given polygon is $75 + 55 + 70 + 50 + 60 = 310$ rods. The longest side is 1 rod longer. Hence, it is equal to $75 + 1 = 76$ rods. Denoting the desired perimeter by p and applying the fact just proved, we have $\frac{p}{310} = \frac{76}{75}$. So that $p = 310 \times \frac{76}{75} = 314.13$ rods.

The commander of a redoubt (that is, a fortified position inclosed by parapets) desires to build a barbed wire fence of seven rows surrounding the redoubt. The redoubt has the shape of a regular hexagon each side of which is 100 feet long. The barbed wire fence is to have the same shape as the redoubt, that is, the fence is also to form a regular hexagon. The sides of the regular hexagon formed by the fence are parallel to the sides of the redoubt. The perpendicular distance between each side of the regular hexagon formed by the fence and the corresponding side of the redoubt is 50 feet. The necessary length of fencing may be found by simply measuring the radius or the apothem of the regular hexagon forming the redoubt. It is necessary, however, to prove first certain properties of regular polygons.



The two regular polygons, P and P' in Figure 159, have the same number of sides. Consider one of the polygons, say P. From any vertex, such as F, draw as many diagonals as possible. Obviously, there is a triangle for each side except the two adjacent to F, namely, EF and AF. So that, if the number of sides of the polygon be denoted by n, there are $(n - 2)$ triangles. The sum of the angles of each triangle is, by Fact 18, a straight angle. Therefore, the sum of the angles of the

polygon, namely, of the $(n - 2)$ triangles, is $(n - 2)$ straight angles. That is,

Fact 103: The sum of the angles of a convex polygon of n *sides is* $(n - 2)$ *straight angles.*

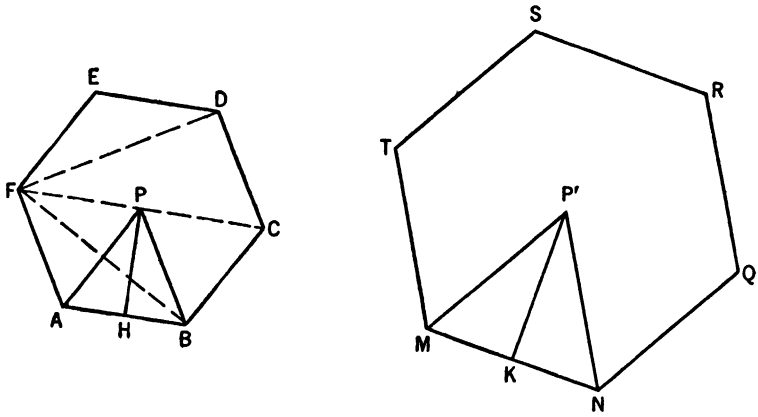


Fig. 159.

It follows, since the number of sides, and hence also the number of angles, of the polygon is n, that

Fact 104: Each angle of a regular polygon of n *sides is equal to* $\frac{(n - 2)}{n}$ *straight angles.*

Hence, each angle of P, Figure 159, is equal to $\frac{(n - 2)}{n}$ straight angles, and each angle of P' is also equal to $\frac{(n - 2)}{n}$ straight angles; for P and P' have the same number of sides, that is, the two polygons are mutually equiangular.

Moreover, the polygons are regular. Hence, by definition, page 348, $AB = BC = CD = DE = EF = FA$, and $MN = NQ = QR = RS = ST = TM$. But if equals are divided by equals, the quotients are equal. So that $\frac{AB}{MN} = \frac{BC}{NQ} = \frac{CD}{QR} = \frac{DE}{RS} = \frac{EF}{ST} = \frac{FA}{TM}$, that is, the corresponding sides of the two polygons are proportional. Therefore, by definition, page 168, P and P' are similar. That is,

Fact 105: Regular polygons of the same number of sides are similar.

CONCERNING PERIMETERS OF REGULAR POLYGONS

From the centers, P and P', of the two regular polygons of the same number of sides, in Figure 159, draw the radii AP, PB, MP', and $P'N$, and the apothems PH and $P'K$. Let p and p', respectively, denote the perimeters of P and P'.

The two regular polygons have the same number of sides; hence, by Fact 105, they are similar. Moreover, by Fact 102, the perimeters, p and p', of the two similar polygons have the same ratio as any two corresponding sides. Hence, $\frac{p}{p'} = \frac{AB}{MN}$.

Clearly, the angles at the center of any regular polygon are equal, for they are corresponding angles of congruent triangles. Further, in Figure 159, angles $APB + PBA + PAB = 180°$, and angles PAB and CBP are equal. So that angles $APB + ABC = 180°$. That is, each angle at the center of any regular polygon is supplementary to an interior angle of the polygon. We have, then, that angles APB and $MP'N$ are equal, for angles with equal supplements are equal. Since radii of the same circle are equal, $AP = PB$ and $MP' = P'N$. Hence, $\frac{AP}{PB} = \frac{MP'}{P'N} = 1$. That is, triangle APB has angle APB equal to angle $MP'N$ of triangle $MP'N$, and the sides including these angles, namely, AP, PB, MP', and $P'N$, proportional. Hence, by Fact 51, the triangles are similar. Therefore, $\frac{AB}{MN} = \frac{AP}{MP'}$.

Moreover, right triangles PHA and $P'KM$ have the acute angles PAH and $P'MK$ equal. Hence, by Fact 49, the triangles are similar, and $\frac{AP}{MP'} = \frac{PH}{P'K}$. Therefore, $\frac{p}{p'} = \frac{AP}{MP'} = \frac{PH}{P'K}$. That is,

Fact 106: The perimeters of two regular polygons of the same number of sides are to each other as their radii or as their apothems.

With these facts, the commander of the redoubt may ascertain the necessary amount of barbed wire as follows. The cen-

ter of the hexagon forming the redoubt can be located by simply finding the intersection of any two diagonals drawn to opposite vertices. Once the center is located, the apothem of the hexagon formed by the redoubt is measured and found to be, in this case, 86 feet. Since the barbed wire fence is to be 50 feet away from the parapet, the apothem of the hexagon formed by the barbed wire is then, obviously, $86 + 50 = 136$ feet. The redoubt measures 100 feet on each side; hence, its perimeter is $6 \times 100 = 600$ feet. Denoting the perimeter of the hexagon formed by the barbed wire by p, we then have, by Fact 106, $\frac{p}{600} = \frac{136}{86}$. Therefore, $p = 600 \times \frac{136}{86} = 950$ feet, approximately. Since seven rows of barbed wire are to be used, the total amount of barbed wire necessary is $7 \times 950 = 6{,}650$ feet, approximately.

AREA OF REGULAR POLYGON

A silo for the preservation of fodder is constructed, whose cross-section has the shape of a regular octagon. To ascertain the amount of fodder it will hold, it is first necessary to determine the amount of floor space available, that is, the area of the regular octagon forming the base.

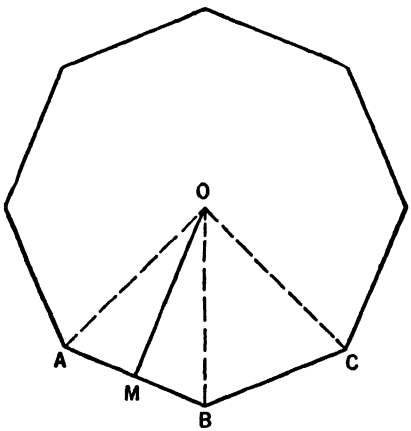


Fig. 160.

Consider, then, the regular polygon O, Figure 160, with apothem OM. Let p denote its perimeter and A its area. Draw the radii OA, OB, OC, and so forth, dividing the polygon into as many congruent triangles as there are sides. Evidently, the apothem is the common altitude of all these triangles. The area of each triangle is, by Fact 59, equal to one-half the apothem multiplied by the base, that is, $\frac{1}{2}OM \times$ the base. So that the sum of the areas of all these triangles is $\frac{1}{2}OM \times$ the sum of all the bases. But the sum of the areas of the triangles is the area of the regular polygon, and the sum of all the

bases is its perimeter. Therefore, if we denote the apothem by a, we have:

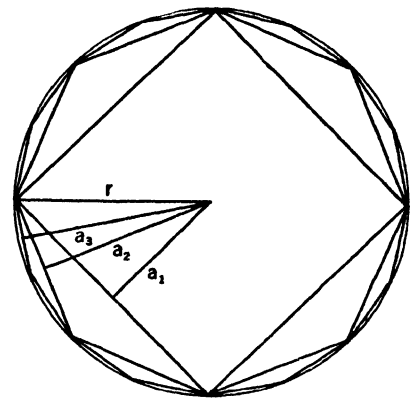
$$A = \tfrac{1}{2}\, ap. \quad \text{That is,}$$

Fact 107: The area of a regular polygon is equal to one-half the product of its perimeter and its apothem.

To find the floor space available, it is therefore sufficient to measure one side and the apothem of the regular polygon forming the base. Thus, assume that the side of the silo in question measures 31 feet and its apothem 38 feet. Then its perimeter is $31 \times 8 = 248$ feet and its area $A = \frac{1}{2} \times 38 \times 248 = 4{,}712$ square feet.

CONCEPT OF LIMIT

The preceding discussion of polygons enables us now to investigate the ideas involved in the measurement of the length and area of a circle.



Suppose that a square is inscribed in a circle. Then, by doubling the number of sides of the inscribed square, a regular inscribed octagon is formed. If this process of doubling the number of sides is continued, we obtain regular inscribed polygons of 16, 32, 64, and so forth, sides. From the figure, the perimeter, apothem, and area of the successive polygons appear to increase and approach nearer and nearer in value to the circumference, radius, and area of the given circle, which remain constantly of the same size. Since we can measure the perimeter, apothem, and area of these regular inscribed polygons, we can, by increasing the number of sides sufficiently, approximate to any degree of accuracy desired the circumference, radius, and area of the given circle.

The circumference of a circle has, therefore, a very special relation to the perimeter of a regular inscribed polygon whose sides are increased indefinitely. Since the perimeter can be made to approach nearer and nearer in value to the circumfer-

ence of the given circle, the difference between the circumference and the perimeter can be made as small as we please by simply increasing sufficiently the number of sides of the given polygon. We then say that the circumference of the given circle is the *limit* of the perimeter of the inscribed regular polygon as the number of sides is increased indefinitely.

The popular idea and the mathematical concept of limit are closely related. Thus, we say that an airplane has a limit as to speed. A manufacturer has a price beyond which he cannot sell if he expects to make a profit. The boundary of a nation is supposed to limit the authority of its government. That is, in everyday life a limit is usually a fixed magnitude or number to which a variable may approach infinitely near, but which ordinarily *it never reaches* or, if it does reach it, *beyond which it cannot pass.*

The mathematical concept, however, differs radically, in the italicized part, from the popular idea of limit. As stated before (Chapter 12, page 179), if we have two variables, such as x and y, and the value of y is determined by the value of x, then the variable y is said to be a function of x. Now *if, as the independent variable* x *approaches some value* a, *the numerical difference between some constant* c *and the function* y *can be made as small as we please and ultimately becomes less than any preassigned positive number, the constant* c *is said to be the limit of the function* y. Hence, in some cases, a limit may be reached and even passed. One of the special meanings in mathematics is identical with the meaning of limit in everyday speech, that is, something sometimes attainable but always impassable. An example of this *special* meaning is a regular polygon inscribed in a circle. By increasing the number of sides of the polygon indefinitely, the area of the inscribed polygon can be made to approach the area of the circle as closely as desired without actually being able to reach it, let alone pass beyond it.

Another illustration of the meaning of limit follows. Suppose that a track inspector on a railroad is to look over carefully the condition of the tracks between two cities, A and C in Figure 161. He decides to inspect the section between city A and town B during the first day. As he begins his tour of inspection, the distance he travels from A toward B is a function of the time elapsed since starting from A. Assuming that he proceeds without interruption, the distance traversed is con-

tinually increasing. Yet meanwhile the distance AB is constant. If we use the letter t to denote the time elapsed since he began his tour of inspection, and S to denote the distance actually traversed, then S is a function of t. Now, assume that it will take the inspector h hours to cover the distance AB. As the time elapsed since he began his tour of inspection, namely, t, approaches h, the difference between the constant distance AB and the distance actually traversed, that is, S, can be made as small as we please. Therefore, the constant distance AB is the limit of the function S.

In this case it is evident that, since it has been assumed that the tour proceeded without interruption, the function may attain its limit. Furthermore, if, upon arriving at B, the track inspector decides to continue further and carry his inspection

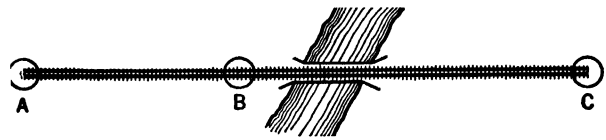


Fig. 161.

beyond the point B, the previously chosen limit may be passed. Similarly, we might be interested in studying the behavior of a function between certain definite values only; in that case, the limit may be reached and even passed if it is desired to carry the study further.

Now suppose, as the inspector continues toward C, that a flood has carried away a bridge spanning a river between B and C. It is evident that he cannot reach C along the tracks. That is, the distance AC cannot be considered in this case as the limit of the function S, for the difference between the distance S traversed by the inspector from A and the constant distance AC cannot be made as small as we please.

In considering the length of a circle, it is important to note that we have assumed the fact that the perimeters of the inscribed polygons approach a definite limit, and that this limit is the circumference of the circle. Although in this case, as may be shown by means of calculus, our instinctive reasoning happens to give us the correct answer, intuition in general should not be trusted.

As an illustration, consider the equilateral triangle ABC,

Figure 162,* whose sides are each one unit in length. Then $AC + CB$ is equal to 2. Bisect AC at D, CB at F, and AB at E, and draw line segments DE and EF. Evidently, the broken line $ADEFB$ is also equal to 2. Continue this process by bisecting the sides of the equilateral triangles ADE and EFB, thus obtaining the four equilateral triangles AMP, PNE, ERQ, and QSB. The broken line $AMPNERQSB$ is undoubtedly also equal to 2. As this process of bisecting the sides is continued, the maximum distance from a point of one of the broken lines $ADEFB$, $AMPNERQSB$, and so forth, from the base AB of the triangle, measured perpendicular to this base, can be made as small as we please. We may then say that the line AB is the limit of the sequence of the broken lines which we have been constructing. But it is easily seen that while the sum of the bases of the triangles is always equal to AB, that is, equal to one, the broken lines formed by the sides of the triangles also remain equal to a constant value, namely, 2. In fact, the difference between the length of the broken lines and the length of the line segment AB is constantly equal to $2 - 1 = 1$, and hence cannot be made as small as we please. Therefore, the length of the broken lines does not approach the length of AB as a limit.

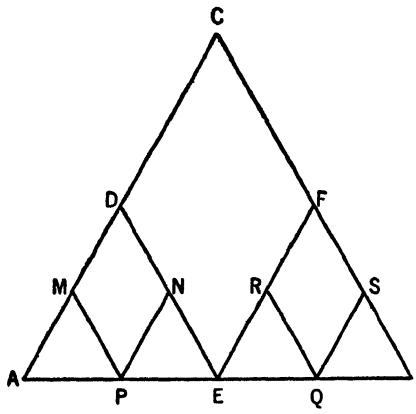


Fig. 162.

We then make the following assumptions:

(*a*) That the limit of the perimeter of a regular inscribed or of a regular circumscribed polygon, as the number of sides is indefinitely increased, is the circumference of the circle.

(*b*) That the area of a circle is the limit of the area of a regular inscribed or of a regular circumscribed polygon, as the number of sides is indefinitely increased.

(*c*) That if the number of sides of a regular inscribed poly-

* For a more detailed discussion of this problem, see J. W. Young's *Fundamental Concepts of Algebra and Geometry*, New York, The Macmillan Company, 1936.

gon is indefinitely increased, the apothem of the polygon approaches the radius of the circle as its limit.

CIRCUMFERENCE OF A CIRCLE

Consider then the circles O and O' in Figure 163. Denote the circumferences by C and C' and the radii by r and r', respectively. Let p and p' be the perimeters of two regular inscribed polygons of the same number of sides.

Since, by Fact 106, the perimeters of two regular polygons are to each other as their radii, we may write $\frac{p}{p'} = \frac{r}{r'}$. Now, if we increase the number of sides of the inscribed polygons

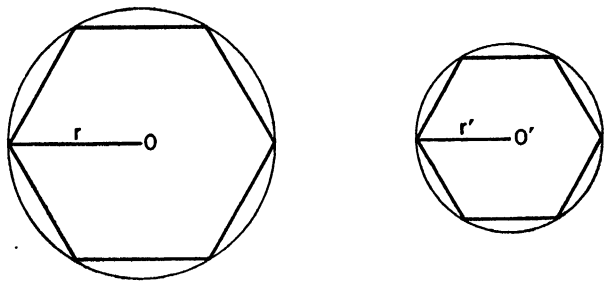


Fig. 163.

indefinitely, p approaches C and p' approaches C', while r and r' remain constant. So that, applying the assumption made in the preceding discussion, page 360, we obtain $\frac{C}{C'} = \frac{r}{r'}$. That is,

Fact 108: The circumferences of two circles are to each other as their radii.

Multiplying the numerator and denominator of a fraction by the same number does not change the value of the fraction. Hence, since $\frac{C}{C'} = \frac{r}{r'}$, we may write $\frac{C}{C'} = \frac{2r}{2r'}$. Multiplying both sides by C', we obtain $C = \frac{2rC'}{2r'}$; and dividing both sides by $2r$, we may write $\frac{C}{2r} = \frac{C'}{2r'}$. In other words,

Fact 109: The ratio of the circumference of any circle to its diameter is constant.

The constant ratio of the circumference of any circle to its diameter is denoted by the Greek letter π (pi). This constant π is intimately connected with one of the three famous problems referred to in the beginning of Chapter 20, namely, the squaring of the circle. To obtain a square equal in area to a given circle, the exact value of π was needed. But although a rational number approximation to the constant π was computed correctly, by means of inscribed polygons, to a number which today would be written with 35 decimal places, no exact value could be obtained, for the simple reason that π is represented arithmetically by a non-ending and non-repeating decimal. With the discovery of the calculus by Newton (1642-1727) and Liebnitz (1646–1716), simple methods for the computation of π have been developed. So that we may now obtain the value of π correct to any desired number of decimal places. The number π is equal to 3.141592653 . . . ; but for computations where great accuracy is not necessary, the value of π is taken approximately as $\frac{22}{7} = 3\frac{1}{7}$.

Since the constant ratio of the circumference of a circle to its diameter is π, we may write in any circle: $C = \pi d$, or $C = 2\pi r$.

Assume that a flat copper ring is to encircle a monumental stone lamp post. The diameter of the lamp post is 3 feet. It is desired to know the length of copper necessary to form the ring. Here $d = 3$. Therefore, $C = \pi d = \pi \times 3 = \frac{22}{7} \times 3 = \frac{66}{7} = 9\frac{3}{7}$ feet, approximately.

AREA OF A CIRCLE

The most malleable and ductile of all metals is gold. It can be beaten into thin leaves only 1/250,000 of an inch thick. On the average, one ounce Troy may be beaten so thin that it will cover 100 square feet. Suppose that one face of each of 150 circular plaques commemorating historical events is to be covered with gold leaf. The diameter of the plaques is 11 inches. To ascertain the number of ounces of gold needed to cover the 150 plaques, it is first necessary to show that,

Fact 110: The area of a circle is equal to one-half the product of its circumference and its radius, or to the product of the square of the radius and the constant number π. That is,

$$A = \tfrac{1}{2} Cr = \pi r^2.$$

Consider any circle O, Figure 164, and let r, C, and A, respectively, denote its radius, circumference, and area. Circumscribe about circle O a regular polygon with perimeter p and area a. Then, by Fact 107, the area of the circumscribed polygon is equal to one-half the product of its perimeter p and its apothem r. That is, $a = \frac{1}{2}pr$. If the number of sides of the circumscribed polygon is increased indefinitely, the perimeter p approaches C and a approaches A, while r remains constant. So that $\frac{1}{2}$ pr approaches $\frac{1}{2}$ Cr, and a approaches A. Therefore, applying the assumption made on page 360, $A = \frac{1}{2}$ Cr. But, by Fact 109, $C = 2\pi r$; hence, $A = \frac{1}{2}$ $Cr = \frac{1}{2} \times 2\pi r \times r = \pi r^2$.

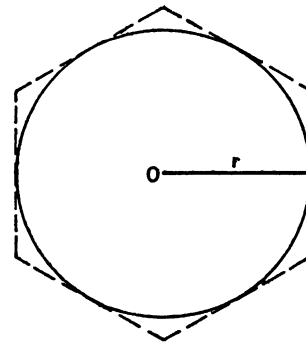


Fig. 164.

The number of ounces of gold necessary to cover the plaques is now easily obtained. Since the diameter of a plaque is 11 inches, its radius is $\frac{11}{2} = 5\frac{1}{2}$ inches. Substituting in the formula for the area, we obtain $A = \pi r^2 = \pi\left(\frac{11}{2}\right)^2 = \frac{22}{7} \times \frac{121}{4} = \frac{1331}{14} = 95.07$ square inches. Since there are 150 plaques, the total number of square inches to be covered is $150 \times 95.07 = 14{,}260$ square inches approximately. There are 144 square inches in a square foot. Hence, the 150 plaques will require $14{,}260 \div 144 = 99$ square feet, approximately, of gold leaf. But 100 square feet weigh one Troy ounce; therefore, the total number of ounces is $\frac{99}{100} = 0.99$ of a Troy ounce of gold, approximately.

The formula for the area of a circle may also be given in terms of the diameter. Thus, since $A = \pi r^2$ and $2r = d$, then $r = \frac{d}{2}$, so that $A = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4} = \frac{1}{4}\pi d^2$.

Now, let A and A', r and r', and d and d', respectively, denote the areas, radii, and diameters of any two circles. Then, by Fact 110, $A = \pi r^2$ and $A' = \pi r'^2$. If equals are divided by equals, the quotients are equal; hence, $\frac{A}{A'} = \frac{\pi r^2}{\pi r'^2}$. Cancelling π, we obtain:

$$\frac{A}{A'} = \frac{r^2}{r'^2}.$$

Similarly, since $A = \frac{1}{4}\pi d^2$ and $A' = \frac{1}{4}\pi d'^2$, then

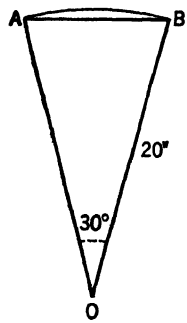
$$\frac{A}{A'} = \frac{\frac{1}{4}\pi d^2}{\frac{1}{4}\pi d'^2}.$$

So that $\frac{A}{A'} = \frac{d^2}{d'^2}$. Therefore,

Fact 111: The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.

SECTOR OF A CIRCLE

The cross section of an airplane part has the shape shown in the figure. The shape of the cross section formed by two radii and the arc of the same circle is called a *sector* of a circle. The radius of this particular circle is 20 inches, and the central angle of the sector, namely, angle AOB, is 30°. To ascertain the air resistance offered by this part, it is necessary to determine the area of the sector.



Since the area of a sector of a circle is a part of the total area of the circle, it is easily seen that *the area of a sector is to the area of the circle as the angle of the sector is to 360°*. Hence, if we denote the area of a sector by K, $\frac{K}{\pi r^2} = \frac{\text{central angle}}{360°}$, so that $K = \frac{\text{central angle}}{360°} \times \pi r^2$.

Applying this formula, the area of this particular cross section is $K = \frac{30}{360} \times \pi \times (20)^2 = \frac{1}{12} \times \frac{22}{7} \times 400 = 104.76$ square inches.

RADIAN

The constant ratio of the circumference of any circle to its diameter, namely, π, provides an additional and very useful method of measuring angles. Since, by Fact 109, $C = 2\pi r$, we may write $\frac{C}{r} = 2\pi$. The right-hand side of this equality,

that is, 2π, is a constant value. Hence, the radius of any circle is contained exactly 2π times in its circumference. This fact and the certainty that a central angle is measured by its intercepted arc suggest the possibility of using the radius as a unit in the measurement of angles as follows.

Construct any circle O, Figure 165, with radius r. From A measure an arc AB equal in length to the radius of the circle.* Draw the radius OB. Then angle AOB, used as the unit of measurement, is called a *radian*. That is, *a radian is the angle subtended at the center of a circle by an arc equal to the radius.*

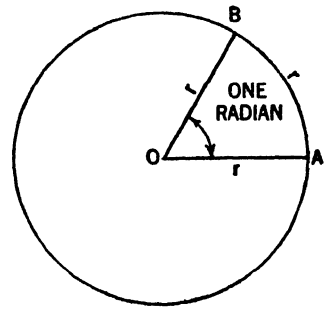


Fig. 165.

It has just been shown that there are exactly 2π radii in the circumference of a circle. Hence, if the radius is taken as the unit of measurement, there are exactly 2π radians round the center of a circle. Since there are 360° round the center of a circle, then 2π radians = 360°, or, dividing both sides by 2, we may write:

$$\pi \text{ radians} = 180°.$$

$$\text{Therefore, 1 radian} = \frac{180°}{\pi} = 57.295° \ldots\ldots = 57° 17' 45''\dagger;$$

$$\text{and, conversely, } \frac{\pi}{180} \text{ radian} = 1°; \text{ that is,}$$

$$0.01745 \ldots \text{ radian} = 1°.$$

That is, to reduce degrees to radians, multiply the number of degrees by $\frac{\pi}{180}$ *or* 0.01745. *The result is, of course, radians.*

To reduce radians to degrees, multiply the number of radians by $\frac{180}{\pi}$ *or* 57.295 *and label the product degrees.*

Thus 60° 32′ 10″, equal to 60.536°, is equal to $60.536 \times 0.01745 = 1.056$ radians approximately. Similarly, 2.3 radians is equal to $2.3 \times 57.295 = 131.7785$ degrees, that is,

* Note that the *arc* AB and NOT the chord AB is equal in length to the radius.
† This value is correct to the nearest second.

131° 46′ 43″ approximately. The accompanying tables on page 368 are convenient as time-savers in making such computations.

LENGTH OF AN ARC

A machine shop is to install two pulleys with radii 1 foot 6 inches and $5\frac{1}{2}$ inches, respectively. The distance between

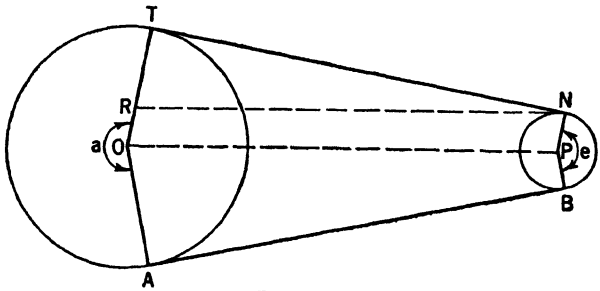


Fig. 166.

the centers of the pulleys, OP in Figure 166, is 6 feet. To ascertain the length of driving belt required to run around the two pulleys, it is necessary to know how to find the length of an arc of a circle.

Consider, then, the two concentric circles shown in Figure 167. By Fact 108, the circumferences of two circles are to each other as their radii. That is, if C and C' denote the respective circumferences of the two given circles, then $\frac{C}{C'} = \frac{OA}{OD}$. Evidently, this fact is also true for any part of the circumference, for dividing both numerator and denominator of the left-hand side of the equality by the same number does not change its value. That is, if $\frac{C}{C'} = \frac{OA}{OD}$, then evidently, $\frac{\frac{1}{6}C}{\frac{1}{6}C'} = \frac{OA}{OD}$, or $\frac{\frac{1}{8}C}{\frac{1}{8}C'} = \frac{OA}{OD}$. Therefore, since a part of the circumference is an arc of the circle, in unequal circles arcs subtending equal angles are to each other as the corresponding radii.

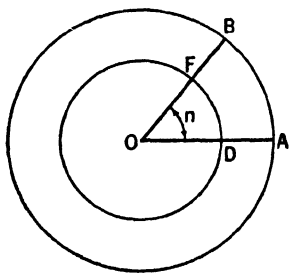


Fig. 167.

Thus, in the figure, $\frac{\text{arc } AB}{\text{arc } DF} = \frac{OA}{OD}$. Now, if OD is taken as the unit radius, arc DF is then, by definition, the radian measure of angle n, that is, arc $DF = n$ radians. Substituting these values in the equality $\frac{\text{arc } AB}{\text{arc } DF} = \frac{OA}{OD}$, we obtain $\frac{\text{arc } AB}{n} = \frac{OA}{1}$. So that, denoting the radius OA by R, we may write:

$$\text{arc } AB = n \cdot R.$$

That is, *the length of any arc is equal to the radius multiplied by the radian measure of the angle subtended by the arc.*

The length of driving belt necessary to run around the two pulleys may be now determined as follows. Through N, Figure 166, draw NR parallel to OP. Lines OT and PN are radii of the two circles, drawn to the points of contact of the common tangent TN. Therefore, by Fact 84, they are perpendicular to TN. It follows, by Fact 11, that OT and PN are parallel. Since NR and OP are also, by construction, parallel, the figure $OPNR$ is a parallelogram. Hence, by Fact 36, opposite sides OR and PN and OP and NR are, respectively, equal. That is, $OR = PN = 5.5$ inches and $NR = OP = 72$ inches. Since OT is 18 inches, then $TR = OT - OR = 18 - 5.5 = 12.5$ inches.

Now, triangle NRT is a right triangle, for OT is perpendicular to TN. Hence, by Fact 70, $(NR)^2 = (TR)^2 + (TN)^2$, or $(72)^2 = (12.5)^2 + (TN)^2$. So that $(72)^2 - (12.5)^2 = (TN)^2$; that is, $5184 - 156.25 = (TN)^2$ and $TN = \sqrt{5027.75} = 70.9$ inches. The length of belt which does *not* touch the two pulleys is therefore $2 \times 70.9 = 141.8$ inches.

To compute the length of belt touching the pulleys, the values of angles a and e in radians must be determined. It is evident that quadrilaterals $OTNP$ and $OABP$ are congruent. Hence, angles TOP and AOP are equal. That is, angle TOP is one-half of angle TOA. Therefore, to determine angle a, it is sufficient to compute angle TOP. The parallel lines NR and OP are cut by the transversal OT. Hence, by Fact 14, corresponding angles TRN and TOP are equal. Angle TRN of right triangle NRT is easily found by using any one of the angle functions. Choosing the tangent ratio, we obtain tan TRN

DEGREES—RADIANS

Deg.	Radians	Deg.	Radians	Min.	Radians	Sec.	Radians
0	0.00000	**60**	1.04720	**0**	0.00000	**0**	0.00000
1	0.01745	61	1.06465	1	0.00029	1	0.00000
2	0.03491	62	1.08210	2	0.00058	2	0.00001
3	0.05236	63	1.09956	3	0.00087	3	0.00001
4	0.06981	64	1.11701	4	0.00116	4	0.00002
5	0.08727	**65**	1.13446	**5**	0.00145	**5**	0.00002
6	0.10472	66	1.15192	6	0.00175	6	0.00003
7	0.12217	67	1.16937	7	0.00204	7	0.00003
8	0.13963	68	1.18682	8	0.00233	8	0.00004
9	0.15708	69	1.20428	9	0.00262	9	0.00004
10	0.17453	**70**	1.22173	**10**	0.00291	**10**	0.00005
11	0.19199	71	1.23918	11	0.00320	11	0.00005
12	0.20944	72	1.25664	12	0.00349	12	0.00006
13	0.22689	73	1.27409	13	0.00378	13	0.00006
14	0.24435	74	1.29154	14	0.00407	14	0.00007
15	0.26180	**75**	1.30900	**15**	0.00436	**15**	0.00007
16	0.27925	76	1.32645	16	0.00465	16	0.00008
17	0.29671	77	1.34390	17	0.00495	17	0.00008
18	0.31416	78	1.36136	18	0.00524	18	0.00009
19	0.33161	79	1.37881	19	0.00553	19	0.00009
20	0.34907	**80**	1.39626	**20**	0.00582	**20**	0.00010
21	0.36652	81	1.41372	21	0.00611	21	0.00010
22	0.38397	82	1.43117	22	0.00640	22	0.00011
23	0.40143	83	1.44862	23	0.00669	23	0.00011
24	0.41888	84	1.46608	24	0.00698	24	0.00012
25	0.43633	**85**	1.48353	**25**	0.00727	**25**	0.00012
26	0.45379	86	1.50098	26	0.00756	26	0.00013
27	0.47124	87	1.51844	27	0.00785	27	0.00013
28	0.48869	88	1.53589	28	0.00814	28	0.00014
29	0.50615	89	1.55334	29	0.00844	29	0.00014
30	0.52360	**90**	1.57080	**30**	0.00873	**30**	0.00015
31	0.54105	91	1.58825	31	0.00902	31	0.00015
32	0.55851	92	1.60570	32	0.00931	32	0.00016
33	0.57596	93	1.62316	33	0.00960	33	0 00016
34	0.59341	94	1.64061	34	0.00989	34	0.00016
35	0.61087	**95**	1.65806	**35**	0.01018	**35**	0.00017
36	0.62832	96	1.67552	36	0.01047	36	0.00017
37	0.64577	97	1.69297	37	0.01076	37	0.00018
38	0.66323	98	1.71042	38	0.01105	38	0.00018
39	0.68068	99	1.72788	39	0.01134	39	0.00019
40	0.69813	**100**	1.74533	**40**	0.01164	**40**	0.00019
41	0.71558	101	1.76278	41	0.01193	41	0.00020
42	0.73304	102	1.78024	42	0.01222	42	0.00020
43	0.75049	103	1.79769	43	0.01251	43	0.00021
44	0.76794	104	1.81514	44	0.01280	44	0.00021
45	0.78540	**105**	1.83260	**45**	0.01309	**45**	0.00022
46	0.80285	106	1.85004	46	0.01338	46	0.00022
47	0.82030	107	1.86750	47	0.01367	47	0.00023
48	0.83776	108	1.88496	48	0.01396	48	0.00023
49	0.85521	109	1.90241	49	0.01425	49	0.00024
50	0.87266	**110**	1.91986	**50**	0.01454	**50**	0.00024
51	0.89012	111	1.93732	51	0.01484	51	0.00025
52	0.90757	112	1.95477	52	0.01513	52	0.00025
53	0.92502	113	1.97222	53	0.01542	53	0.00026
54	0.94248	114	1.98968	54	0.01571	54	0.00026
55	0.95993	**115**	2.00713	**55**	0.01600	**55**	0.00027
56	0.97738	116	2.02458	56	0.01629	56	0.00027
57	0.99484	117	2.04204	57	0.01658	57	0.00028
58	1.01229	118	2.05949	58	0.01687	58	0.00028
59	1.02974	119	2.07694	59	0.01716	59	0.00029
60	1.04720	**120**	2.09440	**60**	0.01745	**60**	0.00029

DEGREES AND DECIMAL FRACTIONS TO RADIANS

Deg.	Radians	Deg.	Radians	Deg.	Radians	Deg.	Radians	Deg.	Radians
10	0.174533	1	0.017453	0.1	0.001745	0.01	0.000175	0.001	0.000017
20	0.349066	2	.034907	.2	.003491	.02	.000349	.002	.000035
30	0.523599	3	.052360	.3	.005236	.03	.000524	.003	.000052
40	0.698132	4	.069813	.4	.006981	.04	.000698	.004	.000070
50	0.872665	5	.087267	.5	.008727	.05	.000873	.005	.000087
60	1.047198	6	.104720	.6	.010472	.06	.001047	.006	.000105
70	1.221731	7	.122173	.7	.012217	.07	.001222	.007	.000122
80	1.396264	8	.139626	.8	.013963	.08	.001396	.008	.000140
90	1.570797	9	.157080	.9	.015708	.09	.001571	.009	.000157

RADIANS—DEGREES

Radians	Degrees	Radians	Degrees	Radians	Degrees	Radians	Degrees
1	57.2958	0.1	5.7296	0.01	0.5730	0.001	0.0573
2	114.5916	.2	11.4592	.02	1.1459	.002	.1146
3	171.8873	.3	17.1887	.03	1.7189	.003	.1719
4	229.1831	.4	22.9183	.04	2.2918	.004	.2292
5	286.4789	.5	28.6479	.05	2.8648	.005	.2865
6	343.7747	.6	34.3775	.06	3.4377	.006	.3438
7	401.0705	.7	40.1071	.07	4.0107	.007	.4011
8	458.3662	.8	45.8366	.08	4.5837	.008	.4584
9	515.6620	.9	51.5662	.09	5.1566	.009	.5157
10	572.9578	1.0	57.2958	.10	5.7296	.010	.5730

RADIANS—DEGREES

Multiples and Fractions of π Radians

Radians	Radians	Degrees	Radians	Radians	Degrees	Radians	Radians	Degrees
π	3.1416	180	$\pi/2$	1.5708	90	$2\pi/3$	2.0944	120
2π	6.2832	360	$\pi/3$	1.0472	60	$3\pi/4$	2.3562	135
3π	9.4248	540	$\pi/4$	0.7854	45	$5\pi/6$	2.6180	150
4π	12.5664	720	$\pi/5$	0.6283	36	$7\pi/6$	3.6652	210
5π	15.7080	900	$\pi/6$	0.5236	30	$5\pi/4$	3.9270	225
6π	18.8496	1080	$\pi/7$	0.4489	25.714	$4\pi/3$	4.1888	240
7π	21.9911	1260	$\pi/8$	0.3927	22.5	$3\pi/2$	4.7124	270
8π	25.1327	1440	$\pi/9$	0.3491	20	$5\pi/3$	5.2360	300
9π	28.2743	1620	$\pi/10$	0.3142	18	$7\pi/4$	5.4978	315
10π	31.4159	1800	$\pi/12$	0.2618	15	$11\pi/6$	5.7596	330

$= \frac{TN}{TR}$, that is, tan $TRN = \frac{70.9}{12.5} = 5.6720$. So that, by the table of tangents, angle $TRN = 80°0'4'' =$ angle TOP.

But angle TOP is one-half of angle TOA; hence, angle $TOA = 2 \times 80°0'4'' = 160°0'8''$. Angle a is, therefore, $360° - 160°0'8'' = 359°59'60'' - 160°0'8'' = 199°59'52''$. Reducing angle a to radians, we obtain $199°59'52'' \times 0.01745 = 199.998 \times 0.01745 = 3.49$ radians. The length of belt touching the larger pulley is the length of the arc subtended by angle a, that is, arc AT. Since arc $AT = a \times OT$, the length of belt touching circle O is $3.49 \times 18 = 62.82$ inches.

The reader may easily show, by extending OP to the right and applying Fact 14, that angle TOA is equal to angle e. Therefore, angle $e = 160°0'8''$. Reducing angle e to radians, we obtain $160°0'8'' \times 0.01745 = 160.002 \times 0.01745 = 2.79$ radians. The length of belt touching the smaller pulley is, therefore, $2.79 \times 5.5 = 15.34$ inches.

Hence, the length of the driving belt around the two pulleys is $141.8 + 62.82 + 15.34 = 219.96$ inches, or 18 feet 4 inches.

AREA OF A SECTOR

A plot of ground in a public park has the shape of a segment of a circle. A *segment of a circle* is a figure composed of an arc of a circle and its chord, such as $ABCDA$ in Figure 168. The plot is to be seeded for grass. To ascertain the number of pounds of seed needed, it is necessary to determine the area of the segment.

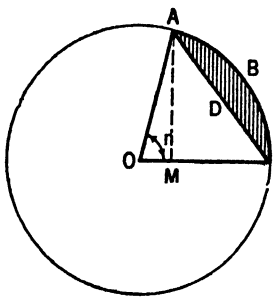


Fig. 168.

The length of an arc of a circle is part of its circumference. It is, then, easily seen that the length of an arc of a circle is to the circumference as the central angle subtended by the arc is to 360°. That is, if l denotes the length of the arc and r the radius of the circle, $\frac{l}{2\pi r} = \frac{\text{central angle}}{360°}$. But on page 364 of this chapter, it was shown that if K represents the area of a

sector and r the radius of the circle, $\frac{K}{\pi r^2} = \frac{\text{central angle}}{360°}$. Therefore, we may write $\frac{K}{\pi r^2} = \frac{l}{2\pi r}$; that is, $K = \pi r^2 \frac{l}{2\pi r} = \frac{1}{2}rl$. In other words, *the area of a sector of a circle is equal to one-half the radius multiplied by the length of the arc.*

The area of the sector $AOCBA$ is, then, equal to $\frac{1}{2}$ arc $AC \times OC$. Denote the radius of the circle, namely, OC, by R. Then, since arc AC is equal to the radius multiplied by the radian measure of the angle subtended by the arc, arc $AC = Rn$. So that we may write: sector $AOCBA = \frac{1}{2}RnR = \frac{1}{2}R^2n$.

Evidently, the area of segment $ABCDA$ is equal to area of sector $AOCBA$ less the area of triangle OAC. The area of triangle OAC is equal to one-half its base OC multiplied by its altitude AM. That is, area of triangle $OAC = \frac{1}{2}OC \times AM = \frac{1}{2}R \times AM$. The triangle OAM is, by construction, a right triangle. Hence, $\sin n = \frac{AM}{OA} = \frac{AM}{R}$. So that $R \sin n = AM$. That is, area of triangle $OAC = \frac{1}{2}R \times R \sin n = \frac{1}{2}R^2 \sin n$.

Therefore, area of segment $ABCDA = \frac{1}{2}R^2n - \frac{1}{2}R^2 \sin n$. Since these two terms contain $\frac{1}{2}R^2$ as a common factor, we may write: area of segment $ABCDA = \frac{1}{2}R^2 (n - \sin n)$.

The area of the plot of ground can now be easily found. Assume that the radius of the circle and the central angle are, respectively, 350 feet and 75°. The central angle is equal to $75 \times 0.01745 = 1.3$ radians, approximately. Hence, the area of the segment is $\frac{1}{2}(350)^2 [1.3 - 0.9659] = 20{,}463.63$ square feet, approximately. Upon ascertaining how much seed is needed per square foot, from some practical horticulturist, the amount to be purchased for this particular plot can then be found by arithmetic.

Problems

1. What is the waste in cutting the largest possible circular plate from a rectangular piece of copper 19 inches wide and 23 inches long?

2. The diameter of a rose window in a cathedral is 20 feet. What is the length of an arc of 80°?

3. The minute hand of a tower clock is 8 feet long. Over what distance will the extremity move in 42 minutes?

4. In a linoleum design a regular hexagon having a perimeter of 36 inches is inscribed in a circle. What is the area of the circle?

5. Three tubular canisters, 5 inches in radius, are to be enclosed in an equilateral triangular prism. What is the width of a side of the container?

6. Taking the radius of the earth to be 3,960 miles, find the length in feet of one second of arc on the equator.

7. A five-inch water pipe (other things, such as friction, being equal) can convey how many times as much water as a $\frac{1}{2}$-inch pipe?

8. The inner and outer diameters of a flat ring are $9\frac{1}{2}$ and 10 inches, respectively. Find the area of the ring.

9. It requires 92 rods of barbed wire to enclose a semicircular field. Find the area of the field.

10. If the diameter of a locomotive wheel is 76 inches, find the number of revolutions per minute made by the wheel when the locomotive is going 58 miles per hour.

11. (The "gear" of a bicycle is the diameter of a wheel one revolution of which would advance it as far as one revolution of the pedal advances the bicycle.)

(*a*) If the diameter is d, and the number of teeth in the front sprocket is m and in the rear sprocket is n, express the gear x as a formula.

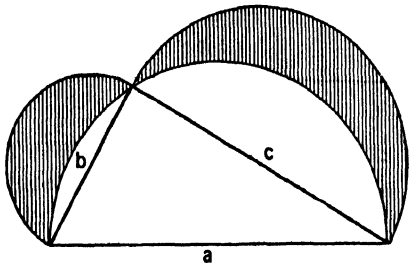
(*b*) If the diameter of the rear wheel is 28 inches, the front sprocket has 22 teeth, and the rear sprocket 8 teeth, what is the gear?

12. In designing a stained glass window for a cathedral, the artist inscribes a rectangle with sides 4 and 7 in a circle. What is the area of this circle?

13. Two circles, with respective radii of 5 and 8 feet, intersect each other. The line connecting the points of intersection is 6 feet long. What is the area included between the two intercepted arcs?

14. In a large mural design, three circles of one, two, and three feet radius, respectively, are in juxtaposition, the circle of two-foot radius being between the other two circles, with the line of centers in a straight line. This line is prolonged. A point P upon it is

found which is the crossing place of two other lines, each tangent to the largest and next largest circles. What is the distance from P to the center of the largest circle?



15. Describe semicircles on the three sides, a, b, and c of a right triangle as diameters, as in the diagram. Prove that the combined areas of the two shaded lunes are equal to the area of the triangle. (NOTE: These are called the *lunes of Hippocrates.*)

23 FIGURE SKATING IN THE SKY

That type of intellectual curiosity which revels in puzzles and riddles for their own sake, without heed to any possible practical use, is exemplified by the Greeks' interest in abstract mathematics. Plato, Euclid, and others scorned utilitarian applications and never remotely suspected that more than two thousand years in the future, engineers, astronomers, physicists, and chemists would find many ways to utilize the relations of various curves.

As already mentioned in Chapter 21, page 335, until John Kepler published his famous laws of planetary motion in 1608 A. D., all planets were thought of as moving in circular paths, or, at any rate, along paths which did not much deviate from circles. Kepler and his teacher, Tycho Brahe, by painstaking observations, noted that the tables in use at that time represented actual motions very imperfectly. Also, they noticed that the amount of deviation in some planets was much greater than in others. They then concluded that the principles on which the tables had been computed were false.

THE ELLIPSE

After Tycho's death, Kepler tried a large number of combinations of oval-shaped curves without success. Eventually, having experimented with most of the more complicated curves, he used the simplest known oval curve, namely, the ellipse. To his amazement and joy he found that, if the sun were assumed to be at one focus, an ellipse accurately marked the path of Mars around the sun.

An ellipse is the curve traced by a point which moves so that the sum of its distances from two fixed points remains constant.

To determine the equation of the curve traced by a point moving under the conditions imposed by this definition, assume any two distinct fixed points in a plane, such as F and

F' in Figure 169. Through these two points draw a line. Designate this line as the X-axis. Draw the Y-axis, through the midpoint of the segment FF', perpendicular to the X-axis. Denote the distances from F and F' to the origin by c. The coordinates of F are $(c, 0)$ and those of F' are $(-c, 0)$. Place the moving point P in a general position, that is, in *any* position, designated by (x, y).

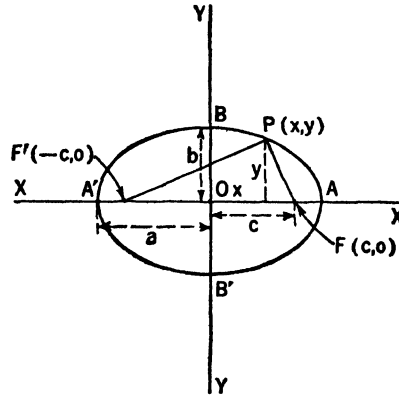


Fig. 169.

The problem as stated imposes the condition that the sum of the distances from the moving point to the two fixed points shall remain constant. That is, $PF + PF'$ = a constant. For convenience, since any number may be expressed as twice another number, choose the constant equal to $2a$. Then,

$$PF + PF' = 2a. \tag{1}$$

To write this condition in terms of the coordinates of P, apply the distance formula $d = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}$ (Chapter 19, page 298).

Then $PF = \sqrt{(x - c)^2 + (y - 0)^2} = \sqrt{(x - c)^2 + y^2}$,

and $PF' = \sqrt{(x + c)^2 + (y - 0)^2} = \sqrt{(x + c)^2 + y^2}$.

Substituting in (1), we obtain:

$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a.$$

To express this equation in a more compact form, transpose one of the radicals; thus,

$$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2}.$$

Square both sides, keeping in mind that the right-hand side of the equation is then a binomial of the type $(a - b)^2 = a^2 - 2ab + b^2$. We obtain:

$$(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + [(x - c)^2 + y^2].$$

Expand, collect terms, and transpose as follows:

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

or $$4a\sqrt{(x-c)^2 + y^2} = 4a^2 - 4cx.$$

Dividing both sides by 4, we may write:

$$a\sqrt{(x-c)^2 + y^2} = a^2 - cx.$$

Squaring both sides again gives:

$$a^2[(x-c)^2 + y^2] = a^4 - 2a^2cx + c^2x^2,$$

or $$a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2x^2.$$

Finally collecting terms, we may write:

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

The figure FPF' is by construction a triangle. Hence, $PF + PF'$ is greater than FF' (Chapter 7, page 107). Since $PF + PF' = 2a$ and $FF' = 2c$, then $2a > 2c$, so that $a > c$. The expression $a^2 - c^2$ is, therefore, a positive number whose square root can be obtained. Hence, for additional simplicity, let $a^2 - c^2 = b^2$. Then,

$$b^2x^2 + a^2y^2 = a^2b^2. \qquad (2)$$

This is the equation of an ellipse whose *center* O is at the origin. The given fixed points F and F' are the *foci*. The longest chord through the center, that is, AA' in the figure, is called the *major axis*. The shortest chord, namely, BB', is the *minor axis*. The major axis is equal to $2a$ and the minor axis is $2b$.

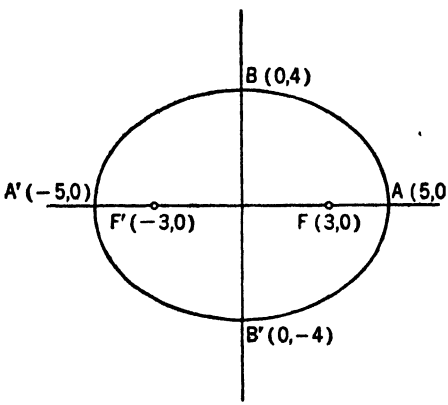


Fig. 170.

The points of intersection of the curve and the coordinate axes are now easily obtained. If $y = 0$, equation (2) reduces to $b^2x^2 = a^2b^2$. So that, dividing both sides by b^2, we obtain $x^2 = a^2$, or $x = \pm a$. Similarly, setting $x = 0$, equation (2) reduces to $a^2y^2 = a^2b^2$, or $y^2 = b^2$ and $y = \pm b$. That is, the curve crosses the X-axis at a distance of $\pm a$ from the origin, and the Y-axis at a distance of $\pm b$ from the origin.

For convenience, both sides of equation (2) may be divided by a^2b^2, giving:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{3}$$

When written in form (3), the intercepts on the coordinate axes are seen to be the square roots of the denominators. Thus, the equation $16x^2 + 25y^2 = 400$ is of type (2). Hence, the curve representing this equation is an ellipse whose center is at the origin. Dividing both sides by 400, we may write $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Here $a^2 = 25$ and $b^2 = 16$. So that the intercepts on the X- and Y-axes are, respectively, $a = \pm\sqrt{25} = \pm 5$ and $b = \pm\sqrt{16} = \pm 4$. Since the intercepts on the X-axis give the greatest chord through the center of the ellipse, the foci must lie on this axis. To locate the foci, we make use of the relation $a^2 - c^2 = b^2$. Substituting in this expression, we obtain $25 - c^2 = 16$; that is, $25 - 16 = c^2$ and $c = \pm\sqrt{9} = \pm 3$. The graph of $16x^2 + 25y^2 = 400$ is shown in Figure 170.

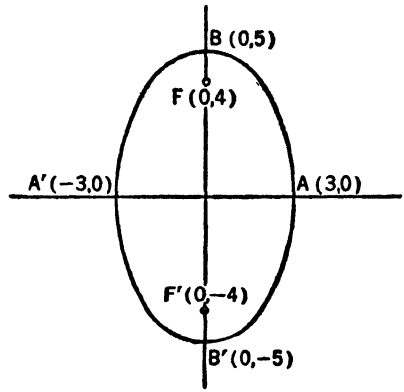


Fig. 171.

Similarly, the equation $25x^2 + 9y^2 = 225$ is represented graphically by an ellipse. Dividing both sides by 225, we may write $\frac{x^2}{9} + \frac{y^2}{25} = 1$. The intercepts on the X- and Y-axes are, therefore, $a = \pm\sqrt{9} = \pm 3$ and $b = \pm\sqrt{25} = \pm 5$. The intercepts on the Y-axis determine the greatest chord through the center of the ellipse. Hence, the major axis is $2b = 2 \times 5 = 10$, and the foci are on the Y-axis. If the foci are on the Y-axis, it may be easily ascertained, in a way precisely similar to that followed in the development of the general equation, that the relation between a, b, and c becomes $b^2 - c^2 = a^2$. Substituting in this expression, we obtain: $25 - c^2 = 9$, that is, $25 - 9 = c^2$ and $c = \pm\sqrt{16} = \pm 4$. The graph of $25x^2 + 9y^2 = 225$ is shown in Figure 171.

CONSTRUCTION OF THE ELLIPSE

To draw the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, mark off the intercepts and the foci as shown in Figure 172. Since $a = 5$, then the sum of the distances from the moving point to the two fixed points is $2a = 2 \times 5 = 10$. Place two pins at the foci F and F'. The distance FF' is by definition equal to $2c = 2 \times 3 = 6$. Therefore, the perimeter of triangle FPF' is $2a + 2c = 10 + 6 = 16$. Cut a string so that its total length, after tying the ends together, is also 16 units. Pass this loop over the pins at F and F' and let it lie on the flat surface of the paper. Place a pencil inside the loop and hold the string taut. The string will then assume the shape of a triangle. If the pencil is moved so as to keep the string taut, the sum of the distances from the pencil's point to F and F' will remain constant and equal to 10. Therefore, the pencil point will trace the ellipse whose equation is $16x^2 + 25y^2 = 400$.

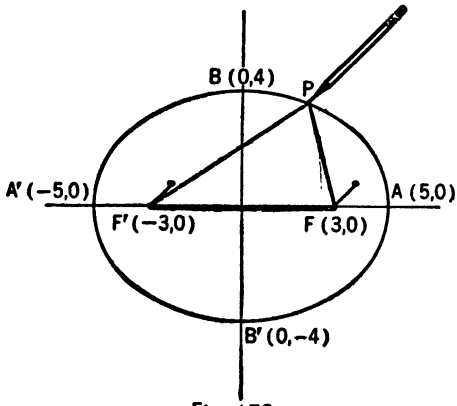


Fig. 172.

A more accurate method by employing compasses may be used to draw an ellipse. For example, to draw $16x^2 + 25y^2 = 400$, describe from F and F' and with a radius of 6 units, four arcs as shown in Figure 173. Then, with F and F' as centers and a radius of 4 units, describe four arcs intersecting the arcs already drawn. Evidently, the sum of the distances from each of these points to F and F' is 10. Hence, the points are

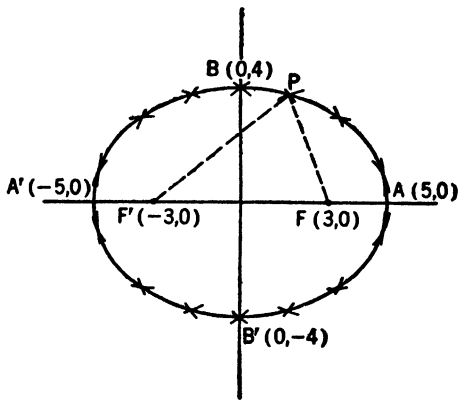


Fig. 173.

on the ellipse. Any number of points may be located in this manner as long as the sum of the radii is equal to $2a = 10$. The curve can then be easily sketched by connecting these points by a smooth curve.

ECCENTRICITY

The reader has undoubtedly noticed the similarity of the equations of the ellipse and of the circle (Chapter 20, page 312). Both curves are represented by equations of the second degree in two variables; moreover, in both equations the terms of second degree form the sum of terms in the squares of the two variables. But while in the equation of the circle the terms of second degree have equal coefficients, in the ellipse the coefficients of second-degree terms are different. That is, if we consider only equations of the second degree in which no xy term appears, *an ellipse is represented by an equation of the second degree in two variables, such as* x *and* y, *in which the terms of the second degree have different positive coefficients and are the sum of the squares of* x *and* y.*

The relation between the circle and the ellipse can be more clearly grasped by considering the ratio $\frac{c}{a}$. This ratio is called the *eccentricity*. The value of c is, as before, the distance between the center and one of the foci, while a is the distance between the center and one of the intercepts on the major axis. Denoting the eccentricity by e, we have, for an ellipse,

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

In this equation, since b is a real number,

$$a^2 - b^2 \leqq a^2,$$

or

$$\sqrt{a^2 - b^2} \leqq a.$$

But

$$e = \frac{\sqrt{a^2 - b^2}}{a}.$$

Comparing the fraction $\frac{\sqrt{a^2 - b^2}}{a}$ with the fraction $\frac{a}{a}$, it may be seen that the numerator of the first, namely, $\sqrt{a^2 - b^2}$, is

* This statement is shown to be true on page 402 of this chapter.

less than or equal to the numerator of the second, that is, a. The denominators of both fractions are equal. Hence,

$$e = \frac{\sqrt{a^2 - b^2}}{a} \leqq \frac{a}{a}.$$

But $$\frac{a}{a} = 1.$$

Therefore, $$e = \frac{\sqrt{a^2 - b^2}}{a} \leqq 1.$$

Now, if $e = 1$, $\frac{\sqrt{a^2 - b^2}}{a} = \frac{a}{a}$. So that $\sqrt{a^2 - b^2} = a$ or $a^2 - b^2 = a^2$. This relation holds only if $b = 0$. But by hypothesis (page 376), $a^2 - c^2 = b^2$, and a is not equal to c. Therefore, b is not equal to zero, and hence,

$$e < 1.$$

Hence, *the eccentricity of an ellipse is always less than unity.*

If the major axis is held at a fixed length, the value of a, in the fraction $\frac{c}{a}$, remains constant. The value of e then varies from 0 to 1 as c goes from 0 to a. Obviously, when c is 0, the foci coincide with the center and the ellipse becomes a circle. If the value of c is a, then $c = \sqrt{a^2 - b^2} = a$. But $\sqrt{a^2 - b^2} = a$ only if b is zero. So that, when the foci coincide with the intercepts on the major axis, the intercepts on the minor axis coincide with the center and the ellipse becomes the straight line AA' (Figure 169). That is, the limiting forms of an ellipse are a circle of diameter $2a$ and the line segment AA'. Therefore, the flatness of an ellipse increases as e increases from 0 to 1.

PRACTICAL APPLICATIONS OF THE ELLIPSE

There are many practical applications of the ellipse. Eyeglasses, machine gears, and arches of bridges are instances from the fields of physics and engineering.

In nature study, the ellipse is indispensable. Planets move in an elliptical orbit around the sun, the latter being at one focus. The path of the earth around the sun is shown in Figure 174. The axis of the earth is not perpendicular to the plane of its orbit. At the summer solstice, June 21st, the North

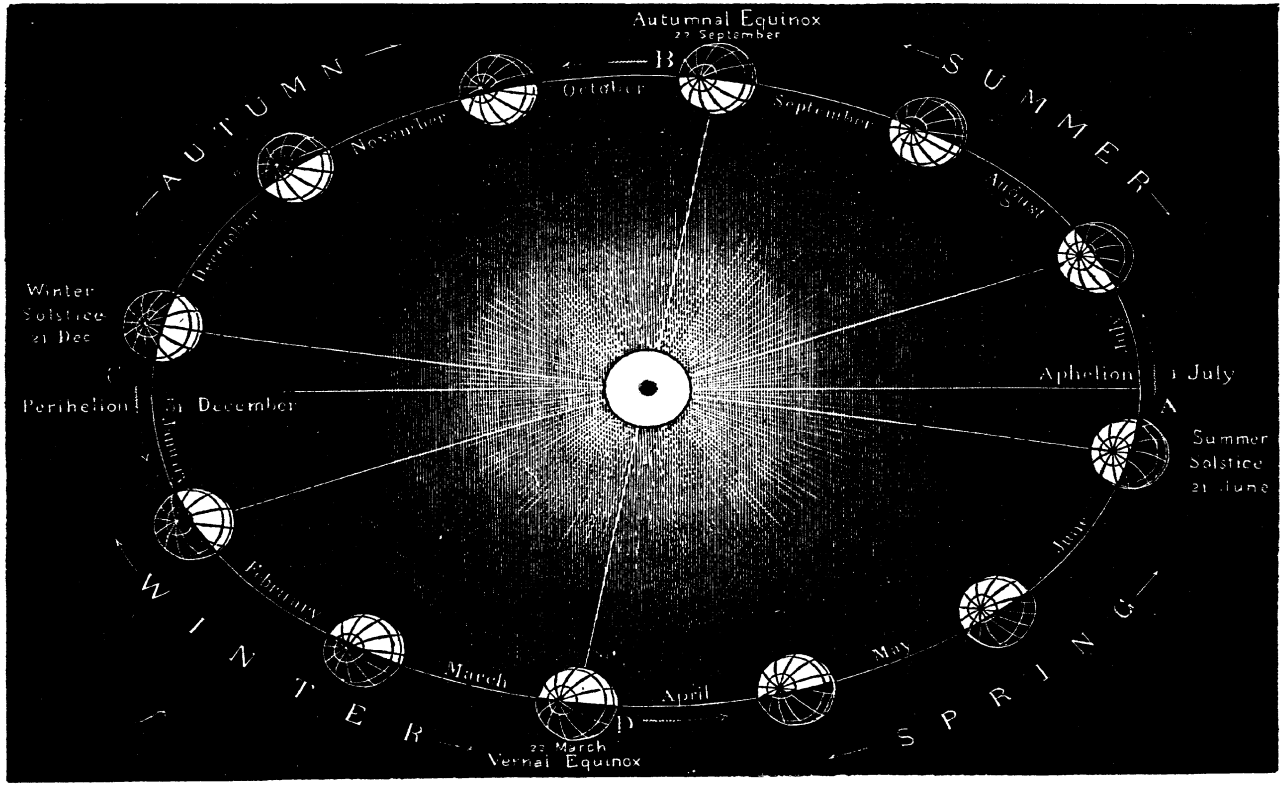


Fig. 174. The Seasons.

From *Recreations in Astronomy*, by Henry White Warren, D.D., New York, Harper & Brothers, 1892.

Pole is inclined toward the sun at an angle of about $23\frac{1}{2}$ degrees (Figure 174 A). The North Pole is then in constant sunlight, while the South Pole is shrouded in darkness. In all portions of the northern hemisphere, day is longer than night, but inhabitants on the southern hemisphere see the sun for less than half a day. Three months later, at the autumnal equinox, September 22nd, the poles of the earth are turned neither toward nor away from the sun, but rather *aside* (Figure 174 B). The world is then seen illuminated equally on each side of the equator, and everywhere daylight and night last for 12 hours. At the winter solstice, December 21st, three months later still, since the axis of the earth preserves the same direction, the North Pole is turned away from the sun and hence is in complete darkness while the South Pole is enjoying continual daylight. Inhabitants of the northern hemisphere get more darkness than light but those of the southern hemisphere see the sun for more than half the day. Finally, at the vernal equinox, March 21st, day and night are again everywhere equal.

The scientist, like Shakespeare's poet, "Doth glance from heaven to earth, from earth to heaven," and from the contemplation of an infinitely vast universe turns his efforts to the study of the smallest particle. The tiny atom, until modern times, was believed to be the ultimate indivisible particle of which all matter is made up. Modern scientists, by means of an electrical gun, subjected the atom to such terrific bombardment that it broke up into minute electrically charged particles called *electrons*. These minute particles or electrons move at enormous speeds in elliptical orbits around the nucleus of the atom (Figure 175). The empty spaces in an atom are so large in comparison to its nucleus that all the nuclei of all the component material of all the ships in the world packed tightly would easily go inside the well of an ordinary fountain pen.

THE HYPERBOLA

No drama in the sky is more splendid or awe-inspiring than a comet. The word *comet*, meaning a "long-haired star," is a misnomer. Comets are bodies very different from stars. Their brilliant nucleus or head is composed of a swarm of solid particles, and the long nebulous appendage which stretches away from the sun, called the *tail*, is caused by gas ejected from the scattered multitude of particles which form its head.

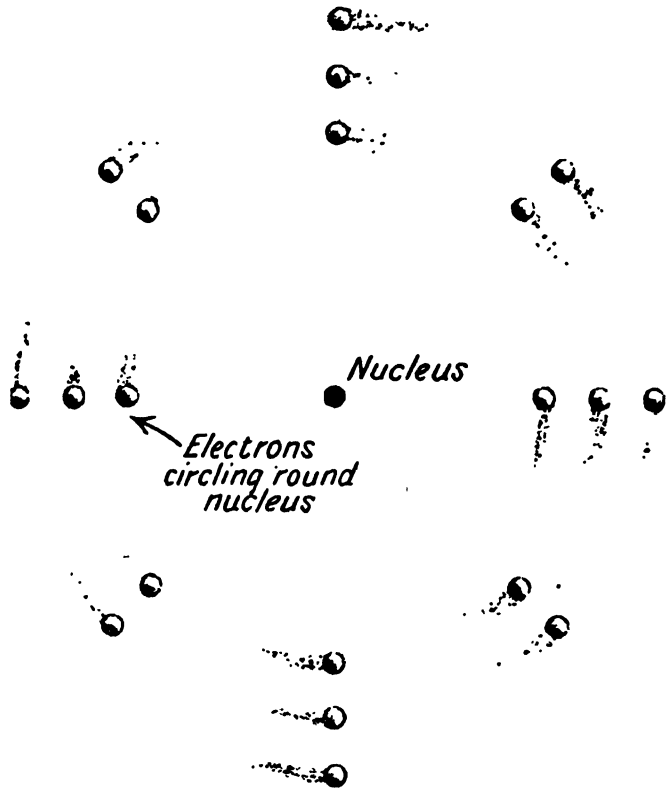


Fig. 175.

This tail, observable only in large comets, is sometimes as long as the distance from the earth to the sun.

In ancient times, comets were regarded with terror. Plagues, war, and death were attributed to their maleficent influence. Yet today it is known that, on the contrary, the earth or any other large planet considerably affects the movements of a comet, and that the mass of planets exerting a gravitational pull on the

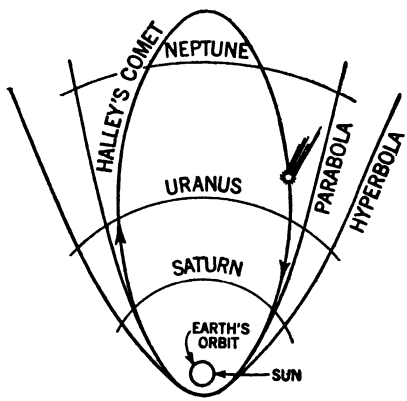


Fig. 176.

comet direct its path through space. Some comets move in elliptical orbits, others trace parabolas, and still others hyperbolas. Comets which travel in elliptical orbits, of course, return. Halley's comet, whose orbit is shown in Figure 176, was last seen in 1910 and will return in 1985. Others, traveling along a parabolic or hyperbolic path, are seen once, then gradually disappear into space never to be seen again.

A hyperbola is a curve traced by a point which moves so that the difference of its distances from two fixed points remains constant.

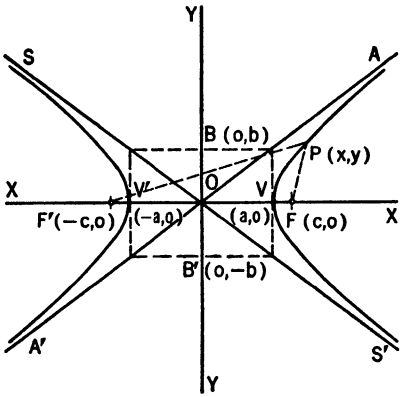


Fig. 177.

To develop the equation of a hyperbola, consider any two fixed points in the plane, such as F and F' in Figure 177. Draw the X-axis through these two points. Through the midpoint of the segment FF', draw the Y-axis perpendicular to the X-axis. Denote the distances from F and F' to the origin by c. The coordinates of F are $(c, 0)$ and those of F' are $(-c, 0)$. Place the moving point $P(x, y)$ in a general position.

Then, from the conditions of the problem, $PF - PF' = a$ constant. Again, for convenience, denote the constant by $2a$, so that

$$PF - PF' = 2a. \tag{4}$$

Applying the distance formula $d = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}$, we obtain

$$PF = \sqrt{(x - c)^2 + (y - 0)^2} = \sqrt{(x - c)^2 + y^2},$$

and $$PF' = \sqrt{(x + c)^2 + (y - 0)^2} = \sqrt{(x + c)^2 + y^2}.$$

Substituting in (4), we may write:

$$\sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = 2a.$$

Transpose $2a$ and the second radical; thus,

$$\sqrt{(x - c)^2 + y^2} - 2a = \sqrt{(x + c)^2 + y^2}.$$

Square both sides, recalling again that $(a - b)^2 = a^2 - 2ab + b^2$. We then obtain:

$$[(x - c)^2 + y^2] - 4a\sqrt{(x - c)^2 + y^2} + 4a^2 = (x + c)^2 + y^2.$$

Expand, collect terms, and transpose as follows:

$$x^2 - 2cx + c^2 + y^2 - 4a\sqrt{(x - c)^2 + y^2} + 4a^2 = x^2 + 2cx + c^2 + y^2,$$

or
$$4a^2 - 4cx = 4a\sqrt{(x - c)^2 + y^2}.$$

Dividing both sides by 4 and squaring again, we obtain:

$$a^4 - 2a^2cx + c^2x^2 = a^2[(x - c)^2 + y^2],$$

or
$$a^4 - 2a^2cx + c^2x^2 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2.$$

Finally, collecting terms, we may write:

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2).$$

The figure FPF' is a triangle. Then, since the difference of two sides of a triangle is less than the third side (Chapter 7, page 107), $PF - PF'$ is less than FF'. But $PF - PF'$ is $2a$, and $FF' = 2c$. Hence, $2a < 2c$, so that $a < c$. The expression $c^2 - a^2$ is, therefore, a positive number whose square root can be easily obtained. Hence, for added simplicity, let $c^2 - a^2 = b^2$. Then,

$$b^2x^2 - a^2y^2 = a^2b^2. \tag{5}$$

This is the equation of a hyperbola whose *center O* is at the origin. The given fixed points F and F' are its *foci*. The points V and V', where the two branches of the hyperbola cross the X-axis, are called the *vertices*. The line joining the vertices, namely, VV', of length $2a$, is the *transverse axis*. The segment BB', of length $2b$, is the *conjugate axis*.

The points of intersection of the curve can now be obtained. If $y = 0$, equation (5) reduces to $b^2x^2 = a^2b^2$, so that $x^2 = a^2$ and $x = \pm a$. If $x = 0$, equation (5) becomes $-a^2y^2 = a^2b^2$. So that, dividing both sides by $-a^2$, we obtain $y^2 = -b^2$, or $y = \pm\sqrt{-b^2}$. The number b^2 is positive, for the square of any real number is always positive. Therefore, $-b^2$ is negative and $\sqrt{-b^2}$ is an imaginary number. That is, the curve does not cross the Y-axis, but it crosses the X-axis at a distance of $\pm a$ from the origin.

The nature of the hyperbola may be arrived at through the

following reasoning. If we solve equation (5) for y, we obtain:

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}.$$

In order to obtain a real number for y, the difference $x^2 - a^2$ must be positive. Whatever may be the values of a and x, a^2 and x^2 are positive because, as is well known, the square of any real number is always positive. Therefore, in order that $x^2 - a^2$ be positive, x^2 must be greater than a^2, and hence x must be greater than a. That is, the curve does not exist in that part of the plane bounded by two lines parallel to the Y-axis and passing through V and V' (Figure 177). As x increases, y also increases, so that the curve continually rises. Evidently, since $x^2 - a^2$ is the difference of two positive numbers, $x^2 - a^2 < x^2$. But $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$, and consequently $y < \frac{b}{a}\sqrt{x^2}$ and $y > -\frac{b}{a}\sqrt{x^2}$. That is, $y < \frac{b}{a}x$ and $y > -\frac{b}{a}x$.

The two equations $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ represent two straight lines passing through the origin with slopes $\frac{b}{a}$ and $-\frac{b}{a}$, respectively (see Chapter 13, page 198). As x increases in value, the hyperbola will tend to approach nearer and nearer to these lines without ever actually reaching them, for the value of y in $y = \frac{b}{a}\sqrt{x^2 - a^2}$ can be made as near to $y = \pm\frac{b}{a}x$ as we please by merely increasing the value of x sufficiently.

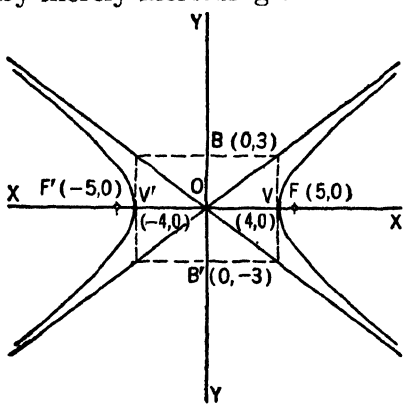


Fig. 178.

These two lines (AA' and SS' in Figure 177), called *asymptotes*, are easily constructed by drawing the diagonals of the rectangle whose sides are $2a$ and $2b$ and whose center is the origin. Obviously, this reasoning holds whether the values of x are positive or negative, since x^2 will always be positive. That is, the curve, as shown in Figure 177, intersects the X-axis

at $(a, 0)$ and $(-a, 0)$ and then approaches nearer and nearer to its asymptotes as it recedes, on both sides, from the Y-axis.

Both sides of equation (5) may be divided by a^2b^2, obtaining:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{6}$$

When written in form (6), the intercepts on the X-axis and the corresponding points on the Y-axis are seen to be the square roots of the denominators. Thus, the equation $9x^2 - 16y^2 = 144$ is of type (5). Hence, the curve representing this equation is a hyperbola. Dividing both sides by 144, we may write $\frac{x^2}{16} - \frac{y^2}{9} = 1$. Here $a^2 = 16$ and $b^2 = 9$. So that the intercepts on the X-axis are $a = \pm\sqrt{16} = \pm 4$ units and the corresponding points on the Y-axis $b = \pm\sqrt{9} = \pm 3$ units away from the origin. Since the curve crosses the X-axis, its transverse axis is $2a = 8$, and the foci must be on this axis. To locate the foci, we make use of the relation $c^2 - a^2 = b^2$. Substituting in this expression, we obtain $c^2 - 16 = 9$, that is, $c^2 = 9 + 16 = 25$ and $c = \pm\sqrt{25} = \pm 5$. To locate the asymptotes, construct a rectangle whose center is the origin and whose sides, parallel to the axes, pass through the points of intersection of the curve with the X-axis and the corresponding points on the Y-axis. The diagonals of this rectangle are the asymptotes. The graph of this hyperbola is shown in Figure 178.

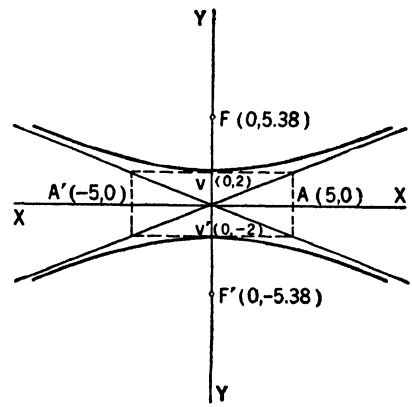


Fig. 179.

If the foci of the hyperbola are chosen on the Y-axis, it is easily seen that the equation, developed by exactly the method of the preceding discussion, is:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \tag{7}$$

Thus, the equation $25y^2 - 4x^2 = 100$ is represented graph-

ically by a hyperbola. Dividing both sides by 100, we obtain $\frac{y^2}{4} - \frac{x^2}{25} = 1$. Evidently, when $x = 0$, $y = \pm 2$; but when $y = 0$, $x = \pm\sqrt{-25}$. Hence, the curve does not cross the X-axis. That is, the intercepts on the Y-axis are $b = \pm\sqrt{4} = \pm 2$ units away from the origin, and the corresponding points on the X-axis are $a = \pm\sqrt{25} = \pm 5$ units away from the origin. Since the curve crosses the Y-axis, its transverse axis is $2b = 2 \times 2 = 4$, and the foci must be on this axis. The relation between a, b, and c now becomes $c^2 - b^2 = a^2$. Substituting in this expression, we obtain $c^2 - 4 = 25$; that is, $c^2 = 25 + 4 = 29$ and $c = \pm\sqrt{29} = \pm 5.38$. To locate the asymptotes, construct a rectangle whose center is the origin and whose sides, parallel to the axes, pass through the points of intersection of the Y-axis and the corresponding points on the X-axis. The diagonals of the rectangle are the asymptotes. The graph of this hyperbola is shown in Figure 179.

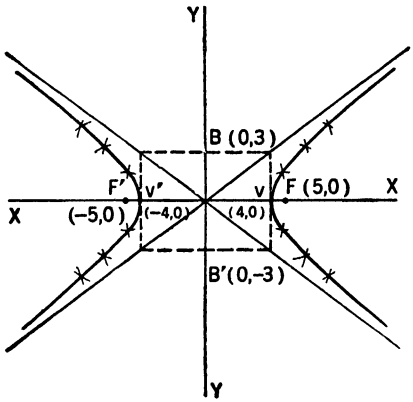


Fig. 180.

As in the case of the ellipse, separate points of a hyperbola may be drawn by means of compasses. Consider the hyperbola $9x^2 - 16y^2 = 144$. From this equation the law of motion may be easily derived. Dividing both sides by 144 gives $\frac{x^2}{16} - \frac{y^2}{9} = 1$. Hence, $a^2 = 16$ and $b^2 = 9$. So that $c^2 = 16 + 9 = 25$ and $c = \pm 5$. The coordinates of the foci are therefore (5, 0) and (−5, 0). Since $a^2 = 16$, $a = \pm 4$. That is, the curve crosses the X-axis at the points (4, 0) and (−4, 0). (See Figure 180.) Since these points are the vertices, they are on the hyperbola. Choosing either vertex, say V (4, 0), its distance from $F'(-5, 0)$ is 9 and from $F(5, 0)$ is 1. The difference 9 − 1 is 8. Therefore, the difference of the distances of any other point on the hyperbola from the foci must also be 8. Thus, to draw the hyperbola $9x^2 - 16y = 144$, describe

from F and F', with a radius of 10 units, four arcs, as shown in Figure 180. Then, with F and F' as centers, describe, with a radius of 2 units, four arcs intersecting the arcs already drawn. Evidently, the difference of the distances from each of these points to F and F' is 8. Hence, the points are on the hyperbola. Any number of points may be located in this manner as long as the difference of the radii used is equal to $2a = 8$. The curve can then be easily sketched by connecting these points by a smooth curve.

ECCENTRICITY OF THE HYPERBOLA

The eccentricity of a hyperbola, as of an ellipse, is the ratio $\frac{c}{a}$, where c is the distance between the center and one of the foci and a is the distance between the center and a vertex. Hence, denoting the eccentricity by e, we have:

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

In this equation, since b is a real number,

$$a^2 + b^2 \geqq a^2,$$

or

$$\sqrt{a^2 + b^2} \geqq a.$$

But

$$e = \frac{\sqrt{a^2 + b^2}}{a}.$$

Comparing the fraction $\frac{\sqrt{a^2 + b^2}}{a}$ with the fraction $\frac{a}{a}$, it may be seen that the numerator of the first, namely, $\sqrt{a^2 + b^2}$, is greater than or equal to the numerator of the second, that is, a. The denominators of both fractions are equal. Hence,

$$e = \frac{\sqrt{a^2 + b^2}}{a} \geqq \frac{a}{a}.$$

But

$$\frac{a}{a} = 1.$$

Therefore,

$$e = \frac{\sqrt{a^2 + b^2}}{a} \geqq 1.$$

If $e = 1$, $\frac{\sqrt{a^2 + b^2}}{a} = \frac{a}{a}$ and $\sqrt{a^2 + b^2} = a$. This relation holds only when $b = 0$. But $b^2 = c^2 - a^2$ (page 385), and c by hypothesis does not equal a. Hence, b is not equal to zero, and consequently $e > 1$. Therefore, *the eccentricity of a hyperbola is always greater than unity.*

If the length of the transverse axis is held fixed, the value of a in the fraction $\frac{c}{a}$ remains constant. The value of e then increases as c is increased from a to any value greater than a. Obviously, if e is made to approach 1, c will approach a in value. Since the value of a is fixed, $c = \sqrt{a^2 + b^2}$ can be made to approach a only if b approaches zero. But as b approaches 0, the altitude BB' (Figure 177) of the rectangle approaches zero, the asymptotes turn toward the X-axis, and the hyperbola flattens. If c is increased indefinitely, the value of e also increases. But since a is constant, $c = \sqrt{a^2 + b^2}$ can be increased only by increasing the value of b. As b is increased, the altitude BB' of the rectangle increases, the asymptotes turn toward the Y-axis, and the hyperbola broadens.

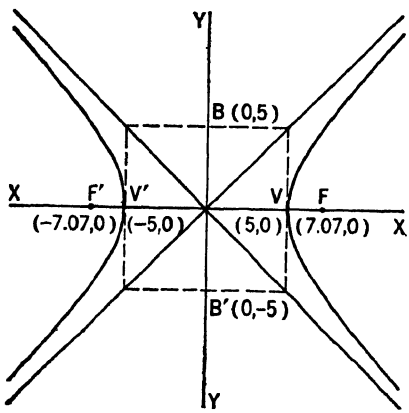


Fig. 181.

RECTANGULAR HYPERBOLA

When the axes of a hyperbola are equal, a is equal to b, and equation (5) may be written $a^2x^2 - a^2y^2 = a^4$, which evidently reduces to:

$$x^2 - y^2 = a^2. \tag{8}$$

The rectangle, formed by drawing lines parallel to the coordinate axes and passing through the vertices and the corresponding points on the conjugate axis, is in this case a square. It may be easily shown that the diagonals of a square are at right angles, that is, that the asymptotes are at right angles. For this reason, the curve is called a *rectangular* hyperbola.

Thus, the equation $x^2 - y^2 = 25$ represents a rectangular hyperbola. Here $a = \pm\sqrt{25} = \pm 5$. Since $a = b$, $c = \sqrt{a^2 + b^2} = \sqrt{2a^2} = a\sqrt{2}$. So that, $c = \sqrt{2} \times 25 = \sqrt{50} = \pm 7.07$. The graph of $x^2 - y^2 = 25$ is shown in Figure 181.

Summarizing the preceding discussion of the hyperbola, it may be noted that this curve is represented by an equation of the second degree in two variables. It may be observed also that if the signs of the coefficients are not taken into consideration, the numerical value of the coefficients of the terms of second degree may be equal or different. However, the signs of the coefficients have to be different. If, as before, we consider only equations of the second degree in which the xy term does not appear, *a hyperbola is represented by an equation of the second degree in two variables, such as* x *and* y, *in which the terms of second degree have opposite signs.**

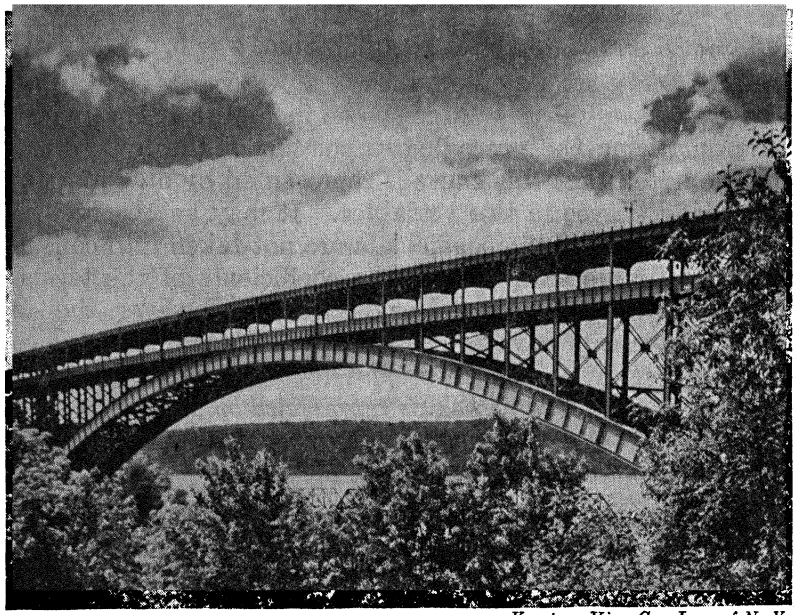
THE PARABOLA

Save for the circle, the simplest and probably the best-known

Golden Gate Bridge, San Francisco, Calif.

curve represented by an equation of the second degree is the parabola. Physical phenomena are represented very often by this curve. The path of an object thrown up in the air, the cable of a suspension bridge, and the paths of some comets are parabolic in shape. Its scientific application in the reflecting surfaces of searchlights and telescopes, in the arches of bridges,

* This general statement is shown to be true on page 402.



Keystone View Co., Inc., of N. Y.

Henry Hudson Bridge, New York

and as the graph of many scientific formulas renders the parabola one of the most useful curves.

A parabola is the curve traced by a point which moves so that its distances from a fixed line and a fixed point are equal.

To determine the equation of the curve traced by a point moving under this law of motion, assume any fixed line (DD', Figure 182) and any fixed point (such as F) on the plane. Through F draw the X-axis perpendicular to DD' and intersecting the latter at A. Draw the Y-axis perpendicular to the X-axis and passing through the midpoint of segment AF. Denote the distance from the origin to F by c. The coordinates of F are $(c, 0)$. Since DD' and the Y-axis are both, by construction, perpendicular to the X-axis, they are,

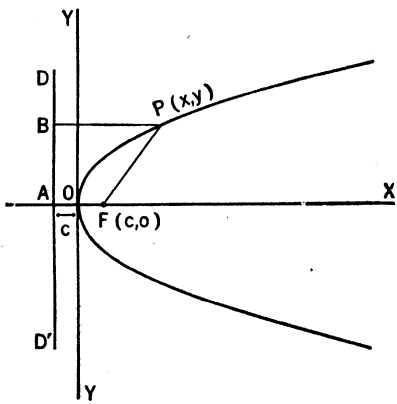


Fig. 182.

by Fact 11, parallel. Therefore, all points on DD' are at a distance of c units from the Y-axis (Fact 39). Place the moving point $P\ (x, y)$ in a general position conforming to the conditions of the problem, that is, equidistant from DD' and F. Since the distance from a point to a line is measured along the perpendicular from the point to the line, draw PB perpendicular to DD'. Draw PF.

The law of motion in the case of a parabola states that the distance from the moving point to the fixed line must be equal to its distance from the fixed point. That is,

$$PF = PB. \tag{9}$$

The lines PB and the X-axis are both, by construction, perpendicular to DD', so that, by Fact 11, they are parallel. Since PB is parallel to the X-axis, points B and P are at the same distance from the X-axis (Fact 39), and the ordinate of B is equal to the ordinate of P. That is, the ordinate of B may be also denoted by y. Furthermore, B is on line DD'; hence, it is at a distance of c units from the Y-axis. Therefore, the coordinates of B are $(-c, y)$. To write the condition expressed by (9) in terms of the coordinates of P, apply the distance formula $d = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}$.

Then, $PF = \sqrt{(x - c)^2 + (y - 0)^2} = \sqrt{(x - c)^2 + y^2}$,

and $PB = \sqrt{(x + c)^2 + (y - y)^2} = \sqrt{(x + c)^2}$.

Substituting in (9) gives:

$$\sqrt{(x - c)^2 + y^2} = \sqrt{(x + c)^2}.$$

Squaring both sides and expanding gives:

$$x^2 - 2cx + c^2 + y^2 = x^2 + 2cx + c^2.$$

Collecting terms, we may write:

$$y^2 = 4cx. \tag{10}$$

This is the equation of a parabola whose *vertex* O is at the origin. The fixed point F is called the *focus*, and the line DD' is the *directrix*. The line passing through the vertex and the focus, in this case the X-axis, is called the *axis* of the parabola.

To ascertain the nature of a parabola, solve equation (10) for y, obtaining:

$$y = \pm\sqrt{4cx}. \tag{11}$$

In order to obtain real values for y, the expression $4cx$ must be positive, for otherwise the value of y would be imaginary. If c is assumed to be positive, then only positive values of x may be taken. That is, if the coefficient of x in (10) is positive, the curve lies on the right-hand side of the Y-axis only. Similarly, if c is assumed to be negative, since $4cx$ must be positive, only negative values of x may be taken. Hence, if the coefficient of x in (10) is negative, the curve lies on the left of the Y-axis only.

Assume in the following discussion that c is positive. If $x = 0$, then $y = 0$. Hence, the parabola intersects the axes only at the origin. It is easily seen from equation (11) that, for each positive value of x, we obtain two values of y numerically equal but opposite in sign. In other words, for each positive value of x we obtain a positive value of y, locating a point in the first quadrant, and a negative value of y locating a point in the fourth quadrant at exactly the same distance from the X-axis as the point already located in the first quadrant. Hence, the part of the curve in the fourth quadrant is an exact reflection of the portion of the curve in the first quadrant.

If the value of x is increased, the value of y also increases. Thus, the curve extends indefinitely to the right of the Y-axis and away from its axis. It is to be noted, however, that if the values of x are very small, the difference between two successive values of y is large; while if the values of x are taken very large, the difference between two successive values of y is small. For example: if c is assumed to be equal to 1, for $x = \frac{1}{16}$, $y = \pm\sqrt{4 \times \frac{1}{16}} = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$; for $x = \frac{1}{4}$, $y = \pm\sqrt{4 \times \frac{1}{4}} = \pm\sqrt{1} = \pm 1$. The difference in the values of x is only $\frac{1}{4} - \frac{1}{16} = \frac{4-1}{16} = \frac{3}{16}$, while the difference in the corresponding values of y is $1 - \frac{1}{2} = \frac{1}{2}$. That is, since, for very small values of x, y increases more rapidly than x, the curve rises very rapidly near the vertex. However, if $x = 1000$, $y = \pm\sqrt{4000} = \pm 63.25$; and if $x = 1001$, $y = \pm\sqrt{4004} = \pm 63.28$. The difference in the values of x is $1001 - 1000 = 1$, while the difference in the corresponding values of y is only $63.28 - 63.25$

$= 0.03$. Thus, since x now increases more rapidly than y, the curve tends to become parallel to its axis.

Evidently if c is assumed to be negative, the same reasoning holds, but the curve lies in the second and third quadrants.

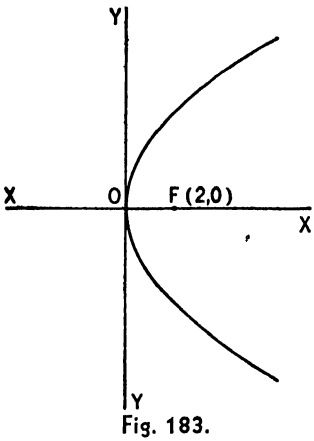


Fig. 183.

The reader may easily show that if the Y-axis is assumed to be the axis of the parabola, then its equation is:

$$x^2 = 4cy. \tag{12}$$

The curve will then lie above or below the X-axis, depending on whether the value of c is positive or negative.

Thus, the equation $y^2 = 8x$ represents a parabola whose vertex is at the origin and whose axis is the X-axis. Since the coefficient of x is positive, the curve lies on the right-hand side of the Y-axis only. Comparing type equation (10) with the given equation, it is evident that $4c = 8$ and $c = 2$. Hence, the coordinates of the focus are (2, 0). The parabola represented by $y^2 = 8x$ may now be plotted by choosing any arbitrary positive values for x and finding, from the given equation, the corresponding values of y. The graph of this equation is shown in Figure 183.

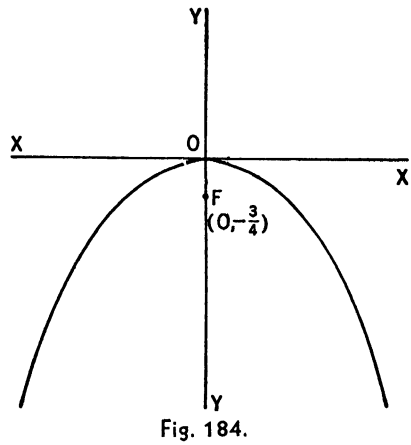


Fig. 184.

Similarly, the equation $x^2 = -3y$ represents a parabola whose vertex is at the origin and whose axis is the Y-axis. Since the coefficient of y is negative, the curve lies wholly below the X-axis. Obviously, by comparing the given equation to type equation (12), $4c = -3$ and $c = -\frac{3}{4}$. Hence, the coordinates of the focus are $(0, -\frac{3}{4})$. Substituting any arbitrary negative values for y in the given equation, the corresponding values of x are obtained. The graph may then be plotted, as shown in Figure 184.

CONSTRUCTION OF THE PARABOLA

To draw a parabola, such as $y^2 = 4x$, locate the focus and draw its directrix. Choose any point, M, Figure 185, on the axis of the parabola and within the curve. That is, if the curve, as in this case, lies to the right of the Y-axis, then choose the point to the right of the vertex. Through point M draw a line MN parallel to the directrix. Then, with a radius equal to AM and with F as a center, draw arcs intersecting MN at P and P'. Then FP and FP' are equal to AM, so that P and P' are, by definition, on the parabola. Any number of points on the curve may be located in this manner. The parabola can then be easily sketched by connecting these points by a smooth curve.

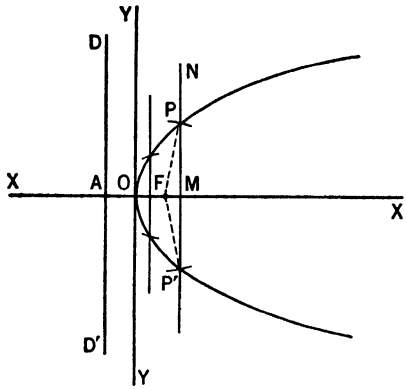


Fig. 185.

Considering, as before, only equations of the second degree in two variables, such as x and y, in which no xy term appears, the type equation of the parabola presents the following characteristics: *A parabola is represented by an equation of the second degree in two variables, such as* x *and* y, *in which only one term of the second degree in either one of the variables is present.**

TRANSLATING THE AXES

In the problems discussed in this chapter, we were at liberty to choose the position of the axes. The axes were then selected so that the resulting equation would have the simplest possible form. Evidently this policy of simplification should be followed whenever possible. In many cases, however, the equation is previously given with respect to a given set of axes. To simplify the given equation, the curve is referred to another set of axes selected so that the equation of the curve, when expressed in reference to the new set of axes, appears in a simple, standard form.

When the axes are moved so that in their new position they are respectively parallel to their original position, the axes are then said to be *translated* from the first to the second position.

* See page 403 of this chapter for the proof of this statement.

If the axes are translated, the relation between the original coordinates of any point on a curve and the new coordinates can be easily obtained. Assume point P in Figure 186 to be any point on any given curve, and denote the coordinates of this point by x and y. That is, $ON = x$ and $PN = y$. Let the new set of axes X' and Y' intersect at O', and assume that the coordinates of this new origin are (h, k). Then $OQ = h$ and $O'Q = k$. Denote the coordinates of P with respect to the new set of axes by x' and y'. That is, $O'M = x'$ and $PM = y'$. Evidently, $ON = QN + OQ$, and $PN = PM + MN$. But, by construction, the figure $O'QNM$ is a rectangle. Hence, by Fact 36, $QN = O'M$ and $MN = O'Q$. Hence, we may write $ON = O'M + OQ$ and $PN = PM + O'Q$. Substituting in these expressions their equivalents in terms of the coordinates of P and O', we obtain the following values:

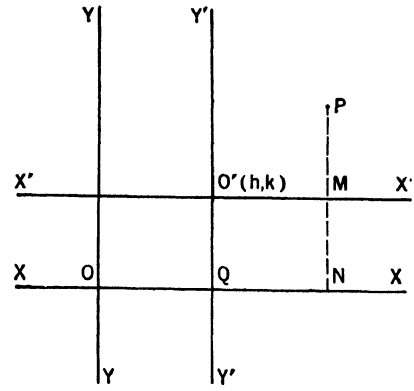


Fig. 186.

$$\left.\begin{aligned} x &= x' + h, \\ y &= y' + k. \end{aligned}\right\} \tag{13}$$

A given equation may be simplified and then its graph plotted by the following method. Suppose that it is desired to draw the curve represented by the equation $9x^2 + 4y^2 + 54x + 16y + 61 = 0$. Rewrite the equation, collecting the terms in x and y, as follows: $9x^2 + 54x + 4y^2 + 16y = -61$, or $9(x^2 + 6x) + 4(y^2 + 4y) = -61$. Completing the squares, as shown in Chapter 16, page 258, we obtain: $9(x^2 + 6x + 9) + 4(y^2 + 4y + 4) = 97 - 61$, or $9(x + 3)^2 + 4(y + 2)^2 = 36$. Evidently, if we substitute $x' = x + 3$ and $y' = y + 2$, the transformed equation is $9x'^2 + 4y'^2 = 36$. This equation is now easily recognized to be in the simple standard form (2). Hence, the curve represented by this equation is an ellipse. Since $x' = x + 3$ and $y' = y + 2$, then $x = x' - 3$ and $y = y' - 2$. Thus, referring to equations (13), $h = -3$ and $k = -2$. The new set of axes are now readily located as in Figure 187.

The transformed equation $9x'^2 + 4y'^2 = 36$ is plotted with reference to the new axes. Dividing both sides by 36, we may write: $\frac{x'^2}{4} + \frac{y'^2}{9} = 1$. Here $a = \pm\sqrt{4} = \pm 2$, and $b = \pm\sqrt{9} = \pm 3$. Since the major axis is $2b = 6$, the foci are determined by means of the relation $b^2 - c^2 = a^2$ (page 376). Substituting in this expression, we obtain: $9 - 4 = c^2$, and $c = \pm\sqrt{5} = \pm 2.24$. To draw the curve, locate these points on the plane, measuring all distances from the X'- and Y'-axes. The graph of this equation is shown in Figure 187. Therefore, the graph of $9x^2 + 4y^2 + 54x + 16y + 61 = 0$ is an ellipse whose center is at $(-3, -2)$ and whose major axis is parallel to the Y-axis.

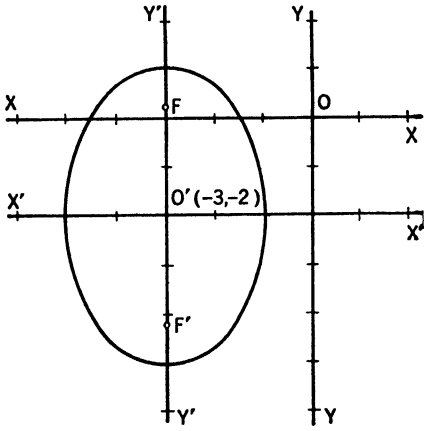


Fig. 187.

