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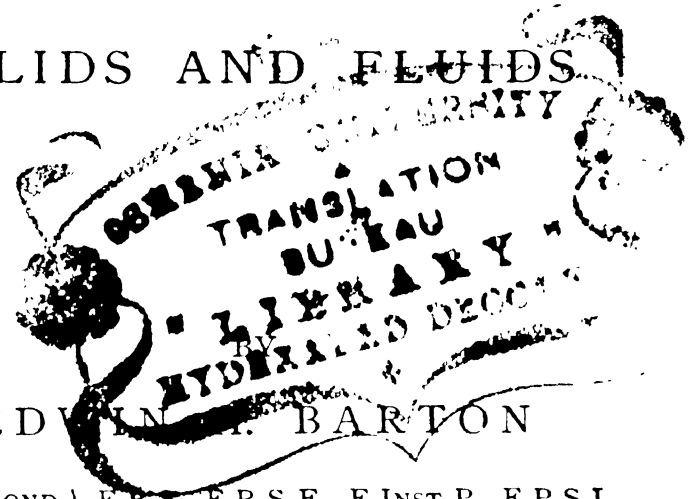
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# Analytical Mechanics

COMPRISING THE

## KINETICS AND STATICS OF SOLIDS AND FLUIDS




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# PREFACE

THIS text-book on theoretical mechanics is intended for those possessing, or concurrently acquiring, an elementary knowledge of the calculus. Within these bounds it is fairly complete, dealing with the kinetics and statics of solids and of fluids. Further, in the kinematical portion, mechanisms and strains are included, and the work closes with a short chapter on elasticity.

In the transition from kinematics to kinetics, Newton's principles and the views subsequently held respecting them are passed in review. This critical treatment culminates in a set of proposed enunciations. To minds thus prepared these enunciations, though brief enough to be easily remembered, may serve to recall a sort of central position of modern thought on dynamical axioms. But no finality has yet been reached on these philosophical topics. Their discussion is accordingly confined to a single chapter. This leaves the formal mathematical developments equally readable to those holding the most diverse views as to the foundations underlying this superstructure.

Probably most users of the book will bring to its study some previous knowledge of the subject, the amount and freshness of which differ widely in individual cases. To provide for such variety of preliminary attainment, the elementary parts are briefly outlined to serve as a revision or reference and for logical completeness. Similarly, parts lying beyond the central scope of the work are often indicated and sources of fuller information quoted.

The work is not written narrowly to any one examination syllabus. But its general scope and treatment will be found to meet the needs of degree candidates of London and other

universities at home and abroad, also of those offering the third and honours stages of the Board of Education. The arrangement is such that any candidate not requiring the whole would usually find his course provided for by a certain selection of chapters taken entire, the others being wholly omitted. This fact conduces to a simplicity and coherence in these special courses which could not be attained if the chapters themselves needed minute subdivision into parts to be read or omitted.

As to order of treatment: after a short introductory part there follow Kinematics, Kinetics, Statics, Hydromechanics, and Elasticity. But the detachment of Chapters II., III., and XI., giving respectively formulae, geometrical basis and physical basis, will enable a student or teacher to take the other sections of the book in a different order if preferred.

Through the body of the work, at frequent intervals, are given sets of examples mostly of a simple character and strictly on the text. At the end occur additional examples of a harder or more varied character, some classified, some miscellaneous, also subjects for essay-writing. These bring the total number of examples almost to eight hundred. It was intended to give hints for the solution of the problems set, but considerations of space forbad the inclusion of such a section in the present volume.

In addition to the great classic authors, too well known to need mention here, many other authorities have been consulted and quoted, as may be seen by the index, in which proper names are italicised.

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To Mr. H. T. H. Piaggio, M.A.(Cantab.), cordial thanks are offered for his careful reading of the proofs, and for his valuable comments thereon.

If any readers noticing errors, omissions, or obscurities would communicate such, their kindness would be highly appreciated, and subsequent editions in consequence improved.

NOTTINGHAM, *July* 1911.

## PREFACE TO SECOND EDITION

IN preparing for the press this second edition, the text has been carefully revised, a number of errors corrected, and alternative proofs added where deemed desirable.

The value of the work has been much enhanced by the addition of nearly three hundred Examples chosen from London examination questions set in the years 1911-1923. The total number of examples included is thus considerably over a thousand. To these a complete set of answers is now given separately at the end.

For kind permission to publish these questions thanks are duly tendered to the University of London. Also for valuable suggestions and help in checking many of the answers, cordial thanks are hereby offered to various colleagues, viz. :—

Professor H. T. H. Piaggio, M.A., D.Sc., Dr. John Marshall, M.A., F.R.S.E., and H. Gwynedd Green, M.A.

While it is thus hoped that no serious slips have escaped detection, readers are assured that their kindness in notifying any error or obscurity, yet remaining, would be warmly welcomed.

NOTTINGHAM, *December* 1923.

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# ANALYTICAL MECHANICS

## PART I.—INTRODUCTORY

### CHAPTER I

#### PRELIMINARY SURVEY

**1. Scope of Mechanics.**—The theory of Mechanics deals with space, time, and mass. But if left with so vague a description it would be found to include both Physics and Chemistry, instead of being but the simpler part of the former as it really is. In order to see more precisely how Mechanics is limited, it will be a convenience to note what the above broad statement might include and how it must then be reduced. This may be done as follows:—Take, in imagination, a number of columns and head each with the name of some material system, such as *a single particle*, *two particles*, *rigid body*, etc. Next, divide the columns into a number of lines or rows, reserving to each row some one definite type of attracting or other influence or constraint, under which the systems might be placed, such as *gravitational attraction*, *one point fixed*, and so forth. We should then obtain in this table a number of spaces or squares each referring to a specified system placed under given conditions. And, with respect to each such square, Mechanics would be concerned with two types of problems, viz.:—(1) What motion ensues for each possible initial configuration or motion? (2) What forces or initial configurations are necessary in order that the subsequent motion may be of some given type, including the special case of rest? Thus, under the first type of problem, we may be asked what happens if a pendulum bob be pulled aside and let go. While, under the second, we may be asked (*a*) what the length of the pendulum must be so that it shall beat seconds, or (*b*) what force is required to keep it pulled aside a given amount.

Now, in our supposed table, the columns and the lines each extend without any definite limit; hence the possible number of squares is doubly infinite, the total number of cases being further complicated by the double or treble nature of the problems attaching to each square.

But all the above applies to the initial vague statement as to the scope of Mechanics, and which really includes also Physics and Chemistry. We exclude these two branches of science by restricting to *their simpler forms* the material systems contemplated. And specially we restrict our attention to systems whose constituent parts are of known shape and number. The line between the systems retained and those excluded is somewhat arbitrary, and when drawn so as to relegate

explained as to be intelligible to those without any previous conversance with them.

**3. Order of Treatment.**—Since we are so much concerned with motions, we shall consider them first apart from their causes. This necessary preliminary branch, called *Kinematics*, occupies Chapters III.-X., forming the second part of the work. Kinematics rests on the conceptions of space and time only, and leads naturally to the third part, called *Kinetics*. In this we study motions, having regard to the circumstances under which they may be expected to occur. But, to form a basis for this, something beyond conceptions of space and time are needed. The conception of mass must be introduced. And for the part it plays we must fall back upon universal experience as generalised, co-ordinated, and formulated by thinkers ancient and modern. A *résumé* of this in Chapter XI. accordingly introduces the third part; the kinetics of particles and rigid solids occupying Chapters XII.-XIV. Statics is next dealt with in Chapters XV.-XVIII., the treatment concluding with brief chapters on *Hydromechanics* and *Elasticity*.

This order has been adopted as appearing on the whole most convenient. But no possible sequence is free from objection. Readers wishing to take the several parts in a different order will find their task simplified (*a*) by the collection of preliminary notions and theorems which occupies Chapter III., and (*b*) by the mathematical formulae given for reference in Chapter II.

EXAMPLES—I.

1. State what you understand by Mechanics, showing clearly how it is distinguished from Physics.
2. Make a scheme of the subdivisions of Mechanics, dealing with the various possible systems and their states of motion or rest.
3. Analyse some familiar mechanical phenomena, indicating the various approximations which may be made in the endeavour to treat it mathematically.

# CHAPTER II

## FORMULAE

**4. Object and Use of this Collection.**—When solving mechanical problems a number of mathematical formulae are needed. Many of these are usually remembered readily as required. But to each student at times there may occur a lapse of memory, or his knowledge in a certain essential may be found incomplete. To obviate the necessity of reference to other books in the first case, and to direct attention to the defect in the second case, the collection of formulae composing this chapter is placed here. No one student is likely to require every one of these formulae. But while only the formulae of an advanced character may be referred to by the stronger readers, the more elementary examples may be useful to those less highly equipped. Further, the collection as a whole serves to indicate the scope of the mathematical knowledge which should be possessed or soon acquired by the student. Thus the ground indicated by these formulae in algebra, plane trigonometry, and plane co-ordinate geometry is supposed already familiar to the student of this book. The study of the elements of the differential and integral calculus must be undertaken concurrently with the reading of this book, if not already possessed. While the systematic study of differential equations may be deferred for a time, though, of course, its possession is a great advantage.

**Algebra.**

*Binomial Expansions.*

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \frac{n(n-1)(n-2)}{\lfloor 3} x^3 + \ldots$$

For $n$, a *positive* integer, the right side is clearly a finite series, and then correctly expresses the left side for any value of $x$.

For $n$, a *negative* integer or *fractional*, the right side is an infinite series; but, *provided* $x<1$ numerically, it is a convergent series which truly expresses the value of the left side.

When $x$ is very small, $h$ say, whose square is negligible in comparison with unity, we may write

$$(1+h)^n = 1 + nh \text{ nearly.}$$

*Exponential Series.*

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots = 2{\cdot}71828183 \text{ nearly.}$$

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots$$

$$a^x = e^{x \log_e a} = 1 + x \log_e a + \frac{x^2 \log_e^2 a}{\lfloor 2} + \ldots$$

*Logarithmic Series, etc.*

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ when } x < 1 \text{ numerically.}$$

$$\log_b N = (\log_a N) \div (\log_a b).$$

$$\log_e N = \frac{\log_{10} N}{\log_{10} e} = 2{\cdot}30258509 \log_{10} N.$$

## 5. Plane Trigonometry.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$
$$\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1.$$
$$\frac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A.$$
$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B.$$
$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B.$$
$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B.$$
$$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B.$$

*Note.*—The angles on the right are the half-sum and half-difference of those on the left.

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

*Solution of Triangles.*

$$A + B + C = \pi \text{ or } 180^\circ.$$
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$
$$a^2 = b^2 + c^2 - 2bc \cos A.$$
$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

where $s$ is the half-sum of the sides $a$, $b$, and $c$.

Area of triangle $= \frac{1}{2}bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}$.

*De Moivre's Expansion.*

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta,$$

where $i = \sqrt{-1}$ and $n$ is any integer positive or negative. When $n$ is fractional one value of the left side is given by the right as it stands, the others being found by writing in it $\theta + 2\pi$, $\theta + 4\pi$, etc., for $\theta$.

Thus, for the cube root of $a + ib$, writing $a = r \cos\theta$, and $b = r \sin\theta$, we have $r^2 = a^2 + b^2$ $\tan\theta = b/a$.

Hence

$$(a+ib)^{1/3}=r^{1/3}(\cos\theta+i\sin\theta)^{1/3}$$
$$=r^{1/3}\left(\cos\frac{\theta}{3}+i\sin\frac{\theta}{3}\right)$$
$$\text{or } r^{1/3}\left(\cos\frac{\theta+2\pi}{3}+i\sin\frac{\theta+2\pi}{3}\right)$$
$$\text{or } r^{1/3}\left(\cos\frac{\theta+4\pi}{3}+i\sin\frac{\theta+4\pi}{3}\right).$$

*Sine and Cosine Series.*

$$\cos\theta=1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\frac{\theta^6}{\lfloor 6}+\ \ldots$$

$$\sin\theta=\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\frac{\theta^7}{\lfloor 7}+\ \ldots$$

*Exponential Values of Sine and Cosine.*

$$2\cos\theta=e^{i\theta}+e^{-i\theta}.$$
$$2i\sin\theta=e^{i\theta}-e^{-i\theta}.$$
$$\cos\theta\pm i\sin\theta=e^{\pm i\theta}.$$

*Hyperbolic Sine and Cosine.*

$$2\cosh\theta=e^{\theta}+e^{-\theta}.$$
$$2\sinh\theta=e^{\theta}-e^{-\theta}.$$

**Spherical Triangle** of spherical angles $A$, $B$, and $C$ and opposite *sides* subtending at the centre of the sphere the *angles* $a$, $b$, and $c$.

$$\cos a=\cos b\cos c+\sin b\sin c\cos A.$$

## 6. Plane Co-ordinate Geometry.

*The Straight Line* may be represented by $y=mx+b$, $\frac{x}{a}+\frac{y}{b}=1$, or $x\cos\alpha+y\sin\alpha=p$.

The general form is $ax+by+c=0$.

The perpendicular on this from $(h, k)$ has the length $l=\frac{ah+bk+c}{\sqrt{a^2+b^2}}$, its equation being $bx-ay+ak-bh=0$.

Two lines through the origin are given by $b^2x^2\pm a^2y^2=0$.

*Polar Equation*, $r\cos(\theta-\alpha)=p$.

*The Circle* of radius $a$ with centre at $(h, k)$ is

$$(x-h)^2+(y-k)^2=a^2.$$

The tangent to it at $(x', y')$ is

$$(x-h)(x'-h)+(y-k)(y'-k)=a^2.$$

*Polar Equation* is $r^2-2Rr\cos(\theta-\alpha)+R^2-a^2=0$ when the radius is $a$ and the centre at $(R, \alpha)$.

*The Parabola* is represented by $y^2=\pm 4ax$, or $x^2=\pm 4ay$, with origin at the vertex.

If the vertex is at $(h, k)$, and the axis along the negative direction of $y$, the equation is

$$(x-h)^2+4a(y-k)=0,$$

the focus being at $(h, k-a)$, and the directrix being $y=k+a$.

The tangent at $(x', y')$ is

$$(x-h)(x'-h)+2a(y-k+y'-k)=0.$$

If $h^2=4ak$, the parabola passes through the origin $P$, and the tangent there is

$$hx-2ay=0.$$

Calling the focus $S$, $R$ denoting the point $(x', y')$ on the curve, and $Q$ the point of abscissa $x'$ on the tangent through the origin $P$, we have $PS=a+k$, $PQ^2=x'^2(a+k)/a$,

and $QR=x'^2/4a$,

whence $PQ^2=4PS.QR$—a useful relation.

*The Ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has at the point $(x', y')$

the tangent $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$.

The semi-axes $a$ and $b$ satisfy $b^2=a^2(1-e^2)=al$, where $e$ is the eccentricity and $l$ the semi-latus rectum.

If the foci are denoted by $S$ and $S'$, the corresponding extremities of the major axis by $A$ and $A'$, $r$ and $r'$ being the focal radii to a point $P$ on the curve, and $p$ and $p'$ the perpendiculars from the foci on the tangent at $P$, we have the following properties:—

$$AS.SA'=b^2=pp',$$

$$p/p'=r/r',$$

and $$r+r'=2a.$$

Also, if $\rho$ be the radius of curvature of the ellipse at $P$, and $c$ the semi-axis conjugate to that through $P$, then

$$c^2=rr',$$

and $$c^3=ab\rho.$$

*The Hyperbola* and its conjugate are represented by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\pm 1,$$

the asymptotes of both being $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.

*General Equation of a Conic.*

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

*Geometrical Relation fulfilled by any Conic.*

$SP/PM$=a constant ratio, $e$ say, where $S$ is the focus, $P$ a point on the conic, and $M$ the foot of the perpendicular from $P$ on the directrix. The conic is an ellipse, parabola, or hyperbola according as $e<1$, $e=1$, or $e>1$.

*Polar Equation of a Conic, its Focus being the Pole.*

$$l/r=1-e\cos\theta.$$

**Solid Co-ordinate Geometry**, the axes being rectangular.

*Co-ordinates of a Point* $(x, y, z)$.

*Equations of a Plane.*

General form, $Ax+By+Cz+D=0$.

Perpendicular form, $lx+my+nz=p$,

where $p$ is the length of the perpendicular from the origin on to the plane, and $l$, $m$, $n$ are the *direction cosines* of that perpendicular (*i.e.* the cosines of its angles with the axes).

*The Straight Line* may be represented as the intersection of two planes of equations, $lx+my=1$ and $ny+pz=1$; or we may write for a line

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n},$$

where $(a, b, c,)$ and $(x, y, z)$ are points on the line whose direction cosines are $l$, $m$, and $n$.

*The Angle between Lines* is given by $\cos\theta=ll'+mm'+nn'$, where $l$, $m$, $n$ are the direction cosines of one line, and $l'$, $m'$, $n'$ those of the other.

*Sphere*, $x^2+y^2+z^2=a^2$.

*Ellipsoid*, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

*Cone*, vertex as origin, $Ax^2+By^2+Cz^2=0$.

*Right Circular Cone*, about $Oz$ as axis, $A(x^2+y^2)-Cz^2=0$.

**7. Differential and Integral Calculus.**—The following list may be regarded as giving on the right the differential coefficients of the functions on the left, or as giving on the left the result of integrating the functions on the right.

*Simple Algebraic Functions*:—

| *Integrals* $y$ or $\int\frac{dy}{dx}dx$ | *Differential Coefficients* $\frac{dy}{dx}$ |
|---|---|
| $x^n$ | $nx^{n-1}$ |
| $e^x$ | $e^x$ |
| $e^{bx}$ | $be^{bx}$ |
| $a^x$ | $a^x\log_e a$ |
| $\log_e x$ | $\frac{1}{x}$ |
| $\log_a x$ | $\frac{1}{x\log_e a}$ |
| $uv$ [1] | $v\frac{du}{dx}+u\frac{dv}{dx}$ |
| $\frac{u}{v}$ | $\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ |
| $\sqrt{x^2+a^2}$ | $\frac{x}{\sqrt{x^2+a^2}}$ |

[1] From this we have, for Integration by parts, $\int vdu=uv-\int udv$.

*Trigonometrical Functions*:—

| *Integrals* $y$ or $\int \frac{dy}{dx} dx$ | *Differential Coefficients* $\frac{dy}{dx}$ |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ |
| $\cot x$ | $-\operatorname{cosec} x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\operatorname{cosec} x$ | $-\operatorname{cosec} x \cot x$ |

*Inverse Functions*:—

| *Integrals* $y$ or $\int \frac{dy}{dx} dx$ | *Differential Coefficients* $\frac{dy}{dx}$ |
|---|---|
| $\sin^{-1} x$ | $\pm \frac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1} x$ | $\mp \frac{1}{\sqrt{1-x^2}}$ |
| $\tan^{-1} x$ | $+ \frac{1}{1+x^2}$ |
| $\cot^{-1} x$ | $- \frac{1}{1+x^2}$ |
| $\sec^{-1} x$ | $\pm \frac{1}{x\sqrt{x^2-1}}$ |
| $\operatorname{cosec}^{-1} x$ | $\mp \frac{1}{x\sqrt{x^2-1}}$ |

**8.** *Hyperbolic Functions*:—

| *Integrals* $y$ or $\int \frac{dy}{dx} dx$ | *Differential Coefficients* $\frac{dy}{dx}$ |
|---|---|
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\operatorname{sech}^2 x$ |
| $\coth x$ | $-\operatorname{cosech}^2 x$ |
| $\operatorname{sech} x$ | $-\operatorname{sech} x \tanh x$ |
| $\operatorname{cosech} x$ | $-\operatorname{cosech} x \coth x$ |

*Harder Miscellaneous Functions*:—

| *Integrals* $y$ or $\int \frac{dy}{dx} dx$ | *Differential Coefficients* $\frac{dy}{dx}$ |
|---|---|
| $\frac{1}{a} \tan^{-1} \frac{x}{a}$ | $\frac{1}{a^2+x^2}$ |

| | |
|---|---|
| $\frac{1}{2a}\log_e\frac{a+x}{a-x}$ | $\frac{1}{a^2-x^2}$, $(x<a)$ |
| $\frac{1}{2a}\log_e\frac{x+a}{x-a}$ | $-\frac{1}{x^2-a^2}$, $(x>a)$ |
| $\sin^{-1}\frac{x}{a}$ | $\pm\frac{1}{\sqrt{a^2-x^2}}$, $(x<a)$ |
| $\cosh^{-1}\frac{x}{a}$ or $\log_e(x+\sqrt{x^2-a^2})$ | $\pm\frac{1}{\sqrt{x^2-a^2}}$, $(x>a)$ |
| $\sinh^{-1}\frac{x}{a}$ or $\log_e(x+\sqrt{x^2+a^2})$ | $\frac{1}{\sqrt{x^2+a^2}}$ |
| $\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\log_e(x+\sqrt{x^2+a^2})$ | $\sqrt{x^2+a^2}$ |
| $\frac{1}{a}\sec^{-1}\frac{x}{a}$ | $\pm\frac{1}{x\sqrt{x^2-a^2}}$, $(x>a)$ |
| $\frac{1}{2a}\log_e\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}$ | $-\frac{1}{x\sqrt{a^2-x^2}}$, $(a>x)$ |
| $\frac{1}{a}\log_e\frac{x}{a+\sqrt{a^2\pm x^2}}$ | $\frac{1}{x\sqrt{a^2\pm x^2}}$ |

*Conception of Definite Integral.*—The following paragraph may be of service to some who have to use definite integrals in mechanics before they have reached them in their systematic study of pure mathematics.

Consider the area OPM between some curve OP, a portion OM of the axis of $x$ and the ordinate MP, and denote this area by

$$u=x^n \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

the curve OP being such as to fulfil this relation.

Then, if MP shifts to M'P' by the very small increment $MM'=h$ or $dx$, we may write

$$\frac{du}{dx}=\frac{\text{Area MPP'M'}}{MM'}=MP=y \quad \ldots\ldots\ldots \quad (2),$$

but by (1) $\frac{du}{dx}=nx^{n-1}$, so $y=nx^{n-1}$ $\ldots\ldots\ldots$ (3).

Take now any value $a$ of $x$, and erect the ordinate $aA$, cutting the curve OP in A. Then the area of $OaA$ is $a^n$, but it may be regarded as made up of vertical strips each of area $yh$ or $ydx$. Thus we have, except for $n=0$, the summational formula,

$$\Sigma_0^a nx^{n-1}h=a^n \quad \ldots\ldots\ldots\ldots \quad (4);$$

or, in the notation of the integral calculus,

$$\int_0^a nx^{n-1}dx=a^n \quad \ldots\ldots\ldots\ldots \quad (5).$$

For $n=0$, the summation yields a logarithm; see Art. 7, line 5.

**9. Differential Equations.**—Many mechanical problems lead to differential equations in which the *variables are separable.* Hence, after the separation, such an equation may be integrated and the solution readily obtained. Thus the differential equation

$$\frac{dy}{dx}+ay+b=0$$

may be written $$\frac{ady}{ay+b}+adx=0.$$

And this integrates to $\log_e(ay+b)+ax=C,$

which is equivalent to $ay+b=Be^{-ax},$

the $B$ and $C$ being constants.

Other differential equations often occur in mechanics whose solutions are very easy. Thus, some may be dealt with by multiplying by an integrating factor or assuming a trial solution of the form $e^{mx}$, or of some other form suggested by mechanical considerations. It must suffice here to notice the following important types, the solutions of which may be verified by differentiation. When any differential equations occur in the subsequent text, they are dealt with simply so as to be understood by students not having any previous knowledge of them, though, of course, such knowledge is a great advantage.

Thus, the differential equation

$$\frac{d^2y}{dx^2}=p^2y$$

is easily found, by trial of $y=e^{mx}$, to be satisfied by

$$y=Ae^{-px}+Be^{px},$$

the quantities $A$ and $B$ being arbitrary constants to be fixed by the initial conditions.

Again, the differential equation

$$\frac{d^2y}{dx^2}+p^2y=0$$

is satisfied by $y=A\sin px+B\cos px,$

$A$ and $B$ being arbitrary constants depending on the initial state. If $x$ refers to time, this is obviously a to-and-fro motion or vibration, the former case being a subsidence if $B$ vanishes.

For the differential equation

$$\frac{d^2y}{dx^2}+2k\frac{dy}{dx}+p^2y=0$$

the solution may be written

$$y=e^{-kx}(A\sin qx+B\cos qx),$$

where $q^2=p^2-k^2$. This gives a diminishing vibration if $x$ is time.

The differential equation

$$\frac{d^2y}{dx^2}+p^2y=f\sin nx$$

is satisfied by $$y=\frac{f\sin nx}{p^2-n^2}.$$

# PART II.—KINEMATICS

## CHAPTER III

### GEOMETRICAL BASIS

**10. Space and Position.**—Geometry treats of space; and the space of ordinary human experience has three dimensions only, often referred to as length, breadth, and thickness. By a process of abstraction we can conceive of space of two dimensions only (or even of one only). We thus have the geometry of two dimensions as well as that of three. Or we have *plane* geometry as well as *solid*. And it is convenient to deal first with the position of points in a plane.

To fix the position of one point relative to another, both being in a given plane, we need either (1) two distances along given directions, or (2) one distance and an angle with a given direction in the plane. These methods form respectively the *cartesian* and *polar* systems of *co-ordinates*. They are illustrated in Fig 1.

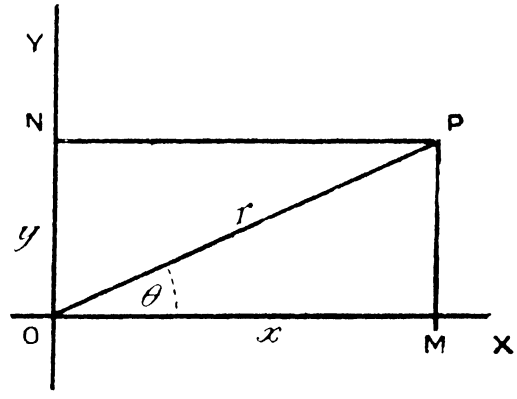



FIG. 1. PLANE CO-ORDINATES.

Thus, on the cartesian system, the position of P with respect to O is fixed by the two distances or co-ordinates, OM called $x$ and ON called $y$, MP and NP being respectively parallel to the two directional axes OY and OX.

On the polar system, P's position is fixed by the distance OP called $r$, and the angle called $\theta$ which OP makes with the fixed direction OX.

It is accordingly obvious that the following relations hold when as usual OX and OY are at *right* angles :—

$$x = r\cos\theta \text{ and } y = r\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = y/x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The pair of equations (1) give the cartesian co-ordinates $x$ and $y$ in terms of the polars, while equations (2) give the polar co-ordinates in terms of the cartesians; hence either transformation can be readily effected.

Looking at the two systems, we see that the specification of the position of a point in a given plane requires two quantities, of which one at least must be a *length*, the other being a length or an *angle*.

**11. Units.**—The facts just noted lead us to inquire how lengths and angles can be measured and specified. Obviously any given length can be specified only by stating the *number* of times it contains some other *given length* taken as a standard. Thus the measure of a length consists of two factors—(1) a pure *number*, and (2) a given length called a *unit*. The unit may be the British yard, foot, inch, mile, etc., or the French metre, centimetre, millimetre, kilometre, etc., as may be convenient. These units are defined by the respective governments by reference to standards which are preserved, as far as may be, in their official departments.

The above remarks as to stating the magnitude of a length apply equally to the measure of an angle. It must consist of two factors (expressed or understood), of which one is a pure number and the other an angle taken as the unit angle. The unit angle may be the degree, of which 360 correspond to one complete revolution, or the radian, which is the angle whose arc equals its radius.

In analytical geometry the units, whether of length or angle, are often omitted, the numbers (or letters representing them) being used alone. It should be borne in mind, however, that such numbers or letters, apart from the unit understood, fail to completely express the length or angle in question, but are simply the working factors with which we are concerned in the analysis.

The complete expression for the measurement of a physical quantity may be represented symbolically. Thus, suppose some length $l$ contains the standard length or unit $[L]$ $m$ times, then we may write

$$l = m[L] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Similarly for an angle $\alpha$ which contains $\theta$ radians, the unit angle or radian being denoted by $[R]$, we have

$$\alpha = \theta[R] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Suppose we change our unit of length to one of a third the size, equation (1) may then be written

$$l = 3m\left[\frac{L}{3}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

thus showing that the new number expressing the given length is increased threefold. Or, generally, the number measuring a given quantity varies inversely as the size of the unit in terms of which it is specified.

Now let it be required to express the angle $\alpha$ in degrees $[D]$, of which 180 correspond to $\pi$ radians. Then $\pi[R] = 180[D]$, which put in (2) gives

$$\alpha = \frac{180}{\pi}\theta[D] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Or in other words, the new units (degrees) being $\pi/180$ times the old units (radians), the new number measuring the given angle $\alpha$ is $180/\pi$ times the old number.

**12. Dimensions of Units.**—Consider now the measurement of

an area, say of the rectangle ONPM in Fig. 1. If $[L]$ be written for the unit of length, the area in question is evidently expressed by

$$a=x[L]\times y[L]=xy[L^2]=xy[A] \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where $[A]$ is written for the unit of area and $a$ for the complete measure of the area.

Now, as before in equation (3), let the unit of length be changed to one a third the size. Then (5) becomes

$$a=3x\left[\frac{L}{3}\right]\times 3y\left[\frac{L}{3}\right]=3^2xy\left[\frac{L^2}{3^2}\right]=9xy\left[\frac{A}{9}\right] \quad . \quad . \quad (6).$$

An inspection of (5) and (6) shows that the unit of area $[A]$ equals the *square* of that of length $[L]$, and that consequently, when $[L]$ is reduced to $[L/3]$, then $[A]$ is reduced to $[A/3^2]$. Hence the new number expressing $a$ is $3^2$ times the old number. If we wish to distinguish between the directions of the lengths in the above two cases, we may replace (5) by

$$a=x[X]y[Y]=xy[XY].$$

The unit of length is called a *fundamental* unit, and that of area a *derived* one, since it depends on the former. Thus we see that if a derived unit equals the $n$th power of a fundamental unit, and the latter is changed in the ratio $r$, the former is changed in the ratio $r^n$. Or, in symbols, if $Q=L^n$, and $L'=rL$, that the corresponding derived unit $Q'$ is expressed by $Q'=(rL)^n=r^nQ$ . . . . . (7). In this case the unit $Q$ is said to be of *n dimensions* in $L$, or to involve $L$ to the $n$th degree.

It is obvious that we may extend this principle to the case where a derived unit is founded upon several fundamental units — each, it may be, raised to a certain power.

Thus, if $P=A^{\alpha}.B^{\beta}.C^{\gamma}$ . . . . . (8),

where $P$ is a unit derived from the units $A$, $B$, $C$, we have

$$a^{\alpha}b^{\beta}c^{\gamma}P=(aA)^{\alpha}(bB)^{\beta}(cC)^{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

$$\text{or,} \qquad P'=a^{\alpha}b^{\beta}c^{\gamma}P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

in which $a^{\alpha}b^{\beta}c^{\gamma}$ is the factor affecting the derived unit when the fundamental units are respectively affected by the factors $a$, $b$, and $c$. It is specially noteworthy that, if any of the indices are *negative*, an increase of the corresponding fundamental unit will involve a decrease of the derived unit.

In equations (8) and (9) the derived unit on the left side is said to be of the dimensions $\alpha$, $\beta$, and $\gamma$ respectively of the three fundamental units on the right side. Thus we may write for the unit of volume

$$V=[L^1][L^1][L^1]=[L^3], \text{ or } V=[XYZ] \quad . \quad . \quad . \quad (11),$$

either of which shows that volume is of *three* dimensions in length.

**13. Displacement.**—Suppose that the position of a point undergoes a definite change. How may this change of position, step, or displacement be specified? Obviously one method is to specify by the usual co-ordinates the initial and final positions, for from these data the

change of position is ascertainable. A more usual way is to specify the step directly and leave the final position to be ascertained if required. But how may it be directly specified? Suppose its magnitude to be 2 inches. We have here the two factors which specify a length, but these fail to specify a displacement. For if the point in question is free to move in any direction, its final position is simply any point on the surface of a sphere of radius 2 inches, and whose centre was the initial point. Or, if it was free to move in a plane only, then its final position is simply any point on a certain circle, namely, where that plane intersects the sphere previously mentioned. Hence, to specify a displacement, we must have something in addition to its magnitude. It is evident that the magnitude and *direction* suffice to specify any displacement of a given point.

On looking back at the specifications of position, it is evident that this method of specifying a displacement is simply the polar-co-ordinate method of specifying a position. For we have simply to take the origin of co-ordinates at the initial position of the point, and then the $r$ and $\theta$, which specify the final position, also specify the displacement, if it is understood to be in the plane of the diagram. Of course, the displacement could be expressed by the equivalent cartesian co-ordinates $x$ and $y$. But, though each system is available for each class of specification, the cartesian system is usually preferable for specifying positions and the polar system for specifying displacements.

**14. Scalars and Vectors.**—We are thus led to observe that though a length may be thought of apart from any definite direction, as for example when we say the earth's diameter is 8000 miles, there are other cases in which the *direction* of a length is just as vital as its magnitude, as in the case of specifying the displacement of a point or other figure. These are examples respectively of the classes of quantities called *scalars* and *vectors*, of which the latter have direction, while the former have not. Many other examples of these two classes of quantities will occur later.

It is evident that vectors may be represented by straight lines. For a vector is specified by magnitude and direction, and the length of the line may represent to a certain scale the magnitude of the vector, while one of the two possible directions along the line, viz. that of the order of naming its terminal letters, represents the direction of the vector. Thus any straight line OP could adequately represent the displacement of a point in the direction OP, and of a magnitude represented by OP on a certain scale. The displacement called PO would be equal in magnitude but opposite in direction to the displacement called OP. Another device is to put an arrow head on the line representing a vector so as to indicate the direction of the vector. It should be noted that some writers use 'direction' with a wider meaning than that employed above, namely, to denote *both* ways along a given line. They then say that the line's 'direction' represents the 'direction' of the vector, which needs also the 'sense' in which the line is supposed drawn to be indicated for the complete specification of the vector. Whichever phraseology is employed, the fact to be borne in mind is that a line fails to represent a vector precisely until an order of naming its

terminal letters (or some equivalent device) is supplied. Thus, if the centre of a sphere is taken as the point through which to draw lines representing various vectors, a vertical diameter is ambiguous, but its upper and lower halves, *if supposed drawn from the centre*, specify vectors equal in magnitude but opposite in direction (or, as some would prefer to say, equal in magnitude and direction but opposite in sense).[1]

**15. Composition of Displacements.**—Suppose a point to suffer the displacement represented in Fig. 2 by OP, and then the displacement represented by OQ. To find its position after both displacements, or to compound them, it is obviously necessary and sufficient to allow the second displacement to operate upon the position due to the first. In other words, we must draw on the figure from P the line PR equal and parallel to OQ. Or, we may complete the parallelogram on OP and OQ by drawing PR and QR parallel to OQ and OP respectively and intersecting at R, thus making PR equal to OQ as well as parallel to it. If we are now asked to state to what the two displacements are equivalent, *i.e.* what is the result of their composition, the reply may be, the point originally at O is, after the two displacements, at R. Or, more formally thus, the composition of the two displacements represented to scale by OP and OQ yields the resultant displacement represented to the same scale by OR. This is a very simple example of the important operation called the *Addition of Vectors*. It may be represented symbolically as follows:—

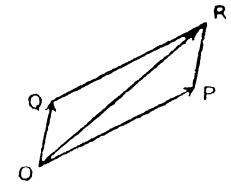



FIG. 2. COMPOSITION OF DISPLACEMENTS.

$$OP \hat{+} OQ = OR \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where the sign over the plus indicates that the addition is *vectorial*, *i.e.* having regard to direction and not simply algebraic. If the displacement OQ occurred first, and then the point at Q received the displacement represented by OP, it would as before be found finally at R, as is obvious. Thus we might write

$$OQ \hat{+} OP = OR \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Other examples of the addition of vectors are the theorems of the parallelogram and triangle of forces, familiar to the student of elementary statics.

**16. Localisation of Vectors.**—A comparison of equations (1) and (2) and the operations they respectively represent will bring out important points as to the component vectors and their results in the two cases. Thus the equations seem to express that the order of quantities added is indifferent, and the result therefore the same, in the two cases. But, on going into details, we see that in (1) OP is applied at O and then OQ is applied at P. Whereas in (2) OP is applied at Q, OQ having been previously applied at O. In other words, the points of application or *localisations* of the vectors were different in

[1] DEFINITION. A *vector* is a quantity which has magnitude and involves one direction only and that to the first positive power.

Vectors compound according to the parallelogram law. Other quantities which involve various directions, but are found on scrutiny to obey the parallelogram law, are often called vectors also.

each case. Again, though the resultant displacements were the same in each case the paths of the points were different, being OPR in (1) and OQR in (2). Hence we see that although the equations (1) and (2) may express all that is needed for certain purposes, they do not give the entire details of what we suppose to have happened.

Vectors may be spoken of as *unlocalised* or *localised* to various degrees, as for example in a point or a line. In concrete cases they are probably always localised to some extent. Thus, if a ship in harbour is said to be raised 10 feet by the tide, the vector has a magnitude 10 feet, a direction vertically upwards, and is *localised* to the *volume* occupied by the ship in question. If one of the masts be similarly referred to, the same vector is localised to the volume of that mast, or if the thickness of the mast is regarded as negligible, the vector is *localised in a line*. Or, again, if the top of the mast be in like manner spoken of, the vector is *localised* in what may be regarded as a *point*.

Hence, in compounding displacements and adding vectors in general, care must be taken that the appropriate degree of localisation is in each case present in the component vectors thus dealt with.

### EXAMPLES—II.

1. Explain the distinction between fundamental and derived units, and show how the size of a unit of one class is related to the sizes of those of the other class.
2. Define the terms *scalar* and *vector*. Give examples of each, and show how to add vectors.
3. Show by illustrations what you understand by the localisation of vectors. What bearing has the localisation of vectors upon their composition?

**17. Time and Motion.**—The physicist, as such, regards time as that familiar though inscrutable *one-dimensional something* which separates changes in bodies and individual sensations and extends without known limits from the past to the future. Just as space is the abstract of all relations of co-existence, so time is the abstract of all relations of sequence. Each is a primary conception that cannot be rendered in terms of anything simpler. When we combine, in a certain manner, the conceptions of space and time, we have the conception of motion, a point being said to move if, at different instants of time, it occupies different positions in space.

To measure a given quantity of time we need a unit of time in which to measure it and a number to express how many of these units the given quantity contains. As units, the familiar second, minute, hour, etc., are used. To fix an instant in time with respect to another instant we need to state only (1) the duration, or quantity of time, separating them, and (2) which instant is the later. Or, we may accomplish both algebraically by prefixing to the statement of the duration the sign of *plus* if the instant to be fixed is later than that taken as origin, or the sign *minus* if it is earlier. This is like fixing the position of a point on a line, no divergence into solid or even plane space being permitted.

**18. Velocity.**—The velocity of a point is its rate of change of position. This is a brief definition serving to introduce the subject which we shall now examine in detail. Thus let a point move from C along the axis OY, and let the times when it has the displacements o, $y_1$, $y_2$, $y_3$, . . . $y$ be respectively o, $t_1$, $t_2$, $t_3$, . . . $t$. Consider the quotients $y_1/t_1$, $y_2/t_2$, $y_3/t_3$, . . . $y/t$. And suppose these quotients all have the same value, $v$ say. Then this common value $v$ measures the *velocity* of the point or its rate of change of position with time while passing from O to P.

In what units is this quantity measured and expressed? To answer this we fall back upon the dimensional equations previously used. Thus, writing $[Y]$ for the unit of *length*, $[T]$ for the unit of time, and $[V]$ for the unit of velocity, we have

$$v[V]=y[Y]\div t[T] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

whence

$$[V]=[Y^1T^{-1}]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Or, the unit of velocity is of *plus one* dimension in length (in some definite direction) and of *minus one* dimension in time. Of course, the actual size of the unit of velocity depends upon the units adopted for length and time.

Let us now consider the significance of the supposed equality of the quotients $y_1/t_1$, $y_2/t_2$, $y_3/t_3$, . . . and $y/t$. It is clearly only when this equality holds that we can apply to the common value $v$ the phrase velocity of the point from O to P, and leave it unqualified by any further restriction.

If, on the other hand, knowing nothing of $y_1$, $y_2$, $y_3$, we simply knew the values $y$ and $t$ having the quotient $v$, we should then say that $v$ was the *mean velocity* of the point from O to P.

Again, if we knew a very large number of such intermediate values of the distances $y_1$, $y_2$, $y_3$, . . . $y_n$ . . ., and their corresponding times $t_1$, $t_2$, $t_3$, . . . $t_n$ . . ., and found all the quotients $y_1/t_1$ . . . $y_n/t_n$ . . . $=v$, we should then say that the velocity of the point between O and P was *uniform* and of the value $v$. The rigour with which the term uniform would apply would depend upon the number of the intermediate positions and times known, and would become absolute only in the ideal case of all such intermediate information being available.

Finally, if the quotients had different values, thus $y_1/t_1=v_1$, $y_2/t_2=v_2$, etc., then the velocity would be *variable*, $v_1$, $v_2$, etc., each expressing the mean velocity over the range in question. But even though the velocity is varying, the idea forces itself upon us that at each instant of time (or position in space) the velocity must have some definite value; just as when a point is moving it has at each instant of time some definite position. How shall we in thought attain and measure this instantaneous value of a varying velocity? Take shorter and shorter durations, $\tau_1$, $\tau_2$, $\tau_3$ . . ., each including the instant in question, suppose the corresponding displacements $\eta_1$, $\eta_2$, $\eta_3$ . . . known, take the respective quotients $\eta_1/\tau_1$, $\eta_2/\tau_2$, $\eta_3/\tau_3$ . . ., and suppose these quotients to continually approach a limiting value $v$, then $v$ is the value sought, and expresses the *instantaneous velocity* at the instant in question.

It is obvious to students of the calculus that an instantaneous velocity is the first differential coefficient of a distance (in some given direction) with respect to the time. Or, in symbols

$$v = dy/dt = \dot{y},$$

where the dot denotes a single differentiation with respect to time.

It should be noted that in scientific usage velocity involves the idea of direction just as displacement does. If we wish to speak of the magnitude of a velocity apart from all idea of its direction we use the word *speed*. Thus if a point described a circle so as to pass over an arc of 10 cm. length in each second, 1 cm. in each tenth of a second and so forth, we should say that the speed was uniform, but that the velocity varied because one of its elements, viz. direction, was continually changing. It should be noted, however, that in speaking of the speed of a point moving along a given path we may use the positive or negative sign to indicate the opposite directions in that path.

**19. Acceleration.**—Since velocities may vary it is incumbent upon us to consider the changes of velocity, and also the *time-rate of change of velocity*, which is called *acceleration*. Just as we passed from the conception of displacement to that of velocity by using time once as a divisor, so we may pass from velocity to acceleration by using time once more as a divisor. Thus, let a moving point be considered, having at times $0, t_1, t_2, t_3, \ldots t$, the velocities $0, v_1, v_2, v_3, \ldots v$, all along the same straight line, OY say. Then, if the quotients $v_1/t_1, v_2/t_2, v_3/t_3, \ldots v/t$ all have the same value $a$, this common value $a$ measures the acceleration of the point, which in this simple case is also directed along the line OY in question.

If it is only known that after time $t$ the velocity has increased by the amount $v$, then the quotient, $v/t = a$ say, measures the *mean acceleration* during the time $t$. If, on the other hand, a large number of corresponding intermediate values of $v$ and $t$ are known, each pair yielding the same quotient $a$, then $a$ measures the *uniform acceleration*, the uniformity being ascertained with more and more rigour as the data increase in number.

If, however, the intermediate values of $v$ and $t$ yield varying quotients the acceleration is *variable*. Its instantaneous value may be ascertained in thought and expressed in symbols, as was done for a velocity. Thus, let shorter and shorter times, $\tau_1, \tau_2$, etc., be taken, each including the instant in question, the corresponding changes in velocity being respectively $v_1 - v, v_2 - v$, etc. Then, if the quotients $(v_1 - v)/\tau_1 = a_1, (v_2 - v)/\tau_2 = a_2$, etc., approach a limiting value $a$, that value $a$ measures the *instantaneous acceleration* at the instant in question Or, in the notation of the calculus

$$a = dv/dt = d^2y/dt^2 = \ddot{y},$$

where the two dots denote two differentiations with respect to time.

From what has gone before it is easily seen that an acceleration is of *plus one* dimension in length and *minus two* dimensions in time.

Or, if its unit is denoted by $[A]$, then

$$a[A]=v[V]\div t[T]=\frac{y}{t^2}[YT^{-2}].$$

Accelerations may accordingly be measured in centimetres (or other units of length) per second per second. It is often convenient to abbreviate the units thus: cm. per sec.$^2$

It should be noted that acceleration is a vector, since it has magnitude and *direction*. If we wish to speak of the magnitude only of an acceleration we may say 'rate of change of speed' or *quickening*. This is, of course, a scalar having magnitude and sign only.

The relations of the various quantities hitherto considered may be exhibited compactly as shown in Table II., their order of development being followed by reading down the columns.

TABLE II. RELATIONS OF KINEMATICAL QUANTITIES.

| QUANTITY | (POSITION) | VELOCITY |
|---|---|---|
| CHANGE OF QUANTITY | DISPLACEMENT | CHANGE OF VELOCITY |
| CHANGE DIVIDED BY TIME | VELOCITY | ACCELERATION |

**20. Displacement Graphs.**—The motion of a point along a straight line may be usefully represented graphically on a displacement-time diagram or by a *displacement graph* as follows:—Take distances along the axis of $y$ to represent the respective displacements, and distances along the axis of $x$ to represent the corresponding times. Then each such pair of co-ordinates will define a point on the diagram, and a continuous line drawn through those points will give the graph required. It should be noted here that some amount of discretion must be exercised in drawing this line between the points furnished by the data of the case, and that only more or less *probability*, but *not certainty*, attaches to any such intermediate portions of the line so drawn. Examples of such graphs are shown in Fig. 3, plotted from the data of Table III.

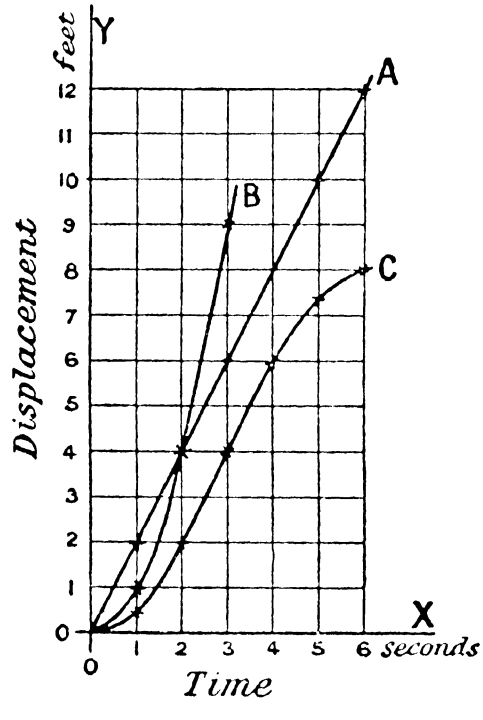



FIG. 3. DISPLACEMENT GRAPHS.

TABLE III. DATA FOR DISPLACEMENT GRAPHS.

| GRAPH A. | | GRAPH B. | | GRAPH C. | |
|---|---|---|---|---|---|
| Displacement in Feet. | Time in Seconds. | Displacement in Feet. | Time in Seconds. | Displacement in Feet. | Time in Seconds. |
| 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 1 | 0·54 | 1 |
| 4 | 2 | 4 | 2 | 2 | 2 |
| | | | | 4 | 3 |
| 10 | 5 | 9 | 3 | 6 | 4 |
| | | | | 7·46 | 5 |
| 12 | 6 | | | 8 | 6 |

Graph A illustrates a uniform velocity or speed of 2 feet per second, and even if only the final point (12, 6) were given we should have a mean velocity of the same amount, though no knowledge of the uniformity. Obviously the equation of the graph is $y=2x$, in which the 2, being the tangent of the angle between the graph and the axis of time, expresses the rate of increase of displacement with time, *i.e.* the speed.

Graph B, for the first second, shows only half the mean velocity represented by A, but by the end of two seconds, where the graphs intersect, their mean velocities are the same. Beyond that point the graph B is higher than A, *i.e.* indicates a higher mean speed from the start than A does. It therefore gives an example of a variable speed. Let us now inquire what is the instantaneous speed at some instant, say two seconds from the start. A glance at the corresponding point on the diagram shows that the speed there is of the order 4 feet per second, or double that of graph A. A closer examination would confirm this result. We may also arrive at the same conclusion by another method. Thus we see from Table III. that graph B has the equation $y=x^2$, hence the tangent to it at the point $(x'y')$ is represented by the equation $\frac{1}{2}(y+y')=xx'$. Thus for the point in question (2, 4) the geometrical tangent to the graph is $y=4x-4$. Hence 4, the co-efficient of $x$, represents the trigonometrical tangent of the angle between the axis of $x$ and the geometrical tangent to the graph at the point in question. In other words, the *slope* of the curve at this place, estimated so as to measure the speed, is represented by the number 4. Similarly after three seconds from the start, when the displacement is 9 feet, we find the speed to be 6 feet per second.

If we apply the method of the calculus we must differentiate $y$ with respect to $x$ in the equation of the graph to find the ratio of the very small increase of displacement $y$ to the corresponding increase of time represented by $x$.

Thus, the graph being $y=x^2$,

we find $dy/dx=2x=v$,

which agrees with the two results already dealt with.

Again, if we ask at what instant is the speed denoted by graph B equal to a certain assigned quantity, say that of graph A, then by the graphical method we must find the point on B which is touched by a line parallel to A. It is evident that point is in the neighbourhood (1, 1), *i.e.* the instant one second from the start, and when the displacement is 1 foot. And this is confirmed by the analysis.

Consider now the graph C; it indicates by the horizontal portions, at times 0 and 6, zero velocities or instantaneous states of rest. It also indicates continuous increase or decrease of speed between these instants, and a maximum speed at three seconds where the graph is steepest. It may accordingly be recognised as representing a motion which at any rate resembles that of a pendulum bob. The equation of the graph may be written $y = 4 - 4\cos(\pi x/6)$ or $y = 8\sin^2(\pi x/12)$. For its slope we have $dy/dx = \frac{2}{3}\pi \sin(\pi x/6)$.

Although the ordinates in the displacement graph only refer to steps taken along a direction otherwise specified, it is clearly legitimate and convenient to let *negative* ordinates mean displacements just opposite in direction to those represented by positive ordinates.

These examples sufficiently illustrate that in a displacement graph slope represents speed, and therefore change of slope per unit distance along the axis of time represents acceleration. But acceleration can be more clearly and simply exhibited on another type of graph now to be dealt with.

**21. Velocity or Speed Graphs.**—Let us now take the velocity of a point moving along a straight line as the quantity to be plotted along the $y$ axis, the time being represented by distances along the $x$ axis as before. The curve thus obtained may be called the *velocity graph* or *speed graph* of the moving point. In order to facilitate comparison of the two types of graph we shall plot the speed graphs corresponding to the data in Table III. and the inferences as to speeds deduced from them. These are shown in Fig. 4.

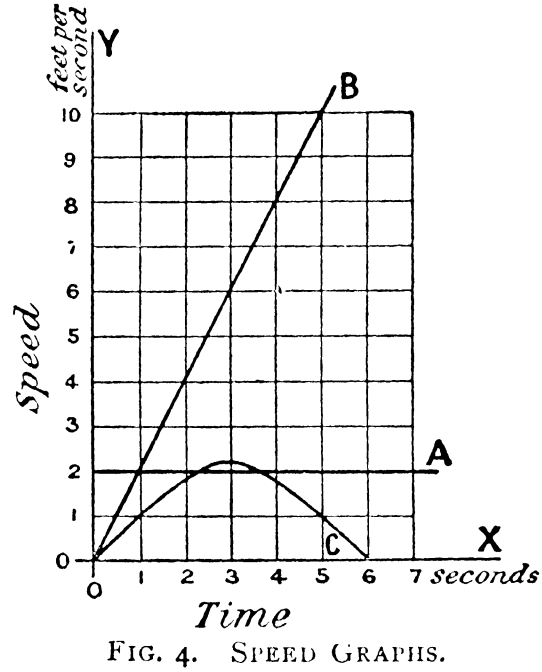



FIG. 4. SPEED GRAPHS.

It is thus seen that graph A, representing a uniform velocity, is now a *horizontal* line, whereas graph B, representing a velocity proportional to the time, is now a *straight* line inclined to the horizontal, *i.e.* it represents a uniform acceleration. Finally, the graph C rises from the origin and afterwards falls to zero. The equations of these three speed graphs are respectively $y = 2$, $y = 2x$, and $y = \frac{2}{3}\pi \sin(\pi x/6)$, as shown in the preceding article.

It may be noted here that in a speed graph the area below the graph between two given ordinates represents the length, distance, or space described by the moving point in the time defined by the ordinates. For each narrow vertical strip of this area has an area which is the product of height into width. But the height of the strip or ordinate represents the instantaneous speed, and the width of the strip or increase of abscissae denotes a short time. Also the product of these is the distance described in that short interval. Hence the whole area, or sum of these strips, represents the sum of such distances, and is therefore the whole distance described in the finite time under consideration.

Thus, the abscissae being times and the ordinates speeds, the slope of the graph represents acceleration and the area the distance described. The speed graph is accordingly often very useful, embracing as it does so conveniently the four quantities with which we are concerned in the case of a moving point.

Though the ordinates in a speed graph denote the speeds of a point along a line whose direction is otherwise specified, it is convenient to use *negative* ordinates to denote speeds in the opposite directions along that same line.

**22. Other Graphs for a Moving Point.**—Since out of four quantities we may choose six different pairs, it is evident that for a moving point we may construct graphs in six different ways, viz. with co-ordinates denoting $s$ and $t$, $v$ and $t$, $a$ and $t$, $v$ and $s$, $a$ and $s$ or $a$ and $v$, where $s$, $t$, $v$, and $a$ denote space, time, velocity, and acceleration respectively.

But most of these other graphs have only a very limited usefulness, or may be described as curious rather than useful. We may perhaps refer to some of them in special cases later. It may be just noted here that plotting accelerations and distances as co-ordinates brings out the fact that in the graph C of Figs. 3 and 4 the acceleration is proportional to the displacement from a certain point, the graph being a straight inclined line cutting the axis of distances at the point 4.

## EXAMPLES—III.

1. Define *velocity*, *uniform* velocity, *mean* velocity, and *instantaneous* velocity.
2. Exhibit in tabular form the relation of displacement, velocity, and acceleration, giving also the dimensions of each.
3. Plot a displacement graph from the following data :—

   *Displacements in yards*, 0, 0, 8, 19, 29, 39, 49, 58, 67, 76, 88, 100, 103.
   *Times in seconds*, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

   If the graph represents a man running a race, indicate what you believe are the start and finish.
4. A point moves so that its displacement after $t$ seconds is $16t^2$. Plot its displacement and speed graphs, showing the corresponding features of each graph, and find its acceleration.
5. Explain the special advantages of a speed graph. Plot one to represent some variable motion, find the space described, and indicate any important features.
6. 'Define the *mean velocity* of a point in any interval of time. Prove by a

graphical method (or otherwise) that if the acceleration be constant the mean velocity is equal to the arithmetic mean of the initial and final velocities, and also to the velocity at the middle instant of the interval. Show by examples that in any other case these three quantities are usually different.' (LOND. B.SC., PASS, MIXED MATH., 1904, I. 1.)

7. 'The times $t_1$, $t_2$, $t_3$ at which a particle moving with constant retardation passes three points $P_1$, $P_2$, $P_3$ of its path are recorded; find the acceleration, and the velocity at $P_2$, having given $P_1P_2=a$, $P_2P_3=b$.' (LOND. B.SC., PASS, APPLIED MATH., 1905, II. 1.)

**23. Composition of Velocities and Accelerations.**—A little reflection will show that the operation of the addition of vectors which is valid for the composition of displacements applies also to the composition of velocities, provided that the point in question is so circumstanced that, no matter how far it goes due to one velocity, it is still equally affected by the other. Thus, after a short time $\tau$, the two velocities $v_1$ and $v_2$ will have imparted to the point displacements in their own directions and of magnitudes $v_1\tau$ and $v_2\tau$ respectively. The position of the point after time $\tau$ is accordingly determined by the vectorial addition of these two displacements, and may be denoted by $v\tau$. Now let the position after a finite time $t$ be considered. It will evidently be determined by the vectorial addition of the displacements $v_1t$ and $v_2t$ along the same directions as at first, and is therefore denoted by $vt$. Hence we have the same diagram as before, but magnified in the ratio of $t$ to $\tau$. Thus the point has uniform velocity $v$, and in the direction first determined. Or, in other words, the point moves with a resultant speed and direction determined by the vectorial addition of its two component velocities.

Similar reasoning would apply to show that two accelerations simultaneously possessed by any point may be compounded by vectorial addition to give the resultant acceleration.

Suppose now that a velocity and an acceleration have to be dealt with, even then one aspect of the problem can be treated by vectorial addition. For the acceleration, though possibly varying in magnitude and direction, will impart to the point in time $t$ some definite velocity, $v_1$ say. Thus, if the original velocity of the body were $v_0$, the final velocity $v$ would be determined in magnitude and direction by the vectorial addition of $v_0$ and $v_1$,

or
$$v=v_0\hat{+}v_1.$$

The aspect of the problem not here dealt with is the path of the point during time $t$. Unless the acceleration is for the *whole* time in the direction of the original velocity, it is evident that the path must be curved. And even if the velocity and acceleration are always collinear, we have still left undetermined the position of the point at each instant of time. Such aspects of the cases will be dealt with in their proper places later.

**24. Vectorial Polygons.**—For the composition of more than two vectors it is evident that we might take any two first and find their resultant, then compound this resultant with a third of the components

for a new resultant, and so forth, until all had been dealt with. A simpler method, however, of compounding $n$ vectors is to construct a polygon $n$ of whose sides taken in order represent in magnitude and direction the vectors to be compounded. Then the remaining side required to close the polygon will, if taken in the reverse order, represent both in magnitude and direction the resultant vector sought: and this holds true whether the components are all in one plane or are in various directions in solid space, *i.e.* the vectorial addition is valid whether the polygon by which it is effected is of necessity plane or *gauche*. This is easily seen by considering the simplest example of a vector, namely, a displacement. Thus, to illustrate the gauche polygon, if three displacements at right angles and of magnitudes equal to 9 inches, $4\frac{1}{2}$ inches, and 3 inches respectively are given to a point at one corner of a brick of that size, the point would be carried to the opposite corner of such a brick if rightly situated with respect to those displacements.

It is sometimes a great convenience to be able to write down the result of the addition of vectors without actually drawing the polygon to scale. The formulae for a plane polygon may be easily derived by reference to Fig. 5.

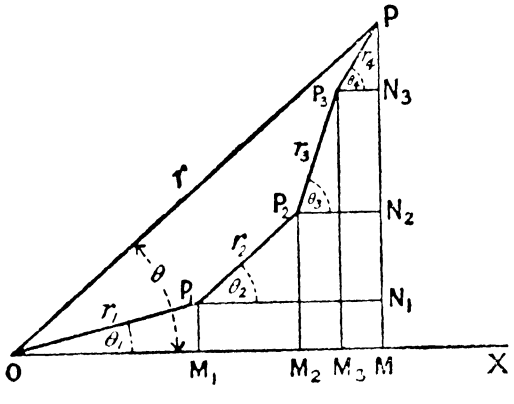



FIG. 5. POLYGON OF VECTORS.

In this figure the component vectors, which may be thought of as displacements, are represented to scale by $OP_1$, $P_1P_2$, $P_2P_3$, $P_3P$. The resultant is obviously represented to the same scale by OP. Further, let the magnitudes and directions of the components be respectively denoted by $r_1\theta_1$, $r_2\theta_2$, $r_3\theta_3$, and $r_4\theta_4$, their resultant being $r\theta$.

Then, by the figure, we have

$$OP^2 = OM^2 + MP^2 \text{ and } \tan MOP = MP/OM \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But $OM = OM_1 + M_1M_2 + M_2M_3 + M_3M$,

and $MP = MN_1 + N_1N_2 + N_2N_3 + N_3P$.

Or, using the symbols—

$$OM = r_1 \cos\theta_1 + r_2 \cos\theta_2 + r_3 \cos\theta_3 + r_4 \cos\theta_4 = \Sigma r \cos\theta \quad . \quad (2),$$

$$\text{and } MP = r_1 \sin\theta_1 + r_2 \sin\theta_2 + r_3 \sin\theta_3 + r_4 \sin\theta_4 = \Sigma r \sin\theta \quad . \quad . \quad (3).$$

Hence, by (2) and (3) substituted in (1), we obtain

$$r^2 = (\Sigma r \cos\theta)^2 + (\Sigma r \sin\theta)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\text{and} \qquad \tan\theta = (\Sigma r \sin\theta)/(\Sigma r \cos\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

In equation (5), giving the value of $\tan\theta$, it should be noted that an ambiguity arises unless the algebraic signs of numerator and denominator of the fraction on the right side be each maintained in their original positions. Thus if the fraction in question is $+9/(+10)$, the angle is uniquely determined as of the order 42°. But if the fraction is

written $+(9/10)$ we are uncertain whether the angle is $42°$ or $222°$ whose tangent should have been preserved in the form $(-9)/(-10)$. Thus the angles $45°$, $135°$, $225°$, and $315°$ are uniquely determined by their tangents if written respectively $(+1)/(+1)$, $(+1)/(-1)$, $(-1)/(-1)$, and $(-1)/(+1)$.

**25. Gauche Polygon of Vectors.**—If the vectors to be compounded form sides of a gauche polygon, then the closing side which represents the resultant will be the diagonal of a parallelepiped whose adjacent edges are built up as were the sides OM and MP of the rectangle in Fig. 5, whose diagonal OP represented the resultant of the plane polygon. This is illustrated in Fig. 6, in which the axis of $z$ shown in perspective is to be understood as at right angles to the plane of $xy$. The separate vectors are not shown in the figure, but are supposed to have magnitudes $r_1$, $r_2$, $r_3$, etc., and to make with the axes of $x$, $y$, $z$ the angles $\alpha_1$, $\beta_1$, $\gamma_1$, $\alpha_2$, $\beta_2$, $\gamma_2$, $\alpha_3$, $\beta_3$, $\gamma_3$, etc., the resultant OP having magnitude $r$, and making with the axes the angles $\alpha$, $\beta$, and $\gamma$ as shown.

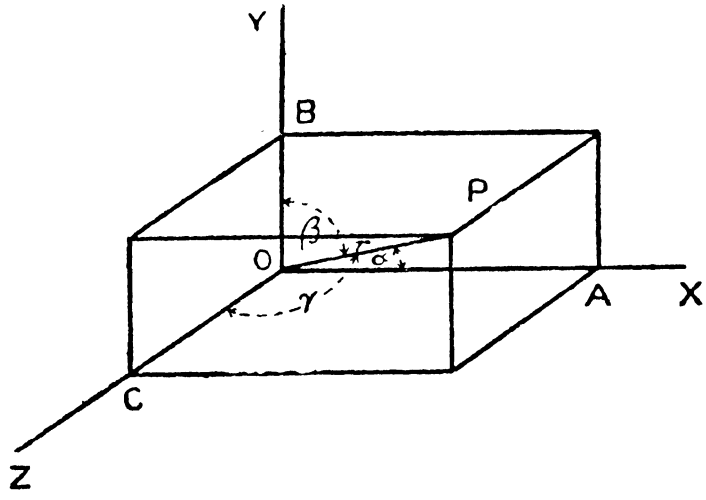



FIG. 6. ADDITION OF VECTORS IN SOLID SPACE.

It is thus seen that OP is the resultant of OA, OB, and OC, *i.e.* its components along the co-ordinate axes are $r\cos\alpha$, $r\cos\beta$, and $r\cos\gamma$ respectively. Similarly each of the component vectors, $r_1$ say, contributed along the axes of $x$, $y$, and $z$ the components $r_1\cos\alpha_1$, $r_1\cos\beta_1$ and $r_1\cos\gamma_1$. Hence we have

$$OP^2=OA^2+OB^2+OC^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and $\cos AOP=OA/OP$, $\cos BOP=OB/OP$, $\cos COP=OC/OP$ (2).

But $$OA=r_1\cos\alpha_1+r_2\cos\alpha_2+\ldots=\Sigma r\cos\alpha \quad . \quad . \quad . \quad . \quad (3),$$

$$OB=r_1\cos\beta_1+r_2\cos\beta_2+\ldots=\Sigma r\cos\beta \quad . \quad . \quad . \quad . \quad (4),$$

$$OC=r_1\cos\gamma_1+r_2\cos\gamma_2+\ldots=\Sigma r\cos\gamma \quad . \quad . \quad . \quad . \quad (5).$$

Thus, by substituting in (1) and (2) the values from (3), (4), and (5), we obtain $r^2=(\Sigma r\cos\alpha)^2+(\Sigma r\cos\beta)^2+(\Sigma r\cos\gamma)^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$,

also $\cos\alpha=(\Sigma r\cos\alpha)/r$, $\cos\beta=(\Sigma r\cos\beta)/r$, and $\cos\gamma=(\Sigma r\cos\gamma)/r$ (7).

It may also be seen by the geometry of Fig. 6 that we have

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

**25a. Moments.** DEFINITION.—The *moment* of a vector with respect to a point is the product of the vector into the perpendicular from that point upon the line in which the vector is localised, and is reckoned positive when the direction of the vector about the point is counter-clockwise.

*Theorem.*—If two vectors are localised at a point O, then the algebraic sum of their moments with respect to any point P in their plane is equal to the moment of their resultant about P.

In Fig. 6A, let OA, OB represent the component vectors, OC their resultant, found by the parallelogram law, and all localised in O; also let P, in the plane of OABC, be the point about which the moments are to be taken.

*Proof.*—Then, by definition, the moment of any vector is represented to scale on the figure by twice the area of the triangle

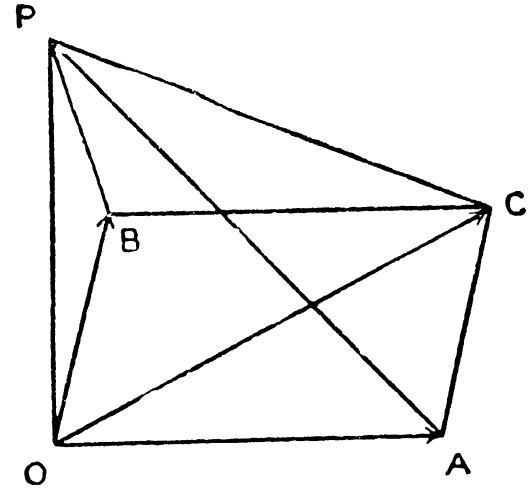



FIG. 6A. THEOREM OF MOMENTS.

whose base in the vector and whose vertex is the point P about which its moment is taken. Thus, half the moment of OC

$$\begin{aligned} &= \text{area of } \triangle OCP = \triangle OBP + \triangle OCB + \triangle BCP \\ &\qquad = \triangle OBP + \triangle OAP \\ &\qquad = \text{half-sum of moments of OB and OA.} \end{aligned}$$

This accordingly establishes that case of the proposition represented in the figure. When P is differently placed the needed demonstration follows in like manner.

**25b. Composition of Angular Velocities.**—Measuring angles in radians, we naturally measure *angular velocities* in radians per second. Thus, if a point P moves with an angular velocity $\omega = \theta/t$ about a given axis from which it is distant by the radius $r$, we have the relations

$$\omega = \theta/t = s/rt \text{ or } s = \omega rt,$$

where $s$ is the arc described by the point in time $t$. Hence in a given time the displacement $s$ is proportional to the product $\omega r$. This

suggests an analogy with the moments of a vector with respect to a point. Thus, referring to Fig. 6A, consider the point P in the plane of the diagram as having an angular velocity about OA and proportional to the length of OA $=\omega_1$ say. Then in time $dt$, P would suffer a displacement perpendicular to the plane of the diagram of amount $ds_1=\omega_1 dt\times$ perpendicular from P upon OA $=dt\times$ double area of OAP. Similarly if the point P had a coexistent angular velocity $\omega_2$ proportional to OB about OB, its displacement in virtue of that in time $dt$ would be given by $ds_2=\omega_2 dt\times$ perpendicular from P upon OB. Hence the sum of these displacements would be represented to a certain scale on the diagram by the sum of the areas OAP and OBP. But, as we have just seen, this sum equals the area of OCP, which consequently represents to the same scale the displacement $ds$ which P would experience in time $dt$ under the influence of an angular velocity about OC of amount $\omega$ proportional to the length OC. Thus, so far as a point P in the plane of the diagram is concerned, the resultant of the coexistent angular velocities whose axes are OA and OB and magnitudes OA and OB respectively is an angular velocity whose axis is OC and whose magnitude is OC.

Suppose now a point P′ is taken out of the plane of the diagram such that PP′ $=p$ is perpendicular to that plane. Then the displacements we have just considered for P would also be the displacements perpendicular to that plane for P′. The displacements parallel to the plane for P′ would obviously be $\omega_1 p dt$, which would appear in the diagram as perpendicular to OA, $\omega_2 p dt$ appearing perpendicular to OB, and $\omega p dt$ appearing perpendicular to OC. Thus these displacements would form in the diagram a parallelogram of the same shape as OACB, but of different size, and with sides perpendicular to the original parallelogram. Hence these component displacements due to $\omega_1$ and $\omega_2$ about OA and OB would give that due to the resultant angular velocity $\omega$ about OC. This, which has been proved for any one point, is obviously true for any assemblage of points.

We have accordingly seen that if two coexistent angular velocities are respectively represented in magnitude and axis by two lines meeting in a point, then the line found as the resultant by the addition of vectors represents in magnitude and axis the resultant angular velocity. In other words, an angular velocity is a vector. It should be observed that associated with a certain direction of drawing or describing the line representing an angular velocity must be associated a certain direction of rotation about that line as axis. We shall adopt the convention that the direction of rotation and corresponding drawing of the axis are those of rotation and advance of a right-handed screw in a stationary nut, as for example in driving an ordinary screw into wood. Thus, in Fig. 6A, OA, OB, and OC each denote an angular velocity which would bring P out from the plane of the diagram towards the reader.

It must be distinctly noted that this relation or theorem as to the composition of *coexistent* angular *velocities* in no wise applies to *successive finite rotations* or *angular displacements.*

**25c. Curvature.**—The *curvature* of a plane curve is defined as the rate of change of its direction per unit length measured along the curve. Thus take two points P and Q, Fig. 6B, an infinitesimal distance $ds$ apart on a plane curve, and let tangents PR, QS and normals PC, QC be drawn through P and Q making angles $d\psi$ with each other, and the normals meeting at C where $CP=CQ=\rho$.

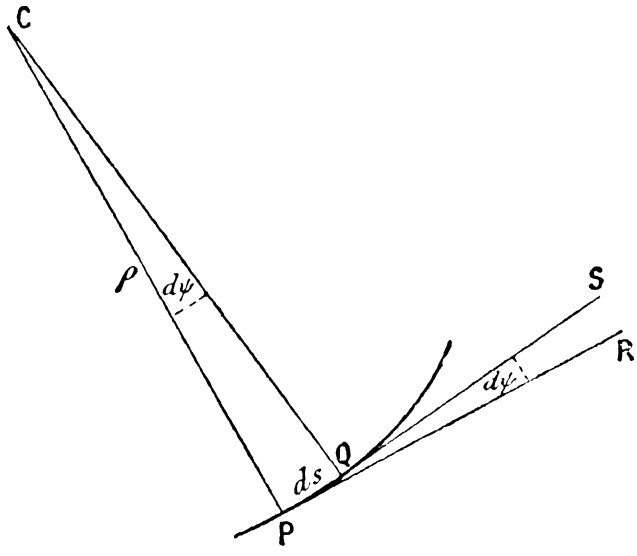


FIG. 6B. CURVATURE.

Then, regarding the tangents, we see by the definition that the curvature at PQ is $d\psi/ds$. But, regarding the normals, we see that $d\psi = ds/\rho$. Hence we obtain

$$\text{curvature} = d\psi/ds = 1/\rho.$$

Or, the curvature is also measured by the reciprocal of $\rho$, a length which is called the *radius of curvature*. The point C where the consecutive normals meet is called the *centre of curvature*.

**25d. Centroids.**—It is often desirable in the various sections of Mechanics to find a centre or point of mean position for a system of points, a line, a surface, a volume, or for some other distributed quantity. The formal development of this subject naturally occurs in Chapter XVII., since its chief mechanical application is to centres of mass, forming part of the section of Statics there treated. But the idea itself is purely geometrical and independent of any such particular application, so must be introduced here. Moreover, this early notice is a convenience, since the conception is required in Chapter XIII. when dealing with the Kinetics of Rigid Bodies.

Obviously the centre of two points is the point of bisection of the straight line joining them. Or, if the abscissae of the points were $x_1$ and $x_2$ and that of their centre $\bar{x}$, we could write $\bar{x} = \frac{1}{2}(x_1 + x_2)$. Thus, for any number $n$ of points each having different abscissae, $\bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots x_n)$. But some of the points might have abscissae of the same value. Thus, let $m_1$ points have the abscissa $x_1$, and $m_2$ points the abscissa $x_2$, and so on; the same notation with $y$'s holding for their ordinates. Then the co-ordinates of their centre would be

$$\bar{x} = \Sigma(mx)/\Sigma m \text{ and } \bar{y} = \Sigma(my)/\Sigma m \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).^1$$

We may extend the application by supposing the $m$'s to refer to the

[1] We may apply this to the resultant $R$ of $n$ vectors acting at a point O. For $R = \sqrt{\{(\Sigma x)^2 + (\Sigma y)^2\}}$, and the centroid G of $n$ unit particles at the ends $(x, y)$ of the vectors is given by $OG = \sqrt{(\bar{x}^2 + \bar{y}^2)}$. Thus, by (1) in the text, $R = n$ OG.

magnitudes of elements of length, area, volume, or other scalar quantity situated at the points defined by the corresponding $x$ and $y$. The point $(\bar{x}, \bar{y})$ so determined is called the *centroid* of the system, figure, or body in question.

If the co-ordinates of points with respect to the centroid are $x'$, $y'$, we have

$$x = \bar{x} + x' \text{ and } y = \bar{y} + y' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Hence $\Sigma mx = \bar{x}\Sigma m + \Sigma mx'$ and $\Sigma my = \bar{y}\Sigma m + \Sigma my'$.

But, by (1), these reduce to

$$0 = \Sigma mx' = \Sigma my' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Thus, if equations (1) are taken as defining the centroid, we may regard (3) as giving some of its properties. It is, however, often convenient to take (3) as giving the definition of the centroid and equations (1) as forming the working rules for the determination of its position.

For quantities distributed throughout space of three dimensions we have simply to add the corresponding equations, involving $z$ by analogy with (1), (2), and (3), thus giving the following additional relations:—

$$\left.\begin{aligned} \bar{z} &= \Sigma mz/\Sigma m \\ z &= \bar{z} + z' \\ \text{and} \quad 0 &= \Sigma mz' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

**26. Constraints and Degrees of Freedom.**—If a point is not constrained in any way, it is said to have *three degrees of freedom*, since it is obviously free to move parallel to the three co-ordinate axes of solid space. If the point is constrained to remain on a plane surface, say that of $xy$, it has then lost one degree of freedom, namely, the motion parallel to the axis of $z$, and retains two degrees of freedom only, namely, those along the axes of $x$ and $y$. Similarly, if the point is constrained to remain in a line, say the axis of $x$, it is evident that it has lost two degrees of freedom and retains only one. With any actual small particle these two cases may be represented by floating on still water and confinement in a straight tube respectively.

Let us now contemplate an extended body whose parts are debarred from any relative motion; this is called a *rigid body*. Then obviously, if it has no constraints whatever, it has what may be called *six degrees of freedom*, viz. translation parallel to each of the three co-ordinate axes and rotation about each of them. It should be noted that what is styled a single degree of freedom of a rigid body may (namely, if it is a rotation) involve the two-dimensional motion of its points. And if with this rotation we combine a translation along the axis of rotation, we have a three-dimensional motion of its points. Whereas two translations and a rotation about the axis perpendicular to both of them involves only a two-dimensional motion, for all the motions occurring are parallel to the plane containing the two translations. Thus the classification of motions according to their constraints and degrees of freedom, though very useful and needing to be borne in mind, differs materially from that according to dimensions in

space, which is usually simpler, and will be followed in developing the subject later.

To take away any one of the original six degrees of freedom of a rigid body, one constraint is needed, so that six such constraints are needed to fix its position. A few illustrations of simple cases may be given here. Let possible translations in three rectangular directions be denoted by $x$, $y$, and $z$, the corresponding possible rotations being $u$, $v$, and $w$.

*Top-spinning on level ground.*—One point of the body touches the $xy$ plane, and so loses motion in the $z$ direction up or down. Degrees of freedom left are $x$, $y$, $u$, $v$, and $w$.

*Top with spike in groove.*—Two points of the body touch the sides of the V-shaped groove parallel to axis of $x$, and so lose motion in directions of $y$ or $z$. Degrees of freedom left are $x$, $u$, $v$, and $w$.

*Top with spike in hole.*—Three points of the body touch the sides of the quasi-conical hole, and so it loses motion in directions $x$, $y$, and $z$. Degrees of freedom left are $u$, $v$, and $w$.

*Beam of Balance.*—Instead of knife edges, let the beam have two blunt screw points, one of which touches at three points in a quasi-conical hole and the other at two points in a V-shaped groove pointing to that hole, *i.e.* along the axis of $x$; it thus loses all three translations and two rotations. The sole degree of freedom left is $u$, or the rotation about the axis of $x$.

*Instrument on 'hole, slot, and plane.'*—Three points of the apparatus touch the sides of a quasi-conical hole, two points touch the sides of a V-shaped groove pointing to that hole, one point rests on the plane containing the hole and groove. In the case of physical and other apparatus requiring levelling and leaving set up without shake, or replacing in exactly the same position after temporary removal; this method is adopted, the blunt points of the levelling screws resting on the 'hole, slot, and plane' respectively.

It may easily be seen that a rigid straight line, if unconstrained, has five degrees of freedom. Thus, denoting the numbers of constraints and degrees of freedom by $C$ and $F$ respectively, we have the following scheme:—

| | |
|---|---|
| *For a point,* | $C+F=3$. |
| *For a rigid straight line,* | $C+F=5$. |
| *For other rigid bodies,* | $C+F=6$. |

Where no great pressures or speeds are to be used, the above arrangements of constraints, called *geometrical clamps*, attain ideal results in spite of the inevitable imperfections of human workmanship or machining. The ordinary arrangements for sliding and rotating motions in machinery involve surfaces which are approximately plane and cylindrical But though such surfaces fail of ideal perfection, they attain an approximation sufficient for the purpose and in combination with a facility for maintaining that degree of accuracy in spite of wear. Thus the physicist and the engineer have different ends in view, and rightly take different methods of reaching them.

## EXAMPLES—IV.

1. A boat steams due north at 15 miles per hour, while a man walks her deck in various directions at 2 miles per hour. Find graphically the man's volocities when his directions of walking the deck are (*a*) south-east, (*b*) north-north west, (*c*) west.
   From what body as base are your velocities reckoned ?
2. Show how to compound graphically three or more velocities. Will this construction serve for any other quantities ? if so, name some such.
3. Obtain analytical expressions for the resultant of any number of coplanar vectors.
4. Determine graphically or analytically the resultant velocity of a point which has simultaneously a vertically upward velocity of 5 cm. per second, a horizontal westerly one of 45 cm. per second, and a north-easterly one of 1000 cm. per second.
5. Taking the point P in a different position from that shown in Fig. 6A (*e.g.* inside the parallelogram OACB), show that the theorem of moments still holds.
6. Establish the construction for the composition of simultaneous angular *velocities*.
7. Consider the turning of a parallelepiped through a right angle first about one axis OA, and then about another axis OB, then reverse the order of turning about these axes. Hence show that the construction of question 6 does *not* apply to the composition of *successive* finite angular displacements.
8. Determine the angular velocities of the earth about two rectangular diameters, one of which meets the surface at Greenwich (lat. $\lambda$) and the other in the meridian of Greenwich, so as to compound to the earth's angular velocity $\omega$ about its polar diameter.
9. Discuss the degrees of freedom of particles and bodies with and without constraints and obtain expressions for the numbers of degrees of freedom of various bodies.
10. If a railway track turns uniformly at the rate of $9^{\circ}\cdot 55$ in a length of one furlong, what is its radius of curvature?
11. Define the term *centroid*, and obtain general expressions to locate it in a plane over which any number of points are distributed.
12. State what arrangements can be made to permit only the following geometrical motions of a rigid body :—(*a*) translation along a given horizontal line, (*b*) rotation about a given horizontal axis, (*c*) rotation about a vertical axis.
13. Name five geometrical and mechanical quantities, giving their *dimensions* and stating for each whether it is a scalar or a vector.

# CHAPTER IV

## RECTILINEAR MOTIONS

**27. Uniform Acceleration: Low Falls.**—In this chapter we restrict ourselves to rectilinear motion; hence, when the acceleration is given as uniform of value $a$, any initial velocity of magnitude $u$ possessed by the point under consideration must be along the same line as the acceleration, though it may have either algebraic sign. Let the velocity have magnitude $v$ at time $t$, and the space described be $s$, then it is required to establish relations between $v$, $u$, $a$, $s$, and $t$, and to use them for the solution of any problem relating to motions of the type in question.

By definition it is clear that in time $t$ the change of velocity occurring is $at$, which must be added to the initial to obtain the final velocity. Thus, we have

$$v=u+at \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

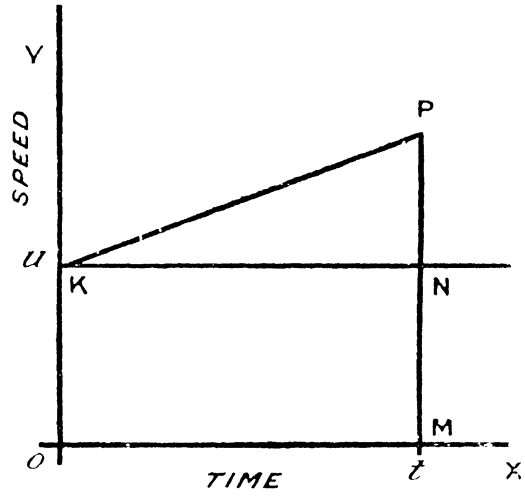



FIG. 7. UNIFORM ACCELERATION.

This is illustrated by the speed graph of Fig. 7, in which OK is the initial speed $u$, MP the final speed equal to MN+NP or $u+at$. It is clear that the equation of KP is $y=u+ax$, so that NP$=at$.

Consider next the space $s$, which is represented in the diagram by the area of KPMO. It is evidently given by the rectangle OKNM plus the triangle KNP, *i.e.* by $ut$ plus $\frac{1}{2}$NP$\times t$. But NP is $at$, thus we have

$$s=ut+\tfrac{1}{2}at^2 \quad . \quad . \quad . \quad (2).$$

On eliminating $t$ between (1) and (2) we have

$$v^2=u^2+2as \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

(For by squaring (1) we find $v^2=u^2+2uat+a^2t^2$,
and from (2) we see that $2as=2uat+a^2t^2$.)

These equations, (1)-(3), are the required relations between the five quantities concerned, and serve for the solution of any cases of rectilinear motion with uniform acceleration. For example, for 'low' falls or rises, *i.e.* motions in a vertical line near the surface of the earth, we may take that surface as the origin and measure $s$ or $v$ upwards as positive, then the acceleration due to the earth being downwards is to

be accounted negative. It has the approximate numerical value 32·2 feet per second per second or 981 centimetres per second per second. Thus denoting either of these positive numbers by '$g$,' as is usual, we must insert $-g$ in the equations instead of $+a$.

If, on the other hand, we find it convenient in other problems to take positive quantities to denote downward displacement, velocities, etc., then the acceleration due to the earth becomes positive, and is accordingly represented by $+g$ in the equations.

EXAMPLES—V.

1. Find the uniform acceleration which in 5 seconds changes an upward velocity of 64 feet per second into a downward velocity of 96 feet per second.
2. A train passes in 18 minutes between two stations 6 miles apart, stopping at each. If the train at first increased its speed uniformly under steam, and then immediately, with steam off and brakes on, decreased its speed uniformly at twice the former rate of increase, find the acceleration and retardation, and make displacement and speed graphs of the journey.
3. What is the rate of increase per foot of the square of the speed of a point under uniform acceleration of 20 ft. per sec.$^2$? and what velocity is attained in 35 feet from rest?

**28. Uniform Acceleration by the Calculus.**—Using the notation of the calculus for the motion of a point whose distance from the origin is $s$ and acceleration $a$, we have

$$d^2s/dt^2 = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Thus, on integrating, we find

$$\int d\left(\frac{ds}{dt}\right) = a\int dt,$$

or

$$ds/dt = at + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where $b$, the constant of integration, is evidently the initial velocity previously denoted by $u$.

By a second integration we have

$$\int ds = \int (at+b)dt,$$

or

$$s = \tfrac{1}{2}at^2 + bt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Obviously equations (2) and (3) of this article correspond respectively with (1) and (2) of article 27.

**29. Acceleration proportional to Displacement. Simple Harmonic Motion.**[1]—We pass now to cases of varying acceleration, taking first that in which it is proportional to the displacement but oppositely directed.

Let the displacement OM, Fig. 8, be denoted by $y$, and suppose

[1] If desired at this early stage, this motion may be treated without the calculus, as is often done by students of physics. (See, *e.g.*, the writer's *Sound*, Arts. 13-16.)

the acceleration to be $-\omega^2 y$. Then we have as the equation of motion of the point M

$$\frac{d^2y}{dt^2}+\omega^2 y=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Now it is obvious that the acceleration, being always opposite to the displacement, will retard every motion of M from the origin and reverse it, thus causing the point M to pass through O in the opposite direction. But on passing through O the displacement reverses, and therefore also the acceleration. Thus this motion from O must be annulled and reversed as before. Hence we see that the motion of M is a to-and-fro movement or vibration along YOY′ about the origin O as centre. The simplest way to express such a motion analytically is by a sine or cosine function of the time. To make such a function as general as possible we need three constants, denoting (i) the *amplitude* ($a$) or maximum value of OM, (ii) an *angular velocity* ($n$ say), and (iii) the *epoch* ($\epsilon$) or phase angle at the beginning of the time. We thus write as a trial solution

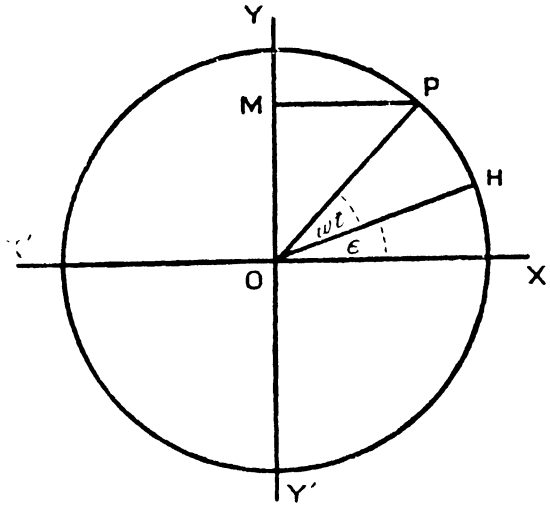



FIG. 8. SIMPLE HARMONIC MOTION.

$$y=a\sin(nt+\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Substituting this in (1) we have

$$(-n^2+\omega^2)a\sin(nt+\epsilon)=0.$$

Hence (2) is a solution of (1) provided that

$$-n^2+\omega^2=0,$$

*i.e.* when

$$n=\pm\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Now reversing the sign of $n$ is only equivalent to changing $\epsilon$ into $\pi-\epsilon$. Thus we may write the solution of (1) as follows:—

$$y=a\sin(\omega t+\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

in which $a$ and $\epsilon$ are as yet undetermined, whereas $\omega$ shows that the motion passes through its complete cycle of changes in the time $2\pi/\omega$. The meanings of the various symbols and the solution itself are illustrated in Fig. 8. In this figure the angle $XOH=\epsilon$, the angle $HOP=\omega t$, and the *auxiliary* circle passing through H and P has centre O and radius $a$. PM is drawn at right angles to YOY′, cutting off the displacement $OM=y$. Hence by construction OM corresponds to the value of $y$ expressed by (4).

It should be noted that by differentiation of (4) twice with respect to time we have

$$\ddot{y}=-\omega^2 a\sin(\omega t+\epsilon),$$

that is, the acceleration is a sine function of the time. Hence if we had begun with that problem instead of acceleration directly proportional to displacement, the solution would have been the same.

**30. Initial Conditions.**—We have seen that of the three constants in the solution only one is determined by the given law of acceleration, the other two are dependent upon the initial values of displacement and speed. Let these be respectively $y_0$ and $v_0$. Then from (4) we have

$$y_0 = a \sin \epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Also, by differentiating (4) with respect to time and then putting $t = 0$ and equating, we obtain for the speed at time $t$

$$\dot{y} = \omega a \cos (\omega t + \epsilon)$$

and

$$v_0/\omega = a \cos \epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Hence by squaring and adding (5) and (6)

we find

$$a^2 = y_0^2 + v_0^2/\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Also, by division of (5) by (6), we have

$$\tan \epsilon = \omega y_0/v_0 \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Thus (7) and (8) when with (4) complete the solution of (1) for any specified initial conditions.

Two cases of special importance and simplicity may be noticed.

*First.* The initial displacement is zero, the initial speed being $v_0$. Then, putting these values in (7) and (8), (4) becomes

$$y = \frac{v_0}{\omega} \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

*Second.* The initial speed is zero, the initial displacement being $y_0$. Then, from (7), (8), and (4), we find as the solution

$$y = y_0 \sin \left( \omega t + \frac{\pi}{2} \right) = y_0 \cos \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

If we illustrate the motion under consideration by a bullet suspended by a cocoon fibre about a metre long, then these two cases of initial conditions correspond (i) to striking the bullet a horizontal blow while it hangs at rest, and (ii) to pulling it aside and letting go while at rest in the displaced position.

In the simple harmonic motions noticed the time of complete execution of the cycle once is called the *period.* Denoting it by $\tau$, we obviously have from (4), etc.

$$\tau = 2\pi/\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

### EXAMPLES—VI.

1. Define simple harmonic motion and obtain expressions for the velocity and acceleration of a point executing it.
2. Given that a particle P in a straight tube has acceleration directed to a fixed point O in the tube and of magnitude proportional to OP, determine the motion in general terms.
3. If the acceleration of a point M in rectilinear motion is always $-16 \times$ OM, where O is a fixed point, find the equation locating M, its initial displacement and velocity being respectively 5 cm. and 10 cm. per sec.