Similarly, assume that the curve represented by $x^2 - 2y - 2x + 5 = 0$ is to be drawn. Rewrite the equation, collecting the terms in x and y, as follows: $x^2 - 2x = 2y - 5$. Completing the square of the left-hand side, we may now write: $x^2 - 2x + 1 = 2y - 5 + 1$, or $(x - 1)^2 = 2(y - 2)$. Substituting $x' = x - 1$ and $y' = y - 2$, the equation becomes $x'^2 = 2y'$. Obviously, this equation is in standard form (12), and is represented by a parabola. Since $x' = x - 1$ and $y' = y - 2$, then $x = x' + 1$ and $y = y' + 2$. Hence, referring to equations (13), $h = 1$ and $k = 2$. The new set of axes are then drawn as in Figure 188. The vertex of the parabola is at the new origin. Therefore, its

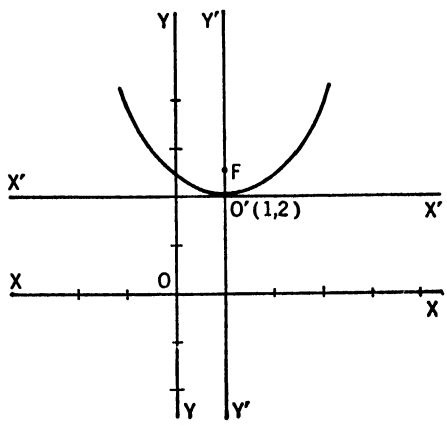


Fig. 188.

coordinates are (1, 2). Since the coefficient of y' is 2, the curve lies above the X'-axis only, and $4c = 2$, so that $c = \frac{1}{2}$. Measuring all distances from the X'- and Y'-axes, the curve is drawn as in Figure 188.

TYPICAL EQUATION OF THE ELLIPSE

The type equations discussed in this chapter can now be easily generalized. Assume that an ellipse is referred to a set of axes, X' and Y' in Figure 189, so that its center coincides with the origin (h, k). The equation of the ellipse with respect to these axes is, by standard form (3), $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$. Since, by equations (13), $x = x' + h$ and $y = y' + k$, then $x' = x - h$ and $y' = y - k$. Hence, the equation of the ellipse may be referred to the X- and Y-axes by merely substituting for x' and y' their values in terms of x, y, h, and k. That is,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is the equation of an ellipse whose center is at (h, k) and whose axes, passing through its center, are parallel to the X- and Y-axes.

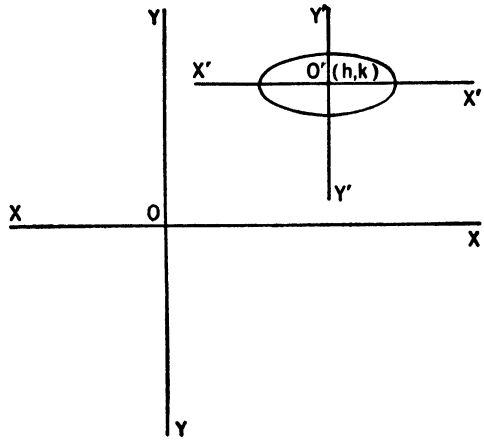


Fig. 189.

Thus, $\frac{(x-2)^2}{4} + \frac{(y+3^2)}{16} = 1$ represents an ellipse whose center is $(2, -3)$ and whose axes are parallel to the X- and Y-axes. To construct the ellipse, locate the point $(2, -3)$, as in Figure 190. Through this point draw two lines respectively parallel to the X- and Y-axes. With these lines as axes and point $(2, -3)$ as the origin, locate $a = \pm\sqrt{4} = \pm 2$ and $b = \pm\sqrt{16} = \pm 4$. The foci are obtained, as usual, from the relation $b^2 - c^2 = a^2$. Thus, $16 - 4 = c^2$, and $c = \pm\sqrt{12} = \pm 3.46$. The curve is then drawn, as shown on page 400.

TYPICAL EQUATION OF THE HYPERBOLA

The equations of the hyperbola can be easily generalized in the same manner. We then obtain:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

and

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

for the general equations of the hyperbola. The first equation represents a hyperbola with center at (h, k), transverse axis parallel to the X-axis, and conjugate axis parallel to the Y-axis. The second equation represents a hyperbola with center at (h, k), transverse axis parallel to the Y-axis, and conjugate axis parallel to the X-axis.

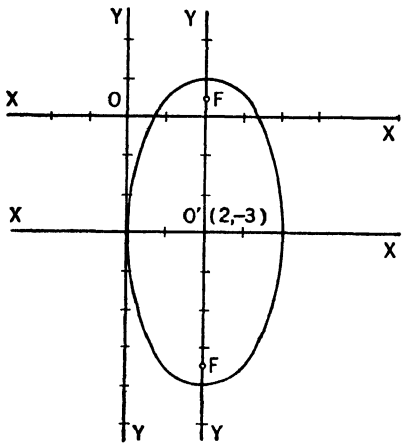


Fig. 190.

Thus, $\frac{(x-5)^2}{49} - \frac{(y+4)^2}{36} = 1$ represents a hyperbola with center at $(5, -4)$, transverse axis parallel to the X-axis, and conjugate axis parallel to the Y-axis. The curve is then plotted, as in the case of the ellipse, with $(5, -4)$ as the origin.

TYPICAL EQUATION OF THE PARABOLA

Similarly, if equations (10) and (12) of the parabola are generalized, we obtain:

$$(y-k)^2 = 4c(x-h)$$

and

$$(x-h)^2 = 4c(y-k).$$

The first equation represents a parabola with vertex at (h, k) and with its axis parallel to the X-axis. The second equation represents a parabola with vertex at (h, k) and with its axis parallel to the Y-axis.

For example, $(y-6)^2 = 16(x-3)$ represents a parabola

with vertex at (6, 3) and with its axis parallel to the X-axis. The value of c is found as before by setting $4c = 16$, so that $c = 4$. Considering (6, 3) as the origin, the focus is located on the axis of the parabola and the curve drawn as previously shown.

APPLICATIONS TO PRACTICAL PROBLEMS

Applications of equations representing the curves discussed in this chapter are legion. The following example illustrates one of the many uses of the parabola in the field of engineering. Assume that in the construction of a suspension bridge it is desired to ascertain the length of the wires running down from the suspension cable to the floor of the bridge. The towers to which the suspension cables are attached are 1,000 feet apart and 104 feet higher than the midpoint of the floor of the bridge. The lowest point on the suspension cable is 4 feet above the midpoint of the bridge. For convenience, in this illustration, assume a level floor bridge. To determine the length of the wires, we may proceed as follows.

Considering the X-axis as representing the floor of the bridge, draw the Y-axis and assume the midpoint of the bridge to be at the origin. Since the lowest point on the cable is 4 feet above the floor, the coordinates of the vertex of the parabola representing the cable are (0, 4). The towers are 1,000 feet apart; hence, they are 500 feet from the center. The tops of the towers are then easily seen to be represented by the points (500, 104) and (−500, 104). Obviously, the axis of the parabola is the Y-axis. Hence, the equation representing the curve of the suspension cable is of the standard form $(x - h)^2 = 4c(y - k)$. Since the coordinates of the vertex are (0, 4), then $h = 0$ and $k = 4$. Thus, the equation becomes $x^2 = 4c(y - 4)$. The parabola must pass through the point (500, 104). Therefore, these values of x and y must satisfy the equation. Substituting in the equation, we obtain $(500)^2 = 4c\,(104 - 4)$; that is, $250{,}000 = 4c(100)$, and $4c = 2500$. That is, the equation representing the curve of the suspension cable is $x^2 = 2500(y - 4)$. To ascertain the length of wire running down from the suspension cable to the floor of the bridge is merely to determine the value of y for any given value of x. Therefore, to determine the length of wire necessary to reach from the suspension cable to a point on the floor of the bridge

50 feet from the center, simply substitute 50 for the value of x in the equation. Thus, $(50)^2 = 2500(y - 4)$, or $2500 = 2500(y - 4)$, so that $1 = y - 4$ and $y = 5$. Hence, a wire 50 feet from the middle running down from the suspension cable to the floor of the bridge must have a length of 5 feet.

GENERAL EQUATION

The general equation of the second degree, in which the xy term does not appear, may be written:

$$Ax^2 + By^2 + Dx + Ey + F = 0. \tag{14}$$

The method of completing the square, factoring, and substituting, developed on page 397, can be easily applied to this general equation. Thus (provided that neither A nor B is zero), completing the square and factoring, we obtain:

$$A\left(x + \frac{D}{2A}\right)^2 + B\left(y + \frac{E}{2B}\right)^2 + F - \frac{D^2}{4A} - \frac{E^2}{4B} = 0.$$

If we substitute $x' = x + \frac{D}{2A}$, $y' = y + \frac{E}{2B}$, and $F_1 = F - \frac{D^2}{4A} - \frac{E^2}{4B}$, the equation reduces to:

$$Ax'^2 + By'^2 + F_1 = 0. \tag{15}$$

If A and B have like signs, equation (15) is of the standard form (2). Hence, the curve represented by (14) is, in that case, an ellipse with its axes parallel to the X- and Y-axes.

If A and B have unlike signs, equation (15) is of the standard form (5) or (7). Therefore, the curve represented by (14) is then a hyperbola with its axes parallel to the coordinate axes.

If the coefficient of x^2 in (14) is zero, that is, if $A = 0$, equation (14) reduces to:

$$By^2 + Dx + Ey + F = 0.$$

By completing the square and factoring, this equation may be written:

$$B\left(y + \frac{E}{2B}\right)^2 + Dx + F - \frac{E^2}{4B} = 0$$

or

$$B\left(y + \frac{E}{2B}\right)^2 + D\left(x + \frac{F}{D} - \frac{E^2}{4BD}\right) = 0.$$

If we substitute $y' = y + \frac{E}{2B}$ and $x' = x + \frac{F}{D} - \frac{E^2}{4BD}$, the equation reduces to:

$$By'^2 + Dx' = 0$$

or

$$y'^2 = -\frac{D}{B}x'. \tag{16}$$

Equation (16) is readily recognized to be of the standard form (10). Therefore, if $A = 0$, the curve represented by (14) is a parabola with its axis parallel to the X-axis.

Similarly, if the coefficient of y^2 in (14) is zero, namely, if $B = 0$, equation (14) reduces to:

$$Ax^2 + Dx + Ey + F = 0.$$

Completing the square, factoring, and substituting, this equation reduces to:

$$x'^2 = -\frac{E}{A}y'. \tag{17}$$

Equation (17) is recognized at once to be of the standard form (12). Therefore, if $B = 0$, the curve represented by (14) is a parabola with its axis parallel to the Y-axis.

Problems

1. In a room with an elliptical floor plan, a sound originating at one focus will, after reflection from the walls, pass through the other focus. Thus, a person standing at one of the two foci may speak in a low voice and be distinctly heard by another person standing at the other focus, although his voice may be inaudible at all points between. If a room with an elliptical cross section is 200 feet long and 30 feet wide, at what distance from the ends of the room should two persons stand so that they will distinctly hear each other?

2. A rectangular hyperbola whose asymptotes, that is, the lines to which the curve approaches nearer and nearer, are the X- and Y-axes is represented by an equation of the form $xy = \frac{a^2}{2}$. The relation between the volume of a gas and the pressure that it sustains, the temperature remaining constant, is also represented by an equation of this type. That is, if p and v, respectively, denote the pressure and the volume, then $pv = k$. This is known as Boyle's Law. For

a certain gas, the volume is 5 cubic feet for 90 pounds per square foot of pressure. Find the relation between v and p. Plot the curve for pressures from 10 pounds per square foot to 90 pounds per square foot.

3. The length of a pendulum (l ft.) varies with the time of a beat thus:

$$t^2 = 0.3084l.$$

Plot a curve showing the time of a beat for lengths up to 3 feet.

4. A stone bridge with a semi-elliptical arch is to be constructed. The span of the bridge is to be 40 feet. If the bridge must have a clearance of 12 feet at a distance of 11 feet from the ends, how high must the arch be at the middle?

5. The headlight of an automobile is a reflector, every longitudinal cross section of which is a parabola. The source of light is placed at the focus of the parabola. Find the position of the source of light if the reflector is 8 inches across the rim and 10 inches deep.

6. The orbit of a certain comet is a rectangular hyperbola with the sun at one of the foci. It is known that the comet passes at a distance of 6 million miles from the sun when crossing the transverse axis of the hyperbola. Find (*a*) the equation of the orbit of the comet; (*b*) the distance from the sun to the comet when the latter is 4 million miles directly to the left of the sun.

7. The earth's orbit is an ellipse with the sun at one of the two foci. The eccentricity of the ellipse is about $\frac{1}{60}$, and the distance from the center of the ellipse to the sun is 1.55 million miles. Find the greatest and the least distances from the earth to the sun.

8. The towers of a bridge, to which the suspension cables are attached, are 70 feet high and 800 feet apart. The lowest point on a suspension cable is 6 feet above the midpoint of the bridge. Assuming a level-floor bridge, find the length of a vertical wire running down from the suspension cable to the floor and 50 feet from the center of the bridge.

9. In order to locate a gun emplacement, two very accurate instruments for recording sound are placed at two points A and B 800 feet apart. The report of the cannon is heard 0.45 of a second earlier at A than at B. Find the equation of the curve depicting the possible positions of the gun emplacement. (*Hint:* Sound travels at about 1120 feet per second. Since the report of the cannon is heard 0.45 of a second earlier at A than at B, the gun is $1120 \times 0.45 = 504$ feet nearer A than B.)

10. To locate the actual position of the gun emplacement of Problem 9, another sound-recording instrument is placed at point C

on a prolongation of the same straight line AB and at a distance of 1800 feet from A measured in the direction from A toward B. Thus, the distance $BC = 1{,}000$ feet. The report of the cannon is heard 1.1608 seconds later at C than at A. To determine the position of the gun emplacement, find the equation of the hyperbola defined by the conditions of this problem. Plot the hyperbola obtained for Problem 9. Then, on the same graph paper, plot the hyperbola of this problem, keeping the position of A fixed and drawing a new set of axes. Note that only the branch of each hyperbola nearer to A is to be taken into consideration. The intersection of the two curves gives the position of the gun emplacement.

24 THE MARCH OF NUMBERS

The language of mathematics has expanded chiefly because of the desire of its devotees to perfect this succinct means of expressing and preserving thought, and only secondarily for the sake of stating the hidden laws of nature that observation, measurement, and experiment have revealed. Devotion to mathematics for its own sake has paved the way for applications in everyday affairs. Production of most forms of wealth rests today, directly or indirectly, upon knowledge expressed in mathematical symbols. Banks and brokerage houses every day use tables the construction of which was calculated by means of the laws of numerical series.

SERIES

A *sequence* is a succession of items arranged in a definite order. For example, the natural numbers, if arranged in order, form the sequence: 1, 2, 3, 4, ... , and so forth. The arrangement $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, $\frac{3}{10,000}$, ... , and so forth, also forms a sequence.

Each quantity in a sequence is called a *term.*

A *series* is the indicated *sum of the terms* of a sequence. Thus, from the above sequences are obtained the series $1 + 2 + 3 + 4 + \ldots$ and $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \ldots$.

A series may have a *limited* or an *unlimited* number of terms. If the series has a limited number of terms, it is called a *finite* series. If the number of terms is unlimited, the series is said to be *infinite.*

Thus, the total number of iron tubes stacked in a prism, Figure 191, if considered by rows, forms a series. There is 1 tube at the top, the next row has 2 tubes, the next 3, and each successive row one more tube than the preceding. That is, the total number of iron tubes in the prism is $1 + 2 + 3 + 4 + \ldots$. The number of terms depends on the number of rows in the prism. Thus, if there are 10 rows of iron tubing,

the total number of tubes in the prism is obtained by finding the sum of the finite series $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$.

The unending and repeating decimal $0.3333 \cdots$ signifies $0.3 + 0.03 + 0.003 + 0.0003 + \cdots$, and may be written in fraction form as $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots$. The series formed by this unending but repeating decimal is an infinite series. The problem of finding the common fraction which is equal to an infinite sum of this type is, contrary to what one might suppose offhand, rather easy. A practical application in finding the present value of an income which, barring accidents, will go on for generations (such as ground rents) is discussed in Chapter 31.

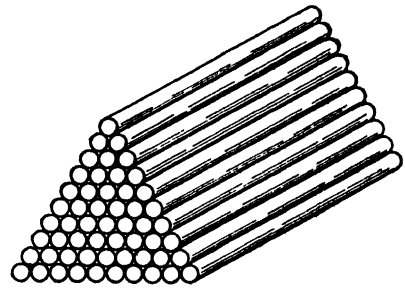


Fig. 191.

It is customary, as seen above, to indicate the existence of further terms by a set of three dots. When the series is finite, the dots are usually closed by an expression representing a last term. For example, the series denoting the total number of tubes stacked in a prism of 10 rows may be written $1 + 2 + 3 + 4 + \cdots + 10$. If the dots are not closed, the series is infinite. Thus, $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$ denotes an infinite series.

Whenever possible, it is very convenient to indicate by means of a formula the manner in which each term of the series is formed. This expression representing the law of formation of the terms is called the *general* or nth term.

In the formula denoting the general term, the letter n is used to represent the number of the term. Thus, the total number of tubes stacked in a prism is given by the series $1 + 2 + 3 + 4 + \cdots$ and so forth, the number of terms in the series depending on the number of rows in the prism. Note that the value of the term in this series is the same as the number of the term itself. That is, the value of the term for $n = 1$ (namely, for the first term) is 1. The value of the term for $n = 2$ (that is, for the second term) is 2. Hence, in general, the value of the nth term is n. That is, the series may be written $1 + 2 + 3 + 4 + \cdots + n + \cdots$.

Similarly, the nth term of the series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \cdots$ may be found as follows. Note that the numerator of the fraction in each term is the same number, that is, 3, but that the denominators are 10, 10^2, 10^3, 10^4, and so forth. It is now easily seen that the exponent of 10 in the denominator corresponds to the number of the term. Thus, the general term is $\frac{3}{10^n}$. Therefore, the series may be written

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \cdots + \frac{3}{10^n} + \cdots .$$

ARITHMETIC PROGRESSION

A series such that each term is derived from the preceding one by adding to it the same fixed amount is called an *arithmetic progression.* The constant amount added may be either positive or negative, integral or fractional. This fixed amount, which may be obtained by subtracting any term from the following term, is called the *common difference.*

Thus, the series $1 + 2 + 3 + 4 + \cdots + n + \cdots$ is an arithmetic progression obtained by adding the common difference 1 to the preceding term. The series $3 + 2 + 1 + 0 - 1 - 2 - 3 - \cdots$ is obtained by subtracting the common difference 1 from the preceding term.

From this definition, it is evident that all that is needed to form an arithmetic progression is a first term and a common difference. Hence, if a denotes the first term and d represents the common difference, the second term, formed by adding the common difference to the first term, is $a + d$. The third term is $(a + d) + d$, or simply $a + 2d$, and the following terms in order are $a + 3d$, $a + 4d$, and so forth. That is, the general form of an arithmetic progression is $a + (a + d) + (a + 2d) + (a + 3d) + \cdots$.

Note that the coefficient of d in any given term is one less than the number of that term. That is, in the second term, namely, for $n = 2$, the coefficient of d is 1. In the third term, that is, for $n = 3$, the coefficient of d is 2. Thus, the coefficient of d for any term is $(n - 1)$. Therefore, the general or nth term of an arithmetic progression may be written $a + (n - 1)d$. Hence, if l denotes the nth term,

$$l = a + (n - 1)d.$$

We can now find any term of an arithmetic progression without going through the long and tedious process of writing all the preceding terms. For example: \$200 if invested at the simple interest rate of 5 per cent for 12 years increases by the constant amount of $200 \times .05 = \$10$ each year. That is, the amounts at the end of each year form the arithmetic progression $210 + 220 + 230 + \cdots$. Evidently the amount at the end of any given year is the value of the corresponding term for that year. Thus, at the end of the third year, the sum amounted to \$230, the value for $n = 3$ in the series. Hence, to find the amount at the end of the twelfth year, we merely have to find the value of the twelfth term. In this series, $a = 210$, $n = 12$, and $d = 10$. Hence, substituting in the formula, we obtain

$$\begin{aligned} l &= a + (n - 1)d \\ &= 210 + 11 \times 10 \\ &= 210 + 110 \\ &= \$320. \end{aligned}$$

That is, \$200 invested at the simple interest rate of 5 per cent for 12 years amounts to \$320.

SUM OF AN ARITHMETIC PROGRESSION

In contrast to the arithmetic progression, just discussed, formed by a sum of money at simple interest at a given rate per cent, consider the arithmetic progression developed in ascertaining the number of iron tubes stacked in a prism. The first problem is solved by determining the value of the term corresponding to the number of years for which the sum has been invested. The second problem, however, requires for its solution the sum of the terms corresponding to the number of rows in the prism.

Since the nth term of an arithmetic progression is represented by l, the term before the nth may be denoted by $(l - d)$, the one before that by $(l - 2d)$, and so forth. Hence, the sum of the first n terms is

$$S_n = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l. \quad (1)$$

To obtain the value of the sum of a long arithmetic progression by means of this expression entails a considerable amount

of work. The value of S_n may be greatly simplified, however, as follows.

The same sum written in the reverse order is

$$S_n = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a. \quad (2)$$

Expressions (1) and (2) may be added without affecting the equality, for if equals are added to equals the sums are equal. Note that in the addition of (1) and (2), d in the second term of (1) cancels with $-d$ in the second term of (2), $2d$ in the third term of (1) cancels with $-2d$ in the third term of (2), and so forth. Thus, we obtain:

$$2S_n = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l) + (a + l).$$

By assumption, the number of terms in this equation is denoted by n; hence, the expression $(a + l)$ occurs n times. Therefore, we may write

$$2S_n = n(a + l), \text{ or}$$
$$S_n = \frac{n}{2}(a + l).$$

The total number of tubes stacked in a prism of 10 rows is now easily obtained. The series $1 + 2 + 3 + 4 + \cdots + n$ represents the number of tubes in a prism. In the particular case in question, the number of rows is 10. Since the number of rows corresponds to the number of terms in the series, $n = 10$. Moreover, $a = 1$ and $d = 1$. Hence, $l = a + (n - 1)d = 1 + 9 \times 1 = 10$ and $S_n = \frac{n}{2}(a + l) = \frac{10}{2}(1 + 10) = 5 \times 11 = 55$ tubes.

ARITHMETIC MEANS

The terms between any two given terms of an arithmetic progression are called *arithmetic means* between the given terms. If only one arithmetic mean is to be found between two given numbers a and b, this arithmetic mean, namely, M, is a third number such that a, M, and b form an arithmetic progression. Thus, 7 is the arithmetic mean of 5 and 9, for $5 + 7 + 9$ form an arithmetic progression. It is easily seen that if a, M, and b form an arithmetic progression, then $a - M$ is the common difference, and so also is $M - b$. Hence, $b - M = M - a$.

Thus, $2M = a + b$ and $M = \frac{1}{2}(a + b)$. That is, *the arithmetic mean between two numbers is one-half their sum.*

For example, if \$200 is invested at the simple interest rate of 5 per cent, the arithmetic progression $210 + 220 + 230 + 240 + \cdots + [210 + (n - 1)(10)]$ represents the amounts at the end of each year.

To ascertain the amount at the end of $3\frac{1}{2}$ years, let $a = 230$ and $b = 240$.

Then $$M = \frac{230 + 240}{2} = \$235.$$

Thus, \$200 invested at the simple interest rate of 5 per cent for $3\frac{1}{2}$ years amounts to \$235.

All the terms between the first and last terms of an arithmetic progression may be considered arithmetic means of the first and last terms. Hence, *any number of arithmetic means may be inserted between two numbers.*

Thus, the total sum to which \$200 invested at 5 per cent simple interest amounts at the end of 3 years and 2 months may be found as follows: The amounts at the end of 3 and 4 years are, as already stated, \$230 and \$240, respectively. Since there are 12 months in a year and \$240 denotes the amount at the end of the fourth year, this sum, namely, \$240, may also be considered as representing the amount at the end of the twelfth month following the third year. Hence, eleven arithmetic means must be inserted between the two given numbers 230 and 240. To determine any one of these arithmetic means, it is only necessary to find the common difference, from the formula $l = a + (n - 1)d$. Since eleven arithmetic means are to be inserted, the number of terms, n, is 13, because the terminal numbers 230 and 240 must be taken into consideration.

In addition, the values $l = 240$ and $a = 230$ are known. Hence, substituting in $l = a + (n - 1)d$, we obtain $240 = 230 + 12d$. Thus, $\frac{240 - 230}{12} = d$, and $d = \frac{10}{12} = \frac{5}{6}$. The amount at the end of the second month may now be obtained by applying again the formula $l = a + (n - 1)d$. This time, however, $a = 230$, $d = \frac{5}{6}$, $n = 3$, and l is the term to be found. Hence, $l = 230 + 2 \times \frac{5}{6} = \$231.67$.

COMPOUND INTEREST—ANNUITIES

When unpaid interest is added to the principal of a debt at definite intervals, and interest is computed on this new augmented principal, the interest is said to be *converted* or *compounded.* Interest charged in this way is called *compound interest.**

The compound amount of any sum may be obtained by tables specially prepared for this purpose or by logarithms. The use of logarithms in obtaining the compound amount of a sum is discussed in Chapter 25.

An *annuity* is a series of equal payments made or due at equal intervals of time. Thus, rent on land or building, regular preferred stock dividends, bond coupons, and income from, or investments for, trust funds are annuities.

Assume that \$300 is invested at the end of each year for 5 years in a trust fund which earns 4 per cent. If this annuity is placed at compound interest, its sum at the end of 5 years may be found.

At the end of the first year, the first investment is made. This amount begins then to earn interest for the four remaining years. The amount to which this first investment accumulates during the next four years is then found as follows. The interest earned by it during the second year is $300 \times .04$. If this amount is added to the principal at the end of the first year, the resulting sum, that is, $300 + (300 \times .04)$, is the amount to which the \$300 deposited at the end of the first year accumulates at the end of the second year. But since the expression $300 + (300 \times .04)$ contains 300 as a common factor, it may be written $300\ (1 + .04) = 300 \times 1.04 = \$312$.

The interest earned during the third year is, then, $312 \times .04$. Adding this amount to what was the principal at the end of the second year, the resulting sum, that is, $312 + (312 \times .04)$, is the amount to which the \$300 deposited at the end of the first year accumulates at the end of the third year. The expression $312 + (312 \times .04)$ contains 312 as a common factor. Hence, it may be written $312\ (1 + .04) = 312\ (1.04)$. Since 312 is equal to $300(1.04)$, the expression $312(1.04)$ is equal to

* For a more detailed discussion of compound interest and annuities, see Moore and Mira, *Practical Business Mathematics*, New York, Longmans, Green and Co., 1937.

$300(1.04)(1.04)$, or $300(1.04)^2$. That is, the \$300 deposited at the end of the first year accumulates to $300(1.04)^2$ at the end of the third year.

Clearly, the same process must be repeated in obtaining the amounts to which the \$300 invested at the end of the first year accumulates at the end of each successive year. Thus, at the end of the fourth year, the \$300 accumulates to $300(1.04)^3$, and at the end of the fifth year the amount is $300(1.04)^4$.

The expression $300(1.04)^4 = 300(1 + .04)^4$ represents the compound amount of \$300 invested at 4 per cent for a period of 4 years.

Hence, in calculating compound interest, there are three quantities to consider, which have the following symbols: P represents the principal, that is, the amount invested; n denotes the number of years; and i is the interest rate per year.

The symbol P is easily identified as the quantity \$300. The expression $(1 + i)$ evidently takes the place of $(1 + .04)$. The letter n obviously corresponds to the exponent 4. Hence, if the letter S is used to represent the compound amount and the given quantities are replaced by the corresponding letters,*

$$S = P(1 + i)^n.$$

To obtain the total amount of the annuity, the amounts to which the \$300 deposited at the end of the second, third, fourth, and fifth years accumulates at the end of the 5 years must be computed also. The whole transaction may be visualized as follows:

	Annual Payment	*No. of Years Accumulated*	*Amounts*
Invested at end of 1st year	\$300	4	$300\ (1.04)^4$
" " " " 2nd "	\$300	3	$300\ (1.04)^3$
" " " " 3rd "	\$300	2	$300\ (1.04)^2$
" " " " 4th "	\$300	1	$300\ (1.04)$
" " " " 5th "	\$300	0	300

The total amount of this annuity is evidently the sum:

$$S = 300 + 300(1.04) + 300(1.04)^2 + 300(1.04)^3 + 300(1.04)^4.$$

* The reader may easily develop this formula by following exactly the same line of reasoning used in finding the compound amount of \$300 invested at 4 per cent for a period of 4 years. It is only necessary to use the literal expressions P, n, and i instead of the given numbers.

GEOMETRIC PROGRESSION

Evidently, if a general formula can be found for the sum of such a series, the work involved in the solution of the problem is greatly simplified.

It is easily seen that each of the terms after the first in the above series may be formed by multiplying the preceding term by 1.04. A series such that each term is derived by multiplying the preceding one by a common fixed number is called a *geometric progression.* The constant number may be either positive or negative, integral or fractional. This fixed number, which may be obtained by dividing any term by the preceding term, is called the *common ratio.* Thus, the series $300 + 300(1.04) + 300(1.04)^2 + 300(1.04)^3 + 300(1.04)^4$ is a geometric progression obtained by multiplying the preceding term by the common ratio 1.04.

To form a geometric progression, we need a term and a ratio. If a denotes a term and r the common ratio, the following term, formed by multiplying the preceding term by r, is ar. The third term is $ar \times r$, or simply ar^2, and the following terms in order are ar^3, ar^4, and so forth. That is, the general form of a geometric progression is $a + ar + ar^2 + ar^3 + \cdots$.

Note that each term contains a as a factor and contains also as a factor a power of r, with an exponent one less than the number of the term. That is, in the second term, namely, for $n = 2$, the exponent of r is 1. In the third term, that is, for $n = 3$, the exponent of r is 2. Thus, the exponent of r for any term is $(n - 1)$. Hence, the general or nth term is ar^{n-1}. We are, as before, considering a finite series. Therefore, if l denotes the nth term,

$$l = ar^{n-1}.$$

The sum of the first n terms of a geometric progression can then be written:

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}. \tag{3}$$

Evidently it will be much more convenient if a more compact form can be found for this sum. Multiplying both sides of (3) by r gives:

$$S_n r = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n. \tag{4}$$

Since if equals are subtracted from equals the remainders are equal, equation (3) may be subtracted from (4). Note that in this subtraction the second term of (3) cancels the first term of (4), the third term of (3) cancels the second term of (4), and so forth. This gives $S_n r - S_n = ar^n - a$. Factoring both sides gives:

$$S_n(r - 1) = a(r^n - 1).$$

Dividing both sides by $(r - 1)$ gives:

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

As already explained, division by 0 is not allowed; therefore, the preceding equation is meaningless if $r = 1$.

The total amount of the annuity of \$300 payable for 5 years at 4 per cent may now be easily found. Since the rate of interest is 4 per cent, the common ratio is, as shown in the preceding discussion, $1 + .04$, that is, $r = 1.04$. The number of years is 5, so that $n = 5$. Finally, $a = 300$. Therefore,

$$S = \frac{300[(1.04)^5 - 1]}{1.04 - 1} = \frac{300(1.21665 - 1)}{.04} = \frac{300(.21665)}{.04} = \$1624.87.$$

PRESENT VALUE OF THE AMOUNT

Principal increases by an accretion of interest until it terminates in the amount. The amount, decreased by the interest, can be conceived of as moving backward in time until it becomes the principal. The principal is merely another word for the *present value* of the amount.

If in the formula for the compound amount, namely, $S = P(1 + i)^n$, each side be divided by $(1 + i)^n$, we have: $\frac{S}{(1 + i)^n} = P$, or reversing, $P = \frac{S}{(1 + i)^n}$. This formula may be written in the form $P = S \times \frac{1}{(1 + i)^n}$.

Now suppose that by a contract an insurance company is obligated to pay \$300 at the end of each year for 5 years. The payments are to begin on January 1, 1943. The beneficiary would like to settle his claim for a lump sum to be paid on January 1, 1942, agreeing with the insurance company upon

an interest rate of 4 per cent. This problem consists of finding the present value of five sums, each payable at a different time in the future. By using the formula just developed for the present value of a compound amount, the whole transaction can be scheduled as follows:

Date Payable	*Present Value* $P = S \times \frac{1}{(1+i)^n}$ *on January 1, 1942*
1st payment \$300, Jan. 1, 1943	$300 \times \frac{1}{(1+.04)}$
2nd payment \$300, Jan. 1, 1944	$300 \times \frac{1}{(1+.04)^2}$
3rd payment \$300, Jan. 1, 1945	$300 \times \frac{1}{(1+.04)^3}$
4th payment \$300, Jan. 1, 1946	$300 \times \frac{1}{(1+.04)^4}$
5th payment \$300, Jan. 1, 1947	$300 \times \frac{1}{(1+.04)^5}$

The present value of this annuity is obviously the sum:

$$A = 300 \times \frac{1}{(1+.04)} + 300 \times \frac{1}{(1+.04)^2} + 300 \times \frac{1}{(1+.04)^3} + 300 \times \frac{1}{(1+.04)^4} + 300 \times \frac{1}{(1+.04)^5}.$$

This series is easily seen to be a geometric progression whose first term is $300 \times \frac{1}{(1+.04)} = \frac{300}{1.04}$ and whose common ratio is $\frac{1}{(1+.04)}$.

Since in this case the common ratio is a fraction less than 1, it is more convenient to express the formula $S_n = \frac{a(r^n-1)}{r-1}$, denoting the sum of a geometric progression, in a different form. Rewrite the formula in the form $S_n = \frac{ar^n - a}{r-1}$. If both numerator and denominator are multiplied by -1, the value of the fraction is not changed. Hence,

$$S_n = \frac{ar^n - a}{r - 1} = \frac{-ar^n + a}{-r + 1} = \frac{a - ar^n}{1 - r}.$$

Factoring a from both terms in the numerator gives:

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Applying this formula to obtain the present value of the annuity under discussion, it is evident that a is $\frac{300}{1.04}$, $r = \frac{1}{1 + .04}$, and n is 5. Thus, denoting the sum by A, we obtain:

$$A = \frac{\frac{300}{1.04}\left[1 - \frac{1}{(1+.04)^5}\right]}{1 - \frac{1}{1 + .04}} = \frac{\frac{300}{1.04}\left[\frac{(1.04)^5 - 1}{(1.04)^5}\right]}{\frac{1 + .04 - 1}{1 + .04}}$$

$$A = \frac{\frac{300}{1.04}\left[\frac{(1.04)^5 - 1}{(1.04)^5}\right]}{\frac{.04}{1.04}} = \frac{300}{1.04} \times \frac{1.04}{.04} \times \frac{(1.04)^5 - 1}{(1.04)^5}$$

$$A = \frac{300[(1.04)^5 - 1]}{(.04)(1.04)^5} = \frac{300(1.21665 - 1)}{(.04)(1.21665)} = \frac{300 \times .21665}{.048666}$$

$$A = \$1{,}335.54.$$

GEOMETRIC MEANS

The terms between any two given terms of a geometric progression are called *geometric means* between the given terms. If only one geometric mean is to be found of any two numbers in a geometric progression, a formula is easily obtained as follows: Assume that $a + c + b$ forms a geometric progression. Then c is the geometric mean of a and b. The common ratio of this progression is obtained by dividing any term by the preceding term. Hence, $\frac{c}{a}$ is the common ratio, and so also is $\frac{b}{c}$. Therefore, $\frac{c}{a} = \frac{b}{c}$. Note that if a, b, and c are positive, c is the mean proportional of a and b. Multiplying both sides by $a \times c$, we obtain $c^2 = ab$. Hence, $c = \pm\sqrt{a \cdot b}$.

In other words, the geometric mean of two numbers is a square root of their product.

Thus, the geometric mean of 3 and 27 is equal to $\pm\sqrt{3 \times 27} = \pm\sqrt{81} = \pm 9$.

Any number of geometric means may be inserted between two given positive numbers. Each term between the first and last terms of a geometric progression may be considered a geometric mean.

Thus, if three geometric means are to be inserted between 5 and 80, we have, considering the two given numbers as the first and last terms, $a = 5$, $l = 80$, and $n = 5$. To form the geometric progression, it is necessary to find r. The formula $l = ar^{n-1}$, for the nth term, may be used. Hence, substituting, $80 = 5r^{5-1}$. Dividing both sides by 5 gives $16 = r^4$. Hence, $\sqrt[4]{16} = r$, and therefore $r = \pm 2$. If $r = 2$, the geometric progression is $5 + 10 + 20 + 40 + 80$. If $r = -2$, the geometric progression is $5 - 10 + 20 - 40 + 80$. Thus, the three geometric means are ± 10, 20, and ± 40.

THE BINOMIAL THEOREM

It seems a far cry from the expansion of a binomial, such as was discussed in Chapters 3 and 16, to the practical, matter-of-fact computation of the compound amount of a sum of money invested at a given rate. But an inspection of the compound interest formula, $S = P(1 + i)^n$, set forth on page 413, which can be restated, using r instead of i, in the form $S = P(1 + r)^n$, immediately shows that the expression $(1 + r)^n$ becomes $(1 + r)^2$ when $n = 2$, and this expression is merely a particular case of $(a + x)^2 = a^2 + 2ax + x^2$. Obviously, in the solution of compound interest problems, a general formula for the expansion of $(a + x)^n$ will, of course, be found very useful.

When $(a + x)$ is raised to successive powers by multiplication, the following results are obtained:

$$(a + x)^2 = a^2 + 2ax + x^2,$$
$$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3,$$
$$(a + x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4,$$
$$(a + x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

The above expressions, if carefully compared, yield the following results:

(1) The number of terms in each expression is one greater than the power to which $(a + x)$ is raised. Thus, for any positive integral value of the exponent, the number of terms is finite.

(2) The coefficient of the first term of the expansion is unity, and the exponent of a is equal to the exponent of the binomial.

(3) The coefficient of the second term is equal to the exponent of the binomial, the exponent of a is one less than in the preceding term, and x appears in this term with an exponent equal to unity.

(4) In each succeeding term, the exponent of a decreases by one while the exponent of x increases by one until in the last term a does not appear as a factor and the exponent of x is equal to that of the binomial.

(5) The coefficient of any given term after the first can be found by multiplying the coefficient of the preceding term by the exponent of a in that term divided by the number of terms preceding the given term. Thus, in the expansion of $(a + x)^5$, the first two terms are $a^5 + 5a^4x$. The coefficient of the third term is equal to $5 \times \frac{4}{2} = 10$, where 5 is the coefficient of the second term, 4 is the exponent of a in the second term, and 2 is the number of terms preceding the third term.

From these observations, a general formula may be inferred. If n represents the exponent of the binomial, we then have $(a + x)^n$. The first term of the expression is evidently a^n. The coefficient of the second term is equal to the exponent of the binomial, that is, n. Since the exponent of a is, in this term, one less than in the preceding, it is then $n - 1$. Hence, the second term is $na^{n-1}x$. To obtain the coefficient of the third term, multiply the coefficient of a in the second term, that is, n, by the exponent of a in that term, namely, $(n - 1)$, divided by the number of terms preceding the third term. That is, the coefficient of the third term is $n \cdot \frac{n-1}{2}$. Since the exponent of a ***decreases*** by one while that of x ***increases*** by one, the exponent of a in the third term is $(n - 1) - 1 = n - 2$, and that of x is 2. Hence, the third term is $\frac{n(n-1)}{2} a^{n-2}x^2$. Similarly, the coefficient of the fourth term is

$$\frac{n(n-1)}{2} \cdot \frac{n-2}{3} = \frac{n(n-1)(n-2)}{2 \cdot 3}.$$

Thus, the fourth term is $\frac{n(n-1)(n-2)}{2 \cdot 3} a^{n-3}x^3$. The following terms are then obtained in a similar manner. In general, then,*

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{2 \cdot 3}a^{n-3}x^3 + \cdots + x^n. \quad \text{(I)}$$

For example: Expand $(x+y)^6$. Here x takes the place of a and y takes the place of x in formula (I). Substituting, we obtain:

$$(x+y)^6 = x^6 + 6x^5y + 6 \cdot \frac{5}{2}x^4y^2 + \frac{6 \cdot 5}{2} \cdot \frac{4}{3}x^3y^3 + \frac{6 \cdot 5 \cdot 4}{2 \cdot 3} \cdot \frac{3}{4}x^2y^4 + \frac{6 \cdot 5 \cdot 4 \cdot 3}{2 \cdot 3 \cdot 4} \cdot \frac{2}{5}xy^5 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{1}{6}y^6.$$

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

Similarly, to expand $(a-b)^5$, first write the binomial $[a + (-b)]^5$ and then apply formula (I). Thus,

$$(a-b)^5 = [a+(-b)]^5 = a^5 + 5a^4(-b) + 5 \cdot \frac{4}{2}a^3(-b)^2 + \frac{5 \cdot 4}{2} \cdot \frac{3}{3}a^2(-b)^3 + \frac{5 \cdot 4 \cdot 3}{2 \cdot 3} \cdot \frac{2}{4}a(-b)^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4} \cdot \frac{1}{5}(-b)^5.$$

$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

Note that if the binomial is the *sum* of two terms, such as $(x+y)^6$, the signs of the terms are all *positive*. However, if the binomial is the *difference* of two terms, such as $(a-b)^5$, the signs are *alternately positive and negative.*

If either term or both terms of the binomial have a coefficient different from 1, the expansion is carried out in the same manner. Thus, to expand $(2x-3y)^4$, first write the binomial $[2x + (-3y)]^4$. Here a is $2x$ and x is $-3y$. Substituting in (I), we obtain:

* The proof of this formula, although not difficult, is rather long. The reader interested in this proof is referred to Hawkes' *Advanced Algebra*, Ginn and Company, Boston, 1928.

$$(2x - 3y)^4 = (2x)^4 + 4(2x)^3(-3y) + 4 \cdot \frac{3}{2}(2x)^2(-3y)^2 +$$

$$\frac{4 \cdot 3}{2} \cdot \frac{2}{3} (2x)(-3y)^3 + \frac{4 \cdot 3 \cdot 2}{2 \cdot 3} \cdot \frac{1}{4}(-3y)^4,$$

$$(2x - 3y)^4 = 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$$

PASCAL'S TRIANGLE

In expressions of the type $(a + x)^n$, the coefficients of the expansion for any positive integral value of n may be obtained by means of a triangular arrangement of numbers called *Pascal's triangle*. The coefficients of the expansion of $(a + x)^n$ for $n = 0$, 1, 2, and 3 are:

Binomial	*Coefficients*			
$(a + x)^0 = 1$	1			
$(a + x) = a + x$	1	1		
$(a + x)^2 = a^2 + 2ax + x^2$	1	2	1	
$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$	1	3	3	1

Note that the coefficient of any term in these expansions may be obtained by adding the number directly above its position to the number at the left of the latter. Thus, the coefficient of the third term in $(a + x)^3$ is $1 + 2 = 3$, where 1 is the number directly above 3 and 2 is the number at the left of 1.

In general, this relationship holds true for any coefficients in the expansion of $(a + x)^n$. Pascal's triangle may then be developed indefinitely for as many values of n as desired. The following arrangement has been carried to $(x + y)^{10}$.

Binomial	*Coefficients*										
$(x + y)^0$	1										
$(x + y)^1$	1	1									
$(x + y)^2$	1	2	1								
$(x + y)^3$	1	3	3	1							
$(x + y)^4$	1	4	6	4	1						
$(x + y)^5$	1	5	10	10	5	1					
$(x + y)^6$	1	6	15	20	15	6	1				
$(x + y)^7$	1	7	21	35	35	21	7	1			
$(x + y)^8$	1	8	28	56	70	56	28	8	1		
$(x + y)^9$	1	9	36	84	126	126	84	36	9	1	
$(x + y)^{10}$	1	10	45	120	210	252	210	120	45	10	1

The numbers in the second column represent the coefficients of the second term in any one expansion. Therefore, these numbers correspond to the desired value of n.

PRACTICAL APPLICATIONS

Now assume that, as in the case of the annuity problem on page 417, it is desired to ascertain the value of $(1 + .04)^5$. Applying the binomial formula (I) and using Pascal's triangle, we obtain:

$$(1 + .04)^5 = (1)^5 + 5(1)^4(.04) + 10(1)^3(.04)^2 + 10(1)^2(.04)^3 + 5(1)\,(.04)^4 + (.04)^5$$

$$(1 + .04)^5 = 1 + .20 + .016 + .00064 + .0000128 + .0000001024$$

$$(1 + .04)^5 = 1.2166529024.$$

In expressions of the type $(1 + i)^n$, the application of the binomial formula (I) yields:

$$(1 + i)^n = (1)^n + n(1)^{n-1}i + n \cdot \frac{n-1}{2}(1)^{n-2}i^2 + \frac{n(n-1)}{2} \cdot \frac{n-2}{3}(1)^{n-3}i^3 + \cdots + i^n.$$

Since the number 1 raised to any power is equal to unity, this expression reduces to:

$$(1 + i)^n = 1 + ni + n \cdot \frac{n-1}{2}i^2 + \frac{n(n-1)}{2} \cdot \frac{n-2}{3}i^3 + \cdots + i^n,$$

which may be written:

$$(1 + i)^n = 1 + ni + ni \times \frac{n-1}{2}i + \frac{n(n-1)}{2}i^2 \times \frac{n-2}{3}i + \cdots + i^n.$$

The following characteristics are easily observed:

(1) The first term is always equal to unity.

(2) The second term is equal to the product of the exponent of the binomial and the rate i.

(3) Any term after the second may be obtained as follows: Reduce the exponent by 1 and divide by a number one less than the number of the term. Multiply the quotient by i.

The value of the term is then the product of this result and the preceding term.

Thus, the value of the third term in the expansion of $(1 + .04)^5$ was found to be .016. This value could be obtained by (3) above. Reducing the exponent by 1, we obtain 4. Since this is the third term, the number one less than 3 is 2. The quotient is, then, $\frac{4}{2} = 2$. Multiplying this quotient by the rate, we obtain $2 \times .04 = .08$. The value of the third term is then the product of this result, namely, .08, by the preceding term, that is, .2. Hence, the third term is equal to $.2 \times .08 = .016$.

The whole operation may be visualized as follows:

$$
\begin{aligned}
(1 + .04)^5 & \\
1 &= 1 \\
5 \times .04 &= 0.2 \\
.2 \times \tfrac{4}{2} \times .04 &= 0.016 \\
.016 \times \tfrac{3}{3} \times .04 &= 0.00064 \\
.00064 \times \tfrac{2}{4} \times .04 &= 0.0000128 \\
.0000128 \times \tfrac{1}{5} \times .04 &= \underline{0.0000001024} \\
& \quad 1.2166529024
\end{aligned}
$$

If accuracy to only two decimal places had been desired, every term after the third could have been dropped, for obviously the fourth term does not affect the value of the second decimal.

Similarly, to determine to what \$500 would amount if invested for 5 years at 2 per cent compounded semiannually, we proceed as follows: A yearly rate of i per cent compounded m times a year produces the same result as does a rate of $\frac{i}{m}$ per cent compounded annually for m years. Thus, a yearly rate of 2 per cent compounded semiannually, for one year, produces the same result as does a rate of 1 per cent compounded annually for 2 years. Therefore, the given problem may be considered as the equivalent of finding what \$500 would amount to if invested for 10 years at 1 per cent compounded annually.

The formula for the compound amount of any sum is $S = P(1 + i)^n$. Here $P = \$500$, $i = .01$, and $n = 10$. Hence, $S = 500(1 + .01)^{10}$. Applying the method previously developed, it is easily seen that:

$$
\begin{aligned}
S &= 500\,(1 + .01)^{10} \\
500 \times 1 &= 500 \\
500 \times 10 \times .01 &= 50 \\
50 \times \tfrac{9}{2} \times .01 &= 2.25 \\
2.25 \times \tfrac{8}{3} \times .01 &= 0.06 \\
.06 \times \tfrac{7}{4} \times .01 &= 0.00105 \\
S &= \$552.31105
\end{aligned}
$$

Since the later terms have no effect on the value of the second decimal, it is not necessary to carry out the calculation further. Therefore, \$500 if invested for 5 years at 2 per cent compounded semiannually amounts to \$552.31.

Again, as the reader may well imagine, the arithmetic of the above calculation can be tremendously simplified by the use of special tables. Treatises on financial mathematics commonly give eight or more tables based upon the binomial theorem, at different rates of interest and for different periods. The same tables can be used in situations where interest is compounded more than once a year.

Problems

1. The rungs of a ladder diminish uniformly from $2'4''$ long at the base to $1'3''$ long at the top. If there are 24 rungs, what is the total length of wood in them?

2. A bomb dropped from an airplane, if air resistance is disregarded, falls 16.1 feet in the first second, 48.3 feet the next, and 80.5 feet the third. (*a*) How far will it fall in 20 seconds? (*b*) How long will it take to fall 550 feet?

3. A contractor has stacked iron pipes in his yard in rows one upon the other, each successive row containing one tube less than the row immediately beneath. If the top row has seven pipes, and there are 1254 pipes in all, how many rows are there? How many pipes are there in the bottom row?

4. Insert between 1 and 36 such a number of arithmetic means that the sum of the resulting arithmetic progression shall be 148.

5. A charity lottery sells tickets numbered consecutively 1, 2, 3, 4, and so forth. If the price in cents of a lottery ticket corresponds to the number of that particular ticket, how many tickets must be sold to realize \$5,005?

6. A cabinet maker, in planning the heights of drawers in a dresser, seeks to insert two geometric means between 8 and 27. What are they?

7. In a trust fund for a hospital \$100,000 is invested at $3\frac{1}{2}$ per cent compounded annually. To what sum will this money amount in 7 years?

8. Mr. G. Trent's son is 5 years old. The father wishes to have \$5,000 available for the lad's education at the time he is 18 years old. How much money must be deposited now if 4 per cent compounded annually is paid by the bank?

9. A man pays \$59.25 per annum for 25 years on his life insurance policy. If the rate is 3 per cent compounded annually, to what sum will this money amount?

10. To provide for laboratory equipment, \$5,000 is left to a university with directions that the sum remain intact for a period of 10 years. If the sum is invested at 4 per cent compounded semi-annually, how much will be available at the end of 10 years?

25

LOGARITHMS

Probably the most important mathematical labor-saving device, since the decimal notation, was discovered by John Napier in 1614 A. D. Impressed by the stupendous quantity of calculations which Kepler and others had to perform in their astronomical work, Napier invented the logarithmic method of computation. Logarithms have enormously lessened the labor of people engaged in numerical computation. The slide rule, based on logarithms, is almost indispensable to anyone whose occupation requires rapid approximate calculation. Very often in business or in scientific work, in considering the relationship of numerical data, *rate* of change is more important than *amount* of change. In such cases the use of logarithms brings out clearly the relative fluctuations of the series compared. In recent times, modern computing machines have brought additional economies in mental labor. If the user of calculating machines understands logarithms, a still further saving of effort is made possible, since complicated multiplication can be reduced to simple addition, division to subtraction, raising to a power to multiplication, and the extraction of a root to division.

$3^0 = 1$
$3^1 = 3$
$3^2 = 9$
$3^3 = 27$
$3^4 = 81$
$3^5 = 243$
$3^6 = 729$
$3^7 = 2{,}187$
$3^8 = 6{,}561$
$3^9 = 19{,}683$
$3^{10} = 59{,}049$
$3^{11} = 177{,}147$
$3^{12} = 531{,}441$
$3^{13} = 1{,}594{,}323$

NATURE OF LOGARITHMS

A logarithm is an exponent. To use logarithms, therefore, a knowledge of the laws of exponents, already discussed in Chapter 3, is indispensable. Consider the small adjoining table, which gives a few numbers expressed as powers of 3. Suppose that we had the task of dividing 531,441 by 19,683. This operation could be written as follows:

$$531{,}441 \div 19{,}683 = 3^{12} \div 3^9 = 3^{12-9} = 3^3 = 27.$$

Thus, a knowledge of the laws of exponents enables one to use a table to lighten the work of division.

The small preceding table could be similarly used for computations involving powers of 9. Thus, $9^4 = (3^2)^4$ or 3^8, and $9^2 = (3^2)^2$ or 3^4; therefore, the following multiplication could be performed by using the same small table as follows:

$$6{,}561 \times 81 = 9^4 \times 9^2 = 9^{4+2} = 9^6 = (3^2)^6 = 3^{12} = 531{,}441.$$

Operations with exponents stress the relation between the base and the exponent.

Thus, $3^5 = 243$ stresses the fact that the base 3 taken as a factor 5 times is 243.

If, in the expression $5^3 = 125$, the exponent 3 is considered as the indicator of an operation that produces 125, then 3 is said to be the logarithm of 125 to the base 5. This relationship is indicated by the following expression:

$$\log_5 125 = 3.$$

Similarly, since $3^4 = 81$, the statement that the logarithm of 81 to the base 3 is 4 is written as follows:

$$\log_3 81 = 4.$$

The logarithm of a given number, therefore, is the exponent that will produce this given number when some preassigned base is used.

In general, if $b^x = N$, then $\log_b N = x$.

Hence, *the logarithm of any number* N *is the exponent* x *to which the base* b *has to be raised to produce the number* N.

COMMON LOGARITHMS

A table similar to the one given above, for powers of 3, could be developed by using any other positive number, except 1, as a base. For convenience, however, since our number system is decimal, that is, uses 10 as the basis in counting, it is advantageous to use 10 also as the base in logarithmic work. Logarithms to the base 10 are called *common logarithms*. Unless otherwise stated, the base 10 is assumed in logarithmic computations. Therefore, it is not necessary to indicate it in writing logarithms. Thus, $\log_{10} 100 = 2$ is written $\log 100 = 2$.

$10^4 = 10{,}000$	$\log 10{,}000 = 4$
$10^3 = 1{,}000$	$\log 1{,}000 = 3$
$10^2 = 100$	$\log 100 = 2$
$10^1 = 10$	$\log 10 = 1$
$10^0 = 1$	$\log 1 = 0$
$10^{-1} = 0.1$	$\log 0.1 = -1$
$10^{-2} = 0.01$	$\log 0.01 = -2$
$10^{-3} = 0.001$	$\log 0.001 = -3$
$10^{-4} = 0.0001$	$\log 0.0001 = -4$

From the definition of logarithms, the above table of powers of 10 and the corresponding logarithmic forms is easily prepared, for as long as the exponent of 10 is an integer, a power of 10 can easily be computed. When the exponent is a fraction, the power of 10 can be ascertained to any desired degree of accuracy. Thus, $10^{3/2} = \sqrt{10^3} = \sqrt{1000} = 31.6228$ approximately and can be found to any number of decimal places. If the exponent is an irrational number, such as $\sqrt{2}$, we meet the expression $10^{\sqrt{2}}$. The $\sqrt{2}$ was seen to be approached, in Chapter 1, page 5, by the sequence of numbers 1.4, 1.41, 1.414, $\cdots$. The expression $10^{\sqrt{2}}$ is also approached by the sequence of numbers $10^{1.4}$, $10^{1.41}$, $10^{1.414}$, $\cdots$. The computation of a number such as $10^{1.41}$ is long and tedious but can be performed, for $10^{1.41} = 10^1 \times 10^{41/100} = 10 \times \sqrt[100]{10^{41}}$.

We therefore assume,* from these considerations, that any positive number can be expressed as a power of ten.

It may be seen in the preceding table that a number between 1,000 and 10,000 will have a logarithm between 3 and 4, and hence it is 3 plus some positive decimal. If the number is between 1 and 10, the logarithm lies between 0 and 1, and hence it is 0 plus some positive decimal. Similarly, the logarithm of a number between 1 and 0.1 lies between 0 and -1, and hence it is -1 plus some positive decimal. Thus, the logarithm of a number, except in the case of an exact power of 10, consists of two parts: (*a*) a whole number or integral part; and (*b*) a positive decimal part. The whole number or integral part of the logarithm is called the *characteristic* of the logarithm, and the positive decimal part is called the *mantissa* of the logarithm. Thus, it can be shown that the log 7,000 = 3.8451 to four deci-

* This assumption is proved in advanced works on mathematics.

mal places. This logarithm consists of two parts, namely, the integer 3 and the positive decimal 0.8451. Hence, the characteristic here is 3 and the mantissa is .8451.

CHARACTERISTIC

The characteristic of the logarithm of a number is determined by inspection, that is, by noting whether the specified number is between 1 and 10, 10 and 100, 100 and 1,000, and so forth; or between 1 and 0.1; 0.1 and 0.01; 0.01 and 0.001; and so forth. For, since the logarithm of a number between 1 and 10 lies between 0 and 1, then it is 0 plus some mantissa. Similarly, the logarithm of numbers between 100 and 1,000, between 1 and 0.1, or between 0.001 and 0.0001 lie, respectively, between 2 and 3, between 0 and -1, and between -3 and -4. Thus, the logarithm of a number:

between 100 and 1,000 is 2. + some mantissa;
between 10 and 100 is 1. + some mantissa;
between 1 and 10 is 0. + some mantissa;
between 1 and .1 is -1. + some mantissa;
between .1 and .01 is -2. + some mantissa;
between .01 and .001 is -3. + some mantissa.

It appears, then, that the characteristic may be either positive or negative, and that it may be determined by noting how many figures there are to the left of the decimal point in the number or how many zeros there are to the right of the decimal point in the decimal fraction. Hence, the following rules for determining the characteristic of the logarithm of a number may be formulated.

1. *The characteristic of the logarithm of a number greater than 1 is positive, and is one less than the number of figures to the left of the decimal point.*

Thus,
log 9.83 = 0 + some mantissa
log 37 = 1 + some mantissa
log 422 = 2 + some mantissa
log 8,700.39 = 3 + some mantissa

2. *The characteristic of the logarithm of a number between 0 and 1 is numerically one more than the number of zeros between the decimal point and the first figure other than zero, and is negative.*

Thus,
log 0.418 = -1 + some mantissa
log 0.000249 = -4 + some mantissa

MANTISSA AND SIGNIFICANT FIGURES

The mantissa, or positive decimal fractional part of the logarithm of a number, is found from the table in the booklet. This table is called a *four-place table* because it gives the mantissas to four decimal places only. The decimal point in front of the mantissa is omitted in the table. Only three pages are needed for this table. Naturally, the more places there are in a logarithmic table, the more accurate will be the results obtained by using it.* The table gives the mantissa for the logarithm of a number of three significant figures.

In ascertaining the mantissa of the logarithm of a number, the *significant figures* of the number are those which remain unaltered if the number is multiplied or divided by any integral power of ten. Thus, the numbers 206, 2,060, 20.6, and 0.00206 have the same significant figures, namely, 2, 0, and 6; therefore, *they all have the same mantissa.* The explanation of this fact is as follows:

In finding the logarithm of 206, the characteristic is seen at once to be 2. The mantissa is found in the table on page 3 of the booklet inside the back cover of this volume. In the column at the extreme left, headed *N*, the first two figures of the number, namely, 20, are found. Run the finger along the same horizontal line to the right until the column headed 6, corresponding to the third figure of the number, is reached. The mantissa is here found to be .3139. Hence,

$$\log 206 = 2.3139$$

or

$$206 = 10^{2.3139}.$$

Applying the law of exponents,

$$2{,}060 = 206 \times 10 = 10^{2.3139} \times 10^{1} = 10^{2.3139+1} = 10^{3.3139}.$$

$$20.6 = 206 \div 10 = 206 \times \frac{1}{10} = 10^{2.3139} \times 10^{-1} = 10^{2.3139-1} = 10^{1.3139}.$$

$$.00206 = 206 \div 10^{5} = 206 \times \frac{1}{10^{5}} = 10^{2.3139} \times 10^{-5} = 10^{2.3139-5} = 10^{-3+.3139}.$$

* James W. Glover's *Tables of Applied Mathematics in Finance, Insurance and Statistics*, published by George Wahr, Ann Arbor, Michigan, 1923, contains a seven-place table of logarithms which is accurate enough for all ordinary purposes.

Hence, we may write:

$$\begin{aligned} \log \quad 206 &= 2.3139 \\ \log \ 2{,}060 &= 3.3139 \\ \log \quad 20.6 &= 1.3139 \\ \log .00206 &= -3 + .3139. \end{aligned}$$

Therefore, *the mantissa of the logarithm of a number is independent of the position of the decimal point in the given number and depends only on the significant figures of the number.*

NEGATIVE CHARACTERISTICS

The characteristic of a positive quantity less than unity is negative; the mantissa, however, is positive. Thus, log $.00206 = -3 + .3139$. In this form, the characteristic is negative but the mantissa is positive. This form of writing the logarithm is conventional and highly convenient in practice because the same mantissa can be used for all positive numbers that have the same significant figures. If the indicated operation, $-3 + .3139$, is carried out, log .00206 is equal to -2.6861. The number -2.6861 is equal to $-2 - .6861$, and the decimal part is now negative. Sometimes it is necessary to perform this operation, but for purposes of computation it is usually very inconvenient. Hence, it is customary to keep the form of a negative characteristic and a positive decimal part or mantissa. For example, if a logarithm such as $-3 + .3139$ is to be divided by 3, it is not correct to divide -3.3139 by 3, for the positive and negative parts would be confused. To avoid this and other confusions that may arise, logarithms having negative characteristics may be written in two different forms.

In the first form, the negative characteristic is indicated by placing the minus sign above the characteristic. Thus, log .00206 may be written $\bar{3}.3139$.

The second form, the most suitable for logarithmic computations, is the one most commonly used. It consists in adding a convenient integer to the logarithm, indicating its subtraction at the same time. This operation does not change the value of the logarithm. Thus,

$$\begin{aligned} \log .00206 &= -3 + .3139 \\ &= -3 + .3139 + 4 - 4 \\ &= 4 - 3 + .3139 - 4 \\ &= 1.3139 - 4. \end{aligned}$$

The number most often added to and subtracted from a logarithm having a negative characteristic is 10. Thus,

$$\begin{aligned}\log .00206 &= -3 + .3139 \\ &= -3 + .3139 + 10 - 10 \\ &= 10 - 3 + .3139 - 10 \\ &= 7.3139 - 10.\end{aligned}$$

USING A TABLE OF LOGARITHMS

A "table of logarithms" contains only the mantissas and not the characteristics. Such tables can be used in two different ways:

(*a*) To find the logarithm of a given number.

(*b*) To find the number whose logarithm is given.

The first procedure (*a*) has already been discussed on page 430. The second procedure (*b*) is explained by the following illustrations.

Find the number whose logarithm is 3.9465. In the table on page 5, the mantissa .9465 is seen to be in the 4 column opposite 88 in the *N* column. Hence, the mantissa .9465 corresponds to the significant figures 884. Since the characteristic is 3, by the rule on page 429 there must be four figures to the left of the decimal point. Therefore, the number whose logarithm is 3.9465 is 8,840.

Similarly, find the number whose logarithm is $9.6990 - 10$. In the table, the mantissa .6990 is in the 0 column opposite 50 in the *N* column. Hence the mantissa .6990 corresponds to the significant figures 500. Since the characteristic is -1, by the rule on page 429 there must be no zeros between the decimal point and the first figure to the right other than zero. Therefore, the number whose logarithm is $9.6990 - 10$, or $\bar{1}.6990$, is 0.5000.

The number corresponding to a given logarithm is called the *antilogarithm* of that number. Thus, 8,840 is the antilogarithm of 3.9465; 0.5000 is the antilogarithm of $\bar{1}.6990$. That is,

$$\text{antilog } 3.9465 = 8{,}840,$$

and

$$\text{antilog } \bar{1}.6990 = 0.5000.$$

INTERPOLATION

The mantissa of a number containing four or more significant figures cannot be found directly from the table on page 3.

This table can be used, however, if the method of approximations called *interpolation* is employed.*

Suppose that a table of logarithms at our disposal gave only the logarithms of 1, 5, 10, 15, 20, and so forth, and that the logarithm of 7 is to be found. The logarithms of 1, 5, 10, 15, and 20 may be plotted, and these points connected by a smooth curve, thus obtaining the graph shown in Figure 192. The accuracy of this graph is limited by the precision of measurement. However, approximate values of the logarithms of numbers that lie between the recorded values may be obtained from this graph. For example, the logarithm of 7 as obtained from this graph is approximately 0.85. This process of reading interpolated values from a graph is known as *graphic interpolation.*

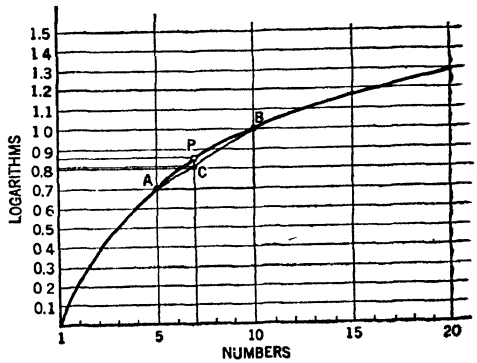


Fig. 192.

When the graph is a curve, for practical purposes, where great accuracy is not necessary, we may regard a short distance along the curve as a straight line. Thus, suppose that the portion of the graph included between points A and B is considered as the straight line AB. Then the logarithm of 7, obtained by reading the amount on the vertical scale corresponding to point C on the straight line AB, is approximately 0.82. The difference between this value and that obtained by graphic interpolation is $0.85 - 0.82 = 0.03$. Evidently, if the points A and B had been chosen nearer to point P, the difference between these readings would be smaller. Hence, for a very short distance along the curve, the error incurred by regarding the curve as a straight line would be so small that, for practical purposes, it may be disregarded. This fact may be used to advantage as follows:

Assume that A and B, Figure 193, are two points on the curve,

* The logarithm of a number is an approximation unless the number is an exact power of ten. Therefore, the method of interpolation is necessary regardless of the size of the table if greater accuracy is desired.

corresponding to any two given positive numbers M and N, whose logarithms appear in the table. Let C be a point on the straight line AB, corresponding to any other given positive number R whose logarithm does not appear in the table and such that the value of R lies between the given positive numbers M and N. To obtain approximately the logarithm of the interpolated number R, it is sufficient to determine the value of CR. The reader may easily show, by applying Fact 36, that $CR = CD + DR$ is also equal to $CD + AM$. Line segment AM is the ordinate of A, and hence its value is equal to the logarithm of M. Since the logarithm of M may be obtained from the table, it is now merely necessary to find the value of CD.

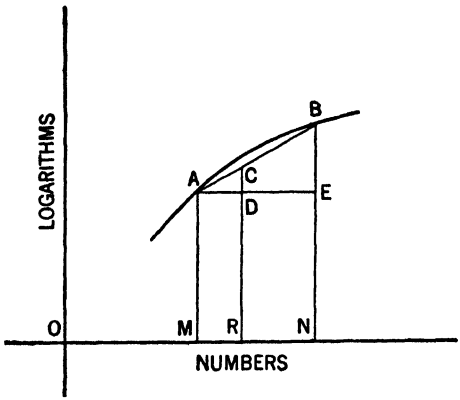


Fig. 193.

All right triangles formed by drawing parallel lines to the axes, using part of the given line AB for the hypotenuse, are similar (Chapter 13, page 194). Hence, $\frac{CD}{AD} = \frac{BE}{AE}$. As before, by applying Fact 36, it may be easily shown that $BE = BN - EN = BN - AM$. Hence, BE is the difference between the logarithms of the given positive numbers M and N, and may be obtained from the table. Similarly, $AE = MN = ON - OM$. That is, AE is the difference between the values of the two given numbers M and N. Finally, $AD = MR = OR - OM$ is the difference between the given number M and the interpolated value R. Since, in the proportion $\frac{CD}{AD} = \frac{BE}{AE}$, the values of AD, BE, and AE are known, the value of CD may be determined. The approximate value of the logarithm of R, that is, the value of CR, can now be obtained by adding the value of CD to the logarithm of M represented by the value of AM.

Summarizing, in the proportion $\frac{CD}{AD} = \frac{BE}{AE}$,

CD = the difference between the logarithm of the smaller given number and the logarithm of the interpolated number;

AD = the difference between the smaller given number and the interpolated value;

BE = the difference between the logarithms of the two given numbers;

AE = the difference between the two given numbers.

Since the values of AD, BE, and AE may be obtained either from the given values or from the table, the value of CD may be easily found without the necessity of drawing a graph. Thus, suppose the logarithm of 45.37 is to be determined. Since 45.37 is between 45.30 and 45.40, the logarithm of 45.37 must lie between the logarithms of 45.30 and 45.40. The mantissa is independent of the position of the decimal point and the characteristic is found at once by inspection. Hence, to find the mantissa of 45.37, the decimal point may, for the time being, be disregarded. From the table, the mantissas corresponding to the significant figures of 4530 and 4540 are, respectively, 6561 and 6571. The difference between the smaller given number and the interpolated value, namely, AD, is $4537 - 4530 = 7$. The difference between the mantissas of the two given numbers, that is, BE, is $6571 - 6561 = 10$. Finally, the difference between the two given numbers, namely, AE, is $4540 - 4530 = 10$. Substituting in the proportion $\frac{CD}{AD} = \frac{BE}{AE}$, we obtain $\frac{CD}{7} = \frac{10}{10}$, or $CD = 7$. Hence, the mantissa of 4537 is $6561 + 7 = 6568$, and the logarithm of 45.37 is 1.6568.

This method, called *linear interpolation*, can be simplified as follows:

Assume that log 38.76 is to be found. Since 38.76 is between 38.80 and 38.70, the logarithm of 38.76 must lie between the logarithms of 38.80 and 38.70. As stated before, the mantissa is independent of the position of the decimal point and the characteristic is found at once by inspection. Therefore, disregarding the decimal point, the work may be arranged:

Number		*Mantissa*	
3880		5888	
10 3876	6	x M	11
3870		5877	

Since 3876 is $\frac{6}{10}$ or $\frac{3}{5}$ of the way from 3870 to 3880, its mantissa is taken as $\frac{3}{5}$ of the way from 5877 to 5888.

That is, $x = \frac{3}{5}$ of 11 = 6.6, or approximately 7.
Hence, $M = 5877 + 7 = 5884$.

Thus, the mantissa of 38.76 is 5884, and therefore log 38.76 = 1.5884.*

If the antilogarithm, that is, the number whose logarithm is $\bar{2}.5617$, is to be found, the same method may be used. Disregarding the characteristic, it is seen from the table that the mantissa 5617 lies between 5611 and 5623. It follows that the number whose mantissa is 5617 lies between the numbers whose mantissas are 5611 and 5623. Arranging the work as before, we get:

Number		*Mantissa*	
3650		5623	
10 N	x	6 5617	12
3640		5611	

Since 5617 is $\frac{6}{12}$ or $\frac{1}{2}$ of the way from 5611 to 5623, the number N is taken as $\frac{1}{2}$ of the way from 3640 to 3650.

That is, $x = \frac{1}{2}$ of 10 = 5.
Hence, $N = 3640 + 5 = 3645$.

Thus, the antilogarithm of $\bar{2}.5617$ is 0.03645, or log 0.03645 = $\bar{2}.5617$.

LOGARITHMS IN COMPUTATION

All that has been said about the subject of logarithms has been merely preliminary, forming a necessary introductory explanation of how logarithms are actually used. The reader who may have thought that the discussion thus far given seemed purely theoretical and of doubtful practical value should now consider the following four principles for using logarithms in calculations. It will be seen in a moment that otherwise harassingly difficult problems can, by logarithms, be easily and quickly solved.

As already stated, a logarithm by definition is an exponent;

* Using a five-place table, log 38.76 = 1.58839. The error due to using a four-place table is the difference between the two logarithms, namely 1.5884 − 1.58838 = 0.00002, which is very slight even in this fifth place.

hence, the laws of operations with exponents will apply to logarithms.

Assume that the product of two numbers is to be determined. Denote the two numbers by a and b and their product by N. Then $N = a \cdot b$.

Now suppose that, in the usual manner, that is, by determining the characteristic and looking up the mantissa in the table, $\log a$ is found to be equal to some number, say x; and, similarly, $\log b$ is found to be equal to some other number y. That is,

let $\log a = x$ and $\log b = y$.

Then, by the definition of a logarithm, if $\log a = x$, the number x is the exponent to which the base 10 must be raised to obtain a. Hence,

$a = 10^x$, and similarly $b = 10^y$.

Thus, $N = a \cdot b = 10^x \cdot 10^y$,

or $N = 10^{(x+y)}$.

Again, by the definition of a logarithm, if $(x + y)$ is the exponent to which 10 must be raised to obtain N, then $(x + y)$ is the logarithm of N to the base 10, that is,

$$\log N = (x + y).$$

But, since $x = \log a$ and $y = \log b$, we may write:

$$\log N = \log a + \log b$$

or

$$\log (a \cdot b) = \log a + \log b.$$

In other words,

I. *The logarithm of a product is equal to the sum of the logarithms of the factors.**

By applying this law, it is possible to reduce multiplication to addition. For example, to find the product $85 \times 4.7 \times .00609$, set N equal to this product. Thus,

$$N = 85 \times 4.7 \times .00609.$$

Evidently,

$$\log N = \log (85 \times 4.7 \times .00609).$$

* Note that this law states that $\log (a \cdot b) = \log a + \log b$ and *not* that $(a \cdot b)$ is equal to $\log a + \log b$. Although this seems obvious, it is a constant source of error with beginners.

Applying the law just proved gives:

$$\log N = \log 85 + \log 4.7 + \log .00609.$$

Arranging the work and determining the values of the logarithms, we may write:

$$\begin{aligned} \log 85 &= 1.9294 \\ \log 4.7 &= 0.6721 \\ \log .00609 &= 7.7846 - 10 \\ \log N &= 10.3861 - 10 \end{aligned}$$

or $$\log N = 0.3861.$$

To find N, determine the number whose logarithm is 0.3861. The mantissa 3861 is found in the table to lie between 3856 and 3874. These mantissas correspond, respectively, to the significant figures 243 and 244. By interpolation, the significant figures corresponding to the mantissa 3861 are found to be 2433. Since the logarithm has 0 as its characteristic, $N = 2.433$.

To determine the quotient of two numbers by means of logarithms, the same procedure as in the preceding proof may be used. That is, assume $N = \frac{a}{b}$, and let:

$$\log a = x \text{ and } \log b = y;$$

then, by the definition of a logarithm,

$$a = 10^x \quad \text{and} \quad b = 10^y,$$

so that $$N = \frac{a}{b} = \frac{10^x}{10^y},$$

or $$N = 10^{x-y}.$$

Again, by the definition of a logarithm,

$$\log N = x - y.$$

Since $x = \log a$ and $y = \log b$, we may write:

$$\log N = \log a - \log b,$$

or $$\log \frac{a}{b} = \log a - \log b.$$

That is,

II. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.**

Thus, to ascertain the value of $75{,}600 \div .919$, let $N = 75{,}600 \div .919$. Then, by the law just proved,

$$\log N = \log 75{,}600 - \log .919.$$

Arranging the work as follows and determining the values of the logarithms, we may write:

$$\log 75{,}600 = 4.8785,$$
$$\log .919 = 9.9633 - 10.$$

To facilitate the subtraction, it is convenient to change the form of $\log 75{,}600 = 4.8785$ by adding and subtracting 10. Thus,

$$\log 75{,}600 = 14.8785 - 10$$
$$\log .919 = \underline{9.9633 - 10}$$
$$\log N = 4.9152.$$

The value of N is now determined by finding the number whose logarithm is 4.9152. From the table and by interpolating, we find $N = 82{,}260$ (*Answer*).

Similarly, to find the value of $(43.5 \times 673) \div (354 \times 738)$, let $N = (43.5 \times 673) \div (354 \times 738)$.

Then, by the second law of logarithms,

$$\log N = \log (43.5 \times 673) - \log (354 \times 738).$$

Applying the first law to the two members on the right-hand side of this equation, we obtain:

$$\log N = (\log 43.5 + \log 673) - (\log 354 + \log 738).$$

Determine the values of these logarithms and arrange the work as follows:

$\log 43.5$	$= 1.6385$	$\log 354$	$= 2.5490$
$\log 673$	$= \underline{2.8280}$	$\log 738$	$= \underline{2.8681}$
log of dividend	$= 4.4665$	log of divisor	$= 5.4171$

* Note that this law states: $\log \left(\frac{a}{b}\right) = \log a - \log b$, and *not* that $\frac{\log a}{\log b}$ is equal to $\log a - \log b$ or that $\frac{a}{b}$ is equal to $\log a - \log b$. Again the reader is cautioned against a mistake very frequently made in learning to work with logarithms.

The logarithm of N is equal to the difference, log of dividend − log of divisor, that is, 4.4665 − 5.4171. To carry out this subtraction, it is important to note that the operation must be performed as indicated, that is, 5.4171 must be subtracted from 4.4665, although the latter figure is the smaller.* This may be easily accomplished by adding and subtracting 10 to the logarithm of the dividend. Thus,

$$\begin{aligned} \text{log of dividend} &= 14.4665 - 10. \\ \text{log of divisor} &= \underline{\quad 5.4171} \\ \log N &= \quad 9.0494 - 10. \end{aligned}$$

The value of N is determined by finding the number whose logarithm is $\bar{1}.0494$. From the table and by interpolating, we find $N = 0.1120$ (*Answer*).

To obtain the power of a number by the method of logarithms, assume, as in previous cases, $N = a^n$, where a, n, and N are any real numbers.

Let	$\log a = x;$
then, by definition,	$a = 10^x$
and	$N = a^n = (10^x)^n$
or	$N = 10^{nx}.$

Again applying the definition of a logarithm, we obtain:

$$\log N = nx.$$

Since $x = \log a$, we may write:

$$\log N = n \log a$$

or

$$\log a^n = n \log a.$$

Hence,

III. *The logarithm of the power of a number is equal to the logarithm of that number multiplied by the exponent indicating the power.*

PRACTICAL APPLICATIONS

By applying the laws just set forth, the calculations involved in the solution of problems in compound interest are greatly re-

* If 4.4665 is subtracted from 5.4171, the indicated operation of division is inverted, that is, instead of determining the logarithm of the quotient (43.5 × 673) ÷ (354 × 738), the logarithm of (354 × 738) ÷ (43.5 × 673) is obtained, which is, of course, an entirely different problem.

duced. Consider the last problem treated in Chapter 24, that is, determine what \$500 would amount to if invested for 5 years at 2 per cent compounded semiannually. As stated before, this problem is equivalent to finding what \$500 would amount to if invested for 10 years at 1 per cent compounded annually. Substituting in the formula $S = P(1 + r)^n$, we obtain:

$$S = 500\ (1 + .01)^{10}$$

or

$$S = 500\ (1.01)^{10}.$$

Then, by the first law of logarithms, we may write:

$$\log S = \log 500 + \log (1.01)^{10}.$$

Applying the third law to the second term on the right-hand side of this equation gives:

$$\log S = \log 500 + 10 \log (1.01).$$

Determining the values of these logarithms, we may write:

$$\begin{array}{rr} \log 500 = 2.6990 & \log 500 = 2.6990 \\ \log 1.01 = 0.0043 & 10 \log 1.01 = \underline{0.0430} \\ & \log S = 2.7420 \end{array}$$

The value of S determined from the table and interpolation is $S = \$552.10$.

Owing to the use of a four-place table, an error of \$0.21 is incurred. However, if a five-place table had been used, the correct answer, namely, \$552.31, would have been obtained.

DEPRECIATION

Time and the wear and tear of use cause a decline in value of most kinds of physical property. Since there should be no impairment of capital of a going business, it is important to know how to set up a reserve fund to offset the losses due to time and wear and tear. *Depreciation* is the decline in value of a physical asset due to these causes. The *book value* of an asset at any time is its original cost minus the reserve set aside to cover its depreciation. When an asset has outlived its usefulness, it may not, even then, be entirely worthless. If it can be put to any other profitable use or sold as junk, this final or residual value is known as the *scrap value*.

An estimate of the probable life of an asset must be made

upon the basis of observation and experience; consequently, no problem can be solved by mere formulas alone apart from actual realities. Records kept during a long period of time show what actually has been the life of various kinds of machinery, structures and equipment.

There are many methods of computing the periodic depreciation charge. Only the two most common methods will here be discussed.

The simplest method of figuring depreciation charge is the *straight line method.* Because of its simplicity, it is probably more often used than all the other methods collectively. Under it, an equal part of the wearing value of the asset is laid aside each year during the life of the asset.

For example, assume that a machine that cost \$3,500 has an estimated life of 5 years and a scrap value of \$350. To obtain the periodic depreciation charge, find the difference between cost and scrap value and divide this result by the number of years that the asset is expected to last. Thus, if D represents the periodic depreciation charge,

$$D = \frac{3500 - 350}{5} = \$630.$$

The second method, often regarded as more scientific, charges to depreciation each year a constant percentage of the book value. It is called the *constant percentage method.* Since the percentage charged is constant, it is figured on the original cost at the end of the first year only. The following year, the payment is reckoned as the same percentage, yet this time not a percentage of the original cost, but of the book value. Thus, if the initial value of a machine is \$2,000 and the percentage used is 20 per cent, the operation is easily visualized as follows:

Depreciation at end of 1st year = 20% of \$2,000 = \$ 400.
Book value at end of 1st year = \$2,000 − \$400 = \$1,600.
Depreciation at end of 2nd year = 20% of \$1,600 = \$ 320.
Book value at end of 2nd year = \$1,600 − \$320 = \$1,280.
Depreciation at end of 3rd year = 20% of \$1,280 = \$ 256.
Book value at end of 3rd year = \$1,280 − \$256 = \$1,024.

In general, if r represents the rate of depreciation and C the original cost, the depreciation at the end of the first year is equal to Cr, and the book value at the end of the first year is

$C - Cr$. This expression contains C as a common factor; hence, it may be written $C(1 - r)$. To obtain the depreciation at the end of the second year, the book value at the end of the first year is multiplied by the constant rate. Hence, $C(1 - r) \cdot r$ denotes the depreciation at the end of the second year, and the book value at the end of the same year is then $C(1 - r) - C(1 - r) \cdot r$. This expression contains $C(1 - r)$ as a common factor, so that it may be written $[C(1 - r)](1 - r)$, or simply $C(1 - r)^2$. Similarly, the depreciation at the end of the third year is $C(1 - r)^2 \cdot r$, and the book value is $C(1 - r)^2 - C(1 - r)^2 \cdot r$. Since this expression contains $C(1 - r)^2$ as a common factor, it may be written $[C(1 - r)^2](1 - r)$, or $C(1 - r)^3$. That is, the book values at the end of the first, second, and third year are, respectively, $C(1 - r)$, $C(1 - r)^2$, and $C(1 - r)^3$.

As a consequence, we infer that the book value at the end of n years is $C(1 - r)^n$. If at the end of n years the machine is scrapped and the residual or scrap value is denoted by S, we have:

$$S = C(1 - r)^n.$$

Thus, suppose that the Day Furniture Company has a machine that cost \$4,500. The probable life of this machine is estimated at 8 years and the residual scrap value as \$500. It is desired to ascertain the rate of depreciation that should be used.

Here $S = \$500$, $C = \$4{,}500$, and $n = 8$. Applying the above formula gives:

$$500 = 4500\ (1 - r)^8;$$

or, dividing both sides by 500,

$$1 = 9(1 - r)^8.$$

Then, by the first law of logarithms, we may write:

$$\log 1 = \log 9 + \log\ (1 - r)^8.$$

Applying the third law to the second term on the right-hand side of this equation gives:

$$\log 1 = \log 9 + 8 \log\ (1 - r).$$

Transposing log 9, we obtain:

$$\log 1 - \log 9 = 8 \log\ (1 - r);$$

and dividing both sides of the resulting equation by 8 gives:

$$\frac{\log 1 - \log 9}{8} = \log (1 - r).*$$

Determining the values of these logarithms, we may write:

$$\log 1 = 0.0000,$$
$$\log 9 = 0.9542.$$

The calculation calls for the subtraction of 0.9542 from 0. Evidently, the result may be written -0.9542. However, in this form the mantissa is negative and therefore cannot be found in the table, since, as has already been pointed out, all the mantissas in the table are positive. To facilitate the calculation involved, we, as before, perform the operation by adding and subtracting 10 to the log 1. Thus,

$$\begin{aligned} \log 1 &= 10.0000 - 10 \\ \log 9 &= \quad 0.9542 \\ \text{Difference} &= \quad 9.0458 - 10 \end{aligned}$$

That is,

$$\frac{9.0458 - 10}{8} = \log (1 - r).$$

To perform the indicated division, two operations are necessary: 9.0458 must be divided by 8, and -10 must also be divided by 8. Obviously, since 8 is not exactly contained in 10, the second operation would result in a decimal. This inconvenience is easily avoided. The logarithm $9.0458 - 10$ is equal to $\bar{1}.0458$. Since any convenient integer may be added to the logarithm if its subtraction is indicated at the same time, we select an integer that is exactly contained in the divisor. Choosing the integer 8, we may write $9.0458 - 10 = \bar{1}.0458 = 7.0458 - 8$, and hence,

$$\frac{7.0458 - 8}{8} = \log (1 - r).$$

The division of 7.0458 by 8 gives 0.8807, and evidently -8 divided by 8 is equal to -1. Thus,

$$.8807 - 1 = \log (1 - r).$$

* Note that $\log (1 - r)$ is *not* equal to $\log 1 - \log r$.

The value of $(1 - r)$ is now determined by finding the number whose logarithm is $\bar{1}.8807$. From the table and by interpolating, we find:

$$.7599 = 1 - r,$$

or

$$r = 1 - .7599 = .2401.$$

That is, the rate of depreciation is 24.01 per cent (*Answer*).

ROOTS BY LOGARITHMS

The third law of logarithms can be easily extended to include the roots of numbers. In the proof of the third law, the exponent n was taken to represent any real number. Evidently, as the reader may easily show, if the exponent is a fraction, the law still holds. That is, if $N = \sqrt[n]{a}$, then, by the definition of fractional exponents given in Chapter 6, page 74, we may write: $N = (a)^{\frac{1}{n}}$; and, applying the third law of logarithms, we obtain:

$$\log N = \frac{1}{n} \log a,$$

or

$$\log \sqrt[n]{a} = \frac{\log a}{n}.$$

Hence,

IV. *The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.*

This operation of finding the root of a number by the method of logarithms has already been performed by the reader although probably without realizing it at the time. In the preceding depreciation problem, to ascertain the rate of depreciation that should be used by the Day Furniture Company, we obtained the expression:

$$1 = 9(1 - r)^8.$$

If in this expression both sides are divided by 9, we obtain:

$$\tfrac{1}{9} = (1 - r)^8.$$

Taking the eighth root of both sides gives:

$$\sqrt[8]{\tfrac{1}{9}} = 1 - r,$$

and

$$r = 1 - \sqrt[8]{\tfrac{1}{9}}.$$

The value of the second term on the right-hand side of this equation can be found by logarithms as follows:

Let $$N = \sqrt[8]{\tfrac{1}{9}}.$$

Then, by the fourth law of logarithms, we may write:

$$\log N = \frac{\log \frac{1}{9}}{8}.$$

Applying the second law to the numerator of the right-hand side gives:

$$\log N = \frac{\log 1 - \log 9}{8}.$$

The reader can easily recognize this expression as identical with the one previously obtained on page 444. Hence,

$$\log N = \frac{7.0458 - 8}{8} = 0.8807 - 1,$$

and $N = .7599.$

Hence, $r = 1 - \sqrt[8]{\tfrac{1}{9}} = 1 - .7599.$

That is, $r = .2401$, or 24.01 per cent (*Answer*).

OPERATIONS WITH NEGATIVE NUMBERS

There is no exponent, positive or negative, in the real number system to which 10 may be raised to get a negative number. That is, a negative number has no "real" logarithm.

However, applying the rules of signs (pages 7–10), operations with negative numbers may easily be performed as follows. Find, by using the rules of signs, whether the final result should be positive or negative. Then, disregarding the signs, effect the calculation by means of logarithms, giving to the answer the sign already obtained for the final result.

Thus, to calculate $(-14.5)(72.6) \div (-5.68)(-39.4)$, consider first the signs only. The operation is

$$\frac{(-) \times (+)}{(-) \times (-)} = \frac{(-)}{(+)} = (-).$$

The final result, therefore, should be negative. Disregarding signs, let:

$$N = \frac{(14.5)(72.6)}{(5.68)(39.4)};$$

then $\log N = (\log 14.5 + \log 72.6) - (\log 5.68 + \log 39.4);$

$\log N = 0.6725.$

Thus, $N = 4.704.$

Therefore, $(-14.5)(72.6) \div (-5.68)(-39.4) = -4.704.$

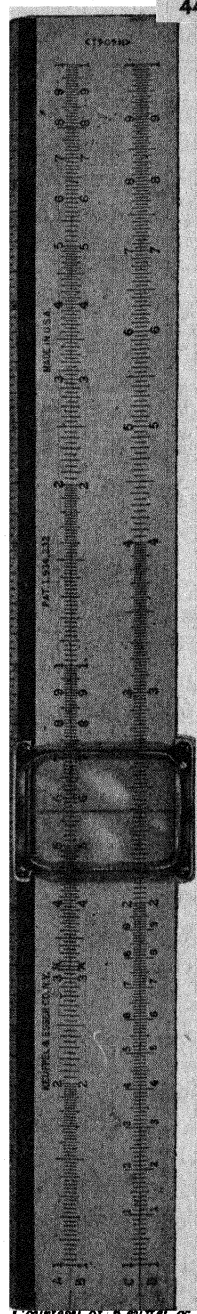
THE SLIDE RULE

The slide rule is an instrument usually in the form of a ruler, containing scales of measurement at the top and bottom and having a movable strip in between, called a *slider*, also containing two scales. The strip can be slid either to the right or to the left. Comparison in vertical position of the numbers on the two sets of scales is effected by a vertical hair line inserted in a small plate of glass or celluloid, fitted onto the rims of the ruler by grooves. This transparent plate is called the *runner*. By its use, a whole series of multiplications, for example, can be performed without recording the intermediate products.

Thus, $3.5 \times 7.1 \times 0.4$ can be done in a few seconds by briskly moving the strip and the runner. The slide rule, said to have been invented by William Oughtred (1574–1660), can be used to multiply, divide, extract roots, and find powers. Its fundamental principle is based on the laws of logarithms.

CONSTRUCTION OF A LOGARITHMIC SCALE

The sketch shown in Figure 195 represents a ruler. On the ruler, points A and K are marked so that AK is of any desired length. This distance AK is taken as the unit length. For convenience, AK is usually taken approximately* 10 inches long. Suppose here that $AK = 10$ inches; then, since AK denotes the unit length, and $\log 10 = 1$, point K can be chosen to represent the log 10. That is, AK = unit length = $\log 10$. On this length AK lay off, from A, the fractional lengths that are represented by the logarithms of the numbers from 1 to 10. Thus, since $\log 2 = 0.3010$, to locate point B, corresponding to $\log 2$, take 0.3010 of 10, that is, 3.01 inches, and measure this distance from A. The in-



Courtesy of Keuffel & Esser Company, New York

Fig. 194.

* Actually, 25 cm.

tervals between the letters, in Figure 195, correspond, then, to the numerical size of the logarithms, as follows:

AB = .3010 of 10	= 3.010 in.	corresponds	to log 2
AC = .4771 of 10	= 4.771 in.	"	" log 3
AD = .6021 of 10	= 6.021 in.	"	" log 4
AE = .6990 of 10	= 6.990 in.	"	" log 5
AF = .7782 of 10	= 7.782 in.	"	" log 6
AG = .8451 of 10	= 8.451 in.	"	" log 7
AH = .9031 of 10	= 9.031 in.	"	" log 8
AI = .9542 of 10	= 9.542 in.	"	" log 9

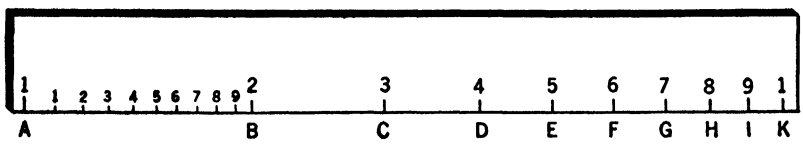


Fig. 195.

Above the letters in the diagram are written the numbers whose logarithms are represented by AB, AC, and so forth. Since log 1 = 0, the number 1 is written over point A. The number 1 is usually called the left-hand *index*. Thus, *the distance from the index 1 to any number is the logarithm of that number.*

The divisions marked off on the logarithmic scale are subdivided in order to locate the logarithms of 1.1, 1.2, and so forth. This subdivision of the scale is done in the same manner. Thus, to mark off the point corresponding to log 1.1 = 0.0414, take 0.0414 of 10, that is, 0.414 of an inch, and measure this distance from A. These subdivisions are further subdivided to obtain the logarithms of three-figure numbers. Beyond this, subdivisions are not usually carried out because they become too small for practical use.

By making AK sufficiently long, the graduations may be increased and a greater degree of accuracy attained in reading off the numbers corresponding to given logarithms. A scale 10 inches long gives the numbers to three figures. When greater accuracy is required, either a longer scale or a more elaborate instrument has to be used.

The usual inexpensive slide rule has four logarithmic scales: A, B, C, and D. (See Figure 194.) Scales C and D are *single* scales and are marked off in the manner already shown. Scales

A and B are *double* scales. These scales are marked off by first dividing the line RP equal to AK, Figure 196, into two equal parts, and then each one of these parts is subdivided as before. That is, the unit length is taken half as long as the unit length in a single scale. Thus, if the unit length in a single scale is 10 inches, then the unit length in the corresponding double scale is 5 inches. Evidently a segment representing a logarithm on a single scale is twice as large as the segment representing the same logarithm on a double scale.

Each part of the double scale is marked exactly the same as a single scale by the numbers from 1 to 10. However, since

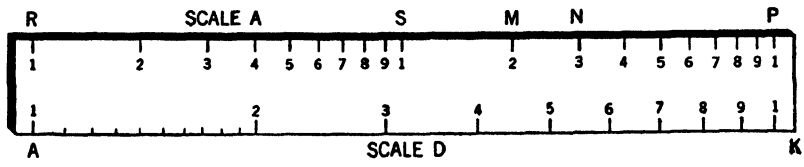


Fig. 196.

the unit length is 5 inches, point M is 1.3010 units away from R. But 1.3010 is the log 20. Hence, the distance RM = 1.3010 units corresponds to log 20. In general, since the distance RS is 1 unit long, the distance from R to any point in the right-hand part is 1 plus the distance from S to that point. From the laws of logarithms, previously discussed, it is easily seen that:

$$\log (10 \times a) = \log 10 + \log a = 1 + \log a.$$

That is, adding 1 to the logarithm of any number gives the logarithm of 10 times that number. Thus, since N is marked off 0.4771 of a unit away from S, it is $1 + 0.4771 = 1.4771$ units away from R. But 0.4771 is the log 3; hence, 1.4771 is the log 30. Therefore, the subdivisions on the right-hand part of the double scale correspond to the logarithms of the numbers which are ten times those of the left-hand part. Thus, scale A reads 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, and so on to 100.

USING A SLIDE RULE

The laws governing calculations with logarithms suggest the mechanical operations which must be performed to compute by means of the slide rule.

By the first law of logarithms, to find the product of two numbers, we add the logarithms of each factor and then find the number whose logarithm is this sum. The distance from the left-hand index, on the slide rule, to any point on the scale represents the logarithm of the number which appears printed on the scale. Hence, by adding the lengths representing the logarithms of two numbers, we find the logarithm of their

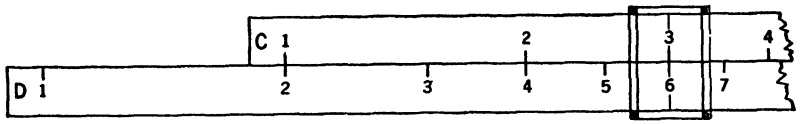


Fig. 197.

product. But since the integer printed on the scale is the number whose logarithm is represented by this length, the product of the two numbers can be read directly from the slide rule.

For example, to multiply 2 by 3, move the strip or slider so that 1 on the left-hand end of the C scale is directly above the large 2 on the D scale. Then move the runner until the hair line is over 3 on the C scale (see Figure 197). Read the answer, 6, beneath the hair line on the D scale.

As another illustration, multiply 2.15 by 7.25. If we set the

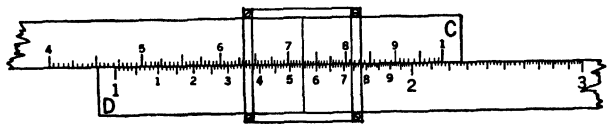


Fig. 198.

left-hand index of the C scale above 2.15, as in the preceding example, we find that 7.25 on the C scale falls out beyond the body of the rule. In a case like this, simply use the *right-hand* index of the C scale. If we set this above the 2.15 on the D scale, and then move the runner to 7.25 on the C scale, the result, 15.6, is found on the D scale beneath the hair line.

Why call the result 15.6 and not 1.56? The answer is that the *slide rule takes no account of decimal points.* Thus, the settings would be identical for all of the following products:

$$2.15 \times 7.25 = 15.5875$$
$$.215 \times .725 = .155875$$
$$21500 \times .0725 = 1558.75$$

The most convenient way to locate the decimal point is to make a mental multiplication, using only the first digits in the given factors. Thus, in 2.15×7.25, one would say mentally, "Two times seven is fourteen." There will be a result somewhat, but not much, larger than 14.

Division is the reverse of multiplication. Therefore, to divide, the steps indicated above are reversed. That is, to find the quotient $15.6 \div 7.25$, locate 15.6 on scale D (see Figure 198). Place the hair line of the runner over 15.6. Move the slider until 7.25 is exactly above 15.6. Glance along scale C to the right-hand index 1. Below this, on scale D, read the answer, 2.15.

In actual practical affairs, very often, the answer is the only thing that is wanted and it is usually wanted speedily. The slide rule enables the calculator to record the intermediate product either by changing the position of the left or right index, or the position of the runner, and then by another shift of position of either the slider or the runner with respect to the D scale, to obtain the desired answer. There is no limit to the chain of multiplications and divisions which can thus be effected. The procedure is almost automatic, except that no decimal points are recorded during the chain of operations. If decimal fractions are involved, the correct location of the decimal point in the final result must be done by the calculator according to the usual rules of arithmetic or their equivalent in terms of the characteristics of the corresponding logarithms.

A very simple illustration of such a combination of multiplications and divisions will now be described step by step. Words are profuse. The operation can be done easily in ten seconds.

Thus, to find the value of $45.50 \div 3.22 \times 5.37$ necessitates two operations, as follows: (a) to divide 45.50 by 3.22, and (b) to multiply the resulting quotient by 5.37.

To divide 45.50 by 3.22, move the runner to 45.50 on scale D. Move scale C until 3.22 falls on the hair line exactly above 45.50. The result of $45.50 \div 3.22$ is shown by the figure on D below the left-hand index 1 on scale C. There is no need of recording this result. Leaving the slider untouched, move the runner to 5.37 on scale C. The answer 7.58 is found below 5.37 on the D scale.

By construction (see page 449), a segment representing a logarithm on a single scale, such as C or D, is twice as large as the segment representing the same logarithm on a double scale, such as A or B.

Applying the third law of logarithms, it is easily seen that $\log a^2 = 2 \log a$. That is, multiplying the logarithm of any number by 2 gives the logarithm of the square of that number. Hence, the numbers on scales A and B are the squares of those directly below on scales C and D.

Thus, to find the square of 32.7, move the runner to 327 on the D scale. Take the reading directly above on the A scale. The answer is 1,070, approximately.

Inversely, the numbers on scales C and D are the square roots of those directly above on scales A and B. Thus, to find the square root of 32.7, move the runner to 327 on the A scale. Directly below on the D scale is found 572. The answer is 5.72, approximately.

SEMI-LOGARITHMIC CHARTS

Businessmen, sociologists, and scientists are often more concerned with comparison of the *ratio* of one item in a series to a preceding item than with comparisons of the *amount* of change between the two items. Ordinary graph paper, usually called "arithmetic paper," easily shows at a glance the amount of change between two items of a series, but does not lend itself to comparison of the ratio of one item to another in a series.

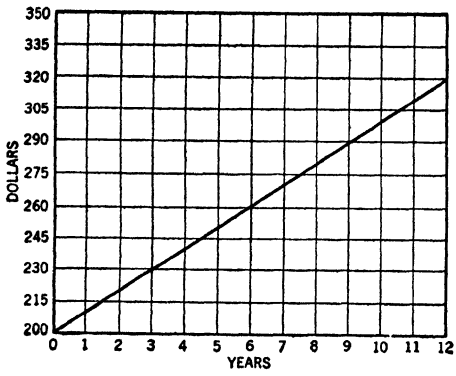


Fig. 199.

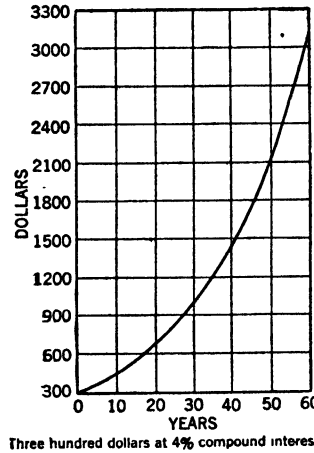
Consider, for example, Figure 204, showing the exports of the United States to Europe and to Africa for the years 1913–1935. Almost anyone looking at this chart would think at once that the exports to Europe from 1914 to 1916 were increasing more rapidly than were those to Africa during the same period. If, however, the same data be plotted on another kind of graph paper known as "semi-logarithmic paper," Figure 205, the ob-

server immediately notices that the exports to Africa from 1914 to 1916 were actually increasing more rapidly than were those to Europe. Semi-logarithmic paper is therefore highly useful to reveal at a glance the relative changes between items in a series.

An arithmetic progression was defined in Chapter 24, page 408, as a series obtained by adding a common difference to the preceding term. In the same chapter, on page 409, it was shown that \$200, if invested at the simple interest rate of 5 per cent for 12 years, yields the following arithmetic progression:

Years	0	1	2	3	4	5	6	7	8	9	10	11	12
Amounts	200	210	220	230	240	250	260	270	280	290	300	310	320

Since the amounts increase each year by the constant value 10, the difference in the amounts per unit change in the years is constant and equal to $\frac{10}{1}$, or simply 10. That is, the rate of change is constant. Therefore (Chapter 13, page 198), the arithmetic series shown in the above table may be represented graphically by a straight line. This series, if plotted on arithmetic paper, describes the straight line shown in Figure 199.



Three hundred dollars at 4% compound interest. Arithmetic Vertical Scale

Fig. 200.

A series such that each term is derived by multiplying the preceding one by a fixed amount was defined, in Chapter 24, page 414, as a geometric progression. A series of this type is illustrated by the annuity discussed in the same chapter on page 413. The amounts at the end of 0, 10, 20, 30, 40, 50, and 60 years are given in the following table:

No. of Years Accumulated	*Amounts*
0	\$ 300.00
10	444.07
20	657.34
30	973.02
40	1,440.31
50	2,132.00
60	3,155.88

This series, when plotted on arithmetic paper, describes the curve shown in Figure 200. Evidently, this curve differs radically from a straight line. However, consider the following table in which the data of the annuity problem have been repeated, and in which the logarithms of the amounts and their differences have been added:

No. of Years Accumulated	*Amounts*	*Logarithms of the Amounts*	*Differences of the Logarithms*
0	$ 300.00	2.47712	
10	444.07	2.64745	.17033
20	657.34	2.81778	.17033
30	973.02	2.98811	.17033
40	1440.31	3.15844	.17033
50	2132.00	3.32877	.17033
60	3155.88	3.49910	.17033

It is easily seen that the logarithms of the amounts differ by a constant quantity, namely, 0.17033. That is, the logarithms form an arithmetic progression. But it has just been shown that an arithmetic progression is represented graphically by a straight line. Hence, if, instead of plotting the amounts in the vertical scale, as in Figure 200, the *logarithms* of these amounts are plotted, as in Figure 201, the resulting graph is a straight line. That is, assume two series of related data, such as amounts in a fund and the corresponding years. Suppose that the items in the first series do not differ by a constant quantity, but that their logarithms do. Items in the second series differ by a constant quantity. Let these two series be plotted on a chart whose vertical scale is logarithmic and whose horizontal scale is arithmetic. If the vertical scale is used in plotting the items of the first series, and if the horizontal scale is used in plotting the items of the second series, the resulting graph is a straight line.

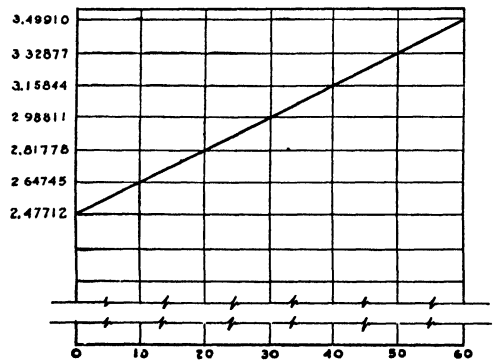


Fig. 201.

To take advantage of this fact, graph paper with logarithmic

ruling on the vertical scale may be used. This graph paper is called *semi-logarithmic paper*. However, if semi-logarithmic paper is not available in the stationery stores, it may be easily built by following the instructions given in this chapter on page 447 on the construction of a logarithmic scale. The scale

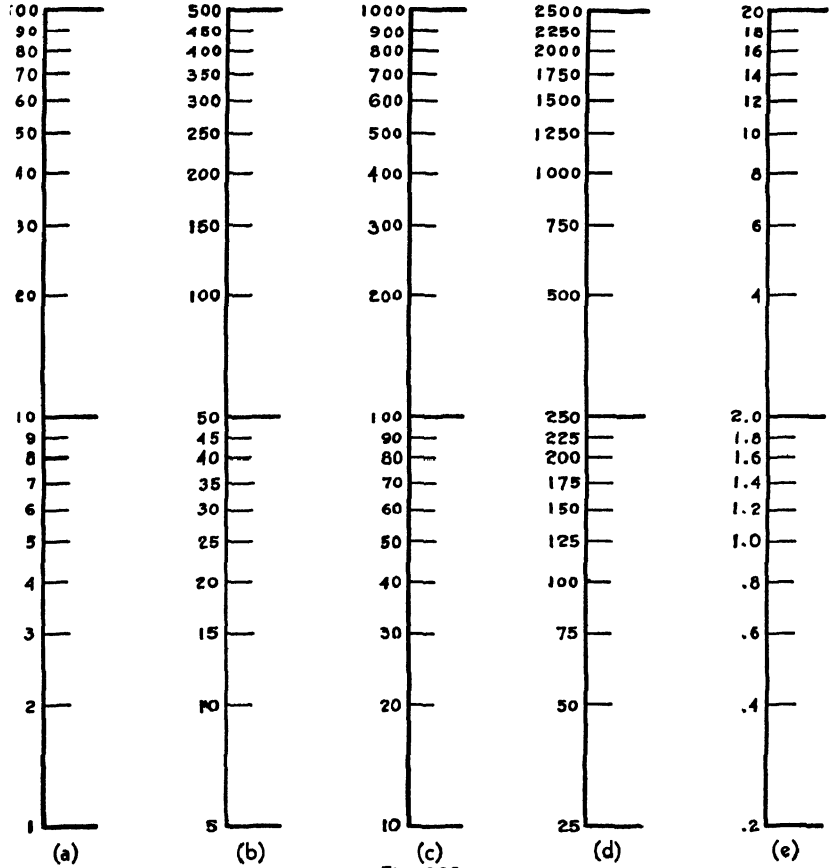


Fig. 202.

on the vertical line in semi-logarithmic paper may have as many sections as needed. Usually not more than two sections or cycles are necessary. In that case, the scale is constructed in exactly the same manner as scale A or B on the slide rule (see page 449).

The reader can easily note that not one of the logarithmic scales, shown in Figure 202, begins with zero. A common logarithm was defined, in this chapter, on page 427, as the exponent

to which 10 must be raised to obtain the given number. It may be readily seen, by referring to the table on page 428, that there is no power of 10 that will produce zero. Thus, $10^{-3} = 0.001$, $10^{-4} = 0.0001$; and if we continue raising 10 to numerically greater negative exponents, we can approach as near as we please to zero, but we can never actually reach it. Therefore, the values of the vertical scale may begin with any positive number, but not with zero.

The scale shown in Figure 202 (*a*) is a two-cycle logarithmic scale of the same type as scale *A* or *B* on the slide rule. As

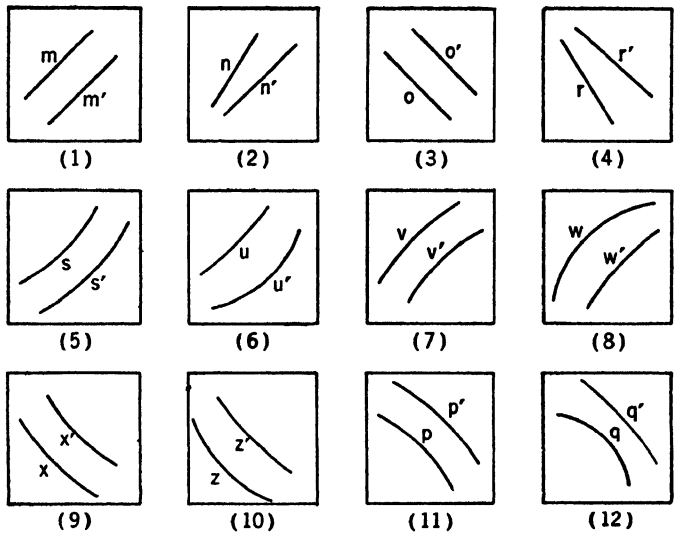


Fig. 203.

previously explained (see page 449), the value at the top of the first cycle is 10 times the value at the bottom of that cycle; and the value at the top of the second cycle is 10 times the value at the top of the first cycle. Thus, in Figure 202 (*a*), the value at the top of the first cycle is 10 while the value at the bottom of that cycle is 1. The value at the top of the second cycle is 100, which is 10 times the value at the top of the first cycle, namely, 10. The other scales shown may be obtained by simply multiplying each of the scale numbers in scale (*a*) by the number chosen to begin the first cycle. Thus, scale (*b*) is obtained by multiplying each of the scale numbers in (*a*) by 5; scales (*c*),

(d), and (e) are obtained by multiplying each of the scale numbers in (a) by 10, 25, and 0.2, respectively.

In the preceding discussion on page 454, it was shown that if two series of related data are such that the ratio of the difference of the logarithms of the items in one series to the difference of the items in the second series is constant, and semi-logarithmic paper is used in plotting the data, the resulting graph is a straight line.

Fig. 204.

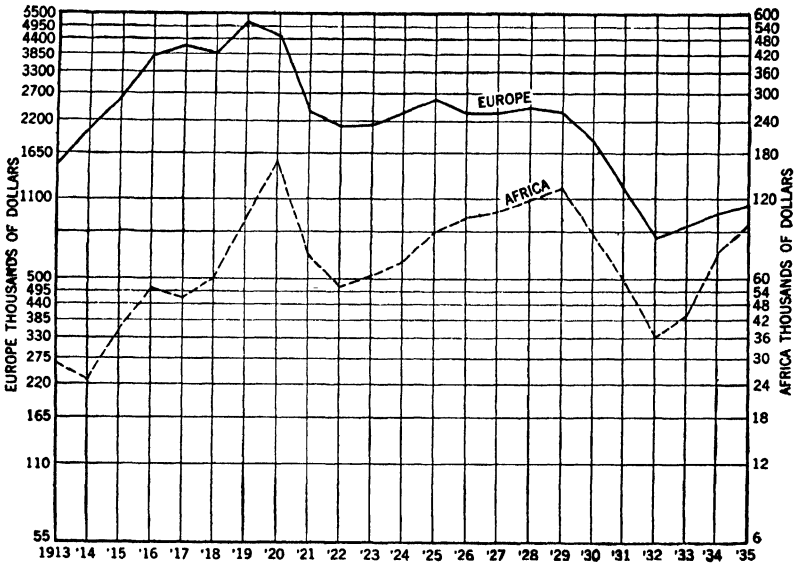
Evidently, by studying the deviations from a straight line of any given data plotted on semi-logarithmic paper, we can ascertain the rate of growth or of decrease of the given data. Moreover, the rates of growth or decrease of two sets of statistical data may be compared. Thus, in Figure 203, assuming that all the data are plotted on semi-logarithmic paper, the following relationships are shown:

(1) Rate of growth of m and m' is constant and the same for both.

(2) Different constant rates of growth, greater for n.

(3) Rate of decrease constant, same for both.

(4) Different constant rates of decrease, greater for r.
(5) Rate of growth increasing, same for both.
(6) Different increasing rates of growth, greater for u'.
(7) Rate of growth decreasing, same for both.
(8) Different decreasing rates of growth, greater for w.
(9) Rate of decrease decreasing, same for both.
(10) Different decreasing rates of decrease, greater for z'.
(11) Rate of decrease increasing, same for both.
(12) Different increasing rates of decrease, greater for q.



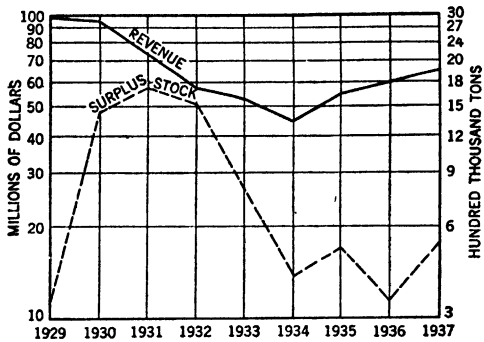
U. S Merchandise Exports to Europe and Africa 1913-1935
Logarithmic Vertical Scale. (Data from the World Almanac)

Fig. 205.

The logarithmic scale may be used to great advantage to bring two series, whose items differ greatly in size, close together for comparison. Since a semi-logarithmic chart shows the rate of increase or decrease, distances above the base line are not important. Hence, two different vertical scales may be used in order to bring out the relative fluctuations in two sets of statistical data.* Thus, Figure 204 shows, on arithmetic paper, the United States merchandise exports to Europe and

* In choosing vertical scales so that two sets of statistical data shall be brought closer together, the set containing items smaller in numerical value should always be plotted *below* the set containing items that are larger in numerical value.

Africa from 1913 to 1935. The fluctuations of the exports to Europe appear to be much greater than those of exports to Africa. However, when the same data are plotted on a semi-logarithmic chart, as shown in Figure 205, the relative fluctuations of the exports to Africa are seen to be actually greater.



Relation of abnormal stocks of sugar in Cuba to revenue of the Cuban Government (Data from Report of the Commission on Cuban Affairs)

Fig. 206.

Since two scales may be used in a semi-logarithmic chart, the rate of increase or decrease of two sets of statistical data expressed in different units may be compared by this type of graph. The chart shown in Figure 206 compares the revenue of the Cuban Government with the accumulation of surplus stock of sugar. It is easily seen that a decline in Government revenue *follows* an accumulation of surplus stocks of this commodity.

SUMMARY

Exponents	*Logarithms*
$a^m \cdot a^n = a^{m+n}$	$\log (ab) = \log a + \log b$
$a^m \div a^n = a^{m-n}$	$\log \left(\frac{a}{b}\right) = \log a - \log b$
$(a^m)^n = a^{mn}$	$\log a^n = n \log a$
$a^{\frac{m}{n}} = \sqrt[n]{a^m}$	$\log \sqrt[n]{a} = \frac{1}{n} \log a$

Problems

1. A cube-shaped vat when full contains 1,792,420 cubic centimeters of an aniline compound. Find the length of one edge of the vat. (*Hint:* Use logarithms to extract the cube root of 1,792,420.)

2. The formula for the volume of a sphere is $V = \frac{4}{3}\pi R^3$, where R is the radius. By the use of logarithms, find to the nearest hundredth the diameter of a sphere whose volume is 856 cubic inches. (Consider $\pi = 3.14159$.)

3. The formula for the volume of a circular cylinder is $V = \pi R^2 H$. Find by logarithms the volume of a boiler whose radius is 5 feet and whose height is 12 feet.

4. If P dollars are placed at i per cent interest for n years and the interest is compounded annually, the amount A at the end of that time is shown by the formula:

$$A = P(1 + i)^n.$$

How long will it take \$2,000 to amount to \$3,079 at 4 per cent, if interest is compounded annually?

5. If P dollars are placed at j per cent interest for n years, and the interest is compounded m times a year, the amount A at the end of that time is shown by the formula:

$$A = P\left(1 + \frac{j}{m}\right)^{mn}.$$

How long will it take \$10,000 to amount to \$13,684 at $3\frac{1}{2}$ per cent if interest is compounded quarterly?

6. The present value, P, at i per cent interest, compounded annually, of an amount S, n years subsequent to the date of the present value, is obtained by the formula:

$$P = \frac{S}{(1 + i)^n}.$$

As of January 1, 1942, compute, at 3 per cent interest compounded annually, the present value of \$1,000,000 due January 1, 1951.

7. The present value, P, at j per cent interest, compounded m times a year, of an amount S, n years subsequent to the date of the present value, is obtained by the formula:

$$P = \frac{S}{\left(1 + \frac{j}{m}\right)^{mn}}.$$

What sum would have had to be invested on July 1, 1922, at $3\frac{1}{2}$ per cent interest, compounded semiannually, to amount to \$1,500,000 on July 1, 1942?

8. If bacilli in a certain culture multiply at the rate of 100 per cent every 24 hours, how many will there be in two weeks, if there are 1,000,000 at the start?

9. Given $A = \pi r^2$; express $\log A$ in terms of $\log \pi$ and $\log r$.

10. In computing the strain on a bridge cable, an engineer obtains the expression $\log n = 1.8769$. How would he find $\log \sqrt[5]{n}$?

11. A surveyor desires to find the value of:

(*a*) log sin 18°.
(*b*) log tan 24°.
(*c*) log cos 41°.

What does he obtain?

12. In estimating some ore deposits, an engineer wishes to find the value of the following expression: $\sqrt[5]{43.87} \div (\tan 19°)^2$. What is the value?

26 SOLID SHAPES

Intellectual knowledge begins with the senses. An object is perceived and then represented in the mind as an image. We see this dog, of this particular breed, color, and size, in a certain direction and at a definite distance. However, when we say "a dog," we apply to this concrete object a symbol of an abstract and general idea. That is, the characteristics common to all dogs, such as having four legs, two ears and eyes, a tail, and so forth, remain prominent. This abstract and general idea is a *concept.* A concept does not refer to a particular object as it is in nature with all its individual qualities, but considers only those characteristics that are absolutely essential to the object and therefore common to all other objects of the same kind. Thus, the concept "sphere" includes qualities essential to all spheres, such as roundness and smoothness, although these characteristics are neither represented nor, indeed, perfectly representable physically. The figures that we draw simply aid in visualizing our concept of the solid shape that exists only in our mind and therefore has no material existence. Since we live in a world of three dimensions, the power of visualizing solid shapes is essential.

REPRESENTATION OF SOLID SHAPES

In analyzing the properties of solids, it is convenient to be able to represent them on paper as accurately as possible. A few suggestions will be of assistance in drawing their figures. The figures are usually supposed to be either to the right or to the left of the eye of the observer and a little below it. Actually, if an object is represented as it appears to the eye, it must be drawn in perspective. However, unless the reader has given some time to the study of drawing, it is much easier to use what is known as "the parallel line method." In this method, a plane, although unlimited in extent, is represented by a parallelogram. Hence, solids that are bounded by planes

are drawn by the use of parallel lines. The figure representing a showcase, page 465, is a good example. The lines representing the receding edges of the showcase are slanting lines. Although, in fact, all angles in the showcase, with the exception of the angles made by the slanting face with the top and base, are right angles, not all angles in the figure are right angles. It should also be noted that two lines that intersect in the drawing do not necessarily intersect in space. Thus, line *SD*, Figure 210, intersects line *RN*, although actually in back of it.

Solid figures may be imagined as opaque or transparent. If considered opaque, the hidden lines should either be entirely left out or should appear as dotted lines. If the solid is considered transparent, these lines should be drawn lighter than the others. All vertical lines are drawn vertical. A horizontal line that does not recede from the eye is represented by a horizontal line. If the horizontal line recedes from the eye, it is drawn slanting. Parallel lines are drawn parallel. A circle, unless viewed from a point directly in front of the center, is represented by an ellipse. In general, each part of the figure is drawn as nearly as possible as it appears to the eye. Hence, if two lines are of the same size but one is either horizontal or vertical while the other recedes from the eye, as in Figure 207, the receding line appears to be shorter, and therefore it is drawn shorter.

DETERMINING PLANES

A *plane* was defined, in Chapter 7, page 96, as generated by a straight line moving laterally in a constant direction. The plane is considered to extend indefinitely in each direction, and is ideally and perfectly flat. Hence, we may say that a plane is a surface such that if any two points in it are taken, the straight line containing them lies wholly in the surface.

A plane is said to be *determined* by given lines or points when it is the *only* plane which contains these given lines or points.

Assuming that a plane can be made to rotate about any straight line in it, as shown in Figure 207, it may be made to take as many different positions as we please. Hence, *a straight line does not determine a plane.*

Assume a point whose position in space is fixed. Assume further that this point is not on a given straight line. If the plane in which this point and line are located is now made to

rotate about the given line, it will no longer contain the given point, for the position of the point in space is fixed. Hence, *a straight line and a point not in the line determine a plane.*

Evidently, *three points not on the same straight line also determine a plane;* for by joining two of the points, we have a straight line and a point not in the line, and it has just been shown that these determine a plane.

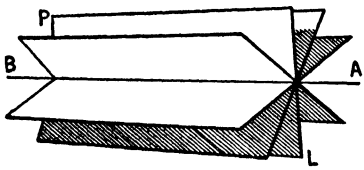


Fig. 207.

It may also be said that *two intersecting lines determine a plane.* For it is easily seen that any two points in one of the two intersecting lines and any point on the other line, not the point of intersection, are three points not on the same straight line, and these have just been shown to determine a plane.

In the definition of parallel lines given in Chapter 7, page 97, the reader may have considered the words "in the same plane" superfluous. However, attention is here called to the fact that two non-parallel lines in space do not have to intersect: they may simply pass each other. Thus, in Figure 208, the line segment AB lying on plane PL and the line segment CD are evidently not parallel, yet they do not intersect, no matter how far they are produced. These non-parallel, non-intersecting lines are called *skew* lines. Therefore, in defining parallel lines in space, it is not sufficient to state, "Lines that do not meet are parallel"; rather, one must say, *Two lines are parallel if they lie in the same plane and do not meet.*

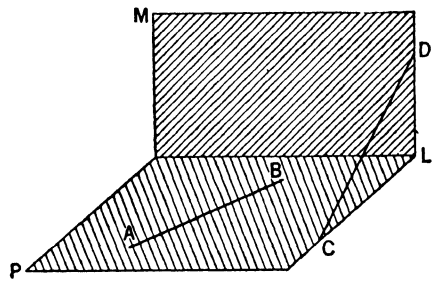


Fig. 208.

Since two parallel lines lie in the same plane and do not meet, then any two points in one of the lines and any point on the other line are not on the same straight line. But three points not on the same straight line determine a plane. Therefore, *two parallel lines determine a plane.*

For convenience, we may summarize the results obtained:

(1) *A straight line and a point not in the line determine a plane.*

(2) *Three points not in the same straight line determine a plane.*

(3) *Two intersecting lines determine a plane.*

(4) *Two parallel lines determine a plane.*

INTERSECTION OF PLANES

The intersection of two lines, curves, or surfaces are those points which they have in common. As a result of this definition, we then assume without proof that *if two planes have one point in common, they have at least two distinct points in common.*

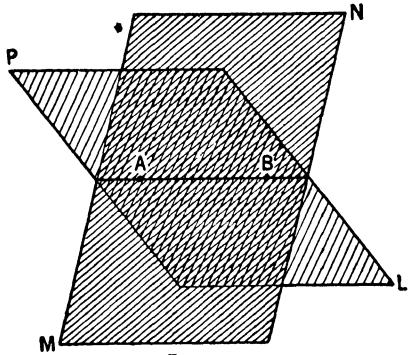


Fig. 209.

Now consider two intersecting planes such as MN and PL in Figure 209. It may be easily shown that:

Fact 112. If two planes intersect, their intersection is a straight line.

By the assumption just made in the preceding paragraph, MN and PL must have at least two points, such as A and B, in common. Then, by the definition of a plane (page 463), the line determined by the two points A and B lies wholly in both planes. Moreover, no point not in the line AB can be in both planes; for, by (1) above, only one plane can contain a straight line and a point not in the line. Therefore, the straight line AB contains all the points common to the two planes, and consequently it is the intersection of MN and PL.

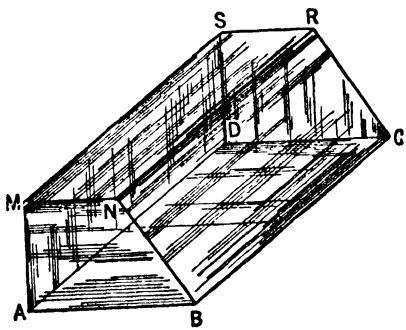


Fig. 210.

The parallel concept developed in the case of lines lying in the same plane can be applied to planes by the statement: *Two planes which do not intersect are parallel.* Since a plane is considered to extend indefinitely, parallel planes have no points in common.

A showcase made of glass has the shape shown in Figure 210. The top $MNRS$ is parallel to the base $ABCD$. To cut the inclined face $NBCR$ from a glass plate, the craftsman simply measures the distance NB, draws NR parallel to BC, and then cuts along NR. The inclined face will then lie flat against the top and base. In order to justify the craftsman's actions, we must show that:

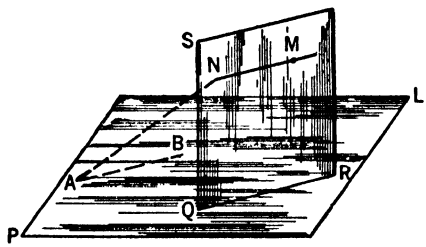
Fact 113. If a plane intersects two parallel planes, the intersections are parallel lines.

The top and base of the showcase are assumed to be parallel; that is, plane MR is parallel to plane AC. These two planes are intersected by the plane NC in two straight lines, represented in Figure 210 by the two line segments NR and BC. For, by Fact 112, the intersection of two planes is a straight line.

Now assume that NR is not parallel to BC. Then, since the line segments NR and BC lie in the same plane NC, they must meet, if extended far enough, at some point P. So that P, being on both line segments NR and BC, would also be in both planes MR and AC. For line segment NR is in plane MR, line segment BC is in plane AC, and a straight line containing any two points in a plane lies wholly in the surface (page 463). But two parallel planes have no points in common. Hence, this contradicts the given condition that plane MR is parallel to plane AC. Therefore, the assumption that the line segments are not parallel is wrong, and NR is parallel to BC.

PARALLEL LINES AND PLANES

The method of ascertaining whether the frame of a picture, placed at a certain height on a wall, is parallel to the plane of the floor is well known. The picture is placed so that the lower edge of the frame is parallel to the intersection of the planes of the wall and floor. The procedure is equivalent to the *construction of a line parallel to a plane* through a point outside the plane.*



* A line and a plane are said to be parallel if they do not intersect.

Thus, assume a point M outside plane PL. Through point M pass a plane SR intersecting PL in some line, such as QR. In the plane SR construct, as shown in Chapter 9, page 129, a line NM through M parallel to QR. To show that NM is parallel to plane PL, we must prove that:

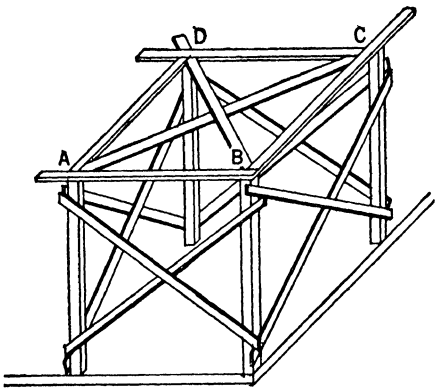


Fig. 211.

Fact 114. If a plane contains only one of two parallel lines, it is parallel to the other line.

If NM is not parallel to plane PL, it must intersect PL at some point A. But since NM is, by construction, parallel to QR, point A cannot be on line QR, for two lines are parallel if they lie in the same plane and do not meet. Through A, as shown in Chapter 9, page 129, draw, in plane PL, line AB parallel to QR. We then have through A two lines AB and NM both parallel to QR. But this is impossible, for through a given point outside a line only one parallel to this line can be drawn (Chapter 7, page 97). Thus, the assumption that NM is not parallel to plane PL contradicts the last statement, and hence it is wrong. Therefore, line NM is parallel to plane PL.

As a deduction from the fact just proved, the reader may easily show that:

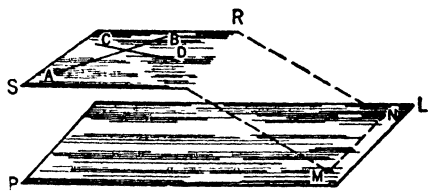


Fig. 212.

Fact 115. If a line is parallel to a plane, the intersection of the plane with a plane passed through the line is parallel to the line.

To erect the wooden form necessary to build a reinforced concrete ceiling, a contractor sets the ceiling form so that it shall be parallel to the floor. By ascertaining that all the supporting columns, shown in Figure 211, have exactly the same height, the contractor reasons that the ceiling form resting on these columns is parallel to the floor. The four columns de-

termine two intersecting lines, represented by the cross beams AC and DB in Figure 211, parallel to the plane of the floor. These two intersecting lines determine the plane of the ceiling form (see page 464). Hence, the contractor's reasoning is based on:

Fact 116. If two intersecting lines are parallel to a plane, their plane is parallel to the plane.

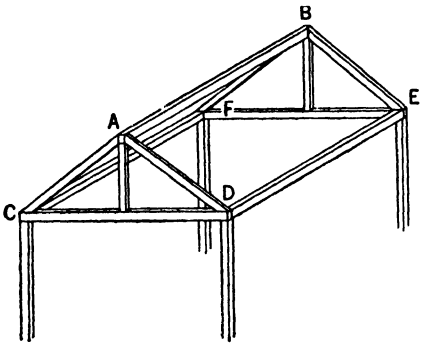


Fig. 213.

Consider the intersecting lines AB and CD, Figure 212, parallel to plane PL, and let SR be the plane determined by these lines.

As before, assume that plane SR is not parallel to plane PL. If not, then SR intersects PL in some line such as MN. Then, by Fact 115, lines AB and CD parallel to plane PL are also parallel to line MN, the intersection of plane SR, passed through these lines, with PL. We then have two lines AB and CD passing through P, both parallel to MN. This is impossible, for it contradicts the fundamental assumption, made in Chapter 7, page 97, that through a given point outside a line only one parallel can be drawn. Hence, the assumption that SR is not parallel to PL is wrong. Therefore, the plane SR, determined by the two intersecting lines AB and CD, is parallel to plane PL.

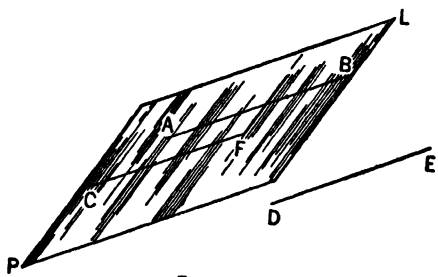


Fig. 214.

The assertion that,

Fact 117. If a plane intersects one of two parallel lines, it intersects the other also,

seems obvious and is, therefore, here merely stated. Its simple proof is left to the reader.

In the construction of a roof truss, such as the one shown in Figure 213, the purlin AB is placed parallel to the horizontal

timbers *CF* and *DE*. The carpenter constructs the framework so that the timbers *CF* and *DE* are parallel; then he erects the purlin *AB* parallel to one of the horizontal timbers and concludes that it must be parallel to the other. His reasoning is based on

Fact 118. If two distinct lines are parallel to the same line, they are parallel to each other.

Consider the lines *CF* and *AB*, Figure 214, both parallel to *DE*. Pass the plane *PL*, determined by *CF* and a point in *AB*.

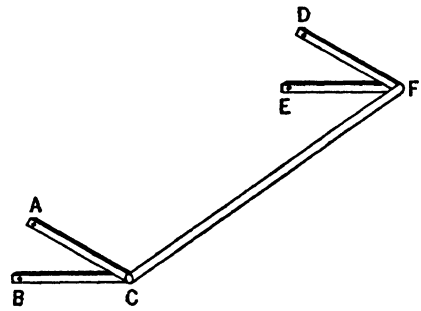


Fig. 215.

If plane *PL* were to cut *AB*, then, by Fact 117, it would have to cut *DE;* and if it cuts *DE*, it would also, by Fact 117, have to cut *CF*. But, by construction, plane *PL* contains *CF*. Hence, *PL* cannot cut *CF*. Thus, the assumption that plane *PL* cuts *AB* is wrong. Since *PL* does not cut *AB*, it must contain *AB*. That is, lines *CF* and *AB* lie in the same plane.

Moreover, *AB* and *CF* cannot intersect, for in that case we would have two intersecting lines parallel to the same line. This is impossible, for through a given point outside a line only one parallel can be drawn (Chapter 7, page 97). Thus, *AB* and *CF* lie in the same plane and do not meet. Hence, by definition, *AB* is parallel to *CF*.

A factory receives an order for a number of steel pieces which are to have the shape shown in Figure 215. The steel piece consists of a circular rod *CF* with two flat steel bars at each end. The bars must be at a given angle. In assembling the complete piece, the mechanic must place the flat ends so that angles *ACB* and *DFE* are equal and their sides parallel and running in the same direction. The mechanic places *AC* and *BC* so as to get the required angle, and then bolts *DF* and *EF* respectively parallel to *AC* and *BC*. He then concludes that angles *ACB* and *DFE* are equal. To justify the mechanic's conclusion, we must show that:

Fact 119. If two angles not in the same plane have their sides

respectively parallel and extending in the same direction from their vertices, they are equal.

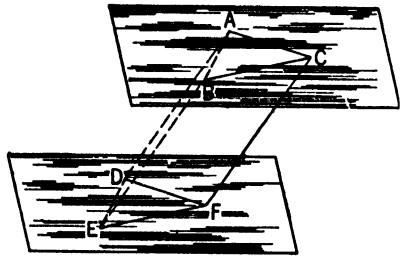


Fig. 216.

Assume that angles ACB and DFE, Figure 216, have AC parallel to DF and BC parallel to EF. Draw CF. Mark off AC equal to DF and BC equal to EF. Draw AB, DE, AD, and BE.

By construction, AC is equal and parallel to DF. Hence, by Fact 40, $ADFC$ is a parallelogram and, by Fact 36, AD is equal and parallel to CF. Similarly, BC is, by construction, equal and parallel to EF. Thus, by Fact 40, $BEFC$ is a parallelogram and, by Fact 36, BE is equal and parallel to CF.

That is, AD and BE are equal and parallel to CF. Therefore, by Fact 118, AD is parallel to BE. Moreover, since AD and BE are equal to CF, they are equal to each other. Thus, AD is equal and parallel to BE. Again applying Fact 40, $ADEB$ is seen to be a parallelogram, and, by Fact 36, DE is equal to AB.

Hence, triangle ACB has its three sides equal respectively to the three sides of triangle DFE. Therefore, by Fact 6, the triangles are congruent, and angle ACB is equal to the corresponding angle DFE.

The method of *constructing a plane parallel to a given plane through a point not in the given plane* was indirectly implied in proving Fact 116. However, because of its importance in the following proofs, it will now be fully explained.

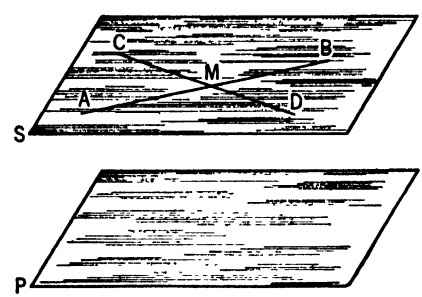
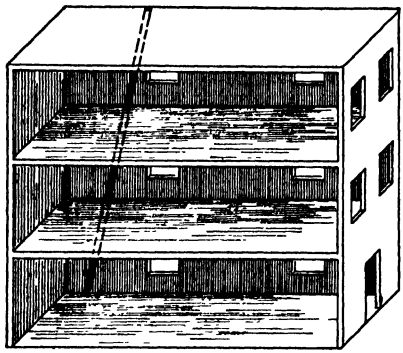


Fig. 217.

Assume that point M, Figure 217, is outside plane PL. As shown on page 466, construct, through M, lines AB and CD parallel to the plane PL. Pass plane SR, determined by the intersecting lines AB and CD [page 465, (3)]. Since AB and

CD are both parallel to plane PL, then, by Fact 116, their plane SR is parallel to PL.



The distance between the floors of a certain warehouse vary as follows: the distance between the first and second floors is 12 feet, between the second and third 11 feet, and between the third floor and the roof 10 feet. A water pipe for a sprinkler system against fires is to run from the ground floor to the roof, making an angle of 75° with the floors. To find the length of pipe required between floors, it is convenient to show that

Fact 120. If two straight lines are cut by three parallel planes, the corresponding segments are proportional.

Consider three parallel planes PL, SR, and MN, Figure 218, cutting the two lines AB and CD at A, E, B and C, H, D, respectively. Pass the planes determined by A, B, D and A, C, D [page 465, (2)]. These planes, by Fact 112, intersect in line AD. Furthermore, the planes determined by A, B, D and A, C, D intersect the given planes in EF, BD, AC, and FH.

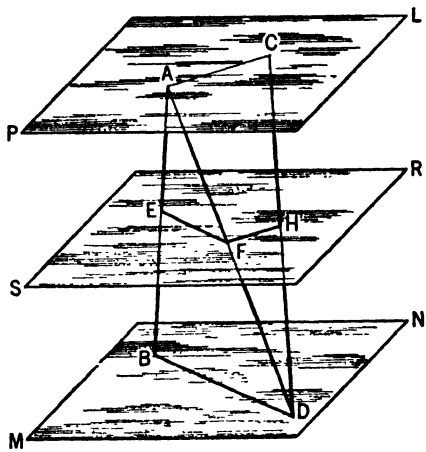


Fig. 218.

The given planes are parallel; hence, by Fact 113, EF is parallel to BD and AC is parallel to FH. Thus, in triangle ACD, line segment FH, parallel to AC, intersects the other two sides. Therefore, by Fact 43, line segment FH divides sides AD and CD proportionally. That is, $\frac{CH}{HD} = \frac{AF}{FD}$. Similarly, in triangle ABD, line segment EF parallel to line segment BD intersects the other two sides. Therefore, again by Fact 43,

$\frac{AE}{EB} = \frac{AF}{FD}$. But things equal to the same thing are equal to each other. Therefore, $\frac{CH}{HD} = \frac{AE}{EB}$.

From this fact it can be inferred that:

Fact 121. If two parallel planes cut a series of concurrent lines, the corresponding segments are proportional.

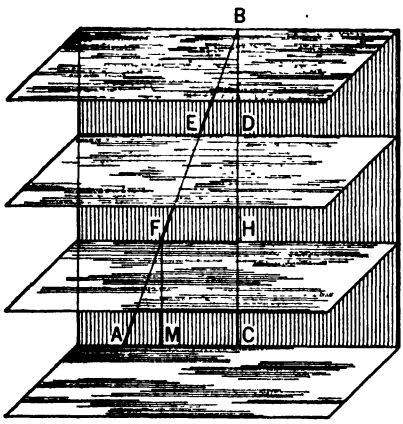


Fig. 219.

The length of the water pipe running along the back wall from floor to floor of the warehouse can now be easily ascertained. In the diagram shown in Figure 219, the different floors are represented by parallel planes. Line segment AB represents the water pipe, while BC is a perpendicular drawn, in the plane of the back wall, from B to AC. Line segment HC is 12 feet, DH is 11 feet, and BD is 10 feet.

To find AF, drop a perpendicular from F to AC. Then line segments FM and HC in the same plane perpendicular to the same line segment AC are, by Fact 11, parallel. Since FH is, by construction, parallel to MC, then $FMCH$ is a parallelogram, and, by Fact 36, FM is equal to HC. In the right triangle AFM, we have FM equal to 12 feet and angle MAF equal to 75°. Line segment AF may now be found by using the sine ratio. Thus, $\sin MAF = \frac{FM}{AF}$, that is, $\sin 75° = \frac{12}{AF}$; hence,

$$AF = \frac{12}{\sin 75°}, \text{ and } AF = \frac{12}{.9659} = 12.42 \text{ feet.}$$

Since the planes of the floors are assumed to be parallel, we have, by Fact 120, $\frac{AF}{HC} = \frac{FE}{DH}$. Substituting known values, we obtain $\frac{12.42}{12} = \frac{FE}{11}$, so that $\frac{12.42 \times 11}{12} = FE$, or $FE =$

11.385 feet. Similarly, $\frac{FE}{DH} = \frac{EB}{BD}$. Hence, $\frac{11.385}{11} = \frac{EB}{10}$, so that $\frac{11.385 \times 10}{11} = EB$, and $EB = 10.35$ feet.

PERPENDICULAR LINES AND PLANES

The concepts and facts developed in the study of plane figures may, in most cases, be carried over to the analysis of lines or shapes of solids in space. Thus, we define the *foot* of a line in space as the intersection of the line with a plane. However, while dealing with plane figures, the words "in a plane," being understood, were sometimes omitted. Hence, to avoid errors, in applying facts developed in the analysis of plane figures, care should be used that all the parts of the figure concerned lie in one plane.

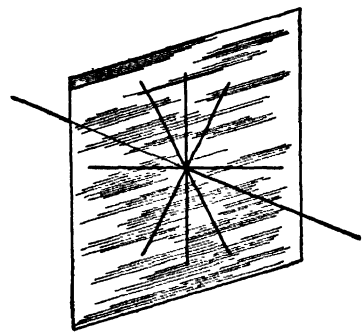


Fig. 220.

For example: the assumption that a given angle can be divided into two equal angles by but one ray (Chapter 7, page 102) led us to the logical conclusion that "at a given point in a straight line, one and only one perpendicular can be erected to the line" (Chapter 7, page 103). This conclusion really means "one and only one perpendicular can be erected *in a plane* to the line." However, lines in space can lie in three dimensions, and hence any number of perpendiculars may be erected at a given point in a straight line. This fact, easily visualized in Figure 220, may be made still clearer by holding two pencils perpendicular to each other. Evidently, as we rotate one of the pencils around the point of contact it remains perpendicular to the other in all its positions.

We then say that *a straight line is perpendicular to a plane if it is perpendicular to every line in the plane passing through its foot.* To show that a line is perpendicular to a plane, it is not necessary to prove directly that the line is perpendicular to *every* line passing through its foot. Fortunately, this proof is unnecessary, for we may pass an unlimited number of lines through the foot of the perpendicular.

A carpenter erects a beam which is to be a support of the roof truss of a cabin. To test whether the beam is perpendicular to the floor of the cabin, he places a carpenter's square with one edge on the floor and the other flat against the beam, as in Figure 221. He then changes the position of the edge of

Fig. 221.

the carpenter's square on the floor and again places the other edge flush against the beam. If, in these two positions, the beam each time is in line with the vertical edge of the carpenter's square, the beam is perpendicular to the plane of the floor.

By proving that

Fact 122. If a line is perpendicular to two lines at their point of intersection, it is perpendicular to any line in their plane through that point,

the carpenter's conclusion can be justified.

Assume the line segment MN, Figure 222, perpendicular to the lines AB and CD at their point of intersection, N.

The intersecting lines AB and CD, by (3) on page 465, determine a plane. Call this plane PL. Through N in plane PL

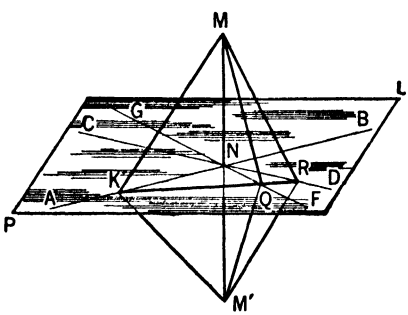


Fig. 222.

draw any other line FG. Draw a fourth line, in plane PL, intersecting AB at K, FG at Q, and CD at R.

Extend the line segment MN to M' so that $MN = M'N$. Draw the line segments MK, MQ, MR, $M'K$, $M'Q$, and $M'R$.

The intersecting lines $M'K$ and MK determine the plane $M'KM$ [see (3), page 465]. Line segment MNM' has two points M and M' which lie in this plane. Therefore, by definition, the line segment MNM' containing these two points lies wholly in this plane $M'KM$. Furthermore, since the points K and N likewise lie in this plane, the line segment KN lies wholly in it. The two line segments KM' and KM, drawn from a point K in the perpendicular AB to the given line segment MM' cut off on this given line segment equal segments, MN and $M'N$, from the foot of the perpendicular. Therefore, by Fact 28, $KM = KM'$.

By similar reasoning, line segments MR and $M'R$ can be proved equal.

Next consider the triangles KMR and $KM'R$. Side $KM = KM'$, $MR = M'R$, and KR is a side common to both. Therefore, by Fact 6, the two triangles are congruent, and their corresponding angles MKR and $M'KR$ are equal.

Finally, consider the triangles KMQ and $KM'Q$. Side $KM = KM'$, KQ is common to both triangles, and the angle MKR = angle $M'KR$. Therefore, by Fact 2, these triangles are congruent and the corresponding sides MQ and $M'Q$ are equal.

Now Q is any point on the line FG. Lines MQ and $M'Q$ drawn from Q to MM' have just been shown to be equal. Therefore, Q is equidistant from M and M'. Point N is also equidistant from M and M' and lies in line FG. Therefore, by Fact 71, FG is a perpendicular bisector of MM'.

Summing up, since FG represents *any* line in the plane PL drawn through the point of intersection of the two lines AB and CD which lie in that plane, the perpendicular MN to these two intersecting lines is also shown to be perpendicular to the line FG.

DIHEDRAL ANGLES

Just as two intersecting lines form four plane angles, two intersecting planes also determine four angles. A figure formed by two intersecting planes is called a *dihedral angle.*

The intersecting planes are the *faces*, and their line of intersection is the *edge* of the dihedral angle. A dihedral angle is usually designated by naming its faces and edge. Thus, the dihedral angle shown in Figure 223 is denoted angle *L-SP-R*. When there is no ambiguity, the dihedral angle may be named by its edge. That is, dihedral angle *L-SP-R*, in Figure 223, may also be designated as dihedral angle *SP*.

To measure a dihedral angle, we proceed as follows: From any point, *M* in Figure 224, in the edge of the dihedral angle, two perpendiculars, *MC* and *MB*, one in each face, are erected to its edge. These two perpendicular lines to *SP* intersect at *M*, and hence [page 465, (3)], they determine a plane. Angle *BMC* is then said to be the *plane angle* of dihedral angle *L-SP-R*. A plane angle may be used to measure a dihedral angle. For, if at any other point, such as *N*, Figure 224, in the edge of the dihedral angle, two other perpendiculars, *ND* and *NF*, one in each face, are drawn to its edge, then, by Fact 11, *ND* and *NF* are respectively parallel to *MC* and *MB*. That is, angles *DNF* and *CMB* have their sides parallel and running in the same direction from their vertices. Therefore, by Fact 119, the two angles are equal. Thus, since any two plane angles of the same dihedral angle are equal, *the plane angle is taken as the measure of the dihedral angle.*

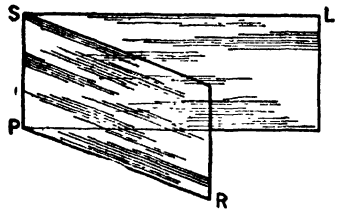


Fig. 223.

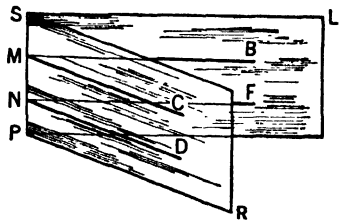


Fig. 224.

Dihedral angles are named, as angles in a plane, by the amount of opening between the two sides of their plane angle. Thus, a dihedral angle whose plane angle is a right angle is a *right dihedral angle*. In this case, the planes are said to be *perpendicular* to each other.

SOLIDS

At the beginning of this chapter, the fact that we deal with concepts and not with actual material objects was stressed. The "solids" which we are now about to discuss are no more solid in the physical sense than is our reflection in a mirror.

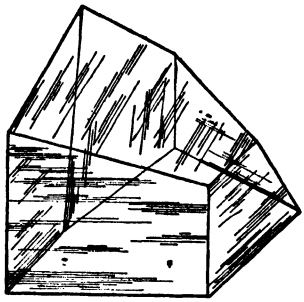
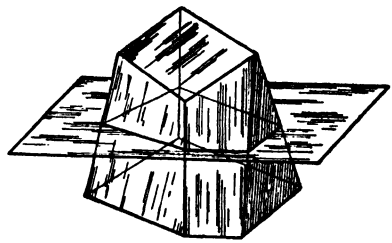


Fig. 225.

We are primarily interested in the *shape* of the solid and in certain characteristics possessed by solids of definite shape. Hence, *we define a solid as any limited portion of space bounded by surfaces.*

If the surfaces bounding the solid are planes, as in Figure 225, the solid is called a *polyhedron.* The intersections of the bounding planes are the *edges,* and the portions of the bounding planes included by the edges are the *faces* of the polyhedron. These faces are, of course, polygons. The intersections of the edges are called the *vertices* of the polyhedron.

A plane passed through a solid cuts from the solid a plane figure. This plane figure, which is the intersection of the plane with the solid, is called a *plane section* of the solid.



PRISM

The solid shown in Figure 226, used very frequently as a container, is called a *prism.* It is defined as a polyhedron two of whose faces are polygons in parallel planes and whose other faces are parallelograms. The faces which lie in the parallel planes, AD and SQ, are its *bases,* and the parallelograms included between the bases, AN, BQ, and so forth, are the *lateral faces.* The intersections of the lateral faces, AM, BN, and so forth, are called the *lateral edges.* Evidently, since, by definition, the lateral faces are parallelograms, these *lateral edges are equal and parallel.* The perpendicular distance, HP, between the planes of its bases is the *altitude* of the prism.

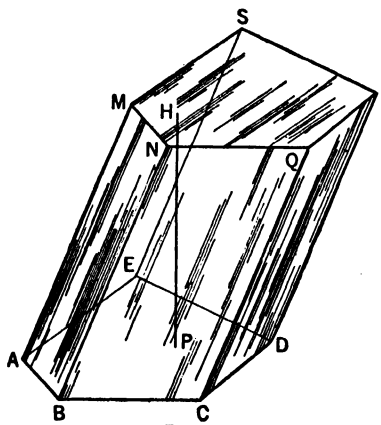


Fig. 226.

Prisms are named according to the number of sides in the polygons of their bases. Thus, a prism is triangular, quadrangular, pentagonal, and so forth, according as its bases are triangles, quadrilaterals, pentagons, and so forth.

A *right section* of a prism is a section made by a plane perpendicular to its lateral edges. Unless otherwise stated, the term *cross section* is used to denote a right section. If the lateral edges of a prism are perpendicular to its bases, the prism is said to be a *right prism.*

Fig. 227.

The log shown in Figure 227 has its two end faces slanting in opposite directions. The log is to be made into a column such that its end faces are parallel and equal polygons. The carpenter makes two parallel cuts at the required distance and concludes that the sections are congruent polygons. His conclusion is based on the following statement:

Fact 123. The sections of a prism cut by two parallel planes intersecting the lateral edges are congruent polygons.

Consider any prism, Figure 228, cut by the parallel planes AC and MP, forming the sections $ABCDE$ and $MNPRS$.

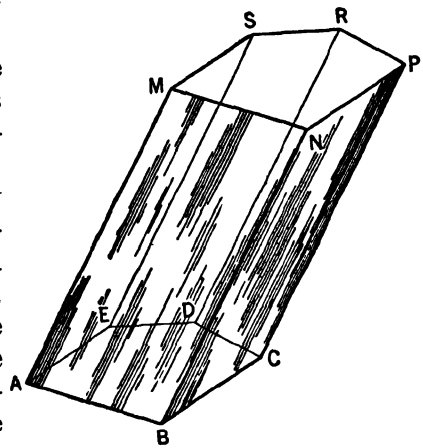
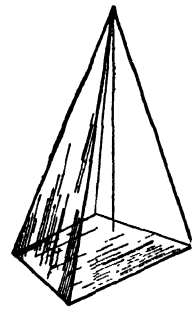


Fig. 228.

The lateral faces are, by definition, page 477, parallelograms. Hence, MN is parallel to AB, NP is parallel to BC, and so forth. Thus, each angle of the polygon forming the upper base has its sides parallel to and running in the same direction as the corresponding angle of the polygon forming the lower base. Therefore, by Fact 119, the angles are equal. That is, angle MNP is equal to angle ABC, angle NPR is equal to angle BCD, and so forth. Moreover, by Fact 36, the opposite sides of the parallelograms forming the lateral faces are equal, so that MN is equal to AB, NP is equal to BC, and so forth. Hence, sections $ABCDE$ and $MNPRS$ have exactly the same size and shape, and therefore are congruent.

PYRAMID

Many famous monuments have made the pyramid a familiar solid. The Pyramids on the Nile, the Monument of Cestius in Rome, and in modern times the regular pyramidal roof of the Washington Monument and many church steeples give evidence of its popularity through the ages.



A *pyramid* is a solid bounded by a polygon of any number of sides and by triangles whose bases are the sides of the polygon and which have a common vertex. The *base* of the pyramid is the face formed by the polygon. The triangles are called the *lateral faces* and the intersections of the lateral faces are the *lateral edges* of the pyramid. The common vertex of the lateral faces is the *vertex* of the pyramid. The perpendicular distance from the vertex to the plane of the base is the *altitude*. Pyramids are named according to the number of sides in the polygon of their bases. Thus, a pyramid is triangular, quadrangular, pentagonal, and so forth, according as its base is a triangle, quadrilateral, pentagon, and so on.

A pyramid whose base is a regular polygon, the center of which coincides with the foot of the perpendicular dropped from the vertex, is said to be a *regular pyramid.* The *slant height* of

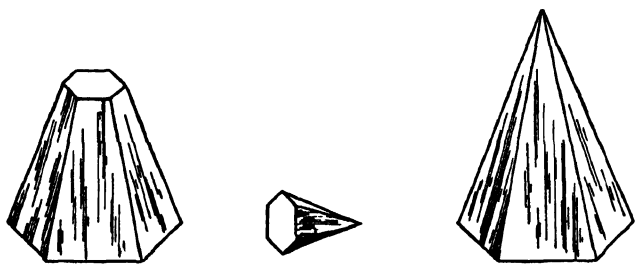


Fig. 229.

a regular pyramid is the altitude of any one of the triangles forming its lateral faces.

A pedestal for a statuette, Figure 229, is to be cut from a piece of wood having the shape of a pyramid. The edges of the pedestal are to be two-thirds of the length of the edges of the original pyramid. To cut the pedestal to its proper di-

mensions, a carpenter measures one of the edges, marks off a point in one of the edges two-thirds of the length from the base, and cuts the pyramid so that the plane of the cut is parallel to the base. He concludes that all the lateral edges of the pedestal are two-thirds of the length of any edge, and that the polygon made by the cut is similar to the base. To justify his reasoning, we must show that:

Fact 124. If a pyramid is cut by a plane parallel to the base, (a) the lateral edges and the altitude are divided proportionally; (b) the section is a polygon similar to the base.

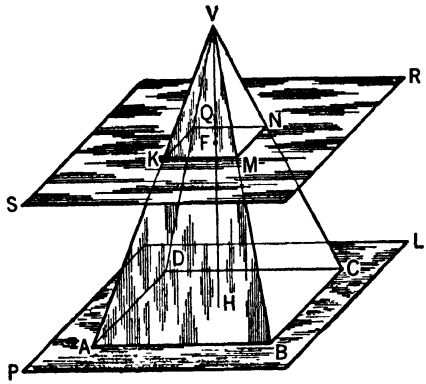


Fig. 230.

Assume that the pyramid V-AC, Figure 230, is cut by a plane SR, parallel to the plane of the base, making the section KN and intersecting the altitude VH at F.

The two parallel planes PL and SR cut the concurrent edges and altitude of the pyramid, so that, by Fact 121, the corresponding segments are proportional. That is,

$$\frac{VK}{VA} = \frac{VM}{VB} = \cdots = \frac{VF}{VH}.$$

In other words, the lateral edges and the altitude of the pyramid are divided proportionally.

Each of the line segments KM, MN, and so forth, divides two sides of the triangles forming the lateral faces proportionally. Hence, by Fact 46, these line segments are respectively parallel to the bases of the triangles. That is, KM is parallel to AB, MN is parallel to BC, and so forth. We then know that angles KMN, MNQ, and so forth, have their sides parallel to and running in the same direction as the sides of angles ABC, BCD, and so forth. Therefore, by Fact 119, angle KMN is equal to angle ABC, angle MNQ is equal to angle BCD, and so forth.

In any one of the lateral faces, say VBC, since MN is parallel to BC, angles VMN and VNM are, by Fact 14, equal, re-

spectively, to angles VBC and VCB. Therefore, by Fact 48, triangle VMN is similar to triangle VBC. Similarly, triangle VNQ is similar to triangle VCD, and so forth. But if two triangles are similar, the corresponding sides are in proportion. Hence, $\frac{MN}{BC} = \frac{VN}{NC}$; $\frac{NQ}{CD} = \frac{VN}{NC}$; and so forth. Since things equal to the same thing are equal to each other, $\frac{MN}{BC} = \frac{NQ}{CD}$; and so forth. That is, the polygon KN, cut by plane SR, and the base AC have the corresponding angles equal and the corresponding sides proportional; therefore, by definition, they are similar.

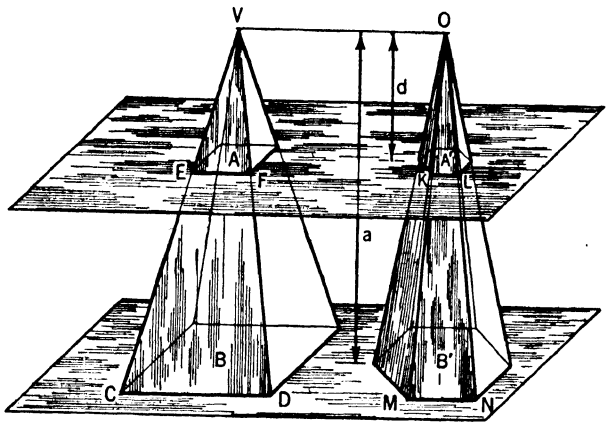


Fig. 231.

The preceding discussion of solid shapes, beside developing the power of visualizing objects in space, has presented the proofs of a few facts necessary for a clear understanding of the formulas to be developed in the next chapter for the areas and volumes of solids. One of the most important of these proofs is the following:

Fact 125. If two pyramids having equal bases and altitudes are cut by a plane parallel to their bases and equidistant from their vertices, the sections formed are equal.

Consider two pyramids, Figure 231, with equal bases B and B' and equal altitudes a. Pass a plane parallel to the bases, cutting the two pyramids at a distance d from their vertices and forming the sections A and A'.

Line segment EF is, by construction, parallel to CD. Hence, by Fact 14, angles VEF and VFE are respectively equal to angles VCD and VDC. Therefore, by Fact 48, triangle VEF is similar to triangle VCD, so that, $\frac{EF}{CD} = \frac{VF}{VD}$. Similarly, $\frac{KL}{MN} = \frac{OL}{ON}$. But, by Fact 124, $\frac{VF}{VD} = \frac{d}{a}$ and $\frac{OL}{ON} = \frac{d}{a}$. Hence, $\frac{VF}{VD} = \frac{OL}{ON}$; and, since things equal to equal things are equal to each other, $\frac{EF}{CD} = \frac{KL}{MN}$. Squaring both sides of this equality gives $\frac{(EF)^2}{(CD)^2} = \frac{(KL)^2}{(MN)^2}$. Again by Fact 124, we have A similar to B, and A' similar to B'. Hence, by Fact 68, $\frac{A}{B} = \frac{(EF)^2}{(CD)^2}$ and $\frac{A'}{B'} = \frac{(KL)^2}{(MN)^2}$. Therefore, $\frac{A}{B} = \frac{A'}{B'}$. But since B is equal to B', we may write $\frac{A}{B} = \frac{A'}{B}$. Therefore, multiplying both sides by B, we obtain $A = A'$.

From this result it may be inferred that:

Fact 126. If a pyramid is cut by two parallel planes, the areas of the sections are proportional to the squares of their distances from the vertex.

Problems

1. An onyx octagonal pyramid, with an altitude of 1 decimeter and a lateral edge 1.054 decimeters in length, is taken to a jeweler, who is asked to make a replica which shall have an altitude of 2.5 centimeters. What is the length of a lateral edge of the replica in centimeters?

2. The Monument of Cestius in Rome is a square pyramid 121.5 feet high with a base edge measuring 98.4 feet.

(*a*) How many square feet are there in the lateral surface of the monument?

(*b*) What is the area of the base of a similar pyramid with an altitude of 40.5 feet?

(*c*) A jeweler desires to make a silver replica of this monument one inch in height. What will be the length of a base edge of the silver replica?

3. A building has six stories, the distance from the roof to the top floor being 9 feet. From the top floor to the next floor beneath is 10 feet, and each succeeding floor downward is 1 foot further in distance from the floor immediately above. A slanting pipe for a sprinkler system is to run from the ground floor to the roof, at an angle of 70° with the floor. Find the length of pipe required between each floor and the next below.

4. The inside dimensions of a trunk are 2 feet, 3 feet, and 5 feet. What is the length of the longest fish pole which will, without bending, go inside this trunk?

5. One part of a quartz crystal is a hexagonal prism with a right section of 10 square centimeters (written 10 cm^2), a lateral edge of 15 centimeters, and a base of 12 cm^2. Find the altitude of the prism.

6. An artificial topaz has the form of a cube 15 millimeters on a side. In planning to cut the topaz into another shape, the jeweler desires to know the length of a diagonal of this cube.

27 VOLUMES— PRISMS AND CYLINDERS

Exchange and traffic between communities existed as far back as the Late Stone Age. At that time, about 4000 B. C., there were neither metal implements nor money. Commerce was carried on by barter. The contents of the jars of Phoenician or Egyptian grain to be exchanged for amber gathered on the shores of the Baltic had to be defined. Even at this early age in his development, man was vitally interested in mensuration.

The first peoples to leave any record of accumulated information were the Babylonians, Assyrians, Egyptians, and Phoenicians, whose genius was practical rather than intellectual and theoretical. The theory of mensuration, however, did not develop until the Greeks subjected such knowledge to rational examination. Formulas developed by the more curious and thoughtful Greeks are today used extensively in engineering, physics, chemistry, and, in fact, almost every commercial project.

The *volume* of a solid is the amount of space included by its bounding surfaces. To measure the volume, a *unit of volume* must be chosen. The fundamental units of our national system are defined in terms of the metric system. The unit of volume in the metric system is the space occupied by the mass of one kilogram of pure water at 4° Centigrade. This unit of volume is known as a *liter*. A liter is approximately equal to one cubic decimeter. In the metric system, measures of volume are simply the cubes of the meter and the cubes of the multiples and subdivisions of the meter. (See Chapter 7, page 92.) That is, one cubic meter is equal to 1,000,000 cubic centimeters.

The amount of matter that a receptacle is able to contain is called its *capacity*. Measures of capacity are commonly used to measure grains or liquids, such as wheat, oil, and so forth.

The metric measures of capacity are given in the following table:

Myrialiter	(Ml)	=	10,000 liters
Kiloliter	(Kl)	=	1,000 "
Hectoliter	(Hl)	=	100 "
Decaliter	(Dl)	=	10 "
Liter	(l)	=	1 liter
Deciliter	(dl)	=	0.1 of a liter
Centiliter	(cl)	=	0.01 " " "
Milliliter	(ml)	=	0.001 " " "

Volume and mass are closely related. In fact, the unit of volume is defined in terms of mass. The *mass* of a body is the amount of matter there is in it, determined by "weighing" the body in a lever balance; that is, by comparing it with some other body whose mass we already know. The *weight* of a body, however, is the measure of the attraction between the body and the earth, determined by "weighing" the body in a spring balance. This attraction varies slightly at different parts of the earth, whereas the amount of matter, or mass, is the same everywhere. The weights of bodies at any one place are directly proportional to their masses; and since the variation in weight of a body with respect to its location on the earth's surface is not very great, the terms which are applied to masses are commonly applied to the corresponding weights also. The *gram*, which is the mass of one cubic centimeter of water at 4° Centigrade, is usually taken as the fundamental unit.

Tonneau or metric ton	(T)	=	1,000,000 grams	=	1,000 Kilograms
Quintal	(Q)	=	100,000 "	=	100 "
Myriagram	(Mg)	=	10,000 "	=	10 "
Kilogram	(Kg)	=	1,000 "	=	1 Kilogram
Hectogram	(Hg)	=	100 "	=	0.1 of a Kilogram
Decagram	(Dg)	=	10 "	=	0.01 " " "
Gram	(g)	=	1 gram	=	0.001 " " "
Decigram	(dg)	=	0.1 of a gram		
Centigram	(cg)	=	0.01 " " "		
Milligram	(mg)	=	0.001 " " "		

Since the units of our national system are defined in terms of the metric system, tables of equivalent cubic measures are also inserted here for reference.

AMERICAN WEIGHTS AND MEASURES

CUBIC MEASURE

1,728 cubic inches = 1 cu. foot
27 cubic feet = 1 cu. yard
7,056 cubic inches = 1 barrel (fruit, vegetables)
5,826 cubic inches = 1 barrel (cranberries)

DRY MEASURE

2 pints = 1 quart
8 quarts = 1 peck
4 pecks = 1 bushel

LIQUID MEASURE

4 gills = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.)
31½ gallons = 1 barrel (bbl.)

AVOIRDUPOIS WEIGHT

437.5 grains = 1 ounce
16 ounces = 1 pound
100 pounds = 1 short hundred-wt.
2,000 pounds = 1 short ton
2,240 pounds = 1 long ton

WOOD MEASURE

16 cubic feet = 1 cord foot
128 cubic feet = 1 cord
24¾ cubic feet = 1 perch of stone or of masonry

TROY WEIGHT

24 grains = 1 pennyweight
480 grains = 1 ounce
12 ounces = 1 pound

APOTHECARIES' WEIGHT

20 grains = 1 scruple
3 scruples = 1 apoth. dram
8 apoth. drams = 1 ounce

Note. The grain and ounce are identical in Troy and apothecaries' weight.

TABLES OF CUBIC MEASURE

AMERICAN EQUIVALENTS OF METRIC SYSTEM

1 cu. millimeter = 0.000061 of a cu. inch
1 cu. centimeter = 0.061 of a cu. inch
1 cu. decimeter = 61.0254 cu. inches or 0.0353 of a cu. foot
1 cu. meter or 1 kiloliter or 1 stere = 35.315 cu. ft. or 1.308 cu. yards

METRIC EQUIVALENTS OF AMERICAN SYSTEM

1 cu. inch = 16.3872 cu. centimeters
1728 cu. inches or 1 cu. ft. = 28.317 cu. decimeters or 0.02831 of a cu. meter
27 cu. ft. or 1 cu. yard = 764.55 cu. decimeters or 0.76455 of a cu. meter or kiloliter
1 barrel (fruit, vegetables) = 0.1156 cu. meter
1 barrel (cranberries) = 0.0955 cu. meter

TABLES OF LIQUID MEASURE

AMERICAN EQUIVALENTS OF METRIC SYSTEM

1 milliliter	=	0.00845	of a gill
1 centiliter	=	0.0845	of a gill
1 deciliter	=	0.8452	of a gill
1 liter	=	1.0567	liq. quarts
1 decaliter	=	10.567	liq. quarts
		or	
		2.6417	gallons
1 hectoliter	=	26.417	gallons
1 kiloliter	=	264.17	gallons

METRIC EQUIVALENTS OF AMERICAN SYSTEM

1	gill	= 118.292	milliliters
4	gills		
	or	= 0.473	of a liter
1	pint		
2	pints		
	or	= 0.9463	of a liter
1	quart		
4	quarts		
	or	= 3.7853	liters
1	gallon	(231	cu. inches)
31.5	gallons		
	or	= 119.23	liters
1	barrel		
2	barrels		
	or	= 238.46	liters
1	hogshead		

Note. 1 gallon of water (U. S.) weighs 8.342 lbs. (avoirdupois).

TABLES OF DRY MEASURE

AMERICAN EQUIVALENTS OF METRIC SYSTEM

1 milliliter	=	0.001816	dry pint
1 centiliter	=	0.01816	dry pint
1 deciliter	=	0.18162	dry pint
1 liter	=	1.8162	dry pints
		or	
		0.9081	dry quart
1 decaliter	=	9.081	dry quarts
		or	
		0.28378	bushel
1 hectoliter	=	2.8378	bushels
1 kiloliter	=	28.378	bushels

METRIC EQUIVALENTS OF AMERICAN SYSTEM

1	dry pint	= 0.5506	of a liter
2	pints		
	or	= 1.1012	liters
1	quart		
8	quarts		
	or	= 8.8096	liters
1	peck		
4	pecks		
	or	= 35.2383	liters
1	bushel		

Note. In the U. S. A., a *heaped bushel*, used for measuring apples, corn in the ear, and so forth, equals 2747.71 cu. inches = 45.03 liters.

A *stricken bushel*, used in measuring grain, equals 2150.42 cu. in. = 35.2383 liters.

A dry quart = 67.2 cu. in. = 1.1012 liters.

A liquid quart = 57.75 cu. in. = 0.946 liter.

TABLES OF AVOIRDUPOIS WEIGHT

AMERICAN EQUIVALENTS OF METRIC SYSTEM

1 milligram	=	0.01543	of a grain
1 centigram	=	0.15432	of a grain
1 decigram	=	1.5432	grains
1 gram	=	15.4324	grains or
		0.03527	of an ounce (avoir.) or
		0.5644	of a dram (avoir.)
1 decagram	=	5.644	drams (avoir.) or
		0.3527	of an ounce (avoir.)
1 hectogram	=	3.527	ounces (avoirdupois)
1 kilogram	=	35.27	ounces (avoirdupois) or
		2.2046	lbs. (avoirdupois)
1,000 kilograms or 1 metric ton	=	2,204.62	lbs. (avoirdupois) or
		1.1023	short tons or
		0.9842	of a long ton

METRIC EQUIVALENTS OF AMERICAN SYSTEM

1 grain	=	64.7989	milligrams
27.34375 grains or 1 dram	=	1.771845	grams
437.5 grains or 16 drams or 1 avoirdupois ounce	=	28.3495	grams
16 ounces av. or 1 avoirdupois pound	=	0.45359	of a kilogram
2,000 avoirdupois pounds or 1 short ton	=	907.1848	kilograms
2,240 pounds or 1 long ton	=	1,016.047	kilograms or
		1.016	metric tons

Note. A cubic foot of water contains 7.4825 gallons and weighs 62.427 lbs. avoirdupois = 28.317 kg.

To determine the contents of an open vessel, it is possible to fill the vessel with water and then dip out the water by means of a container whose capacity is known, such as a gallon can. If the vessel does not contain an exact number of gallons, we can compute its volume to any degree of accuracy desired by merely choosing a unit small enough. The volume would then be a different number, for the unit of measure is different. Evidently, the smaller the unit, the greater the accuracy obtainable. Even if the vessel has no common unit of measure, that is, if its volume is represented by an irrational number, we can, as shown in the case of an area (Chapter 16, page 255), approximate its volume to any desired degree of accuracy.

TABLES OF TROY WEIGHT

AMERICAN EQUIVALENTS OF METRIC SYSTEM

1 milligram	=	0.01543 of a grain
1 centigram	=	0.15432 of a grain
1 decigram	=	1.54323 grains
1 gram	=	15.43235 grains
1 decagram	=	6.43015 pennyweight or
		0.3215 of a Troy ounce
1 hectogram	=	64.3015 pennyweight or
		3.215 Troy ounces
1 kilogram	=	32.1507 Troy ounces or
		2.6792 Troy pounds

METRIC EQUIVALENTS OF AMERICAN SYSTEM

1 grain	=	64.7989 milligrams
24 grains or 1 pennyweight	=	1.555 grams
20 pennyweight or 1 ounce Troy	=	31.10348 grams
12 ounces Troy or 1 lb. Troy	=	373.24177 grams

Note. In weighing diamonds, pearls, or jewels, the unit is the *carat* (of 1877) = 3.168 grains. The metric carat = 3.08647 grains, or 200 milligrams.

The term *carat* also expresses the number of parts in 24 that are pure gold. Thus, 14-carat gold would be $\frac{14}{24}$ pure gold and $\frac{10}{24}$ alloy.

TABLES OF APOTHECARIES' WEIGHT

AMERICAN EQUIVALENTS OF METRIC SYSTEM

1 milligram	=	0.01543 of a grain
1 centigram	=	0.15432 of a grain
1 decigram	=	1.54323 grains
1 gram	=	15.43235 grains or
		0.03215 of a Troy ounce
1 decagram	=	154.32 grains or
		0.3215 of a Troy ounce

METRIC EQUIVALENTS OF AMERICAN SYSTEM

1 grain	=	64.7989 milligrams
20 grains or 1 scruple	=	1.2959 grams
3 scruples or 1 dram	=	3.8879 grams
480 grains or 8 drams or 1 ounce	=	31.10348 grams

Note. The ounce and pound are identical in apothecaries' and Troy weight.

TABLES OF COMPARISON OF AMERICAN SYSTEMS OF WEIGHT

Avoirdupois		Troy	
1 grain	=	1	grain
437.5 grains or 1 ounce av.	=	0.9115	of a Troy ounce
7,000 grains or 16 ounces av. or 1 lb. avoir.	=	14.5833	Troy ounces or
		1.21528	Troy lbs.
2,000 lbs. av. or 1 short ton	=	2,430.56	Troy lbs. or
		907.1848	kilograms
2,240 lbs. av. or 1 long ton	=	2,722.22	Troy lbs. or
		1,016.047	kilograms

Troy		Avoirdupois	
1 grain	=	1	grain
480 grains or 1 ounce	=	1.09712	ounces avoir.
5,760 grains or 12 ounces Troy or 1 lb. Troy	=	13.1657	ounces avoir. or
		0.822857	of a lb. avoir.

Note.				
	1 long ton (avoirdupois)	=	1.12	short tons (avoirdupois)
	1 short ton (avoirdupois)	=	0.89287	of a long ton (avoirdupois)
	1,000 kilograms or 1 metric ton	=	2,204.62	lbs. avoir.
			2,679.23	lbs. Troy
			0.9842	long ton
			1.1023	short tons

Therefore, *we define the volume of a solid as the number of times it contains a given solid arbitrarily chosen as the unit of volume.*

PRISMS

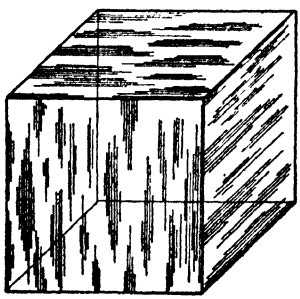


Fig. 232.

The unit of volume is a prism all of whose faces are squares. This type of prism, shown in Figure 232, is a *cube*. A cube each side of which is one square unit, that is, a cube 1 unit wide, by 1 unit long, by 1 unit high, is called a *cubic unit*. Thus, if the unit of length is 1 inch, then the unit of volume is a cube 1 inch wide, by 1 inch long, by 1 inch high, or a cubic inch.

Now consider a prism, Figure 233, all of whose faces are rectangles, that is, a *rectangular prism*. The small cube at the right represents the unit of volume, and the rectangular prism is 3 units wide, by 5 units long, by 4 units high. Obviously, it takes 3×5

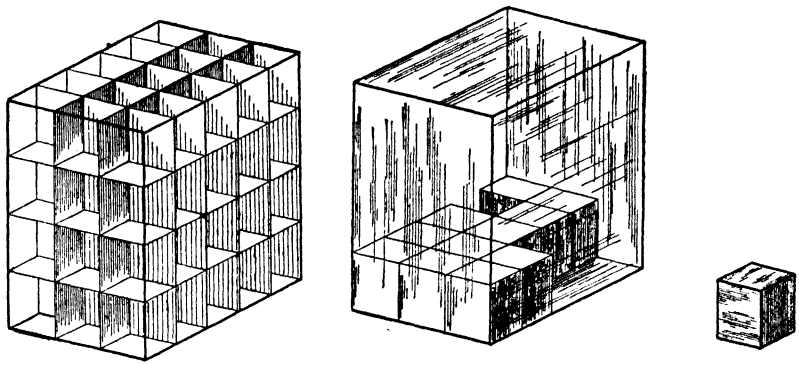


Fig. 233.

or 15 small cubes to cover the base with one layer. Since there are 4 layers of 15 cubes each, the rectangular prism contains 15×4 or 60 cubic units. If the unit chosen had not been contained an exact number of times in the rectangular prism, then, as already explained, the volume of the prism could, nevertheless, be obtained to any degree of accuracy. Therefore, *we assume that the volume of a rectangular solid is the product of its three dimensions.*

Since, by Fact 52, the product of the width by the length is the area of the base of the rectangular solid, we may restate the preceding assumption as follows:

Fact 127. The volume of a rectangular solid is the product of its base and altitude.

Since a cube is a rectangular prism all of whose edges are equal, it follows that:

Fact 128. The volume of a cube is equal to the cube of one edge.

The development of the necessary formulas for the measurement of the volumes of various solids is based on a principle stated by Bonaventura Cavalieri (1598–1647). To get a clear conception of Cavalieri's theorem, suppose that we have a number of cards in the shape of rectangles and all of exactly the same size and thickness. The cards may be divided into

three equal piles in the sense that each pile contains the same number of cards. If the first pile is arranged so that each card is directly above the other, the cards assume the shape of a rectangular solid, shown in Figure 234 (*a*). The other two piles may be arranged in the shapes shown in Figure 234 (*b*) and (*c*),

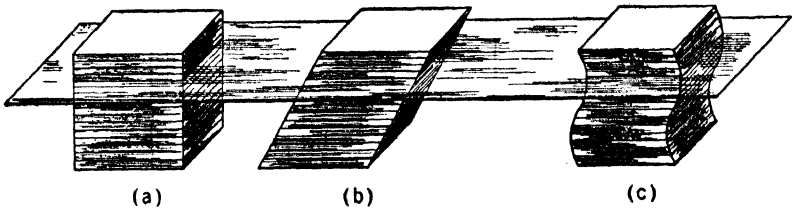


Fig. 234.

or the shape of some other irregular solid. It is easily seen that, since each pile contains the same number of cards, each solid must have the same height and volume. Moreover, if a plane is passed parallel to and at the same distance from their respective bases, the sections made by this plane are always equal, for the cards are, of course, all rectangles of exactly the same size.

Now assume that we have a pile of cards in the form of a

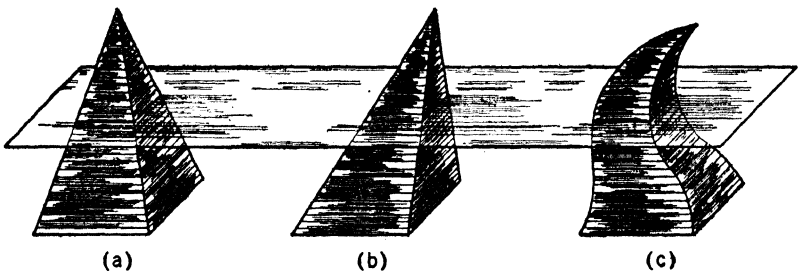
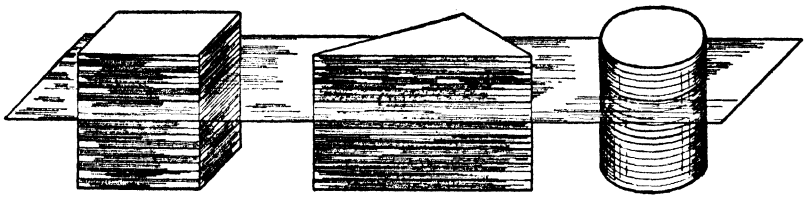


Fig. 235.

pyramid, as shown in Figure 235 (*a*). The shape of the pyramid may be distorted, as shown in (*b*) and (*c*), without affecting the volume of the solid. As before, each solid will have the same height, and the sections made by a plane passed parallel to and at the same distance from their bases will be always equal.

Furthermore, the cards need not be of the same shape. For example, consider a rectangular, a circular, and a triangular pile, each of the same height. Suppose that all cards, regard-

less of their shape, have exactly the same thickness and the same surface area. Thus, if the rectangular card is 14″ by 11″, the triangular card may have a base 28″ long and an altitude of 11″, and the circular card may have a radius of $\sqrt{\frac{154}{\pi}} = 7''$,



approximately. Each one of these cards will have the same volume. Hence, a pile of the rectangular cards will have the same volume as a pile of an equal number of circular or triangular cards. In general,

Fact 129 (Cavalieri's theorem). If in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, the volumes of the solids are equal.

A rigorous proof of this theorem is beyond the scope of this book. However, the underlying idea may be easily grasped. Consider two solids, Figure 236 (*a*), whose bases lie in the same plane and which have the same altitude. Denote the volumes of these solids by P and Q. Suppose that this time we select the cards exactly equal in thickness and of the proper size so that when placed inside of the solids, as shown in the figure, they fit the contours of the solids as closely as possible. These cards will then form two solids whose volumes would be approximately equal to the volumes of the given solids. The difference between the volumes of the given solids and the volumes of the solids formed by the cards would consist of the empty spaces between the cards and the solids. These empty spaces are represented, in Figure 236 (*a*), by the parts shaded very dark.

Assume that the bases of the given solids have the same area and suppose that, if a plane is passed parallel to the bases, the cards cut off, one from each solid, are equal in area. That is,

if the cards in each solid were numbered, starting from the bases, the first card in P would have the same area as the first card in Q, the second card in P would have the same area as

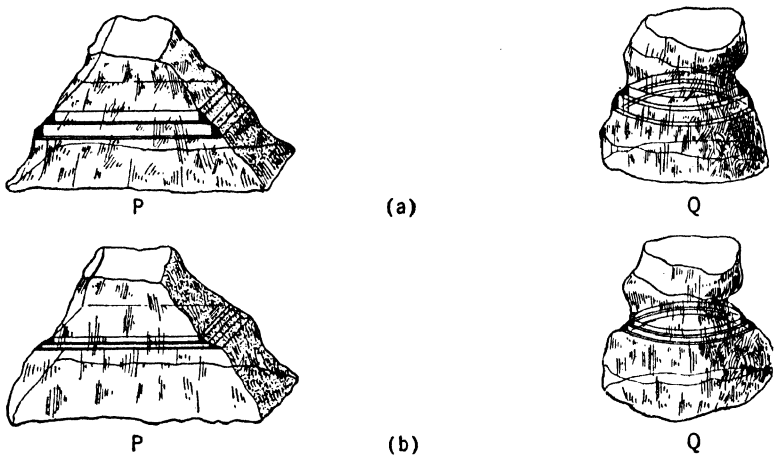


Fig. 236.

the second card in Q, and so forth. Each one of these cards, having equal surface areas and the same thickness, would have the same volume, that is, card one in P is equal in volume to card one in Q, and so forth. Since the solids are of the same height, and the cards are all of exactly the same thickness, each solid will contain the same number of cards. Therefore, if P' and Q', respectively, denote the volumes of the solids formed by the cards, $P' = Q'$.

Now suppose that the cards are chosen in exactly the same manner but half as thick as before, as shown in Figure 236 (*b*). It is easily seen that the empty spaces, shown by the part shaded very dark in the figure, will be collectively much smaller, and therefore the volumes of the solids formed by this new set of cards will be much closer to the volumes of the given solids. Theoretically, we can continue indefinitely this process of choosing cards half as thick as the last set of cards used; and evidently, the thinner the cards, the nearer we approach the volumes of the given solids. That is, we can make the difference between the volumes of the solids formed by the cards and the volumes of the given solids as small as we please. In other words, P' approaches P as its limit and Q' approaches Q

as its limit. (See Chapter 22, page 358.) But, since $P' = Q'$, we conclude that $P = Q$.*

As a deduction from Cavalieri's theorem, the reader can easily show, by applying Fact 123, that:

Fact 130. Two prisms having equal bases and equal altitudes are equal in volume.

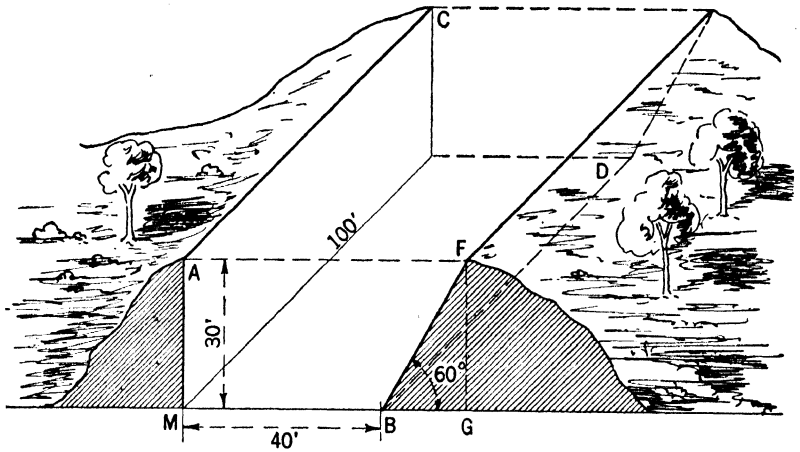


Fig. 237.

A railway cut, Figure 237, 30 feet deep, is to be made with one side vertical and the other inclined 60° to the horizontal. The bottom is to be 40 feet wide. To ascertain the number of cubic feet of earth that must be removed for a cut 100 feet long, we must first show that:

Fact 131. The volume of any prism is equal to the product of its base and altitude.

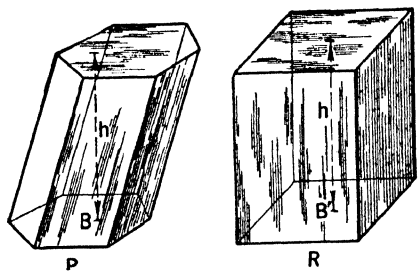


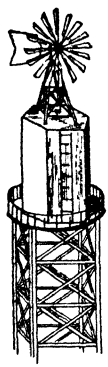
Fig. 238.

Consider any prism P, Figure 238, with base B and altitude h. Erect a rectangular solid R, having the same altitude h and such that its base lies in the same plane and is equal in area

* This conclusion is based on the fact that "If two variables are always equal and each approaches a limit, the limits are equal." This fact is proved in advanced works on mathematics.

to B. Since, by Fact 130, two prisms having equal bases and altitudes are equal in volume, the volume of the given prism P is equal to the volume of the rectangular solid, R. But, by Fact 127, the volume of the rectangular solid is equal to $B' \cdot h$. Therefore, the volume of any prism P is equal to $B \cdot h$.

To ascertain the number of cubic feet of earth which must be removed for the railway cut, it is only necessary to notice that the cut has the shape of a prism whose bases, AB and CD, Figure 237, are trapezoids. Therefore, the volume of the cut is equal to the product of the area of the trapezoid $AMBF$ by the altitude BD. Since, by Fact 62, the area of a trapezoid is equal to one-half the product of its altitude and the sum of its bases, to determine the area of $AMBF$, we must find the width of the cut at the top. From F drop a perpendicular FG to the horizontal line. By construction, $AMGF$ is a rectangle; hence, by Fact 36, $AM = FG = 30'$, and $AF = MG = 40' + BG$. The length of BG can be easily found by using the cotangent ratio. In right triangle BFG, $\cot FBG = \frac{BG}{FG}$, or $\cot 60° = \frac{BG}{30}$, so that $30 \times \cot 60° = BG$. Hence, $BG = 30 \times 0.5774 = 17.32$ feet. Therefore, the width of the cut at the top, that is, AF, is equal to $40 + 17.32 = 57.32$ feet. The area of the trapezoid $AMBF$ is then equal to $\frac{1}{2}(30)(40 + 57.32) = 1459.80$ square feet. The number of cubic feet of earth to be removed is thus equal to $1459.80 \times 100 = 145{,}980$ cubic feet.



A water tank, 10 feet high, is to have the shape of a right prism whose base is a regular hexagon with sides 4 feet long. The materials for the base cost 25 cents per square foot; for the sides, 10 cents per square foot; and for the top, 15 cents per square foot. To ascertain the cost of the materials for the tank, it is convenient to show that:

Fact 132. The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.

Consider any prism, A-D', Figure 239, and denote the length of a lateral edge by l. Pass a plane perpendicular to the lateral edges of the prism, forming the right section $NRSPMQ$.

Then AA', perpendicular to plane NP, is perpendicular to RS; for a line is perpendicular to a plane if it is perpendicular to every line in the plane passing through its foot (Chapter 26, page 473). Similarly, BB' is perpendicular to SP, CC' is perpendicular to PM, and so forth. The lateral faces of the prism are, by definition (Chapter 26, page 477), parallelograms, each having one of the lines RS, SP, PM, and so forth, as its altitude and a lateral edge as its base. Hence, by Fact 54, area of $AB' = AA' \cdot RS$; area of $BC' = BB' \cdot SP$, $\cdots$; area of $FA' = FF' \cdot NR$. Thus, if we denote the lateral area of the prism by S, we may write $S = AA' \cdot RS + BB' \cdot SP + \cdots + FF' \cdot NR$. But the lateral edges of a prism are equal (Chapter 26, page 477). Hence, substituting the length of a lateral edge, namely, l, in the preceding equation gives $S = l \cdot RS + l \cdot SP + \cdots + l \cdot NR$, or $S = l(RS + SP + PM + MQ + QN + NR)$. The sum $RS + SP + PM + MQ + QN + NR$ is the perimeter of the right section. Therefore, denoting the perimeter of the right section by p, we obtain $S = l \cdot p$.

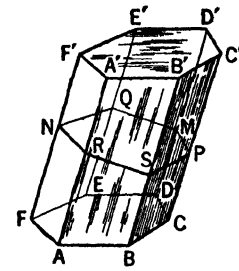


Fig. 239.

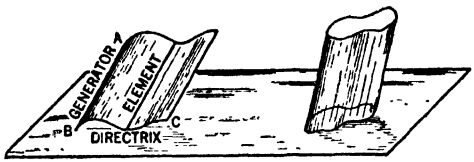
The cost of the water tank may now be ascertained as follows. Since the tank is to have the shape of a right prism, then, by definition, the base is a right section and the length of a lateral edge is equal to the altitude, that is, 10 feet. The base is a regular hexagon with sides 4 feet long, so that its perimeter is $4 \times 6 = 24$ feet. Hence, the lateral area is, by Fact 132, $24 \times 10 = 240$ square feet. The cost of the sides is, then, $240 \times 0.10 = \$24$.

To determine the cost of the base and top, it is necessary to find the area of a regular hexagon forming the bases of the prism. The area of a regular polygon is, by Fact 107, equal to one-half the product of its perimeter and its apothem. To ascertain the length of the apothem, draw, as shown in Figure 155, any two radii of the regular hexagon to consecutive vertices, forming an equilateral triangle each side of which is 4 feet long. Hence, a line CD bisecting the vertex angle ACB is, by Fact 4, perpendicular to the base and bisects the base. That is, triangle ACD is a right triangle with its hypotenuse $AC = 4$ feet and $AD = 2$ feet. Applying Fact 70, we obtain $(AC)^2 =$

$(AD)^2 + (CD)^2$, or $16 = 4 + (CD)^2$; hence $12 = (CD)^2$, and $CD = \sqrt{12} = 3.46$ feet. The area of the regular hexagon forming the bases of the prism is, therefore, $\frac{1}{2}(6 \times 4)(3.46) = 41.52$ square feet. The cost of the base is $41.52 \times 0.25 = \$10.38$, and the cost of the top is $41.52 \times 0.15 = \$6.23$. Therefore, the cost of the material for the water tank is $\$24 + \$10.38 + \$6.23 = \$40.61$ (*Answer*).

CYLINDERS

The production of merchandise of standard quality at the lowest price possible is one of the fundamental principles of sound business practice. Obviously, it is "good business" to sell a product in a container that will require the least amount of material to contain a given quantity. A cylindrical can requires less material for its construction than a prism of the same volume. Hence, fruits, vegetables, and many liquid products are sold in cylindrical containers.



(*a*) Fig. 240. (*b*)

Some curved surfaces may be defined as generated by a straight line which is thought of as moving and changing its direction continuously. Suppose that line AB, Figure 240 (*a*), cuts a fixed plane curve BC not in the same plane with the straight line. If AB is allowed to move so that it always remains parallel to its first position and always intersects the fixed curve BC, it will generate a curved surface called a *cylindrical surface*. The moving line is called the *generator*. An *element* of a cylindrical surface is the generator in any particular position. The fixed curve is called the *directrix* of the surface. The directrix will be assumed to be a closed curve, Figure 240 (*b*), so that the surface generated by the moving straight line will be always considered, in this book, as a closed cylindrical surface. We then *define a cylinder as a solid bounded by a cylindrical surface and two parallel planes cutting the elements of the surface.*

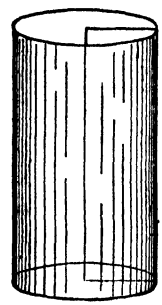


Fig. 241.

The notion of such terms as *base, altitude, right section,* and *right* and *oblique* in connection with cylinders corresponds to the definition of similar terms in a prism. Thus, a cylinder whose bases are perpendicular to its elements is called a *right cylinder.* A cylinder with a circular base is a *circular cylinder.* A right circular cylinder may be thought of as generated by a rectangle rotated about one of its sides; hence, a right circular cylinder is called a *cylinder of revolution.* (Figure 241)

As in the case of a prism,

Fact 133. The sections of a cylinder made by two parallel planes cutting all elements are congruent.

Consider any cylinder, Figure 242. Let A, B, and C be any three points in the intersection of the cylinder by PL, and AD, BF, and CM be elements of the surface. Draw AB, BC, CA, DF, FM, and MD. Then, by the definition of a cylinder, AD, BF, and CM are equal and parallel. Therefore, by Fact 40, the quadrilaterals $ABFD$, $BCMF$, and $ACMD$ are parallelograms. Hence, by Fact 36, $AB = DF$, $CB = MF$, and $CA = MD$. That is, triangle ABC has its three sides respectively equal to the sides of triangle DFM. Therefore, by Fact 6, the triangles are congruent. Now, if the lower section is placed on the upper so that triangle ABC falls on triangle DFM, points A, B, and C will coincide with D, F, and M. Hence, all points on the lower section coincide with points on the upper section, and therefore the sections are congruent.

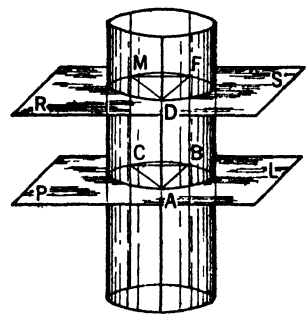
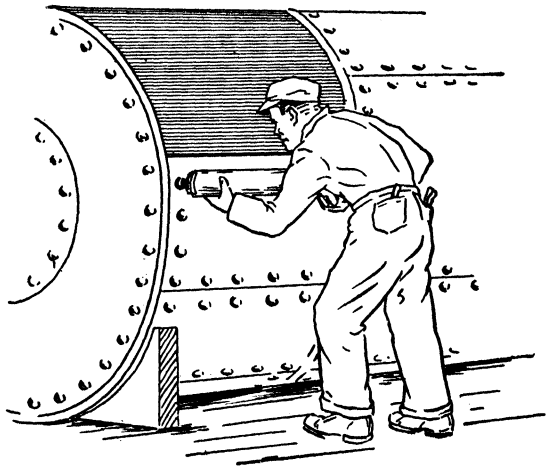


Fig. 242.

Suppose that a boiler maker is to construct a vertical steam boiler with capacity of 396 cubic feet and such that the diameter of the base is 6 feet. In figuring the cost of the material necessary, two questions immediately arise:

(1) What must be the height of the boiler?
(2) How much material is required to build it?

To answer the first question, it is necessary to find a formula for the volume of a cylinder. Consider any circular cylinder, Figure 243, with base B and altitude h. Erect a prism having the same altitude h and such that its base P lies in the same



plane and is equal in area to B. This construction is easily effected by using the formulas for the areas of a circle and a rectangle developed in Chapter 22, Fact 110, and Chapter 16, Fact 52. Pass any plane parallel to the lower bases of the solids, cutting the cylinder in section A and the prism in section S. Then, by Fact 133, $A = B$, and by Fact 123, $S = P$. But by construction, $B = P$. Therefore, $A = S$. That is, the sections made by planes parallel to and at the same distance from the respective bases of the cylinder and prism are always equal. Therefore, since the altitudes of the two solids are also equal, then, by Cavalieri's theorem, Fact 129, the volumes of the solids are equal. But the volume of the prism is, by Fact 131, equal to $P \cdot h$. Therefore, since $B = P$, the volume of the cylinder is $B \cdot h$. In other words,

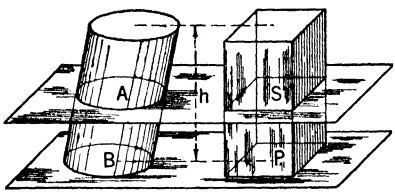


Fig. 243.

*Fact 134. The volume of a circular cylinder is equal to the product of its base and altitude.**

The area of a circle is, by Fact 110, equal to πr^2. Hence, if V and h, respectively, denote the volume and altitude of a right circular cylinder, we may write $V = \pi r^2 h$.

* This fact can be demonstrated, by the theory of limits, to be true for *any* cylinder.

By substituting known values in this formula, the height of the steam boiler may now be found. Thus, the diameter of the base is 6 feet, hence its radius is 3 feet; furthermore, $V = 396$ and $\pi = \frac{22}{7}$. Therefore, $396 = \frac{22}{7}(3)^2h$, so that $396 = \frac{198}{7} \cdot h$, and $\dfrac{396 \times 7}{198} = h$ or $h = 14$ feet.