4. A point executes a simple harmonic motion of amplitude 3 cm. in a period $4\pi$ seconds. Find (*a*) the maximum velocity, (*b*) the velocity at half-displacement, (*c*) the acceleration at full displacement, (*d*) the acceleration per cm. displacement.

**31. Composition of Collinear Simple Harmonic Motions.**—A kinematical problem of considerable interest and importance is presented in the composition of two or more simple vibrations. The case in which they are collinear naturally falls in the present chapter, and we shall first treat two vibrations only, their periods being equal.

Thus, let the component vibrations have displacements $u$ and $v$ along the axis of $y$, and be expressed by

$$u = a \sin(\omega t + \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$v = b \sin(\omega t + \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Then their resultant, of displacement $y$, is given by

$$\left.\begin{aligned} y &= u + v, \\ \text{or} \quad y &= r \sin(\omega t + \theta) \text{ say} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

We are here assuming that the resultant vibration is of the same type as its components, an assumption which remains to be justified or condemned. To test the matter, expand the right sides of (1), (2), and (3) and equate. We thus find that the assumption leads to

$$\begin{aligned} &(a \sin\alpha + b \sin\beta)\cos\omega t + (a\cos\alpha + b\cos\beta)\sin\omega t \\ &\quad = r\sin\theta\cos\omega t + r\cos\theta\sin\omega t \quad . \quad . \quad . \quad . \quad . \quad (4). \end{aligned}$$

But this equation has to hold for every value of $t$. We may accordingly equate the coefficients of $\cos\omega t$ and those of $\sin\omega t$. We thus obtain two equations, namely

$$r\sin\theta = a\sin\alpha + b\sin\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$r\cos\theta = a\cos\alpha + b\cos\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Whence, squaring and adding,

$$r^2 = a^2 + b^2 + 2ab\cos(\alpha - \beta) \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Also, dividing (5) by (6)

$$\tan\theta = \frac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Equations (7) and (8) show that for any real values of $a$, $b$, $\alpha$ and $\beta$, corresponding real values are possible for $r$ and $\theta$. Accordingly the assumption in the lower line of equation (3) is justified and, together with (7) and (8), affords the solution sought.

It is seen from (7) that the resultant amplitude $r$ usually lies between the limits $a \pm b$, reaching them for $\alpha - \beta = 0$ and $\pi$ respectively. Further, it may be seen from (8) that $\theta = (\alpha + \beta)/2$ when $a = b$.

For more than two collinear vibrations of same period we see from (5) and (6) that it is easy to generalise and write

$$r\sin\theta = \Sigma a\sin\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$r\cos\theta = \Sigma a\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

the component amplitudes and phases being denoted respectively by $a_1, a_2, a_3 \ldots$ and $\alpha_1, \alpha_2, \alpha_3 \ldots$

Thus, squaring and adding, we have

$$r^2=(\Sigma a \sin \alpha)^2+(\Sigma a \cos \alpha)^2 \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Again, by division, we have

$$\tan \theta=(\Sigma a \sin \alpha)/(\Sigma a \cos \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

**32. Graphical Composition.**—The composition of collinear vibrations may be illustrated or performed graphically, and this view of the matter well deserves notice.

Fig. 9 illustrates the composition of two simple harmonic motions supposed to occur along YOY′, their components being as already specified in equations (1) and (2) of article 31. The ordinate OF represents the displacement $u$ at the instant $t=0$ due to one vibration, and is the projection of OP of length $a$ and inclination $\alpha$ to OX. Similarly, OG gives the value $v$ of the other displacement at the same instant, being the projection of OQ of length $b$ and inclination $\beta$. The ordinate OH of length $y$ is the sum of OF and OG, and is also the projection upon YOY′ of OR, the diagonal of the parallelogram upon OP and OQ. Hence OH represents at $t=0$ the sum of the component displacements. But since the periods of the two component vibrations are equal, the radii OP and OQ must move with equal angular velocities in describing the auxiliary circles through P and Q corresponding to the two vibrations in question. Hence the angle $POQ=\alpha\sim\beta$ must remain constant as well as the lengths OP and OQ themselves. Thus the parallelogram OPQR remains of fixed size and shape. Therefore H, the projection of R, executes simple harmonic motion upon YOY′ of amplitude $OR=r$ say, the phase angle at $t=0$ being $XOR=\theta$ say. Further, it is easy from the figure to confirm or obtain the relations analytically deduced in article 31 and expressed in equations (7) and (8).

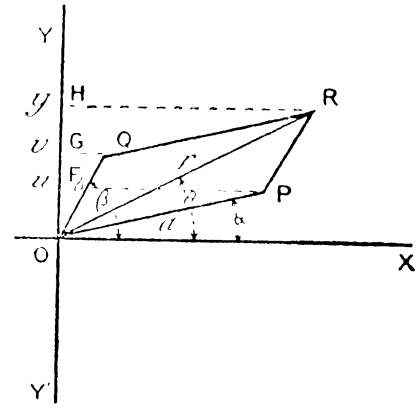



FIG. 9. GRAPHICAL COMPOSITION OF TWO COLLINEAR VIBRATIONS.

**33.** For the construction of the above figure it is evident that OR could have been obtained with fewer lines by putting PR of length $b$ and inclination $\beta$ with OX instead of first drawing OQ and then completing the parallelogram. We should in that case draw *one half* only of the parallelogram to obtain R instead of both halves.

This, which is a small matter when only two components are concerned, is a distinct advantage when three or more component vibrations are to be dealt with.

This method is illustrated for four collinear vibrations in Fig. 10. We suppose the vibrations to occur along YOY′, the component displacements being represented by the projections upon that line of $OP_1$, $P_1P_2$, $P_2P_3$, and $P_3P_4$. The resultant vibration is accordingly represented by the projection upon YOY′ of $OP_4$, the line which closes

the polygon. If the components are as specified by the right side of equations (9) and (10) in article 31, it is seen that (11) and (12) give the amplitude and angle for the resultant.

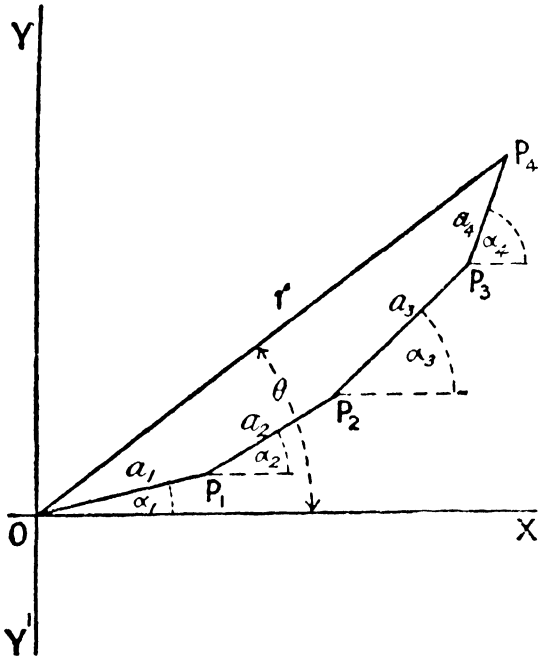



FIG. 10. GRAPHICAL COMPOSITION OF FOUR COLLINEAR VIBRATIONS.

Thus the resultant of collinear simple harmonic motions of equal periods may be obtained by the addition of vectors according to the parallelogram or polygon method, the vectors for components and resultant being in each case the radii of the corresponding auxiliary circles at some one instant, say for $t=0$.

In other words, if the amplitudes and phase angles of component collinear simple harmonic motions be represented respectively by the lengths and inclinations of the sides of a polygon, then shall the closing line of the polygon represent by its length and inclination the amplitude and phase angle of the resultant simple harmonic motion, which is of the same period as its components.

For the composition of rectangular vibrations see Chapter v.

## EXAMPLES—VII.

1. Establish the general expression for the resultant of two collinear simple harmonic motions of the same period.
2. Compound two collinear simple harmonic motions of equal periods, their amplitudes being 2 and 3 cm. and their phase angles $\pi/4$ and $\pi/3$ respectively.
3. Confirm graphically the results obtained for the preceding example.
4. Compound analytically or graphically collinear vibrations of equal periods whose amplitudes are 8, 6, 4, and 2, their phase angles being 0°, 30°, 45°, and 60° respectively.

**34. Acceleration inversely as Distance Squared.**—We now consider a second example of acceleration varying with position, but this time it is inversely as the second power instead of directly as the first.

As this case is of great importance and occurs early in the course we shall treat it first by elementary methods. Thus, students only just starting the calculus may defer the analytical method till a second reading.

Let the acceleration be towards a fixed point O in the line along

which the point P under notice is to move. And let P and P′, distant $s$ and $s'$ from O, be two very near positions of the point at times $t$ and $t'$, the corresponding velocities being $v$ and $v'$. Then, by definition of velocity, we have

$$\frac{v'+v}{2}(t'-t)=s'-s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Now since the acceleration is towards O, and we are taking distances, velocities, and accelerations positive when from it, we may write $-\mu/s^2$ for the acceleration at P, where $\mu$ is some constant. Similarly the acceleration at P′ is $-\mu/s'^2$. Thus, for the *mean* acceleration while PP′ is described, we may write and equate the two expressions

$$\frac{v'-v}{t'-t}=-\frac{\mu}{ss'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Hence, multiplying these two equations, we obtain

$$v'^2-v^2=2\mu\left(\frac{1}{s'}-\frac{1}{s}\right) \quad . \quad . \quad . \quad . \quad . \quad (3).$$

This is the general expression for the very small step PP′. Let us now take a finite step PQ, where $OQ=S$ and the velocity at Q is $V$. Divide this step PQ into a large number of very small ones, the intermediate distances and velocities being $s_1, s_2, s_3, \ldots s_n$ and $v_1, v_2, \ldots v_n$.

Then from (3) we may write

$$v_1^2-v^2=2\mu\left(\frac{1}{s_1}-\frac{1}{s}\right)$$

$$v_2^2-v_1^2=2\mu\left(\frac{1}{s_2}-\frac{1}{s_1}\right)$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$v_n^2-v_{n-1}^2=2\mu\left(\frac{1}{s_n}-\frac{1}{s_{n-1}}\right)$$

$$V^2-v_n^2=2\mu\left(\frac{1}{S}-\frac{1}{s_n}\right).$$

And, on adding these, we see that on each side terms cancel out diagonally, giving

$$V^2-v^2=2\mu\left(\frac{1}{S}-\frac{1}{s}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

On comparing with (3) we see that the relation for a small step is valid for a finite one also.

Suppose now that $V=0$ for $S=r$, then (4) gives the general formula for a fall from rest—

$$v^2=2\mu\left(\frac{1}{s}-\frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

We may now obtain the same result by the calculus. Thus the equation of motion is

$$-\frac{\mu}{s^2}=\frac{dv}{dt}=\frac{ds}{dt}\cdot\frac{dv}{ds}=v\frac{dv}{ds}.$$

Separating the variables and integrating between the appropriate limits we have

$$\int_0^v v\,dv = -\mu \int_r^s \frac{ds}{s^2},$$

whence
$$v^2 = 2\mu\left(\frac{1}{s} - \frac{1}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

in agreement with (5). If $r=\infty$, the corresponding $v$ is $\pm\sqrt{2\mu/s}$.

**35. High Falls.**—We shall see later, in the section on Attractions (Chapter XVI.), that this law of acceleration is that which applies outside a spherical gravitating body of uniform density, or concentric shells, each of which is of uniform density, its centre being the point O. We may accordingly apply it as an approximation to what would happen in the case of a body falling to the earth from a great or infinite distance, all resistances being supposed negligible. In this case it is convenient to express the general constant $\mu$ in terms of the particular one $g$ giving the acceleration on the earth's surface, and $R$ the radius of the earth. The relation is evidently $-\mu/R^2 = -g$ or $\mu = gR^2$.

Hence, if $V_\infty$ is the velocity acquired in an unresisted fall to the earth's surface from an infinite height, we have

$$V_\infty^2 = 2gR \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Suppose now the velocity $V$ is acquired at the earth's surface by a fall from a height $h$ above the surface. Then by (6) introducing $g$ we find

$$V^2 = 2gR^2\left(\frac{1}{R} - \frac{1}{R+h}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Further, we may put this in the forms

$$V^2 = \frac{2gh}{1+h/R} = 2gh\left(1 - \frac{h}{R}\right) \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad (9).$$

the latter expression being an approximation obtained by neglecting $h^2/R^2$ in comparison with unity. This accordingly applies where the height is not too great.

Obviously, if $h/R$ is negligible compared to unity, the square of the velocity reduces to the familiar $2gh$ as for uniform acceleration of magnitude $g$.

**36. Time of Fall.**—We have hitherto dealt with the relations between velocity and distance. Let us now change to space and time. Thus, from equation (6) of article 34, remembering $v = ds/dt$, taking the square root, rearranging and integrating, we have

$$\int_r \frac{ds}{\sqrt{1/s - 1/r}} = \sqrt{2\mu}\int_0^t dt = \int_r^s \frac{\sqrt{r}\,ds}{\sqrt{r-s}} \quad . \quad . \quad . \quad . \quad . \quad (10).$$

To evaluate the right-hand integral, put $s = r\cos^2\theta$, then $\sqrt{r-s} = \sqrt{r}\sin\theta$, $ds = -2r\cos\theta\sin\theta\,d\theta$, and the lower limit becomes zero in the new integral. Hence

$$t\sqrt{2\mu}=r^{3/2}\int_0^\theta(1+\cos 2\theta)d\theta=r^{1/2}(r\theta+r\sin\theta\cos\theta).$$

Or,
$$t=\sqrt{\frac{r}{2\mu}}\left\{r\cos^{-1}\sqrt{\frac{s}{r}}+\sqrt{s(r-s)}\right\} \quad . \quad . \quad (11),$$

giving the time of a fall from rest at distance $r$ to the distance $s$ under the acceleration $-\mu/s^2$.

If we have $r=\infty$, all times of fall to any finite distance $s$ become infinite also. But we may obtain the times $(t-t_0)$ over the distance $(s_0-s)$ as follows:—Beginning with equation (6), and putting $r=\infty$, we obtain

$$v=-\sqrt{\frac{2\mu}{s}}=\frac{ds}{dt}.$$

Hence, separating the variables and integrating

$$\int_{s_0}^{s}s^{1/2}ds=-\sqrt{2\mu}\int_{t_0}^{t}dt, \text{ or } t-t_0=\frac{\sqrt{2}}{3\sqrt{\mu}}(s_0^{3/2}-s^{3/2}) \quad . \quad . \quad (12).$$

EXAMPLES—VIII.

1. Establish the general relation between velocity and space for a point moving along a straight line under acceleration inversely as the square of its distance from a point in that line.
2. From the result obtained for question 1, pass to the relation between space and time.
3. Find the velocity acquired by a fall to the earth's surface from rest at a height of 400 miles, the earth's radius being reckoned 4000 miles, and $g$ as 32·2 ft./sec.$^2$
4. Calculate the time for the fall of question 3.
5. Determine the vertical velocity which, in the absence of resistances, would suffice to carry a particle away from the earth.

**37. Acceleration diminished proportionally to Speed: Mist.**—We now treat cases in which the acceleration over the given region is uniform except as it is changed by the speed of the moving point or body. And in the first place this change of acceleration shall be a diminution proportional to the speed. This applies to very small bodies and to very slow-moving bodies falling through the air. In these cases the uniform acceleration in the region in question is due to gravity and the diminution of the acceleration to the resistance of the air. Thus tiny spherules of water, as in the case of mist or very fine rain, are always falling, with respect to the air surrounding them, but are also resisted so that their speed relative to the air is never great.

*Speed and Time.*—Let the space co-ordinate $s$, the speed $v$, and the acceleration $a$ be all reckoned positively in the same direction. Also let the diminution of acceleration be such that at speed $k$ it equals $a$, and therefore suffices to annul the acceleration which affects bodies at rest. Then at speed $v$ the diminution of the acceleration will be $av/k$. Thus we may write the equation of motion in the form

$$\frac{dv}{dt}=\frac{a}{k}(k-v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let the particle start from rest at the origin of co-ordinates when $t=0$. Then, separating the variables in (1) and integrating, we have

$$\frac{a}{k}\int_0^t dt=\int_0^v \frac{dv}{k-v},$$

and, on evaluating,

$$\frac{at}{k}=\log_e\left(\frac{k}{k-v}\right) \quad . \quad . \quad . \quad . \quad . \quad (2),$$

which gives the time $t$ in terms of the speed $v$. If we wish to have $v$ given explicitly in terms of $t$, we may raise $e$ to the powers given by each side of equation (2). Thus

$$e^{at/k}=\frac{k}{k-v}.$$

Whence $$v=k(1-e^{-at/k}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

We see from either (2) or (3) that the speed $v$ only rigorously reaches its limiting value $k$ in an infinite time. But for $a=g=981$ cm./sec.$^2$ and $k=0{\cdot}981$ cm./sec., we have $e^{-at/k}=e^{-1000t}$.

Hence when $t=$ one hundredth of a second $v$ will differ from $k$ by only $e^{-10}$ out of 1, *i.e.* by less than one part in twenty thousand.[1]

**38.** *Speed and Space.*—We have obtained relations between $v$ and $t$, and now pass to obtain them between $v$ and $s$, the speed and space passed over. Thus, referring to equation (1), we see that the first term may be transformed as follows :—

$$\frac{dv}{dt}=\frac{ds}{dt}\cdot\frac{dv}{ds}=v\frac{dv}{ds}.$$

The whole equation may accordingly be rewritten

$$v\frac{dv}{ds}=\frac{a}{k}(k-v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Thus, separating the variables and integrating, we have

$$\frac{a}{k}\int_0^s ds=\int_0^v \frac{v\,dv}{k-v}=-\int_0^v \frac{v-k+k}{v-k}\,dv.$$

Hence, on evaluating, we find

$$\frac{a}{k}s=-v+k\log_e\frac{k}{k-v} \quad . \quad . \quad . \quad . \quad . \quad (5),$$

which gives the space $s$ in terms of the speed $v$ acquired in it, the start being from rest.

Here again it is seen that the rigorous limiting value of the speed is only attained after an infinite space is passed over. But with the numerical values $k=\cdot 981$ and $a=981$, as previously used, a very small distance suffices for an approach to the limiting speed. Thus, put-

[1] For an isolated sphere of radius $r$ in fluid of viscosity $\eta$, Stokes showed that the resistance $=6\pi r\eta$ times its speed. Hence for limiting speed $k$ under gravity $g$, we find for a sphere of density $\rho$ in fluid of density $\sigma$, $k=\frac{2}{9}\frac{r^2g}{\eta}(\rho-\sigma)$. For a drop of water a hundredth of a millimetre radius, in air, we find $k=1{\cdot}2$ cm./sec. nearly.

ting the second term on the right side of (5) equal to $k$, *i.e.* making $v=k\left(1-\frac{1}{e}\right)$, which is distinctly more than half its limiting value, we have

$$s=k^2/ae=\cdot 962/981\times 2\cdot 718\ldots=1/2{,}772 \text{ of a cm.}$$

Thus less than four ten-thousandths of a centimetre are passed over, while the speed attains nearly two-thirds of its limiting value.

**39. Diminution proportional to Square of Speed: Hailstone.**—In the case of quicker-moving bodies, such as large raindrops, hailstones, or shot, the diminution of the acceleration is approximately as the square of the speed. It thus furnishes us with another slightly different problem for attack. Taking as before the space $s$ and speed $v$ positive when reckoned in the same direction as the acceleration $a$, and again indicating by $k$ the limiting value of the speed, we see that the acceleration is diminished by the amount $av^2/k^2$ when the speed is $v$.[1]

*Speed and Time.*—The equation of motion may accordingly be written

$$\frac{dv}{dt}=\frac{a}{k^2}(k^2-v^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Separating the variables and integrating we have

$$a\int_0^t dt=k^2\int_0^v \frac{dv}{k^2-v^2}=\frac{k}{2}\int_0^v\left(\frac{1}{k-v}+\frac{1}{k+v}\right)dv,$$

which, on evaluation, yields

$$t=\frac{k}{2a}\log_e\frac{k+v}{k-v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

thus giving the time $t$ in terms of the speed $v$.

If the speed is required explicitly in terms of the time we can transform the equation exponentially as before (see equation (2) of article 37). We thus obtain

$$v=k\frac{e^{at/k}-e^{-at/k}}{e^{at/k}+e^{-at/k}}=k\tanh(at/k) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

as the relation required.

**40.** *Speed and Space.*—If, however, the speed is required as a function of the distance, or *vice versa*, go back to equation of article 39, and note, as before, that

$$dv/dt=vdv/ds.$$

Hence (1) may be transformed into

$$v\frac{dv}{ds}=\frac{a}{k^2}(k^2-v^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

[1] For aeroplane bombs at heights from 10,000 to 34,000 feet, the limiting values, $k$ of the speeds, vary between 600 and 1100 ft./sec.—Dr. G. N. Pell, *Aeronautics. R.* and *M.* 340, June 1919.

Whence, on separating the variables and integrating, we find

$$\frac{a}{k^2}\int_0^s ds=\int_0^v \frac{vdv}{k^2-v^2}=-\tfrac{1}{2}\int_0^v \frac{d(k^2-v^2)}{k^2-v^2}.$$

And therefore on evaluating

$$s=\frac{k^2}{2a}\log_e\frac{k^2}{k^2-v^2} \quad \ldots\ldots\ldots \quad (5),$$

which gives the distance $s$ passed over in acquiring from rest the speed $v$. If now we wish to express the speed explicitly in terms of the distance, take exponential values of each side as before. We thus have

$$v^2=k^2(1-e^{-2as/k^2}) \quad \ldots\ldots\ldots \quad (6).$$

**41. Rise and Fall of Shot.**—Let us now consider the rise and fall of a body in a region where the uniform acceleration is changed by an amount proportional to the square of the speed, that change being always opposed to the speed so as to diminish its numerical value.

*Rise.*—Take the origin of co-ordinates where the particle starts upwards at $t=0$ with a speed $U$, and let it reach a height $s$ and have speed $u$ at time $t$, its utmost height being $S$ with zero speed at time $T$. Thus though the space and speed are reckoned positively upwards, the acceleration $g$ due to gravity, and the diminution of the speed $gu^2/k^2$ due to air resistance, are both downwards.

*Speed and Time.*—The equation of motion for the ascent may accordingly be written

$$\frac{du}{dt}=-\frac{g}{k^2}(k^2+u^2) \quad \ldots\ldots\ldots \quad (1).$$

Thus, separating the variables and integrating, we have

$$\frac{g}{k^2}\int_0^t dt=-\int_U^u \frac{du}{k^2+u^2}$$

Whence

$$\frac{gt}{k^2}=-\frac{1}{k}\left[\tan^{-1}\frac{u}{k}\right]_U^u,$$

or

$$t=\frac{k}{g}\left\{\tan^{-1}\frac{U}{k}-\tan^{-1}\frac{u}{k}\right\} \quad \ldots\ldots \quad (2)$$

Thus, at the summit of the motion, we have

$$\frac{gT}{k}=\tan^{-1}\frac{U}{k},$$

or

$$T=\frac{k}{g}\tan^{-1}\frac{U}{k} \quad \ldots\ldots\ldots \quad (3).$$

This ends the ascent in terms of speed and time to which (1) applies. But before taking the descent we may with advantage take the ascent again in terms of

*Speed and Space.*—Thus, transforming the first term of (1), we may write

$$u\frac{du}{ds}=-\frac{g}{k^2}(k^2+u^2) \quad \ldots\ldots\ldots \quad (4).$$

Separating the variables and integrating gives

$$\frac{g}{k^2}\int_0^s ds = -\tfrac{1}{2}\int_U^u \frac{2u\,du}{k^2+u^2}.$$

Whence
$$\frac{gs}{k^2} = -\tfrac{1}{2}\Big[\log(k^2+u^2)\Big]_U^u,$$

or
$$s = \frac{k^2}{2g}\log_e\frac{k^2+U^2}{k^2+u^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Thus, at the summit of the motion, we have

$$S = \frac{k^2}{2g}\log_e\frac{k^2+U^2}{k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

which completes the consideration of the ascent.

**42.** *Fall.*—Leaving the zero and co-ordinates of space and time as before, we will now write $v$ for the *numerical* value of the speed in the descent, so that $v$ is positive throughout the fall, just as $u$ was in the rise. Thus the falling speed $v$ is increased by gravity $g$ and diminished by the air resistance $gv^2/k^2$.

*Speed and Time.*—Thus, we may write as the equation of motion for the descent

$$\frac{dv}{dt} = \frac{g}{k^2}(k^2-v^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

On separating the variables as before and integrating we find

$$\frac{g}{k^2}\int_T^t dt = \int_0^v \frac{dv}{k^2-v^2} = \frac{1}{2k}\int_0^v\left(\frac{1}{k-v}+\frac{1}{k+v}\right)dv.$$

Whence
$$\frac{g}{k^2}(t-T) = \frac{1}{2k}\log\frac{k+v}{k-v}.$$

Thus
$$t = T+\frac{k}{2g}\log_e\frac{k+v}{k-v},$$

or by use of (3)
$$t = \frac{k}{g}\tan^{-1}\frac{U}{k}+\frac{k}{2g}\log_e\frac{k+v}{k-v}\Bigg\} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

*Speed and Space.*—Let us now treat the fall in terms of speed and space passed over. Then on simply reversing the sign of $ds$ in equation (4) of article 40, or deducing it from (7) of the present article, with similar regard to the fact that $v = -ds/dt$, we have as the equation of motion

$$-v\frac{dv}{ds} = \frac{g}{k^2}(k^2-v^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Thus, by the usual steps, we have in succession

$$-\frac{g}{k^2}\int_S^s ds = \int_0^v \frac{v\,dv}{k^2-v^2},$$

$$-\frac{g}{k^2}(s-S) = \tfrac{1}{2}\log\frac{k^2}{k^2-v^2},$$

and
$$s = S-\frac{k^2}{2g}\log_e\frac{k^2}{k^2-v^2} \quad . \quad . \quad . \quad . \quad (10).$$

When the shot again reaches the origin, *i.e.* for $s=0$, let the speed be $V$; then we have

$$S=\frac{k^2}{2g}\log_e\frac{k^2}{k^2-V^2} \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Equations (6) and (11) enable us to find a relation between $U$, the *speed of ascent* through a given point, and $V$, the speed of *repassage downwards* through the same point. For obviously by the two equations for $S$ we have

$$\frac{k^2+U^2}{k^2}=\frac{k^2}{k^2-V^2}.$$

Whence

$$\frac{1}{V^2}=\frac{1}{U^2}+\frac{1}{k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

or

$$V=\frac{k}{\sqrt{1+k^2/U^2}} \quad . \quad . \quad . \quad . \quad . \quad (13).$$

These equations show that $V$ cannot exceed $k$, and will only reach it, for $U=\infty$. Previous equations, as (8) and (10), show that the speed approaches $k$, but could only equal it after infinite time and at infinite distance. Thus, any upward speed is annulled under the prescribed conditions, the state of rest being changed to a downward speed which increases but more and more slowly as the 'limiting' or 'terminal' speed $k$ is approached, and in such wise that this speed can never be overpassed.

If $T'$ denote the whole time of rise and fall from the origin with speed $U$ upwards to the same point again with speed $V$ downwards, we have from the first form of (8) for the time of the fall

$$T'-T=\frac{k}{2g}\log_e\frac{k+V}{k-V} \quad . \quad . \quad . \quad . \quad . \quad (14),$$

in which $V$ is defined by (12) and (13).

**43. Alternative Expressions.**—We have thus given time and space in terms of the speed. If the speed is required in terms of time or space there is no difficulty in obtaining the relations analytically. Thus, from equation (2) of article 41, writing $\tan\Omega=U/k$ and $\tan\omega=u/k$, we have

$$gt/k=\Omega-\omega,$$
$$\tan\omega=\tan(\Omega-gt/k),$$

and

$$\frac{u}{k}=\frac{U/k-\tan gt/k}{1+(U/k)\tan gt/k} \quad . \quad . \quad . \quad . \quad . \quad (15),$$

giving the ascending speed in terms of the time.

Again, from (5) we have by the exponential transformation

$$u^2+k^2=(k^2+U^2)e^{-2gs/k^2} \quad . \quad . \quad . \quad . \quad . \quad (16),$$

giving the ascending speed in terms of the space risen.

For the descent we have from (8) of article 42 by the exponential transformation

$$(k-v)e^{2g(t-T)/k}=(k+v).$$

Whence

$$\frac{v}{k}=\frac{1-e^{-2g(t-T)/k}}{1+e^{-2g(t-T)/k}}=\tanh\{g(t-T)/k\} \quad . \quad . \quad . \quad (17),$$

giving the descending speed in terms of the time.

Again, from equation (10) of article 42, by the same method, we find

$$\left.\begin{aligned} k^2 - v^2 &= k^2 e^{-2g(S-s)/k^2}, \\ \text{or} \qquad v^2 &= k^2\left(1 - e^{-2g(S-s)/k^2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (18),$$

giving the descending speed in terms of the space $(S-s)$ fallen through. It should be remembered that $s$ is reckoned positively if upwards, and cannot in that direction exceed $S$. Hence when the origin is passed through again in the descent, $s$ changes to a negative value. Thus $S-s$ increases from zero at the start of the descent to $S$ at the point of projection, and thenceforward increases towards infinity, being positive all through.

**44. Graphical Treatment.**—We have thus expressed speeds of

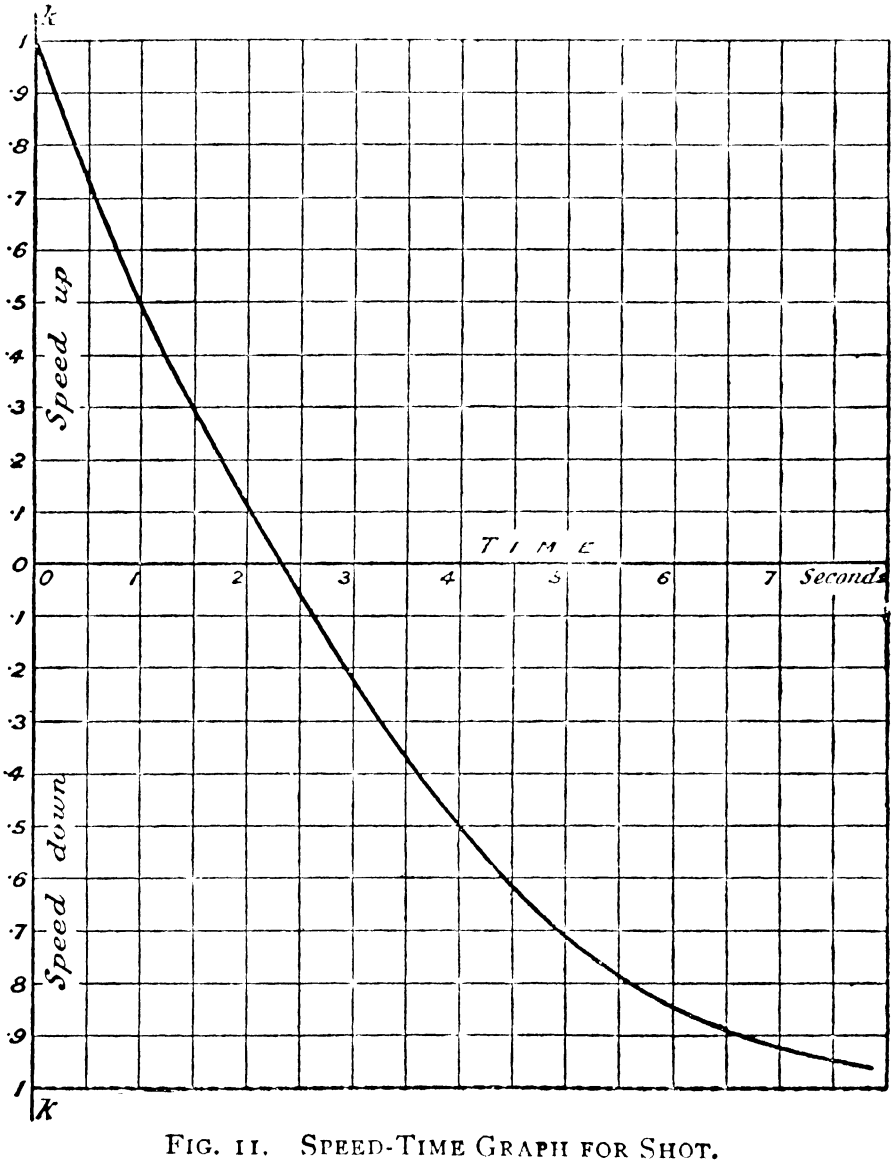



FIG. 11. SPEED-TIME GRAPH FOR SHOT.

rise and fall in terms of the time occupied and distance traversed and *vice versa*. But no general direct relation between space and

time has been given. Neither is it easy to obtain such relations analytically.

But since time and space have been given each in terms of the speed, we can assign to the speed a number of possible values, place them in a column, and then in neighbouring columns insert the corresponding values of time and space calculated from the formulae already developed. We should thus derive a number of corresponding values

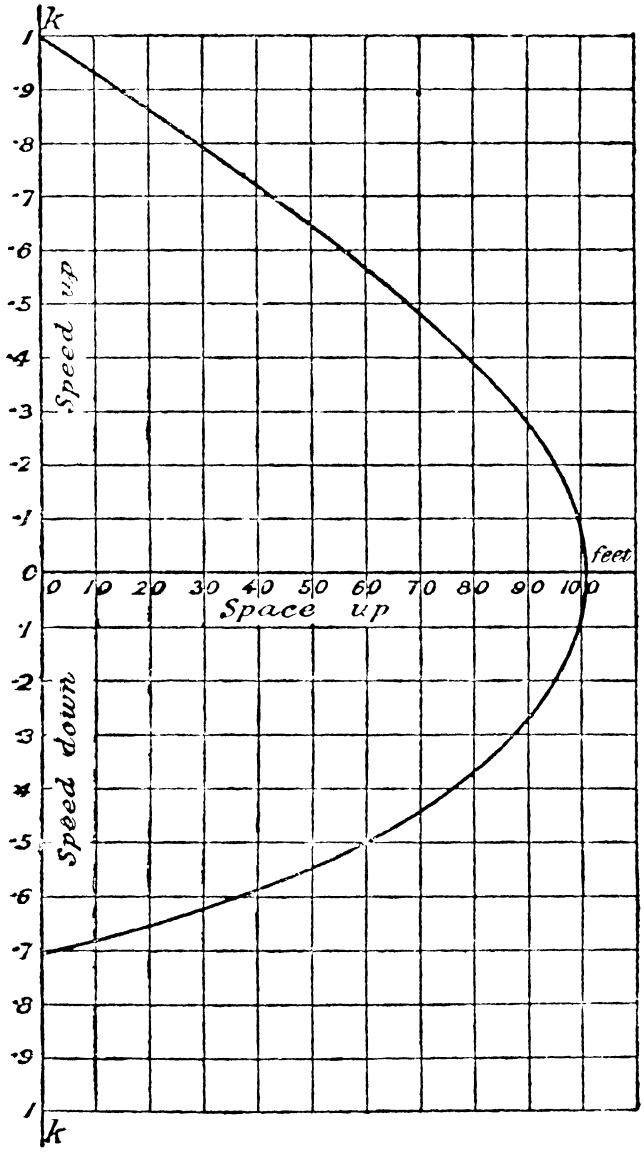



FIG. 12. SPEED-SPACE GRAPH FOR SHOT.

of space and time, and could then plot a space-time graph. Or, we could begin by plotting a speed-time graph and a speed-space graph from the formulae. Then, selecting any one value of the speed on the ordinates of each, their abscissae would give an ordinate and an abcissa for a third curve forming a space-time graph of the motion. This graphical method of expressing the meaning of the equations obtained,

and deriving a new relation from them, is exhibited in Figs. 11, 12, and 13, which will repay careful examination.[1]

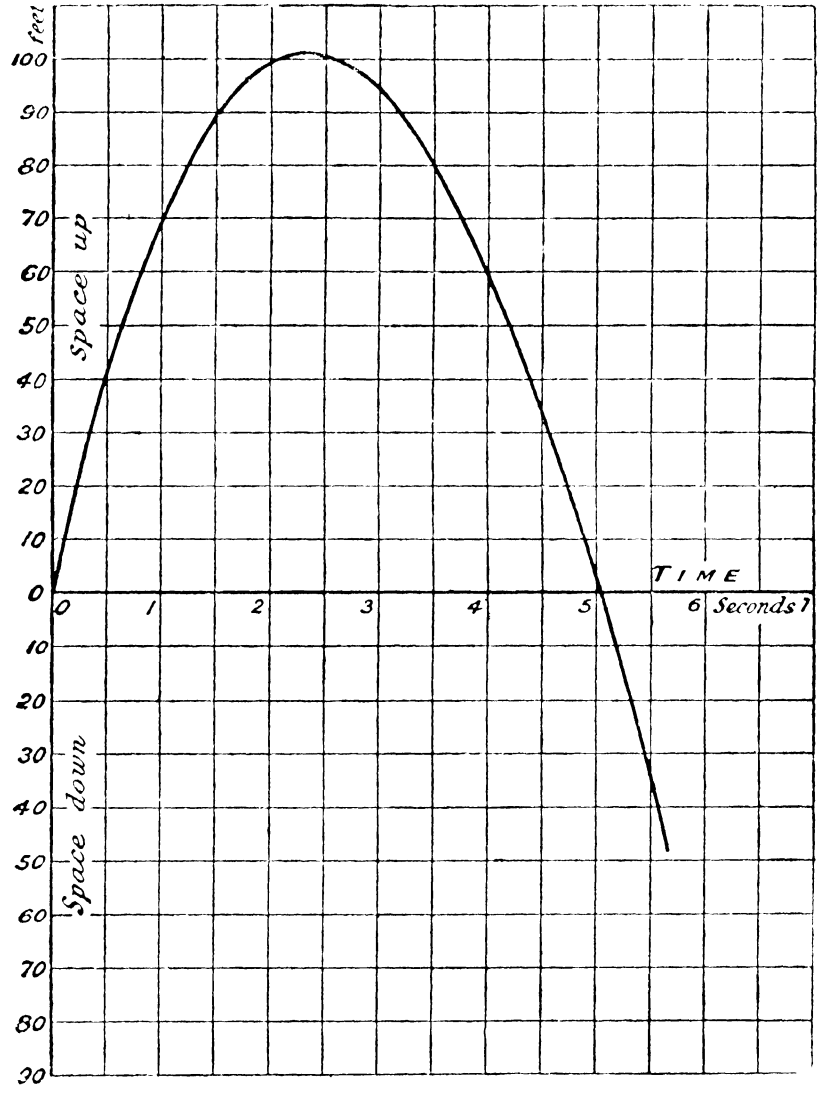



FIG. 13. SPACE-TIME GRAPH FOR SHOT DERIVED FROM THE FORMER TWO.

## EXAMPLES—IX.

1. 'Investigate the motion of a heavy particle which falls vertically from rest in a medium whose resistance is proportional to the velocity.'
(LOND. B.SC., PASS, APPLIED MATH., 1909, II. 5.)
2. 'The position of a point which describes a straight line is defined by its distance $x$ from a fixed point of the line. Show that its acceleration is $d^2x/dt^2$. The motion of a particle projected with velocity $V$ is retarded at a rate which varies as the velocity. Find the time which elapses before its velocity is halved, given that the retardation is $\lambda$, when the velocity is $V$.' (LOND. B.SC., PASS, APPLIED MATH., 1906, II. 1.)
3. For a point with initial velocity $U$ vertically upwards under constant

[1] In these the limiting speed $k$ is 96·6 ft./sec., which is about the value for a golf ball (Tait). The speed of projection is also equal to $k$.

acceleration downwards and retardation proportional to velocity squared, find the relations between speed and time and speed and space.

4. For the point of the previous question trace its motion after the summit of its path is reached, and find the speed $V$ when at its original level again.

**45. Acceleration varying with Displacement and Speed: Damped Vibration.**—Let us now consider the motion of a point whose acceleration is the sum of two parts which are directly proportional to its displacement and its speed respectively, but each oppositely directed. Then that part which is opposite and proportional to the displacement would, if operating alone, yield a simple harmonic motion. But this motion will evidently be diminished by the continuous operation of the other part, which is opposed to the speed and proportional to it. These reflections give us a clue to the motion, which we shall presently see is that diminishing vibration called a *damped* simple harmonic motion.

Let the displacement of the point be called $y$, and let its acceleration be $-(p^2y+2\kappa dy/dt)$. We can then write as the equation of motion

$$\frac{d^2y}{dt^2}+2\kappa\frac{dy}{dt}+p^2y=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The simplest way of representing analytically a damped vibration is to insert in its coefficient a factor of the form $e^{-mt}$. We accordingly try as a solution

$$y=ae^{-mt}\sin(qt+\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

in which the unknown $q$ is purposely written instead of $p$ to provide for a possible difference of period between the ensuing motion and that which would occur if $\kappa$ were absent; $a$ and $\epsilon$ are also inserted to make the expression as general as possible. By substituting (2) in (1) and differentiating we find it is satisfied, provided that $m=\kappa$ and

$$q^2=p^2-\kappa^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

We may accordingly write as the solution sought

$$y=ae^{-\kappa t}\sin(qt+\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

in which $q$ is defined by (3), $a$ and $\epsilon$ being dependent only on the initial conditions. These equations indicate that if $\kappa$ is small the factor $e^{-\kappa t}$ may be appreciable, while $q$ is practically $p$.

It can be shown that the motion analytically expressed by (4) may also be regarded as the projection upon the axis of $y$ of a point $P$ which describes with *angular* velocity $q$ the logarithmic spiral

$$r=ab^{-\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

in which $\log_e b=\kappa/q$ and the angle $\theta$ of the polar co-ordinates is reckoned from the inclination $\epsilon$ to the axis of $x$. If $e^{\lambda}=b^{\pi}$, $\lambda$ is sometimes called the *logarithmic decrement*; in other cases the phrase is applied to $\mu$ where $e^{\mu}=b^{2\pi}$. Thus the term denotes the logarithm to the base $e$ of the ratio of one amplitude to the next on the other side or the same side respectively. For the further development of this aspect of the subject of damped vibrations the reader is referred to the writer's *Text-Book on Sound*, Arts. 56-58.

**46. Acceleration varying with Time, Place, and Speed: Forced**

**Vibration.**—Having considered cases where the acceleration depends upon position only, upon speed only, and upon both, we now finally treat a case where it depends upon time also in addition to the other two variables. We shall thus suppose the acceleration to be made up of three terms, two of them being as in the previous article, the new or third term being a sine function of the time. We may accordingly write for the equation of motion

$$\frac{d^2y}{dt^2}+2\kappa\frac{dy}{dt}+p^2y=f\sin nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Or in words, the acceleration is made up of three parts, of which one is a sine function of the time of amplitude $f$ and period $2\pi/n$, another is $-p^2$ times the displacement, and the other is $-2\kappa$ times the speed of the moving point or particle in question. Now the remark at the end of article 29 shows that the part of the acceleration which is a sine function of the time would of itself produce a motion in which the displacement is a sine function of the time of the *same* period, *opposite* phase, and generally of different amplitude. We may well doubt if like results will follow now where the acceleration has in addition two other terms, but it is clearly easy to test the matter by trying as a solution a sine function of the period of the fluctuating part of the acceleration and of undetermined amplitude and phase. Thus let us try as a solution of (1) the expression

$$y_1=a\sin(nt-\delta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Then, inserting it in the left side of (1), performing the differentiations, regarding $nt$ on the right side of (1) as $(nt-\delta)+\delta$, and expanding, we obtain

$$\begin{aligned}&(p^2-n^2)a\sin(nt-\delta)+2\kappa na\cos(nt-\delta)\\&=f\cos\delta\sin(nt-\delta)+f\sin\delta\cos(nt-\delta).\end{aligned}$$

But since for a valid solution this equation must hold for every instant of time, it breaks up into two on equating the coefficients of $\sin(nt-\delta)$ and $\cos(nt-\delta)$. We accordingly derive from it

$$2\kappa na=f\sin\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$(p^2-n^2)a=f\cos\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Thus by (3)

$$a=\frac{f\sin\delta}{2\kappa n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and by taking the quotient of (3) and (4) we see that

$$\tan\delta=\frac{2\kappa n}{p^2-n^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Hence using (5), (6), and (2) we see that

$$y_1=\frac{f\sin\delta}{2\kappa n}\sin(nt-\delta)=\frac{f\sin(nt-\delta)}{\sqrt{(p^2-n^2)^2+(2\kappa n)^2}} \quad (7)$$

is *a* solution of (1) when $\delta$ has the value given in (6).

**47. Complete Solution.**—But this, though *a solution*, is not necessarily *the complete solution*. Thus, if we write

$$y_2=ae^{-\kappa t}\sin(qt+\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $q^2=p^2-\kappa^2$ we see, from equations (1), (3), and (4) of article 45, that $y_2$ put in the left side of equation (1) of the present article would reduce to zero. Thus if $y_2$ were added to $y_1$ it would not disturb the solution, for $y_1$ in the left side equals the fluctuating term on the right side of the equation, while $y_2$ on the left gives zero.

Indeed, while not disturbing the solution, the presence of $y_2$ serves to complete it, for it contains two arbitrary constants $a$ and $\epsilon$, and the theory of differential equations shows that this is the number required in the *complete* solution of equation (1). A little reflection will show that these two constants are the unknowns to be determined by the displacement and speed respectively of the vibrating point at the instant $t=0$.

We may accordingly write as our complete general solution of (1) the expression which is the sum of $y_1$ and $y_2$ from (7) and (8), viz.

$$y=\frac{f\sin\delta}{2\kappa n}\sin(nt-\delta)+ae^{-\kappa t}\sin(qt+\epsilon) \quad . \quad . \quad . \quad . \quad (9),$$

in which $\tan\delta=\frac{2\kappa n}{p^2-n^2}$, $q^2=p^2-\kappa^2$, $a$ and $\epsilon$ are arbitrary.

**48. Discussion of Solution.**—Of these two parts of the motion shown on the right side of (9), $y_1$ and $y_2$ respectively, the first, $y_1$, depends, as we have seen, solely on the fluctuating or periodic part of the acceleration, it is the response of the moving point to that acceleration, and is maintained by it of the same period and ceases if it is withdrawn. It is called a *forced* vibration, being of the period of this imposed acceleration, and not that due to the accelerations dependent on displacement and speed. The second term, or $y_2$, is the vibration of damped harmonic type studied in article 45, and is the *free* or *natural* vibration peculiar to the conditions which impose the accelerations given as dependent on displacement and speed. These natural vibrations might be present prior to the application of the periodic part of the acceleration, and the resultant of the forced and natural vibrations is competent to represent any initial conditions since the amplitude $a$ and phase $\epsilon$ are arbitrary. It should be noticed, however, that if the periodic part of the acceleration lasts long enough the *natural* vibration represented by $y_2$, being of the damped or diminishing type, will practically disappear, leaving the forced vibration alone in the field. If then, after a time, the periodic acceleration were withdrawn, the displacement and speed obtaining at that instant would have to be regarded as a new *initial* state giving corresponding values for amplitude and phase of the *natural* vibrations, which would thenceforward ensue and continue till they died away by the effect of the damping factor $e^{-\kappa t}$.

It is seen by equations (6) and (7) that the *nearer* $p$ and $n$ are to each other the *greater* is the amplitude of the forced vibration. Indeed, but for the presence of $\kappa$, the amplitude would become infinite for $n=p$. In acoustics, wireless telegraphy, and wireless telephony this is a fact of far-reaching importance, referred to under the terms *resonance*, *tuning*, etc. For a fuller treatment of the subject of forced vibrations, sym-

metrical and asymmetrical, and with single and double forcing, the interested reader may consult articles 91-116 of the author's work on *Sound*.[1]

It may be noted, in conclusion, that all the examples dealt with in this chapter are special cases of the one general differential equation $d^2s/dt^2 = C + S + V + T$, where $s$ is the space co-ordinate, $C$ is a constant, and $S$, $V$, and $T$ are functions of the space, velocity, and time respectively.

## EXAMPLES—X.

1. A point moves in a straight line with an acceleration whose components are respectively proportional and opposite to its displacement and speed. Write down the corresponding equation of motion and solve it.
2. If a point executing simple harmonic motion becomes subject to a further acceleration opposing its motion and directly proportional to its speed, what changes follow in the amplitude and the period? Can one of these quantities suffer an appreciable change while the other is practically unaffected?
3. A rectilinear motion is executed under an acceleration whose components are numerically $-169$ times the displacement and $-10$ times the velocity respectively. Find the equation, period, and logarithmic decrement of the motion.
4. Plot a displacement-time graph of the motion of question 3, putting $a = 10$ and $\epsilon = 0$.
5. Discuss the case of rectilinear motion in which the acceleration has components depending on the displacement, the speed, and the time, and show what will follow after the cessation of the third component of the acceleration.
6. If a point initially without displacement or speed becomes subject to acceleration represented by
$$\ddot{y} + 169y = 25 \sin 12t,$$
show that its motion may be thenceforward expressed by
$$y = \sin 12t - \tfrac{12}{13} \sin 13t.$$
7. Show that all the cases of motion dealt with in the present chapter may be represented by one differential equation, giving illustrative examples, and indicating the character of the solutions.

[1] Also *Phil. Mag.* E. H. Barton and Dr. H. M. Browning, Oct. 1917; Jan., Feb., July, and Aug. 1918; April, July, and Sept. 1919; Nov. 1920; July 1921; Sept. 1922; and Sept. 1923.

# CHAPTER V

## PLANE MOTIONS OF A POINT

**49. Uniform Acceleration: Projectile.**—In dealing with the motions of a point in two dimensions we begin with the case in which the acceleration is uniform over the space under consideration. It thus applies approximately to the motion of an ideal projectile whose path is so small that no variation in gravity need be considered, the projectile itself being regarded as a mere point or particle devoid of rotation and suffering no resistance from the air.

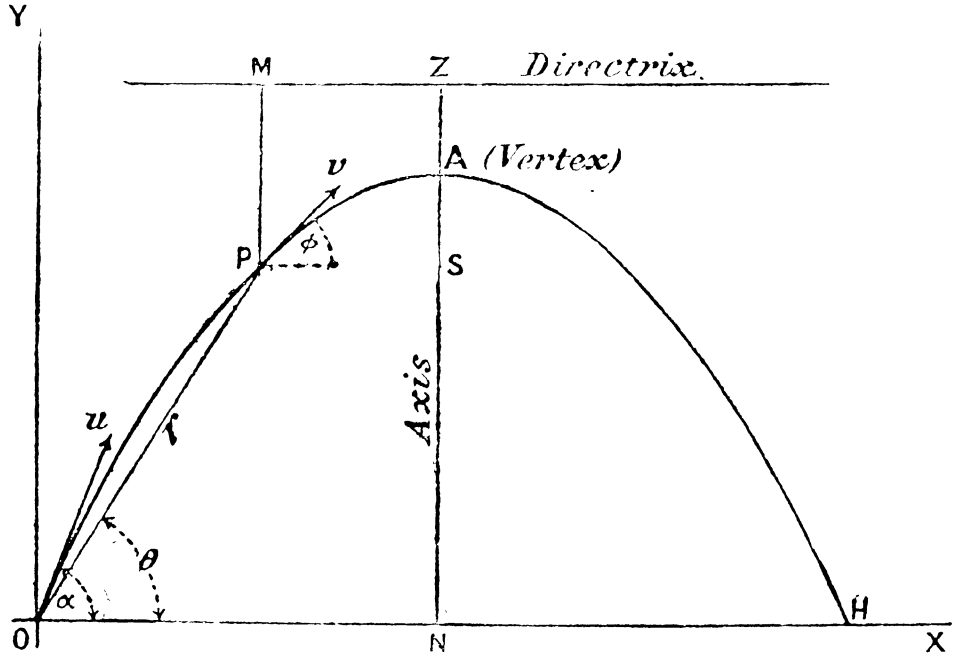



FIG. 14. PARABOLIC TRAJECTORY.

Referring to Fig. 14, let the origin of co-ordinates O be the point of projection, the axes of $x$ and $y$ being respectively horizontal and vertical. Then the general conditions of the problem are expressed by stating that the horizontal acceleration is zero and that the vertical acceleration is $-g$. Or, if we denote the co-ordinates of any point P in the trajectory by $x$ and $y$, the component velocities and accelerations by $\dot{x}$, $\dot{y}$, and $\ddot{x}$, $\ddot{y}$, we have as the acceleration components

$$\ddot{x}=0, \ddot{y}=-g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Thus, if the angle of projection is $\alpha$ and the initial speed $u$, the velocity components after time $t$ are given by

$$\dot{x}=u\cos\alpha \text{ and } \dot{y}=u\sin\alpha-gt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Or, if $v$ is the magnitude of the velocity at P and $\phi$ its inclination with the horizontal, we have

$$v^2=\dot{x}^2+\dot{y}^2 \text{ and } \tan\phi=\dot{y}/\dot{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

For the cartesian co-ordinates of P at time $t$ we have from (2)

$$x=ut\cos\alpha \text{ and } y=ut\sin\alpha-\tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

The corresponding polar co-ordinates are obviously given by

$$r^2=x^2+y^2 \text{ and } \tan\theta=y/x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

*Summit.*—To find the co-ordinates of the summit of the trajectory we must write $\dot{y}=0$ in (2) and obtain from (4) the corresponding values of $x$ and $y$.

Thus we find from (2) that for the summit $t=(u\sin\alpha)/g$, which put in (4) gives

$$x=\frac{u^2\sin 2\alpha}{2g}=\text{ON} \text{ and } y=\frac{u^2\sin^2\alpha}{2g}=\text{NA} \quad . \quad . \quad . \quad (6).$$

*Range and Time of Flight.*—To obtain the range OH on the horizontal plane we write $y=0$ in (4) and obtain the corresponding value of $x$. We see that the time of flight is

$$T=\frac{2u\sin\alpha}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

and that the range is given by

$$X=\frac{u^2\sin 2\alpha}{g}=\text{OH} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Thus, for a given value of $u$, the maximum range is obviously obtained for $\alpha=\pi/4$, under the ideal conditions supposed to exist.

**50. Equation of Trajectory.**—From equations (4), eliminating $t$ we have

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha},$$

or

$$\left(x-\frac{u^2 2\sin\alpha\cos\alpha}{2g}\right)^2=-4\left(\frac{u^2\cos^2\alpha}{2g}\right)\left(y-\frac{u^2\sin^2\alpha}{2g}\right) \quad . \quad (9),$$

equations which show the path of the particle to be a parabola of latus rectum $(2u^2\cos^2\alpha)/g$, and with directrix and focus respectively above and below the summit A by the distance $(u^2\cos^2\alpha)/2g$. The equation to the directrix may accordingly be written

$$y'=u^2/2g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

the co-ordinates of the focus being

$$\frac{u^2\sin 2\alpha}{2g} \text{ and } \frac{u^2}{2g}(\sin^2\alpha-\cos^2\alpha)=\frac{u^2(-\cos 2\alpha)}{2g} \quad . \quad . \quad (11),$$

and those of the vertex as already shown in equations (6).

*Velocity due to Fall from Directrix.*—Referring now to equations (2), (3), and (4), we see that

$$v^2=u^2-2gy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

But by (10) this could be put in the form

$$v^2=2g(y'-y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

*i.e.* the velocity at any point P in the trajectory is that which would be acquired in a free fall from rest in the directrix to the level of the point in question, for $y'-y$ is obviously the length of MP on Fig. 14.

**51. Range on an Incline.**—The range $R$ of a projectile on an incline $\theta$ may be found from the polar co-ordinates given by equations (4) and (5) by putting the given incline in the expression for $\tan\theta$, and thence deducing $R$ and the time of flight $T$.

Thus
$$R=\frac{x}{\cos\theta}=\frac{uT\cos\alpha}{\cos\theta} \quad \ldots\ldots\ldots (14).$$

Also
$$\frac{y}{x}=\frac{u\sin\alpha-\frac{1}{2}gT}{u\cos\alpha}=\tan\theta,$$

or
$$T=\frac{2u}{g}\cdot\frac{\sin(\alpha-\theta)}{\cos\theta} \quad \ldots\ldots\ldots (15).$$

Whence by (15) in (14) we obtain

$$R=\frac{u^2}{g\cos^2\theta}\{\sin(2\alpha-\theta)-\sin\theta\} \quad \ldots\ldots (16).$$

Thus for a given speed $u$ of projection the range on the incline $\theta$ is a maximum for $2\alpha-\theta=\pi/2$, that is, for an angle of elevation which bisects the angle between the incline and the vertical.

An alternative method is to take new axes of co-ordinates parallel and perpendicular to the incline. The accelerations are then

$$\ddot{x}=-g\sin\theta \text{ and } \ddot{y}=-g\cos\theta \quad \ldots\ldots (17).$$

Still retaining $\alpha$ as the angle with the horizontal made by the direction of projection we have for the co-ordinates at time $t$

$$\left.\begin{aligned} x'&=ut\cos(\alpha-\theta)-\tfrac{1}{2}gt^2\sin\theta \\ \text{and}\quad y'&=ut\sin(\alpha-\theta)-\tfrac{1}{2}gt^2\cos\theta \end{aligned}\right\} \quad \ldots\ldots (18).$$

Thus, writing $y'=0$, we obtain again $T$ as in equation (15), and this substituted in the expression for $x'$ gives for the range sought $x'=R$ as in equation (16).

## EXAMPLES—XI.

1. Obtain, both in cartesian and in polar co-ordinates, general expressions for the velocity and position of a point moving in a vertical plane under uniform vertical acceleration.
2. Show that the trajectory of an unresisted shot is a parabola, find its equation, and write down its focus, directrix, and latus rectum.
3. With a given initial velocity of projection determine the angle of elevation for maximum range on an incline.
4. 'A gun is firing from the sea-level out to sea. It is then mounted in a battery $h$ feet higher up and fired at the same elevation $\alpha$.
'Show that its range is increased by the fraction

$$\tfrac{1}{2}\left\{\left(1+\frac{2hg}{V^2\sin^2\alpha}\right)^{\frac{1}{2}}-1\right\}$$

of itself, $V$ being the velocity of projection.'
(LOND. B.SC., PASS, APPLIED MATH., 1908, II. 2.)

5. 'Prove that the least energy of projection of a particle, in order that it may have a given horizontal range, is such as would carry it vertically to a height equal to half the range.'

(LOND. B.SC., PASS, MIXED MATH., 1904, I. 4.)

6. 'A bullet describing a nearly horizontal path with a constant retardation moves over a space of 500 feet whilst its velocity falls from 1200 f.s. to 1000 f.s.; where was it when its velocity was 1100 f.s.?'

(LOND. B.A., PASS, MIXED MATH., 1906, I. 5.)

7. 'Give a simple geometrical construction for the velocity at any time of a point whose acceleration is constant in magnitude and direction.

'Find by kinematical principles an expression for the radius of curvature at any point of a parabola.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, II. 3.)

8. 'A projectile has a horizontal range of 150 yards, and the time of flight is 5 seconds; find the velocity of projection, assuming that the resistance of the air may be neglected.'

(LOND. B.A., PASS, APPLIED MATH., 1906, I. 7.)

**52. Constrained Motion in a Region of Uniform Acceleration.**—We now consider the plane motion of a point or particle in a region of uniform acceleration, but with conditions imposed which constrain it to a specified path along which we have accordingly only a component of the acceleration which obtains in free space.

**Motion down Incline.**—Thus, if a particle be constrained to motions at an angle $\theta$ with the direction of the acceleration $a$ which obtains in the region, we have along the direction of motion (by the resolution of vectors) an acceleration of value $a \cos \theta$. There is accordingly a motion with uniform acceleration of this amount, and the case falls under the methods of articles 27 and 28. In particular, if the motion in question is down a slope inclined at $\alpha$ with the horizontal, and the acceleration is that due to gravity, then the acceleration down the slope is obviously $g \sin \alpha$. It is, of course, supposed that the constraint in question imposes no check in any way upon the motion except that of keeping it confined to a given path.

Suppose a particle to slide a distance $s$ from rest down an incline of angle $\alpha$ with the horizontal, the vertical height descended being $h$ and the speed acquired $v$. Then we have

$$v^2 = 2as \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$= 2g \sin \alpha . s$$
$$= 2g(h/s)s$$

or
$$v^2 = 2gh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

independent entirely of the inclination.

**53. Simple Pendulum in Small Arcs.**—Take now the case of a particle confined to motions near the lowest point of a vertical circle under the uniform acceleration of gravity. This is most easily realised by attaching a shot or bob by a fine thread to a fixed point, the arrangement being known as the simple pendulum. The limitation to a vertical circle instead of to a spherical surface is then obtained by the method of starting the pendulum.

Referring to Fig. 15, let S denote the point of suspension, P the bob,

and let the thread SP have length $l$. Consider it when SP makes as shown an angle $\theta$ with the vertical SO. Then the component of gravity which is effective, being along the arc at P, is $-g\sin\theta$. But since we are limiting the motion to very small arcs, we may write this $-g\theta$ nearly. Let the displacement OP measured along the *arc* be $s$, then $\theta = s/l$. We accordingly have as our approximate equation of motion

$$\frac{d^2s}{dt^2} + \frac{g}{l}s = 0 \quad . \quad . \quad . \quad . \quad . \quad (1).$$

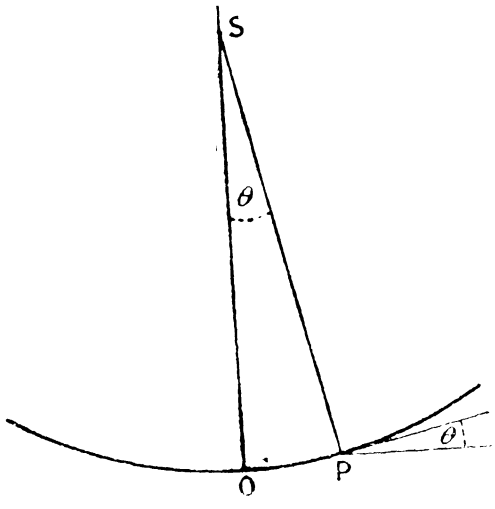



FIG. 15. SIMPLE PENDULUM.

But this is equivalent to equation (1) of article 29, hence the solution may be written

$$s = s_0 \sin(\sqrt{g/l}\, t + \epsilon) \quad . \quad . \quad (2).$$

This shows that along these small arcs the motion is simple harmonic of period

$$\tau = 2\pi\sqrt{l/g} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Thus the period varies as the square root of the length, and is independent of the amplitude to this approximation.

In equation (2) $s_0$ and $\epsilon$, the amplitude and phase, are arbitrary constants to be determined by the initial conditions as in article 30.

**54. Simple Pendulum in Finite Arcs.**—Let us now consider the case in which the arcs or angular displacements of the pendulum are not so small as to admit of writing $\sin\theta = \theta$. Then, again referring to Fig. 15 and the notation of article 53, we have as the equation of motion

$$l\frac{d^2\theta}{dt^2} + g\sin\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

And let the initial conditions be

$$\text{for } t = 0, \qquad \frac{d\theta}{dt} = 0 \text{ and } \theta = \alpha \quad . \quad . \quad . \quad . \quad . \quad (2),$$

that is, the pendulum is let go from rest with an angular displacement $\alpha$ radians. It is required to find the period $\tau$ corresponding to this amplitude $\alpha$.

Multiply (1) by $2d\theta$ and integrate then we obtain

$$l\int 2\frac{d\theta}{dt}d\left(\frac{d\theta}{dt}\right) + 2g\int \sin\theta\, d\theta = 0,$$

$$\text{or} \qquad l\left(\frac{d\theta}{dt}\right)^2 - 2g\cos\theta + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Applying (2) to (3) we find that the integration constant is $C = 2g\cos\alpha$. Equation (3) may therefore be written

$$l\left(\frac{d\theta}{dt}\right)^2 - 2g(\cos\theta - \cos\alpha) = 0,$$

or
$$\frac{d\theta}{dt}=-\sqrt{\frac{2g}{l}(\cos\theta-\cos\alpha)} \quad . \quad . \quad . \quad . \quad . \quad (4),$$

in which the negative sign of the square root is chosen because the angle $\theta$ decreases as time increases. Separating the variables and integrating over a quarter of the period $\tau$, starting from the equilibrium position, we obtain

$$\int_0^{\tau/4} dt=+\sqrt{l/2g}\int_0^{\alpha}\frac{d\theta}{\sqrt{\cos\theta-\cos\alpha}} \quad . \quad . \quad . \quad . \quad (5).$$

The algebraic sign before the square root is now changed to positive because the limits of integration have been written o to $\alpha$ instead of $\alpha$ to o as they were supposed to actually occur.

Using the identity $1-\cos\theta=2\sin^2\theta/2$, and the same for $\alpha$, we may rewrite (5) in the form

$$\int_0^{\tau/4} dt=\sqrt{l/4g}\int_0^{\alpha}\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (6).$$

**55. Transformation of the Integral.**—To deal with the integral on the right of (6) it is desirable to introduce a new variable $\phi$, defined by

$$\sin\frac{\theta}{2}=\sin\frac{\alpha}{2}\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

This relation is obviously legitimate, since $\theta$ is never greater than $\alpha$. The transformation shown by (7) obviously affects the function to be integrated, the differential, and the limits. We accordingly note that

$$\left.\begin{aligned}\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}&=\sin\frac{\alpha}{2}\cos\phi\\ d\theta&=\frac{2\sin\frac{\alpha}{2}\cos\phi\,d\phi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\phi}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The limits o and $\alpha$ for $\theta$ become o and $\pi/2$ for $\phi$.

Hence, substituting from (8) in (6), we transform the integral to

$$\frac{\tau}{4}=\tfrac{1}{2}\sqrt{l/g}\int_0^{\pi/2}\frac{2d\phi}{\sqrt{1-k^2\sin^2\phi}} \quad . \quad . \quad . \quad . \quad . \quad (9),$$

where
$$k=\sin\alpha/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Obviously, for $k=0$, (9) becomes $\tau=2\pi\sqrt{l/g}$, as found before for infinitely small arcs.

**56. Evaluation of the Integral.**—To evaluate (9) note first that by the binomial theorem we have the expansion

$$(1-k^2\sin^2\phi)^{-1/2}=1+\tfrac{1}{2}k^2\sin^2\phi+\frac{1\cdot3}{2\cdot4}k^4\sin^4\phi+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}k^6\sin^6\phi\ \ldots(11).$$

Also, by the integration of even powers of sines, we have

$$\int_0^{\pi/2} \sin^{2m}\phi \, d\phi = \frac{1\cdot3\cdot5\ldots(2m-1)}{2\cdot4\cdot6\ldots 2m}\cdot\frac{\pi}{2} \quad . \quad . \quad . \quad . \quad (12).$$

Hence (9) becomes

$$\tau = 2\pi\sqrt{l/g}\left\{1+(\tfrac{1}{2})^2k^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2k^4+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2k^6\ldots\right\} \quad (13),$$

giving the period in terms of the amplitude.

It may be noted that, if $h$ is the height fallen through by the pendulum bob from its highest to its lowest point, we have

$$k^2 = \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} = \frac{h}{2l} \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Hence, by help of (14), (13) gives $\tau$ in terms of $l$ and $h$.

In many cases, though the oscillations occurring are not infinitely small, they are fairly so, and it is then usual to stop at the term involving $k^2$ in (13). Thus to this approximation, and writing $\alpha/2$ for $\sin\alpha/2$, we have

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{h}{8l}\right) = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\alpha^2}{16}\right) \text{ nearly} \quad . \quad (15).$$

Various relations may be obtained geometrically for a pendulum swinging in finite arcs. Some examples are given later to afford the student exercise in such problems.

**57. Motion in a Vertical Circle.**—Reverting now to equation (9) of article 55, we see that the integral may be immediately evaluated if $k^2 = 1$, *i.e.* if $\alpha = \pi$. This corresponds with a motion of a particle from the highest point of a vertical circle under gravity. It is convenient, however, to go back to the form of equation (6) of article 54 and rewrite it without the limits of integration. We thus have

$$2\int dt = \sqrt{l/g}\int \frac{d\theta}{\cos\theta/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

This gives

$$t = (\sqrt{l/g})\log_e\left(\tan\frac{\pi+\theta}{4}\right) + \text{const.} \quad (17).$$

It should be observed that, since $\log\tan\pi/2 = \infty$, it would take an infinite time for the particle to just reach, or to fall from rest at, the highest point.

If, after starting from rest at the top, a fall is noted from $\theta_1$ to $\theta_2$ then, by (17), the corresponding time $t_{12}$ is given by

$$t_{12} = \sqrt{l/g}\left(\log_e\tan\frac{\pi+\theta_1}{4} - \log_e\tan\frac{\pi+\theta_2}{4}\right) \quad . \quad . \quad (18).$$

The speed at any point, whether the fall started at the top or elsewhere, may be found by equation (4) of article 54.

## EXAMPLES—XII.

1. Show that a particle descending a smooth curve in a vertical plane under gravity will experience the same change of velocity as in a vertical descent between the same levels.

2. Find the approximate period of a simple pendulum from the differential equation of its *small* motions.

3. Fit the equation which expresses the displacement of a simple pendulum (*a*) to the case where it is pulled aside $a$ and let go, and (*b*) to the case where the bob receives a velocity $v$ at the central position.

4. Discuss the equation of motion of a simple pendulum when executing finite arcs and find the period for an amplitude of 5°.

5. 'Prove that the period of a complete oscillation of a simple pendulum of length $l$ is $2\pi\sqrt{(l/g)}$. If the bob of a pendulum 100 feet long be drawn aside from the position of rest through a space of 3 feet and then released, find the velocity with which it passes through the equilibrium position.'

(LOND. B.SC., PASS, MIXED MATH., 1904, I. 6.)

6. 'Prove that the length of the pendulum to beat seconds in London is 39·14 inches.

'To gain or lose one second in one hour, or 24 hours in a clock, the length must be altered 0·02175, or 0·000906 inch.'

(LOND. B.SC., PASS, MIXED MATH., 1903, I. 9.)

7. 'Prove that the time of a single swing of a plummet at the end of a thread $l$ feet long is

$$\pi\sqrt{\frac{l}{g}} \text{ seconds}$$

when the oscillations are small, and that for the plummet to beat seconds the length of the thread must be 39·14 inches.

'Prove that as the plummet swings through the arc $BAB'$ of an angle $2a$ from $B$ to $B'$ on the horizontal chord $BDB'$ of the circle of which $ADE$ is the vertical diameter, the point $Q$ on the circle on the diameter $AD$ will follow $P$ at the same level with velocity

$$\sqrt{\frac{g}{l}}\, AD \cdot \frac{PE}{AE};$$

and thence show that the time of a swing lies between

$$\pi\sqrt{\frac{l}{g}} \text{ and } \pi\sqrt{\frac{l}{g}} \sec \tfrac{1}{2}a.'$$

(LOND. B.SC., PASS, MIXED MATH., 1901, II. 6.)

8. 'Prove that the time of swing of a pendulum $l$ feet long is undistinguishable from $\pi\sqrt{(l/g)}$ seconds when the oscillation is small.

'If a light is placed at $E$, the upper end of the vertical diameter $AE$ of the circle on which the plummet oscillates, to throw the shadow $T$ of the plummet on the floor, moving from $F$ to $F'$, and if $TR$ is the ordinate of the circle on the diameter $FF'$, prove that the velocity of $R$ varies as $ET$, and fluctuates beween

$$\tfrac{1}{2}FF'\sqrt{\frac{g}{l}} \text{ and } \tfrac{1}{2}FF' \cos\frac{a}{2}\sqrt{\frac{g}{l}},$$

$2a$ denoting the angle of oscillation, not restricted to be small; and thence show that the period of oscillation lies between

$$2\pi\sqrt{\frac{l}{g}} \text{ and } 2\pi \sec\frac{a}{2}\sqrt{\frac{l}{g}},$$

(LOND. B.SC., PASS, MIXED MATH., 1902, II. 8.)

9. 'Prove, by analogy with Harmonic Vibration, that the bob $P$ of a circular pendulum of length $l$, oscillating through a finite arc $BAB'$, moves with velocity

$$n\sqrt{(BP.PB')},\ n^2=g/l.$$

'If $AE$ is the vertical diameter of the circle on which $P$ moves, and if $EP$ drawn from the highest point $E$ cuts the horizontal chord $BB'$ in $R$, prove that the velocity of $R$ is

$$n\frac{EB}{EP}\sqrt{(BR.RB')},$$

and that the period of oscillation lies between the limits

$$\left(1 \text{ and } \frac{EA}{EB}\right)2\pi\sqrt{\frac{l}{g}}.'$$

(LOND. B.SC., PASS, MIXED MATH., 1903, II. 9.)

**58. Motion in a Vertical Cycloid.**—A cycloid is a curve described by a point in the circumference of a circle which rolls without sliding upon a fixed straight line. The point is called the *tracing point*, the circle the *generating circle*, and the straight line the *base*.

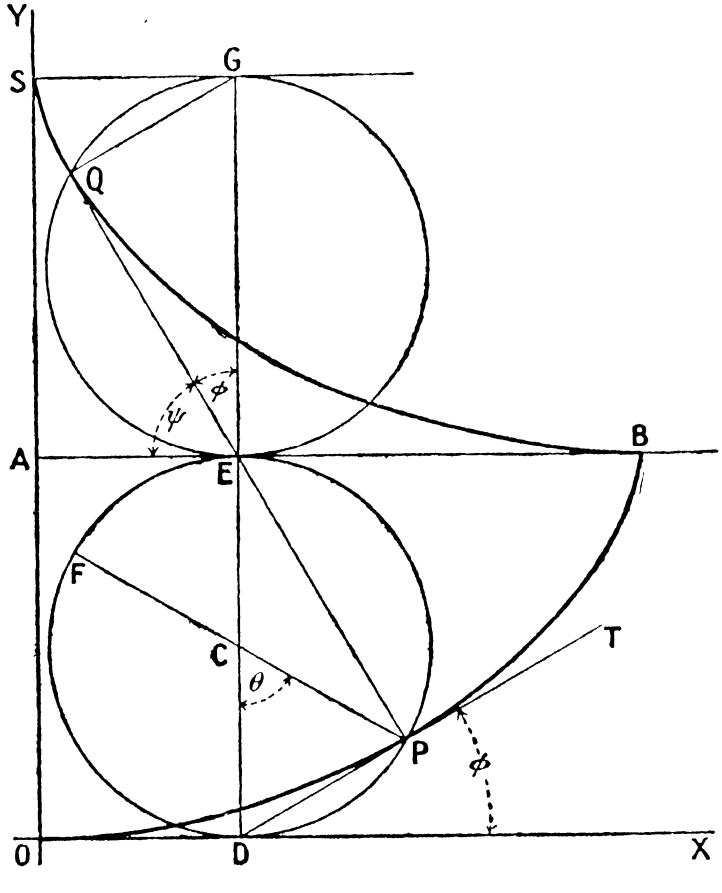



FIG. 16. THE CYCLOID.

Thus, referring to Fig. 16, P is the tracing point, DPEF is the generating circle with centre at C, and AEB is the base of the cycloid, part of which is shown by OPB. It is at once obvious that a cycloid consists of a number of precisely similar portions, and certain points called *vertices* are most remote from the base, O being a vertex in the figure; while other points called *cusps* lie on the base, B being a cusp of the cycloid OPB in the figure. The lines through the vertices at right angles to the base are called *axes*, one of which, AO, is shown in the figure as an axis of OPB.

*Intrinsic Equation of the Cycloid.*—Taking the origin of co-ordinates at the vertex O in Fig. 16, and the axis of $x$ parallel to the base as

shown, let the co-ordinates of the tracing point P be $(x, y)$, it being understood that P was at O when the opposite end of the diameter F was at A. Then we can readily obtain the intrinsic equation of the cycloid. For taking angle $DCP=\theta$ and $CP=a$, the co-ordinates of P are obviously

$$x=a(\theta+\sin\theta) \text{ and } y=a(1-\cos\theta) \quad . \quad . \quad . \quad (1).$$

Hence, on differentiating, we have

$$\left.\begin{aligned} dx^1 &= a(1+\cos\theta)d\theta = 4a\cos^2\frac{\theta}{2}d\frac{\theta}{2} \\ \text{and} \qquad dy^1 &= a\sin\theta\, d\theta = 4a\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\frac{\theta}{2} \end{aligned}\right\} \quad . \quad (2).$$

Thus on division, and calling the inclination to the horizontal at $P=\phi$, we have

$$\tan\phi=dy/dx=\tan\theta/2 \text{ or } \phi=\theta/2 \quad . \quad (3).$$

Also, using this value in (2), we see that

$$(ds)^2=(dx)^2+(dy)^2=16a^2\cos^2\phi\,(d\phi)^2,$$

or

$$ds=+4a\cos\phi\, d\phi \quad . \quad . \quad . \quad (4).$$

Thus integrating, and remembering that the value of the arc $s$ and that of the angle $\phi$ vanish together, we have

$$\int_0^s ds=4a\int_0^\phi \cos\phi\, d\phi,$$

or

$$s=4a\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

the intrinsic equation required.

**59. Period of Cycloidal Oscillation.**—Suppose now that a particle is constrained to describe a cycloidal curve in a vertical plane with the axis vertical and vertex downward. Then, calling the displacement from the vertex along the arc $s$, we have the component acceleration along the curve due to gravity denoted by $-g\sin\phi$. Thus, by (5), the equation of motion of the particle may be written

$$\frac{d^2s}{dt^2}+\frac{g}{4a}s=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

But the solution of this, as already seen, is of the form

$$s=s_0\sin\left(\sqrt{g/4a}\,t+\epsilon\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

thus giving a simple harmonic motion of period

$$\tau=2\pi\sqrt{4a/g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

entirely *independent of amplitude.*

**60. Cycloidal Pendulum.**—The constraint on the particle to make it describe a cycloid might be in the form of a tube, along the smooth interior of which the particle slides. A special property of the cycloid enables us, however, to conveniently regard it from another point of view. By unwinding a thread from one curve, a second curve is

[1] Some writers studiously avoid the *differentials* $dx$ and $dy$ and use always the differential coefficients $dx/d\theta$, $dy/d\theta$, etc. Others use freely the separate symbols $dy$ and $dx$, presumably, on the understanding that they represent *small increments whose ratio* is the *limit* of $\delta y/\delta x$ as $\delta x$ *approaches zero.* For the *formal* justification of the separate use of $dy$ and $dx$, modern works on the calculus should be consulted.

obtained called the *involute* of the first, which is itself termed the *evolute* of the second. Thus, to describe a cycloid by the unwinding of a thread from some curve, we have to determine what that curve is; in other words, we have to find what is the evolute of the cycloid. It may easily be shown that it is two halves of an equal cycloid. Thus, referring again to Fig. 16, the evolute of the half-cycloid OPB is that half-cycloid SQB shown above it. And a thread fixed at S and wrapped round Q to B, if unwrapped while kept stretched and carrying a pencil starting at B, would describe the half-cycloid BPO.

To establish this it is necessary and sufficient to show

(i) that PQ lies along the normal to the cycloid OPB at P;
(ii) that the length PQ is the radius of curvature of the cycloid OPB at P;
(iii) that PQ is tangential at Q to the cycloid SQB; and
(iv) that the length PQ equals the length from Q to B along the cycloid SQB.

(i) We have shown, in equation (3), that $\phi = \theta/2$, but it is evident also from the geometry of the figure that PED is $\theta/2$ also. Thus PQ is perpendicular to the tangent PT or is along the normal to the curve.

(ii) Again, from equation (4), and calling the radius of curvature of the cycloid $\rho$, we have

$$\rho = ds/d\phi = 4a\cos\phi = 2\mathrm{PE} = \mathrm{PQ} \text{ say} \quad . \quad . \quad . \quad . \quad (9).$$

Thus Q is defined as lying along PE produced so that EQ=EP. It is thus on the circumference of the circle GQE, equal to DPEF, and standing vertically over it. Now if this upper circle rolls along GS it is evident that Q will reach S, because it is exactly like the point F in the lower circle, which by rolling reaches A. Accordingly the vertex of the upper cycloid is at B, the cusp of the lower one.

(iii) PQ is readily seen to be tangential to the cycloid SQB at Q since, by construction, it is at right angles to the normal QG, the angle GQE being in a semicircle.

(iv) Finally, to establish the fourth point, let the angle with the horizontal made by the upper cycloid at Q be $\psi$, and the length from B to Q along the curve be $s'$. Then we have by (5)

$$s' = 4a\sin\psi = 4a\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Or, on comparing with (9),

$$s' = \rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

as needed to be shown.

Thus the length of the thread which wraps along the cycloid from cusp S to vertex B is seen from its central position SAO on the figure to be double the diameter of the generating circles. Or, referring to equations (10) or (5), and writing $\pi/2$ for the angle and $l$ for the length of the thread, we have

$$l = 4a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Thus, putting this value in the expression for the period, equation (8) becomes

$$\tau = 2\pi\sqrt{l/g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

EXAMPLES—XIII.

1. Draw a cycloid by carefully rolling on a straight edge a disc of card carrying a pencil on its circumference. Indicate on this diagram the *base*, a *vertex*, a *cusp*, and an *axis*. Also state the lengths of the base and the curve from cusp to cusp in terms of the radius of the generating circle.
2. Obtain the *intrinsic* equation of the cycloid from its definition.
3. Calculate the period of oscillations in a cycloid under uniform acceleration parallel to an axis and directed from the base towards the vertex.
4. Show that a cycloid is the involute of a precisely equal cycloid, and indicate on a carefully drawn figure the relation of the two curves.
5. Assuming that a cycloid is the involute of another, find the period of a cycloidal pendulum, and explain without mathematical symbols why the period is independent of the amplitude in this case, though it is not for a simple pendulum.
6. 'Establish the isochronous property of cycloidal motion.
'Show that if the particle oscillates from cusp to cusp, the direction of motion rotates with constant angular velocity.'
(LOND. B.SC., PASS, APPLIED MATH., 1906, II. 5.)
7. 'A circle rolls on the inside of a fixed circle of twice its size; prove that every point on the circumference of the rolling circle describes a diameter of the fixed circle.'
(LOND. B.SC., PASS, APPLIED MATH., 1905, II. 5.)

**61. The Brachistochrone.**—A notable problem in the history of mechanics is that of the curve of swiftest descent, or the *brachistochrone*. To deal with it in its general form requires the calculus of variations, which is beyond the scope of the present work. We shall accordingly restrict the treatment to the problem in its simplest form, which may be stated as follows:—

*Enunciation of Problem.*—Two points being given which are neither in a vertical nor in a horizontal line, to find the curve joining them down which a particle sliding under gravity, and starting from rest at the higher, will reach the lower in the least possible time.

In the *first* place, we can see that the required curve must lie in the vertical plane containing the two points. For if it deviated therefrom it would thereby both make the acceleration along the path less and the path longer; hence for both reasons the time would be greater.

*Secondly*, if the time through the entire curve is a minimum, each portion of the curve must be such that a change in it would increase the time in that portion.

**62.** *Problem Attacked.*—We have now to find by a simple method some clue to the type of curve required. For this purpose we use the almost self-evident fact, that if a curve exists of minimum time of descent it must be possible to draw near it on each side curves for which the times are slightly greater but equal to each other. And what applies to the whole curve applies to elementary portions of it. Referring, then, to Fig. 17, let H and R be near points in the brachistochrone, HKR and HLR being very near alternative paths down which the times are equal, the distance KL being very small in comparison

with HK and KR. Then the element of the brachistochrone required must lie between HKR and HLR. Let KL be on the same level; then in the paths HK and HL, since the speeds are everywhere the same at the same levels (see equation (2), article 52), the average speeds down each path must be equal. Hence the time in either path equals the length of that path divided by the same value, $v$ say, which is the average speed. Thus the extra time, $\tau$ say, in HK over that in HL equals the extra path length divided by $v$. Let fall from L the perpendicular LM on HK; then since KL is very small compared with KH, HM = HL nearly, and the extra distance in question is denoted by KM, which equals KL cos $\phi$, where $\phi$ is the angle HKL. We accordingly have

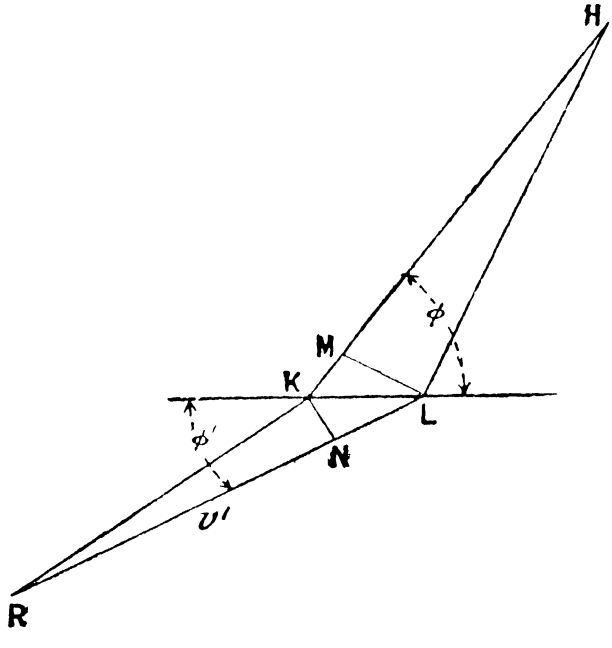



FIG. 17. DERIVATION OF BRACHISTOCHRONE.

$$\tau = \frac{KL \cos \phi}{v} \quad \ldots\ldots\ldots \quad (1).$$

Again, for the lower half of the figure the speeds at any level are the same in the two paths, hence the average speed in KR or LR may be denoted by the same quantity, $v'$ say. Also, drawing KN perpendicular to RL, we have extra time $\tau'$ in path LR over that in KR equal to extra distance LN divided by average speed $v'$ in either path. Or, writing $\phi'$ for the angle KLR, we have

$$\tau' = \frac{KL \cos \phi'}{v'} \quad \ldots\ldots\ldots \quad (2).$$

But, by supposition, $\tau$ and $\tau'$ are equal; hence (1) and (2) give

$$\frac{\cos \phi}{v} = \frac{\cos \phi'}{v'} \quad \ldots\ldots\ldots \quad (3).$$

The reader should note that it is by no means implied that the times of descending the paths HM and HL are equal, neither is the speed over MK equal to $v$ (but possibly double this). It is simply the *extra* time in the one path over that in the alternative one, that is, extra path length divided by that *average* speed between the levels H and KL (or KL and R) which is the same for both parts above and below KL. Again, the speeds over the elements MK and LN are not distinctly different, but differ by an infinitesimal quantity only; but neither of these speeds are represented by $v$ nor by $v'$, which are only average speeds above and below KL as defined.

**63. Equation of Curve.** — Now let the points K and L of Fig. 17 approach and coalesce so that the one path substituted for the two is now an element of the curve sought. Then the property possessed by the paths in common is possessed by this also. But the $\phi$ and $\phi'$ now apply to the single inclinations of the upper and lower parts, the speeds $v$ and $v'$ as before being average speeds in those upper and lower parts. Thus we may represent by UVW in Fig. 18 two consecutive elements of the curve. In the upper element UV the average speed is $v$ and the path inclined $\phi$ to the horizontal, in the lower element VW the average speed is $v'$ and the inclination $\phi'$. And to these elements equation (3) still applies. Hence we may write as the equation to the curve which is obtained by infinitely reducing the lengths of UV and VW

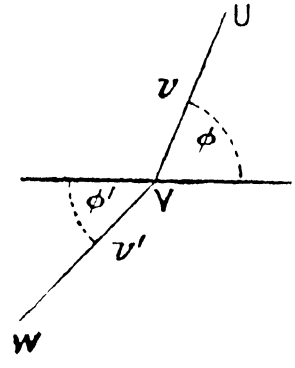


FIG. 18. ELEMENTS OF BRACHISTOCHRONE.

$$v \propto \cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

**64. Cycloid is a Brachistochrone.**—We can now easily show that this equation is satisfied by the cycloid with base horizontal, vertex downwards, the start being from a cusp. Thus in Fig. 16 a particle descending under gravity from rest at B would reach P quicker along the cycloid than by any other path. To establish this note first that $v^2 = 2gh$ if $h$ is the vertical height descended through in acquiring the speed $v$ (see equation (2), article 52).

Hence (4) may be written

$$h \propto \cos^2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

We have accordingly to find from the equations to the cycloid values for $h$ and $\phi$.

Obviously $h$ the depth of P below AB equals OA minus the ordinate $y$ of P, or

$$h = 2a - y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

And from equation (5) of article 58 we have

$$s = 4a \sin\phi = 4a\,dy/ds \quad . \quad . \quad . \quad . \quad (7).$$

Thus, by integration, we obtain

$$\int_0^s 2s\,ds = 8a \int_0^y dy,$$

or

$$s^2 = 8ay \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Hence (6) and (8) yield

$$h = 2a - y = 2a(1 - s^2/16a^2) \quad . \quad . \quad (9).$$

And by (7) $\cos^2\phi = 1 - \sin^2\phi = 1 - s^2/16a^2$ . . . . . (10).

Thus (9) and (10) show that the cycloid satisfies the condition (5) for the brachistochrone, the fall being from a *cusp*.

**65. Construction for the Cycloid as Brachistochrone.**—Suppose with a given initial point B a brachistochrone is required to pass through a lower point R, join BR by a straight line, and describe on a

horizontal base through B and in the vertical plane containing R a cycloid with any generating circle of radius $a$. Let this cycloid cut BR in P. Then, since all cycloids, like all circles, differ from each other only in size, to make a cycloid pass through R we need only to change $a$ in the ratio BP to BR. Thus let the radius of the generating circle for the required cycloid to pass through R be $r$. Then we have

$$r = a\,\mathrm{BR}/\mathrm{BP} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

## EXAMPLES—XIV.

1. Define the *brachistochrone*, and show that every point of the curve must satisfy the condition $v \propto \cos\phi$, $v$ being the speed of the point describing it where it is inclined $\phi$ to the horizontal.
2. Prove that a certain part of a cycloid in a specified position forms a brachistochrone, and construct the curve properly for two points 10 feet apart, the line joining them being inclined at an angle of 45°.

**66. Central Acceleration Proportional to Radius.**—We now consider the plane motions which may be executed by a point P subject to an acceleration directed to a fixed point O in the plane and proportional to the distance OP.