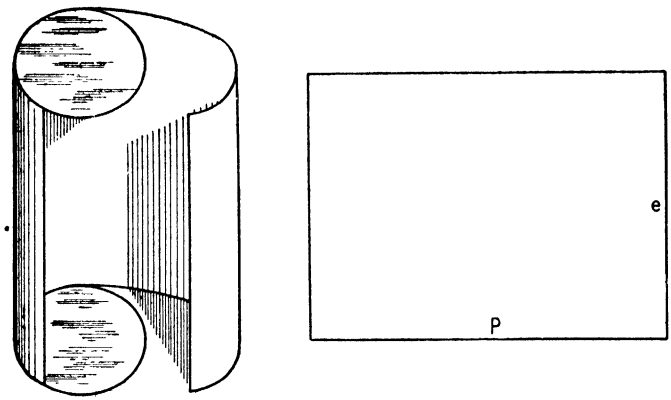


Fig. 244.

To compute the amount of material required for the construction of the steam boiler, it is essential to develop a formula for the lateral area of a cylinder. A right circular cylinder may be constructed by taking two equal circular pieces of cardboard, and then, as shown in Figure 244, pasting a heavy piece of paper, of the proper size, about the circular pieces forming the bases. The piece of paper is then the lateral surface of the solid. It is easily seen that, if the paper is unwrapped and spread out flat, it becomes, as shown in the figure, a rectangle whose width e is equal to an element of the cylinder and whose length p is the perimeter of a base of the cylinder. But the area of a rectangle is, by Fact 52, equal to the product of its base and altitude, that is, $e \cdot p$. Hence, the lateral area of the right circular cylinder is also equal to $e \cdot p$. In other words,

*Fact 135. The lateral area of a right circular cylinder is equal to the product of the perimeter of its base and an element.**

The perimeter of a circle, that is, its circumference, is, as shown in Chapter 22, page 362, equal to $2\pi r$; and an element of

* By the theory of limits it can be shown that the lateral area of *any* cylinder is equal to the product of the perimeter of a right section and an element.

a right circular cylinder is, by definition, equal to its altitude. Therefore, if S and h, respectively, denote the lateral area and altitude of a right circular cylinder, we may write $S = 2\pi rh$.

The total amount of material required for the construction of the steam boiler may now be ascertained as follows. The lateral area of the steam boiler is equal to $2\pi rh = 2\pi(3)(14) = 264$ square feet. The area of a base is $\pi r^2 = \pi(3)^2 = 28.28$ square feet. Therefore, the total area is $264 + 2(28.28) = 320.56$ square feet.

Before going on with the discussion of other solids in the following chapter, it is well to pause here and give the reader an opportunity to reinforce his recollection of the ideas just set forth by solving the following problems.

Problems

1. A cistern holds 1 ton of water. How many gallons can it contain?

2. What is the weight of a cube of ice 4 feet on a side, if ice weighs 92 per cent as much as water?

3. How much material was used in manufacturing 1,000 gross of solid celluloid dice if each die has an edge of 0.25 inch?

4. Find the volume and total surface area of the largest cube of marble that can be cut from a circular cylinder whose diameter is 14 inches and whose height is 15 inches.

5. A cylindrical well is dug, 40 feet deep and 7 feet in diameter. What will be the cost of lining it with stone 1 foot in thickness, if the stone costs, including labor, \$10 per cubic yard?

6. How many gallons of water will be contained in a cylindrical tank 10 feet high, having a diameter of 12 feet? What is the weight of this water?

7. How many square feet of tin would be needed to line the above cylinder?

8. A piston chamber 10 inches long has a volume of 282.74 cubic inches. What is the diameter of the chamber?

9. As a result of landslides, a railway found it necessary to reduce the angle of inclination of one bank of a railway cut from an original angle of 45° to a new angle of 30°. The bank, as it was originally, was 300 feet long and had a slant height of 60 feet. It is planned to surface the area of the new bank with paving stones. (*a*) What

is the area? (*b*) How much earth was removed from the original bank of the cut?

10. Three rifles are stacked during bivouac with their butts forming the vertices of an equilateral triangle whose vertices are three feet apart. Assuming that the rifles are four feet in length, and that the muzzles extend three inches beyond the vertex of the pyramid formed, what is the height of each muzzle from the ground?

11. A silversmith is designing a pedestal in the form of a prism, the perimeter of whose base is to be 14 inches. The edge of the prism is to be 4 inches. Compute the lateral area.

12. Another pedestal, in the form of a regular octagonal pyramid has an altitude of 15 feet and a lateral edge of 17 feet. Find the length of an edge of the base.

13. If a square Egyptian pyramid measures 138 meters along one side of its base and has a slant height of 120 meters,

(*a*) What is the altitude of the pyramid?

(*b*) What is the lateral area of the pyramid?

(*c*) If, at a point on each edge 20 meters from the vertex, lines were drawn parallel to the bases and all material above them were removed, what would be the area of the top of the remaining monument?

14. A church spire, pyramidal in form, and with its base a regular hexagon, each side 9 feet long, is to be painted. The slant height of the spire is 70 feet. If paint costs at the rate of 4 cents per square foot of surface, what will be the total cost of the paint required?

15. Inside the above spire there are two flat platforms at distances from the vertex of 18 and 30 feet, respectively. What are the areas of these platforms?

28 VOLUMES, CONTINUED—PYRAMIDS, CONES, AND SPHERES

The famous pyramids of Egypt are by far the largest and most impressive monuments left by ancient peoples. Laid out with strict regard to the cardinal points of the compass, more than 3,000 years before Christ, they served as the tombs of kings. The skill shown in their construction implies a remarkable knowledge of mathematics and astronomy for that period. The Egyptian pyramids are square at the base. The largest, believed to have been built by King Cheops, covers 13 acres and stands 481 feet high.

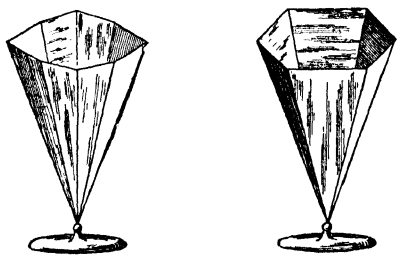


Fig. 245.

The following practical problem will show how the computation of the volume of a pyramid can be applied to commercial affairs. A silversmith is to construct a set of drinking cups in the form of an inverted octagonal pyramid, Figure 245. The cups are to have the same volume and height as a drinking cup furnished as a model, which has the form of an inverted hexagonal pyramid. To obtain accuracy, the silversmith computes the area of the hexagon forming the brim of the model drinking cup and then determines the size of an octagon having the same area. He concludes that the two cups will have the same volume. The silversmith's conclusion is based on:

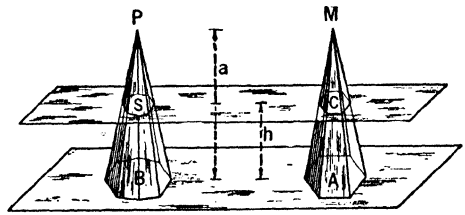


Fig. 246.

Fact 136. Two pyramids having equal bases and equal altitudes are equal in volume.

Assume that any two pyramids, P and M, Figure 246, have the same altitude a and bases of equal area, B and A, resting on the same plane. Pass a plane parallel to and at any distance h from their bases, cutting the pyramids in sections S and C. Then, by Fact 125, sections S and C are equal in area, and therefore, by Cavalieri's theorem (Fact 129), the volumes of the two pyramids are equal.

The gateposts shown in Figure 247 consist of a rectangular prism, made of limestone, and a capstone in the form of a regular square pyramid made of marble. The base of the pyramid forming the capstone measures 15 inches on each side, and the altitude is 10 inches. The cost of shipping the capstones depends on their weight. Marble weighs, on the average, 168 pounds per cubic foot. Therefore, in ascertaining the freight charges, we must consider the volume of this pyramid.

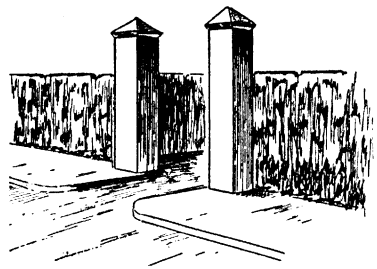


Fig. 247.

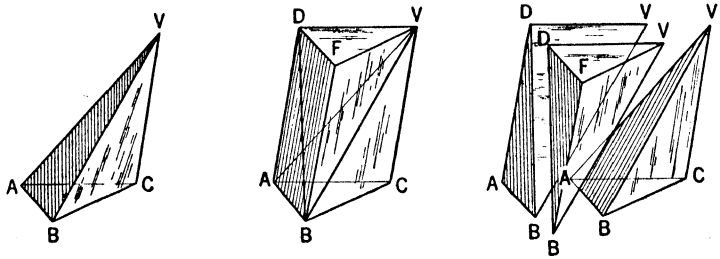


Fig. 248.

Consider any triangular pyramid, V-ABC, Figure 248 (a), with altitude h. Construct the prism shown in Figure 248 (b) as follows. Through edge AB pass a plane parallel to edge VC. Pass another plane through the vertex V parallel to the base ABC, and let AD, BF, DV, and FV be the intersections of these planes with the planes of the lateral faces VBC and VAC. This prism has the same base ABC and altitude h as the given pyramid. Pass the plane VDB dividing the quadrangular pyramid V-$FDAB$ into the two triangular pyramids V-DFB and V-DAB, as shown in Figure 248 (c).

By definition, Chapter 26, page 477, the lateral face *ABFD* of prism *DFV-ABC* is a parallelogram. Therefore, by Fact 36, $AD = FB$ and $AB = DF$. That is, triangle *ADB* has its three sides *AD*, *AB*, and *DB*, respectively, equal to the three sides *FB*, *DF*, and *DB* of triangle *FDB*. Therefore, by Fact 6, triangle *ADB* is congruent to triangle *FDB*. The perpendicular distance from *V* to plane *ABFD* can be considered as the altitude of the two pyramids *V-DAB* and *V-DFB*. Thus, these pyramids have equal bases, *ADB* and *FDB*, and equal altitudes. Therefore, by Fact 136, they are equal in volume. That is, *V-DAB* = *V-DFB*.

Pyramid *V-DFB* may be denoted *B-VFD*, where *VFD* is considered its base and *h* is its altitude. The sections *VFD* and *ABC* of prism *DFV-ABC*, cut by two parallel planes, intersect all of the lateral edges. Therefore, by Fact 123, they are congruent. Hence, pyramids *B-VFD* and *V-ABC* have equal bases, *VFD* and *ABC*, and the same altitude *h*. Therefore, by Fact 136, they are equal in volume. That is, *B-VFD* = *V-ABC;* and since *V-DFB* is identical with *B-VFD*, we may write *V-DFB* = *V-ABC*. But *V-DAB* = *V-DFB*. Therefore, *V-DAB* = *V-DFB* = *V-ABC*.

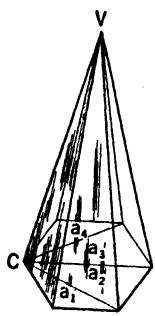


Fig. 249.

It is easily seen that, since these three equal pyramids form the prism *DFV-ABC*, pyramid *V-ABC* is equal to $\frac{1}{3}$ of the prism *DFV-ABC*. But the volume of prism *DFV-ABC* is, by Fact 131, equal to $h \times ABC$. Therefore, the volume of *V-ABC* is $\frac{1}{3}h \times ABC$. That is,

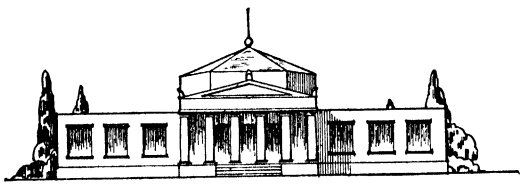
Fact 137. The volume of a triangular pyramid equals one-third of the product of its base and altitude.

Any pyramid whose base is not a triangle can be divided into triangular pyramids. Draw from any vertex, *C* in Figure 249, all the diagonals of the base, dividing it into triangles; and then pass the planes determined by these lines of division and the edge *VC*. Each of these triangular pyramids has an altitude equal to the altitude of the pyramid. Hence, by Fact 137, their respective volumes are $\frac{1}{3}a_1h$, $\frac{1}{3}a_2h$, $\frac{1}{3}a_3h$, and $\frac{1}{3}a_4h$. Obviously, the volume of the given pyramid is $\frac{1}{3}a_1h + \frac{1}{3}a_2h + \frac{1}{3}a_3h + \frac{1}{3}a_4h$, or $\frac{1}{3}h(a_1 + a_2 + a_3 + a_4)$. But the sum of the areas of the triangles forming the bases of these pyramids is equal to the area of the base

of the given pyramid. Hence, if B represents the area of the base of the given pyramid, $a_1 + a_2 + a_3 + a_4 = B$. Therefore, the volume of the given pyramid is equal to $\frac{1}{3} B \cdot h$. That is,

Fact 138. The volume of any pyramid equals one-third of the product of its base and altitude.

The weight of the gatepost's capstone can now be determined. Since the base is a square 15 inches on each side, the area of the base is equal to $(15)^2 = 225$ square inches. The height of the capstone is 10 inches. Therefore, its volume is $\frac{1}{3}(225)(10) = 750$ cubic inches. Marble weighs 168 pounds per cubic foot, and a cubic foot contains 1728 cubic inches. Hence, the weight of the capstone is $\frac{750}{1728} \times 168 = 72.91$ pounds.



The dome of a state capitol has the shape of a regular hexagonal pyramid whose lateral faces are to be covered with hammered Cathedral wire glass costing 40 cents per square foot. A side of the base of the regular hexagonal pyramid measures 20 feet, and its height is 8 feet. To ascertain the cost of the glass covering the dome, it is necessary to determine the lateral area of a pyramid.

Consider any regular pyramid, V-FC, Figure 250, with slant height l. It may be easily shown that the altitudes of the lateral faces of a regular pyramid are equal. By definition, Chapter 22, page 479, the slant height of a regular pyramid is the altitude of any one of the triangles forming its lateral faces. Hence, the slant height, l, is the altitude of each one of the triangles forming the lateral faces of the pyramid.

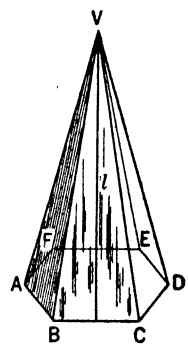


Fig. 250.

The area of triangle VBC is, by Fact 59, equal to $\frac{1}{2} l \cdot BC$. Similarly, the area of triangle $VCD = \frac{1}{2} l \cdot CD$, area of triangle $VDE = \frac{1}{2} l \cdot DE$, and so forth. Obviously, the lateral area, S,

is equal to the sum of the areas of the lateral faces. Therefore, $S = \frac{1}{2}l \cdot BC + \frac{1}{2}l \cdot CD + \frac{1}{2}l \cdot DE + \frac{1}{2}l \cdot EF + \frac{1}{2}l \cdot FA + \frac{1}{2}l \cdot AB$, or $S = \frac{1}{2}l[BC + CD + DE + EF + FA + AB]$. But if we denote the perimeter of the base by p, the sum $BC + CD + DE + EF + FA + AB = p$. Therefore, $S = \frac{1}{2}l \cdot p$. That is,

Fact 139. The lateral area of a regular pyramid equals the product of one-half the slant height and the perimeter of the base.

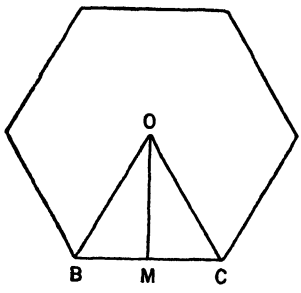


Fig. 251.

The cost of the glass necessary to cover the dome is now found as follows. The regular hexagon forming the base is shown in Figure 251. Angle $BOC = \frac{1}{6}(360°) = 60°$, and the two radii OB and OC are equal. Hence, by Fact 3, angle OBC = angle OCM. Since, by Fact 18, $180° = 60° +$ angle $OBC +$ angle OCM, we may write $120° = 2$ angle OBC, and $60° =$ angle OBC. Therefore, angle OBC = angle $OCM = 60°$. Draw OM perpendicular to BC. Then, by Fact 4, $BM = \frac{1}{2}BC = 10$ feet. That is, right triangle OMB has angle $OBM = 60°$ and side $BM = 10$ feet. To find OM, we may use the tangent ratio. Thus, $\tan OBM = \frac{OM}{BM}$; that is, $\tan 60° = \frac{OM}{10}$, or $10 \tan 60° = OM = 10(1.7321) = 17.32$ feet.

By definition, Chapter 26, page 479, the center of the base of a regular pyramid coincides with the foot of the perpendicular dropped from the vertex. Hence, the slant height l, OM, and the altitude OV form the right triangle VMO shown in Figure 252. Since $OM = 17.32$ feet and $OV = 8$ feet, then, by Fact 70, $l = \sqrt{(17.32)^2 + (8)^2} = \sqrt{363.98} = 19.08$ feet.

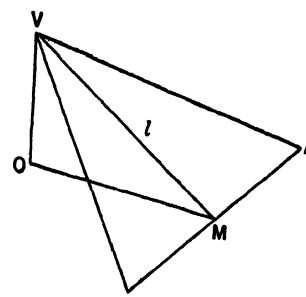


Fig. 252.

The lateral area of the pyramid is then equal to $\frac{1}{2}l \cdot p = \frac{1}{2}(19.08)(6 \times 20) = 1144.80$ square feet. The glass costs 40 cents per square foot. Therefore, the cost of the glass covering the dome is $1144.80 \times .40 = \$457.92$.

FRUSTUM OF A PYRAMID

The solid included between the base of a pyramid and a section formed by passing a plane parallel to the base, as in Figure 253, is called a *frustum of a pyramid.*

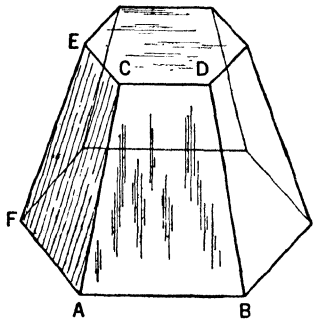
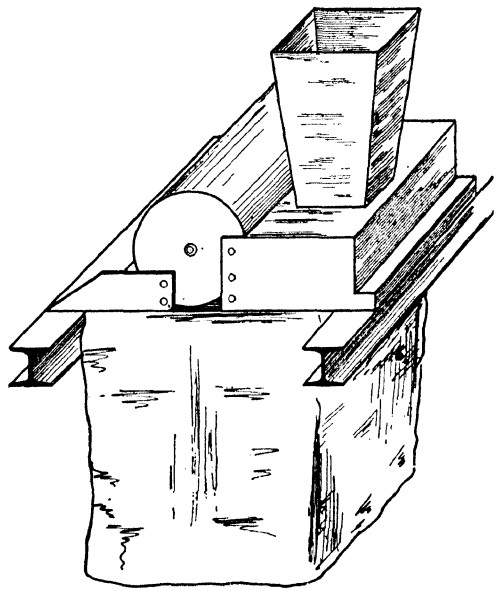


Fig. 253.

Any side of the upper base of the frustum of a regular pyramid, such as CD in Figure 253, is parallel to the corresponding side, AB, of the lower base. For CD and AB lie in the same plane $ABDC$, forming a lateral face of the frustum. Hence, if CD were not parallel to AB, they would intersect, which is impossible, for they also lie in the parallel planes forming the upper and lower bases. That is, the lateral faces of a frustum are trapezoids. The lower base of the frustum of a regular pyramid is, by definition, a regular polygon. Therefore, by Fact 124, the upper base is also a regular polygon. Hence, if a lateral face, such as $ABDC$, were folded over along AC, and if AB were made to fall along AF, CD would coincide with CE. Therefore, $ABDC$ would coincide with $ACEF$. That is,



Fact 140. The lateral faces of a frustum of a regular pyramid are congruent trapezoids.

A farmer prepares fodder by grinding corn in a mill. He wishes to make a hopper for the mill, which shall hold 12 bushels and be in the form of an inverted frustum of a square pyramid. The smaller base of the frustum must be 2 feet on

each side, and the height of the hopper is to be 3 feet. To ascertain the measurements of the hopper and the amount of material needed for its construction, it is necessary to determine the volume and area of the frustum of a pyramid.

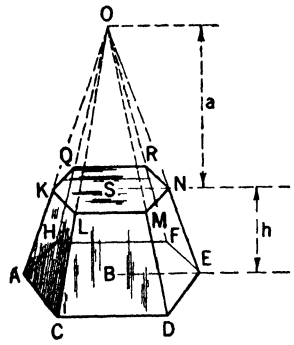


Fig. 254.

Consider any frustum of a pyramid, KE in Figure 254, with lower base B, upper base S, and altitude h. Complete the pyramid O-HD, of which the frustum KE is a part. Denote by p the volume of the small pyramid O-QM, whose altitude is a. Then the altitude of O-HD is $h + a$. Let V and P, respectively, represent the volumes of the frustum KE and of the pyramid O-HD.

From the figure, it is easily seen that $V = P - p$. Expressing this equality in terms of the dimensions, we may write:

$$V = \tfrac{1}{3}B(h + a) - \tfrac{1}{3}S \cdot a, \text{ or } V = \tfrac{1}{3}[B \cdot h + B \cdot a - S \cdot a]$$
$$= \tfrac{1}{3}[B \cdot h + a(B - S)].$$

The pyramid O-HD may be considered as cut by the two parallel planes HD and QM. Hence, by Fact 126, we may write $\dfrac{a^2}{(h+a)^2} = \dfrac{S}{B}$. Taking the square root of both sides gives $\dfrac{a}{h+a} = \dfrac{\sqrt{S}}{\sqrt{B}}$, or $a\sqrt{B} = h\sqrt{S} + a\sqrt{S}$. Transposing $a\sqrt{S}$ to the left-hand side of this equation and factoring, we obtain $a(\sqrt{B} - \sqrt{S}) = h\sqrt{S}$. Then, dividing both sides by $\sqrt{B} - \sqrt{S}$, $a = \dfrac{h\sqrt{S}}{\sqrt{B} - \sqrt{S}}$.

Substituting this value of a in the equation $V = \tfrac{1}{3}[B \cdot h + a(B-S)]$, we get:

$$V = \tfrac{1}{3}\left[B \cdot h + \frac{h\sqrt{S}}{\sqrt{B} - \sqrt{S}}(B - S)\right].$$

The expression $B - S$ may be considered as the difference of the squares of two numbers. That is, $B - S$ may be written $(\sqrt{B})^2 - (\sqrt{S})^2$. In this form, $B - S = (\sqrt{B})^2 - (\sqrt{S})^2$

$= (\sqrt{B} + \sqrt{S})(\sqrt{B} - \sqrt{S})$. (See Chapter 3, page 35.) Hence,

$$V = \tfrac{1}{3}\left[B \cdot h + \frac{h\sqrt{S}}{\sqrt{B} - \sqrt{S}}(\sqrt{B} + \sqrt{S})(\sqrt{B} - \sqrt{S})\right],$$

$$V = \tfrac{1}{3}[B \cdot h + h\sqrt{S}(\sqrt{B} + \sqrt{S})],$$

$$V = \tfrac{1}{3}[B \cdot h + h\sqrt{BS} + S \cdot h],$$

and $$V = \tfrac{1}{3}h[B + S + \sqrt{BS}].$$

In other words,

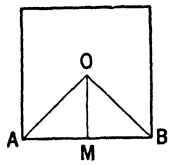
Fact 141. The volume of a frustum of a pyramid is equal to one-third the product of the altitude and the sum of the upper base, the lower base, and the mean proportional between the two bases.

The measurements of the hopper are now found as follows. In the table on page 487, a heaped bushel, used for measuring corn, is given equal to 2747.71 cubic inches. Since the hopper is to contain 12 bushels, and 1728 cubic inches equals one cubic foot, the volume of the hopper is $\frac{2747.71}{1728} \times 12 = 19.08$ cubic feet. The area of the smaller base is $2 \times 2 = 4$ square feet. Denoting a side of the larger base by x, its area may be represented by x^2. Substituting known values in the formula just developed for the frustum of a pyramid, we obtain $19.08 = \frac{1}{3}(3)[x^2 + 4 + \sqrt{4x^2}]$, or $19.08 = x^2 + 4 + 2x$. Transposing 19.08 gives $0 = x^2 + 2x - 15.08$. Applying the formula for a quadratic (Chapter 17, page 269), $x = \frac{-2 \pm \sqrt{4 + 60.32}}{2} = \frac{-2 \pm \sqrt{64.32}}{2} = \frac{-2 \pm 8.02}{2}$. Evidently, the negative value does not apply to this problem. Therefore, the side of the larger base of the hopper is equal to $\frac{-2 + 8.02}{2} = 3.01$ feet.

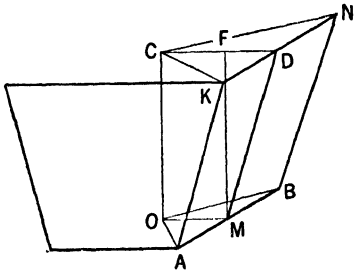
To ascertain the amount of material needed to construct the hopper, it is, as already stated, necessary to determine the lateral area of a frustum of a regular pyramid. The reader may easily show that:

Fact 142. The lateral area of the frustum of a regular pyramid is equal to one-half the product of the slant height and the sum of the perimeters of the bases.

Hence, if S denotes the lateral area, l the slant height, P the perimeter of the lower base, and R the perimeter of the upper base, we may write $S = \frac{1}{2}l(P + R)$.



(a)



(b)

Fig. 255.

In order to apply this formula, it is first necessary to find the slant height of the frustum. The base of the hopper is, as shown in Figure 255 (a), a square 2 feet on each side. Angle $AOB = \frac{1}{4}(360°) = 90°$, and the two radii OA and OB are equal. That is, triangle AOB is an isosceles right triangle. Hence, angle OAB = angle $OBA = 45°$. Draw OM perpendicular to AB. Then, by Fact 4, $AM = \frac{1}{2}AB = 1$ foot. Furthermore, triangle OMA is a right triangle with angle $OAM = 45°$. Hence, angle AOM is also equal to 45°. Therefore, by Fact 26, $AM = OM = 1$ foot.

Draw, as in Figure 255 (b), a perpendicular from O to the upper base of the inverted frustum. Then, since the hopper is a frustum of a regular pyramid, the foot of this perpendicular, namely, C, is the center of the upper base. Draw the radii CK and CN and the apothem CD. As in the case of the lower base, $KD = \frac{1}{2}KN = \frac{1}{2}(3.01) = 1.50$ feet, and $CD = KD = 1.50$ feet. Draw the perpendicular MF. Then $MF = OC = 3$, and $CF = OM = 1$ foot. Hence, $FD = CD - CF = 1.50 - 1 = 0.50$. That is, in right triangle MFD, $MF = 3$ and $FD = 0.50$. Therefore, by Fact 70,

$$MD = l = \sqrt{(3)^2 + (0.50)^2} = \sqrt{9 + 0.25} = \sqrt{9.25} = 3.041 \text{ feet.}$$

The lateral area of the hopper is, therefore, $S = \frac{1}{2}l(P + R) = \frac{1}{2}(3.041)(8 + 12.04) = 30.47$ square feet. That is, 30.47 square feet of material are necessary to construct the hopper (*Answer*).

CONES

A surface generated by a straight line which constantly

touches a fixed plane curve and passes through a fixed point not in the plane of the curve is called a *conical surface* [Figure 256 (*a*)].

The definitions of such terms as *generator*, *directrix*, *vertex*, *element*, *frustum*, and so forth, applied to cones correspond to the definitions, already stated, for other solids. Thus, an *element* is the generator of the conical surface in any particular position. The *surface* consists of two parts or *nappes*, one above and the other below the vertex. A conical surface may be easily visualized by holding a pencil at some point near the middle and moving the point so that it constantly touches a given fixed plane curve.

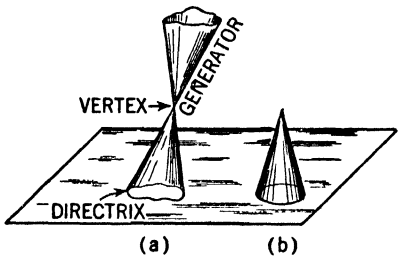


Fig. 256.

When the fixed plane curve, namely, the *directrix*, is a closed curve and the conical surface is bounded by a plane, as in Figure 256 (*b*), a solid is formed. This solid is called a *cone*. The conic surface is the *lateral surface* of the cone, and the plane surface is the *base* of the cone. The *altitude* is, as usual, the perpendicular distance from the vertex to the base.

The most common cones have circular bases and are called *circular cones*. The line from the vertex to the center of the base of a circular cone is called the *axis*. If the axis is perpendicular to the base, the cone is a *right circular cone*. A right circular cone can be thought of as generated by a right triangle rotated about one of its sides, Figure 257; hence, a right circular cone is called a *cone of revolution*. The length of an element of a right circular cone is called its *slant height*.

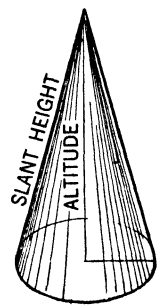


Fig. 257.

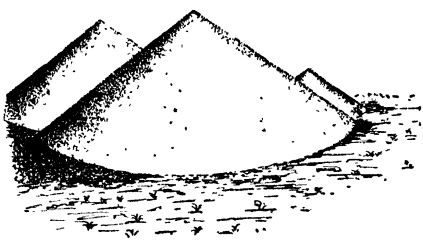


Fig. 258.

A series of sand piles, in the shape of right circular cones, as in Figure 258, have been dumped next to a construction job. The superintendent,

desiring to ascertain the number of cubic feet of sand in each pile, measures the slant height, and by means of two flat boards held tangent to the circular base determines the diameter. Suppose that the slant height and diameter of a sand pile are found to be 4 and 6 feet, respectively. To ascertain the cubic feet of sand in the conical pile, it is necessary to determine the volume of a circular cone in terms of its dimensions.

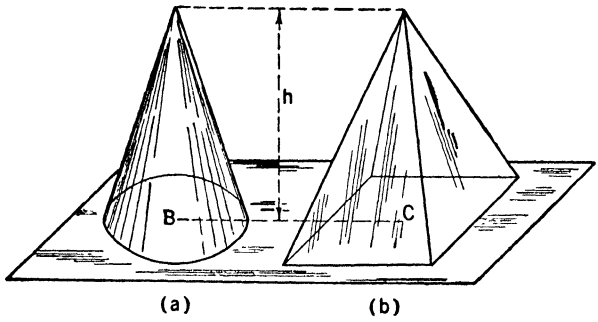


Fig. 259.

Consider any circular cone, Figure 259 (*a*), with base B and altitude h. By means of the formulas for the areas of a circle and a square, Facts 110 and 53, construct in the plane of the base of the cone a square equal in area to B. It is easily seen that, by repeating this procedure as often as necessary, a pyramid, Figure 259 (*b*), may be erected having the same altitude, h, as the cone and such that the sections made by planes parallel to and at the same distance from their respective bases are always equal. Therefore, by Cavalieri's theorem, Fact 129, the volumes of the two solids are equal. But the volume of the pyramid is, by Fact 138, equal to $\frac{1}{3}S \cdot h$. Therefore, since $B = S$, the volume of the cone is $\frac{1}{3}B \cdot h$. That is,

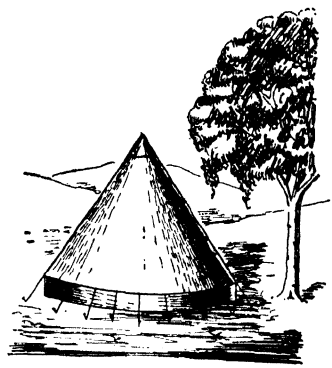
*Fact 143. The volume of a circular cone is equal to one-third the product of its base and altitude.**

The area of a circle is, by Fact 110, equal to πr^2. Hence, if the altitude is denoted by h, the volume of a circular cone may be written $V = \frac{1}{3}\pi r^2 h$.

The sand pile is a right circular cone whose slant height is 4

* As in the case of a cylinder, it can be shown, by the theory of limits, that Fact 143 holds for any cone.

feet and whose base has a radius of $\frac{1}{2}(6) = 3$ feet. Since a right circular cone can be considered as generated by a right triangle rotated about one of its sides, Figure 257, then, by Fact 70, $(4)^2 = (3)^2 + h^2$, or $16 - 9 = h^2$; that is, $7 = h^2$, and $h = 2.65$. Substituting known values in the formula for the volume of a cone, we obtain $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3)^2(2.65)$, or 24.97 cubic feet.



A tent manufacturing concern has an order for a large number of conical tents 16 feet in diameter and 12 feet high. To compute the cost of a tent, it is necessary to ascertain the amount of canvas required. That is, the lateral area of a right circular cone, in terms of its dimensions, must be known.

Assume that the conical surface of a right circular cone is slit along an element, as shown in Figure 260 (*a*). If the surface is

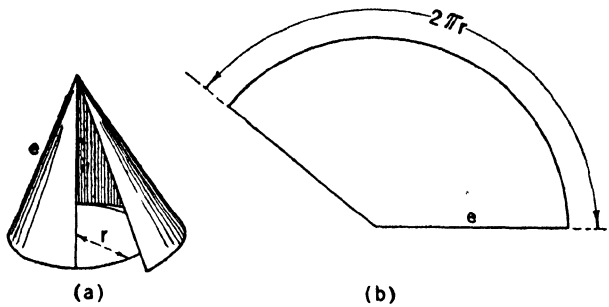


Fig. 260.

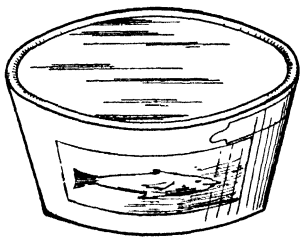
then spread out flat, as in (*b*), it becomes a sector of a circle. The radius of the circle, namely, e, is an element of the cone and the length of the arc is equal to the circumference of the cone's base. Hence, if r denotes the radius of the base of a right circular cone, the length of the arc in (*b*) is $2\pi r$ (Chapter 22, page 362). But the area of a sector of a circle is equal to one-half the radius multiplied by the length of the arc (Chapter 22, page 371). Therefore, the lateral area of a right circular cone is $\frac{1}{2}e(2\pi r)$ or $S = \pi re$. That is,

Fact 144. The lateral area of a right circular cone is equal to

one-half the product of the perimeter of the base and the slant height.

The amount of canvas required for the construction of a tent is now found as follows. Since the tent forms a right circular cone, the slant height, altitude, and radius form a right triangle. Therefore, by Fact 70, $e^2 = (8)^2 + (12)^2 = 208$; that is, $e = \sqrt{208} = 14.42$ feet. Substituting known values in the formula for the lateral area of a right circular cone, we obtain $S = \pi re = \pi(8)(14.42) = 362.56$ square feet.

FRUSTUM OF A CONE



An exporter of food products buys a number of containers, each in the form of a frustum of a cone, at a profitable price. The diameters of the bases are 3 and 6 inches, respectively, and the height is 2 inches. To set a price upon his product, the exporter desires to know the capacity of the containers.

Consider the frustum of a circular cone, Figure 261 (*a*), with base B and altitude h. From the discussion on the volume of

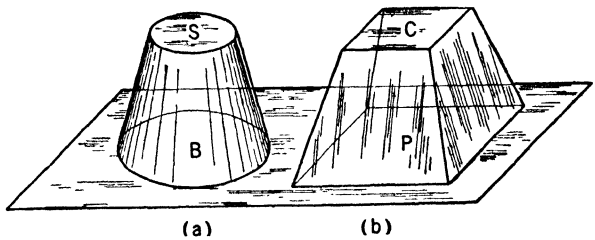


Fig. 261.

a cone, page 514, it is easily seen that a frustum of a pyramid may be constructed, Figure 261 (*b*), with the same altitude h as the frustum of the cone, whose base P lies in the same plane and is equal in area to B, and such that the sections made by planes parallel to and at the same distance from their respective bases are always equal. Therefore, again by Fact 129, the volumes of the two solids are equal. But the volume of the frustum of the pyramid is, by Fact 141, equal to $\frac{1}{3}h[P + C + \sqrt{PC}]$. Therefore, since $P = B$ and $C = S$, the volume of a

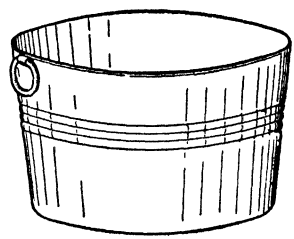
frustum of a circular cone may be written $V = \frac{1}{3}h[B + S + \sqrt{BS}]$. That is,

Fact 145. The volume of a frustum of a circular cone is equal to one-third the product of the altitude and the sum of the upper base, the lower base, and the mean proportional between the two bases.

If r and r_1, respectively, denote the radii of the lower and upper bases of the frustum of the cone, then, by Fact 110, $B = \pi r^2$ and $S = \pi r_1^2$. Hence, $\sqrt{BS} = \sqrt{(\pi r^2)(\pi r_1^2)} = \pi r r_1$. Substituting these values in the formula for the volume of the frustum of a circular cone, we obtain $V = \frac{1}{3}h[\pi r^2 + \pi r_1^2 + \pi r r_1]$, or $V = \frac{1}{3}\pi h[r^2 + r_1^2 + r r_1]$.

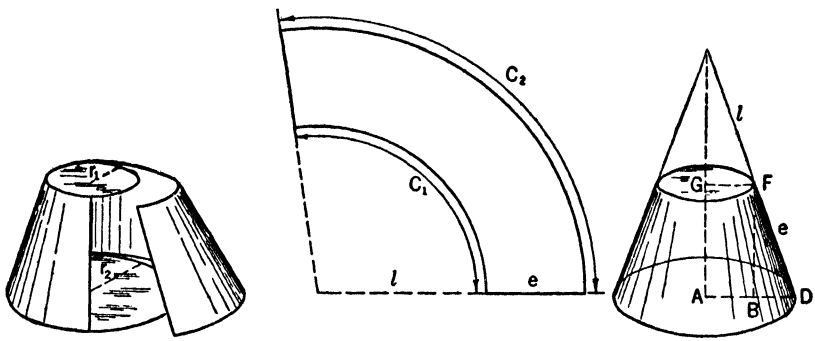
The capacity of the container is now found by merely substituting known values in the above formula. Thus,

$$V = \tfrac{1}{3}\pi(2)[(3)^2 + (1.5)^2 + (3)(1.5)] = 33 \text{ cubic inches.}$$



A manufacturer has an order for 500 washtubs. The washtubs have the shape of a frustum of a cone 15 inches in diameter at the base, 19 inches in diameter across the rim, and 10 inches deep. To order the necessary amount of material, the lateral area of a frustum of a cone must be determined.

The lateral surface of the frustum of a right circular cone may be assumed to be slit along an element, as in Figure 262 (*a*).



(a) (b) (c)

Fig. 262.

If the surface is then spread out flat, as in (*b*), it becomes a sector of a circular ring. Complete the cone of the frustum, Figure 262 (*c*), and denote the slant height of the completed cone by $l + e$. It is easily seen, Figure 262 (*b*), that the lateral area of the frustum, that is, the area of the ring sector, is equal to the area of the sector of the circle whose radius is $l + e$ less the area of the sector of the circle whose radius is l. The area of a sector of a circle is equal to one-half the radius multiplied by the length of the arc (Chapter 22, page 371). The length of the smaller arc is the circumference of the upper base of the frustum, and the length of the other arc is the circumference of the lower base. Let C_1 and C_2, respectively, denote the circumferences of the upper and lower bases of the frustum. Then, if S represents the lateral area of the frustum, we may write $S = \frac{1}{2}(l + e)C_2 - \frac{1}{2}lC_1$, or $S = \frac{1}{2}C_2l + \frac{1}{2}C_2e - \frac{1}{2}C_1l$.

The arcs C_1 and C_2 have a common angle. But in unequal circles, arcs subtending equal angles are to each other as the corresponding radii (Chapter 22, page 366). Therefore, $\frac{C_2}{C_1} = \frac{l + e}{l}$, or $C_2l = C_1(l + e)$. Substituting this value of C_2l in the expression obtained in the preceding paragraph for the lateral area of the frustum, we obtain $S = \frac{1}{2}C_1(l + e) + \frac{1}{2}C_2e - \frac{1}{2}C_1l = \frac{1}{2}C_1l + \frac{1}{2}C_1e + \frac{1}{2}C_2e - \frac{1}{2}C_1l$; that is, $S = \frac{1}{2}C_1e + \frac{1}{2}C_2e$, or $S = \frac{1}{2}e(C_1 + C_2)$. In other words,

Fact 146. The lateral area of a frustum of a right circular cone is equal to one-half the product of the slant height and the sum of the circumferences of the bases.

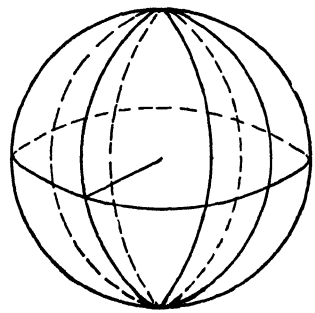
If r_1 and r_2, respectively, denote the radii of the upper and lower bases of the frustum, $C_1 = 2\pi r_1$ and $C_2 = 2\pi r_2$ (Chapter 22, page 362). Substituting these values in the formula $S = \frac{1}{2}e(C_1 + C_2)$, we obtain $S = \frac{1}{2}e(2\pi r_1 + 2\pi r_2)$, which reduces to $S = \pi e(r_1 + r_2)$.

The amount of material required for a washtub is now easily obtained. Since the diameter of the smaller base is 15 inches, $r_1 = 7.5$ inches. The diameter across the rim is 19 inches; hence, $r_2 = 9.5$ inches. To determine the slant height, erect a perpendicular BF, Figure 262. Since by construction $ABFG$ is a rectangle, then, by Fact 36, $AB = GF$ and $AG = BF$. The altitude of the cone is 10 inches. Hence, $BF = 10$ inches.

Furthermore, $BD = AD - AB$. Therefore, since $AD = r_2 = 9.5$, and $AB = GF = r_1 = 7.5$, $BD = 9.5 - 7.5 = 2$ inches. Applying Fact 70, $(FD)^2 = (BF)^2 + (BD)^2$; that is, $(FD)^2 = (10)^2 + (2)^2 = 104$, so that the slant height of the frustum $e = FD = \sqrt{104} = 10.2$ inches. Substituting known values in $S = \pi e(r_1 + r_2)$ gives $S = \pi(10.2)(7.5 + 9.5) = 544.97$ square inches. The area of the base of the washtub is, by Fact 110, equal to $\pi(7.5)^2 = 176.78$ square inches. Therefore, the amount of material required for a washtub is $544.97 + 176.78 = 721.75$ square inches or 5.01 square feet (*Answer*).

SPHERES

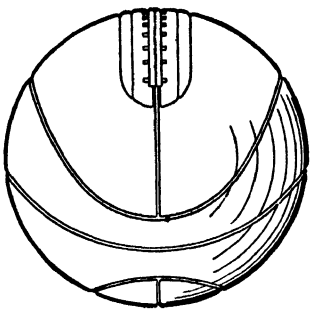
The material sphere is probably the most familiar of the common solids. Used extensively in sports, baseball, tennis, bowling, and so forth, it becomes a subject for serious study when we consider that the world in which we live, the sun, and the planets are spherical.



Mathematically, a *sphere* is defined as a closed surface every point of which is equidistant from a fixed point called the *center*. The solid bounded by such a surface is also often called a sphere. Terms such as *radius*, *diameter*, *chord*, and so forth, as applied to the sphere, are defined in the same sense as for the circle. Thus, a radius of a sphere is a straight line segment connecting its center with any point on the surface. Obviously, all radii of the same sphere are equal. The sphere may also be considered as generated by the complete rotation of a semicircle about a diameter.

Every section made by a plane passed through a sphere is a *circle*. If the plane passes through the center of the sphere, the plane section is a *great circle;* otherwise, the section is a *small circle*. Clearly, any plane passing through the center of the sphere contains a diameter. Hence, all great circles of a sphere are equal, have for their common center the center of the sphere, and have for their radius the radius of the sphere. A great circle bisects the surface of a sphere. One of the two

equal parts into which the sphere is divided by a great circle is called a *hemisphere.*



A manufacturer of sport goods has to order the material necessary for a number of basketballs 6 inches in diameter. To ascertain the number of square inches of material required for the manufacture of a basketball, a formula for the surface area of a sphere must be determined.

Consider any sphere of radius r and surface area S, Figure 263. Divide the great circle $QBGPQ$ into 16 equal parts and draw the chords of these arcs. Then, by Fact 101, these chords form a regular inscribed polygon. Draw lines KN and BP. The reader can easily show that, by construction, KN is parallel to BP. Through KN and BP pass two parallel planes cutting the sphere in two small circles of radii r_1 and r_2, and denote the distance between the two planes by h. Now assume that the trapezoid $BPNK$ is rotated about DH, forming the frustum of a right circular cone. The lateral area of this frustum is *approximately* equal to the surface area of the sphere included between the two parallel planes. The lateral area s of the frustum is, therefore, by Fact 146, equal to $\frac{1}{2}BK(2\pi r_1 + 2\pi r_2)$, or

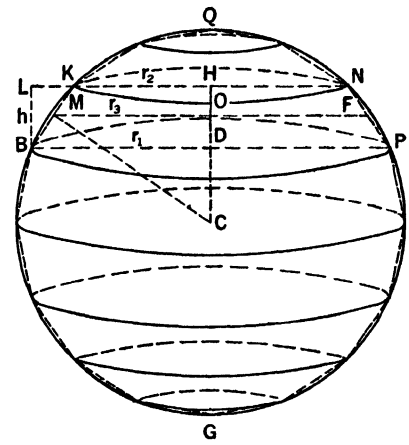


Fig. 263.

$$s = 2\pi BK\left(\frac{r_1 + r_2}{2}\right). \tag{1}$$

Draw CM perpendicular to chord BK. Then, by Fact 80,

CM bisects BK. Draw MF parallel to BP, and NX parallel to BK, Figure 263 (b). Since $BM = MK$, then, by Fact 41, $NY = YX$ and $NF = FP$. Draw FZ parallel to NX; then, again by Fact 41, $XZ = ZP$. That is, $XZ = \frac{1}{2}XP$. By construction, $XYFZ$ is a parallelogram; hence, by Fact 36, $YF = XZ$. Therefore, $YF = \frac{1}{2}XP$. Furthermore, also by Fact 36, $MY = BX = KN$. Hence, we may write $MY = \frac{1}{2}(2BX)$ or $MY = \frac{1}{2}(BX + KN)$. Therefore, $MF = MY + YF = \frac{1}{2}(BX + KN) + \frac{1}{2}XP$; that is, $MF = \frac{1}{2}(BX + XP + KN)$, or $MF = \frac{1}{2}(BP + KN)$. Since line HC, Figure 263 (a), perpendicular to chord KN, is, by Fact 16, also perpendicular to MF and BP, then, by Fact 80, HC bisects all these chords. Thus, we may write $r_3 = \dfrac{r_1 + r_2}{2}$. Substituting in (1), we obtain:

$$s = 2\pi BKr_3. \tag{2}$$

Consider triangles BLK and CMO. Line CM is perpendicular to BK, and OH is perpendicular to LH. Hence, by Fact 64, angle LKB is equal to angle MCO. Thus, by Fact 49, triangles BLK and CMO are similar, and, therefore, $\dfrac{r_3}{CM} = \dfrac{h}{BK}$, or $r_3 = \dfrac{CM \cdot h}{BK}$. Substituting in (2) gives:

$$s = 2\pi \cdot BK \cdot \frac{CM \cdot h}{BK},$$

or

$$s = 2\pi(CM)(h). \tag{3}$$

By connecting the vertices as shown in the figure and passing parallel planes, the surface area of the sphere can be approximated by adding the lateral areas of the frustums of right circular cones included between these planes. Let h_1, h_2, and so forth, denote the respective altitudes of the frustums. Since the inscribed polygon is regular, all its sides are at the same distance CM from the center. Hence, the sum S' of the lateral areas of the inscribed frustums may be written:

$$S' = 2\pi(CM)(h_1 + h_2 + \cdots + h_n).$$

But it is easily seen from the figure that the sum of the altitudes of these frustums is equal to the diameter of the sphere.

Thus, if R denotes the radius of the sphere, $h_1 + h_2 + \cdots + h_n = 2R$, and we may write:

$$S' = 2\pi(CM)(2R),$$

or

$$S' = 4\pi(CM)(R).$$

If, as in the discussion of Cavalieri's theorem, page 493, we increase indefinitely the number of these frustums, by repeatedly doubling the number of sides of the inscribed polygon, it is easily seen that S' approaches the surface area of the sphere S as its limit, while CM approaches R as its limit. We then make the following assumptions:*

(*a*) That the limit of the sum of the lateral areas of the inscribed frustums, as the number of these frustums is thus increased, is the surface area of the sphere.

(*b*) That the radius of the sphere is the limit of the perpendicular line drawn from the center of the sphere to an element of an inscribed frustum as the number of these frustums is thus increased.

Therefore, $S = 4\pi(R)(R)$;

that is,

$$S = 4\pi R^2.$$

Fact 147. The surface area of a sphere is equal to the area of four of its great circles.

The manufacturer of sport goods can ascertain the material required for the manufacture of a basketball 6 inches in diameter by merely substituting in the formula $S = 4\pi R^2$. Thus, $S = 4\pi(3)^2 = 113.14$ square inches.

The portion of the surface of a sphere included between two parallel planes which intersect the sphere is called a *zone*. If we consider equation (3), it is clear that, since CM approaches R, the area of a zone is:

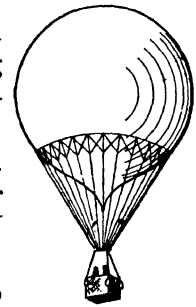
$$Z = 2\pi R \cdot h.$$

Fact 148. The surface area of a zone equals the product of the circumference of a great circle of the sphere and its altitude.

An observation balloon has a spherical gas bag 15 feet in diameter. The amount of gas necessary to fill the bag can be easily ascertained by developing a formula for the volume of a sphere.

* These assumptions can be proved true by the theory of limits.

Consider any sphere, Figure 264 (*a*), with radius R and volume V. Pass a plane S through the center of the sphere. On this plane construct the solid shown in Figure 264 (*b*) as follows. By rotating a square of side R, generate a cylinder of revolution such that its altitude is R, and its base, lying on plane S, has a radius also equal to R. Remove from this cylinder the right circular cone generated by rotating the triangle MKV about KV.



Pass a plane L through both solids parallel to plane S and at any distance $BC = DV$ from their bases. Plane L cuts the sphere in the small circle

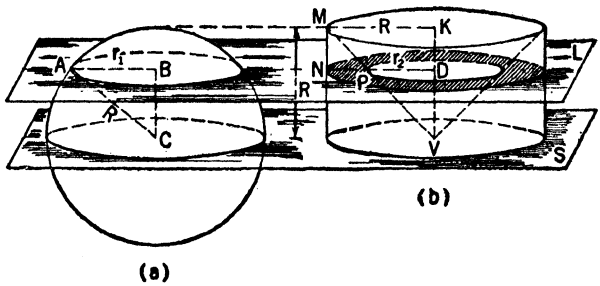


Fig. 264.

whose radius is $AB = r_1$, and the other solid in a section bounded by an outer circle of radius $DN = MK = R$ and an inner circle whose radius is $DP = r_2$. Denote the area of the small circle whose radius is $AB = r_1$ by A. Then, by Fact 110,

$$A = \pi r_1^2. \tag{4}$$

Applying Fact 70 to right triangle ABC, Figure 264 (*a*), we have $(AC)^2 = (AB)^2 + (BC)^2$, or $R^2 = r_1^2 + (BC)^2$, so that

$$r_1^2 = R^2 - (BC)^2. \tag{5}$$

In right triangle MKV, Figure 264 (*b*), $DP = r_2$ is, by construction, parallel to $MK = R$. So that, by Fact 14, corresponding angles KMV and DPV are equal; and hence, by Fact 49, right triangle MKV is similar to right triangle PDV. Therefore, $\frac{KV}{MK} = \frac{VD}{DP}$. But, by construction, $KV = MK =$

R, so that $\frac{KV}{MK} = \frac{R}{R} = 1$. Hence, $1 = \frac{VD}{DP}$, or $DP = VD$; that is, $r_2 = VD$. Furthermore, also by construction, $BC = VD = r_2$. Substituting in (5) gives:

$$r_1^2 = R^2 - r_2^2;$$

and again, substituting this value of r_1^2 in (4), we have:

$$A = \pi(R^2 - r_2^2). \tag{6}$$

The area of the section made by plane L on the solid shown in Figure 264 (*b*) is equal to the area of the outer circle less the area of the inner circle. The radius of the outer circle is $DN = MK = R$, and the radius of the inner circle is $DP = r_2$. Hence, if we denote the area of this section by A_1 and apply Fact 110, we obtain:

$$A_1 = \pi R^2 - \pi r_2^2,$$

or

$$A_1 = \pi(R^2 - r_2^2).$$

Therefore, $A_1 = A$. That is, the hemisphere cut by plane S and the solid shown in Figure 264 (*b*) have equal altitudes, and the sections made by planes parallel to and at the same distance from their respective bases are always equal. Therefore, by Cavalieri's theorem, Fact 129, the volumes of the solids are equal. The volume of the solid shown in Figure 264 (*b*) is equal to the volume of the cylinder less the volume of the right circular cone. Hence, if we denote the volume of the solid shown in Figure 264 (*b*) by V_1 and apply Facts 134 and 143, we may write:

$$V_1 = (\pi R^2)(R) - \tfrac{1}{3}(\pi R^2)(R).$$

That is,

$$V_1 = \tfrac{2}{3}\pi R^3.$$

Therefore, the volume of the hemisphere is also $\frac{2}{3}\pi R^3$; and the volume V of a sphere with radius R is:

$$V = \tfrac{4}{3}\pi R^3.$$

In other words,

Fact 149. The volume of a sphere is equal to the product of $\frac{4}{3}\pi$ *and the cube of its radius.*

To obtain the volume of a spherical gas bag 15 feet in diameter is now merely a matter of substituting the proper values

in the above formula. Thus, $V = \frac{4}{3}\pi(7.5)^3 = 1{,}767.86$ cubic feet (*Answer*).

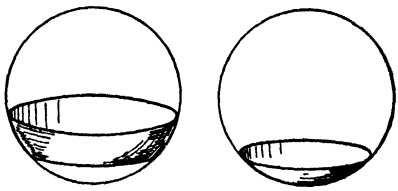
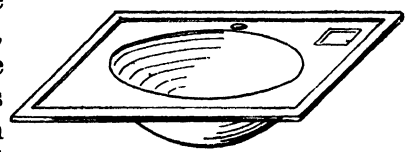


Fig. 265.

SPHERICAL SEGMENT

The solid bounded by a zone and the planes of its bases is a *spherical segment.* As shown in Figure 265, a spherical segment may have one or two bases.

A manufacturer of plumbing fixtures designs a stamped metal lavatory as shown below. The slab measures 23 by 21 inches and the bowl, in the shape of a segment of a sphere, measures 14 inches across the top and is 6 inches deep at its greatest depth. To ascertain the amount of water the bowl will hold, a formula for the volume of a segment of a sphere is necessary.



The volume of a segment of a sphere can be determined by following the same reasoning used in developing a formula for the volume of a sphere. Instead of passing only one plane parallel to the base of the solids as in Figure 264, simply pass two parallel planes. The distance between the two planes determines the altitude of the spherical segment. The reader may then show that:

Fact 150. The volume of a spherical segment of two bases is $V = \frac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)$, *where* h *is the altitude and* r_1 *and* r_2 *are, respectively, the radii of the bases of the segment.*

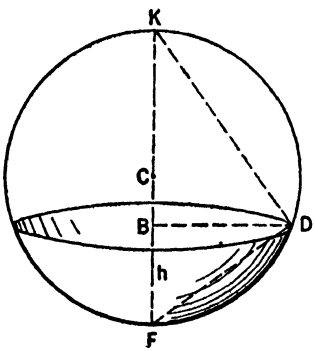


Fig. 266.

For a spherical segment of one base, the radius of the upper base, namely, r_1, is equal to zero. Therefore,

$$V = \tfrac{1}{6}\pi h(3r_2^2 + h^2).$$

This formula may be given in terms of the radius of the sphere

as follows. Consider the segment of one base, shown in Figure 266, with altitude $BF = h$ and radius of base $BD = r_2$. Denote the radius of the sphere by R. Then $KC = R$ and $BC = FC - BF = R - h$. Hence, $KB = KC + CB = R + R - h = 2R - h$. Draw KD and FD. Triangle KDF is, by Fact 90, a right triangle. Thus, BD, drawn perpendicular from the vertex of the right angle to the hypotenuse KF, is, by Fact 69, the mean proportional between the segments KB and BF. That is, $\frac{KB}{BD} = \frac{BD}{BF}$, or $KB \cdot BF = (BD)^2$. Substituting in terms of r_2, h, and R, we obtain $(2R - h)(h) = r_2^2$.

This value of r_2^2 may now be substituted in

$$V = \tfrac{1}{6}\pi h(3r_2^2 + h^2).$$

Thus,
$$\begin{aligned} V &= \tfrac{1}{6}\pi h[3(2R - h)(h) + h^2] \\ &= \tfrac{1}{6}\pi h[6Rh - 3h^2 + h^2] \\ &= \tfrac{1}{6}\pi h[2h(3R - h)]. \quad \text{That is,} \end{aligned}$$

Fact 151. The volume of a spherical segment of one base is $V = \frac{1}{3}\pi h^2(3R - h)$, *where* h *is the altitude of the segment and* R *is the radius of the sphere.*

The contents of the bowl may now be obtained by substituting known values in one of these formulas. The conditions of the problem state that the bowl is a spherical segment with an altitude of 6 inches and a radius of the base equal to 7 inches. Substituting in $V = \frac{1}{6}\pi h(3r_2^2 + h^2)$ gives:

$$V = \tfrac{1}{6}\pi(6)[3(49) + 36] = 575.14 \text{ cubic inches.}$$

From the table on page 487, 1 gallon = 231 cubic inches. Therefore, the bowl contains $575.14 \div 231 = 2.49$ gallons (*Answer*).

SPHERICAL SECTOR AND CONE

If a sector of a circle is rotated about an axis which passes through the center of the circle but which does not cut the sector, a solid called a *spherical sector* is generated [Figure 267 (*b*) and (*d*)]. The bounding surfaces of a spherical sector are a zone, which is called the *base*, and either one or two conical surfaces. If the spherical sector has only one conical surface, the solid is called a *spherical cone* [Figure 267 (*b*)].

The volume of a spherical cone may be determined. Con-

sider the spherical cone shown in Figure 267 (*b*) generated by rotating the sector *ACB*, shown in (*a*), about *BC*. It is easily seen that the volume of the spherical cone is equal to the volume of the cone *ACD* and the volume of the spherical segment *ABD*. Denote the radius of the sphere, namely, *CD*, by *R;* the radius of the base of the cone, that is, *OD*, by *r;* and let *h*

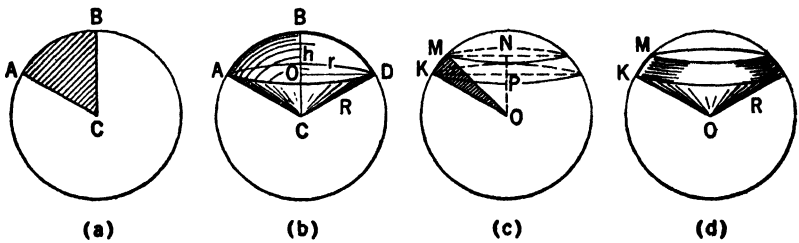


Fig. 267.

represent the altitude *OB* of the spherical segment. Then, applying Facts 143 and 151, we obtain:

$$V = \tfrac{1}{3}\pi r^2(OC) + \tfrac{1}{3}\pi h^2(3R - h). \tag{7}$$

From the figure, $OC = BC - OB$. Since $BC = R$ and $OB = h$, then $OC = R - h$. In right triangle DOC, $(CD)^2 = (OC)^2 + (OD)^2$ [Fact 70]. Or $R^2 = (R - h)^2 + r^2$. Hence, $R^2 - (R - h)^2 = r^2$. Substituting these values of r^2 and OC in (7) gives:

$$\begin{aligned}
V &= \tfrac{1}{3}\pi[R^2 - (R - h)^2](R - h) + \tfrac{1}{3}\pi h^2(3R - h)\\
&= \tfrac{1}{3}\pi[R^2 - R^2 + 2Rh - h^2](R - h) + \tfrac{1}{3}\pi h^2(3R - h)\\
&= \tfrac{1}{3}\pi(2Rh - h^2)(R - h) + \tfrac{1}{3}\pi h^2(3R - h)\\
&= \tfrac{1}{3}\pi(2R^2h - Rh^2 - 2Rh^2 + h^3) + \tfrac{1}{3}\pi h(3Rh - h^2)\\
&= \tfrac{1}{3}\pi h(2R^2 - 3Rh + h^2) + \tfrac{1}{3}\pi h(3Rh - h^2)\\
&= \tfrac{1}{3}\pi h(2R^2 - 3Rh + h^2 + 3Rh - h^2)\\
&= \tfrac{1}{3}\pi h(2R^2)\\
&= \tfrac{2}{3}\pi R^2h.
\end{aligned}$$

That is,

Fact 152. The volume of a spherical cone is $V = \frac{2}{3}\pi R^2h$, *where* h *is the altitude of the spherical segment which forms the base of the cone and* R *is the radius of the sphere.*

It may be easily seen, in Figure 267 (*c*) and (*d*), that the volume of a spherical sector is the difference between two spherical cones. Denote the altitude of the outer cone, namely, *OP*,

Figure 267 (*c*), by h_1 and the altitude of the inner cone, that is, *ON*, Figure 267 (*c*), by h_2. Let the respective volumes of the two cones be V_1 and V_2. Then, applying Fact 152, we have:

$$V_2 = \tfrac{2}{3}\pi R^2 h_2$$

and $$V_1 = \tfrac{2}{3}\pi R^2 h_1.$$

Therefore, $$V = \tfrac{2}{3}\pi R^2 h_2 - \tfrac{2}{3}\pi R^2 h_1;$$

that is, $$V = \tfrac{2}{3}\pi R^2 (h_2 - h_1).$$

If the altitude of the zone, namely, *PN*, Figure 267 (*c*), is denoted by h, then $h = h_2 - h_1$. Hence, $V = \tfrac{2}{3}\pi R^2 h$. In other words,

Fact 153. The volume of a spherical sector is $V = \tfrac{2}{3}\pi R^2 h$, *where* h *is the altitude of the zone which forms the base of the sector and* R *is the radius of the sphere.*

Problems

1. An obelisk in Central Park, New York City (often called "Cleopatra's Needle"), consists of the frustum of a pyramid surmounted by a small pyramid. The frustum has square bases, the upper one measuring 4 feet on a side and the lower base $7\frac{1}{2}$ feet on a side. The altitude of the frustum is 61 feet. Find the weight of the frustum, if the stone of which it is made weighs 172 pounds per cubic foot.

2. If the lateral surface of this frustum is to be covered with a glycerine preparation to keep water out of the fissures in the stone, what is the approximate area to be covered?

3. An indoor skating rink with an area of 1800 sq. yds. has a concrete flooring 12 inches thick. How much concrete is required?

4. During a storm $1\frac{1}{2}$ inches of rain fell. How many gallons of water fell on a 200-acre park?

5. A circular tunnel under a river is 1 mile long and has an inside diameter of 15 feet. If it fills up with mud, how many cubic yards will there be to be excavated?

6. The standard unit used in calculating the electrical resistance of a wire is a piece called a "mil-foot"; this piece is $\frac{1}{1,000}$ of an inch in diameter and one foot long. The word *mil* means one-thousandth. The cross-sectional area of a wire whose diameter is one mil is one *circular mil.*

(*a*) If wire #30 has a diameter of 10.03 mils, what is the total volume of 10 miles of this wire?

(*b*) If the wire weighs 0.3042 lbs. per 1,000 linear feet, what is the length of a side of a cube of copper which is equal in weight to 10 miles of this wire?

7. If pure gold is worth $35 a Troy ounce, and a cubic foot of pure gold weighs 1204 avoirdupois pounds, how big a cube could be made from one billion dollars' worth of gold?

8. The crater of a certain volcano is conical in shape and is 1 mile in diameter. The greatest depth is 2,000 feet. If the level of the molten lava is 500 feet below the surface of the crater, how many cubic yards of lava are in the crater of the volcano?

9. How many square yards of canvas will be required to make a conical circus tent 50 feet high and 100 feet in diameter if 10 per cent of the material is allowed for cutting and fitting?

10. If a cubic foot of ivory weighs 114 pounds, find the weight of 100 gross of billiard balls $2\frac{1}{2}$ inches in diameter.

11. If gold leaf costs 20 cents per square inch, what will it cost to cover the hemispherical dome of a mausoleum whose diameter is 18 feet?

12. An airplane is out of sight of land on a flight from New York to London. At a height of 2,000 feet, how much of the ocean's surface can be seen? At an altitude of 10,000 feet? (*Hint:* Take the diameter of the earth as 7,920 miles.)

29 THE HIERARCHY OF BALANCES

The problems of engineering, physics, astronomy, and so forth, involve space and time, and very often lead to equations of some degree higher than the second. To solve problems involving equations of degree higher than the second requires a more extensive knowledge of mathematics. However, graphic representation affords a simple and useful way of attacking such problems. The possibility of associating an equation with a curve has been demonstrated in the preceding chapters. The elementary method of tracing a curve consists in determining values of x and y satisfying the equation, plotting the corresponding points, and joining these points by a smooth curve. This method merely indicates that the curve passes through all points whose coordinates are determined but sheds no light on the nature of the curve *between* any two of the successive points plotted. This lack of information may be the cause of serious error. To eliminate this possibility of error, the general properties of the curve may be determined by a discussion of the equation before plotting.

The following properties of the curve of an equation can usually be obtained by a preliminary discussion:

(I) The intercepts of the curve;
(II) The symmetry of the curve;
(III) The extent of the curve.

INTERCEPTS

The *x-intercepts* are the distances from the origin cut off by a curve on the X-axis, that is, the abscissas of the points where a curve crosses the X-axis. Similarly, the distances from the origin cut off by a curve on the Y-axis are the *y-intercepts.* Therefore, the y-intercepts are the ordinates of the points where a curve crosses the Y-axis.

Since the ordinate of any point on the X-axis is zero, to find the x-intercepts we simply substitute 0 for y in the equation

and solve for x. Likewise, the abscissa of any point on the Y-axis is zero; hence, to find the y-intercepts, put $x = 0$ in the equation and solve for y. Thus, the curve $x^2 + y^2 = a^2$ crosses the X-axis at $(\pm a, 0)$ and the Y-axis at $(0, \pm a)$.

SYMMETRY

Two points, A and B, Figure 268, are said to be *symmetric with respect to a point* O, when O bisects the line joining A and B. *A curve is symmetric with respect to a point* O, Figure 268, if to each point A on the curve there corresponds another point B such that O is the midpoint of AB. The point O is called the *center of symmetry*. Thus, the circle, ellipse, and hyperbola are symmetric with respect to their centers.

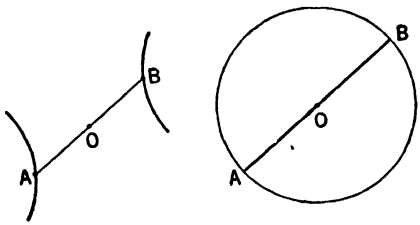


Fig. 268.

If the two points, A and B, Figure 269, are symmetric with respect to the origin, then the point O bisects the line AB, that is, $OA = OB$; hence, the coordinates of A must be numerically equal to the coordinates of B and opposite in sign. Now, if a curve is symmetric with respect to the origin, then to any point $A(x, y)$ on the curve there corresponds another point $B(-x, -y)$ also on the curve. Since both points lie on the curve, their coordinates must satisfy the equation of the curve. Therefore, *if an equation is unchanged when both* x *and* y *are replaced by* $-x$ *and* $-y$, *respectively, the curve is symmetric with respect to the origin.* Thus, the equation of the curve shown in Figure 269 is $y = x^3$. Substituting $-x$ for x and $-y$ for y gives $-y = (-x)^3$, or $-y = -x^3$; that is, $y = x^3$. Since the equation is unchanged, the curve represented by $y = x^3$ is symmetric with respect to the origin. It is easily seen that the equation is unchanged by the simultaneous substitution of $-x$ and $-y$ for x and y, *when each term is of even degree, or when each term is of odd degree, in* x *and* y.

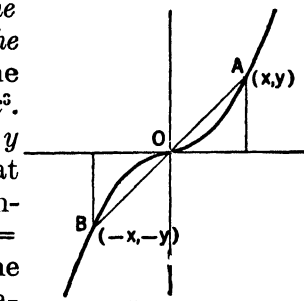


Fig. 269.

If a line AB, Figure 270, is the perpendicular bisector of a line

CD, joining points C and D, the two points C and D are said to be symmetric with respect to AB. The line AB is called the *axis of symmetry*. A curve is symmetric with respect to an axis if to each point C on the curve there corresponds another point D also on the curve such that the axis is the perpendicular bisector of CD.

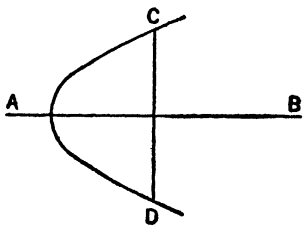


Fig. 270.

Applying this definition it may be easily seen that:

(1) *If an equation is unchanged when* y *is replaced by* $-y$, *the curve is symmetric with respect to the X-axis.*

(2) *If an equation is unchanged when* x *is replaced by* $-x$, *the curve is symmetric with respect to the Y-axis.*

Thus, $y^2 = 4x - 5$ represents the parabola shown in Figure 271 (see Chapter 23, page 400). Substituting $-y$ for y gives $(-y)^2 = 4x - 5$, or $y^2 = 4x - 5$. The equation is unchanged by the substitution; therefore, the parabola is symmetric with respect to the X-axis. In plotting the graph of this equation, it is then sufficient to limit our calculations to points in the first quadrant, for the part of the curve lying in the fourth quadrant is an exact reflection of the portion lying in the first quadrant.

Similarly, if $-x$ is substituted for x in the equation $3y^3 + x^2 = 0$, we obtain $3y^3 + (-x)^2 = 0$, or $3y^3 + x^2 = 0$. The equation is unchanged by the substitution; therefore, the curve represented by this equation is symmetric with respect to the Y-axis.

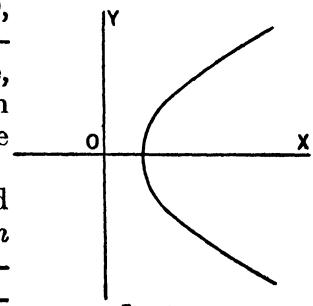


Fig. 271.

Clearly, an equation is unchanged when $-y$ is substituted for y *when only even powers of* y *appear in the equation.* Likewise, an equation is unchanged when $-x$ is substituted for x *when only even powers of* x *appear in the equation.* Hence, to ascertain whether the graph of an equation is symmetric to the origin, to the X-axis, or to the Y-axis, it is sufficient to note the exponents of x and y.

For example, the symmetry of the graph represented by

$x^2 + y^2 + 4x = 10$ can be determined as follows. Not all terms are of even degree in x and y, nor are all odd; hence, the graph is not symmetric with respect to the origin. An odd power of x, namely, $4x$, occurs; therefore, the curve is not symmetric with respect to the Y-axis. Since only even powers of y appear in the equation, the curve is symmetric with respect to the X-axis.

EXTENT

If we know beforehand that a certain curve does not go beyond a definite point or line, we can restrict our inquiry. That is, in plotting the graph of an equation, if the extent of the curve can be ascertained, a lot of unnecessary work may be eliminated. The extent of a curve may often be determined by solving for each variable in terms of the other. If a radical of even index appears, certain values of a variable may give imaginary values for the other variable. These values must be excluded, because coordinates are real numbers. Furthermore, the solution for one of the variables in terms of the other may result in a fraction whose denominator contains the other variable. In that case, one of the variables may increase indefinitely as we approach a definite value of the other.

DISCUSSION OF AN EQUATION

Consider the equation $3y^3 + x^2 = 0$, already mentioned in the preceding pages, and let us determine its graph.

Putting $y = 0$, we find $x = 0$, the intercept on the X-axis. Substituting 0 for x gives $y = 0$, the intercept on the Y-axis. Thus, the origin is a point on the curve.

The equation contains no odd powers of x; hence, the curve is symmetric with respect to the Y-axis.

Solving for x, we obtain $x^2 = -3y^3$ or $x = \pm\sqrt{-3y^3}$, that is, $x = \pm y\sqrt{-3y}$. When positive values are substituted for y, the expression $-3y$ is negative and x is imaginary. Therefore, all positive values of y must be excluded; that is, the curve does not exist above the X-axis. For each negative value of y, two values of x are determined, namely, a positive and a negative value. Therefore, the curve exists in the third and fourth quadrants. Since the curve is symmetric with respect

to the Y-axis, we may restrict our inquiry to the fourth quadrant only.

The curve may now be plotted by substituting in the equation negative values of y, determining the values of x, plotting the corresponding points, and joining these points by a smooth curve. However, a further study of the equation is advisable. The curve evidently touches the origin but does not cross the

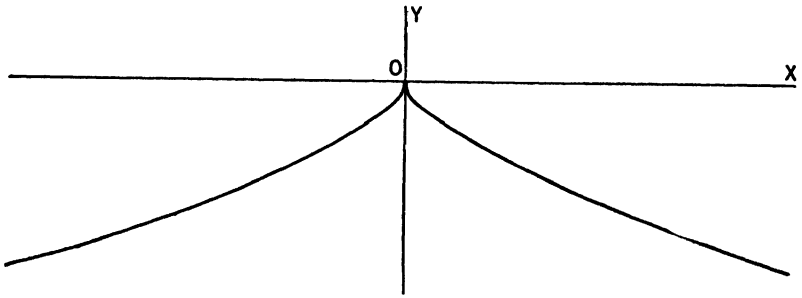


Fig. 272.

X-axis. The behavior of the curve near the origin is therefore an important factor. If y is assumed to be equal to $-\frac{1}{27}$, the corresponding value of x is found to be $\pm\frac{1}{81}$. Putting $y = -\frac{1}{243}$, the value of x is $\pm\frac{1}{2187}$. The difference in the values of x is $\frac{1}{81} - \frac{1}{2187} = \frac{27-1}{2187} = \frac{26}{2187}$, while the difference in the values of y is: $-\frac{1}{27} - \left(-\frac{1}{243}\right) = \frac{-9+1}{243} = \frac{-8}{243} = \frac{-72}{2187}$. That is, if the values of x are very small, the difference between two successive values of y is large by comparison. Therefore, the curve is concave downward assuming the shape shown in Figure 272.

FACTORABLE EQUATIONS

If the product of two or more factors is equal to zero, the equation is satisfied whenever any one of the factors is zero. Thus, $x^2 - y^2 = 0$ can be reduced to $(x - y)(x + y) = 0$. Now, any values of x and y that will make any one of these fac-

tors equal zero satisfies the equation. Therefore, the curve representing the equation is composed of the distinct curves obtained by putting each one of the factors equal to zero. But, $x - y = 0$ and $x + y = 0$ are represented by two straight lines. Therefore, $x^2 - y^2 = 0$ is represented by the two straight lines shown in Figure 273.

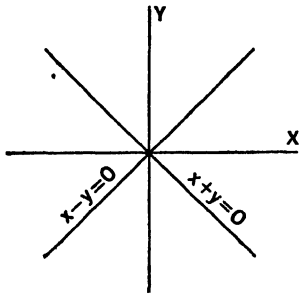


Fig. 273.

SOLUTION OF EQUATIONS OF HIGHER DEGREE

As stated at the beginning of this chapter, there is no easy method of solving equations of degree higher than the second, except the simple cases of factoring discussed in Chapter 4, page 48. The real roots of an equation, however, may be approximately determined by drawing a graph.

Consider the equation $x^3 - 2x - 5 = 0$. To solve this equation is to find the values of x for which the function $y = x^3 - 2x - 5$ will be zero. Obviously, the values of x for which y will be zero are the x-intercepts of the curve determined by the function $y = x^3 - 2x - 5$. That is, the abscissas of the points in which the graph of $y = x^3 - 2x - 5$ crosses the X-axis are the roots of the equation $x^3 - 2x - 5 = 0$.

Naturally, in solving for the roots of an equation, we are primarily interested in the graph near the x-intercepts. Moreover, it is important to ascertain that all the points of intersection, that is, all the real roots, are shown. In determining the points of intersection, an analysis of the equation will undoubtedly save time and labor.

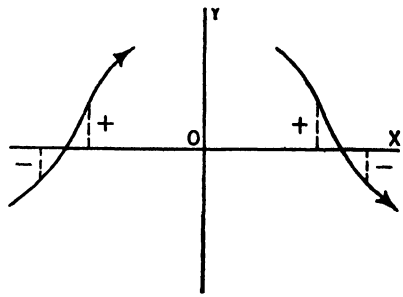


Fig. 274.

Suppose that a curve is considered as always drawn from left to right, as in Figure 274. If the curve comes up from below and crosses the X-axis, any point on the curve to the left of the point of intersection must have a negative ordinate, while a point to the right of the point of intersection has a positive ordinate. Similarly, if the curve comes down and crosses

the X-axis, a point on the curve to the left of the point of intersection has a positive ordinate while a point to the right of the point of intersection has a negative ordinate. That is, if the curve of an equation crosses the X-axis, the values of y must show a change in sign. Therefore, in forming the table of values to plot the curve of a given equation, if after a certain positive value of x it is seen that the values of y will continually increase or continually decrease, we can conclude that the curve will not cross the X-axis for any greater value of x. Similarly, if after a certain negative value of x it is seen that the values of y will continually increase or continually decrease, the curve will not cross the X-axis for any smaller value of x.

Thus, it may be seen, by studying the function $y = x^3 - 2x - 5$, that, for any value of x greater than 3, the values of y will continually increase. Furthermore, for any negative value of x, the function takes the form $y = (-x)^3 - 2(-x) - 5$, or $y = -x^3 + 2x - 5$. In order to obtain positive values of y, the expression $-x^3 + 2x$ must be positive and greater than 5; that is, $-x^3 + 2x > 5$, or $(-x)(x^2 - 2) > 5$. Now, $(-x)(x^2 - 2)$ can be positive only if $(x^2 - 2)$ is negative, that is, if $x^2 - 2 < 0$ or $x^2 < 2$. Thus, the negative value of x chosen must be such that $x^2 < 2$. But, for any negative value of x such that $x^2 < 2$, the expression $x^2 - 2$ is evidently less than 2, and of course the factor $(-x)$ is also less than 2. Hence, their product, that is, $(-x)(x^2 - 2)$, cannot be greater than 5. Therefore, for any negative value of x, the value of y will also be negative. That is, the curve does not cross the negative side of the X-axis, and hence the equation $x^3 - 2x - 5 = 0$ has no negative real roots.

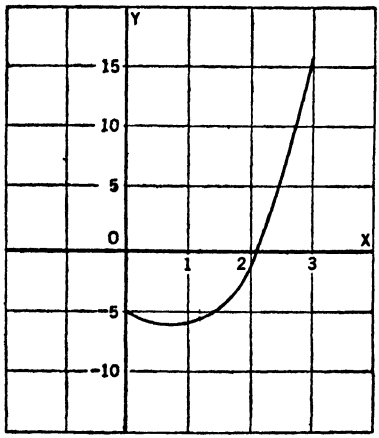


Fig. 275.

The real roots of the equation can, therefore, only occur for positive values of x between 0 and 3. Calculating values of y, by substituting values of x between 0 and 3 in the equation $y = x^3 - 2x - 5$, we obtain the adjoining table.

y	-5	$-5\frac{7}{8}$	-6	$-4\frac{5}{8}$	-1	$5\frac{5}{8}$	16
x	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3

Plotting these values and carefully drawing the graph, Figure 275, the curve is seen to intersect the X-axis in one point. At this point, $y = 0$, the abscissa must be such as to make $x^3 - 2x - 5$ equal to 0. That is, this abscissa is a root of the equation $x^3 - 2x - 5 = 0$. The value of the root, as obtained from the graph, is approximately 2.1.

SIMULTANEOUS SOLUTIONS

The simultaneous solution of two equations of higher degree may be easily performed by the use of graphs. As already stated in Chapter 15, page 242, the coordinates of each point of intersection must satisfy both equations. Therefore, the solutions correspond to the common points of the two curves. Thus, to solve $x^2 + y^2 = 16$ and $y^2 = 6x$ simultaneously, plot the curves of the two equations. The graph of $x^2 + y^2 = 16$ is a circle with center at the origin and radius of 4 (Chapter 20, page 312). The equation $y^2 = 6x$ is represented by a parabola whose axis is the X-axis and whose vertex is at the origin (Chapter 23, page 393). The graphs of these two equations are shown in Figure 276. From the figure, the points of intersection are $(2, 3\frac{1}{2})$ and $(-2, 3\frac{1}{2})$, approximately.

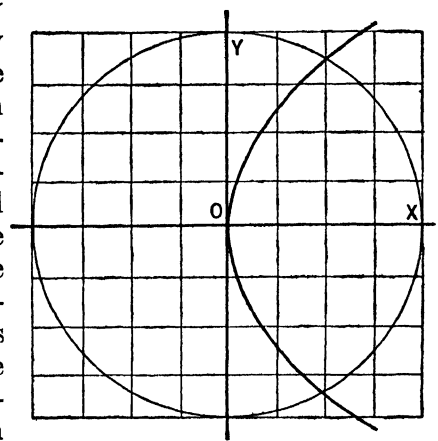


Fig. 276.

EXPONENTIAL EQUATIONS

It has been observed that many natural phenomena follow more or less closely the relationship expressed by the *exponential equation* $y = kb^x$. Consequently, this relationship is often called the "law of organic growth." It finds applications in physics to express the relation between the height above sea

level and atmospheric pressure; in chemistry to show the progress of a chemical action; in biology to depict the growth of the number of bacteria in a culture; and, in fact, in almost all fields of applied science.

Suppose that the law of increase for a certain kind of bacteria is $y = kb^x$ and assume that $k = 1$ and $b = 3$, that is, $y = 3^x$.

If $x = 0$, $y = 3^0 = 1$. Hence, the curve crosses the Y-axis at the point (0, 1). In general, for $k = 1$ and any value of b except $b = 0$, $y = b^0 = 1$. Hence, the curve of $y = kb^x$ crosses the Y-axis at (0, 1). When $y = 0$, $0 = kb^x$, which is impossible for any definite value of x. Hence, the curve does not cross the X-axis.

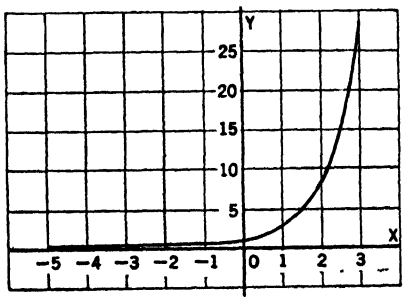


Fig. 277.

The curve is not symmetric to any of the axes or to the origin, since substituting $-x$ for x, $-y$ for y, or both $-x$ and $-y$ for x and y changes the equation.

When $k = 1$ and b is positive, as in the case $y = 3^x$, the value of y is positive for all integral values of x. That is, no integral value of x can make y negative. Thus, when $x = 2$, $y = 3^2 = 9$; when $x = -2$, $y = 3^{-2} = \dfrac{1}{3^2} = \frac{1}{9}$. If the value of x is a fraction, such as $\frac{1}{2}$, we obtain $y = b^{\frac{1}{2}}$. The value of y may be readily found by means of logarithms; for, if $y = b^{\frac{1}{2}}$, then $\log y = \frac{1}{2} \log b$ (see Chapter 25, page 445). Since negative numbers have no real logarithms (Chapter 25, page 446), negative values of y are excluded. Therefore, the curve exists only above the X-axis. Moreover, as the positive values of x are increased, the values of y also increase, so that the curve continually rises from left to right. But as the negative values of x are numerically increased, the values of y decrease. Thus, when $x = -2$, $y = 3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$; when $x = -4$, $y = 3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$. Clearly, by increasing numerically the negative values of x, the values of y can be made as small as we

please. However, no matter how much the negative values of x are increased numerically, y cannot be made equal to zero. Therefore, the curve approaches nearer and nearer to the negative side of the X-axis without ever crossing it.

The coordinates of points on the curve may be found by taking logarithms of both sides of the equation $y = 3^x$. Hence, $\log y = x \log 3$, or $\log y = 0.4771\ x$. Substituting values of x in this equation, we obtain the following table:

x	-5	-4	-3	-2
$\log y$	$\bar{3}.6145$	$\bar{2}.0916$	$\bar{2}.5687$	$\bar{1}.0458$
y	0.004	0.012	0.038	0.111

x	-1	0	1	2	3
$\log y$	$\bar{1}.5229$	0	0.4771	0.9542	1.4313
y	0.333	1	3	9	27

Plotting these values and drawing a smooth curve through the points, we obtain the graph of $y = 3^x$ shown in Figure 277.

The curve of $y = kb^x$, when $k = -1$ or is negative, may be developed by a similar discussion. It may be easily seen that this curve will continually rise from right to left while it approaches nearer and nearer to the positive side of the X-axis. The graph of $y = -3^x$ is shown in Figure 278.

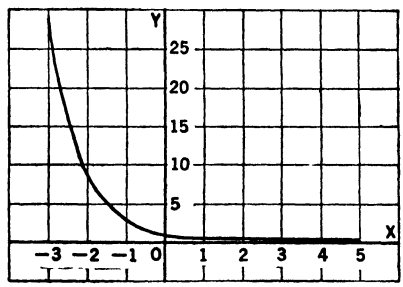


Fig. 278.

LOGARITHMIC EQUATIONS

The equation $A = P(1 + i)^n$ represents the compound amount of P dollars, that is, the amount to which a principal of P dollars will accumulate, at a compound interest rate of i per annum, over a period of n years. (See Chapter 24, page

413.) Thus, if the principal is one dollar, the compound interest rate is 4 per cent per annum, and s denotes the amount, $s = 1.04^n$. If the sum is considered as a function of the time, the values of n are plotted on the horizontal axis and the corresponding values of the sum s are plotted along the vertical axis. In this case, we obtain a curve of the type $y = kb^x$. The reader may ascertain this fact by merely comparing the equations $s = 1.04^n$ and $y = kb^x$, or by plotting the graph of $s = 1.04^n$. However, if the time is made a function of the sum, we may write $\log s = n \log 1.04$, that is, $\log s = 0.017n$, $n = \frac{1}{0.017} \log s$, or $n = 58.82 \log s$. This equation is of the type $y = \log x$ and is called a *logarithmic equation*. If $k = 1$, the equation $y = kb^x$ reduces to $y = b^x$. Thus, $y = b^x$ and $y = \log_b x$ are exponential or logarithmic, depending only on the point of view. As might be expected, the graph of $y = \log_b x$ differs from the curve of $y = b^x$ only in its relation to the axes. That is, the curve of the logarithmic equation may be obtained by simply interchanging the X- and Y-axes.

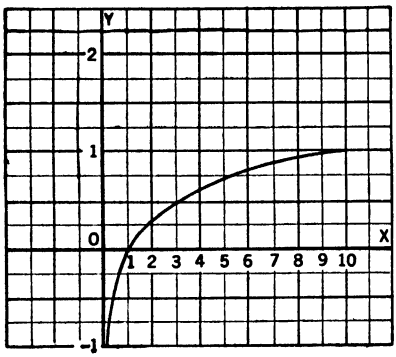


Fig. 279.

Consider, then, the equation $y = \log x$. When $y = 0$, $0 = \log x$ and $x = 1$. Hence, the curve crosses the X-axis at the point (1, 0). When $x = 0$, $y = \log 0$ or $10^y = 0$, which is impossible for any definite value of y. Therefore, the curve does not cross the Y-axis.

The curve of this equation is not symmetric to the axes or the origin, because substituting $-x$ for x, $-y$ for y, or both $-x$ and $-y$ for x and y changes the form of the equation.

Since negative numbers have no real logarithms (Chapter 25, page 446), negative values of x are excluded. That is, the curve does not exist to the left of the Y-axis. For all positive values of x greater than 1, the values of y are positive and increase with the values of x. Hence, the curve after crossing the X-axis at (1, 0) rises continually from left to right. Values of x between 0 and 1 give negative values of y. Thus, when $x = 0.5$, $y = \log 0.5 = \bar{1}.6990$, that is, $y = -1.0000 + .6990 = -0.3010$. The smaller the value of x chosen, the larger

numerically the negative values of y become. Hence, the curve extends indefinitely downward, approaching nearer and nearer to the Y-axis. The graph of $y = \log x$ is shown in Figure 279.

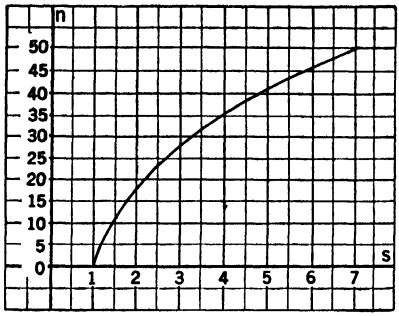


Fig. 280.

The curve of $n = 58.82 \log s$ may be developed by a similar discussion. Its graph is shown in Figure 280. Since negative values of n will not be applied in this case, the portion of the curve below the X-axis is not shown. This graph may be used for approximate results in compound interest problems where the rate is 4 per cent per annum. Thus, to determine the number of years required for a principal of \$150 to accumulate to \$250 at 4 per cent per annum, substitute in the formula $A = P(1 + i)^n$, obtaining $250 = 150(1.04)^n$, or $\frac{250}{150} = (1.04)^n$ and $1.67 = (1.04)^n$. Comparing this equation with $s = 1.04^n$, it is easily seen that $s = 1.67$. From the graph, when $s = 1.67$, the value of n is 13 years, approximately.

TRIGONOMETRIC EQUATIONS

A stretched wire or a tuning fork, when tapped, will vibrate. The vibrations push back the air in all directions, thus giving to the air particles which lie about the wire or fork a great outward velocity. These air particles collide with other particles, thereby imparting to these second particles an outward motion which these second particles in turn pass on to other particles, and so forth. That is, the motion started by the vibrating fork travels from particle to particle, but the particles themselves move only very short distances. The phenomenon may be easily visualized by the following mechanism. Suppose a series of balls to be hanging from a frame as shown in Figure 281. If the first ball is raised and then allowed to fall

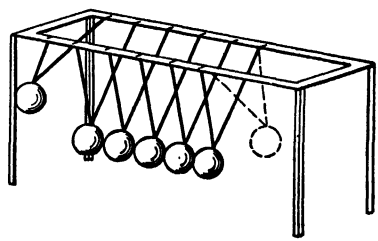


Fig. 281.

against the second, the motion will be transmitted from ball to ball, the last ball moving with virtually the same velocity as that originally possessed by the first ball, while the other balls which transmitted the motion give up the whole of their motion almost as soon as received. When the motion of the air particles reaches our ear drums, it produces, by tapping against them, the sensation of sound.

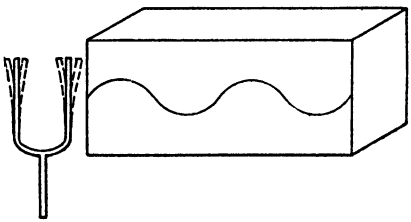


Fig. 282.

If a tuning fork is held near an oscillograph, the air vibrations caused by the fork are recorded in the form of a graph. The graph resulting from a vibrating tuning fork is shown in Figure 282.

Now consider the equation $y = \sin x$, and let us determine its curve. As already explained (Chapter 22, page 365), angles may be measured in degrees or radians. For convenience, in plotting the curve of a trigonometric function, radians are used. When $x = 0$, $y = \sin 0$ and $y = 0$. Thus, the curve crosses the Y-axis at the point (0, 0). If $y = 0$, $0 = \sin x$. The sines of 0, π, 2π, 3π, 4π, and so forth, are equal to zero. Furthermore, the sine of x is also equal to zero for $-\pi$, -2π, -3π, -4π, and so forth. Therefore, the curve crosses the X-axis in an indefinite number of points at a distance of π from one to the next.

If $-x$ is substituted for x, or $-y$ is substituted for y, the equation is changed. Hence, the curve is not symmetric with respect to the axes. Replacing x and y by $-x$ and $-y$, we obtain $-y = \sin(-x)$. But if two angles x and $-x$ are numerically equal, it is easily seen that if one is in the first quadrant, the other is in the fourth; if one is in the second quadrant, the other is in the third. Therefore, by referring to the table in Chapter 14, page 235, it follows that $\sin(-x) = -\sin x$. Hence, we may write $-y = \sin(-x) = -\sin x$ or, changing signs, $y = \sin x$. That is, the equation is not changed by the simultaneous substitution of $-x$ and $-y$ for x and y, respectively. Therefore, the curve is symmetric with respect to the origin.

Since the sine of any angle, however great, can be obtained, the curve extends indefinitely along the X-axis on both sides

of the origin. From the definitions of the trigonometric functions, given in Chapter 14, page 211, it is clear that the sine of angles between 0 and $\frac{\pi}{2}$ varies from 0 to 1; between $\frac{1}{2}\pi$ and π, it varies from 1 to 0; between π and $\frac{3}{2}\pi$, it varies from 0 to -1; and between $\frac{3}{2}\pi$ and 2π, it varies from -1 to 0. Evidently, these values repeat themselves as larger angles are considered. That is, the value of y in $y = \sin x$ is not greater than 1 nor less than -1. Therefore, the curve does not extend above the line $y = 1$ nor below the line $y = -1$. The curve then repeats itself at each interval of 2π. This property of a curve is expressed by the statement, "The curve of $y = \sin x$ is a *periodic curve* with a *period* of 2π."

Since the curve has a period of 2π, it is sufficient to calculate values of y for values of x between 0 and 2π. Substituting in the equation $y = \sin x$, we obtain the following table.

x	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
y	0	.5	.707	.866	1	.866	.707	.5	0

x	$\frac{7}{6}\pi$	$\frac{5}{4}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{7}{4}\pi$	$\frac{11}{6}\pi$	2π
y	$-.5$	$-.707$	$-.866$	-1	$-.866$	$-.707$	$-.5$	0

Choosing a unit on the Y-axis of the same length as a radian, and plotting, the curve shown in Figure 283 is obtained.

The resemblance with the curve traced by the sound waves of a tuning fork is immediately noted. The regularity of the waves determines the quality of a sound. A musical note is produced by a source sending out waves at absolutely regular intervals. If, on the other hand, the waves are irregular, our nerve-cells are disturbed and we experience the disagreeable sensation of noise. The principal characteristics of musical tones are *intensity*, *pitch*, and *quality*.

The intensity of a sound depends upon the amplitude of its vibration and the distance from the source. The *amplitude* of a periodic function is the greatest absolute value* of its ordinates. Thus, in the function $y = \sin x$, the values of y vary from $+1$ to -1. Hence, the amplitude of this function is 1.

* See page 563.

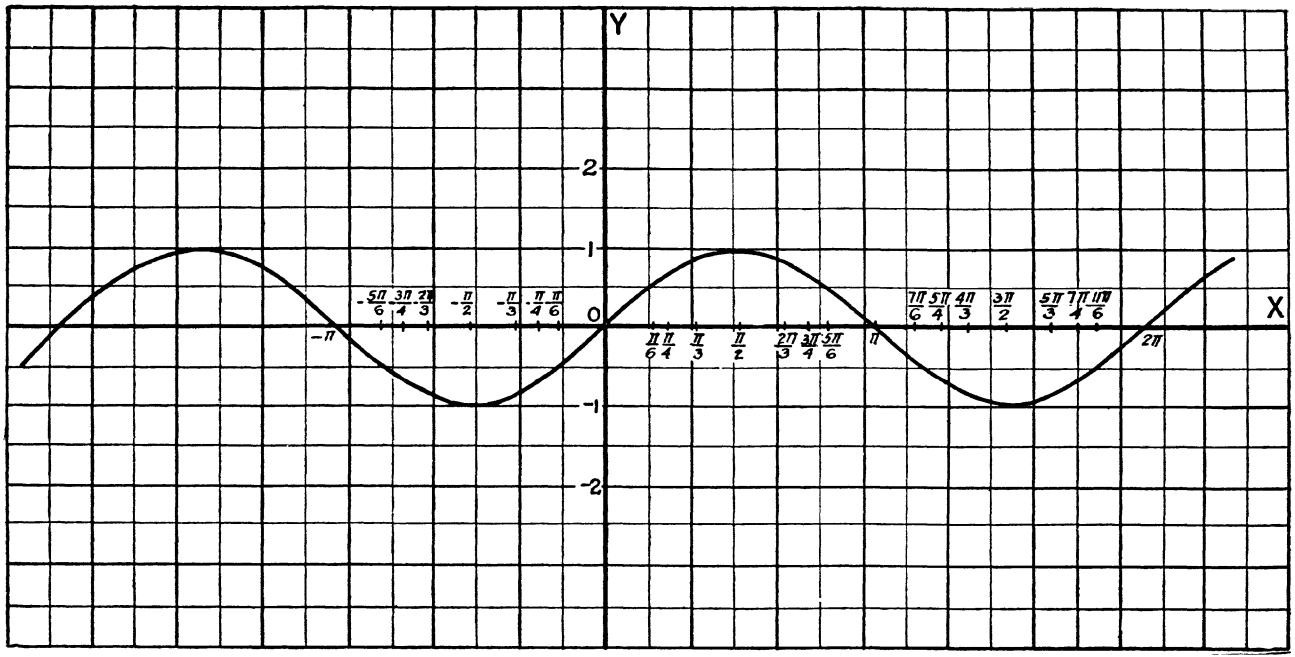

Y
X
2
1
0
-1
-2
$-\frac{5\pi}{6}$ $-\frac{3\pi}{4}$ $-\frac{2\pi}{3}$ $-\frac{\pi}{2}$ $-\frac{\pi}{3}$ $-\frac{\pi}{4}$ $-\frac{\pi}{6}$
$-\pi$
$\frac{\pi}{6}$ $\frac{\pi}{4}$ $\frac{\pi}{3}$ $\frac{\pi}{2}$ $\frac{2\pi}{3}$ $\frac{3\pi}{4}$ $\frac{5\pi}{6}$
π
$\frac{7\pi}{6}$ $\frac{5\pi}{4}$ $\frac{4\pi}{3}$ $\frac{3\pi}{2}$ $\frac{5\pi}{3}$ $\frac{7\pi}{4}$ $\frac{11\pi}{6}$
2π

Fig. 283.

In general, in an equation of the type $y = n \sin kx$, the values of y will range from $+n$ to $-n$ inclusive. Thus, the greatest absolute value of y, namely, n, is its amplitude.

A vibrating body sends out sound waves in every direction. These waves form a spherical shell having the vibrating body as its center (see Figure 284). The spherical shell becomes larger and larger as the sound waves are transmitted from particle to particle; as the wave is transmitted from particle to particle, it loses in intensity.

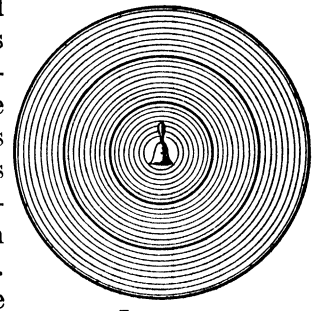


Fig. 284.

The *pitch* of a tone is its relative position in the musical scale. The number of vibrations made per second by the vibrating body producing the sound determines its pitch. If the vibrations are rapid, the pitch is high; while if the vibrations are slow, the pitch is low. Thus, as is well known, the speed of an electric motor can be judged by the pitch of its sound. The wave length of a tone of low pitch is greater than that of a sound having a high pitch. Hence, a tone may be measured by the period of its wave.

COMPOSITE EQUATIONS

Most sounding bodies vibrate not only as a whole, but in various parts. This fact enables us to distinguish the sound of a piano, violin, or clarinet, although the notes emitted by these instruments be of the same pitch and loudness. These three instruments when producing a note of the same pitch and loudness give a *fundamental* tone of the same wave length. However, the partial vibrations produce various *overtones* which give the sound its *quality*. That is, the quality of a sound is determined by the number and prominence of its perceptible overtones. The graph of a tone rich in quality is a complex one. However, it may be readily plotted by the method of *addition of ordinates*.

A tuning fork produces a graph of the simple form shown in Figure 283. If two tuning forks of slightly different pitch are placed near each other, the interference and coincidence of the sound waves produce variations in the intensity of the sound.

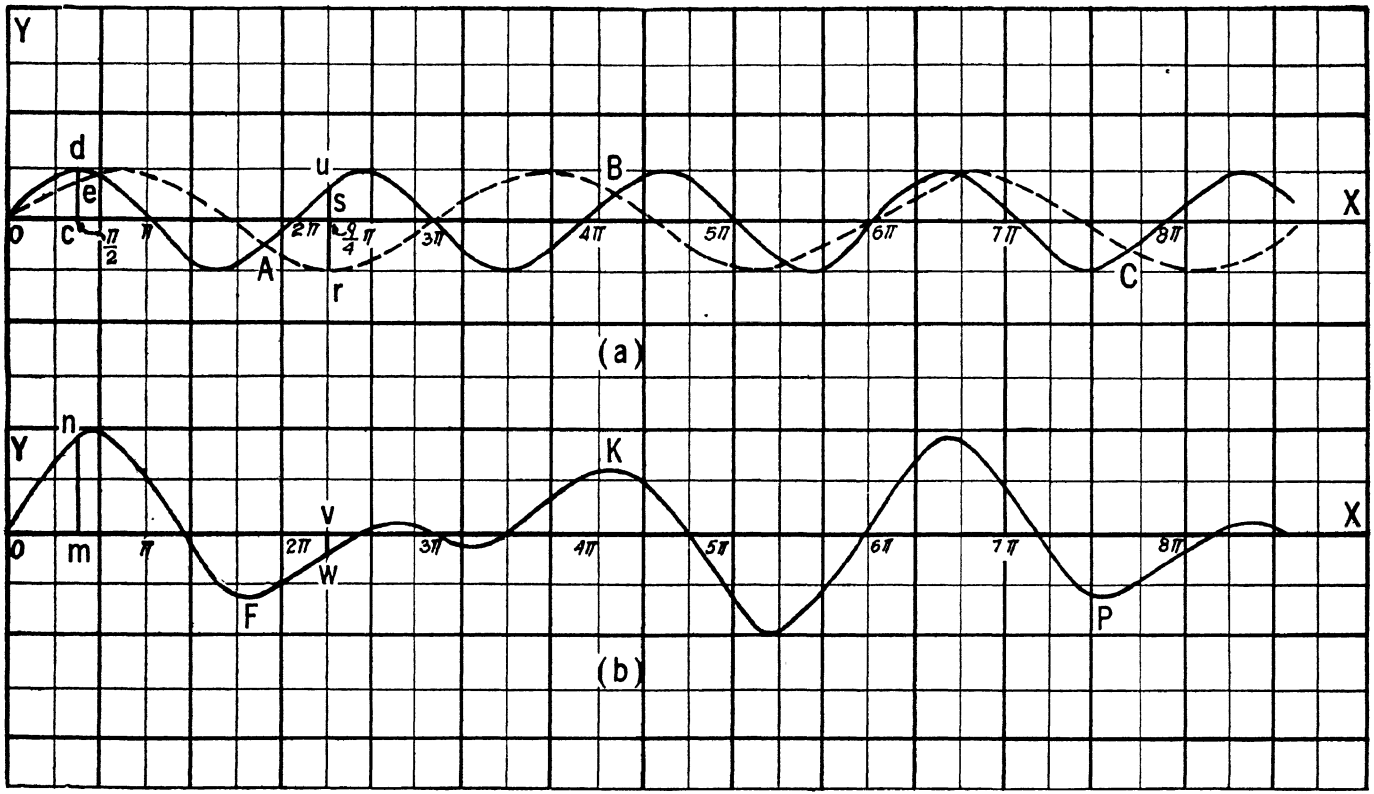

Y
d
e
u
s
B
X
o
c
π/2
π
A
2π
9/4 π
r
3π
4π
5π
6π
7π
8π
C
(a)
n
Y
K
X
o
m
π
v
2π
W
3π
4π
5π
6π
7π
8π
F
P
(b)

Fig. 285.

That is, the smooth, even tone of a tuning fork will be replaced by a throbbing or pulsating one due to the alternate reinforcement and neutralization of the sounds produced by the two forks. These *beats* can be easily detected by the ear.

Suppose that the sound waves sent out by the two vibrating bodies are represented, respectively, by the equations $y_1 = \sin x$ and $y_2 = \sin \frac{2}{3}x$. Following the method shown on page 543, plot the curves of these two equations, as in Figure 285 (*a*), using the same scale for both. From the figure, it is easily seen that the waves will interfere from A to B, while they will reinforce each other from B to C. The equation of the sound waves resulting from the simultaneous vibrations of the two bodies is the sum $y = y_1 + y_2$ or $y = \sin x + \sin \frac{2}{3}x$. To obtain the curve of this equation, add the ordinates of $y_1 = \sin x$ to the corresponding ordinates of $y_2 = \sin \frac{2}{3}x$. Thus, for $x = \frac{\pi}{2}$ the value of y_1 is $\sin \frac{\pi}{2}$, that is, $y_1 = +1$; while the value of y_2 is $\sin \frac{2}{3}\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{3}$, or $y_2 = +.866$. The value of $y = \sin x + \sin \frac{2}{3}x$, for $x = \frac{\pi}{2}$, is therefore $y = +1 + .866 = +1.866$. Graphically, this value can be obtained by measuring ce and cd in Figure 285 (*a*). Their sum is mn in Figure 285 (*b*).

Similarly, when $x = \frac{3}{4}\pi$, measure with the compasses sr and su. To su add $(-sr)$. Mark off the result, $-vw$, in Figure 285 (*b*). The curve of $y = \sin x + \sin \frac{2}{3}x$ is shown in Figure 285 (*b*). The beat will obviously occur between K and P, as that part of the curve diverges the greatest distance from the X-axis, while from F to K there will be very little sound.

Problems

1. The equation $S = C(1 - r)^n$ represents the book value of an asset depreciating at a rate of r per annum, when the original cost is C and its probable life is n years. Thus, if the cost is one dollar, the rate of depreciation is 10 per cent, and v denotes the book value,

$$v = .90^n.$$

Discuss this equation and plot its graph.

2. Discuss and sketch the graphs of the following trigonometric equations:

(*a*) $y = \tan x$, (*b*) $y = \sec x$.

3. The structural engineer frequently desires to ascertain the maximum deflection of a beam. In finding the maximum deflection of a 15-foot beam, loaded in a certain way, it is necessary to solve the equation:

$$4x^3 - 90x^2 + 540x - 1125 = 0.$$

Find the positive root of this equation.

4. A sum of money is invested at simple interest. The principal and its interest for one year will amount to \$520. If the sum were increased by \$50 and the rate of interest by 1 per cent, the amount would be \$577.50. Find the principal and the interest.

5. After traveling 7 hours, a passenger bus is still 120 miles from its destination. If the bus had traveled 3 miles per hour faster, it could have made the entire trip in 1 hour less than the scheduled time. Find the distance and the actual speed for the entire trip.

6. The population growth of the United States between 1895 and 1935 is expressed by the exponential function:

$$y = 71.98 \times 10^{.00664x},$$

where x is the number of years after 1895 and y is the population in millions.

(*a*) Plot the graph of this equation.

(*b*) From the graph, estimate the population in 1918.

(*c*) If we assume that the growth of the population will continue to be represented by this function, find from the equation what the population of the United States was in 1940.

7. The book value of an asset depreciating at a constant annual rate is given by the formula:

$$S = C(1 - r)^n.$$

A machine costing \$14,000 is estimated to depreciate at the rate of 10 per cent per annum. Draw a graph showing the book value of the machine during its probable life of 20 years. Find, from the graph, the book value at the end of 17 years. Check your result by computing this value from the graph obtained in Problem 1.

8. In the interpretation of numerical data, the most widely used curve is perhaps "the probability curve," also known as "the normal curve of error." This curve is represented by the equation:

$$y = ke^{-ax^2},$$

where k and a are constants determined from the numerical data

and e is also a constant equal to 2.71828. The constant e is the base of the natural or Naperian system of logarithms.

A statistician, desiring to ascertain the number of hats of each size which should be manufactured by his concern, obtains the following equation:

$$y = (252.51)[(2.72)^{-1.14x^2}].$$

Discuss this equation and plot its graph.

9. The equation

$$y = .3 \sin 1900t + .5 \sin 1500t$$

represents the single sound emitted by a musical instrument. Discuss this equation and plot its graph.

10. If the electromotive force producing an alternating current is cut off, the current does not vanish instantly, but tends to die out. The die-out phenomena of an alternating current can be represented by a curve that is a very close approximation to the graph of

$$i = e^{-at} \sin kt,$$

where i denotes the intensity of the current, t the time elapsed since the electromotive force was cut off, a and k are constants determined from the numerical data, and e, the base of the natural system of logarithms, is equal to 2.71828. The intensity of a certain current at any instant varies with the time according to the law:

$$i = e^{-\frac{1}{2}t} \sin \pi t.$$

Discuss this equation and plot its graph.

30 COMPLEX NUMBERS

The rules of operation for signed numbers were *postulated* simply for the convenience resulting from the use of the + and − symbols to distinguish quantities of opposite character. These laws, explained in Chapter 1, made necessary still another extension of the concept of number. The square of any real number, whether positive or negative, is positive or zero. That is, $(+2)^2 = (+2)(+2) = +4$, and $(-2)^2 = (-2)(-2) = +4$. Hence, $\sqrt{-4}$ is not equal to any real number. For, let $b = \sqrt{-4}$. Then $b^2 = -4$, which, as already shown, cannot be possible. This difficulty evidently arises whenever an even root of a negative number is to be extracted.

THE IMAGINARY UNIT

This situation was difficult to deal with until a new number, $i = \sqrt{-1}$, defined to be such that $i^2 = -1$, was introduced. This number i is called the *imaginary unit.* Here, as in several other cases, mathematics, owing to the permanence of usage, has adopted a name which is somewhat confusing and unsatisfactory. Although i is called the imaginary unit, it has just as much actual existence, as an idea, as the so-called real numbers. However, the number $i = \sqrt{-1}$ differs from any real number not only in magnitude, but also in character. That is, $\sqrt{-1}$ cannot be said to be greater than any particular real number for the same reason that a barrel cannot be compared in magnitude with a yard.

The number i is easy to operate with by adhering, as far as possible, to the rules already set forth for operating with real numbers.

Thus,

$$i + i = 2i,$$
$$i + i + i = 3i,$$
$$\underbrace{i + i + i + \cdots + i}_{n \text{ terms}} = ni.$$

Multiplication of real numbers, as already stated on page 10, is in accord with the following rule of signs:

$$(+1) \times (+1) = +1 \qquad (-1) \times (+1) = -1$$
$$(-1) \times (-1) = +1 \qquad (+1) \times (-1) = -1$$

Similarly, with i we have:

$$i \times (+1) = \sqrt{-1}$$
$$i \times (-1) = -\sqrt{-1}$$
$$i \times 0 = 0$$
$$i \times i = i^2 = \sqrt{-1} \times \sqrt{-1} = -1$$
$$i \times i \times i = i^3 = \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} = -\sqrt{-1}$$

Powers of i occur frequently. The following table deserves careful study:

$i^1 = i$	That is	$(\sqrt{-1})^1 = \sqrt{-1} = i$
$i^2 = -1$		$(\sqrt{-1})^2 = (\sqrt{-1})(\sqrt{-1}) = -1$
$i^3 = i^2 \cdot i = (-1)(i) = -i$		$(\sqrt{-1})^3 = (\sqrt{-1})^2(\sqrt{-1}) = (-1)(\sqrt{-1}) = -i$
$i^4 = i^3 \cdot i = (-i)(i) = +1$		$(\sqrt{-1})^4 = (\sqrt{-1})^3(\sqrt{-1}) = (-\sqrt{-1})(\sqrt{-1}) = +1$
$i^5 = i^4 \cdot i = (+1)(i) = +i$		$(\sqrt{-1})^5 = (\sqrt{-1})^4(\sqrt{-1}) = (+1)(\sqrt{-1}) = +i$
$i^6 = i^5 \cdot i = (i)(i) = -1$		$(\sqrt{-1})^6 = (\sqrt{-1})^5(\sqrt{-1}) = (\sqrt{-1})(\sqrt{-1}) = -1$
$i^7 = i^6 \cdot i = (-1)(i) = -i$		$(\sqrt{-1})^7 = (\sqrt{-1})^6(\sqrt{-1}) = (-1)(\sqrt{-1}) = -i$
$i^8 = i^7 \cdot i = (-i)(i) = +1$		$(\sqrt{-1})^8 = (\sqrt{-1})^7(\sqrt{-1}) = (-\sqrt{-1})(\sqrt{-1}) = +1$

It will be noted, from this table, that *odd* powers of i are equal to $-i$ or $+i$, and that *even* powers of i are equal to -1 or $+1$.

Any integral power of i can readily be determined by applying the laws of exponents (see page 459). Thus,

$$i^{15} = i^8 \cdot i^7 = (+1)(-i) = -i.$$

PURE IMAGINARY NUMBERS

Some further comment is required to develop the number system now in use.

A *pure imaginary number* is the product of a real number and the imaginary unit. Thus, $4i$, $-5i$, and $\pm 29i$ are pure imaginary numbers.

In general, bi, when b represents any real number, is called a pure imaginary number.

To be consistent with the rules of operation with radicals developed in Chapter 6, we write:

$$\pm\sqrt{-b^2} = \pm\sqrt{b^2(-1)} = \pm\sqrt{b^2}\sqrt{-1} = \pm bi,$$

and say that a pure imaginary number is *positive* or *negative* according as the radical sign is preceded by a positive or negative sign. Thus, $(4i)^2 = 16i^2 = 16(-1) = -16$, so that $\sqrt{-16} = \pm 4i$. That is, in general, $\pm\sqrt{-b^2} = \pm bi$.

COMPLEX NUMBERS

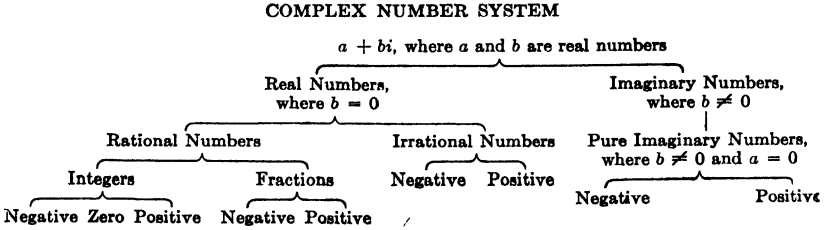
Any real number a connected by the $+$ or $-$ sign with a pure imaginary number forms an expression $a + bi$, which is called a *complex number*. Thus, $5 + 3i$ and $7 - 8i$ are complex numbers. Note that a complex number $a + bi$ consists of two parts which are of different kinds: the real part a and the pure imaginary part bi. In a complex number, the numbers a and b may be any real numbers whatever, including 0.

If $b = 0$, the complex number $a + bi$ reduces to $a + (0)(i) = a + 0 = a$. Hence, in this case we have a real number. For this reason, the complex number system includes the real numbers.

If b is *not* equal to zero, the complex number $a + bi$ is called an *imaginary number*. Note that the class of imaginary numbers contains the pure imaginary numbers; for if $a = 0$ and $b \neq 0$, the complex number $a + bi$ reduces to bi, a *pure* imaginary number.

For example, in the complex number $a + bi$, if $b = 0$ and $a = -7$, we have $-7 + (0)(i) = -7$, which is a real number. If $a = 5$ and $b = -3$, we have $5 - 3i$, which is an imaginary number. Finally, if $a = 0$ and $b = \sqrt{2}$, we have $0 + \sqrt{2}\,i = \sqrt{2}\,i$, which is a pure imaginary number.

The system of complex numbers may be summarized as follows:



OPERATIONS WITH PURE IMAGINARIES

Addition and subtraction of pure imaginaries are defined as follows:

$$ai \pm bi = (a \pm b)i.$$

Thus $9i + 3i = (9 + 3)i = 12i.$

and $10i - 4i = (10 - 4)i = 6i.$

The multiplication of pure imaginary numbers differs somewhat from the familiar rules of real numbers.

Since, by definition,

$$\sqrt{-1} \cdot \sqrt{-1} = i \cdot i = i^2 = -1,$$

then

$$\sqrt{-a} \cdot \sqrt{-b} = \sqrt{a} \cdot i \cdot \sqrt{b} \cdot i = \sqrt{ab} \cdot i^2 = \sqrt{ab}(-1) = -\sqrt{ab}.$$

That is: *The product of pure imaginary numbers with like signs is a negative real number.*

Moreover,

$$-\sqrt{-a} \cdot \sqrt{-b} = -\sqrt{a} \cdot i \cdot \sqrt{b} \cdot i = -\sqrt{ab} \cdot i^2 = -\sqrt{ab}(-1) = +\sqrt{ab}.$$

Hence, *The product of pure imaginary numbers with unlike signs is a positive real number.*

For example,

$$-\sqrt{-4} \cdot -\sqrt{-25} = (-\sqrt{4}) \cdot (i) \cdot (-\sqrt{25}) \cdot (i) = (-2)i(-5)i = 10i^2 = 10(-1) = -10.$$

$$-\sqrt{-9} \cdot \sqrt{-16} = (-\sqrt{9})(i)(\sqrt{16})(i) = (-3)i(4)i = -12i^2 = -12(-1) = +12.$$

Note that $\sqrt{-a}$ times $\sqrt{-b}$ is *not* equal to $\sqrt{(-a)(-b)}$, which is equal to $\sqrt{ab}$. In order to avoid this mistake, before performing any operation with a pure imaginary number, it should be written in the form $\sqrt{a}\, i$.

The division of pure imaginary numbers is defined as follows:

$$\sqrt{-a} \div \sqrt{-b} = \sqrt{a} \cdot i \div \sqrt{b} \cdot i = \frac{\sqrt{a} \cdot i}{\sqrt{b} \cdot i} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

Hence: *The quotient of pure imaginary numbers is a real number.*

Thus: $\sqrt{-75} \div \sqrt{-3} = \sqrt{75}\, i \div \sqrt{3}\, i = \dfrac{\sqrt{75}\, i}{\sqrt{3}\, i} = \sqrt{\dfrac{75}{3}}$

$= \sqrt{25} = 5.$

OPERATIONS WITH IMAGINARY NUMBERS

Real numbers are as different from imaginary numbers in kind and character as chairs from tables. In order to have:

$$a \text{ chairs} + b \text{ tables} = c \text{ chairs} + d \text{ tables},$$

it is necessary and sufficient that $a = c$ and $b = d$. Similarly, in order to have an imaginary number $a + bi$ equal another imaginary number $c + di$, a must equal c and b must equal d.

Leaving aside this analogy, consider two imaginary numbers $a + bi$ and $c + di$.

$$\begin{aligned} &\text{If } a + bi = c + di, \text{ then} \\ &\quad a - c = di - bi \text{ or} \\ &\quad a - c = (d - b)i. \end{aligned}$$

But $(a - c)$ is a real number and $(d - b)i$ is a pure imaginary number. Therefore, $\boldsymbol{a - c = (d - b)i}$ *only if* $\boldsymbol{(a - c)}$ *and* $\boldsymbol{(d - b)}$ *are both equal to zero.* But if $a - c = 0$, then $a = c$; and if $d - b = 0$, then $d = b$.

Therefore, *two imaginary numbers* $\boldsymbol{(a + bi)}$ *and* $\boldsymbol{(c + di)}$ *are equal when and only when* $\boldsymbol{a = c}$ *and* $\boldsymbol{b = d}$.

This reasoning governs the operations of addition and subtraction of any two imaginary numbers. That is, *to add or subtract imaginary numbers, add or subtract separately the real and the pure imaginary parts.* Thus,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

For example,

$$(3 - 2i) + (-7 + 4i) = (3 - 7) + (-2 + 4)i = -4 + 2i$$

and

$$(7 + 22i) - (3 + 6i) = (7 - 3) + (22 - 6)i = 4 + 16i.$$

To multiply the imaginary number $a + bi$ by $c + di$, *consider these numbers as polynomials in* i, bearing in mind that $i^2 = -1$. Thus:

$$\begin{aligned} (a + bi)(c + di) &= ac + cbi + adi + bdi^2 \\ &= ac + (cb + ad)i + bd(-1) \\ &= (ac - bd) + (cb + ad)i. \end{aligned}$$

Therefore, the product of the imaginary number $(a + bi)$ by $(c + di)$ is the imaginary number $(ac - bd) + (cb + ad)i$. For example:

$$(3 + 2i)(5 - 3i) = [(3)(5) - (2)(-3)] + [(5)(2) + (3)(-3)]i$$
$$= (15 + 6) + (10 - 9)i = 21 + i.$$

If two imaginary numbers differ only in the sign of their pure imaginary parts, such as $a + bi$ and $a - bi$, they are said to be *conjugate* imaginary numbers.

Applying the laws of addition and multiplication to conjugate imaginary numbers gives:

$$(a + bi) + (a - bi) = (a + a) + (b - b)i = 2a + 0i = 2a$$
and $$(a + bi)(a - bi) = [(a)(a) - (b)(-b)] + [(a)(b) + (a)(-b)]i$$
$$= (a^2 + b^2) + (ab - ab)i = a^2 + b^2;$$

it is evident that: *The sum and the product of conjugate imaginary numbers are real numbers. Such a product, moreover, is not negative.*

Thus: $(5 + 3i) + (5 - 3i) = (5 + 5) + (3 - 3)i = 10 + 0i = 10$
and $(5 + 3i)(5 - 3i) = 25 + 9 = 34.$

Similarly,

$$x^2 + y^2 = (x + yi)(x - yi).$$

Making use of this fact, the quotient of an imaginary number $a + bi$ by another $c + di$ can be expressed by an imaginary number.

Thus, given $a + bi$ to be divided by $c + di$, that is,

$$\frac{a + bi}{c + di},$$

multiply both numerator and denominator by the conjugate of the denominator; hence,

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)}.$$

Then,
$$\frac{a + bi}{c + di} = \frac{(ac + bd) + (cb - ad)i}{c^2 + d^2},$$

or
$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{cb - ad}{c^2 + d^2}i.$$

The result is an imaginary number of the form $x + yi$, where

$$x = \frac{ac + bd}{c^2 + d^2} \text{ and } y = \frac{cb - ad}{c^2 + d^2}.$$

For example,

$$\frac{7 - 3i}{4 + 2i} = \frac{7 - 3i}{4 + 2i} \cdot \frac{4 - 2i}{4 - 2i} = \frac{28 - 26i + 6i^2}{16 + 4} = \frac{22 - 26i}{20} = \frac{11}{10} - \frac{13}{10}i.$$

Note that, in the four operations just defined for imaginary numbers, if the pure imaginary parts are set equal to zero, the numbers become real, and the result of the operation on the imaginary numbers reduces to the same operation on real numbers. Thus:

$$(a + bi)(c + di) = (ac - bd) + (cb + ad)i$$

reduces, by setting $b = d = 0$, to:

$$a \cdot c = a \cdot c;$$

and

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{cb - ad}{c^2 + d^2}i$$

also reduces, by setting $b = d = 0$, to:

$$\frac{a}{c} = \frac{ac}{c^2} = \frac{a}{c}.$$

GRAPHIC REPRESENTATION OF IMAGINARY NUMBERS

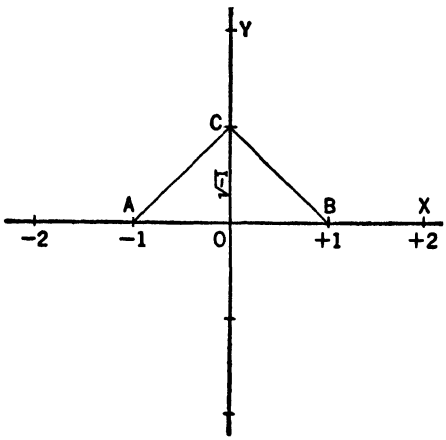


Fig. 286.

The graphic representation of imaginary numbers is made possible by using Fact 69 (*b*), namely, "If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the perpendicular is the mean proportional between the segments of the hypotenuse."

In Figure 286, where, as usual, the X- and Y-axes are perpendicular to each other, select a convenient unit of length and divide each coordinate axis into segments of unit length. Draw the triangle ABC as shown in the figure.

By construction, the triangles COB and COA are isosceles right triangles. Therefore, by Fact 3, angle BCO = angle OBC, and each of these angles, by Fact 20, is 45°. Similarly, angle $ACO = 45°$. Therefore, angle $ACB = 90°$ and the triangle ABC is a right triangle. Thus, CO, by Fact 69(b), is the mean proportional between AO and OB. That is, $CO = \sqrt{AO \times OB}$.

Since $AO = -1$ and $OB = +1$,

then $$CO = \sqrt{(-1)(+1)} = \sqrt{-1} = i.$$

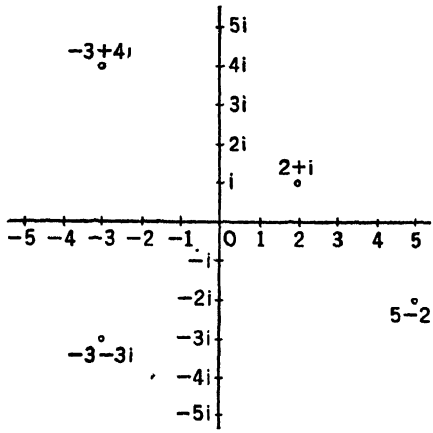


Fig. 287.

Applying this discovery (first noted in about the year 1800 A. D.), assign to the origin the number 0, and the real numbers to points on the X-axis. On the Y-axis, associate the points of subdivision with the corresponding pure imaginary numbers, as shown in Figure 287, so that the point one unit above the X-axis, on the Y-axis, represents i; the point 2 units below, $-2i$; and so on.

Then the X-axis is called the *real* axis and the Y-axis the *imaginary* axis. Thus, the imaginary numbers $2 + i$, $-3 + 4i$, $5 - 2i$, $-3 - 3i$ are represented by points on the plane as in Figure 287.

The graphical illustration of an imaginary number as a point on a plane leads to the following interpretation of the operation of addition.

Let the point A (Figure 288) represent the imaginary number $a + bi$. Then, $OM = a$ and $AM = bi$. In a similar manner, let B represent $c + di$, so that $ON = c$ and $BN = di$.

Draw the parallelogram $OASB$, and the lines SP and

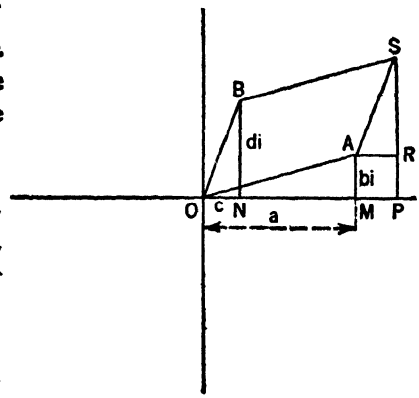


Fig. 288.

AR so that SP is perpendicular to the X-axis and the line AR is parallel to MP. Then, since $OB = AS$ and the angle NOB = the angle RAS, the triangles NOB and RAS are congruent (Fact 24). Therefore, $ON = AR = MP$, and $BN = SR$.

Now, $SP = SR + RP = BN + AM = di + bi = (d + b)i$
and $OP = OM + MP = OM + ON = a + c$.

But by definition point S represents the imaginary number whose real part is OP and whose pure imaginary part is SP. Hence, the point S is the imaginary number $(a + c) + (d + b)i$.

Finally, since $(a + bi) + (c + di) = (a + c) + (b + d)i$, then S is the imaginary number which represents the sum of A and B. That is, *to add graphically two imaginary numbers* A *and* B, *connect the points representing* A *and* B *on the plane with* O, *the origin. With* OA *and* OB *as sides, draw a parallelogram. The fourth vertex,* S, *represents the sum of* A *and* B.

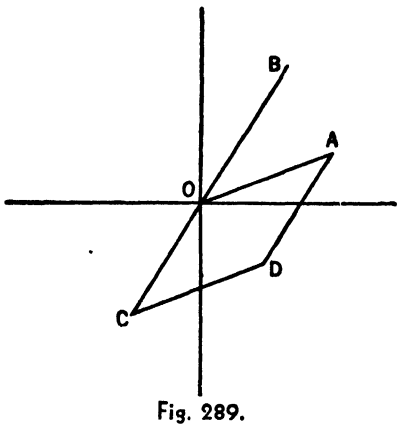


Fig. 289.

The difference, $A - B$, between two imaginary numbers may be obtained graphically as follows:

Connect the point A *with the origin (Figure 289). Draw a line from* B *through the origin to* C, *making* $OC = OB$. *With* OA *and* OC *as sides, draw a parallelogram. The fourth vertex,* D, *represents the difference between* A *and* B.

The proof is similar to the one just given for the sum of two imaginary numbers and is left to the reader as a simple exercise.

Problems

1. Is $\sqrt{-1}$, or i, greater or less than -1? Than 1?
2. Prove that $\frac{1}{i^3} = -i$.
3. Give the value of $(3 - 4i)^2$.
4. Solve $\sqrt{56 - 90i}$.

5. Add $(2 + 3i)$, $(3 - 4i)$, and $(6 + 5i)$.

6. Simplify $(3 + 2i) \div (5 - 4i)$.

7. Show that two imaginary numbers can be equal only if their real parts are equal, and also their pure imaginary parts.

8. Evaluate $(3 - i)^3$.

9. (*a*) Add graphically $(2 + i)$ to $(3 + 4i)$.
   (*b*) Add graphically $(-2 - 2i)$ to $(2 + i)$.
   (*c*) Add graphically $(-3 - 2i)$ to $(1 - 3i)$.

10. (*a*) From $(3 + 4i)$ subtract graphically $(2 + 3i)$.
    (*b*) From $(2 + 4i)$ subtract graphically $(-5 - i)$.
    (*c*) From $(-2 + 3i)$ subtract graphically $(-1 - 4i)$.

31

LIMITS

The fundamental concepts of mathematics hitherto developed are indispensable tools. Without an adequate understanding of these concepts, the investigator in any field of science or business is like a blind man groping in a dark closet for a hat that isn't there. All measurements in science are statistical in character.

Steel measuring tapes used by surveyors are usually 100 feet long and graduated to 0.01 of a foot. If, when we measure a certain distance, we say that it is 52.15 feet long, all we can mean is that this distance is nearer to the mark 52.15 ft. than it is to the mark 52.16 ft. or 52.14 ft. With a finer scale, a figure to the nearest thousandth of a foot could be obtained. A length of 52.15 ft., measured with a steel tape graduated to hundredths of a foot, really means 52.15 ± 0.005 feet. Moreover, variations in temperature cause a slight variation in the length of the tape. These tapes are said to be 100 feet long at 62° F., with a probable error of about 0.0000065 foot per degree Fahrenheit. But corrections for temperature are uncertain, since the temperature of the tape in an open field cannot be determined with any great accuracy. Hence, it is deceptive to state that a certain distance, measured under normal conditions with a surveyor's tape, is *exactly* 52.15 feet in length.

It seems, therefore, that there are two kinds of arithmetic, exact and approximate. Exact arithmetic is possible only when the quantities dealt with can be expressed as common fractions. In scientific or business calculations, exact accuracy is seldom obtainable. Fractions of a penny are neglected. Decimals are very often closed down whether the operation be theoretically complete or not. Thus, a particular fraction may run to 20 places; but if we want only such accuracy as is represented by four places, we close it there and take no interest in the 16 figures dropped.

Cases abound in which exact answers cannot be obtained no matter how far an indicated operation may be carried out. The square root of 2 cannot be exactly obtained as a terminating decimal. There is no integer or rational fraction whose square is 2. If we take the number 2 and apply to it the well-known method for extracting the square root of a number, everything goes smoothly, except that the process is unending, so that we cannot close the operation.

But in very many cases, a definite value may be obtained for what appears to be an unending operation. The unending decimal $0.3333 \cdots$ is, we say, equal to $\frac{1}{3}$. The notation $\frac{1}{3} = 0.3333 \cdots$ is interpreted to signify $\frac{1}{3} = .3 + .03 + .003 + .0003 + \cdots$, and may be expressed in fraction form:

$$\tfrac{1}{3} = \tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \tfrac{3}{10{,}000} + \cdots.$$

Thus, an unending operation, namely, the sum $\frac{3}{10} + \frac{3}{100} + \cdots$, has the definite value $\frac{1}{3}$.

PERPETUITIES

There are many incomes which, barring accidents or unforeseen events, go on for generations. The income from ground rent in a settled community will never cease, so long as the rental asked is sufficiently reasonable to attract occupants. Most endowments are planned so that the income will be received indefinitely. The periodic sums needed for replacement of armaments seem at this writing to be also of this nature! In some countries, corporations issue bonds, interest on which must be paid indefinitely, but the principal does not have to be returned. Notable examples are British consols and French rentes.

Income or outgo, according as we take the point of view of creditor or debtor, constitutes an annuity during a term that has no end. This particular type of annuity is called a *perpetuity*.

Since payments of a perpetuity continue forever, it is impossible to compute its final amount. The present value of a perpetuity, however, has a very definite meaning, and may be expressed by formula.

The present value of an annuity may be obtained, as already explained, from the formula $S = \dfrac{a(1 - r^n)}{1 - r}$ representing the

sum of a geometric progression. A perpetuity is, by definition, an annuity during a term without end. Thus, a perpetuity is represented by a geometric progression with an unlimited number of terms, that is, by an infinite geometric progression. To find the present value of a perpetuity consists of ascertaining the value of the sum $S = \frac{a(1 - r^n)}{1 - r}$, where the value of n is unlimited.

In this expression for the sum S, the only term affected by the value of n is r^n. The common ratio r is easily identified as $\frac{1}{1 + i}$ (see Chapter 24, page 415), where i is the rate of interest per year. Evidently, this ratio is a fraction less than 1. Hence, if $1 + i$ be represented by the letter s, the expression r^n may be written $\left(\frac{1}{s}\right)^n$. Consider, therefore, the values assumed by a fraction of the type $\left(\frac{1}{s}\right)^n$ as n is increased without limit.

Say that we begin with $n = 2$ and then proceed with $n = 3$, $n = 4$, and so forth, and for the sake of simplicity we assume $s = 2$. Substituting these values for s and n, we obtain the following sequence: $(\frac{1}{2})^2$, $(\frac{1}{2})^3$, $(\frac{1}{2})^4$, $\cdots$, that is, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\cdots$; later the terms become $(\frac{1}{2})^{12}$, $(\frac{1}{2})^{13}$, $\cdots$, or $\frac{1}{4096}$, $\frac{1}{8192}$, $\cdots$; and still later on, $(\frac{1}{2})^{17}$, $(\frac{1}{2})^{18}$, $\cdots$, namely, $\frac{1}{131,072}$, $\frac{1}{262,144}$, $\cdots$, and so forth. Here we have an endless series of values of the fraction marked by the characteristic that each term is less than the one preceding it. But though the terms decrease, every term is greater than 0. But here we stop; the number 0 is the greatest number of which it can be said that every term of the series is greater than it. Thus, in this series, for all positive values of n, the greater the value of n, the smaller the difference between the fraction and zero.

If we choose a very small positive number, say 0.00003, we can find a value of n large enough to make the value of the fraction lie between 0 and 0.00003. It is only necessary to choose $n = 17$, so that $(\frac{1}{2})^{17} = \frac{1}{131,072} = 0.000007$ approximately.

Evidently, no matter how small a positive number is chosen, a large enough value of n can be found to make the value of the

fraction lie between 0 and the chosen positive number. This number, 0, we call the *limit* of $(\frac{1}{2})^n$ as n becomes endlessly greater and greater.

In general, *when the successive values of a variable* v *approach nearer and nearer a constant quantity* c *in such a way that the absolute value* of the difference* $v - c$ *finally becomes and remains less than any preassigned positive number, however small, the constant* c *is called the limit of the variable* v.

It is now seen that as long as s is greater than unity, the limit of $\left(\frac{1}{s}\right)^n$, as n increases without limit, is zero.

The value of the expression $S_n = \dfrac{a(1 - r^n)}{1 - r} = \dfrac{a - ar^n}{1 - r}$, representing the sum of the first n terms of a geometric progression, depends on the value of n. That is, S_n is a function of n. As n increases without limit, $r^n = \left(\frac{1}{s}\right)^n$ approaches zero as its limit. The product ar^n evidently also approaches zero as its limit, and the values of S_n approach nearer and nearer to the constant quantity $\dfrac{a - 0}{1 - r} = \dfrac{a}{1 - r} = S$. Since, as n increases without limit, the absolute value of the difference between S_n and the constant S can be ultimately made less than any preassigned positive number, however small, then by the definition of a limit, given above, the limit of S_n is S. This limit is called *the sum of an infinite geometric progression.* Thus, in symbols,

$$S = \frac{a}{1 - r}.$$

This formula, when applied to perpetuities, may be made still more simple. If we refer back to the problem of finding the present value of an annuity (page 561), the symbol a can be easily identified as $\dfrac{R}{1 + i}$, where R denotes the rent, that is, the value of each periodic payment, and i represents the annual interest rate. The symbol r has already been seen to corre-

* By the *absolute value* of a quantity is meant the numerical value of the quantity regardless of sign. For example, the absolute value of $+6$ is 6, and of -6 is 6.

spond to $\frac{1}{1+i}$. Hence, if A is used to denote the present value of a perpetuity:

$$A = \frac{\frac{R}{1+i}}{1-\frac{1}{1+i}} = \frac{\frac{R}{1+i}}{\frac{1+i-1}{1+i}} = \frac{\frac{R}{1+i}}{\frac{i}{1+i}} = \frac{R}{1+i} \times \frac{1+i}{i} = \frac{R}{i}.$$

Thus, assume that an owner of real estate property receives an income of \$4,800 a year at the end of each year. Provided that current money rates are 4 per cent, the value of this property is easily found. The rent $R = 4{,}800$ and the annual interest rate $i = .04$. Hence, $A = \frac{4800}{.04} = \$120{,}000$.

REPEATING DECIMALS

Consider now the problem of finding the common fraction equal to an unending but repeating decimal, referred to in the beginning of this chapter. The repeating decimal $0.333 \cdots$, equal to $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$, is an infinite decreasing geometric progression such that the first term $a = \frac{3}{10}$ and the common ratio $r = \frac{1}{10}$. Hence, its sum is:

$$S = \frac{a}{1-r} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{10-1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{3}{10} \times \frac{10}{9} = \frac{3}{9} = \frac{1}{3}.$$

Similarly, the repeating decimal $0.4545 \cdots$ may be written $0.45 + 0.0045 + \cdots$. Here $a = 0.45$ and $r = 0.01$. Therefore,

$$S = \frac{0.45}{1-0.01} = \frac{0.45}{0.99} = \frac{45}{99} = \frac{5}{11}.$$

In the repeating decimal $0.4545 \cdots$, it is the sequence of digits, namely, 45, which repeats. A so-called repeating decimal, however, may contain, immediately to the right of the decimal point, one or more digits which do not repeat. Thus, to take a simple illustration, in the number $0.8666 \cdots$ we consider the decimal $0.0666 \cdots$; and, after finding its value, add 0.8 to the result. Thus, $0.0666 \cdots$ is equal to $0.06 + 0.006 + 0.0006 + \cdots$. So that, $a = 0.06$ and $r = 0.1$. Hence,

$$S = \frac{0.06}{1 - 0.1} = \frac{0.06}{0.9} = \frac{6}{90} = \tfrac{1}{15}.$$

Therefore, $0.8666 \cdots$ is equal to $\frac{8}{10} + \frac{1}{15} = \frac{24 + 2}{30} = \frac{26}{30} = \frac{13}{15}$.

LIMIT OF A FUNCTION

The concept of a function has been stressed almost constantly throughout this text. When we merely wish to indicate that a quantity y is a function of a variable x, without determining in any way the particular form of this function, we write:

$$y = f(x).$$

This is read, "y is a function of x," or "the function f of x," or simply, "f of x".* The notation $y = f(x)$ may therefore represent an equation such as $y = x^3 - 6x + 7$, or $y = \tan x$, or $y = b^x$, and so forth.

To distinguish between different functions, other letters may, of course, be used. Thus, $F(x)$, $f_1(x)$, and $\phi(x)$ are, respectively, read, "large f of x," "f sub-one of x," and "phi of x."

The value of $f(x) = x^2 - 5x + 3$, when $x = 4$, is denoted by the symbol $f(4)$. Hence, $f(4) = (4)^2 - 5(4) + 3 = -1$. Similarly, if $F(x) = \sin x$, $F(0) = \sin 0$; that is, $F(0) = 0$. In general, the value of $f(x)$, when $x = a$, is denoted by the symbol $f(a)$.

The notation:

$$\lim_{x \to a} f(x) = L$$

is read, "The limit of the function f of x, as x approaches a, is L."

From the definition of a limit, given on page 563, the above notation states that if, when x approaches a, the absolute value of the difference between $f(x)$ and some constant L, that is, the absolute value of $f(x) - L$, can ultimately be made smaller than any preassigned positive number, however small, the function $f(x)$ approaches the limit L.

In some cases, as already explained in Chapter 22, page 358, the variable actually reaches its limit, while in other cases it does not. Thus, if $f(x) = x^2 - 5x + 3$, the limit of $f(x)$, as x

* Note that the notation $f(x)$ does *not* mean some quantity f multiplied by x.

approaches 5, is simply $(5)^2 - 5(5) + 3 = 3$. That is,

$$\lim_{x \to 5} (x^2 - 5x + 3) = 3.$$

In this case, the function evidently reaches its limit. On the other hand, the limit of the infinite series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$ was found to be, on page 564, equal to $\frac{1}{3}$. Clearly, no matter how many terms of this series be added together, their sum will never equal $\frac{1}{3}$. Nevertheless, it is important to note that, although the limit $\frac{1}{3}$ can never be reached by the addition of any number of terms, however large the number of terms added, its limit is not approximately $\frac{1}{3}$, but *exactly* $\frac{1}{3}$. The important fact is that the limit is defined exactly. Hence, $S = \frac{a(1 - r^n)}{1 - r}$ has a perfectly definite limit. As n becomes greater and greater, the sequence of sums approximates its limit, but the limit itself, namely, $\frac{a}{1 - r}$, is exact.

We saw, in Chapter I, page 10, that division by zero is not a legitimate operation. However, it is *very important* to note what happens to the fraction $\frac{v}{x}$ when x approaches zero. Two cases arise:

(*a*) v does not approach zero; and
(*b*) v also approaches zero.

INFINITY

If v approaches some definite value c, different from zero, as x approaches zero, the indicated division $\frac{v}{x}$ is possible as long as x does not *equal* zero. But when $v = c$ and $x = 0$, the fraction $\frac{v}{x}$ takes the form $\frac{c}{0}$. This expression $\frac{c}{0}$ is not a definite number and therefore has no meaning. If, for convenience, we let v assume the value of its limit, we may consider the function $y = \frac{c}{x}$ for values of x near zero. When $x = \frac{1}{2}$, $y = c \div \frac{1}{2} = c \times \frac{2}{1} = 2c$; if $x = \frac{1}{10}$, $y = c \div \frac{1}{10} = c \times \frac{10}{1} = 10c$; if $x = \frac{1}{100}$, $y = c \div \frac{1}{100} = c \times \frac{100}{1} = 100c$. It is easily seen

that $\frac{c}{x}$ becomes indefinitely large as x approaches zero. This is indicated by writing:

$$\lim_{x \to 0} \frac{c}{x} = \infty.$$

The symbol ∞, read "infinity," does not represent a number, but a variable which *increases without limit.* No one number is infinity; that is, there is no greatest number beyond which another number does not go. Therefore, for any value of x, a function is never infinite: it is either finite or it is not defined at all. The notation $\lim_{x \to 0} \frac{c}{x} = \infty$ merely indicates that as the denominator of a fraction, whose numerator is different from zero, approaches zero, the value of the fraction becomes large beyond any fixed amount. That is, the fraction "approaches infinity" or "becomes infinite." The statement "equals infinity" should therefore be avoided as meaningless.

It may now be readily seen that in case (*a*), the fraction $\frac{v}{x}$ may be made indefinitely large as x approaches zero, whatever value, different from zero, v may approach. Therefore, the fraction in this case does not approach a limit.

In case (*b*), where the numerator and denominator both approach zero, the ratio of the limits takes the form $\frac{0}{0}$, which has no value, that is, no meaning whatever. However, the limit of this ratio may exist. Thus, consider the function:

$$f(x) = \frac{x^3 - 64}{x - 4}.$$

This fraction may be factored and written:

$$f(x) = \frac{x^3 - 64}{x - 4} = \frac{(x - 4)(x^2 + 4x + 16)}{x - 4}.$$

We may be tempted to divide the numerator and denominator of this fraction by $x - 4$. But if $x = 4$, $x - 4 = 0$; that is, for the value $x = 4$, division by $x - 4$ is division by zero, and division by zero is not valid. That is, the identity obtained by dividing both terms of the fraction by $x - 4$,

namely, $x^2 + 4x + 16$, is true for all values of x different from 4, but is useless for $x = 4$. Therefore, $f(x)$ has a definite value for every value of x except 4. For $x = 4$, the $f(x)$ takes the form $\frac{0}{0}$, which, as already stated, is meaningless. However, we can divide both terms of the fraction by any number not equal to zero, and $x - 4$ is not zero so long as x is not equal to 4. Therefore, for all values of x except the value $x = 4$,

$$f(x) = \frac{x^3 - 64}{x - 4} = x^2 + 4x + 16. \tag{1}$$

Now consider the $f(x)$ for values of x near 4. The equality $f(x) = x^2 + 4x + 16$ holds true for all values of x near 4 no matter how small the difference between x and 4 may be, as long as there is a difference. As the values of x are taken nearer and nearer to 4, it is easily seen that the values of $f(x)$ in its simpler form, namely, $x^2 + 4x + 16$, will approach closer and closer to $(4)^2 + 4(4) + 16 = 48$. Obviously, since for all values of x, except the value $x = 4$, the equality (1) is valid, the values of

$$f(x) = \frac{x^3 - 64}{x - 4},$$

as x approaches 4, will also approach nearer and nearer to 48. Thus, no matter how small a positive number is chosen, a value of x sufficiently near to 4 can be found to make the difference between $\frac{x^3 - 64}{x - 4}$ and 48 smaller than the chosen positive number. Therefore,

$$\lim_{x \to 4} \frac{x^3 - 64}{x - 4} = \lim_{x \to 4} (x^2 + 4x + 16) = 48.$$

RATE OF CHANGE

The concept of limit is one of the most important concepts in mathematics. It is the basis for the development of the mathematics necessary to the study of motion. Therefore, it furnishes an instrument that has made possible the solution of problems involving continuous rate of change.

Suppose that an automobile starting at 9 A. M. arrives at its destination, 60 miles away, at 11 A. M. Its average speed

is the distance traveled divided by the time taken to travel this distance, that is, $\frac{60}{2} = 30$ mi/hr. The symbol Δ, introduced in Chapter 13, page 197, is read "the change in." If s and t, respectively, denote distance and time, Δs represents the change in the distance, while Δt is read "the change in the time."* The average speed may then be represented by the expression $\frac{\Delta s}{\Delta t}$. The average speed gives no information whatever about the velocity at any particular time during the trip. It merely states that if the automobile were to travel at a constant speed of 30 mi/hr., it would cover 60 miles in 2 hours. It is clear, however, that the car might have been traveling at 20 miles per hour at 9.20 A. M., at 40 mi/hr. at 10 A. M., and so forth, depending on traffic conditions along the route.

INSTANTANEOUS SPEED

If, beginning at 10 A. M., we carefully note that the distance traveled by the car during the next 5 minutes is 3 miles, then the average speed for these 5 minutes of the trip is $\frac{3}{5}$ of a mile per minute, or $\frac{3}{5} \times 60 = 36$ mi/hr. Barring accidents, this average speed should be a closer approximation to the velocity of the automobile at 10 A. M. than the average speed of 30 mi/hr. obtained for the entire trip. If we know that the car travels 308 feet during the 5 seconds immediately following 10 o'clock, then the average speed for this interval, namely, $\frac{308}{5} = 61.6$ ft/sec. or $\frac{61.6 \times 60 \times 60}{5280} = 42$ mi/hr., is a still closer approximation to the velocity of the car at exactly 10 o'clock. Now suppose that we cut up the interval of time into tenths of seconds. The average speed for the interval between 10 A. M. and $\frac{1}{10}$ of a second afterward will not be appreciably different from the velocity at exactly 10 o'clock. Finally, by taking smaller and smaller intervals of time, we can bring the average speed closer and closer to the speed at the given instant, in fact, as close as we please. Therefore, we conclude that if the interval of time is made to approach zero, the average speed approaches the instantaneous speed as its limit. This statement may be expressed symbolically:

* The reader's attention is again called to the fact that Δs and Δt are each *single symbols* and *not* the product of Δ by s or Δ by t.

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \text{instantaneous speed},$$

read, "The limit of the average speed, as the change in the time approaches zero, is the instantaneous speed."

Assume that the distance (s ft.) traveled by a car in t minutes is given by the equation $s = 120\,t^2 - \frac{2}{3}\,t^3$. To find the average speed during an interval of 0.1 of a minute beginning at the instant $t = 9$, it is necessary to find the values of Δs (change in the distance) and Δt (change in the time). Evidently, the change in the time is 0.1 of a minute, that is, $\Delta t = 0.1$. The change in the distance, namely, Δs, may be found as follows:

$$\text{At } t = 9, \qquad s = 120(9)^2 - \tfrac{2}{3}(9)^3 = 9234 \text{ ft.} \tag{1}$$

The distance traveled depends on the time; that is, distance is a function of the time. Hence, a change in the time will cause a corresponding change in the distance. That is, if we change the time from $t = 9$ to $t + \Delta t = 9 + .1 = 9.1$, the distance changes by some amount, Δs. Thus, if the distance corresponding to $t = 9$ is $s = 9234$ ft., the distance corresponding to $t + \Delta t = 9.1$ we may denote by $s + \Delta s$. Hence,

$$\text{at } t + \Delta t = 9.1,\ s + \Delta s = 120(9.1)^2 - \tfrac{2}{3}(9.1)^3 = 9434.8194 \text{ ft.} \tag{2}$$

The change in the distance may now be easily obtained by subtracting (1) from (2). That is,

$$\begin{aligned} s + \Delta s &= 9434.8194 \quad (2)\\ s \quad\ \ &= 9234 \quad (1)\\ \hline \Delta s &= 200.8194 \text{ ft.} \end{aligned}$$

Thus, the average speed is:

$$\frac{\Delta s}{\Delta t} = \frac{200.8194}{.1} = 2008.194 \text{ ft/min.}$$

Now consider any interval beginning at $t = 9$, and let Δt denote the length of the interval. Then the interval ends at $t + \Delta t = 9 + \Delta t$. Let us apply the same process of reasoning to obtain the average speed for *any* interval beginning at $t = 9$. As before,

$$\text{at } t = 9, \quad s = 120(9)^2 - \tfrac{2}{3}(9)^3 = 9234 \text{ ft.} \tag{3}$$

$$\text{at } t + \Delta t = 9 + \Delta t, \quad s + \Delta s = 120(9 + \Delta t)^2 - \tfrac{2}{3}(9 + \Delta t)^3. \tag{4}$$

Expanding this last expression and collecting terms give:

$$s + \Delta s = 9234 + 1998\Delta t + 102(\Delta t)^2 - \tfrac{2}{3}(\Delta t)^3.$$

Subtracting $s = 9234$

$$\Delta s = 1998\Delta t + 102(\Delta t)^2 - \tfrac{2}{3}(\Delta t)^3.$$

The average speed may now be obtained by dividing both sides of this equation by Δt. Thus,

$$\frac{\Delta s}{\Delta t} = 1998 + 102(\Delta t) - \tfrac{2}{3}(\Delta t)^2.$$

This is the average speed for *any* interval beginning at $t = 9$. As a check, let $\Delta t = 0.1$; then,

$$\frac{\Delta s}{\Delta t} = 1998 + 10.2 - .006 = 2008.194 \text{ ft/min.},$$

which, of course, agrees with the result previously obtained.

The ratio:

$$\frac{\Delta s}{\Delta t} = 1998 + 102\Delta t - \tfrac{2}{3}(\Delta t)^2$$

is merely the average speed over the interval Δt. If we now let Δt approach zero, the change in the distance, namely, Δs, also approaches zero, and the ratio of the limits takes the form $\frac{0}{0}$, which, as we have already repeatedly stated, has no meaning. However, the limit of the ratio $\dfrac{\Delta s}{\Delta t}$ does exist. In fact, it is, as shown on page 570, the instantaneous speed. Now, as Δt approaches zero, $102\Delta t$ approaches zero as its limit. This fact may be easily seen by assuming values for Δt very near zero and noting that the sequence formed by the corresponding values of $102\Delta t$ has zero as its limit. Similarly, as Δt approaches zero, $\frac{2}{3}(\Delta t)^2$ will also approach zero as its limit. Therefore, if we let v denote the instantaneous speed, the limit approached by the average speed is precisely:

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = v = 1998 \text{ ft/min.}$$

That is, the instantaneous speed at $t = 9$ is 1998 feet per minute.

Thus far we have thought of the time t as being fixed at some value such as 9. But evidently a great deal of effort

may be saved by developing a general formula for the speed at any instant whatever. Consider any interval beginning at t, where t is any instant whatever, and let Δt denote the length of the interval. The interval, then, ends at $t + \Delta t$. As before, we first obtain an expression for the average speed, namely, $\frac{\Delta s}{\Delta t}$, and then find the limit of this ratio as Δt approaches zero. From the given conditions, we have:

$$\text{at } t, \quad s = 120\,t^2 - \tfrac{2}{3}t^3. \tag{5}$$

Since the distance is a function of the time, a change in the time will cause a corresponding change in the distance. That is, if we change the time from t to $t + \Delta t$, the distance changes by some amount Δs. Thus, if to t corresponds the distance s, to $t + \Delta t$ corresponds a distance which we may denote by $s + \Delta s$. Hence,

$$\text{at } t + \Delta t, \quad s + \Delta s = 120(t + \Delta t)^2 - \tfrac{2}{3}(t + \Delta t)^3.$$

Expanding and collecting terms gives:

$$s + \Delta s = 120t^2 + (240t - 2t^2)\Delta t + (120 - 2t)(\Delta t)^2 - \tfrac{2}{3}(\Delta t)^3 - \tfrac{2}{3}t^3$$

Subtracting:

$$\begin{array}{lll} s & = 120t^2 & - \tfrac{2}{3}t^3 \\ \hline \end{array}$$

gives

$$\Delta s = (240t - 2t^2)\Delta t + (120 - 2t)(\Delta t)^2 - \tfrac{2}{3}(\Delta t)^3.$$

Dividing both sides of this equation by Δt, we obtain:

$$\frac{\Delta s}{\Delta t} = 240t - 2t^2 + (120 - 2t)\Delta t - \tfrac{2}{3}(\Delta t)^2. \tag{6}$$

This is the equation for the average speed for *any* interval beginning at *any* time. The reader may check the accuracy of the answer previously obtained, namely, 2008.194 ft/min., by substituting in this equation the values $t = 9$ and $\Delta t = .1$. If we now let Δt approach zero, it is easily seen that the limits of the last two terms in (6) is zero. Therefore,

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = v = 240t - 2t^2.$$

To check the answer previously obtained, let us find the instantaneous speed for $t = 9$. Substituting in the above equation gives:

$$v = 240(9) - 2(9)^2 = 1998 \text{ ft/min.}$$

THE DERIVATIVE

The reader undoubtedly realizes that this process of reasoning may be applied to any problem involving continuous rate of change. The notion of speed is merely a special case of the more general idea of rate of change. Thus, we may speak of the rate of consumption of fuel by an automobile, or of the rate at which the volume of a certain weight of gas changes with the pressure. Since all problems involving rate present the same argument, consider the more general case where y is any function $f(x)$. Then, if x changes by any amount Δx, y will change by some amount Δy. If we then let Δx approach zero, Δy will also approach zero, but the ratio $\frac{\Delta y}{\Delta x}$ may approach some definite limit. *If, as* Δx *approaches zero, the ratio* $\frac{\Delta y}{\Delta x}$ *approaches a definite limit, we call this limit the derivative of* y *with respect to* x, *and denote it by* $\frac{dy}{dx}$. That is,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

The derivative, therefore, may be interpreted as the *instantaneous rate of change*, or, in the special case of speed, as the instantaneous speed. Many formulas have been derived to obtain the derivatives of different types of functions in a precise and quick way. The scope of this book, however, permits the treatment of only the most elementary type.

DERIVATIVE OF kx^n

Consider the function:

$$y = kx^n,$$

where n is a positive integer. Let x change by any amount Δx; then y will change by some amount Δy. Hence,

$$y + \Delta y = k(x + \Delta x)^n.$$

Applying the Binomial Theorem, Chapter 24, page 420, we obtain:

$$y + \Delta y = k\left[x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{1.2}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n\right],$$

or

$$y + \Delta y = kx^n + knx^{n-1}\Delta x + k\frac{n(n-1)}{1.2}x^{n-2}(\Delta x)^2 + \cdots + k(\Delta x)^n.$$

Subtracting

$y = kx^n$

gives

$$\Delta y = knx^{n-1}\Delta x + k\frac{n(n-1)}{1.2}x^{n-2}(\Delta x)^2 + \cdots + k(\Delta x)^n.$$

Dividing both sides of this equation by Δx to obtain the average rate gives:

$$\frac{\Delta y}{\Delta x} = knx^{n-1} + k\frac{n(n-1)}{1.2}x^{n-2}\Delta x + \cdots + k(\Delta x)^{n-1}.$$

The second term and every term after the second contain Δx as a factor. Thus, as Δx approaches zero, all the terms in the expression, except the first, approach zero as their limits. Hence,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = knx^{n-1}.$$

That is, *the derivative of a function* $y = kx^n$, *where* n *is a positive integer, is equal to the product of the coefficient* k, *the exponent of the given power* n, *and* x *raised to a power one less than* n.*

We are now in a position to obtain certain derivatives at sight. Thus, the distance that a body falls in t seconds is given by the formula:

$$s = 16t^2.$$

The velocity at any instant may be determined by finding the derivative of the distance with respect to the time, namely, $\frac{ds}{dt}$. Applying the rule just developed,

$$\frac{ds}{dt} = (16)(2)t^{2-1} = 32t.$$

The instantaneous speed of a freely falling body, say 5 seconds after being dropped, is then:

$$\frac{ds}{dt} = 32(5) = 160 \text{ ft/sec.}$$

* This formula can be shown to hold also for any fractional or negative powe of x.

TANGENT TO A CURVE

The intuitive idea of tangency defined in Chapter 20, page 322, will undoubtedly lead us to the conclusion that the line PT, shown in Figure 290, seems to be tangent to the curve at point P, although it cuts the curve in two points. As stated in Chapter 20, page 322, the definition of a tangent to a circle does not apply to all curves. The limit concept furnishes the necessary tool for a general definition of tangency which may be applied to all curves.

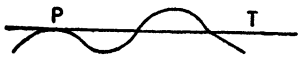


Fig. 290.

Consider any function $y = f(x)$, and assume that the curve shown in Figure 291 represents its graph. Let $P(x, y)$ be any point on the curve. If the value of x is changed by any amount Δx, then the corresponding value of y, found from the relation $y = f(x)$, will change by some amount Δy. Thus, the values $x + \Delta x$ and $y + \Delta y$ define any other point Q also lying on the given curve. Let a secant be drawn through P and Q. From the figure, it is easily seen that the ratio $\frac{\Delta y}{\Delta x}$ is the slope of the secant chord PQ. Now let Δx approach zero and consider its effect on the graph. The point B approaches point A through the positions C, D, and so forth. These points, C, D, and so forth, determine definite values of x which define the corresponding points Q_1, Q_2, and so forth, on the curve. That is, as Δx approaches zero, the point Q moves along the curve, approaching P as its limiting position. But as Q approaches P, the secant through P and Q approaches as its limiting position the line PT. This line PT is defined as the *tangent* to the curve at P.

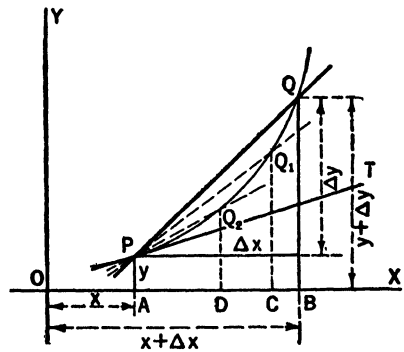


Fig. 291.

SLOPE OF A CURVE

If, as Δx approaches zero, the secant chord approaches the tangent line PT as its limiting position, then the slope of the secant chord PQ must approach the slope of the tangent line

PT as its limit. This slope of the tangent to the curve at any point is defined as *the slope of the curve* at that point.

The slope of the secant chord PQ is $\frac{\Delta y}{\Delta x}$. When Δx approaches zero, the slope of the secant chord PQ approaches the slope of the curve at P as its limit. But this limit, namely, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, is by definition the derivative. Therefore, *the derivative of a function is the slope of the curve.*

Hence, to find the slope of the curve representing the equation $y = 3x^2$ at any point, simply find the derivative of y with respect to x. Thus, if $y = 3x^2$, $\frac{dy}{dx} = 6x$. If the slope of the curve at the point (2, 12) is desired, substitute the value of x, namely, 2, in the expression $\frac{dy}{dx} = 6x$. Thus, $\frac{dy}{dx} = 6(2) = 12$. That is, the slope of the curve $y = 3x^2$ at (2, 12) is 12.

DERIVATIVE OF A SUM

If the function is represented by a polynomial in x such as

$$y = 3x^2 - 6x + 10,$$

consider each term as representing some particular function of x. Thus, let $y_1 = 3x^2$, $y_2 = -6x$, and $y_3 = 10$. We may then write:

$$y = y_1 + y_2 + y_3.$$

Now, if x changes by any amount Δx, y_1 changes by some amount Δy_1, y_2 changes by some amount Δy_2, y_3 changes by some amount Δy_3, and y changes by some amount Δy. Hence,

$$y + \Delta y = y_1 + \Delta y_1 + y_2 + \Delta y_2 + y_3 + \Delta y_3.$$

Subtracting $y = y_1 + y_2 + y_3$

gives $\Delta y = \Delta y_1 + \Delta y_2 + \Delta y_3$

Dividing both sides of this equation by Δx gives:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y_1}{\Delta x} + \frac{\Delta y_2}{\Delta x} + \frac{\Delta y_3}{\Delta x}.$$

Since y, y_1, y_2, and y_3 are all functions of x, the limits of each of the above ratios are the respective derivatives of each of the functions. Hence,

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx} + \frac{dy_3}{dx}.$$

That is, *the derivative of the sum of several functions is the sum of their respective derivatives.*

The derivative of

$$y = 3x^2 - 6x + 10$$

may therefore be easily obtained by applying the rule for differentiating kx^n to the first two terms. The derivative of the third term is easily seen to be zero by the following reasoning. The derivative is the instantaneous rate of change. A constant quantity does not change. Therefore, its derivative or instantaneous rate of change is zero. Hence, if

$$y = 3x^2 - 6x + 10$$
$$\frac{dy}{dx} = 6x - 6.$$

The present book is now ending, yet the subject of which it treats has no end. Merely to explore further the topics already outlined would easily fill another volume of the present size. The reader can easily imagine familiar methods of analysis being applied in further detail to the simple plane figures and solids mentioned. The law of compound interest required a volume of 1200 pages for adequate discussion. Everywhere in mathematics there is room for an endless proliferation of ideas. The higher fields studied in college, such as calculus, differential equations, theory of equations, and functions of a complex variable, are themselves only elementary steps toward other boundless fields of research. No limits have been set to the range and scope of the human mind. Entirely new fields of mathematics will in due time be revealed to us by men of genius. More than any other science, this one is international and catholic in its human aspect, owing its pre-eminence not to a handful of talented people but to many thousands of investigators throughout the ages. The mental toil thus gladly given to posterity inspires profound gratitude for such a human heritage. If fledglings today can fly higher and farther and more easily than those of yore, it is because they have been borne aloft by eagles.

Problems

1. St. John's Hospital receives by bequest a fixed income of \$5,600 a year. Current money rates are 5 per cent per annum. What is the capitalized value of this bequest?

2. Mr. R. Eckley wills to his heirs a perpetual income of \$10,500 a year. Requiring ready cash, the heirs assign their rights to receive this perpetuity for \$262,500. What effective rate of interest was assumed in obtaining this present value?

3. Find the common fraction equivalent to:
   (*a*) $0.222\cdots$ (*c*) $0.5666\cdots$
   (*b*) $0.285714285714\cdots$ (*d*) $0.9444\cdots$

4. Find the limits of the following functions:
   (*a*) $\lim_{x\to 2}(3x^2 - 6x + 7)$. (*c*) $\lim_{x\to 5}\dfrac{x^2 - 2x - 15}{x - 5}$.
   (*b*) $\lim_{x\to 4}\dfrac{x^2 - 16}{x^2 - 4x}$. (*d*) $\lim_{x\to 0}\dfrac{\cos x}{\cot x}$.

5. If a ball is thrown straight up with an initial speed of 96 feet per second, its height at any instant is given by the equation:

$$s = 96t - 16t^2.$$

Find: (*a*) the average speed from $t = 2$ to $t = 3$;
(*b*) the average speed during any interval Δt;
(*c*) the instantaneous speed when $t = 2$.

6. A freight train of ten box cars, moving through still air, is resisted by a force given by the expression:

$$R = 0.23\ V^2,$$

where R is the resistance in pounds per ton and V is the speed in miles per hour. Find: (*a*) the average rate of change of R; (*b*) the instantaneous rate of change for $V = 40$ miles per hour.

7. A spherical balloon is being inflated with gas. Find an expression for the rate of change of the volume with respect to the radius. How fast is the volume changing when the radius is 7 feet?

8. The intensity of light on an object is given by the formula:

$$I = \frac{k}{x^2},$$

where I is the intensity, x the distance of the object from the source, and k a constant determined by experiment. The intensity of light on a certain object varied with the distance to the object as shown by the expression:

$$I = \frac{800}{x^2},$$

or $$I = 800x^{-2}.$$

(*a*) Find the rate of change in the intensity with respect to the distance when $x = 4$.

(*b*) Find the slope of the curve represented by the given equation when $x = 10$.

9. It has been shown, in Chapter 17, page 274, that the maximum or minimum values of a function may be found approximately from its graph. A tangent line drawn at a maximum point (*A* in Figure 292) or at a minimum point (*B* in Figure 292) is parallel to the *X*-axis. That is, the tangent line is horizontal. Since, by definition (Chapter 13, page 196), the slope of a line is the rise or fall of the line per horizontal unit, it is obvious that the slope of a horizontal line is zero. But the slope of the tangent to the curve at any point is the derivative of the function. Therefore, at any maximum or minimum point on a continuous curve, the derivative is zero. Thus, *to find the maximum or minimum value of a function,* $y = f(x)$, *we simply set* $\frac{dy}{dx} = 0$, *solve for* x, *and substitute this value of* x *in the given equation.*

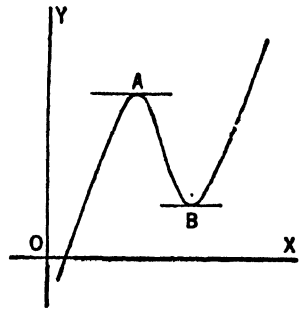


Fig. 292.

A rectangular bin with an open top and square base is to be so constructed as to have a capacity of 62.5 cubic feet. Find the dimensions of the bin that will require the least amount of material for its construction.

10. Under certain conditions, the height of a projectile fired straight up is given by the expression:

$$s = -16t^2 + 120t + 400,$$

where s is the height in feet and t is the time in seconds. Find: (*a*) when the projectile was highest, and how high; (*b*) when the projectile hit the ground; and (*c*) with what velocity the projectile hit the ground.

ADDITIONAL
EXERCISES AND PROBLEMS

ADDITIONAL EXERCISES AND PROBLEMS

CHAPTER 1

Page 2

EXERCISE A. Perform the indicated operations:

(1) $16 + 4 \div 2 - 8 \times 2$
(2) $36 \div 6 + 3 \times 8 - 8$
(3) $9 \times 8 \div 4 + 3 - 6 \div 3$
(4) $11 + 8 \times 3 \div 4 - 5$
(5) $7 \times 5 + 83 - 192 \div 3$
(6) $73 - 13 \times 5 + 583 \div 11$

Pages 6-9

EXERCISE B. Perform the indicated operations:

(1) $(+3) + (+7)$
(2) $(-3) + (+7)$
(3) $(+3) + (-7)$
(4) $(-3) + (-7)$
(5) $(+9) + (-27)$
(6) $(-32) + (+8)$
(7) $(-3) + (+2)$
(8) $(+20) + (-33)$
(9) $(+7) - (+3)$
(10) $(+7) - (-3)$
(11) $(-7) - (+3)$
(12) $(-7) - (-3)$
(13) $(+42) - (+58)$
(14) $(-70) - (-24)$

Pages 9-10

EXERCISE C. Perform the indicated operations:

(1) $(+3) \times (-4)$
(2) $(-4) \times (+3)$
(3) $(+5) \times (+6)$
(4) $(-5) \times (-6)$
(5) $(-1) \times (+1)$
(6) $(+36) \div (-6)$
(7) $(-24) \div (+6)$
(8) $(-66) \div (-11)$
(9) $(+162) \div (-9)$
(10) $(-102) \div (+3)$

Pages 11-14

PROBLEMS.

(1) The fingerprints of *A* in the rogues' gallery are found to agree with the bloody marks on an envelope lying beside the body of a murdered man. What is this kind of evidence called? If the jury, relying on this evidence, finds *A* guilty of murder, what kind of reasoning is used? What fundamental fact gives the evidence probative force?

(2) Two Germans are caught in England: one in Birmingham and one in Liverpool. Each German enemy, when caught, is in the act of putting dynamite under a bridge. One man is in civilian clothes, the other in a German uniform. Is there any reason for inflicting different penalties in the two cases? What implied reasoning is involved?

(3) Was it logical in Dickens' time to imprison a debtor until his debt was paid?

(4) The numbers 3, 5, and 7 are *prime numbers;* that is, they have no inte-

gral factors (except, of course, themselves and 1). Since 5 is 2 more than 3, and 7 is 2 more than 5, can we conclude that 9, which is 2 more than 7, is also a prime number?

(5) If the disease *pellagra* were found to be twice as frequent among Negroes as among whites, could we infer that the disease has some connection with the color of the skin?

(6) The microscope has not revealed any germs which cause colds. Is it safe to conclude that colds are not caused by germs?

(7) An insurance official, in discussing the risks of loss by fire, stated that they were greater in times of depression than in times of prosperity. What kind of reasoning supported his statement?

CHAPTER 2

Pages 17-19

PROBLEMS.

Choose letters and express the following rules of computation as formulas:

(1) The area of a rectangle is equal to the product of its length by its width.

(2) The volume of a rectangular solid is equal to the product of its length, width, and height.

(3) The surface of a cube is 6 times the square of one edge.

(4) The diameter of a circle is 2 times the length of its radius.

(5) The perimeter of a rectangle is twice its length plus twice its width.

(6) The simple interest on a sum of money is the product of the principal, rate, and time.

(7) The volume of a pyramid is one-third the product of its base and height.

Pages 19-20

Express each of the following statements in the form of an equation:

(8) Fifty-three divided by a certain number is 5.

(9) Seven times what number is 16?

(10) By what number must 28 be divided so that the result will be 7?

(11) A city wishes to build a subway spur. It has 4 million dollars in the treasury and expects to receive 3 million more in new real estate taxes, but will still lack 2 million dollars of having one-third the cost of the project.

(12) Five times the age of a man less 100 is equal to 130.

(13) Company A had 5 times as many machine guns as Company B; but after the former transferred 6 machine guns to the latter, each company had the same number.

(14) John and David have \$100 between them. John has \$10 more than three times the amount that David has.

(15) A number is just as much below 100 as it is above 36.

(16) An airplane travels y miles per hour. How long does it take to fly from New York to Mexico City, a distance of 2,700 miles?

Page 21

EXERCISE A. State the degree of each of the following equations:

(1) $x = 2y$

(2) $x^2 - 2xy = 0$

(3) $xy = 75$

(4) $x^3y^3 = 6x^4 - 2xy^3 + y^4$

(5) $7x^3 - 2x^2y^2 + x^4y + 17 = 0$

(6) $x^2 + 2xy + y^2 = 15$

Pages 21-22

EXERCISE B. Add the following expressions:

(1) $12x - 20y + 2c$, $24x - 6c + 5y$, $3c - 15x - 27y$
(2) $b - 3a + c$, $3a - 6c + 4b$, $a - 2c + b$
(3) $8k^2 - 6k + 10$, $7k + 2 - k^2$, $3k + 5k^2 - 3$
(4) $4x^2 - 10 + x$, $2 - 4x - 3x^2$, $-2x^2 - 5x - 7$
(5) $4a^2b^2 - 3ab + \frac{1}{4}$, $2ab - 4a^2b^2 + 7$, $6ab + \frac{1}{4}$
(6) $x^2 + x - \frac{1}{4}$, $3x + \frac{2}{3} - x^2$, $6x - \frac{1}{3}$
(7) $2a + 3c + \frac{2}{3}x$, $-7a + c$, $-5c - a - \frac{1}{3}x$
(8) $a^2x^2 + 2a^2x + \frac{1}{2}x$, $-a^2x - 4a^2x^2 - \frac{1}{4}x$
(9) $4d^2k - 7k^3 - 2$, $5d^2k + k^3 + 4$
(10) $xy^3 + 2x^2y - \frac{1}{2}y^2$, $-6x^2y + 3y^2 + 3xy^3$

Pages 22-23

EXERCISE C.

(1) Subtract $4mns^2 - 2m^2s$ from $12\,m^2s - 6\,mns^2$.
(2) From $27a + b$ subtract $42a - b$.
(3) Subtract $7pg + 9sg - kx$ from $7k^2x - 4pg - 3sg$.
(4) From $5a + 2b - x$ subtract $a - 3b + \frac{1}{2}x$.
(5) Subtract $abk - 2a^2bk - 5a^2b^2k$ from $11a^2b^2k + 2a^2bk + 4abk$.
(6) Subtract the sum of $2y + 10k + 1$ and $-3y - k$ from $y^2 - v - 3k$.
(7) Subtract $25x^2 - x + 1$ from $19x^2 + x - 4$.
(8) Subtract $12w + 6w^2t$ from $6w^2t^2 - 4w + wt$.
(9) What must be added to $a^2 - 9a + 4$ to make $5a$?
(10) What must be added to $7z^2 + rx + 5$ to make $z + r^2x$?

Pages 23-26

EXERCISE D. Evaluate the following expressions:

(1) $23 + 12 \div 3 - 8 \times 4 + 18 \div 6$
(2) $(23 + 12 \div 3 - 8) \times 4 + 18 \div 6$
(3) $(23 + 12) \div 3 - (8 \times 4 + 18) \div 6$
(4) $(3a - 4) - (a - 5)$
(5) $(3d + 4) + (d + 5)$
(6) $(xy - r) - (7r^2 - r + xy)$
(7) $(ab + 4) - (4y + 4 + ab)$
(8) $(-x + t) + (7x - 7t - 6)$
(9) $(5k + 2m + s) - (2m + s)$
(10) $(r + s + t) - \frac{1}{4}(r + s + t)$
(11) $(10a^2b - 2ab) - (3ab - 10a^2b)$
(12) $(6apg - 2ad + 7) - (7 - 2ad + 4apg)$
(13) $(17kt + r^2) - (2k - r^2 + 15kt)$

Pages 23-26

EXERCISE E. Simplify:

(1) $10a + [7a + (a - y)]$
(2) $9y - [2y + (y - 4)]$
(3) $10b - [4b - (5b - 9)]$
(4) $24c - [9d - (4d - 10c)]$
(5) $[12s - (2a + 7s)] - (5s - 2a)$

Page 26

PROBLEMS.

(17) What must be added to ab to make k?
(18) If ab is one factor of x, what is the other factor?
(19) In n years a ship will be 30 years old. How old is it now?
(20) If n men take 11 days to assemble a machine, how long will one man take?
(21) How far will an electron travel in k hours at the rate of n miles per hour?
(22) By how much does $5k - 9$ exceed $2k + 3$?
(23) What number must be taken from $x - 5y$ to leave $y + x$?
(24) If a debt is shared equally among n persons, and each pays \$9.25, how many cents does the debt amount to?
(25) In how many days will k ships consume n tons of oil, if one ship consumes n tons a day?
(26) By what number must 239 be divided to give a quotient of 11 and a remainder of r?
(27) If the divisor is ab, the quotient d, and the remainder f, what is the dividend?

CHAPTER 3

Pages 28-30

EXERCISE A. Simplify each of the following expressions:

(1) $2^2 \times 2^3$
(2) $(-2)^2 \times (-2)^2$
(3) $10a^5 \times 6a^2$
(4) $-2ab \times 3a^2b^3$
(5) $(2k^2m) \times (-4k^2m^3)$
(6) $(3x)^2$
(7) $(3 \times 2)^2$
(8) $(2mn)^3$
(9) $(-a)^2 \times (-a^2)$
(10) $(-ab)^2 \times (-ab^2)$

Pages 30-31

EXERCISE B. Simplify:

(1) $(2^3)^2$
(2) $(a^3)^2$
(3) $(-5m^2)^3$
(4) $(a^2b)^2$
(5) $(3m^2n^2)^3$
(6) $(a^m)^2$
(7) $(a^m \cdot a^n)^3$
(8) $(x^2 \cdot x^a)^m$
(9) $5^3 \div 5^2$
(10) $a^5 \div a^2$
(11) $27a^2b^3 \div 9ab$
(12) $a^{m+1} \div a^m$
(13) $b^{2y} \div b^{y-1}$
(14) $(a^n)^2 \div a^{n-2}$
(15) $k^3 \cdot k^x \div k^{3-x}$
(16) $k^{x+6} \div k^x \cdot k^5$
(17) $1^0 + 5d^0 - (5d)^0$
(18) $(k^x)^3 \div (k^{x-3})^0$
(19) $(b^2 \cdot b^x)^0 \div (a^{2x})^0$
(20) $a^x \cdot a^{x-1} \div (a^{3x-7})^0$

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EXERCISE C. Simplify:

(1) $2^{-2} \times 2^3$
(2) $(-a)^{-4}$
(3) $x \div x^{-1}$
(4) $(x^0)^{-1}$
(5) $(\frac{1}{2})^{-2}$
(6) $(\frac{1}{5})^{-3}$
(7) $4(\frac{1}{2})^{-2}$
(8) $k^3 \cdot k^{-x} \div k^{3-x}$
(9) $(k^x)^{-3} \div k^{-3}$
(10) $\left(\frac{a}{b}\right)^{-2} \div \frac{a^{-3}}{b^{-4}}$

Pages 33-35

EXERCISE D. Perform the indicated multiplications:

(1) $(8x^2y)(2yx^3)$
(2) $(5a^2b)(a - b)$
(3) $(a + b^2)(a + b)$
(4) $(3k^3n^4 + 5kn - 10)(2kn^2)$
(5) $(2x^2 - 3x + 7)(x + 5)$
(6) $(4a^2 - 10b + 1)(3a + 2b)$
(7) $x^2(x + 1)(-5x + 5)$
(8) $(10a^3b + 7ab^4)(2a^2 - b)$
(9) $(6x^2 + 2xy - 3y^2)(x + y)$
(10) $(5x^3 + 2x^2y - 4xy^2 + y^3)(x - y)$
(11) $(x - y)(x + y)$
(12) $(a + b)(a + b)$
(13) $(x - 3)(x - 3)$
(14) $(x - y)(x + y)(x^2 + y^2)$
(15) $(2x - 3)(2x + 3)$
(16) $(a^x + b^y)(a^x + b^y)$
(17) $(x^a - y^b)(x^a + y^b)$
(18) $(a - b)(a + b)(a^2 - b^2)$
(19) $(a^2b - 3ab^3)(a^2b - 3ab^3)$
(20) $(x^k - y^t)(x^k + y^t)$

Pages 35-37

EXERCISE E. Perform the indicated divisions:

(1) $(2a^2 + 3a - 2) \div (a + 2)$
(2) $(6y^3 - 11y^2 + 2) \div (2y - 1)$
(3) $(6x^2 + 19x - 7) \div (3x - 1)$
(4) $(9x^2 - 30x + 25) \div (3x - 5)$
(5) $(a^3 - 2a^2b + 2ab^2 - b^3) \div (a - b)$
(6) $(9a^5b^3 - 12a^3b^5) \div 3a^4b^4$
(7) $(ax^5 + bx^3 + cx) \div x^2$
(8) $(a^3 - a^2b + 2b^3) \div (a + b)$
(9) $(y^3 + y^2 - y + 2) \div (y^2 - y + 1)$
(10) $(x^4 - 3x^3 + x^2 + 2x - 1) \div (x^2 - x + 2)$
(11) $(4x^2 - 9) \div (3 + 2x)$
(12) $(36b^2 - 25y^2) \div (5y + 6b)$
(13) $(x^2 + 2xy + y^2 - z^2) \div (x + y - z)$
(14) $(a^2 - 2ab + b^2 - k^2) \div (a - b + k)$
(15) $(9x^2 - 25y^2 + 10yz - z^2) \div (3x + 5y - z)$
(16) $(a^4 - b^4) \div (a + b)$
(17) $(x^3 + 8) \div (x + 2)$
(18) $(x^3 + y^3) \div (x + y)$
(19) $(a^3 - 8) \div (a - 2)$
(20) $(x^4 - y^4) \div (x - y)$

CHAPTER 4

Page 41

EXERCISE A. State which of the following are equations of identity and which are equations of condition:

(1) $(a - b)^2 = a^2 - 2ab + b^2$
(2) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
(3) $6x - 7 = 0$
(4) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
(5) $a^3 + 3a^2 + 6a = 7$
(6) $a^2 - 2ab + b^2 = 0$
(7) $a^2 - b^2 = (a + b)(a - b)$
(8) $x^2 - 9 = 0$
(9) $3x^3 - 6x + 5 = 2x - 3$
(10) $3x^2 - 5x + 7 = 2x^2 - 3x + 22$

Pages 41-42

EXERCISE B. Solve the following equations:

(1) $7x = 4x + 9$
(2) $8x - 15 = 65$
(3) $6x - 11 = 2x + 5$
(4) $9x - 8 = 4x + 22$
(5) $x + 6 = 6 - a$
(6) $bx = 6b$
(7) $2x^2 + 12 - 11x = 75 + 2x^2 - 2x$
(8) $bx = 6$
(9) $5x = 5a - 5$
(10) $16x = 4 - 20a$

(11) $3y + 15 = y + 45$
(12) $5d - 6(d - 5) = 2(d + 5) + 5(d - 4)$
(13) $x - [3 + \{x - (3 + x)\}] = 5$
(14) $(t + 1)(2t + 3) = 2(t + 1)^2 + 8$
(15) $(y + 1)^2 + 2(y + 3)^2 = 3y(y + 2) + 35$
(16) $0.5x - 0.03 = 0.3x + 0.17$
(17) $0.9x - 0.45 = 0.75 - 3.1x$
(18) $ax - a^2 + 2ab = b^2 + bx$
(19) $6(x - 1) - [3x - (5 - x)] = 7$
(20) $3(y - 2)^2 = (3y - 5)(y + 3)$

Pages 42-45

PROBLEMS.

(1) Divide 21 into two parts such that 5 times the smaller part shall exceed the greater by 3.

(2) Find two numbers whose sum is 180 and whose difference is 62.

(3) One plot of ground is half as large as another. Their combined surface area is 45 square feet. What is the area of each?

(4) Divide \$50 among A, B, and C so that A may receive \$12.50 more than B and \$19.25 more than C.

(5) Divide \$100 among A, B, C, and D so that A receives \$30 less than B, \$5 more than C, and \$15 less than D.

(6) A triangular piece of ground requires 43 yards of fencing. The first side is 4 yards shorter than the second, and the third side is 1 yard less than the second. What is the length of each side?

(7) A sum of money, amounting to \$2.85, consists entirely of dimes and quarters. There are 18 coins in all. How many dimes and how many quarters?

(8) A man is 31 years old and his son is 5. In how many years will the man be twice as old as his son?

(9) What two numbers are those whose sum is 15 and the difference of whose squares is 45?

(10) A clerk sold a certain number of typewriters at \$45 each and 3 times as many slide rules at \$12 each, realizing \$972 from the sale. How many typewriters and how many slide rules did he sell?

(11) A platoon of 40 soldiers is to be divided into two groups such that one group shall contain 12 more than the other. How many soldiers will there be in each group?

(12) Five hundred crates are to be loaded upon 2 ships. One ship has engaged 3 times as many longshoremen as the other. How many crates should be sent to each ship so that the work will be evenly divided among the men?

(13) Find 4 consecutive numbers whose sum is 458.

(14) Find 4 consecutive numbers such that the product of the first and fourth shall be less by 2 than the product of the second and third.

(15) A father agreed to employ his son for 40 days on condition that for every day he worked \$3.50 would be paid to him, but that for every day absent he should forfeit \$2. At the end of the period, he received \$85. How many days did he work, and how many days was he absent?

(16) For every nickel a boy saved, his uncle put a dime in his bank, and his father a quarter dollar. At the end of a year, the boy had a bank account \$60 larger than at the outset. How much had each person contributed?

(17) The length of a room is 6 feet greater than its width. If each dimension were to be increased by 3 feet, the area of the room would be increased by 90 square feet. What is the present area of the room and its dimensions?

(18) How many pounds of tea at 50 cents a pound must be added to 24 pounds of tea at 20 cents a pound to make a mixture worth $27\frac{1}{2}$ cents a pound?

(19) The tickets of admission to a moving picture are 35 cents each for adults and 20 cents each for children. At the end of a day, it is found that $265.40 was paid and 1,027 tickets sold. How many adults and how many children attended?

(20) A French peasant woman brings to market a certain number of eggs, which she counts on selling at 60 centimes apiece. Six eggs are broken, however. Yet she succeeds in selling the others at 75 centimes apiece, and consequently she obtains 120 centimes more than expected. How many eggs did she have at the outset?

Pages 45-47

EXERCISE C. Factor the following expressions:

(1) $2x^2 + 7x$
(2) $9x^2 - 21x^3$
(3) $10a^2b^3 + 6ab^5$
(4) $30ab^3x^2 - 42a^2b^2x^3 + 18a^3b^2x$
(5) $2b^3x^2 - 6ab^2x^3 + 30ab^3x^3$
(6) $22ak^2w^3 + 11a^3w^2 - 33k^3w^3$
(7) $91xyz - 65xy + 39xz$
(8) $15w^2xy^2z + 21wxz - 30wxyz$

Pages 47-48

(21) Mr. Owen borrowed $375 on October 15 and paid it back with interest at 5 per cent on January 15. Find the amount he had to pay.

(22) Mr. Demarest invested $6,000, part in one security paying 4 per cent and the other in a mortgage paying 6 per cent. In one year, his total income from these two investments was $270. How much was invested at each rate?

(23) Mr. Young invested $5,000, part in stocks paying 5 per cent and the other in bonds paying 4 per cent. In six months he sells his stocks and bonds, receiving $5,110. How much was invested at each rate?

(24) Acme's Department Store allowed 2 per cent on customers' deposit accounts. Mrs. Smith kept a deposit of $325. She was credited with $2\frac{2}{3}$ dollars interest. What was the period covered by this credit?

(25) A customer owed $275 on May 14 but did not pay until October 14. If interest at 6 per cent was charged, how much money discharged the indebtedness?

Pages 48-52

EXERCISE D. Separate into their simplest factors:

(1) $ax - ay + bx - by$
(2) $ab - ax - mb + mx$
(3) $x^2 - xy - 3x + 3y$
(4) $m^2 + mn - am - an$
(5) $p^2 - 5p + ap - 5a$
(6) $2c - a + 4c^2 - 2ca$
(7) $1 - x + y - xy$
(8) $3a^3 - 15a + 10c - 2a^2c$
(9) $15xy^2 - 9y^2z - 35xy + 21yz$
(10) $ax - a - bx + b - cx + c$
(11) $x^2 - y^2$
(12) $x^2 - 16$
(13) $a^2 - 9$
(14) $25 - y^2$
(15) $36m^2n^2 - 16t^2$
(16) $(a + b)^2 - c^2$
(17) $a^2 - (b - c)^2$
(18) $a^2 - (a - c)^2$
(19) $y^4 - 4y^2$
(20) $s^3 - 16s$
(21) $t^3 - t^2 - t + 1$
(22) $a^2 - b^2 + a - b$
(23) $25x^{2n} - 1$
(24) $36a^{2m} - 9b^{2n}$
(25) $81cx^{2a} - cx^{2n}$
(26) $a^3 + b^3$
(27) $125x^3 + 8y^3$
(28) $x^6 + y^6$
(29) $m^3 + 27$
(30) $x^5 + 1$
(31) $a^5 - b^5$
(32) $m^3 - 8$
(33) $x^3 - a^3$
(34) $x^5 - 32$

(35) $a^5 - 1$
(36) $x^4 - y^4$
(37) $16a^4 - 81b^4$
(38) $m^6 - n^6$
(39) $81 - x^4$
(40) $1 - a^6$

CHAPTER 5

Page 55

EXERCISE A:

(1) Change $\frac{a}{5x}$ to a fraction whose denominator is $20x$.

(2) Change $\frac{x}{3y}$ to a fraction whose denominator is $9y^2$.

(3) Change $\frac{3a}{4x}$ to a fraction whose denominator is $12x^2$.

(4) Change $\frac{2a}{7y}$ to a fraction whose denominator is $21xy$.

(5) Change $\frac{x-2}{x-5}$ to a fraction whose denominator is $(x-5)^2$.

(6) Change $\frac{3x-1}{2x+3}$ to a fraction whose denominator is $(2x+3)^2$.

(7) Change $\frac{m-2}{6-m}$ to a fraction whose denominator is $6m - m^2$.

(8) Change $\frac{a+b}{2b}$ to a fraction whose numerator is $a^2 - b^2$.

(9) Change $\frac{2x-3}{x+y}$ to a fraction whose denominator is $x^2 - y^2$.

(10) Change $\frac{2x-3}{2x-1}$ to a fraction whose numerator is $4x^2 - 6x$.

Pages 55-57

EXERCISE B. Reduce to their lowest terms:

(1) $\frac{ab}{bx}$
(2) $\frac{ax^2}{xy}$
(3) $\frac{abc^2}{bc}$
(4) $\frac{a^2x^3}{abx^5}$
(5) $\frac{a^2bc^2}{a^3bc}$
(6) $\frac{x^2y^3}{ax^2y^4}$
(7) $\frac{9a^2bm^2}{18ca^3m^3}$
(8) $\frac{16xz^2b}{32yx^2z}$
(9) $\frac{15x^2y^2z^2}{21y^3za^2}$
(10) $\frac{35x^4y^3z}{40x^2y^5}$
(11) $\frac{x^{m+1}y^3}{x^my^2}$
(12) $\frac{x^2 - y^2}{x^2 + 2xy + y^2}$

(13) $\dfrac{x^2 - 2ax + a^2}{x^3 - a^3}$

(14) $\dfrac{x^2 - 9}{x^2 + 6x + 3}$

(15) $\dfrac{x^3 - 4x + 2}{x^2 - 4}$

(16) $\dfrac{4a^2 - 9c^2}{8a^3 + 27c^3}$

(17) $\dfrac{3x^2y - 6xy}{x^4y - 8xy}$

(18) $\dfrac{25a^3 - ac^2}{125a^4 + ac^3}$

(19) $\dfrac{x^4yz - y^5z}{3x^6y + 3y^7}$

(20) $\dfrac{2a^2b^2 - 8b^4}{4a^3b - 32b^4}$

Pages 57-59

EXERCISE C. Change to an integral or mixed expression:

(1) $\dfrac{x^3 - 3x^2 + x - 1}{x^2}$

(2) $\dfrac{xy - ay - ca + a^2}{y}$

(3) $\dfrac{a^2 - a - 17}{a - 4}$

(4) $\dfrac{x^2 - 2x - 25}{x + 3}$

(5) $\dfrac{a^3 - 2ac - c^3}{a - c}$

(6) $\dfrac{x^3 + y^3}{x^2 - xy + y^2}$

(7) $\dfrac{a^3 - 5a^2 + 3a + 6}{a - 2}$

(8) $\dfrac{x^3 + 9x^2 + 24x + 22}{x + 3}$

(9) $\dfrac{x^4 + 3x^2y^2 + y^4}{x^2 + y^2}$

(10) $\dfrac{4a^2 + 22a + 25}{2a + 4}$

Pages 57-59

EXERCISE D. Change to fractional forms:

(1) $x - \dfrac{y}{3}$

(2) $a + \dfrac{b}{2}$

(3) $2x + \dfrac{a^2 - x^2}{x}$

(4) $\dfrac{x + y}{2} - y$

(5) $2a + \dfrac{ac - 2}{c}$

(6) $a + x + \dfrac{x^2}{a - x}$

(7) $a + 3 - \dfrac{a^2}{a - 2}$

(8) $x^2 - xy + y^2 - \dfrac{y^3}{x + y}$

(9) $a^2 + ac + c^2 + \dfrac{a^3}{a - c}$

(10) $x + y - \dfrac{2xy}{x + y}$

Pages 59-60

EXERCISE E. Change to fractions having positive numerators and denominators:

(1) $\dfrac{-2}{5}$

(2) $\dfrac{-5}{-7}$

(3) $\dfrac{3}{-4}$

(4) $\dfrac{-x - a}{3x}$

(5) $\dfrac{-4x}{-x - y}$

(6) $\dfrac{-a - c}{c + x}$

(7) $\dfrac{-6m}{-x-m}$ (8) $\dfrac{-7(x-y)}{9(-x-a)}$

Pages 59-60

EXERCISE F. Prove the following identities:

(1) $\dfrac{-x}{x-a} = \dfrac{x}{a-x}$ (4) $\dfrac{a}{(x-y)(a-y)} = -\dfrac{a}{(y-x)(a-y)}$

(2) $\dfrac{-c}{x-c+y} = \dfrac{c}{c-x-y}$ (5) $\dfrac{a-b}{(x-y)(z-x)} = \dfrac{a-b}{(y-x)(x-z)}$

(3) $\dfrac{7}{x(y-a)} = -\dfrac{7}{x(a-y)}$ (6) $\dfrac{a-b}{x(x-y)} = \dfrac{b-a}{x(y-x)}$

Pages 60-65

EXERCISE G. Perform the indicated operations:

(1) $\dfrac{x-a}{ax} + \dfrac{a-y}{ay}$

(2) $\dfrac{ac}{a-c} + a$

(3) $\dfrac{x+1}{x^2+x+1} + \dfrac{1}{x-1}$

(4) $\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y}$

(5) $a + b - \dfrac{a^2+b^2}{a-b}$

(6) $\dfrac{2x^2+y^2}{x^2-y^2} - 2$

(7) $\dfrac{2a+1}{3} + \dfrac{a-2}{4} - \dfrac{a-3}{6}$

(8) $\dfrac{5-x}{2} + \dfrac{x-2}{6} - \dfrac{x-4}{9} + \dfrac{2-3x}{4}$

(9) $\dfrac{2-6c}{5} - \dfrac{c-1}{3} + \dfrac{4c-1}{6} - \dfrac{5c-3}{10}$

(10) $\dfrac{x-5}{3} - \dfrac{x-4}{8} + 3 - \dfrac{x+9}{6}$

(11) $\dfrac{1-2a}{5} + \dfrac{2a-1}{4} - \dfrac{2a-a^2+1}{8}$

(12) $\dfrac{2-x^2+x}{2} - \dfrac{2-x+x^2}{6} - \dfrac{1-2x-2x^2}{3}$

(13) $\dfrac{3x}{5} + \dfrac{y}{2} - 3 + \dfrac{1}{y}$

(14) $\dfrac{x+y}{y-x} + \dfrac{y-x}{x+y} + \dfrac{2xy}{x^2-y^2}$

(15) $2 - \dfrac{ac-2bc+ab}{c^2}$

(16) $3x + \dfrac{5}{xy} - \left(2x + \dfrac{3}{xy}\right)$

(17) $\dfrac{c+1}{c^2-9} - \dfrac{6}{c+5} + \dfrac{10}{c+3}$

(18) $x - \dfrac{x^2+a^2}{x-y} + y$

(19) $\dfrac{1}{a} + 1 + \dfrac{2a}{1+a} - 2$

(20) $\dfrac{x}{x-2} - \dfrac{x-2}{x+2} - \dfrac{3}{4-x^2}$

(21) $\dfrac{5a+2}{a^2-9} + \dfrac{2}{a-3} - \dfrac{3}{2-a}$

(22) $\dfrac{x(x+y)}{y-x} - \dfrac{3xy-x^2}{x-y} + 4y$

(23) $\dfrac{a}{b(b-a)} + \dfrac{2b}{a(a-b)} + \dfrac{1}{b-a}$

(24) $\dfrac{1}{x^3+8} - \dfrac{1}{8-x^3} + \dfrac{1}{4-x^2}$

(25) $\dfrac{2}{a} + \dfrac{3}{1-2a} - \dfrac{2a-3}{4a^2-1} - \dfrac{1}{1+2a}$

(26) $\dfrac{3x}{(x-2y)^2} + \dfrac{2x+y}{(x+y)(x-2y)} - \dfrac{5}{x+y}$

(27) $\dfrac{x}{x-y} - \dfrac{x}{x+y} - \dfrac{2xy}{x^2+y^2} - \dfrac{4xy^3}{x^4-y^4}$

(28) $\dfrac{1}{x-2} + \dfrac{1}{x+2} - \dfrac{x}{x^2+4} - \dfrac{x^3+3x}{x^4-16}$

(29) $\dfrac{a+c}{a-c} - \dfrac{a-c}{a+c} - \dfrac{4ac}{a^2+c^2} + \dfrac{8ac^3}{a^4-c^4}$

(30) $\dfrac{x^2+y^2}{x^3-y^3} + \dfrac{x+y}{x^2+xy+y^2} - \dfrac{2}{x-y}$

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EXERCISE H. Perform the indicated operations:

(1) $xy \times \dfrac{y}{x}$

(2) $ab \times \dfrac{mn}{ab^2}$

(3) $\dfrac{3}{2x^2} \times x^2y$

(4) $\dfrac{5}{a^2b^2} \times b^2c$

(5) $\dfrac{3xy}{4ac} \times \dfrac{2b}{3x^2}$

(6) $\dfrac{6ab}{7xy} \times \dfrac{2ax}{3b^2}$

(7) $\dfrac{3xy}{10z^2} \times \dfrac{5yz}{x^3}$

(8) $\dfrac{7ac}{10b^2} \times \dfrac{4bc}{a^3}$

(9) $\dfrac{3}{a^2c^2} \times c^2b$

(10) $\dfrac{3xy}{2ac} \times \dfrac{4ax}{9y^2}$

(11) $\dfrac{11xy^2}{4ac} \times \dfrac{8a^2c}{11y^3}$

(12) $\dfrac{a-4}{a+2} \times \dfrac{4-a^2}{16-a^2}$

(13) $\dfrac{cd^2}{14-6c} \times \dfrac{21-9c}{c^2d}$

(14) $\dfrac{x^4-y^4}{x+y} \times \dfrac{x-y}{x^2+y^2}$

(15) $\dfrac{(x-y)^2}{x+y} \times \dfrac{y}{x^2-xy} \times \dfrac{(x+y)^2}{x^2-y^2}$

(16) $\dfrac{x-3}{a+5} \times \dfrac{2a^2-50}{3x^2-27} \times \dfrac{6}{ax-5x}$

(17) $\dfrac{x^3-27}{x^3+8} \times \dfrac{x^2-2x+4}{x^2+3x+9}$

(18) $\dfrac{a^2+6a+9}{a-2} \times \dfrac{a^2-4a+4}{a+3}$

(19) $\dfrac{x^2-3x+9}{2x-6} \times \dfrac{x^2-9}{x^3+27}$

(20) $\dfrac{x^4+x^2y^2+y^4}{x^4-xy^3} \times \dfrac{y}{x^2-xy+y^2}$

(21) $\dfrac{x-5y}{3a-c} \times \dfrac{3x}{x^2-5xy} \times \dfrac{9a^2-c^2}{6}$

(22) $\dfrac{81-x^4}{27+x^3} \times \dfrac{3+x}{9-x^2} \times \dfrac{9-3x+x^2}{9+6x+x^2}$

(23) $\dfrac{(x-y)^2}{x+y} \times \dfrac{y}{x^2-xy} \times \dfrac{(x+y)^2}{x^2-y^2}$

(24) $\left(\dfrac{5}{7x}-1\right) \times \dfrac{21x^2y}{25-49x^2} \times \dfrac{5+7x}{2y^2}$

(25) $\left[a - \dfrac{a^3}{a^2+b}\right] \cdot \left[\dfrac{a}{b} + \dfrac{1}{a}\right]$

(26) $\left(a+1+\dfrac{1}{a}\right)\left(a-1+\dfrac{1}{a}\right) \times \left[\dfrac{a^2(a^2-1)}{a^6-1}\right]$

(27) $\left(\frac{x}{y}+1\right)\left(1-\frac{y}{x}\right)\left(1-\frac{x^2+y^2}{x^2-y^2}\right)$

(28) $\frac{20a+50b}{20a^2-125b^2}\times\frac{6a^2b-15ab^2}{6a^2b^2}\times\frac{ab}{a+b}$

(29) $\frac{a-c}{a^2+ac}\times\frac{a^4-c^4}{(a-c)^2}\times\frac{a^2}{a^2+c^2}$

(30) $\frac{x-y}{7xy}\times\frac{x^2-y^2}{(x-y)^2}\times\frac{1}{x+y}\times\frac{21x^2y^2}{3xy}$

Pages 67-69

EXERCISE I. Perform the indicated operations:

(1) $1\div\frac{x}{y}$

(2) $5\div\frac{a^3}{a^2}$

(3) $1\div\frac{x+2}{x-3}$

(4) $3\div\frac{x^3y}{x^2}$

(5) $1\div\frac{x-y}{x+y}$

(6) $\frac{2xy}{7ac}\div\frac{4xy^2}{21ac^2}$

(7) $\frac{5a^2}{7xy}\div\frac{10a^3}{21x^2y^2}$

(8) $\frac{55xyz}{13}\div 11xy$

(9) $\frac{7xy}{5x^2z^2}\div\frac{21y^2}{10x^2}$

(10) $\frac{14a^3}{21b^3}\div\frac{4a^2b^2}{7x}$

(11) $\frac{x^2-y^2}{x+y}\div\frac{x^2-xy}{y}$

(12) $(6x-3)\div\frac{2x-1}{5x}$

(13) $\left(\frac{1}{a}-\frac{1}{b}\right)\div\left(\frac{a}{b}-\frac{b}{a}\right)$

(14) $\left(\frac{1}{k}-k\right)\div\left(\frac{1}{k}+1\right)$

(15) $\left(\frac{1}{x^2}-1\right)\div\left(1+\frac{1}{x}\right)$

(16) $\left(1-\frac{1}{a^2}\right)\div\left(1-\frac{1}{a}\right)$

(17) $\left(\frac{x-5}{2x}-\frac{11}{2}+\frac{25}{x}\right)\div\left(\frac{9-2x}{x}\right)$

(18) $\frac{1}{xy}\div\left(\frac{1}{x}-\frac{1}{y}\right)$

(19) $a\left(\frac{a}{x}-1\right)\div\left(\frac{a^2}{x^2}-1\right)$

(20) $\left(n+\frac{y^2}{n-y}\right)\div\left[n-\frac{y^2(n-y)}{n^2+y^2}\right]$

(21) $(x^3-y^3)\div\frac{x^2+xy+y^2}{x+y}$

(22) $(x^5-y^5)\div\frac{x^3-y^3}{x+y}$

(23) $(x^n-y^n)\div\frac{6x^n-6y^n}{2a}$

(24) $\left(d+\frac{1}{d}\right)\div\left(d+\frac{1}{d}-1\right)$

(25) $\left(\frac{2a+b}{a+b}-1\right)\div\left(1-\frac{b}{a+b}\right)$

(26) $\left(\frac{1}{x^3}+1\right)\div\left(1-\frac{1}{x^2}\right)$

(27) $\left(\frac{a}{2}-\frac{a+1}{3}\right)\left(1-\frac{a}{a-2}\right)\div\frac{a-2}{a}$

(28) $\left(\frac{1}{a^2}-a^2\right)\left(\frac{1}{a^2}+a^2\right)\div\frac{1-a^4}{a^4}$

(29) $\left(\frac{x^2+xy+y^2}{x^2-xy+y^2}\div\frac{x+y}{x-y}\right)\times\left(\frac{18x^3+18y^3}{6x^3-6y^3}\right)$

(30) $\left[\left(1-\frac{25}{9m^2}\right)\div\frac{3m+5}{27m^3}\right]\times\left[\frac{1-\frac{1}{m^2}}{3m(3m-5)}\right]$

Pages 69-72

PROBLEMS:

(1) Divide 39 into two parts such that $\frac{1}{5}$ of the first part is equal to $\frac{3}{4}$ of the second.