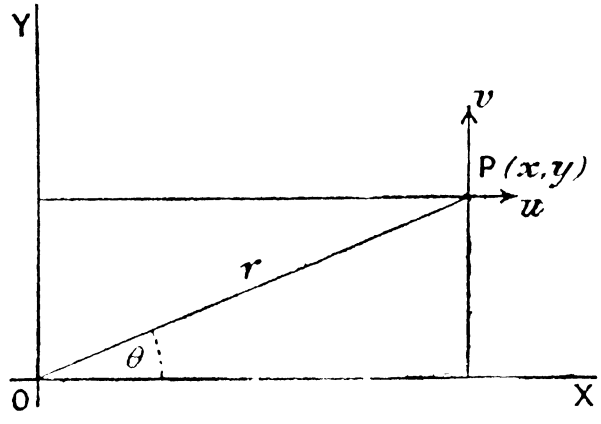



FIG. 19. CENTRAL ACCELERATION PROPORTIONAL TO RADIUS.

Thus, referring to Fig. 19, let the point P have co-ordinates $x$ and $y$, speed components $u$ and $v$ parallel to OX and OY respectively, and let the central acceleration be $-p^2r$. Then, denoting by $\ddot{x}$ and $\ddot{y}$ the accelerations parallel to the co-ordinate axes, we have as our equations of motion

$$\ddot{x} = -p^2 r\cos\theta = -p^2 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\ddot{y} = -p^2 r\sin\theta = -p^2 y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

But these equations are the same in form as (1) of article 29; hence by (4) of the same article the solutions may be written

$$x = a\sin(pt+\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$y = b\sin(pt+\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

In these equations the constants $a$, $\alpha$, $b$, and $\beta$ are to be determined from the initial displacements and velocities as shown in article 30.

The motion is therefore to be regarded as represented by equations (3) and (4), in which all the quantities are known. We thus see that it consists of two simple harmonic motions at right angles to each other, of same period, but differing in amplitude and phase. The problem accordingly reduces to the composition of rectangular vibrations.

**67. Composition of Rectangular Vibrations.**

*Equal Periods.*—By changing the origin of time equations (3) and (4) of the last article transform to

$$x=a\sin(pt+\delta) \text{ and } y=b\sin pt \quad . \quad . \quad . \quad . \quad (5),$$

where $\delta=\alpha-\beta$. Thus expanding the expression for $x$ and using that for $y$, we eliminate $t$ between the two equations, and obtain

$$\frac{x}{a}-\frac{y}{b}\cos\delta=\sqrt{1-\frac{y^2}{b^2}}\sin\delta.$$

Or, on rationalising,

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\delta+\frac{y^2}{b^2}=\sin^2\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

which is the equation of the path of the point P.

*Case I.*—For $\delta=\pi/2$, this becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

the equation of an ellipse with axes along OX and OY.

*Case II.*—For $\delta=0$ or $\pi$, (6) becomes

$$\left(\frac{x}{a}\mp\frac{y}{b}\right)^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

which represents two coincident straight lines through the origin, sloping one way or the other according to the sign in the brackets, which again depends upon the value of $\delta$.

In the general case (6) represents an ellipse with inclined axes, which evidently, however, lies inside and is tangential to the rectangle of sides $2a$ and $2b$, their equations being $x=\pm a$, $y=\pm b$.

**68.** *Different Periods.*—So far we have supposed the acceleration to be directed towards the centre. It then follows that the acceleration parallel to $x$ or to $y$ is the same for the same displacements in each direction. If, however, the acceleration per unit displacement parallel to the $x$ axis is $n^2$ times its value parallel to the $y$ axis, it is seen from equations (1)-(4) of article 66 that the period of the vibration parallel to the $x$ axis will be $1/n$th of its value parallel to the $y$ axis. Thus, in this case, to find the resultant motion we should have to eliminate $t$ between equations of the form

$$\left.\begin{aligned} x&=a\sin(npt+\delta)\\ \text{and}\qquad y&=b\sin pt \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

This is not easy *analytically* unless $n$ is a small integer or a vulgar fraction whose numerator and denominator are small integers. In any case, however, the resultant motion may be found graphically by taking a series of values of $t$ and plotting the corresponding values of $x$ and $y$ from (9). In these circumstances the acceleration is not central, and the resulting motion cannot be an ellipse.

But as the case of unequal periods is chiefly of interest in physics we leave the reader to follow it up in the text-books devoted to that branch of science. (See, for example, the writer's *Sound*, Arts. 32-37, or Barton and Black's *Practical Physics*, Expt. 12.)

EXAMPLES—XV.

1. Show that under central acceleration varying directly as the distance the plane motion of a point is in general an ellipse. To what may this reduce in special cases?

2. 'Show that two simple harmonic motions of the same period and in the same line are equivalent to a single simple harmonic motion of the common period, and give expressions for the amplitude and phase of the resultant motion in terms of the amplitudes and phases of the components.

'Exhibit in diagrams the resultant of two S.H. motions in perpendicular directions whose amplitudes are equal, and whose periods are as 2 : 1, when the phases of the two components differ by $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the longer period respectively.'

(LOND. B.SC., PASS, APPLIED MATH., 1905, II. 3.)

3. 'The co-ordinates of a point at time $t$ are

$$x = a\cos 2nt, y = a\sin nt.$$

'Prove that the point describes an arc of a parabola. Determine the co-ordinates of the ends of this arc and the velocity of the point when passing the vertex.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, II. 2.)

**69. Tangential and Normal Accelerations.**—Suppose a point is describing a plane curve with any continuously changing speed. Then

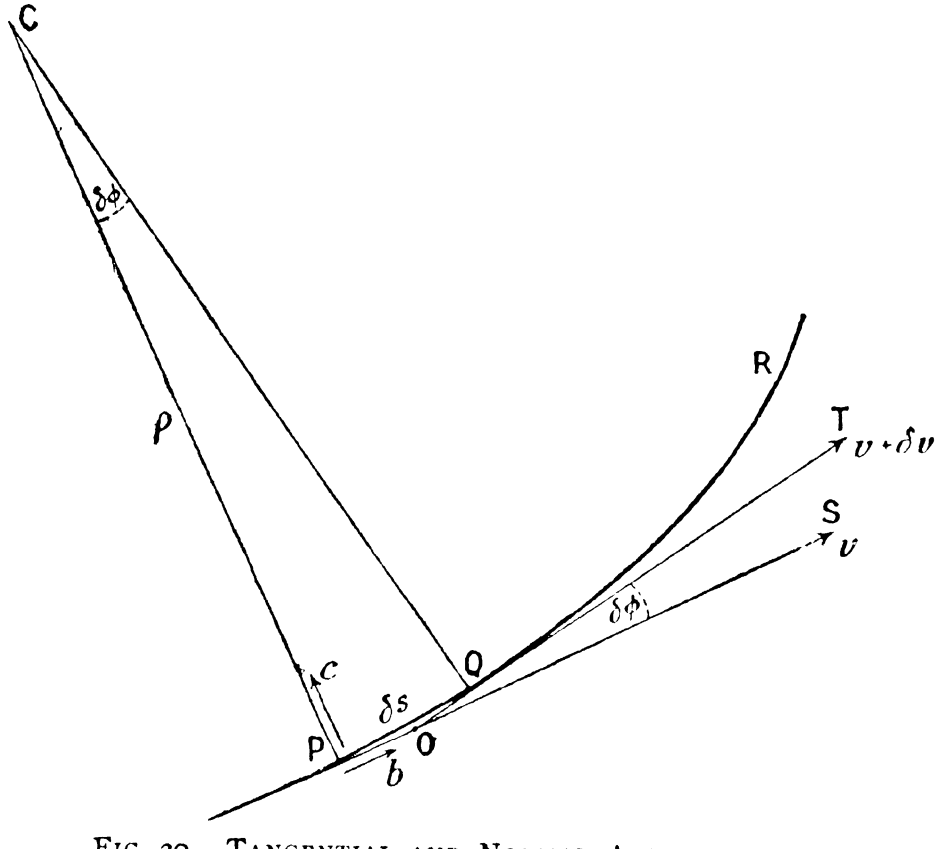



FIG. 20. TANGENTIAL AND NORMAL ACCELERATIONS.

the total acceleration to which it is subject may be regarded as the resultant of two component accelerations respectively tangential and normal to the curve at the place occupied by the point at the instant in

question. It is almost obvious that the tangential component is concerned only with the change in speed and the normal component only with the change in direction of that speed. Thus these two components of the acceleration correspond to the components which we recognise as characterising the velocity.

Let it be required to express these component accelerations in terms of the other circumstances of the motion.

Referring to Fig. 20, let a point describe the plane curve PQR, having at P the velocity OS of magnitude $v$, and at the new point Q, at a time $\delta t$ later, the velocity OT of magnitude $v+\delta v$. Also let the angle SOT between the tangents at P and Q and the equal angle PCQ between the normals be $\delta\phi$, the radii of curvature PC and QC being denoted by $\rho$ and the small arc PQ by $\delta s$.

Then if $b$ is the tangential acceleration we have on resolving parallel to PS

$$b = \text{the limit of } \frac{(v+\delta v)\cos\delta\phi - v}{\delta t} = \frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Again, resolving perpendicular to PS, *i.e.* along PC, we have for $c$, the normal acceleration,

$$c = \text{the limit of } \frac{(v+\delta v)\sin\delta\phi}{\delta t} = v\frac{d\phi}{dt} = v\frac{ds}{dt}\cdot\frac{d\phi}{ds} = v^2/\rho \quad . \quad . \quad (2),$$

since $\dfrac{d\phi}{ds} = 1/\rho$ and $ds/dt = v$. The quantity $d\phi/ds$, or rate of change of direction per unit length along the arc, is called the *curvature*, and is, as we have just seen, the reciprocal of the radius of curvature. The centre of curvature denoted by C in the figure is the intersection of consecutive normals. Thus, if the total acceleration affecting a moving point is at any time normal to the direction of the motion, and of value $c$, we have no change in its speed, but only a curvature of the path produced, whose radius is

$$\rho = v^2/c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Denoting by $\Omega$ the angular velocity $d\phi/dt$, and remembering that $v/\rho = ds/\rho dt = d\phi/dt$, (2) may be written

$$c = v\Omega = \rho\Omega v/\rho = \rho\Omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

forms which are often very useful.

**70. The Hodograph.**—Instead of using the analytical method of the preceding article for the component accelerations, it is often convenient to represent graphically the total or resultant acceleration derived from the consecutive velocities. This is done in a very elegant manner by the use of a curve called the *hodograph*, introduced into kinematics by Sir W. R. Hamilton.

Referring again to Fig. 20, we see that the effect in time $\delta t$ of the total acceleration is to change the velocity OS into OT. Hence, by the addition of vectors, the effect in question must be represented by ST. The same would apply to any other lines all drawn from O and representing the velocities of the point in the path PQR. Thus, if such lines were drawn from a point O called the *pole*, and their ends S, T, etc.

connected by a continuous curve, the speed of describing this curv called the *hodograph* would represent the total acceleration of the poin describing the path PQR. For, taking the first element ST, we have

ST represents the total acceleration $\times \delta t$.

$\therefore$ speed of describing $ST = ST/\delta t$, which represents total acceleration (1)

And this is the fundamental property of the hodograph which makes it so useful. If a point move so that when its inclination to a fixed line is $\theta$ its speed is $v = f(\theta)$, it is obvious that the polar equation of the hodograph may be written

$$r = kf(\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where $k$ is some constant chosen for convenience when drawing to scale.

**71. Uniform Circular Motion.**—As the simplest example of the use of the hodograph, let us apply it to find the acceleration of a point describing a circle of radius $\rho$ at uniform angular speed $\omega$. Then the linear speed is always $\rho\omega = v$ say. The polar equation of the hodograph may in consequence be written $r = v$. Or, the hodograph is another circle described in the same direction and in the same time, but the corresponding points in each are distant a quarter of the circumference. These points are clearly seen from Fig. 21, in which the circle of radius $\rho$ at the left shows the actual path and the circle of radius $v$ at the right shows the hodograph. The time $\tau$ of describing each is obviously given by

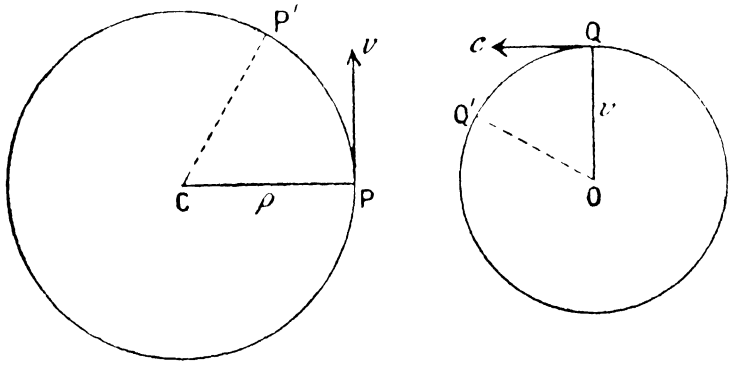



FIG. 21. UNIFORM CIRCULAR MOTION.

$$\tau = 2\pi/\omega = 2\pi\rho/v = 2\pi v/c \quad . \quad . \quad . \quad . \quad . \quad (3),$$

in which $c$, the *total acceleration* of the point in the actual path, is represented by the *linear speed* of describing the hodograph. Thus we have

$$\rho/v = v/c,$$

or

$$c = v^2/\rho = \rho\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

And, since the total acceleration is here entirely normal we see that this result agrees with (2) and (4) of article 69. The points P' and Q' are another pair of corresponding points in path and hodograph.

Sometimes from the conditions of the motion the hodograph is virtually given, and its use forms the readiest means of investigating the speed and direction of the moving point after a given time. Thus, for an unresisted projectile, the hodograph is evidently a vertical straight line described downwards with uniform speed numerically equal to $g$, the acceleration due to gravity.

**72. Conical Pendulum.**—We may approximately realise the above case of uniform circular motion in a horizontal plane by the arrangement known as the conical pendulum. In this a small bob P is attached by a strong fine thread to a fixed point S, just as in the case of the simple pendulum, but the motion imparted to P at the start is such as to obtain in a horizontal plane a circular motion, of radius $\rho$ say, described with angular velocity $\omega$. But this, as we have just seen, is only possible under the influence of a central acceleration of value $\rho\omega^2$, which in the present case must be derived as a component from the acceleration $g$ due to gravity. To find the relations which correspond to this motion, let the length SP of the thread be $l$ and the angle it makes with vertical $\theta$. Then, resolving the vertical acceleration $g$ into two components, one along the thread and the other of value $c$ horizontally to the centre (see Fig. 22), we have

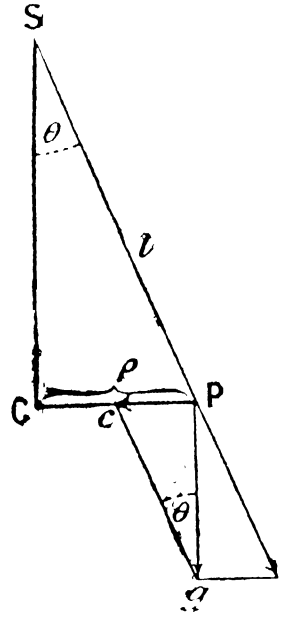


FIG. 22. CONICAL PENDULUM.

$$c/g = \tan\theta = \mathrm{CP}/\mathrm{SC} = \rho/l\cos\theta \quad . \quad . \quad . \quad (5).$$

But since $c = \rho\omega^2$, we obtain from (5)

$$\omega^2/g = 1/l\cos\theta, \text{ or } \omega = \sqrt{g/l\cos\theta} \quad . \quad . \quad (6).$$

Hence the period of rotation $\tau$ is given by

$$\tau = 2\pi/\omega = 2\pi\sqrt{(l\cos\theta)/g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

The component of $g$ along the thread is, of course, ineffective, as the thread is supposed inextensible.

## EXAMPLES—XVI.

1. 'Define the *hodograph* of a moving point. Find the hodograph in the case of a projectile moving under gravity, and the law of its description.
'Sketch the hodograph, as accurately as you can, in the case of a simple pendulum swinging through a *finite* angle.'
(LOND. B.SC., PASS, APPLIED MATH., 1905, II. 2.)

2. 'Define the hodograph of a moving point, and prove that the velocity in the hodograph varies as the acceleration in the original orbit.
'Hence (or otherwise) prove that the accelerations of a point along and at right angles to its direction of motion are $\dot{v}$ and $v\dot{\phi}$ respectively, $v$ being the velocity at any instant and $\phi$ the angle which this velocity makes with a fixed line in the plane of the motion.'
(LOND. B.SC., PASS, APPLIED MATH., 1908, II. 1.)

3. 'Obtain expressions for the tangential and normal components of the acceleration of a particle which is describing a plane curve.
'Prove that, if these two components are constant throughout the motion, the angle $\psi$ through which the direction of motion turns in a time $t$ is given by $\psi = A\log(1 + Bt)$.'
(LOND. B.SC., PASS, APPLIED MATH., 1900, II. 2.)

4. 'If a particle describes an ellipse under the action of a force directed towards its centre, find its velocity at any point of its path, and show that the ellipse is its own hodograph.'
(LOND. B.SC., PASS, APPLIED MATH., 1900, II. 3.)

**73. Angular and Areal Velocities and their Relations.**—It is now desirable to consider more generally the relation between angular and linear volocities and to introduce the conception of an *areal* (or *sectorial*) velocity. Thus let a point move in the plane curve PQR, Fig. 23, and have at P the linear speed $v$ and, with respect to the fixed point H, the angular velocity $\omega$. Also, let the rate V, at which the radius vector HP describes area as P moves in the curve, be called the *areal velocity* (or *sectorial velocity*) of the moving point P with respect to the fixed point H. Let the infinitesimal path length PQ be called $ds$, the angle PHQ be $d\theta$, the radius vector HP be denoted by $r$, the angle UPT between it produced and the direction of P's motion be $\phi$, and let $p$ denote the length of the perpendicular HN upon the tangent to the curve at P. Then the length of the perpendicular PM upon HQ is $ds \sin\phi$, and the area HPQ $= dS$ say, described or swept out by the radius vector in time $dt$ as P moves to Q, is given by

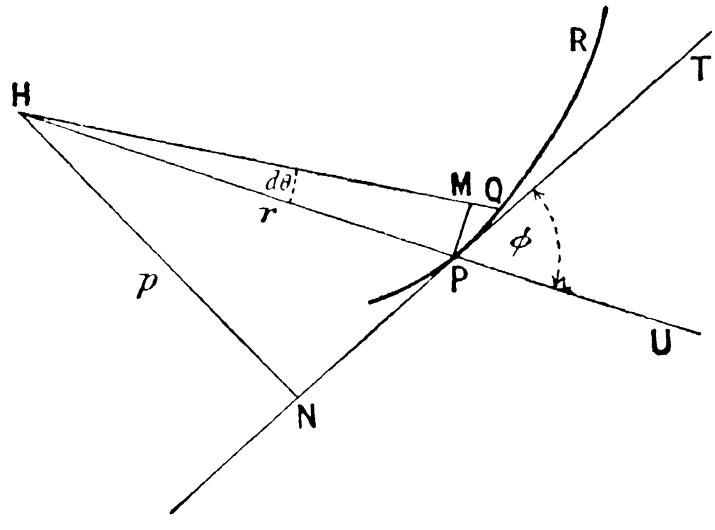



FIG. 23. ANGULAR AND AREAL VELOCITIES.

$$dS = \tfrac{1}{2} r\,ds \sin\phi = \tfrac{1}{2} p\,ds.$$

Similar expressions are easily obtained for the angular displacements and velocity and the areal velocity. These are collected in Table II.

TABLE II. LINEAR, ANGULAR, AND AREAL VELOCITIES.

| | LINEAR | ANGULAR | AREAL |
|---|---|---|---|
| DISPLACEMENT ELEMENTS | $ds$ | $d\theta = \dfrac{ds \sin\phi}{r} = p\,ds/r^2$ | $dS = \tfrac{1}{2} r\,ds \sin\phi = \tfrac{1}{2} p\,ds$ |
| VELOCITIES | $v = ds/dt = \dot{s}$ | $\omega = \dot{\theta} = \dfrac{v \sin\phi}{r} = pv/r^2$ | $V = \dot{S} = \tfrac{1}{2} rv \sin\phi = \tfrac{1}{2} r^2\omega = \tfrac{1}{2} pv = \tfrac{1}{2} h$ say |

**74. Radial and Transversal Velocities and Accelerations.**—Let $r$ and $\theta$ be the polar co-ordinates of a moving point P referred to the fixed axes or 'frame' OX, OY. And suppose $u$ and $v$ to be the radial

and transversal velocities of the point, that is, the component velocities respectively parallel and perpendicular to OP. We shall also denote by $f$ and $j$ respectively the component radial and transversal accelerations, which are effective on the moving point and change its velocity in magnitude or direction with respect to the fixed axes or 'frame' XOY. Then let it be required to express $u$, $v$, $f$, and $j$ in terms of $r$ and $\theta$ and their differential coefficients.

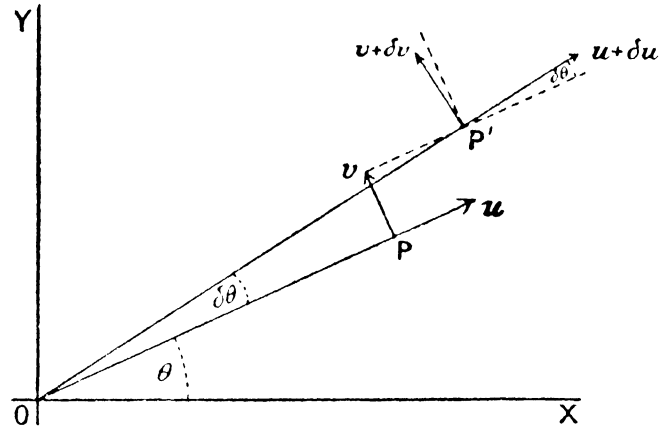



FIG. 24. RADIAL AND TRANSVERSAL VELOCITIES AND ACCELERATIONS.

Referring to Fig. 24, let P $(r, \theta)$ be the position of the moving point at time $t$ and P' $(r+\delta r, \theta+\delta\theta)$ be that at time $t+\delta t$. Then we have by the figure

$$u = \text{the limit of } \frac{(r+\delta r)\cos\delta\theta - r}{\delta t} = \frac{dr}{dt} = \dot{r} \quad \ldots\ldots (1)$$

and $$v = \text{the limit of } \frac{(r+\delta r)\sin\delta\theta}{\delta t} = \frac{r\,d\theta}{dt} = r\dot{\theta} \quad \ldots\ldots (2).$$

Again, if $u+\delta u$ and $v+\delta v$ be the radial and transversal velocities of the point when at P' at time $t+\delta t$, we have from the diagram the radial acceleration given by

$$f = \text{the limit of } \frac{(u+\delta u)\cos\delta\theta - u - (v+\delta v)\sin\delta\theta}{\delta t}$$

$$= \frac{du}{dt} - v\frac{d\theta}{dt} \quad \ldots\ldots (3).$$

Also, from the diagram, we have for the transversal acceleration

$$j = \text{the limit of } \frac{(v+\delta v)\cos\delta\theta - v + (u+\delta u)\sin\delta\theta}{\delta t}$$

$$= \frac{dv}{dt} + u\frac{d\theta}{dt} \quad \ldots\ldots (4).$$

By use of (1) and (2) in (3) and (4) these latter transform into the compact expressions

$$f = \ddot{r} - r\dot{\theta}^2 \quad \ldots\ldots (5)$$

and $$j = r\ddot{\theta} + 2\dot{r}\dot{\theta} \quad \ldots\ldots (6).$$

Thus, though $\dot{r}$ truly represents the velocity of P along OP, $\ddot{r}$ does *not* represent the *acceleration* of P's motion with respect to OX and OY in this direction, but it represents only the *rate of change of rate of change of* OP. This will be made clearer by the following illustrations.

**75. Circle uniformly described.**—As an example of radial and

transversal velocities and accelerations, consider first uniform circular motion. In this case $\dot{r}=u=0$, $v$ is constant, so $\ddot{\theta}=0$. Then we see that these agree with (1) and (2), and that from (5) and (6) we obtain

$$f=-r\dot{\theta}^2=-r\omega^2 \text{ and } j=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Thus $j$ and $f$ here agree with our $b$ and $c$ of articles 69 and 71, as they should do for the circle with constant speed.

**Straight Line uniformly described.**—Consider as a second example the point P moving with constant speed $V$ in the straight line $x=a$; or, in polar co-ordinates

$$r\cos\theta=a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$a\tan\theta=Vt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

We then obtain from these by elimination and differentiation

$$\left.\begin{aligned} &\dot{r}=V\sin\theta, \qquad \dot{\theta}=aV/r^2, \\ &\ddot{r}=a^2V^2/r^3, \text{ and } \ddot{\theta}=-(2aV^2\sin\theta)/r^3 \end{aligned}\right\} \quad . \quad . \quad (10).$$

Thus, substituting these values in (5) and (6), we get

$$f=a^2V^2/r^3-ra^2V^2/r^4=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$j=-\frac{r2aV^2\sin\theta}{r^3}+\frac{2aV^2\sin\theta}{r^2}=0 \quad . \quad . \quad . \quad . \quad (12).$$

Or in words, the accelerations are zero in each direction; as should be the case, since the velocity varies neither in magnitude nor direction.

### EXAMPLES.—XVII.

1. Exhibit in tabular form the relations between the linear, angular, and sectorial displacements and velocities of one point about another.
2. Obtain expressions for the radial and transversal velocities and accelerations of a point in terms of its polar co-ordinates $r$ and $\theta$.
3. A point moves from rest with acceleration $g$ vertically downwards. If, at any instant $t$, its polar co-ordinates in a vertical plane are $(r, \theta)$, their initial values being $(a, 0)$, state in terms of these the radial and transversal accelerations of the point.
4. 'Two points are moving with uniform velocities $u$, $v$ in perpendicular lines OX, OY, the motions being towards O; when $t=0$ they are at distances $a$, $b$ respectively from O. Calculate the angular velocity of the line joining them at time $t$, and show that this angular velocity is greatest when

$$t=\frac{au+bv}{u^2+v^2}.$$

(LOND. B.SC., PASS, APPLIED MATH., 1907, II. 1.)

5. If the radial and transverse velocities of a point in plane motion are both constant, show that in time $t$ the angle described by the radius vector and by the direction of motion may be expressed by

$$\psi=A\log(1+Bt).$$

**76. Areal Velocity is Constant under any Central Acceleration.**—When a moving point describes a plane curve with an acceleration

towards a fixed point in the plane, then its areal or sectorial velocity about that fixed point is constant. This theorem, though so simple, is yet of such fundamental importance in connection with planetary motion that we shall offer three proofs of it, one geometrical and two analytical.

*Geometrical Proof.*—Referring to Fig. 25, let the moving point have at P a velocity $u$ represented by OS, and at the very near point Q reached in time $\delta t$ let it have a velocity $v$ represented by OT, O being the intersection of the tangents to the path at P and Q. Then ST will represent the vector which, added to OS, gives the resultant OT, *i.e.* ST represents in magnitude and direction the change of velocity produced in time $\delta t$ by the acceleration, $f$ say, directed towards the fixed point.

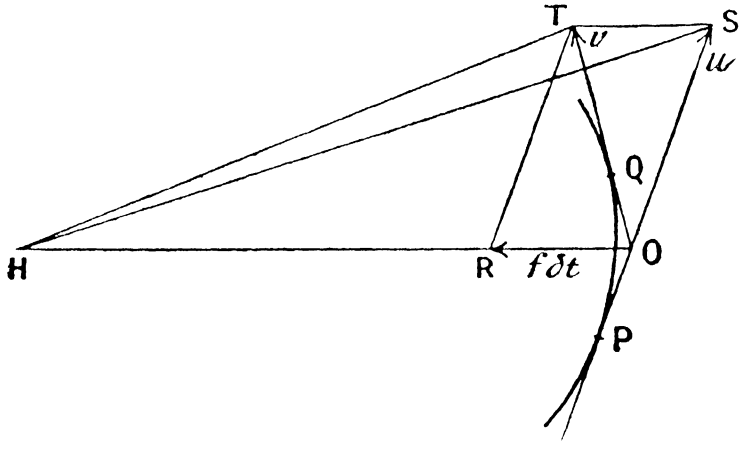



FIG. 25. AREAL VELOCITY CONSTANT IN CENTRAL ORBIT.

Thus a line from O parallel to ST will pass through the fixed point, H say. Take on OH, OR = ST = $f\delta t$, and join TR, TH, and SH. Then, if the initial and final areal velocities at P and Q are respectively $U$ and $V$, we have by definition and the figure

$$U = \triangle \text{HOS}$$

and

$$V = \triangle \text{HOT}.$$

But the triangles HOS and HOT are equal, being on the same base HO and between the same parallels HO and TS. Therefore $V = U$, as was to be proved.

**77. Proof by Moments.**—Turning now to analytical methods for a proof of the constancy of areal velocity in a central orbit, we first recall the theorem of article 25*a*, viz. that the moment of a resultant localised vector about a point equals the algebraic sum of its component vectors about that point, all being in one plane.

For the resultant localised vector in the present case take the acceleration of the moving point P directed to a fixed point, O say. Then its moment about O being zero, the sum of the moments for its components, however chosen, will be zero also. As components take the cartesian co-ordinates of the moving point P $(x, y)$, the origin being at O. Then using dots for differentiations with respect to time, we have

$$x\ddot{y} - y\ddot{x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

or

$$\frac{d}{dt}(x\dot{y} - y\dot{x}) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Whence, on integrating,

$$x\dot{y} - y\dot{x} = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

But the left side of (3) represents the algebraic sum of the moments of the component velocities of P about O. Hence by the theorem of moments it represents the moment about O of P's velocity, which is thus seen to be a constant. But this moment is double the areal velocity. So, if the linear velocity of P is $v$, and the perpendicular upon its direction from O is $p$, the areal velocity about O being $V$, we have

$$2V = pv = h \text{ say, a constant} \quad . \quad . \quad . \quad . \quad (4).$$

**78. Proof by Radial and Transversal Accelerations.**—Recalling now the expressions obtained in article 74 for the radial and transversal accelerations, we can easily give another proof of the constancy of areal velocity in the description of a central orbit. Thus from equation (6) of article 74 we have for the transversal acceleration

$$j = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

But this quantity is here zero, because the acceleration is wholly radial. Also the right side has in the brackets $r^2\dot{\theta}$, which is twice the areal velocity. Hence with the former notation (5) becomes

$$\frac{d}{dt}(pv) = \frac{dh}{dt} = \frac{d}{dt}(2V) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

In other words, the areal velocity is constant, as shown by equation (4).

**79. Differential Equation of Orbit.**—Consider the case where a point P moves in a plane under an acceleration of numerical value $f$ directed towards a fixed point S in the plane, and let it be required to find the differential equation of its path or orbit.

We again use the equations (5) and (6) at the end of article 74, which for our present case become, S being the origin of $r$,

$$\ddot{r} - r\dot{\theta}^2 = -f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The second of these, as we have previously seen, gives at once

$$r^2\dot{\theta} = h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$h$ being the constant value equal to double the areal velocity of P about S. But these equations contain the variables $r$, $\theta$, and $t$, from which we must eliminate $t$. Thus

$$\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \dot{\theta}\frac{dr}{d\theta} = \frac{h}{r^2} \cdot \frac{dr}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Now introduce the new variable $u$, defined by

$$u = 1/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

so that $du = -dr/r^2$ and $dr = -du/u^2$.

Then (4) becomes

$$\dot{r} = -h\,du/d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

So, on differentiating again, we obtain

$$\ddot{r} = -h(d^2u/d\theta^2)\dot{\theta} = -h^2u^2(d^2u/d\theta^2) \quad . \quad (7).$$

Hence (3), (6), and (7) substituted in (1) give

$$\frac{d^2u}{d\theta^2} + u = \frac{f}{h^2u^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

the differential equation required, in which $1/u(=r)$ and $\theta$ are the polar co-ordinates of the orbit and $f$ is the acceleration towards the origin. Thus when $f$ is given in terms of $u$ the orbit can be determined. Or, on the contrary, if the orbit is given, the form of $f$ as a function of $r$ can be found. We shall return to these cases a little later.

**80. Curvature of Orbit.**—Suppose we now wish to obtain an expression for the central acceleration in terms of the areal velocity, the radius vector, and the perpendicular $p$ upon the tangent. For this purpose we need to use the relation $rdr=\rho dp$, where $\rho$ is the radius of curvature of the orbit at the point defined by $r$ and $p$. This relation is proved in text-books on the calculus, but because of its importance, and also to save references, it will be briefly given here also.

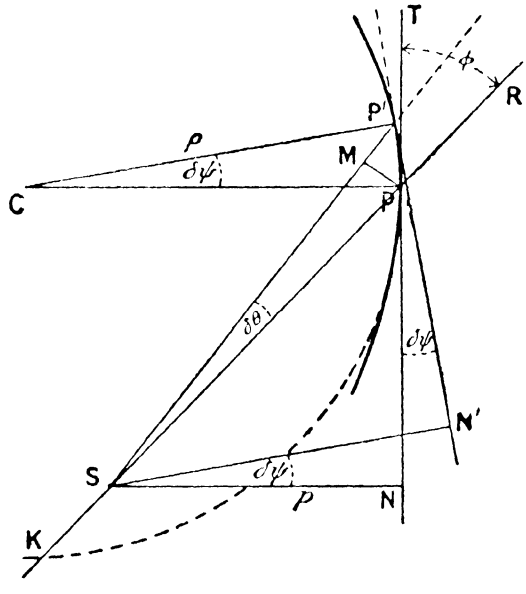



FIG. 26. CURVATURE OF ORBIT.

Thus, referring to Fig. 26, P and P', $\delta s$ apart, are consecutive positions of the point describing the orbit under acceleration to S, SN is the perpendicular $p$ on the tangent, C the centre of curvature of the orbit at P, $\phi$ the angle between the radius vector SPR, and the direction NPT of the point's motion, $\delta\psi$ the angle subtended at C by PP'. Then from the figure we have the following five relations:—

$$p=r\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$\delta\phi=\delta\psi-\delta\theta,\ \delta\psi=\delta s/\rho,\ \mathrm{PM}=r\delta\theta=(\sin\phi)\delta s,\ \delta r=\delta s\cos\phi \quad (2).$$

Thus, by differentiating the first and using it and the others, we obtain

$$dp=r\cos\phi\,d\phi+\sin\phi\,dr=r\cos\phi(d\psi-d\theta)+(p/r)dr$$

$$=r\cos\phi\left(\frac{ds}{\rho}-\frac{\sin\phi ds}{r}\right)+\frac{p}{r}dr=rdr\left(\frac{1}{\rho}-\frac{p}{r^2}+\frac{p}{r^2}\right),$$

or $$\rho dp=rdr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

as was to be shown, thus giving the curvature in terms of $r$ and $p$.[1]

**81. Formula for Central Acceleration.**—Referring again to Fig. 26, we see that the normal component along PC of the acceleration $f$ to S is given by

$$c=f\sin\phi,$$

or, using (1), $$c=fp/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

[1] Or more briefly thus:—$\delta p=(r\cos\phi)\delta\psi$ and $\delta r=\delta s\cos\phi$; hence $dp/dr=rd\psi/ds=r/\rho$.

But in article 69 equation (2) showed that

$$c = v^2/\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Thus from (4) and (5), and writing as before $pv = h$, we have

$$f p/r = v^2/\rho = h^2/p^2\rho, \text{ so that } f = \pm h^2 r/p^3\rho \quad . \quad . \quad . \quad (6),$$

the plus and minus signs being used to suit cases of either curvature Further, using in addition equation (3) of article 80 we obtain

$$f = \frac{h^2}{p^3} \cdot \frac{dp}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a),$$

equations (6) and (6*a*) being the desired expressions giving the central acceleration in terms of the areal velocity, radius vector, and perpendicular on tangent.

**82. Velocity in Orbit.**—In Fig. 26, let the osculating circle to the orbit at P cut PS (produced if necessary) at K, and let PK be denoted by $q$. It is called the *chord of curvature* in the direction of the acceleration. We see from the figure that its length is given by

$$\text{PK} = 2\text{CP} \cos \text{CPK},$$

or

$$q = 2\rho \sin \phi = 2\rho p/r \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Now by (4) and (5) of article 81

$$v^2 = f\rho p/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Hence

$$v^2 = fq/2 = 2f(q/4) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

But this is the velocity acquired from rest under an acceleration $f$ while passing over a space $q/4$. Hence the velocity of the moving point when at P in the orbit is that which it would acquire by moving from rest under a constant acceleration $f$ equal to that at P, through a space equal to one quarter o the chord of curvature in the direction PS.

## EXAMPLES—XVIII.

1. Show, both graphically and analytically, that under central acceleration of any law the sectorial velocity of the moving point about the centre of acceleration is constant.
2. Obtain the differential equation of the orbit described by a point under central acceleration of any law.
3. Derive an expression for the curvature of an orbit in terms of the radius vector $r$ and the perpendicular from the centre of acceleration on to the direction of motion of the moving point.
4. Establish a formula giving the central acceleration in terms of the constant sectorial velocity, the radius vector, and the perpendicular upon the tangent. Also show that the dimensions of the expression so obtained are those of a linear acceleration.
5. Find a general relation between the velocity of a point describing any orbit and the *chord of curvature* in the direction of the acceleration.
6. 'A point moves so that the radius vector describes equal areas in equal times. What may be inferred as to the acceleration under which the path is described?

   'If the path is an ellipse, and the centre of force is the centre of the curve, show that the eccentricity is $\sqrt{[1 - (V'/V)^2]}$ where $V$, $V'$ are the maximum and minimum velocities in the orbit.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, II. 4.

**83. Orbits under Natural Law.**—Let us now consider the orbits which may be described under the natural law of central acceleration, viz.

$$f \propto 1/r^2, \text{ or } f = \mu u^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $\mu$ is a constant, $u$ being as before equal to $1/r$.

Then the differential equation of the orbit, given by (8) of article 79, may be written

$$\frac{d^2(u-\mu/h^2)}{d\theta^2} + (u-\mu/h^2) = 0 \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Then, by comparison with article 29, we see that the solution may be written

$$u - \mu/h^2 = A \cos(\theta - \gamma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

in which $A$ and $\gamma$ are arbitrary and depend on the initial conditions.

Thus, on writing

$$l = h^2/\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and using $e$ for $-Al$, we may transform (3) into

$$l/r = 1 - e\cos(\theta - \gamma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

which is the well-known polar equation of a conic, the focus being the pole, $l$ being the semi-latus rectum, and $e$ the eccentricity.

Thus, for the natural law of central acceleration, that of the inverse square of the distance, the orbit is a conic; but whether an ellipse, a parabola, or a hyperbola depends upon whether the value of $e$ is less than, equal to, or greater than unity. And the value of $e$ depends in turn upon the initial circumstances of the motion. Before, however, discussing this further by the purely analytical method, let us take another and more geometrical point of view, applying the hodograph to the problem of planetary orbits.

**84. Natural Orbits by Hodograph, which is a Circle.**—To show that the hodograph is a circle we have to prove that its curvature $d\psi/ds$ is constant. We accordingly need the values of $\delta s$ and $\delta\psi$, the element of the curve and the angle between the tangents at its

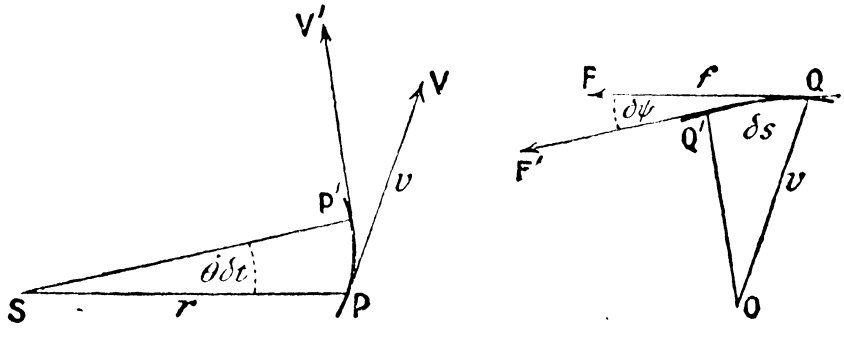



FIG. 26A. CORRESPONDING ELEMENTS OF ORBIT AND HODOGRAPH.

extremities. By the principle of the hodograph (article 70) the element $\delta s$ of its arc described in time $\delta t$, shown by QQ′ in Fig. 26A, represents in magnitude and direction the change of velocity of the point in

that time $\delta t$, and while describing the corresponding element PP′ of the path. Thus, for our present case, we have the relations

$$\delta s = f\delta t = \mu\delta t/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where $f$ is the acceleration along PS.

Further, since each element of the hodograph is parallel to the acceleration obtaining for the corresponding portion of P, the tangent to the hodograph at Q in our case is *parallel* to the radius vector PS$=r$. Thus QF and PS both change direction with the same angular velocity $\dot\theta$. But, by article 76, the areal velocity $\frac{1}{2}r^2\dot\theta$ of P about S is constant$=h/2$ say; hence the angular velocity of the radius vector and of the tangent to the hodograph are each given by $\dot\theta = h/r^2$. Thus, in the time element $\delta t$, the angle $\delta\psi$ through which the tangent QF to the hodograph turns is

$$\delta\psi = \dot\theta\delta t = h\delta t/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Hence, by (6) and (7), the curvature and radius of curvature of the hodograph are respectively

$$\delta\psi/\delta s = h/\mu = 1/\rho \text{ and } \rho = \mu/h \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

But these values are constant. Hence, for orbits described under a central acceleration proportional to the inverse square of the distance, *the hodograph is a circle.*

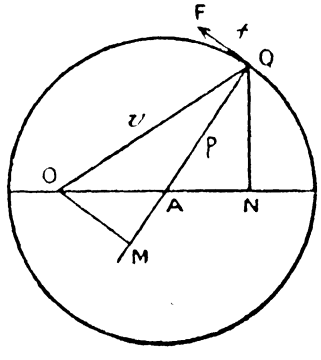



FIG. 27. CIRCULAR HODOGRAPH FOR PLANETARY ORBIT.

The pole of the hodograph may, however, be inside the circle, on the circumference, or outside.

Suppose first the pole to be inside, as represented by O in Fig. 27, and consider the point Q on the hodograph.

Then AQ, the radius of the circle, is $\rho = \mu/h$; OQ$=v$, the velocity of the point in the orbit; QF, the velocity in the hodograph, is the acceleration $f$ of the point in the orbit, and so is parallel to the radius vector.

Thus we see that the velocity $v$ in the orbit, represented by OQ, may be regarded as the resultant of two parts OA and AQ, each constant in magnitude, OA being fixed in direction also; while AQ, always perpendicular to the radius vector, moves round, Q describing the circle, while the actual point P describes the orbit. This consideration is very useful in the graphical treatment of certain problems.

**85. Orbit a Conic.**—We may also resolve the velocity $v$ in a different manner. Thus, in Fig. 27, let fall OM perpendicular to QA, produced if necessary; also make QN perpendicular to OAN. Then OM represents that component of the velocity of the point P which is parallel to FQ. But FQ is parallel to the radius vector to P. Hence OM represents the speed with which the radius vector changes in length. Or, in symbols

$$\text{OM} = dr/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Again, NQ represents the component, perpendicular to the fixed line OAN, of the velocity of the point P in the orbit. So, if the co-ordinate of P perpendicular to OAN is denoted by $y$, we have

$$NQ = dy/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

But we see by the figure that

$$OM/NQ = OA/AQ = \text{a constant, } e \text{ say} \quad . \quad . \quad . \quad (11).$$

Thus by (9), (10), and (11)

$$dr/dy = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

So, on integrating, we have

$$r = ey + b = e\left(y + \frac{b}{e}\right) = ey' \quad . \quad . \quad . \quad . \quad . \quad (13),$$

showing by the focus-directrix property that the orbit is a conic, $b$ being the constant of integration. It is clear from (13) that this conic is an ellipse, a parabola, or a hyperbola according as $e$ is less than, equal to, or greater than unity; or, by (11), according as O is within the circle, on the circumference, or outside it; that is to say, according to the velocity OQ at a given instant.

**86. Alternative Proof that Natural Orbit is a Conic.**—We can also show that the orbit is a conic by use of the properties referred to at the end of article 84, viz. that the velocity of P is resolvable into two parts each fixed in amount, one in direction also, and the other always perpendicular to the radius vector.

Thus, in Fig. 28, let P be the point describing the orbit under acceleration, $f = \mu/r^2$, directed to S, the origin of co-ordinates. Let PV be the velocity $v$ of P. Then it may be regarded as consisting of the two parts PA′ and A′V, equal and parallel respectively to OA and AQ in Fig. 27. Hence, if VN′ is let fall perpendicular to PA′N′, which is parallel to SX, we see that $\angle VA'N' = \angle PSY' = \beta$ say, $A'V = \rho$, $PA' = e\rho$. Thus, if $x$ and $y$ are the co-ordinates of P, $\dot{x}$ and $\dot{y}$ the components of its velocity, we have by the figure

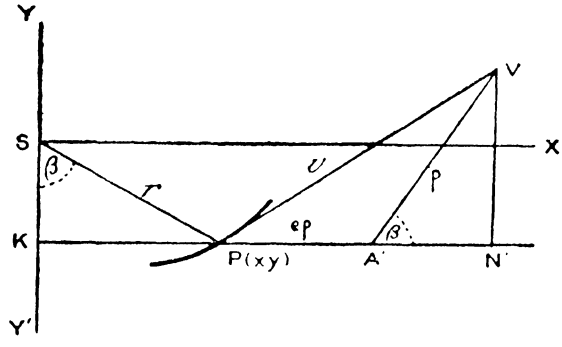



FIG. 28. PLANETARY ORBIT IS A CONIC.

$$\dot{x} = PA' + A'N' = PA' + A'V\cos\beta,$$

or
$$\dot{x} = \rho(e - y/r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Also
$$\dot{y} = N'V = \rho x/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

Thus, since
$$r^2 = x^2 + y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

we have, by differentiation and use of (14) and (15),

$$r\dot{r} = x\dot{x} + y\dot{y} = x\rho e = er\dot{y},$$

or
$$r = e\dot{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

So, by integrating, we have as before

$$r = ey + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

giving the orbit as a conic by its focus-directrix property, $b$ being the constant of integration.

A comparison of Figs. 27 and 28 shows that the line OAN in the hodograph is parallel to the directrix of the conic, and therefore at right angles to its axis.

**87. Hyperbolic Orbit and its Hodograph.**—From what we have seen as to the relations of the hodograph and planetary orbits, it is easy to draw the two diagrams and mark off on them a series of positions for the points Q and P that correspond to each other. So for the simpler cases of the ellipse and parabola this exercise will be left to the student. But, because of its special interest in one particular, the pair of associated diagrams will be given for the case in which the orbit is a hyperbola.

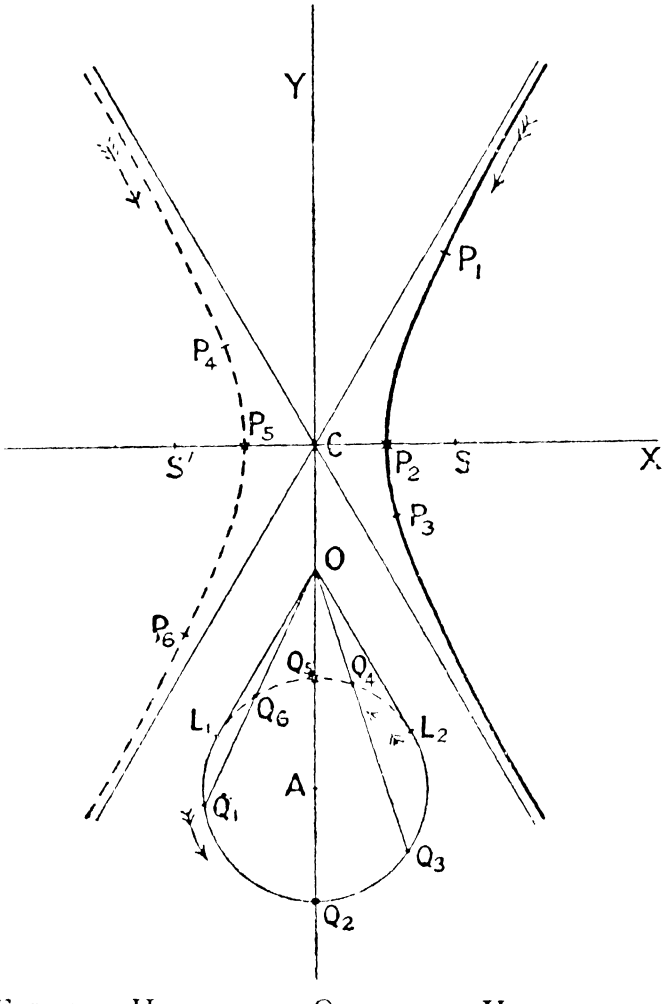



FIG. 29. HYPERBOLIC ORBIT AND HODOGRAPH.

Fig. 29 represents this case, for $e=2$, so the equation of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1$$

becomes

$$\frac{x^2}{a^2}-\frac{y^2}{3a^2}=1 \quad . \quad . \quad (19).$$

Also, the semi-latus rectum $l=a(e^2-1)$ becomes $3a$, and the asymptotes are

$$y=\pm\frac{b}{a}x=\pm\sqrt{3}x \quad (20),$$

and therefore make angles of 60° with the axis of $x$.

The P's with subscripts on the hyperbolic orbit correspond with the Q's with the same subscripts on the circular hodograph in the lower part of the figure, whose pole is O, where OA equals twice the radius of the circle. It will be seen that the right branch of the hyperbola is described downwards by P, while the lower part of the hodograph from $L_1$ to $L_2$ is described counter-clockwise by Q. This is effected under acceleration *towards* S. Then, while Q describes the upper part of the hodograph (again between the tangents from O) still counter-clockwise,

P describes the left branch of the hyperbola still downwards. But it will be noticed that the *smallest* value of the speed of P is reached at $P_6$, corresponding to $OQ_6$. Hence this left branch is described under acceleration *from* S. Indeed, this may be seen directly from the figure. For, not only is the tangent at any P parallel to the corresponding OQ, since both represent the velocity of P; but also the tangent at any Q is parallel to the corresponding PS (or SP), since both represent the direction of the acceleration. For example, the *forward* tangent at $Q_6$ represents the acceleration in the same direction *from* S to $P_6$. Thus the full lines in orbit and hodograph correspond to *attraction to S*, whereas the broken lines in each curve correspond to *repulsion from S*.

EXAMPLES—XIX.

1. Assuming the differential equation for an orbit described under central acceleration, prove that for the natural law the orbit is a conic. State also what decides the type of the conic.
2. Prove that for a planetary orbit the hodograph is a circle.
3. Having given that the hodograph is a circle, show that the planetary orbit is a conic.
4. With pins at the foci and a loop of thread round, draw an ellipse of eccentricity 1/2 to represent an orbit. Draw also the circular hodograph, mark its pole, and indicate several pairs of corresponding points on the orbit and hodograph.
5. Draw as carefully as you can the two branches of an equilateral hyperbola to represent possible orbits, and show the corresponding hodograph. Explain under what conditions each branch of the hyperbola may be described and which parts of the hodograph apply to each such orbit.
6. Draw a circular hodograph and the corresponding parabolic orbit, marking corresponding pairs of points on each.

**88. Initial Conditions.**—Let us now return to the general solution in article 83 of the differential equation of an orbit under the inverse square law and determine the arbitrary constants $e$ and $\gamma$ which depend on the initial conditions. We rewrite the solution in the form (see equations (4) and (5) of article 83)

$$u=\frac{\mu}{h^2}\{1-e\cos(\theta-\gamma)\} \quad . \quad (1).$$

Let the initial conditions be—

$$\left.\begin{array}{l}\text{For } \theta=0,\ 1/u=r=c,\\ \text{and velocity of P}=\\ v_0 \text{ at angle } \beta \text{ with}\\ \text{radius vector}\end{array}\right\} \quad . \quad (2).$$

This initial state is represented in Fig. 30, in which SP is the initial position of the radius vector of length $c$, PV represents the velocity $v_0$, making the angle $KPV=\beta$, Q is the position of P after the time $\delta t$, and

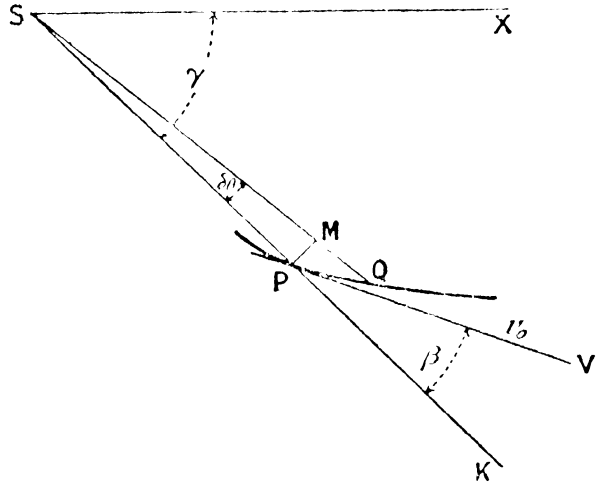



FIG. 30. INITIAL CONDITIONS OF ORBITAL MOTION.

PM is a perpendicular let fall upon SQ, so that $MQ/MP=\cot\beta$ nearly.

Then by reference to the figure and definition of areal velocity we have

$$h=v_0 c\sin\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Again, by (2) in (1) we have

$$\frac{1}{c}=\frac{\mu}{h^2}(1-e\cos\gamma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Thus (3) and (4) give on elimination of $h$ between them

$$v_0^2 c\sin^2\beta-\mu=-\mu e\cos\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

We now need another equation giving $e\sin\gamma$, then we can solve for $e$ and $\gamma$ as required. Referring to Fig. 30, we see that MQ is $\delta r$, and thus is equal to $-(\delta u)/u^2$ or $-c^2\delta u$, since $r$ is initially $c$. Again, $MP=c\delta\theta$, therefore we have

$$MQ/MP=\cot\beta=-c^2\delta u/c\delta\theta=-c\,du/d\theta \quad . \quad . \quad . \quad . \quad . \quad (6).$$

But from (1), by differentiation, we obtain

$$\frac{du}{d\theta}=\frac{\mu}{h^2}e\sin(\theta-\gamma)=-\frac{\mu}{h^2}e\sin\gamma \text{ for } \theta=0 \quad . \quad . \quad . \quad (7).$$

Hence, from (6) and (7) we obtain

$$\frac{\cot\beta}{c}=\frac{\mu}{h^2}e\sin\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Thus, using (3) again to eliminate $h$ we find

$$v_0^2 c\sin\beta\cos\beta=\mu e\sin\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

So on squaring and adding (5) and (9) we have

$$\left.\begin{aligned} e^2&=1+\frac{v_0^4c^2\sin^2\beta}{\mu^2}-\frac{2v_0^2c\sin^2\beta}{\mu},\\ \text{or}\quad e^2&=1+\frac{v_0^2c^2\sin^2\beta}{\mu^2}\left(v_0^2-\frac{2\mu}{c}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Also, taking the quotient of (9) by (5) we find

$$\tan\gamma=\frac{v_0^2c\sin\beta\cos\beta}{\mu-v_0^2c\sin^2\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Thus it follows from the second form of (10) that the conic is an ellipse, parabola, or hyperbola

according as $v_0^2<,=,\text{or}>\frac{2\mu}{c}$ . . . . . (12).

**89. Velocity and Period in Elliptic Orbit.**—When the orbit is an ellipse of semi-axes $a$ and $b$ we have the semi-latus rectum given by

$$l=a(1-e^2)=h^2/\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Thus, by (3) and (13) we find

$$a(1-e^2)=\frac{v_0^2c^2\sin^2\beta}{\mu}.$$

But by (10)

$$a(1-e^2)=a\left(\frac{2v_0^2c\sin^2\beta}{\mu}-\frac{v_0^4c^2\sin^2\beta}{\mu^2}\right).$$

Hence, equating the right sides of these we obtain

$$v_0^2=\mu\left(\frac{2}{c}-\frac{1}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

As we may consider any point in the orbit the point of projection, we have for the linear velocity $v$ at any instant when the radius vector has the value $r$, the expression

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

Again, the period $\tau$ of describing the ellipse is area divided by areal velocity. Thus using (13), and remembering that $b^2=a^2(1-e^2)=al$, we have

$$\left.\begin{aligned}\tau=\frac{\pi ab}{h/2}=\frac{2\pi ab}{\sqrt{\mu l}}=\frac{2\pi ab}{\sqrt{\mu b^2/a}}=\frac{2\pi}{\sqrt{\mu}}a^{3/2}\\ \text{or} \qquad \tau^2\propto a^3,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16),$$

for $\mu$ constant.

By putting (15) in (16) to eliminate the semi-axis-major $a$, we find

$$\tau^2=\frac{4\pi^2\mu^2r^3}{(2\mu-rv^2)^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

showing that the periodic time is independent of the angle of projection ($\beta$ of last article) provided the conditions are such as to give an elliptic orbit.

**90. Focal Acceleration, Period, and Velocity for an Elliptic Orbit.** —We have considered the orbits possible for a central acceleration varying as the inverse square of the distance. Let us now regard the matter from the other standpoint, and supposing that a point is known to describe an ellipse with constant areal velocity about a focus, let it be required to find the direction and law of the acceleration. This gives the essential features of the astronomical problem of the elliptic orbit of the earth round the sun.

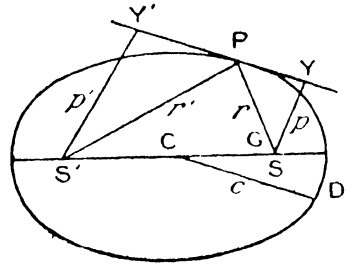



FIG. 31. FOCAL ACCELERATION FOR ELLIPTIC ORBIT.

Thus, let the ellipse in Fig. 31 represent the orbit of semi-axes $a$ and $b$, semi-latus rectum $l$, and let $c$ denote the semi-diameter CD conjugate to CP, and $\rho$ the radius of curvature at P. Draw the tangent at P, and let fall upon it from the foci S and S' the perpendiculars SY and S'Y' of lengths $p$ and $p'$ respectively.

Then, by the known properties of the ellipse, we have

$$\rho=c^2/ab,\ p/r=p'/r',\ pp'=b^2,\ rr'=c^2,\ b^2=al, \text{ and } r+r'=2a \quad . \quad (18).$$

$$\text{Thus } \frac{p}{r}=\frac{p'}{r'}=\sqrt{\frac{pp'}{rr'}}=\frac{b}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

Also if $h/2$ is the constant areal velocity of P about S, and $f$ the acceleration directed to S, we have by equation (6) of article 81, and using (18) and (19),

$$f=\frac{h^2r}{\rho p^3}=h^2r\left(\frac{ab}{c^3}\right)\left(\frac{c^3}{b^3r^3}\right)=\frac{h^2a}{b^2r^2}=\frac{h^2}{l}\cdot\frac{1}{r^2}=\frac{\mu}{r^2} \quad . \quad . \quad (20),$$

where $\mu$ is written for $h^2/l$.

Thus the acceleration is directed towards the focus S, about which the areal velocity is constant, and is seen to vary inversely as the square of the radius vector from that focus.

**Period.**—For the period $\tau$ of description of the orbit we have

$$\tau=\frac{\pi ab}{h/2}=\frac{2\pi a\sqrt{al}}{\sqrt{\mu l}}=\frac{2\pi}{\sqrt{\mu}}a^{3/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

**Velocity.**—Also, remembering that $pv=h$, and using (18), we have for the velocity at any point in the orbit

$$v^2=\frac{h^2}{p^2}=\mu\frac{l}{p^2}=\mu\frac{b^2}{ap^2}=\mu\frac{pp'}{ap^2}=\mu\frac{r'}{ar}=\mu\frac{2a-r}{ar},$$

or

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

as previously found in equation (15) of article 89. This result may also be easily obtained by using $pv=h$ and the tangential-normal equation for an ellipse, the focus being a pole, which is

$$\frac{l}{p^2}=\frac{2}{r}-\frac{1}{a}.$$

It may be noticed that articles 69, 76, 81, 84, 85, 87, and 90 give the more elementary parts of central accelerations in a form practically free from the calculus. It may, therefore, be an advantage to some students to take these articles consecutively, omitting the other parts till these are understood.

**90a. Simultaneous Elliptic Orbits.**—Although an intrusion upon the pure kinematics with which the present part of the book is concerned, it is perhaps as well to note here that, strictly speaking, we have not in nature a point S occupied by one body void of acceleration while the other body at P describes an orbit round it. On the contrary, the acceleration of the body at P should be reckoned with respect to a point G on SP, and then the body at S has also an acceleration towards G which may be regarded as at rest. Further, these two simultaneous accelerations are proportional to the segments into which SP is divided. Thus at any instant the acceleration of P towards G is to that of S towards G as GP is to GS. And each acceleration is inversely proportional to the square of SP as it changes from instant to instant. Hence we have in any actual case an ellipse described by S about G as a focus, while P describes the similar ellipse also about G as a focus.

In the case of the sun and the earth or other planets G almost coincides with S. Assuming this to be so, we have the acceleration $\mu/r^2$ for any of them, and consequently from equation (21)

$$\tau^2\propto a^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22a),$$

which is one of Kepler's laws symbolically expressed. In the case of

equal double stars G would be midway in SP. These points may receive further attention when we deal with attractions in Chapters XI. and XVI.

**91. Focal Acceleration for any Conic.**—Let the polar equation of the conic be

$$lu = 1 - e\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23),$$

$l$ being the semi-latus rectum, $u$ the reciprocal of the radius vector $r$, and $e$ being the eccentricity, and let it be described with areal velocity $h/2$ about the pole. Then we have, on differentiating,

$$l\frac{d^2u}{d\theta^2} = e\cos\theta.$$

So

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

But by equation (8) of article 79 the left side of (24) equals $f/h^2u^2$ where $f$ is the acceleration directed to the pole or origin.

Thus

$$f = h^2/lr^2 = \mu/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),$$

that is, the acceleration varies inversely as the square of the radius vector independently of the value of $e$, and therefore this is the law whether the conic is an ellipse, parabola, or hyperbola.

## EXAMPLES—XX.

1. Discuss the initial conditions of a point describing an orbit under central acceleration of the natural law, and show how to determine the type of conic described and its orientation.
2. Prove that when an ellipse is described with constant areal velocity about a focus the acceleration is directed to that focus and varies inversely as the square of that focal distance.
3. Obtain expressions for the velocity at a point in an elliptic orbit under focal acceleration and for the period.
4. Derive expressions for the velocity at a point in a parabolic orbit and in a hyperbolic orbit.
5. 'If the velocity of a planet describing a circular orbit about the sun be suddenly diminished by a slight amount, show by a figure the relation of the new orbit to the old one.

   'Will the periodic time be increased or diminished?'

   (LOND. B.A., PASS, APPLIED MATH., 1906, I. 10.)

   *Ans.* Point where the velocity is checked becomes the end of the major axis of new elliptic orbit, the sun being at the distant focus. The periodic time is diminished according to the expression $\tau/\tau_0 = (2 - v^2/v_0^2)^{-3/2}$, where the subscripts o refer to original period and velocity.
6. 'Prove the formula $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$ for the velocity at any point of an elliptic orbit described under a central acceleration $\mu$(distance)$^{-2}$. Obtain corresponding formulae for parabolic and hyperbolic orbits described under the same attraction.

   'Show that, if a parabolic orbit and rectangular hyperbolic orbit have

the same latus rectum, the velocity at the end of the latus rectum is $\sqrt{3/2}$ times as great in the latter case as in the former.'

(LOND. B.SC., PASS, APPLIED MATH., 1908, II. 4.)

7. 'Investigate the law of force tending to the focus under which an ellipse may be described, and show that the periodic times in the orbits described by particles projected from the same point with the same velocity are equal.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, II. 4.)

8. 'A particle P describes a parabola under the action of a force tending to the focus; determine the law of force, and prove that the kinetic energy is inversely proportional to the distance from the focus.

'Prove also that, if M is the foot of the perpendicular from P on the directrix, the motion of M is the same as that of a particle constrained to slide without friction along the directrix and attracted to the focus by a force proportional to the inverse fifth power of the distance from the focus.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, II. 4.)

**91*a*. Alternative Proof for Velocity in Elliptic Orbit.**—If already conversant with the kinetics and attractions of particles, the student may prefer to proceed as follows. By the conservation of energy it is known that the sum of the kinetic and potential energies of the planet is constant. And, by (12) of Art. 355, the latter is $-\mu m/r$, where $m$ is the mass of the planet. Thus

$$\tfrac{1}{2}mv^2-\mu m/r=\text{constant, or}$$

$$v^2=2\frac{\mu}{r}+C \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

To evaluate the constant $C$, take the instant when the planet is at an end of the minor axis of the elliptic orbit and when therefore $r=a$. Then, since $pv=h$, we have by using some properties of the ellipse

$$v^2=\frac{h^2}{p^2}=\frac{\mu l}{b^2}=\frac{\mu b^2}{b^2 a}=\frac{\mu}{a}=2\frac{\mu}{a}+C \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Whence $C=-\mu/a$; and putting this in (1) we find for the required law of velocity,

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

# CHAPTER VI

## PLANE ROTATIONAL MOTIONS

**92. Uniform Angular Acceleration.**—Consider an assemblage of points in a plane, each of which moves in the plane with the same uniform angular velocity $\omega$ about the same fixed axis perpendicular to that plane, no other motion but this being possessed by such point. Then obviously the relative distances of any of the pairs of points would remain unchanged. Further, if this common angular velocity were changed in any way to some other value $\omega'$, so that at each instant its value was the same for every particle, still the relative distances of the particles would be unchanged by the motion about the axis. Such an assemblage of particles under the conditions named may accordingly be referred to as a *rigid body* or *system*, since it changes neither shape nor size. Strictly speaking, no absolutely rigid bodies are ever met with, but the conception and the term are useful, and the state of things in question is often closely approximated to.

If the assemblage of points or particles, instead of being confined to a plane, are distributed in solid space and all move parallel to a fixed plane with the same uniform angular velocity about a fixed axis perpendicular to that plane, we have, as before, a rigid system in plane rotational motion.

Let us now suppose that a rigid body has an initial angular velocity $\omega_0$ about a certain axis fixed with respect to the body and also with respect to our co-ordinate axes, and let the body have also a uniform angular acceleration $\alpha$ about this axis of motion. It is required to determine the subsequent motion.

A little reflection will suffice to show that this case is closely analogous to that of the rectilinear motion of a point under uniform linear acceleration. Hence by the meanings of angular velocity and angular acceleration (rate of increase of angular velocity) we obtain at once a set of equations for rotations like those obtained in article 27 for translations. Thus, using $\theta$ for angle described in time $t$ and repeating the former equations, we have the two analogous sets as follows :—

| *Translations.* | *Rotations.* | |
|---|---|---|
| $v=v_0+at$ | $\omega=\omega_0+\alpha t$ . . . | (1), |
| $s=v_0t+\frac{1}{2}at^2$ | $\theta=\omega_0t+\frac{1}{2}\alpha t^2$ . . | (2), |
| $v^2=v_0^2+2as$ | $\omega^2=\omega_0^2+2\alpha\theta$ . . | (3). |

Further, if the perpendicular distance of any particle from the axis of rotation is $r$, it is obvious that the relations between its linear and angular displacements, velocities, and accelerations may be written

$$r=s/\theta=v/\omega=a/\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

These equations accordingly serve to solve any problems concerned with the uniform acceleration of rigid bodies about a *single axis fixed both in space and in the body.*

**93. Angular Acceleration proportional to Displacement.**—Consider a rigid body capable of motion about a fixed axis under conditions which impose an angular acceleration proportional but opposite to its angular displacement $\theta$. Then, using dots to denote differentiation with respect to time, we have for the equation of motion

$$\ddot{\theta}+p^2\theta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The solution of this (see article 29) may be written

$$\theta=\theta_0 \sin(pt+\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

in which $\theta_0$ and $\epsilon$, the amplitude and phase angle, are to be determined by the initial conditions as shown in article 30.

We thus see that the motion performed under the specified conditions is a *simple harmonic rotation* of period given by

$$\tau=2\pi/p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

## EXAMPLES—XXI.

1. With angular acceleration 2 radians per sec.,$^2$ find (1) the angular velocity 5 sec. after it was 3 radians per sec.; (2) the angle described in that 5 sec.; and (3) the increase in the square of the angular velocity while one revolution is described.
2. If the angular velocity of a rigid system about a fixed axis increases from 12 to 13 radians per sec. while it turns through $2\frac{1}{2}$ radians, what was the angular acceleration if uniform?
3. Find the time required to describe 28 radians with an initial angular velocity of 3 radians per sec. under an angular acceleration of 2 radians per sec.$^2$
4. A rigid figure turning about a fixed axis is subject to angular acceleration $-9$ times its angular displacement. If displaced through 30° and let go, what will be the subsequent motion?

**94. Composition of Coplanar Rotation and Translation.**—The composition of rotations about a given fixed axis is clearly a matter of algebraic addition, so may be dismissed without further remark. But the case of a rotation or angular displacement, and a translation or linear displacement, requires consideration. Thus, in Fig 32, let the linear displacement or translation of a point A in the body be $AA'=s$, and let the angular displacement or rotation of the body be $\theta$ also in the plane of the diagram.

*Construction.*—Bisect AA′ in M, and draw MR and AC at right angles to AA′. Also draw AB, making the angle $-\theta/2$ with AC, and produce BA to meet MR in R. Draw RA′B′. Then A′B′ is the new

position of the line in the body originally at AB. That is, A'B' represents the effect on AB of the rotation and translation which were to be compounded.

*Proof.*—By construction RA'=RA, and is inclined at an angle $\theta$ with it. Hence by rotation about R, the line AB moves to the position A'B', in which the point A' has the prescribed translation AA'=$s$, and the line AB has the prescribed rotation BRB'=$\theta$. We see by the figure that $s/2 = MR \tan \theta/2 = AR \sin \theta/2$, and that the angle MAR is the complement of $\theta/2$.

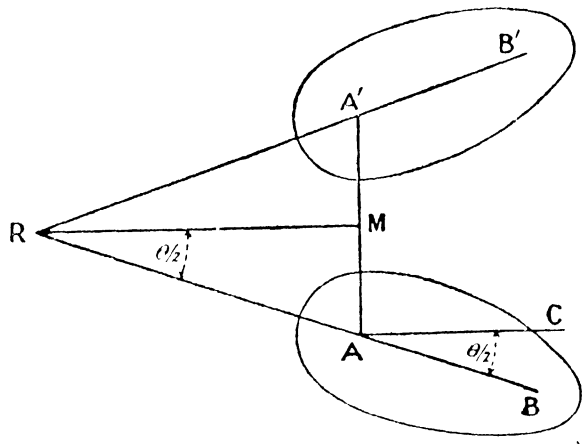



FIG. 32. ROTATION AND TRANSLATION COMPOUNDED.

**95. Instantaneous Centre of Rotation: Rolling Motion.**—The foregoing example of composition shows that a coplanar rotation and translation are equivalent to an equal pure rotation about a particular fixed point. Thus, as regards initial and final positions, any prescribed displacement in a plane may be regarded as a rotation about some point. (In the special case where the motion is translation only this point is at infinity.) Hence, by taking the steps small enough, we may approximate to any motion in a plane by a series of motions, each of pure rotation, about certain points. Each such point, while in use, is called the *instantaneous centre of rotation.* Thus any finite motion of a rigid system in a plane could be regarded as a series of infinitesimal rotations about an infinite number of instantaneous centres of rotation. The locus of the instantaneous centre forms a curve in the plane called the *space centrode.* Its locus in the rigid body or system forms another curve called the *body centrode.* Thus, the instantaneous centre of rotation may be regarded as the point of contact of the space centrode and the body centrode. Hence the whole motion of the body may be represented as the *rolling* of the body centrode on the space centrode, and this conception is often of great value. Thus, as the rolling proceeds, that mathematical point called the instantaneous centre has in general a velocity both in space and in the body. But the *point in the body* which for the instant coincides with the instantaneous centre, has for that instant *no velocity either in space or in the body.* And this distinction must be clearly grasped. As an example, in Fig. 16 of article 58, consider the circle EPDF as rolling along the straight line AEB, and let its centre advance at uniform speed $u$. Then AEB is the space centrode and EPDF the body centrode. Also E is the instantaneous centre of rotation in the position of the figure. And clearly the point E will advance along the straight line and along the circle with uniform speed $u$. But consider now the point P, fixed with respect to the circle and carried with it so as to reach B, a cusp of the cycloid which it is describing as the circle

rolls. Then we see that B is always at rest in space, that P is always at rest with respect to the circle, and, when it reaches B, the point P is *at rest in space also.* For it is then at the cusp of the cycloid, and is at the pause before describing the next branch of the curve. But, although E moves along the straight line at the uniform speed $u$, without any check or pause, it passed through B just when P coincided with it. And as soon as E has got beyond B, P has moved perpendicularly away from AEB, and so has ceased to coincide with the instantaneous centre. Thus P's first motion, after the pause at B, destroyed its claim to be the instantaneous centre, because it took it out of contact with the line AEB.

If the rigid system extends into the three dimensions of space, but all the motions are still parallel to one plane, it is more fitting to speak of the *instantaneous axis* of rotation, and of the *space axode* and *body axode*, which are the loci in space and in the body of the instantaneous axis.

**96. Analysis of Plane Motion of a Rigid Body.**—In article 94 we dealt with the composition of a rotation and a coplanar translation. Let us now take a different aspect of the matter and, the initial and final positions of a line in the system being specified, find the centre of rotation, also the equivalent rotation and translations.

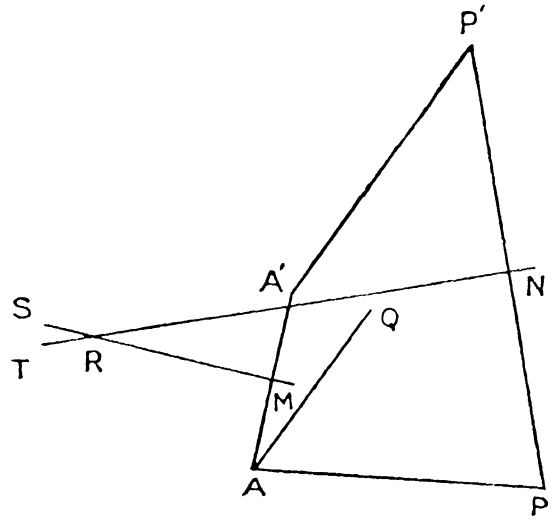



FIG. 33. ANALYSIS OF PLANE MOTION OF A RIGID BODY.

Thus, in Fig. 33, let AP and A′P′ be respectively the initial and final positions of a line in the rigid body. Join AA′, bisect it in M, and draw MS at right angles to AA′. Then obviously the centre of rotation must lie in MS. Similarly by joining PP′, bisecting it in N′, and making NT at right angles to PP′, we have another line on which the centre must lie.

It is accordingly at R, the intersection of MS and NT. We may therefore describe the motion from AP to A′P′ as a rotation about R through the angle ARA′= PRP′=PAQ, where AQ is parallel to A′P′. But we may also describe the motion from AP to A′P′ as *either* (1) a rotation through the angle PAQ *about A* together with a translation of A from A to A′; *or* (2) a rotation through the angle PAQ *about P* together with a translation of P from P to P′.

When AA′ and PP′ are parallel, the above construction for R obviously fails. In this event two cases arise according as AP and A′P′ are parallel or not. In the latter case, illustrated by Fig. 34, R is found as the intersection of AP and A′P′.

Whereas when the initial and final positions of the lines are parallel as well as those which represent the displacements of their ends as

AP and the dotted line A′P″ in Fig. 34, the centre of rotation is at infinity, viz. at the intersection of MU and SV, the perpendiculars through the middle points of AA′ and PP″. In other words, as is obvious, the motion in question is one of pure translation only, the displacement of each point being equal to AA′ or PP″.

If the character of a rigid body's motion is variable, it is obvious that the instantaneous centre of rotation lies in the line which bisects at right angles any known small displacement of a point in the body. Thus, the intersection of two such lines gives the instantaneous centre of rotation. A consideration of the circumstances of the case, or repeated constructions, then determine the space centrode as the locus of this centre. The body centrode is often found most easily by 'fixing' the body and supposing the 'space' to move.

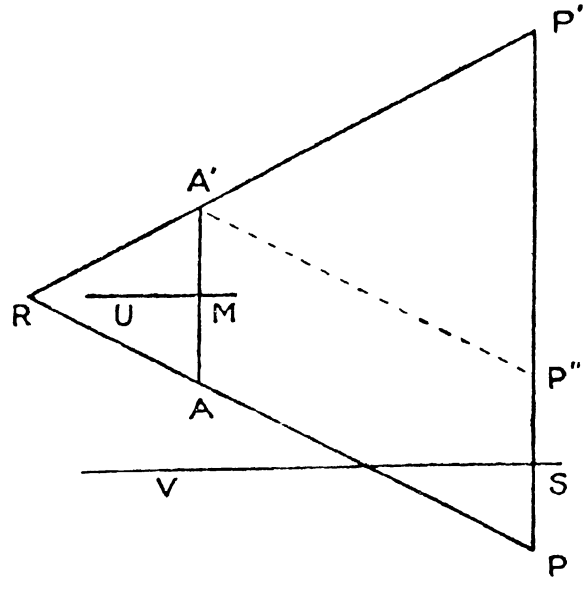



FIG. 34. SPECIAL CONSTRUCTION FOR CENTRE OF ROTATION.

Examples of the application of these principles will be found in connection with mechanisms treated in Chapter IX.

**97. Composition of Linear and Angular Velocities.**—Velocities, whether linear or angular, being the rates of increase of the corresponding displacements per unit time, the results obtained for the composition of translations and rotations still hold, but admit of some simplification when the velocities in question are instantaneous ones and not mean velocities merely.

Thus, referring to the end of article 94 and supposing that in Fig. 32 the line AA′ shrinks to the infinitesimal element $ds$ described in time $dt$, we have

$$ds/2 = \mathrm{AR}\,d\theta/2 \text{ and } \mathrm{MAR} = \pi/2 \text{ nearly.}$$

So, if the linear and angular velocities are respectively $v$ and $\omega$, we obtain

$$ds/dt = \mathrm{AR}\,d\theta/dt,$$

or

$$\mathrm{AR} = v/\omega.$$

That is, the resultant of an angular velocity $\omega$ about a given axis and a linear velocity $v$ in the perpendicular plane is an equal angular velocity $\omega$ about a parallel axis distant $v/\omega$, such distance being perpendicular to that of the linear velocity $v$ and in the direction shown in Fig. 32.

We see that if $v$ is in the positive direction along the axis of $y$ and

$\omega$ denotes the positive rotation in the plane of $xy$, that the distance AR is in *the negative* direction along the axis of $x$.

Conversely, any given angular velocity $\omega$ (say in the $xy$ plane) may be resolved into an equal angular velocity about a parallel axis distant $p$ (say positively along the $x$ axis), together with a linear velocity $v = +p\omega$, perpendicular to the axes of rotation and to the direction of $p$ (*i.e.* positively along the $y$ axis).

The student should carefully note that although he is provided with the power of compounding a translation and a rotation into an equal rotation about some other point, it is not advisable in every case to use it. On the contrary, it will sometimes be found preferable to solve a problem by directing attention to the specified components of translation and rotation especially if accelerations be required. (See EXAMPLES—XXII.)

**98. Composition of Angular Velocities about Fixed Parallel Axes.**—Let us now determine the resultant of two component angular velocities $\omega_1$ and $\omega_2$ about parallel axes which intersect the perpendicular plane at A and B a distance $p$ apart. Take a point C on the straight line joining AB, and consider the motions there. Then by the previous article we have

$\omega_1$ about A $=\omega_1$ about C and linear velocity $(+\omega_1.\mathrm{AC})$ perpendicular to AB,

also $\omega_2$ about B $=\omega_2$ about C and linear velocity $(-\omega_2.\mathrm{CB})$ perpendicular to AB.

Now let C be so chosen that the linear speeds there are equal as well as opposite. Then we have as the resultant motion an angular velocity about C of value

$$\omega = \omega_1 + \omega_2 \quad \ldots\ldots\ldots\ldots \quad (1),$$

C being defined by $\mathrm{AC}.\omega_1 = \mathrm{CB}.\omega_2$, or

$$\frac{\mathrm{AC}}{\omega_2} = \frac{\mathrm{CB}}{\omega_1} = \frac{\mathrm{AB}}{\omega_1 + \omega_2} \quad \ldots\ldots\ldots\ldots \quad (2).$$

And, if other points are taken in the plane, it will be found that the motions due to the resultant angular velocity about C is the resultant of the motions due to the component angular velocities about A and B.

It should be carefully noted that these results do *not* apply to *successive finite* angular displacements but to *simultaneous velocities.*

**99. Linear and Angular Accelerations.**—As, in article 97, we passed from linear and angular displacements to the corresponding velocities, so we may pass from velocities to accelerations, the same laws of composition and analysis still holding true.

**100. Analytical Treatment of Coplanar Motions.**—Let us now treat the coplanar motions of a rigid system or body analytically. Referring to Fig. 35, let the motions occur in the $xy$ plane, let $x'$, $y'$ be the co-ordinates of a definite point O′ in the body and $x$, $y$ those of any other point P. Take also axes O′X′ and O′Y′ meeting in O′, *fixed in the body and moving with it.* Also let P have co-ordinates $\xi$ and $\eta$ referred to these moving axes O′X′ and O′Y′, and at time $t$ let these axes

be inclined at the angle $\psi$ with the fixed axes OX and OY. Then we have from the figure

$$x=x'+\xi\cos\psi-\eta\sin\psi,\ y=y'+\xi\sin\psi+\eta\cos\psi.$$

Thus, by differentiation, remembering $\xi$ and $\eta$ are constant, we obtain

$$\left.\begin{aligned}\dot{x}&=\dot{x}'-(\xi\sin\psi+\eta\cos\psi)\omega'\\ \text{and}\quad \dot{y}&=\dot{y}'+(\xi\cos\psi-\eta\sin\psi)\omega'\end{aligned}\right\}\quad\ldots\ldots\quad(1),$$

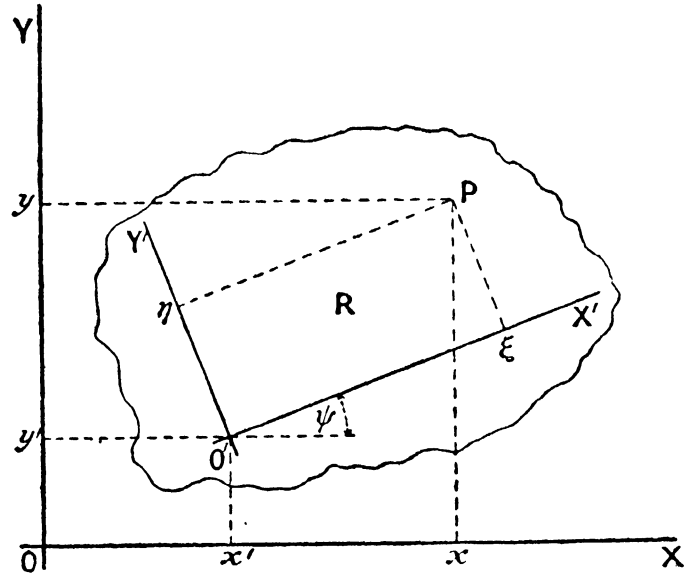



FIG 35. COPLANAR MOTIONS OF RIGID SYSTEM.

where $\omega'=d\psi/dt$=angular velocity of body about a perpendicular axis through the point $x'y'$. These may be put in the form

$$\dot{x}=\dot{x}'-(y-y')\omega' \text{ and } \dot{y}=\dot{y}'+(x-x')\omega'\quad\ldots\quad(2).$$

These show that the velocity of the point $xy$ is made up of two parts, viz. (i) one of translation expressed by $\dot{x}'$ and $\dot{y}'$, and (ii) one of rotation in the $xy$ plane about the point $x'y'$ with velocity $\omega'$, expressed by $-(y-y')\omega'$ along the $x$ axis and $+(x-x')\omega'$ along the $y$ axis.

For any other definite point O'' in the body, having co-ordinates $x''$, $y''$, we have similarly

$$\dot{x}=\dot{x}''-(y-y'')\omega'' \text{ and } \dot{y}=\dot{y}''+(x-x'')\omega''\quad\ldots\quad(3),$$

$\omega''$ being the angular velocity about the point $(x'', y'')$. Equating the values of $\dot{x}$ in (2) and (3) and those of $\dot{y}$, we obtain

$$\left.\begin{aligned}\dot{x}'&=\dot{x}''-(y'-y'')\omega''+(y-y')(\omega'-\omega'')\\ \text{and}\quad \dot{y}'&=\dot{y}''+(x'-x'')\omega''-(x-x')(\omega'-\omega'')\end{aligned}\right\}\quad\ldots\quad(4).$$

But, by analogy with (2) and (3), we see that

$$\left.\begin{aligned}\dot{x}'&=\dot{x}''-(y'-y'')\omega''\\ \text{and}\quad \dot{y}'&=\dot{y}''+(x'-x'')\omega''\end{aligned}\right\}\quad\ldots\quad(5).$$

Hence comparing (4) and (5) we see that

$$\omega''=\omega'=\omega \text{ say}\quad\ldots\quad(6).$$

Or, in words, the angular velocity is the same whatever the point through which the axis is supposed to pass.

The linear velocity of the axis, on the other hand, depends upon the position of that axis. Thus from (5) and (6) we have

$$\left.\begin{aligned} &\dot{x}'-\dot{x}''+(y'-y'')\omega=0 \\ \text{and}\quad &\dot{y}'-\dot{y}''-(x'-x'')\omega=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (7).$$

**101. Instantaneous Centre; Body and Space Centrodes.** — By putting $\dot{x}=\dot{y}=0$ in equation (1) we obtain the co-ordinates of the instantaneous centre of rotation R referred to the axes X'OY' fixed in the body. Thus, calling these co-ordinates $\xi_0$ and $\eta_0$, and remembering $\omega'=\omega$, we have

$$\left.\begin{aligned} &\xi_0 \sin\psi+\eta_0\cos\psi=\dot{x}'/\omega \\ \text{and}\quad &\xi_0\cos\psi-\eta_0\sin\psi=-\dot{y}'/\omega, \\ \text{whence}\quad &\xi_0=(\dot{x}'\sin\psi-\dot{y}'\cos\psi)/\omega \\ \text{and}\quad &\eta_0=(\dot{x}'\cos\psi+\dot{y}'\sin\psi)/\omega \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8)$$

Similarly, putting $\dot{x}=\dot{y}=0$ in (2) and writing $x_0$ and $y_0$ for the corresponding values of $x$ and $y$, we obtain the co-ordinates of the instantaneous centre R referred to the axes XOY with respect to which the body moves. Thus we have

$$x_0=x'-\dot{y}'/\omega \text{ and } y_0=y'+\dot{x}'/\omega. \quad \ldots\ldots\ldots \quad (9).$$

If $x'$, $y'$, and $\psi$ are known functions of the time $t$ (and therefore also $\dot{x}'$, $\dot{y}'$, and $\omega$), we could substitute and eliminate $t$ between the two equations (8), and thus determinate a relation between $\xi_0$ and $\eta_0$. This would be the equation of the curve described by the instantaneous centre of rotation with respect to the body. In other words, it would specify the *body centrode* already referred to in article 95.

Again, on substituting the known functions of $t$ in the two equations (9) and eliminating $t$ between them, we should have the equation of the curve described by the instantaneous centre with respect to the fixed axes XOY. In other words, we should obtain the *space centrode*.

**102. Summary of Coplanar Motions.**—The chief results we have obtained for the coplanar motions of a rigid body or system, together with others which easily follow from them, may now be summarised as follows:—

1. The motion of a rigid body parallel to a given fixed plane at any instant consists in the general case of—

(i) An angular velocity about an axis perpendicular to the plane and passing through any arbitrary point in the body, the magnitude $\omega$ of this velocity being independent of the position of the axis; and

(ii) A linear velocity $v$ parallel to the plane, the magnitude and direction of which are dependent on the position of the axis. (See articles 94, 96, and 97, also 100.)

2. At each instant there is, in general, an axis called the *instantaneous axis of rotation*, such that the whole motion of the body is one of rotation about it with angular velocity $\omega$. (The intersection of this axis with the plane in which the motion occurs or to which it is referred is called the *instantaneous centre of rotation.*)

In other words, a motion of rotation only with angular velocity $\omega$ about one point is equivalent to one of rotation with same angular

speed $\omega$ about another point $x$ distant, together with a linear velocity $x\omega$ parallel to the axis of $y$, *i.e.* perpendicular to that of $x$. (See articles 95 and 101.)

3. Two coexisting angular velocities round parallel axes fixed in space are equivalent to a single angular velocity equal to their algebraic sum about an axis parallel to and in the plane of the two former, and dividing the distance between them inversely as the component velocities. (See article 98.)

4. Two equal and opposite angular velocities whose common magnitude is $\omega$, round parallel axes $p$ apart, are equivalent to a linear velocity $p\omega$, perpendicular to the plane of the axes. (See equation (2) of article 98.)

## EXAMPLES—XXII.

1. 'Prove that any displacement of a plane figure in its own plane is equivalent to a rotation about some finite or infinitely distant point.
'If the figure be rotated through 90° about a fixed point $A$, and then through 90° (in the same sense) about a fixed point $B$, the result is equivalent to a rotation of 180° through a certain fixed point $C$; find the position of $C$.'
(LOND. B.SC., PASS, APPLIED MATH, 1905, II. 4.)

2. 'Prove that when a lamina is moving in its own plane there is in general one point of it which is instantaneously at rest.
'A rod moving in a vertical plane with one end on the ground remains constantly in contact with a small peg. Construct geometrically the tangent to the path of any point of the rod; and show that, when the inclination of the rod to the vertical is $a$, the velocity of the point which is moving vertically is to the velocity of the point which is moving horizontally in the ratio of tan $a$ to unity.'
(LOND. B.SC., PASS, APPLIED MATH., 1907, II. 3.)

3. 'Es giebt fur jede Bewegung eines Systems in einer Ebene immer zwei bestimmte Curven oder Polbahnen, eine feste in der Bewegungsebene und eine in der beweglichen Ebene, welche beide den Pol enthalten.
'Translate, prove, and give illustrations of the proposition.'
(LOND. B.SC., PASS, APPLIED MATH., 1906, II. 7.)

4. 'Explain what is meant by relative velocity and relative acceleration.
'A circle of radius $a$ rolls on a fixed horizontal straight line with velocity $v$ and acceleration $f$. Find the accelerations of the highest and lowest points of the circle.'
(LOND. B.SC., PASS, APPLIED MATH., 1907, II. 2.)