Solution

Let $x = \frac{1}{5}$ of the first part $= \frac{3}{4}$ of the second part.
Then, $5x$ = first part, and $\frac{3}{2}x$ = second part.
Therefore, $5x + \frac{3}{2}x = 39.$
Whence, $10x + 3x = 78,$
$13x = 78$
$x = 6$
$5x = 30$, the first part
and $\frac{3}{2}x = 9$, the second part.

(2) Separate 7500 ball bearings into two groups such that $\frac{1}{5}$ of the greater shall be 500 less than $\frac{1}{3}$ of the smaller.

(3) Divide 650 into two parts such that, if the smaller is divided by the greater, the quotient is $\frac{4}{9}$.

(4) A grocer buys eggs at the rate of 5 for 8 cents. He sells one-sixth of them at the rate of 4 for 10 cents and the remainder at the rate of 5 for 10 cents, and makes $2.90 by the transaction. How many eggs did he buy?

(5) At what time between 1 and 2 o'clock are the hands of a watch opposite each other?

Solution

Let x = the number of minute spaces that the minute hand travels after 1 o'clock before the two hands are opposite each other.

Then, $\frac{x}{12}$ = the number of minute spaces that the hour hand travels in the same time.

Since the two hands are 5 minute spaces apart at 1 o'clock and 30 minute spaces apart when they are opposite each other,

$$(x - 5) - \frac{x}{12} = 30.$$

Therefore, $x = 38\frac{2}{11}$, the number of minutes after 1 o'clock.

(6) At what time between 7 and 8 o'clock are the hands of a watch together?

(7) At what time after 8 o'clock and before 9 o'clock are the hands of a watch at right angles to each other?

(8) At what time between 3 and 4 o'clock are the hands of a watch opposite each other?

(9) It takes a carpenter 9 days' less time than an apprentice to build a garage. If the two work together, they can build a garage in one-third the time it takes the apprentice. How long would it take each alone?

Solution

Let x = the number of days it would take the carpenter alone.
Then, $x + 9$ = the number of days it would take the apprentice alone
and $\frac{x + 9}{3}$ = the number of days it would take both working together.

Whence, $\frac{1}{x}$ = part of the work the carpenter can do in 1 day,

$\frac{1}{x+9}$ = part of the work the apprentice can do in 1 day,

and $\frac{\frac{1}{x+9}}{3} = \frac{3}{x+9}$ = part of the work both working together can do in 1 day.

Therefore, $\frac{1}{x} + \frac{1}{x+9} = \frac{3}{x+9}$.

Solving, $x = 9$, the number of days it would take the carpenter,
and $x + 9 = 18$, the number of days it would take the apprentice.

(10) Mr. Harris, alone, can mow the grass on his lawn in 5 days. His older son, alone, takes 2 days longer. His younger son, alone, takes twice as long as his father. How long will it take all of them working together to complete the mowing?

(11) It required 60 men working 32 days to build an embankment to help control a flood. If, at the very start of operations, 14 men had been removed to another job, how much longer would it have taken to build the embankment?

(12) A tank can be filled by one pipe in 8 hours or by another pipe in 12 hours. How long would it take to fill the tank if both pipes are open?

(13) A and B can dig a ditch in 8 days, and A, alone, can do it in 14 days. How long would it take B alone to dig this ditch?

(14) How long will it take two pipes to fill a tank if one alone can fill it in 7 hours and the other in 21 hours?

(15) Collins alone requires 22 hours to paint a fence. His neighbor, Foster, can paint the same fence, alone, in 18 hours. How long will it take Collins to finish the job after both have worked 6 hours?

(16) Two pipes are connected with a tank. The larger one can fill it in 6 hours; the small one can empty it in 8 hours. With both pipes open, how long will it be before the tank will fill?

(17) The velocity of a freight train is 10 miles per hour less than the speed of a passenger train. If the freight train travels 210 miles in the same time that the passenger train goes 270 miles, find the rate of each.

Solution

Let x = rate of freight train.
Then $x + 10$ = rate of passenger train.
Since distance = rate × time,

$\frac{210}{x}$ = time taken by freight train to travel 210 miles,

$\frac{270}{x+10}$ = time taken by passenger train to travel 270 miles.

Therefore, $\frac{210}{x} = \frac{270}{x+10}$,

$x = 35$ miles per hour, the rate of the freight train,
and $x + 10 = 45$ miles per hour, the rate of the passenger train.

(18) The rate of an express train is $\frac{1}{6}$ that of a slow train, and it covers 550 miles in 12 hours' less time than the slow train. Find the rate of each train.

(19) An airplane traveling at 200 miles an hour crosses a lake in 45 minutes' less time than another airplane traveling at 150 miles an hour. How wide is the lake?

(20) A train 1,320 feet long, running at 15 miles an hour, will pass completely through a tunnel in 9 minutes. How long is the tunnel? (1 mile = 5,280 ft.)

(21) An automobile and a motorcycle leave the same place at the same time to go to a point 100 miles distant, the latter traveling $\frac{5}{3}$ times as fast as the former. The motorcycle makes the trip, and, returning, meets the automobile $2\frac{1}{2}$ hours from the time they started. Find the rate of each.

(22) An airplane whose rate is 250 miles an hour starts 42 minutes after another airplane and overtakes it in 1 hour and 48 minutes. Find the rate of the slower plane.

(23) A 6-gallon mixture of nitric and picric acid contains 60 per cent nitric acid. How much picric acid must be added to cause the mixture to contain only 50 per cent nitric acid?

Solution

Since the given mixture contains nitric and picric acids only, and 60 per cent of it is nitric acid, then it contains 100 per cent − 60 per cent = 40 per cent of picric acid.

That is, $\frac{40}{100}(6) = 2.4$ gallons of picric acid in original mixture.

Let x = number of gallons of picric acid to be added.

Then $x + 6$ = number of gallons of new mixture,

and $x + 2.4$ = number of gallons of picric acid in the new mixture.

Therefore, $$\frac{2.4 + x}{x + 6} = \frac{50}{100}.$$

Whence, $x = 1\frac{1}{6}$ gallons.

(24) How much water must be added to 90 lbs. of a 10 per cent salt solution, to reduce the liquid to an 8 per cent salt solution?

(25) At the battle of Trafalgar, warships carried cannon made of bronze. Gun metal for these cannons usually consisted of 10 per cent tin and 90 per cent Mexican copper. If (disregarding loss of metal in melting) a cannon weighing 3,500 lbs. is melted, how many pounds of copper must be added in order to form a new alloy containing 6 per cent of tin?

(26) A candy manufacturer mixes 20 lbs. of chocolate and nuts. The chocolates cost 30 cents a pound while the nuts cost 45 cents a pound. How many pounds of each kind must he use in order to sell the mixture at 36 cents a pound?

(27) A cask contains 50 gallons of a mixture of wine and water of which 40 per cent is wine. Another cask contains 30 gallons of a mixture of wine and water of which 30 per cent is wine. How many gallons must be drawn from each cask to form a 30-gallon mixture containing 33 per cent of wine?

(28) A milkman has 2,000 quarts of milk that test 5 per cent butter fat, but the city in which he sells his milk requires only 4 per cent butter fat. How many quarts of cream testing 24 per cent butter fat may he separate from the milk and still satisfy the city requirements?

(29) The tens' digit of a number expressed by two digits exceeds the units' digit by 2. If the number decreased by 3 is divided by the sum of the digits, the quotient is 6. Find the number.

Solution

Let x = the digit in units' place.
Then $x + 2$ = the digit in tens' place,
so that $10(x + 2) + x = 11x + 20$ = the number.

Therefore, $$\frac{11x + 20 - 3}{x + (x + 2)} = 6;$$

whence, $x = 5$
and $x + 2 = 7$.

Therefore, the number is 75.

(30) The units' digit of a number expressed by two digits exceeds the tens' digit by 3. If the number is divided by the sum of the digits, the quotient is 4. Find the number.

(31) The tens' digit of a number expressed by two digits exceeds the units' digit by 7. If the number decreased by 11 is divided by the difference of the digits, the quotient is 10. Find the number.

(32) The tens' digit of a number expressed by two digits exceeds the units' digit by 4. If the number increased by 2 is divided by the sum of the digits, the quotient is 8. Find the number.

CHAPTER 6

Pages 74-75

EXERCISE A. Write the following with fractional exponents:

(1) $\sqrt{a}$
(2) $\sqrt[3]{18}$
(3) $\sqrt[4]{3}$
(4) $\sqrt[3]{8}$
(5) $\sqrt[4]{32}$
(6) $\sqrt{xy}$
(7) $\sqrt[3]{x}$
(8) $\sqrt[5]{ab}$
(9) $\sqrt[5]{m}$
(10) $\sqrt[3]{5a}$

Pages 74-75

EXERCISE B. Express the following with radical signs:

(1) $10^{1/2}$
(2) $13^{1/3}$
(3) $9^{1/4}$
(4) $7^{1/6}$
(5) $35^{1/5}$
(6) $a^{1/3}$
(7) $x^{1/4}$
(8) $(ab)^{1/2}$
(9) $m^{1/7}$
(10) $(6y)^{1/5}$

Pages 74-75

EXERCISE C. Find the value of:

(1) $9^{1/2}$
(2) $8^{1/3}$
(3) $16^{1/4}$
(4) $36^{1/2}$
(5) $27^{1/3}$
(6) $32^{1/5}$
(7) $81^{1/4}$
(8) $25^{1/2}$
(9) $64^{1/3}$
(10) $125^{1/3}$

Pages 75-80

EXERCISE D. Find the value of:

(1) $\sqrt{169}$
(2) $\sqrt{324}$
(3) $\sqrt{576}$
(4) $\sqrt{784}$
(5) $\sqrt{225}$
(6) $\sqrt{\frac{25}{81}}$
(7) $\sqrt{\frac{16}{49}}$
(8) $\sqrt{\frac{1}{100}}$
(9) $\sqrt{\frac{9}{64}}$
(10) $\sqrt[3]{\frac{8}{27}}$

Pages 75-80

EXERCISE E. By means of the table, find the value of:

(1) $(.35)^2$
(2) $\sqrt{5.6}$
(3) $\sqrt{6.7}$
(4) $\sqrt{4300}$
(5) $\sqrt{3700}$
(6) $\sqrt[3]{790}$
(7) $\sqrt[3]{0.45}$
(8) $\sqrt[3]{2.8}$
(9) $\sqrt{.0073}$
(10) $\sqrt[3]{8.7}$

Pages 80-82

EXERCISE F. Express the following with radical signs:

(1) $x^{2/3}$
(2) $k^{1/7}$
(3) $9xy^{7/3}$
(4) $9(xy)^{7/3}$
(5) $(9xy)^{7/3}$
(6) $k^{1/2}\,b^{5/3}$
(7) $k^{1/6}\,b^{5/6}$
(8) $5x^{\frac{m}{n}}$

Pages 80-82

EXERCISE G. Express the following with fractional exponents:

(1) $\sqrt[4]{d^3}$
(2) $\sqrt[5]{x^2}$
(3) $5\sqrt[a]{m^2}$
(4) $\sqrt[4]{a^5}\sqrt[5]{d^4}$
(5) $\sqrt[n]{6k^4}$
(6) $\sqrt[x]{ab^xy}$
(7) $\sqrt[3]{27k^m}$
(8) $\sqrt[4]{m^2n^3}$
(9) $\sqrt{x^{-1}y^3}$
(10) $\sqrt{\frac{a}{b}}$

Pages 80-82

EXERCISE H. Simplify the following expressions:

(1) $49^{1/2}$
(2) $64^{1/2}$
(3) $36^{0.5}$
(4) $(\frac{1}{4})^{-1/2}$
(5) $8^{1/3}$
(6) $16^{3/4}$
(7) $8^{2/3}$
(8) $9^{3/2}$
(9) $27^{1/3}$
(10) $27^{2/3}$
(11) $16^{-1/2}$
(12) $32^{0.2}$
(13) $81^{1/4}$
(14) $x^{1/2}\cdot x^{-1/6}$
(15) $x^{2/3}\sqrt[3]{x}$
(16) $c^{1/2} \div c^{1/6}$
(17) $\sqrt[4]{x} \div x^{-1/4}$
(18) $\sqrt{a} \div a^{-1/2}$
(19) $(3^{3/8})^{-1/3}$
(20) $a^{1/2} \div \sqrt[3]{a}$

Pages 80-82

EXERCISE I. Perform the indicated operations:

(1) $(a^{2/3} + a^{1/3}b^{1/3} + b^{2/3})(a^{1/3} - b^{1/3})$
(2) $(4x^{-4/3} - 6x^{-2/3} + 9)(2x^{-2/3} + 3)$
(3) $(a^{1/2} - b^{1/2})^2$
(4) $(x^{2/3} + 10)(x^{-1/2} - 1)$
(5) $(m^{2/3} + n^{1/2})^2$
(6) $(x^{2/3} - y^{2/3})(x^{1/3} + y^{1/3})$
(7) $(x^{-9/4} + 2x^{-3/2} + 4x^{-3/4} + 10)(x^{-3/4} - 2)$
(8) $(2a^{-1} - 8 - 3a)(4a^{-1} + 5)$
(9) $(a^{1/3} + a^{1/6}b^{1/6} + b^{1/3})(a^{1/3} - a^{1/6}b^{1/6} + b^{1/3})$
(10) $x^{-1} \cdot \dfrac{7}{x^{-4/5}}$
(11) $(x^2 + y^2) \div (x^{2/3} + y^{2/3})$
(12) $(x - y) \div (x^{1/3} - y^{1/3})$
(13) $(1 - x) \div (1 - x^{1/2})$
(14) $(ax - bx) \div (\sqrt{a} - \sqrt{b})$
(15) $\left(1 - \dfrac{1}{x}\right) \div (1 + \sqrt{x})$
(16) $\left(\dfrac{a}{b} - \dfrac{x}{y}\right) \div \left(\sqrt{\dfrac{a}{x}} + \sqrt{\dfrac{b}{y}}\right)$
(17) $(x\sqrt[3]{x} - y\sqrt[3]{y}) \div (\sqrt[3]{x} - \sqrt[3]{y})$
(18) $\left(x\sqrt{\dfrac{x}{y}} - y\right) \div \left(1 - \sqrt{\dfrac{y}{x}}\right)$
(19) $(x^{4/3} - y^{4/3}) \div (x^{1/3} + y^{1/3})$
(20) $(3 - 4x^{-1} + x^{-2}) \div (x^{-1} - 3)$

Page 82

EXERCISE J. Simplify the following:

(1) $\sqrt{32}$
(2) $\sqrt{162}$
(3) $\sqrt[3]{16}$
(4) $\sqrt{63}$
(5) $\sqrt[3]{320}$
(6) $\sqrt[3]{625}$
(7) $\sqrt{\frac{1}{3}}$
(8) $\sqrt{\frac{2}{5}}$
(9) $\sqrt{\frac{3}{4}}$
(10) $\sqrt[3]{\frac{5}{9}}$
(11) $\sqrt[3]{\frac{4}{5}}$
(12) $\sqrt{\frac{3}{8}}$
(13) $\sqrt{27x^3}$
(14) $\sqrt{16y}$
(15) $\sqrt{75a^3}$
(16) $\sqrt[3]{16x^6y^4}$
(17) $\sqrt[3]{54x^6 - 162x^3}$
(18) $\sqrt{12a^4 - 8a^3}$
(19) $\sqrt{\dfrac{1}{x^5}}$
(20) $\sqrt{\dfrac{2x^3}{y}}$
(21) $\sqrt{\dfrac{4}{5a^3}}$
(22) $\sqrt[3]{\dfrac{x}{2y^2}}$
(23) $\sqrt{\dfrac{a+b}{a-b}}$
(24) $(x + y)\sqrt{\dfrac{x-y}{x+y}}$

Page 83

EXERCISE K. Simplify:

(1) $\sqrt{12}+\sqrt{48}-\sqrt{27}$

(2) $\sqrt{245}+\sqrt{80}-\sqrt{125}$

(3) $\sqrt{128}-\sqrt{98}+\sqrt{18}$

(4) $\sqrt{75a}+\sqrt{48a}-\sqrt{27a}$

(5) $\sqrt[3]{250x^2}-\sqrt[3]{54x^5}-\sqrt[3]{16x^2}$

(6) $\sqrt[3]{-108x^4}+\sqrt[3]{500x^4}+\sqrt[3]{32x^4}$

(7) $\sqrt{\frac{5a^2}{7x^2}}-\sqrt{\frac{45a^2}{7x^2}}+\sqrt{\frac{125a^2}{7x^2}}$

(8) $\sqrt{\frac{x}{yz}}-\sqrt{\frac{y}{xz}}+\sqrt{\frac{z}{xy}}$

(9) $\begin{cases}(a-b)\sqrt{\dfrac{80(a+b)}{a-b}} \\ -\ (a+b)\sqrt{\dfrac{20(a-b)}{a+b}}\end{cases}$

(10) $\begin{cases}\sqrt{50x^4-75x^3+25x^2} \\ -\sqrt{32x^4-48x^3+16x^2}\end{cases}$

Pages 83-86

EXERCISE L. Reduce the following radicals to the same degree:

(1) $\sqrt{3}$ and $\sqrt[3]{5}$

(2) $\sqrt{2}$ and $\sqrt[4]{3}$

(3) $\sqrt[3]{x^2y}$ and $\sqrt[4]{x^4y^3}$

(4) $\sqrt{2}$ and $\sqrt[4]{xy}$

(5) $\sqrt[3]{5}$ and $\sqrt[4]{7}$

(6) $\sqrt{ab}$, $\sqrt[4]{bc}$, and $\sqrt[4]{ac}$

(7) $\sqrt[3]{3d}$, $\sqrt[4]{2k}$, and $\sqrt[6]{6s}$

(8) $\sqrt[3]{2}$, $\sqrt[4]{8}$, and $\sqrt[6]{20}$

(9) $\sqrt[4]{1-a}$ and $\sqrt[4]{1+a}$

(10) $\sqrt[4]{x+y}$ and $\sqrt[4]{x-y}$

Pages 83-86

EXERCISE M. Rationalize the denominators:

(1) $\dfrac{6}{2+\sqrt{2}}$

(2) $\dfrac{14}{3-\sqrt{2}}$

(3) $\dfrac{1}{\sqrt{3}+\sqrt{2}}$

(4) $\dfrac{4}{\sqrt{7}-\sqrt{3}}$

(5) $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

(6) $\dfrac{\sqrt{a+b}}{\sqrt{a-b}}$

(7) $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

(8) $\dfrac{x-\sqrt{y}}{x+\sqrt{y}}$

(9) $\dfrac{\sqrt{a+2}+2}{\sqrt{a+2}-2}$

(10) $\dfrac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$

Pages 83-86

EXERCISE N. Perform the indicated operations:

(1) $\sqrt{xy}\times\sqrt{yz}$

(2) $\sqrt{xy^2}\div\sqrt{yz}$

(3) $\sqrt{a^3b}\times\sqrt{ac}$

(4) $\sqrt[3]{25x^2y^2}\div\sqrt{5xy}$

(5) $\sqrt{3ab}\times\sqrt[3]{9ab}$

(6) $\sqrt{x+y}\div\sqrt{x-y}$

(7) $\sqrt[4]{x^2y^2}\times\sqrt[3]{2x^2y^2}$

(8) $\begin{cases}\sqrt[4]{(x-y)^{-2}}\times\sqrt[4]{x^2y^2} \\ \times\sqrt{x-y}\end{cases}$

(9) $(\sqrt{25}-\sqrt{5})\div\sqrt{5}$

(10) $\sqrt[3]{x^2-y^2}\div\sqrt[4]{x-y}$

Pages 86-88

EXERCISE O. Solve the following equations:

(1) $\sqrt{x+6} = 5$
(2) $\sqrt{a-5} = 8$
(3) $\sqrt[3]{x-2} = 3$
(4) $\sqrt{x} - 5 = 10$
(5) $\sqrt[3]{x^3 - a^3} = a$
(6) $\sqrt{a+21} = \sqrt{a} + 3$
(7) $\sqrt{x^2 - 25} = x - 1$
(8) $\sqrt{2x-3} + \sqrt{2x+5} = 8$
(9) $\sqrt{x^2 - 7x + 12} = x - 3$
(10) $\sqrt{x^2 - 8x + 17} = x - 5$

CHAPTER 7

Pages 90-92

EXERCISE A. Express the given measures in the equivalent American standards

(1) 8m. to feet
(2) 12Dm. to rods
(3) 183.45m. to yards
(4) 785.63Km. to miles
(5) 185cm. to inches
(6) 7.05Hm. to rods
(7) 19mm. to inches
(8) 13.56Mm. to furlongs
(9) 87.23dm. to feet
(10) 1.82m. to inches

Pages 90-92

EXERCISE B. Express the given measures in the equivalent metric standards:

(1) 12 feet to meters
(2) 15 yards to meters
(3) 28 rods to decameters
(4) 69 inches to centimeters
(5) 356 miles to kilometers
(6) $3\frac{3}{4}$ inches to millimeters
(7) 6 furlongs to kilometers
(8) 3.47 feet to centimeters
(9) 30 feet to decameters
(10) $13\frac{1}{2}$ inches to decimeters

Pages 90-92

PROBLEMS:

(1) A lot requires 200 meters of wire fencing. Find the cost of the wire fence at 25 cents a foot.

(2) An airplane travels 1,000 kilometers in $2\frac{1}{2}$ hours. What is the airplane's speed in miles per hour?

(3) The Meca Department Store ships 350 yards of rayon silk and 205 yards of velvet to South America. In making out the bill of lading and invoice, the measures must be changed to the metric system. Express these measures in metric standards.

(4) A South American importer receives the following quotations for wire fencing: 78 cents per meter from a European concern and 23 cents per foot from an American firm. Which is the better offer?

(5) A lot of woolen cloth is invoiced at Le Havre, France, as 83.56 meters at $0.58 a meter. Find the amount of duty paid in New York at 18 cents a yard and 25 per cent ad valorem. (*Ad valorem* means according to the invoice value.)

Pages 92-98

PROBLEMS:

(6) State how a point is required to move on a plane in order to generate: (*a*) a straight line; (*b*) a curved line.

(7) Draw: (*a*) a line segment AB; (*b*) line CD; (*c*) a curved line EF.

(8) Through a given point draw two straight lines. Can these two lines meet at another point?

(9) Locate two points on a plane. (*a*) How many different straight lines can be drawn through these two points? (*b*) How many different curved lines can be drawn through the two given points?

(10) (*a*) How many points are required to determine a straight line? (*b*) Why are two sights necessary for a rifle?

(11) Draw three line segments on a plane, no two of which will meet no matter how far they are produced. If, on the same plane, another line segment is drawn intersecting one of the three line segments, how many of the other line segments will it intersect if extended far enough?

Pages 98-108

EXERCISE C. Classify the following angles:

(1) 35°10′
(2) 275°
(3) 103°6′
(4) 90°
(5) 54°50′
(6) 76°54′
(7) 180°
(8) 53°7′12″
(9) 126°52′48″
(10) 312°

Pages 98-108

EXERCISE D. State the relationship between the following angles:

(1) 35°10′ and 54°50′
(2) 103°6′ and 76°54′
(3) 53°7′12″ and 126°52′48″
(4) 23°10′5″ and 66°49′55″
(5) 113°35′47″ and 66°24′13″

Pages 98-108

EXERCISE E. Find the complements of the following angles:

(1) 23°12′
(2) 44°23′15″
(3) 35°0′53″
(4) 62°5′34″
(5) 51°16′41″

Pages 98-108

EXERCISE F. Find the supplements of the following angles:

(1) 120°
(2) 135°
(3) 98°53′
(4) 114°34′17″
(5) 53°0′38″

Pages 98-108

EXERCISE G. Express the following angles in degrees, minutes, and seconds:

(1) 73°.28
(2) 43°.35
(3) 108°.62
(4) 15°.14
(5) 37°.01

Pages 98-108

EXERCISE H. Express the following angles in degrees only:

(1) 22°12′
(2) 51°10′48″
(3) 64°25′12″
(4) 86°32′24″
(5) 93°0′54″

Pages 98-108

PROBLEMS:

(12) State how a point is required to move on a plane in order to generate a circle.

(13) The circumferences of two circles measure 126 inches and 84.6 inches, respectively. (*a*) What is the length of an arc of the larger circle containing 10°? (*b*) What is the length of an arc of the smaller circle containing 10°?

(14) One degree at the equator, namely, at a circle passing around the earth's surface halfway between the poles, is approximately 69.1 miles long. At the circle of latitude passing through New York City, that is, at a circle on the earth's surface passing through New York City such that every point of it is equally distant from the equator, a degree is approximately 52.3 miles. How many more miles would an airplane making a trip around the world following the equator have to travel than an airplane making a trip around the world following New York City's circle of latitude?

(15) The spoke of a wheel turns through an angle of 175°. What fraction of a revolution does the wheel make?

(16) (*a*) How long does it take the minute hand of a clock to turn through an angle of 90°? Of 60°? Of 150°? (*b*) How long does it take the hour hand to turn through 75°? Through 262°30′?

(17) Find the number of degrees in the angle included by the hands of a watch at 4 o'clock, 9 o'clock, 6 o'clock, 1 o'clock, and 3 o'clock.

(18) Two angles are both equal and supplementary. What kind of angles are they?

(19) The sum of an obtuse angle and a right angle is what kind of an angle?

(20) The difference of an obtuse angle and a right angle is what kind of an angle?

(21) Find the number of degrees in an angle if it is three times its complement.

(22) How many degrees are there in an angle if it is one-fifth of its complement?

(23) An angle is double its supplement. Find its magnitude.

(24) Find the magnitude of an angle if it is one-fourth of its supplement.

(25) Find the number of degrees in an angle which exceeds its complement by 20°.

(26) An angle is 30° less than its supplement. Find the magnitude of the angle.

(27) The complement and supplement of an angle are together equal to 120°. How many degrees are there in this angle?

(28) Draw two angles which have a common side but which are not adjacent.

(29) Which of the following sets of numbers can be used as the lengths of the sides of a triangle? (*a*) 3, 4, 5. (*b*) 15, 20, 39. (*c*) 41, 12, 35.

(30) Classify the triangles whose angles are equal to: (*a*) 15°, 83°, and 82°; (*b*) 35°, 55°, and 90°; (*c*) 10°, 60°, and 110°; (*d*) 60°, 60°, and 60°.

(31) Angle a is the complement of angle b, angle m is the complement of angle n, and angle a = angle m. Prove that angle b = angle n.

(32) Angle a is the supplement of angle b, angle m is the supplement of angle n, and angle b = angle n. Prove that angle a = angle m.

(33) When are two triangles congruent?

(34) Given two parallel line segments AB and CD. Draw line EF intersecting AB at M and CD at N, so that angle CNM = angle NMB. Prove angle AME = angle DNF.

(35) Given line segment MN perpendicular to line segment AB at N. Draw line segments NC and ND so that angle CNB = angle DNA. Prove angle MNC = angle MND.

(36) Given triangle ABC with base AB. Extend AB to E and CB to D. Draw BF parallel to AC, so that angle CAB = angle FBE, and angle ACB = angle CBF. State the reasons for the following statements:

(a) angle CBA = angle EBD.
(b) angle CBE = angle CAB + angle ACB.
(c) angle ABD = angle FBE + angle FBC.
(d) angle DBF is greater than angle EBD.
(e) angle CAB = supplement of angle ABF.
(f) angle DBF is greater than angle CAB.

CHAPTER 8

Pages 110-111

PROBLEMS:

(1) In an isosceles triangle, ABC, side AC is equal to side CB. Through C draw a line segment, CD, meeting the base AB at D and bisecting angle C. Prove that CD bisects the base AB.

(2) Divide a straight line segment, AB, by the two points C and D, into three equal parts, so that $AC = CD = DB$. At A and B, draw AE and FB respectively perpendicular to AB and such that $AE = FB$. Prove that $EC = DF$.

(3) Connect two equal line segments AB and DC by another line segment AC so that angle BAC = angle ACD. Draw the line segments AD and BC. Prove that $AD = BC$.

(4) Two line segments AB and CD intersect at E so that $AE = EB$ and $CE = ED$. Draw AD and CB. Prove $AD = CB$.

(5) In an isosceles triangle ABC, side AB is equal to side AC. Through A draw a line segment, AD, meeting the base BC at D and bisecting angle A. From any point P on line segment AD draw BP and PC. Prove angle PBC = angle PCB.

(6) Draw any angle BAC and its bisector AD. Mark off on the sides of this angle the equal segments AB and AC. Through any point D on the bisector AD draw the line segments BD and DC. Prove that the two triangles formed, namely, triangles DBA and DCA, are congruent.

(7) Draw any angle BAC. From B draw a line segment, BE, meeting AC at E. Through C draw line segment CD meeting AB at D and so that $AE = AD$ and $BD = CE$. Prove that $BE = CD$.

(8) In any isosceles triangle ABC, such that $AB = AC$, let E and D, respectively, denote the midpoints of AB and AC. Draw BD and CE. Prove $BD = CE$.

(9) In any isosceles triangle ABC, such that $AB = AC$, let E and D, respectively, denote the midpoints of AB and AC. Draw BD and CE. Prove that, if BD is perpendicular to AC, and CE is perpendicular to AB, the triangle is equilateral.

(10) In an equilateral and equiangular triangle ABC, let M, N, and P, respectively, denote the midpoints of AB, BC, and CA. Draw MN, NP, and MP. Prove that triangle MNP is equilateral.

Pages 111-112

PROBLEMS:

A carpenter is erecting a roof truss in the shape of an isosceles triangle ACB. The two equal sides AC and CB are called the *rafters*, and the base AB is the *tie beam*.

(11) From the vertex C, the carpenter constructs the *sag tie* to the midpoint of the tie beam. Prove that the sag tie bisects the angle at the vertex C and is perpendicular to the tie beam.

(12) From the vertex C, the carpenter is to erect two ties, CD and CE, to the tie beam, AB, so that AD is equal to EB. Prove that the two ties CD and CE are equal.

(13) Instead of the two ties of Problem 12, two *struts*, DE and FE, from the midpoints, D and F, of the rafters AC and CB to the midpoint, E, of the tie beam AB are to be constructed. Prove that the two struts DE and FE are equal.

(14) A design consists of an equilateral triangle ABC. Points D, E, and F are on AB, BC, and AC, respectively, and $AF = CE = DB$. Prove that triangle DEF is equilateral.

Pages 112-113

PROBLEMS:

(15) The triangles ABC and ADC have a common base AC. Side AB is equal to side DC and AD is equal to BC. Prove that angle ABC is equal to angle ADC.

(16) Prove that two points each equidistant from the ends of a line segment determine the perpendicular bisector of this line segment.

(17) Prove that two triangles are congruent if two sides and the median to one of these sides are equal, respectively, to two sides and the corresponding median of the other.

(18) A mosaic design has two equal circles, whose centers are O and O', intersecting at A and B. Prove that angle OAO' is equal to angle OBO'.

(19) In the design of Problem 18, line segment OO' intersects AB at D. Prove that AD is equal to DB.

(20) Prove that if a side of an isosceles triangle and the line which joins the midpoints of the equal sides are equal, respectively, to a side of another isosceles triangle and the line joining the midpoints of its equal sides, the two isosceles triangles are congruent.

Pages 113-114

PROBLEMS:

(21) In a triangle ABC, with base AC, draw from B a line segment BD meeting AC at D. Extend the base AC through A to any point F. Prove that angle FAB is greater than angle DBC.

(22) In a triangle ABC, with base AC, draw from C a line segment CD meeting AB at D. From A draw another line segment AE meeting CD at E. Prove that angle AEC is greater than angle ABC.

Pages 114-116

PROBLEMS:

(23) Given a triangle NMP with base NP. Extend NP through P to any point B, NP through N to any point A, MN through N to any point E, MN through M to any point F, MP through P to any point G, and MP through M to any point H. Draw line CD through M. Prove that AB is parallel to CD if:

(*a*) angle $HMC = 70°$ and angle $HPA = 70°$

(*b*) angle $NMP = 60°$, angle $FMD = 45°$, and angle $HPA = 70°$

(*c*) angle $CME = 50°$, angle $NMP = 60°$, and angle $HPA = 70°$

(*d*) angle $HPB = 110°$ and angle $HMC = 70°$
(*e*) angle $FNA = 130°$, angle $DMG = 70°$, and angle $HMF = 60°$

(24) Two line segments AB and CD intersect at P so that AP is equal to PB and CP is equal to PD. Prove that CB is parallel to AD.

(25) Given two lines AB and CD cut by a transversal EF so that the pair of alternate interior angles AEF and EFD are equal. Prove that the lines EG and FH, bisecting angles AEF and EFD, are parallel.

(26) Draw two line segments AB and CD intersecting at E and so that $AE = EB$ and $CE = ED$. Draw AC, CB, BD, and AD. Prove that AD is parallel to CB and AC is parallel to BD.

(27) Draw a triangle ABC with AC as its base. Draw another triangle ADC so that DC is equal to AB and AD is equal to BC. Prove that AD is parallel to BC and AB is parallel to DC.

Pages 116-119

PROBLEMS:

(28) In triangle ABC, angle B is twice angle C, and angle A is three times angle B. Find the angles of the triangle.

(29) In an isosceles triangle ABC, with base AC, extend AB through B to any point D. Through B draw line BE parallel to the base AC. Prove that BE bisects angle DBC.

(30) In any triangle ABC, with base AB, extend AC through A to any point D, AC through C to any point E, BC through C to any point F, and AB through B to any point G. Find the value of:

(*a*) $\angle BAC$ if $\angle CBA = 60°$ and $\angle ACB = 50°$
(*b*) $\angle ECB$ if $\angle CAB = 50°$ and $\angle CBA = 60°$
(*c*) $\angle FCE$ if $\angle CBG = 110°$ and $\angle CAB = 50°$
(*d*) $\angle BAD$ if $\angle FCE = 40°$ and $\angle CBA = 60°$
(*e*) $\angle ECB$ if $\angle CBG = 140°$ and $\angle BAD = 120°$

(31) How many degrees are there in each angle of an isosceles right triangle?

(32) A contractor in building a scissors truss constructs an isosceles triangular frame ABC with base BC. From B he erects a beam BE bisecting angle ABC and meeting AC at E. Prove that another beam CD erected from C to AB, bisecting angle ACB and meeting AB at D must equal BE.

(33) Show that in the scissors truss of Problem 32, if the beam BE is erected from B to AC meeting AC at E and making any angle EBA with AB, then the other beam CD erected from C to AB meeting AB at D and making an angle ACD with AC will be equal to BE provided that angles EBA and ACD are equal.

(34) A roof truss has the shape of an isosceles triangle. Show that two beams constructed from the midpoint of the base to the equal sides, making equal angles with the base, are equal.

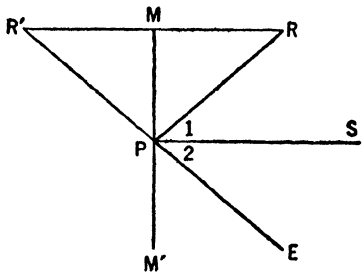
(35) Given any line segment BC. Through the midpoint, D, of BC draw an oblique line AF. From B and C construct two line segments, BM and CN, perpendicular to AF and meeting AF at M and N, respectively. Prove that BM is equal to CN.

Pages 119-121

PROBLEMS:

(36) Line RP (see page 608) represents a ray of light from an object R striking a mirror MM' at P. Line PS is perpendicular to MM'. The ray of light RP is reflected to the eye at E so that angle 1 is equal to angle 2. Line segment RM is drawn perpendicular to MM'. Line segments EP and RM are extended,

through P and M, respectively, until they meet at R'. Show that the distance MR' at which the object appears to be behind the mirror is equal to the actual distance MR at which the object is in front of the mirror.



(37) A roof truss in the shape of an isosceles triangle, ABC, with base BC, has a rafter, EF, parallel to the base, BC, supported by two beams, BD and CD, bisecting the base angles, B and C, and meeting the rafter EF at D. Prove that the rafter EF is equal in length to BE plus FC.

(38) In any isosceles triangle ABC, with base BC, mark off any point D on AB. Extend AC through C to a point E so that CE is equal to BD. Draw DE intersecting BC at F. Prove that DF is equal to FE.

(39) Prove that the altitude to the base of an isosceles triangle divides the figure into two congruent triangles.

(40) In a triangle, the line segments drawn from the base angles to the opposite sides are respectively perpendicular to these sides and are also equal to each other. Prove that the triangle is isosceles.

CHAPTER 9

Pages 123-124

PROBLEMS:

(1) A diemaker is required to make a triangular punch. The triangular section is to be an equilateral triangle $1\frac{1}{2}$ inches on each side. Construct the triangular section.

(2) Construct a triangle equal to a given triangle. In how many ways may this be done?

(3) A carpenter is to construct a gable-roofed shed. The section has the shape of a triangle with a base 24 feet wide and base angles of 30° and 45°, respectively. Let 1 inch equal 12 feet and construct this section. How long are the rafters?

Pages 125-126

PROBLEMS:

(4) Draw two intersecting lines. Construct the lines bisecting the angles formed by these lines.

(5) Locate by construction a point within the acute angle ABC which is equidistant from the sides of the angle and at a given distance s from the vertex of the angle.

(6) A *cam* is a revolving plane used to modify the motion of a machine. The shape and dimensions of a cam depend upon the use for which it is designed. The section of a heart-shaped cam may be constructed as follows: Draw a line segment AB. Divide AB into any number of equal parts, say 9, and denote these points by C, D, E, and so forth, respectively. With A as center and AC, AD, AE, and so forth, as radii, draw a series of concentric circles. With a protractor draw a line segment perpendicular to AB at A. With the compasses divide each 90° angle into four angles by successively bisecting the proper angles. Beginning at C, locate the points where the consecutive circles and consecutive

rays intersect. Draw a smooth curve through these points. The smooth curve is the section of the heart-shaped cam. Choosing AB five inches long, construct the section of a heart-shaped cam.

(7) The distance from the top of a pole, AB, to a point, C, cannot be measured directly; but the distance from the bottom of the pole to point C, that is, AC, is known. Show that the distance BC may be obtained as follows: Extend AC through A to a point D so that AD is equal to AC. The length of BD is the required distance.

Pages 126-128

PROBLEMS:

(8) Draw a line segment $2\frac{7}{16}$ inches long. Bisect this line segment by measuring. Check the accuracy of your measurement by bisecting this line segment with the compasses.

(9) Given an obtuse triangle, construct (*a*) its altitudes; (*b*) the perpendicular bisectors of its sides.

(10) Construct a square the total length of whose sides is 8 inches.

(11) A baseball diamond is a square 90 feet on a side. Construct its plan, using 1 inch equal to 30 feet. What is the distance from home plate to second base?

(12) By the use of ruler and compasses, find a point equidistant from three given points not on the same straight line.

(13) Prove that two points each equidistant from the extremities of a line segment determine the perpendicular bisector of the line.

(14) Given a circle with center O. Let P be any point on its circumference. Construct a line perpendicular to the radius OP at P. This line is called the *tangent* to the circle at P.

Pages 128-131

PROBLEMS:

(15) Construct a quadrilateral such that its opposite sides are parallel and equal. Let the sides be 2 inches and 3 inches long, respectively, and one of its angles equal to 50°. Construct its altitudes.

(16) Through the vertex of an isosceles triangle construct a line parallel to its base.

(17) Construct two parallel lines and the bisectors of two interior angles on the same side of the transversal. Are these bisectors perpendicular to each other? Why?

(18) What fact does a carpenter apply when he lays off parallel lines on a board by moving an arm of a carpenter's square along a straight edge of the board and marking lines along the other arm?

(19) Construct a triangle having given (*a*) three sides; (*b*) two angles and the included side; (*c*) two sides and the included angle.

(20) Construct a triangle whose sides are 2, 3, and 4. Find the length of the median drawn to the smallest side. Find the length of its altitudes.

Pages 131-135

PROBLEMS:

(21) If 3 is added to each member of the inequality $7 > 4$, how will the two inequalities subsist? How will they subsist if 3 is subtracted from each member?

(22) If 5 is added to each member of the inequality $x - 5 > y$, how will the

two inequalities subsist? What is the effect of subtracting 5 from each member of this inequality?

(23) What must be done to the sign of a term if it is transposed from one member of an inequality to the other?

(24) If both members of the inequality $8 > 7$ are multiplied by 4, how will the two inequalities subsist? How will they subsist if both members are divided by 4?

(25) How will both members of the inequality of Problem 24 subsist if the multiplier or divisor is -4?

(26) If each member of the inequality $6 > 5$ is added to the corresponding member of the inequality $3 > 2$, how will the three inequalities subsist?

(27) If each member of the inequality $-5 < -2$ is added to the corresponding member of the inequality $-7 < -4$, how will the three inequalities subsist?

(28) If each member of the inequality $6 > 4$ is divided by -2, how will the two inequalities subsist?

(29) If each member of the inequality $-10 < -5$ is divided by -5, how will the two inequalities subsist?

(30) If each member of the inequality $7 > 6$ is subtracted from the corresponding member of the equation $15 = 15$, how will the resulting inequality subsist?

(31) If each member of the inequality $-5 > -8$ is subtracted from the corresponding member of the equation $9 = 9$, how will the resulting inequality subsist?

(32) Given any quadrilateral $ABCD$ with base AB. Let P be any point on CD. Prove that the perimeter of $ABCD$ is greater than the perimeter of APB.

(33) Given any triangle ABC with base AC. Let D be any point on AB and E be any point on BC. Prove that $AB + BC$ is greater than $AD + DE + EC$.

Pages 135-138

PROBLEMS:

(34) A quadrilateral $ABCD$, with base AB, has AB for its longest side and CD for its shortest. Prove that angle ADC is greater than angle ABC.

(35) Which is the longest side of a right triangle? of an obtuse triangle?

(36) A quadrilateral whose opposite sides are parallel is called a parallelogram. If in a parallelogram $ABCD$ side AB is greater than AD, prove that the diagonal DB does not bisect angle ADC, but that angle ADB is greater than angle CDB.

(37) In a triangle ABC, with base AB, angle A is bisected by a line meeting CB at D. Prove that AC is greater than CD and that AB is greater than DB.

Pages 138-142

PROBLEMS:

(38) Given a triangle ABC with base AB. Mark off a point D within the triangle so that CD is equal to CB. Draw AD. Prove that AB is greater than AD.

(39) In a triangle ABC, with base AB, a line segment, CD, is drawn from the vertex C to the base AB so that AD is equal to DB and angle CDB is obtuse. Prove that CB is greater than AC.

(40) In a triangle ABC, with base AB, mark off a point D within the triangle. Draw the line segments AD, BD, and CD. Prove that the sum of these line segments is greater than one-half the perimeter of the triangle.

(41) In an isosceles triangle ABC, with base AB, mark off point D on AB so that $AD < DB$. Draw CD. Prove that angle BCD is greater than angle ACD.

(42) In a quadrilateral $ABCD$, side AD is equal to side BC and angle DCB is greater than angle ADC. Prove that DB is greater than AC.

(43) In the quadrilateral $ABCD$, of Problem 42, having proved DB greater than AC, prove that angle DAB is greater than angle CBA.

(44) In a triangle ABC, with base AB, side AC is greater than AB. On AC and AB, respectively, mark off points D and E so that CD is equal to BE. Prove that CE is greater than BD.

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EXERCISE A. Solve the following inequalities:

(1) $2x + 5 > 9$
(2) $3x + 4 < 10$
(3) $32 - 5x < 7$
(4) $2 - 7x < 16$
(5) $x^2 - 8x > -16$
(6) $x^2 + 6x > -9$

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EXERCISE B. Solve the following inequalities:

(1) $5x + 4 > 3x - 16$; $5x - 1 < 6x + 4$
(2) $7x + 3 > 4x + 12$; $6x - 3 < 4x + 27$
(3) $5x - 7 > 13$; $32 + 2x > 5x + 5$
(4) $3\frac{1}{2}x + 5 > 2x + 2$; $\frac{11}{15}x + \frac{2}{5} < 2\frac{1}{15} + \frac{2}{3}x$

(45) Prove that the sum of the squares of any two unequal numbers is greater than twice their product

(46) A wholesaler, who sells coffee by the bag, sells two brands of coffee at \$18 and \$16 a bag, respectively. His total weekly sales for coffee is between \$4,590 and \$4,600. He sells 28 bags more of the \$16 kind than of the \$18 coffee. How many bags of each did he sell?

CHAPTER 10

Pages 144-146

EXERCISE A. Find the ratio of:

(1) 2 to 4; 9 to 27
(2) 5 to 6; 3 to 5
(3) 12 to 24; 4 to 20
(4) $1\frac{2}{3}$ to $2\frac{1}{2}$; $2\frac{4}{5}$ to $3\frac{3}{5}$
(5) $2\frac{1}{4}$ to $3\frac{1}{2}$; $1\frac{1}{3}$ to $1\frac{1}{6}$
(6) 6 quarts to 17 gallons
(7) 30 cents to 2 dollars
(8) 50 minutes to $2\frac{1}{2}$ hours
(9) $2a$ to $4a$; $4m$ to $9m$
(10) $6x$ to $18y$; $5x^2m$ to $9xm^2$
(11) $(1 - x)$ to $(1 - x^2)$
(12) $\frac{3x}{4y}$ to $\frac{9x}{8y}$

Pages 144-146

PROBLEMS:

(1) The addition of aluminum to fluid iron is recommended for castings that are subjected to a high pressure. For 100 lbs., the proportion of aluminum recommended is $3\frac{1}{2}$ ounces What is the ratio of aluminum to iron?

(2) Silver coin is considered an alloy of silver and copper in the ratio of 37 to 3. How many pounds of copper would be necessary to alloy with 6,586 lbs. of silver?

(3) W. Hitchcock and J. Stewart invested \$45,650. Hitchcock received \$10,956 as his share of the profits, while Stewart received \$7,304. What amount of the capital invested was contributed by each partner?

(4) Jewellers' gold is considered an alloy of gold and copper in the ratio of

3 to 1; while gold coin is an alloy of gold and copper in the ratio of 11 to 1. How many more pounds of gold would be necessary to make gold coin out of 2,560 lbs. of jewellers' gold?

Pages 146-147

EXERCISE B. Find the unknown term in each of the following proportions:

(1) $8 : 6 = 4 : d$
(2) $25 : b = 50 : 42$
(3) $a : 4 = 6 : 15$
(4) $3 : 8 = c : 13$
(5) $4 : 3 = 16 : x$
(6) $0.5 : 0.2 = x : 4$
(7) $\frac{2}{5} : \frac{3}{2} = 4\frac{1}{3} : x$
(8) $(x - 10) : 3 = (x - 16) : 5$
(9) $\frac{1}{2} : 0.36 = x : 4\frac{1}{5}$
(10) $\frac{1}{2} : x = x : \frac{1}{8}$

Pages 147-149

PROBLEMS:

(5) If 4 men finish a job in 15 days, how long would it take 7 men to finish a similar job?

(6) Twenty pounds of rice cost \$1.25. How much would 37 pounds cost?

(7) On a certain map of the United States, a line $1\frac{9}{16}$ inches long represents 100 miles. The distance from Los Angeles, California, to Santa Fe, New Mexico, is represented on that map by a line $8\frac{3}{4}$ inches long. What is the distance in miles from Los Angeles to Santa Fe?

(8) On the map mentioned in Problem 7, what is the distance in miles represented by a line 1 inch long?

(9) A bankrupt owes \$10,353 28. His creditors agree to accept \$7,764 96 as full payment on this obligation. How much on a dollar will his creditors receive?

(10) James Clifford saved \$72.10 for his vacation. During the first five days of his vacation, he spends \$25.75. If he continues to spend at the same rate, how long a vacation may he have?

(11) A. Morley and M Logar rent a farm. The total rent paid by the two men is distributed in the ratio of 7 to 11 If Morley's share of the rent is smaller than Logar's and he pays \$500, how much rent must Logar pay?

(12) If an automobile travels at the rate of s miles in t hours, how many hours will it take to travel d miles at the same rate?

(13) A number of laborers, working 8 hours a day, finish a certain job in 14 days. How long would it take this party of laborers, working 7 hours a day, to finish the same job?

(14) A spring yields 65 gallons of water in 10 minutes. How many gallons more will it yield in $13\frac{1}{2}$ minutes?

Pages 149-152

EXERCISE C. Prove that if $a : b = c : d$, the following proportions are true:

(1) $b : d = a : c$
(2) $d^2 : b^2 = c^2 : a^2$
(3) $1 : a = d : bc$
(4) $b + d : a + c = b - d : a - c$
(5) $dc : ab = c^2 : a^2$

Pages 149-152

EXERCISE D. Find the mean proportional between:

(1) 3 and 12
(2) 81 and 4
(3) 9 and 49
(4) 0.16 and 169
(5) $\frac{3}{5}$ and $\frac{4}{15}$
(6) 64 and 25

Pages 152-156

PROBLEMS:

(15) It takes an airplane whose speed is 210 miles per hour 2 hours to cover a certain distance. How long will it take another plane whose speed is 275 miles per hour to cover the same distance?

(16) Working on the basis of 8 hours a day and a 6-day week, a factory can produce 30 tanks per week. How long will it take to fill an order for 250 tanks if the men work 4 hours extra each day?

(17) C. Harris buys 28 cases of wine at \$3.60 per case and makes a profit of \$33 60. What profit should Harris make if he buys 100 cases of wine at \$3.48 per case?

(18) It is found by experiment that, the temperature remaining the same, the greater the pressure acting upon a gas, the smaller the volume. If at a certain temperature the volume of a gas is 6 cubic feet at a pressure of 15 lbs. per square inch, what would be the volume of this gas if, while the temperature remains the same, the pressure is raised to 20 lbs. per square inch?

(19) W. Sears bought a store for \$90,000. A year later, M. Judd joined Sears in the business, investing \$45,000 to buy the building. A year after Judd joined Sears, A. Levin joined the firm, investing \$36,900 to expand; and two years later G. Stern joined the company by investing \$34,200. Five years after Sears had bought the store, the total accumulated profit for the five years amounted to \$232,470. How much of the total profit should each receive?

(20) P. Dempsey and M. Martin invested \$28,000 and \$37,600, respectively, as silent partners in a business administered by M. Kellogg and R. Hanavan. Kellogg and Hanavan were to receive 60 per cent and 40 per cent, respectively, of 50 per cent of the profits; the remaining profit was to be divided between Dempsey and Martin in proportion to their investments. If the total profits made were \$32,000, how much of the profits did each receive?

CHAPTER 11

Pages 157-159

PROBLEMS:

(1) Prove that if the diagonals of a parallelogram are perpendicular to each other, the figure is a rhombus.

(2) Prove that if a diagonal bisects an angle of a parallelogram, the figure is a rhombus.

(3) Prove that the diagonals of a rectangle are equal.

(4) Prove that if the diagonals of a parallelogram are equal, the figure is a rectangle.

(5) The diagonals of a quadrilateral bisect each other. Prove that the figure is a parallelogram.

(6) In a parallelogram $ABCD$, let E and F represent the midpoints of BC and AD, respectively. Draw AE and FC. Prove that AE is parallel to FC.

Pages 159-161

PROBLEMS:

(7) To lay railroad tracks, a rod 4′ 8½″ long is used. This rod is fitted over a rail so that it is perpendicular to this rail at two different places. The other rail is then laid at this exact distance. Prove that tracks so laid out must be parallel.

(8) In a triangle, a line segment is drawn joining the midpoints of two sides. Prove that this line segment is parallel to the third side and equal to one-half of it.

(9) Prove that if the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

(10) Two opposite angles of a parallelogram are bisected. Prove that these bisectors are parallel. (*Hint:* Use the fact stated in Problem 9.)

(11) The midpoints of the sides of any quadrilateral are joined in succession. Prove that a parallelogram is formed. (*Hint:* Use the fact stated in Problem 8.)

(12) In a trapezoid, a line segment is drawn joining the midpoints of the non-parallel sides. Prove that this line, called the *median*, is parallel to the bases and equal to one-half their sum. (*Hint:* Use the fact stated in Problem 8.)

Pages 161-162

PROBLEMS:

(13) Prove that the base angles of an isosceles trapezoid are equal.

(14) The base angles of a trapezoid are equal. Prove that the trapezoid is isosceles.

(15) Show that in a parallelogram each pair of adjacent angles is supplementary.

(16) If one angle of a parallelogram is 45°, find the other angles.

(17) Prove Fact 42, that is, that a line parallel to one side of a triangle, bisecting a second side, bisects the third side

(18) Perpendiculars are dropped from the midpoints of two sides of a triangle to the third side. Prove that these perpendiculars are equal. (*Hint:* Use the fact stated in Problem 8.)

Pages 162-165

PROBLEMS:

(19) In a trapezoid $ABCD$, line segment EF is drawn parallel to one of its bases, meeting the non-parallel sides at E and F. Prove that EF divides its non-parallel sides proportionally

(20) In a triangle ABC, with base AB, a line segment, DE, intersecting AC and CB at D and E, respectively, is drawn parallel to AB.

(*a*) If $AD = 3$, $DC = 9$, $CE = 6$, find EB
(*b*) If $AC = 12$, $AD = 3$, $CB = 16$, find EB.
(*c*) If $AC = m$, $DC = n$, $EB = p$, find CE.
(*d*) If $AC = a$, $DC = b$, $CB = c$, find CE.

Pages 165-167

PROBLEMS:

(21) Draw a line segment and then determine $\frac{2}{3}$ of the line

(22) In a scalene triangle ABC, divide the base AB into five equal parts.

(23) Construct a fourth proportional to three lines $\frac{2}{3}$ in., 1 in., and $1\frac{1}{3}$ in. long.

(24) The perimeter of a rhombus is given by the length of a line AB. An angle of the rhombus is 45°. Construct the rhombus.

Pages 167-168

PROBLEMS:

(25) In an isosceles triangle ABC, with base AB, the equal angles, A and B, are respectively bisected by line segments AD and BE meeting AC at E and BC at D. Prove that DE is parallel to the base of the triangle.

(26) In a square $ABCD$, a line is drawn bisecting a diagonal and a side. Prove that this line is parallel to the base of the square.

(27) Prove that if the midpoints of the sides of a parallelogram are connected in order, the figure formed is a parallelogram.

Pages 168-171

PROBLEMS:

(28) Prove that two triangles are similar if their corresponding sides are parallel each to each

(29) Prove that two triangles are similar if their corresponding sides are perpendicular each to each.

(30) Construct a right triangle ABC, with hypotenuse AB, so that AB is its base Draw a line segment AE perpendicular to AB at A, and extend BC through C meeting AE at E. Draw another line BD perpendicular to AB at B and extend AC through C, meeting BD at D Prove $(AB)^2 = AE \times DB$.

(31) A method to locate, approximately, the distance between two inaccessible points by means of a drawing board may be used as follows: Assume that the inaccessible points are A and B. Place the drawing board horizontally at B and mark off a point b on the drawing board directly above B. On the drawing board, draw a line bA' of indefinite length toward A, and another line bC' toward some accessible point C. Measure the distance BC and move the drawing board to point C Place the board horizontally and so that a point c on bC' is directly above C and the line bC' is in line with BC. Draw a line segment ca from c toward A meeting bA' at a. Prove that $AB = \dfrac{BC \times ab}{bc}$.

(32) Prove that a line parallel to the base of a triangle and intersecting the other two sides cuts off a triangle similar to the given triangle.

(33) The distance across a river may be found as follows: Locate two points A and B directly opposite each other on opposite banks of the river. Lay out a line segment BC, of any convenient length, perpendicular to AB. At C construct a line segment CD, of any convenient length, perpendicular to BC. From D, sighting on A, locate point E on BC in line with D and A. Measure BE, EC, and CD. Prove that $AB = \dfrac{BE \times DC}{EC}$.

(34) In a triangle ABC, with base AB, perpendiculars AD and BE are drawn to BC and AC, respectively, intersecting at F. Prove that triangles BFD and AFE are similar

(35) The height of a building or any other object may be approximately found as follows: Assume that the height of an object is represented by a vertical line AB. At any convenient distance BM from the object and measured along the ground, place a mirror M horizontally. Stand at some distance MC from the mirror, away from the object along line segment BM extended, so that the image of the top of the object is visible in the mirror Let your height be represented by a vertical line CD. It is shown in physics that a ray of light from the top of the object is reflected by the mirror to your eyes in such a way that angles BMA and CMD are equal. Prove that $AB = \dfrac{BM \times CD}{MC}$.

(36) In a trapezoid $ABCD$, with lower base AB, draw the diagonals AC and DB meeting at E. Prove that triangles AEB and DEC are similar.

Pages 172-175

PROBLEMS:

(37) **Prove that the corresponding medians of similar triangles have the same ratio as any two corresponding sides.**

(38) In a trapezoid $ABCD$, with bases AB and CD, the non-parallel sides are produced until they meet at E. Prove that the two triangles formed are similar.

(39) Construct a right triangle ABC, with hypotenuse AB, so that AB is also its base. Draw CD perpendicular to AB. Prove that the product of the sides equals the product of the hypotenuse and the altitude upon the hypotenuse.

(40) In Problem 38, given $AB = 7$, $CD = 5$, and $BC = 3$; find EC.

(41) Prove that if two triangles are similar, their bases and altitudes are in proportion.

CHAPTER 12

Pages 177-179

PROBLEMS:

(1) The following table gives the estimated population of various countries. Construct a bar chart from these data

POPULATION OF VARIOUS COUNTRIES

Source: 1941 *World Almanac*

Countries	*Population*	*Countries*	*Population*
Australia	6,997,326	Hungary	13,507,993
Austria	7,009,014	Italy	44,557,000
Canada	11,012,734	Japan	72,222,700
China	422,707,868	United Kingdom	46,213,169
France	42,014,594	United States	131,409,881
Germany	79,375,281	U. S. S. R.	192,695,710

(2) The foreign trade of the United States with its best customers during 1939 is shown in the following table Draw: (*a*) a bar chart of exports; (*b*) a bar chart of imports.

UNITED STATES FOREIGN TRADE

Source: 1941 *World Almanac*

Countries	*Exports* ($1,000)	*Imports* ($1,000)	*Countries*	*Exports* ($1,000)	*Imports* ($1,000)
United Kingdom	505,226	149,669	Netherlands	96,809	28,930
Canada	493,450	340,066	Sweden	96,661	42,218
Japan	231,405	161,195	Mexico	83,177	56,319
France	181,825	62,375	Cuba	81,644	104,930
Philippines	100,017	91,927	Brazil	80,440	107,243

(3) The total value, expressed in thousands of dollars, for the countries given in Problem 2 is 1,950,654 for exports. Calculate the per cent of total exports for each country. Construct a table of these data and draw a pie diagram

(4) The total value, expressed in thousands of dollars, for the countries given in Problem 2 is 1,144,872 for imports. Calculate the per cent of total imports for each country. Construct a table of these data and draw a pie diagram.

Page 179

EXERCISE A. In the following examples, state which is the independent variable, which is the dependent variable, and which variable is a function of the other:

(1) Cost of sending a letter by mail and its weight.

(2) The radius of a circle and its area.
(3) The pressure of water and the depth below sea level.
(4) The price of a manufactured article and the cost of production.
(5) The age of the insured and the life insurance premium.
(6) The distance required to stop an automobile and its speed.
(7) Cost of production and cost of labor.
(8) Weight of a person and his height.
(9) Cost of raw material and cost of production.
(10) A man's health and the food he eats.

Pages 179-181

PROBLEMS:

(5) Plot the points: (3, 5), (−2, 4), (2, 3), (−1, −6), (−7, 9), (5, −3), (3, 0), (−4, −3), (0, −4), (7, −2), and (−5, 0).

(6) Draw the triangle which has the following points as vertices: (3, 4), (−5, 2), and (−2, −1).

(7) Draw the quadrilateral which has the following points as vertices: (4, 0), (−5, 6), (−3, −4), and (2, −5).

(8) What is the characteristic of the coordinates of all points: (*a*) On the X-axis? (*b*) On the Y-axis? (*c*) On a line through the origin making an angle of 45° with the positive side of the X-axis? (*d*) On a line through the origin making an angle of 135° with the positive side of the X-axis? (*e*) On a line parallel to the Y-axis two units to the left of it? (*f*) On a line parallel to the X-axis one unit above it?

(9) (*a*) If the abscissa of a point is 0, where does it lie? (*b*) If the ordinate of a point is 0, where does it lie? (*c*) If the abscissa and the ordinate of a point are equal, where does it lie? (*d*) If the abscissa and the ordinate of a point are numerically equal but opposite in sign, where does it lie?

Pages 181-189

PROBLEMS:

(10) A manufacturer finds that a machine can be made to turn out from 10 to 100 articles a day The cost of manufacturing, selling price, and total profit are shown in the following table:

Number of Articles	*Total Cost*	*Selling Price at $1.10*	*Total Profit*
20	$31.00	$ 22 00	−$ 9.00
30	31 50	33.00	1.50
40	33.00	44.00	11.00
50	35 50	55.00	19.50
60	40.50	66.00	25.50
70	49 00	77.00	28.00
80	60 00	88.00	28.00
90	74.50	99 00	24.50
100	91.00	110.00	19.00

Draw on the same graph paper a sales graph, a cost graph, and a profit graph.
(*a*) Find the selling price of 46 articles.
(*b*) Find the cost of 61 articles.
(*c*) How many articles can be manufactured at a cost of $48?
(*d*) What is the selling price, as obtained from the graph, of 52 articles?
(*e*) How many articles may be sold for $60?

(*f*) For how many articles are the cost and the selling price the same?

(*g*) What is the most profitable output for the machine?

(11) An empty tank is being filled by a faucet supplying 3 gallons of water per minute. After this faucet has been running 4 minutes, a second faucet is turned on which independently supplies 2 gallons of water per minute. When the two faucets have been running together for 7 minutes, an outlet is opened, but both faucets continue to run. The tank is empty at the end of 25 minutes from the start. Draw a graph representing the amount of water in the tank at any instant. (*a*) How many gallons were there in the tank 10 minutes from the start? (*b*) 20 minutes from the start? (*c*) At how many minutes from the start did the tank contain 33 gallons? (*d*) About how many gallons of water per minute flow out of the outlet?

(12) A man riding a motorcycle travels from a place A to another place B, 150 miles away, in 3 hours. When 90 miles away from A, he passes an automobile going in the same direction. The man riding the motorcycle remains at B $\frac{1}{2}$ hour and then resumes his trip towards C, 150 miles away from B. When he is $2\frac{1}{2}$ hours away from B, the automobile passes him The cyclist arrives at C 5 hours after leaving B. Find: (*a*) the time at which the automobile left A; (*b*) the time at which the automobile arrived at C.

(13) The graph of the equation $y = x^2$ shows the relationship between any number, x, and its square, y. Construct a graph for values of x from 0 to 10

(*a*) From the graph find the approximate value of the square of 3.7. Of 7.2.

(*b*) From the graph find the approximate value of the square root of 32. Of 70.

(14) The reciprocal of a given number is 1 divided by the number. Thus, the reciprocal of 4 is $1 \div 4 = .25$. Draw a graph showing the reciprocals of numbers from 1 to 100.

(*a*) From the graph find the approximate value of the reciprocal of 18. Of 87.

(*b*) From the graph find the approximate value of the number whose reciprocal is 0.03. Of the number whose reciprocal is 0.019.

(15) The volume of a sphere is obtained from the formula $V = \frac{4}{3}\pi R^3$, where $\pi = 3.14159$ or approximately $\frac{22}{7}$. Draw a graph showing the volumes of spheres whose radius R varies from 0 to 10.

(*a*) From the graph find the approximate value of the volume of a sphere of radius 4.5. Of radius 9.25.

(*b*) From the graph find the approximate value of the radius of a sphere whose volume is 27.6. Whose volume is 199.

CHAPTER 13

Pages 191-194

PROBLEMS.

(1) Draw the graph, discuss fully, and prove that the loci of the following equations are straight lines:

(*a*) $y = 3x - 5$ (*c*) $y = \frac{2}{3}x + 2$

(*b*) $y = 2x + 7$ (*d*) $y = -\frac{4}{5}x + 6$

(2) An airplane 50 miles east of San Diego, Cal., starts to travel east at a uniform rate of 250 miles per hour. Express its distance, y, east of San Diego as a function of the time (x hours). Draw the graph of this function and prove that it is a straight line. At what time is the plane $487\frac{1}{2}$ miles from San Diego? What is the significance of the graph that extends below the X-axis?

(3) Does the point (2, 1) lie on the graph of the equation $y = 4x - 7$?

(4) Does the graph of the equation $y = 3x - 5$ pass through the point $(1, -2)$? Through the point $(-2, 1)$?

(5) The amount of \$100 at 4 per cent simple interest after t years is $S = 100 + 4t$. Plot the graph of this function and prove that it is a straight line. What is the amount when $t = 5$? In how many years will the amount be \$156?

Pages 194-195

PROBLEMS.

(6) The volume, V, of a sphere of radius R is approximately given by the formula $V = \frac{88}{21} R^3$. A spherical balloon is being inflated. Find the average rate of change of the volume for the interval between $R = 1$ and $R = 3$.

(7) The repulsion R (in dynes) between a pair of electrical charges varies with their distance s (in centimeters) according to the law expressed by the formula $R = \frac{20}{s^2}$. Find the average rate of change of the repulsion for the interval between $s = 2$ and $s = 6$. What is the meaning of the negative answer?

(8) The distance s (in feet) through which a body falls in t seconds is given approximately by the formula $s = 16t^2$. Find approximately the average speed of a bomb dropped from an airplane for the interval between $t = 10$ and $t = 11$.

(9) The amount of water g (in gallons) in a tank h hours after an inlet is opened is given by the equation $g = 50h + 75$. Find the average rate of change of the amount of water in the interval between $h = 1$ and $h = 2\frac{1}{2}$.

(10) The distance l (in feet) moved t seconds after the brakes have been applied on a loaded freight car, moving on a straight and level track under certain conditions, is approximately given by the formula $l = 44t - 2t^2$. Find the average speed in the interval between $t = 8$ and $t = 10$.

Pages 195-197

EXERCISE A. Find the slopes of the lines joining the following pairs of points:

(1) $(1, 1)$, $(4, 3)$
(2) $(-2, 1)$, $(1, 5)$
(3) $(1, 6)$, $(7, 1)$
(4) $(-3, -2)$, $(5, 4)$
(5) $(-4, 5)$, $(3, -1)$
(6) $(-2, -1)$, $(4, -7)$

Pages 195-197

PROBLEMS.

(11) Prove that the points $(0, -3)$, $(2, 1)$, $(3, 3)$, and $(5, 7)$ lie in a straight line.

(12) Prove that the points $(-2, 8)$, $(0, 7)$, $(4, 5)$, and $(6, 4)$ lie in a straight line.

(13) An equilateral triangle has two vertices on the X-axis and its third vertex in the first quadrant. Find the slopes of its sides.

(14) The base of a square is on the X-axis. Find the slopes of the diagonals of this square.

(15) An automobile starts from a point 30 miles away from New York City on the road to Chicago. It arrives at Chicago, 801 miles away from the starting point, in 26 hours and 42 minutes of traveling time. Express the distance, s, that the automobile is away from New York City as a function of the time t. Plot the graph of this function. From the graph find the slope of the straight line representing this function. Check your answer by finding from the equation the coordinates of any two points and using these values to compute the slope. Express the slope of this line in terms of distance and time.

(16) A cistern that contains 500 gallons of water is being emptied at the rate of 25 gallons per hour. Express the number of gallons in the cistern as a

function of the time. Draw the graph and find the slope of the straight line representing this function. Check your answer by finding two pairs of values that satisfy the equation and computing the slope. Express the slope in terms of gallons of water and time.

Pages 197-199

EXERCISE B. Find the equations and plot the graphs of the straight lines given by each of the following sets of conditions:

(1) The slope = 2 and the Y-intercept = 1.
(2) The slope = $\frac{1}{2}$ and the Y-intercept = -3.
(3) The slope = $-\frac{2}{3}$ and the Y-intercept = 4.
(4) The slope = -1 and passing through the origin.
(5) The slope = $\frac{1}{3}$ and passing through the point $(-3, 5)$.
(6) The slope = $-\frac{3}{4}$ and passing through the point $(-8, 8)$.
(7) The slope = $-\frac{5}{7}$ and passing through the point $(14, 2)$.
(8) Passing through the points $(2, 1)$ and $(-3, -5)$.
(9) Passing through the points $(-4, 5)$ and $(6, -1)$.
(10) Passing through the points $(1, 5)$ and $(-3, 5)$.
(11) Passing through the points $(2, -3)$ and $(2, 7)$.
(12) Parallel to the X-axis and the Y-intercept = 3.
(13) Parallel to the Y-axis and the X-intercept = -5.
(14) Coinciding with the X-axis.
(15) Coinciding with the Y-axis.

Pages 197-199

PROBLEMS:

(17) Find the equations of two lines parallel to the X-axis and at a distance from it numerically equal to 3.

(18) Find the equations of two lines parallel to the Y-axis and at a distance from it numerically equal to 2.

(19) In the general equation $y = mx + b$, what is the effect of changing b while m is held fixed? What is the effect of changing m while b remains constant?

(20) Find the slope and the Y-intercept and plot the graph of: (*a*) $y = 2x - 5$; (*b*) $y = -\frac{3}{4}x + 7$; (*c*) $y = \frac{3}{5}x - 2$.

Pages 199-201

EXERCISE C. Find the slope and the Y-intercept and plot the graph of each of the following equations:

(1) $3x - y - 6 = 0$
(2) $2x + 5y - 6 = 0$
(3) $x - 2y = 0$
(4) $2y - 3x + 1 = 0$
(5) $y - x = 2$
(6) $6 - x = 0$
(7) $y - 4 = 0$
(8) $2x - 3y - 8 = 0$

Pages 199-201

PROBLEMS:

(21) A straight line intercepts the X-axis at the point $(a, 0)$ and the Y-axis at the point $(0, b)$. Find the equation of the line.

(22) Find the slope of a line if its X- and Y-intercepts are, respectively, $(a, 0)$ and $(0, b)$.

(23) A straight line passes through the points (x_1, y_1) and (x_2, y_2). Find the equation of the line.

(24) A straight line has a slope equal to m and passes through the point (x_1, y_1). Find the equation of this line.

(25) Find the equation that must be satisfied by the coefficients in the equation $Ax + By + C = 0$ if the line:

(*a*) Is parallel to the X-axis.
(*b*) Is perpendicular to the X-axis.
(*c*) Passes through the origin.
(*d*) Has its Y-intercept equal to 3.
(*e*) Has equal intercepts.
(*f*) Has a slope equal to 1.

Pages 202-204

EXERCISE D. From the table on page 204, find the angles whose tangents are:

(1) 0.194
(2) 1.376
(3) 0.81
(4) 14.301
(5) $2\frac{1}{10}$
(6) $\frac{7}{10}$
(7) $1\frac{3}{8}$
(8) $\frac{9}{10}$

Pages 202-204

EXERCISE E. From the table on page 204, find the tangent of each of the following angles:

(1) 87°
(2) 45°
(3) 57°
(4) 30°
(5) 60°
(6) 13°

Pages 204-206

PROBLEMS:

(26) Find the angle of inclination of a line whose slope is (*a*) $\frac{7}{100}$; (*b*) $\frac{3}{8}$; (*c*) $\frac{27}{40}$; (*d*) $5\frac{29}{100}$.

(27) Find the equations of the following lines:

(*a*) Passing through (1, 3) and making an angle of 50° with the X-axis.
(*b*) Passing through (1, 3) and making an angle of 50° with the Y-axis.
(*c*) Passing through the origin and making an angle of 45° with the X-axis.
(*d*) Passing through $(-5, 2)$ and having an angle of inclination equal to 135°.
(*e*) Having an inclination of 40° and X-intercept = 5.
(*f*) Passing through (1, 2), the inclination of which is the angle whose tangent is 0.781.

(28) The base of an equilateral triangle joins the points (1, 2) and (3, 0). Find the equations of its sides.

(29) An observer standing on the ground 185 ft. away from the foot of a tower finds that the angle of elevation of the top of the tower is 39°. Find the height of the tower.

(30) The angle of elevation of the sun is 25° and the shadow cast by a building is 180.25 ft. long. How high is the building?

(31) The span of a roof truss, that is, the distance from center to center of the supporting walls, is 50 ft. The height of the truss or rise is 14.425 ft. Find the angle of inclination of the rafters.

(32) An observer on a balloon 2,000 ft. above sea level and directly above a gun emplacement of a shore battery finds that the angle of depression of a ship at sea is 8°. Find the distance from the gun emplacement to the ship.

(33) The tread (horizontal part) in a step of a certain steel stairway is 10 inches and the riser (vertical part) is 8.39 inches. Find the angle of inclination of the stairway.

(34) The angle of inclination of a stairway is 43° and the tread is 9 inches. Find the height of the riser.

(35) A captive balloon is 3,000 ft. high and the angle of elevation to the balloon is 70°. How far away is the balloon?

(36) A submarine cable runs 50° north of east of a certain shore station. A repair ship reports the cable broken at a point 15 miles north of the station. How far east of the station is the break in the cable?

(37) A straight road has an average grade of $3\frac{1}{2}$ per cent; that is, it rises 3.5 ft. for every 100 ft. measured along a horizontal line. Find the angle of inclination of the road.

(38) A surveyor wishes to determine the width of a river. Choosing any point, say A, directly opposite his position, B, on the bank of the river, he measures a convenient distance BC, for example, 100 ft. long, at right angles to line segment AB. Setting up his transit at C, he then measures angle BCA. Find the width, AB, of the river if angle BCA is equal to 58°.

CHAPTER 14

Pages 209-211

EXERCISE A. In each of the following right triangles, ABC, the three sides are given. Find the value of the six trigonometric functions of angle A.

(1) $a = 9,\ b = 12,\ c = 15$
(2) $a = 4,\ b = 3,\ c = 5$
(3) $a = 15,\ b = 8,\ c = 17$
(4) $a = 12,\ b = 16,\ c = 20$
(5) $a = 9,\ b = 40,\ c = 41$
(6) $a = 12,\ b = 5,\ c = 13$

Pages 209-211

EXERCISE B. With a protractor construct a right triangle containing the given acute angle. Measure the sides and find the six trigonometric functions.

(1) 50°
(2) 45°
(3) 30°
(4) 60°
(5) 40°
(6) 70°

Pages 209-211

EXERCISE C. Construct a right triangle containing the given acute angle. Measure the remaining side and find the other five functions.

(1) $\sin \theta = \frac{3}{5}$
(2) $\cos \alpha = \frac{12}{13}$
(3) $\tan \beta = \frac{9}{40}$
(4) $\cot \phi = 2$
(5) $\sec \theta = 5$
(6) $\csc \alpha = \frac{13}{6}$

Pages 209-211

PROBLEMS:

(1) In right triangle ABC, $a = \sqrt{3}$, $b = 1$, and $c = 2$. Find the values of the six trigonometric functions of (a) angle A; (b) of angle B. (c) What is the relationship of the functions of angle A to those of angle B?

(2) Given a right triangle, ABC, find the missing sides if (a) $\sin A = \frac{3}{5}$ and $c = 10$; (b) $\cos A = \dfrac{1}{\sqrt{2}}$ and $b = 5$; (c) $\tan A = \dfrac{3.5}{5}$ and $a = 7$.

Pages 209-213

EXERCISE D. In the following examples, the given point is on the rotating ray. Locate the point on the plane, measure its distance from the origin, and find the value of the trigonometric functions for the positive angle formed by the rotating ray and the positive side of the X-axis. Do not reduce the answers to decimal form.

(1) (3, 4)
(2) (−3, 4)
(3) (5, −5)
(4) (−2, −3)
(5) (−5, 12)
(6) (−5, 3)
(7) (−6, −6)
(8) (1, 1)

Pages 209-213

EXERCISE E. For which quadrants is:

(1) $\tan \theta$ positive?
(2) $\sin \theta$ negative?
(3) $\sec \theta$ negative?
(4) $\csc \theta$ positive?
(5) $\cos \theta$ negative?
(6) $\cot \theta$ positive?
(7) $\sin \theta$ positive?
(8) $\tan \theta$ negative?

EXERCISE F. In what quadrant is the rotating ray if:

(1) $\sin \phi$ is positive and $\cot \phi$ is negative?
(2) $\cos \phi$ is positive and $\csc \phi$ is negative?
(3) $\tan \phi$ is positive and $\sec \phi$ is negative?
(4) $\sin \phi$ is negative and $\cos \phi$ is positive?
(5) $\sin \phi$ and $\cos \phi$ are both positive?

EXERCISE G. Using the table on page 215, find the value of the sine, cosine, and tangent of the following angles:

(1) $25°$ (2) $60°$ (3) $135°$ (4) $-240°$ (5) $150°$ (6) $-315°$ (7) $115°$ (8) $-220°$

PROBLEMS:

(3) What other function always has the same sign as (*a*) $\sin \theta$? (*b*) $\cos \theta$? (*c*) $\tan \theta$?

(4) If the center of a watch is made to coincide with the origin while the point marked XII lies on the positive side of the X-axis, (*a*) what negative angle does the minute hand describe in 40 minutes? (*b*) What positive angle is then formed? (*c*) Give the sine, cosine, and tangent of these angles.

(5) What is the relationship between:

(*a*) Sin θ and sin $(180° - \theta)$?
(*b*) Cos θ and cos $(180° - \theta)$?
(*c*) Tan θ and tan $(180° - \theta)$?
(*d*) Cot θ and cot $(180° - \theta)$?
(*e*) Sec θ and sec $(180° - \theta)$?
(*f*) Csc θ and csc $(180° - \theta)$?

Pages 213-216

EXERCISE H. Express each of the following functions as another function of its complementary angle:

(1) $\sin 15°$ (2) $\cos 60°$ (3) $\tan 20°$ (4) $\cot 67° 30'$ (5) $\sec (90° - 18°)$ (6) $\csc 12° 27'$ (7) $\cot (90° - 57°)$ (8) $\cos (90° - a)$

Pages 216-217

EXERCISE I. Find the value of sine, cosine, and tangent of the following angles:

(1) $195°$ (2) $290°$ (3) $240°$ (4) $300°$ (5) $210°$ (6) $-70°$ (7) $330°$ (8) $225°$ (9) $315°$ (10) $130°$

Pages 217-218

EXERCISE J. Find the value of the following trigonometric functions:

(1) $\sin 20° 19'$
(2) $\cos 70° 34'$
(3) $\tan 190° 11'$
(4) $\cot 0° 47'$
(5) $\cos 316° 33'$
(6) $\tan 12° 30' 20''$
(7) $\cot 209° 27' 6''$
(8) $\sin 142° 00' 13''$
(9) $\cos 52° 17' 30''$
(10) $\sin 275° 05' 46''$

Pages 217-218

EXERCISE K. Find the smallest positive value of θ if:

(1) $\sin \theta = 0.1867$
(2) $\cot \theta = 0.3358$
(3) $\cos \theta = 0.9912$
(4) $\sin \theta = 0.6230$
(5) $\tan \theta = 1.3085$
(6) $\cos \theta = -0.5013$
(7) $\cot \theta = -4.1565$
(8) $\sin \theta = -0.8699$
(9) $\tan \theta = 0.5157$
(10) $\cot \theta = 0.7010$

Pages 217-218

EXERCISE L. In what quadrant is the rotating ray if:

(1) $\tan \theta > 0$ and $\sin \theta < 0$
(2) $\cos \theta > 0$ and $\sin \theta < 0$
(3) $\sin \theta > 0$ and $\sec \theta < 0$
(4) $\csc \theta < 0$ and $\cot \theta > 0$
(5) $\sec \theta > 0$ and $\tan \theta < 0$

Pages 217-218

EXERCISE M. Prove the following identities:

(1) $\sin (180° + \theta) = -\sin \theta$
(2) $\sin (180° - \theta) = \sin \theta$
(3) $\cos (360° - \theta) = \cos \theta$
(4) $\cot (270° - \theta) = \tan \theta$
(5) $\cos (90° + \theta) = -\sin \theta$
(6) $\tan (270° + \theta) = -\cot \theta$
(7) $\cos (180° - \theta) = -\cos \theta$
(8) $\cot (180° + \theta) = \cot \theta$
(9) $\sec (90° - \theta) = \csc \theta$
(10) $a \sin (90° + \theta) + b \sin (90° - \theta) = (a + b) \cos \theta$

Pages 217-218

EXERCISE N. Assuming that the following angles are all acute, find their values if:

(1) $\sin 32° = \cos \theta$
(2) $\cos 2\theta = \sin 3\theta$
(3) $\tan 5\theta = \cot 4\theta$
(4) $\sec \theta = \csc 43°$
(5) $\sin 7\theta = \cos 3\theta$
(6) $\tan 65° = \cot 5\theta$
(7) $\csc 12\theta = \sec 42°$
(8) $\cot 13\theta = \tan 5\theta$

Pages 217-218

PROBLEMS:

(6) What is the relationship between the trigonometric functions of: (*a*) θ and $90° - \theta$? (*b*) θ and $-\theta$? (*c*) θ and $180° + \theta$? (*d*) θ and $360° - \theta$?

(7) What are the largest and smallest values of: (*a*) $\sin \theta$? (*b*) $\cos \theta$?

(8) What trigonometric functions have their products equal to unity when taken in pairs?

Pages 218-220

PROBLEMS: Applications to Surveying

(9) In a railroad survey, an engineer at a point A finds that the angle of elevation to another point, B, 804 feet away from A, is 18°. How much higher is B than A?

(10) The rafters of a gable roof truss are 25 feet long and are set with a

pitch (angle of elevation) of 32°. Find the height of the truss and the width of the house.

(11) A ladder 28 feet long just reaches a window ledge. If the ladder makes an angle of 76° with the floor, how high is the window ledge?

(12) From a point P on one side of a river and directly opposite a point M on the other side, a surveyor laid out a line PN, 75 feet long and perpendicular to PM. At N, by means of a transit, he then found angle PNM to be 63° 10′. Find the width of the river.

(13) To measure the height of a building, two convenient points A and B in line with a point on the base of the building are chosen so that AB can be easily measured. The angles of elevation at A and B are found to be 35° 20′ and 43° 10′, respectively. If the distance AB measures 100 ft., what is the height of the building?

(14) To find the distance from a gun (G) to a target (T), an observer at P found, by a range finder, PG equal to 1.12 miles, PT equal to 1.83 miles, and angle GPT equal to 71° 40′. Find the distance GT from the gun to the target.

Pages 220-222

PROBLEMS: Applications to Physics.

(15) A force of 300 lbs. acts in a direction inclined at an angle of 32° with the horizontal. Find the horizontal and vertical components of the force.

(16) A force of 500 lbs. has a horizontal component of 300 lbs. Find the direction at which the force is acting and its vertical component.

(17) The horizontal and vertical components of a force are 50.75 lbs. and 132.12 lbs., respectively. Find the direction at which the force is acting and its magnitude.

(18) A balloon rising vertically at a uniform velocity of 653 ft. per minute encounters a wind blowing horizontally at the rate of 21 miles per hour. Find the angle at which the balloon will rise and its velocity.

(19) A diving airplane descends at a velocity of 400 miles per hour while making an angle of 70° with the horizontal. Find its vertical and horizontal speeds.

(20) The rafter of a roof truss is inclined at an angle of 40° with the horizontal beam. The supporting wall on which they both rest exerts a force of 200 lbs. Find the forces acting along the rafter and the horizontal beam.

(21) A sign weighing 150 lbs. is supported by two wires, each inclined 34° 20′. Find the pull or tension in each wire.

Pages 222-227

PROBLEMS: Applications to Navigation

(22) The latitude and longitude of San Francisco, Cal., are, respectively, 37° 47′ 28″ N and 122° 25′ 43″ W, while those of Honolulu are 21° 18′ 17″ N and 157° 52′ 17″ W. Find: (*a*) the direction a plane must follow to fly from San Francisco to Honolulu; (*b*) the flying distance.

(23) The latitude and longitude of Rio de Janeiro, Brazil, are, respectively, 22° 54′ 24″ S and 43° 10′ 21″ W, while those of La Plata, Argentina, are 34° 54′ 30″ S and 57° 55′ 56″ W. Find: (*a*) the direction a ship must follow in sailing from Rio de Janeiro to La Plata; (*b*) the distance from Rio de Janeiro to La Plata.

(24) The latitude and longitude of New York City are, respectively, 40° 48′ 35″ N and 73° 57′ 30″ W, while those of Liverpool, England, are 53° 24′ 05″ N and 3° 4′ 20″ W. Find: (*a*) the direction a ship must follow in sailing from New York to Liverpool; (*b*) the distance from New York to Liverpool.

Pages 227-232

PROBLEMS: Applications to Astronomy—Sine and Cosine Laws

(25) The distance of Venus (V) and the Earth (E) from the Sun (S) are 67.2 and 92.9 million miles, respectively. The angle SEV is called the angle of *elongation* of the planet. On a certain date, the angle of elongation of Venus is found to be 41° 10′. Find the distance from the Earth to Venus on that date.

(26) On a certain date the distance from the Earth (E) to Mars (M) is known to be 145.8 million miles. The distance from the Earth to the Sun (S) is 92.9 million miles and the angle SEM is found to be 68° 40′. Find the distance from Mars to the Sun.

(27) A surveyor, running a line 20° north of east, finds that he must change his course because of a small pond. From a point P he alters his direction directly north and measures a distance of 500 feet to another point M. At this point M he follows a line 30° south of east for a distance of 800 feet to point N on the original course. Find the distance PN.

(28) An engineer, in order to ascertain the width of a lake, sets up the transit at a point P, 500 yards from one end of the lake and 750 yards from the other end. He observes that the angle subtended by the lake is 105° 20′. Find the width of the lake.

(29) An airplane pilot on a reconnaissance flight, traveling at a constant height, passes directly over a town. At this point the angle of depression to a fort outside of the town is 25°. After traveling for 3 minutes at a constant speed of 200 miles an hour in the direction of the fort, he finds that its angle of depression is now 70°. Find the distance from the town to the fort.

(30) An enemy ship is observed 72° 30′ east of north of a shore station. At the same time, another shore station 3 miles directly east of the first locates the enemy ship 27° 40′ west of north. Find the distance from the shore stations to the ship.

(31) Two forces of 25 lbs. and 40 lbs., respectively, have an included angle of 60°. Find the magnitude of the resultant force. (*Hint:* The magnitude and direction of the resultant force are given by the length and direction of the diagonal of the parallelogram two of whose sides are the given forces.)

(32) Chicago, Ill., Detroit, Mich., and Cleveland, Ohio, may be considered as the vertices of a triangle. Let C, D, and L, respectively, denote the positions of Chicago, Detroit, and Cleveland. Since Detroit is further north than Chicago or Cleveland, and the latter is further east than Chicago, denote the base by CL and the vertex of the triangle by D. If the distance $CD = 238$ miles approx., $DL = 123$ miles approx., and $CL = 316$ miles approx., find the direction of Detroit from an imaginary line connecting Chicago to Cleveland.

(33) Two forces of 30 lbs. and 50 lbs. have a resultant of 70 lbs. Find their included angle.

(34) In a railroad survey, an engineer is to determine the height (TB) of a mountain. Choosing a point, S, directly south of B, he finds that the angle of elevation, BST, to the top of the mountain is 75° 40′. While still at S, he lays out a line SP directly east, that is, perpendicular to SB, and 400 ft. long. At P he measures angle SPB and finds it equal to 43° 10′. Find the height of the mountain.

(35) An engineer at a point P, on a hillside, finds the angle of elevation to the top (T) of a tower, standing on the hillside, to be 29° 10′. From P the angle of elevation to another point, C, directly in line with the tower and 500 ft. from P, is found to be 10° 20′. At C the angles of elevation to the top (T) and bottom (B) of the tower are found to be 38° 40′ and 17° 30′, respectively. Find the height of the tower.

Pages 232-235

EXERCISE O. Find the value of each of the following expressions:

(1) $2 \cos 0° - 5 \sin 270°$
(2) $4 \csc 90° + 3 \sin 0° - 5 \sec 180°$
(3) $\sec 180° - 8 \tan 360° + \sin 90°$
(4) $\cot 270° + 7 \cos 180° - 3 \csc 270°$
(5) $5 \sin 180° - 6 \cos 360° + 7 \sec 0°$

EXERCISE P. Find the value or values of θ if:

(1) $\sin \theta = -1$
(2) $\sec \theta = -1$
(3) $\csc \theta = 1$
(4) $\cot \theta = \pm\infty$ and $\cos \theta = 1$
(5) $\sin \theta = 1$ and $\tan \theta = \pm\infty$
(6) $\tan \theta = 0$ and $\csc \theta = \pm\infty$

CHAPTER 15

Pages 238-240

EXERCISE A. Solve each of the following systems of equations by the addition and subtraction method:

(1) $\begin{cases} x + y = 5 \\ x - y = 3 \end{cases}$

(2) $\begin{cases} 2u + w = 17 \\ u + w = 12 \end{cases}$

(3) $\begin{cases} 2a - b = 5 \\ 5a + 4b = 19 \end{cases}$

(4) $\begin{cases} x + y = -9 \\ 7x - 3y = 47 \end{cases}$

(5) $\begin{cases} 2m - 5k = 24 \\ 3m + 5k = 11 \end{cases}$

(6) $\begin{cases} d + f = 0 \\ 4d - 3f = 6 \end{cases}$

(7) $\begin{cases} 10s - 3z = -6 \\ 7s + 4z = 8 \end{cases}$

(8) $\begin{cases} 19x + 13y = 83 \\ 13x + 19y = 77 \end{cases}$

(9) $\begin{cases} x + \dfrac{y}{3} = 8 \\ \dfrac{x}{5} + 2y = 19 \end{cases}$

(10) $\begin{cases} 93a + 71b = 350 \\ 90a + 74b = 344 \end{cases}$

(11) $\begin{cases} \dfrac{x}{2} - \dfrac{2y}{3} = -2 \\ \dfrac{3x}{4} - \dfrac{y}{3} = 3 \end{cases}$

(12) $\begin{cases} x + y = 5(a + b) \\ x - y = a - b \end{cases}$

(13) $\begin{cases} \dfrac{6}{x} + \dfrac{3}{y} = 4 \\ \dfrac{10}{x} - \dfrac{6}{y} = 3 \end{cases}$

(14) $\begin{cases} \dfrac{5}{x} - \dfrac{6}{y} = -1 \\ \dfrac{15}{x} + \dfrac{3}{y} = 4 \end{cases}$

(15) $\begin{cases} x + y = a^2 - b^2 \\ x - y = (a - b)^2 \end{cases}$

(16) $\begin{cases} \dfrac{2}{x} + \dfrac{6}{y - 1} = 3 \\ \dfrac{8}{x} - \dfrac{15}{y - 1} = -1 \end{cases}$

PROBLEMS:

(1) A retail merchant bought 3 dozen collars and 5 dozen shirts for \$61.20. Three days later he bought, at the same price, 7 collars and 26 shirts for \$24.80. What was the price of each?

(2) A theater allowed the use of its premises for a charity benefit. One thousand seats were available. Some tickets were sold at \$2.00 apiece, and others at \$1.25 apiece. The total receipts were \$1400. How many tickets of each kind were sold?

(3) A bricklayer and his assistant had a contract to do a certain piece of work. If both had worked together, they could have done the work in 6 days. But the bricklayer worked 8 days and the assistant finished the work in 3 days. In how many days could each have done the work?

(4) The buyer for a furniture store has \$73 with which to buy 30 chairs. He finds that some cost \$2 each and some cost \$3 each. How many of each kind can he purchase?

(5) If three times the units' digit of a number is subtracted from five times the tens' digit the difference is 8, and when 18 is added to the number the order of the digits is reversed. Find the number.