5. 'Translate and prove the following statements:—
'Jede Bewegung einer ebenen Figur auf einer festen Ebene von der Stellung $S$ auf Zeit $t$ zu der Stellung $S'$ auf Zeit $t'$ ist einer einzigen Drehung gleichwertig. Die Bewegung einer solchen Figur auf einer Ebene kann sich als Rollen einer zur beweglichen Figur gehörigen Kurve auf einer festen Kurve darstellen.'
(LOND. B.SC., PASS, APPLIED MATH., 1908, II. 3.)

6. 'Explain the terms (i) angular velocity, (ii) instantaneous centre, in the case of a lamina moving in its own plane.
'The centre of a disc falls vertically with constant acceleration, while the disc rotates in its own plane (which is vertical) with constant angular velocity. Prove that the locus in space of the instantaneous centre is a parabola.'
(LOND. B.SC., PASS, APPLIED MATH., 1909, II. 1.)

7. The centre of a lamina moves in its own plane with constant linear speed $u$, while it rotates in its own plane with uniform acceleration $a$. Show that the space centrode is the rectangular hyperbola $xy = u^2/a$.
8. A ladder of length $L$ rests on horizontal ground and leans against a vertical wall. Show that as it moves parallel to a plane perpendicular to the intersection of ground and wall, its space centrode is a quadrant of radius $L$ and its body centrode a semicircle of diameter $L$.
9. A joiner's square is moved in a plane with its blade in contact with one fixed circle and its head or stock in contact with another fixed circle. Show that the space and body centrodes are similar curves, the former having double the curvature of the latter.
10. Show how to compound velocities about fixed axes parallel or inclined.
11. Discuss analytically the subject of coplanar motions.

# CHAPTER VII

## SOLID MOTIONS OF A POINT

**103. Solid Co-ordinates.**—To represent the position of a point in space of three dimensions, in addition to the rectangular co-ordinates $x$ and $y$, we may use another, $z$, parallel to OZ and therefore perpendicular to XOY, and thus expressing the perpendicular distance of the point from the $xy$ plane. We are then adopting the *solid cartesian* system of co-ordinates, which is the one most frequently employed.

Taking another view of the matter, we see that this system defines a point as the intersection of three planes respectively perpendicular to the three axes of co-ordinates. For to say that the co-ordinates of a point are $a$, $b$, and $c$ is equivalent to saying that it is on each of the three planes $x=a$, $y=b$, and $z=c$, *i.e.* on the planes parallel to YOZ, ZOX, and XOY, and distant from them by the respective amounts $a$, $b$, and $c$.

Obviously any three intersecting surfaces would define a point, and other systems of co-ordinates are founded upon this principle. Thus, on the *polar* system of solid co-ordinates a point is specified as the intersection of a sphere, cone, and axial plane; on the *cylindrical* system a point is given as the intersection of a cylinder with planes through and perpendicular to its axis. But these systems will be but little used in this work; we leave the further consideration of them to the instances of their occurrence.

**104. Right-handed System.**—Reverting now to the cartesian system of co-ordinates, we have still to decide which way to draw the axis of $z$ with respect to those of $x$ and $y$. In this work we shall adopt the *right-handed* system, as uniformly adopted by Lord Kelvin for sixty years (see *Baltimore Lectures*, pp. 445-6), and as advocated and used by Maxwell (*Electricity and Magnetism*, 1873, § 23). In this system 'the motions of translation along any axis and of rotation about that axis' are 'assumed to be of the same sign when their directions correspond to those of the translation and rotation of an ordinary or right-handed screw.

'For instance, if the actual rotation of the earth from west to east is taken positive, the direction of the earth's axis from south to north will be taken positive, and if a man walks forward in the positive direction, the positive rotation is in the order, head, right-hand, feet, left-hand.

'If we place ourselves on the positive side of a surface, the positive direction along its bounding curve will be opposite to the motion of the hands of a watch with its face towards us.'

Hence if (as usually seems to the writer most natural and convenient) the axis of $x$ is drawn horizontally to the right, and that of $y$ vertically upward on the blackboard, just as in plane geometry, then the axis of $z$ must be supposed to project from the board towards the spectator, and may be represented in perspective accordingly. Also the positive direction of rotation, as always used in trigonometry, from $x$ to $y$, or counter-clockwise, corresponds on this right-handed system with the positive direction of advance along the axis of $z$. Of course, the axes may be turned about into any convenient position provided their mutual relation be preserved. 'Thus, if $x$ is drawn eastward and $y$ northward, $z$ must be drawn upward.' These two positions are illustrated in Fig. 36. Of course, they may be shown in other ways to suit the case in hand. It should be noted that the positive direction of rotation in any co-ordinate plane is that which corresponds with the cyclical order of the letters indicating the axes in

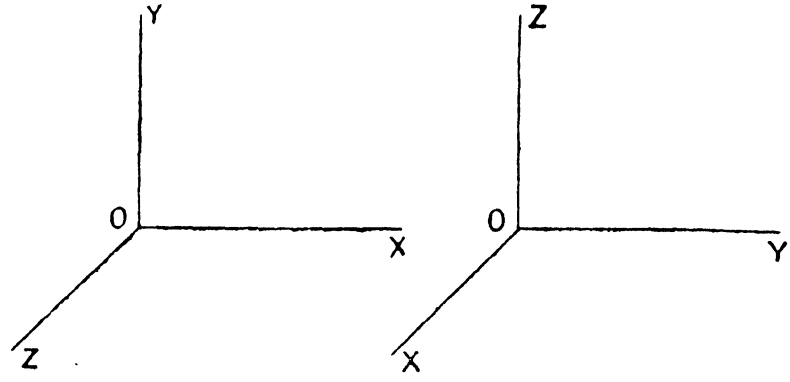



FIG. 36. RIGHT-HANDED CO-ORDINATE AXES.

that plane, as YZ, ZX, XY. Also translation along the positive direction of the remaining axis OX, OY or OZ in each case bears the right-handed relation to that rotation.

**105. The Straight Line.**—If a straight line pass through two points $(a, b, c)$ and $(x, y, z)$ distant $r$ apart and make with the co-ordinate axes angles whose cosines are $l$, $m$, and $n$ respectively, it may easily be seen that we have

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=r \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and

$$l^2+m^2+n^2=1 \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Equation (1) is useful as giving the distance between two points whose co-ordinates are known, (2) gives the equations of a straight line in space of three dimensions, $(x, y, z)$ being the current co-ordinates, while (3) gives an important relation between what are called the *direction cosines* of the line.

This brief notice will suffice to introduce the system of solid geometry and render what follows intelligible. For further information recourse must be had to the text-books on the subject.

EXAMPLES.—XXIII.

1. State the relations between the three rectangular cartesian axes which form a right-handed system, and sketch such a set in various positions.
2. Show in perspective the points P and Q, having the co-ordinates (1, 1, 1) and (4, 3, 2) respectively.
3. Write the equations of the lines OP, OQ, and PQ, O being the origin and P and Q as defined in question 2.
4. Find an expression for the angle $\theta = POQ$ as defined by questions 2 and 3 (see Chapter II.).

**106. Cylindrical Motion.**—Consider the motion of a point P on the curved surface of a right circular cylinder of radius $c$ with axis along the axis of $z$, and let $\theta$ represent the angle between the fixed plane ZOX through the axis of $z$ and another plane containing that axis and the point P. This is shown in Fig. 37.

Let us note expressions for the velocities and accelerations as the point moves in any way on the cylindrical surface.

The velocities parallel to the axis, to the tangent to the base, and to its radius, *i.e.* parallel respectively to PW, PT, and PC, are easily seen to be

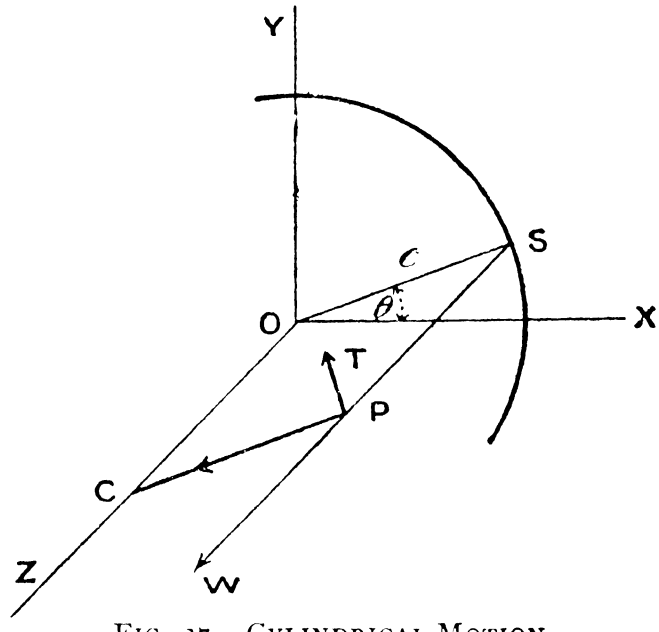



FIG. 37. CYLINDRICAL MOTION.

$$\dot{z},\ c\dot{\theta}, \text{ and zero} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The accelerations in the same directions are respectively

$$\ddot{z},\ c\ddot{\theta}, \text{ and } -c\dot{\theta}^2 \text{ (see articles 69 and 71)} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus, the magnitude of the resultant or total velocity $v$ is given by

$$v^2 = \dot{z}^2 + c^2\dot{\theta}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Also, that of the resultant or total acceleration $a$ is given by

$$a^2 = \ddot{z}^2 + c^2\ddot{\theta}^2 + c^2\dot{\theta}^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

It is also evident that the quotient (any component by resultant) gives the cosine of the angle between the two; hence the direction is determined.

**107. Conical Motion.**—Consider now the motion of a point P on the surface of a right circular cone of semi-vertical angle $\alpha$, let P s

distance from the vertex be $r$, and let the axial plane containing P make an angle $\theta$ with a fixed axial plane. This arrangement is represented in Fig. 38, in which the axis of the cone is ONG, the fixed axial plane ONA and that through P contains Q also, PNOQ being a rectangle. PT is a tangent to the right section of the cone at P.

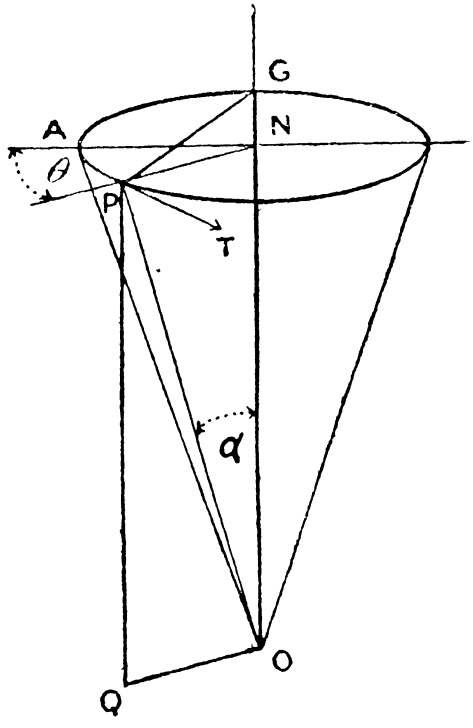



FIG. 38. CONICAL MOTION.

Leaving the expression of the velocities as an exercise for the student, let us consider the accelerations in various directions.

Thus, the *axial* acceleration along QP parallel to ON is

$$\frac{d^2}{dt^2}\mathrm{ON}=\frac{d^2}{dt^2}(r\cos\alpha)=\ddot{r}\cos\alpha\ .\quad(1).$$

The *tangential* or *transversal* acceleration along PT perpendicular to the plane ONP is, by equation (6) of article 74,

$$r\sin\alpha.\ddot{\theta}+2\dot{r}\sin\alpha.\dot{\theta}=\sin\alpha(r\ddot{\theta}+2\dot{r}\dot{\theta})(2).$$

The *radial* acceleration along NP is that of Q along OQ, so by equation (5) of article 74 is expressed by

$$\ddot{r}\sin\alpha-r\sin\alpha.\dot{\theta}^2\quad\ldots\quad(3).$$

To find the acceleration along the generator OP, multiply equation (1) by cos $\alpha$ and equation (3) by sin $\alpha$ and add, for each of these products gives a component acceleration along OP. Hence, acceleration along the *generator* OP is

$$\ddot{r}-r\sin^2\alpha.\dot{\theta}^2\quad\ldots\ldots\ldots\quad(4).$$

Again, to find the acceleration perpendicular to a generator and outwards in the axial plane, we have to multiply equation (3) by cos $\alpha$ and take from this the product of equation (1) by sin $\alpha$. Thus we have acceleration along the *normal* GP is

$$-r\sin\alpha\cos\alpha.\dot{\theta}^2\ .\ .\ .\ .\ .\ .\quad(5).$$

**108. Spherical Motion.**—In dealing with conical motion the semi-vertical angle $\alpha$ was constant, while the distance $r$ from the vertex was variable. It is obvious that if we substitute for $\alpha$ the variable angle $\phi$ and for $r$ the constant length $a$, we shall have passed from conical motion to that on the surface of a sphere of radius $a$. These co-ordinates defining the position of a point on the sphere are shown in Fig. 39, also the lines parallel to which we shall consider the accelerations. In this diagram ARBOQ represents the horizontal diametral plane of the sphere, ONCG the vertical axis. The variable plane OCPR makes the angle $\theta$ with the fixed plane OCA, while OP makes the angle $\phi$ with OC. PQ and PN are perpendicular respectively to OR and OC, while PT is perpendicular to the plane OCPR containing

the four lines just mentioned, and PG in the plane OCP is perpendicular to OP. The lines OA, OB, OC, OP, and OR being all radii of the sphere are each of length $a$.

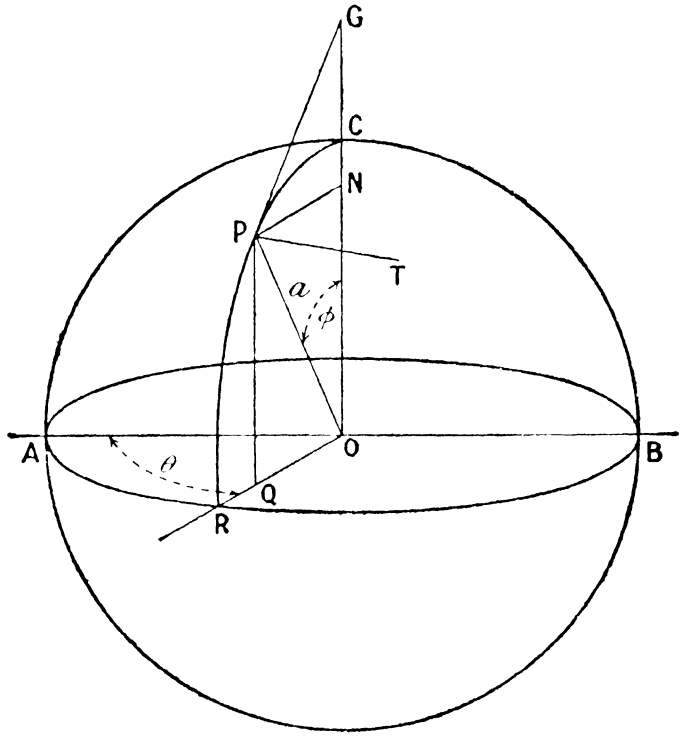



FIG. 39. SPHERICAL MOTION.

Consider now the motion of the point P on the surface of the sphere, its position being defined by $\theta$ and $\phi$. Then adopting what we have just established for conical motion, we easily obtain from the figure the accelerations in three directions at right angles, viz. ON, NP, and PT. Thus, the acceleration parallel to ON

$$= \frac{d^2}{dt^2}(a\cos\phi) = \frac{d}{dt}(-a\sin\phi.\dot{\phi}) = -a\cos\phi.\dot{\phi}^2 - a\sin\phi.\ddot{\phi} \quad (1).$$

The acceleration parallel to NP

$$= \frac{d^2}{dt^2}(a\sin\phi) - a\sin\phi.\dot{\theta}^2 = \frac{d}{dt}(a\cos\phi.\dot{\phi}) - a\sin\phi.\dot{\theta}^2$$

$$= -a\sin\phi.\dot{\phi}^2 + a\cos\phi.\ddot{\phi} - a\sin\phi.\dot{\theta}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The acceleration parallel to PT, *i.e.* perpendicular to OCPR (see equation (6) of article 74)

$$= a\sin\phi.\ddot{\theta} + 2(a\cos\phi.\dot{\phi})\dot{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Equation (2) multiplied by cos $\phi$, less equation (1) multiplied by sin $\phi$, gives the acceleration parallel to GP, which

$$= a\ddot{\phi} - a\sin\phi\cos\phi.\dot{\theta}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Lastly, the sum of the products, equation (1) multiplied by cos $\phi$ and (2) by sin $\phi$ gives the acceleration parallel to OP

$$= -a\dot{\phi}^2 - a\sin^2\phi.\dot{\theta}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

On replacing the constant radius $a$ by a variable, and differentiating

it when necessary, these results could be extended *to any* motion in space of three dimensions.

### EXAMPLES—XXIV.

1. Obtain general expressions for the accelerations in the principal directions for a point moving on the surface of—
   A cylinder;
2. A cone;
3. A sphere.
4. A point moves with constant speed $v$ on the surface of a right circular cylinder of radius $a$ and traces out a regular helix whose axial advance per radian is $p$. Find the velocities and accelerations of the point radially, tangentially, and axially.
5. A circular wire 10 cms. in radius is in a vertical plane and rotates with constant velocity 3 radians per second about its vertical diameter; at a certain instant a bead on the wire is 30° from the top of the circle, and is moving downwards along the wire with speed 4 cm./sec. and acceleration 1 cm./sec.$^2$ Find the vertical and radial accelerations of the bead.

**109. Spherical Motions under Uniform Acceleration.**—The full treatment of this problem is beyond the scope of the present work; it must accordingly suffice to indicate here the chief features of the motion executed under these conditions. It is easily seen to be the general case in three dimensions of which the simple pendulum was the special case in a vertical plane. The present problem is, therefore, often known as the *spherical pendulum*. Referring to Fig. 40, we take the uniform acceleration $g$ to be vertically downwards parallel to OCG. The notation and lettering are as in Fig. 39, but the figure is now inverted. Since the particle P is supposed constrained to remain on the spherical surface, it has no motion normal to the surface, and we are therefore concerned chiefly with its component accelerations in two directions *tangential* to that surface at P. It is convenient to take these directions in the horizontal and vertical planes through P, thus giving as the tangents PT and PG respectively. The vertical acceleration at P being along PM has clearly no component in the perpendicular direction PT. We can accordingly equate to zero the corresponding expression. Thus from equation (3) of article 108 we have

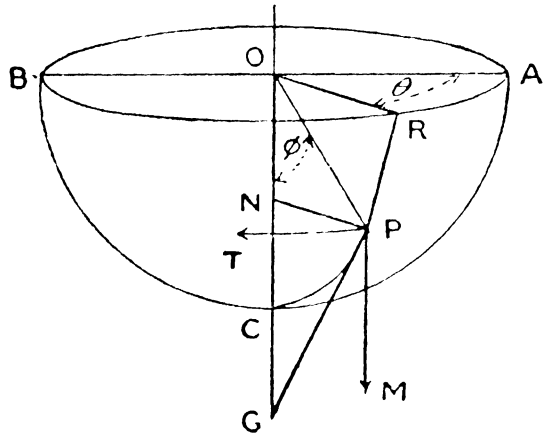



FIG. 40. SPHERICAL PENDULUM.

$$a \sin \phi . \theta + 2a \cos \phi . \dot{\phi} \dot{\theta} = 0,$$

where $a$ is the radius of the sphere. If now we write $NP = r = a \sin\phi$, the above equation becomes

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\cdot\frac{d}{dt}(r^2\dot{\theta}) = 0.$$

Hence
$$r^2\dot{\theta} = a^2 \sin^2\phi.\dot{\theta} = h \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $h$ is double the areal velocity of the horizontal radius vector NP about the vertical axis of the sphere OC.

We have now to consider the component acceleration along PG, the other tangent. It is clearly $g \cos MPG = g \sin\phi$. We may accordingly change its algebraic sign and equate to the expression for the acceleration along GP given in equation (4) of article 108. We thus have

$$a\ddot{\phi} - a \sin\phi \cos\phi.\dot{\theta}^2 = -g \sin\phi.$$

Multiplying by $2a d\phi$ and integrating, this becomes

$$2a^2\int\frac{d\phi}{dt}d\left(\frac{d\phi}{dt}\right) + 2a^2\dot{\theta}^2\int\sin\phi\, d(\sin\phi) = 2ga\int d(\cos\phi),$$

or
$$a^2\dot{\phi}^2 + a^2 \sin^2\phi.\dot{\theta}^2 = C + 2ga\cos\phi,$$

where $C$ is the integration constant. By use of (1) this may be written in the more useful form

$$a^2\dot{\phi}^2 + \frac{h^2}{a^2 \sin^2\phi} = C + 2ga\cos\phi \quad . \quad . \quad (2).$$

Equations (1) and (2), giving $\ddot{\phi}$ and $\dot{\theta}$ in terms of $\phi$ and constants, express the relations that must be fulfilled at every instant during the motion.

To obtain a statement of the motion in any given case we must now insert the initial conditions. Thus at $t = 0$ let the particle be at

$$\phi_0 = \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

with no vertical motion, so that $\phi$ is not changing, or

$$\dot{\phi}_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and with a horizontal speed $u$. Thus, if the angular speed in the horizontal plane about OC is then $\dot{\theta}_0$, we have $u = a \sin\alpha\, \dot{\theta}_0$, or, from (1),

$$h = ua \sin\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

Thus, substituting (3), (4), and (5) in (2), we obtain

$$u^2 = C + 2ga\cos\alpha \quad . \quad . \quad . \quad . \quad (6),$$

which defines $C$ in terms of the initial circumstances. Inserting this value of $C$ in equation (2), and using (5) also, we have

$$a^2\dot{\phi}^2 + u^2\left(\frac{\sin^2\alpha}{\sin^2\phi} - 1\right) = 2ga(\cos\phi - \cos\alpha) \quad (7),$$

expressing $\ddot{\phi}$ in terms of $\phi$ and the initial state of things.

To find the highest and lowest places, put $\ddot{\phi} = 0$ in (7), when we see that either $\phi = \alpha$ as in (3), or else we may reduce the equation to the forms

$$u^2\frac{\sin^2\alpha - \sin^2\phi}{\cos\phi - \cos\alpha} = 2ga \sin^2\phi$$

and
$$u^2(\cos\phi + \cos\alpha) = 2ga(1 - \cos^2\phi) \quad . \quad (8),$$

from the latter of which the factor $(\cos\phi - \cos\alpha)$ is removed.

It accordingly gives other values for $\cos\phi$. It is easy to see that equation (8) gives one value for $\cos\phi$ lying between $-1$ and $+1$. Thus, for example, if we make $\cos\phi$ change continuously from $-1$ to $+1$, the left side of (8) changes continuously from the negative value $-u^2(1-\cos\alpha)$ to the positive value $+u^2(1+\cos\alpha)$. The right side of (8) for the same range of values of $\cos\phi$ changes from zero to the maximum positive value $2ga$ and falls again to zero. Hence for some value of $\cos\phi$ between $\pm 1$ the equation is satisfied. That is to say, there is some value, $\beta$ say, of $\phi$ other than $\alpha$, for which, in equation (7), the angular velocity $\dot{\phi}$ vanishes. Hence the whole motion of P lies between the two horizontal circles on the sphere defined by $\alpha$ and $\beta$.

**110. Alternative View of Spherical Pendulum.**—A useful insight into the character of the motion by a spherical pendulum is obtained by the following alternative view of the matter:—Take two vertical planes at right angles to each other through the centre of the sphere as shown by AOA'C and BOB'C in Fig. 41. Now suppose a pendulum suspended at O and of length $l$ (the radius of the sphere) to perform simultaneously oscillations of amplitudes CP and CQ in these two planes respectively. Also suppose that both these amplitudes, though in the figure shown large for clearness' sake, are really so small that the period of vibration is in each case represented with sufficient approximation by the simple expression given in article 53, viz.

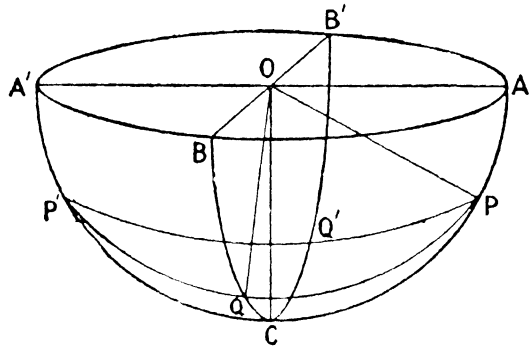



FIG. 41. SPHERICAL PENDULUM BY RECTANGULAR VIBRATIONS.

$$\tau = 2\pi\sqrt{l/g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

On these assumptions we may regard the actual motion of P on the spherical surface as represented with close approximation by the resultant of two rectangular simple harmonic vibrations of equal periods. But this is an ellipse, in the general case which we now require (as was shown at the end of article 67). Further, since this ellipse, or *quasi-ellipse*, is to be traced on a spherical surface with its centre C at the lowest point on the sphere, it is evident that the ends PP' of the major axis will be higher than QQ', the ends of the minor axis. Hence the special extreme values $\alpha$ and $\beta$ of $\phi$ found in article 109 correspond to the angles COP and COQ in Fig. 41.

**111. Rotation of Quasi-Ellipse.**—Let us now proceed to a further approximation, and suppose that while the amplitude CQ is still small and the period $\tau$ represented by (9), that the amplitude CP is so large that the period of the vibration in this plane has a period

$$\sigma > \tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

$\sigma$ being the value of the period for finite arcs given by the analysis of articles 54-56. To consider the effect of this on the motion, take a plan of the quasi-ellipse described. This is shown by the broken line in Fig. 42, the point moving round in the clockwise direction as indicated by the arrows. Suppose the moving point to start from P, then when it next approaches that position it will have completed its vibration in the vertical plane through QCQ′ in the time $\tau$, *a little earlier* than the instant at which it will have completed its vibration in the vertical plane through PCP′, for this takes the slightly greater period $\sigma$. The moving point accordingly crosses the line PCP′ at R a little inside P, and reaches its maximum elongation at S a little on the Q-side of P, as shown by the full line in Fig. 42. In other words, the quasi-ellipse *rotates* about the vertical axis of the sphere in the *direction of its description* by the moving point. Further, the angle PCS, whose magnitude depends partly on $(\sigma - \tau)$, is itself described in the time $\sigma$ nearly. Thus the curve is not immediately re-entrant, but forms a series of loops in radial symmetry about the centre C.

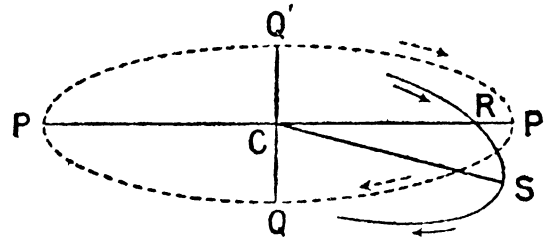


FIG. 42. ROTATION OF QUASI-ELLIPSE.

It can be shown by higher analysis that the rate of this rotation of the quasi-ellipse is proportional to its area. By an elementary view of the matter it is at once seen that the speed of rotation is increased by an increase of the larger amplitude CP, since this augments the excess of the periods $\sigma - \tau$, thus shifting R and S each farther from P. The increased speed of rotation when the amplitude CQ is increased, and therefore R shifted nearer P, is accounted for by the fact that the ellipse is now broader at the ends, and S is still shifted farther from P in spite of the approach of R.

Of course, when the quasi-ellipse becomes a horizontal circle the rotation of the figure ceases to be of consequence, so the two views are harmonised.

The rotation vanishes only when the area of the ellipse vanishes, *i.e.* when the motion is initially confined to a vertical plane. And this accords with our elementary treatment of the problem.

The phenomenon of the rotation of this quasi ellipse in the direction of its description may be easily noticed in the case of a plummet suspended from a fixed point and properly started. With a pendulum started truly in a vertical plane, Foucault showed by the apparent shift of that plane that the earth was rotating.

**112. Accelerations of a Point moving in any Curve.**—A plane curve has at each point a tangent, and a normal in that plane and a radius of curvature along that normal. But what is called a *tortuous* curve has also a change of its plane by rotation about the tangent; it therefore extends to three dimensions in space, and is the general

example of a curve. The *tortuosity* of a curve is measured by the rate of rotation of the plane about the tangent per unit length along the curve. Thus, if this angle is denoted by $\phi$, the length along the curve by $s$, and the tortuosity by $\sigma$, we have

$$\sigma = d\phi/ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

In this general type of curve the normal in the plane of the curve is called the *principal normal*, and that at right angles to that plane is called the *binormal*.

Let us now find expressions for the accelerations along the tangent, the principal normal, and the binormal, these three directions being at right angles to each other. Suppose the moving point P to have the cartesian co-ordinates $x$, $y$, and $z$. Then we have

$$\frac{dx}{dt} = \frac{dx}{ds} \cdot \frac{ds}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus, differentiating again, we find for the acceleration parallel to the $x$ axis

$$\frac{d^2x}{dt^2} = \frac{d^2s}{dt^2} \cdot \frac{dx}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2x}{ds^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Similarly for the other accelerations, we have

$$\frac{d^2y}{dt^2} = \frac{d^2s}{dt^2} \cdot \frac{dy}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2y}{ds^2} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and

$$\frac{d^2z}{dt^2} = \frac{d^2s}{dt^2} \cdot \frac{dz}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2z}{ds^2} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

But

$$dx/ds, \; dy/ds, \text{ and } dz/ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

are the direction-cosines of the tangent; also if $\rho$ is the radius of curvature at P, then it may be shown that

$$\rho d^2x/ds^2, \; \rho d^2y/ds^2, \text{ and } \rho d^2z/ds^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

are the direction-cosines of the principal normal. Hence equations (3), (4), and (5), interpreted by the expressions (6) and (7), show that the resultant acceleration of the point P is compounded of the acceleration

$$d^2s/dt^2 \text{ along the tangent} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 \text{ in the direction of the principal normal} \quad . \quad (9).$$

Thus the acceleration along the binormal is zero, as there is no other component left over from equations (3), (4), and (5).

## EXAMPLES—XXV.

1. Show that a point under uniform acceleration but constrained to remain on the surface of a sphere describes a spherical *quasi*-ellipse provided the displacements are all small.
2. In the case of a point describing under uniform acceleration a very elongated path on a spherical surface, show that this quasi-ellipse, instead of being re-entrant, rotates slowly in the sense of description.
3. For a point describing any curve obtain general expressions for its accelerations along the tangent, the principal normal, and the binormal.

# CHAPTER VIII

## SOLID ROTATIONAL MOTIONS

**113. Displacements with One Point Fixed: Euler's Theorem.**—If a rigid body move with one point O fixed, it is evident that a spherical surface S fixed in the body will move on a concentric spherical surface fixed in space, the common centre of the spheres being the fixed point O. Thus, the position of the body is specified by stating the positions on the fixed spherical surface of two points, A and P say, on the spherical surface S fixed in the body, and therefore moving with it. Thus the most general displacement possible to a rigid body when it has one point fixed may be described as the displacement of the two points from the positions A and P to A′ and P′ respectively, all four points being on the fixed spherical surface. The statement that this most general displacement is expressible as a *determinate angular displacement about a determinate axis* constitutes the theorem due to Euler and revived by Rodrigues.

To establish it let the points A, A′, P, and P′ be as shown in Fig. 43, O being the common centre of the fixed and movable spheres. Draw the great circles AA′ and PP′, bisect AA′ at M and PP′ at N, through M and N draw the great circles MR and NR perpendicular respectively to AA′ and PP′, and intersecting at R; join OR, and draw the great circles RA and RA′. Then the rotation or angular displacement ARA′ about the axis OR shall bring the body from the position defined by AP to that defined by A′P′. For on joining R to P and P′ by great circles, we see that the spherical triangles RAP, RA′P′ have all their similarly lettered sides equal, each to each; thus their angles are equal, and the angle ARP equals the angle A′RP′. To each of these equals add the angle PRA′. Then the angles ARA′ and PRP′ are seen to be equal. Hence the angular displacement ARA′ about the axis OR which carries A to A′ also carries P to P′.

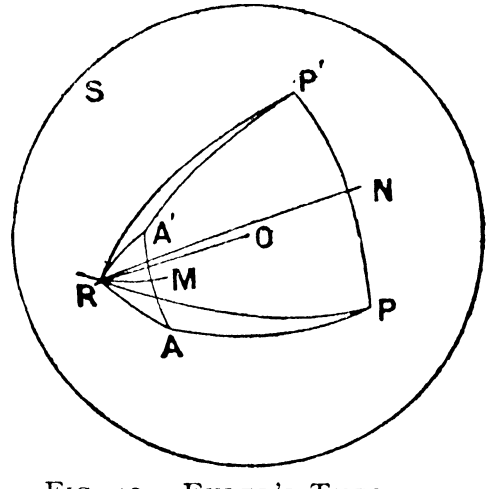


FIG. 43. EULER'S THEOREM.

**114.** If the displacements AA′ and PP′ are very small in comparison with OR, the radius of the sphere, then all the spherical triangles

involved become practically plane. The proposition then reduces to a combined translation AA′ and rotation through the angle ARA′ of a plane figure moving in its own plane. The student may accordingly compare Fig. 43 of the present article with Fig. 33 of article 96.

If AA′ and PP′ are parallel the preceding construction obviously fails, but the intersections of PA and P′A′ each produced give as their intersection the point R required, as in the analogous case in article 96. But, unlike that case of plane motion, the great circles through PA and P′A′ will *always intersect*, so that Euler's theorem holds *without exception.*

Another view of the matter is to represent the whole displacement as the resultant of two angular displacements about different axes meeting in O. Thus, first rotate the body through the angle AOA′ about an axis through O and perpendicular to the plane AOA′. This would bring A to A′ and P to some point, $p$ say, on the surface of the sphere; and A′$p$ would equal AP, and therefore equal A′P′. Thus a second rotation about OA′ through the angle $p$A′P′ would bring $p$ to P′, and accordingly complete the specified displacement. This shows incidentally that two angular displacements about intersecting axes are equivalent to a resultant angular displacement about another axis through the intersection of the former axes. The composition of angular displacements will be referred to again in article 116.

**115. Rodrigues' Co-ordinates.**—The axis OR of resultant rotation in Euler's theorem may be defined by its direction cosines $\lambda$, $\mu$, and $\nu$, referred to fixed axes OX, OY, and OZ, and the amount of rotation or angular displacement about OR may be denoted by $\chi$. Then the four quantities $\lambda$, $\mu$, $\nu$, and $\chi$ are called *Rodrigues' co-ordinates.* They are, of course, reducible to three by the relation

$$\lambda^2+\mu^2+\nu^2=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Take a sphere of unit radius with its centre at O as shown in Fig. 44, and let the intersection of the fixed axes and OR with its surface be the points X, Y, Z, and R. Also let three rectangular lines OA, OB, and OC moving with the body coincide in its initial or zero position with OX, OY, and OZ respectively.

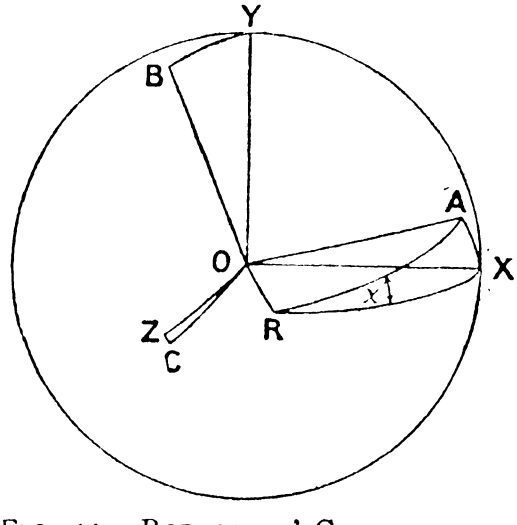



FIG. 44. RODRIGUES' CO-ORDINATES.

Let it be required to express Rodrigues' co-ordinates in terms of the final positions of OA, OB, and OC. Since the sphere is of unit radius the angle subtended at the centre by any arc of a great circle is equal to that arc and, for the sake of brevity, may be accordingly denoted by it. Thus the angle XOA may be referred to as XA simply, and so for all other angles and arcs

The final positions of OA, OB, and OC may be denoted by their direction cosines, nine in number. But these nine quantities are reducible to three independent ones. For we have three relations between the direction cosines of the form

$$\cos^2 XA + \cos^2 YA + \cos^2 ZA = 1 \quad . \quad . \quad . \quad . \quad (2),$$

the other two referring in like manner to the lines OB and OC. Then, because the three lines are always at right angles, we have three equations of the form

$$\cos BC = \cos XB \cos XC + \cos YB \cos YC + \cos ZB \cos ZC = 0 \quad . \quad (3),$$

the other two referring to the angles between the pairs of lines OC, OA and OA, OB.

Thus it will suffice to express $\lambda$, $\mu$, $\nu$, and $\chi$ in terms of any three of the nine direction cosines, provided those three define the final positions of OA, OB, and OC as shown in Fig. 44.

Referring to the figure, we see that in the spherical triangle XRA, the angle $XRA = \chi$, and the sides containing this angle RX and RA are equal, since

$$\lambda = \cos RX = \cos RA.$$

But it is shown in spherical trigonometry that in any spherical triangle $ABC$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

the small letters $a, b, c$ denoting the *sides* of the triangle and the corresponding large letters the opposite angles.

Thus, applying this relation to the isosceles triangle XRA, we obtain

$$\cos XA = \cos^2 RX + \sin^2 RX \cos \chi,$$

or

$$\cos XA = \lambda^2 + (1 - \lambda^2) \cos \chi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

We may then by symmetry write the two corresponding equations

$$\cos YB = \mu^2 + (1 - \mu^2) \cos \chi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

$$\cos ZC = \nu^2 + (1 - \nu^2) \cos \chi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Adding equations (4), (5), and (6), and using equation (1), we find

$$2 \cos \chi = \cos XA + \cos YB + \cos ZC - 1 \quad . \quad . \quad . \quad (7).$$

Then, using this in (4), (5), and (6) in succession, we have

$$\left.\begin{aligned} K\lambda^2 &= 1 + \cos XA - \cos YB - \cos ZC \\ K\mu^2 &= 1 - \cos XA + \cos YB - \cos ZC \\ K\nu^2 &= 1 - \cos XA - \cos YB + \cos ZC \end{aligned}\right\} \quad . \quad . \quad (8),$$

where

$$K = 3 - \cos XA - \cos YB - \cos ZC \quad . \quad . \quad . \quad (9).$$

Thus (7), (8), and (9) give $\chi$, $\lambda$, $\mu$, and $\nu$ in terms of the cosines of the angles between the original and final directions of the three axes OA, OB, and OC, and so solve the problem.

**116. Successive Finite Angular Displacements: Rodrigues' Theorem.**—A body has two angular displacements, *first*, about an axis OA through an angle $\alpha$; and *second*, a *subsequent* displacement about an axis OB through an angle $\beta$, and both these axes are *fixed in space*. It is required to compound these angular displacements or rotations.

The above theorem, being due to Rodrigues, is called after his

name by Routh, whose treatment is followed here with but little modification.

Let the axes OA, OB be drawn from O so that their directions correspond on the right-handed system with the directions of the rotations $\alpha$ and $\beta$ about those axes. And let A and B lie on the surface of a sphere whose centre is at O, as shown in Fig. 45.

On the surface of this sphere draw the great circle AB, and on the sphere make the spherical angle BAC equal to $\alpha/2$ and in a direction *opposite* to that of the rotation $\alpha$ round OA. Also make the angle ABC equal to $\beta/2$ and in the *same* direction as the rotation $\beta$ about OB, and let the great circles AC and BC intersect at C. Further, make the angles BAC′, ABC′ respectively equal to BAC, ABC but on the other side of AB, as shown by broken lines on the figure, and join OA, OB, OC, and OC′.

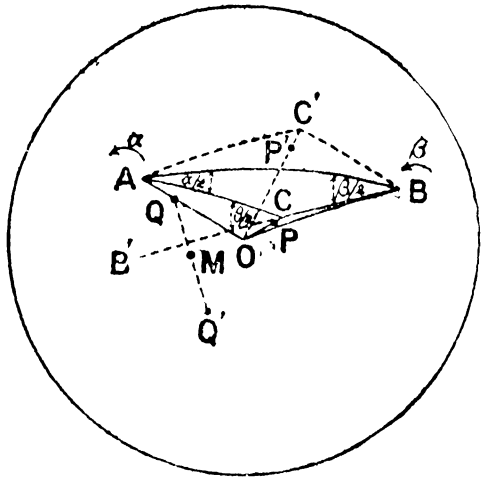



FIG. 45. RODRIGUES' THEOREM.

The rotation $\alpha$ round OA will then, by construction, carry any point P in OC into the straight line OC′, and the subsequent rotation $\beta$ about OB will carry the point P back into its original position in OC. Thus the points in OC are unmoved by the double rotation, and OC is therefore the axis of the single rotation by which the given displacement of the body may be constructed. The straight line OC is called the *resultant axis of rotation.*

If the order of the rotations were reversed, so that the body was rotated first about OB through the angle $\beta$, and then about OA through the angle $\alpha$, the resultant axis would be OC′.

**117.** To find the magnitude $\theta$ of the rotation or angular displacement about the resultant axis OC, note that if a point Q be taken in OA, it is unmoved by the rotation $\alpha$ about OA, and the subsequent rotation $\beta$ about OB will bring it into the position Q′, where QQ′ is bisected at right angles at M by the plane OBC. But the rotation $\theta$ about OC must give Q the same displacement; hence in the standard case $\theta$ is twice the external angle between the planes OCA and OCB.

If the order of the rotations were reversed, the rotation about the resultant axis OC′ would be twice the external angle of C′, which is the same as that at C. So that though the position of the resultant axis of rotation depends on the order of rotation, the resultant angle of rotation is independent of that order.

As regards the *final position* to which it leads, an angular displacement represented by twice any internal angle of the spherical triangle ABC is equivalent and opposite to that represented by twice the corresponding external angle. For since the sum of the internal and external

angles is $\pi$, a rotation through twice the internal angle ACB would be $(2\pi - \theta)$, while that through twice the corresponding external angle ACB' would be $\theta$. And it is evident that a rotation through the angle $2\pi$ cannot alter the final position of any point of the body.

Hence the rule given by Routh for compounding finite rotations.

'If ABC be a spherical triangle, a rotation round OA from C to B through twice the internal angle at A, followed by a rotation round OB from A to C through twice the internal angle at B, is equal and *opposite* to a rotation round OC from B to A through twice the internal angle at C.

'It will be noticed that the order in which the axes are to be taken as we travel round the triangle is opposite to that of the rotations.'

*Axes fixed in the Body.*—If the axes OA, OD were fixed in the *body*, the rotation $\alpha$ about OA would bring OD into a position OD' say. Then the body could be brought from its initial to its final position by the specified rotations about OA and OD' respectively fixed in *space*. Hence with OD' substituted for OD, the preceding construction will suffice for the resultant axis and the rotation about it.

### EXAMPLES—XXVI.

1. Show that any displacement of a rigid body with one point fixed may be represented by a certain angular displacement.
2. If you are given the initial and final positions of three mutually rectangular lines in a rigid body which meet in a fixed point, obtain expressions for the direction cosines of the axis and the angular displacement about it which would represent the displacement in question.
3. If two specified finite angular displacements occur successively about determinate axes fixed either in space or in the rigid body, show how to obtain the resultant angular displacement.

**118. Composition of Angular Velocities about any Axes meeting in one Point.**—If instead of a finite angular displacement about the axis OA followed by another about OB, or *vice versa*, as just treated, we had an infinitesimal angular displacement about one axis followed by an infinitesimal one about the other axis, it is clear that the points C and C' of Fig. 45 would coalesce; or in other words, the resultant axis OC would lie in the plane of the component axes OA and OB. If now the body in question has component angular velocities about the axes OA and OB, the consequent angular displacements about each axis in time $dt$ can be taken and considered in either order instead of simultaneous. We are thus led to the composition of angular velocities about a *pair* of meeting axes, a theorem that was established for convenience sake so far back as article 25*b*, the point of view there being, however, somewhat different. We then saw that the component angular velocities obeyed the usual law of *vector* addition, which is in contrast with the case for the composition of finite successive angular displacements just treated in articles 116 and 117.

Hence, if we have three coexistent angular velocities about three mutually perpendicular axes, the single angular velocity equivalent to them is determined in magnitude, and the direction of the axis found by

the corresponding vector addition in space of three dimensions. Thus, let the angular velocities about OX, OY, and OZ be $a$, $b$, and $c$ respectively, and let the resultant angular velocity be $\Psi$ about the resultant axis OR, whose direction cosines are $\lambda$, $\mu$, and $\nu$. Then we have

$$\Psi^2=a^2+b^2+c^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

also

$$\Psi=a/\lambda=b/\mu=c/\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

If, instead of three component angular velocities about the coordinate axes, we have any number of coexistent angular velocities about any axes passing through O, take one as a type and call the angular velocity $\omega$ and the direction cosines of its axis $l$, $m$, and $n$. Then we may resolve this angular velocity into three component angular velocities $l\omega$, $m\omega$, and $n\omega$ about OX, OY, and OZ respectively. Treating all the others in like manner, and taking the algebraic sum of all such components about any one axis, we may write

$$\Sigma(l\omega)=a,\ \Sigma(m\omega)=b, \text{ and } \Sigma(n\omega)=c \quad . \quad . \quad . \quad (3).$$

Hence, putting the values from (3) in (1) and (2), we obtain the resultant angular velocity of magnitude $\Psi$ about the axis, whose direction cosines are $\lambda$, $\mu$, and $\nu$.

In dealing with a numerical case it is well to arrange the quantities in seven columns headed, $\omega$, $l$, $m$, $n$, $l\omega$, $m\omega$, and $n\omega$. Then on casting up the last three columns we have $a$, $b$, and $c$ according to equation (3), and ready for insertion in equations (1) and (2).

*The Composition of Angular Accelerations about meeting Axes* obviously follows the same law as that of angular velocities, and so calls for no further treatment.

**119. Composition of an Angular Velocity and an Angular Acceleration about meeting Axes.**—Suppose now we have an angular velocity $\omega$ about OX and a coexistent angular acceleration about an axis passing through O, and it is required to compound them. The first step is to resolve the acceleration into two components $\alpha$ and $\beta$ about OX and OY respectively, the plane XOY being taken so as to contain the axis of the given angular acceleration. Then $\omega$ and $\alpha$ are easily dealt with by the formulae of article 92.

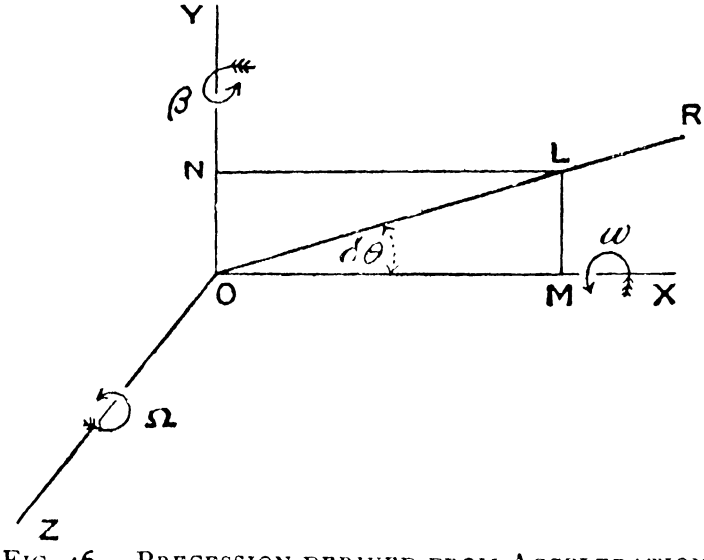



FIG. 46. PRECESSION DERIVED FROM ACCELERATION.

We have therefore to consider here the composition of the angular velocity $\omega$ about OX with the angular acceleration $\beta$ about OY, which leads us to a somewhat new con-

ception. Let us begin to consider the motion at the instant $t=0$. Then at this instant the angular velocity possessed by the body is $\omega$ about OX, and is represented by OM in Fig. 46, there being at that instant no angular velocity about OY. After the infinitesimal time $\delta t$, the angular acceleration $\beta$ about OY will have produced the angular velocity $\beta\delta t$ about OY, this being denoted by ON on the figure. Hence to find the state of things at the instant $t=\delta t$ we have to compound the angular velocities represented by OM and ON. Their resultant of magnitude $\Psi$ about OR inclined $\delta\theta$ with OX is represented by OL and, according to article 118, is expressed in magnitude and direction by

$$\Psi^2=\omega^2+(\beta\delta t)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\mu=\cos \mathrm{YOL}=\sin \mathrm{XOL}=\beta\delta t/\Psi \quad . \quad (2).$$

Since there is no acceleration to produce angular velocity about the axis of $z$, the third direction cosine, $\nu$, is zero; that is, the resultant axis OR is in the $xy$ plane.

In the limit where the square of $\beta\delta t$ vanishes in comparison with $\omega^2$, have from (1)

$$\Psi=\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

And from (2) and (3), when the sine of XOL may be assimilated to the angle,

$$\delta\theta=\beta\delta t/\omega \text{ or, in the limit,}$$

$$\beta/\omega=d\theta/dt=\Omega \text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where $\Omega$ represents in radians per second the angular velocity of OR in the plane of XOY, *i.e.* about the axis of $z$.

**120. Precessional Motion.**—Hence, the initial effect on the angular velocity $\omega$ of the perpendicular acceleration $\beta$ may be represented by leaving the magnitude of $\omega$ unchanged, while changing the direction of its axis by a *rotation* $\Omega$ radians per second in the plane of the axes of $\omega$ and $\beta$ and from $\omega$ towards $\beta$. This *rotation* of the axis is called *precession*, and $\Omega$ is called the *rate of precession*. It is useful and suggestive to rewrite equation (4) in the form

$$\beta=\omega\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and to compare it with equation (4) of article 69. We then see that there is an analogy between the familiar conception of uniform circular motion and the present new phenomenon of precession. Thus the two equations under consideration may be put in words as follows :—

| *Uniform Circular Motion.* | *Precession.* |
|---|---|
| Without changing its magnitude, to rotate a *linear* velocity $v$ at angular velocity $\Omega$ requires a perpendicular *linear* acceleration $v\Omega$. | Without changing its magnitude, to rotate an *angular* velocity $\omega$ at angular velocity $\Omega$ requires a perpendicular *angular* acceleration $\omega\Omega$. |

We may further notice as to the relation of the directions of angular velocity, angular acceleration, and precession that (for the right-handed system of axes used in this work) if the first two are positive about the

axes $x$ and $y$ respectively, the last is positive about the axis of $z$. These directions are indicated by arrows in the figure.

It should be noted that, in the ideal case just considered, the axis of rotation precesses not only in space, *i.e.* with respect to the co-ordinate axes, but *with respect to the body also*; for we postulated neither velocity nor acceleration round the axis OZ, so there can be no rotation of the body itself about OZ.

Thus, if at the instant $t=0$ a certain point K is on the axis of rotation and therefore on the axis OX, it does not remain on the axis of rotation and describe a circle in the plane XOY. This would involve an angular velocity of the body about OZ, which does not exist, as it was not there at first and there has been nothing to generate it. On the contrary, when the instantaneous axis of rotation has passed from the axis of $x$, the point K would be no longer on the axis of rotation but would be circling round that axis.

It may naturally be asked how could the motion under discussion be realised. The following scheme would accomplish it:—Let the body be a sphere with centre at the origin of co-ordinates and the diameter KOK′ lying initially along the axis of $x$; let two cylinders with axes parallel to OZ touch the sphere at K and K′ and be capable of a motion round OZ. Give to the sphere the angular velocity $\omega$ about the diameter KOK′ coincident with the axis of $x$, then let the cylinders *roll upon the sphere without slipping as though the sphere were at rest*, and so that their axes move round OZ with the angular velocity $\Omega=\beta/\omega$. Then, though the sphere itself has no rotation about OZ, its instantaneous axis of rotation which replaces KOK′ would be moving round OZ at angular velocity $\Omega$ with respect both to space and *to the sphere itself*.

**121. Rotation about a Moving Axis.**—Let us now consider the subject of articles 119 and 120 from another point of view, as this throws an additional light on the matter. We now start by supposing the body to have an angular velocity $\omega$ about an axis OA, which axis has itself a precession or angular velocity $\Omega$ about a fixed perpendicular axis OZ. Let it be required to find what angular acceleration, if any, this velocity and precession entail. Let OA move in the plane XOY, and consider the moving axis OA when it makes an angle $\Omega t$ with the fixed axis OX as shown in Fig. 47.

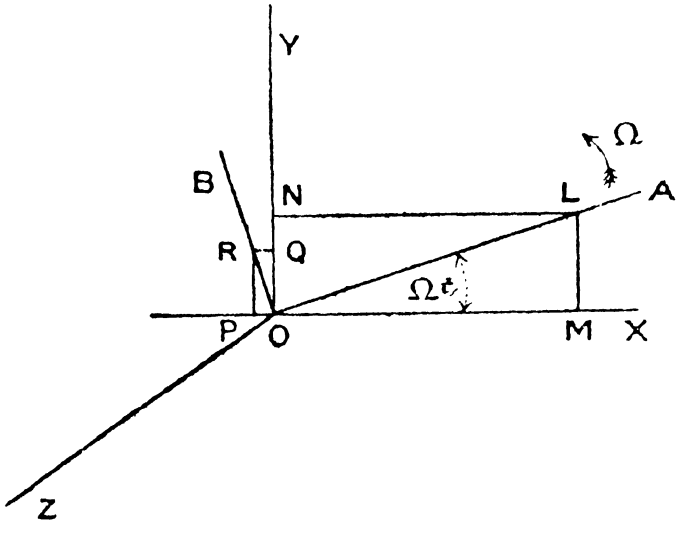



FIG. 47. ACCELERATION ENTAILED BY PRECESSION.

Let OL on the axis OA represent the angular velocity $\omega$. Then, on the same scale, $OM = \omega \cos \Omega t$ and $ON = \omega \sin \Omega t$ represent at the instant in question the component angular velocities about the axes OX and OY respectively. Hence, by differentiation, we may obtain the angular accelerations $\xi$ and $\eta$ about these axes. We thus have

$$\xi = -\Omega\omega \sin \Omega t = OP \text{ on Fig. 47} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\eta = \Omega\omega \cos \Omega t = OQ \quad \text{do.} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Whence the resultant acceleration, found by compounding the vectors OP and OQ, is given by OR on the axis OB, defined in magnitude $\beta$ and direction YOB by

$$\beta^2 = \xi^2 + \eta^2 = \omega^2\Omega^2, \text{ or } \beta = \omega\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\tan YOB = \tan \Omega t = \tan XOA, \text{ or } \angle YOA = \angle XOA \quad . \quad (4),$$

*i.e.* OB remains perpendicular to OA, and therefore rotates with angular velocity $\Omega$ about the axis OZ.

Thus the result may be summed up in words as follows:—If a body has an angular velocity $\omega$ about an axis OA, which axis rotates at angular velocity $\Omega$ about OZ perpendicular to OA, then there is an angular acceleration $\beta = \omega\Omega$ about the axis OB, which rotates at angular velocity $\Omega$ about OZ so as to be always at right angles both to it and to OA.

This proposition holds whether (i) the axis of rotation OA moves both in space and in the body, or (ii) moves in space only, being fixed in the body. But, whichever is the state of things at starting, so it will remain if there is no angular acceleration about OZ to change the zero or uniform rotation which is initially postulated.

The cases of motion considered in this and the two previous articles are somewhat ideal, and the student is warned against hastily concluding that they apply rigorously to any special case he may have in mind in which the constraints and masses involved may need somewhat detailed consideration. The subject will be referred to again in Chapter XIV.

**122. General Precessional Rotation.**—The example already considered illustrates precession in its simplest form, the axis of rotation moving in a plane because it is at right angles to the axis OZ about which the precession occurs. In the general case of precessional rotation the axis of rotation OC may make any angle ZOC with the axis of precession; it accordingly follows that OC describes a conical surface of semi-vertical angle ZOC. It is also easily seen that any desired relation between the angular velocity $\omega$ about OC and the rate of precession $\Omega$, or angular velocity of OC about OZ, can be represented by the rolling of a moving circular cone of axis OC and semi vertical angle $\theta$ on a fixed circular cone of axis OZ and semi-vertical angle $\phi$, the sum of $\theta$ and $\phi$ being ZOC and their common vertex being at O. This is shown in Fig. 48, representing the plane ZOC, which accordingly contains also the line OI along which

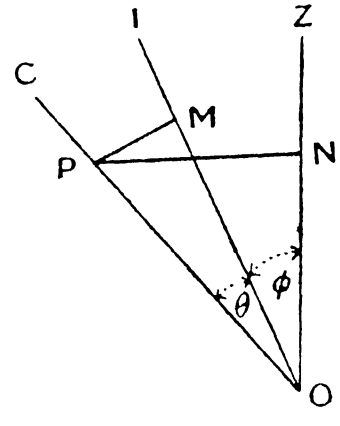



FIG. 48. PRECESSIONAL ROTATION REPRESENTED BY ROLLING CONES.

the two cones are instantaneously in contact. Suppose OC is advancing from the plane of the diagram towards the reader. Then, since the moving cone is rotating about the instantaneous axis OI, the point P will in time $\delta t$ move perpendicularly to the diagram a distance

$$MP\omega\delta t = \omega r \sin\theta\delta t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $r$ denotes OP. Similarly, as OC is moving round OZ at angular velocity $\Omega$, in time $\delta t$ the point P will move perpendicularly to the diagram through the distance

$$NP\Omega\delta t = \Omega r \sin(\theta+\phi)\delta t \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Hence, equating these two expressions for the element of path described by P, we have

$$\omega \sin\theta = \Omega \sin(\theta+\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

For the simpler case of the previous articles, in which the conical surface described by the moving axis is a plane, we have $\theta+\phi=\pi/2$. Hence, equation (3) reduces to

$$\omega \sin\theta = \Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Comparing this with equation (4) of article 119 we find

$$\sin\theta = \beta/\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

which with

$$\phi = \pi/2 - \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

gives the angles $\theta$ and $\phi$ for the rolling cones to represent this simple precessional motion when $\omega$ and $\beta$ are known.

To find the angular acceleration in the general case of precession in which the axis of rotation describes a conical surface, we need general expressions for the angular accelerations about moving axes, which is dealt with in the next article.

**123. Angular Accelerations about Moving Axes.**—Take a system of rectangular moving axes OABC, with their origin O fixed, their angular velocities about the *positions instantaneously occupied* by OA, OB, and OC being respectively $\theta_1$, $\theta_2$, and $\theta_3$. Let the body or figure under consideration have angular velocities $\omega_1$, $\omega_2$, $\omega_3$ about these moving axes, all as represented in Fig. 49. It is required to obtain expressions for the angular accelerations $\alpha$, $\beta$, $\gamma$ about the axes OA, OB, and OC respectively.

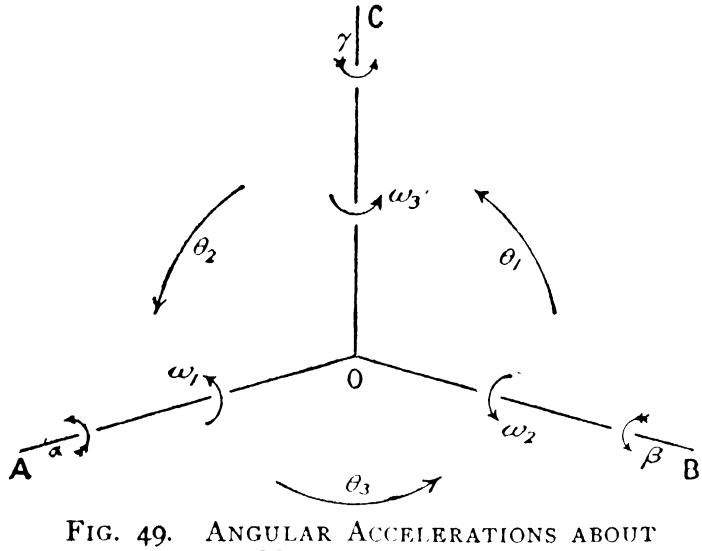



FIG. 49. ANGULAR ACCELERATIONS ABOUT MOVING AXES.

It must be noted that though the six angular velocities referred to are all about the instantaneous positions of the moving axes, the velocities are all of magnitude reckoned with respect to axes fixed in space. Thus, if $\theta_3$ had the magnitude 0·1, this would mean that at the instant in question the axes OA and OB were moving in their own plane away from axes OX and OY fixed in space at the rate of 0·1 radian per second. Again, if $\omega_3$ had the value 0·5, that would mean that with respect to the *fixed* axes OX and OY the body rotated at the speed 0·5 radian per second about OC. Thus the body would have the angular velocity $(\omega_3 - \theta_3) = 0{\cdot}4$ with respect to OA and OB.

Consider first the angular acceleration $\beta$ about OB. By reference to equation (3) of article 121, it is clear that it will contain the term $+\omega_1\theta_3$, since we have an angular velocity $\omega_1$ precessing towards OB at the rate $\theta_3$.

But by a second application of the same principle to the axis OC, it is clear that the acceleration about OB must have another term $-\omega_3\theta_1$, due to the angular velocity $\omega_3$ precessing at rate $\theta_1$ from OB. There is also the term $\dot{\omega}_2$ contributed by the rate of increase, if any, of the velocity about the axis OB itself. We accordingly obtain for $\beta$ the algebraic sum of the three terms just mentioned. But it is evident from the symmetry of the notation and the figure that the other accelerations may be written down by suitably changing the subscripts to $\omega$ and $\theta$.

We accordingly obtain for the angular accelerations about the moving axes the following expressions, each consisting of the rate of increase of one angular velocity and a pair of products of the other angular velocities concerned :—

$$\left.\begin{aligned} \alpha &= \dot{\omega}_1 - \omega_2\theta_3 + \omega_3\theta_2 \\ \beta &= \dot{\omega}_2 - \omega_3\theta_1 + \omega_1\theta_3 \\ \gamma &= \dot{\omega}_3 - \omega_1\theta_2 + \omega_2\theta_1 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1).$$

These results are valid for any actual case whether the moving axes move *both in space and in the body* or always coincide with certain lines *fixed in the body*. In either case the relations between the $\omega$'s and the $\theta$'s will follow from the conditions prescribed. Thus, if the axis OA is fixed in the body, then $\theta_2 = \omega_2$, $\theta_3 = \omega_3$ ; hence $\alpha = \dot{\omega}_1$.

**124. Angular Acceleration for Steady Precession.**—We are now prepared to deal with the determination of the angular acceleration occurring in the general precessional rotation of article 122. We suppose the body to have an angular velocity of constant magnitude $\omega$ about an axis OC *fixed in the body*, while this axis describes a conical surface by moving with an angular velocity of constant magnitude $\Omega$ about a fixed axis OZ, to which it is inclined at the constant angle $\theta$. We take OZ vertically up and other fixed axes OX and OY horizontally as shown in Fig. 50; also two moving axes at right angles to OC, namely OB horizontal and OA in the same meridian as C. It will be convenient to regard OX, OY, OZ, OA, OB, and OC as each of unit length, so that O is the centre of a unit sphere on whose surface the other six points always lie.

Then, using the notation of the preceding article, we easily see that $\omega_1=\theta_1$, $\theta_2=\omega_2=0$, $\omega_3=\omega$, $\dot{\omega}_3=\dot{\omega}=0$. We have thus to find $\theta_3$ and $\theta_1$. Suppose that in time $\delta t$ C moves to the near position C'. Then the displacement CC' may be regarded (1) as the result of the angular velocity $\Omega$ about OZ, the radius being $\sin\theta$, or (2) as the result of the angular velocity $\theta_1$ (or $\omega_1$) about OA, the radius being OC, which equals unity. We accordingly find that $-\Omega\sin\theta=\theta_1=\omega_1$, the minus sign occurring because the displacement due to a positive value of $\Omega$ is negative, while that to $\theta_1$ or $\omega_1$ is of the same sign as the velocity.

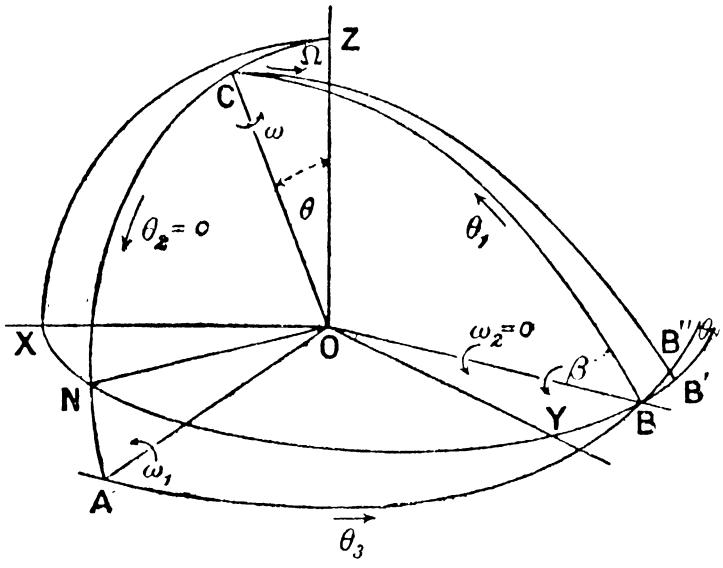



FIG. 50. ANGULAR ACCELERATION FOR STEADY PRECESSION.

Again, in time $\delta t$, let B move to B', then BB' has the value $\Omega\delta t$, since it is described by unit radius about OZ. Now consider OC the polar axis of our unit sphere, and draw the quadrant from C to B; also another meridian from C through B', and an arc of an equator through A and B cutting the meridian CB' in B''. Then the angle between the two equatorial arcs BB' and BB'' is that between their respectively polar axes OZ and OC, namely $\theta$. Hence $BB''=BB'\cos\theta$. And, since the arc BB'' is described in time $\delta t$ with the angular velocity $\theta_3$ and unit radius about OC, we have

$$\theta_3\delta t=BB''=BB'\cos\theta=(\Omega\delta t)\cos\theta, \text{ or } \theta_3=\Omega\cos\theta.$$

We may accordingly write the following scheme of values:—

About the axes:— OA, OB, and OC;

the angular velocities of the *body* are $\omega_1=-\Omega\sin\theta,\ \omega_2=0,\ \omega_3=\omega$ . . (2);

the angular velocities of the *axes* are $\theta_1=-\Omega\sin\theta,\ \theta_2=0,\ \theta_3=\Omega\cos\theta$ (3).

We have also $\dot{\omega}_1=\dot{\omega}_2=\dot{\omega}_3=0$ . . . . . (4).

Hence, to determine the angular accelerations, we substitute from equations (2), (3), and (4) in the right sides of equations (1) of article 123. We thus find

$$\alpha=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

$$\beta=(\omega-\Omega\cos\theta)\Omega\sin\theta \quad . \quad . \quad . \quad (6),$$

and $$\gamma=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

It is seen that if $\theta=\pi/2$, (6) reduces to

$$\beta=\omega\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

in agreement with the results of articles 119 and 121 for the case first considered.

Further, (6) shows that

if $\omega = \Omega \cos\theta$, then $\beta = 0$ . . . . (9).

**125. Most General Motion of Rigid Body with One Point Fixed.**—The example of precession considered in article 122, in which the rolling cones are both circular, though representing any purely precessional rotation, falls short of complete generality. Now, if the fixed point of the rigid body is taken as the centre of a fixed spherical surface, it is obvious that the most general motion of that body can be represented by the most general motion of a spherical figure on this fixed spherical surface. But as in the case of a plane moving in a plane (dealt with in articles 94-96 and 101), so here this most general motion of the spherical cap on the fixed sphere may be represented by the rolling of some line, fixed relatively to the moving cap, on some other line, fixed on the stationary spherical surface; these lines being respectively the body and space centrodes, both, however, being spherical now instead of plane. Extending our thoughts to the interior of the sphere, it is evident that the body and space centrodes are simply the base lines of the moving and fixed cones (with common vertex at the centre of the sphere) which, by the rolling of one on the other, represent the motion of the rigid body in question. These guiding cones are not necessarily circular, nor need their bases be bounded by re-entrant curves. On the contrary, the term cone is here to be understood in its widest sense as a conical surface generated by a straight line passing through a fixed point and a fixed line circular, re-entrant, intersecting and re-entrant or even non-re-entrant.

With these provisos then, the *most general* motion of a rigid body with *one point fixed* may be represented by the *rolling of a cone fixed in the body on a cone fixed in space*, the fixed point of the body being the common vertex of the two cones.

## EXAMPLES—XXVII.

1. Discuss the composition of angular velocities about intersecting axes.
2. If a simple pendulum were set vibrating at one of the geographical poles of the earth, how would you expect its plane of vibration to move with respect to the earth? How would the plane of vibration of a pendulum in latitude $\lambda$ appear to move with respect to the earth's surface?
3. Trace the analogy between the effect of an angular acceleration on a perpendicular angular velocity and the corresponding case of linear acceleration and velocity.
4. What acceleration, if any, is needed to maintain an angular velocity of constant magnitude about an axis which is itself rotating so as to describe a plane?
5. Show that the motion of a body consisting of rotation about an axis which is describing a conical surface may be constructed as the rolling of one cone on another, and obtain the relations between the various quantities involved.

6. Obtain the complete scheme of expressions for the angular accelerations about a set of rectangular moving axes with fixed origin in terms of the various velocities involved.
7. Assuming the general expressions for the angular accelerations about moving axes, find the acceleration required to maintain a steady conical precessional motion. Under what special conditions may this acceleration vanish, and when may it reduce to a simpler form?
8. Explain carefully how to construct the most general motion of a rigid body with one point fixed.

**126. General Displacement of a Rigid Body.**—We now pass to the consideration of the most general displacement possible to a rigid body having *no point fixed*, and shall closely follow the treatment adopted in Routh's *Rigid Dynamics* (i. pp. 186-8, sixth edition, 1897). It is first necessary to establish the following *proposition*:—

*Enunciation.*—Every displacement of a rigid body may be represented by a combination of (1) a motion of translation, or *linear displacement*, whereby every point in it has a displacement equal in magnitude and direction to those of any assumed point P rigidly connected with the body; and (2) a motion of rotation or *angular displacement* of the whole body about some axis through this assumed point P, which may be referred to as the *base point*.

*Proof.*—It is evident that the change from initial to final position may be effected by shifting P from its old to its new position, P′ say, by a motion of translation of the body as a whole, and then retaining P′ as a fixed point while moving any two other points of the body not in a straight line with P′ into their final positions.

But any displacement of a rigid body with one point fixed has been shown in article 113 to be equivalent to a determinate angular displacement about some axis through that fixed point.

This accordingly establishes the proposition.

Since the above displacements, linear and angular, are quite independent, their order may be reversed, *i.e.* we may rotate the body first and then translate it. Further, the motions might occur simultaneously.

It may easily be seen that any point P could be chosen as the base point of the double operation, the translation and rotation being defined accordingly. Hence the given displacement may be constructed in an infinite variety of ways.

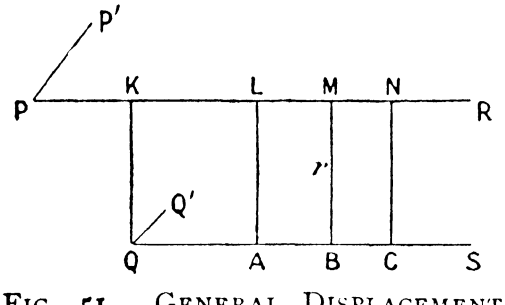



FIG. 51. GENERAL DISPLACEMENT OF A RIGID BODY: AXES PARALLEL.

**127. Change of Base Point: Axes Parallel.**—Let us now find the relations between the axes and angular displacements when different points P, Q are taken as base points.

Suppose that the displacement in question may be represented (*a*) by the angular displacement $\theta$ about an axis PR together with the linear displacement PP′, or (*b*) by the angular displacement $\phi$ about an

axis QS together with the linear displacement QQ′ as indicated in Fig. 51, in which PP′ and QQ′ are not necessarily in the plane of the diagram.

Consider the method (*a*) of constructing the total displacement. Then it is clear that any point, B say, has two displacements—(1) a translation equal and parallel to PP′, and (2) a motion through an arc in a plane perpendicular to the axis of rotation PR, the length of this arc being $r\theta$ where $r$ is the length of the perpendicular BM let fall from B on to the axis PR. This arc is zero only when $r$ is zero, that is, when the point B is on the axis PR. We accordingly conclude that the only points whose displacements are the same as that of the base point *lie on the axis of rotation* corresponding to that base point.

Thus, any line all points of which have the same displacement *must be an axis of rotation* for any point in it taken as base point (1).

Through the second base point Q draw QABC parallel to PR. Then, for each of these points Q, A, B, C, etc., on this parallel, the displacements by the method (*a*) are a translation PP′; and an arc (in planes perpendicular to PR) of magnitude $r\theta$ where $r$ is the length of any perpendicular QK, AL, BM, CN, etc., between the parallels PR and QABC. Hence, by the statement (1) above, we see that QABC parallel to PR must be the axis of rotation QS corresponding to the base point Q. Thus, as P and Q are quite general, we conclude that

the axes of rotation corresponding to *all base points are parallel* (2).

**128. Rotations Equal.**—We have still to find the relation between the angular displacements $\theta$ and $\phi$ which occur about the parallel axes PR and QS respectively in the methods (*a*) and (*b*) of representing the total displacements. For this purpose let us take a new diagram in a plane perpendicular to these axes PR and QS, as shown in Fig. 52.

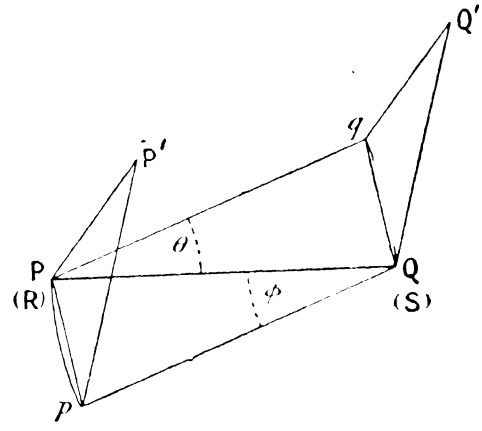



FIG. 52. GENERAL DISPLACEMENT OF A RIGID BODY: ANGLES EQUAL.

Then, by method (*a*) of constructing the displacement, we see that the rotation through the angle $\theta$ about PR carries Q to $q$, where the chord $Qq = 2r \sin \theta/2$. The whole displacement QQ′ of Q is accordingly the resultant of this and the displacement PP′ of P. Or, in symbols,

$$QQ' = Qq \hat{+} PP' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It should be noted that P′ and Q′ are not necessarily in the plane of the diagram of Fig. 52, any more than they were in that of Fig. 51.

Again, by method (*b*) of constructing the displacement, the rotation $\phi$ about QS carries P to $p$, the chord $Pp$ being $2(-r) \sin \phi/2$. Thus,

the whole displacement PP′ of P is the resultant of that chord and QQ′, giving the equation

$$PP' = P\rho \hat{+} QQ' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Hence (3) and (4) yield

$$Qq \hat{+} P\rho = 0, \text{ or } \theta = \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Therefore, since P and Q are quite general, we have the result that the *angular displacements corresponding to all base points are equal.*

**129. Axial Projections Equal.**—From equation (3) we may find the translation and rotation for any given base point, Q say, when those for any other base point P are known. For, since the displacement $Qq$ is produced by rotation about the axis PR, it must occur in a plane perpendicular to PR, and consequently its projection upon PR is zero. Thus, taking projections upon PR, from equation (3) we find

Projection of QQ′ upon PR = projection of PP′ upon PR . (6).

Hence, *the projections upon the axis of rotation of the displacements of all points of the body are equal.*

Thus QQ′ is fully determined. And we already know that QS is parallel to PR, also that $\phi = \theta$. Hence, referring to Fig. 52, if P′ is at any distance, 3 cm. say, from the plane of the diagram towards the reader, so is Q′ at that same distance and in the same direction.

If the projections of PP′ and QQ′ upon PR are each zero, then all axial projections of displacements are zero, and the whole displacement reduces to an example of coplanar rotation and translation, as already treated in articles 94-102.

**130. Central Axis: Twists and Screws.**—It is often important to choose a base point such that the direction of translation may lie along the axis of rotation. We shall now examine how this may be done.

Let the specified displacement of the body be a rotation $\theta$ about PR and a translation PP′. And, if possible, let this displacement be represented by a rotation $\phi$ *about* QS and a translation QQ′ *along* QS. These lines are shown in Fig. 53, in which also P′L and $P\rho$ are drawn perpendicular to PR, and therefore parallel to each other; M and N are the middle points of $P\rho$ and LP′ respectively.

Then, from articles 127-129 we have the following relations:—

$$QS \text{ is parallel to } PR \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

$$\phi = \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

and $$QQ' = PL = MN \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Also, with the former notation, $P\rho$ will represent the displacement of P due to the rotation $\phi = \theta$ about QS.

Thus, $$Q\rho = QP \quad \Big)$$

and $$P\rho = 2QP \sin \theta/2 = LP' \Big) \quad . \quad . \quad . \quad (10).$$

Hence, QS lies in the plane QQ′NM which bisects LP′ at right angles; it is also parallel to PR and at a perpendicular distance PQ from it, such that $2PQ \sin \theta/2 = LP'$. And this forms the solution of the problem. The distance of the central axis QS may be stated by giving MQ or NQ′, which are the perpendiculars from the plane PLP′; thus

$2MQ\tan\theta/2 = LP'$. As to the direction of this distance, it should be noted that NQ′ must be measured from the middle point of LP′ in the direction in which that middle point is moved by its rotation round PR.

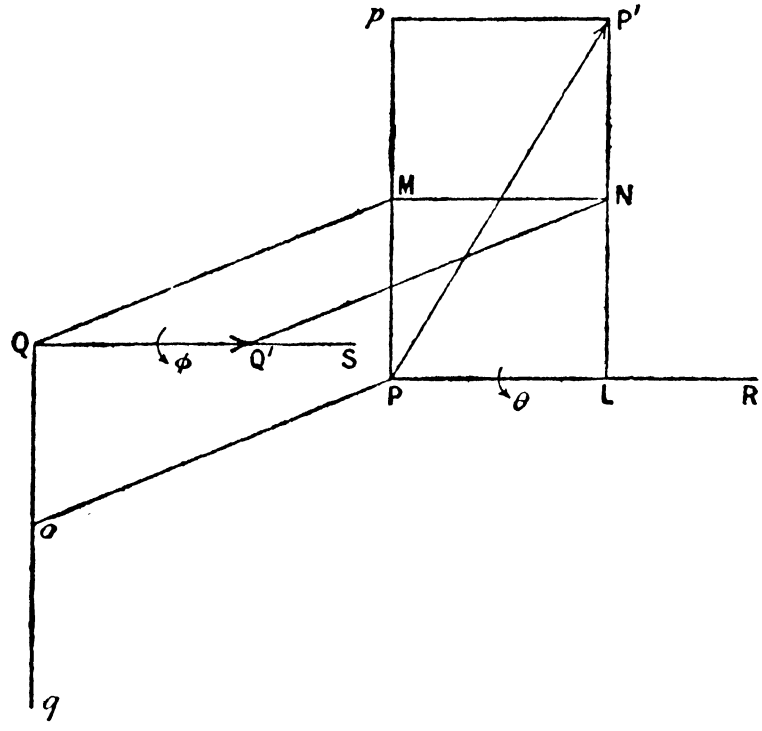



FIG. 53. DETERMINATION OF CENTRAL AXIS.

Having found the only possible position of QS, it is still desirable to show that the linear displacement is really QQ′ *along* QS. The rotation $\theta$ about PR will carry Q through an arc $Qq$, whose chord is parallel to P′L and equal to $2PQ\sin\theta/2$. That is, this chord is equal to P′L. Then, taking the displacement PP′ in the two steps P$p$ and $p$P′, the first step brings $q$ back to Q, and the second step carries it from Q to Q′ along QS, as was to be shown.

It thus follows that *the most general displacement of a rigid body can be represented by a rotation about some straight line and a translation parallel to that same line.*

Such a displacement is like that of a nut on a screw, and is called a *twist*. Hence we may say that a rigid body may pass from one given position to any other by means of a twist.

In order to specify a screw we must state (i) the *direction* and *position* of the line round which the rotation is effected, *i.e.* the *central axis* or *axis of the screw*; and (ii) the *ratio of the translation to the rotation*, which occur along and about that axis. This ratio is a linear magnitude called *the pitch of the screw*. In the present theoretical work the pitch of the screw is measured in units of length *per radian* of rotation; in workshop practice it is measured in units of length *per revolution* of the thread.

To specify a twist we must further state the *amplitude*, *i.e.* the

magnitude of the rotation or angular displacement occurring on the screw in question about which the twist is effected.

It should be noted that the twist by which a rigid body may pass from one position to another is, in general, unique. For, if possible, let there be two central axes PR and QS; see Fig. 51 of article 127. Then by that article these axes are parallel. Also, taking PR as the central axis, the displacement of any point A on QS is found by turning the body round PR and moving it parallel to PR. Thus A has one displacement perpendicular to the plane PRA, *i.e.* to QS, and another parallel to QS, and accordingly cannot move solely along QS. Hence QS *cannot be a central axis.* When the rotations are indefinitely small, the construction to find the central axis is simply stated by Routh somewhat as follows:—

Let the displacement be represented by a rotation $\omega dt$ about an axis PR and a translation $vdt$ in the direction PP'. Measure a distance $y=v(\sin \mathrm{P'PR})/\omega$ from P perpendicular to the plane P'PR on that side of the plane towards which P' is moving. A straight line parallel to PR through the extremity of $y$ is the central axis.

This construction may be illustrated by reference to Fig. 53 of the present article. From this figure it may be seen that when PP' shrinks indefinitely, MQ and PQ each coalesce with P$o$, which corresponds with Routh's construction for the central axis QS.

**131. General Motion of a Rigid Body.**—On consideration of article 126, it may be seen that the most general *motion* of a rigid body, being a succession of an infinite number of elementary displacements of the most general type, may be represented by *the rolling of a cone, fixed in the body, on a cone unattached to the body*, the latter cone having a motion of *pure translation*, the two cones having their vertices in coincidence. For this rolling of cones gives at each instant the combination of translation of a point and rotation about an axis through it, together with a possible variation of every element of that motion.

The motion of the body would be completely determined by (i) the dimensions of the two cones and their initial positions, (ii) the path and velocity at each instant of their common vertex, and (iii) the rate of rolling at each instant of the *body cone* on the *moving space cone.* Hence the whole motion falls into two parts, that of a point and that of rolling cones.

## EXAMPLES—XXVIII.

1. Discuss the general displacement of a rigid body with no point fixed, showing that if the base point is moved the axes are parallel and the rotational displacements equal.
2. Reduce the most general displacement of a rigid body to a rotation about a determinate axis and a translation parallel to it.
3. A point P in a rigid body has a displacement $5\sqrt{2}$ cm. at an angle of 45° with the horizontal axis, about which it rotates through 60°. Draw a diagram indicating (1) the central axis, (2) the displacement along it, and (3) the rotation about it.

4. Establish the proposition that any displacement of a rigid body may be represented as a twist about a screw.
5. Show that the most general motion of a rigid body may be constructed as the rolling of a body cone on a space cone which has a motion of translation only, the vertices of the cones being always in contact.

**132. Velocity of any Point of a Rigid Body in most general Motion.**—We have already seen (in article 126) that the most general *displacement* of a rigid body may be represented by a linear displacement or translation of a base point, O say, and an angular displacement or rotation about some axis through O. Further, we have seen (in article 131) that the most general *motion* of a rigid body is that of translation of some point O combined with a rotation about some axis through O. But the magnitude and direction of the linear velocity of O may change from instant to instant, also the magnitude of the angular velocity and the direction of its axis through O may change in like manner. We may compactly provide for these changes by supposing the magnitude and direction of each of these vectors to be given by their rectangular components. It is then a problem, in terms of those components, to specify the velocity of *any point* in the body. This we now treat, following the method of Routh.

Choose any three rectangular axes OX, OY, OZ, meeting at the base point O and *moving with* O, *but keeping their directions fixed in space*. Let $u$, $v$, $w$ be the components of the linear velocity of O and $\xi$, $\eta$, $\zeta$ be those of the angular velocity of the body. The usual conventions as to the relation of the positive directions of these six components apply as shown in Fig. 54.

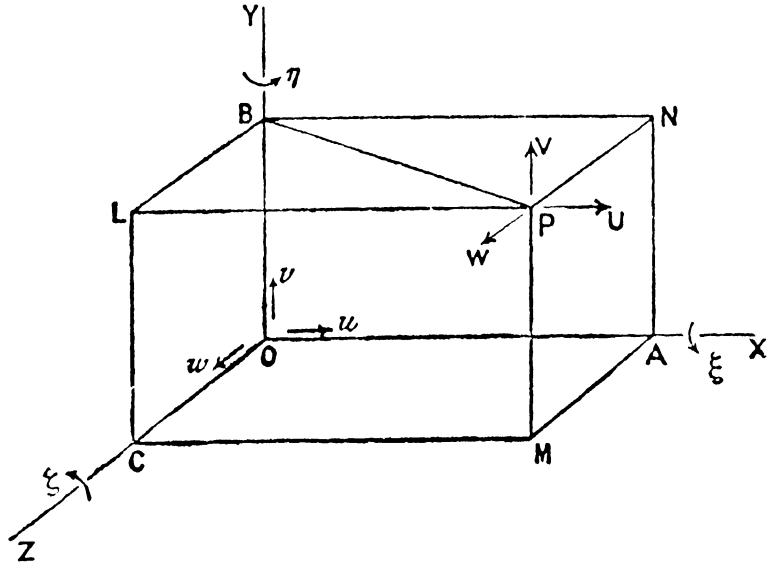



FIG. 54. VELOCITY OF ANY POINT IN A RIGID BODY.

Let $U$, $V$, $W$ be the velocity components of P, its co-ordinates being $x$, $y$, $z$. Consider first the expression for $U$, the $x$ component of P's velocity. It obviously consists of three terms, viz. those due to translation of O in the OX direction and to the rotations about OY and OZ. These are seen to be $u$, $+\eta z$, and $-\zeta y$ respectively, and so

give the first equation in the following scheme, of which the others follow also from the figure, or may be written down by observing the cyclical order of the letters :—

$$\left.\begin{aligned} U &= u + \eta z - \zeta y \\ V &= v + \zeta x - \xi z \\ W &= w + \xi y - \eta x \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Each of the products in the above equations expresses that part of the linear velocity in the given direction, which is due to the angular velocity denoted by the Greek letter and the perpendicular distance denoted by the co-ordinate.

Suppose now it is desirable to change the base point from O to O′, whose co-ordinates are $a$, $b$, and $c$, the *axes at* O′ *being parallel* to the *former set at* O. Let the linear and angular velocity components for O′ be respectively $u'$, $v'$, $w'$ and $\xi'$, $\eta'$, $\zeta'$. The motion of P must be the same as before ; hence, for the very same values of $U$, $V$, and $W$ as in (1), we may now write, by analogy, the new expressions

$$\left.\begin{aligned} U &= u' + \eta'(z-c) - \zeta'(y-b) \\ V &= v' + \zeta'(x-a) - \xi'(z-c) \\ W &= w' + \xi'(y-b) - \eta'(x-a) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2),$$

in which $x$, $y$, and $z$ are the co-ordinates of the point P referred still to the original axes.

Let us now equate the right side of any line of (1) to the corresponding part of (2). Take, say, the third line, then

$$w + \xi y - \eta x = w' + \xi'(y-b) - \eta'(x-a) \quad . \quad . \quad (3).$$

But this equation holds for any position of P, thus we may change any one co-ordinate in any way we please with or without alteration of the others. If, therefore, we put $y=0$, in the above equation (3), it still holds. But we have thereby reduced the left side by $\xi y$ and the right side by $\xi' y$. Hence these two angular velocities are equal. And this can in like manner be shown to hold for the others. Hence

$$\xi' = \xi, \ \eta' = \eta, \text{ and } \zeta' = \zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Or, in words, whatever the base point chosen, the component angular velocities for a given resultant motion remain the same.

By substitution of (4) in (2) and equating the result to (1) we obtain the linear velocity components of the new base point O′ in the form

$$\left.\begin{aligned} u' &= u + \eta c - \zeta b \\ v' &= v + \zeta a - \xi c \\ w' &= w + \xi b - \eta a \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

It is seen that this just agrees with what we should obtain directly from (1).

**133. Resultant Twisting Velocity.**—Let the motion of a rigid body be specified by the linear velocities ($u$, $v$, $w$) of some base point O, and the angular velocities ($\xi$, $\eta$, $\zeta$) about axes (OX, OY, OZ) meeting in O and moving with it but keeping their directions fixed in space. It is required to find the central axis, the linear velocity along it, and the angular velocity round it. In other words, the three rectangular

screws and the twisting velocities about them being given, it is required to determine the resultant or equivalent screw and the resultant twisting velocity which occurs about it.

In the treatment of this problem we shall again follow Routh's method. Let the direction cosines of the central axis be $\lambda$, $\mu$, $\nu$, the linear velocity along it be $V$, and the angular velocity round it be $\Omega$. Then we have to determine these five quantities and make certain deductions from them. Let P be any point on the central axis, then if P were chosen as base point, the components of the angular velocity would be the same as for the base point O (equation (4) of article 132). Also we have seen (articles 25*b* and 118) that angular velocities are vectors, and therefore compounded by vectorial addition; hence we obtain

$$\Omega^2=\xi^2+\eta^2+\zeta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and $$\lambda=\xi/\Omega,\ \mu=\eta/\Omega,\ \nu=\zeta/\Omega \quad . \quad . \quad . \quad . \quad (2),$$

or $$\Omega=\xi/\lambda=\eta/\mu=\zeta/\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Again, we have seen (in articles 129-130) that the velocity of every point resolved in a direction parallel to the central axis is the same and equal to that along the central axis.

We accordingly obtain by projection

$$V=u\lambda+v\mu+w\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Taking the product of (3) and (4) we have

$$V\Omega=u\xi+v\eta+w\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Also, dividing (5) by (1), we obtain as the *pitch* of the equivalent screw round which the resultant twist occurs

$$\frac{V}{\Omega}=\frac{u\xi+v\eta+w\zeta}{\xi^2+\eta^2+\zeta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Let $x$, $y$, $z$ be the co-ordinates of any point P on the central axis. Then the linear velocity of P is along the axis of rotation. Hence its components, given by equation (1) of article 132, are proportional to the direction cosines $\lambda$, $\mu$, $\nu$ of the central axis, and accordingly proportional to the components $\xi$, $\eta$, $\zeta$ of the angular velocity $\Omega$ about that axis.

We may accordingly write

$$\frac{u+\eta z-\zeta y}{\xi}=\frac{v+\zeta x-\xi z}{\eta}=\frac{w+\xi y-\eta x}{\zeta}=\frac{V}{\Omega} \quad . \quad . \quad (7)$$

And these form the *equations of the central axis.*

Hence equations (1), (2), (4), (6), and (7) present the solution of the problem under consideration.

If we shift the base point, or change the direction of the axes, it may be shown from (5) that the value of $V\Omega$ remains constant. The product $V\Omega$ may therefore be called *the invariant of the components.* The resultant angular velocity has already been seen to be constant, and may be called *the invariant of the rotation.*

If the motion is such that

$$V\Omega=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

then it follows that either $V=0$ or $\Omega=0$. Accordingly equation (8)

expresses the condition that the motion is equivalent *either* to one of *translation only or* to one of *rotation only.* For either motion to be not zero we must, of course, have its components *not all zero.*

The foregoing very brief introduction to the subject of screws and twists must suffice here. For fuller treatment the reader is referred to the works on Dynamics by Routh and by Williamson and Tarleton, also to the original memoirs of Sir Robert Ball, to whom the theory of screws is principally due.

## EXAMPLES—XXIX.

1. Obtain expressions for the velocity components of any point of a rigid body in the most general motion possible to it.
2. Having given that a certain point O in a rigid body has linear velocity components $u$, $v$, $w$ parallel to the axes OX, OY, OZ, and that the body also has angular velocity components $\xi$, $\eta$, $\zeta$ about these axes, it is required to determine the resultant motion as a twist about a screw.

# CHAPTER IX

## MECHANISMS

**134. Subdivisions and Treatment.**—We now pass to the consideration of deformable figures, a simple treatment of which will occupy this chapter and the next. A fairly comprehensive scheme indicating a set of possible subdivisions of this subject is given in Table III. Thus by supposing any one of the six types of systems 1-6 to be subject

TABLE III. KINEMATICS OF DEFORMABLE FIGURES.

DEFORMABLE SYSTEMS :—

A. *Mechanisms, or Partially Deformable Figures.*

1. Inextensible Cords and Membranes.
2. Incompressible Fluids.
3. Linkages, etc., with Rigid Links.

B. *Elastic Bodies, or Generally Deformable Substances.*

4. Extensible Cords and Membranes.
5. Compressible Fluids.
6. Elastic Solids.

POSSIBLE DEFORMATIONS AND MOTIONS :—

*a.* Displacements and Strains.
*b.* Steady Flow or Currents.
*c.* Reciprocating Motions and Vibrations.
*d.* Wave Motions.
*e.* Vortical Motions.

in turn to each of the five types of motion *a-e*, where such motions are possible to them, we obtain the various subdivisions.

Some points in this full scheme have, however, been already sufficiently touched upon under other headings. Others again lie beyond the scope of this work. Of the remainder, the various displacements and motions of mechanisms occupy this chapter, the strains of elastic bodies being dealt with in the next. The consideration of a few points of an advanced character will be deferred to those later chapters, where they are needed in connection with the corresponding kinetical or statical problems.

It may be well to explain here that the '*partially* deformable

figures' in the table denote those bodies or systems which by their nature or arrangement absolutely preclude or render negligibly small certain conceivable deformations, other deformations being possibly very large. The '*generally* deformable figures,' on the other hand, are those of a nature such that no conceivable deformation is thereby precluded or kept negligibly small, though some deformations may still be much smaller than others. Thus, the extensible cords could suffer much greater deformation by bending them than by stretching, though the latter is not prohibited as for the inextensible cords.

### 135. Inextensible Cords.

*Multiplied Cord.*—The most striking mechanical use of a cord which is not appreciably extensible but very readily flexible occurs when it is passed round and round two parallel cylinders or similar bodies, one end of the cord being fastened to one cylinder, while the other is free. We then have an example of the *reduplication* of a cord or of the *multiplied* cord. Regarding the cylinder to which one end of the cord is fastened or fixed, we may inquire what is the ratio of the displacement $s$ of the free end of the cord to the displacement $r$ of the centre of the moving cylinder, round which say $n$ plies of the cord pass, all practically parallel to each other. It is easily seen that we have the relation

$$s = nr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

since for every element of the cylinder's displacement an equal element of each of the $n$ plies of cord is set free, which total to a displacement $nr$ of the free end.

If the displacement $r$ affects the free end of another multiplied cord, we have only to repeat the equation (1) to find the final displacement ratio. Or, we may combine two or more such equations in the form

$$s = n_1 r_1 = n_1(n_2 r_2) = n_1 n_2 (n_3 r_3) = \text{etc.} \quad . \quad . \quad . \quad . \quad (2),$$

where $n_1$ is the number of plies of the cord whose end has the displacement $s$ corresponding to the displacement $r_1$ of the centre of the first moving cylinder; $n_2$ is the number of plies of the cord attached to the centre of the first cylinder, whose displacement $r_1$ corresponds to the displacement $r_2$ of the centre of the second cylinder round which the $n_2$ plies pass; and so on for the others.

The above considerations form the first step in the theory of that simple machine or mechanism often miscalled 'the pulley.' The machine in question is seen to derive its essential character from the kinematics of the *multiplied cord*, which settles its *displacement ratio* as shown above. The other details of the machine, such as the provision of a pulley to lessen friction, are really only *modifications* of the ideal case now under notice, though they may be of great practical importance.

If, in the case of equation (1), we inverted matters and regarded the cylinder to which the cord was fastened as moving, the other being

our basis of reference, then the $n$ would be replaced by $(n+1)$. Thus we might write, using accents for distinction,

$$s' = (n+1)r' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

**136. Connecting Belts.**—In the case where an endless cord or belt passes over two cylinders or other shaped wheels, we are concerned with the velocity ratio of the two wheels thus connected. The motion is supposed to occur without slipping between the wheels and the belt. Thus, if $a$ and $b$ are those radii of the wheels at which no slipping occurs, and $s$ is the length of the belt passing for the angles of rotation $\theta$ and $\phi$ respectively, we have

$$a\theta = s = b\phi,$$
$$a\dot{\theta} = \dot{s} = b\dot{\phi},$$

or

$$\dot{\theta}/\dot{\phi} = \theta/\phi = b/a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

That is, the angular velocities of the belt-connected wheels are inversely as their effective radii.

The above is on the supposition that the belt is uncrossed or open, so that the angular velocities have the same sign. If the belt is crossed the displacements and velocities are of opposite sign, so that we should have

$$\dot{\theta}/\dot{\phi} = \theta/\phi = -b/a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a).$$

**137. Analytical Conditions of Inextensibility.**—Consider an element $\delta s$ of a cord, and let the tangential velocity of the cord be $\dot{s}$ at one end of the element and $\dot{s}+\delta\dot{s}$ at the other. Then, in time $\delta t$, the tangential displacements of the two ends will be $\dot{s}\delta t$ and $(\dot{s}+\delta\dot{s})\delta t$. Thus the total increase of length of the element is the difference of these quantities, viz. $\delta\dot{s}\delta t$. Hence the rate of increase of length per unit length per second will be

$$\delta\dot{s}/\delta s, \text{ or } d\dot{s}/ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and, for an inextensible cord, this must always be zero.

Taking cartesian components of velocity, we see that the tangential velocity of the cord at a point is

$$\dot{s} = \dot{x}\frac{dx}{ds} + \dot{y}\frac{dy}{ds} + \dot{z}\frac{dz}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

since the terms on the right are the projections on the cord of the components of velocity. Hence the general analytical condition for the inextensibility of a cord may be written

$$\frac{d\dot{s}}{ds} = \frac{d\dot{x}}{ds}\cdot\frac{dx}{ds} + \frac{d\dot{y}}{ds}\cdot\frac{dy}{ds} + \frac{d\dot{z}}{ds}\cdot\frac{dz}{ds} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

**138. Incompressible Fluids.**—Imagine a chamber prismatic or cylindrical throughout, having two portions, of which one is small and the other large in cross-section, the respective areas being $a$ and $b$. Let pistons fit tightly in these two portions and enclose between them a volume $v$ of incompressible fluid. Then we have, as the expression of the condition of incompressibility, $v$ = constant. Hence, if a normal

displacement $s$ of the smaller piston occurs, and a simultaneous displacement $r$ in the same sense of the larger piston, we must have

$$as=br, \text{ or } \dot{s}/\dot{r}=s/r=b/a \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

which gives the displacement and velocity ratio of the pistons as the inverse ratio of their areas. The arrangement is illustrated in Fig. 55, in which $AA'=s$ and $BB'=r$, the lines at A, A', B, B' representing the inner faces of the pistons.

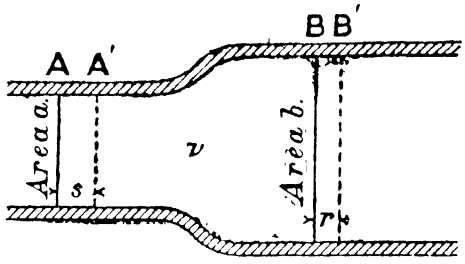



FIG. 55. DISPLACEMENTS OF INCOMPRESSIBLE FLUID.

It is easily seen that this relation is the essential principle of the hydraulic press and the so-called hydrostatic paradox.

EXAMPLES—XXX.

1. Discuss the subdivisions of mechanisms, and draw up a scheme showing the chief topics needing treatment.
2. Sketch three different arrangements, each showing an application of the multiplied cord, and calculate the velocity ratio for each.
3. Obtain the analytical condition of inextensibility of a cord in motion and passing round curves in solid space.
4. Explain the so-called hydrostatic paradox, and obtain the velocity ratio of pump plunger and ram in a hydraulic press.

**139. Links and their Relative Motion.**—As its name implies, a *linkage* is an aggregate of separate parts called *links*, whose nature, connection, and motion must now be studied. The term linkage is, however, usually restricted to a certain class of connections which is but one example of a more general type, called *a kinematic chain* by Reuleaux, whose classic treatment of this subject will, in the main, be followed here. In a kinematic chain the connections of the parts may be of the most general type, and when one part of the chain is fixed the arrangement is called a *mechanism*. Hence the title of the present chapter.

Let us first suppose the links to be rigid, and then inquire how they may be connected, and what relative motions are thereby permitted. Consider the following simple examples of *pairs of elements*, which represent the types of contact of the touching parts of the adjacent links:—

1. Square prism in square hole.
2. Cube on plane.
3. Circular cylinder in circular hole.
4. Circular cylinder in circular hole, but with collars to stop end play.
5. Cylinder with generator touching plane.
6. Cylinder with generator and base touching perpendicular planes.
7. Sphere touching plane.
8. Sphere touching two perpendicular planes.
9. Screw in nut.
10. Spherical cap on sphere.
11. Spherical cap on sphere, but working against a circular rib.
12. Spherical cap on sphere, but working on a pivot.

It will easily be seen that the relative motions permitted are *rectilinear* in example 1, *plane* in examples 2, 4, 6, 11, and 12, but are *solid* in examples 3, 5, 7, 8, 9, and 10.

**140. Lower Pairing of Links.**—On further examination of the twelve examples just given it may be noticed that some pairs are in contact throughout the surfaces of identical geometrical form, both being plane, or the one convex and the other equally concave (often termed *solid* and *hollow*). Such pairing of links is called by some writers *lower pairing*, and is illustrated by examples 1, 2, 3, 4, 6, 9, 10, 11, and 12. When, on the other hand, the contact is along lines or points only, the arrangement of links is termed *higher pairing*. This form of connection is illustrated by examples 5, 6, 7, and 8, and will be dealt with in article 141.

Reverting now to the lower pairing, we see that in examples 1, 4, 9, 11, and 12 only one degree of freedom is left. Such cases of lower pairing are said to be *closed*. In examples 2, 3, and 10 the pairs were not closed, for in each case two or more degrees of freedom were left.

We have yet to assign names to the closed examples of lower pairing. Following the nomenclature of Reuleaux and others, we call these respectively

(*a*) A *sliding* pair. (Example 1.)
(*b*) A *turning* pair. (Example 4.)
(*c*) A *screw* pair. (Example 9.)

**141. Higher Pairing of Links.**—Passing now to the higher pairing of examples 5, 6, 7, and 8 of article 139, we find that a greater variety of relative motions is possible. But at each instant the moving link, being supposed a rigid body, is generally rotating about some axis. It may also have a motion of translation. The consideration of these two possible motions affords a clue in the examination of the matter in hand.

Thus, *simple rolling* occurs if the instantaneous axis lies in the common tangent plane at the point of instantaneous contact. But, when the instantaneous axis is the common normal at the point of contact, *simple spinning* occurs. If, however, the relative motion is such that the instantaneous axis passes through the point of contact, but is neither in nor perpendicular to the tangent plane, then that motion is *combined rolling and spinning*. Lastly, if the instantaneous axis does not pass through the point of contact, there is *sliding* combined, it may be, with rolling or spinning according to the inclination of the axis.

Thus if rolling, spinning, and sliding be denoted by the letters $R$, $N$, and $D$ respectively, then the possibilities of motion for a sphere on a plane would be denoted by ($R$), ($N$), ($D$), ($R$ and $N$), ($N$ and $D$), ($D$ and $R$), and ($R$, $N$, and $D$).

Similarly, the possible motions of a cylinder with a generator in contact with a plane would be denoted by ($R$), ($D$), ($N$ and $D$), ($D$ and $R$), and ($R$, $N$, and $D$).

**142. Non-Rigid Links.**—In the classification shown in article 134,

Table III., *A.* 3, the term linkage was restricted to deformable aggregates of rigid parts, but in the classification adopted by Reuleaux the term linkage includes all the partially deformable figures of Table III. Hence Reuleaux calls ropes, belts, chains, and the cylinders on which they wrap, examples of *tension pairing*, since no motion can be communicated except by pulling. On the same principle the water in a hydraulic press is called by Reuleaux an example of *pressure pairing* since the desired motion of the piston can only be produced by pressure. Thus, these *non-rigid or deformable links* give examples of motions which can be communicated in one way only, and not in the reverse way also through a given single link, whereas in the case of rigid links both motions may be transmitted through any single link.

Simple cases of these non-rigid links have already been dealt with in articles 135-138.

**143. Plane Linkages.**—We now commence the treatment of that most important class of kinematic chains consisting solely of rigid links whose whole motion is plane and the relative motion of adjacent links always an angular one. In other words, we have now a number of rigid links in a plane, all their contacts being turning pairs. This is the form of kinematic chain called a *linkage*; or, if one link is fixed, a *linkwork*. The discussion of various types of this class extends to article 157.

*Inversions.*—To represent the relative motion of the links we must reckon the displacements from some frame of reference. Now in any use of a linkage some one link is usually fixed. Hence, the problem of its motions is in that case obviously simplified, if lines in the fixed link are chosen as the frame of reference or axes of co-ordinates. If now, instead of the previous one, a second link is fixed and the motion of the others redrawn with reference to it, we may have a motion apparently quite different from the first, and perhaps, in some respects, really so. The linkage is now said to be *inverted*. Thus we have for a given linkage first, second, third, etc., *inversions*, the motions being perhaps apparently very different in each case.

For, it should be noticed, that although the inversion of a pair of adjacent links may cause no alteration in the relative motion of that pair, yet it may and generally does alter the motion of the moving link with respect to other bodies.

**144. Criterion of Deformability and Rigidity.**—If we consider linkages or jointed frames typified by the capital letters V, Δ, N, and W, it is evident that the first is *deformable*, the second is *rigid*, while the third and fourth have motions which are *indeterminate*, if any one link is fixed and only one displacement is specified. Again, if we have a quadrilateral linkage, first without diagonals, second with a diagonal, and third with both diagonals, it is obvious that we have three examples which are respectively deformable, just rigid, and over-rigid. In the last case we see that there is a redundant link, or one beyond the number necessary to make the arrangement just rigid.

We are here almost solely concerned with those arrangements of

links which are deformable, and that in a determinate manner specifiable by the relative position of one pair of adjacent links.

It is of interest, however, to seek an analytical criterion for the deformability or rigidity of a plane linkage. Thus, following somewhat the method of Henrici and Turner (*Vectors and Rotors*, pp. 183-4, 1903), let us consider $n$ points confined to a plane but without other constraint, then they possess two degrees of freedom each or $2n$ in all. Next let these points coincide with the centres of the eyes of rigid links, each link having two such eyes and two only. Then each link will, in general, introduce a constraint and remove one degree of freedom. Thus, if there are $m$ bars fulfilling this general condition, the number of degrees of freedom of the $n$ points is expressed by $2n-m$. But a rigid body in plane motion has three degrees of freedom, viz. two translations and one rotation. Hence if the aggregate of links is just rigid, we have

$$2n-m=3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

We may now find a relation between $n$ and $m$ depending upon the number of links meeting at an eye. Thus, of the $n$ points or eye centres, let a number $h_1$ each have only one link there, let $h_2$ points each have two links meeting there, let $h_3$ be joints of three links each, and so forth.

Then $$n=h_1+h_2+h_3+h_4+h_5+h_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Also, since each bar has two eyes, we have for the number of bars

$$m=\tfrac{1}{2}\{h_1+2h_2+3h_3+4h_4+5h_5+6h_6+\ldots\} \quad (3).$$

Hence, by (2) and (3) in (1) we have

$$\tfrac{3}{2}h_1+h_2+\frac{h_3}{2}+0-\frac{h_5}{2}-h_6-\ldots=3 \quad . \quad . \quad . \quad . \quad (4),$$

expressing the condition that the frame is just rigid. If the left side of (4) exceeds 3, then the linkage is deformable; if the left side of (4) falls short of 3, then the frame is over-rigid. It must not be supposed, however, that its degrees of freedom have fallen below 3, because the member or link added beyond those necessary for rigidity only forbade a motion which was already forbidden, and consequently introduced no new constraint. Thus equation (4), or cases where the left side has a value differing from 3, must be interpreted with caution, as they only apply to certain assumed possibilities which, though usual, are not universal.

For another way of obtaining a criterion of determinate deformability the interested student is referred to S. Dunkerley's *Mechanism*, § 9.

**145. Use of Instantaneous Centres of Rotation.**—Linkages are used to derive from one motion another motion of a different kind or magnitude. Hence, we must be able to deal with this derivation or conversion and determine the ratios of the displacements and velocities of the various parts. For this purpose we may begin at the fixed link and pass to those immediately connected to it; we can thus find the motions of certain points in other links not directly connected to the

fixed one. Then, knowing the instantaneous coplanar displacements of two such points, we can find their instantaneous centre of rotation, and thus determine the relative displacements of all points in the link in question. The conceptions here involved have already been dealt with in articles 95-97 and 101. The method of applying these principles will become sufficiently clear as the various typical cases are dealt with in order.

Where a linkage is used to produce some particular motion of one point, say, *e.g.*, a straight-line motion, a special treatment may be necessary to demonstrate this property.

### EXAMPLES—XXXI.

1. Give examples of the contact of rigid pieces or links which illustrate relative motions that are rectilinear, plane, and solid, and also lower and higher pairing.
2. What is meant by tension pairing and what by pressure pairing? Draw some example of a mechanism illustrating one of these connections, and calculate the velocity ratio involved.
3. Define plane *linkage* and *linkwork*, also state what is meant by the *inversion* of a linkage, giving drawings in illustration.
4. Discuss the various possible states of an aggregate of rigid links as to deformability, rigidity, etc., and obtain an analytical criterion for them.
5. What method is available for the determination of the motions of the various points in any moving part of a linkwork? Illustrate your answer by a diagram.

**146. Quadric Linkages.**—Let us now consider the relative motions of a plane quadric linkage having joints which admit of rotation only. A typical case is obtained if the links have lengths as 2 and 4 for one pair of opposite sides, 5 and 6 for the other pair, as illustrated in Fig. 56 by KLMN.

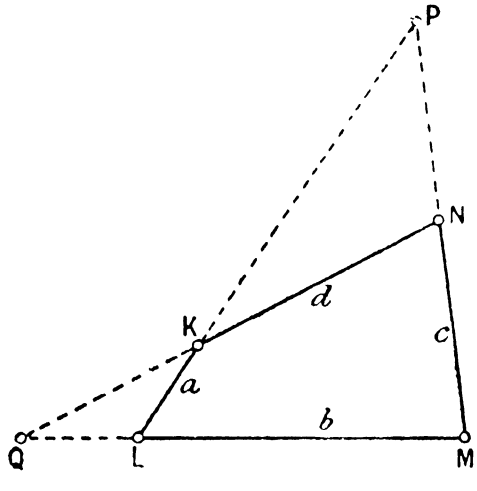



FIG. 56. QUADRIC LINKAGE AND INSTANTANEOUS CENTRES.

The instantaneous centres of relative rotation of adjacent links obviously coincides with that of the pin which connects them. Hence we have immediately four of the required centres at the angular points K, L, M, N. It can easily be shown that the instantaneous centre for either pair of opposite sides is the point of intersection of the remaining sides. Take for example the side LM as fixed, then K must move at right angles to KL, whence it follows that the instantaneous centre of KN lies on LK, produced if necessary. Similarly since N must move at right angles to MN, the instantaneous centre of KN lies along MN, produced if necessary. Thus the centre sought is the point P where LK and MN intersect. In precisely the same way it is seen that Q, the intersection

of ML and NK, is the instantaneous centre of relative rotation of KL and MN. It is noteworthy that if any three adjacent links be taken the three instantaneous centres of relative rotation of the three pairs, which can be taken from that set of three, all lie on a straight line through the two joints of the three links. Thus, choosing the three links KL, LM, and MN, or $a$, $b$, and $c$, the three pairs are $ab$, $bc$, and $ca$, and the three corresponding instantaneous centres are L, M, and Q, all lying on the straight line LM, produced where necessary. The same applies to the other three sets of three adjacent links.

Suppose now that LM is the fixed link or *frame*, then we may call LK the *crank* since complete rotation is possible to it, MN the *lever* since it can only move to and fro in a limited arc, while NK may be called the *coupler* since it couples the crank and lever (see Fig. 57). If, on the understanding of LM fixed, we examine the six instantaneous centres, it is evident that only two, L and M, remain stationary, K moves in a circle with the crank round L, N swings with the lever in a limited circular arc round M, while P and Q describe curves which may be found by the following construction:—Draw N displaced to N′ in the circle round M as centre, K displaced to K′ in the circle round L, and making K′N′ = KN. Then producing MN′ and LK′ to their intersection we have P′, the new position of P. In other words, we have P′, the instantaneous centre for LM and K′N′. It will be found that this locus of P may be a curve of two branches which intersect at M and have points at infinity. It is obviously the *space centrode* for the motion of KN.

In like manner, by taking the locus of P *as though* KN *were fixed*, we should obtain the *body centrode* for KN moving with LM fixed. We might also find the locus of Q, but it would not represent either a body or space centrode while LM is fixed.

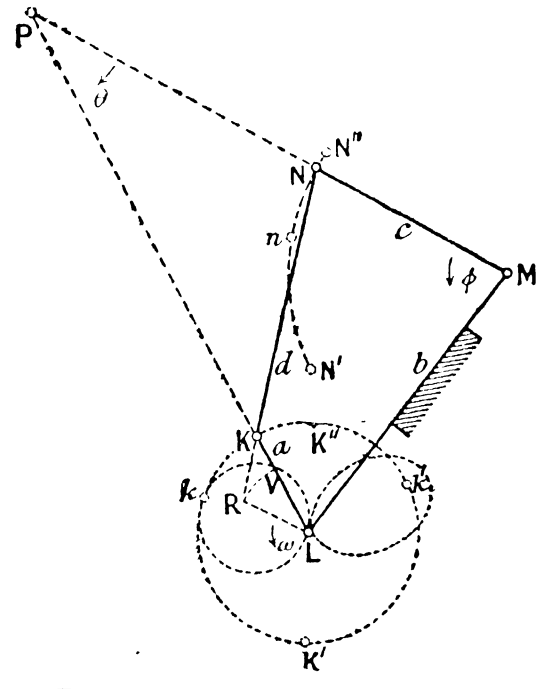



FIG. 57. VELOCITY RATIOS OF QUADRIC LINKWORK.

**147. Velocity Ratios: Polar Diagrams.**—Continuing our supposition that, in the quadric linkwork, LM is fixed as shown in Fig. 57, let us determine the linear and angular velocity ratios for the lever MN = $c$ and the crank LK = $a$.

Let $\theta$, $\phi$, and $\omega$ represent the instantaneous angular velocities of the coupler KN about P, of the lever MN about M, and of the crank LK about L respectively. Also let $u$ and $v$ be the instantaneous linear velocities of N and K. Then we have, from the figure,

$$u=\theta.\mathrm{PN}=\phi.\mathrm{MN} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and $$v=\theta.\mathrm{PK}=\omega.\mathrm{LK} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

Hence $$\frac{u}{v}=\frac{\mathrm{PN}}{\mathrm{PK}} \text{ and } \frac{\phi}{\omega}=\frac{\mathrm{LK.PN}}{\mathrm{MN.PK}} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Now produce NK to meet in R the line LR parallel to MN. Then the triangles PNK and LRK are similar, the order of the letters expressing the corresponding corners. Also describe with L as centre the arc RV cutting KL in V. Then (3) becomes

$$u/v=\mathrm{LR/LK}=\mathrm{LV/LK} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and $$\phi/\omega=\mathrm{LR/MN}=\mathrm{LV/MN}. \quad . \quad . \quad . \quad . \quad (5).$$

Thus we see from (4) that if LK describes the circle with uniform motion, and the radius LK represents to some scale the linear speed $v$ of K, then the radius LV represents to the same scale the instantaneous linear speed of N.

Again, from (5) we have that if MN represents to some scale the angular velocity of LK, then LV represents to the same scale the instantaneous angular velocity of MN.

It is thus clear that if a number of positions of the linkage were drawn and the corresponding positions of V found, we could determine the *locus* of V, which locus is called a *polar diagram* of velocity. In the present case the polar diagram resembles an asymmetrical figure eight as shown in the diagram.

**148. Analytical Conditions for the Crank and Lever.**—The letters N′ and K′ in Fig. 57 show the lowest positions of the lever and crank, N″ and K″ the highest positions, while the points $n$, $k$, and $k'$ show the one position of the lever and the two of the crank where the latter is perpendicular to the coupler.

It may be noticed that NM is like one-half of the beam of a beam engine, NK being like the connecting rod, and KL representing the engine crank. The fixed points M and L being the axes of beam and crank shaft respectively, and the 'link' LM indicated in the figure by a straight line, is the equivalent of any convenient form of framing and foundation of the engine.

The analytical conditions as to lengths of links to permit the complete rotation of the crank may be written—

(1) for passing the lower point K′ in the circle, $(a+b)\not>(c+d)$
(2) for passing the upper point K″ in the circle, $(a+d)\not>(b+c)$ (6),

*provided* that, as here supposed,

$$c<b,\ c<d, \text{ and } a<c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

A surer and safer method of ascertaining the possible motions is the graphical one developed in article 152.

It is evident from the above inequalities that we should have nothing essentially new to notice if $d$ were now fixed instead of $b$. In fact, both inversions would form what we may call the *crank and lever* form of the quadric linkage.

**149. Double-Crank Linkwork.**—If we now consider for the third

inversion of our quadric linkage that in which the shortest link, KL=$a$, is fixed, we find that new properties appear. Indeed both the links $b$ and $d$ immediately attached to the fixed link $a$ are capable of complete rotation. We may accordingly call this mechanism a *double-crank* linkwork. This is represented in Fig. 58, the links being of the same sizes as before, KL being now the fixed link or frame, $b$ and $d$ the cranks, and $c$ the coupler.

It is obvious from the figure that for the proportions in use the conditions for passing round at the right and left sides may be written—

$$c \nless a+b-d \quad . \quad . \quad . \quad (8)$$

and

$$c \nless a+d-b \quad . \quad . \quad . \quad (9)$$

respectively. And these are equivalent to the inequalities (6) of article 148.

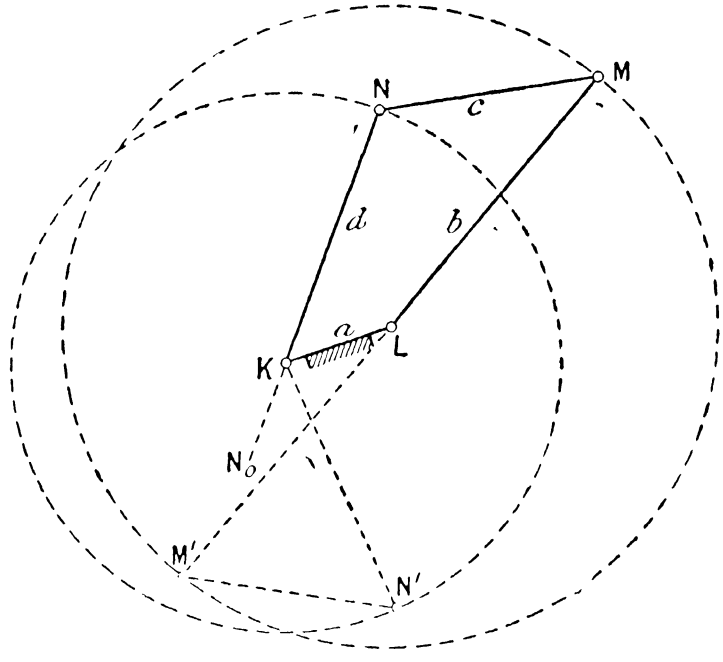



FIG. 58. DOUBLE-CRANK LINKWORK.

It may be further seen from the figure (1) that the linkwork in the lower part of its revolutions assumes the *crossed form* shown dotted by KLM′N′, and (2) that while the crank $b$ passes through two right angles between M and M′, the link $d$ passes through an angle which differs from two right angles by the very appreciable angle $N_0KN'$.

This double-crank linkwork in the asymmetrical form as illustrated occurs in feathering paddle wheels of steamboats, and also in what is known as a drag-link coupling, used occasionally on steam engines.

In the symmetrical form, in which $b=d$ and $a=c$, it occurs in the coupled driving wheels of locomotives, and may be called *parallel cranks*. In this case $a$ and $c$ are greater than $b$ and $d$.

**150. Change Points and Dead Points.**—If in a quadric linkwork we have the link $a$ fixed

and either $a+b=c+d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$,

or $a+d=b+c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$,

then it is of the double-crank type, but exhibits a special peculiarity. For, when the links are in one straight line, it may be easily seen that while it is possible for the two cranks to rotate in the same senses, it is also possible for them to rotate in *opposite* senses. In other words, the linkwork possesses a *change point* when the links are all in one straight line, as from that configuration; it may pass by similar

rotation of the cranks to the form of an open quadrilateral; or it may pass by *dissimilar* rotation of its cranks to the form of a *crossed* quadrilateral. The former is shown by the full lines in Fig. 59 and the latter by the partly dotted lines, the change point occurring when all the links are in line with the fixed link *a*. The linkwork is drawn with the links *a*, *b*, *c*, and *d* proportional to 6, 3, 5, and 2 respectively, so that the equality (11) is fulfilled.

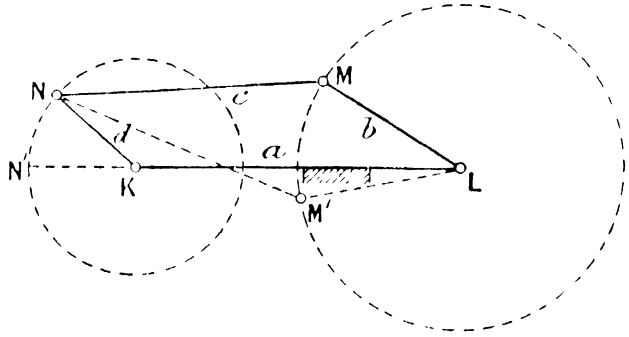



FIG. 59. CHANGE POINT OF A LINKWORK.

Thus, the term change point may be defined as one at which a lack of constraint in the linkwork allows it to pass over into another configuration. The necessary constraint may be supplied by duplicating the linkage, the cranks of the second being at an angle with the first.

A *dead point* occurs if the motion of a certain link, say a crank, cannot follow from another, say a coupler, although, if the motion of the crank is first produced, that of the coupler quite readily follows. It is evident that when, in Fig. 59, all the links lie along the line KL, we should have a dead point at N'. For no motion of the coupler produced at the end remote from N would then move the crank KN, though any motion of the crank KN would be readily followed by the corresponding motion of the coupler NM.

Hence, in the present linkwork, the dead point occurs where the change point does, as is often the case. But the two are essentially different. For the change point is present when it is possible to pass over into a different configuration or to refuse to so pass over, and this state of things is independent of which link is supposed to move the other. Whereas the dead point occurs when the configuration of the linkage is such that motion is possible if a certain link is the driver, but is impossible if another is the driver, and this state of things has nothing to do with possible changes of configuration at the point.

**151. Watt's Parallel Motion.**—We now consider the fourth inversion of the quadric linkage. In this the fixed link is usually the largest, the two adjacent links being each levers, swinging but not able to perform complete rotations, and the fourth link is a coupler joining the ends of the levers and crossing the fixed link or frame when the latter is represented by a straight line. This inversion may accordingly be termed the *double-lever* form of the quadric linkage.

If the fixed link is called *c*, the conditions that the adjacent links *b* and *d* should be unable to rotate completely round it may be derived from (8) and (9) of article 149 by interchanging the *c* and *a* and using the sign 'less than' instead of its negative. We thus have the inequalities

$$\left.\begin{aligned} a+b<c+d \\ a+d<b+c \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (12),$$

where also $a<b$, $a<d$, and $c>a$ . . . . . (13).

Such a linkwork was used by Watt in the beam engine to make one point of the connecting rod or coupler travel approximately in a straight line, the arrangement being still known as Watt's *parallel motion.* It is of interest to inquire into the necessary conditions as to the point in the coupler which gives the best approximation.

The diagram necessary for this is shown in Fig. 60, in which the links $a$, $b$, $c$, and $d$ have lengths respectively 3, 4, 11, and 8, and are represented by KL, LM, MN, and NK, the link $c$ or MN being fixed. Watt usually had the two levers $b$ and $d$ equal, but they are made

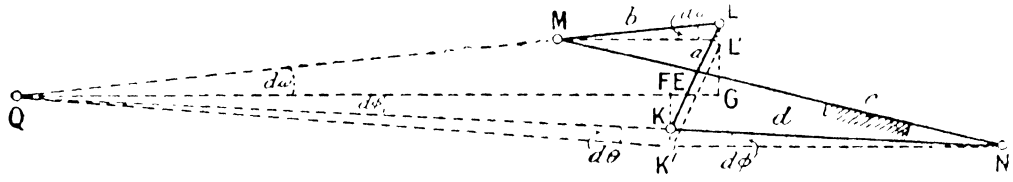



FIG. 60. WATT'S PARALLEL MOTION.

different in the diagram for the sake of generality. Of course, the frame represented by the link $c$ is not in practice straight as shown, but we are only concerned with the length MN.

First consider the linkwork in the position shown by K'L'MN, in which the links $b$ and $d$ are parallel and horizontal. Let a displacement now occur so that the levers rise through the small angles $d\omega$ and $d\phi$ to the positions ML and NK respectively. Then, on producing LM and NK to their intersection Q, we obtain the instantaneous centre of rotation of the link KL (or $a$) with respect to the fixed link MN (or $c$). Now when the links $b$ and $d$ were parallel both were horizontal, and Q was at infinity in the horizontal direction. Thus the initial motion of every point in the coupler KL was vertical. But that point in the connector which still has a vertical motion in the displaced position is clearly the point E where the horizontal through Q intersects KL. To determine the position of E in KL we may proceed as follows:—By consideration of the small angles at M and N, we have

$$\frac{d\phi}{d\omega}=\frac{K'K/KN}{L'L/LM}\doteqdot\frac{QK\,d\theta}{KN}\cdot\frac{LM}{QL\,d\theta}\doteqdot\frac{LM}{KN} \quad \ldots \quad (14),$$

in which $d\theta$ is the small angle subtended at Q by KK' and LL', and $\doteqdot$ denotes 'equals nearly.' It is easily seen that since $d\phi$ and $d\omega$ are small Q is far away, and therefore QK $\doteqdot$ QL, hence the last ratio at the right in (14).

Again, by considering the small angles at Q, we have

$$\frac{d\phi}{d\omega}=\frac{KF}{QF}\div\frac{LG}{QG}\doteqdot\frac{KF}{LG}=\frac{KE}{LE} \quad \ldots\ldots\ldots \quad (15).$$

Hence $KE/LE\doteqdot LM/KN$ . . . . . (16);

or, the segments of the couplers are inversely as the levers which it couples.

Thus, as might have been anticipated, if the levers $b$ and $d$ are equal so also are the segments into which E divides the coupler $a$, and this is the usual arrangement in actual practice.

The motion of the point E is only approximately rectilinear for a small distance. Its whole motion, if the levers swing in large arcs, is a kind of lemniscoid or asymmetrical figure of eight such that the centre portion of one part is nearly straight, the other central part being more distorted.

EXAMPLES—XXXII.

1. Draw a plane linkage of sides 3, 5, 6, and 7, and indicate the 6 instantaneous centres of rotation.
2. Show a crank and lever linkwork with sides 3, 5, 6, and 7, and obtain for it the polar diagram of velocity ratios.
3. Exhibit the linkage of the previous questions inverted so as to become a double-crank linkwork, and state the analytical conditions for complete rotation of each crank.
4. Distinguish between dead points and change points. Sketch an illustration of a dead point which is not a change point and a dead point which is also a change point.
5. Draw a Watt's parallel motion with horizontal levers of lengths 4 and 6 and a vertical coupler of length 3. Find the point in the coupler whose motion is most nearly straight, and obtain its locus for a 30° oscillation of the long lever.

**152. Graphical Criterion for Rotation in Linkworks.**—We have several times given inequalities upon whose fulfilment depends the possibility or impossibility of the complete rotation of a certain link. But these may prove troublesome to remember and apply, and a slight slip with respect to one of them may lead to an error. It is as well therefore to note that the subject lends itself to an extremely simple geometrical construction as follows:—Let $b$ be the frame, $c$, $d$, and $a$ the other links, their lengths being given. It is required to determine whether $a$ and $c$ can rotate completely. Lay off the link $b$ to scale, on a horizontal line, say. Taking M, the junction of $b$ and $c$, as centre, describe circles of radii $c+d$ and $(c\sim d)$. Then it is clear that the annular space between these circles represents the whole space which can be reached by K, the junction of $d$ and $a$, by all the possible motions of the links $c$ and $d$ about M, their point of attachment to the frame or fixed link $b$. Hence, to test the possibility of the complete rotation of the link $a$ or LK, we have simply to describe a circle of

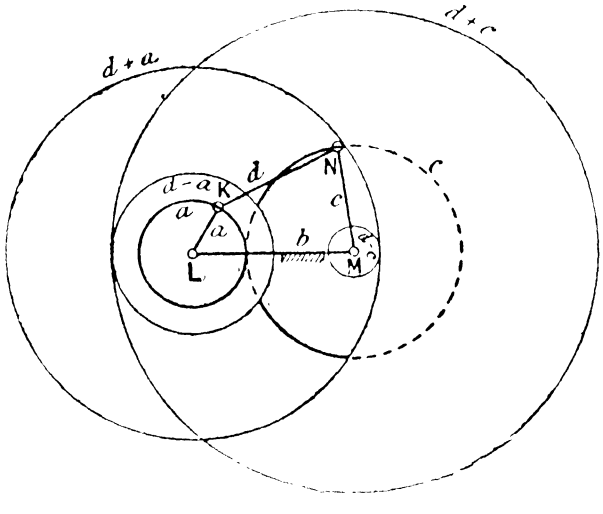



FIG. 61. GRAPHICAL CRITERION FOR ROTATION IN LINKWORKS.

radius $a$ with L as centre and note if this circle anywhere passes beyond the bounds of the annular space previously described. If it does, then complete rotation of $a$ is impossible; if it does not, then complete rotation is possible. This is illustrated in Fig. 61, in which case $a$ can rotate completely, so is a crank.

Similarly, to test the possibility of $c$'s rotation, we describe a circle of radius $c$ with centre at M and note if it lies entirely within the annular space formed by concentric circles about L, whose radii are respectively $(a+d)$ and $(a\sim d)$. As shown in the figure, the complete rotation of $c$ is clearly impossible. The parts of $c$'s rotation which are precluded are shown by dotted lines. The linkwork is accordingly of the crank and lever type. Indeed its proportions are just those of Fig. 57 in article 147.

**153. The Pantograph.**—A linkwork consisting of a parallelogram with a side produced and a fifth link fixed is used for copying drawings to a different scale, and is termed a *pantograph*. Its arrangement is illustrated by Fig. 62, from which its essential properties are easily established. In this figure KFG is the fixed link, the four moving links being KL, LMS, MN, and NK, of which KLMN form a parallelogram of sides $a$ and $b$ respectively. In the position shown, KRS is a straight line, and it always remains so. For *first*, the lengths KL, LS, RM, MS are all constant, and *second*, the two angles marked, viz. KLM and RMS, always remain equal, since KLMN is a parallelogram of fixed sides. Hence, the ratios KL : LS and RM : MS being once equal (when KRS is straight) always remain so, *i.e.* KRS remains straight.

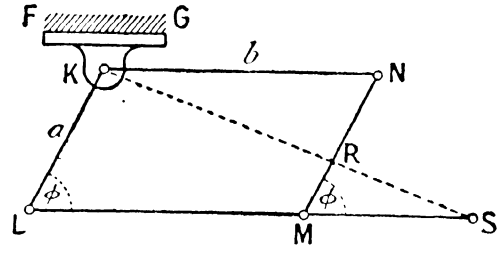


FIG. 62. THE PANTOGRAPH.

Further, the ratio KR : KS, or say $r/s$, remains constant, for it is seen to be equal to LM : LS. Thus while the point R traces about K as pole the curve whose polar equation is $r=f(\theta)$, the point S simultaneously traces the same curve, and similarly placed about K, but enlarged in the ratio $s/r$.

It may be noted here that in actual practice for beam engines Watt's parallel motion was combined with the kind of pantograph just dealt with, so that one point, like E in Fig. 60, being constrained to an approximate straight-line motion, a second point copied it as K would copy R, in Fig. 62, if S were fixed.

**154. Peaucellier's Cell.**—This eight-part linkwork due to M. Peaucellier was devised in 1864, and solves the problem of drawing accurately a straight line by geometrical means. It is represented in Fig. 63, and, while the point B describes the circle GBA, the point C describes the straight line HC at right angles to AFGH.

The link AF is fixed, and the equal link FB rotates about F. The two links AD and AE are equal, and finally the four links BD, DC,

CE, and EB are all equal. Hence, by symmetry, ABC is always a straight line, also B and C always lie on a circle whose centre is at D, thus by the properties of the circle

$$AB.AC = AD^2 - BD^2 = \text{a constant} \quad . \quad . \quad . \quad . \quad (1).$$

Produce AF to H, cutting at G the circle ABG whose centre is F, and

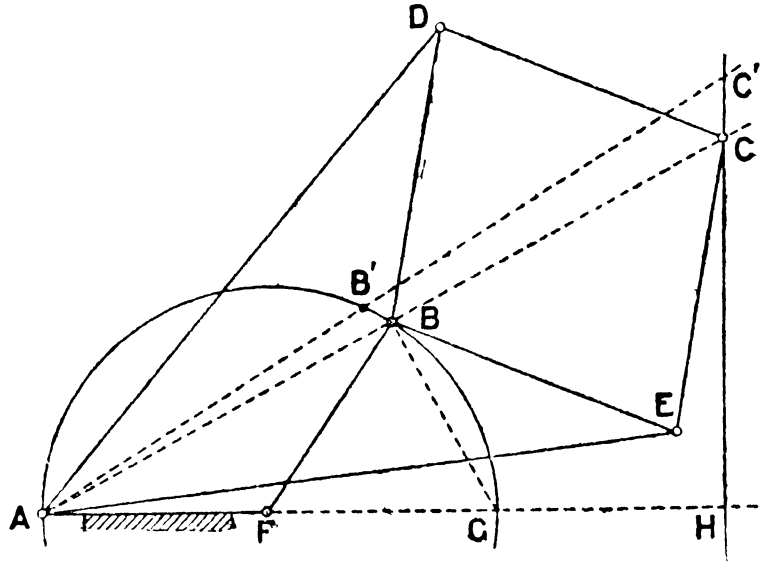



FIG. 63. PEAUCELLIER'S CELL.

let fall from C on AH the perpendicular CH. Then we have by the construction

$$AB/AG = \cos BAH = AH/AC.$$

Or $$AG.AH = AB.AC.$$

Thus by (1) we see that AH is a constant . . . . . (2).

Hence the conditions imposed on the points A, B, C by the linkwork are precisely those which correspond to the *accurate* description of the *straight line* CH by C, while B describes the circle of radius FB = FA about the centre F. Thus C' and B' show another pair of corresponding positions of the tracing points C and B. It is obvious that if A passes inside the rhombus BCDE, the point C still traces a straight line.

**155. Hart's Cell.**—Let us now consider some of the properties and uses of a linkage in the form of a contra-parallelogram as shown by KLMN in Fig. 64, in which KL = MN and KN = LM.

The contra-parallelogram may be regarded as derived from the ordinary parallelogram by folding one half through two right angles about a diagonal. Or, it may be considered as consisting of the inclined sides and diagonals of a symmetrical trapezoid. Thus LN is parallel to KM. To examine the motion and properties of the contra-parallelogram we may draw certain lines which, though not corresponding to any material in the links themselves, behave exactly like the rhombus in Peaucellier's cell (shown in Fig. 63). Thus in Fig. 64 bisect KN at B, LM at C, LN at D, and KM at E. Then, by construction and the properties of similar triangles, we see that BDCE is

and always remains a rhombus, its sides being also of constant length, namely, each equal to the half of either of the links KL or MN. Bisect KL in A, then by similar triangles ABC is a straight line. Attach at AB the equal links AF and BF, AF being fixed, also draw CH perpendicular to AF. Then the six-part linkwork, consisting of AF, FB, and the contra-parallelogram KLMNK, constitutes the *Hart Cell.* It has the property that as B moves in the circle BA of centre F and radius FB, the point C traces accurately the straight line CH at right angles to the fixed link AF. That it possesses this property is easily seen by comparing in Figs. 63 and 64 the points AFBDCE. In the former figure these points are connected by the eight actual links of the Peaucellier cell, while in the latter figure, showing the Hart cell, beyond the links AF and FB we have the contra-parallelogram KLMNK which, as already shown, serves the purpose of maintaining the distances required, though there are no links at AD, AE, BD, DC, CE, and EB, as in the Peaucellier arrangement.

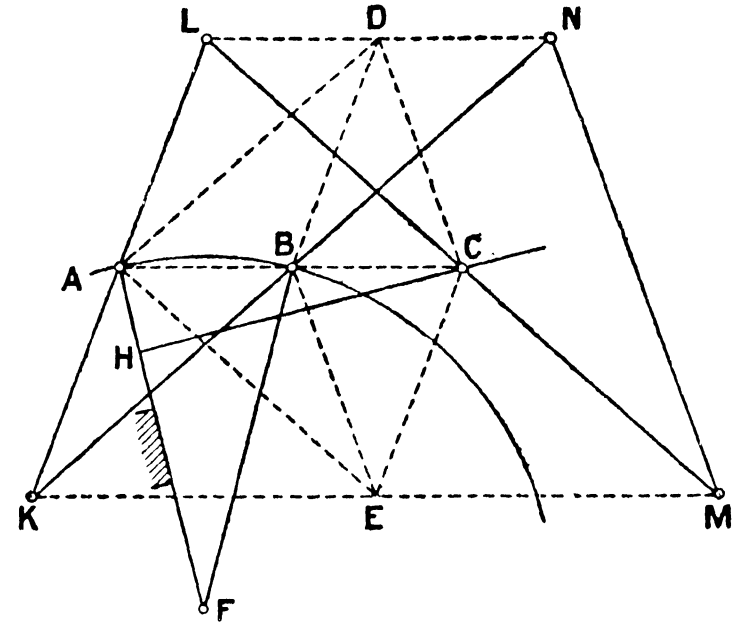



FIG. 64. HART'S CELL WITH CENTRAL POINTS.

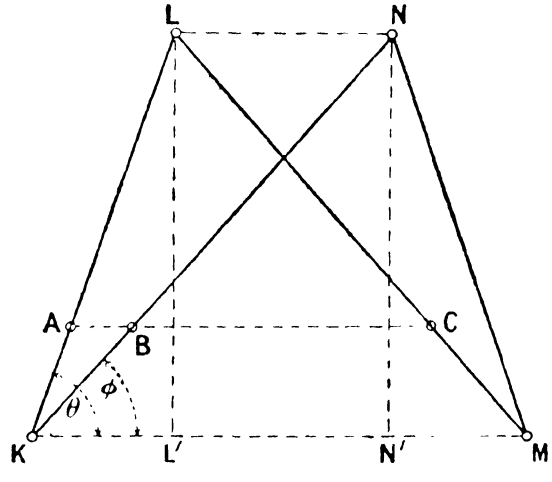



FIG. 65. HART'S CELL WITH GENERAL RATIO.

**156. Alternative Proportions for Hart's Cell.**—The proportions and use of the Hart cell just discussed are not the only ones possible. The points A, B, and C might have been taken so as to divide KL, KN, and ML in some other constant ratio instead of being points of bisection as chosen. Thus, let $KA/KL = KB/KN = MC/ML = n$. Then, it is seen from the properties of similar triangles that ABC are in a straight line parallel to KM and LN as shown in Fig. 65. We can also easily show that AB.AC is constant.

Thus, using again the properties of similar triangles, we have

$$AB=nLN,\ AC=(1-n)KM,$$

or
$$AB.AC=n(1-n)KM.LN \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

But from the figure we have

$$\left.\begin{aligned} KM&=KN'+N'M=b\cos\phi+a\cos\theta \\ \text{and}\quad LN&=KN'-KL'=b\cos\phi-a\cos\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where $a=KL=MN,\ b=KN=LM,\ \theta=LKM,\ \phi=NKM.$

Also
$$a\sin\theta=L'L=N'N=b\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Thus, substituting (4) and (5) in (3), we obtain

$$AB.AC=n(1-n)(b^2-a^2)=\text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Thus, as shown for the Peaucellier cell in equation (2) of article 154, if B moves in a circle passing through A, its centre being at F, C moves in a straight line CH at right angles to AF. (See also Figs. 63 and 64.)

**157. Parallelogram Linkages.**—If we have a parallelogram of sides $a$ and $b$ including an angle $\theta$, the diagonals being $c$ and $d$, the ordinary expressions for each of the triangles into which the diagonals divide it give

$$c^2=a^2+b^2-2ab\cos\theta,$$
$$d^2=a^2+b^2-2ab\cos(\pi-\theta).$$

Thus
$$c^2+d^2=2(a^2+b^2)=\text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Hence if the sides are equal and the diagonals lie along the co-ordinate axes and are denoted by $x$ and $y$, we have

$$x^2+y^2=\text{constant},$$

and
$$x\,dx+y\,dy=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

or
$$dy/dx=-x/y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

which gives a useful relation between the corresponding small changes in the respective diagonals.

If we have a succession of rhombuses, the sides of one figure passing for an equal length beyond the crossing to form half of the next figure, we obtain the linkage called *the lazy-tongs*. If the succession of figures lies along the axis of $x$, which accordingly coincides with one diagonal of each, then obviously a given change $dy$ in the $y$ diagonal results in changes of magnitude $dx$ in *each* of the $x$ diagonals, which lie end to end. Hence at a distance of $m$ rhombuses from the origin we shall have a displacement $+m\,dx$ consequent upon the change $-dy$ in any one of the $y$ diagonals. This we might express by writing

$$dX=m\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

or
$$X=mx.$$

A familiar example of a parallelogram linkwork is afforded by the *parallel cranks* of coupled driving wheels on locomotives.

## EXAMPLES—XXXIII

1. Apply the graphical criterion for a double-crank linkwork to a linkage of sides 2, 4, 5, and 6. Will it still be a double-crank linkwork (i) if each side is increased by 2, and (ii) if each side is doubled?

2. Describe a form of pantograph, and establish its property of copying any curve on a different scale.
3. Explain carefully the linkwork called Peaucellier's cell, and show that one point of it draws a straight line while another describes a circle.
4. Show that the property of Peaucellier's cell is attained by Hart's cell with fewer links.
5. Sketch a pair of lazy-tongs, and find its velocity ratio for the position shown.

**158. Slider Crank Chain: Instantaneous Centres.**—This important kinematic chain is derived from the quadric linkage by substituting a *sliding pair* for *one* of the four turning pairs. As each of the four links is fixed in succession it produces four distinct mechanisms, which we shall deal with presently. We may, however, first note with advantage the positions of the six instantaneous centres of the relative motions of the links in this slider chain. Thus, using the same notation as in article 146 for the quadric linkage, and the same methods as there employed, we obtain the results shown in Fig. 66. In this figure the links *d*, *a*, *b*, and *c* are united by turning pairs at K, L, and M respectively, the corresponding instantaneous centres being obviously their centres of junction denoted by these letters. The link or block *c* shown by M slides on the link or bar *d* denoted by KM; hence the instantaneous centre for the relative motion of *c* and *d* is on the line drawn through M at right angles to KM, but is situated at infinity in either direction along that line. It is accordingly indicated on the diagram by N at $\pm\infty$. Take next the centre for *b* and *d*. It is evidently at P, the intersection of KL with the line through M at right angles to KM. For L must move at right angles to KL and the centre lie along KL, while M moves along KM, and the centre accordingly lies along the perpendicular to KM at M. Finally, consider the centre of instantaneous rotation of *a* with respect to *c*. If we imagine the block *c* fixed, K must move along KM, so the centre in question lies along the perpendicular to KM at K, while L must move at right angles to ML, so the centre lies along ML, produced if necessary. The instantaneous centre sought is therefore as shown by Q. Thus of the six centres five, K, L, M, P, and Q may usually be shown in the diagram, but the sixth, N, corresponding to the sliding pair of the block *c* on the bar *d*, is always inaccessible.

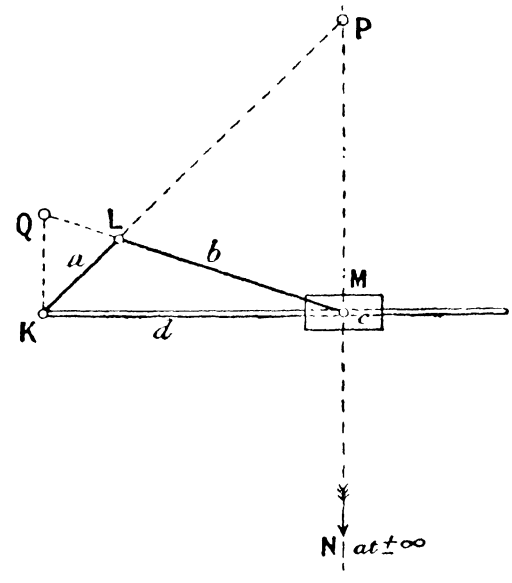



FIG. 66. SLIDER CRANK CHAIN AND INSTANTANEOUS CENTRES.

**159. Velocity Ratios obtained Analytically.**—We shall now con-

sider what may be called the first inversion of the slider crank chain, in which the link or bar $d$ is fixed, and so becomes the frame. The resulting mechanism then illustrates the case of the direct-acting engine, the block $c$ becoming the cross-head, $b$ the coupler or connecting rod, and $a$ the crank. The important relations as to displacement and velocity between the crank and coupler will be first treated analytically and afterwards graphically. Referring to Fig. 67, let the crank KL, of length $a$, make at a given instant the angle $\theta$ with the frame KOMX, the inclination of the coupler LM of length $b=na$ being at the same instant $\phi$. Let fall the perpendicular LL′ from L to the frame KOX, where O denotes the central point in the traverse or stroke of M, and denote OM by $x$. Then we have by construction

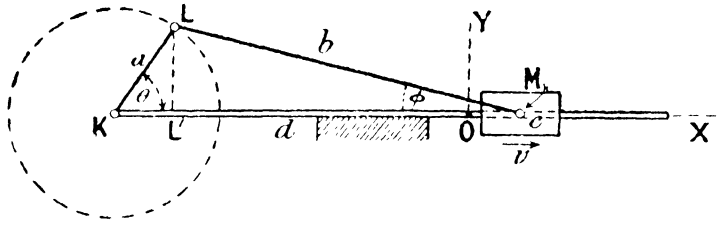



FIG. 67. ANALYTICAL DIAGRAM FOR DIRECT-ACTING ENGINE.

$$L'L/a=\sin\theta=n\sin\phi,$$

whence
$$n\cos\phi=\sqrt{n^2-\sin^2\theta}\quad\ldots\ldots\ldots\quad(1).$$

Also
$$x=KM-KO=KM-LM$$
$$=KL'+L'M-LM$$
$$=a\cos\theta+na\cos\phi-na\quad\ldots\ldots\quad(2).$$

Thus (1) in (2) gives
$$x=a(\cos\theta-n+\sqrt{n^2-\sin^2\theta})\quad\ldots\ldots\quad(3).$$

Now differentiate (3) with respect to time, writing $v$ for the linear speed of M and $\omega$ for the angular speed of L. We thus obtain

$$v=-a\omega\sin\theta\left(1+\frac{\cos\theta}{\sqrt{n^2-\sin^2\theta}}\right)\quad\ldots\quad(4).$$

And if $n^2$ is large compared with unity, as is usually the case, this reduces to the approximate formula

$$v=u\sin\theta\left(1+\frac{\cos\theta}{n}\right)\quad\ldots\ldots\ldots\quad(5),$$

in which $u$ is the linear speed of the crank pin L without regard to sign.

Referring to (3), we see that $a\cos\theta$ would be the displacement for simple harmonic motion, so that the other terms $a(\sqrt{n^2-\sin^2\theta}-n)$ are corrections for the obliquity of the connecting rod or coupler. Of course, if $n$ is $\infty$ these corrections reduce to zero. And quite low values of $n$ make the corrections small; thus if $n$ is only 5, even then $\sqrt{n^2-\sin^2\theta}-n$ never exceeds $-0{\cdot}11$.

**160. Velocity Ratios Graphically Treated.**—Referring now to Fig. 68, we will treat the same problem graphically. Thus, the link $d$ being the frame as before, and $b$ the coupler, we produce the line KL

of the crank and draw from M a perpendicular MP to KM, thus finding at their intersection P the instantaneous centre for $b$'s motion with respect to $d$. Thus, the ratio of M's velocity to that of L or $v/u$ is given by the ratio PM/PL. Produce ML to meet at V the line KV drawn perpendicular to KM. Then the triangles PLM and KLV are similar.

Hence

$$\frac{v}{u}=\frac{\mathrm{PM}}{\mathrm{PL}}=\frac{\mathrm{KV}}{\mathrm{KL}} \quad \ldots\ldots\ldots \quad (6).$$

Accordingly if $u$ is constant and represented to scale by KL, $v$, at the instant to which the diagram applies, is represented to the same scale by KV. As it is inconvenient to have any variable angle like that shown between the lengths KL and KV, we may describe about K as centre with radius KV the arc VR, thus transferring V to R. It is thus seen that R is one point of the polar diagram of velocities of M.

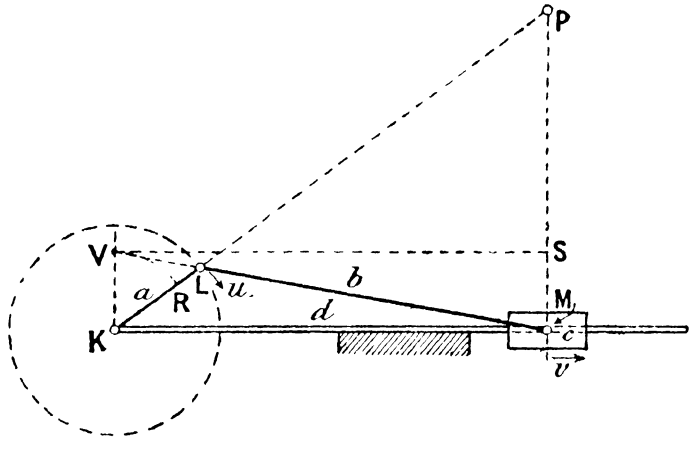



FIG. 68. GRAPHIC TREATMENT FOR DIRECT-ACTING ENGINE.

Another method of graphically exhibiting the velocity of M is to draw VS parallel to KM, cutting MP in S. Then S is one point of a cartesian diagram of velocities in which the abscissae are the *displacements* of M and the ordinates the velocities of M.

**161. Other Inversions of the Slider Crank.**—In articles 159 and 160 we have considered what may be called the first inversion of the slider crank chain, namely, that in which the bar $d$ is fixed, the resulting mechanism being that utilised in the direct-acting engine. Three new mechanisms result if each of the other links is in succession fixed. Thus if the link $b$ is fixed (see Figs. 66, 67, or 68), we obtain the mechanism used in the oscillating engine, in which $a$ is the crank as before, $d$ is now the piston rod, and $c$ is the oscillating cylinder.

Again, if the link $a$ is fixed, we obtain the Whitworth quick-return motion, in which the link $b$ may rotate *uniformly*, while $d$ moves with a variable motion in complete rotation if $b>a$, but with a swinging motion only if $a>b$.

Lastly, if $c$ is fixed, we have the mechanism used in the pendulum pump. The link $c$ here takes the form of steam and water cylinders and their connecting framework, $d$ is the piston, rod, and plunger, while $a$ takes the form of a small flywheel, of which the junction of $a$ and $d$ moves vertically, while the junction of $a$ and $b$ rotates about the

former point and swings about the junction of $b$ and $c$. It is this swinging or *pendulum* motion of the link $b$ which gives its name to the mechanism.

The displacement and velocity ratios in any of these mechanisms yielded by the various inversions of the slider crank may be dealt with as already explained in articles 159 and 160 for the first inversion.

The fuller discussion of the subject belongs rather to special treatises such as the classic by Reuleaux (translated by A. B. W. Kennedy), or the excellent and more recent *Kinematics of Machines* by R. J. Durley, which should be consulted by those wishing for further information.

**162. Screw Pairs.**—The last example of lower pairing that we shall consider is afforded by screw pairs, in which the relative motion of the parts so paired is obviously a *twist* as defined in article 130. Kinematic chains involving screw pairs frequently occur in machines. Cases of special interest are presented when two screw threads of *different* pitches, $p$ and $q$ say, are cut upon the same cylinder, each such thread engaging an appropriate nut, the nuts sliding without turning in a frame which also carries the screw and allows it to turn without sliding. We have thus a chain of four links involving two screw pairs, two sliding pairs, and one turning pair. Hence on turning the screw, while one nut advances endwise the distance $p$ the other advances the distance $q$, so that the relative motion of the nuts is $(p \sim q)$. An example of this character is often seen in the screw couplings of railway carriages, in which $q = -p$, or the screw threads are of the same pitch numerically but of opposite *hands*.

If we extend our survey so as to include fluid links, we may note as further examples of screw pairs—(i) ships' screw propellers, (ii) turbine water wheels, (iii) windmills, (iv) the rifle barrel and its projectile, and (v) steam turbine engines. For in each of these cases we have screw surfaces in use whose relative motions are accordingly of the type called a twist. In case (iv) the screw pair consists of the projectile and the rifled bore of the barrel, the fluid link being the gas which drives the projectile endwise. In each of the other cases the fluid link assumes the form of one surface of the screw pair so as to fit the other surface, which is of solid material.

**163. Higher Pairing.**—We now conclude the treatment of kinematic chains by a brief reference to examples of higher pairing, the contact between the elements here allowing them greater freedom of relative motion because they touch only along lines or points as already mentioned in articles 139-141.

Taking first examples of plane motion, the case of toothed wheels formed from right circular cylinders needs consideration. These are called by engineers *spur* wheels. It is shown in technical treatises that if the teeth of both wheels in gear are involutes of the circle of constant obliquity, or have their faces formed of appropriate epicycloidal curves, and their flanks of corresponding hypocycloidal curves, then the velocity ratio of the pair is constant and inversely as the radii of certain circles known as the *pitch* circles. These circles are concentric with

the wheels, and lie rather nearer the tips of the teeth than their root. They are purely geometrical lines, *but are in contact* at the pitch point when the wheels are running in gear, and the *pitch* of each wheel (that is, sum of tooth and space) is measured on this circle, and is, of course, of the same value in each of the wheels gearing together. Hence the radii of pitch circles of any two wheels which are to gear together are proportional to their respective numbers of teeth. We accordingly have the working formula for relative speeds:—Angular velocity ratio of gearing wheels is the *inverse ratio* of their numbers of teeth.

It is evident that this still applies approximately if the connection is not by direct meshing and contact of the teeth but by the intervention of a chain, as is usual in pedal bicycles. But the conditions for accurate *constancy* of velocity ratio may not be the same as before, and are perhaps rarely fulfilled in either case in actual practice.

Other examples among mechanisms of higher pairing and plane motion are afforded by cams, ratchets, locks, and escapements.

The case of the possible motion of an inclined ladder is an example of plane motion and higher pairing, and is easily dealt with by reference to its instantaneous centre.

Perhaps the commonest examples of higher pairing in solid motion are given by (i) *bevel wheels*, in which the teeth are formed on cones, their axes being inclined and intersecting; and (ii) the *worm and wheel*, in which the axes are at right angles and not intersecting. The worm is simply a short screw, and the worm wheel resembles a spur wheel, but has the teeth set obliquely (and often hollowed out also) to suit the worm with which it gears.

It is evident that the velocity ratio for bevel wheels is simply the inverse ratio of their teeth numbers, while that for the worm and wheel is the inverse ratio of teeth number and number of threads. Thus if the wheel has fifty teeth and the worm but a single thread, the worm turns round fifty times to the wheel's once.

Many other examples of both higher and lower pairing are dealt with in treatises on mechanisms, but are beyond the scope of the present work. Students requiring further information on this subject may with advantage refer to Dunkerley's *Mechanism*.

## EXAMPLES—XXXIV.

1. Explain by a diagram what is a slider crank chain, and find its instantaneous centres.
2. Sketch a slider crank chain with a coupler four times the length of the crank, and obtain an expression for its velocity ratio. What does this approximate to when the coupler is much longer?
3. Obtain a graph for the velocity ratio of a slider crank chain whose coupler is six cranks long.
4. What forms are assumed by the slider crank in other inversions?
5. Give several familiar examples of screw pairs, and state the velocity ratios which hold.
6. Enumerate and discuss several examples of higher pairing.

# CHAPTER X

## STRAINS

**164. Simple Strains.**—In dealing with elastic bodies the terms *stress* and *strain* were introduced in 1854 by the late Professor Rankine and have been found very useful. In the following year Kelvin modified the original usage as regards stress, and gave[1] definitions of both terms, that for strain being as follows:—

DEFINITION.—'A strain is any definite alteration of form or dimensions experienced by a solid.'

It is easily seen that a slightly modified form of this definition will allow us to apply the term to the compression or dilation of a fluid. The term has been so used by Kelvin and Tait in their *Natural Philosophy* (vol. i. p. 116, 1890), and we shall follow that precedent here. It is obvious that in the case of a fluid we cannot so easily identify the individual particles in their primitive and strained positions, but we can note the *volume* change, and that is all we require. Indeed for our present purpose we may say that a *strain* is a change in the dimensions of any given figure.

The mathematical theory of elasticity and even that of strains, which forms the kinematical preliminary to it, are both beyond the scope of this work. The subject of strains will accordingly be considered here at first in a very restricted form, the lines of a fuller treatment being just indicated later.

Imagine a unit cube with its edges parallel to the co-ordinate axes and centre at the origin. Now let all lines in the cube parallel to the axis of $x$ be elongated by the very small amount $a$, so that the faces parallel to the $yz$ plane each move normally from it by the distance $a/2$, remaining parallel to their former positions. Also let it be understood that this elongation of lines parallel to the $x$ axis occurs proportionally throughout the length of each line as well as uniformly over the $yz$ faces. Let a similar uniform and proportional small elongation of amount $e$ occur parallel to the $y$ edges, and finally one of amount $i$ parallel to the $z$ edges. Then, if the primitive position of a point P in the unstrained cube has co-ordinates $(x, y, z)$, and shifts to P′ with co-ordinates $(x', y', z')$ in consequence of the strain, the operations we have described may be represented by the equations

$$\left.\begin{aligned} x'-x &= ax \\ y'-y &= ey \\ z'-z &= iz \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1).$$

[1] *Encyclopaedia Britannica*, ninth edition, vii. p. 819.

Thus the first three vowels of the alphabet here denote the *fractional elongations* (or briefly elongations) occurring parallel to the axes $x$, $y$, and $z$ respectively.

The form of these equations shows that the elongation occurs proportionally along each line, and also that each face moves normally to its own plane and remains parallel to itself, for the change of $x$ depends on the $x$ co-ordinate alone and not on $y$ or $z$, and so for the other co-ordinates.

As will be seen more fully later, the above strain is of the type called *homogeneous*, because it is all over alike. It is also called a *pure* strain, that is, *devoid of rotation* of the body as a whole, because the three diameters of the cube elongate simply without rotation. Lines *inclined* to the diameters may change their inclination even in this case.

Further, since the directions or principal axes of elongations are parallel to the co-ordinate axes, the strain as expressed by the equations (1) assumes a very simple analytical form, and can be very easily dealt with.

**Fractional Change of Volume.**—If we now consider a parallelepiped of edges $x$, $y$, and $z$, in the unstrained state, we see that by (1) it has edges $(1+a)x$, $(1+e)y$, and $(1+i)z$ in the strained state; hence its strained volume is $(1+a)(1+e)(1+i)xyz=(1+a+e+i)xyz$ nearly, if the elongations are so small that their products are negligible, which we shall always suppose to be the case unless otherwise stated. Hence, to this approximation, $\delta$, the fractional change of volume, is given by the sum of the elongations, or

$$\delta=a+e+i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus, the *condition* for *no change* of volume is obviously

$$a+e+i=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a).$$

**165. Typical Pure Strains.**—A few strains that it is important to notice are collected in Table IV. as given in article 68 of the writer's *Sound*.

TABLE IV. TYPICAL PURE STRAINS.

| CASES. | ELONGATIONS PARALLEL TO AXES OF | | | STRAINS. |
|---|---|---|---|---|
| | $x$ | $y$ | $z$ | |
| 1 | $a$ | $e$ | $i$ | General Pure Strain about co-ordinate axes. |
| 2 | $a$ | 0 | $i$ | Plane Strain. |
| 3 | 0 | $e$ | 0 | Simple Elongation. |
| 4 | $d$ | $d$ | $d$ | Uniform Dilatation, $\delta=3d$. *Shape unaltered.* |
| 5 | $d$ | 0 | $d$ | Uniform Plane Strain. |
| 6 | $e$ | 0 | $-e$ | Simple Shear. $\delta=0$ *Volume unchanged* $\chi=e-(-e)=2e$ |
| 7 | $a$ | $-i$ | $-i$ | Elongation $a$ with lateral contractions $i$, or *axial* strain. |

The first strain in Table IV. is the general pure strain with elongations parallel to the axes of co-ordinates, the second the same reduced to two dimensions, the third an elongation merely. All these have involved change of both size and shape; the fourth is a case in which, though the size is increased, the *shape remains unaltered.* The fifth is but a simpler case of the second. The sixth, called a *simple shear*, is of special importance, since it presents the case of a change of shape *without change of volume*, the $e$ being supposed small. Of course, the *negative elongation* denoted by $-e$ in the $z$ column represents a contraction parallel to the $z$ axis and equal in amount to the elongation occurring parallel to the $x$ axis. The seventh and last case in the table is also specially important, since it is what occurs when certain solids are pulled parallel to one axis, the other two axes being unacted upon. Thus, as shown by the symbols, an endwise elongation $a$ is accompanied by $-i, -i$, *i.e.* lateral contractions, in the two perpendicular directions. The ratio of $i$ to $a$ in these cases is called *Poisson's ratio*, and will here be denoted by $\sigma$; thus

$$\sigma = i/a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

The algebraic difference of the elongations of a very small simple shear equals what is called the *amount of the shear*, a term which will be explained more fully later. Hence, denoting it by $\chi$, we have

$$\chi = 2e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

**166. Composition and Resolution of Small Coaxial Pure Strains.**—We may now take a few simple cases of composition and resolution of these simple small pure strains with elongations parallel to the co-ordinate axes. To compound two strains means to find the resultant strained state when two such strains are successively applied to the same figure. Now it is shown by the mathematical theory of elasticity that in general two pure strains result in a strain which is not pure; in other words, two pure strains give as their resultant a third pure strain, *plus a rotation.* Moreover, it is not indifferent whether the pure strain or the rotation be first applied, the two not being commutative. But with our present limitation to strains along the same co-ordinate axes this difficulty will not trouble us, as, in that case, two or more pure strains will give a pure strain as their resultant. Further, since the applications of elongations $d$ and $e$ to any axis means first multiplying all lengths parallel to it by $(1+d)$ and then by $(1+e)$, the resultant will be the factor $(1+d)(1+e)$. But since $d$ and $e$ are each supposed to be very small, the resultant factor is $(1+d+e)$ nearly. That is, the resultant elongation is the simple *sum* of the two or more *component small* elongations. We may thus write down the component elongations in order of the axes along which they occur and add them for the resultant elongations.

**167. Plane Strains.**—Take first the case of compounding a uniform plane strain and a shear so as to produce a general plane strain; *i.e* referring to Table IV., we are to compound strains 5 and 6 to build up strain 2.

We thus have the components ($d$, 0, $d$)

and $(e, 0, -e)$

with which to build up $(a, 0, i)$.

Hence $d+e=a$ and $d-e=i$,

or $2d=a+i=\delta$ . . . . . (5),

and $2e=a-i=\chi$ . . . . . (6).

Thus the general plane strain resolves into a uniform plane dilation whose elongation is *half the sum* of the initial elongations, and a simple shear whose elongation is *half their difference*. Or, taking the quantities on the right sides of (5) and (6), we may say that a plane strain involves a fractional increase of volume equal to the *sum* of the elongations, and a shear whose amount is their *difference*.

**168. Uniform Dilatation and Two Shears.**—Let us now build up the first strain in the table from a uniform dilation and two shears by the following scheme:—

Components $\left\{\begin{array}{l}(d, \quad d, \quad d)\\ (e_1, \quad 0, -e_1)\\ (e_2, -e_2, \quad 0)\end{array}\right.$

Resultant $(a, \quad e, \quad i)$

Then by addition we have $d+e_1+e_2=a$,

$d \quad -e_2=e$,

and $d-e_1 \quad =i$.

Whence $3d=a+e+i=\delta$,

$3e_1=a+e-2i=3(d-i)$,

and $3e_2=a-2e+i=3(d-e)$, . . . (7).

Since we have thus built up with the given strains, the most general one in the table, it is evident that any other strain can be made from these by giving suitable values to $a$, $e$, and $i$. Hence we have the important kinematical theorem that *any of the strains in Table IV.* can be built up of *one* strain involving change of *size only*, and *two* others, each of which involves change of *shape only*. Thus, for the third case in the table, a simple elongation $(0, e, 0)$, we have from (7), putting $a=i=0$,

$$d=e_1=e/3 \text{ and } e_2=-2e/3 \text{ . . . . . (8).}$$

**169. Composition of Axial Strains.**—We now take three strains of the type in the last line of Table IV. with which to build up any of those in the table. We begin therefore by obtaining the first case, which being general includes all the rest. Taking the value of $\sigma$ the same throughout, we have accordingly the following scheme:—

Components $\left\{\begin{array}{l}(\quad a_1, -\sigma a_1, -\sigma a_1)\\ (-\sigma a_2, \quad a_2, -\sigma a_2)\\ (-\sigma a_3, -\sigma a_3, \quad a_3)\end{array}\right.$

Resultant $(a, \quad e, \quad i)$

Hence by addition $a_1-\sigma a_2-\sigma a_3=a$,

$-\sigma a_1+a_2-\sigma a_3=e$,

and $-\sigma a_1-\sigma a_2+a_3=i$.

Whence, on solving for $a_1$, $a_2$, and $a_3$, we find

$$\left.\begin{aligned} a_1(1+\sigma)(1-2\sigma) &= a(1-\sigma)+(e+i)\sigma \\ a_2(1+\sigma)(1-2\sigma) &= e(1-\sigma)+(i+a)\sigma \\ a_3(1+\sigma)(1-2\sigma) &= i(1-\sigma)+(a+e)\sigma \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9).$$

Thus, to build up the simple elongation $e$ along the $y$ axis from three axial strains, we have merely to write $a=0$ and $i=0$ in (9). We then find as the solution

$$\left.\begin{aligned} a_1(1+\sigma)(1-2\sigma) &= e\sigma \\ a_2(1+\sigma)(1-2\sigma) &= e(1-\sigma) \\ a_3(1+\sigma)(1-2\sigma) &= e\sigma \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

This result will enable us to find in a later chapter what lateral forces must be applied to prevent contraction or bulging. (See equation (10) of article 462.)

Again, to analyse a uniform dilatation $(d, d, d)$ into three axial strains, we write $a=e=i=d$ in (9) and obtain the solution

$$a_1=a_2=a_3=d/(1-2\sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Lastly, let us compound axial strains so as to result in the simple shear $(e, 0, -e)$. Then, writing these values on the right side of (9) instead of $(a, e, i)$, we have

$$\left.\begin{aligned} a_1 &= \phantom{-}e/(1+\sigma) \\ a_2 &= \phantom{-}0 \\ a_3 &= -e/(1+\sigma) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

**170. Shear viewed as a Sliding.**—It is now important to take an entirely different view of the strain called a simple shear. We have hitherto regarded it as consisting of an elongation ($e$) with equal contraction ($-e$) at right angles. But it is also possible to regard it as *a progressive relative sliding of undistorted planes*; for, strange as it may appear at first, these two statements of the matter lead to precisely the same type of strain. The agreement of these two descriptions can best be traced out by reference to the Figs. 69, 70, and 71.

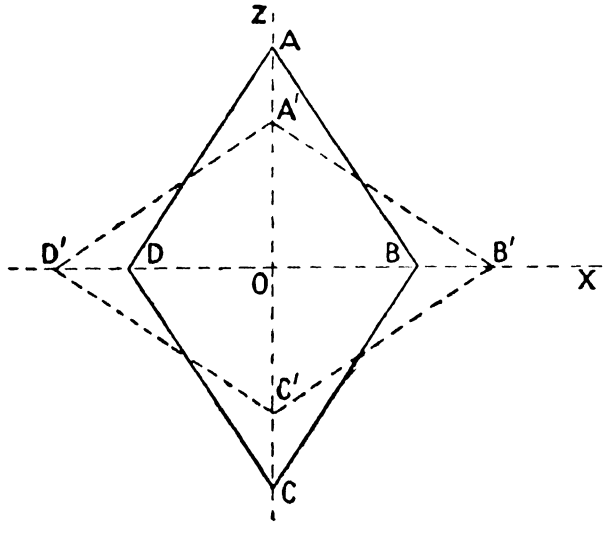



FIG. 69. RHOMBUS BEFORE AND AFTER A SIMPLE SHEAR.

In Fig. 69 a rhombus before shear is represented by the full lines ABCD, and after the shear $(e, 0, -e)$ by the dotted lines A'B'C'D'. Though the elongations are supposed to be very small, they are represented large in the figure for clearness' sake. The primitive or unstrained rhombus has semi-axes $OA=1+e$ and $OB=1$. Thus, to our usual approximation for small strains, the semi-axes of the final or strained rhombus are $OA'=(1-e^2)=1$ nearly and $OB'=1+e$. Hence

the strained rhombus has axes and sides of the same size as the original one; the angles at corresponding letters with and without accents are accordingly interchanged. Thus, the acute angles at A and C become obtuse ones at A′ and C′ exactly equal to those originally at B and D, which by the strain have become the acute ones at B′ and D′ equal to those at A and C.

It accordingly follows that by a proportionate sliding of all lines parallel to one side, say BA, in the direction from C to D while BA is itself fixed, we can change the original figure to the shape of the final one, but by this sliding we should also *displace* the *centre* of the figure and *rotate the axes*. This is easily seen by reference to Fig. 70, in which as before ABCD represents the original rhombus, and ABC′D′ now represents the final form and position of the equal rhombus after the strain, BA being held fixed. In this case we must not regard the figure as a linkwork which swings with links BC and AD of fixed lengths about B and A as centres. On the contrary, we must regard the solid body, which the figure represents, as divided into infinitely thin plane layers parallel to BA and perpendicular to the diagram and sliding parallel to BA, as though it were a pile of paper or a thick book whose covers are AB and CD, which remain the *same distance* apart during their relative motion, each such sheet of paper or leaf being *undistorted* as it slides.

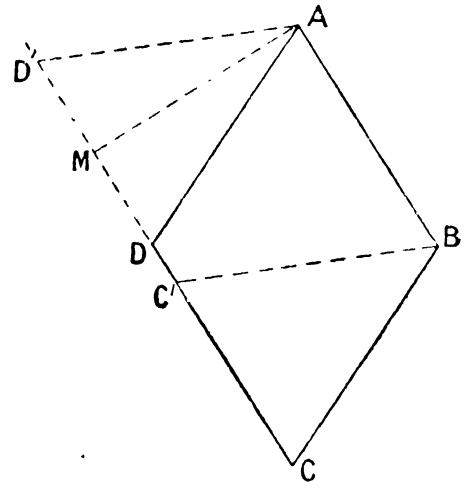



FIG. 70. SIMPLE SHEAR AS A SLIDING PARALLEL TO BA.

**Amount of Shear.**—This new view of the simple shear also leads to a new mode of measuring it. Thus, referring to Fig. 70, the sliding distance DD′ divided by the perpendicular distance AM is a measure of the shear, and is called its *amount*. Hence we may say generally, the *amount of a shear* is the amount of relative *sliding* of parallel undistorted planes *per unit distance apart*. Though this definition shows that the amount of the shear, is strictly twice the tangent of the angle MAD, it is evident that for very small shears we may also regard it as the value in *radians* of the angle DAD′.

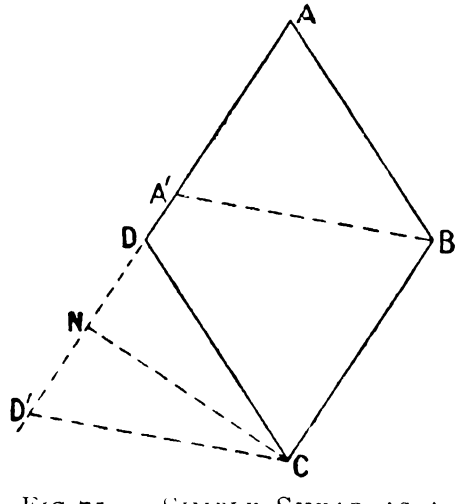
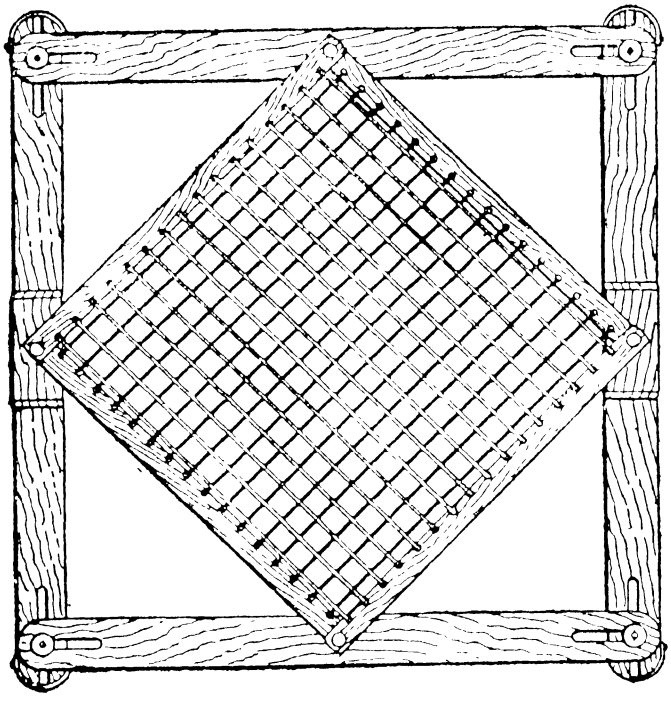


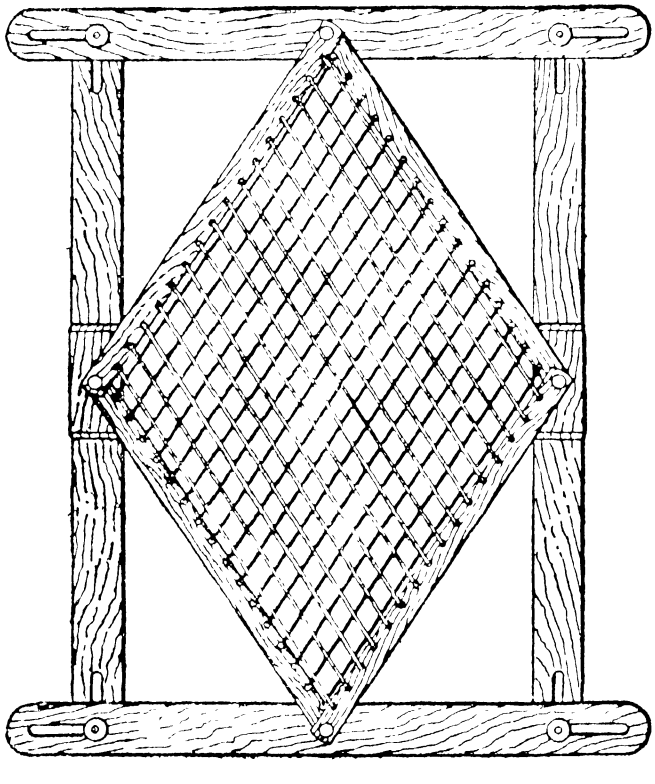

FIG. 71. SIMPLE SHEAR AS A SLIDING PARALLEL TO BC.

**171. A Shear presents two Slidings.**—We have still to notice that the shear may be viewed as a sliding of undistorted planes parallel to the

other pair of sides of the original rhombus. Thus if, as shown in Fig. 71, we keep BC at rest, we may pass from the primitive rhombus ABCD to the strained one A'BCD' by the proportionate sliding of undistorted planes in the direction AD parallel to BC.



*A*. UNSTRAINED POSITION.



*B*. STRAINED POSITION.

FIG. 72. A SHEAR PRESENTS TWO SLIDINGS.

The amount of the shear is now the quotient DD' divided by CN, and is obviously of the same value as before.

The fact that a shear presents these *two slidings of undistorted planes occurring simultaneously* and parallel respectively to AB and to BC may be illustrated to a class by the frame, of which a photographic reproduction is given in Fig. 72.

The outer part is of mahogany with brass bolts and lock nuts at the corners, which have slots to allow opposite sides of the rectangle to be approached or separated by the hands, thus representing the first view of the shear. The inner rhombus is of brass, and represents also the shear on the first view, the elongations and contractions being as in the rhombus ABCD of Fig. 69. Across this rhombus, parallel to one pair of sides, a number of *white* cords pass to represent the sliding of one set of undistorted planes. In the direction parallel to the other pair of sides *black* cords pass to represent the other set of undistorted planes and their simultaneous

sliding in another direction. If the model is held in front of a brown or grey background both sets of cords are seen simultaneously; or, if held in front of a black board or white screen the white or black cords are respectively brought into prominence as desired, the other set being at the time scarcely noticeable. It should be borne in mind that this model is to represent the simultaneous double sliding of two sets of undistorted planes, and must not be supposed as correctly representing *every* feature of a shear. Thus the brass rhombus causes the parallel cords when sliding to slightly change their distance apart, which is unlike the actual shear.

**172. The Amount of a Small Shear is twice its Elongation.**—Denoting now the amount of a small shear by $\chi$, its elongations being $(e, 0, -e)$, we can easily show, by reference to Figs. 69 and 70 of article 170, that $\chi = 2e$, *i.e.* the *amount* of the shear is *double the elongation*, or *equals their algebraic difference*. Thus, referring to Fig. 70 and then to Fig. 69, we have

$$\chi/2 = \tan \text{MAD(Fig. 70)} = \tan (\text{OBA} - \text{OAB})\text{(Fig. 69)}.$$

But, on Fig. 69,

$$\tan \text{OBA} = 1 + e \text{ and } \tan \text{OAB} = 1/(1+e) = 1 - e \text{ nearly.}$$

Hence, by the formula for the tangent of the difference of two angles

$$\chi/2 = \frac{1+e-(1-e)}{1+(1+e)/(1+e)} = \frac{2e}{2} = e,$$

or
$$\chi = 2e = e - (-e) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

as was to be shown.

It should also be noticed from Figs. 69, 70, 71, and 72 that the plane of the elongation and contraction which describe the shear in the first way is also the plane of the slidings which specify it in the second way. This plane is called the *plane of the shear*. The *axes of the shear* are those of the elongation and contraction, and are obviously, for the small shear under consideration, *inclined at angles of* 45° *to the lines of sliding*, which are the intersections of the plane of the shear with the parallel undistorted planes that slide.

## EXAMPLES—XXXV.

1. Define *strain*, and state the condition for a strain to be *pure*. Give two examples of strains, and find in each case the volume change involved.

2. Specify five different strains, expressing each by its elongations along the co-ordinate axes. Show that any homogeneous strain may be made by compounding a strain involving change of *size* only with others, each involving change of *shape* only. Give a numerical example.

3. Show that three axial strains are competent to build up any homogeneous strain whatever.

4. Find the axial strains to produce—

   (*a*) A uniform dilation whose fractional volume change is 0·006.
   (*b*) A simple shear of which the amount is 0·006.
   (*c*) A simple elongation equal to 0·001.

   Take $\sigma = 0{\cdot}2$ throughout, and check your results by addition of the axial strains obtained.

5. Explain carefully how a shear may be viewed as a progressive sliding of parallel planes without other distortion of those planes. Prove also that for a small shear its amount is twice each of the elongations involved.

**173. Homogeneous Strains.**—It will be well now to slightly broaden our view and pass from the pure strains hitherto considered to homogeneous strains, which form a rather more general class. The distinction between the two was just mentioned in article 164; we may now define as follows, quoting Kelvin and Tait:—

DEFINITION.—'If, when the matter occupying any space is strained in any way, all pairs of points of its substance which are initially at equal distances from one another in parallel lines remain equidistant, it may be at an altered distance; and in parallel lines, altered, it may be, from their initial direction; the strain is said to be homogeneous.'