Pages 240-241

EXERCISE B. Solve each of the following systems of equations by the substitution method:

(1) $\begin{cases} 4a - 2d = 10 \\ a + d = 7 \end{cases}$

(2) $\begin{cases} 3m + n = 3 \\ 2m - 3n = -20 \end{cases}$

(3) $\begin{cases} 4k - 4s = -26 \\ 4k + 3s = 23 \end{cases}$

(4) $\begin{cases} 3x - y = -2 \\ x + 5y = 2 \end{cases}$

(5) $\begin{cases} 0.6u + 0.1w = -3 \\ 0.9u - 0.2w = 13 \end{cases}$

(6) $\begin{cases} 0.3m + 0.2r = 1.2 \\ 0.1m + 0.6r = 2 \end{cases}$

(7) $\begin{cases} -5s - 0.4t = 0 \\ 0.3s - t = -51.2 \end{cases}$

(8) $\begin{cases} 9a + 16b = 7 \\ a - b = \frac{1}{12} \end{cases}$

(9) $\begin{cases} \dfrac{x}{15} = \dfrac{y}{21} \\ \dfrac{7x}{5} - \dfrac{3y}{7} = 4 \end{cases}$

(10) $\begin{cases} \dfrac{a+b}{11} - \dfrac{2a-b}{8} = -1 \\ \dfrac{a+3b}{2} + \dfrac{2a-b}{4} = 9 \end{cases}$

(11) $\begin{cases} \dfrac{m}{2} - \dfrac{n+32}{4} = 12 \\ \dfrac{3m-2n}{5} - 25 = -\dfrac{n}{8} \end{cases}$

(12) $\begin{cases} \dfrac{x}{6} + \dfrac{2v}{3} = 6 \\ \dfrac{x}{8} + 4v = 15 \end{cases}$

(13) $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \frac{1}{2} \\ \dfrac{1}{x} - \dfrac{1}{y} = \frac{1}{6} \end{cases}$

(14) $\begin{cases} \dfrac{2}{3m} + \dfrac{1}{2n} = 9 \\ \dfrac{5}{6m} + \dfrac{4}{5n} = 13 \end{cases}$

(15) $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{a} \\ \dfrac{1}{x} - \dfrac{1}{y} = a \end{cases}$

(16) $\begin{cases} \dfrac{4}{x-1} + \dfrac{6}{y+1} = 5 \\ \dfrac{2}{x-1} + \dfrac{54}{y+1} = 11 \end{cases}$

Pages 240-241

PROBLEMS:

(6) A certain number of men agree to share equally the expenses for a fishing trip. If each paid \$5.50, there would be 2 dollars lacking; but if each paid \$6.00, there would be 1 dollar left over. Find the number of persons and the expense per person for the fishing trip.

(7) A steamer that travels with the current 500 miles in 16 hours and 40 minutes requires 25 hours to travel back to its starting point. Assuming the rate of the current to remain constant, find the speed of the steamer in still water and the rate of the current.

(8) A merchant buys some coffee for 23 cents a pound and some for 28 cents a pound. He sells a mixture of the coffees for 29 cents a pound and gains 20 per cent on the selling price. In what proportion does he mix them?

(9) Sea water weighs 1.025 kilograms per liter; it contains, by weight, 2.8 per cent of salt. How much water must be taken to obtain, by evaporation, 1 liter of water containing 5 per cent of salt? (Assume that 1 liter of pure fresh water weighs 1 kilogram.)

(10) Two machines working together can do a certain piece of work in 12 days. After one machine has worked alone for 5 days, the other machine finishes the work in 26 days. Find the time in which each machine alone can do the work.

Pages 241-242

EXERCISE C. Solve each of the following systems of equations by the comparison method:

(1) $\begin{cases} 7x - 3y = 4 \\ 7y - 3x = 0.4 \end{cases}$

(2) $\begin{cases} 3m + 2r = 12 \\ m + 6r = 14 \end{cases}$

(3) $\begin{cases} v + 3w = -1.5 \\ 7v - 2w = 1 \end{cases}$

(4) $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 2 \\ 2x + 3y = 60 \end{cases}$

(5) $\begin{cases} 1.2a + 3.5b = 0.827 \\ 2.8a + 1.5b = 1.063 \end{cases}$

(6) $\begin{cases} \frac{x}{5} + \frac{y}{2} = 5 \\ x - y = 4 \end{cases}$

(7) $\begin{cases} ax + by = 1 \\ a_1x + b_1y = 2 \end{cases}$

(8) $\begin{cases} s = 3t \\ \frac{s}{3} + t = 34 \end{cases}$

(9) $\begin{cases} \frac{a+b}{a-b} = \frac{7}{3} \\ \frac{a+1}{b+1} = \frac{13}{5} \end{cases}$

(10) $\begin{cases} 3v + \frac{1}{2}u = 17 \\ \frac{v}{5} - \frac{u}{4} = 0 \end{cases}$

(11) $\begin{cases} \frac{8.4}{x} - \frac{4.5}{y} = 3 \\ \frac{4.9}{x} - \frac{2.5}{y} = 2 \end{cases}$

(12) $\begin{cases} \frac{a}{x} + \frac{b}{y} = 1 \\ \frac{c}{x} + \frac{d}{y} = 2 \end{cases}$

(13) $\begin{cases} \frac{w}{3} + \frac{y}{4} = 0 \\ 3w - 7y - 37 = 0 \end{cases}$

(14) $\begin{cases} \frac{2x+7y}{4} - \frac{x+7}{6} = 4 \\ \frac{2x+7y}{6} - \frac{x+7}{3} = 0 \end{cases}$

(15) $\begin{cases} \dfrac{3s-4}{5} + \dfrac{4t-5}{3} = 3(s-t) \\ \dfrac{5s+2}{3} - \dfrac{7t-3}{4} = 2(s-t) \end{cases}$ (16) $\begin{cases} \dfrac{15}{x-2} + \dfrac{6}{y-1} = 5 \\ \dfrac{10}{x-2} - \dfrac{21}{y-1} = -5 \end{cases}$

Pages 241-242

PROBLEMS:

(11) A river boat travels downstream for 50 miles in 3 hours and 20 minutes; but it requires 5 hours and $33\frac{1}{3}$ minutes to return upstream to its starting point. Assuming the rate of the current to remain constant, find the velocity of the river boat in still water and the rate of the current.

(12) A merchant has some domestic tobacco at \$28 a hundred pounds and imported tobacco at \$56 a hundred pounds. How many pounds of each kind must he take to make a mixture of 147 lbs. worth \$36 a hundred pounds?

(13) In a factory where 2,050 men and women are employed, the average daily pay for the men is \$6.25 and for the women \$4.75. If \$12,287.50 is paid daily for labor, find the number of men and the number of women employed.

(14) Five years ago Jim was $\frac{1}{2}$ as old as John, and 10 years hence he will be $\frac{5}{7}$ as old as John. Find their ages.

(15) The price of 2 lbs. of sugar and 5 lbs. of coffee is \$1.70. If the price of sugar increased 20 per cent and that of coffee decreased 10 per cent, the cost of the same amounts of each would be \$1.59. Find the cost of a pound of each.

Pages 242-243

EXERCISE D. Solve each of the following systems of equations by the graphic method:

(1) $\begin{cases} x + y = 2 \\ 3x - 2y = 1 \end{cases}$

(2) $\begin{cases} a + 3b = 21 \\ 4a - b = 19 \end{cases}$

(3) $\begin{cases} y = 3x - 5 \\ y = -x + 2 \end{cases}$

(4) $\begin{cases} y = \frac{1}{2}x - 1 \\ y = -\frac{3}{4}x + 3 \end{cases}$

(5) $\begin{cases} x = 5y + 7\frac{1}{2} \\ x = 2y + 9 \end{cases}$

(6) $\begin{cases} y = 4x - 7 \\ 2x + 3y = -4 \end{cases}$

(7) $\begin{cases} 5x + 13y - 4 = 0 \\ 3x + y - 11 = 0 \end{cases}$

(8) $\begin{cases} 3x + 4y - 12 = 0 \\ x - 5y - 5 = 0 \end{cases}$

Pages 242-243

PROBLEMS:

(16) A man has \$10,000 invested in a fund on which he would receive \$200 a year interest. He draws the \$10,000 out, investing a part in bonds paying 3 per cent and the rest in stock that yields 4 per cent. By making the change, his yearly income is increased by \$140. Find the amount invested in bonds and the amount invested in stock.

(17) A purse containing 33 coins holds only nickels and dimes. The total value of the coins is \$2.55. How many coins of each kind are in the purse?

(18) A foreman paid 15 mechanics and 5 helpers \$870 for 6 days' work. A few days later, with wages remaining at the same rate, he paid 8 mechanics and 7 helpers \$396 for 4 days' work. Find the daily wage of each.

(19) The length l of a wire depends upon the tension p as in the formula $l = mp + b$, where m and b are constants. For a certain wire it is found that if $p = 200$, $l = 190.2$; and that if $p = 350$, $l = 190.35$. Find m and b by the graphic method.

(20) The height (h in inches) of the mercury barometer depends upon the pressure (p in lb. per sq. in.) of the atmosphere as in the formula $h = mp + b$. It is known by experiment that if $p = 14$, $h = 28.50$; and that if $p = 14.74$, $h = 30$. Find m and b by the graphic method.

Pages 243-246

EXERCISE E. Solve each of the following systems of equations by the use of determinants:

(1) $\begin{cases} x + y = 11 \\ x - y = -3 \end{cases}$

(2) $\begin{cases} 3x + 2y = 3 \\ 3x - y = 3 \end{cases}$

(3) $\begin{cases} x - 2y = 13 \\ x - y = 9 \end{cases}$

(4) $\begin{cases} 8x + 3y = -12 \\ 4x + y = -2 \end{cases}$

(5) $\begin{cases} y = 6x - 31 \\ 2y = 9x - 44 \end{cases}$

(6) $\begin{cases} 13x + 10y - 59 = 0 \\ 11x - 9y - 15 = 0 \end{cases}$

(7) $\begin{cases} 5x - 3y - 5 = 0 \\ 3x + 11y - 67 = 0 \end{cases}$

(8) $\begin{cases} 15x + 12y = -78 \\ 9x - 8y = 14 \end{cases}$

Pages 243-246

PROBLEMS:

(21) At a benefit bazaar 985 ice cream cones were sold, some at 5 cents each and others at 10 cents each. The total receipts were \$68.50. How many cones were sold at each price?

(22) A staff officer starts, in his automobile, traveling at an average speed of 35 miles per hour, from Richmond, Va., on the road to Washington, D. C., 109 miles away. At the same time a courier on a motorcycle starts from Washington toward Richmond at an average speed of 45 miles per hour. How far from Richmond will the courier meet the staff officer's car?

(23) A man invests \$7,500, part at 3 per cent and the rest at 6 per cent, and obtains an average of $4\frac{2}{5}$ per cent on his money. Find the amount invested at each rate.

(24) A wine merchant has two kinds of brandy: the one costs \$3 per gallon, the other \$7 per gallon. He wishes to mix both sorts together in such quantities that he may have 300 gallons, each gallon of the mixture to be worth, without profit or loss, \$4. How much of each sort must he take to make up this mixture?

(25) Two pumps are discharging water from a tank. The first works 4 minutes and the second 9 minutes, and they pump 3,700 gallons of water; later the first works 5 minutes and the second 8 minutes, and they pump 3,650 gallons of water. Find the capacity of each pump per minute.

Pages 246-250

EXERCISE F. Solve each of the following systems of equations by the use of determinants and interpret the results geometrically:

(1) $\begin{cases} x - 2y = -6 \\ 3x + 4y = 32 \end{cases}$

(2) $\begin{cases} 2x - 5y = -25 \\ x + y = 12 \end{cases}$

(3) $\begin{cases} 3x + 2y = 57 \\ 6x + 4y = 2 \end{cases}$

(4) $\begin{cases} 2x + 3y = 39 \\ 4x - 6y = -54 \end{cases}$

(5) $\begin{cases} 7x - 2y = 3 \\ 21x - 6y = 9 \end{cases}$

(6) $\begin{cases} 5x - 9y = 8 \\ 3x + 4y = 33 \end{cases}$

(7) $\begin{cases} 8x + 12y = 7 \\ 6x + 9y = 11 \end{cases}$

(8) $\begin{cases} 15x - 5y = 10 \\ 6x - 2y = 4 \end{cases}$

Pages 246-250

PROBLEMS:

(26) In the system of equations $2x - 3y = 5$, $6x + ky = 8$, find k so that (a) the lines shall be parallel; (b) the lines shall intersect.

(27) In the system of equations $3x - 5y = 8$, $6x - 10y = k$, find k so that the lines shall be concurrent.

(28) The sides of a triangle have for their equations $5y + x = 23$, $7y + 8x = 19$, and $2y + 7x = 29$. What are the coordinates of the vertices of this triangle?

(29) Find the equation of the straight line passing through $(5, -2)$ and $(3, 4)$, using the equation $Ax + By + C = 0$. (*Hint:* Solve for $\frac{A}{B}$ and $\frac{C}{B}$.)

(30) Given the determinant $\begin{vmatrix} 5 & 9 \\ 2 & 4 \end{vmatrix}$, form a new determinant: (a) by multiplying the elements of the first column by 3; (b) by interchanging the rows with the columns; (c) by interchanging the second row with the first. Find the value of each of these determinants and compare these values with the value of the given determinant.

Pages 250-251

EXERCISE G. Solve each of the following systems of equations:

(1) $\begin{cases} x + 3y + z = 14 \\ x + y + 3z = 16 \\ 3x + y + z = 20 \end{cases}$

(2) $\begin{cases} 3x - y + z = 6 \\ 2x - 2y + z = 15 \\ x - 3y + 3z = -2 \end{cases}$

(3) $\begin{cases} 4x - 5y + 2z = 11 \\ 6y - 5x + 2z = 8 \\ 8y - 2z - 9x = -22 \end{cases}$

(4) $\begin{cases} 3x - 5y + z = 1 \\ x + 15y - 2z = 3 \\ 5x - 6y + 3z = 12 \end{cases}$

(5) $\begin{cases} x - y = 2 \\ y - z = 3 \\ x + z = 9 \end{cases}$

(6) $\begin{cases} x + y = 9 \\ x - y = 3 \\ x - z = 4 \end{cases}$

(7) $\begin{cases} \frac{1}{x} + \frac{1}{y} = 5 \\ \frac{1}{x} + \frac{1}{z} = 6 \\ \frac{1}{y} + \frac{1}{z} = 7 \end{cases}$

(8) $\begin{cases} \frac{1}{z} + \frac{1}{y} = a \\ \frac{1}{z} + \frac{1}{x} = b \\ \frac{1}{x} + \frac{1}{y} = c \end{cases}$

Pages 250-251

PROBLEMS:

(31) A merchant has three kinds of coffee. He can sell 3 pounds of the first kind, 5 pounds of the second, and 2 pounds of the third for \$3.02; or 7 pounds of the first kind, 4 pounds of the second, and 3 pounds of the third for \$4.18. If a pound of the first kind is worth 8 cents more than $\frac{2}{3}$ of a pound of the second kind, and $\frac{1}{2}$ of a pound of the third kind is worth 13 cents less than a pound of the second kind, find the value of 1 pound of each kind.

(32) A man invests \$10,865, part in government bonds at 3 per cent, another part in industrial bonds at 6 per cent, and the rest in mortgages at 8 per cent.

His total yearly income from these three investments is $651.10. The combined income from the government and industrial bonds is $196.90 less than the income from the mortgages. Find the amount of each investment.

(33) A purse, containing 19 coins whose total value is $2.35, holds only nickels, dimes, and quarters. The number of nickels and dimes together is 4 more than twice the number of quarters. How many coins of each kind are there in the purse?

(34) A machine shop has three lathes of different sizes. Lathe No. 1 and lathe No. 2 can do a piece of work together in 6 days; lathe No. 1 and lathe No. 3 together can do the same piece of work in 12 days; and lathe No. 2 and lathe No. 3 can do it together in 9 days. How long will it take each alone to do the work?

(35) A quantity of water sufficient to fill three jars of different sizes will fill the smallest jar 4 times; the largest jar twice with 2 gallons to spare; or the second jar 3 times with 1 gallon to spare. Find the capacity of each jar.

CHAPTER 16

Pages 253-256

EXERCISE A. Find the areas of the rectangles having the following measurements:

(1) 2 by 15 rods
(2) 6.4 by 4.6 meters
(3) 90 feet by 20 yards
(4) 10.06 meters by 21.7 meters
(5) 70 yards by 70 meters
(6) 57 rods by 47 rods

Pages 253-256

PROBLEMS:

(1) How many hectares are there in a township?

(2) The area of Switzerland is 41,346 square kilometers; that of Maryland, 12,327 square miles. Which is larger and by how many square kilometers?

(3) Texas has an area of 265,986 square miles; Argentina, 2,806,000 square kilometers. Which is larger and how many times larger?

(4) One country taxes lace imports at 90 per cent per square yard; another country, at 85 per cent per square meter. Compare the two tariff rates.

(5) How many hectares will equal 40 acres?

(6) How many acres will equal 40 hectares?

(7) How many square inches of gold leaf are there in 220 square millimeters?

(8) How many square meters are there in an acre?

(9) How many square yards are there in a hectare?

(10) A surveyor wishes to ascertain the area of a rectangular field. Representing the rectangle by $ABCD$, so that AB is the base, he sets up the transit at A and finds angle $CAB = 28°$ and side $AB = 450.8$ ft. Find the area of this field.

(11) The owner of a rectangular field, the area of which is 15,000 square yards, wishes to determine its length and width. On arriving at the field, he finds that a direct accurate measurement of these distances is not possible; but he ascertains that the angle made by a side of the rectangle forming the field and a diagonal is 15° 32′. Find the length and width of the field.

(12) The area of a rectangular field is known to be 147.2 acres. Owing to the nature of the terrain, an engineer, to determine the length of the sides of the rectangular field, locates the point of intersection of its two diagonals and ascertains that the angle formed by these diagonals is 71° 18′. Find the length and width of the field.

Pages 256-259

EXERCISE B. Complete the square:

(1) $x^2 - 6x$ (3) $x^2 + 3x$ (5) $x^2 - 5x$
(2) $x^2 + 10x$ (4) $x^2 - 7x$ (6) $x^2 + 8x$

Pages 259-263

EXERCISE C. Solve the following equations by completing the square:

(1) $x^2 - 4x - 5 = 0$
(2) $x^2 - 2x = 3$
(3) $y^2 + y - 90 = 0$
(4) $x^2 - 8x + 7 = 0$
(5) $x^2 - 14x = -33$
(6) $y^2 + 8y + 7 = 0$
(7) $x^2 + 77 = 18x$
(8) $x^2 - 7x + 10 = 0$
(9) $y^2 - 20y + 91 = 0$
(10) $x^2 - 26x = 87$
(11) $x^2 + x = 72$
(12) $x^2 + 8x = 128$
(13) $y^2 - 20y + 51 = 0$
(14) $x^2 + 20x + 84 = 0$
(15) $y^2 + 16y = 57$
(16) $16x^2 + 40x + 21 = 0$
(17) $9y^2 - y = 2$
(18) $5x^2 + 27x + 10 = 0$
(19) $2y^2 - 7y + 3 = 0$
(20) $35x^2 + 11x = 6$
(21) $48x^2 + 58x + 15 = 0$
(22) $3y^2 + 16y = 35$
(23) $2x^2 - x = 14$

(24) $\dfrac{4}{x+3} - \dfrac{40}{x-3} = 3$

(25) $\dfrac{15}{x-1} - \dfrac{6}{x+1} = 1$

(26) $\dfrac{3}{x+2} - 4 = \dfrac{35}{x-2}$

(27) $\dfrac{13}{x-5} - 2 = \dfrac{9}{x+5}$

Pages 259-263

PROBLEMS:

(13) A stationer bought a number of fountain pens for \$100. Had they cost \$1.00 apiece less, he would have obtained five more for the money. Find the cost of each.

(14) A swimming pool can be filled by two pipes running together in $22\frac{1}{2}$ hours. The larger pipe would fill the pool in 24 hours less than the smaller one. Find the time taken by each alone.

(15) A patio can be paved with 600 square tiles of a certain size; if each tile were 1 inch longer each way, 384 tiles would be required to pave the same patio. Find the length of each tile.

(16) Two dealers, A and B, have 30 batteries between them. They sell at different prices, but each receives the same sum. If A had sold his at B's price, he would have received \$320. If B had sold his at A's price, he would have received \$245. How many batteries did each dealer have?

(17) What is the price of a dozen eggs if two more in \$1.84 worth lowers the price 2 cents per dozen?

Pages 263-265

EXERCISE D. Solve the following equations by factoring:

(1) $x^2 - x - 6 = 0$
(2) $y^2 + 11y + 30 = 0$
(3) $x^2 + x = 12$
(4) $x^2 - 4x - 21 = 0$
(5) $y^2 + 5y = 36$
(6) $x^2 = 11x - 28$
(7) $y^2 - 13y + 40 = 0$
(8) $x^2 + 5x = 84$
(9) $x^2 - 3x = 0$
(10) $5x^2 = 6x$
(11) $y^2 + 23y = 108$
(12) $x^2 + 18x + 72 = 0$

(13) $y^2 = 7y + 78$
(14) $6x^2 - 5x - 6 = 0$
(15) $20x^2 - 47x + 21 = 0$
(16) $10y^2 + 23y + 12 = 0$
(17) $2x^2 = 17x + 55$
(18) $7x^2 - 43x + 6 = 0$
(19) $9y^2 + 34y = 8$
(20) $16x^2 + 46x - 35 = 0$

Pages 263-265

PROBLEMS:

(18) A rectangular field has an area of 192,500 sq. ft. and a perimeter of 1800 ft. Find the dimensions of the field.

(19) A man riding a bicycle a distance of 72 kilometers would arrive 30 minutes earlier if he traveled 2 kilometers per hour faster. Find the rate at which he traveled.

(20) Two men working together can complete a piece of work in 6 days. If it would take one man 5 days longer than the other to do the work alone, in how many days can each man do the work alone?

(21) A vintner had 64 gallons of wine in a vat. He withdrew a certain number of gallons of pure wine and replaced them with water. He then withdrew from the mixture as many gallons as he drew before of pure wine, when it was found that the vat contained only 36 gallons of pure wine. How many gallons of pure wine were withdrawn at each draught?

(22) The Darby Investment Company invests \$1600 at simple interest. At the end of the first year it withdraws \$40. The remainder is kept invested at the same rate for another year. At the end of this second year \$78 is added to the fund, making \$1800 in all. What was the rate of interest?

Page 266

EXERCISE E. Solve the following equations:

(1) $x^4 - 25x^2 + 144 = 0$
(2) $x^4 - 29x^2 + 100 = 0$
(3) $x^4 - 81 = 0$
(4) $x^4 - 34x^2 + 225 = 0$
(5) $x^4 - 58x^2 + 441 = 0$
(6) $16x^4 - 153x^2 + 81 = 0$
(7) $9x^4 - 229x^2 + 100 = 0$
(8) $x^{1/4} + 4x^{1/2} - 5 = 0$
(9) $x^{2/3} - 2x^{1/3} - 15 = 0$
(10) $x^{1/3} - 3x^{1/6} + 2 = 0$
(11) $x - 3x^{1/2} - 28 = 0$
(12) $x^3 - 3x^{3/2} + 2 = 0$

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PROBLEMS:

(23) A man worked a certain number of days for \$40. He now earns \$1 per day more than he did, and works 2 days less to obtain the same sum of money. How many days did he work?

(24) A painting that was 8 inches by 10 inches was placed in a frame of uniform width. The area of the frame was 8 sq. in. more than the area of the painting. Find the width of the frame.

(25) A railway train traveled 5 miles per hour slower than usual and was one hour late in making a run of 210 miles. Find the usual speed of the train.

(26) After Mr. Wolf, a mechanic, had worked alone on a piece of work for 3 days, Mr. Sullivan, another mechanic, joined him and they finished the work $5\frac{1}{3}$ days later. If Mr. Sullivan had worked alone, he would have required 3 days less than Mr. Wolf to do the whole piece of work. Find the time each man would have required to do the work alone.

(27) A man bought some bonds for \$10,000. When the price advanced \$10 on each bond, he sold all the bonds but two for \$8,080. How many bonds did he buy?

CHAPTER 17

Pages 268-270

EXERCISE A. Solve the following equations by the quadratic formula:

(1) $x^2 + 4x = 77$
(2) $x^2 + 11x + 24 = 0$
(3) $x^2 - 5x = 36$
(4) $x^2 - 17x + 66 = 0$
(5) $6y^2 + 40 = 31y$
(6) $5x^2 + 43x + 24 = 0$
(7) $8y^2 + y = 30$
(8) $15 + 17x = 4x^2$
(9) $5y^2 = 8y + 21$
(10) $2(x - 3) = 3(x + 2)(x - 3)$
(11) $196x^4 - 113x^2 + 36 = 0$
(12) $7x^{1/2} - 16x^{1/4} = 15$
(13) $144x^4 - 25x^2 + 1 = 0$
(14) $16x^{2/3} + 16x^{1/3} = 5$
(15) $(x - 1)^2 - 7(x - 1) + 10 = 0$
(16) $(x - 2)[(x - 2) + 2] = 35$
(17) $6(y + 4)^2 - (y + 4) = 7$
(18) $x^2 + bx + c = 0$
(19) $\dfrac{1}{1 + x} - \dfrac{1}{3 - x} = \frac{6}{35}$
(20) $\dfrac{4}{y - 1} - \dfrac{5}{y + 2} = \dfrac{3}{y}$

Pages 268-270

PROBLEMS:

(1) A plumber buys \$300 worth of valves. If he had obtained 10 more valves for the same money, each valve would have cost \$1 less. How many valves were bought and what was the price per valve?

(2) Two laborers A and B work for a certain time, B working 2 days more than A. The latter, that is, A, earns \$84 and B earns \$80. If each had worked as many days as they did, but each at the other's daily wage, their joint earnings would have totaled \$2.00 more. How many days did each work and at what rate of pay?

(3) A museum has two funds invested at simple interest. The first produces annually \$336. The second fund, which is \$1200 smaller than the first, produces \$21 less annually although the rate is $\frac{1}{4}$ per cent higher than the rate on the first fund. What are the two funds and the two rates of interest?

(4) An airplane traveled 510 miles from Chicago to Detroit and back in 2 hours and 8 minutes. On the return trip, the speed was 30 miles per hour faster. Find the speed on each trip.

(5) A rectangular tract of land, covering an area of 64,400 sq. yds., is divided into four equal rectangular blocks by two streets, each 90 ft. wide, running through it at right angles. The area of the four rectangular blocks is 50,000 sq. yds. Find the dimensions of the rectangular tract of land.

Pages 270-272

EXERCISE B. Without solving, find the nature of the roots of the following equations:

(1) $x^2 + 7x - 18 = 0$
(2) $9x^2 - 12x + 4 = 0$
(3) $2x^2 + 6x + 5 = 0$
(4) $x^2 + 6x - 12 = 0$
(5) $x^2 + 3x + 2 = 0$
(6) $x^2 - 4x + 5 = 0$
(7) $25x^2 + 30x + 9 = 0$
(8) $5x^2 + 4x - 5 = 0$
(9) $49x^2 - 56x + 16 = 0$
(10) $25x^2 - 9 = 0$

Pages 272-277

EXERCISE C. By means of graphs, find the maximum and minimum values of the following functions and determine the roots of the corresponding quadratic equations:

(1) $y = x^2 + 2x - 8$
(2) $y = 4 - 3x - x^2$
(3) $y = 4x + 5 - x^2$
(4) $y = 2x^2 - x - 15$
(5) $y = 4x^2 - 27x + 35$
(6) $y = 44x - 117 - 4x^2$

Pages 272-277

PROBLEMS:

(6) The Board of Trade Building in Chicago is 612 ft. high. An antiaircraft gun is placed on its roof and fires directly upward with an initial velocity of 960 ft. per second. Using the formula given in Problem 6, page 279, find: (*a*) the maximum height attained by the projectile; (*b*) how many seconds after being fired the projectile reached its maximum height; (*c*) how many seconds after being fired the projectile hit the ground.

(7) A rectangle is to have a perimeter of 64 inches. For what dimensions will its area be greatest?

(8) A long sheet of tin 6 inches wide is to be made into a gutter, whose cross section is a rectangle, by turning strips up vertically along the two sides. How many inches should be turned up at each side to secure the greatest carrying capacity, that is, the greatest sectional area?

(9) A printed page is to allow 30 sq. in. for printed matter and have a margin of 1 inch at each side and $\frac{5}{8}$ in. at top and bottom. Find the dimensions of the page that will require the least amount of paper.

(10) The area of a rectangular field bordered on one side by a straight river bank is 128 sq. yds. If no fence is needed along the bank of the river, find the dimensions of the field that will require a minimum amount of fencing.

Pages 277-278

EXERCISE D. Solve the following systems of equations:

(1) $\begin{cases} x^2 + y^2 = 153 \\ y - 4x = 0 \end{cases}$

(2) $\begin{cases} x^2 + y^2 = 10 \\ 3x^2 - 5y^2 = 22 \end{cases}$

(3) $\begin{cases} 5x^2 - y^2 = 9 \\ xy = 18 \end{cases}$

(4) $\begin{cases} 2x^2 - y^2 = 4 \\ y - 2x = 2 \end{cases}$

(5) $\begin{cases} y^2 = 4x \\ x - 3y = 0 \end{cases}$

(6) $\begin{cases} 3x^2 - 2y^2 = 40 \\ 2x^2 + 3y^2 = 44 \end{cases}$

(7) $\begin{cases} x^2 + 2y^2 = 96 \\ y^2 = 2x \end{cases}$

(8) $\begin{cases} x^2 + 9y^2 = 5 \\ 3y - x = 3 \end{cases}$

(9) $\begin{cases} x^2 + y^2 = 26 \\ 2x^2 + 3y^2 = 53 \end{cases}$

(10) $\begin{cases} 4x^2 + 5y^2 = 216 \\ xy = 14 \end{cases}$

Pages 277-278

PROBLEMS:

(11) If the usual speed of a motor bus is increased 5 miles an hour, 1 hour and 36 minutes will be saved in going from Boston to Cincinnati. The trip will require 2 hours and 24 minutes longer if the usual speed is decreased 6 miles an hour. What is the distance between the two cities and the usual speed of the bus?

(12) Two rectangles contain the same area, 480 square meters. The difference of their lengths is 10 meters and of their breadths 4 meters. Find the dimensions of their sides.

(13) Twenty persons contributed \$800 toward a fund for a college library. The men contributed 3 times as much as the women, each man giving \$25 more than each woman. What did each man and what did each woman contribute?

(14) Morton and Rush working together finished a piece of work in a certain number of days. The total wages received by the two men amounted to \$120, each man receiving the same amount. Rush earned \$1 per day more than

Morton. If Morton worked 3 days more than Rush, find the number of days each worked and their respective rates of pay.

(15) A sum of money invested at simple interest for one year amounted to \$12,480. If the rate had been 1 per cent higher and the principal \$2400 less, the interest received for one year would have been the same. Find the principal and the rate of interest.

CHAPTER 18

Pages 280-283

PROBLEMS:

(1) The sides of a parallelogram are 20 and 25, respectively, and include an angle of 30°. Find the area of the parallelogram.

(2) A *rhombus* is a parallelogram all of whose sides are equal. One side of a rhombus is 16 inches and one of its angles is 38°. Find the area of the rhombus.

(3) A rectangle and a parallelogram each have an altitude equal to one-half their common base. Prove that the figures have the same area.

(4) Two adjacent sides of a parallelogram are 10″ and 12″ and their included angle is 60°. Find the base of a rectangle whose altitude is $5\sqrt{3}$ and whose area is equal to the area of the parallelogram.

(5) The altitudes of two parallelograms, with equal bases, are 5 and 7 inches, respectively. Find the ratio of their areas.

Pages 283-284

PROBLEMS:

(6) An engineer is to survey a field having the shape of a parallelogram. He chooses any point within the parallelogram and measures the distances from this point to the sides. Prove that the area of the parallelogram is equal to twice the sum of the areas of either pair of triangles formed by joining the point within the parallelogram to its vertices.

(7) A parallelogram whose sides are equal is called a *rhombus* Prove that the area of a rhombus is equal to half the product of its diagonals.

(8) Prove that the area of a triangle is equal to one-half the product of any two sides multiplied by the sine of their included angle.

(9) Prove that, if from the midpoint of one leg of a trapezoid line segments be drawn to the opposite vertices, the area of the triangle formed is equal to one-half the area of the trapezoid.

(10) A surveyor, to determine the area of a field having the shape of a trapezoid $ABCD$, locates the midpoint, P, of side AD. Setting up the transit at P, he ascertains its distances to the opposite vertices, namely, PC and PB, to be 180 ft. and 220 ft., respectively, while angle CPB is 40°. Using the facts stated in Problems 8 and 9, find the area of the field.

(11) The surveyor, ascertaining the area of the field having the shape of a trapezoid $ABCD$ (given in Problem 10), could have determined the length of side BC, the distance from this side to the midpoint of side AD, and have applied the fact that "the area of a trapezoid is equal to the product of one side by the distance from this side to the midpoint of the other side." (*a*) Prove this fact. (*b*) Find the area of the trapezoid of Problem 10 if side BC is 145 ft. and the distance from BC to the midpoint of AD is 175.55 ft.

(12) Prove that the perpendicular drawn from the vertex of an isosceles triangle to the base bisects the vertex angle and the base. Using this fact, find the area of an isosceles triangle whose vertex angle is 100° and whose base is 50 ft. long.

(13) Find the area of an isosceles right triangle whose hypotenuse is 25 inches.

(14) Find the area of an equilateral triangle whose altitude is 10 ft.

(15) Two streets intersect at right angles. A third street cuts the other two at angles of 40° and 50°. The shortest side of the triangular plot is 300 ft. Find the area of the plot.

(16) Prove that the areas of two triangles that have an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.

(17) The corresponding bases of two triangles, having equal altitudes, are 8″ and 10″. The area of the smaller is 108 sq. in. Find the area of the larger triangle and the altitude of each.

(18) The corresponding altitudes of two triangles having equal bases are 5″ and 7″. The area of the larger is 105 sq. in. Find the area of the smaller and the base of each.

Pages 284-285

PROBLEMS:

(19) The altitude of a triangle is 15 inches and its base is 40 inches. Find the area of the trapezoid formed by a line parallel to the base of the triangle and 6 inches from the vertex.

(20) Prove that if the non-parallel sides of a trapezoid are equal, the diagonals also are equal.

(21) The cross section of the roof of a barn has the shape of an isosceles trapezoid surmounted by an isosceles triangle. The bases of the trapezoid are 50 ft. and 34 ft., respectively, and its altitude is 14 ft. The altitude of the triangle is 7 ft. Find the area of the cross section.

(22) One gallon of paint covers 400 sq. ft. on weather boarding. Find the number of gallons necessary to paint the outside of the roof of the barn of Problem 21 if the barn is 300 ft. long.

(23) Prove that a line drawn from the midpoint of one of the non-parallel sides of a trapezoid parallel to the base is equal to one-half the sum of its sides.

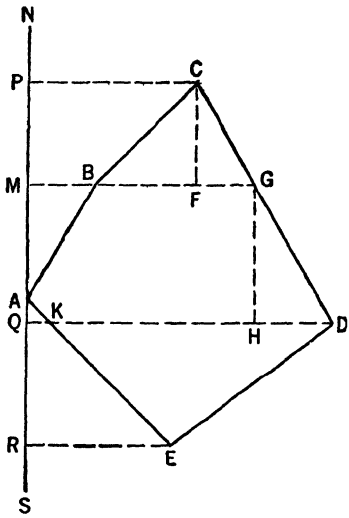
(24) Prove that the area of a trapezoid is equal to the product of its altitude and its median, namely, the line segment joining the midpoints of the non-parallel sides. (*Hint:* Use the fact stated in Problem 12, Chapter 11, page 614.)

Page 285

PROBLEMS:

(25) A field has the shape of an irregular quadrilateral $ABCD$. In surveying this field, an engineer at A finds that AB runs 60° north of east and that it measures 40 rods, while AD runs 70° south of east and measures 45 rods. Moving his transit to B, he finds that BC runs 70° south of east and measures 25 rods. Finally, setting up the transit at D, he finds that DC runs 76° 10′ north of east and measures 55.1 rods. Find the area of the field.

(26) The plan of the cellar of a house forms an irregular polygon $ABCDEFGH$. Line segment AH runs directly east and is 50 ft. long; AB is perpendicular to AH at A, runs directly north, and is 35 ft. long; BC is perpendicular to AB, runs directly east and is 14 ft. long; CD is perpendicular to BC, runs directly north, and is 25 ft. long; DE is perpendicular to DC, runs directly east, and is 30 ft. long; EF is perpendicular to DE, runs directly south, and is 30 ft. long; FG is perpendicular to EF, runs directly east, and is 6 ft. long; GH is perpendicular to FG and AH. Find the area of the cellar.



(27) A field has the shape of the irregular polygon shown in the figure. $AM = 24$ rods; $MB = 14$ rods; $AP = 45$ rods; $PC = 35.5$ rods; $QD = 64$ rods; $RE = 30$ rods; $AQ = 6$ rods; $AR = 30$ rods; angle $FCG = 30°$. Find the area of this field.

Pages 285-287

PROBLEMS:

(28) Draw any angle ABC. On AB choose any point D. Draw FG, through D, perpendicular to BC, and construct MD perpendicular to AB at D. Which angle on the diagram is equal to angle MDF? Prove it.

(29) Draw any angle ABC. Extend BC to D and draw DF perpendicular to DC. From F drop FK perpendicular to AB extended if necessary. Prove that angle ABC is equal to angle DFK.

(30) The carrying capacity of a triangular manger depends on the area of its cross section. The cross section of a certain manger is an isosceles right triangle whose area is 8 sq. ft. and whose side is 4 ft. A farmer wishes to build another manger, similar in shape, but having an area of 6 sq. ft. Find the length of a side of this manger.

(31) The corresponding sides of two similar triangles have the ratio 3 to 5, and the sum of their areas is 136 sq. inches. Find the area of each.

(32) The area of a certain triangle is 64 sq. ft. Lines drawn parallel to the base divide one side into four equal segments. Find the areas of the four parts of the triangle.

Pages 287-289

PROBLEMS:

(33) Prove that the areas of two similar polygons have the same ratio as the squares of two corresponding diagonals.

(34) The areas of two similar hexagons are $22\frac{1}{2}$ sq. inches and 90 sq. inches. If a side of the first hexagon is 3 in., find the corresponding side of the second.

(35) A redoubt is a fortified position entirely inclosed by defensible parapets. The commander of a battle area wishes to build a redoubt that will contain 9360 sq. ft. This redoubt is to be similar to another having the shape of a polygon whose area is 18,750 sq. ft. and a side of which measures 300 ft. Find the length of the corresponding side of the redoubt to be built.

(36) Prove that the areas of two similar polygons have the same ratio as the squares of their perimeters.

Pages 290-291

PROBLEMS:

(37) An officer of engineers, mapping a piece of land lying between a straight road and a curved stream, obtains the data given in the accompanying table. This table shows the distance from the road to the stream, s yards, at various points d yards from one end of the road. Map the land and find its approximate area.

s	0	17	56	109	174	200	184	132	96	60	0
d	0	10	20	30	40	50	60	70	80	90	100

(38) Find the area bounded by $x^2 = 8y$, the X-axis, and the ordinate $x = 8$.

(39) Find the area bounded by $xy = 10$, the X-axis, and the ordinates $x = 1$ and $x = 10$.

(40) The speed of an automobile, v ft. per sec., varies with the time, t sec., as in the following table. Find the total distance traveled by the automobile during the interval covered by the table. (*Hint:* Assume that the average speed for any interval is represented by the average height of the ordinate for that interval, and apply the formula rate $\times$ time = distance.)

t	0	1	2	3	4	5	6	7	8	9	10
v	0	12.6	17.9	21.9	25.3	28.3	31	33.5	35.8	37.9	40

(41) The speed of a locomotive, v miles per hour, varied with the time, t hours, as in the following table. Find the total distance traveled by the locomotive.

t	0	1	2	3	4	5	6	7	8	9	10
v	0	21.6	38.4	50.4	57.6	60	57.6	50.4	38.4	21.6	0

Pages 291-292

PROBLEMS:

(42) Construct a triangle equal in area to a rectangle whose base is 10 inches and whose altitude is 4 inches.

(43) Construct a triangle equal in area to a regular pentagon each side of which is 5 inches.

(44) Construct a triangle equal in area to a regular hexagon each side of which is 3 feet.

CHAPTER 19

Pages 293-295

PROBLEMS:

(1) Find the area of an equilateral triangle whose side is a.

(2) Find the area of an isosceles trapezoid whose bases are 8 and 16 and whose side is equal to 5.

(3) Prove that the area of a triangle whose perimeter is $2s$ is equal to $\sqrt{s(s-a)(s-b)(s-c)}$.

(4) Using the formula given in Problem 3, find the area of a triangle whose sides are: (*a*) 3, 8, and 9; (*b*) 4, 5, and 7.

(5) Carpenters obtain the strongest rectangular beam that can be cut from a given circular log as follows: The diameter of the circular face of the log is trisected. At these two points, perpendiculars to the diameter are drawn so that they run in opposite directions and intersect the circle at two points. The ends of the diameter and the two points of intersection of the perpendiculars to the diameter are the vertices of the rectangular beam. Show that the ratio of the depth to the breadth is approximately 7 to 5.

(6) Prove that the area of a square is equal to one-half the area of the square on its diagonal. (*Hint:* Use this fact to solve Problem 7, Chapter 19, page 309.)

(7) The height of a mountain on the moon was first approximately measured by Galileo (1564-1642) as follows: Assume the moon to be represented by a circle with center O. Draw a vertical diameter and shade a semicircle to represent the dark half of the moon. At the upper point, N, where the diameter intersects the circle, draw a line, NT, perpendicular to the diameter, to represent a ray of light from the sun. Choose a point M on line NT and on the same side of the shaded semicircle, to represent the peak of the mountain. Draw OM intersecting the circle at P. Let r denote the radius of the moon, h represent the height of the mountain, that is, PM, and d be the distance NM from the vertical diameter dividing the light from the dark half of the moon to the mountain peak M. Show how Galileo, by using the known radius of the moon and measuring d, was able to compute the height of the mountain.

(8) Two shafts, A and C, are located 6 inches between centers. A gear on shaft A measures 6 inches in diameter, and another gear on shaft C measures 4 inches in diameter. A third shaft, B, is to be located so that a gear on this shaft will contact the gears on shafts A and C, the center line BC to be perpendicular to the center line AC. Find the distance BC.

(9) By applying the theorem of Pythagoras, namely, Fact 70, and the definitions of the trigonometric functions, prove that:

$$(a)\ \sin^2\theta + \cos^2\theta = 1$$
$$(b)\ \sec^2\theta - \tan^2\theta = 1$$
$$(c)\ \csc^2\theta - \cot^2\theta = 1$$

Pages 295-296

PROBLEMS:

(10) Find the values of the following expressions by inserting the values of the trigonometric functions:

(*a*) $5 \tan 45° - 6 \sin 30° + \sec 60°$
(*b*) $7 \cos 45° + 8 \cot 60° - 2 \csc 30°$
(*c*) $5 \sin 60° - 2 \sec 45° + 3 \tan 30°$
(*d*) $3 \sec 30° - 8 \cos 60° + 5 \cot 45°$
(*e*) $(x + y)^2 \sin 30° - (x - y)^2 \tan 60° - \sqrt{3}\, xy \tan 45°$

(11) Compute the functions of 45° from an isosceles right triangle whose equal sides are 5.

(12) Compute the functions of 30° and 60° from an equilateral triangle whose equal sides are 8.

(13) How long a shadow is cast by a chimney 100 ft. high when the sun is 30° above the horizon?

(14) A captive balloon has 3,000 ft. of stretched cable out. If the cable makes an angle of 45° with the horizontal, find the height of the balloon.

Pages 296-298

PROBLEMS:

(15) Find the distances between the following pairs of points:

(*a*) (3, 5), (7, 8) (*d*) (−1, 7), (3, −2)
(*b*) (−2, 1), (2, 6) (*e*) (6, 4), (2, −1)
(*c*) (−3, 2), (5, −1) (*f*) (−2, 0), (− 6, −3)

(16) The vertices of a triangle are (−3, −2), (−1, 5), and (4, −1). Find the area of the triangle by means of the formula given in Problem 3 of this chapter on page 641.

(17) Find the area of the triangle whose vertices are (−2, 3), (6, 5), and (3, −2). Use the formula given in Problem 3 of this chapter on page 641.

(18) Show that the following points are the vertices of an isosceles triangle: (*a*) (−2, 2), (3, 7), and (3, −3); (*b*) (−3, 4), (4, 5), and (2, −1).

(19) Show that the following points are the vertices of a right triangle: (*a*) (1, 3), (−1, 2), and (2, 1); (*b*) (2, 1), (4, 7), and (5, 0).

(20) Find the center of the circle passing through the three points (3, −6), (5, −4), and (1, 2). [*Hint:* By definition, these three points on the circumference must be equally distant from the center. Let (x, y) represent the coordinates of the center; obtain the distance of each point from (x, y) and form two equations. Solve these equations for x and y.]

Pages 298-301

PROBLEMS:

(21) Find the equations of the lines through the given point that are parallel and perpendicular, respectively, to the given line:

(*a*) (2, 1), $5y + 6x = 30$ (*d*) (−2, −2), $y - 2 = 0$
(*b*) (−1, 2), $2x - y + 2 = 0$ (*e*) (2, 3), $2y + 3x + 6 = 0$
(*c*) (3, −1), $x - y = 2$ (*f*) (2, −3), $x + 5 = 0$

(22) Three vertices of a parallelogram are (2, 1), (5, 5), and (11, 5). Find the fourth vertex.

(23) Show that the following points are the vertices of a right triangle: (1, 3), (6, 5), and (5, −7).

(24) The equations of the sides of a quadrilateral are: $3y - 4x = 12$, $4y + 3x = 16$, $3y - 4x - 16 = 0$, and $4y + 3x + 36 = 0$. What kind of a quadrilateral is it?

(25) Show that the condition that the lines $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$ are perpendicular is $AA_1 + BB_1 = 0$.

(26) Show that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, and that $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

(27) Using the trigonometric identities given in Problem 9 of this chapter on page 642 and other identities given in this chapter and in the preceding problem, prove:

(*a*) $\sin^2\theta\,(1 + \cot^2\theta) = 1$
(*b*) $\cot^2\theta\,(\sec^2\theta - 1) = 1$
(*c*) $\sin\theta\cot\theta\sec\theta = 1$
(*d*) $\sin\theta\cos\theta\,(\tan\theta + \cot\theta) = 1$
(*e*) $\cos\theta + \tan\theta\sin\theta = \sec\theta$
(*f*) $(\sec\theta - 1)(1 + \cos\theta) = \tan\theta\sin\theta$
(*g*) $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$
(*h*) $\tan^2\theta\csc^2\theta = 1 + \tan^2\theta$

(*i*) $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$

(*j*) $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\csc\theta$

Pages 301-302

PROBLEMS:

(28) Find the coordinates of the midpoints of the line segments from:
(*a*) (−2, 1) to (5, 3) (*c*) (−4, 2) to (3, 1)
(*b*) (1, 4) to (5, −3) (*d*) (−6, 3) to (2, 3)

(29) Find the coordinates of the midpoints of the sides and the lengths of the medians of the triangle whose vertices are: (1, 3), (3, −1), and (−1, −3).

(30) The center of a circle is (2, 1) and a point on its circumference is (4, 5). Find the other end of the diameter through this point.

Pages 302-304

PROBLEMS:

(31) Find the equation of the locus of a point which moves so that it is always equidistant from the points: (*a*) (2, 3) and (−5, 1); (*b*) (−3, 4) and (1, −6).

(32) Find the equation of the locus of a point which moves so that: (*a*) its distance from (6, 0) is always twice its distance from (2, 0); (*b*) its distance from (0, 12) is always twice its distance from (0, 4).

(33) A point moves so that its distance from (−2, 3) is always equal to 4. Find the equation of its locus.

(34) Find the equation of the locus of a point which moves so that it is always equidistant from the line $x + 3 = 0$ and the point (3, 0).

(35) A point moves so that it is always equidistant from the line $y - 5 = 0$ and the point (2, 1). Find the equation of its locus.

(36) A point moves so that its distance from the point (9, 0) is three times its distance from the line $x = 3$. Find the equation of its locus.

(37) A point moves so that its distance from the point (0, 4) is one-half its distance from the line $y = 9$. Find the equation of its locus.

(38) A point moves so that the sum of its distances from (3, 0) and (−3, 0) is always equal to 5. Find the equation of its locus.

(39) Two vertices of a right triangle are (−2, 1) and (2, 6), with the right angle at (2, 6). Find the equation of the locus of the third vertex.

(40) The base of a triangle joins (−1, 4) and (3, 1), and the altitude is $3\frac{1}{2}$. Find the equation of the locus of the third vertex.

Pages 304-309

PROBLEMS:

(41) Prove that the bisectors of the angles of a triangle meet in a point which is equidistant from the three sides. (Use deductive reasoning.)

(42) Prove that the medians of a triangle meet in a point which is two-thirds of the distance from each vertex to the midpoint of the opposite side. (Use the analytic method.)

(43) Prove that the perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the three vertices. (Use the analytic method.)

(44) Prove that the altitudes of a triangle meet in a point. (Use the analytic method.)

(45) Prove that the bisectors of two exterior angles and the bisector of the third interior angle of a triangle meet in a point. (Use deductive reasoning.)

(46) Two forts are placed 20 miles apart on a straight shoreline. Each fort has a gun with a range of 16 miles. Find, by analysis, the point nearest to the shore at which an enemy ship could be placed out of the range of the guns.

Prove the facts in Problems 47-63 analytically:

(47) The diagonals of a rectangle are equal.

(48) The line segment joining the midpoints of two sides of a triangle is parallel to the base and equal in length to one-half the third side.

(49) The diagonals of a rhombus are perpendicular to each other.

(50) A quadrilateral whose diagonals bisect each other at right angles is a rhombus.

(51) An isosceles triangle has two equal medians.

(52) A triangle having two equal medians is isosceles.

(53) If the diagonals of a rectangle are perpendicular to each other, the rectangle is a square.

(54) The diagonals of a square are perpendicular to each other.

(55) The distance between the midpoints of the non-parallel sides of a trapezoid is equal to one-half the sum of the parallel sides.

(56) The line segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

(57) A quadrilateral whose diagonals bisect each other is a parallelogram.

(58) The line segments joining the midpoints of adjacent sides of a quadrilateral form a parallelogram.

(59) The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

(60) If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

(61) The line segments joining the midpoints of the adjacent sides of any rectangle form a rhombus.

(62) The diagonals of an isosceles trapezoid are equal.

(63) A trapezoid whose diagonals are equal is isosceles.

CHAPTER 20

Pages 311-315

EXERCISE A. Write the equations of the following circles:

(1) Center (2, 1), radius 5.
(2) Center (−3, −5), radius 3.
(3) Center (4, −2), radius 6.
(4) Center (−6, 2), radius 4.
(5) Center (1, 0), radius 7.
(6) Center (0, 5), radius 2.

Pages 311-315

EXERCISE B. Write the equation of a circle with:

(1) Center at the origin and radius 3.
(2) Center (3, −1) and passing through the origin.
(3) Center (0, 4) and passing through (−3, 2).
(4) Center on the X-axis and radius 4.
(5) Center on the Y-axis and radius a.
(6) Center on the X-axis, radius a, and passing through the origin.

Pages 311-315

EXERCISE C. Locate the center, find the radius, and draw the graph of the following circles:

(1) $x^2 + y^2 - 6x + 4y = 3$

(2) $x^2 + y^2 + 10x + 8y = 8$
(3) $x^2 + y^2 - 6y = 0$
(4) $x^2 + y^2 - 6x + 20 = 0$
(5) $9x^2 + 9y^2 + 18x - 72y + 128 = 0$
(6) $36x^2 + 36y^2 - 144x - 120y = 845$
(7) $16x^2 + 16y^2 - 24x = 667$
(8) $100x^2 + 100y^2 - 300x - 120y + 41 = 0$

Pages 311-315

PROBLEMS:

(1) The diameter of a circle meets the circumference at the points $(-1, 4)$ and $(7, -2)$. Find the equation of the circle.

(2) Find the equation of the circle that passes through $(1, 0)$ and is concentric with the circle $x^2 + y^2 + 4x - 8y = 29$.

(3) A point moves so that the ratio of its distances from the points $(a, 0)$ and $(-a, 0)$ is constant and equal to c. Find the equation of its locus and draw its graph for the case $c = 1$.

(4) A point moves so that the square of its distance from a fixed point is numerically equal to its distance from a fixed line. Find the equation of its locus and determine the kind of curve described by the moving point.

(5) A point moves so that the sum of the squares of its distances from the points $(a, 0)$ and $(-a, 0)$ is constant and equal to c^2. Find the equation of its locus and draw its graph for the case $c^2 = 10a^2$.

(6) Prove analytically that a line from the center of a circle bisecting any straight line-segment joining the extremities of an arc is perpendicular to it.

(7) Prove analytically that the length of a perpendicular from any point on the circumference of a circle to a diameter is a mean proportional between the segments into which it divides the diameter.

Pages 315-317

EXERCISE D. Find the equation of the circle passing through the following points:

(1) $(3, 4), (-4, 3), (2, 3)$
(2) $(-4, 2), (6, 1), (1, 1)$
(3) $(3, 1), (1, 5), (-2, 2)$
(4) $(1, -2), (2, 3), (3, -2)$
(5) $(3, 2), (2, 3), (4, -5)$
(6) $(2, -1), (-3, 1), (5, 2)$
(7) $(-3, 2), (7, 3), (-11, 1)$
(8) $(-1, -2), (2, -3), (3, 1)$
(9) $(1, -3), (3, 4), (4, -1)$
(10) $(-6, -2), (3, 3), (-7, 5)$

Pages 315-317

PROBLEMS:

Find the equations of the following circles:

(8) Passing through the origin, radius 10, and abscissa of center 8.
(9) Passing through the origin, radius 13, and ordinate of center 5.
(10) Having the line joining $(8, 5)$ and $(0, -3)$ as diameter.

Hint: In Problems 11, 12, 13, and 14, use the fact stated in Problem 20, Chapter 19, page 643.

(11) Passing through $(1, 4)$ and $(5, 2)$ and having its center on the line $y - 2x + 3 = 0$.

(12) Passing through $(-8, 2)$ and $(-2, 6)$ and having its center on the line $3y + x = 27$.

(13) Passing through $(-5, 2)$ and $(-4, -5)$ and having a radius of 5.

(14) Passing through $(7, -14)$ and $(-10, -7)$ and having a radius of 13.

(15) Find the equation of the diameter with slope $\frac{1}{2}$ of the circle $x^2 + y^2 + 8x + 4y = 29$.

(16) A military airfield is to be built at the same distance from Atlanta, Ga., Birmingham, Ala., and Chattanooga, Tenn. Atlanta is 140 miles from Birmingham, Chattanooga is 135 miles from Birmingham, and Atlanta is 99 miles from Chattanooga. (*a*) Find, by an accurate construction, at what distance from these cities the airfield should be located. (*b*) Choosing the point at which the airfield should be located as the origin, find the equation of the circle passing through the three cities.

(17) A portion of the circular rim of a driving wheel was brought to a foundry. The chord subtended by the portion of the circular rim was 18 inches and the perpendicular distance from the midpoint of this chord to the rim was 4 inches. Find, by an accurate construction, the diameter of the driving wheel. Check your result by computation.

Pages 317-322

PROBLEMS:

(18) Prove that if the perpendiculars drawn from a point in the circumference upon two radii are equal, the point bisects the arc intercepted by the radii.

(19) A chord is drawn from an extremity of the diameter of a circle. Prove that a radius parallel to the chord bisects the arc intercepted by the diameter and the chord.

(20) Prove that if two equal circles intersect, the arcs between the points of intersection are equal.

(21) The extremities of two diameters of a circle are joined. Prove that the figure thus formed is a parallelogram.

(22) Prove that two diameters perpendicular to each other divide a circle into four equal parts.

(23) Prove that if a circle is described on the radius of another circle as a diameter, any chord of the greater circle passing through the point of contact of the circles is bisected by the smaller circle.

(24) Prove analytically that if two diameters of a circle are perpendicular to each other and have their ends connected, the figure thus formed is a square.

(25) Prove that the perpendiculars drawn from the midpoint of an arc of a circle intercepted by two radii to these radii are equal.

(26) The floor of a bridge spanning a stream has the shape of an arc of a circle having a radius of 40 ft. The guard rail is also in the shape of an arc of a circle, concentric with the circle whose arc forms the floor of the bridge, and having a radius of 43 ft. If the span of the bridge is 30 ft., find the height of the end posts of the guard rail.

(27) The wheel of a railroad car, 33 inches in diameter, had a piece chipped off causing a portion of the circular surface to become flat. The flat part of the wheel measured 2 inches in length. To what diameter must the wheel be ground to return it to a circular shape?

(28) Prove that if a diameter bisects a chord, it is perpendicular to it.

(29) Prove that the line connecting the midpoints of two parallel chords passes through the center.

(30) Prove that all the diameters of a circle are equal.

(31) Prove that if through any point in a radius two chords are drawn making equal angles with the radius, these chords are equal.

(32) Prove that chords drawn from the extremities of a diameter, making equal angles with the diameter, are equal.

(33) The diameter of a circle is perpendicular to a chord. Prove that the line segments drawn from one end of the diameter to the extremities of the chord are equidistant from the center.

Pages 322-326

PROBLEMS:

(34) Prove that a chord is parallel to the tangent drawn at the midpoint of its arc.

(35) Prove that the bisector of the angle formed by two tangents drawn to a circle from an external point passes through the center of the circle.

(36) Two circles touch each other externally at a point P. A common external tangent* touches the circles at M and N. Prove that angle MPN is a right angle.

(37) The sides of a quadrilateral are all tangents to a circle. Prove that the sum of two opposite sides of the quadrilateral is equal to the sum of the other two sides.

(38) Three circles touch each other externally. Prove that their three common internal tangents† meet in a point equidistant from the points of contact.

(39) Two circles are tangent externally. Prove that the common internal tangent bisects the common external tangent.

(40) The sides of a trapezoid are all chords of a circle. (*a*) Prove that the base angles are equal. (*b*) Prove that the trapezoid is isosceles.

(41) A pipe, running along a floor and against a wall, is to be boarded over by placing a board lengthwise against the pipe, touching the wall and floor. The diameter of the pipe is 12 inches. Find the width of the board.

(42) Circular railroad curves are sometimes laid out by the method of "offsets from the tangent" as follows: Assume the horizontal line AB is tangent to a circular curve at A. A distance d is measured along the tangent from the beginning of the curve at A to some point C. The offset is the distance from C (measured along a perpendicular to AB) to the curve. Show that if r is the radius of the curve, then the offset is $r - \sqrt{r^2 - d^2}$.

CHAPTER 21

Pages 328-336

PROBLEMS:

(1) The vertex of an isosceles triangle, constructed with the diameter of a circle as the base, is outside the circle. Prove that the exterior segments of the sides of the triangle are equal.

(2) An angle ABC, inscribed in a circle, is bisected by a chord BD. From D another chord DE is drawn parallel to AB. Prove that BC is equal to DE.

(3) Two chords AB and CD intersect on a point P. Prove that the triangles APC and BPD are similar.

(4) From a point, P, on the circumference of a circle, a chord PB and a diameter PC are drawn. Prove that the radius that bisects the arc BC is parallel to PB.

(5) Prove that the sum of the number of degrees in an angle formed by two tangents and the number of degrees in the smaller arc intercepted is equal to a straight angle.

(6) Prove that if the diameters of two intersecting circles be drawn from one point of intersection, the line segment joining the extremities of the diameters passes through the other point of intersection.

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* A tangent which does not cut the line segment joining the centers of two circles is a common external tangent.

† A tangent which cuts the line segment joining the centers of two circles is a common internal tangent.

(7) Prove that if the sides of a parallelogram are all chords of a circle, the parallelogram is a rectangle.

(8) Two circles are tangent externally. Two line segments are drawn through the point of contact meeting the circumferences of the circles. Prove that the triangles formed by connecting the extremities of the line segments are similar.

(9) The sides of an isosceles triangle are all chords of a circle. Prove that the tangent to the circle at the vertex of the triangle is parallel to its base.

(10) Two secants are drawn through the point of contact of two tangent circles. Connect the points in which the secants cut the circles. Prove that the figure contains two parallel lines.

(11) Two circles are tangent to each other internally. Prove that if chords be drawn from the point of tangency, the line segments joining the extremities of the chords are parallel.

(12) The sides of an equilateral triangle are all chords of a circle. Prove that the tangents drawn through the vertices form another equilateral triangle.

(13) Two tangents to a circle form an angle of 46°. Find the intercepted arcs.

(14) The smaller arc intercepted by two secants which meet at an angle of 32° is 83°. Find the number of degrees in the greater intercepted arc.

(15) AB and AC are chords of a circle on opposite sides of the center. Prove that the triangle formed by connecting the midpoints of the arcs AB and CA is isosceles.

(16) To lay out a line perpendicular to another given line, surveyors sometimes use the following method: Assume that line segment AB is given. Select any convenient point C, not on AB, at any distance, say 50 ft., from B. With one end of the 50-ft. tape at C, swing the other end to intersect line segment AB, extended if necessary, at D. Holding the end of the tape at C, swing the other end of the tape to locate another point E in line with CD. Prove that BE is the required perpendicular.

(17) The position of a ship off shore may be determined as follows: Let A, B, and C represent three stations located along a straight line on the shore so that B is between A and C. Denote the position of the ship by S. Angles BSC and BSA are observed from the ship. The locations of A, B, and C are obtained from the chart. The position of the ship S is then obtained by the intersection of two circles passing through A, B, S and B, C, S, respectively. To get the two circles it is, of course, necessary to determine their centers. Find the centers of the circles if angle BSC is 50° and angle BSA is 80°.

(18) If in Figure 140, page 333, a tangent to the curve is drawn at D intersecting CV at T, angle VTD is called the "intersection angle." (*a*) Show that the intersection angle equals twice the sum of the "deflection angles." (*b*) As shown in Chapter 21, page 333, the degree of curve is determined by the central angle subtended by a chord of 100 ft. If the intersection angle is 67° 12′ and the degree of curve is 4°, find the length of the curve.

(19) In railroad surveying it is sometimes desirable to make the degree of curvature less at the beginning of a curve than at its end. For this purpose a "compound curve," made up of two or more arcs of circles of unequal radii having a common tangent at their point of junction, is used. Assume that a compound curve, ABC, is made up of two arcs, AB and BC, of circles having OA and CD as radii, respectively, and so that OA is greater than CD. Draw tangents at A and C intersecting at T. Show that the intersection angle at T (see Problem 18 above) equals the sum of the central angles of arcs AB and BC. (*Hint:* Draw the common tangent at B and use the fact stated in Problem 5, page 648.)

Pages 336-341

PROBLEMS:

(20) Inscribe a circle in a given triangle.

(21) Using the formula given in Problem 3, Chapter 19, page 641, show that the radius, r, of an inscribed circle is equal to:

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

(22) At a given point on a given circle draw a tangent to the circle.

(23) Through a given point outside a given circle draw tangents to the circle.

(24) Construct a line tangent to a given circle and parallel to a given line.

(25) Construct a line tangent to a given circle and perpendicular to a given line.

(26) Prove that if two opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.

(27) Construct a common tangent to two given circles.

(28) The driving wheel of a locomotive is 72 inches in diameter and the trailer wheel is 36 inches in diameter. The distance from center to center of the axles of these wheels is 160 inches. Find, by construction, the distance between the two points where the wheels touch the rails. Check by calculating the answer.

(29) The cross section of a steel rod has the shape of an equilateral triangle 8 inches on a side. The rod is to be machined so that it will have a cylindrical section. Find, by construction, the largest diameter to which the section can be turned. Check by calculating the answer.

(30) The design of a frieze consists of three circles placed tangent to each other, in order of size, so that their centers lie on the same straight line and a line tangent to the largest and smallest is also tangent to the middle circle. The diameter of the smallest circle is 1 inch and that of the middle circle is $1\frac{1}{2}$ inches. Find, by construction, the diameter of the largest circle.

(31) In making a die for punching pieces, an equilateral triangle is to be cut out from a rectangular steel plate. The mechanic marks the outline of the triangle on the plate. To remove most of the metal by drilling, a circle is inscribed in the triangle and three additional smaller circles each tangent to the sides of the triangle and to the inscribed circle are marked on the steel plate. The side of the triangle is 5 inches. Find the diameters of the circles, that is, the diameters of the drills to be used.

Pages 341-345

PROBLEMS:

(32) AD is the diameter of a circle, and A, B, C, and D are points taken in succession on its circumference. Prove that chord AC is greater than chord AB.

(33) An obtuse triangle, a right triangle, and an acute triangle are inscribed in the same circle. Which triangle has the longest side? Prove it.

(34) The sides of a trapezoid are all chords of a circle. Prove that the bases are unequally distant from the center.

(35) An obtuse triangle, ABC, is inscribed in a circle O. Construct a line segment, OM, perpendicular to side AC opposite the obtuse angle, and draw another line segment, ON, perpendicular to side AB. Prove that ON is greater than OM.

(36) Two chords intersect at the midpoint of one of them, but not of the other. Prove that the chords are unequal.

(37) Two inscribed triangles have their vertices at the center of the circle. The altitude of the first is twice that of the second. Prove that the second triangle has the greater base.

(38) Prove that the shortest chord that can be drawn through any point, not the center, within a circle is perpendicular to the radius through that point.

(39) Prove that if two circles are tangent externally, the portion of their

common external tangent included between the points of tangency is a mean proportional between their diameters.

(40) Prove that tangents to two intersecting circles, drawn from any point in the common chord produced, are equal.

(41) The distance from the center of a chord 2 ft. long to the center of its arc is 9 inches. Find the radius of the circle.

(42) Find the length of a tangent from a point 4 inches from a circle whose diameter is $40\frac{1}{2}$ ft.

(43) The radius of a circle is 12 inches. A tangent to the circle from an external point is 9 inches long. Find the distance from the center of the circle to the external point.

(44) Two intersecting railroad tracks are to be connected by a circular track, as in Figure 129. The distance from the point of intersection to the point where the curve joins the tangent track is 130 ft., while the distance between the points of tangency is 240 ft. Find the radius of the circular track.

(45) The line segment joining the centers of two circles is intersected by their common internal tangent. If the line segment joining the centers is 14 inches, and the radii are 4 and 3 inches, respectively, find the segments of the line of centers made by the common internal tangent.

(46) The line segment joining the centers of two circles is 26 inches; their radii are 4 inches and 14 inches, respectively, and the portion of their common external tangent included between the points of tangency is 24 inches. Extend the line segment joining the centers until it meets the common external tangent. (*a*) Find the distance from the point of tangency of the smaller circle to the point where the line of centers and the tangent meet. (*b*) Find the distance from the center of the larger circle to the point where the line of centers and the tangent meet.

CHAPTER 22

Pages 347-352

PROBLEMS:

(1) Prove that if two diagonals of a regular pentagon intersect, the larger segment of either equals one side of the pentagon.

(2) Prove that an equiangular polygon circumscribed about a circle is a regular polygon.

(3) Prove that the angles at the center of any regular polygon are equal, and that each is supplementary to an interior angle of the polygon.

(4) Prove that the radius drawn to any vertex of an inscribed regular polygon bisects the angle at that vertex.

(5) Prove that the side of a circumscribed equilateral triangle is equal to twice the side of the inscribed equilateral triangle.

(6) Show that the ratio of the area of a circumscribed equilateral triangle to the area of an inscribed equilateral triangle is 4 to 1.

(7) Prove that a regular hexagon may be inscribed in a circle as follows: Draw any radius OA. With A as center and radius equal to OA, construct an arc intersecting the circle at B. Draw line segment AB. Then AB is a side of a regular hexagon.

(8) Find the area of a regular hexagon inscribed in a circle of radius r.

(9) Prove that tangents to a circle at the midpoints of the arcs of the sides of a regular inscribed polygon form a regular circumscribed polygon whose sides are parallel to the sides of the inscribed polygon and whose vertices lie on radii produced of the inscribed polygon.

(10) Prove that lines drawn from each vertex of a regular inscribed polygon

to the midpoints of the arcs of adjacent sides of the polygon form a regular inscribed polygon of double the number of sides.

(11) A cylindrical piece of iron is to be cut so that one end is a regular hexagon whose side is $\frac{1}{2}$ inch. Find the size of the diameter of the smallest cylindrical piece that may be used.

(12) Prove that the area of an inscribed regular hexagon is the mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

Pages 352-357

PROBLEMS:

(13) Find the sum of the interior angles of (*a*) a pentagon; (*b*) a hexagon; (*c*) an octagon; (*d*) a decagon.

(14) Find one angle of a regular (*a*) pentagon; (*b*) hexagon; (*c*) octagon; (*d*) decagon.

(15) A regular polygon has an exterior angle equal to 18°. How many sides has it?

(16) Find the number of sides of a polygon if the sum of its interior angles (*a*) equals twice the sum of its exterior angles; (*b*) equals three times the sum of its exterior angles.

(17) How many sides has a regular polygon each of whose interior angles equals five times the adjacent exterior angle?

(18) Prove that the areas of two regular polygons of the same number of sides are to each other as the squares on any two corresponding sides.

(19) A silo to preserve fodder has the shape of a regular polygon one side of which is 6 ft. Another silo, having a regular polygon of the same number of sides for its cross section, has a side equal to 8 ft. Find the ratio between the areas of their cross sections.

(20) Prove that the area of a square constructed on a diameter of a circle is twice the area of the inscribed square.

(21) Find the area of a regular triangle inscribed in a circle whose radius is r.

(22) Prove that the area of a polygon circumscribed about a circle is equal to one-half of the product of the perimeter of the polygon and the radius of the circle.

(23) Two logs are 8 in. and 10 in., respectively, in diameter. Find the ratio between the area of the cross section of the largest hexagonal beam that can be cut from the first log and that of the largest hexagonal beam that can be cut from the second log.

Pages 357-364

PROBLEMS:

(24) The diameters of two wheels are 8 and 10 inches, respectively. Find the ratio of their circumferences.

(25) The radius of a circle is 6 inches. Find the difference between the areas of an inscribed regular hexagon and an inscribed regular triangle.

(26) A man traveling on the earth's surface along the equator goes around the world. A plane flying at a constant altitude of 4000 feet undertakes the same trip. If the radius of the earth is approximately 3960 miles, how much farther would the plane have to travel than the man?

(27) If the diameter of an auto wheel is 28 inches, find the number of revolutions per minute made by the wheel when the auto is going 70 miles per hour.

(28) A stained glass window has the form of a square, each side of which is 14 inches long, with four semicircles each having a side of the square as its diameter. How many square inches of glass are there in this window?

(29) A circular flower bed 20 ft. in diameter is surrounded by a walk 5 ft. wide. How many square feet are there in the surface of the walk?

(30) The sides of a triangle are 8, 21, and 25 feet. Find the radius of the circle whose area is equal to the area of this triangle. (*Hint:* Use the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, given in Chapter 19, Problem 3, page 641.)

(31) The vertices of a triangle are the three points (0, 0), (8, 0), and (8, 8). What is the area of the circle passing through these three points?

(32) A semicircular field is enclosed by $82\frac{1}{2}$ rods of fencing. What is the area of this field?

(33) A circular mine field is to be divided into two equal parts by means of a concentric circle. Show that the diameters of the two circles are approximately in the ratio of 7 to 5.

(34) A circular inch is defined as the area of a circle one inch in diameter and equals 0.7854 sq. in. Compute in circular inches the area of the end of a steel cable 4 inches in diameter.

(35) Show that if n = number of sides of a regular polygon, a = length of a side of the regular polygon, R = radius of the circumscribed circle, r = radius of the inscribed circle, and A = area of the polygon, then,

$$a = 2R \sin \frac{180°}{n} = 2r \tan \frac{180°}{n};$$

$$R = r \sec \frac{180°}{n} = \frac{a}{2} \csc \frac{180°}{n};$$

$$r = R \cos \frac{180°}{n} = \frac{a}{2} \cot \frac{180°}{n};$$

$$A = \frac{na^2}{4} \cot \frac{180°}{n} = \frac{naR}{2} \cos \frac{180°}{n} = nr^2 \tan \frac{180°}{n}.$$

(36) Using the formulas given in the preceding problem, find R, r, and A for an equilateral triangle measuring 6 inches on a side.

Pages 364-371

PROBLEMS:

(37) Two tangents to a circle with a radius of 11 inches form an angle of 80°. Find the length of the minor arc intercepted.

(38) The area of a sector is 76 π and the angle of the sector is 70°. Find the radius of the circle.

(39) In a circle whose radius is 30, what is the length of an arc intercepted by a central angle of 40°?

(40) A circle has two perpendicular radii OM and ON. Draw the chord MN. With this chord as diameter, draw a second circle. Prove that the area of the triangle ONM is equal to the area of that part of the second circle which lies outside the first circle.

(41) In the design of a floor mosaic, there are three circles, each one tangent to the other two, and each with a radius of 15 inches. Compute the area enclosed by these three circles.

(42) A crescent $ABCD$ is bounded by a semicircle, ADC, whose radius is 6 meters and an arc, ABC, whose radius is 12 meters. Find the area of the crescent.

(43) Two cannon are 5 miles apart. Each cannon has a range of 5 miles. Calculate the area exposed to the fire of both cannon.

(44) Show that one degree of arc of any circle of latitude is equal in length

to the product of the length of one degree along the equator and the cosine of the latitude. (This fact was assumed and used in Chapter 14, page 225.)

(45) A degree of arc at the equator is 68.704 miles. The latitude and longitude of Columbus, O., are approximately 40° N. and 83° W., while those of Philadelphia, Pa., are approximately 40° N. and 75° 17′ W. Find the air distance between Columbus and Philadelphia along the 40th parallel of latitude. (*Hint:* Use fact stated in Problem 44.)

(46) The latitude and longitude of New Orleans, La., are approximately 30° N. and 90° W., while those of Cairo, Egypt, are approximately 30° N. and 31° 15′ E. Find the air distance between New Orleans and Cairo along the 30th parallel of latitude. (*Hint:* Use fact stated in Problem 44.)

CHAPTER 23

Pages 374-377

EXERCISE A. Find the vertices and foci, and sketch the curve:

(1) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

(2) $25x^2 + 9y^2 = 225$

(3) $x^2 + 4y^2 = 9$

(4) $\frac{x^2}{4} + \frac{y^2}{9} = 1$

(5) $\frac{x^2}{36} + \frac{y^2}{4} = 1$

(6) $25x^2 + 16y^2 = 400$

(7) $9x^2 + 64y^2 = 576$

(8) $\frac{x^2}{25} + \frac{y^2}{169} = 1$

Pages 374-377

EXERCISE B. Find the equation and sketch the curve of the ellipse with center at the origin, having given:

(1) $a = 2$ and $b = 5$
(2) $b = 7$ and $c = 5$
(3) $a = 13$ and $c = 12$
(4) Vertices $(\pm 7, 0)$, foci $(\pm 2\sqrt{6}, 0)$
(5) Vertices $(\pm 5, 0)$ and $(0, \pm 3)$
(6) Major axis 7, distance between foci 5
(7) Passing through $(-4, 0)$ and $(2, -6)$
(8) Passing through $(-6, 2)$ and $(4, -3)$

Pages 374-377

PROBLEMS:

(1) Find the distance from each of the foci of the ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$ to a point on the ellipse whose abscissa is 5.

(2) The minor axis of an ellipse is 4, and the distance from the origin to a focus is $\frac{1}{6}$ its semimajor axis. Find the equation of the ellipse.

(3) The sum of two sides of a triangle is 18 and its base is 14. Find the equation of the locus of the vertex of the triangle.

(4) Find the equation of a circle that passes through four points on the ellipse $\frac{x^2}{49} + \frac{y^2}{24} = 1$ which are at the same distance from the Y-axis as the foci.

(5) Show that the length of the line segment from a focus of an ellipse to an end of the minor axis is equal to the semimajor axis.

(6) Find the equation of the locus of the midpoints of the ordinates of the points on the circle $x^2 + y^2 = r^2$.

Pages 378-382

PROBLEMS:

(7) Construct an ellipse whose foci are 3 inches apart and whose major axis measures 10 inches.

(8) Construct an ellipse whose foci are 24 units apart and whose minor axis measures 10 units.

(9) Construct an ellipse whose major and minor axes are 10 and 6 units, respectively.

(10) A contractor is to lay out an elliptical pool 15 feet long by 9 feet wide (*a*) Find the distance between two stakes driven in the ground so that the total length of a loop of cord, after tying the ends together, would be just sufficient to mark the outline of the elliptical pool. (*b*) Find the total length of the cord.

(11) Write the equation of an ellipse with center at the origin, major axis on the X-axis, and having (*a*) its minor axis 6 units long and eccentricity of $\frac{2}{3}$; (*b*) foci $(\pm 5, 0)$ and eccentricity of $\frac{3}{5}$.

(12) Find the equation of an ellipse with center at the origin, major axis on the Y-axis, eccentricity $\frac{4}{5}$, and passing through the point (10, 0).

(13) Find the equation of an ellipse with center at the origin, major axis on the X-axis, eccentricity $\frac{1}{2}$, and passing through the point $(-2, 4)$.

(14) A bridge with a semi-elliptical arch is to be constructed. The arch of the bridge is to have a span of 50 ft. and a height of 15 ft. The level roadway is 2.5 ft. above the center of the arch. Find, at 5-ft. intervals, the vertical distance from the roadway to the arch.

(15) The Colosseum or Flavian amphitheatre in Rome, inaugurated by Titus in A. D. 80, has an elliptical plan, with its major axis 615 ft. long and its minor axis 510 ft. long. Its arena, also elliptical in form, is 281 ft. long and 177 ft. wide. Find the equations of the two ellipses, locate their foci, and sketch the curves.

(16) Sound travels 1,090 ft. per second, approximately. The cross section of a large hall is elliptical in shape. If the hall is 3,000 ft. long by 1,800 ft. wide, how much sooner will sound travel from one focus to the other directly than it will if reflected from the walls? (*Hint:* See Chapter 23, page 403, Problem 1.)

(17) Mercury's orbit is an ellipse with the sun at one of its foci. The eccentricity of the ellipse is approximately $\frac{1}{5}$ and the distance from the center of the ellipse to the sun is 7.5 million miles. Find the equation of its orbit, locate the foci, and sketch the curve.

(18) Find the equation of the locus of a point that moves so that its distance from the point $(-1, 0)$ is one-half of its distance from the line $x = 6$.

Pages 382-388

EXERCISE C. Locate the center, vertices, and foci, draw the asymptotes, and sketch the curve of the following hyperbolas:

(1) $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(2) $4y^2 - 5x^2 = 20$

(3) $\frac{y^2}{144} - \frac{x^2}{25} = 1$

(4) $6x^2 - 3y^2 = 18$

(5) $\frac{y^2}{4} - \frac{x^2}{36} = 1$

(6) $x^2 - 16y^2 = 16$

(7) $\frac{x^2}{36} - \frac{y^2}{100} = 1$

(8) $2y^2 - 4x^2 = 8$

Pages 382-388

EXERCISE D. Find the equation, draw the asymptotes, and sketch the curve of the hyperbola with center at the origin, having given:

(1) $a = 2$, $b = 5$, and transverse axis on the Y-axis.
(2) $a = 4$, $b = 7$, and conjugate axis on the Y-axis.
(3) $a = 3$, $b = 1$, and transverse axis on the X-axis.
(4) $a = 9$, $b = 6$, and conjugate axis on the X-axis.
(5) Distance between foci 26; transverse axis on the Y-axis and equal to 24.
(6) Passing through $(-1, 6)$, $(4, -3)$ and transverse axis on the X-axis.
(7) Passing through $(0, 8)$, $a = 6$, and transverse axis on the Y-axis.
(8) Distance between foci 14, $a = 4\frac{1}{3}$, and transverse axis on the X-axis.

Pages 382-388

PROBLEMS:

(19) Find the equation of the locus of a point that moves so that the difference of its distances from the points $(0, 4)$ and $(0, -4)$ is 4.

(20) Find the equation of the locus of a point that moves so that its distance from the point $(0, -4)$ is twice its distance from the line $y = 2$.

(21) Find the equation of the locus of the center of a circle tangent to the circle $x^2 + y^2 + 2x = 0$ and passing through $(1, 0)$.

(22) Prove that if $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the equation of a hyperbola, then the difference of the distances of any point on its curve from the foci is equal to $2a$.

(23) Prove analytically that a line parallel to an asymptote of a hyperbola intersects the curve in one and only one point.

Pages 388-391

PROBLEMS:

(24) Construct a hyperbola: (*a*) whose eccentricity is $1\frac{3}{2}$ and whose conjugate axis is 5; (*b*) whose transverse and conjugate axes are 8 and $3\frac{1}{3}$, respectively.

(25) Construct a hyperbola: (*a*) whose foci are $6\frac{1}{4}$ units apart and whose transverse axis is 5; (*b*) whose foci are 10 units apart and whose eccentricity is $\frac{5}{3}$.

(26) Find the equation of a hyperbola with center at the origin, having given: (*a*) eccentricity 2 and transverse axis on the X-axis and equal to 4; (*b*) eccentricity $\frac{4}{3}$ and transverse axis on the Y-axis and equal to 9.

(27) Find the equation of a hyperbola with center at the origin, having given: (*a*) eccentricity $\frac{5}{4}$ and transverse axis on the Y-axis and passing through $(-1\frac{1}{2}, 3\frac{1}{3})$; (*b*) eccentricity $\frac{5}{3}$ and conjugate axis on the Y-axis and passing through $(1\frac{8}{10}, -1)$.

(28) Show that the eccentricity of a rectangular hyperbola is $\sqrt{2}$.

(29) (*a*) Show that if in the equation of a rectangular hyperbola, namely, $x^2 - y^2 = a^2$, the expression $x' \cos 45° - y' \sin 45°$ is substituted for x, and $x' \sin 45° + y' \cos 45°$ is substituted for y, the resulting equation is $x'y' = \frac{a^2}{2}$.

(*b*) Discuss the equation $x'y' = \frac{a^2}{2}$, and show that it represents a rectangular hyperbola whose asymptotes are the coordinate axes.

(30) Prove that in any hyperbola the distance from a focus to an asymptote is equal to its semi-conjugate axis.

(31) Prove that in any hyperbola the distance from the center to the foot of the perpendicular from a focus to an asymptote is equal to its semi-transverse axis.

(32) A bridge with a hyperbolic arch is to be constructed. The arch of the bridge is to have a span of 300 ft. and a height of 75 ft. The level roadway is

30 ft. below the center of the arch. Find, at 50-ft. intervals, the vertical distance from the roadway to the arch.

(33) In order to locate a gun emplacement, three sound instruments, A, B, and C, are placed along a straight line and so that $AB = 2{,}000$ ft. and $BC = 2{,}500$ ft. The report of the cannon is heard $1\frac{1}{4}$ seconds earlier at A than at B, and $4\frac{1}{8\,6}$ seconds later at C than at A. Determine the position of the gun emplacement. (*Hint:* See Chapter 23, page 404, Problems 9 and 10.)

Pages 391-396

EXERCISE E. Find the vertex, focus, and directrix and sketch the curve:

(1) $y^2 = 8x$
(2) $x^2 = 12y$
(3) $y^2 - 10x = 0$
(4) $x^2 = -2y$
(5) $5y^2 + 12x = 0$
(6) $8y - 3x^2 = 0$
(7) $7x^2 + 12y = 0$
(8) $2y^2 - 9x = 0$

Pages 391-396

EXERCISE F. Find the equation and sketch the curve of the parabola with vertex at the origin, having given:

(1) Focus $(-5, 0)$
(2) Axis on X-axis and passing through $(2, -4)$
(3) Axis on Y-axis and passing through $(-8, 4)$
(4) Directrix $2x + 3 = 0$
(5) Directrix $4y - 9 = 0$
(6) Axis on X-axis and directrix passing through $(-3, 7)$
(7) Axis on Y-axis and focus lying on $3x + 5y - 9 = 0$
(8) Axis on X-axis and focus lying on $2y - x + 3 = 0$

Pages 391-396

PROBLEMS:

(34) Find the equation of the locus of a point that moves so that its distance from $(0, -3)$ is always equal to its distance from the line $x - 3 = 0$.

(35) Find the equation of the locus of the center of a circle passing through $(5, 0)$ and tangent to the line $x + 5 = 0$.

(36) Find the equation of the straight line passing through the focus of the parabola $y^2 = 12x$ and making an angle of 45° with the axis of the parabola.

(37) Find the length of a side of an equilateral triangle having one vertex at the origin and inscribed in the parabola $x^2 = 4cy$.

(38) Prove that the lines drawn from the point of intersection of the directrix and the axis of a parabola to the ends of the chord passing through the focus of a parabola, parallel to the directrix and terminated by the curve, are perpendicular to each other.

(39) Construct a parabola: (a) whose focus is $(3, 0)$ and which has its vertex at the origin; (b) whose directrix is $y - 2 = 0$ and which has its vertex at the origin.

(40) Construct a parabola: (a) having its vertex at the origin and $2c = 9$; (b) whose focus is $(5, 1)$ and vertex $(2, 1)$.

(41) A chord passing through the focus of a parabola parallel to the directrix and terminated by the curve is called the *latus rectum*. Show that the length of the latus rectum of a parabola whose vertex is at the origin is equal to $4c$.

(42) Find the equation of the circle having the latus rectum of the parabola $x^2 = 4cy$ as a diameter. (*Hint:* See the preceding problem.)

(43) The *focal radius* of a point on a parabola is its distance from the focus. Show that the focal radius of a point $P_1(x_1, y_1)$ that lies on the parabola $y^2 = 4cx$ is numerically equal to $x_1 + c$.

(44) The towers supporting a suspension bridge are 1,000 ft. apart and rise 90 ft. above the level roadbed. The lowest point of the parabola formed by a suspension cable is 10 ft. above the level roadbed. Find the equation of the curve of the cable.

Pages 396-399

PROBLEMS:

(45) Find the coordinates of the points (5, 7), (−3, 4), (6, 0), and (2, −4) referred to parallel axes through (2, 3).

(46) Find the coordinates of the points (4, 2), (−3, 5), (−6, −5), and (1, −2) referred to parallel axes through (−3, −3).

Pages 396-399

EXERCISE G. Transform the following equations to a standard form by translating the axes. Sketch the curve.

(1) $9x^2 + 25y^2 - 36x - 150y + 36 = 0$
(2) $x^2 + y^2 + 2x - 8y + 1 = 0$
(3) $y^2 - 8x + 6y = 7$
(4) $3x^2 - 6y^2 - 30x - 48y = 3$
(5) $36x^2 + 36y^2 - 180x - 168y + 97 = 0$
(6) $225x^2 + 100y^2 - 1650x + 440y + 2609 = 0$
(7) $4y^2 - 7x^2 - 20y - 70x = 178$
(8) $2x^2 - 4x + 20y + 37 = 0$

Pages 396-399

PROBLEMS:

(47) Find the equation of the locus of a point that moves so that its distance from the point (−3, 2) is always equal to its distance from the line $x = 2$. Translate the axes and sketch the curve.

(48) Find the equation of the locus of a point that moves so that its distance from the point (2, 4) is two-thirds of its distance from the point (2, 9). Translate the axes and sketch the curve.

(49) Find the equation of the locus of a point that moves so that its distance from the point (2, 3) is always twice its distance from the X-axis. Translate the axes and sketch the curve.

(50) Find the equation of the locus of a point that moves so that its distance from the point (−1, 2) is always one-half of its distance from the line $x = 8$. Translate the axes and sketch the curve.

Pages 399-403

EXERCISE H. Locate the center, vertices, and foci, draw the asymptotes, and sketch the curve of the following equations:

(1) $\dfrac{(x+3)^2}{9} + \dfrac{(y+2)^2}{16} = 1$

(2) $(x+2)^2 = 12(y-5)$

(3) $\dfrac{(x-3)^2}{36} - \dfrac{(y+1)^2}{64} = 1$

(4) $(y+3)^2 = 3(x-4)$

(5) $\dfrac{(x-3)^2}{9} + \dfrac{(y-4)^2}{4} = 1$

(6) $x^2 + 18(y-5) = 0$

(7) $\dfrac{(x+3)^2}{9} - \dfrac{y^2}{25} = 1$

(8) $\dfrac{(x+3)^2}{4} + \dfrac{y^2}{25} = 1$

Pages 399-403

EXERCISE I. Write the equations, in typical form, of each of the following curves, having given:

(1) Focus (3, 1); directrix $y = 7$.
(2) Intercepts on major axis (1, 3) and (−11, 3); intercepts on minor axis (−5, −1) and (−5, 7).
(3) Intercepts on transverse axis (1, 0) and (1, −8); foci (1, 1) and (1, −9).
(4) Foci (1, −2) and (−7, −2); eccentricity $\frac{2}{5}$.
(5) Foci (−2, 8) and (−2, −2); eccentricity $\frac{5}{4}$.
(6) Center (2, −1); passing through (6, 2); eccentricity 0.
(7) Vertex (−3, −3); directrix $x = 0$.
(8) Foci (3, −1) and (3, 7); eccentricity $\frac{4}{5}$; tangent to Y-axis

Pages 399-403

EXERCISE J. State, by inspection, the type of curve represented by the following equations:

(1) $x^2 + 3y^2 - 6x + 9y = 10$
(2) $x^2 - 5x + 3y = 9$
(3) $x^2 + 3x + y^2 - 2y = 0$
(4) $3x^2 - 3y^2 + 6y = 5$
(5) $y = x^2 - 3x + 5$
(6) $y^2 + 2y - x^2 + x = 4$
(7) $3x^2 - 6x + 3y^2 = 0$
(8) $2y^2 + 5x^2 - 3x = 15$

Pages 399-403

PROBLEMS:

(51) The Golden Gate Bridge has a channel span of 4,200 ft. Each suspension cable in the channel span forms a parabolic arc whose ends are 475 ft. higher than its lowest point. Assuming a level roadway and the lowest point of the suspension cable to be 10 ft. above the floor, find an equation for the curve, choosing the level roadway as the X-axis.

(52) The following table gives the eccentricity and the semimajor axis of the ellipses forming the orbits of the four inner planets. Find the equation of the orbit of each.

Planet	*Eccentricity*	*Semimajor Axis (Millions of miles)*
Mercury	$\frac{1}{5}$	36
Venus	$\frac{17}{200}$	67.2
Earth	$\frac{1}{60}$	92.9
Mars.....	$\frac{9}{100}$	141.5

(53) The height of an object dropped from a building 728 ft. high, at any time t, is given by the formula $s = -16t^2 + 728$. Locate the vertex and focus and sketch the curve.

(54) The George Washington Bridge has a channel span of 3,500 ft. Each suspension cable in the channel span forms a parabolic arc whose ends are 325 ft. higher than its lowest point. Assuming a level roadway and the lowest point of the suspension cable to be 8 ft. above the floor, find an equation for the curve, choosing the level roadway as the X-axis

CHAPTER 24

Pages 406-408

EXERCISE A. Find the general or nth term of the following series:

(1) $2 + 4 + 6 + 8 + \cdots$
(2) $1 + 4 + 9 + 16 + \cdots$
(3) $2 + 4 + 8 + 16 + \cdots$
(4) $3 + 6 + 9 + 12 + \cdots$
(5) $\frac{2}{3} + \frac{4}{3} + 2 + \frac{8}{3} + \cdots$
(6) $1 + 2 + 4 + 6 + \cdots$
(7) $1 + 3 + 5 + 7 + \cdots$
(8) $2 + 6 + 12 + 20 + \cdots$
(9) $\frac{1}{2} + 1 + 1\frac{1}{2} + 2 + \cdots$
(10) $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \cdots$

Pages 408-411

EXERCISE B. Find the last term and the sum of each of the following arithmetic progressions:

(1) $1 + 2 + 3 + 4 + \cdots$ to 25 terms
(2) $2 + 4 + 6 + 8 + \cdots$ to 15 terms
(3) $1 + 3 + 5 + 7 + \cdots$ to 14 terms
(4) $3 + 6 + 9 + 12 + \cdots$ to 18 terms
(5) $\frac{1}{2} + 1 + 1\frac{1}{2} + 2 + \cdots$ to 12 terms
(6) $-2 - 1 + 0 + 1 + \cdots$ to 17 terms
(7) $-\frac{9}{2} - \frac{7}{5} + \frac{17}{10} + \frac{24}{5} + \cdots$ to 10 terms
(8) $4 - 1 - 6 - 11 - \cdots$ to 14 terms
(9) $5\frac{1}{4} + 4 + 2\frac{3}{4} + 1\frac{1}{2} + \cdots$ to 16 terms
(10) $1\frac{1}{2} - 1 - 3\frac{1}{2} - 6 - \cdots$ to 9 terms
(11) $-\frac{1}{12} + \frac{1}{4} + \frac{7}{12} + \frac{1}{12} + \cdots$ to 11 terms
(12) $9 + 8\frac{1}{3} + 7\frac{2}{3} + 7 + \cdots$ to 13 terms
(13) $-1 - \frac{2}{5} + \frac{1}{5} + \frac{4}{5} + \cdots$ to 15 terms
(14) $-7\frac{1}{4} - 6\frac{1}{2} - 5\frac{3}{4} - \cdots$ to 8 terms
(15) $\sqrt{\frac{1}{3}} + 2\sqrt{\frac{1}{3}} + \sqrt{3} + 4\sqrt{\frac{1}{3}} + \cdots$ to 9 terms
(16) $-\sqrt{\frac{5}{9}} + 2\sqrt{\frac{1}{5}} + \sqrt{5} + 8\sqrt{\frac{1}{5}} + \cdots$ to 12 terms

Pages 408-411

PROBLEMS:

(1) The first term of an arithmetic progression is 62, the last term is 7, and the common difference is -5. Find the number of terms.

(2) What term of the series $3 + 7 + 11 + 15 + \cdots$ is equal to 59?

(3) The last term of an arithmetic progression of 23 terms is 5 and the common difference is $\frac{1}{6}$. Find the first term and the sum.

(4) Find the arithmetic progression of 10 terms whose fifth term is 1 and whose eighth term is 2.

(5) When $d = -4$, $a = 101$, and $l = 1$, find n and S_n.

(6) Insert five numbers between 5 and 7 so that the seven numbers will form an arithmetic progression.

(7) Insert three terms between each two consecutive numbers of the arithmetic progression $1 + 7 + 13$ so that the progression thus formed will be arithmetic.

(8) Insert n numbers between a and b so that the $n + 2$ numbers will form an arithmetic progression.

(9) The sum of an arithmetic progression of six terms is 66. The sum of the squares of these six terms is 1,006. What is the first term and the difference of the arithmetic progression?

(10) The sum of four numbers in arithmetic progression is 20 and their product is 384. Find the numbers.

(11) Two cities proceed to build up a cash surplus against their outstanding bonds. One city sets aside \$270,000 the first year, \$210,000 the second year, and so on, at the same diminishing rate. The second city sets aside \$180,000 each year. In how many years (disregarding interest) will the two funds be equal?

(12) A debt can be discharged in a year by paying \$1 the first week, \$3 the second week, \$5 the third week, and so on. What is the last payment and the total indebtedness?

(13) Two trucks start from the same factory at the same time and travel in the same direction. A goes 300 miles per day. B goes 150 miles the first day and increases the rate of travel thereafter 25 miles each day. How far apart will they be at the end of 9 days, and which will be in advance?

(14) In an endurance test, 100 oranges are placed in a line exactly 2 yards apart, the first orange at 2 yards from a barrel. What distance must an athlete run, starting from the barrel, to gather them up singly and return with each orange to the barrel?

(15) Some beer kegs are tiered in a brewery. One is on top, two directly beneath, three in the third row, and so on. The bottom row contains 21 kegs. How many are there in all?

(16) If \$800 is invested at the simple interest rate of 3 per cent per annum, what will be the amount in $7\frac{1}{2}$ years?

(17) If \$500 is invested at the simple interest rate of 4 per cent per annum, what will be the amount in 5 years and 9 months?

(18) A club deposits its funds in a local bank which agrees to pay 2 per cent simple interest per annum. If the fund is \$1,600, what would be the interest for 11 months?

(19) Geologists have ascertained that within the earth the temperature increases about 0.00032° Centigrade per centimeter of depth. Assuming the surface temperature to be 10° Centigrade, and disregarding the amount of heat being conducted out of the earth, what is the temperature at the bottom of a boring 2,000 meters deep?

(20) A weather expert ascertains that during the first 15 days of August the average daily temperature rises one-half of a degree Centigrade, and during the last 16 days it declines one-fourth of a degree Centigrade. Find the maximum and the average temperature for August, assuming that on the first day of the month the average temperature was 14° Centigrade.

Pages 412-418

EXERCISE C. Find the last term and the sum of the following geometric progressions:

(1) $1 + 3 + 9 + 27 + \cdots$ to 10 terms
(2) $6 + 4 + \frac{8}{3} + \frac{16}{9} + \cdots$ to 8 terms
(3) $\frac{2}{5} - 2 + 10 - 50 + \cdots$ to 6 terms
(4) $20.25 + 13.5 + 9 + 6 + \cdots$ to 9 terms
(5) $25 - 15 + 9 - 5.4 + \cdots$ to 6 terms
(6) $2 + 4 + 8 + 16 + \cdots$ to 12 terms
(7) $-1 + 2 - 4 + 8 - \cdots$ to 11 terms
(8) $15 + 5 + \frac{5}{3} + \frac{5}{9} + \cdots$ to 6 terms
(9) $8 - 4 + 2 - 1 + \cdots$ to 12 terms
(10) $36 - 6 + 1 - \frac{1}{6} + \cdots$ to 6 terms
(11) $18 + 3 + \frac{1}{2} + \frac{1}{12} + \cdots$ to 5 terms
(12) $\frac{1}{27} + \frac{1}{9} + \frac{1}{3} + 1 + \cdots$ to 7 terms
(13) $\sqrt{\frac{1}{8}} + \sqrt{\frac{1}{2}} + \sqrt{2} + \sqrt{8} + \cdots$ to 9 terms
(14) $3 - \sqrt{3} + 1 - \sqrt{\frac{1}{3}} + \cdots$ to 11 terms
(15) $2 + \sqrt{8} + 4 + \sqrt{32} + \cdots$ to 12 terms
(16) $\sqrt{\frac{1}{3}} + \sqrt{2} + 2\sqrt{3} + 6\sqrt{2} + \cdots$ to 8 terms

Pages 412-418

PROBLEMS:

(21) The first and seventh terms of a geometric progression are 64 and 1. Find the progression.

(22) The fourth and eighth terms of a geometric progression are $\frac{9}{25}$ and $\frac{25}{9}$. Find the progression.

(23) The ratio of a geometric progression is 1.5. The sum of four terms is 65. Find the first and last terms.

(24) The sum of a geometric progression is 775, the ratio is 5, and the last term is 625. Find the first term.

(25) The ratio of a geometric progression is 2, the last term is 30, and the sum is 75. Find the first term.

(26) Find four numbers in geometric progression such that the sum of the first two is 12 and the sum of the last two is 108.

(27) The first term of a geometric progression is 10, the last term is $50\frac{5}{8}$, and the sum is $131\frac{7}{8}$. Find the ratio.