*Properties of Homogeneous Strain.*—'Hence if any straight line be drawn through the body in its initial state, the portion of the body cut by it will continue to be a straight line when the body is homogeneously strained. For, if ABC be any such line, AB and BC, being parallel to one line in the initial, remain parallel to one line in the altered, state; and therefore remain in the same straight line with one another. Thus it follows that a plane remains a plane, a parallelogram a parallelogram, and a parallelepiped a parallelepiped.'[1]

Further, similar and similarly situated figures in the primitive state remain similar and similarly situated figures after a homogeneous strain. The lengths of parallel lines in the body are all altered in the same proportion. Thus, any plane figure changes to another plane figure, which is a magnified or diminished orthographic projection of the first on some plane. Accordingly, an ellipse remains an ellipse and an ellipsoid remains an ellipsoid. For, since the sections of an ellipsoid are all ellipses, they can only be changed to other ellipses, which are therefore the sections of another ellipsoid. The circle and sphere are here each included in the more general terms ellipse and ellipsoid.

In particular, let us notice that a circle becomes an ellipse in which any pair of conjugate diameters were perpendicular diameters of the circle. But the major and minor axes of the ellipse are perpendicular conjugate diameters, and therefore have *not* been changed in *mutual* inclination by the strain, though they may have been moved thereby *from their original directions* when perpendicular diameters of the circle.

**174. Strain Ellipsoid.**—Take next a sphere in the primitive or

[1] *Natural Philosophy*, Part I., articles 155-156, p. 116, 1890.

unstrained figure, and describe a cube about it. Then the homogeneous strain will change the sphere to an ellipsoid and the circumscribing cube to a parallelepiped, not, however, in general to a rectangular one. But the six points of contact of the sphere and the cube, being the ends of three rectangular diameters of the sphere, become the ends of three conjugate diameters of the ellipsoid, and are still the points of contact of the inscribed and circumscribed figures. Hence if, at the outset, we rightly chose the orientation of the cube with respect to the strain about to occur, then that cube would become a rectangular parallelepiped touching the inscribed ellipsoid at the ends of its three perpendicular conjugate diameters, *i.e.* its principal axes. We must remember, too, that whatever happens to a sphere in one part of a body experiencing a homogeneous strain, happens also to any other sphere anywhere else in that body.

That ellipsoid which is produced by a homogeneous strain from a portion of the body initially spherical is called *the strain ellipsoid.* Thus, as we have seen, the principal axes of this ellipsoid being derived from perpendicular diameters of a sphere have suffered *no change in their mutual inclination*, but they have, in general, experienced *a common change* in their directions from the configuration which they had when diameters of the primitive sphere.

But we have also seen (in articles 113-115) that a rigid body with one point fixed can pass from one position to any other by rotation through a calculable angle about a specified axis through the fixed point. Hence, there must be one line in the figure devoid of change in inclination. And this line is the axis about which the three rectangular axes may be regarded as rotating from their original to their final position without any change in their mutual inclination.

The three principal axes of the strain ellipsoid are called the *principal axes* of the strain by which it was derived from a sphere. The *principal elongations* of a strain are those which occur along these axes. When the strain is as general as possible for the given axes, all the elongations are different. Hence, the principal axes of the corresponding ellipsoid are all different. Thus, if the sphere were of unit radius and the elongations $(a, e, i)$ where $a > e > i$, the semi-axes of the ellipsoid in order of decreasing magnitude are $1+a$, $1+e$, and $1+i$, along the axes of $x$, $y$, and $z$ say. Then, along the axis of $x$ the elongation is a maximum, along that of $z$ a minimum, while along the axis of $y$ the elongation is a minimum for all directions in the $xy$ plane, but a maximum for all directions in the $yz$ plane. A contraction is to be counted as a negative elongation, and the above statement taken in the algebraic sense.

If two of the elongations are equal, the ellipsoid becomes an ellipsoid of revolution, *i.e.* a spheroid, oblate, or prolate. If all three elongations are equal, it becomes a sphere, and the strain is a uniform dilation or contraction.

**175. Analytical Representation of Homogeneous Strains.**—Let us now express a homogeneous strain by a set of equations. Take O, the

origin of cartesian axes, at a point of the body that remains unmoved by the strain. Let $x$, $y$, $z$ be the co-ordinates of any point P before the strain, and $x'$, $y'$, $z'$ those of P', its new position in consequence of the strain, which we will suppose to be as general as possible subject to its being homogeneous. Now we have seen that any three perpendicular diameters of a sphere in the primitive state become three conjugate diameters of the strain ellipsoid, and are consequently in general changed in length and in mutual inclination. Hence to fully express a homogeneous strain we need to indicate what becomes of three lines not originally coplanar. For simplicity's sake we will take these along the co-ordinate axes and of unit length. Obviously each line can suffer a change both of length and of inclination, and the latter needs two angles to specify it. We accordingly need three constants to state what happens to each of our three unit lines, *i.e.* *nine* constants in all. We may conveniently take these as constants expressing the displacements of the ends of the three unit lines parallel to each of the co-ordinate axes respectively.

Thus, let the end of the unit line from the origin along the axis of $x$ shift by $a$, $d$, and $g$ parallel to the axes $x$, $y$, and $z$ respectively. Let the end of the unit line along the $y$ axis have like shifts $b$, $e$, and $h$. Finally, let the end of the unit line along the $z$ axis have shifts $c$, $f$, and $i$. Then, if we multiply the shifts for a unit line by the value of a co-ordinate in the same direction, $x$ say, we should obtain the shifts for the end of a line of original length $x$. This therefore applies to the $x$ co-ordinate of P, similar remarks holding for the $y$ and $z$ co-ordinates. But the strained values and positions of these three co-ordinates meeting in O, and originally $x$, $y$, and $z$, are the three adjacent edges of the oblique parallelepiped whose opposite corner is P', the strained position of P.

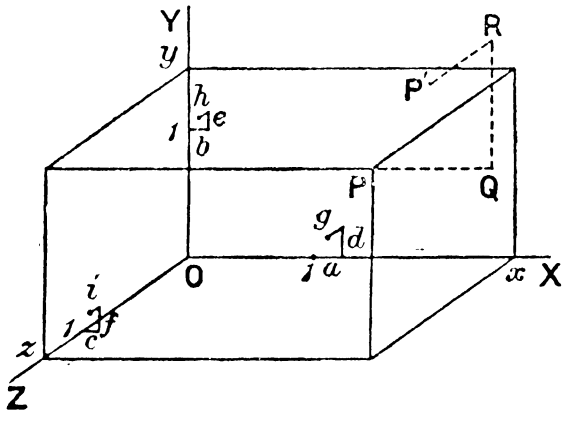



FIG. 73. NINE CONSTANTS OF HOMOGENEOUS STRAIN.

The shifts from P to P' parallel to the fixed co-ordinate axes are accordingly given by the expressions

$$\left.\begin{aligned} x'-x &= ax+by+cz \\ y'-y &= dx+ey+fz \\ z'-z &= gx+hy+iz \end{aligned}\right\} \quad \ldots\ldots \quad (14).$$

It is seen that the coefficients on the right side are the first nine letters of the alphabet taken in order. We also see from (14) that the shifts of any point in a body experiencing a homogeneous strain are *linear functions of its co-ordinates.* The equations can easily be verified by reference to Fig. 73, which shows the meaning of the nine constants

and represents by PQ, QR, and RP′ the values of $x'-x$, $y'-y$, and $z'-z$ respectively.

Of the nine constants in (14), it should be noted that the vowels $a$, $e$, $i$ denote the elongations parallel to the axes of $x$, $y$, $z$, as in Table IV. and elsewhere. The consonants $b$, $c$, $d$, $f$, $g$, $h$, on the other hand, show the amounts of the shears reckoned as the relative slidings of the three pairs of planes to which the co-ordinate axes are normal, one of each of these pairs of planes having slidings in the two directions parallel to its edges. Thus, referring again to Fig. 73, $d$ shows the amount of the shear suffered by the body by the sliding of the plane $Px$ parallel to the axis of $y$, *i.e.* $d$ is the relative slide parallel to $y$, of planes parallel to $yz$, per unit distance apart along the axis of $x$. Similarly $g$ is the amount of the shear reckoned as the sliding of the same planes parallel to $z$, and so on for the other four consonants, as may be seen from the figure.

**176. Rotation in Homogeneous Strains.**—Let us now inquire if the homogeneous strain expressed by (14) involves any rotation; if so, what modification in it would correspond to the elimination of that rotation, and so reduce it to a pure strain. To deal generally with the problem requires the use of solid analytical geometry and the discussion of the resulting cubic equation. But the following simple geometrical treatment gives a useful preliminary insight into the matter.

Consider a cube of the unstrained substance, and take co-ordinate axes parallel to its edges with origin at its centre, also let its sides be of length two units. Take a section of this cube in the $xy$ plane as represented in Fig. 74. Hence, since $z=0$ in the plane of the diagram, the equations (14) reduce to

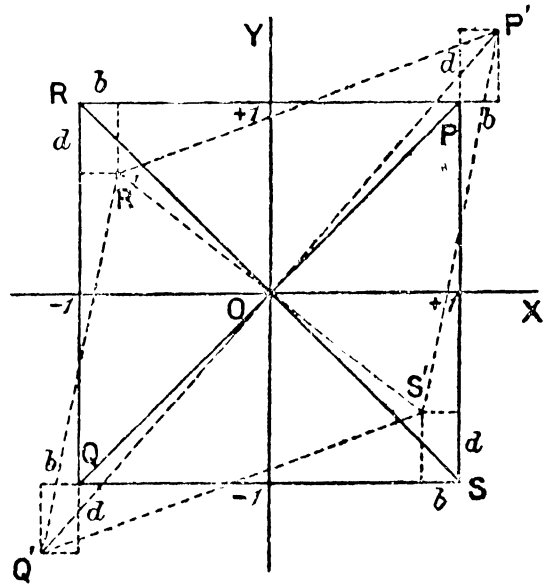



FIG. 74. UNEQUAL SLIDES INVOLVE ROTATION.

$$\left.\begin{aligned} x'-x &= ax+by \\ y'-y &= dx+ey \\ z'-0 &= gx+hy \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (15).$$

But as the constants $a$ and $e$ express elongations *without rotation* along the rectangular axes of $x$ and $y$ respectively, we may omit them from our present consideration. Further, since the components of $z'$ will only involve a rotation of our sectional plane about the axes of $y$ and $x$ respectively, and be invisible in the diagram, we ignore them also, and confine our attention to the constants which may express rotation

in the plane of the diagram that is about O (or about the axis OZ). Hence for this plane case (15) finally reduces to

$$x'-x=by \text{ and } y'-y=dx \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Thus, following the equations (16), the square PQRS, as shown in Fig. 74, is changed by the strain to the parallelogram P'Q'R'S'. And it is seen that not only are the diagonals of the square PQ and RS each rotated, but that they are *both rotated in the same direction*, viz. counter-clockwise, if $d>b$. Now, if the original square had been subject to unequal elongations parallel to the axes of $x$ and $y$, the diagonals would have been each rotated, but in opposite directions; P and S approaching and P and R separating, or *vice versa*. Thus we see that *opposite* rotations of certain lines originally perpendicular are consistent with a pure strain. But rotations of such lines in the *same* way are *not* consistent with a pure strain, for they clearly involve a *rotation on the whole* in the direction in question.

**177. Conditions for Pure Strain.**—If now $d$ and $b$ interchange the values assigned to them in the diagram (Fig. 74); or, keeping the same values, the positions of these consonants in equations (16) are interchanged; then, in either of these equivalent cases, the pure strain involved is the same in character and magnitude as before, and the rotation is the same in numerical value, but *reversed* in sign.

Hence, if $d$ and $b$ have the same values, the rotation, if any, is still reversed by their interchange. But since the interchange of equals has no effect, there cannot be in that case any rotation to reverse. This could also be easily seen by drawing $d$ equal to $b$ on Fig. 74, when obviously the diagonal PQ would be simply lengthened without rotation, and the diagonal RS shortened only, also without rotation.

Referring again to equations (14) of article 175, we have now shown that the equality of $d$ and $b$ means no rotation about OZ of the sections parallel to the plane of XY. Similarly, therefore, the equality of $g$ and $c$ would mean no rotation about OY of the section parallel to the plane of ZX. Finally, the equality of $h$ and $f$ would correspond to no rotation about OX of the planes parallel to YZ. Thus, with all the three equalities fulfilled, we have a strain devoid of rotation. We may accordingly write as the *conditions for a pure strain*

$$d=b,\ g=c, \text{ and } h=f \quad . \quad . \quad . \quad . \quad . \quad (17),$$

the *pure strain* being itself analytically expressed by

$$\left.\begin{aligned} x'-x&=ax+by+cz\\ y'-y&=bx+ey+fz\\ z'-z&=cx+fy+iz \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Thus, the *nine* constants for the homogeneous strain (equation (14) of article 175) being reduced by *three* on introduction of the conditions for a pure strain of equation (17), leave us the *six* constants of equation (18). And it may easily be seen that six consonants are necessary and sufficient to specify any quite general pure strain, since three constants are needed for the three principal elongations, and three more to define the principal axes about which they occur. For, with respect to the

co-ordinate axes, two angles give one principal axis of elongation, and a third angle then suffices to fix the other two principal axes of elongation, since all three are mutually perpendicular.

**178. Pure Strain analytically derived from Homogeneous Strain.**—Let us now derive the conditions for a pure strain in a more formal analytical manner. In the primitive state, let $x$, $y$, $z$ be the co-ordinates of a point P distant $r$ from the origin, and, by the strain, let this become P′ of co-ordinates $x'$, $y'$, $z'$ distant $r'$ from the origin, but *without angular displacement* of the line OP, which accordingly suffers elongation only in the ratio $r : r' = 1 : \lambda$ say. Then we have

$$x'/x = y'/y = z'/z = \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

Also, both before and after the strain, the line OP, or OP′, has direction cosines

$$x/r,\ y/r,\ z/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20);$$

or, the same letters with accents all through.

If we were to write $x' = x\lambda$, etc., from (19) in equations (14) of article 175, we should obtain three linear equations, and on eliminating from them the two ratios of $x$, $y$, and $z$ we should leave a single equation, namely, a cubic in $\lambda$, which must accordingly have three roots. Of these roots one must be real, and the other two may be both real or both imaginary.

But, for our present purpose, it is unnecessary to write these equations and derive the cubic. We need only to take the case in which the three roots are real, $\lambda_1$, $\lambda_2$, $\lambda_3$ say. These correspond to three directions along which elongation free from rotation occurs, and we shall further suppose them to be mutually perpendicular, points on each line being denoted by $x$, $y$, $z$, and $r$, with subscripts 1, 2, and 3 like the $\lambda$'s. We have accordingly to determine the condition that the roots should be real and correspond to mutually perpendicular lines.

Then, making in equation (14) the substitutions $x' = \lambda x$, etc., for two of the lines, and writing $a$ for $(1+a)$, $\epsilon$ for $(1+e)$, and $\iota$ for $(1+i)$, we find

$$\left.\begin{aligned} \lambda_2 x_2 &= a x_2 + b y_2 + c z_2 \\ \lambda_2 y_2 &= d x_2 + \epsilon y_2 + f z_2 \\ \lambda_2 z_2 &= g x_2 + h y_2 + \iota z_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

and

$$\left.\begin{aligned} \lambda_3 x_3 &= a x_3 + b y_3 + c z_3 \\ \lambda_3 y_3 &= d x_3 + \epsilon y_3 + f z_3 \\ \lambda_3 z_3 &= g x_3 + h y_3 + \iota z_3 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Now multiply the three equations of (21) by $x_3$, $y_3$, and $z_3$ respectively and add the results. From this sum subtract that obtained by multiplying the three equations of (22) by $x_2$, $y_2$, and $z_2$ and adding. We thus find as the difference of these two sums

$$\begin{aligned} (h-f)(y_2 z_3 - y_3 z_2) + (c-g)(z_2 x_3 - z_3 x_2) + (d-b)(x_2 y_3 - x_3 y_2) \\ = (\lambda_2 - \lambda_3)(x_2 x_3 + y_2 y_3 + z_2 z_3) \quad . \quad . \quad (23). \end{aligned}$$

But, remembering that the cosine of the angle between two lines is the sum of the products of their corresponding direction cosines, and

noting (20), we see that the condition for perpendicularity of the lines $r_2$ and $r_3$ is

$$x_2x_3+y_2y_3+z_2z_3=0 \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Hence the right side of (23) vanishes, and consequently the left side also.

The conditions that $r_1$ is perpendicular to each of the lines $r_2$ and $r_3$ must also be introduced, and may be written

$$\left.\begin{aligned} x_1x_2+y_1y_2+z_1z_2&=0\\ x_1x_3+y_1y_3+z_1z_3&=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (25).$$

From these two equations (25), by the ordinary algebraic elimination, we obtain

$$\frac{x_1}{y_2z_3-y_3z_2}=\frac{y_1}{z_2x_3-z_3x_2}=\frac{z_1}{x_2y_3-x_3y_2} \quad . \quad . \quad . \quad (26).$$

Substituting (24) and (26) in (23) we have

$$(h-f)x_1+(c-g)y_1+(d-b)z_1=0 \quad . \quad . \quad . \quad . \quad (27).$$

But, since the order of subscripts is indifferent, this relation must hold for the co-ordinates of points on each of the three mutually rectangular lines along which elongation occurs free from angular displacement. This can be true only when each of the coefficients of $x$, $y$, and $z$ vanish. We accordingly find as the condition for *the reduction of a homogeneous to a pure strain*

$$h=f,\ c=g, \text{ and } d=b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

And this agrees with (17) of article 177, and reduces the expression of the strain to the form shown in (18), requiring six constants only.

**179. Pure Strain along Co-ordinate Axes.**—Of the six constants for a pure strain, we have seen that three were required to define the principal axes of elongation. Hence, if these are chosen as the co-ordinate axes, the corresponding defining constants disappear, and therefore, in order to fully specify the strain, we then need only the three constants which express the principal elongations. We have thus returned to the simple case with which we began in article 164 as expressed in equation (1). This result may be deduced analytically also from the equations of article 178. Thus, if we put $r_1$, $r_2$, $r_3$ along the axes of $x$, $y$, and $z$ respectively, we have

$$y_1=z_1=0,\ x_2=z_2=0,\ x_3=y_3=0.$$

Then by regarding equations (21), (22), and (28), it may be seen

$$\left.\begin{aligned} &\text{that} && d=g=b=h=c=f=0\\ &\text{and} && \lambda_1=a=1+a,\ \lambda_2=e=1+e,\ \lambda_3=\iota=1+i \end{aligned}\right\} \quad . \quad (29).$$

We may accordingly compactly summarise, as in Table v., the characteristics of the homogeneous and pure strains already noticed, and may refer to any one of these strains by simply quoting the corresponding set of co-efficients.

TABLE V. COEFFICIENTS FOR HOMOGENEOUS STRAINS.

| THE BODY OF THE TABLE SHOWS THE COEFFICIENTS OF THE CO-ORDINATES $x$, $y$, AND $z$ TO EXPRESS— | GENERAL HOMOGENEOUS STRAIN | | | GENERAL PURE STRAIN | | | PURE STRAIN ALONG THE CO-ORDINATE AXES | | |
|---|---|---|---|---|---|---|---|---|---|
| | $x$, | $y$, | $z$. | $x$, | $y$, | $z$. | $x$, | $y$, | $z$. |
| $x'-x$, | $a$ | $b$ | $c$ | $a$ | $b$ | $c$ | $a$ | 0 | 0 |
| $y'-y$, | $d$ | $e$ | $f$ | $b$ | $e$ | $f$ | 0 | $e$ | 0 |
| and $z'-z$. | $g$ | $h$ | $i$ | $c$ | $f$ | $i$ | 0 | 0 | $i$ |

EXAMPLES—XXXVI.

1. Define a homogeneous strain, and enumerate some of its properties. A bar of india-rubber about three inches long and half an inch square is in turn pulled, bent, and twisted in the fingers; describe the strains in each case, showing by sketches what becomes of straight lines, circles, etc., drawn in the substance in its unstrained state.
2. Show that a homogeneous strain can be specified by stating the figure produced by it from a sphere in the primitive body.
3. Define *strain ellipsoid*, and explain how it is derived, illustrating your answer by some simple examples.
4. Represent a homogeneous strain by a set of equations involving nine constants, and illustrate your answer by a sketch showing the meanings of each constant.
5. Show that a homogeneous strain of the most general type involves rotations in addition to a change in dimensions, and simplify the expressions for the strain so as to remove the rotations.
6. Represent by equations homogeneous strains requiring for their specification 9, 6, and 3 constants respectively, and show by diagrams and descriptions what each class of equations really denotes.
7. Show analytically that the nine constants of a homogeneous strain reduce to six if rotations are absent, and to three if the co-ordinate axes are taken along the principal elongations.

**180. Equation of the Strain Ellipsoid.**—The principal axes of elongation being now taken as the co-ordinate axes, the equation of the strain ellipsoid assumes in consequence a simple form. Thus if, in the primitive figure, we take a sphere of unit radius, we may denote it by the equation

$$x^2+y^2+z^2=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

Then, on subjecting it to the strains of elongations $a$, $e$, $i$, this sphere becomes the strain ellipsoid, whose equation is

$$\left.\begin{aligned} \frac{x^2}{(1+a)^2}+\frac{y^2}{(1+e)^2}+\frac{z^2}{(1+i)^2}&=1 \\ \text{or} \qquad \frac{x^2}{\alpha^2}+\frac{y^2}{\epsilon^2}+\frac{z^2}{\iota^2}&=1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (31),$$

of semi-axes $\alpha$, $\epsilon$, $\iota$, which equal $(1+a)$, $(1+e)$, and $(1+i)$ respectively.

By giving to $a$, $e$, and $i$ any of the values shown in Table IV. of

article 165, we obtain the strain ellipsoid corresponding to the particular type of strain in question. Thus for case 1 we have the general type, the ellipsoid having three unequal axes as shown in (31). In case 2 only two of the axes are changed from their original unit values; in case 3 only one of them is changed. For case 4 the ellipsoid is a sphere. For case 5 the sections parallel to the $zx$ plane are circles, the semi-axes along $y$ remaining of its original unit value. For the simple shear shown in case 6 the axis of $y$ remains unchanged, that of $x$ has the elongation $e$, and that of $z$ the equal *contraction*, or *negative* elongation $-e$. Finally, in case 7 we have an elongation $a$ along the $x$ axis, and contractions $i$ along the other two axes. Thus for cases 3, 5, and 7 we have ellipsoids of revolution.

**181. Cone of Given Constant Elongation.**—Let us now consider the possible directions in which the elongation has a given constant value, say $1:\lambda$. These will evidently be given by the directions from the origin to the intersection of a sphere of radius $\lambda$ with the strain ellipsoid of semi-axes $\alpha$, $\epsilon$, and $\iota$. Hence we may write the equation of this sphere in the form

$$\frac{x^2}{\lambda^2}+\frac{y^2}{\lambda^2}+\frac{z^2}{\lambda^2}=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

Subtracting (31) from this we obtain, for the required locus, the equation

$$x^2\left(\frac{1}{\lambda^2}-\frac{1}{\alpha^2}\right)+y^2\left(\frac{1}{\lambda^2}-\frac{1}{\epsilon^2}\right)+z^2\left(\frac{1}{\lambda^2}-\frac{1}{\iota^2}\right)=0 \quad . \quad . \quad (33),$$

or

$$Ax^2+By^2+Cz^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

These last two equations represent a *general* conical surface with vertex at the origin. Various special cases need notice.

*Case I.—Let* $\alpha>\lambda>\epsilon=\iota$. The ellipsoid is then one of *revolution* about the axis of $x$, and from (33) we see that the locus becomes

$$Ax^2-B(y^2+z^2)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

which is a *right circular* cone about the axis of $x$, as might have been anticipated on geometrical grounds.

If we now reduce $\lambda$ so as to equal $\epsilon$ and $\iota$, it is evident that the cone shrinks to two planes coincident with the plane of $yz$ and represented by the equation

$$x^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (35a).$$

*Case II.—Let* $\alpha>\lambda=\epsilon>\iota$. The ellipsoid is now general, and the elongation $\lambda$ equals the *medium* principal elongation. Then we see from (33) that the locus reduces to

$$Ax^2-Cz^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

That is, the cone reduces to two planes intersecting on the axis of $y$. But, since these planes are also sections of the sphere of radius $\lambda$, we see that they are *the two central circular sections* of the ellipsoid.

*Case III.—Let* $\alpha=(1+e)$, $\lambda=\epsilon=1$, and $\iota=1-e$, $e$ being very small. Then this gives us the simple shear of elongations $e$ in the plane of $zx$. Referring again to (33), we see that the locus now reduces to

$$x^2-z^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

which represents two planes intersecting each other *perpendicularly* on the axis of $y$, and inclined at angles of 45° to the axes of $z$ and $x$. Further, since these planes are the intersections of a sphere of radius $\lambda = 1$, they are the two central *circular sections* of the ellipsoid and of their *primitive size*. And this fact, of no distortion caused by the strain, holds for *all planes parallel* to those of (37).

**182. Shear Ellipsoid derived by Slidings.**—But we have seen in articles 170 and 171 that a simple shear may be viewed as a proportionate sliding of undistorted planes parallel to each other. Hence the strain ellipsoid representing a simple shear should be capable of derivation from the primitive sphere by this method of sliding, and we may now easily see that this is the case. As a preliminary, let us note that all the sections of the ellipsoid parallel to the central circular sections are also circular, since all parallel sections of an ellipsoid are similar. It remains then to show that the circular sections of the sphere and ellipsoid by a given plane are equal, and that the same *amount of shear* suffices to slide any such section of the sphere to the position it must occupy as the corresponding section of the ellipsoid. To illustrate these points clearly in a diagram it is desirable to deal with a finite shear of elongational *ratios* $\epsilon$, 1, and $1/\epsilon$ along the axes $x$, $y$, and $z$ respectively. A section in the $zx$ plane of the primitive sphere and the strain ellipsoid is shown in Fig. 75.

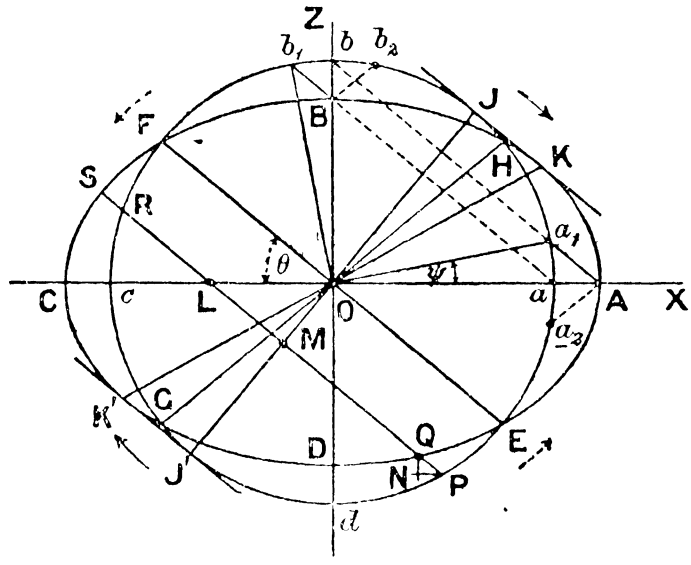



FIG. 75. STRAINED ELLIPSOID DERIVED BY SLIDINGS.

In this figure the central circular sections are shown by EOF and GOH. Draw OJ perpendicular to EOF and meeting the circle in J, then draw through J the line JK parallel to EOF and tangential to the circle at J and to the ellipse at K, also join KO and produce to K'. Then KOK' and EOF are conjugate diameters. Hence OJ and OK bisect in the sphere and ellipse respectively all chords parallel to EOF, one of the central circular sections. Thus the amount of the shear when viewed as a sliding parallel to EOF, as shown by the arrows, is measured by JK/OJ. Draw $Aa_1$ and $Bb_1$ parallel to EOF and through the extremities of the major and minor semi-axes of the ellipse Then we see that the primitive lines $a_1Ob_1$ must *rotate* clockwise to their final positions AOB through the angle $AOa_1$ or $BOb_1$. Thus the

sliding of the parallel undistorted planes involves a rotation and derives A from $a_1$ and B from $b_1$; whereas, if the same strain ellipsoid is derived from the sphere by the elongations $1 : \epsilon$ parallel to the axis of $x$ and the contraction $\epsilon : 1$ parallel to the axis of $z$, the A is derived from $a$ and B from $b$, the strain being pure, that is, devoid of rotation. It should be noted that in either case lines parallel to the axis of $y$ perpendicular to the plane of the figure suffer no elongation or contraction. Thus, for the pure strain, unit lengths along the axes of $x$, $y$, and $z$ change to lengths $\epsilon$, 1, and $1/\epsilon$ respectively, these three rectangular axes suffering no angular displacements. Whereas for the shear when occurring as a system of slidings parallel to EOF, only the axis of $y$ remains without angular displacement, and is the axis about which the lines $Oa_1$ and $Ob_1$ rotate to the final positions OA and OB.

Of course, the slidings might occur parallel to the other circular section GOH, as shown by the dotted arrows, in which case the points A and B would be derived from $a_2$ and $b_2$ respectively, and the rotation involved is equal and opposite to that in the former case.

Thus, if a diametral section, like EOF, of the primitive sphere is maintained at rest, the shear produced by a set of slidings involves a *rotation* in the sense of those slidings in addition to the pure strain as expressed by the elongation ratios. If, on the other hand, a point on the surface of the sphere, say J, is kept at rest while a shear occurs by these slidings, then we have a *shift* of the centre besides the above rotation and the pure strain.

**183. Analytical Treatment of Shear Ellipsoid.**—In the preceding article certain relations between the circle and the ellipse were referred to in general terms and without any formal proof. Some of them rest on well-known properties and need no further proof, others requiring proof may be treated in various ways. The ordinary cartesian treatment will be outlined here and the necessary quantitative relations established.

The primitive sphere of unit radius and the strain ellipsoid of semi-axes $\epsilon$, 1, and $1/\epsilon$ have for their sections in the $zx$ plane the respective equations

$$z^2 + x^2 = 1 \quad \ldots\ldots\ldots \quad (38),$$

and

$$\epsilon^2 z^2 + x^2/\epsilon^2 = 1 \quad \ldots\ldots\ldots \quad (39).$$

The equations of EF and GH derived from these are respectively

$$\epsilon z + x = 0 \text{ and } \epsilon z = x \quad \ldots\ldots\ldots \quad (40).$$

The radius OJ, perpendicular to EF, is

$$z = \epsilon x \quad \ldots\ldots\ldots \quad (41),$$

J has co-ordinates

$$(1/\sqrt{\epsilon^2+1},\ \epsilon/\sqrt{\epsilon^2+1}) \quad \ldots\ldots\ldots \quad (42),$$

and JK, parallel to EF, and tangential to the circle at J, is

$$\epsilon z + x = \sqrt{\epsilon^2+1} \quad \ldots\ldots\ldots \quad (43).$$

But this fulfils the condition for tangency to the ellipse, which it touches at the point K, whose co-ordinates are

$$(\epsilon^2/\sqrt{\epsilon^2+1},\ 1/\epsilon\sqrt{\epsilon^2+1}) \quad \ldots\ldots\ldots \quad (44).$$

Thus, the tangents of the angles XOK, XOH, and XOJ are respectively

$$1/\epsilon^3,\ 1/\epsilon, \text{ and } \epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Also JK is $\epsilon - 1/\epsilon$, and since OJ is unity, we see that *the amount of the shear* is given by

$$\chi = JK/OJ = \epsilon - 1/\epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46).$$

Thus, if the elongational ratio $\epsilon = 1 + e$, where $e$ is vanishingly small, we find

$$\chi = 1 + e - 1/(1+e) = 2e \text{ nearly} \quad . \quad . \quad . \quad (47),$$

as shown in article 172 of equation (13).

Also, we see by (45) that where $\epsilon$ is practically unity, as for the small shear just considered, the lines OK, OH, OJ all coalesce and make with the axes of $z$ and $x$ the angle $\pi/4$. Hence we see, as before, that the two sets of planes of no distortion are perpendicular to one another and bisect the angles between the axes of elongation and contraction.

That the shear is of the same amount everywhere really follows from the fact that we could make the sphere and ellipse of any size we like within the volume subject to the strain in question, but we can also prove it analytically thus. Take the line PQRS parallel to EF, then its equation may be written

$$\epsilon z + x + l = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48),$$

where $l = LO$, and the difference of the $x$'s for either (i) P and Q or (ii) R and S is, by (38), (39), and (48),

$$l(\epsilon^2 - 1)/(\epsilon^2 + 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

Thus, denoting by $\theta$ the angle between EF and OC, we have

$$\tan\theta = \epsilon,\ \cos\theta = 1/\sqrt{\epsilon^2+1},\ \sin\theta = \epsilon/\sqrt{\epsilon^2+1},$$

and so find $PQ = NP/\cos\theta = l(\epsilon^2-1)/\sqrt{\epsilon^2+1}$,

and $OM = l\sin\theta = l\epsilon/\sqrt{\epsilon^2+1}$.

Hence, the amount of the shear which carries P to Q and R to S is given by

$$\gamma = PQ/OM = (\epsilon^2-1)/\epsilon = \epsilon - 1/\epsilon \quad . \quad . \quad . \quad . \quad (50),$$

and is thus seen to be independent of $l$ and to agree with (46).

Consider now the rotation involved when the shear is produced by slidings parallel to EF, which remains at rest. It is easily seen that A is derived from $a_1$ and B from $b_1$, the equations of $Aa_1b$ and $aBb_1$ being respectively $\epsilon z + x = \epsilon$ and $\epsilon z + x = 1$ . . . . . (51).

The co-ordinates of $a_1$ and $b_1$ are also seen to be

$$\left(2\epsilon/(\epsilon^2+1),\ (\epsilon^2-1)/(\epsilon^2+1)\right) \text{ and } \left(-(\epsilon^2-1)/(\epsilon^2+1),\ 2\epsilon/(\epsilon^2+1)\right);$$

thus $Oa_1$ and $Ob_1$ are at right angles, and the angle $\psi$, through which each rotates about the axis of $y$ from its initial to its final position, is given by

$$\tan\psi = (\epsilon^2-1)/2\epsilon = (\epsilon - 1/\epsilon)/2 = \chi/2 \quad . \quad . \quad . \quad (52).$$

Therefore, in the case of a vanishingly small shear of elongation $e$, and with slidings parallel to EF, we should have the rotation expressed by

$$\tan\psi = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

**184. Composition of Pure Strains and Rotations.**—Although the composition of strains and rotations in general (*i.e.* when they are about axes inclined in any way) lies beyond the scope of this work, it seems desirable to point out that the resultant of two finite pure strains occurring in succession may involve a *rotation* in addition to the distortion. Hence on afterwards applying another pure strain to undo that already produced, we reach the striking result that three successive pure finite strains may yield a *rotation only*, without relative change of the parts of the figure. This is shown by Tait in his *Dynamics* as follows:—

Let the first pure strain be represented by

$$\left.\begin{aligned} x' &= \alpha x + by + cz \\ y' &= bx + \epsilon y + fz \\ z' &= cx + fy + \iota z \end{aligned}\right\} \quad \ldots\ldots\ldots (54),$$

and the second pure strain by

$$\left.\begin{aligned} x'' &= \alpha' x' + b' y' + c' z' \\ y'' &= b' x' + \epsilon' y' + f' z' \\ z'' &= c' x' + f' y' + \iota' z' \end{aligned}\right\} \quad \ldots\ldots\ldots (55),$$

in which the Greek letters $\alpha$, $\epsilon$, and $\iota$ represent as before the elongation ratios, and are respectively equal to $1+a$, $1+e$, and $1+i$.

Hence, by substituting in (55) the values of $x'$, $y'$, and $z'$ expressed in (54), we can express $x''$, $y''$, and $z''$ in terms of $x$, $y$, and $z$. In other words, we can express the resultant of the two pure strains when applied in the order given. To show that this resultant involves a rotation, we need only fill in two of the coefficients. Thus

$$\left.\begin{aligned} x'' &= (\ldots\ldots)x + (\alpha' b + b'\epsilon + c' f)y + (\ldots)z \\ y'' &= (b'\alpha + \epsilon' b + f' c)x + (\ldots\ldots)y + (\ldots)z \\ z'' &= (\ldots\ldots)x + (\ldots\ldots)y + (\ldots)z \end{aligned}\right\} \quad \ldots (56).$$

Now the criterion of a pure strain is that three equalities exist between the nine constants as proved in articles 177 and 178, shown in Table V. article 179, and as illustrated by the $b$, $c$, and $f$ in (54) and (55). But it is clear that in the general case these equalities will not be satisfied in (56), consequently a rotation is, in general, involved. For example, an elongation along one axis followed by an equal contraction along another axis *not at right angles* to the first involves a rotation.

The reader should also note that the resultant of a pure strain and a rotation usually depends on the *order* in which they occur, for these operations are *not* in general *commutative*.

**185. Restricted Strains.**—We have hitherto dealt with strains chiefly in three dimensions, and supposed that the individual points of the primitive and strained figures can be identified and followed during the occurrence of the strain. Of course, these assumptions apply only to solid bodies extending throughout this tri-dimensional space. But the modifications required for other cases need only brief mention. Thus, for *fluids*, the strain reduces to a change of *volume* simply, and it may become impossible to identify any points and follow them in their

motion during the strain. Again, in the case of thin extensible *membranes*, we are almost reduced to a surface, and often a plane surface, in which case we have only *plane* strains to consider. Lastly, for elastic *cords* of negligible thickness, if straight, we have simple *elongation*.

If, however, the cord passes round constraints so as to occupy solid space, it is sometimes of importance to find an expression for its rate of stretching per unit length per unit time. Equation (7) of article 137 was simply this rate of stretching equated to zero. Hence we have for our present case where it may be finite

$$\frac{d\dot{s}}{ds}=\frac{d\dot{x}}{ds}\cdot\frac{dx}{ds}+\frac{d\dot{y}}{ds}\cdot\frac{dy}{ds}+\frac{d\dot{z}}{ds}\cdot\frac{dz}{ds} \quad . \quad . \quad . \quad . \quad . \quad (57).$$

We have now dealt sufficiently for our purpose with pure and homogeneous strains. Whatever theory of *heterogeneous* strains may be needed later will be then developed as required, and with immediate reference to the special problem under discussion.

Similar remarks apply to other possible motions of deformable bodies as vibrations and waves, for since we are only subordinately concerned with them their kinematics may be taken along with the kinetics of each such problem.

## EXAMPLES—XXXVII.

1. Define *strain ellipsoid*, and write down its equations for (*a*) a uniform dilation of 0·003 fractional volume change, (*b*) a shear of amount 0·004, (*c*) an axial strain of elongation 0·008 and lateral contractions 0·002.
2. In the case of a unit sphere in the primitive body becoming an ellipsoid in the strained body, find the locus of the constant elongations, and discuss some of the chief cases which arise.
3. What special significance have the two circular sections of the strain ellipsoid in a certain case?
4. Draw carefully the strain ellipsoid for a shear of elongations one-tenth and amount one-fifth nearly, showing how it may be regarded as derived from the unit sphere by progressive slidings of undistorted planes.
5. Show analytically that the shear ellipsoid may be derived from the sphere by an elongation and equal perpendicular contraction, or by slidings at angles of 45° with the above directions and of an amount double the elongation.
6. Show that the successive application of two pure strains may result in a rotation. Is this possible if each of the pure strains involved only elongations about the same co-ordinate axes?

# PART III.—KINETICS

## CHAPTER XI

### PHYSICAL BASIS

**186. Physical Conceptions.**—In Chapter III. the fundamental conceptions or intuitions of space and time laid the geometrical basis for the kinematical development which followed in Chapters IV.-X. In these the combination of displacements, velocities and accelerations was sometimes seen to tally more or less closely with various natural occurrences familiar or rare. But the accelerations were in all cases simply postulated without any conditions being prescribed under which alone they might be expected to obtain. And that which was supposed to move was often only a mathematical point, linear, plane or solid figure or a combination of such ideal parts.

In order, therefore, to make our description of phenomena at once more extensive and more precise, we must now introduce new and physical conceptions. We must conceive *bodies* as moving and know something as to the *conditions* under which their accelerations may be expected to occur. Then, having laid this physical basis in the present chapter, we can build upon it in combination with the kinematical theorems already developed, any subsidiary experimental data being introduced where required.

The chief new physical conceptions that need introduction here are the following four:—

(i) *Mass*, or *Inertia*;

(ii) *Gravitation*, or the tendency of all particles to approach each other;

(iii) *Friction*, or resistance to relative sliding of bodies in contact; and

(iv) *Elasticity*, or resistances to change of volume or shape.

Two limiting cases of elasticity are especially noteworthy. If the resistances to change of shape and volume are both infinite, the body is said to be *rigid*. If the resistance to change of shape entirely vanishes, the substance is a frictionless *fluid*.

Following the fundamental physical conception of mass and arising from it, others naturally occur, being products and quotients of the new and old quantities, such as force, momentum, work, etc., and needing definition merely. The consideration of some of these can be deferred till they are needed in the development of the subject. But one of them, *force*, had better be taken now along with *mass*. And as

this conception of *mass* was only slowly developed, and very various views have been held concerning it and force, a brief historical account may be useful, in which of necessity other topics must be referred to.

**187. Mechanics before Newton.**—The earliest mechanical theories related wholly to statics of solids and fluids, dynamics being founded by Galileo (1564-1642), and continued by Huyghens (1629-1695).

Galileo experimentally investigated the motion of falling bodies, timing by a water-clock the rolling of a ball down an inclined groove. He thus discovered that the distances descended on a given incline were proportional to the squares of the corresponding times. This he had previously shown theoretically would be the case if the velocity was simply proportional to the time. He thus established that bodies fall with constant acceleration, *acceleration* being an entirely new conception to which he was led by this investigation.

He further showed that bodies by falling acquired a velocity such that they were able to ascend to the level from which they fell. If this ascent were inclined and made less and less steep, the time of ascent was more and more increased. So that if the slope were continually diminished, this time might be indefinitely prolonged. He thus groped his way to the fundamental conception of inertia.

For motion from rest with constant acceleration, Galileo thus used the two relations which, in our notation, are written

$$v = at, \text{ and } s = at^2/2.$$

The importance of the third relation, derived from these, namely :— $v^2 = 2as$, was perceived by Huyghens, who thus laid the foundation for important advances. For, soon after Galileo, it was noticed that a body having velocity had a 'something' in virtue of which a resistance could be overcome. Was this something, this efficacy, proportional to the velocity simply or to its square? Huyghens seemed to have seen quite clearly that a doubled velocity enabled a body to ascend against its weight for a *double time*, but through a *fourfold distance*. So that the efficacy as regards time is proportional to the velocity simply; but, as regards distance, is proportional to the square.

Huyghens solved problems on the dynamics of several connected bodies, whereas Galileo restricted himself to a single body. Thus the compound or bar pendulum was treated, the centre of oscillation determined, and the acceleration of gravity found by pendulum observations.

But it should be noted that throughout this period before Newton, the conception of mass had not been clearly formed. 'It did not occur to Galileo that mass and weight were different things. Huyghens, too, in all his considerations, puts weights for masses. . . .'

**188. Newton's Principles.**—To Sir Isaac Newton (1642-1726) we owe (i) the enumeration of those principles of mechanics which still form the basis of its formal development, and (ii) the discovery of universal gravitation. Perhaps it was in connection with the latter that the distinction between weight and mass was first felt; weight being

something different for a given body on the earth and on the moon, but its mass, the difficulty of starting it and of stopping it, being the same everywhere.

The enumeration of the mechanical principles Newton arranged in a number of definitions, axioms or laws, and corollaries, all interpersed with remarks. These were contained in his *Principia*,[1] written in Latin; hence the very different versions of them which appear in various text-books. The definitions and laws are given here with an indication of the scope of the corollaries and some of the more important explanatory remarks, the English version of Evans and Main being chiefly followed.

**189. Newton's Definitions.**

'DEFINITION 1.—*Quantity of matter is the measure of it arising from its density and bulk conjointly.*

'This *quantity of matter* is, in what follows, sometimes called the body, or mass. It is known for each body by means of its weight; for it has been found, by very accurate experiments with pendulums, to be proportional to the weight.

'DEF. 2.—*The quantity of motion of a body is the measure of it, arising from its velocity and the quantity of matter conjointly.*

'DEF. 3.—*The innate force of matter is its power of resisting, whereby every body, so far as depends on itself, perseveres in its state, either of rest, or of uniform motion in a straight line.*

'This is always proportional to the body, and differs in no respect from the inertia of the mass, except in the manner of viewing it. To the inertia of matter is due the difficulty of disturbing bodies from their state of rest or motion; on which account the innate force may be called by the very suggestive name, *force of inertia.*

'DEF. 4.—*An impressed force is an action exerted on a body, tending to change its state either of rest or of uniform motion in a straight line.*

'DEF. 5.—*A centripetal force is one by which bodies are drawn, impelled, or in any other way tend from all parts towards some point as centre.*

'Of this kind is *gravity*, by which bodies tend to the centre of the earth; *magnetic force*, by which iron approaches a magnet; and that force, whatever it may be, by which the planets are perpetually drawn away from rectilinear motions and forced to revolve in curves.

The quantity of this centripetal force is of three kinds, absolute, accelerative, and motive.

'DEF. 6.—*The absolute quantity of a centripetal force is a measure of it which is greater or less according to the efficacy of the cause which propagates it from the centre through the regions of space all round it.*

'Just as *magnetic force* is greater in one magnet and less in another, according to the mass of the magnet, or the intensity of its magnetism.

'DEF. 7.—*The accelerative quantity of a centripetal force is a*

[1] *Philosophiae Naturalis Principia Mathematica* (London, 1686).

*measure of it proportional to the velocity which it generates in a given time.*

'Just as the *power of the same magnet* is greater at a less distance, less at a greater. Or, as *gravitating force* is greater in valleys, less on the peaks of high mountains, and so less the greater the distance from the earth; but at equal distances the same on all sides, because it accelerates equally all falling bodies.

'DEF. 8.—*The motive quantity of a centripetal force is a measure of it proportional to the motion which it generates in a given time.*

'Just as *weight* is greater in a greater mass, less in a less mass; and, in the same, is greater near the earth, less in remote space.

'These quantities of forces may for brevity be called *motive*, *accelerative*, and *absolute* forces; and, for the sake of distinctness, may be ascribed severally to the bodies which tend to the centre, to the positions of the bodies, and to the centre of the forces; so that, in fact the motive force is ascribed to the body as if it were the effort of the whole composed of the efforts of all its parts; the accelerative force to the position of the body, as if there were diffused from the centre to all places around it, some power efficacious towards moving bodies which are in those places; and the absolute force of the centre, as if at this point there were situated something which was the cause of motive forces being propagated through space in all directions; whether that cause be some central body (just as a magnet is at the centre of magnetic force, or the earth at the centre of gravitating force) or any other cause which is not ascertained. This is simply a mathematical conception; the physical causes and seats of the forces are not here considered.'

## 190. Newton's Axioms, or Laws of Motion.

'LAW I.—*Every body perseveres in its state of rest, or of uniform motion in a straight line, except in so far as it is compelled to change that state by forces impressed on it.*

'Projectiles persevere in their motions, except in so far as they are retarded by the resistance of the air, and driven downwards by the force of gravity. A hoop, whose parts continually draw each other from their rectilinear motions by cohesion, ceases to roll only in consequence of its motion being retarded by the air. But the larger bodies of planets and comets, whose motions, both progressive and circular, take place in less resisting spaces, retain these motions longer.

'LAW II.—*Change of motion is proportional to the moving force impressed, and takes place in the straight line in which that force is impressed.*

'If a force produce any motion, twice the force will produce twice the motion, thrice the force three times the motion, whether it has been impressed all at once, or by successive gradations. And this motion (since it must always take place in the same direction as the force which produces it) is—if the body was originally in motion—added to

its original motion if that motion was in the same direction, subtracted from it if in the opposite; or if in an inclined direction, is added to it in an inclined direction, and compounded with it, the position of the body being determined by the motion in such direction.

'LAW III.—*An action is always opposed by an equal reaction; or, the mutual actions of two bodies are always equal and act in opposite directions.*

'Whatever presses or pulls something else, is pressed or pulled by it in the same degree. If a man presses a stone with his finger, his finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse will be (so to speak) drawn back equally towards the stone: for the rope being stretched at both ends will by the same attempt to relax itself urge the horse towards the stone and the stone towards the horse; and will impede the progress of one as much as it promotes the progress of the other. If a body impinge on another and by its force change the motion of the other in any way, the latter will in its turn (on account of the equality of the mutual pressure) undergo the same change of motion in a contrary direction. To these actions are equal the changes, not of velocities, but of motions; that is, in bodies not hindered in their motions by other forces. For the changes of velocities, which also take place in the same direction, are—since the motions are changed equally—reciprocally proportional to the bodies. This law holds also in attractions.'

**191. Newton's Corollaries.**—To his three laws Newton appended six corollaries. Of these the first and second relate to the principle of the parallelogram of forces (or impulses), the third to the conservation of momentum in spite of mutual actions, the fourth to the inability of mutual actions to disturb the motion of the centre of gravity of the system, while the fifth and sixth refer to relative motions.

**192. Newton's Disciples and Critics.**—From the foregoing translation of the definitions and laws laid down by Newton, in this one branch of his varied activities, some notion may be formed of his transcendant greatness. The doctrines thus formulated have been accepted as a sufficient and satisfactory basis of dynamics and statics by numerous writers, including the authors (Kelvin and Tait) of the classic work on *Natural Philosophy*.

On the other hand, some, while agreeing in the main with the information Newton gave, have seriously criticised the verbal forms in which he gave it.

Others have gone further than this, and taken a distinctly different view of most of the points in question.

But possibly Newton would have been unintelligible to his contemporaries had his thought and language been abreast of the most advanced thought of the present day.

Of matters so fundamental, probably no statements logically perfect can be humanly invented. Certainly no such statement can be at once brief and full, conveying to friends and foes alike the self-same

message. Yet the brevity seems highly desirable, to facilitate memorising and quotation. Hence, for any audience in any age, the best attainable enunciations of such principles are probably those that briefly convey the essential truths to the audience in view, even though they may be slightly redundant in parts needing emphasis, or logically incomplete in others of minor importance. Thus, much of the modern endeavour to modify Newton's enunciations must not be taken either as any disparagement of his greatness or as any claim to present-day superiority. It is rather an attempt to restate very similar contents in forms more suitable to the audiences now addressed.

Before going into details, four general lines of criticism of Newton's enunciations may be noticed.

*Firstly*, it has been the task of modern criticism to disentangle the mere definitions from the statements of natural fact.

*Secondly*, it has been urged that the first law is logically unnecessary because only a special case of the second. Some defend the first law as being necessary to remove the pre-conceived notions from men of Newton's time, while others regard it as permanently necessary.

*Thirdly*, there is a strong body of opinion that Newton's definition of mass is incomplete and illogical, and must be replaced by one in which the ratio of masses is the negative inverse ratio of mutual accelerations. On this view Newton's third law becomes unnecessary.

*Fourthly*, it has been pointed out that since all motion is relative, force is relative also, and that accordingly to complete the laws some statement is necessary as to the base, axes, or frame of reference to which they must be referred, and for which alone they are valid.

We may now fitly pass into details, noticing at some length the criticism and constructive scheme of Mach, and more briefly the views of some other writers.

**193. Criticisms by Mach.**—In his *Science of Mechanics* (Prague, 1883, American Edition, Chicago, 1902), Dr. Ernst Mach, after devoting fifty pages to the achievements of Newton, passes on to a synoptical critique of the Newtonian Enunciations. After quoting the Definitions, Mach writes (p. 241 of American Edition):—'Definition 1 is, as has already been set forth, a pseudo-definition. The concept of mass is not made clearer by describing mass as the product of the volume into the density, as density itself denotes simply the mass of unit of volume. The true definition of mass can be deduced only from the dynamical relations of bodies.

'To Definition 2, which simply enunciates a mode of computation, no objection is to be made. Definition 3 (inertia), however, is rendered superfluous by Definitions 4-8 of force, inertia being included and given in the fact that forces are accelerative.

'Definition 4 defines force as the cause of the acceleration, or tendency to acceleration, of a body. The latter part of this is justified by the fact that in the cases also in which accelerations cannot take place, other attractions that answer thereto, as the compression and distension, etc. of bodies occur. The cause of an acceleration towards

a definite centre is defined in Definition 5 as centripetal force, and is distinguished in 6, 7, and 8 as absolute, accelerative, and motive. It is, we may say, a matter of taste and of form whether we shall embody the explication of the idea of force in one or in several definitions. In point of principle, the Newtonian definitions are open to no objection.

Mach then quotes the laws and deals with them and the appended corollaries as follows:—'We readily perceive that Laws I. and II. are contained in the definitions of force that precede. According to the latter, without force there is no acceleration, consequently only rest or uniform motion in a straight line. Furthermore, it is wholly unnecessary tautology, after having established acceleration as the measure of force, to say again that change of motion is proportional to the force. It would have been enough to say that the definitions premised were not arbitrary mathematical ones, but correspond to properties of bodies experimentally given. The third law apparently contains something new. But we have seen that it is unintelligible without the correct idea of mass, which idea, being itself obtained only from dynamical experience, renders the law unnecessary.

'The first corollary really does contain something new. But it regards the accelerations determined in a body $K$ by different bodies $M$, $N$, $P$, as *self-evidently* independent of each other, whereas this is precisely what should have been explicitly recognised as a *fact of experience*. Corollary Second is a simple application of the law enunciated in Corollary First. The remaining corollaries, likewise, are simple deductions, that is, mathematical consequences, from the conceptions and laws that precede.

'Even if we adhere absolutely to the Newtonian points of view, and disregard the complications and indefinite features mentioned, which are not removed but merely concealed by the abbreviated designations "Time" and "Space," it is possible to replace Newton's enunciations by much more simple, methodically better arranged, and more satisfactory propositions. Such, in our estimation, would be the following.'

**194. Enunciations by Mach.**

'*a. Experimental Proposition.*—Bodies set opposite each other induce in each other, under certain circumstances to be specified by experimental physics, contrary accelerations in the direction of their line of junctions. (The principle of inertia is included in this.)

'*b. Definition.*—The mass-ratio of any two bodies is the negative inverse ratio of the mutually-induced accelerations of those bodies.

'*c. Experimental Proposition.*—The mass-ratios of bodies are independent of the character of the physical states (of the bodies) that condition the mutual accelerations produced, be those states electrical, magnetic, or what not; and they remain, moreover, the same, whether they are mediately or immediately arrived at.

'*d. Experimental Proposition.*—The accelerations which any number of bodies $A$, $B$, $C$, . . . induce in a body $K$, are independent of each other. (The principle of the parallelogram of forces follows immediately from this.)

'*e. Definition.*—Moving force is the product of the mass-value of a body into the acceleration induced in that body.

'Then the remaining arbitrary definitions of the algebraic expressions "momentum," "vis viva," and the like, might follow. But these are by no means indispensable. The propositions above set forth satisfy the requirements of simplicity and parsimony which, on economico-scientific grounds, must be exacted of them. They are, moreover, obvious and clear; for no doubt can exist with respect to any one of them either concerning its meaning or its source; and we always know whether it asserts an experience or an arbitrary convention.'

**195. The Tribute of Mach to Newton.**—'Upon the whole, we may say, that Newton discerned in an admirable manner the concepts and principles that were *sufficiently assured* to allow of being further built upon. It is possible that to some extent he was forced by the difficulty and novelty of his subject, in the minds of the contemporary world, to great amplitude, and, therefore, to a certain disconnectedness of presentation, in consequence of which one and the same property of mechanical processes appears several times formulated. To some extent, however, he was, as it is possible to prove, not perfectly clear himself concerning the import, and especially concerning the source of his principles. This cannot, however, obscure in the slightest his intellectual greatness. He that has to acquire a new point of view naturally cannot possess it so securely from the beginning as they that receive it unlaboriously from him. He has done enough if he has discovered truths on which future generations can further build. For every new inference therefrom affords at once a new insight, a new control, an extension of our prospect, and a clarification of our field of view. Like the commander of an army, a great discoverer cannot stop to institute petty inquiries regarding the right by which he holds each post of vantage he has won. The magnitude of the problem to be solved leaves no time for this. But, at a later period, the case is different. Newton might well have expected of the two centuries to follow that they should further examine and confirm the foundations of his work, and that, when times of greater scientific tranquillity should come, the principles of the subject might acquire an even higher philosophical interest than all that is deducible from them. Then problems arise like those just treated of, to the solution of which, perhaps, a small contribution has here been made. We join with the eminent physicists, Thomson and Tait, in our reverence and admiration of Newton. But we can only comprehend with difficulty their opinion that the Newtonian doctrines still remain the best and most philosophical foundation of the science that can be given.'

**196. Retrospect by Mach.**—'If we pass in review the period in which the development of dynamics fell,—a period inaugurated by Galileo, continued by Huyghens, and brought to a close by Newton,—its main result will be found to be the perception, that bodies mutually

determine in each other *accelerations* dependent on definite spatial and material circumstances, and that there are *masses*. The reason the perception of these facts was embodied in so great a number of principles is wholly an historical one; the perception was not reached at once, but slowly and by degrees. In reality only *one* great fact was established. Different pairs of bodies determine, independently of each other, and mutually, in themselves, pairs of accelerations, whose terms exhibit a constant ratio, the criterion and characteristic of each pair.

'Not even men of the calibre of Galileo, Huyghens, and Newton, were able to perceive this fact at once. Even they could only discover it piece by piece, as it is expressed in the law of falling bodies, in the special law of inertia, in the principle of the parallelogram of forces, in the concept of mass, and so forth. To-day, no difficulty any longer exists in apprehending the unity of the whole fact. The practical demands of communication alone can justify its piecemeal presentation in several distinct principles, the number of which is really only determined by scientific taste. What is more, a reference to the reflections above set forth respecting the ideas of time, inertia, and the like, will surely convince us that, accurately viewed, the entire fact has, in all its aspects, not yet been perfectly comprehended.

'The point of view reached has, as Newton expressly states, nothing to do with the "unknown causes" of natural phenomena. That which, in the mechanics of the present day, is called *force* is not a something that lies latent in the natural processes, but a measurable, actual circumstance of motion, the product of the mass into the acceleration. Also, when we speak of the attractions or repulsions of bodies, it is not necessary to think of any hidden causes of the motions produced. We signalise by the term attraction merely an actually existing *resemblance* between events determined by conditions of motion and the results of our volitional impulses. In both cases either actual motion occurs, or when the motion is counteracted by some other circumstance of motion, distortion, compression of bodies, and so forth, are produced' (*Science of Mechanics*, Chicago, 1902, p. 246.)

### EXAMPLES—XXXVIII.

1. On what points are physical conceptions needed to enable us to pass from kinematics to kinetics?
2. What do you know of mechanics before the time of Newton?
3. Give an outline of Newton's *definitions* and critically examine them.
4. State Newton's *laws of motion* and carefully comment upon them.
5. Enumerate some respects in which Mach criticises Newton's principles.
6. What enunciations does Mach propose in place of Newton's laws of motion?
7. Write a critical essay on Mach's position with respect to Newton's principles of mechanics.

**197. Karl Pearson's View.**—The attitude of Professor Pearson to the laws of motion as shown in his *Grammar of Science* (London, 1892) is, in some respects, distinctly radical, and may have to wait

long for wide-spread adoption. It must accordingly suffice to quote here his own summary of the chapter of about fifty pages in which his five laws of motion are developed and those of Newton are criticised.

'SUMMARY.

'The physicist forms a conceptional model of the universe by aid of corpuscles. Those corpuscles are only symbols for the component parts of perceptual bodies, and are not to be considered as resembling definite perceptual equivalents. The corpuscles with which we have to deal are ether-element, prime-atom, atom, molecule, and particle. We conceive them to move in the manner which enables us most accurately to describe the sequences of our sense-impressions. This manner of motion is summed up in the so-called laws of motion. These laws hold in the first place for particles, but they have been frequently assumed to be true for all corpuscles. It is more reasonable, however, to conceive that a great part of mechanism flows from the structure of gross "matter."

'The proper measure of mass is found to be a ratio of mutual accelerations, and force is seen to be a certain convenient measure of motion, and not its cause. The customary definitions of mass and force, as well as the Newtonian statement of the laws of motion, are shown to abound in metaphysical obscurities. It is also questionable whether the principles involved in the current statements as to the superposition and combination of forces are scientifically correct when applied to atoms and molecules. The hope for future progress lies in clearer conceptions of the nature of ether and of the structure of gross "matter."'

**198. Love's Treatment.**—In his *Theoretical Mechanics* (Cambridge, 1897), Prof. A. E. H. Love expresses indebtedness to Kirchhoff, Pearson, and Mach, and adopts for the basis of the subject a set of rules, of which he then speaks as follows:—

'The system of definitions and rules which we have laid down lead to a system of differential equations for determining the motions, relative to a frame, of a system of particles, or of a body or a system of bodies, conceived to be made up of particles. It may be regarded as a purely ideal system, and its validity is unaffected by the question whether it has or has not any relation to the observed motions of natural bodies. The subject, so treated, is known as Rational Mechanics. The objects of which it treats are pure objects of thought. Its development consists in the logical deduction of particular results from the general principles laid down.

'The application of Rational Mechanics to the formulation of the Laws that govern the motions of natural bodies consists in the statement that it is possible to assign masses to the bodies and to choose a frame of reference determined by parts of natural bodies, such that the observed motions of natural bodies, relative to the frame, obey the Laws of Rational Mechanics with certain limits of exactness; that in fact the observed motions coincide with the motions described in the

phraseology of Rational Mechanics so closely that no discrepancy can be observed.'

**199. Lodge on Axioms.**—Sir Oliver Lodge, in a paper to the Physical Society of London (*Proc.*, vol. xii. pp. 291-292, 1893) on *The Foundations of Dynamics*, spoke as follows on fundamental laws or axioms:—'The setting forth of an axiom I regard as a kind of challenge, equivalent to the statement—"Here is what seems to me to be a short summary of a universal truth; disprove it if you can. I cannot prove it; it is too simple and fundamental for proof; I can only adduce hundreds of instances where it holds. I have indeed critically examined a few special cases and never found it fail, but a single contrary instance will suffice to overthrow it; hence, though it be hard to prove, yet if not true its disproof should be easy: find that contrary instance if you can! If no disproof is forthcoming for a few generations, the axiom is likely to get accepted. Meanwhile its undeniable simplicity is a practical advantage, even though in the course of centuries a flaw or needful modification in its statement may be discovered."'

Of this nature are some of the principles already given, and the following also by Newton on gravitation which now needs notice.

**200. Universal Gravitation.**—Kepler (1571-1630) analysed the observations of Tycho Brahé to find the true motion of Mars. After years of labour emerged his first two laws, and subsequently (in 1618), his third, which relates to other planets. Kepler's laws may be stated as follows (see *Pioneers of Science* by Sir Oliver Lodge, 1893, p. 56)—

LAW I. Planets move in ellipses, with the sun in one focus.

LAW II. The radius vector sweeps out equal areas in equal times.

LAW III. The square of the time of revolution of each planet is proportional to the cube of its mean distance from the sun.

From these laws and other considerations of his own, Newton passed to his grand conception of universal gravitation. This idea is to be gathered from various parts of the *Principia* and by Tait (*Properties of Matter*, p. 113, 1890) is expressed thus:—*Law of Gravitation.* 'Every particle of matter in the universe attracts every other particle with a force whose direction is that of the line joining the two, and whose magnitude is directly as the product of their masses, and inversely as the square of their distance from each other.'

It is now very questionable whether the law of inverse squares holds for small distances of the order of those between adjacent molecules. But the law still serves our purpose, as we are here concerned only with much greater distances.

**201. Friction: Coulomb, Morin, Beauchamp Tower.**

Leaving for a little the most fundamental bases of mechanics, let us now notice in order the subsidiary matters enumerated in article 186. The essential laws of friction for *dry* surfaces, with which we are chiefly concerned, seem to have been first enunciated by Coulomb in 1781, and

were confirmed by the experiments of Morin, 1830-1834. For knowledge respecting *lubricated* surfaces we are greatly indebted to the experiments of Beauchamp Tower (see *Proc. Inst. of Mechl. Engineers*, 1883-1888). The subject of friction is fully treated by Prof. J. Goodman (*Mechanics applied to Engineering*, 1908, pp. 240-260), from whose tabulated comparison of dry and lubricated surfaces the statements in Table VI. are abridged.

TABLE VI.—ON FRICTION.

| Dry Surfaces. | Lubricated Surfaces. |
|---|---|
| 1. The frictional resistance between surfaces in relative motion is *nearly proportional to the* normal force (or *total pressure*) between the two surfaces. The upper limit of the ratio, resistance ÷ normal force, is called the coefficient of friction ($\mu$). | 1. The frictional resistance is *almost independent of the pressure* with bath lubrication, and approaches the behaviour of dry surfaces as the lubrication becomes meagre. |
| 2. The frictional resistance is *nearly independent of the speed* for low pressures. For high pressures it tends to decrease as the speed increases. | 2. The frictional resistance *varies directly as the speed* for low pressures. But for high pressures the friction is very great at low velocities, becoming a minimum at about 5/3 ft. per sec., and then increases nearly as the square root of the speed. |
| 3. The frictional resistance is not greatly affected by the temperature. | 3. The frictional resistance depends more upon the temperature than upon any other condition. |
| 4. The frictional resistance depends *largely* on the nature of the material of the rubbing surfaces. | 4. The frictional resistance with a flooded bearing depends *but slightly* upon the nature of the material of the rubbing surfaces. |
| 5. The friction of *rest is slightly greater* than that of motion, but may be reduced by vibration, to that lower value. | 5. The friction of *rest is enormously greater* than that of motion, especially with thin lubricants, which are then probably squeezed out. |
| 6. When the pressure between the surfaces becomes excessive, seizing occurs. | 6. When the pressure becomes excessive (requiring much higher pressure than for dry surfaces) the lubricant is squeezed out and seizing occurs. |
| 7. The frictional resistance is greatest at first, and rapidly decreases with the time after the two surfaces are brought together, perhaps due to polishing. | 7. The frictional resistance is least at first, and rapidly increases with the time after the two surfaces are brought together, perhaps due to partial squeezing out of the lubricant. |

Since we are here concerned chiefly with dry surfaces, paragraphs 1, 2, 4, and 5 in the first column are the most important. And, owing to the remark about vibration under 5, we may always use the smaller value for the frictional resistance.

**202. Laws of Hooke and Boyle.**—We must now pass to the necessary physical conceptions respecting the simple elastic bodies with which we have to do. For our purpose these are sufficiently expressed (i) for gases, by Boyle's Law, and (ii) for solids and liquids, by a generalisation of Hooke's Law.

What we now know as Hooke's law was expressed by him in Latin in the form 'ut tensio sic vis.' This may be translated *as the extension so is the force.* Let us pass from this particular case of strain to strain in general. And let the single word *stress* be used to denote *any set of equilibrating forces applied to a body, i.e.* any set of forces which would have no apparent effect on a rigid body. Then we may restate the law in a generalised form thus, *strain is proportional to the corresponding stress.* This law only holds within very small limits, which we shall suppose are not passed over in the strains and stresses under treatment. Hence, for a given substance and a given type of stress and strain, the quotient, stress divided by strain, is a constant which measures the particular elasticity of the substance in question.

Obviously, therefore, the elastic behaviour of bodies, under the limitations mentioned, depends upon and is sufficiently specified by its various elastic constants, the details of which we shall examine later.

These constants are called *moduli of elasticity*, as bulk modulus, or bear special names, *e.g. rigidity.*

For gases, Boyle's law states that the *volume* of a given mass of gas, kept at a *constant temperature*, is *inversely as its pressure.* This is only a very approximate statement, but is near the truth for moderate pressures and for temperatures far above the point of liquefaction of the gas in question. It accordingly serves our purpose here.

**203. Relative Character of Motion and Mechanics.**—Since motion is relative, force and mechanics usually have a like relative character. Hence, in each class of problems, it is important to notice that the laws of motion, gravitation, etc., should be construed with respect to axes appropriate to the phenomena under discussion.

It may be difficult to give an instruction, at once general and precise, as to the choice of co-ordinate axes. Yet no difficulty in this respect usually occurs in the development of the subject. Especially is this the case if, at the outset, the fact has been recognised that discretion in this matter must be exercised. We then obtain, for each problem, the system of mechanics possessing just that kind of relative character which it needs.

Thus, for terrestrial motions, comprised within a few miles and a few minutes, the co-ordinate axes may be fixed in the earth. This gives us at once the ordinary mechanics of the factory, the field, the road and the railway, which may be called terrestrial mechanics.

When concerned with planetary rotations or their orbital motions, or in dealing with Foucault's pendulum, the tides, the seasons, etc., the axes may be directed by the so-called fixed stars, yielding a system that might be called planetary mechanics.

Hence, proceeding in this manner, as larger spaces and longer times were surveyed, a number of systems of mechanics might be in turn developed, each suitable and sufficient for certain problems, each

succeeding system embracing new problems and conferring a deeper insight into the old ones.

We might thus repeatedly approach, though perhaps never attain, a system of mechanics deservedly regarded as absolute and universal.

In the meantime, the relative systems serve for all practical purposes, although perhaps the formulation of an absolute one may be a legitimate problem for philosophy. [1]

### EXAMPLES—XXXIX.

1. What is Karl Pearson's attitude towards Newton's principles of mechanics?
2. What do you understand by *rational mechanics*?
3. Explain precisely what you mean by an axiom, and state on what understanding it is accepted.
4. Enunciate the law of universal gravitation. Do you believe it applies under all conditions?
5. Give a brief outline of what is known about friction.
6. State and explain the laws of Hooke and Boyle.
7. Explain what you mean by the relative character of mechanics and give illustrations. Can we reach or approach an absolute mechanics?

**204. Measurement of Time.**—Similar remarks to those in Art. 203 apply to the measurement of time. For all ordinary purposes we may be content to adopt the second of mean solar time as the unit of time; just as, for terrestrial mechanics, we may fix our axes of co-ordinates in the earth. But for astronomical purposes sidereal time is used. And, if we wish to express the retardation of rotation of the earth (due to the tidal friction acting on it like a band brake), it is evident we must go a step further and choose some measurer of time which is believed or imagined to suffer no change throughout centuries, as *e.g.* a perfect clock truly rated by the earth centuries ago. Kelvin and Tait suggest (*Natural Philosophy*, Art. 406) as such a timekeeper 'a carefully arranged metallic spring, hermetically sealed in an exhausted glass vessel.' The period of vibration corresponding to a given spectral line (say the $D_1$ line for sodium) might also be used, and has been suggested by Maxwell (*Electricity and Magnetism*, vol. i. Art. 4, page 3, Oxford, 1873; see also Kelvin and Tait's *Natural Philosophy*, part i., Art. 223, p. 227, Cambridge, 1890).

**205. Attitude towards Physical Axioms.**—We may now fitly revert to the subject of axioms and the place they fill, which was just referred to in Art. 199. Though such physical laws or axioms cannot

[1] The General Theory of Relativity, developed by A. Einstein and others, makes any one relative system as valid as another, though some may be more convenient. But though this modern theory has thrown much light on the relations of space, time, gravitation and other physical quantities, it seems to have pushed the ideal of an absolute system farther out of reach rather than brought it nearer. This book, however, is scarcely the place for any attempt at a *résumé* of Relativity, for the theory in question scarcely affects the ordinary slow-motion mechanics with which we are here chiefly concerned.

be formally proved, we may legitimately trust them provisionally, at any rate as first approximations. But since their acceptance is based on their inherent probability and the lack of any disproof, it must be noted that our trust in them should be coextensive with the experience upon which their acceptance is based. Any pushing of belief in them beyond such limits should be of the nature of an experiment. Thus, the Newtonian principles are accepted as consistent with an experience having certain bounds of space and time and relating to gross or ponderable matter with speeds within certain limits. May we push them and legitimately build upon them outside these limits? May we apply them to that medium called the ether, conceived as co-extensive with the physical universe and supposed endowed with inertia and elasticity but not with gravitation? May we apply them to the very small corpuscles or electrons of modern science, moving at speeds comparable with that of light? This may be done, but only tentatively and in an exploring manner, the results being continually tested by experimental checks. In fact we are now in a position to appreciate the following quotation from E. Mach (*Science of Mechanics*, pp. 237-238, Chicago, 1902):—

'The most important result of our reflection is, however, that precisely the apparently simplest mechanical principles are of a very complicated character, that these principles are founded on uncompleted experiences, nay on experiences that never can be fully completed, that practically, indeed, *they are sufficiently secured*, in view of the tolerable stability of our environment, to serve as the *foundation of mathematical deduction*, but that they can by no means themselves be regarded as mathematically established truths but only as *principles that not only admit of constant control by experiment but actually require it*.'

**206. Mass at High Speeds.**—In illustration of the preceding article we may note that the electron (or elementary negative electric charge) now so prominent in physical research is believed to behave as having different inertias at different very high speeds, this inertia being moreover different along and perpendicular to the direction of motion. According to the theories of Prof. H. A. Lorentz of Leiden, if the so-called longitudinal and transverse masses at speed $v$ are denoted by $m_1$ and $m_2$ respectively, and that at infinitesimal speeds by $m_0$ we have $m_1 = m_0/\gamma^3$ and $m_2 = m_0/\gamma$, where $\gamma = \sqrt{1 - v^2/c^2}$, $c$ being the speed of light. The motions are here reckoned with respect to the ether which, on this view, is considered not to share any of the translatory motions of gross matter.

It is possible that the ether, if in the above sense 'stagnant,' may prove to be the best base in which to fix our co-ordinate axes for an absolute system of mechanics.

**207. Quantities usually proportional to Mass.**—It is perhaps desirable to note here that though Newton defined mass as quantity of matter proportional to product of density and volume, most other writers have regarded mass as the inertia or measure of sluggishness of a body. And it thus appears in the definitions of Mach and of Pearson.

Really, in the general use of the terms ounce, pound, or ton, for mass or weight, we have more or less clearly in our minds several different quantities which we often vaguely assume or believe to be proportional. Thus, in buying food of a standard quality we are concerned with its nutritive or heat-generating properties and believe these quantities to be proportional to its weight. We might say we were here, if anywhere, concerned with *quantity of matter*. Again, if material is disposed in different places on a cricket-bat or a golf-club, we may be chiefly concerned with the *inertia* of that material, whether wood or metal. Thirdly, if we hang a piece of iron or lead to balance a sash window or pull a door to, we are concerned with the *weight* of that iron or lead. There are other cases in which we are concerned with the *elastic* resistance of a given piece of a certain substance, or with its usefulness as a conductor of heat or electricity, etc. Now the first four quantities are the most important to us mechanically, viz.: *quantity of matter*, *inertia*, *weight*, and *elasticity*. The first three we often take to be strictly proportional. The last we take to be proportional to any one of the others for a given definite shape in the simplest cases, like the stretching of a wire of given length.

But, since it may prove that none of these quantities are strictly proportional in any perfectly general view of matters, it must be clearly grasped in which sense the word mass is used in mechanics. It is used fundamentally in this work to denote *inertia* or sluggishness of a body to change its velocity in magnitude or direction. It is so used in the theory of Lorentz just noticed in Art. 206. From which it appears that at those high speeds, mass (or inertia) is *not* proportional to quantity of matter, *if* electrons are counted as matter and quantity is gauged by number of electrons, as Nature's identical elemental units.

**208. Retrospect.**—It is now time to glance in thought over the various views advanced as to the basis of mechanics, and to endeavour to extract some simple rules for guidance, some sufficient foundation to build upon. Nothing can be said or written on this topic which is not open to attack from several quarters. Nothing can be stated which shall be at once full, precise and brief. Yet it seems desirable that some short enunciations should be made which, to the sympathetic student, will convey a view of the foundations of mechanics that, while yielding much to modern criticism, both as to form and substance, avoids going to an extreme in any direction. With much diffidence, an attempt in this direction is submitted in the next article. Each statement is accompanied by a brief symbolic expression of the law or definition under its title. The notation in each case will be readily understood from the context.

Of course no permanent or widespread value can attach to any such brief statements. At best, they can but suit a limited number for a limited time. To such, for the present, it is hoped they may be of service. Possessing no possible permanency, they, or any of like nature, should be under constant criticism and revision. And if their presence here serves to stimulate an interest in the subject which shortly leads to their supercession, their formulation will not have been in vain.

**209. Brief Enunciation of Chief Mechanical Bases.**

| | |
|---|---|
| 1. *Law of Motion.* $(-a \propto a')$ | Accelerations occur only in opposite pairs, whose ratios are constant for given particles. |
| 2. *Definition of Mass.* $(m/m' = -a'/a)$ | The masses of particles are positive constants, inversely as their mutual accelerations. |
| 3. *Definition of Force.* $(F = ma)$ | Force is the product, mass into acceleration, and has the direction of the acceleration. |
| 4. *Law of Gravitation.* $(F \propto mm'/r^2)$ | Every particle attracts every other with a force, along their joining line, directly as the product of their masses, and inversely as their distance squared. |
| 5. *Law of Friction.* $(T \not> \mu N)$ | With dry surfaces in contact, relative sliding is resisted by tangential forces, whose limiting values are proportional to the normal forces. |
| 6. *Law of Elasticity.* (*Strain* $\propto$ *Stress*) | Within narrow limits, the strain of a body is proportional to the equilibrating set of forces, or *stress*, applied to it. |
| 7. *Gaseous Law.* ($pv =$ const. for $t$ const.) | The volume of a given mass of gas is inversely as its pressure at constant temperature. |
| 8. *Choice of Axes.* | Since motion is relative, force and mechanics are relative also. Hence, the foregoing and any problems based upon them, should be referred to axes which, in each case, yield a mechanics most appropriate to the phenomena under discussion. |

**210. Concluding Remarks.**—We may now with advantage glance over the enunciations of the previous article and note certain points concerning them. In the first single statement the endeavour is made to embody all we know, of an experimental or axiomatic nature, as to motion generally. It contains within itself Newton's first law and the qualitative aspect of his third law. It also asserts that proportional aspect of all the simultaneous accelerations possible to any given pair of mutually interacting particles, which forms the basis of the modern definition of mass, as inertia. This definition accordingly follows; mass being stated to be a positive constant characteristic of any given particle, and such that the products, mass into acceleration, for any pair of interacting particles have opposite signs but equal magnitudes. This supplies the quantitative aspect of Newton's third law, and so completes the statement of its substance.

It is a convenience to have a name for these opposite but equal products, and this is next supplied by the modern definition of force, which replaces Newton's eighth definition and second law, those two being practically identical.

But, by this definition, force is seen to be a vector, hence the law of

addition of vectors naturally applies. We thus have, at once, the triangle, parallelogram and polygon of forces without further proof.

It has not been considered necessary to include among these brief enunciations any specific statement either (i) as to the mass of any particle or body being the sum of the masses of its parts, or (ii) as to the mass of any particle being the same whether derived directly or indirectly by comparison with some given particle. For, it is supposed all through that the system being enunciated is a self-consistent one, and also that it satisfies any experimental checks that can be applied to it. These suppositions imply that the second point holds. As to the first point, it may be naturally inferred that the quantity called mass, since primarily it expresses the *sluggishness* of a particle and is characteristic of it, will increase to some larger quantity of the same kind characteristic of the sluggishness of a body when conceived as built up of particles. Also, since mass is a positive constant without reference to direction; and force, the product of mass and acceleration, has the direction of the acceleration; we see that mass is a *scalar* quantity. Hence masses, being scalars, are susceptible only of arithmetical addition. Thus, consistently with the foregoing enunciations, we obtain the mass of a body by the simple arithmetical addition of the masses of its parts. And, unless this agreed with experimental checks, the enunciations would be invalid as a description applicable to the physical universe as perceived by means of our senses. Hence, though it is conceivable that the masses of particles might not add to that of the body, the fact that they do so add may be naturally inferred from the enunciations without formal statement.

It should also be noticed here that, in the laws of motion and gravitation and in the definition of mass, only *particles* are mentioned. Then, the behaviour of particles being axiomatically formulated, that of extended bodies and systems of various constitutions is to be analytically derived.

Further, it is specifically mentioned in the fourth statement that the gravitational accelerations between particles occur in the line joining them. But, in accordance with the general principles adopted in these brief enunciations, the fact that in the so-called contact of particles the consequent opposite accelerations occur along the same line, is not mentioned in the Law of Motion, but is left to the natural inference of each reader.

Of the remaining enunciations, Nos. 5-7 call for little remark. Obviously those readers omitting parts of the present course may omit the corresponding enunciations. But the *last*, on the choice of axes, may need noting by those not concerned with some of the intermediate ones. This eighth statement is purposely placed last for two reasons: First, because it may apply to all the rest; Secondly, in accordance with the principle that although such a proviso may be needed somewhere, it should not be obtruded too early, and thus introduce difficulty in the preliminary axioms, where all should be kept as broad and simple as possible.

We may perhaps with advantage note here an application of this

principle of choice of axes, expressed in 8, to the law of gravitation given in 4. Thus, to reduce the proportionality to an equality, introduce a constant $\gamma$ and write $F=\gamma mm'/r^2$. Then by the definition of force in 3, we see that the accelerations of $m$ and $m'$ are respectively $\gamma m'/r^2$ and $\gamma m/r^2$. In other words, the accelerations have magnitudes inversely as the masses, which agrees with the definition of mass in 2. We thus see that these accelerations are *not* reckoned for each mass relatively to the other, but are each reckoned with respect to an *origin* which lies between them and divides their distance apart *inversely* as the masses. We shall see afterwards that this point is called the centre of mass or centre of gravity of the two bodies. Obviously the relative acceleration of the two masses is the sum of their separate accelerations relative to the same point between them. Thus adding the above expressions, we obtain for this total or mutual acceleration the value $\gamma(m+m')/r^2$.

The units used for measuring any of these mechanical quantities will be dealt with as occasion arises throughout the rest of the work.

Some of the topics of the present chapter are of a highly controversial nature. Thus, a partial discussion of them here and there throughout the book might well prove irritating to some readers and make the corresponding parts of the detailed treatment less acceptable. Accordingly, to obviate this drawback, the discussion of the physical bases of mechanics has been confined to this single chapter, from which mathematical deductions are excluded. Hence, the remainder of the work, presenting the formal developments of kinetics and statics, is left equally open to all classes of teachers and students whatever their views or convictions on the debatable matters underlying them.

**211. Bibliography.**—Considerations of space preclude any further treatment of these and kindred topics, for which the reader is referred to the following works, placed in alphabetical order of their authors:—

H. HERTZ. *Principien der Mechanik* (Leipzig, 1894.)
SIR OLIVER LODGE. *The Foundations of Dynamics* (Proc. Phys. Soc. London, 12 pp., 289-236, 1894.)
A. E. H. LOVE. *Theoretical Mechanics* (Cambridge, 1897.)
ERNST MACH. *The Science of Mechanics* (Chicago, 1902.)
SIR ISAAC NEWTON. *Philosophiae Naturalis Principia Mathematica* (London, 1686.)
KARL PEARSON. *The Grammar of Science* (London, 1900.)
H. POINCARÉ. *Science and Hypothesis* (London, 1905.)
BERTRAND A. W. RUSSELL. *The Principles of Mathematics* (Cambridge, 1903).
HERBERT SPENCER. *First Principles.* (London, 1898.)
ALEXANDER ZIWET. *The Relation of Mechanics to Physics* ('Science, 23 pp., 49-56. New York, Jan. 12, 1906.)

## EXAMPLES—XL.

1. State how time is usually measured and by what inaccuracy the method is affected. What other methods have been proposed with the view of diminishing the present inaccuracy?

2. What do you consider should be our attitude towards mechanical and physical axioms?
3. Explain how the inertias of an electron vary with its speed.
4. What three mechanical quantities are often tacitly assumed to be proportional to one another? Give examples showing the distinctions between them.
5. Do you consider the mechanical creed as enunciated by Newton needs revision? If so, state in your own words by what you would replace it. If not, defend Newton from his various critics.
6. Translate and comment upon :—

L'accélération d'un corps est égale à la force qui agit sur lui divisée par sa masse.

Cette loi peut-elle être vérifiée par l'expérience? Pour cela, il faudrait mesurer les trois grandeurs qui figurent dans l'énoncé ; accélération, force et masse.

J'admets qu'on puisse mesurer l'accélération, parceque je passe sur la difficulté provenant de la mesure de temps. Mais comment mesurer la force, ou la masse? Nous ne savons même pas ce que c'est.

Qu'est-ce que la *masse*? C'est, répond Newton, le produit du volume par la densité. Il vaudrait mieux dire, répondent Thomson et Tait, que la densité est le quotient de la masse par le volume. Qu'est-ce que la *force*? C'est, répond Lagrange, une cause qui produit le mouvement d'un corps ou qui tend à le produire. C'est, dira Kirchhoff, produit de la masse par l'accélération. Mais alors, pourquoi ne pas dire que la masse est le quotient de la force par l'accélération?'

(LONDON B.Sc., PASS, APPLIED MATH., 1908, III. 7.)

# CHAPTER XII

## KINETICS OF PARTICLES

**212. Mass brought into Equations.**—To pass from the kinematics of a point to the kinetics of a particle we suppose our point to be endowed with mass, $m$ say, and multiply the appropriate kinematical equations throughout by that mass ($m$).

Thus, for the rectilinear motion of a particle with uniform acceleration, we take (from article 27) the kinematical equations representing the increase of velocity from $u$ to $v$ in time $t$ under acceleration $a$; also the increase of the square of this velocity while describing space $s$. These may be written

$$v-u=at \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and

$$v^2-u^2=2as \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Now, multiplying by $m$ and writing $F$ for the force concerned in place of the product $ma$, we obtain

$$mv-mu=Ft \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mu^2=Fs \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

We see that these equations involve certain products, and since these often occur names have been adopted for them which are now given and defined, each being accompanied by a convenient symbol shown in brackets.

*Momentum.*—The product of a mass into its linear velocity is called its linear momentum ($mv=P$).

*Impulse.*—The product of a force into its duration is called its impulse ($Ft=Q$; or, if $F$ is the *variable* value of the force during the time $\tau$, the impulse is $Q=\int_0^\tau F dt$.)

*Kinetic Energy.*—Half the product of a mass into its velocity squared is called its kinetic energy ($\frac{1}{2}mv^2=T$).

*Work.*—The product force into distance described by the accelerated particle in the direction of the force is called work ($Fs=W$; or, if $F$ is the value of the *variable* force over the space $s$, $W=\int_0^s F ds$).

Using these ideas and symbols, we may put (3) and (4) in the forms

$$P-P_0=Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and $$T - T_0 = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where the zero subscripts denote initial values of the quantities in question.

We may state the above important relations verbally thus:—*Change of momentum equals impulse* and *Change of kinetic energy equals work*. The first of these is equivalent to Newton's second law, but may be here regarded as derived from the single law of motion and the definitions of mass and force of article 209. The second statement is important in connection with the *conservation of energy*, that fundamental doctrine so well known to physicists.

From (5) we see that $Q$ is of the same nature as $P$, *i.e.* mass into velocity. And we have hitherto supposed $P$ to change by change of velocity only, the mass being constant. But it is evident that $P$ might, under some circumstances, change by a *change of mass*, all these increments of mass having the *same velocity change*, from o to $v$ say. Or, both changes may occur together. We might thus write as a more general relation

$$dQ = Fdt = mdv + vdm = d(mv) = dP, \text{ whence } F = dP/dt \quad . \quad (7).$$

That is, *force is the time-rate of increase of momentum.* Similarly, from (6), we find $F = dT/ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$; or, *force is the space-rate of increase of kinetic energy.*

Some other current forms of speech used in connection with the above quantities and relations may be noted here. The acceleration $a$ of a particle of mass $m$ is said to be *due to the action* of the force $F = ma$. The particle, assimilated to a point, is called the *point of application* of the force. In the case of more extended bodies, the *point* of application becomes a *surface or volume of application*, according as the force is applied by contact of gross matter over a surface or by other means throughout a volume, as in the case of an attraction like gravitation.

The change of momentum from $P_0$ to $P$ is said to be *due to* or *produced by* the impulse $Q$, which equals $P - P_0$.

The change of kinetic energy from $T_0$ to $T$ is said to be *due to* or *produced by the expenditure of work* $W$, which equals $T - T_0$.

Some writers object to these expressions as implying *causal* relations of which we have no proof. But it is difficult to avoid these current and convenient expressions, even though we may regard all mechanics as *purely descriptive of observed phenomena.*

The relations involving impulse and work were derived from those for *uniform* acceleration, and so correspond to a *uniform* force, but we may now with advantage remove that restriction. Thus, if $F$ is the *variable* force for time $t$ and space $s$, the speed meanwhile changing from $u$ to $v$, we may write

$mdv/dt = F = mvdv/ds$. From these, multiplying up and integrating, we obtain

$$m(v - u) = \int_0^t Fdt = Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

and $$\tfrac{1}{2}m(v^2-u^2)=\int_0^s Fds=W. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Thus, on eliminating $m$ between them, we have the general relation

$$W=Q\frac{u+v}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11);$$

or, the *work of a variable force* is its *impulse* multiplied by the arithmetic *mean of the initial and final velocities.*

**213. Choice of Units of Mass, Force, etc.**—The above new quantities need appropriate units for their measurement. These units are not entirely settled by the relations between the quantities expressed above in words or symbols; but, certain units being chosen, the others then naturally follow from the relations in question. Thus, corresponding to the three indefinable quantities with which mechanics is concerned, viz. Space, Time, and Matter, we may choose units of length, time, and mass; the corresponding units of force, momentum, and impulse, energy and work, then follow. The units of length, time, and mass are then called *fundamental* units and the others *derived* units. Or, a different selection of fundamental units may be made, say length, time, and force, and then the others derived from them. We shall use both plans, and in the above order, but it will be well first to consider the relation between the mass and weight of a body near the earth's surface.

Let a particle of mass $m$ fall freely at some place on the earth and be found to have an acceleration $g$. Then by our definition the force concerned, called its weight $w$, is given by

$$w=mg, \text{ so that } m=w/g \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

The acceleration $g$ and the weight $w$ of a given particle or body are both known to vary in different parts of the earth, as we shall see later, but the quotient $w/g=m$, called the mass of the particle or lump, is believed to be practically constant, as stated in the definition of mass. (See article 209.) Now, if our units of length and time were such that $g$ were unity at some place, then, for that place, the mass $m$ and the weight $w$ for any body would be represented by the same number. But this is not the case for any system of units of length and time at present in vogue. Hence in all such systems the weight of a body is not represented by the same number as its mass, while we retain the definition $F=ma$. Thus, while it would be a great simplicity to retain this definition, have a standard lump of stuff as the unit of mass, and its weight at some standard place as the unit of force, this is incompatible with our present units of length and time. There are thus three simple courses open to us in choosing units of mass and force.