(28) Insert three geometric means between 4 and 324.

(29) Insert six terms between a^7 and b^7 so that the eight terms will form a geometric progression.

(30) If 6 is a geometric mean between two numbers whose sum is 13, what are the numbers?

(31) The arithmetic and geometric means of two quantities are A and G, respectively. Prove that the two quantities are given by the formula

$$A \pm \sqrt{(A + G)(A - G)}.$$

(32) John Briggs made annual deposits of \$150 at the end of each year for four years. If he made no withdrawals during this period, and his bank paid 3 per cent per annum compounded annually, how much money would he have at the end of four years?

(33) A ball falls from a height of 75 feet and rebounds 30 feet the first time, 12 feet the second time, and so on. What is the whole distance through which the ball will have passed after rebounding four times?

(34) What amount should now be invested with the Excelsior Trust Co. to obtain an annuity of \$2,400 a year for six years, if the trust company pays 5 per cent per annum compounded annually?

(35) To provide for the college education of a son, R. Shea makes a contract with an insurance company in order to obtain an annuity of \$800 a year for four years, beginning one year after the date of issue. If interest is at 4 per cent per annum compounded annually, how much should R. Shea pay?

(36) R. Burns purchased a house and lot for \$14,000. He paid \$8,000 in cash and agreed to pay the balance, with interest at 4 per cent per annum compounded annually, in five equal annual installments. Find the size of the annual payment.

(37) Ten years after R. Burns had bought the house and lot, he decided to sell them. He received two offers to buy them. The first was an offer of \$9,000 cash and \$6,691.30 to be paid five years later. The second was for \$8,000 cash and \$1,572.20 a year for the next five years. Which of these offers, on the basis of 4 per cent per annum compounded annually, was most advantageous?

Pages 418-424

EXERCISE D. Expand each of the following expressions by the binomial theorem:

(1)	$(a + b)^6$	(5)	$(1 + .04)^6$	(9)	$(6a - 7x)^5$	(13)	$\sqrt{x - y}$
(2)	$(x - n)^8$	(6)	$(1 - .02)^7$	(10)	$(1 - a^{-1})^6$	(14)	$(1 - .015)^{3/2}$
(3)	$(a + x)^7$	(7)	$(3 - 2x)^5$	(11)	$(a + b)^{2/3}$	(15)	$(2\sqrt{2} - a\sqrt{x})^5$
(4)	$(1 + .03)^5$	(8)	$(4x + 3y)^4$	(12)	$\sqrt{(a - b)^3}$	(16)	$(\sqrt[3]{x^2} - \sqrt[4]{a^3})^4$

Pages 418-424

PROBLEMS:

(38) Benjamin Franklin bequeathed \$5,000 to Philadelphia. If kept invested

at 3 per cent compounded annually for 150 years, what sum would have accrued to the city?

(39) By saving 1 cent the first week, 2 cents the second week, 4 cents the third week, and so on, doubling the amount each week, how much would be saved the last week of the year?

(40) A man pays \$250 annually for 20 years on his life insurance policy. If the rate is $3\frac{1}{2}$ per cent compounded annually, to what sum will the money amount?

(41) M. Eckles expects to retire at age 65. What sum of money should he invest at age 35 at 4 per cent compounded annually so that on reaching 65 he will have \$10,000 available?

(42) W. Green holds a contract whereby he is to receive 20 payments of \$100 each every three months. If Mr. Green invests these amounts as soon as he receives them, at 4 per cent per annum compounded quarterly, how much money would he have immediately after receiving the last payment?

(43) J. Roberts, aged 21, is to receive the principal of a trust fund of \$50,000 on reaching age 30. How large a sum of money could Roberts receive every six months, beginning six months from today, with interest at 5 per cent per annum compounded semiannually, so that when he reaches age 30 the principal would be exhausted?

(44) On reaching his 65th birthday, M. Eckles deposits his \$10,000 in a bank paying 5 per cent per annum compounded semiannually. What is the largest sum he can withdraw every six months for the next 10 years, if, in doing so, he consumes part of his capital each six months, and if his capital is exhausted at the end of 10 years?

(45) The Cooper Aircraft Company wishes to accumulate a fund of \$200,000 to expand its research department. The directors wish to have this amount accumulated in 7 years. If the payments toward this fund are made every three months and these deposits are invested in securities yielding 6 per cent per annum compounded quarterly, how large a sum must be invested every three months?

CHAPTER 25

Pages 426-427

EXERCISE A. Using the table of powers of 3 given on page 426, perform each of the following indicated operations:

(1)	$6{,}561 \div 729$	(4)	$3^{13} \div 3^8$	(7)	$9^6 \div 9^3$	(10)	$9^2 + 3^4$
(2)	$2{,}187 \times 27$	(5)	$3^7 \times 243$	(8)	$27^2 \div 3^5$	(11)	$3^{11} - 3^9$
(3)	$(729)^2 \div (243)^2$	(6)	$9^3 \times 9^2$	(9)	$9^5 \div 2{,}187$	(12)	$(9^4 - 9^6) \div 3^8$

Pages 426-427

EXERCISE B. Translate each of the following expressions into logarithmic notation:

(1)	$2^7 = 128$	(6)	$16^{1/2} = 4$
(2)	$5^5 = 3125$	(7)	$125^{1/3} = 5$
(3)	$4^3 = 64$	(8)	$4^{3/2} = 8$
(4)	$7^4 = 2401$	(9)	$10^{1.3979} = 25$
(5)	$10^3 = 1000$	(10)	$10^{-3} = 0.001$

Pages 426-427

EXERCISE C. Translate each of the following expressions into exponential notation:

(1)	$\log_2 512 = 9$	(6)	$\log_{16} 2 = 0.25$
(2)	$\log_5 625 = 4$	(7)	$\log_{25} 5 = 0.5$
(3)	$\log_3 729 = 6$	(8)	$\log_{81} 27 = 0.75$
(4)	$\log_7 343 = 3$	(9)	$\log_{10} 0.01 = -2$
(5)	$\log_4 1024 = 5$	(10)	$\log_{10} 45.5 = 1.6580$

Pages 426-427

EXERCISE D. Find the logarithm of each of the following numbers to the base indicated:

(1) $\log_3 81$
(2) $\log_8 64$
(3) $\log_5 625$
(4) $\log_4 256$
(5) $\log_9 729$
(6) $\log_8 2$
(7) $\log_{81} 9$
(8) $\log_{16} 64$
(9) $\log_{10} 0.001$
(10) $\log_{10} 0.1$

Pages 426-427

PROBLEMS:

(1) If 8 is the base, what is the number whose logarithm is $\frac{2}{3}$?
(2) If 3 is the logarithm of 216, what number was used as the base?
(3) What numbers cannot be used as the base of a system of logarithms? Why?

Pages 427-429

PROBLEMS:

(4) By how much is the logarithm of a number increased if the number is multiplied by 10? By 10,000?
(5) By how much is the logarithm of a number decreased if the number is divided by 100? By 1,000?
(6) The log 37.5 = 1.5740. What is the log 3.75? The log 3750?
(7) The log 8.47 = 0.9279. What is the log 84.7? The log 0.847?
(8) The log 4250 = 3.6284. What is the log $4\frac{1}{4}$? The log $\frac{17}{4}$?

Pages 429-431

EXERCISE E. Give the characteristic of the logarithm of each of the following numbers:

(1) 315
(2) 6
(3) 38
(4) 5179
(5) 73.8
(6) 214
(7) 0.33
(8) 2.09
(9) 0.07
(10) 40.3
(11) 0.0403
(12) 0.00109

Pages 429-431

EXERCISE F. Give the significant figures in each of the following numbers:

(1) 435
(2) 43,500
(3) 0.0304
(4) 20.001
(5) 2,000
(6) 4.03
(7) 0.003
(8) 0.5003

Pages 429-431

EXERCISE G. Indicate which of the following pairs of numbers have the same mantissa:

(1) {256, 0.2056}
(2) {0.0076, 70,600}
(3) {9,008, 90.08}
(4) {3,450, 34.5}
(5) {49,500, 0.00495}
(6) {2,057, 0.0257}

Pages 429-431

EXERCISE H. Find the logarithm of each of the following numbers:

(1) 1,080
(2) 851
(3) 1.73
(4) 5.34
(5) 0.1
(6) 1.0
(7) 0.0966
(8) 20,100
(9) 0.00348
(10) 47.9
(11) 0.605
(12) 0.0007

Pages 431-432

EXERCISE I. Write the logarithm of each of the following numbers in three forms:

(1) log 0.244
(2) log 0.0043
(3) log 0.0573
(4) log 0.0009
(5) log 0.193
(6) log 0.00352

Pages 431-432

EXERCISE J. Find the numbers whose logarithms are:

(1) 2.9031
(2) 3.3345
(3) $\bar{2}.8998$
(4) 9.5944–10
(5) 0.2945
(6) 4.0086
(7) $\bar{1}.8998$
(8) 1.1584
(9) 8.9420–10
(10) 9.7050–10
(11) 0.6776
(12) 7.4886–10

Pages 432-436

EXERCISE K. Find the logarithm of each of the following numbers:

(1) 457.6
(2) 76.54
(3) 6.457
(4) 0.2765
(5) 5.893
(6) 36.01
(7) 1,642
(8) 0.008129
(9) 931.8
(10) 50.07
(11) 70,040
(12) 625,300

Pages 432-436

EXERCISE L. Find the numbers whose logarithms are:

(1) 3.0775
(2) 1.6979
(3) 9.4125–10
(4) 0.8695
(5) $\bar{3}.0059$
(6) 2.7942
(7) $\bar{1}.2890$
(8) 4.9496
(9) 8.5610–10
(10) 2.1999
(11) 7.7507–10
(12) $\bar{2}.3984$

Pages 436-438

EXERCISE M. Find the value of each of the following products by logarithms:

(1) 24.8×6.32
(2) 3.14×41.6
(3) 0.507×13.5
(4) 72.66×8.32
(5) 9.678×26.4
(6) 39.04×0.0017
(7) 0.4674×0.704
(8) 0.00911×5.257
(9) 0.6725×0.008043
(10) $65.72 \times 0.4008 \times 2.635$
(11) $3.246 \times 79.61 \times 0.01564$
(12) $0.8637 \times 1.718 \times 0.09223$

Pages 438-440

EXERCISE N. Find the value of each of the following quotients by logarithms:

(1) $79.5 \div 2.11$
(2) $1162 \div 5.77$
(3) $864 \div 0.483$
(4) $316.8 \div 9.57$
(5) $718.5 \div 0.462$
(6) $0.592 \div 0.309$
(7) $0.624 \div 0.0271$
(8) $98 \div 8743$
(9) $0.05104 \div 0.7396$
(10) $(8.3 \times 3.7) \div 9.5$
(11) $(25.17 \times 0.625) \div 45.83$
(12) $65.49 \div (0.08012 \times 1200)$

Page 440

EXERCISE O. Using logarithms, find the value of each of the following expressions:

(1) $(5\frac{2}{3})^2$
(2) $(8.354)^3$
(3) $(0.5287)^4$
(4) $4.73 \times (79.7)^2$
(5) $(1.736)^3$
(6) $(65 \div 21)^5$
(7) $(32.18 \div 92.15)^2$
(8) $(517.3 \div 27.38)^3$
(9) $[(73.49 \times 0.0513) \div 35.62]^2$
(10) $[61.3 \times (8.136)^2] \div (13.5)^3$
(11) $(34.78)^2 \div [(85.47)^3 \times (0.0453)^2]$
(12) $[\{16.7 \times (0.304)^2\} \div 63.8]^2$

Pages 440-445

PROBLEMS:

(9) What amount will M. Lawrence have to invest, at $3\frac{1}{2}$ per cent compounded quarterly, in order to have \$7,500 available in 5 years?

(10) T. Black invested \$5,000. Eight years later his investment amounted to \$7,110.50. What rate of interest, compounded annually, did T. Black realize on his investment?

(11) R. West deposited \$1,200 in a bank which pays 4 per cent per annum compounded semiannually. In how many years will this original deposit grow to \$2,180?

(12) J. Hall invested \$10,000 for his children, aged 2 and 9, at 4 per cent compounded quarterly. What sum should be given to the elder child upon reaching the age of 21 so that a similar amount may be paid the younger child upon his attaining his majority seven years later?

(13) R. Morrissey received \$120 monthly as the rental on some property that he owned in Maine. He invested this at the end of every three months at 5 per cent per annum compounded quarterly. What was the amount of his holdings at the end of 10 years? (*Hint:* See Chapter 24, page 415.)

(14) S. Dowling makes a contract whereby he (or his heirs) will receive 20 equal semiannual payments of \$1,500 beginning six months after the date of contract. If interest is at 4 per cent per annum compounded semiannually, how much should Dowling pay six months before the first payment is due? (*Hint:* See Chapter 24, page 417.)

(15) D. Woods purchased an automobile for \$1,000. Assuming the rate of depreciation to be 35 per cent per annum, what is the value of Mr. Woods' automobile six years after the date of purchase?

(16) An electric motor was installed at a cost of \$3,750. Experts estimated its probable life at 25 years, and its scrap value at \$75. (*a*) What sum would be set aside each year for depreciation by the straight line method? (*b*) What percentage of the book value would be set aside each year by the constant percentage method?

(17) The Pratt Publishing Co. purchased a printing press for \$6,000. Estimating the rate of depreciation at 20 per cent per annum and the residual scrap value at \$400, what is the probable life of the printing press?

(18) R. J. Johnson & Co. sold a building erected 20 years ago for \$35,000 cash. What was the original cost of the building, if the rate of depreciation is estimated at 10 per cent per annum?

(19) The Milltex Co. purchased a machine for \$11,000. The probable life of this machine was estimated at 15 years and its scrap value at \$550. Find the sum set aside each year for depreciation by the Milltex Co. on the basis of the straight line method. Five years after the date of purchase, the machine was destroyed by fire. The insurance company offered to settle the claims of the Milltex Co. on the basis of the constant percentage method of depreciation. Calculate the difference between the insurance company's offer and the book value of the machine as computed by the Milltex Co.

Pages 445-446

EXERCISE P. Using logarithms, find the value of each of the following expressions:

(1) $\sqrt{51.6}$

(2) $\sqrt{8765}$

(3) $\sqrt[3]{31.7}$

(4) $\sqrt[3]{673.4}$

(5) $\sqrt[4]{795.7}$

(6) $\sqrt[5]{2683}$

(7) $\sqrt{91.2 \times 4.23}$

(8) $\sqrt{7.46 \times 0.0237}$

(9) $\sqrt[3]{0.396 \times 0.9426}$

(10) $\sqrt{\dfrac{6.438}{30.52 \times 0.0467}}$

(11) $\sqrt[3]{\dfrac{83.49}{217.2 \times 0.587}}$

(12) $\sqrt[4]{\dfrac{(9.734)^2 \times 5.387}{(8.453)^3}}$

Pages 445-446

EXERCISE Q. Using logarithms, find the value of each of the following expressions:

(1) $(21.3) \times (-42.5)$

(2) $(-67.89) \div (314.7)$

(3) $(-5.312)^5$

(4) $\sqrt[3]{-3.164}$

(5) $\dfrac{(-71.73)(-0.872)^2}{(-5.825)^3}$

(6) $\sqrt[5]{\dfrac{(-915.6)^2}{(-0.8158)^3}}$

Pages 445-446

PROBLEMS:

(20) The time, t sec., of oscillation of a pendulum of length l ft. is given by the formula $t = 2\pi\sqrt{\dfrac{l}{32.16}}$. Find the time of oscillation of a pendulum 39.37 inches long.

(21) The diameter of the pipe required for a pump is given by the formula $D = 4.95\sqrt{\dfrac{G}{V}}$, in which G = gallons per minute and V = velocity in ft. per min. Find D when $G = 117$ and $V = 20$.

(22) The relation between the flange-lap, L, of a wheel, the depth, f, of the wheel-flange, and the radius, r, of the wheel-tread is given by the formula $L = 2\sqrt{(r+f)^2 - r^2}$. Find the flange-lap when $f = \frac{5}{8}$ in. and $r = 16\frac{1}{2}$ in.

(23) The slenderness, K, of a column is given by the formula $K = \sqrt{\dfrac{E}{p}}$, in which E = elastic modulus of material and p = average unit load on column. Find K when $E = 30{,}000{,}000$ and $p = 150$.

(24) The elevation of the outer rail on a railroad curve is given by the formula

$$e = \frac{4.9}{\sqrt{1 + 223.64\dfrac{R^2}{S^4}}},$$

in which R = radius of the curve in ft. and S = speed in miles per hour. Find the elevation of the outer rail when $R = 1042$ ft. and $S = 40$ miles per hour.

Pages 446-452

PROBLEMS:

(25) Construct a logarithmic scale of one cycle 9 inches long.

(26) Using Figure 194, page 447, approximate the value of: (*a*) $\sqrt{35}$; (*b*) $(22)^2$; (*c*) $\sqrt{22.7}$; (*d*) $(34.2)^2$; (*e*) $\sqrt{451}$; (*f*) $(59.5)^2$.

(27) Construct a logarithmic scale of one cycle 6 inches long. Begin the scale with the number 12. (*Hint:* On a piece of plain paper draw a vertical line 6 in. long. From the two end points of this line draw two horizontal lines. Place the 9-in. logarithmic scale, constructed for Problem 25, diagonally across the piece of paper so that the two end points of the scale fall on the horizontal lines. From each subdivision of the logarithmic scale, draw horizontal lines to intersect the vertical line.)

(28) Construct a logarithmic scale of one cycle 11 inches long. Begin the scale with the number 20. (*Hint:* Place a piece of plain paper diagonally across the paper containing the 9-in. logarithmic scale drawn for Problem 25. From the points of division of the logarithmic scale, draw horizontal lines to intersect the edge of the sheet of plain paper.)

(29) Using the method shown in Problem 27, construct a logarithmic scale of two cycles.

Pages 452-459

PROBLEMS:

(30) The values obtained from the equation $y = 2x - 2$ form an arithmetic progression. Plot the graph of this equation (*a*) on arithmetic paper; (*b*) on semi-logarithmic paper.

(31) The values obtained from the equation $y = 2^x$ form a geometric progression. Plot the graph of this equation (*a*) on arithmetic paper; (*b*) on semi-logarithmic paper.

(32) Why does the graph of a geometric progression plotted on semi-logarithmic paper appear as a straight line?

(33) Why can there be no zero in a logarithmic scale?

(34) What are the advantages of using two (or more) vertical scales on a semi-logarithmic chart?

(35) A curve is plotted on semi-logarithmic paper. What would be your interpretation of the graph if: (*a*) the curve bends upward? (*b*) the curve is descending and is nearly straight? (*c*) the curve bends downward? (*d*) two curves m and n bend downward and m is steeper than n?

(36) Explain how a slide rule 25 cm. long can be used to draw a semi-logarithmic chart on a sheet of paper 8 by 11 inches in size.

(37) The equation $y = ae^{mx}$, where a and m are constants and $e = 2.718$ approximately, expresses what is often called "the law of organic growth" or the "compound interest law." Assuming any values for a and m, say 2 and 1, respectively, plot the graph of this equation on semi-logarithmic paper.

(38) The following data give the estimated population of the United States from 1870 to 1940:

UNITED STATES POPULATION

Census Year	Population	Census Year	Population
1870	38,558,371	1910	91,972,266
1880	50,155,783	1920	105,710,620
1890	62,947,714	1930	122,775,046
1900	75,994,575	1940	131,669,275

Plot these data on semi-logarithmic paper. By carefully noting the tendency of the slope of the curve and assuming that the causes of the growth of the population in the past will continue in the future, extend the graph. From the extended graph, estimate, that is, *forecast*, the population of the United States for 1950.

(39) Business affairs have a tendency to follow the law of growth given in Problem 37. It cannot be expected, however, that any given business will follow this law indefinitely. A slowing-up process takes place and a *saturation point* is reached. The following table gives the national income from agriculture from 1932 to 1939:

NATIONAL INCOME FROM AGRICULTURE

Year	Income (in $1,000,000)	Year	Income (in $1,000,000)
1932	2,551	1936	5,970
1933	3,419	1937	6,378
1934	4,553	1938	5,432
1935	5,276	1939	5,750

Plot these data on semi-logarithmic paper. Is the rate of increase increasing or decreasing? In what year was the saturation point reached?

(40) The following table gives the trade of continental United States from 1921 to 1934:

TRADE OF CONTINENTAL UNITED STATES

Year	Value of Exports		Year	Value of Exports	
	Foreign Countries (in $100,000)	*U. S. Possessions (in $100,000)*		*Foreign Countries (in $100,000)*	*U. S. Possessions (in $100,000)*
1921	4,428	193	1928	5,039	272
1922	3,780	183	1929	5,147	279
1923	4,109	224	1930	3,771	253
1924	4,521	240	1931	2,370	212
1925	4,839	245	1932	1,563	172
1926	4,730	262	1933	1,627	175
1927	4,785	271	1934	2,081	201

Compare the fluctuations of these data when plotted (*a*) on arithmetic paper; (*b*) on semi-logarithmic paper.

CHAPTER 26

Pages 462-465

PROBLEMS:

(1) How many planes can be passed through two points? Through three points in the same straight line?

(2) If two lines do not meet, are they necessarily parallel? What is wrong with the statement: "Lines that do not meet are parallel"?

(3) Why is it that a three-legged table will always stand firmly on a level floor while a four-legged one will sometimes be unsteady?

(4) Draw a rectangular box: (*a*) as seen from a point to the left of it; (*b*) as seen from a point directly above it.

(5) Draw three planes: (*a*) intersecting in one line; (*b*) intersecting, but not having a common line of intersection.

(6) What is the least number of planes that can enclose a space?

(7) What kind of surface does a moving line generate: (*a*) if it always passes through a given point and always intersects a given line; (*b*) if it always intersects two intersecting lines?

(8) How many planes are determined by five lines passing through a common point and so that no three of these lines lie in the same plane?

Pages 465-466

PROBLEMS:

(9) How many planes can be passed meeting a given plane in a given line?

(10) Is every closed four-sided figure necessarily a plane figure?

(11) Two parallel planes cut off equal lengths on two lines. (*a*) Are the lines necessarily parallel? (*b*) Do the lines necessarily intersect?

(12) Two surfaces intersect in a straight line. Is it necessary that both of them be planes?

(13) Is the converse of Fact 112 true?

(14) Prove that parallel line segments intercepted between parallel planes are equal.

Pages 466-473

PROBLEMS:

(15) How many lines may be drawn through a given point parallel to a given plane?

(16) A line is parallel to a plane. To how many lines in the given plane may this line be parallel?

(17) What condition is necessary for a straight stick to be parallel to its own shadow on level ground?

(18) Under what conditions may a line be parallel to three planes?

(19) How many planes may be passed through a given point parallel to a given line?

(20) Through a given point construct a line parallel to a given plane and meeting a given line.

(21) A given line is parallel to a given plane. Prove that a line parallel to the given line drawn through any point of the plane lies wholly in the plane.

(22) Given two intersecting planes. Prove that if a line of one is parallel to a line of the other, then each of these lines is parallel to the line of intersection.

(23) Two intersecting lines are respectively parallel to, but not coplanar with, two other intersecting lines. Prove that the plane determined by the first pair is parallel to the plane determined by the second pair.

(24) What is the locus of all lines drawn through a given point parallel to a given plane?

(25) Prove that if a plane intersects one of two parallel lines it intersects the other also.

(26) Prove that if a line intersects one of two parallel planes, it intersects the other also.

(27) Prove that if a plane intersects one of two parallel planes, it intersects the other also.

(28) Prove that if two planes are parallel to the same plane, they are parallel to each other.

(29) If the angles of Fact 119 have their sides parallel, but extend in opposite directions from their vertices, what is the relation between the angles?

(30) Prove that through a given point not lying in a plane, one and only one plane can be passed parallel to that plane.

(31) How many planes can be passed through a given point parallel to two given parallel planes?

(32) Construct a line through a given point parallel to two given intersecting planes. (*Hint:* Pass the plane determined by the point and the intersection of the planes.)

(33) Given two lines not in the same plane. Prove that through either of these lines one and only one plane can pass parallel to the other line.

(34) Given two nonparallel lines in space. Prove that through a given point one and only one plane can be passed parallel to the given lines.

(35) Prove that if two parallel planes cut a series of concurrent lines, the corresponding segments are proportional. (*Hint:* Pass a plane through the common point parallel to one of the given planes.)

(36) A bookcase is arranged so that the spaces between its shelves are 10 in., 16 in., 8 in., and 8 in., respectively. If a slanting board is nailed diagonally across the back for re-enforcement and cuts off a segment of 25 in. between the first two shelves, what segments are cut off between the remaining shelves?

Pages 473-475

PROBLEMS:

(37) Prove that if a line is perpendicular to two lines at their point of intersection, it is perpendicular to their plane.

(38) Prove that all the lines perpendicular to a line at a point lie in a plane perpendicular to that line at that point. (*Hint:* Pass the plane determined by two perpendicular lines passing through the point, and use the indirect method.)

(39) A right angle is rotated about one of its sides. What does the other side generate?

(40) Construct a plane containing a given point and perpendicular to a given line. Justify the construction. (*Hint:* If the point is on the given line, construct two lines each perpendicular to the given line at the point. When the point is not on the given line, drop a perpendicular from the point to the given line, and at the point of intersection draw another line perpendicular to the given line.)

(41) Prove that if two planes are perpendicular to the same line, the planes are parallel.

(42) Prove that if a line is perpendicular to one of two parallel planes, it is perpendicular to the other also. (*Hint:* Assume the line perpendicular to plane M. Show that it intersects the other plane, N. Through the point of intersection of the line with plane N, pass two lines lying in this plane. Pass the planes determined by these lines and the given line.)

(43) Will a straight rod held perpendicular to the floor of a room also be perpendicular to the ceiling?

(44) Prove that the locus of points in space equidistant from two given points is the plane which bisects perpendicularly the line segment joining the two given points. (*Hint:* Draw the line segment determined by the two given points. Through the intersection of this line segment with the given plane, draw two line segments lying in the plane. Connect the end points of these line segments to the two given points.)

(45) Prove that the locus of all points equidistant from the vertices of a triangle is a line through the center of the circumscribed circle and perpendicular to the plane of the triangle.

Pages 475-476

PROBLEMS:

(46) Is it possible to form angles other than plane angles by two lines, one in each face of the dihedral angle and intersecting its edge at the same point? Give some illustrations.

(47) Prove that if from any point in a face of a dihedral angle a perpendicular is drawn to each face, the angle formed between these perpendiculars is equal to the plane angle of the given dihedral angle.

(48) Prove that if two dihedral angles are congruent, their plane angles are equal. (*Hint:* Use superposition.)

(49) Prove that if a line is perpendicular to a plane, every plane containing the given line is perpendicular to the plane.

(50) Prove that if a line is perpendicular to a plane, every plane parallel to the line is perpendicular to the plane. (*Hint:* Use the fact stated in Problem 49.)

(51) Prove that if a plane is perpendicular to the line of intersection of two planes, it is perpendicular to each of the planes.

(52) Three lines are perpendicular to each other at a common point. Prove that the three planes determined by the three pairs of lines are perpendicular to each other.

(53) Prove that if two planes are perpendicular to each other, a line drawn in one perpendicular to their intersection is perpendicular to the other. (*Hint:* Use fact stated in Problem 37.)

Pages 476-478

PROBLEMS:

(54) What is the least number of faces that a polyhedron can have? The least number of edges? The least number of vertices? The greatest number of faces?

(55) Show that each side of the upper base of a prism is necessarily parallel to a side of the lower base.

(56) Show that the upper and the lower bases of a prism must have the same number of sides.

(57) Are the altitudes of two prisms having equal lateral edges necessarily equal?

(58) Prove that the section of a prism made by a plane cutting the prism and parallel to one of the lateral edges is a parallelogram.

(59) A parallelepiped is a prism whose bases are parallelograms. Prove that the four diagonals of a parallelepiped meet in a point.

Pages 479-482

PROBLEMS:

(60) For what kind of pyramid may any one of the faces be taken as the base?

(61) A pyramid is cut by a plane parallel to the base and halfway between the base and the vertex. Find the ratio of the area of the section to the area of the base.

(62) A regular square pyramid has a side of its base equal to 4 inches and the dihedral angle made by a face with the base equal to 60°. Find the altitude of the pyramid.

(63) A pyramid is cut by a plane parallel to the base. If the altitude of the

pyramid is 4 inches and the area of the section cut by the plane is equal to half the area of the base, how far from the vertex of the pyramid is the plane?

(64) The perimeters of the bases of two pyramids having equal altitudes are equal. Prove that if the two pyramids are cut by a plane parallel to their bases and equidistant from their vertices, the sections formed have equal perimeters.

CHAPTER 27

Pages 484-490

PROBLEMS:

(1) An automobile holds 18 gallons of gasoline. How many liters does it hold?

(2) A man weighs 195 lbs. How many kilograms does he weigh?

(3) What is the weight in kilograms of a cubic foot of water?

(4) Mr. Lopez in Argentina orders 500 tons of coal from Norfolk, Va. He means metric tons, but receives short tons. If coal is worth $5 per long ton, to what, if any, rebate is he entitled?

(5) A physician orders 10 grams of strychnine. The druggist sends 3 drams. What is the over-weight or under-weight of the strychnine received?

(6) If wheat weighs 60 pounds per bushel, how many bushels would be equivalent to a metric ton?

(7) If a steel rail weighs 110 lbs. per yard, what is its weight in kilograms per meter?

(8) An American agrees to pay $1200 for a two-carat diamond. The French vendor sends him one weighing two metric carats. How many grains is it under-weight, from the American's point of view?

(9) Give the equivalence in liters of 100 dry quarts and 100 liquid quarts.

(10) If the maximum speed allowed on Mexican highways is 80 kilometers per hour, how many miles per hour are legal for an autoist?

(11) A jeweler uses 100 ounces of pure gold to make wedding rings of 14-carat fineness. How many ounces of alloy will be used? If the rings average 100 grains in weight and retail at an average of $11 apiece, and gold costs $35 an ounce, what is the gross profit on the gold?

(12) How many cubic meters are there in a cord of wood?

Pages 491-498

PROBLEMS:

(13) The altitude of a regular triangular prism is 6 inches and an edge of its base is 9 inches. (*a*) Find its volume. (*b*) Find its total surface.

(14) The diagonal of a cube is 9 inches. (*a*) Find its volume. (*b*) Find its total surface. (*c*) What is the angle between the diagonal of this cube and the diagonal of one of the faces?

(15) Find the length of the diagonal of a rectangular solid whose edges are 9, 12, and 15.

(16) If the inside edge of a cubical tank is 5 feet, find the number of barrels of water it will hold when full.

(17) A common brick is 2 by 4 by 8 inches. Find the number of bricks in a solid pile 17 by 6 by 20 feet.

(18) How many cubic yards of soil will be required to fill in a lot 75 ft. front by 125 ft. deep, if the surface is to be raised 4 feet in the rear and gradually sloped to the front, where it is to be 1 foot deep?

(19) A fall of $1\frac{1}{8}$ inches of rain occurs. How many barrels is this fall per square rod?

(20) The volume of a box 10 by 12 by 15 inches is to be increased approxi-

mately 91.5 cubic inches by adding the same amount to each dimension. What should this amount be?

(21) What is the weight of a hexagonal steel prism 6 ft. high and with a base whose perimeter is 36 inches? (The average weight of a cubic foot of steel is 490 lbs.)

(22) What is the weight of a hexagonal bar of bronze 18 ft. long if the radius of the hexagon is $1\frac{1}{32}$ inches? (The average weight of a cubic foot of bronze is 529 lbs.)

(23) On a base 4 by 5 feet, how high must a prismatical tank be to contain 15 long tons of water?

(24) An open rectangular tank is to have a square base and to contain 1,496.5 gallons of water. Find the most economical dimensions. (*Hint:* Solve by means of a graph, as shown in Chapter 17, page 275.)

(25) A closed rectangular tank is to have a square base and to contain 320 cubic feet. Find the least possible cost, if the base costs \$2 per square foot, the sides \$1 per square foot, and the top \$0.50 per square foot. (*Hint:* See Problem 24.)

Pages 498-502

PROBLEMS:

(26) What is the diameter of a cylindrical tank 6 feet high containing 400 gallons of water?

(27) How many gallons of water are contained by a cylindrical tank with a diameter of 40 inches and a height of $5\frac{1}{2}$ feet?

(28) A cylindrical smoke stack is to be made by bending a rectangular sheet of tin. The smoke stack is to have an inside diameter of 12 inches and a height of 10 feet. How much material will be necessary if the sheets of tin are 14 inches wide, and if 2 inches are allowed for overlapping?

(29) A blacksmith places a band of metal around a hub that is 18 inches in diameter. If the cross section of the band is a circle 2 inches in diameter, (*a*) how long a piece of metal is necessary? (*b*) if this band weighs $\frac{1}{4}$ lb. per cubic inch, what is its total weight? (*Hint:* To find the length of the piece of metal, use the circle formed by a line along the middle of the bar.)

(30) Assuming the weight of steel to be 490 lbs. per cubic foot, what is the weight of 1,000 feet of wire 0.3648 inches in diameter?

(31) Assuming the weight of copper to be 555 lbs. per cubic foot, what is the weight of one mile of copper wire 0.12849 inches in diameter?

(32) What is the capacity of a cylindrical tank 10 ft. long and having an inside diameter of 5 ft.? What are the dimensions of a cubical tank of the same capacity?

(33) An octagonal prism of steel 4 ft. long, and such that the radius of the octagon is $\frac{3}{16}$ of an inch, is bored throughout its length by a drill $\frac{1}{4}$ of an inch in diameter. What is the weight of the steel removed? Of the remaining barrel? (The average weight of a cubic foot of steel is 490 lbs.)

(34) An open cylindrical copper tank is to be made by bending a rectangular copper sheet. The tank is to have an inside diameter of 4 ft. and a capacity of 352 cubic ft. The sheets of copper are $2\frac{1}{2}$ ft. wide and $\frac{1}{4}$ of an inch thick. If sheet copper costs 22 cents a pound, what will be the cost of the material required? (The average weight of a cubic foot of copper is 555 lbs.)

(35) A hexagonal column is cut from a cylindrical oak log 18 inches in diameter and 12 ft. long. Find the weight of the column. (The average weight of a cubic foot of oak is 48 lbs.)

(36) A cylindrical piece of steel, with a diameter of 14 inches, is 40 inches high. A wedge is cut from this cylinder so that the angle, made by the straight

edges of the ends of the wedge meeting at the center of the circular base, is 45° Find the volume and total surface of the wedge.

(37) A cylindrical piece of steel, with a diameter of 18 inches, is 28 inches long. A flat surface 6 inches across is to be lathed throughout its length. Find the weight of the steel removed. (The average weight of a cubic foot of steel is 490 lbs.)

(38) An open cylindrical tank is to contain approx'ı ıtely 535 gallons of water. Find the most economical dimensions. (*Hint:* Solve y means of a graph, as shown in Chapter 17, page 275.)

(39) A closed cylindrical tank is to contain 1,617 cubic ft. The base costs \$1.50 per square foot, the lateral surface \$3.00 per square foot, and the top \$0.75 per square foot. Find the least possible cost. (*Hint:* See Problem 38.)

CHAPTER 28

Pages 504-512

PROBLEMS:

(1) Two pyramids have equal volumes. One has a square base 54 sq. in. in area; the base of the other is a right triangle with sides 9, 12, and 15 in., respectively. If the altitude of the first is 16 in., what is the altitude of the other?

(2) Find the volume of a pyramid if the base is an equilateral triangle 10 in. on a side and its altitude is 8 ft.

(3) Compare the volumes of a prism and a pyramid having the same base and altitude.

(4) Prove that the ratio of the volumes of two pyramids: (*a*) is equal to the ratio of their respective altitudes if their bases are equal; (*b*) is equal to the ratio of their respective bases if their altitudes are equal.

(5) Prove that the ratio of the volumes of two pyramids is equal to the ratio of the respective products of their bases and altitudes.

(6) The base of a regular pyramid is a square 6 ft. on a side. If the altitude of the pyramid is 8 ft., find: (*a*) the dihedral angle between a lateral face and the base; (*b*) the volume of the pyramid.

(7) The base of a pyramid has an area of 54 sq. in. The altitude of the pyramid is 16 in. (*a*) Find the area of a section midway between the vertex and the base and parallel to the base. (*b*) What are the volumes of the two portions of this pyramid?

(8) The base of a regular pyramid is a hexagon whose area is 4 sq. ft. Find the volume of the pyramid if a lateral edge is 5 ft.

(9) A silver replica of a steeple has the shape of a hc'l¬w sq are pyramid each of whose lateral edges is 6 ft. and the perimeter of v .ose b se is 240 in. The silver replica of the steeple has a uniform thickness of $\frac{1}{8}$ in. Find its weight. (The average weight of a cubic inch of silver is 0.379 lb.)

(10) The base of a pyramid is a square, and each of the lateral faces of the pyramid is an equilateral triangle, the sides being 12 ft. long. (*a*) What is the angle of inclination of the lateral faces with the plane of the base? (*b*) What is the angle of inclination of each lateral edge with the plane of the base? (*c*) If the pyramid is cut by a plane parallel to the base and 3 ft. from the vertex, what are the respective volumes of the two portions of the pyramid?

(11) A regular pyramid 4 in. high has a square base whose area is 16 sq. in. Find the total area of the pyramid.

(12) A church steeple has the shape of a regular hexagonal pyramid. A lateral edge of the pyramid is 38 ft. and each side of the base is $7\frac{1}{2}$ ft. The average covering capacity of buff paint on weatherboarding is 400 sq. ft. per

gallon, one coat. Find the number of gallons necessary to give the steeple one coat of paint.

(13) A marble monument has the shape of a frustum of a regular pyramid with a square base. The monument is 9 ft. high and the sides of the bases are 3 and 2 ft., respectively. Find the weight of the monument. (The average weight of one cubic foot of marble is 168 lbs.)

(14) What is the lateral area of a regular pyramid whose base is a square 12 in. on a side and whose slant height is 10 in.? If a plane is passed parallel to the base and 4 in. from the vertex, what is the lateral area of the frustum?

(15) The base of a regular pyramid is an equilateral triangle 6 cm. on a side. A plane parallel to the base cuts off a frustum whose lateral area is 50 sq. cm. and whose upper base is 2 cm. on a side. What is the altitude of the frustum?

Pages 512-519

PROBLEMS:

(16) The area of the base of a circular cone is 51 sq. in. and its altitude is 10 in. Find the volume of the cone.

(17) The radius of the base of a right circular cone is 3 in. and its slant height is $3\sqrt{5}$ in. Find the volume of the cone.

(18) A right circular cone 15 in. in altitude and with a circular base 10 in. in diameter has a hole 2 in. in diameter bored through the center from vertex to base. Find the volume of the solid remaining.

(19) Large egg coal weighs 38 lbs. per cubic foot. A quantity of such coal is dumped in a pile having the shape of a right circular cone 12 ft. high. An element of the conical pile makes an angle of 30° with the plane of the base. How many tons of coal are in the pile?

(20) Prove that the ratio of the volumes of two circular cones: (*a*) is equal to the ratio of their respective altitudes if their bases are equal; (*b*) is equal to the ratio of the respective radii of their bases if their altitudes are equal.

(21) A piece of tin in the form of a sector of a circle of radius 6 in. is rolled into a cone. If the central angle of the sector is 300°, what is the volume of the cone?

(22) An ink bottle has the shape of an open right circular cylinder. A cap having the shape of an inverted right circular cone with an opening at its vertex is screwed on the ink bottle so that the vertex of the cone is inside of the bottle. If the proper dimensions are used, when the bottle is filled level with the vertex of the conical cap, it can be turned upside down without any ink spilling. Prove that the smallest depth the cone may have is $\frac{3}{8}$ the depth of the bottle.

(23) The perimeter of a right circular cone is 12 in. and the slant height is 1 ft. Find the lateral area of the cone.

(24) The lateral area of a right circular cone is 36 sq. in. What is the slant height of the cone if the perimeter of the base is 9 in.?

(25) How far from the top must a conical tent be cut in order to cut the cloth in half?

(26) A cylindrical chimney 12 ft. high with a diameter of 8 in. has a conical cap 10 in. high, the diameter of whose base is 10 in. Find the total amount of square ft. of tin required to make this chimney.

(27) Cones generated by the revolution of similar right triangles are called *similar cones of revolution.* Prove that the lateral areas of two similar cones of revolution have the same ratio as (*a*) the squares of their altitudes; (*b*) the squares of their slant heights; (*c*) the squares of the radii of their bases.

(28) A right circular cone of altitude 16 in. is cut by a plane parallel to the base and 7 in. from the vertex. If the areas of the bases of the frustum thus formed are 49 sq. in. and 81 sq. in., what is the volume of the frustum?

(29) How many square inches of tin are required to make a funnel whose top and bottom have diameters of 4 in. and 6 in., respectively, and whose slant height is 12 in.?

(30) A pail is 11 in. in diameter at the bottom and 13 in. on top. If the slant height is 12 in., (*a*) what is the number of gallons of water it will hold; (*b*) how many square feet of material are required to make this pail?

Pages 519-528

PROBLEMS:

(31) How many square feet of copper will be required to cover a hemispherical dome 30 ft. in diameter?

(32) Prove that the surface areas of two spheres are to each other as (*a*) the squares of their respective radii; (*b*) the squares of their respective diameters.

(33) Assuming the radius of the earth to be 3,960 miles, find its surface area.

(34) A hundred gross billiard balls $2\frac{1}{2}$ in. in diameter are to be painted. Assuming the average covering capacity of white paint to be 500 sq. ft. per gallon, one coat, how many gallons are required to paint the hundred gross billiard balls?

(35) What is the area of the surface of a zone of altitude 10 in. if the radius of the sphere is 50 in.?

(36) A hollow metal sphere 10 in. in diameter is used as a float for an automatic switch that keeps the water level in a tank within a given height. If the metal sphere sinks to a depth of 4 in., what is the area of the wetted surface?

(37) Prove that the area illuminated by a point of light s ft. away from the surface of a sphere of radius r ft. is given by the formula $\frac{2\pi s r^2}{s + r}$. Find the area illuminated by a point of light 30 ft. away from the surface of a sphere 14 ft. in diameter.

(38) How many cubic feet of gas are necessary to inflate a spherical balloon to a diameter of 60 inches?

(39) Prove that the volumes of two spheres are to each other as the cubes of their respective radii.

(40) The average weight of a cubic foot of copper is 555 lbs. What is the weight of 8 solid spheres of this metal having a diameter of 12 in.?

(41) What is the weight of 100,000 steel ball bearings $\frac{1}{4}$ in. in diameter? (The average weight of a cubic foot of steel is 490 lbs.)

(42) An ornamental spherical ball made of bronze has a diameter of 6 ft. and a uniform thickness of $\frac{1}{2}$ in. The average weight of a cubic foot of bronze is 529 lbs. Find the weight of this ball.

(43) The core for a cast iron piece has the shape of a spherical segment of two bases. The diameters of the upper and lower bases are 2 ft. and 6 ft., respectively, and the distance between the bases is 3 ft. If the average weight of a cubic foot of sand is 100 lbs., find the weight of the core.

(44) What is the volume of a spherical segment of one base if the altitude of the segment is 9 in. and the radius of the sphere is 4 ft.?

(45) The bronze base of a statuette has the shape of a spherical segment of one base. The metal has a uniform thickness of $\frac{1}{8}$ in.; the diameter of the inner sphere is $4\frac{1}{2}$ in.; and the altitude of the segment formed by the inner shell is 2 in. Find the weight of the base. (The average weight of a cubic foot of bronze is 529 lbs.)

(46) A hollow glass bowl in the shape of a spherical segment of one base 10 in. high is used as an aquarium. The glass bowl has an inner radius of 6 in. and is filled with water to a depth of 8 in. (*a*) Find the volume of the water.

(*b*) The round bottom of this aquarium is filled with pebbles and sand to a depth of 2 in. and then water is poured to within 2 in. from the top. Disregarding the water present throughout the sand and pebbles, find the volume of the water.

(47) Find the volume of a spherical sector of a sphere 20 in. in diameter, if the altitude of the cone that forms the base of the sector is 3 in.

(48) The stem of a silver candlestick has the shape of a spherical cone. The altitude of the conical part is 6 in. and the radius of the base is $\frac{3}{4}$ in. If the metal has a uniform thickness of $\frac{1}{16}$ in., find the material required to make 100 stems.

(49) A buoy has the shape of a spherical cone. Find the material required for a buoy if the radius of the sphere is 3 ft., the altitude of the zone is 6 in., and the metal has a uniform thickness of $\frac{1}{2}$ in.

(50) The radius of a sphere is 5 in. Find the volume of a spherical sector of this sphere if the radii of the circular bases of the inner and outer cones are $1\frac{1}{4}$ in. and $1\frac{1}{2}$ in., respectively. (*Hint:* Use trigonometric functions.)

CHAPTER 29

Pages 530-534

EXERCISE A. Discuss each of the following equations and draw their graphs:

(1) $y = 8x^3$
(2) $xy^2 = 1$
(3) $64x^3 = y^2$
(4) $y(4 - x) = 30$
(5) $y(x - 2)(x - 3) = 2x$
(6) $y(x^2 - 9) = 10x$
(7) $x^2 + 4xy + y^2 = 4$
(8) $x^2 + xy + y^2 + 6x = 0$
(9) $x^2 - xy + 2y^2 + 6y = 0$
(10) $x^2 + 2xy - 3y^2 = 4$
(11) $y = x(x^2 - 9)$
(12) $y^2 = x^2(8x + 1)$
(13) $y^2 = x^2(1 - x^2)$
(14) $y^2(4 - x) = x^2(4 + x)$
(15) $xy^2 - 2x - 4 = 0$
(16) $xy = 0$
(17) $x^{1/2} + y^{1/2} = 8^{1/2}$
(18) $y^2(2a - x) = x^3$ (Cissoid of Diocles)
(19) $y^2(a - x) = x^2(a + x)$ (Strophoid)
(20) $x^{2/3} + y^{2/3} = a^{2/3}$ (Hypocycloid of 4 cusps)
(21) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ (Lemniscate of Bernoulli)
(22) $y(x^2 + a^2) = a^3$ (Witch of Agnesi)
(23) $x^3 + y^3 = 3axy$ (Folium of Descartes)
(24) $x^2(3a - x) = y^2(x + a)$ (Trisectrix of Maclaurin)
(25) $x^2 + y^2 + ax = a\sqrt{x^2 + y^2}$ (Cardioid)
(26) $(y - x^2)^2 = x^5$

Pages 534-535

EXERCISE B. Discuss each of the following equations and draw their graphs:

(1) $9x^2 - y^2 = 0$
(2) $xy + 3x = 0$
(3) $x^2 + 2xy = 0$
(4) $y^2 - 4xy = 0$
(5) $x^2 - x - 2 = 0$
(6) $y^2 + xy - 6x^2 = 0$
(7) $3x^2 - 5xy - 2y^2 = 0$
(8) $x^2 - 2xy - 15y^2 = 0$
(9) $x^2 + 2xy + y^2 + x + y = 0$
(10) $4x^2 + 8xy + 4y^2 + 4x + 4y + 1 = 0$

Pages 535-537

EXERCISE C. Discuss each of the following equations and find their roots by drawing the necessary portions of their graphs:

(1) $x^3 - 6x^2 + 11x = 6$
(2) $x^3 + 3x^2 + 3x + 2 = 0$
(3) $x^3 - x^2 - 3x = 9$
(4) $x^3 - x^2 - 4x + 4 = 0$
(5) $x^3 + x^2 = 12x$
(6) $x^3 - 7x^2 + 16x = 8$
(7) $x^3 = 18$
(8) $6x^3 - 15x^2 - 27x + 54 = 0$
(9) $x^3 - 7x + 13 = 0$
(10) $x^4 + x^3 - 3x^2 - 4x = 4$

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EXERCISE D. Solve the following pairs of equations graphically and check your results by solving algebraically:

(1) $\begin{cases} y + 2x = 3 \\ y = 3x - 2 \end{cases}$

(2) $\begin{cases} 3y - 2x = 4 \\ 4x^2 + y^2 = 16 \end{cases}$

(3) $\begin{cases} y = 2x - 1 \\ x^2 + y^2 = 5 \end{cases}$

(4) $\begin{cases} x^2 - 16y = 0 \\ y - x = 0 \end{cases}$

(5) $\begin{cases} 3y - 2x = 9 \\ x^2 + 12 = 4x + 4y \end{cases}$

(6) $\begin{cases} 8y - x^2 = 8 \\ 9x^2 - 4y^2 + 36x + 24y = 36 \end{cases}$

(7) $\begin{cases} x^2 + y^2 - 4x + 6y = 12 \\ x = y^2 - 4y + 3 \end{cases}$

(8) $\begin{cases} 4x^2 + y^2 - 24x + 8y = 12 \\ 16x^2 - y^2 - 96x - 8y + 144 = 0 \end{cases}$

(9) $\begin{cases} 9x^2 + 16y^2 - 36x - 32y = 92 \\ x^2 + y^2 - 4x - 2y = 4 \end{cases}$

(10) $\begin{cases} xy + 3y + 4x = 8 \\ x^2 + y^2 + 6x + 8y = 11 \end{cases}$

Pages 537-539

EXERCISE E. Discuss each of the following equations and draw their graphs:

(1) $y = 10^x$
(2) $A = (1.05)^n$
(3) $y = 2^x$
(4) $y = -2^x$
(5) $s = (1.04)^n$
(6) $y = 3^{-x}$
(7) $v = (1.02)^{-n}$
(8) $y = e^x$, where $e = 2.718$
(9) $y = e^{-x}$, where $e = 2.718$
(10) $v = (1.03)^{-n}$

Pages 539-541

EXERCISE F. Discuss each of the following equations and draw their graphs:

(1) $y = 2 \log x$
(2) $y = \log x^3$
(3) $n = 77.9 \log s$, *i. e.*, $s = (1.03)^n$
(4) $y = \log \sqrt{x}$
(5) $n = 47.19 \log s$, *i. e.*, $s = (1.05)^n$
(6) $y = \log (1 + x)$
(7) $y = \log x^{-2}$
(8) $n = -58.82 \log v$, *i. e.*, $v = (1.04)^{-n}$
(9) $n = -116.3 \log v$, *i. e.*, $v = (1.02)^{-n}$
(10) $y = 2.303 \log x$, *i. e.*, $x = e^y$

Pages 541-545

EXERCISE G. Discuss each of the following equations and draw their graphs:

(1) $y = \cos x$
(2) $y = \tan x$
(3) $y = \cot x$
(4) $y = \sec x$
(5) $y = \csc x$
(6) $y = \sin 2x$
(7) $y = 2 \sin x$
(8) $y = \cos \frac{1}{2}x$
(9) $y = \tan \frac{\pi x}{4}$
(10) $y = \cot \pi x$
(11) $y = 2 \sec \frac{\pi x}{3}$
(12) $y = 4 \csc \frac{1}{2}x$
(13) $y = 3 \sin \frac{\pi x}{2}$
(14) $y = 2 \cos \frac{\pi x}{5}$
(15) $y = 4 \cot \frac{\pi x}{6}$

Pages 545-547

EXERCISE H. Discuss each of the following equations and draw their graphs:

(1) $y = \sin 2x + \sin \frac{1}{2}x$

(2) $y = \sin x + \cos x$

(3) $y = x + \sin x$

(4) $y = \frac{1}{2}x + \cos 2x$

(5) $y = \frac{1}{2}x + \sin \frac{\pi x}{2}$

(6) $y = x^2 + \sin 2x$

(7) $y = \frac{1}{2}x - 4 \cos \frac{\pi x}{4}$

(8) $y = e^x + \cos x$

(9) $y = e^x - 2 \sin x$

(10) $y = \log x + \sin x$

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PROBLEMS:

(1) The period of revolution of a planet depends only upon the major axis of its orbit. This law of dependence may be expressed by the formula $p^2 = ka^3$, where p is the period of revolution, k is a constant, and a is the semimajor axis. Discuss the equation and draw its graph.

(2) The *Law of Universal Gravitation* is usually stated as follows: "Every particle of matter in the universe attracts every other particle with a force that varies directly as the product of the masses of the particles and inversely as the square of the distance between them." If the product of the masses remains constant and is denoted by k, this law may be expressed by the formula $F = \frac{k}{d^2}$. Discuss the equation and plot its graph.

(3) The velocity v in feet per second of a body that has fallen s feet is approximately given by the formula $v = 8s^{1/2}$. Discuss the equation and plot its graph.

(4) The resultant displacement, at a given point in a medium, caused by the interference of two sound waves may be expressed by the formula

$$y = 2 \sin 2\pi\left(200t - \frac{20}{11}\right) + 2 \sin 2\pi\left(200t + \frac{20}{11}\right).$$

Discuss the equation and plot its graph.

(5) The *Catenary* is the curve assumed by a perfectly flexible cord when its ends are fastened at two points and the weight of a unit length of the cord is constant. This curve is represented by the equation $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$, where $e = 2.718$ and a is constant. Assuming a to have any value, say 2, discuss the equation and plot its graph by the method of addition of ordinates.

(6) The law of variation of the atmospheric pressure may be expressed by the formula $p = 0.000185e^{-x}$. Taking $e = 2.718$, discuss the equation and plot its graph.

(7) The intensity of a sound varies inversely as the square of its distance from the sounding body. If I denotes the intensity, k is a constant, and d represents the distance from the sounding body, this law may be expressed by the formula $I = \frac{k}{d^2}$. Discuss the equation and plot its graph.

(8) The final temperature of a thermometer bulb exposed to the air under certain conditions may be expressed by the formula $0.0235s = 3.689 - 2.303 \log T$, where T denotes the final temperature and s the time in seconds. Taking $e = 2.718$, discuss the equation and plot its graph.

(9) Find the equation of the locus of a point that moves so that the product of its distances from two fixed points is equal to a constant. Discuss the equation and plot its graph. This curve is known as the *oval of Cassini*. (*Hint:* Choose the two fixed points on the X-axis and equally distant from the origin, and let the constant be b^2.)

(10) Find the equation of the locus of a point, P, that moves so that its distances from the origin, namely, $O(0, 0)$, and a point $A(a, 0)$ satisfy the relation $AP = \pm b\,OP \pm c$, where b and c are constants. Discuss the equation and plot its graph. This curve is known as the *Cartesian oval.*

CHAPTER 30

Pages 550-551

EXERCISE A. Simplify:

(1)	i^9	(4)	i^{10}	(7)	i^{19}	(10)	$\sqrt{16i^2}$	(13)	$\sqrt{4i^6}$
(2)	$\sqrt{-4}$	(5)	$\sqrt{-9x^2}$	(8)	i^{25}	(11)	i^0	(14)	$\sqrt{-25a^2b^2}$
(3)	$\sqrt{-12}$	(6)	$\sqrt{-a^2}$	(9)	$\frac{1}{i^5}$	(12)	$\frac{1}{i^{15}}$	(15)	$\sqrt{-i^2}$

Pages 550-551

PROBLEMS:

(1) Show that, in general, even powers of i are equal to $+1$ or -1. (*Hint:* An even power of i may be written i^{2n}.)

(2) Show that, in general, odd powers of i are equal to $+i$ or $-i$. (*Hint:* An odd power of i may be written i^{2n+1}.)

Pages 551-552

EXERCISE B. Simplify:

(1) $3i^3 + 9i^4$

(2) $\sqrt{-16} + \sqrt{-25}$

(3) $8i - 4i^3$

(4) $\sqrt{4i^2} + \sqrt{-9i^4} + \sqrt{-25}$

(5) $3i \cdot 6i$

(6) $\sqrt{-36} \cdot \sqrt{-16} \cdot \sqrt{-28}$

(7) $\sqrt{-32} \div \sqrt{-2}$

(8) $\sqrt{-18} \div \sqrt{-3}$

(9) $\sqrt{16i^2} \cdot \sqrt{4i^6}$

(10) $\sqrt[4]{16i^8} \cdot \sqrt[3]{8i^6} \cdot (-\sqrt[6]{64i^6})$

(11) $\sqrt{-x^4} \div \sqrt{-x^2}$

(12) $(\sqrt{-36} \cdot \sqrt{-9}) \div 9i$

(13) $(4i + 3i)^2$

(14) $(10i - 8i)^2$

(15) $(7i^4 + 3i^2)^{1/2}$

(16) $(12i^3 + 4i^6)^{1/2}$

(17) $\dfrac{6i + 4i}{6i - 4i}$

(18) $\dfrac{7i^3 + 5i^5}{5i^5 - 7i^3}$

(19) $\dfrac{\sqrt{-a^4} - \sqrt{-b^4}}{\sqrt{-a^2} + \sqrt{-b^2}}$

(20) $\dfrac{36i + 25i^3}{4i^5 + 3i^3}$

(21) $[3\sqrt{-4x^2} + 4\sqrt{-16x^4}]^2$

Pages 552-556

EXERCISE C. Simplify:

(1) $(6 + 7i) + (7 + 6i)$

(2) $(8 - 4i) - (4 + 8i)$

(3) $(16 - 2i) + (7 - 5i)$

(4) $(16x - 11iy) + (8x + 3iy)$

(5) $(5x^2 - \sqrt{-16x^4}) + (7x^2 - \sqrt{-4x^4})$

(6) $(7x^2 + \sqrt{-4x^2}) - (5x^2 - \sqrt{-16x^2})$

(7) $16i - (8i - 5)$

(8) $(7x^2 - 8i^5) - (6x^2 + 7i^3)$

(9) $(5 + 8i)(8 + 5i)$

(10) $(6 - \sqrt{-4})(2 + \sqrt{-36})$

(11) $(3 + 2i)(3 - 2i)$

(12) $(8i^2 - 9i^3)(9i^2 - 8i^3)$

(13) $(a + bi)^2$

(14) $(a - bi)^2$

(15) $(\sqrt{16x^2} + 3\sqrt{-25})(x - 4\sqrt{-9})$

(16) $(3x - 2i^2y)(2x + 3i^3y)$

(17) $(3 + 4i)^3$

(18) $(7 + 5i)(7 - 5i)$

(19) $(x + iy)(x - iy)$

(20) $(ix + y)(ix - y)$

(21) $(5 - 3i)(7 + 4i)$

(22) $(5 + 3i)^2$

(23) $(1 + i)^2(1 - i)^2$

(24) $\sqrt{24 + 7i} - \sqrt{24 - 7i}$

(25) $(1 + i)^3$

(26) $(a + bi)^3$

(27) $(1 + i)^4$

(28) $(a + bi)^4$

(29) $(a + bi)^3 + (a - bi)^3$

(30) Factor $a^3 + b^3$

(31) $\sqrt[3]{-1}$

(32) $\dfrac{4}{5i}$

(33) $\dfrac{i^5}{i^7}$

(34) $\dfrac{2 + i}{2 - i}$

(35) $\dfrac{3}{3 + i\sqrt{2}}$

(36) $\dfrac{4}{2x + \sqrt{-4y^2}}$

(37) $\dfrac{6}{\sqrt{6} - i\sqrt{3}}$

(38) $\dfrac{x}{(x - i)^2}$

(39) $\dfrac{4 + \sqrt{-9}}{3 - \sqrt{-16}}$

(40) $\dfrac{6 - \sqrt{3i^2}}{\sqrt{3} + 6i}$

(41) $\dfrac{3i^2 + 2i}{3i^2 - 2i}$

(42) $\dfrac{x - iy}{x + iy}$

(43) $\dfrac{i^2}{(x + i)^2}$

(44) $\dfrac{\sqrt{-16}}{\sqrt{2y} - \sqrt{-4}}$

(45) $\dfrac{\sqrt{-9}}{(2 + i)^2}$

(46) $\dfrac{\sqrt{4} + i\sqrt{9}}{\sqrt{9} + i\sqrt{4}} + \dfrac{\sqrt{4} - i\sqrt{9}}{\sqrt{9} - i\sqrt{4}}$

(47) $\dfrac{\sqrt{x + y} + i\sqrt{x - y}}{\sqrt{x + y} - i\sqrt{x - y}}$

(48) $\dfrac{x + iy}{ix - y}$

Pages 556-558

EXERCISE D. Represent graphically the following expressions:

(1) $2 + 4i$
(2) $3 - 4i$
(3) $-2 + 2i$
(4) $2 - 2i$
(5) $-2 - 4i$
(6) $(1 + i) + (i + 1)$
(7) $(2 + 3i) + (3 + 2i)$
(8) $(3 + 6i) - (5 + 2i)$
(9) $(4 + 3i) + (-2)$
(10) $(-3 + i) - (4 + 3i)$
(11) $(7 - 3i) - 2(1 - i)$
(12) $(-6 + 2i) + (4 + 5i)$
(13) $(3i^2 - 2i^3) + (4i^2 - 3i^3)$
(14) $2(1 + 2i) - (2 + 4i)$
(15) $4(i^4 - \sqrt{-3}) - (i^2 + \sqrt{-3})$

Pages 556-558

EXERCISE E. Solve the following equations. Check results.

(1) $7x + 7i^2 = 0$
(2) $x^2 + 16i^4 = 0$
(3) $x^2 - 2ix - 1 = 0$
(4) $x^2 + 2ix + 24 = 0$
(5) $12x^2 + 4x + 4 = 0$
(6) $x^2 + x + 1 = 0$
(7) $4x^2 - 8x + 5 = 0$
(8) $4x^2 - 12x + 7 = 0$

Pages 556-558

PROBLEMS:

(3) Prove that the difference, $A - B$, between two imaginary numbers may be obtained graphically as follows: Connect the point A with the origin. Draw a line from B through the origin to C making $OC = OB$. With OA and OC as sides, draw a parallelogram. The fourth vertex, D, represents the difference between A and B.

(4) Find the square root of $32 + 4i$. (*Hint:* Let x and y represent any two real numbers, and solve the equation $\sqrt{32 + 4i} = x + iy$.)

(5) Find the square root of: (*a*) $-7 - 24i$; (*b*) $16 - 6i$; (*c*) $15 - 4i$; (*d*) $a + bi$.

(6) Find three roots of the equation $x^3 - 1 = 0$, and represent the roots graphically.

(7) Find four roots of the equation $x^4 - 16 = 0$, and represent the roots graphically.

CHAPTER 31

Pages 560-565

EXERCISE A. Find the common fraction equivalent to:

(1) $0.666\cdots$
(2) $0.222\cdots$
(3) $0.555\cdots$
(4) $0.1818\cdots$
(5) $0.5454\cdots$
(6) $0.5757\cdots$
(7) $0.148148\cdots$
(8) $0.259259\cdots$
(9) $0.108108\cdots$
(10) $0.1666\cdots$
(11) $0.13636\cdots$
(12) $0.41666\cdots$

Pages 560-565

EXERCISE B. Find the sum of each of the following infinite series:

(1) $1 + \frac{1}{2} + \frac{1}{4} + \cdots$
(2) $1 - \frac{1}{3} + \frac{1}{9} - \cdots$
(3) $8 + 2 + \frac{1}{2} + \cdots$
(4) $9 - 6 + 4 - \cdots$
(5) $6 + 5 + 4\frac{1}{6} + \cdots$
(6) $64 + 24 + 9 + \cdots$
(7) $30 + 12 + 4\frac{4}{5} + \cdots$
(8) $1.4 + 0.28 + 0.056 + \cdots$
(9) $2 + 0.8 + 0.32 + \cdots$
(10) $4.8 - 2.88 + 1.728 - \cdots$

Pages 560-565

PROBLEMS:

(1) Under the terms of the will of a patron, a research laboratory is granted a perpetual income of $20,000 a year payable at the end of each year. If cur-

rent money rates are 4 per cent per annum, what is the present value of this bequest?

(2) A realty company receives \$15,000 in rentals at the end of every three months. Assuming that the property can always be rented at these rates, and assuming that current rates are 6 per cent per annum compounded quarterly, what is the capitalized value of the property?

(3) In order to receive for himself and his descendants a perpetual income of \$5000 a year, what sum must Walter Haggerty invest at 5 per cent per annum?

(4) A graduate of a certain university presented the trustees with \$28,000 in order to provide for a fellowship in mathematics worth \$980 a year. At what annual rate of interest must this endowment be invested in order to obtain the required yearly income?

(5) A college receives by bequest a perpetual income of \$12,000 a year. Requiring ready cash in order to erect a library building, the college assigns its right to receive this perpetuity for \$400,000. What annual rate of interest was assumed in obtaining this present value?

(6) A ball falling from a height of 50 feet rebounds two-thirds of the previous fall each time it strikes the ground. How far will it travel before coming to rest?

(7) A pendulum swings through an arc 36 inches long. Each time that it swings, it travels through $\frac{7}{8}$ of the distance covered by the previous arc. How far will it travel before coming to rest?

(8) A skier runs 60 feet during the first second after reaching the foot of a hill. His distance decreases by 40 per cent during each second thereafter. How far will he run before coming to rest?

(9) A boy standing on the roof of a building 25 feet high throws a ball up to a height of 30 feet from the roof. If, after each fall, the ball rebounds to a height of $\frac{4}{7}$ of the distance from which it last fell, how far will it travel before coming to rest?

Pages 565-568

PROBLEMS:

(10) Given $f(x) = 2x^3 - 3x + 6$. Find: (a) $f(0)$; (b) $f(2)$; (c) $f(-1)$; (d) $f(a)$.

(11) Given $f(\theta) = \theta - \sin\theta$. Find: (a) $f(0)$; (b) $f\left(\frac{\pi}{2}\right)$; (c) $f\left(\frac{3\pi}{2}\right)$; (d) $f(\beta)$.

(12) Given $f(x) = 2x - \log(x + 1)$. Find: (a) $f(1)$; (b) $f(0)$; (c) $f\left(\frac{1}{x}\right)$; (d) $f(x + \Delta x)$.

Pages 565-568

EXERCISE C. Find the value of each of the following expressions:

(1) $\lim_{x\to 3}(3x^3 - 6x + 7)$

(2) $\lim_{x\to 2}\left(\frac{x^3}{4} + 3x - 15\right)$

(3) $\lim_{x\to 4}\frac{x-1}{\sqrt{x}}$

(4) $\lim_{x\to 2}\frac{x-2}{\sqrt{x-2}}$

(5) $\lim_{x\to 0}(7x^3 - 8)^{2/4}$

(6) $\lim_{x\to 2}\frac{x^2-4}{x-2}$

(7) $\lim_{x\to 3}\frac{x^2-x-6}{x-3}$

(8) $\lim_{x\to 2}\frac{x^3-x-6}{x^2-x-2}$

(9) $\lim_{x\to\infty}\frac{2x^3-6x+5}{3x^3-7x+4}$ (See next page.)

(*Hint:* Divide each term of the numerator and denominator by x^3, the highest power of x present in either. Find the limit of the resulting expression.)

(10) $\lim_{x\to\infty} \dfrac{x-10}{2x}$

(11) $\lim_{x\to\infty} \dfrac{x+5}{10x}$

(12) $\lim_{x\to\infty} \dfrac{6x^2-8x+7}{2x^2-9}$

(13) $\lim_{x\to\infty} \dfrac{5x^3-7x^2-5x-2}{2x^3+3x^2-8x-14}$

(14) $\lim_{x\to 0} \dfrac{3x^3-6x^2}{x^4+5x^2}$

(15) $\lim_{\theta\to 0} \csc\theta \sec\theta$

(16) $\lim_{\beta\to\frac{\pi}{2}} \dfrac{\sin\beta}{\cos\beta}$

(17) $\lim_{\phi\to 0} \dfrac{\sin\phi}{\cot\phi}$

(18) $\lim_{\alpha\to\frac{\pi}{2}} \dfrac{\csc\alpha}{\cot\alpha}$

(19) $\lim_{\theta\to\infty} \dfrac{\cos\theta}{\theta}$

Pages 565-568

PROBLEMS:

(13) Discuss the behavior of $\sin\dfrac{\pi}{\theta}$ near the origin.

(14) Prove that as $\beta\to 0$, the function $\cos\dfrac{\pi}{\beta}$ oscillates between -1 and $+1$, without approaching any limit.

(15) Discuss the behavior of $\tan\dfrac{\pi}{x}$ near the origin.

(16) Prove that the $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$. (*Hint:* Choose two points on a circle such that the chord joining the two points subtends an angle $2\theta < \pi$. Show that as $\theta\to 0$, the limit of the ratio of the chord to the arc it subtends is 1. Express the chord in terms of the radius and the sin θ, and denote the length of the arc in terms of radians.)

Pages 568-572

EXERCISE D. In each of the following expressions, find the average rate for the interval indicated and the instantaneous rate at the beginning of the interval:

(1) $y = 5x^2 - 6$ from $x = 3$ to $x = 3.1$

(2) $y = 3x^2 - 2x + 7$ from $x = 2$ to $x = 2.3$

(3) $y = \dfrac{8}{x}$ from $x = 4$ to $x = 4.2$

(4) $y = \dfrac{2x}{x-1}$ from $x = 2$ to $x = 2.5$

(5) $y = 5x - 7x^2$ from $x = 0$ to $x = 0.3$

(6) $y = 4 + 3x - 5x^2$ from $x = 1$ to $x = 1.4$

(7) $y = \dfrac{1}{x^2}$ from $x = 5$ to $x = 5.2$

(8) $y = \dfrac{x-1}{2x^2}$ from $x = 1$ to $x = 1.3$

Pages 568-572

PROBLEMS:

(17) Given a sphere of radius R. (*a*) Find the average rate of change of the surface of the sphere with respect to its radius if the latter changes from 5 to 5.2. (*b*) Find the average rate of change of the surface of the sphere with respect to its radius if the latter changes from 5 to any other length. (*c*) Find the instantaneous rate of change of the surface of the sphere with respect to its radius when the latter is 5. (*d*) Find the average rate of change of the surface of the sphere with respect to its radius and check the answer obtained for (*a*). (*e*) Find the instantaneous rate of change of the surface of the sphere with respect to its radius and check the answer obtained for (*c*).

(18) Given a circular cylinder whose height is equal to the radius of its base. (*a*) Find the average rate of change of the volume of the cylinder with respect to its radius if the latter changes from 2 to 2.3. (*b*) Find the average rate of change of the volume of the cylinder with respect to its radius if the latter changes from 2 to any other length. (*c*) Find the instantaneous rate of change of the volume of the cylinder with respect to its radius when the latter is 2. (*d*) Find the average rate of change of the volume of the cylinder with respect to its radius and check the answer obtained for (*a*). (*e*) Find the instantaneous rate of change of the volume of the cylinder with respect to its radius and check the answer obtained for (*c*).

(19) The distance (s feet) traveled by an automobile in t minutes, on a certain trip, is given by the equation $s = 250t - \frac{5}{4}t^2$. (*a*) Find the average speed during an interval of one-half minute beginning at $t = 4$. (*b*) Find the average speed during any interval beginning at $t = 4$. (*c*) Find the instantaneous speed at $t = 4$. (*d*) Find the average speed during any interval beginning at any time, and check the answer obtained for (*a*). (*e*) Find the instantaneous speed at any time and check the answer obtained for (*c*).

(20) The volume of a balloon being inflated with gas is $V = 3104t^3$, where V = volume in cubic feet and t = time in minutes. (*a*) Find the average rate of change of the volume during any interval beginning at any time. (*b*) Find the instantaneous rate of change of the volume at any time.

(21) The time of vibration, or *period*, of a pendulum may be approximately expressed by the formula $t = 1.11\sqrt{l}$, where t = time in seconds and l = length in feet. (*a*) Find the average rate of change of t with respect to l. (*b*) Find the instantaneous rate of change of t with respect to l.

Pages 573-574

EXERCISE E. Find the derivative of each of the following functions:

(1)	$y = x^3$	(6)	$y = -0.9x^{1/3}$	(11)	$y = x^{-3}$
(2)	$y = 3x^4$	(7)	$y = 27x^{2/3}$	(12)	$y = 7x^{-5}$
(3)	$y = -0.8x^3$	(8)	$y = 16x^{3/5}$	(13)	$y = x^{-3/2}$
(4)	$y = 6x$	(9)	$y = 3x^{-2}$	(14)	$y = -5x^{-3/5}$
(5)	$y = 2x^{1/2}$	(10)	$y = -5x^{-4}$	(15)	$y = \pi x^{-5/7}$

Pages 573-574

PROBLEMS:

(22) Show that the derivative of the area of a circle, with respect to its radius, is its circumference.

(23) Show that the derivative of the volume of a sphere, with respect to its radius, is its surface.

(24) A circular metal plate, with radius R, is being expanded by heat. Find the rate at which the area is being expanded when $R = 5$.

(25) The edge of a cube is increasing. Find the rate at which (*a*) the volume, (*b*) the total surface area, (*c*) the diagonal, of the cube increases when the edge increases.

(26) Sand poured from a spout forms a conical pile whose height is always equal to the diameter of its circular base. Find the rate at which the volume is changing with respect to the diameter of its circular base. How fast is the volume increasing when the height is 10 ft.?

Pages 575-579

EXERCISE F. Find by differentiation the slope of the tangent line of the curves represented by each of the following equations at the point indicated:

(1) $y = 4x^2$, where $x = 3$
(2) $y = x^2 - 4x$, where $x = 4$
(3) $y = \sqrt{x}$, where $x = 9$
(4) $y = 8x^3$, where $x = 2$
(5) $y = 2x^{3/2}$, where $x = 4$
(6) $y = x^3 - 6x$, where $x = 1$
(7) $y = 5 + 8x^2 - 0.5x^3$, where $x = 4$
(8) $y = 0.25x^3 - 1.5x^2 + 2x - 7$, where $x = 4$

Pages 575-579

EXERCISE G. Find by differentiation the points where the tangent line of the curves represented by each of the following equations is parallel to the X-axis. Check the result by sketching the graph. (*Hint:* See Problem 9, page 579.)

(1) $y = x^2 - 4x - 5$
(2) $y = 10 + 8x - 2x^2$
(3) $y = \frac{1}{3}x^3 - 4x$
(4) $y = \frac{2}{3}x^3 - 2x^2 - 6x$
(5) $y = x^2 + \frac{64}{x} + 16$
(6) $y = x^4 - 4x$
(7) $y = x^3 + 3x^2$
(8) $y = \frac{1}{4}x^4 - 8x^2$

Pages 575-579

PROBLEMS:

Solve the following problems by the method explained in Problem 9, page 579:

(27) Problems 7, 8, 9, and 10 of Chapter 17, page 637.

(28) Problems 38 and 39 of Chapter 27, page 675.

(29) The base of a chest is a square. The bottom and back are made of cedar, the remainder of mahogany. If mahogany is three times as expensive as cedar, find the most economical dimensions.

(30) The strength of a rectangular beam is given by the formula $S = kbd^2$, where S = strength of beam, b = breadth, d = depth, and k is a constant. Find the dimensions of the strongest beam that can be cut from a log 18 inches in diameter.

(31) Find the largest rectangle that can be inscribed in a circle.

(32) Find the volume of the largest container that can be made by cutting equal squares out of the corners of a piece of tin 12 by 8 inches and turning up the sides.

(33) Find the largest right circular cone that can be inscribed in a sphere of radius $2\sqrt{3}$.

APPENDICES

Appendix A

GREEK ALPHABET; LIST OF SYMBOLS; METHOD OF EXTRACTING SQUARE ROOT

GREEK ALPHABET

Letters		*Names*	*Letters*		*Names*
A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ϵ	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ, s	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

LIST OF SYMBOLS
Used in This Volume

$+$	Plus, positive, add.
$-$	Minus, negative, subtract.
$\times$, $\cdot$	Multiplied by, times.
$\div$, $/$, $\frac{}{}$	Divided by. Thus, $6 \div 3 = 6/3 = \frac{6}{3} = 2$.
$:$, $-$	Is *to*, ratio. Thus, $5 : 3 = \frac{5}{3}$; read, "5 is to 3."
$=$	Equals, is equal to.
$\neq$	Is not equal to.
$>$	Is greater than. Thus, $9 > 7$.
$<$	Is less than. Thus, $7 < 9$.
$\gtrless$	Is either greater or less than.
$\geqq$	Is either greater than or equal to.
$\leqq$	Is either less than or equal to.
$\sqrt[n]{a^m} = a^{\frac{m}{n}}$	The nth root of a to the mth power.
$a^{-n} = \frac{1}{a^n}$	Read aloud either, "a to the minus nth power," or "the reciprocal of the nth power of a."
$( )$	Parentheses.
$[ ]$	Brackets.
$\{ \}$	Braces.
$^\circ$	Degrees.
$'$	Minutes or feet.
$''$	Seconds or inches.
π	The ratio of the circumference of a circle to its diameter. $\pi = 3.14159265 \cdots$, usually taken as 3.1416, and for approximate results as $\frac{22}{7}$.
$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \lvert a\ b \rvert$	The $a\ b$ determinant.
$\lvert n \rvert$	The absolute value of the number n.
$\therefore$	Therefore.
$x \to a$	As x approaches a.
Δ	The change in.
∞	Infinity.
$\frac{dy}{dx}$	The derivative of y with respect to x.

TO FIND THE SQUARE ROOT OF A NUMBER

ILLUSTRATIVE EXAMPLE: Find the value of $\sqrt{5337.912}$ to three decimal places.

	53	37.	91	20	00	73.061
$7^2 =$	49					
	4	37				
143	4	29				
		8	91	20		
14606		8	76	36		
			14	84	00	
146121			14	61	21	
				22	79	

Step 1. Beginning at the decimal, separate the number into groups of two figures each. Indicate groups from the decimal point to the left for the integral part of the root, and from the decimal point to the right for the decimal part of the root. Ciphers may be added as needed to obtain as many decimal places in the answer as are asked for by the problem. Any single figure remaining at the left of the integral part, after the groups have been indicated, is counted as a group. Each group of the decimal part must contain two figures.

Step 2. Find the greatest number, 7, whose square, 49, is less than the first left-hand group, 53. Place it, that is, 7, as the first figure of the root. Subtract its square, 49, from the left group, 53. To the remainder, 4, annex the two figures of the second group, 37, making 437 the dividend.

Step 3. Double the first figure of the root for a trial divisor, $7 \times 2 = 14$. Find how many times the latter, 14, is contained in the dividend, 437, exclusive of the right-hand figure, 7. That is, 14 is contained 3 times in 43. Set the figure 3, representing that number of times, as the second figure of the root, and annex it to the right of the trial divisor, 14, forming the real divisor, 143.

Step 4. Multiply the real divisor, 143, by the last figure of the root, namely, 3, and subtract the product, 429, from the dividend 437. To the remainder, 8, annex the two figures of the next group, and proceed as before.

Note. If the trial divisor is not contained in the dividend, place a cipher in the root, bring down the next group, and double the root already found for the new trial divisor. Thus, in the above example, doubling the root 73 gives 146 for the trial divisor. The dividend 891, less the right-hand figure, is 89. The trial divisor 146 is not contained in 89; hence, place a cipher in the root, making it 730. Bring down the next group, 20, making the dividend 89120. Double the root already found, that is, $730 \times 2 = 1460$; then divide 1460

into the dividend less the right-hand figure. Place 6 as the next figure of the root and proceed as before.

Should the product of the last figure of the root by the real divisor be found greater than the dividend, erase the last figure of the root from both the root and the divisor, and substitute a figure small enough so that the product of the last figure of the root by the new real divisor is less than or equal to the dividend.

Appendix B
MENSURATION FORMULAS

Area of a rectangle whose base is **b** and altitude **h**	$A = b \cdot h$
Area of a square whose side is **s**	$A = s^2$
Area of a parallelogram whose base is **b** and altitude **h**	$A = b \cdot h$
Area of a triangle whose base is **b** and altitude **h**	$A = \frac{1}{2} b \cdot h$
Area of a trapezoid whose parallel sides are b_1 and b_2 and altitude **h**	$A = \frac{1}{2} (b_1 + b_2)h$
Area of a polygon whose apothem is **a** and perimeter **p**	$A = \frac{1}{2} p \cdot a$
Circumference of a circle whose radius is **r** and diameter **d**	$C = 2\pi r = \pi d$
Area of a circle whose circumference is **C**, radius **r**, and diameter **d**	$A = \frac{1}{2} Cr = \pi r^2 = \frac{1}{4} \pi d^2$
Length of an arc of a circle whose radius is **r**, where **n** is the angle between the radii in radians	$L = r \cdot n$
Area of a sector of a circle whose radius is **r**, where **L** is the length of the arc and **n** is the angle between the radii in radians	$A = \frac{1}{2} r \cdot L = \frac{1}{2} r^2 \cdot n$
Area of a segment of a circle whose radius is **r**, where **n** is the angle between the radii in radians	$A = \frac{1}{2} r^2 (n - \sin n)$
Volume of a regular prism whose altitude is **h** and area of base is **B**	$V = B \cdot h$
Volume of a cube whose edge is **e**	$V = e^3$
Lateral area of a regular prism whose lateral edge is **l** and perimeter of a right section is **p**	$S = l \cdot p$
Volume of a right circular cylinder the radius of whose base is **r** and altitude **h**	$V = \pi r^2 h$
Lateral area of a right circular cylinder the radius of whose base is **r** and altitude **h**	$S = 2\pi rh$
Volume of a pyramid whose altitude is **h** and area of base is **B**	$V = \frac{1}{3} Bh$
Lateral area of a regular pyramid whose slant height is **l** and perimeter of base is **p**	$S = \frac{1}{2} l \cdot p$
Volume of the frustum of a regular pyramid whose altitude is **h**, area of upper base is **S**, and area of lower base is **B**	$V = \frac{1}{3} h [B + S + \sqrt{BS}]$

Lateral area of the frustum of a regular pyramid whose slant height is l, perimeter of upper base is R, and perimeter of lower base is P	$S = \frac{1}{2} l (P + R)$
Volume of a right circular cone whose altitude is h and radius of base is r	$V = \frac{1}{3} \pi r^2 h$
Lateral area of a right circular cone the length of whose element is e, altitude is h, and radius of base is r	$S = \pi r e = \pi r \sqrt{r^2 + h^2}$
Volume of the frustum of a circular cone whose altitude is h, area of upper base is S, and area of lower base is B	$V = \frac{1}{3} h [B + S + \sqrt{BS}]$
Lateral area of the frustum of a right circular cone the length of whose element is e, radius of upper base is r_1, and radius of lower base is r_2	$S = \pi e (r_1 + r_2)$
Surface area of a sphere whose radius is r and whose diameter is d	$S = 4\pi r^2 = \pi d^2$
Surface area of a zone whose altitude is h and radius of the sphere is r	$Z = 2\pi rh$
Volume of a sphere whose radius is r and diameter d	$V = \frac{4}{3} \pi r^3 = \frac{1}{6} \pi d^3$
Volume of a spherical segment of two bases whose altitude is h, radius of upper base is r_1, and radius of lower base is r_2	$V = \frac{1}{6} \pi h [3r_1^2 + 3r_2^2 + h^2]$
Volume of a spherical segment of one base whose altitude is h and radius of the sphere is r	$V = \frac{1}{3} \pi h^2 (3r - h)$
Volume of a spherical cone where the altitude of the spherical segment forming the base is h and r is the radius of the sphere	$V = \frac{2}{3} \pi r^2 h$
Volume of a spherical sector where h is the altitude of the zone which forms the base and r is the radius of the sphere	$V = \frac{2}{3} \pi r^2 h$

Appendix C

LIST OF THE FACTS REGARDING SPATIAL RELATIONSHIPS

Note: The page number indicates where the fact is first introduced or proved.

CHAPTER 7

1. If two straight lines intersect each other, the vertical angles are equal. (p. 105)

CHAPTER 8

2. If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles are congruent. (p. 111)

3. If two sides of a triangle are equal, the angles opposite these sides are equal. (p. 112)

4. The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base. (p. 112)

5. An equilateral triangle is equiangular. (p. 112)

6. Two triangles are congruent if the three sides of one are equal respectively to the three sides of the other. (p. 113)

7. An exterior angle of a triangle is greater than either opposite interior angle. (p. 114)

8. When two lines are cut by a transversal, if a pair of alternate interior angles are equal, the two lines are parallel. (p. 115)

9. When two lines are cut by a transversal, if the alternate interior angles are unequal, the two lines are not parallel. (p. 115)

10. When two lines are cut by a transversal, if two corresponding angles are equal, or if two interior angles on the same side of the transversal are supplementary, the lines are parallel. (p. 115)

11. Two distinct lines in the same plane, perpendicular to the same line, are parallel. (p. 116)

12. Not more than one perpendicular can be drawn to a given line from a given external point. (p. 116)

13. If two parallel lines are cut by a transversal, the alternate interior angles are equal. (p. 117)

14. If two parallel lines are cut by a transversal, the corresponding angles are equal. (p. 118)

15. If two parallel lines are cut by a third straight line, the sum of the two interior angles on the same side of the transversal is equal to two right angles. (p. 118)

16. If a straight line is perpendicular to one of two parallel lines, it is also perpendicular to the other. (p. 118)

17. If two different lines are parallel to the same line, they are parallel to each other.
Distinct lines parallel to parallel lines are parallel.
Distinct lines perpendicular to parallel lines are parallel. (p. 118)

18. The sum of the interior angles of a triangle is a straight angle. (p. 119)

19. An exterior angle of a triangle is equal to the sum of the two opposite interior angles. (p. 119)

20. In a right triangle, the sum of the two acute angles equals one right angle. (p. 119)

21. If an acute angle of one right triangle equals an acute angle of another right triangle, the remaining acute angles of the two triangles are equal. (p. 119)

22. If two angles of one triangle are equal respectively to two angles of a second triangle, the third angles are equal. (p. 119)

23. If two angles and a side of one triangle are equal, respectively, to two angles and the corresponding side of another triangle, the triangles are congruent. (p. 119)

24. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal, respectively, to the hypotenuse and an acute angle of the other. (p. 119)

25. Each angle of an equilateral triangle contains 60°. (p. 119)

26. If two angles of a triangle are equal, the sides opposite these angles are equal. (p. 120)

27. Two right triangles are congruent if the hypotenuse and a leg of one are equal, respectively, to the hypotenuse and a leg of the other. (p. 121)

CHAPTER 9

28. If two straight line segments drawn from a point in a perpendicular to a given line cut off on the given line equal segments from the foot of the perpendicular, they are equal. (p. 126)

29. If one side of a triangle is greater than a second side, the angle opposite the first side is greater than the angle opposite the second side. (p. 136)

30. If one angle of a triangle is greater than a second angle, the side opposite the first angle is greater than the side opposite the second angle. (p. 137)

31. The perpendicular is the shortest line segment that can be drawn from a given point not on a given straight line, to this line. (p. 137)

32. If a line segment is the shortest that can be drawn from a given point to a given line, it is the perpendicular from the point to the given line. (p. 138)

33. If two triangles have two sides of one equal, respectively, to two sides of the other, and the included angles unequal, the triangle which has the greater included angle has the greater third side. (p. 139)

34. If two triangles have two sides of one equal, respectively, to two sides of the other and the third sides unequal, the triangle which has the greater third side has the greater angle opposite the third side. (p. 140)

35. If two straight line segments drawn from a point in a perpendicular to a given line, cut off on the given line unequal segments from the foot of the perpendicular, the more remote of the two given straight line segments is the greater. (p. 142)

CHAPTER 11

36. The opposite sides of a parallelogram are equal and the opposite angles are equal. (p. 159)

37. A diagonal divides a parallelogram into two congruent triangles. (p. 159)

38. Segments of parallel lines cut off by parallel lines are equal. (p. 159)

39. Two parallel lines are everywhere equidistant. (p. 159)

40. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram. (p. 161)

41. If three or more parallels cut off equal segments on one transversal, they cut off equal segments on any transversal. (p. 162)

42. A line parallel to one side of a triangle, bisecting a second side, bisects the third side. (p. 162)

43. A line parallel to one side of a triangle and intersecting the other two sides divides these two sides proportionally. (p. 163)

44. A line parallel to the base of a triangle cuts the other two sides so that one side is to a segment of that side as the other side is to the corresponding segment of the second side. (p. 164)

45. Three or more parallel lines cut off proportional segments on any two transversals. (p. 165)

46. If a line divides two sides of a triangle proportionally, it is parallel to the third side. (p. 168)

47. If two triangles have the three angles of one equal, respectively, to the three angles of the other, the triangles are similar. (p. 171)

48. If two triangles have two angles of one equal, respectively, to two angles of the other, the triangles are similar. (p. 171)

49. If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar. (p. 171)

50. If two triangles have their sides respectively proportional, the triangles are similar. (p. 174)

51. If two triangles have an angle of one equal to an angle of the other and the sides including these angles proportional, the triangles are similar. (p. 175)

CHAPTER 16

52. The area of a rectangle is the product of its base and altitude. (p. 254)

53. The area of a square is equal to the square of one of its sides. (p. 256)

CHAPTER 18

54. The area of a parallelogram is equal to the product of its base and altitude. (p. 282)

55. Two parallelograms having equal bases and equal altitudes are equal in area. (p. 282)

56. The ratio of the areas of two parallelograms having equal bases is equal to the ratio of their respective altitudes. (p. 283)

57. The ratio of the areas of two parallelograms, having equal altitudes, is equal to the ratio of their respective bases. (p. 283)

58. The ratio of the areas of any two parallelograms is equal to the ratio of the respective products of their bases and altitudes. (p. 283)

59. The area of a triangle is one-half the product of its base and altitude. (p. 283)

60. Two triangles having equal bases and equal altitudes are equal in area. (p. 284)

61. The ratio of the areas of two triangles having equal bases is equal to the ratio of their respective altitudes.

The ratio of the areas of two triangles having equal altitudes is equal to the ratio of their respective bases.

The ratio of the areas of any two triangles is equal to the ratio of the respective products of their bases and altitudes. (p. 284)

62. The area of a trapezoid is equal to one-half the product of its altitude and the sum of its bases. (p. 285)

63. If two angles have their sides respectively parallel, they are either equal or supplementary. (p. 286)

64. If two angles have their sides respectively perpendicular, they are either equal or supplementary. (p. 286)

65. The areas of two similar triangles are to each other as the squares of any two corresponding sides. (p. 287)

66. If two polygons are similar, they can be divided into the same number of triangles similar each to each and similarly placed. (p. 288)

67. If two polygons can be divided into the same number of triangles, similar each to each and similarly placed, the polygons are similar. (p. 288)

68. The areas of two similar polygons are to each other as the squares of any two corresponding sides. (p. 289)

CHAPTER 19

69. If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse,

(*a*) The two triangles thus formed are similar to the given triangle and similar to each other;

(*b*) The perpendicular is the mean proportional between the segments of the hypotenuse. (p. 294)

70. The square on the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. (p. 295)

71. The locus of points equidistant from two given points is the perpendicular bisector of the line joining them. (p. 306)

72. The locus of points within an angle equidistant from the sides is the ray that bisects the angle. (p. 306)

73. The diagonals of a parallelogram bisect each other. (p. 308)

CHAPTER 20

74. Through any three given points not lying in a straight line, one circle and only one can be drawn. (p. 317)

75. The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the three vertices. (p. 317)

76. If in the same circle, or in equal circles, two central angles are equal, the arcs which they intercept are equal. (p. 318)

77. If in the same circle, or in equal circles, two arcs are equal, the central angles which they determine are equal. (p. 318)

78. If in the same circle, or in equal circles, two chords are equal, the arcs which they determine are equal. (p. 319)

79. If in the same circle, or in equal circles, two arcs are equal, their chords are equal. (p. 320)

80. The diameter perpendicular to a chord of a circle bisects the chord and the arcs determined by the chord. (p. 320)

81. The perpendicular bisector of a chord of a circle passes through the center of the circle. (p. 321)

82. If in the same circle, or in equal circles, two chords are equidistant from the center, they are equal. (p. 321)

83. If in the same circle, or in equal circles, two chords are equal, they are equidistant from the center. (p. 322)

84. A tangent to a circle is perpendicular to the radius drawn to the point of contact. (p. 323)

85. A line perpendicular to a radius at its extremity on the circle is tangent to the circle. (p. 324)

86. Tangents drawn to a circle from an external point are equal. (p. 325)

87. Two parallel chords intercept equal arcs on a circle. (p. 326)

CHAPTER 21

88. An angle inscribed in a circle is measured by one-half its intercepted arc. (p. 329)

89. An angle formed by two secants, by a secant and a tangent, or by two tangents, intersecting outside a circle, is measured by one-half the difference of the intercepted arcs. (p. 337)

90. An angle inscribed in a semicircle is a right angle. (p. 331)

91. An angle formed by a tangent and a chord drawn from the point of contact is measured by one-half the intercepted arc. (p. 334)

92. An angle formed by two chords intersecting within a circle is measured by one-half the sum of the intercepted arcs. (p. 336)

93. If in the same circle, or in equal circles, two minor arcs are unequal, the greater arc determines the greater chord; and conversely, if two chords are unequal, the greater chord determines the greater minor arc. (p. 341)

94. If in the same circle, or in equal circles, two chords are unequal, the longer chord is nearer the center. (p. 341)

95. If in the same circle, or in equal circles, two chords are unequally distant from the center, the chord nearer the center is the longer. (p. 341)

96. If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other. (p. 342)

97. If, from a point outside a circle, a tangent and a secant are drawn to the circle, the tangent is the mean proportional between the secant and its external segment. (p. 343)

CHAPTER 22

98. A circle can be circumscribed about any regular polygon. (p. 349)

99. A circle can be inscribed in any regular polygon. (p. 349)

100. An equilateral polygon inscribed in a circle is regular. (p. 350)

101. If a circle is divided into any number of equal parts, the chords of these arcs form a regular inscribed polygon, and the tangents drawn at the points of division form a regular circumscribed polygon. (p. 351)

102. The perimeters of two similar polygons have the same ratio as any two corresponding sides. (p. 352)

103. The sum of the angles of a convex polygon of n sides is $(n - 2)$ straight angles. (p. 354)

104. Each angle of a regular polygon of n sides is equal to $\frac{(n-2)}{n}$ straight angles. (p. 354)

105. Regular polygons of the same number of sides are similar. (p. 355)

106. The perimeters of two regular polygons of the same number of sides are to each other as their radii or as their apothems. (p. 355)

107. The area of a regular polygon is equal to one-half the product of its perimeter and its apothem. (p. 357)

108. The circumferences of two circles are to each other as their radii. (p. 361)

109. The ratio of the circumference of any circle to its diameter is constant. (p. 361)

110. The area of a circle is equal to one-half the product of its circumference and its radius, or to the product of the square of the radius and the constant π. That is: $A = \frac{1}{2} Cr = \pi r^2$. (p. 362)

111. The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters. (p. 364)

CHAPTER 26

112. If two planes intersect, their intersection is a straight line. (p. 465)

113. If a plane intersects two parallel planes, the intersections are *parallel* lines. (p. 466)

114. If a plane contains only one of two parallel lines, it is parallel to the other line. (p. 467)

115. If a line is parallel to a plane, the intersection of the plane with a plane passed through the line is parallel to the line. (p. 467)

116. If two intersecting lines are parallel to a plane, their plane is parallel to the plane. (p. 468)

117. If a plane intersects one of two parallel lines, it intersects the other also. (p. 468)

118. If two distinct lines are parallel to the same line, they are parallel to each other. (p. 469)

119. If two angles not in the same plane have their sides respectively parallel and extending in the same direction from their vertices, they are equal. (p. 469)

120. If two straight lines are cut by three parallel planes, the corresponding segments are proportional. (p. 471)

121. If two parallel planes cut a series of concurrent lines, the corresponding segments are proportional. (p. 472)

122. If a line is perpendicular to two lines at their point of intersection, it is perpendicular to any line in their plane through that point. (p. 474)

123. The sections of a prism cut by two parallel planes intersecting the lateral edges are congruent polygons. (p. 478)

124. If a pyramid is cut by a plane parallel to the base,

(*a*) The lateral edges and the altitude are divided proportionally;

(*b*) The section is a polygon similar to the base. (p. 480)

125. If two pyramids having equal bases and altitudes are cut by a plane parallel to their bases and equidistant from their vertices, the sections formed are equal. (p. 481)

126. If a pyramid is cut by two parallel planes, the areas of the sections are proportional to the squares of their distances from the vertex. (p. 482)

CHAPTER 27

127. The volume of a rectangular solid is the product of its base and altitude. (p. 491)

128. The volume of a cube is equal to the cube of one edge. (p. 491)

129. Cavalieri's theorem: If, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, the volumes of the solids are equal. (p. 493)

130. Two prisms having equal bases and equal altitudes are equal in volume. (p. 495)

131. The volume of any prism is equal to the product of its base and altitude. (p. 495)

132. The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge. (p. 496)

133. The sections of a cylinder made by two parallel planes cutting all elements are congruent. (p. 499)

134. The volume of a circular cylinder is equal to the product of its base and altitude. (p. 500)

135. The lateral area of a right circular cylinder is equal to the product of the perimeter of its base and an element. (p. 501)

CHAPTER 28

136. Two pyramids having equal bases and equal altitudes are equal in volume. (p. 504)

137. The volume of a triangular pyramid equals one-third of the product of its base and altitude. (p. 506)

138. The volume of any pyramid equals one-third of the product of its base and altitude. (p. 507)

139. The lateral area of a regular pyramid equals the product of one-half the slant height and the perimeter of the base. (p. 508)

140. The lateral faces of a frustum of a regular pyramid are congruent trapezoids. (p. 509)

141. The volume of a frustum of a pyramid is equal to one-third the product of the altitude and the sum of the upper base, the lower base, and the mean proportional between the two bases. (p. 511)

142. The lateral area of the frustum of a regular pyramid is equal to one-half the product of the slant height and the sum of the perimeters of the bases. (p. 511)

143. The volume of a circular cone is equal to one-third the product of its base and altitude. (p. 514)

144. The lateral area of a right circular cone is equal to one-half the product of the perimeter of the base and the slant height. (p. 515)

145. The volume of a frustum of a circular cone is equal to one-third the product of the altitude and the sum of the upper base, the lower base, and the mean proportional between the two bases. (p. 517)

146. The lateral area of a frustum of a right circular cone is equal to one-half the product of the slant height and the sum of the circumferences of the bases. (p. 518)

147. The surface area of a sphere is equal to the area of four of its great circles. (p. 522)

148. The surface area of a zone equals the product of the circumference of a great circle of the sphere and its altitude. (p. 522)

149. The volume of a sphere is equal to the product of $\frac{4}{3}\pi$ and the cube of its radius. (p. 524)

150. The volume of a spherical segment of two bases is $V = \frac{1}{6}\pi h\,(3r_1^2 + 3r_2^2 + h^2)$, where h is the altitude and r_1 and r_2 are, respectively, the radii of the bases of the segment. (p. 525)

151. The volume of a spherical segment of one base is $V = \frac{1}{3}\pi h^2\,(3r - h)$, where h is the altitude of the segment and r is the radius of the sphere. (p. 526)

152. The volume of a spherical cone is $V = \frac{2}{3}\pi r^2 h$, where h is the altitude of the spherical segment which forms the base of the cone and r is the radius of the sphere. (p. 527)

153. The volume of a spherical sector is $V = \frac{2}{3}\pi r^2 h$, where h is the altitude of the zone which forms the base of the sector and r is the radius of the sphere. (p. 528)

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