1. Choose a standard body as the unit of mass, and let the unit of force follow from the definition $F=ma$. We thus have the weight in these units of force given by $w=mg$. That is, the weight of a body is expressed by a number $g$ times the number which expresses its mass.

Hence the unit of force is $1/g$th of the weight of the unit mass. The centimetre-gram-second system and the so-called British absolute system are both of this type.

2. Choose a standard body and specify a place at which its weight shall be the unit of force, and let the unit of mass follow from $F=ma$. Thus as before $w=mg$ or $m=w/g$. Hence in these units the number expressing the mass of a body is $1/g$th of the number which expresses its weight at the standard place, $g$ being the acceleration of gravity there. That is, the unit of mass is $g$ times that of the standard whose weight is unit force. The British engineers' system is of this type.

3. Choose a standard body as unit mass, its weight at a specified place as unit force, and abandon the definition $F=ma$, replacing it by $F \propto ma$ or $F=kwa$. Then $k$ would have to be $1/g$, and the force would be given by $F=wa/g$.

The use of any system of units requires care in passing from the theoretical expressions to the concrete ones, namely, the care that all the units introduced are of the same system. But the adoption of this third choice for a system of units calls for greater care, for it involves a change in the definition of force (from an equality to a proportionality), and thus changes all the theoretical expressions which explicitly or implicitly involve force. It will accordingly be no further considered here, but attention confined mainly to the first and second methods of choice of units, though occasionally special units may be introduced to suit special problems.

**214. Established Systems of Units.**—In Table VII. are given the chief units on the three systems already referred to, while Table VIII. gives the ratios for conversion from each system to the others.

Of these systems, the *C. G. S.* was fixed by an International Conference at Paris in 1875, the *standard* of length being the metre, which is the length at the temperature of melting ice between the ends of a platinum rod, made by Borda and preserved in the Bureau des Archives in Paris; the standard of mass being the *kilogramme des archives*, made in platinum by Borda and preserved in the Conservatoire des Arts et Métiers, Paris.

The British standards of length (bronze) and mass (platinum) are the yard and pound respectively, and are in the charge of the Warden of the Standards.

TABLE VII. MECHANICAL UNITS.

| Systems of Units. | Units in each System | | | | | |
|---|---|---|---|---|---|---|
| | Length. | Time. | Mass. | Force. | Momentum and Impulse. | Kinetic Energy and Work. |
| International or *C. G. S.* $g=981$ nearly. | Centimetre | Second of Mean Solar Time. | Gram. | Dyne $=1$ gm. cm./sec.$^2$ $=\frac{1}{g}$ of a gm. wt. | A gram at a cm. per sec. or a dyne for a second. | Erg $=\frac{1}{2}$ gm.cm.$^2$/sec.$^2$ $=1$ cm. dyne. |
| British or *F. P. S.* $g=32{\cdot}2$ nearly. | Foot. | Second of Mean Solar Time. | Pound. | 'Poundal' $=1$ lb. ft./sec.$^2$ $=\frac{1}{g}$ of a lb. wt. | A pound at a foot per sec. or a poundal for a second. | Foot Poundal $=\frac{1}{2}$ lb. ft.$^2$/sec.$^2$ |
| Engineers' or *F. S. S.* $g_0=32{\cdot}1912$. | Foot. | Second of Mean Solar Time. | 'Slug' $=g_0$ lbs. | Weight of a Pound at Sea-level in London. 1 lb. wt. $=g_0$ poundals. | A slug at a foot per sec. or a lb. wt. for a second. | Foot Pound Weight $=\frac{1}{2}$ slug ft.$^2$/sec.$^2$ |

TABLE VIII. CONVERSION OF UNITS.

| Metric to British. | British to Metric. |
|---|---|
| 1 cm. $=0{\cdot}03280899$ ft. | 1 ft. $=30{\cdot}48$ cm. |
| 1 gm. $=0{\cdot}0022046$ lb. | 1 lb. $=453{\cdot}59265$ gm. |
| 1 dyne $=0{\cdot}0000723432$ poundal. | 1 poundal $=0{\cdot}0310644$ lb. wt. |
| $=0{\cdot}0000022473$ lb. wt. | $=13{,}823$ dynes. |
| | 1 lb. wt. $=32{\cdot}1912$ poundals. |
| | $=444{,}979$ dynes. |

In the two British systems of units, the first in Table VII., often referred to as the British *Absolute*, takes the pound as the unit of mass, the unit of force being called the *poundal*, a term suggested by the late Professor James Thomson (see Kelvin and Tait's *Natural Philosophy*, Part I., p. 229, 1890). In the British *engineers'* system the pound weight at sea-level in London is taken as the unit force, the unit mass being 32·1912 pounds, for which unit the term *slug* was suggested by Professor A. M. Worthington (see his *Dynamics of Rotation*, p. 9, footnote,

1904). It seems desirable to note here that some writers who freely use one or other of these systems object to the terms poundal and slug and prefer to leave the units in question without a name. In the table *F. S. S.*, after the title engineers', signifies *foot, slug, second.*

Corresponding to these two British systems there are two Continental ones (*M. K. S.*) using the metre, kilogram, and second as the units from which all else are derived. But the one system (called absolute) takes the kilogram as the unit of mass, derives $10^5$ dynes as the unit force and $10^7$ ergs, called a *joule*, as the unit of work. The other system takes the weight of a kilogram as the unit of force, and then derives 9·81 kilograms as the unit of mass.

Much controversy has arisen as to the relative advantages of some of these systems of units. The view here taken is that, in spite of any personal preference, it is incumbent on serious students to become conversant with the chief systems that have attained any considerable vogue.

### EXAMPLES—XLI.

1. Accepting the definitions of mass and force, transform the kinematic equations of rectilinear motion under uniform acceleration to the corresponding case of kinetics of a particle. Define carefully any new quantities that now enter into the equations.
2. From the fundamental equations of kinetics of a particle derive two new expressions for a force.
3. Describe carefully what you mean by an impulse, and show to what other quantity it may be equated. Obtain the value of an impulse as the area of a curve, and show approximately the proportions of such curves (1) for the blow of the racquet on a tennis ball, and (2) for the blow of a hammer on a nail in hard wood.
4. Assuming that force is equal (or proportional) to mass into acceleration, derive and critically discuss the systems of mechanical units in use among the English-speaking nations.
5. 'A shot having given size and shape, how will its penetrative power depend (1) on its weight, and (2) on its velocity, the resistance to penetration being supposed uniform? Give reasons for your answer.
   'A bullet fired with a velocity of 2000 f.s. penetrates to a depth of 18 inches in wood; what would be its velocity of emergence if fired through a board 1 inch thick?'
   (LOND. B.A. AND B.SC., PASS, MIXED MATH., 1904, I. 3.)
6. 'Determine the tension in any position of the thread by which a body is whirled round in a vertical circle.
   'Prove that in a bicycle track looped round in a complete vertical circle of 30 feet diameter, the velocity at the highest point should be due to a fall of 7 feet 6 inches, or about 15 miles an hour; and the reaction of the ground on entering the circular track at the lowest point will be increased to about six-fold.'
   (LOND. B.A. AND B.SC., PASS, MIXED MATH., 1902, I. 8.)

**215. Potential Energy and Transformations to and from Kinetic.** —Consider a particle of mass $m$ moving in a region where it will have uniform acceleration $a$. Let it have at P, Fig. 76, a velocity $u$ in the direction of the acceleration and $w$ at right angles thereto, its kinetic

energy being then $T_0=\frac{1}{2}m(u^2+w^2)$. When it has moved through a space $s$ parallel to the direction of the acceleration, let it be at Q, with velocity components $v$ and $w$ as shown in the figure, its kinetic energy being now $T$. Then we have

$$T-T_0=\tfrac{1}{2}m(v^2+w^2)-\tfrac{1}{2}m(u^2+w^2)=\tfrac{1}{2}m(v^2-u^2),$$

or
$$T-T_0=mas=Fs=W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $F$ is the force $ma$ and $W$ is called the work done on the particle by the *field* while it moves from P to Q. Also, by suitably changing our equations, we should obtain the result tnat the reversed velocity at Q would restore the particle to P with the reverse of its original velocity. That is, the kinetic energy would here change from $T$ to $T_0$ while the work $W$ was done by the particle against the field, for the reversals of the velocities have no effect on their squares, which alone enter into the expressions for kinetic energies. And, since the cross velocity $w$ disappears from equation (1), it is clear that what we have obtained applies not only to P and Q but to any other corresponding pairs of points on their levels. But though the particle in passing from Q to P loses kinetic energy to the amount $T-T_0$, it acquires an advantage of position in the field, which enables it to gain the like amount of kinetic energy on passing back to the level of Q. This advantage of position is called *potential energy*, and increases to the extent of the work done by the body against the field. Thus, calling the potential energies of the body $V_0$ and $V$ at the levels of P and Q respectively, we have

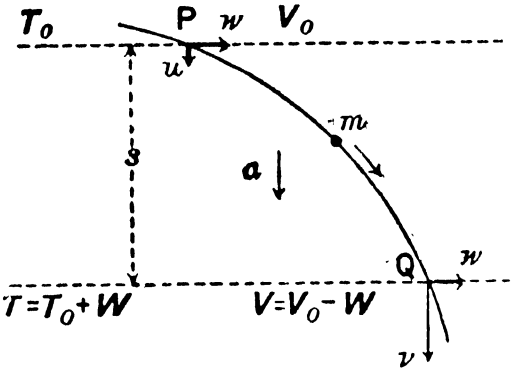



FIG. 76. TRANSFORMATION OF ENERGY.

$$V_0=V+W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

But by (1) this becomes $V_0=V+T-T_0$.

Hence
$$T+V=T_0+V_0=\text{constant} \quad . \quad . \quad . \quad (3).$$

This is an example of the conservation of energy, the energy of one kind which is lost by the body being exactly balanced by the energy of the other kind gained by it. Also during the motion either from P to Q or in the reverse direction, we have the *transformation* of energy occurring from potential to kinetic or the reverse. These transformations and reverses occur automatically in many familiar instances, *e.g.* a stone thrown into the air, a pendulum in motion. Here, in the rise, kinetic energy is lost and potential energy is gained; at the summit of the motion the transformation ceases for an instant; in the fall the reverse transformation from potential to kinetic energy occurs. In the case of a pendulum there is also a reversal of the transformation to the opposite kind at the lowest point where the velocity is greatest.

**216. Work in Oblique Displacements.**—Let us now consider the

work involved when a particle is constrained to move at an angle $\theta$ with the force $F$. Then, if the element of work $dW$ corresponds to the actual displacement $ds$, whose component is $dy$, in the direction of the force $F$, we have

$$dW = Fdy = F\cos\theta\, ds = F'ds \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where $F' = F\cos\theta$.

This is illustrated by Fig. 77, in which the force $F$ acts parallel to PN, along which $y$ is measured, but the particle is constrained to move along PQR, along which $s$ is measured. Thus if PQ is $ds$ and PM is $dy$, then $dW$ represents the work from P to Q. It is seen that the $F'$ in (4) is the component of the force along the path; hence we may state the equation in words thus:—When a particle is constrained to describe a specified path under a given force, the element of work is expressible by any of the three following products:—(i) the total force into the component displacement in direction of force; (ii) the total force into the total displacement into the cosine of the angle between them; (iii) the total displacement into the component force in direction of displacement.

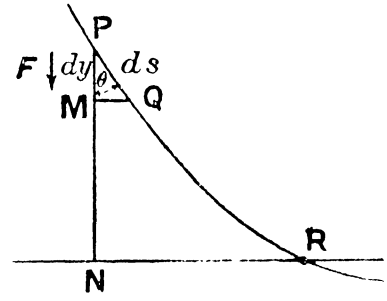


FIG. 77. WORK IN OBLIQUE DISPLACEMENT.

The total work along any finite portion of the path, say P to R, is given by the corresponding integrals, viz.

$$W = \int Fdy = \int F\cos\theta\, ds = \int F'ds \quad . \quad . \quad . \quad . \quad (5).$$

If $F$ varies from point in magnitude or direction or both, $F$ and $\theta$ may be expressed in terms of $y$ or $s$ and the evaluation effected. If $F$ is constant in magnitude and direction and $\theta$ varies simply owing to the curvature of the constrained path, we may take the first expression to the right of (5), and find

$$W = \int_0^y Fdy = Fy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where $y$ is the length of PN.

It is seen that this use of the constrained path corresponds in this respect with the motion (supposed free) considered in article 215.

**217. Direct Impact.**—Let a perfectly smooth particle of mass $m$ with velocity $u$ strike a second similar particle of mass $m'$ with velocity $u'$, both velocities being along the same line which is also the line of centres of the particles when in contact. Then by Newton's laws or the definition of mass in article 209, the accelerations of the particles are in the negative inverse ratio of their masses. Thus from the equation of article 209 (2) we have

$$m/m' = -a'/a \text{ or } ma + m'a' = 0 \quad . \quad . \quad . \quad . \quad . \quad (1),$$

the $a$'s denoting the accelerations of the particles. But since this relation holds for any and every instant, the total changes of velocity must be proportional to these accelerations. Hence we have

$$m(v-u) + m'(v'-u') = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, the algebraic sum of the changes of momenta is zero. Or we may write this in the form

$$mv+m'v'=mu+m'u' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

which is equivalent to the statement:—*The algebraic sum of the final momenta equals that of the initial.* This is often referred to as the principle of the *conservation of momentum.*

The passage from equation (1) to (2) could be expressed fully in symbols thus:—

$$-\frac{m}{m'}=\frac{a'}{a}=\int a'dt \div \int adt=\frac{v'-u'}{v-u}.$$

Equation (3) is, however, insufficient to determine the two final velocities $v$ and $v'$. We accordingly require another condition, which is supplied by stating the ratio of the velocity of separation to that of approach. For this ratio, called the *coefficient of restitution*, is found to be approximately a constant for given bodies. Thus, denoting it by $e$, we have

$$-v+v'=e(u-u') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

In deriving the above equations it should be noted that all velocities are reckoned positive when in one given direction and negative if in the opposite direction. Care must be exercised if the conditions of any problem are stated in terms which violate this convention.

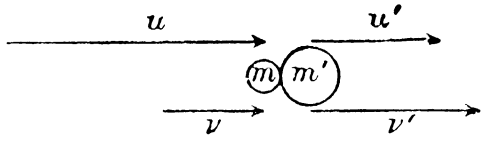



FIG. 78. DIRECT IMPACT.

Fig. 78 illustrates the case of direct impact in question, the velocities before impact being inserted above the particles and those after impact below them, any example being directly solvable from equations (3) and (4).

**218. Oblique Impact.**—To pass from the case of direct impact of smooth particles to that of oblique, it is obvious we have simply to compound with the velocities along the line of centres that which occurs at right angles thereto and which is not changed by the impact. Denoting these cross components by $w$'s, and retaining the previous notation, we have the complete scheme as shown in Fig. 79.

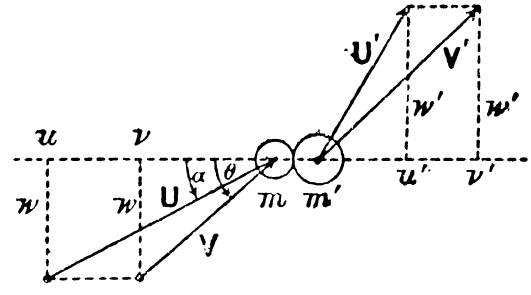



FIG. 79. OBLIQUE IMPACT.

In this case, therefore, the $v$'s are found as for the direct impact, and the unchanged $w$'s being compounded with them give the final velocities in magnitudes and directions, though often it is just as convenient to retain the expressions for the components simply.

Where the actual velocities are required, denoting them by capital

letters, and calling the angles with the line of centres before and after impact $\alpha$ and $\theta$ respectively, we obviously have

$$u = U\cos\alpha \text{ and } w = U\sin\alpha \quad . \quad . \quad . \quad . \quad . \quad (5),$$

also $$V^2 = v^2 + w^2 \text{ and } \tan\theta = w/v \quad . \quad . \quad . \quad . \quad . \quad (6),$$

and similarly with the accented letters.

**219. Loss of Kinetic Energy at Impact.**—Let the kinetic energy of the two particles before impact be $T_0$ and after impact be $T$, and for the sake of generality let the impact be oblique. Then we are concerned with the difference

$$\begin{aligned} T_0 - T &= \tfrac{1}{2}m(u^2 + w^2 - v^2 - w^2) + \tfrac{1}{2}m'(u'^2 + w'^2 - v'^2 - w'^2) \\ &= \tfrac{1}{2}m(u^2 - v^2) + \tfrac{1}{2}m'(u'^2 - v'^2) \quad . \quad . \quad . \quad . \quad . \quad (7). \end{aligned}$$

In transforming this expression to determine its sign we shall need the following relations, the first two of which are readily derived from equations (3) and (4):—

$$m'(u' - v') = -m(u - v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

$$v - v' = -e(u - u') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Then by elimination of $v'$ between (8) and (9) we find

$$u - v = \frac{m'}{m + m'}(u - u')(1 + e) \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Then using (8), (9), and (10) in turn we transform (7) as follows:—

$$\begin{aligned} 2(T_0 - T) &= m(u^2 - v^2) + m'(u'^2 - v'^2) \\ &= m(u - v)(u + v) - m(u - v)(u' + v') \\ &= m(u - v)(u - u' + v - v') \\ &= m(u - v)(u - u')(1 - e) \\ &= m\frac{m'}{m + m'}(u - u')(1 + e)(u - u')(1 - e). \end{aligned}$$

Or, finally, $$2(T_0 - T) = \frac{mm'}{m + m'}(u - u')^2(1 - e^2) \quad . \quad . \quad . \quad . \quad . \quad (11).$$

The loss of kinetic energy is thus seen to be expressible by the product of three factors, of which the first, involving the masses, is essentially positive; the second, being a square of real quantities, is positive or zero; while the third, being the defect of $e^2$ from unity, is positive or zero, for $e$ cannot overstep unity, and indeed never quite reaches it for any known bodies. It may be remarked also that if the second factor, $(u - u')^2$, were zero, that would correspond to equality of initial velocities; in other words, to the absence of impact. We may accordingly say that for all cases of impact between particles or bodies there is a loss of kinetic energy so far as it is estimated by the motion of the bodies considered as entities and moving altogether. If, however, we take the more searching view of the matter which falls within the domain of physics, we find there is no loss of energy. That which has disappeared when we regard only the motions of the bodies as wholes, is compensated by an equal quantity of energy of motion of the separate molecules or atoms, etc., of the body, this form of energy being called heat. Or, we may say briefly, the *molar* kinetic energy

which has disappeared is not lost but merely transformed into the same quantity of *molecular and atomic* kinetic energy. That heat is generated by impact is easily verified by hammering a nail on an anvil.

This loss could also have been calculated by use of equation (11) of article 212, the value of the impulse Q being, of course, $m(u-v)$.

**220. Impact of a Molecule.**—In connection with the kinetic theory of gases, it is of interest to examine the impact of a perfectly elastic particle or molecule of evanescent mass $m$ with a body of ordinary mass $M$, such as the wall of a containing vessel. Let the velocities of $m$ before and after impact be $u$ and $v$ and those of $M$ be $U$ and $V$, all along the line normal to the surfaces in contact at impact. Then we have $m/M=0$ and $e=1$. Thus equation (3) of article 217, on dividing out by $M$, reduces to

$$V=U \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

And (4) of the same article then becomes

$$-v+U=u-U,$$

or

$$u+v=2U \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

That is, the large mass suffers no change of its velocity by the impact of the indefinitely small mass, while the latter's velocities are such as to make their arithmetic mean equal to the unchanged velocity of the large mass.

Hence if the large mass is at rest, it remains so, and the velocity of the small mass is reversed in direction without change of magnitude, for

$$u+v=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Again let a mass $M$ with initial speed $V$ strike directly a mass $m$ at rest, and for the pair let $e=0$ and their final common velocity be $v$. Then we find that the ratio (loss of kinetic energy of $M$) ÷ (gain of kinetic energy of $m$) $=2+m/M$. Thus for equal masses this ratio is 3, and for vanishingly small $m$ it is 2.

### EXAMPLES—XLII.

1. Define *Potential Energy*, and give two or more examples of motions in which the *transformation* of energy occurs, also one in which it does not.
2. State precisely what you mean by *work*, and find an expression of the work of an oblique displacement against a given force.
3. Masses of 5 and 12 lbs. have initial velocities in the same direction and along the same line of 40 and 30 ft./sec. If the coefficient of restitution between them is 0·8, find their respective velocities after impact.
4. If the initial velocity of the larger mass in question 3 were reversed, find what the final velocities would become, the other data being unchanged.
5. If any collision occurs between two smooth spheres of equal mass and coefficient of restitution unity, one of them being initially at rest, show that their paths after the impact are at right angles to one another.
6. Prove that a loss of mechanical kinetic energy occurs at any impact between real bodies.
7. 'Show that a redistribution of momentum takes place in the collision of two bodies; and calculate the change of velocity in the impact of two inelastic bodies.

'Prove that the collision energy in foot-tons of two ships of tonnage $W_1$ and $W_2$ due to a relative velocity $V$ feet/second is

$$\frac{W_1 W_2}{W_1 + W_2} \cdot \frac{V^2}{2g}.$$

(LOND. B.SC., PASS, MIXED MATH., 1903, II. 5.)

**221. Angle and Cone of Friction.**—Let a body be placed on a rough surface and be acted on by a force $F$ inclined $\theta$ to the normal to the surfaces in contact as in Fig. 80.

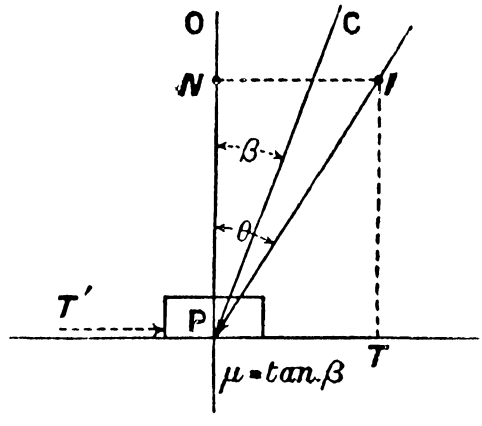



FIG. 80. FRICTION.

Then the normal and tangential components are given by

$$N = F\cos\theta \text{ and } T = F\sin\theta,$$

whence

$$T/N = \tan\theta \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Suppose the coefficient of friction to be $\mu = \tan\beta$, then by the laws of friction (see articles 201 and 217) we have

$$T'/N \not> \tan\beta \quad . \quad . \quad . \quad . \quad (2),$$

where $T'$ is the frictional resistance, which acts so as to prevent relative motion if possible, and is as large as necessary for this under the limitation expressed by (2).

Hence, if $\tan\theta > \tan\beta$, then $T > T'$ . . . . . (3), and there will be *acceleration* of the body no matter how *small* $F$ may be. On the other hand,

$$\text{if } \tan\theta \not> \tan\beta, \text{ then } T \not> T' \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and there is *no acceleration* of the body no matter how *great* the force $F$ may be.

The angle $\beta$ defined by $\tan\beta = \mu$ . . . . . (5) is called the *angle of friction*, and the cone whose semi-vertical angle is $\beta$ and axis the normal to the surfaces at the point of contact is called the *cone of friction* or *friction cone*. Since the vertical makes the same angle with the normal to a surface that the surface makes with the horizontal, it is evident that a surface can be inclined from the horizontal up to the angle of friction before a body will start to slide down it under gravity. So the angle of friction is sometimes called the *angle of repose*. This relation also furnishes a ready means of finding the values of $\beta$ and $\mu$ for a given pair of surfaces.

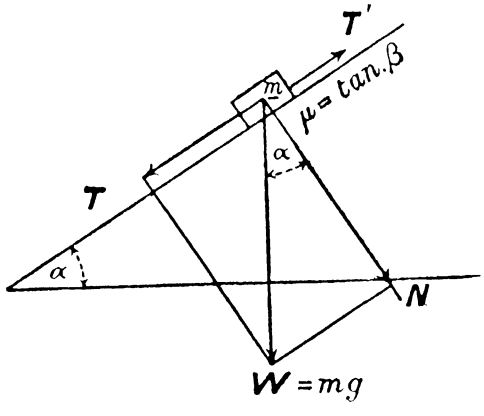



FIG. 81. MOTION ON ROUGH INCLINE.

**222. Motion on Rough Incline.**—Consider a body placed on a rough surface inclined at an angle $\alpha$ with the horizontal, the coefficient of

friction being $\mu=\tan\beta$, where $\beta<\alpha$. Then it is obvious that the body will have, down the plane, an acceleration whose amount will determine the motion for any given initial conditions. Let us therefore, by reference to Fig. 81, find this acceleration, which will be denoted by $a$, the mass being $m$.

The weight $W=mg$ may be resolved into components normal and tangential to the plane, thus giving

$$N=mg\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

and

$$T=mg\sin\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

Then the maximum tangential force due to friction, which is all called into play when there is relative motion, is given by

$$T'=\mu N=\mu mg\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Thus the resultant moving force on the body is $T-T'$, which, by definition of force, equals mass into acceleration.

Hence

$$a=\frac{T-T'}{m}=g(\sin\alpha-\mu\cos\alpha),$$

or

$$a=\frac{g}{\cos\beta}\sin(\alpha-\beta) \quad . \quad . \quad . \quad . \quad (9).$$

**223. Atwood's Machine.**—This machine consists essentially of two equal masses connected by a thread which passes over a pulley freely rotating on a horizontal axis, one of the masses starting with an overweight which is afterwards removed at a chosen position by an adjustable ring, the motion being finally checked by an adjustable stage. It is used to illustrate uniform acceleration and to estimate the value of $g$, the acceleration near the earth's surface. We have here to determine the relation between $g$ and the acceleration $a$ when the overweight is in use, and to find also the tension of the thread. Let the total masses at the overweight side be $M_1$ and at the other side $M_2$ and the tension of the thread be $T$. Then, neglecting the masses of the thread and pulley, also the stiffness of the thread and the friction of the axle, we may proceed as follows:—The larger mass $M_1$ will descend with acceleration under the resultant force of its weight minus the tension of the thread, while the smaller mass $M_2$ will ascend with the same acceleration due to the excess of the tension over its weight. And by definition each force is the product of the acceleration and the mass concerned. Hence

$$M_1g-T=M_1a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and

$$T-M_2g=M_2a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

So, by addition, we eliminate $T$ and obtain

$$(M_1-M_2)g=(M_1+M_2)a,$$

or

$$a=\frac{M_1-M_2}{M_1+M_2}g \quad . \quad . \quad . \quad . \quad (3).$$

Then, putting this value of $a$ in (1) or (2), we find for the tension of the thread

$$T=\frac{2M_1M_2g}{M_1+M_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Here, if the $M$'s are in grams, $g$ is about 981 cm./sec.$^2$ and $T$ is expressed in dynes.

If the masses are expressed in pounds, $g$ is about 32·2 ft./sec.$^2$ and $T$ is given in poundals. Whereas if the masses are in slugs of 32·1912 pounds each, $g$ is again about 32 ft./sec.$^2$ and $T$ is given in pounds weight at sea-level in London. In equation (3) the $a$ is given in the same units as those in which $g$ is expressed.

**224. Friction allowed for in Atwood's Machine.**—While still neglecting the masses of the thread and pulley, let us now make an allowance for the stiffness of the thread and the friction of the pulley axle in its bearings. Suppose these to be balanced by the addition on the descending side of a mass $m$ and the subtraction from the ascending side of the same mass, so that the total mass moved is as before. Let now the tensions of the thread on the descending and ascending sides be $T_1$ and $T_2$ respectively, then we have the arrangement as shown in Fig. 82.

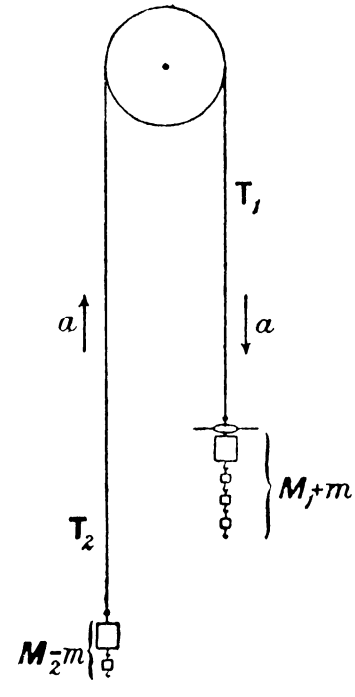



FIG. 82.
ATWOOD'S MACHINE WITH FRICTION.

Then, considering in turn the descending and ascending masses and the resistances at the pulley, we have the three following equations:—

$$(M_1+m)g-T_1=(M_1+m)a \quad . \quad . \quad (5),$$
$$T_2-(M_2-m)g=(M_2-m)a \quad . \quad . \quad (6),$$
$$T_1-T_2=2mg \quad . \quad . \quad . \quad . \quad (7).$$

Hence, on addition of the three equations, we have

$$\left.\begin{aligned}(M_1-M_2)g&=(M_1+M_2)a,\\ a&=\frac{M_1-M_2}{M_1+M_2}g\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8),$$

or

as before in (3) of article 223. But the tensions are now different from each other and from their former value. Thus from (5) and (8) we have

$$T_1=(M_1+m)(g-a)=\frac{2(M_1+m)M_2}{M_1+M_2}g. \quad . \quad . \quad (9).$$

Similarly, we find that

$$T_2=\frac{2(M_2-m)M_1}{M_1+M_2}g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Comparing these tensions with that when friction was supposed negligible we see that, since the acceleration is the same, we ought to have

$$\frac{T_1}{M_1+m}=\frac{T}{M_1} \text{ and } \frac{T_2}{M_2-m}=\frac{T}{M_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

and these checks are satisfied by the values in (4), (9), and (10).

**225. Motion of Connected Particles on Rough Inclines.**—Consider now the case of two particles or bodies connected by a thread and

each resting on a rough incline as shown in Fig. 83. We shall suppose the masses of thread and pulley and the effect of friction of pulley axle negligible.

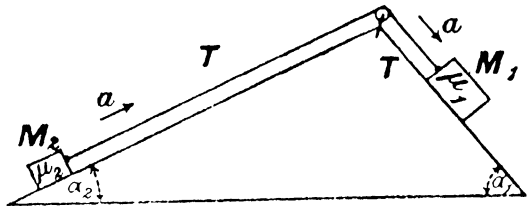



FIG. 83. CONNECTED PARTICLES ON INCLINES.

Also let us at first suppose that the masses, inclinations, and roughnesses are such that motion with acceleration $a$ occurs to the right, $M_1$ descending the incline $\alpha_1$ and dragging $M_2$ up the incline $\alpha_2$, the coefficients of friction being $\mu_1$ and $\mu_2$, and the tension in the thread $T$. Then considering each mass in turn, we have

$$M_1 g(\sin\alpha_1 - \mu_1\cos\alpha_1) - T = M_1 a \quad . \quad . \quad . \quad . \quad . \quad (12),$$

and

$$T - M_2 g(\sin\alpha_2 + \mu_2\cos\alpha_2) = M_2 a \quad . \quad . \quad . \quad . \quad . \quad (13),$$

whence, by addition and transformation, we have

$$\frac{a}{g} = \frac{M_1(\sin\alpha_1 - \mu_1\cos\alpha_1) - M_2(\sin\alpha_2 + \mu_2\cos\alpha_2)}{M_1 + M_2} \quad . \quad . \quad (14).^1$$

And, by substitution of this in (12) or (13), we find

$$T = \frac{M_1 M_2 g}{M_1 + M_2}(\sin\alpha_1 + \sin\alpha_2 - \mu_1\cos\alpha_1 + \mu_2\cos\alpha_2) \quad . \quad . \quad (15).$$

*I. One Mass descending vertically drags the other on a Horizontal Plane.*—The case just dealt with is very general, and by insertion of suitable values for the angles applies to various special cases. Thus for the present case we have only to write $\alpha_1 = 90^\circ$ and $\alpha_2 = 0$ in equations (14) and (15), and we find

$$\frac{a}{g} = \frac{M_1 - M_2\mu_2}{M_1 + M_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

and

$$T = \frac{M_1 M_2 g}{M_1 + M_2}(1 + \mu_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

results which are easily obtained by direct consideration of this problem. We can, of course, pass to the case of a smooth horizontal plane by writing $\mu_2 = 0$ in (16) and (17).

*II. Both Masses hang vertically.*—We now write $\alpha_1 = \alpha_2 = 90^\circ$, and find on substitution

$$a = \frac{M_1 - M_2}{M_1 + M_2} g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

and

$$T = \frac{2 M_1 M_2 g}{M_1 + M_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

which, by their agreement with (3) and (4), serve as an additional check.

[1] If on working this for any given numerical case the value comes *negative*, this must *not* be taken as the correct value for reversed acceleration. For a reversed acceleration involves reversed frictional forces, which would change the value of $a$ since the gravity forces are not reversed. The equations would need rewriting on the supposition that the acceleration is reversed, and if the value is then positive, it is correct.

## EXAMPLES—XLIII.

1. Explain the phrases *angle of friction* and *cone of friction*, and illustrate by a sketch.
2. Discuss the sliding of a body down a rough incline when by a thread and pulley it pulls another body up a second incline also rough. Check your result by reducing the second plane to a horizontal one.
3. 'Describe the use of Atwood's machine for the experimental verification of the Laws of Motion.'

(LOND. B.SC., PASS, MIXED MATH., 1902, II., 1st part of 6.)

4. Show how to allow for friction in Atwood's machine.

**226. Pendulum in Accelerated Chamber.**—A simple pendulum of length $l$ hangs from the roof of a chamber which has accelerations $a$ vertically upwards and $b$ horizontally, let it be required to find the circumstances of the motion in the plane of $a$ and $b$, and the tension $T$ of the thread.

A brief consideration will serve to show that the zero position will be displaced and the value of the effective $g$, and therefore also of the period $\tau$, altered. But perhaps a detailed examination is desirable. The case is represented in Fig. 84, in which S denotes the point of suspension, P the bob of mass $m$, SL and SM vertical and horizontal lines which move parallel to themselves. The pendulum is shown displaced from the vertical by the angle $\theta$, and is supposed to have angular velocity and acceleration $\dot{\theta}$ and $\ddot{\theta}$ respectively.

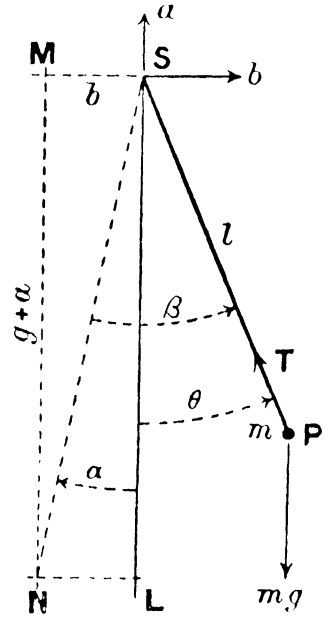



FIG. 84. PENDULUM IN ACCELERATED CHAMBER.

It will be convenient to write expressions for the component radial and transverse accelerations and equate their values to the quotients (resultant force/mass) for the corresponding directions. Thus from equations (5) and (6) of article 74 we have for radial and transverse accelerations about a fixed point the general expressions

$$f=\ddot{r}-r\dot{\theta}^2 \text{ and } j=r\ddot{\theta}+2\dot{r}\dot{\theta}.$$

In the present case of $r=l=$constant, these would become

$$f'=-l\dot{\theta}^2 \text{ and } j'=l\ddot{\theta}$$

if $S$ were at rest. Thus, taking into account the accelerations $a$ and $b$, we have for the total radial acceleration in the direction S to P and the corresponding force divided by mass the expressions

$$-a\cos\theta+b\sin\theta-l\dot{\theta}^2=\frac{-T+mg\cos\theta}{m},$$

or

$$T/m=(g+a)\cos\theta-b\sin\theta+l\dot{\theta}^2 \quad . \quad . \quad . \quad (1).$$

We have similarly for the transverse acceleration, reckoned in the direction from SL to P, and the corresponding force divided by mass

$$a\sin\theta+b\cos\theta+l\ddot{\theta}=\frac{-mg\sin\theta}{m},$$

or

$$(g+a)\sin\theta+b\cos\theta+l\ddot{\theta}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

**227.** The similarity of form of these equations suggests the possibility of a simplification, and obviously in (2) on putting $\theta=0$ the remainder of the equation defines the displaced zero line which takes the place in the moving chamber of the vertical in the chamber when at rest. And on examination of equations (1) and (2) we find they are susceptible of the form

$$T/m=g'\cos\phi+l\dot{\phi}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and $$g'\sin\phi+l\ddot{\phi}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where $$g'=\sqrt{(g+a)^2+b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

$$\phi=\theta+\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

and $$\tan\alpha=b/(g+a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Thus, as the new variable angle $\phi$ exceeds the former one $\theta$ by $\alpha$, we see from (4) that an angle must be measured from the vertical in the *negative* direction and equal numerically to $\alpha$ to represent the zero line. This is shown by SN in Fig. 84, in which SL and SM are proportional to $g+a$ and $b$ respectively.

If we now confine ourselves to small oscillations about this zero line, for which $\sin\phi=\phi$ nearly, equation (4) is replaced by

$$g'\phi+l\ddot{\phi}=0, \text{ or } \ddot{\phi}=-\frac{g'}{l}\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Hence the motion is simple harmonic of period

$$\tau'=2\pi\sqrt{l/g'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

and is fully represented by

$$\phi=\phi_0\cos(\sqrt{g'/l}\,t+\delta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

In this the $\phi_0$ and $\delta$ are arbitrary constants to be determined by the initial conditions. Thus, if the pendulum start from rest with amplitude $\beta$, we have for $t=0$, $\phi=\beta$ and $\dot{\phi}=0$. Hence therefore $\phi_0=\beta$ and $\delta=0$.

Hence (10) becomes

$$\phi=\beta\cos(\sqrt{g'/l}\,t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

the velocity being given by

$$\dot{\phi}=-\beta\sqrt{g'/l}\sin(\sqrt{g'/l}\,t) \quad . \quad . \quad . \quad . \quad (12).$$

**228. Results and Applications.**—Thus, the initial conditions being known, equations such as (11) and (12) substituted in (3) give the tension at any instant, *e.g.* for $t=0$, $\dot{\phi}=0$, $\phi=\beta$, and $T=mg'\cos\beta$.

Hence, to sum up, when a pendulum makes small oscillations in an accelerated chamber, they are performed

(i) about a *displaced zero* defined by (6) and (7);
(ii) in a *disturbed period* $\tau'$ given by (9);
(iii) as though gravity had the *disturbed value* $g'$, in (5); and
(iv) the *tension of the thread* follows from this disturbed $g'$, as shown in (11), (12), and (13).

We have accordingly confirmed by strict analysis the view which might have been conjectured at the outset, and may be stated thus:—

To find the behaviour of a pendulum in an accelerated chamber compound the acceleration due to gravity with the *reversed* acceleration of the point of suspension; this gives in direction and magnitude the

*disturbed or effective gravity*. Then the oscillations occur about the direction of this effective gravity as zero position, the period and tension being expressed in terms of this new gravity just as they are for an ordinary pendulum with the actual gravity.

Obviously this examination applies to a pendulum inside the carriage of a mountain railway when starting or stopping, also to a carriage or cage slipping under gravity down a steep incline and whether or not it is pulling another up. Of course, uniform velocity of the carriage along a straight line has no influence on the pendulum.

In the case of a train, with acceleration $b$, on a horizontal straight track, we have only to put $a=0$ and let Fig. 84 be the longitudinal section of the carriage.

For the case of a lift, with vertical acceleration $a$, we have simply to put $b=0$. We may note also that the zero line is not now displaced. Thus, the presence of the vertical acceleration *a alone* has no power to displace the zero line, although it *can alter* the displacement produced by the horizontal acceleration. See equation (7).

**229. Pendulum in Carriage round a Curve: Elevation of Exterior Rail.**—For a train moving at uniform speed $v$ along horizontal rails curved to a radius $R$, we must take Fig. 84 to be a cross section of the carriage or an end view inside. Then, putting $a=0$ and $b=v^2/R$, we find by (7)

$$\tan\alpha=v^2/Rg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

And, since this is the angle of the effective gravity to the vertical, it should be the angle of elevation of the road crosswise, with the horizontal. Or, in other words, the exterior rail should be elevated $\alpha$ as seen from the interior one, in order that the train may proceed with half load on each rail as if going straight along a level track. This result could have been seen from the first, but serves here as a useful check on the methods of the pendulum problem.

## EXAMPLES—XLIV.

1. 'A railway carriage is moving with a constant acceleration of $a$ feet per second per second along a straight road, and a particle is suspended by a fine thread from the roof of the carriage. The acceleration ceases at a certain point, while the rectilinear motion continues; then the carriage moves with a velocity $v$ f./s. on a curved part of the line where the radius of curvature is $\rho$ feet.

'Trace the motion of the pendulum through these changes.'

(LOND. B.SC., PASS, MIXED MATH., 1904, II. 8.)

2. 'Prove that a man, weighing $w$ lb., moving about in the cage of a lift equilibrated by a counterpoise of $W$ lb., will experience an apparent relative field of gravity

$$2Wg/(2W+w).'$$

(LOND. B.SC., PASS, MIXED MATH., 1902, II., 2nd part of 6.)

3. 'The gauge of a railway being 4 feet 8 inches, calculate the elevation (in inches) of the outer above the inner rail, so that there shall be no flange pressure when trains travel at a speed of 30 miles per hour, at a curved part of the line where the radius of curvature is $\frac{1}{4}$ mile.

'Show that if trains travel at this part of the line with a speed of 45 m./h.,

there will be a flange pressure against the outer rail equal to $11W/192$, where $W$ = weight of train ; and if with a speed of 20 m./h., a flange pressure against the inner rail equal to $11W/432$.'

(LOND. B.SC., PASS, MIXED MATH., 1904, II. 5.)

**230. Motion of Three Connected Masses.**—Let us now consider the behaviour of the masses in an Atwood's machine (see article 223) when any point of the ascending thread suddenly picks up a mass $m$ previously stationary. This could be accomplished by a loop, hook, or cross bar on the thread, but the picking-up arrangement, of whatever form, will be supposed of negligible mass. Thus, retaining the previous notation and neglecting friction, we have the mass $M_1$ descending and the other mass $M_2$ ascending at equal speeds, $u$ say, when the mass $m$ is suddenly caught up at a point on the ascending portion of the thread. This is analogous to an impact between the masses $M_1$ and $m$ with speeds $u$ and zero respectively. But, instead of an impulsive pressure between the bodies while in contact and approaching, we have now an impulsive tension on the connecting thread which the bodies are tending to separate. Since we shall suppose the thread practically inextensible, their velocities after the jerk will be equal, $v$ say.

Hence we may equate the impulsive tension to each change of momentum which occurs. Thus, with an obvious notation

$$\int_0^\tau F dt = mv = M_1(u-v) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

so that

$$\tau = \frac{M_1 u}{M_1+m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

But as the velocity of $m$ and $M_1$ decreases from $u$ to $v$, the thread between $m$ and $M_2$ slackens, since the latter mass has suffered no change of its velocity, which is $u$. Let us suppose that the conditions are such that the thread becomes tight again after the lapse of time $t$. Then the masses $m$ and $M_2$ will each have ascended through the same space, $s$ say, since $m$ became attached. Expressing these values in the form $ut+\frac{1}{2}at^2$ for each body, we have the relation

$$s = vt + \tfrac{1}{2}\frac{M_1-m}{M_1+m}gt^2 = ut - \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad (3).$$

This would serve to determine $t$, and therefore also the speeds $U_1$ of the masses $M_1$ and $m$ and $U_2$ of the mass $M_2$ when the thread again became tight. There would then be a second jerk, this time involving the whole thread, but with different impulsive tensions above and below $m$. Immediately after this jerk the same speed would be common to all the three masses.

Denoting it by $V$, we have

$$(M_1+M_2+m)V = (M_1+m)U_1 + M_2U_2 \quad . \quad . \quad . \quad (4).$$

From this instant the acceleration would be given by

$$A = \frac{M_1-M_2-m}{M_1+M_2+m}g \quad . \quad . \quad . \quad . \quad . \quad (5).$$

We have supposed all through that these phenomena can occur before $m$ reaches the pulley.

**231. Chain falling on Table.** — A uniform chain or cord is supposed to be suspended by its upper end over a horizontal massive table with which its lower end is in contact. The chain is then let go, and it is required to determine the pressure on the table at any instant.

We suppose the table to have such a large mass as to be practically unmoved by the impact of the chain. Thus, by the definition of force as the product of mass into acceleration (*or*, rate of change of momentum), the table at any instant acts upon the arriving chain with the force needed to destroy its momentum. It also supports the portion of chain which has previously arrived. Let the length of this part be $s$, the mass of the chain per unit length be $\sigma$, and the velocity of the part above the table be $v$. Then the mass of chain arriving in time $\delta t$ is $\sigma v \delta t$, so the momentum of it is $\sigma v^2 \delta t$. And this is destroyed in time $\delta t$. Hence force required to stop the arriving chain at this instant is

$$F = \sigma v^2 = \sigma 2gs \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

remembering that $v$ is due to the free fall through the height $s$. But the weight of the chain already on the table is

$$W = \sigma g s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus the total pressure is

$$P = F + W = 3\sigma g s = 3W \quad . \quad . \quad . \quad . \quad . \quad (3).$$

**232. Chain uncoiling from Table.**—Let a length $x$ of a chain hang vertically at one side of a pulley and a length $c$ hang vertically on the other side, beyond which the remainder of the chain lies coiled on a table. It is required to determine the motion of the chain and the tension where it leaves the table; $x$ is supposed equal to or greater than $c$, both being measured to the top of the pulley, which is of negligible mass and free to turn on a horizontal axis.

Let the chain have mass $\sigma$ per unit length and the tension at the table be $T$. Then, equating weight less tension to mass into acceleration, we have

$$(x-c)\sigma g - T = (x+c)\sigma \dot{v} \quad . \quad . \quad . \quad . \quad . \quad (4).$$

But in time $\delta t$ a length $v\delta t$ of mass $\sigma v \delta t$ will be removed from the table and jerked to a speed $v$. And the momentum acquired equals the impulse $T\delta t$. Hence we have

$$T = \sigma v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Putting this in (4), and remembering that $\dot{v} = v dv/dx$, we have on rearranging

$$(x+c)v\frac{dv}{dx} + v^2 = g(x-c) \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Multiplying by the integrating factor $2(x+c)$ gives

$$2(x+c)^2 v\frac{dv}{dx} + 2(x+c)v^2 = 2g(x^2-c^2),$$

or

$$\frac{d}{dx}\left[(x+c)^2 v^2\right] = 2g(x^2-c^2) \quad . \quad . \quad . \quad . \quad (7).$$

Thus, on multiplying by $dx$ and integrating, we have

$$(x+c)^2v^2=2g\left(\frac{x^3}{3}-c^2x\right)+C \quad . \quad . \quad (8),$$

where $C$ is the constant of integration to be determined by the initial condition. Thus, if $x$ is only infinitesimally greater than $c$ at the start, we have then $v=0$, and find from (8) that

$$C=\frac{4}{3}gc^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Hence in this case of starting from rest, with $x=c$, (8) becomes

$$(x+c)^2v^2=\frac{2}{3}g(x-c)^2(x+2c) \quad . \quad . \quad (10),$$

giving $v$ for any other value of $x$. Then (10) in (5) gives the corresponding value of the tension.

**233. Fall of Growing Raindrop.**—Neglecting the resistance of the air, let it be required to determine the motion of a raindrop which, by condensation of stationary vapour upon it, has a radius increasing uniformly with the time; *i.e.* rate of increase of volume proportional to surface.

At time $t$, let the radius be $a+bt$ and its speed vertically downward be $v$. Then writing the density $\rho$ and equating the rate of increase of momentum to its weight, we have

$$\frac{d}{dt}\left[\frac{4}{3}\pi\rho(a+bt)^3v\right]=g.\frac{4}{3}\pi\rho(a+bt)^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

or
$$d[(a+bt)^3v]=g(a^3+3a^2bt+3ab^2t^2+b^3t^3)dt.$$

Whence, on integrating, we obtain

$$(a+bt)^3v=\frac{g}{4b}(4a^3bt+6a^2b^2t^2+4ab^3t^3+b^4t^4),$$

or
$$v=\frac{g}{4b}\left[a+bt-\frac{a^4}{(a+bt)^3}\right] \quad . \quad . \quad . \quad . \quad . \quad (2),$$

the constant of integration being zero for $v=0$ when the radius was $a$.

**234. Slip of Snow on a Slope.**—Consider the case of a uniform layer of snow on a slope inclined $\alpha$ to the horizontal, the adhesion being just sufficient to hold it while at rest.

Next, suppose the upper line of snow next the ridge (of roof or mountain-side) to be moved downwards, the friction between it and the slope being negligible. Then the snow will move down from the top and start other portions till the whole mass is sliding down. Let us find the circumstances of this motion. Consider a portion of the snow of length $b$ parallel to the ridge, and suppose a breadth $x$ of it down the slope to have started and to have velocity $v$.

Then, neglecting friction, the equation of motion may be written

$$\frac{d}{dt}(\sigma bxv)=\sigma bxg\sin\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where $\sigma$ is the mass of snow per unit area. If $a$ be written for $g\sin\alpha$, this becomes

$$x\frac{dv}{dt}+v^2=ax \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

But $$\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=\frac{vdv}{dx}=\frac{1}{2}\frac{d(v^2)}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

So, introducing (5) in (4) and multiplying by $2x$, we obtain

$$\frac{x^2dv^2}{dx}+2xv^2=2ax^2,$$

or $$d(x^2v^2)=2ax^2dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Whence, by integration, we obtain

$$v^2=2\frac{a}{3}x=\frac{2}{3}(g\sin\alpha)x \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

The constant of integration is zero, since $v$ and $x$ vanish together. We thus see that, in this imaginary ideal case, the acceleration is one-third that of a compact body sliding freely down the same slope.

The substance of articles 230-234 is derived from the elegant treatment of these topics by Dr. Besant (*Dynamics*, London, 1885).

**235. Note on Vibrations.**—The chief vibrations of a particle with which we are concerned have been dealt with already in Chapter IV. (see articles 29-33, 45-48) and Chapter VII. (see articles 109-111). The conditions under which some of them are possible may be inferred from the later treatment of elasticity in Chapter XXI.

In the case of forced vibrations (articles 46-48) we may approximately realise the phenomena in question by attaching a small bob of mass $m$ to a very large one of mass $M$, both the pendulums being of about the same length and period.[1] Then, if the large bob be set in oscillation, the consequent obliquity of the suspension of the smaller one below will supply the impressed force which gives to that small bob its acceleration varying as a sine function of the time, in addition to its own acceleration proportional and opposite to its own displacement. But, in this actual case, we must note that with masses of a finite ratio the small mass $m$ will, by its motion, *react* upon the larger mass $M$. It is therefore only as the ratio $M/m$ approaches infinity that we approach the ideal case of article 46, in which the impressed acceleration is unaffected by the forced vibrations of the particle.

## EXAMPLES—XLV.

1. A ring weight is let fall so as to strike and lodge on the small ascending weight of an Atwood's machine. Discuss the motion which ensues in the case where the small weight plus the ring weight are together heavier than the large weight of the machine.
2. Show that the acceleration of an ideally simple avalanche is of the order one-third that of an ordinary compact solid on the same slope.
3. Find the pressure exerted at any instant by a chain falling on a table.
4. Determine the velocity of a raindrop which picks up stationary moisture so that its radius grows proportionally to the time.

[1] Or better by suspending pendulums from a cord stretched so as to be nearly horizontal.

# CHAPTER XIII

## PLANE KINETICS OF RIGID BODIES

**236. Accelerated Rotation of a Rigid Body about a Fixed Axis.**—Let us consider the rotation of a rigid body about a fixed axis under the action of forces. Then because the body is supposed rigid all the motions are parallel to a given plane, which is taken as that of the diagram in Fig. 85, the fixed axis being perpendicular to it at O. Consider first a particle of mass $m_1$ at a point whose projection is $P_1$, distant perpendicularly $r_1$ from the axis and having a tangential acceleration $a_1$, which is of course in, or parallel to, the plane of the diagram and perpendicular to $OP_1$. Then the tangential force on this particle is given by

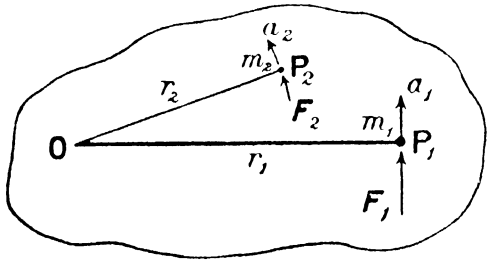



FIG. 85. ROTATION OF RIGID BODY.

$$F_1 = m_1 a_1 \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and is in the direction of the acceleration $a_1$.

Similarly, for another particle at $P_2$, we may write

$$F_2 = m_2 a_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and so forth for all the particles in the body under examination. But it is obvious that equations of this kind cannot be added arithmetically, for the corresponding pairs of $a$'s and $F$'s are in various directions, though all parallel to the plane of the diagram. A little alteration, however, enables us to effect this simple addition. We note *first* that, though the linear accelerations denoted by the $a$'s may all have different magnitudes and directions, the *angular* acceleration about O of all the particles is the same in magnitude and direction, and may be denoted by $\alpha = a_1/r_1 = a_2/r_2$, etc.

Thus $\alpha$ may be made a common factor on the right sides of the equations. *Secondly*, let us pass from the forces $F_1$, $F_2$, etc., to their moments about O, which are represented by $F_1 r_1$, $F_2 r_2$, etc. Then, since each of these moments is a product of two perpendicular vectors, each may be represented by a vector perpendicular to the plane of its components. But the components are in the plane of the diagram, so the resultant is perpendicular to that plane. Further, the direction of this resultant or moment vector is related to the direction of the force about O, as the advance of a right-handed screw is related to its rotation. Thus the products $F_1 r_1$, $F_2 r_2$, etc., may be added algebraically, for they are all representable by vectors along the same line, viz. the perpendicular to the plane of Fig. 85, and out towards the reader if the moment is counter-clockwise.

Lastly, when the equations are thus transformed, we have on the right side, as coefficients of the angular acceleration $\alpha$, terms of the

form $m_1r_1^2$, $m_2r_2^2$, etc. But it is known by the theory of vectors that the product of collinear vectors is a scalar quantity, hence all these terms may be simply added.

We accordingly find, from (1) and (2),

$$\left.\begin{aligned} F_1r_1 &= m_1r_1^2\alpha \\ F_2r_2 &= m_2r_2^2\alpha \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (3).$$

Whence, on addition

$$\Sigma(Fr) = \alpha\Sigma mr^2 \quad \ldots\ldots\ldots\ldots \quad (4).$$

Now the forces denoted by the $F$'s on each of the particles of the body have been hitherto treated as the resultant tangential forces on these particles. So each $F$ may be made up of one force $F_e$ due to external bodies, and another force $F_i$ due to the interaction of the particles of the body itself. We should accordingly have

$$F = F_e + F_i \quad \ldots\ldots\ldots\ldots \quad (4a).$$

But, since these internal forces of mutual interaction occur in pairs whose members are applied along the same line, opposite in direction but equal in magnitude, it is obvious that in the summation their effect will disappear, *i.e.*

$$\Sigma(F_ir) = 0 \quad \ldots\ldots\ldots\ldots \quad (4b).$$

Hence the summation in the left side of (4) may be taken as applying to the external forces just as though the others did not exist.[1]

**237. Moment of Inertia and Torque.**—Equation (4) contains, under the signs of summation, two new quantities of great importance which need definition and symbols to represent them. On the left side we have $\Sigma(Fr)$, which is the sum of the moments of all the forces about the axis through O. We call this briefly the total or resultant *torque* about O and denote it by $G$. On the right side we have $\Sigma(mr^2)$, which is called the *moment of inertia* of the body about the axis through O, and may be denoted by $I$. Thus (4) may be rewritten in the form

$$G = I\alpha \quad \ldots\ldots\ldots\ldots \quad (5).$$

We may further write from (5) and (4) as analytical definitions of moment of inertia

$$I = G/\alpha = \Sigma(mr^2) \quad \ldots\ldots\ldots\ldots \quad (6).$$

The first of these forms (6) may be regarded as the dynamical definition, like (5). Whereas the second expresses $I$ in terms of the masses $m$ of the particles and their perpendicular distances $r$ from the axis, and may be called the geometrical definition. It, of course, forms the basis for evaluations of the moment of inertia of any given body, while the first indicates the dynamical significance of the quantity itself.

The meaning of moment of inertia will become clearer when we introduce it in the equations of uniformly accelerated rotation. Note first how the dynamical equations (3) to (6) of article 212 were derived

[1] There are also radial forces $mr\omega^2$ on each particle of mass $m$ at radius $r$ and angular speed $\omega$; but their resultant, if any, is met by the reaction at the axle. They make no contribution to the angular acceleration, since the moment of each about the axis is zero.

from (1) and (3) of article 27, by the introduction of mass. Next, refer to equations (1) and (3) of article 92 and multiply throughout by $I$, writing $G$ for $I\alpha$. We then obtain

$$I\omega - I\omega_0 = I\alpha t = Gt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

and $$\tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2 = I\alpha\theta = G\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Equation (7) presents two new products which require names. The first, $I\omega$, is the *simple value here assumed* by the quantity called *moment of momentum* or *angular momentum*, and which is fully expressed by

$$\Sigma m(\dot{y}x - \dot{x}y) = H \text{ say.}$$

The second, $Gt$, is called *impulsive torque* or impulsive couple, the former being the more general and inclusive term; we may denote it by $J$.

The products occurring in (8), though made up of new factors, are easily seen to be kinetic energy $T$ and work $W$. We may accordingly rewrite (7) and (8) in the forms

$$H - H_0 = J \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a),$$

and $$T - T_0 = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a).$$

The kinematical and dynamical relations for uniformly accelerated translations and rotations about a *single fixed axis* may now be summarised as given in Table IX. This shows very clearly that the moment of inertia of a body about a given axis plays the same part in rotation about that axis when fixed as the mass plays in the simple translations of a particle.

TABLE IX. TRANSLATIONS AND ROTATIONS.

| TYPE OF RELATIONS. | TRANSLATIONS ONLY. | ROTATIONS ABOUT FIXED AXIS. |
|---|---|---|
| Kinematical | $v - v_0 = at$,<br>$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = as$. | $\omega - \omega_0 = \alpha t$,<br>$\tfrac{1}{2}\omega^2 - \tfrac{1}{2}\omega_0^2 = \alpha\theta$. |
| Dynamical | $mv - mv_0 = mat = Ft$,<br>or $P - P_0 = Q$.<br>$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = mas = Fs$,<br>or $T - T_0 = W$. | $I\omega - I\omega_0 = I\alpha t = Gt$,<br>or $H - H_0 = J$.<br>$\tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2 = I\alpha\theta = G\theta$,<br>or $T - T_0 = W$. |

**Radius of Gyration.**—It is often convenient to express the moment of inertia as the product of the total mass $M$ of the body and the square of a length $k$ called the *radius of gyration*. We accordingly have the relations

$$\text{or} \qquad \left.\begin{aligned} \Sigma mr^2 &= (\Sigma m)k^2, \\ I &= Mk^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

**237a. Rotation about fixed Axis: General Treatment.**—Let us now consider the rotation of a rigid body about a fixed axis, the point of view being more general. Let the axis of rotation be that of $OZ$ perpendicular to the plane of $xy$ depicted in Fig. 85A.

In the rigid body take a particle of mass $m$ at $P$, whose co-ordinates are $(x, y, z)$, and let the projection of OP in the $xy$ plane have length $r$ inclined $\theta$ to OX, as shown. Then, as the body rotates, $r$ will remain constant for the given particle, while $\theta$, $x$, and $y$ will change. We shall denote the angular velocity $\dot{\theta}$ by $\omega$ and the angular acceleration $\theta$ by $\alpha$.

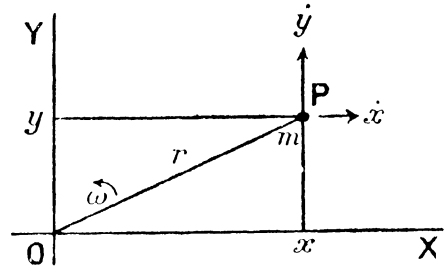


FIG. 85A. GENERAL ROTATION ABOUT FIXED AXIS.

Then, since the angular momentum $H$ about any axis is the sum of the moments about it of all the linear momenta, and the latter are expressed by the algebraic sums of their rectangular components, we have for the angular momentum about OZ

$$H=\Sigma m(\dot{y}x-\dot{x}y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But, on reference to the figure, we see that

$$\left.\begin{aligned} x&=r\cos\theta, & y&=r\sin\theta \\ \dot{x}&=-r\sin\theta.\dot{\theta}=-\omega y, & \dot{y}&=r\cos\theta.\dot{\theta}=\omega x \end{aligned}\right\} \quad (2).$$

Hence by (2), (1) becomes

$$\begin{aligned} H&=\Sigma m(\omega x^2+\omega y^2) \\ &=\omega\Sigma m(x^2+y^2), \end{aligned}$$

or

$$H=I\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Thus, on differentiating (3), we have

$$\dot{H}=I\dot{\omega}=I\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

But, on differentiating (1), we find

$$\dot{H}=\frac{d}{dt}\Sigma m(\dot{y}x-\dot{x}y)=\Sigma m(\ddot{y}x-\dot{x}y) \quad . \quad . \quad . \quad (5).$$

Hence, on writing for the products, mass into acceleration, the symbols for the corresponding force components, $X$ and $Y$ due to external bodies and $X'$ and $Y'$ due to mutual interactions, we find from (5)

$$\begin{aligned} \dot{H}&=\Sigma\{(Y+Y')x-(X+X')y\} \\ &=\Sigma(Yx-Xy)+\Sigma(Y'x-X'y), \end{aligned}$$

or

$$\dot{H}=\Sigma(Yx-Xy)=G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

since the summation for the internal forces vanishes. Thus, (4) and (6) give

$$G=I\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)^1$$

in agreement with (5) of article 237.

We shall see later that, if rotations are occurring about various axes, (5) and (6) still hold, but not (7), for $H$ is then no longer reducible to $I\omega$.

**238. Parallel Axes Theorem.**—As a preliminary to the evaluation

[1] *Alternative Proof in Polar Co-ordinates.* $H=\Sigma(mr\dot{\theta}r)=(\Sigma mr^2)\dot{\theta}=I\omega$. Also since the radial acceleration, $-r\dot{\theta}^2$, has no moment about O; we have for the impressed torque, $G=\Sigma(mr\ddot{\theta}r)=(\Sigma mr^2)\dot{\omega}=I\alpha$.

of the moments of inertia of typical figures, several theorems are needed, and will now be given.

Consider the moments of inertia $K_0$ and $K$ of a body about two parallel axes, the first being that of $z$ and the second passing through a point A on the axis of $x$. Let Fig. 86 represent the $xy$ plane, C being the origin of co-ordinates. Suppose the body of total mass $M$ to have a particle of mass $m$ at B$(xy)$, the sides of the triangle A, B, C being denoted by the corresponding small letters $a$, $b$, $c$ as usual.

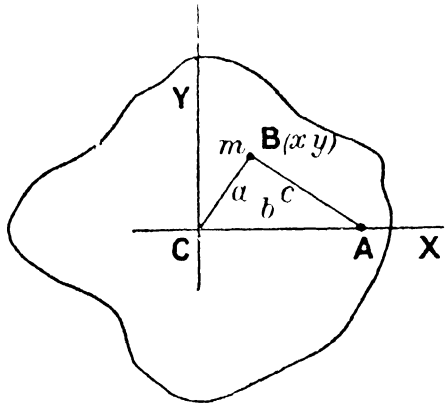



FIG. 86. PARALLEL AXES THEOREM.

Then, beginning with the geometrical definition of moment of inertia, and using the well-known trigonometrical relation, we have

$$K=\Sigma mc^2=\Sigma m(a^2+b^2-2ab\cos C)$$
$$=\Sigma ma^2+\Sigma mb^2-2b\Sigma ma\cos C,$$

or,
$$K=K_0+Mb^2-2b\Sigma mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Hence, if the axes are so chosen that

$$\Sigma mx=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

equation (1) becomes

$$K=K_0+Mb^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

If, in addition, we have also $\Sigma my=0$, the axis of $z$ passes through the point defined by

$$\Sigma mx=\Sigma my=\Sigma mz=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

which point is termed the *Centre of Mass* of the body, and possesses various important properties, as we shall see later.

In this case equation (3) embodies the theorem to be established, which may be worded as follows :—

*Theorem.*—The moment of inertia of any body about any axis is the sum of that about a parallel axis through its centre of mass plus the product of the mass of the body and the square of the perpendicular distance between the axes.

Hence, in evaluating the moments of inertia of typical bodies, it often suffices to take the axis through the centre of mass and with the required orientation. The value for any parallel axis then follows from this theorem.

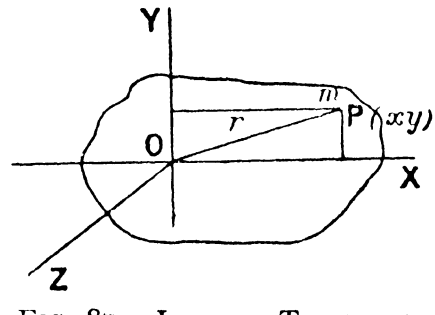



FIG. 87. LAMINA THEOREM.

**239. Lamina Theorem.**—Let the body be in the form of a thin plane lamina and take the axes of $xy$ in this plane as shown in Fig. 87.

Let the moments of inertia about the perpendicular axes $x$ and $y$ in the plane of the lamina be $I$ and $J$ respectively and $K$ denote that about the axis of $z$ through the same origin but perpendicular to the plane of the lamina. Then it is required to show that $I+J=K$.

Take a point P$(xy)$ distant $r$ from O, and let a particle of mass $m$ be situated there. Then we have

$$K=\Sigma mr^2=\Sigma m(x^2+y^2)$$
$$=\Sigma my^2+\Sigma mx^2,$$

or
$$K=I+J \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

as required.

We see at once from this that if different axes OX′ and OY′ are taken in the plane and through O, though the moments of inertia about them may change, their sum remains constant. For

$$I'+J'=K=I+J \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

**240. Rectangular Axes Theorem for any Body.**—Suppose we have now a body with any distribution of matter in solid space, the moments of inertia about the axes of $x, y,$ and $z$ being $I$, $J$, and $K$. Let any radius vector of length $r$ be drawn from the origin to the point P, where there is a particle of mass $m$. Then it is required to show that $I+J+K=2\Sigma mr^2$.

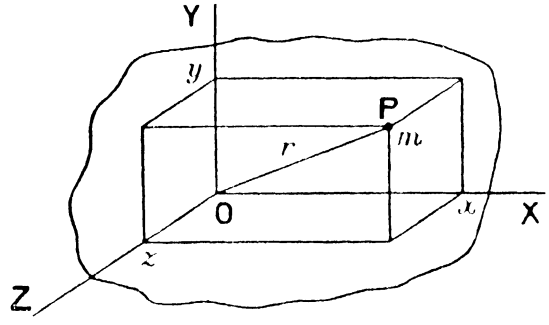


FIG. 88. RECTANGULAR AXES THEOREM.

Let the co-ordinates of P be $x$, $y$, and $z$ as shown in Fig. 88.

Then we have by definition and the geometry of the figure

$$I=\Sigma m(y^2+z^2),$$
$$J=\Sigma m(z^2+x^2),$$
$$K=\Sigma m(x^2+y^2).$$

Hence, by addition, we obtain the result

$$I+J+K=2\Sigma m(x^2+y^2+z^2)=2\Sigma mr^2 \quad . \quad . \quad (7),$$

as sought.

## EXAMPLES—XLVI.

1. For the rotation of a rigid body about a fixed axis obtain an expression analogous to $F=ma$ for the kinetics of a particle. Explain carefully what quantities now replace $F$ and $m$.
2. Define moment of inertia, torque, angular momentum, and impulsive torque; and write equations exhibiting relations between these quantities and others involving kinetic energy of rotation and the corresponding work.
3. Show that of all parallel axes in a body that about which the moment of inertia is a minimum passes through the centre of mass. Also find a general relation between all the moments of inertia about these axes.
4. Any number of particles of equal mass are arranged equidistantly on the circumference of a circle and rigidly connected to the centre by bars of negligible mass. Show that about a central axis perpendicular to the plane of the circle the moment of inertia is only half that for a parallel axis through a particle.
5. Show that for a square lamina of uniform thickness and density the moment of inertia about any central axis in the plane of the lamina is the same. Prove the corresponding relation for a regular octagonal or duodecagonal lamina.

6. Consider the rotation of a uniform filament about perpendicular axes through the centre and through an end, and thence show that the moment of inertia for the latter case is one-third mass into square of length.

**241. Theorem for Oblique Axis.**—Let us now determine the relation between the moment of inertia of a body about any axis through the origin, the moments of inertia about the co-ordinate axes, and other necessary constants being known. Let $I$ denote the moment of inertia about the axis OO′, whose direction cosines are $\lambda$, $\mu$, and $\nu$. Let a particle of mass $m$ be situated at the point P, whose co-ordinates are $x$, $y$, and $z$, and join OP, as shown in Fig. 89. Let fall the perpendicular PN upon OO′, also draw PM parallel to the axis of $z$ meeting the $xy$ plane in M, and ML parallel to the axis of $y$ meeting the axis of $x$ in L.

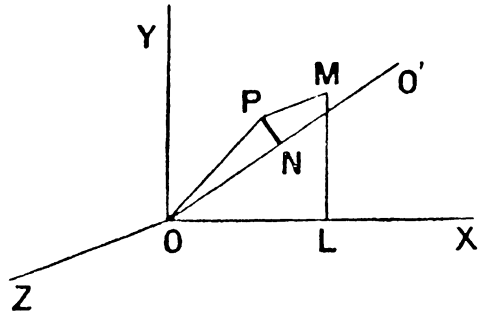



FIG. 89. MOMENT OF INERTIA ABOUT AN OBLIQUE AXIS.

We may note some relations that will be required as we proceed. Thus, we have at once by the geometry of the figure

$$OP^2=x^2+y^2+z^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Also, by regarding ON as the projection upon OO′ of either OP or its components OL, LM, and MP, we see that

$$ON=\lambda x+\mu y+\nu z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Finally,

$$\lambda^2+\mu^2+\nu^2=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Then, by definition, we have

$$\begin{aligned} I&=\Sigma mNP^2=\Sigma m(OP^2-ON^2)\\ &=\Sigma m\{x^2+y^2+z^2-(\lambda x+\mu y+\nu z)^2\}\\ &=\Sigma m\{(x^2+y^2+z^2)(\lambda^2+\mu^2+\nu^2)-(\lambda x+\mu y+\nu z)^2\}, \end{aligned}$$

or

$$\begin{aligned} I&=\Sigma m(y^2+z^2)\lambda^2+\Sigma m(z^2+x^2)\mu^2+\Sigma m(x^2+y^2)\nu^2\\ &\quad-2\Sigma myz\mu\nu-2\Sigma mzx\nu\lambda-2\Sigma mxy\lambda\mu \quad . \quad . \quad . \quad . \quad . \quad (11). \end{aligned}$$

In the three summations of the first line of (11) we recognise the moments of inertia of the body about the co-ordinate axes, which we may denote by $A$, $B$, and $C$ respectively. The second line of (11) contains three other summations, $\Sigma myz$, etc., in which the products of co-ordinates occur instead of their squares. These are called *products of inertia*, and will be denoted by $D$, $E$, and $F$ respectively. Thus (11) may be written briefly

$$I=A\lambda^2+B\mu^2+C\nu^2-2D\mu\nu-2E\nu\lambda-2F\lambda\mu \quad . \quad . \quad (12).^1$$

[1] *Momental Ellipsoid.* In any rigid body let a radius vector OQ of direction cosines $\lambda\mu\nu$ move in any way about the given point O, and in each position be of such length that the moment of inertia of the body about OQ may be proportional to the inverse square of its length, or say $I=MK^4/OQ^2$. Then, since $x=\lambda.OQ$, etc., we get from (12) as the locus of Q, $MK^4=I.OQ^2=Ax^2+By^2+Cz^2-2\,Dyz-2\,Ezx-2\,Fxy$ (12*a*). This is called the *momental ellipsoid* at O. For a lamina, the section of the ellipsoid in the plane of the lamina is called the *momental ellipse*.

The foregoing follows the treatment of Routh, who calls this the *theorem of the six constants.*

When three rectangular lines meeting in a given point of a body are such that, if taken as co-ordinate axes, we have

$$\Sigma myz = \Sigma mzx = \Sigma mxy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

then these are said to be the *principal axes* at the given point. Also, the moments of inertia about the principal axes at any point are called the *principal moments of inertia* at that point.

The constants in (12) are reduced from six to three if the co-ordinate axes are taken so as to be the principal axes at the origin. In this case (12) becomes

$$I = A\lambda^2 + B\mu^2 + C\nu^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

In some simple cases we can determine the position of these axes by considerations of symmetry. Thus, if a body is symmetrical about the plane of $yz$, then for every particle of mass $m$ at $(x, y, z)$ there is an equal mass at $(-x, y, z)$. Hence the summations $\Sigma mzx$ and $\Sigma mxy$ would vanish. If, in addition, the body were symmetrical about the $zx$ plane, we should have a particle at $(x, y, z)$ balanced by one at $(x, -y, z)$, so that $\Sigma myz$ would vanish also. Hence the conditions of (13) are fulfilled, and (14) becomes valid for the case in point.

**242. Typical Moments of Inertia Evaluated.**—The moment of inertia of a *continuous* body, often hitherto represented by $\Sigma mr^2$, is really an integral, and must usually be treated as such and evaluated accordingly. For when the distance from the axis varies continuously the small particle $m$ at any point must then be replaced by the product of the density of the material, and the infinitesimal space occupied by the portion of it under consideration. Then the space integral must be taken between such limits as will include the body under treatment

In different cases different kinds of space and density occur. Thus, for a wire or filament, we are concerned with length and linear density. For a thin sheet or shell we have surface and surface density. And for any ordinary solid extended in the three dimensions we have volume and volume density. Some cases of moment of inertia require rather complicated working, while some are so simple that they can be written down at once. We shall begin with such simple ones, and using the theorems as required proceed to the more difficult examples. Students, without knowledge of the integral calculus, may evaluate most moments of inertia by taking on faith the formula (except for $n=0$)

$$\Sigma_b^a r^{n-1} s = \frac{a^n - b^n}{n},$$

where $a$ and $b$ are the limits of $r$, and $s$ is a small increase of $r$, which is the distance from the axis.

*Circular Wire and Cylindrical Shell.*—Let the mass be $M$ and the radius $a$, the moment of inertia about the geometrical axis being $I$. Then obviously since the $r$ in $\Sigma mr^2$ is everywhere the same, we have

$$I = Ma^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

**243. Filament about Perpendicular Axis.**—Take now the case of a straight filament OA of mass $M$, and length $a$ about a perpendicular axis through one end O. Then the linear density is $M/a$, and we may take as our element PQ of length $dx$ situated at $x$ from the origin where the filament meets the axis OY about which the moment of inertia is to be taken, as shown in Fig. 90.

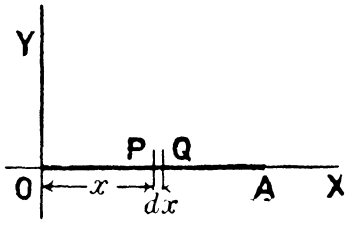



FIG. 90. MOMENT OF INERTIA OF FILAMENT.

Then the mass of our element is $Mdx/a$, and this replaces the $m$ in $\Sigma mr^2$, the $r^2$ being replaced by $x^2$. Further, the limits of integration are clearly o and $a$. Thus we have for the moment of inertia

$$I=\frac{M}{a}\int_0^a x^2dx=\frac{M}{a}\cdot\frac{a^3}{3}=\frac{1}{3}Ma^2 \quad \ldots\ldots\ldots (2).$$

It is obvious that this result applies equally to a rectangular lamina of mass $M$ about one edge as axis, like a door of width $a$ and negligible thickness turning on its hinges. For, the lamina may be regarded as made up of a large number of filaments whose masses add to give that of the lamina, the moments of inertia similarly adding to the expression in (2).

Again, equation (2) expresses the moment of inertia of a filament of length $2a$ about a perpendicular axis through its *centre*, provided the *total mass* is $M$. It should be noted that in this case the linear *density* is $M/2a$, just *half* the previous value.

Suppose now we take a filament of length $a+b$ about a perpendicular axis through the point leaving $a$ and $b$ on either side, the total mass being still $M$. We then have for the moment of inertia

$$I=\frac{M}{a+b}\int_{-a}^{+b}x^2dx=\frac{M}{a+b}\left(\frac{a^3+b^3}{3}\right)=\frac{M}{3}(a^2-ab+b^2) \quad . \quad (3).$$

Obviously the cases covered by (2) and (3), where the axis passes through the filament instead of at one end, apply equally to a rectangular lamina about an axis in its plane *parallel* to one edge instead of *coincident* with an edge.

**244. Lamina and Parallelepiped.**—Consider now a rectangular lamina of mass $M$ with edges $2a$ and $2b$ parallel to the axes of $x$ and $y$ respectively, the origin being the centre of the figure. Then, if the moments of inertia about the axes of $x$, $y$, and $z$ are $I$, $J$, and $K$ respectively, we have by (2) of last article and the lamina theorem (see equation (5), article 239)

$$I=\tfrac{1}{3}Mb^2,\ J=\tfrac{1}{3}Ma^2,\ K=\tfrac{1}{3}M(a^2+b^2) \quad \ldots\ldots (4).$$

But it is obvious that the last of these results, giving $K$, would apply equally if we passed from the single lamina to a parallelepiped of any thickness, $2c$ say, and of *total* mass $M$. Thus, by symmetry, we have for the parallelepiped

$$\left.\begin{aligned} I&=\tfrac{1}{3}M(b^2+c^2),\\ J&=\tfrac{1}{3}M(c^2+a^2),\\ \text{and}\quad K&=\tfrac{1}{3}M(a^2+b^2)\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (5).$$

**245. Circular Disc about Axis and Diameter.**—Consider a circular disc or lamina of mass $M$, radius $a$, surface density $\sigma$, and let it be required to find its moments of inertia $I$ about a diameter and $K$ about the axis perpendicular to its plane. If we take a second diameter perpendicular to the first, it is obvious from symmetry that the moment of inertia $J$ about this is equal to $I$. Hence the lamina theorem applied to the disc yields

$$2I=K \quad \ldots\ldots\ldots \quad (6).$$

It is simpler now to determine $K$ directly by integration and deduce $I$, thus reversing the procedure followed for the rectangular lamina.

For our element we take a ring of radius $r$ and radial width $dr$, as shown in Fig. 91.

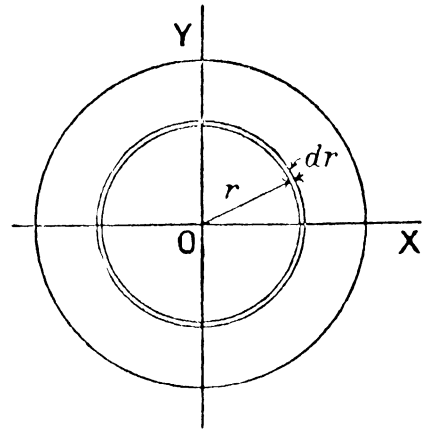



FIG. 91. MOMENT OF INERTIA OF DISC.

The area of this element is $2\pi r dr$, its mass $\sigma$ times this, and its moment of inertia $r^2$ times the previous product. Hence we have

$$dK=(2\pi r dr)\sigma r^2=2\pi\sigma r^3 dr.$$

The limits of integration are o and $a$, thus we obtain

$$K=2\pi\sigma\int_0^a r^3 dr=\frac{2\pi\sigma a^4}{4}=\tfrac{1}{2}Ma^2 \quad \ldots\ldots \quad (7).$$

Whence by (6) we see that

$$I=\tfrac{1}{4}Ma^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (8).$$

It is obvious that (7) will apply also to a cylinder of any length rotating about its geometrical axis. For a cylindrical tube of radii $a$ and $b$ outside and inside and density $\rho$, we have from (7)

$$K'=\tfrac{1}{2}\pi\sigma(a^4-b^4)=\tfrac{1}{2}\{\pi\sigma(a^2-b^2)\}(a^2+b^2)=\tfrac{1}{2}M(a^2+b^2) \quad (7a).$$

**245a. Elliptic Lamina**—Consider now a very thin lamina of mass $M$ in the form of an ellipse with semi-axes $a$ and $b$ along the axes of $x$ and $y$ respectively, and let the corresponding moments of inertia be $I$ and $J$, that about the axis of $z$ being $K$. Then, since this elliptical lamina differs from a circular one of radius $b$ only in the extension of material parallel to the axis of $x$, which makes no difference to the moment of inertia about this axis, we have from (8)

$$I=\tfrac{1}{4}Mb^2 \quad \ldots\ldots\ldots \quad (9).$$

Similarly

$$J=\tfrac{1}{4}Ma^2 \quad \ldots\ldots\ldots \quad (10).$$

Hence, by the lamina theorem,

$$K=\tfrac{1}{4}M(a^2+b^2) \quad \ldots\ldots\ldots \quad (11),$$

which is seen to agree with (7) if $b=a$.

**246. Spherical Shells and Solid Sphere.**—Consider a very thin spherical shell of radius $a$ and total mass $M$, and let $I$, $J$, and $K$ denote its moments of inertia about three rectangular axes meeting at its centre. Then by the three axes theorem, equation (7) in article 240, we have

$$I+J+K=2\Sigma mr^2=2Ma^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

since every $r=a$. But, by symmetry, $I$, $J$, and $K$ are all equal, so that each is one-third their sum. Thus we have

$$I=J=K=\frac{2}{3}Ma^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

giving the moment of inertia of the shell about any diameter.

We may now easily pass to the moment of inertia about its diameter of a solid homogeneous sphere of radius $a$ and density $\rho$ by using a shell of radius $r$ and radial thickness $dr$ as our element. The area of this shell is $4\pi r^2$, its volume $dr$ times this, and its mass $\rho$ times the product. Hence by (13) we have

$$dI=\frac{2}{3}(4\pi r^2 dr\rho)r^2=\frac{8}{3}\pi\rho r^4 dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Thus, taking the integral between the limits 0 and $a$, we have

$$I=\frac{8}{3}\pi\rho\int_0^a r^4 dr=\frac{8}{3}\pi\rho\frac{a^5}{5}=\frac{2}{5}\left(\frac{4}{3}\pi a^3\rho\right)a^2,$$

or

$$I=\frac{2}{5}Ma^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where $M$ is the mass of the solid sphere.

For a shell of finite thickness we have obviously only to integrate the same expression from the internal radius, $b$ say, to the external radius $a$. Thus we have

$$I=\frac{8}{3}\pi\rho\int_b^a r^4 dr=\frac{8}{3}\pi\rho\frac{a^5-b^5}{5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16);$$

or, introducing the mass $M$ of the shell, which is $\frac{4}{3}\pi\rho(a^3-b^3)$, this becomes

$$I=\frac{2}{5}M\frac{a^5-b^5}{a^3-b^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

**247. Right Prism about Perpendicular Axis.**—Consider the moment of inertia of any right prism of homogeneous material about any axis perpendicular to its geometrical axis. Take the axis of $z$ along the axis of rotation and the axis of $x$ parallel to the geometrical axis of the prism, the plane of $xy$ being that of the diagram Fig. 92, in which ABCD shows the prism, GG′ its geometrical axis.

Suppose a particle of mass $m$ of the body to be at the point P of co-ordinates $(x, y, z)$, then for the moment of inertia about the axis of $z$ we have

$$K=\Sigma m(x^2+y^2)$$
$$=\Sigma mx^2+\Sigma my^2,$$

or

$$K=F+S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

where $F$ and $S$ represent respectively the moments of inertia about the axis of $z$ of the bodies produced by condensing the prism *first* to the *filament* FF′ along the axis of $x$, and *second* to the *slice* SS′ in the $yz$ plane. Thus, if for the prism itself were substituted these two ideal figures, the moment of inertia would be unaltered, although the mass would be doubled.

Usually the $K$ is required for the case of O at the centre of mass; the corresponding expressions for $F$ and $S$ are then simpler than for an asymmetrical case.

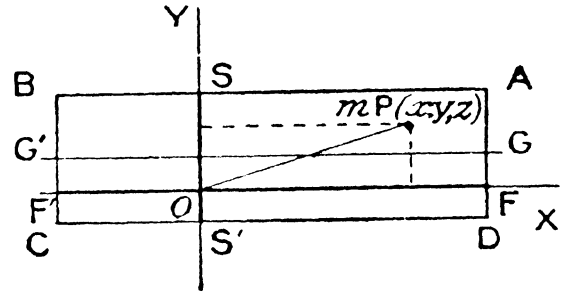



FIG. 92. MOMENT OF INERTIA OF PRISM.

Thus, for the moment of inertia of a right circular cylinder of mass $M$, length $2a$, and radius $c$ about a central axis perpendicular to its geometrical axis, we find from (2), (8), and (18)

$$K=\frac{1}{3}Ma^2+\frac{1}{4}Mc^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

Also, for a rectangular prism of length $2a$ and width $2b$ about a central axis perpendicular to these directions, we find, from the principle of this article, the result already given in equation (5), viz.

$$K=\frac{1}{3}M(a^2+b^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

## EXAMPLES—XLVII.

1. Find the moments of inertia about its three edges of a brick of size 9 inches by 4½ inches by 3 inches, the mass being 9 lbs.
2. Show by direct integration that the moment of inertia of a uniform cylinder about its axis is half mass into radius squared. Thence show that for a disc about a diameter the moment of inertia is one quarter mass into radius squared.
3. Find about a diameter and about a tangent the moments of inertia of a spherical shell and of a solid sphere, the densities being uniform in each case.
4. Obtain a general expression for the moments of inertia of any prism about a central axis perpendicular to the geometrical axis.
5. Find the moment of inertia of a right circular cone of uniform density about an axis through the vertex parallel to the base.
6. Treat the same cone as in No. 5 about a diameter of its base.

**248. Triangular Lamina about Axis parallel to Base.**—Let the triangular lamina be represented by ABC in Fig. 93, its sides being $a$, $b$, and $c$, its mass $M$, and its surface density $\sigma$.

As seen in the figure, the corner C is taken as the origin of co-

ordinates, the axis of $x$ being along the side $a$. Let us first find the moment of inertia $I$ about the axis of $x$. Take as the element the strip PQ of ordinate $y$ and width $dy$. Then, if the perpendicular EA of the

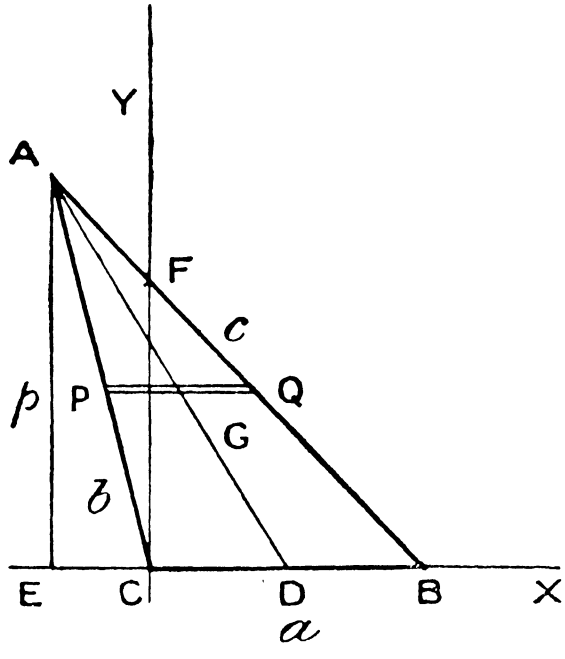



FIG. 93. MOMENTS OF INERTIA OF TRIANGLE.

triangle is denoted by $p$, the length of the strip PQ is $a(p-y)/p$. Thus we have

$$I=\frac{a\sigma}{p}\int_0^p(p-y)y^2dy=\frac{a\sigma}{p}\left[\frac{py^3}{3}-\frac{y^4}{4}\right]_0^p$$
$$=\frac{a\sigma}{12}p^3,$$

or
$$I=\frac{1}{6}Mp^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

Take now a parallel axis through the centre of mass G, which is easily shown to be one-third up the median DA. Then, calling the corresponding moment of inertia $I_0$, we have from the parallel axes theorem

$$I=I_0+M\left(\frac{p}{3}\right)^2,$$

or
$$I_0=Mp^2\left(\frac{1}{6}-\frac{1}{9}\right).$$

So
$$I_0=\frac{1}{18}Mp^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Further, take a parallel axis through A, and call this moment of inertia $I'$. Then we have again by the theorem

$$I'=I_0+M\left(\frac{2}{3}p\right)^2=Mp^2\left(\frac{1}{18}+\frac{4}{9}\right),$$

or
$$I'=\frac{1}{2}Mp^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

**249. Triangular Lamina about Coplanar Axes perpendicular to Base.**—Still referring to Fig. 93, let us denote CE by $\gamma$ and CF by $q$. Then, the moment of inertia about the axis of $y$ being called $J$, we have from (21)

$$J=\frac{1}{6}\left(\frac{q\gamma\sigma}{2}\right)\gamma^2+\frac{1}{6}\left(\frac{qa\sigma}{2}\right)a^2$$

$$=\frac{1}{6}\cdot\frac{q(\gamma+a)\sigma}{2}\cdot\frac{\gamma^3+a^3}{\gamma+a},$$

or

$$J=\frac{1}{6}M(a^2-a\gamma+\gamma^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Transferring now to a parallel axis through G, whose abscissa is

$$\frac{a}{2}-\frac{1}{3}\left(\gamma+\frac{a}{2}\right)=\frac{1}{3}(a-\gamma),$$

we find for the moment of inertia

$$J_0=M\left(\frac{a^2-a\gamma+\gamma^2}{6}-\frac{a^2-2a\gamma+\gamma^2}{9}\right),$$

whence

$$J_0=\frac{M}{18}(a^2+a\gamma+\gamma^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Passing now to the parallel axis through A, and calling the moment of inertia $J'$, we find

$$J'=J_0+M\left(\frac{2\gamma+a}{3}\right)^2$$

$$=M\left(\frac{a^2+a\gamma+\gamma^2}{18}+\frac{4\gamma^2+4\gamma a+a^2}{9}\right),$$

whence

$$J'=\frac{M}{6}(a^2+3a\gamma+3\gamma^2)$$

or

$$J'=\frac{M}{6}(\beta^2+\beta\gamma+\gamma^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

where $\beta$ is written for $a+\gamma$, *i.e.* for EB on Fig. 93.

**250. Triangular Lamina about Axes perpendicular to its Plane.**—With the usual notation, let $K$ denote the moment of inertia of the lamina about the axis of $z$ perpendicular to its plane, $K_0$ and $K'$ standing for the moments about parallel axes through the centre of mass G and through the vertex A respectively. Then, by the lamina theorem, we have

$$K_0=I_0+J_0=\frac{M}{18}(p^2+a^2+a\gamma+\gamma^2)$$

$$=\frac{M}{36}(a^2+p^2+\gamma^2+p^2+a^2+2a\gamma+\gamma^2),$$

or

$$K_0=\frac{M}{36}(a^2+b^2+c^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

To obtain $K'$ from this result and the parallel axes theorem, let us denote by $m$ the median AD. Then we may easily see that

$$4m^2=2b^2+2c^2-a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

Thus
$$K' = K_0 + M.AG^2$$
$$= M\left(\frac{a^2+b^2+c^2}{36} + \frac{4m^2}{9}\right).$$

Whence by (28)
$$K' = \frac{M}{12}(3b^2+3c^2-a^2) \quad \ldots\ldots (29).$$

We may obtain the same result by using the relation $K' = I' + J'$.

Also by either of these methods, or by symmetry of notation merely, we find
$$K = \frac{M}{12}(3a^2+3b^2-c^2) \quad \ldots\ldots (30).$$

**251. Direct Methods for Triangle about an Axis perpendicular to its Plane.**—By this method it will be convenient to take the axis through the vertex A of the triangle and take this point for the origin of co-ordinates, though the moment of inertia will still be called $K'$ to agree with the previous notation.

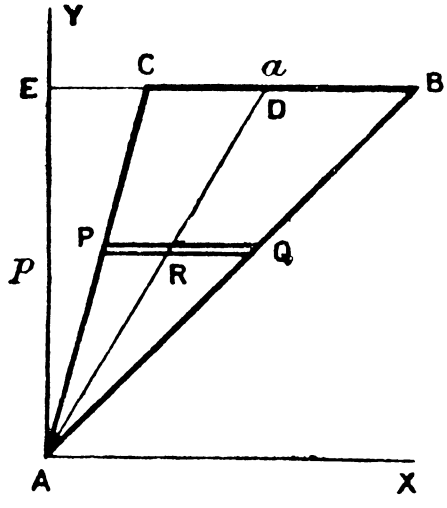


FIG. 94. DIRECT METHOD FOR TRIANGLE.

Thus, referring to Fig. 94, we take the strip PQ parallel to the axis of $x$ as our element. Its ordinate being $y$, its length is obviously $ay/p$. Its moment of inertia about an axis perpendicular to the plane of the diagram and passing through its middle point R is accordingly given by
$$\frac{1}{3}\left(\frac{aydy\sigma}{p}\right)\left(\frac{ay}{2p}\right)^2.$$

But, for its moment of inertia about the parallel axis through A, we must add the product mass of the element into the square of AR. Now AR is evidently given by $my/p$, where $m$ as before denotes the median AD. Hence
$$dK' = \left(\frac{aydy\sigma}{p}\right)\left\{\frac{1}{3}\left(\frac{ay}{2p}\right)^2 + \left(\frac{my}{p}\right)^2\right\}.$$

Then, integrating between the limits o and $p$,
$$K' = \frac{a\sigma}{12p^3}(a^2+12m^2)\int_0^p y^3 dy$$
$$= \frac{ap\sigma}{12}\frac{(a^2+12m^2)}{4}$$
$$= \frac{M}{24}(a^2+6b^2+6c^2-3a^2),$$
or
$$K' = \frac{M}{12}(3b^2+3c^2-a^2) \quad \ldots\ldots (31),$$
which agrees with (29).

We may also with advantage solve this problem by use of a double integral. Thus taking as our element an infinitesimal rectangle of edges $dx$ and $dy$ situated at $(x, y)$, its moment of inertia about the axis of $z$ is $(x^2+y^2)$ times its mass. Also the limits of $x$ are given by the equations $x=\gamma y/p$ and $x=\beta y/p$ of the lines AC and AB, where $\gamma=\text{EC}$ and $\beta=\text{EB}=\gamma+a$. We therefore find

$$\begin{aligned}
K' &= \sigma\int_0^p\int_{\gamma y/p}^{\beta y/p}(x^2+y^2)\,dx\,dy \\
&= \sigma\int_0^p\left[\frac{x^3}{3}+xy^2\right]_{\gamma y/p}^{\beta y/p}dy \\
&= \sigma\int_0^p\left(\frac{\beta^3-\gamma^3}{3}\cdot\frac{y^3}{p^3}+\frac{\beta-\gamma}{p}\cdot y^3\right)dy \\
&= \sigma\cdot\frac{\beta-\gamma}{3p^3}(\beta^2+\beta\gamma+\gamma^2+3p^2)\int_0^p y^3\,dy \\
&= \frac{\sigma a}{3p^3}(\beta^2+\beta\gamma+\gamma^2+3p^2)\frac{p^4}{4} \\
&= \frac{\sigma a p}{2}\cdot\frac{\beta^2+\beta\gamma+\gamma^2+3p^2}{6} \\
&= M\frac{2\beta^2+2\beta\gamma+2\gamma^2+6p^2}{12} \\
&= \frac{M}{12}\left\{3(p^2+\gamma^2)+3(p^2+\beta^2)-(\beta^2-2\beta\gamma+\gamma^2)\right\}.
\end{aligned}$$

Hence, as before in (31), we have

$$K'=\frac{M}{12}(3b^2+3c^2-a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

**252. Triangle about Central Perpendicular Axis by Dimensional Method.**—We may now illustrate the use of the method of dimensions or dynamical similarity in calculating moments of inertia. Referring to Fig. 95, let ABC be the triangular lamina of mass $M$, its moment of inertia about an axis through its centre of mass G and perpendicular to its plane being $K_0$. Take the middle points DEF of the sides and join them to each other and to A, B, and C. Then we have four triangles all similar, each of half the linear dimensions of the original triangle, and therefore of one quarter the area and mass $(M/4)$. Hence, if we write $k$ for the radius of gyration of the triangle ABC, we have

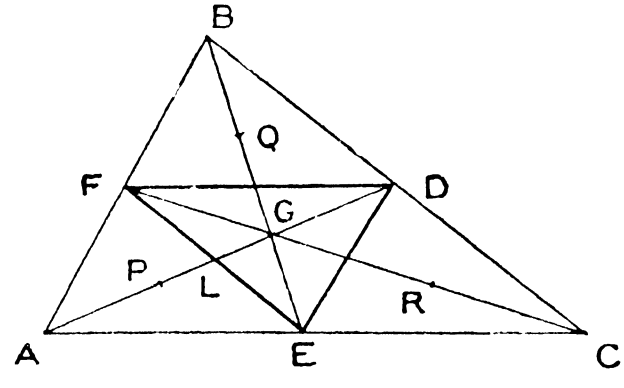



FIG. 95. MOMENTS OF INERTIA OF TRIANGLE BY DIMENSIONAL METHOD.

$$K_0=Mk^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33).$$

Whereas the moments of inertia of the small triangles about axes perpendicular to their planes and passing through their centres of mass G, P, Q, and R, will be expressed by

$$\frac{1}{4}M\left(\frac{k}{2}\right)^2 = K_0/16 \quad \ldots\ldots\ldots\ldots (34).$$

Hence, using the theorem of parallel axes, we can build up the moment of inertia $K_0$ of the whole triangle from those of its component triangles each of one-fourth the mass.

We thus have

$$K_0 = \frac{K_0}{16} + \left(\frac{K_0}{16} + \frac{M}{4}\cdot GP^2\right) + \left(\frac{K_0}{16} + \frac{M}{4}\cdot GQ^2\right) + \left(\frac{K_0}{16} + \frac{M}{4}\cdot GR^2\right),$$

or $3K_0 = M(GP^2 + GQ^2 + GR^2) \quad \ldots\ldots\ldots\ldots (35).$

We can now with advantage transform this expression by noting some of the geometrical properties of the figure.

Thus $AL = \frac{1}{2}AD = LD$,

$PL = \frac{1}{3}AL = \frac{1}{6}AD$,

$LG = \frac{1}{3}LD = \frac{1}{6}AD$,

$$GP = PL + LG = \tfrac{1}{3}AD = GD = \frac{m}{3} \text{ say.}$$

Hence $$GP^2 = GD^2 = \frac{4}{36}m^2 = \frac{2b^2 + 2c^2 - a^2}{36} \quad \ldots\ldots\ldots (36).$$

And obviously similar expressions may be written for GQ, GR, etc., by interchanging the letters.

$$\left.\begin{aligned} \text{Whence } GP^2 + GQ^2 + GR^2 &= \frac{a^2 + b^2 + c^2}{12} \\ &= GD^2 + GE^2 + GF^2 \end{aligned}\right\} \quad \ldots\ldots\ldots (37).$$

Substituting (37) in (35) we have

$$K_0 = \frac{M}{3}(GD^2 + GE^2 + GF^2) \quad \ldots\ldots (38),$$

which shows that the moment of inertia in question is equivalent to that of particles each of *one-third the mass* of the lamina and placed at the *middle points of the sides*.

Again, by (37) in (35) we find

$$K_0 = M\frac{a^2 + b^2 + c^2}{36} \quad \ldots\ldots\ldots (39)$$

in agreement with (27) of article 250.

**253. Routh's Rule.**—The following very useful rule is given by the late E. J. Routh in his *Rigid Dynamics*, and is a valuable help in recalling many of the results established.

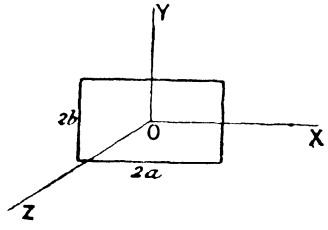
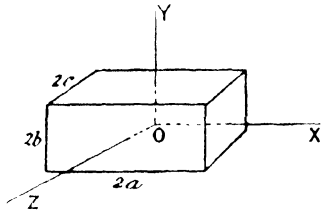
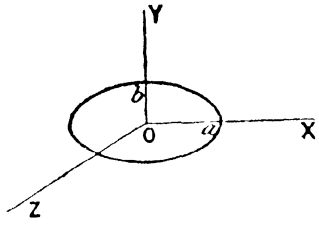
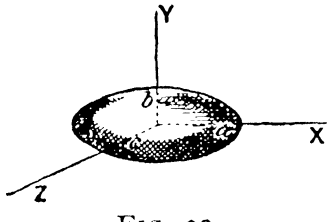
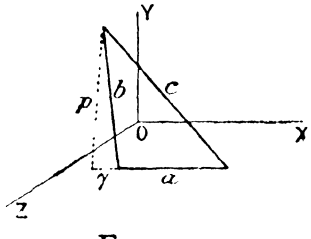
Moment of inertia about an axis of symmetry

= (mass × sum of squares of perpendicular semi-axes) ÷ (3, 4, or 5).

The divisor is to be 3, 4, or 5, according as the body is a rectangular or elliptical lamina, or an ellipsoidal solid.

The chief typical cases of moments of inertia already dealt with are summarised in Table x. and the included Figs. 96-100. All the axes, about which the moments of inertia are taken, pass through the centres of mass of the bodies dealt with.

TABLE X. TYPICAL MOMENTS OF INERTIA.

| BODIES AND AXES. | MOMENTS OF INERTIA. | | |
|---|---|---|---|
| | $I_0$ about OX. | $J_0$ about OY. | $K_0$ about OZ. |
| Rectangular Lamina in $xy$ plane.   FIG. 96. | $\frac{M}{3}b^2$ | $\frac{M}{3}a^2$ | $\frac{M}{3}(a^2+b^2)$ |
| Parallelepiped.   FIG. 97. | $\frac{M}{3}(b^2+c^2)$ | $\frac{M}{3}(c^2+a^2)$ | $\frac{M}{3}(a^2+b^2)$ |
| Elliptical Lamina in $xy$ plane.   FIG. 98. | $\frac{M}{4}b^2$ | $\frac{M}{4}a^2$ | $\frac{M}{4}(a^2+b^2)$ |
| Ellipsoid.   FIG. 99. | $\frac{M}{5}(b^2+c^2)$ | $\frac{M}{5}(c^2+a^2)$ | $\frac{M}{5}(a^2+b^2)$ |
| Triangular Lamina in $xy$ plane.   FIG. 100. | $\frac{M}{18}p^2$ | $\frac{M}{18}(a^2+a\gamma+\gamma^2)$ | $\frac{M}{36}(a^2+b^2+c^2)$ |

**254. Graphical Method for Moments of Inertia of Laminae.**—In problems as to the bending of beams of various cross sections we require what is usually called the moment of inertia of those sections. That is, we need to know the moment of inertia of a lamina of uniform surface density and in the shape of the section in question.

In some cases the graphical method which follows is very useful for this purpose, as it reduces the calculation to mechanical drawing and arithmetic, though the proof of the method requires more than this.

Let us first consider a certain possible relation between two plane figures and the method of deriving one from the other. In Fig. 101 the original figure is an oval, of which we take an element PQ parallel to the axis of $x$ and of width $dy$. Let us then mark off on PQ a smaller element P'Q', whose length is $y/b$ of PQ, where $b$ is here the extreme width of the figures from OX. It is evident that P'Q' can be found graphically by drawing the parallel lines P$p$, Q$q$ and then the converging lines $p$P'O, $q$Q'O. Drawing a line through all such points as P'O', we obtain the *first* derived figure. By dealing with P'Q' as we have just dealt with PQ, it is obvious that we may obtain points P''Q'', such that P''Q'' is $y/b$ of P'Q', and therefore is $y^2/b^2$ of PQ. Drawing a line through all such points as P''Q'', we have the *second derived* figure. Both these figures have valuable properties with respect to the axis OX, in relation to which they were drawn. Thus, let the areas of the original and first and second derived figures be respectively $A$, $A'$, and $A''$, and the corresponding masses of some lamina occupying those positions be $M$, $M'$, and $M''$, then we have

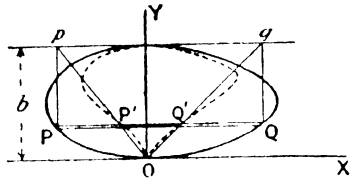



FIG. 101. RELATED FIGURES.

$$M/A = M'/A' = M''/A'' = \sigma \text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40),$$

where $\sigma$ is the surface density of the material forming any one of these laminae.

Also, if the lengths of the strips PQ, P'Q', and P''Q'' at ordinate $y$ are called $x$, $x'$, $x''$ respectively, we have their rule of derivation

$$x'/x = y/b = x''/x', \text{ or}$$
$$xy^2 = bx'y = b^2x'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41).$$

Let us now obtain equivalent expressions for $I$, the moment of inertia of the lamina about the axis OX, the radius of gyration being denoted by $k$. Then, by the definitions and use of (40) and (41), we obtain the following, in which $y'$ is written for the ordinate of the centre of mass of the lamina, occupying the first derived figure :—

$$\left.\begin{aligned} I &= \sigma \int_0^b xy^2 dy = Mk^2 = \sigma A k^2 \\ &= b\sigma \int_0^b x'y dy = bM'y' = b\sigma A'y' \\ &= b^2\sigma \int_0^b x'' dy = b^2 M'' = b^2 \sigma A'' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (42).$$

Thus $I = b^2 M'' = b^2 \sigma A''$ . . . . . (43),

and $k^2 = b^2 A''/A$ . . . . . (44),

which shows the utility of this graphical method when the figures are somewhat irregular. The areas required may be estimated by counting the squares they occupy on squared paper or determined by a planimeter.

In Fig. 101 the axis OX touches one side of the original figure and the constant $b$ is taken so as to just reach the other side, but these conditions are not necessary. The axis OX may be taken anywhere about which the moment of inertia is required, and the distance $b$ may have any convenient value without impairing the validity of the equations, provided only that the integrals are taken between the proper limits. Hence if the figure extended only between the ordinates $e$ and $f$, where $b > f$ say, $e$ and $f$ would need substituting for o and $b$ in the limits of integrations in (42), all else in (42) to (44) remaining the same.

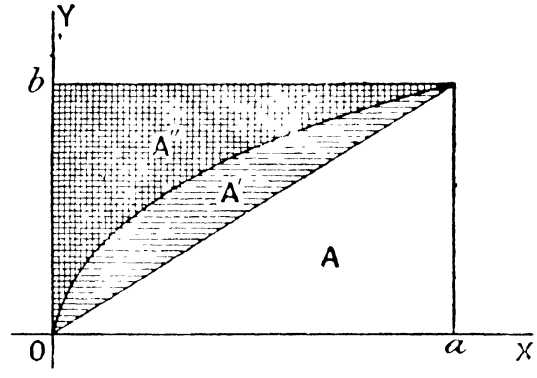



FIG. 102. FIRST AND SECOND DERIVED FIGURES.

The shapes of the first and second derived figures are shown by single and double shading in Fig. 102, the original figure being a rectangle of height $b$ and length $a$.

The line limiting the first figure is given by $x' = ay/b$, while the equation of the parabolic curve for the second is given by

$$x'' = x'y/b = ay^2/b^2, \text{ or } y^2 = b^2 x''/a.$$

It is seen that the point O is here taken at one side, which is quite legitimate.

We may find occasion to apply this method later. For fuller details, with method of moving the pole O and many illustrations, the reader is referred to more technical works, such as Professor Arthur Morley's *Strength of Materials*.

In connection with the torsional pendulum, dealt with a little further on in this chapter, we shall see how the moment of inertia of any body whatever may be experimentally determined.

## EXAMPLES—XLVIII.

1. Obtain expressions for the moments of inertia of a triangular lamina about axes (i) parallel to a side and (ii) perpendicular to a side.
2. Find, by any method, the moments of inertia of a uniform triangular lamina about axes perpendicular to its plane.
3. Give Routh's rule for the moments of inertia of symmetrical figures, and show in tabular form the typical bodies and their moments of inertia about their chief axes.
4. Explain how the moment of inertia of an irregular lamina or surface about an axis in its plane may be obtained by a graphical method.

**255. Well Roller and Bucket.**—The descent of a bucket down a well under gravity affords a simple example of the uniformly accelerated rotation of a body of revolution, namely, the roller. Let the bucket have mass $m$, the roller mass $M$, radius $r$ (to the centre of the rope), and radius of gyration $k$. Suppose at first that the mass of the rope is negligible, also the resistances due to its stiffness or the friction of the axle in its bearings. Then the equations of linear and angular motion may be written

$$mg-T=ma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$Tr=Mk^2\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where $T$ is the tension of the rope and $a$ and $\alpha$ the linear and angular accelerations occurring in bucket and roller respectively. Dividing (2) by $r$, and noting that $\alpha=a/r$, we find

$$T=Mk^2a/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Then, adding (1) and (3), we obtain

$$mg=(m+Mk^2/r^2)a,$$

or

$$a=\frac{mg}{m+Mk^2/r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Substituting this value of $a$ in (3), we find

$$T=\frac{Mk^2mg}{mr^2+Mk^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

**256. Motion modified by Friction of Axle.**—Let us now introduce the coefficient of friction $\mu$ of the axle in the bearings, the radius being $\rho$. Then the reaction of the bearings on the axle, if taken to be vertical, is $Mg+T$, the frictional resistance $\mu$ times this, and the torque due to friction $\rho$ times the product. In reality the axle on turning would *roll up* in the bearings a little before slipping, and the above expressions would need modification for strictness. This aspect may be dealt with rigorously later; the approximation is sufficient for the present case, where the friction only introduces a slight modification in the motion.

Hence, to our present approximation, we may write the equations of motion thus:—

$$mg-T=ma, \text{ or } T=m(g-a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

and

$$Tr-(Mg+T)\rho\mu=Mk^2\alpha=Mk^2a/r,$$

or

$$T=\frac{M(k^2a+gr\rho\mu)}{r(r-\rho\mu)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Then, equating the right sides of (6) and (7), we find

$$a=g\frac{mr^2-(M+m)r\rho\mu}{mr^2+Mk^2-mr\rho\mu} \quad . \quad . \quad . \quad (8).$$

And by (8) in (6) we have

$$T=mg\frac{M(k^2+r\rho\mu)}{mr^2+Mk^2-mr\rho\mu} \quad . \quad . \quad (9).$$

It is easily seen that, if the friction is negligible, we recover the previous values of $a$ and $T$ in (4) and (5) by writing $\rho\mu=0$ in (8) and (9).

**257. Atwood's Machine allowing for Inertia of Pulley.**—In articles 223-224 Atwood's machine was considered without and with friction. But, in each case, the mass of the pulley was supposed negligible. We now proceed to allow for this, friction being supposed absent. Let us write for the larger and smaller suspended masses $M_1$ and $M_2$ respectively, for that of the pulley $M$, its radius of gyration being $k$ and the radius to the centre of the thread $r$. Further, let the tensions of the thread supporting the two masses be $T_1$ and $T_2$. Then, for the equations of motion, we may write as follows :—

$$M_1 g - T_1 = M_1 a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$T_2 - M_2 g = M_2 a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

$$(T_1 - T_2) r = M k^2 a = M k^2 a / r,$$

or

$$T_1 - T_2 = M k^2 a / r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Hence, by addition of (1), (2), and (3), we eliminate the $T$'s and find

$$(M_1 - M_2) g = (M_1 + M_2 + M k^2 / r_2) a,$$

or

$$a = \frac{(M_1 - M_2) g}{M_1 + M_2 + M k^2 / r^2} \quad . \quad . \quad . \quad (4).$$

Then, by (4) in (1) and (2) successively, we obtain

$$T_1 = M_1 g \frac{2M_2 + M k^2 / r^2}{M_1 + M_2 + M k^2 / r^2} \quad . \quad . \quad (5),$$

and

$$T_2 = M_2 g \frac{2M_1 + M k^2 / r^2}{M_1 + M_2 + M k^2 / r^2} \quad . \quad . \quad (6).$$

Here again it is seen that these expressions reduce to the simpler ones of article 223, if the inertia of the pulley is ignored by writing $M k^2 / r^2 = 0$ in (4), (5), and (6).

## EXAMPLES—XLIX.

1. Determine the accelerations and tension when a mass of 10 lbs. hangs by a cord from a solid cylindrical roller of 1 foot diameter and 30 lbs. mass. (Take $g = 32{\cdot}2$ ft./sec.$^2$)
2. In the previous problem, show how to allow for the friction of the axle, assume values for the radius of the axle and its coefficient of friction, and find the new accelerations and tension.
3. Suppose the roller of the previous questions to be replaced by a roller and fly-wheel, and the apparatus to be used for finding $g$; investigate the required relations, and indicate the procedure.
4. In an Atwood's machine the pulley is a single disc of mass 100 grams and runs on ball bearings, the moving masses being 510 and 490 grams. Find the linear acceleration and the error in the determination of $g$ if the pulley's mass is ignored.

**258. Compound Pendulum.**—Having considered, sufficiently for our purpose, uniformly accelerated motion about a fixed axis, we now pass to examples of variable accelerations, namely, those cases which occur under gravity or elastic conditions.

We take first the compound or physical pendulum, which is a bar or other rigid body suspended on a fixed horizontal axis at the end, or elsewhere, and capable of oscillating freely about this axis. For dynamical considerations it is characterised by three important constants: its mass M, the distance $h$ from its axis S to its centre of mass G, and the radius of gyration $k$ about a parallel axis through G (see Fig. 103, which shows the vertical plane of oscillation).

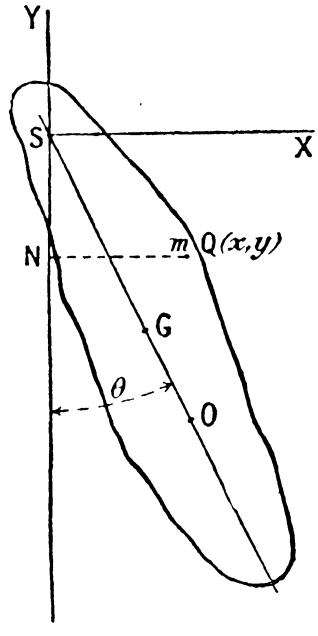



FIG. 103. COMPOUND PENDULUM.

To find the total torque about the axis S when SG is inclined at $\theta$ with the vertical, consider a particle $m$ which is then situated at Q distant horizontally QN$=x$ from the vertical SY. Then the torque due to this particle is $-mgx$.

Thus the total torque is given by

$$G=-g\Sigma mx=-g\bar{x}\Sigma m=-Mgh\sin\theta.$$

Or, if only small oscillations are considered, and for them we write $\sin\theta=\theta$, then we have the approximation

$$G=-Mgh\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But, as seen before, torque=moment of inertia into angular acceleration.

Hence, using the parallel axes theorem for the moment of inertia about S, we find

$$G=M(h^2+k^2)\ddot{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus, equating the right sides of (1) and (2), we have

$$(h^2+k^2)\ddot{\theta}+gh\theta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Further, on writing $h'=k^2/h$, or $k^2=hh'$ . . . . . (4), equation (3) may be put in the form

$$\ddot{\theta}=-\frac{g}{h+h'}\theta=-\omega^2\theta \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Hence, by article 29, we see that the motion is simple harmonic of period given by

$$\tau=2\pi/\omega=2\pi\sqrt{\frac{h+k^2/h}{g}}=2\pi\sqrt{\frac{h+h'}{g}} \quad (6).$$

By comparison of (3), (4), and (6) with article 53 on the simple pendulum we see that

$$h+k^2/h=h+h'=l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where $l$ is the length of the *equivalent* simple pendulum, *i.e.* the one of equal period with the compound one under discussion.

Hence, if we lay off $h'$ from G to O along SG produced, then SO$=l$.

In other words, O is such a point that, if all the material of the pendulum were collected there and yet connected by massless rigid bonds to the axis S, we should have an ideal simple pendulum that

would oscillate in the same time as our actual compound pendulum. This point O defined by

$$SG \times GO = k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

(SGO being straight) is called the *centre of oscillation.*

Thus, though the simple pendulum is an ideal one which can never be realised, we have found how to specify its equivalent length in terms of the constants of any rigid body which may be experimented with.

**259. The Centres of Oscillation and Suspension are Convertible.** —Let us now suppose the pendulum suspended about a parallel axis through O, its period being denoted by $\tau'$. Then by (6) and (4) we have

$$\left.\begin{aligned} \tau' &= 2\pi\sqrt{\frac{h'+k^2/h'}{g}} = 2\pi\sqrt{\frac{h'+h}{g}} \\ \text{or} \qquad \tau' &= \tau \end{aligned}\right\} \quad . \quad . \quad . \quad (9).$$

We thus see that the periods are equal about parallel axes through a given point of suspension S and through the corresponding point of oscillation O. It is, however, easily seen (as pointed out by Professor Gray in his *Dynamics and Properties of Matter*, pp. 147-149) that the parallel axes for the given period $\tau$ are not confined to the *points* S and O. On the contrary, they may move parallel to themselves to *any positions distant h and h'* respectively from G. For, from (4), (6), and (9), we see that precisely the same expressions hold for $\tau$ in whatever directions $h$ and $h'$ are measured in the plane of Fig. 103. But, of course, it is only when $h$ and $h'$ are taken in *opposite* directions from G that SO represents their *sum* and is the length of the *equivalent* simple pendulum.

Thus it is correct to state that if the period of a compound pendulum about an axis through S is $\tau$, and SO, being SG produced, is the length of the equivalent simple pendulum, then the period of the compound pendulum about a parallel axis through O is $\tau$ also.

But it is *not* correct to state that any two points on a line through G, about parallel axes through which the periods are equal, are distant apart by the length of the equivalent simple pendulum. For this to hold, the points S and O must be on the straight line through G and on *opposite* sides of G. For evidently a point O' may be found between S and G and distant $h'$ from G, about a parallel axis through which the period will be $\tau$, but SO' will be $h-h'$ instead of $h+h'$; and it is the latter and not the former which is equal to $l$, the length of the equivalent simple pendulum (see Fig. 104 in article 260).

We may also state this in an analytical form; thus from (7), putting $x$ for $h$, we have

$$x^2 - xl + k^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

$$\text{whence} \qquad x = \frac{l + \sqrt{l^2 - 4k^2}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

showing that $h$ has in general *two values* for any given $l$ or $\tau$.

Since for oscillations $x$ must be real, we note that the limiting value of $l$ is the *minimum*

$$l = 2k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

And for this value of $l$ we have

$$x = l/2 = k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

This, therefore, corresponds to the *minimum* period, which by (13) or (12) is

$$\tau = 2\pi\sqrt{2k/g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

**260. Variation of Period with Axis.**—To study the variation of the period $\tau$ with distance $h$ of the axis, it is convenient to write $y$ and $x$ for these quantities and plot the curve defined by them as co-ordinates. Thus, from equation (6) of article 258, we derive

$$gxy^2 = 4\pi^2(x^2 + k^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

in which $y$ is the period and $x$ is the distance GS from the centre of mass to the axis of suspension.

Differentiating (15) with respect to $x$, we find

$$\frac{dy}{dx} = \frac{\pi(x^2 - k^2)}{x\sqrt{gx(x^2 + k^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

This shows that as $x$ increases from o to $k$, $dy/dx$ is negative, or $y$ is decreasing; for $x = k$, $dy/dx = o$, or $y$ is stationary; as $x$ increases beyond $k$, $dy/dx$ is positive, and therefore $y$ increases indefinitely with $x$. Thus the stationary value of $y$ for $x = k$ (see GK in Fig. 104) is a minimum as shown before in (12) and (13), the corresponding value of $y$ being that given in (14).

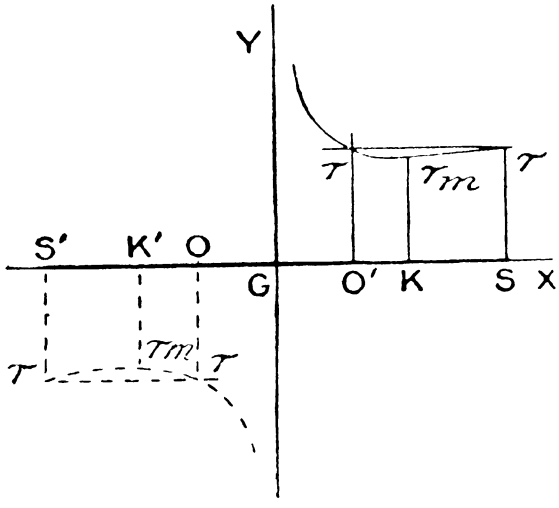



FIG. 104. VARIATION OF PERIOD OF COMPOUND PENDULUM.

Equation (15) shows that for $x = o$, $y = \infty$, as we should expect, for there is no torque to produce motion when the pendulum is disturbed. In fact, the meaning of disturbed position disappears when S coincides with G.

We see also from equation (16) that for $x = o$ the value of $dy/dx$ is $\pm\infty$. Thus the curve asymptotes to the axis OY.

If we write $x$ negative in (15), $y$ becomes imaginary, which may be interpreted that oscillations are no longer performed with the pendulum the same way up as at first. The pendulum topples over, a new $x$, again positive, may be taken in the inverted position and the periods found as before. It is convenient, however, in the diagram to show an inverted curve for this portion, as indicated by the broken line in Fig. 104. In this figure $SO = l = S'O'$, the length of the equivalent simple pendulum, while $GK = GK' = k$ is the distance of the axis from the centre of mass to give minimum period of oscillation. The figure is drawn on the assumption that the pendulum is a uniform bar of length SS' and of negligible thickness. Thus

$$GO = GO' = h' = h/3 = \tfrac{1}{3} \text{ of } GS, \text{ and } GK = GK' = k = h/\sqrt{3}.$$

And the ratio of periods with axis at K and at S is $\sqrt{2\sqrt{3}}/2=0{\cdot}93$ nearly.

**261. Rigid Pendulum by Energy.**—We may now take the motion of a compound or rigid pendulum as an example of potential and kinetic energy and the transformation from one kind to another. Thus, consider the pendulum to have zero potential energy when the centre of mass G is vertically below the axis of suspension S, and take from G as origin the axis of $x$ along SG produced, the axis of $y$ being perpendicularly to the right. (See Fig. 105.) Let there be a particle of mass $m$ at P, whose co-ordinates are $(x, y)$. Then in the standard position this particle is at a distance $h+x$ below S. But when the pendulum is displaced through an angle $\theta$ as shown in the figure, the particle is below S by the smaller distance $(h+x)\cos\theta-y\sin\theta$. Thus, the potential energy of the pendulum in the displaced position is given by

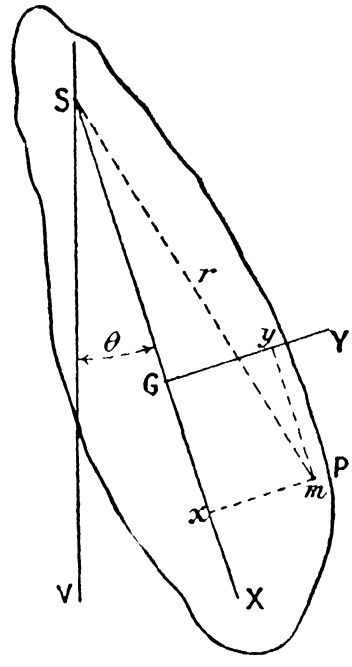



FIG. 105. PENDULUM BY ENERGY.

$$V=\Sigma mg\{(h+x)(1-\cos\theta)+y\sin\theta\}$$
$$=gh(1-\cos\theta)\Sigma m+g(1-\cos\theta)\Sigma mx+g\sin\theta\Sigma my.$$

But $\Sigma m=M$, the mass of the pendulum, and $\Sigma mx=0=\Sigma my$, since G is the centre of mass. Hence the above becomes

$$V=Mgh(1-\cos\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

Again, since the pendulum is rigid, the angular velocity of every point in it is the same, $\dot\theta$ say. Hence, writing $r$ for SP, the linear velocity of the mass $m$ at P is $r\dot\theta$ and its kinetic energy $\frac{1}{2}m(r\dot\theta)^2$. Thus, we have for the kinetic energy of the pendulum

$$T=\Sigma\tfrac{1}{2}m(r\dot\theta)^2=\tfrac{1}{2}\dot\theta^2\Sigma mr^2.$$

Or, writing as before $k$ for the radius of gyration of the pendulum about a parallel axis through G distant $h$ from S, this becomes

$$T=\tfrac{1}{2}M(h^2+k^2)\dot\theta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

But the sum of $T$ and $V$ is constant, as the pendulum is supposed to move under gravity without any other supply or withdrawal of energy. Hence, from (17) and (18) we obtain

$$(h^2+k^2)\dot\theta^2+2gh(1-\cos\theta)=\text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

Thus, on differentiating with respect to the time, and dividing out by $\dot\theta$, we have

$$(h^2+k^2)\ddot\theta+gh\sin\theta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

And, on writing $\sin\theta=\theta$ nearly for small oscillations, this reduces to the approximate equation (3) of article 258.

The inquiry as to initial conditions and oscillations in finite arcs given in articles 30 and 54-56 for rectilinear oscillations and for the

simple pendulum, etc., apply equally to the compound one of equivalent length $l=h+h'$, and so need not be repeated here.

**262. Torsional Pendulum.**—Imagine a body suspended from a fixed point by a wire or other elastic arrangement which introduces a torque proportional and opposite to any angular displacement $\theta$ of the body about a vertical axis. Then, if the body is symmetrical about its vertical axis of suspension, and is displaced and let go, it will obviously oscillate. For the body has the same relation to its vertical axis as the compound pendulum with small amplitudes had to its horizontal axis. Thus, if the body has moment of inertia $I$ about the vertical axis, and the restoring couple is $E\theta$ for a displacement $\theta$, the equation of motion is

$$I\ddot{\theta}+E\theta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Hence the solution may be written

$$\theta=\theta_0\cos(2\pi t/\tau+\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

the period being given by

$$\tau=2\pi\sqrt{I/E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Or, for a given suspension of elasticity independent of the mass hung on it, as is the case for a single wire, we may write

$$I=E\tau^2/4\pi^2=c\tau^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where $c$ is a constant.

**263. Moment of Inertia Table.**—By using as a torsional pendulum a cylindrical disc and stalk and mounting upon it other bodies of known and unknown moments of inertia, the latter can be found in terms of the former by timing the periods in each state. Thus, by reference to Fig. 106, we see the pendulum in three states: the bare table, the table and a body K of known (or easily calculable) moment of inertia, and finally with a body X of unknown moment of inertia. Let the moments of inertia of the whole suspended masses in the three cases be $I_1$, $I_2$, and $I_3$ and the respective periods $\tau_1$, $\tau_2$, and $\tau_3$ as shown.

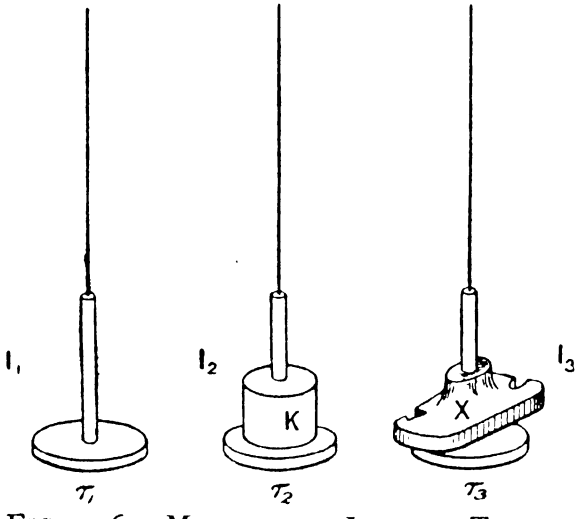


FIG. 106. MOMENT OF INERTIA TABLE.

Then, from (4), we have for the three cases

$$I_1=c\tau_1^2,\ I_2=c\tau_2^2,\ \text{and}\ I_3=c\tau_3^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Whence

$$\frac{I_3-I_1}{I_2-I_1}=\frac{\tau_3^2-\tau_1^2}{\tau_2^2-\tau_1^2}=\frac{\text{moment of inertia of X}}{\text{that of K}} \quad . \quad . \quad . \quad . \quad . \quad (6),$$

thus giving the required ratio in terms of the periods observed.

## EXAMPLES—L.

1. A circular hoop of radius $h$ of uniform wire hangs on a knife edge. Find the periods of its oscillations (i) in its own plane and (ii) perpendicular to that plane. Thence show that the addition of weights at the bottom point of the hoop is without effect on one period but increases the other.
2. Explain what you mean by the centre of oscillation of a pendulum, and find its position
   (i) for a thin uniform rod suspended at one end, and
   (ii) for a uniform disc oscillating about a tangent.
3. Establish the theorem of the convertibility of the centres of suspension and oscillation.
4. Investigate the variation of the period of a pendulum with the position of its axis, and illustrate your answer by curves for some actual pendulum.
5. Obtain by any method the period of small oscillations of a rigid pendulum.
6. Find the period of oscillation of a unifilar torsional pendulum, the torque per radian being assumed. Thence show how the moment of inertia of any irregular body can be determined.
7. 'Find the smallest time of oscillation of a uniform bar of length $2a$ about an axis fixed perpendicularly to the bar at any point in its length.
   'Find the same for a uniform plate in the form of an equilateral triangle, the axis to be fixed perpendicularly to the plane of the plate.
   'For what position of the axis is the time greatest?'
   (LOND. B.SC., PASS, APPLIED MATH., 1906, III. 5.)
8. 'If at equal intervals, $a$, there are fixed $n$ equal particles on a straight rigid wire $AB$, whose mass is negligible (there being no particle at $A$), and a pendulum is made by suspending the system from $A$, prove that it will oscillate in the same time as a simple pendulum of length $(2n+1)a/3$. Is this true if the wire is replaced by a flexible thread? (Give reasons.)' (LOND. B.SC., PASS, APPLIED MATH., 1907, III. 4.)
9. 'Show that the time of oscillation of a compound pendulum is $2\pi\sqrt{k^2/gh}$ where $h$ is the distance of the centre of gravity from the axis of suspension and $k$ is the radius of gyration of the pendulum about that axis.
   'A heavy particle is attached to a straight compound pendulum at a distance $x_1$ from the axis of suspension, and the length of the simple equivalent pendulum is then $l_1$; it is then attached to a point distant $x_2$ from the axis, and the length of the simple equivalent pendulum is now $l_2$. Show that if it be entirely removed, the length of the simple equivalent pendulum will be
   $$\frac{x_2^2 l_1 - x_1^2 l_2 - l_1 l_2 (x_2 - x_1)}{(x_2^2 - x_1^2) - x_2 l_2 + x_1 l_1},$$
   (LOND. B.SC., PASS, APPLIED MATH., 1909, III. 3.)
10. 'Explain the phrase *simple equivalent pendulum* as applied to the motion of a heavy rigid body about a fixed horizontal axis. Find the length of the simple equivalent pendulum when a uniform circular disc moves about a fixed horizontal axis in its plane, in terms of the radius of the disc and the distance of the axis from the centre of the disc.'
    (LOND. B.SC., PASS, APPLIED MATH., 1910, III. 2.)

**264. General Plane Motion of a Rigid Body.**—Let us now consider any rigid body, its general motions of translation, rotation, or both combined all parallel to a given plane, the momentum and energy possessed by it, the impulses, forces, and torques acting upon it, and the work done on it by those forces.

Let the motions and forces be all parallel to the $xy$ plane. At time $t$ let the centre of mass G of the body be at $\bar{x}$, $\bar{y}$, and let there be a particle of mass $m$ of the body at P, whose co-ordinates are $x$, $y$, $z$ with respect to the fixed axes XOY, and $a$, $b$, $z$ with respect to the axes X'GY', which remain always parallel to the fixed ones but move with G. Further, let $r$ be the length of the projection of GP upon the $xy$ plane, and let this projection make the angle $\theta$ with GX'. Then, on reference to Fig. 107, and remembering that for a given particle $r$ is invariable, but that $a$ and $b$ vary with time, we can find the following preliminary relations, in which $u$ and $v$ are written for the component velocities of G and $\omega$ for $\dot{\theta}$:—

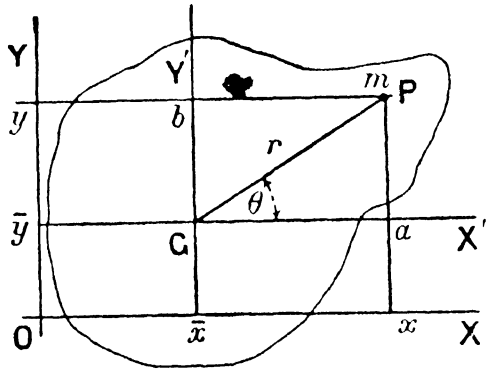



FIG. 107. PLANE MOTIONS OF RIGID BODY.

$$x=\bar{x}+a,\ y=\bar{y}+b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$\Sigma ma=0=\Sigma mb \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

$$a=r\cos\theta,\ b=r\sin\theta,\ a^2+b^2=r^2 \quad . \quad . \quad . \quad . \quad (3),$$

$$\dot{a}=-r\sin\theta.\dot{\theta}=-\omega b,\ \dot{b}=r\cos\theta.\dot{\theta}=\omega a \quad . \quad . \quad (4),$$

$$\dot{x}=u-\omega b,\ \dot{y}=v+\omega a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

$$\ddot{x}=\dot{u}-\dot{\omega}b-\omega^2 a,\ \ddot{y}=\dot{v}+\omega a-\omega^2 b \quad . \quad . \quad . \quad . \quad (6),$$

Not only in these differentiations but throughout the subsequent work it must be remembered that, since the body is supposed *rigid*, $r$ does *not* vary with *time* for a given $m$ at P, but only varies from point to point, *i.e.* with $m$ in the summation over the body. Further, while $\theta$, $\dot{\theta}$, all the $x$'s, $y$'s, and their derivatives may vary with time, $\bar{x}$, $\bar{y}$, $u$, $v$, $\omega$, and $\dot{\omega}$ do *not* vary with $m$, the particle under consideration.

*Position of Centre of Mass.*—Multiplying (1) by $m$ and summing over the body, we have

$$\Sigma mx=\bar{x}\Sigma m+\Sigma ma.$$

Remembering that the last term is zero, and using $M$ for $\Sigma m$, the mass of the body,

we have $M\bar{x}=\Sigma mx,$

and similarly $M\bar{y}=\Sigma my$ $\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$

**265. Linear Momentum and Acceleration represented by those of Total Mass at Centre of Mass.**—Multiplying (5) by $m$ and summing we find for the linear momentum of the body parallel to $x$

$$\Sigma m\dot{x}=u\Sigma m-\omega\Sigma mb=Mu.$$

Let us suppose that the particle $m$ acquired its velocity $\dot{x}$ under

the action of an impulse during the short time $t_1$, then this impulse may be represented as the product of a certain force into the time $t_1$. Further, this force may possibly consist of two parts, one $X$ being referable to the action of some external body on the particle in question, and another $X'$ being referable to the interaction between particles in the rigid body itself. Thus the sum of the impulses may be represented by

$$\Sigma(X+X')t_1=\Sigma Xt_1+\Sigma X't_1.$$

But the last term here vanishes in the summation over the whole body, since for every interaction between a pair of particles the members of the pair exert equal and opposite impulses upon each other.

Thus we see that when a rigid body is moving in any way parallel to a fixed plane, the total linear momentum in a given direction is equal to that of the total mass concentrated at its centre of mass, and that the sum of the impulses in a given direction experienced by all the particles reduces to the sum of the impulses due to external bodies. We may accordingly write

$$\Sigma Xt_1=Mu \text{ and } \Sigma Yt_1=Mv \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

the body being supposed to start from rest.

We may make these more general by supposing the velocities $u$ and $v$ to receive the small increment $du$ and $dv$ in the time $dt$. We then have

$$\Sigma Xdt=Mdu \text{ and } \Sigma Ydt=Mdv. \quad . \quad . \quad . \quad . \quad . \quad (9).$$

On dividing out by $dt$, these became

$$\Sigma X=M\dot{u} \text{ and } \Sigma Y=M\dot{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

giving the forces and accelerations concerned. It is instructive, however, to make a separate examination for the acceleration. Thus from (6) we have

$$\Sigma m\ddot{x}=\dot{u}\Sigma m-\dot{\omega}\Sigma mb-\omega^2\Sigma ma\ ;$$

or, since the summations involving $b$ and $a$ vanish,

$$\Sigma m\ddot{x}=M\dot{u}.$$

Similarly $\Sigma m\ddot{y}=M\dot{v}$ $\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$

Then, dealing with forces as with impulses, the internal reactions disappear in the summation over the body, and we have the expressions of (10) as previously found.

**266. Angular Momentum of Rigid Body.**—Let us now consider the moment of the momentum of a particle about OZ, or its *angular momentum* about OZ, the third rectangular axis through O. This is the moment about O of the linear momentum of the particle. But as we have seen in article 25*a*, the moment of a vector about a given point is equal to the algebraic sum of the moments of its components about that point. Hence, referring again to Fig. 107, we have for the moment of $\dot{y}$ about O the product $+\dot{y}x$, and for the moment of $\dot{x}$ about O the product $-\dot{x}y$, the positive direction of rotation being counter-clockwise. Thus, for the angular momentum of the whole rigid body about OZ, we may write

$$\begin{aligned}\Sigma m(\dot{y}x-\dot{x}y) &= \Sigma m\{(v+\omega a)(\bar{x}+a)-(u-\omega b)(\bar{y}+b)\}\\ &= \Sigma m(v\bar{x}-u\bar{y})+\omega\Sigma m(a^2+b^2)\\ &\quad +(v+\omega\bar{x})\Sigma ma-(u-\omega\bar{y})\Sigma mb\\ &= (v\bar{x}-u\bar{y})\Sigma m+\omega\Sigma mr^2,\end{aligned}$$

or $$\Sigma m(\dot{y}x-\dot{x}y)=M(v\bar{x}-u\bar{y})+K_0\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

where $K_0$ is the moment of inertia of the body about an axis GZ′ through the centre of mass and parallel to OZ. Hence, we see that the angular momentum of a rigid body about a given axis equals that of the whole mass at the centre of mass plus that of the body about a parallel axis through the centre of mass. (See also article 238.)

If we now turn to the impulses under which these momenta may be supposed to have been produced from rest in time $t_1$, we have, in the former notation, for their torque with respect to OZ

$$\Sigma(Yt_1x-Xt_1y)=\Sigma(Y\bar{x}-X\bar{y})t_1+\Sigma(Ya-Xb)t_1 \quad . \quad . \quad . \quad . \quad (13).$$

Thus, the moments of the impulses or the impulsive torques split up in the same way as the momentum. Further, on comparing (12) and (13) and recalling the fundamental relation, impulse equals change of momentum, we see that these equations agree term to term. We may accordingly write

$$\Sigma(Yx-Xy)t_1=\Sigma m(\dot{y}x-\dot{x}y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

$$\Sigma(Y\bar{x}-X\bar{y})t_1=M(v\bar{x}-u\bar{y}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

and $$\Sigma(Ya-Xb)t_1=K_0\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

It should be further noted that (15) splits into two equivalent terms on each side, giving

$$\Sigma Xt_1=Mu \text{ and } \Sigma Yt=Mv \quad . \quad . \quad . \quad . \quad . \quad (17).$$

Thus, equating the left sides of (12) and (13), we may say that a given angular momentum about OZ is acquired under a certain impulsive torque about OZ. Or, using (16) and (17), we may say that the velocity of the centre of mass is due to the linear impulse and the angular momentum about it due to the corresponding impulsive torque.

**267. Growth of Angular Momentum.**—We may now notice that the rate of increase of the angular momentum of a rigid body also falls into two terms, as is the case with the momentum itself. Thus, we may take the differential with respect to time of the angular momentum just found, or we may take the moment about OZ of the product (mass into acceleration) of any particle, and then sum for the whole body. Let us adopt this full method first and use the other as a check on the result. As before with momentum, we may replace the moment of an acceleration by those of its components. Hence we may write as follows:—

Rate of Increase of Angular Momentum of Body about OZ =

$$\begin{aligned}\Sigma m(\ddot{y}x-\ddot{x}y) &= \Sigma m\{(\dot{v}+\dot{\omega}a-\omega^2b)(\bar{x}+a)-(\dot{u}-\dot{\omega}b-\omega^2a)(\bar{y}+b)\}\\ &= \Sigma m(\dot{v}\bar{x}-\dot{u}\bar{y})+\dot{\omega}\Sigma m(a^2+b^2)\\ &\quad +(\dot{v}+\dot{\omega}\bar{x}+\omega^2\bar{y})\Sigma ma-(\dot{u}-\dot{\omega}y+\omega^2\bar{x})\Sigma mb,\end{aligned}$$

or $$\Sigma m(\ddot{y}x-\ddot{x}y)=M(\dot{v}\bar{x}-\dot{u}\bar{y})+K_0\dot{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

And this answers the check of differentiating the equation (12) with respect to time.

Turning now to the forces under whose action these accelerations occur, we see that their moment about OZ may be expressed and split up as follows :—

$$\Sigma(Yx-Xy)=\Sigma(Y\bar{x}-X\bar{y})+\Sigma(Ya-Xb) \quad . \quad . \quad . \quad (19).$$

And here again, not only are the left sides of (18) and (19) equal, but also the terms on the right are equal, each to each. Thus, on splitting the equations involving $\bar{x}$ and $\bar{y}$, we have

$$\Sigma X=M\dot{u},\ \Sigma Y=M\dot{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

$$\Sigma(Ya-Xb)=K_0\dot{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

We have usually hitherto treated the total mass $M$ as invariable. But in certain problems particles become adherent on a rigid body or leave it, so it is desirable also to provide for a variable $M$. Thus, differentiating (17) and (16) with respect to $t$ on this understanding, we have

$$\Sigma X dt=d(Mu)=Mdu+udM,\ \Sigma Y dt=d(Mv)=Mdv+vdM \quad (22),$$

and $$\Sigma(Ya-Xb)dt=d(K_0\omega)=K_0d\omega+\omega dK_0 \quad . \quad . \quad . \quad . \quad . \quad (23).$$

Hence, when forces are absent, we have

$$Mu=\text{constant},\ Mv=\text{constant} \quad . \quad . \quad . \quad . \quad . \quad (24).$$

And when torques are absent

$$K_0\omega=\text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Equations (24) and (25) are the symbolical expressions of the very important principles called the

*Conservations of Linear and Angular Momenta.*

**268. Kinetic Energy of Plane Motion of a Rigid Body.**—Consider now the kinetic energy $T$ of any rigid body whose centre of mass G has the velocity components $u$ and $v$, the body also rotating about GZ′, parallel to OZ, with angular velocity $\omega$ (see Fig. 107). Then, with the previous notation, and the relations of article 264, we have

$$\begin{aligned}2T&=\Sigma m(\dot{x}^2+\dot{y}^2)=\Sigma m\{(u-\omega b)^2+(v+\omega a)^2\}\\&=(u^2+v^2)\Sigma m+\omega^2\Sigma m(a^2+b^2)\\&\quad+2v\omega[\Sigma ma]-2u\omega[\Sigma mb].\end{aligned}$$

Or, since the last two terms in square brackets vanish, we may write

$$T=\tfrac{1}{2}M(u^2+v^2)+\tfrac{1}{2}K_0\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

Thus, the kinetic energy also splits into two terms, one of *translation* expressing the energy of the whole mass as if it were a particle moving with G, the centre of mass, and the other expressing the energy of *rotation* of the actual rigid body about an axis through its centre of mass and parallel to the original axis.

Turn now to the work $W$ which is expended in generating the above kinetic energy in the rigid body. Each element of the work may be expressed as the product of the force into the displacement, in that direction, of the particle. But, if these displacements occur in the small time $t_1$ we may express these small changes of $x$, $y$, and $\theta$ by $x_1=\dot{x}t_1$, $y_1=\dot{y}t_1$, and $\theta_1=\omega t_1$. Thus, using the relations of article 264, we find

$$W=\Sigma(Xx_1+Yy_1)=\Sigma(X\dot{x}+Y\dot{y})t_1$$
$$=\Sigma\{X(u+\dot{a})+Y(v+\dot{b})\}t_1$$
$$=ut_1\Sigma X+vt_1\Sigma Y+\omega t_1\Sigma(Ya-Xb).$$

Thus $$W=\bar{x}_1\Sigma X+\bar{y}_1\Sigma Y+\theta_1\Sigma(Ya-Xb) \quad . \quad . \quad . \quad . \quad (27).$$

This accordingly shows that the work divides into two parts, viz. (i) that of the resultant force into the linear displacement of the centre of mass, and (ii) that of the resultant torque about a parallel axis through the centre of mass into the angular displacement of the body.

On comparison of (26) and (27) we may see that they are not only equivalent in the aggregate but also term by term, each to each.

**269. Independence of Translation of Centre of Mass and Rotation about it.**—Thus the foregoing articles 264-268 have established the complete independence of the translation of the centre of mass of a rigid body moving in a given plane, and its rotation about an axis through the centre of mass and perpendicular to that plane. In other words, (i) The Linear Acceleration of the Centre of Mass of a Rigid Body is determined by the external forces as though the whole mass were concentrated at that point and the forces were applied there parallel to their actual directions.

Or $$M\dot{u}=\Sigma X \text{ and } M\dot{v}=\Sigma Y \quad . \quad . \quad . \quad . \quad . \quad (28).$$

(ii) The Angular Acceleration of a Rigid Body about any axis through the Centre of Mass is determined by the moments of the external forces about that axis, the whole body retaining its actual mass and shape, and the forces being restricted to their actual lines of action.

Or $$K_0\dot{\omega}=\Sigma(Ya-Xb) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

Hence the translation of the Centre of Mass is independent of any rotation about it, and the rotation about it is independent of any translation it may experience.

The statements just made have direct reference to the subject of article 267, but the relations established in articles 265-268 are all interdependent. They accordingly represent different aspects of that property of the centre of mass whose most memorable feature is here emphasised.

It is perhaps desirable to warn the student that though it is often best to use the property of independence, still this is by no means invariably the case, the method of estimating moments of inertia and forces about some other axis not through the centre of mass being sometimes shorter.

A certain class of problem in which the principle is easily applicable is that of evaluating the reactions at the axle of a rotating rigid body, taken say as the origin of co-ordinates. For, taking moments about the origin, we find the angular acceleration and therefore the linear transversal acceleration of the centre of mass. Then, by the principle of energy, and knowing from the independence of translation and rotation that the loss of potential energy is simply that of the whole mass as if at its centre of mass, we may find the angular velocity. And this gives the radial linear acceleration of the centre of mass.

Finally, knowing all accelerations of the centre of mass, and again using the independence theorem, we find the total force components, and thence obtain the reactions sought. (See Examples—LI., Nos. 11, 12, and 13.)

**270. Centre of Percussion.**—Let a rigid body have a fixed axis and let it receive a blow or impulse such that there is no impulsive pressure on the axis, then the point where the direction of the blow meets the plane through the axis and the centre of mass is called the *centre of percussion* of the body with respect to that axis.

Thus, referring to Fig. 108, let the fixed axis be that of $z$ through S and perpendicular to the plane of the diagram. Let the blow be along QP, of magnitude $Xt_1$, and produce no impulsive pressure on the axis. Then the centre of percussion is the point P where QP meets the $yz$ plane through S and the centre of mass G.

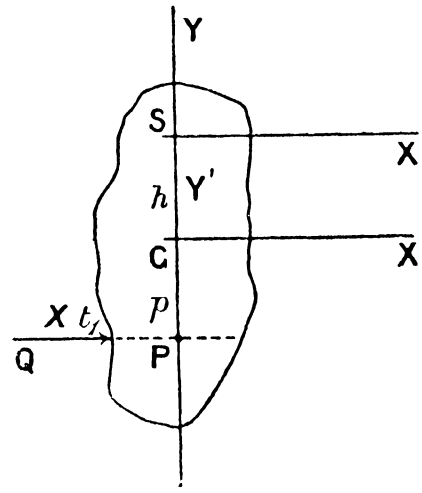



FIG. 108. CENTRE OF PERCUSSION.

To determine the position of P, let SG$=h$, GP$=p$, let the mass of the body be M and $k$ its radius of gyration about GZ′ parallel to SZ.

Then, writing $\omega$ for the angular velocity immediately after the blow, and applying (17) and (16) of article 266, we have

$$Xt_1 = Mh\omega \text{ and } Xt_1 p = Mk^2\omega,$$

since there are to be no impulses at the axis.

Whence $$p = k^2/h \text{ or } k^2 = hp \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But this result gives for $p$ the same value as $h' =$ GO for the centre of oscillation (equation (4) and Fig. 103 of article 258). Hence P and O coincide for the given S. It should be noted that some writers call any point along QP (Fig. 108) a centre of percussion. Also, as we saw in article 259, the period of the pendulum is constant for the axis through O or any parallel axis distant $h'$ from G. Thus it is only the one point P on QP where it meets SG produced, and the one point O on the circle of radius $h'$ round G, that coincide.

If a body at rest and quite free receives a blow, the axis about which it begins to turn is called the *axis of spontaneous rotation*. We may easily see that it corresponds with the fixed axis in the case just considered. Thus again referring to Fig. 108, we imagine the blow $Xt_1$ struck as before and introduce the condition that S does not move. After the blow let the linear velocity of G be $u$ along GX′ and the angular velocity be $\omega$. Then

$$Xt_1 = Mu \text{ and } Xt_1 p = Mk^2\omega,$$

and the velocity of S along SX is

$$u - \omega h = \frac{Xt_1}{M}\left(1 - \frac{hp}{k^2}\right).$$

But this is to be zero, hence

$$k^2 = hp \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

as before in (1).

**271. Fall of Trap Door from Vertical Position.**—It is instructive to consider also cases where the impulsive pressures at the axis do not vanish. A simple example of this is presented by the fall of a trap door from the vertical to the horizontal position, considering the door to be a uniform lamina turning freely about a horizontal hinge at one edge and stopped suddenly by an impulse along the edge parallel to the line of the hinges. Thus, referring to Fig. 109, let the motion be in the plane of $x$, $y$, the hinges being at S, the centre of mass falling from rest at G′ in the quadrant to G, the edge B′ striking the frame at B. Let the trap door acquire in its fall an angular velocity $\omega$ and be arrested by two impulses $D$ and $E$ acting vertically at B and S. Then, calling the mass of the door M and its half width $h$=SG, the square of its radius of gyration about a parallel axis through G is $k^2 = h^2/3$. We thus see that its potential energy when upright is $Mgh$ (see equation (7), article 264) reckoned from S; and that its kinetic energy when down, just before striking the blow, is $\frac{1}{2}M(h^2+k^2)\omega$. Hence, equating the loss of potential to the gain of kinetic energy, we have

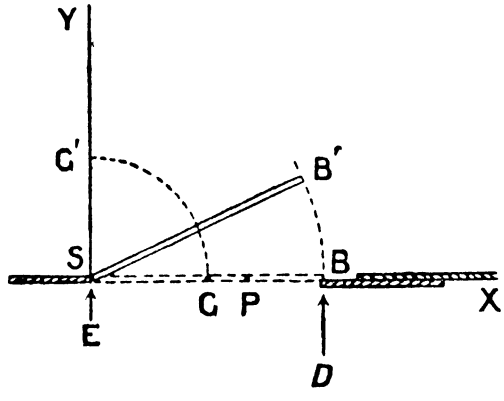



FIG. 109. FALL OF TRAP DOOR.

$$2gh = (h^2+k^2)\omega^2 = 4h^2\omega^2/3,$$

or

$$\omega = \sqrt{3g/2h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Considering the stopping of the door by the blows $D$ and $E$, we have from (17) and (16) of article 266

$$D+E = M\omega h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and

$$Dh - Eh = Mk^2\omega,$$

or

$$D-E = M\frac{h}{3}\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Thus, by addition and subtraction of (4) and (5) and using (3), we find

$$D = \frac{2}{3}M\sqrt{3gh/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

and

$$E = \frac{1}{3}M\sqrt{3gh/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

These two blows are evidently equivalent to a single one $D+E$ applied at P, where

$$GP = p = k^2/h = h/3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

as might be expected.

**272. Fall of Trap Door from Horizontal Position.**—Referring again to Fig. 109, let us now suppose the support at B to be suddenly removed when the door is at rest in the horizontal plane. We may then inquire what is the force exerted by the hinges, and what is the angular acceleration, each *immediately after* removal of the support.

Let this initial force at the hinges S have components $X$ and $Y$, and let the initial angular acceleration be $\gamma$.

Then, by (28) and (29) of article 269, we have

$$X=0,\quad Y-Mg=M(h\gamma) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

and

$$Y(-h)=(Mh^2/3)\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

since $K_0=Mh^2/3$. From these we find

$$Y=Mg/4 \text{ and } \gamma=-3g/4h \quad . \quad . \quad . \quad . \quad . \quad (11).$$

The initial linear acceleration of the centre of mass is accordingly

$$-3g/4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

We might also treat this problem by taking moments about S, yielding

$$-Mgh=M(h^2+h^2/3)\gamma,$$

which with (10) determines $Y$ and $\gamma$ as before.

Before the removal of the support at B, it is obvious from symmetry that the support at each edge was $Mg/2$. It is thus noteworthy that the removal of one support has the immediate effect of changing the force exerted by the other from a half to a quarter of the weight of the door.

**272a. Sudden Fixation of a Point in a Rotating Body.**—The foregoing example of the trap door illustrate the type of problem in which some point or line is suddenly fixed or suddenly freed, the motion being under gravity. If the motions occur in a horizontal plane we may have a point given which is to be suddenly fixed, the subsequent velocity and the impulse to be found. Or, we may have a relation between velocities before and after given, and from that be required to determine what point is fixed.

It is clear that the impulse acts through the point which becomes fixed, so has no moment about any axis through that point. This principle is very useful in attacking such cases.

Thus let a uniform thin rod of mass $M$ and length $2h$ be rotating about its centre in a horizontal plane with angular velocity $\omega_0$, when by the sudden fixing of a point P in it the angular velocity is reduced to half. Find the position of P and the impulse $Q$ exerted there when P is fixed.

At the instant in question let the rod lie along the axis of $y$ with centre at the origin and P at $(0, y)$ and be rotating counter-clockwise. Then, equating angular momenta about P before and after the fixation, we have $(Mh^2/3)\omega_0=M(y^2+h^2/3)\omega_0/2$, whence $y^2=h^2/3$ or $y=(0{\cdot}577\ldots)h$. Accordingly the centre of the rod starts off with velocity $\omega y=\omega_0 h/2\sqrt{3}$, parallel to the axis of $x$.

Thus the impulse is given by $Q=M\omega_0 h/2\sqrt{3}$. As a check, we may note that the moment about O of this impulse reduces $\omega_0$ to $\omega_0/2$.

**273. Ballistic Pendulum.**—This device invented for the determination of the velocity of projectiles, though now superseded by other contrivances, still presents a mechanical interest. It is shown diagrammatically in Fig. 110. Imagine a rigid pendulum of mass $M$ free to turn about a horizontal axis at S, and when at rest to receive, at B situated $b$ below S, a bullet of mass $m$ and horizontal velocity $u$. The bullet may be received on a metal plate and be shattered, or in a chamber of sand or other soft material in which it is embedded. In either case the pendulum receives an impulse during a negligibly short time, so may be considered as starting from its *vertical* position with an angular velocity $\omega$ and finally attaining a displacement $\theta$. Let $k'$ be the radius of gyration of the pendulum about S, and let $h$ be the distance from S to G, the centre of mass. It is required to determine $u$, the speed of the bullet, its mass $m$ being negligible in comparison with $M$, that of the pendulum which is made large to avoid undue swings.

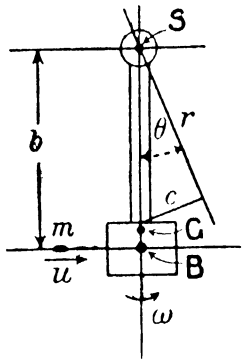



FIG. 110. BALLISTIC PENDULUM.

Then, since we may regard the pendulum at rest and the projectile approaching it as forming a system subjected to no external forces having a moment about the axis at S, the angular momentum or moment of momentum about this axis cannot be changed by the impact. (See equation (25) of article 267.)

But the moment of momentum before impact is that of the bullet only, and that after impact is, to our approximation, that of the pendulum only. Hence we have

$$mub = Mk'^2\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

To find the relation between $\omega$ and $\theta$, we note that the pendulum starts with a certain kinetic energy, and as it swings this is gradually changed into potential energy. Hence we may equate the gain of potential energy to the loss of kinetic, thus obtaining

$$Mgh(1-\cos\theta) = \tfrac{1}{2}Mk'^2\omega^2,$$

or $$2gh(2\sin^2\theta/2) = k'^2\omega^2,$$

or $$2\sqrt{gh}\sin\theta/2 = k'\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Then (1)÷(2) gives

$$u = \left(\frac{2Mk'\sqrt{gh}}{b}\right)\frac{\sin\theta/2}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

the quantity in brackets being a constant for the pendulum, the second factor expressing the variables for the projectile under examination.

Sometimes the angle $\theta$ is estimated by allowing the pendulum to pull out a cord or tape. Thus, if a length $c$ is pulled out at radius $r$ on swinging through the angle $\theta$, we have

$$\sin\theta/2 = c/2r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Then (3) becomes

$$u = \left(\frac{Mk'\sqrt{gh}}{br}\right)\frac{c}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

showing that $u$ varies as $c$. Hence if the tape were graduated uniformly the values of $u$ could be found on multiplying by a constant and dividing by the mass of the bullet. Or, for a given $m$, the tape could be graduated to read values of $u$ directly for the pendulum in question.

By allowing the pendulum to swing we could observe the period, given by

$$\tau = 2\pi k'/\sqrt{gh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Thus, introducing this value to eliminate $k'$, (5) becomes

$$u = \left(\frac{\tau Mgh}{2\pi br}\right)\frac{c}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

in which $Mgh$ is the torque exerted about S by the pendulum in a horizontal position, and so is easily ascertainable.

For fuller details as to construction and use of ballistic pendulums the student may consult Routh's *Rigid Dynamics*, i. pp. 98-101, 1897; and Sir G. Greenhill's *Notes on Dynamics*, pp. 190-191, 1908.

## EXAMPLES—LI.

1. Establish the independence of rotation of a rigid body and the translation of its centre of mass as regards linear momenta and kinetic energy.
2. Show that the angular momentum of a rigid body, about an axis perpendicular to the plane of its motions of rotation and translation, may be divided into two parts, one being that of the body as though condensed to a particle at its former centre of mass G, and the other being that of rotation of the actual body about a parallel axis through G as though it were stationary.
3. 'A uniform bar $AB$, 6 feet long, mass 20 lbs., hangs vertically from a smooth horizontal axis fixed at $A$; it is struck normally at a point 5 feet below $A$ by a blow which would give a mass of 2 lbs. a velocity of 30 feet per second; find the impulse received by the axis, and the angle through which the bar will rise.'
(LOND. B.SC., PASS, APPLIED MATH., 1906, III. 6.)
4. 'A system of particles is moving in one plane. Show how to compound their momenta, and reduce the result to its simplest form.
'Three equal particles are attached to the corners of an equilateral triangular area $ABC$, whose mass is negligible, and the system is rotating about $A$. $A$ is released, and the middle point of $AB$ is suddenly fixed. Prove that the angular velocity is unaltered.'
(LOND. B.SC., PASS, APPLIED MATH., 1905, III. 6.)
5. 'If a rigid body is revolving with angular velocity $\omega$ about a fixed axis, find—
($a$) its kinetic energy;
($b$) its moment of momentum about the axis.
'A uniform rigid bar, $AB$, is rotating in a smooth horizontal plane about its centre with angular velocity $\omega$; if suddenly a point $P$ in the bar is fixed, find the position of $P$ for which the new angular velocity will be $\frac{1}{2}\omega$.' (LOND. B.SC., PASS, APPLIED MATH., 1907, III. 2.)
6. 'Obtain a formula for the kinetic energy of a rigid body in terms of the motion of its centre of mass and its motion relative to the centre of mass.
'At a point $P$ of a uniform circular hoop there is attached a particle of mass equal to that of the hoop. The hoop rolls, in a vertical plane, on a perfectly rough horizontal table. Prove that if the system starts from

rest when $P$ is at the highest point, the angular velocity $\omega$ when the radius to $P$ makes an angle $\theta$ with the downward vertical will be given by

$$\omega^2=\frac{g}{a}\cdot\frac{1+\cos\theta}{2-\cos\theta},$$

(LOND. B.SC., PASS, APPLIED MATH., 1908, III. 4.)

7. 'A disc of mass $M$ and radius $a$ is moving in its plane, the velocity of its centre being $u$ and the spin about its centre $\omega$. One of the points of the disc distant $a/2$ from the centre in a direction at right angles to the direction of motion of the centre is suddenly fixed. Show that the loss of energy is

$$\frac{1}{12}M[2u\pm a\omega]^2.'$$

(LOND. B.SC., PASS, APPLIED MATH., 1908, III. 6.)

8. 'Obtain an expression for the kinetic energy of a lamina moving in its own plane about a fixed point.

'A uniform rod of weight $W$, free to turn about a fixed smooth pivot at one end, is held horizontally and released. Prove that when, in the subsequent motion, the rod makes an angle $\theta$ with the vertical, the pressure on the pivot is

$$\tfrac{1}{4}W\sqrt{(1+99\cos^2\theta)}.'$$

(LOND. B.SC., PASS, APPLIED MATH., 1909, III. 5.)

9. 'In connection with the uniplanar motion of a rigid body, explain what is meant by the principle of the independence of the motions of translation and rotation.

'A lamina moves in its own plane, being subject to a single force constant in magnitude and direction which always acts at the same point of the lamina. Find the motion.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, III. 5.)

10. Explain the construction and action of a ballistic pendulum, and obtain a formula involving the quantities concerned.

11. Obtain the following expressions for the reactions on the axle of a rigid body swinging from rest at the inclination of its centre line $a$ above the horizontal, to $\theta$ above the horizontal :—

$$X=\frac{Wh}{l}(-2\sin a\cos\theta+3\sin\theta\cos\theta),$$

$$Y=\frac{Wh}{l}\left(\frac{l}{h}-2\sin a\sin\theta+3\sin^2\theta-1\right),$$

where $W$ is the weight of the body, $h$ the distance of centre of mass from axis, and $l$ the length of the equivalent simple pendulum.

12. If, in question 11, the body starts from rest in the vertically upright position, show that, for any shaped body whatever, the maximum horizontal force exerted towards the side to which the body swings occurs at $\theta=+63^\circ\cdot25$, and that the maximum horizontal force exerted in the opposite direction is for $\theta=-34^\circ$.

13. For the case of question 12, plot curves for $X$ and $Y$, the angles $\theta$ being the other co-ordinate.

**274. Pure Rolling down an Incline by Energy.**—We now pass to the consideration of motions which are special examples of combined translation and rotation. Let us take first the rolling, *without slipping*, of a solid of revolution down an incline. Then, the solid and incline being specified, the accelerations are required. It is instructive to treat this problem several ways, each possessing its own advantages. We

commence by applying the conservation of energy, equating the potential energy lost by the body in a descent to the kinetic energy gained. Let the body on the incline start from rest in the position shown in Fig. 111, where it makes contact at C.

Let the body have mass $M$, and moment of inertia about its geometrical axis $K_0 = bMr^2/c$, where $b$ and $c$ are mere numbers depending on the form of the body and $r$ is the radius of the rolling part. Then its moment of inertia about a parallel axis through C is $K = (b+c)Mr^2/c$. Let the plane make an angle $\alpha$ with the horizontal, and let $x$ represent distances upward on the slope, $u$ linear speed, and $a$ acceleration. Also let $\omega$ and $\gamma$ be the angular velocity and acceleration in the plane of $xy$.

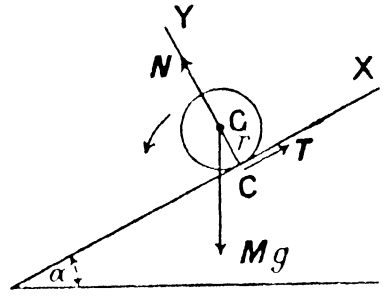



FIG. 111. BODY ROLLING DOWN INCLINE.

Then, if the body rolls till the point of contact C has moved a distance $x$ down the plane, it will have lost potential energy of the amount $Mgx \sin \alpha$. But, if its angular velocity in the meantime changes from zero to $\omega$, its kinetic energy will be increased by $\frac{1}{2}K\omega^2$. For the axis through C perpendicular to the plane of the diagram is the instantaneous axis of rotation. Thus equating we find

$$Mgx \sin \alpha = \tfrac{1}{2}\left(\frac{b+c}{c}Mr^2\right)\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But, if the motion is pure rolling, we have in addition the geometrical relation

$$\omega = -u/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus (2) in (1) gives

$$2gx \sin \alpha = (b+c)u^2/c,$$

or

$$u^2 = 2\left(\frac{-cg \sin \alpha}{b+c}\right)(-x) \quad . \quad . \quad . \quad . \quad (3).$$

And this shows that the motion involves uniform accelerations, given by

$$-a = \frac{cg \sin \alpha}{b+c} = \gamma r \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

If we had considered the kinetic energy made up of the two terms, $\frac{1}{2}Mu^2 + \frac{1}{2}K_0\omega^2$, and then applied (2), we should have obtained (3), and have illustrated the use of (26) in article 268.

**275. Rolling down Incline by Direct Moments.**—We now attack the same problem of pure rolling by taking the moments of the external forces about the instantaneous axis of rotation, *i.e.* the axis of $z$ through $C$ and perpendicular to the diagram (Fig. 111). Thus, equating resultant torque to the product of moment of inertia and angular acceleration, we find

$$Mg(r \sin \alpha) = \frac{b+c}{c}Mr^2\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

We have also the geometrical condition for no slipping

$$\gamma = -a/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Hence (6) in (5) yields

$$g \sin \alpha = -\frac{b+c}{c} a.$$

Thus the accelerations are given by

$$-a = \frac{cg \sin \alpha}{b+c} = \gamma r \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

In cases where the distance between the instantaneous centre and the centre of mass is *changing* another term is needed in the above analysis. It is then simpler to use the method of article 276, which is invariably safe.

**276. Rolling by Properties of Centre of Mass.**—We again consider the problem of the pure rolling of a body down an incline, applying this time the properties of the centre of mass summarised in article 269. Still referring to Fig. 111, we take successively the forces parallel to $y$ and $x$ and the moments about the perpendicular axis through G, equating each to the appropriate product, of inertia and acceleration factors. Thus

$$\Sigma Y = N - Mg \cos \alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

$$\Sigma X = T - Mg \sin \alpha = Ma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

$$\Sigma(Yx' - Xy') = Tr = \left(\frac{b}{c} Mr^2\right)\gamma \quad . \quad . \quad . \quad (10),$$

where $x'$ and $y'$ are here used for the co-ordinates of the points of application of the forces.

We have also the condition expressing no slipping

$$\gamma = -a/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Then (11) in (10) gives

$$-T = \frac{b}{c} Ma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

and this added to (9) yields the accelerations

$$-a = \frac{cg \sin \alpha}{b+c} = \gamma r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Again (13) in (12) determines the frictional force

$$T = \frac{b}{b+c} Mg \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Thus, though this method may be longer, it affords a closer insight into the phenomena, and gives the values of $N$ and $T$, the normal and tangential reactions at the incline on the rolling body.

**277. Condition for Pure Rolling on Incline.**—When the rolling body is also just on the point of *sliding* on the incline the full frictional resistance is called into play, so that we have $T = \mu N$ where $\mu$ is the coefficient of friction.

Hence, by (8), we have

$$T = \mu Mg \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

Thus, equating the right sides of (14) and (15), we have the condition

for the limit between pure rolling and rolling combined with sliding, namely,

$$\mu \cos \alpha = \frac{b}{b+c} \sin \alpha,$$

or
$$\frac{\mu}{\tan \alpha} = \frac{b}{b+c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Hence, for pure rolling,

$$\frac{\mu}{\tan \alpha} \nless \frac{b}{b+c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

The values of the linear acceleration and this limiting relation are given for a few typical figures in Table XI., also the constants in the expression $bMr^2/c$ for their moments of inertia.

TABLE XI. PURE ROLLING DOWN INCLINES.

| FIGURE ROLLING. | CONSTANTS IN MOMENT OF INERTIA. | | LIMIT FOR PURE ROLLING. | ACCELERATION RATIO. |
|---|---|---|---|---|
| | $b$. | $c$. | $\frac{\mu}{\tan \alpha} = \frac{b}{b+c} =$ | $a/(g \sin \alpha) = \frac{c}{b+c} =$ |
| Cylindrical Shell . . . | 1 | 1 | 1/2 | 1/2 |
| Solid Cylinder . . . . | 1 | 2 | 1/3 | 2/3 |
| Spherical Shell . . . | 2 | 3 | 2/5 | 3/5 |
| Solid Sphere . . . . | 2 | 5 | 2/7 | 5/7 |

**278. Combined Rolling and Sliding down Incline.**—When the condition expressed in the inequality (17) is violated sliding occurs. Hence the geometrical relation expressed in equation (11) of article 276 no longer holds. But since the full frictional resistance is called into play we may replace it by

$$T = \mu N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

where $\mu$ is the coefficient of friction. We have thus for the solution of the present problem this equation and (8), (9), and (10) of article 276.

By substituting in (10) the value of $T$ from (18) and (8), we find

$$\gamma = \frac{c\mu g \cos \alpha}{br} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

Then the value of $T$ put in (9) gives

$$a = -g(\sin \alpha - \mu \cos \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

If at time $t$ from rest the linear and angular velocities are $u$ and $\omega$, we have

$$u = at = -gt(\sin \alpha - \mu \cos \alpha) \quad . \quad . \quad . \quad . \quad . \quad (21),$$

and
$$\omega = \gamma t = \frac{c\mu g t \cos \alpha}{br} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Also the speed of sliding of the body over the plane at the part of contact is

$$u + \omega r = -gt(\sin \alpha - \frac{b+c}{b} \mu \cos \alpha) \quad . \quad . \quad . \quad . \quad . \quad (23).$$

It may be noted that a body in pure rolling motion has a smaller linear acceleration than if sliding without friction. Here, where there is sliding combined with rolling, the linear acceleration is precisely that of a body sliding simply down the rough incline. We may naturally inquire whence comes the energy of rotation. The answer is easily found in the saving of work against friction which that rotation effects. For the kinetic energy of rotation after time $t$ is

$$\tfrac{1}{2}K_0\omega^2=\tfrac{1}{2}\left(\frac{b}{c}Mr^2\right)\frac{c^2\mu^2g^2t^2\cos^2\alpha}{b^2r^2}=\tfrac{1}{2}Mg^2t^2\frac{c\mu^2\cos^2\alpha}{b} \quad . \quad . \quad (24).$$

And the work against friction saved by the rotation is resistance into space, or

$$\mu N\left(\frac{\gamma r}{2}t^2\right)=\mu Mg\cos\alpha\frac{c\mu g\cos\alpha}{2b}t^2=\tfrac{1}{2}Mg^2t^2\frac{c\mu^2\cos^2\alpha}{b} \quad . \quad . \quad . \quad (25).$$

Or, to test the matter a second way, the potential energy $Mg(-\frac{1}{2}at^2)\sin\alpha$ lost in descending for a time $t$ may be equated to the sum of the kinetic energy gained, $\frac{1}{2}M(at)^2+\frac{1}{2}K_0(\gamma t)^2$, and the work done against friction, which is $\frac{1}{2}\mu N(-u-\omega r)t$. The result is an identity, the value of each side being $\frac{1}{2}Mg^2t^2(\sin^2\alpha-\mu\sin\alpha\cos\alpha)$.

**279. Frictional Couple for Axle in Bearings.**—We may now fitly consider the value of the torque or couple required to overcome the friction of a cylindrical axle in somewhat loose cylindrical bearings. A little reflection will show that the contact between axle and bearing will not be at the lowest points of each when the axle is turning, for the axle will at first roll up on the inside of the bearings without slipping, and only when a certain steepness of slope is reached by the point of contact will slipping begin. Let this occur when the angle with the horizontal is $\theta$, and to maintain the axle in uniform rotation let the torque or couple $G$ be required, consisting of two equal unlike parallel forces of value $F$ and each applied at a perpendicular distance $p$ from the axis. Then it is required to determine $\theta$ and $G$ in terms of the mass $M$ of the axle and its loads, $r$ its radius, and $\mu(=\tan\beta)$ the coefficient of friction between the axle and its bearings.

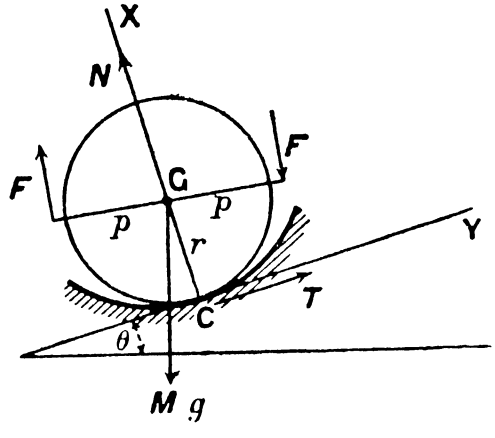



FIG. 112. FRICTIONAL COUPLE FOR AXLE.

Let Fig. 112 represent the position and distribution of forces when slipping has just commenced and the rotation is uniform. Then, resolving parallel to the axes of $y$ and $x$, and taking torques about the axis through the centre $G$ perpendicular to the plane of the diagram, we have

$$N=Mg\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$T=\mu N=Mg\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$
$$G=2Fp=Tr=\mu Mgr\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Thus (2)÷(1) gives

$$\tan\theta=\mu=\tan\beta, \text{ or } \theta=\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

as might have been anticipated.

Putting this value in (3) we find for the torque

$$G=Mgr\sin\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

which is slightly less than if the contact had been at the lowest points of axle and bearing.

**280. Sphere with Initial Rotation on Rough Level Plane.**—Considering now a solid sphere, we will imagine it to have an initial rotation about a horizontal axis but no translation, and to be then gently placed on a rough level plane and immediately let go. It is required to determine the subsequent motion.

Let the motion be in the $xy$ plane as represented in Fig. 113, the sphere having mass $M$, radius $r$, and initial angular velocity $\omega_0$, the coefficient of friction between it and the plane being $\mu$ and the reactions between them being $N$ and $T$. Then, since the only forces and torques available are finite, finite changes of linear and angular velocities can occur only in finite times. But at the start there is at the point of contact a slip of the sphere over the plane at the velocity $-\omega_0 r$. And this can only be removed in a finite time $t$ say, during which the full friction must be called into play. We accordingly have the following equations of motion during this stage of the phenomena. (See article 269.)

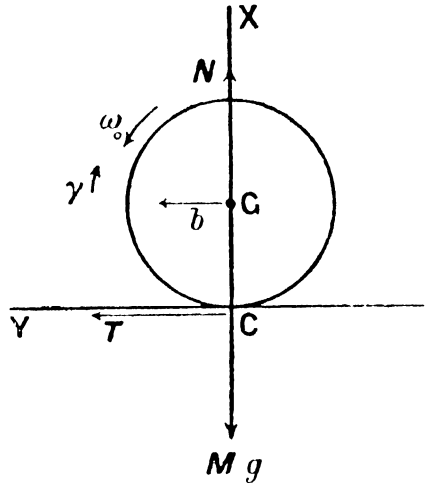



FIG. 113. SPHERE WITH INITIAL ROTATION.

$$T=\mu N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$N-Mg=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

$$T=Mb \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$T(-r)=\frac{2}{5}Mr^2\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where $b$ is the horizontal linear acceleration of the centre of mass G and $\gamma$ is the angular acceleration of the sphere about an axis through G and perpendicular to the plane of the diagram. These equations give

$$b=\mu g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and

$$\gamma=-5\mu g/2r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

But if at time $t$ the linear and angular velocities are $v$ and $\omega$, and slip ceases then, we have

$$v-\omega r=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Hence, substituting from (5) and (6), and remembering that the initial values of $v$ and $\omega$ were 0 and $\omega_0$, we obtain

$$\mu g t - (\omega_0 - 5\mu g t/2r)r = 0.$$

Whence
$$t = \frac{2\omega_0 r}{7\mu g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

And by (5), (6), and (8) we find
$$v = \frac{2}{7}\omega_0 r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

$$\omega = \frac{2}{7}\omega_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

The space passed over in time $t$ is given by
$$s = \frac{2\omega_0^2 r^2}{49\mu g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

It is noteworthy that $v$ and $\omega$ are independent of $\mu$, but that $t$ and $s$ diminish with increasing $\mu$.

We have now to inquire what occurs after the instant when the slip ceases. While the slip speed was negative the frictional force $T$ was positive, the linear acceleration $b$ was positive, and the angular acceleration $\gamma$ was negative. Suppose first that when the slip ceases $T$ still retains *any positive* value. Then obviously the accelerations retain their former signs and the speed of slip of the form given in (7) becomes *positive*. But considerations of friction show that this involves a *negative* $T$ contrary to the hypothesis, which is therefore untenable. Imagine secondly that when the slip ceases the frictional force $T$ suddenly assumes some *negative* value. That would reverse the accelerations, making the linear acceleration negative and the angular acceleration positive. In other words, it would put us back to the state of things obtaining a little before the slip ceased. But we have already seen that the state of things in question involved a negative slip and a *positive* $T$, again contrary to hypothesis. Hence $T$ must be zero, both accelerations henceforth zero (in the absence of rolling friction and air resistance), and the velocities expressed by (9) and (10) are thus retained.

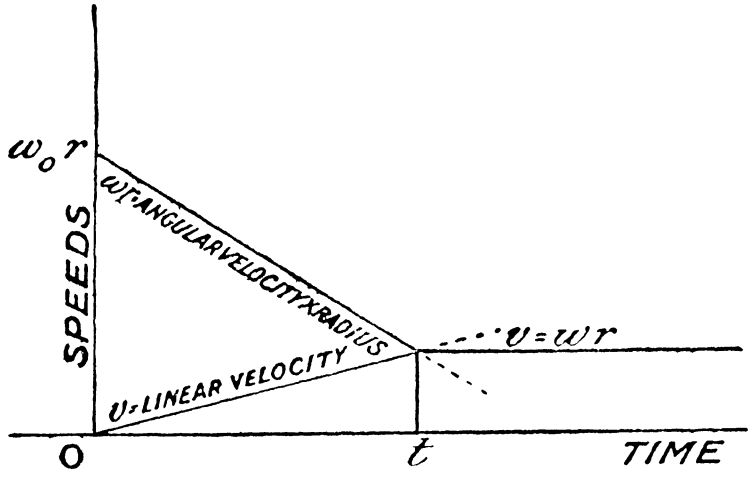



FIG. 114. SPEEDS OF SPHERE WITH INITIAL ROTATION.

The phenomena of this problem can be illustrated graphically by a speed-time diagram as in Fig. 114. The fact that the force $T$ must vanish after the time $t$ may also be seen from this diagram. For the difference of the ordinates of the two speed graphs expresses the slip speed. And beyond the point where this slip vanishes at time $t$, if the accelerations remain as before the graphs cross, the slip speed changes

sign ; hence the frictional force would have to be reversed and the accelerations reversed instead of remaining as before.

**281. Cylinder with Initial Rotation.**—It is easily seen that if the body were a solid cylinder instead of a sphere the moment of inertia is $Mr^2/2$, and we should have

$$b=\mu g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a),$$

$$\gamma=-2\mu g/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a),$$

$$t=\omega_0 r/3\mu g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a),$$

$$v=\omega_0 r/3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a),$$

$$\omega=\omega_0/3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a),$$

and

$$s=\omega_0^2 r^2/18\mu g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11a).$$

**282. Alternative Treatment of Initial Rotation and Proof by Energy.**—If we wish simply to find the ratio of the angular rotation $\omega$ to that $\omega_0$ when the body is initially placed on the rough level plane, we may note the following instructive method :—Having seen that the body will finally roll without slipping, we observe that the three forces available, $Mg$, $N$, and $T$, all act through C, the point or line of contact, and therefore have no moment about it. Hence the angular momentum about this axis remains constant. To express the initial value of this angular momentum we use the theorem established in article 266, which shows that it is the sum of the moment of momentum about the given axis of the whole body as if at the centre of mass, and the angular momentum of the actual body about a parallel axis through the centre of mass. So in this problem the sum reduces to the second term only. The angular momentum when rolling has commenced is obviously the angular velocity multiplied by the moment of inertia about the tangent of contact with the plane. Hence, taking any body whatever of moment of inertia $bMr^2/c$ about a parallel axis through the centre of mass, $b$ and $c$ being pure numbers, we have

$$\frac{b}{c}Mr^2\omega_0=\left(\frac{b}{c}+1\right)Mr^2\omega,$$

or

$$\omega/\omega_0=b/(b+c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

which confirms (10) and (10*a*), for the right side of (12) becomes $2/7$ for a sphere and $1/3$ for a solid cylinder.

We may also check this result by consideration of energy. Thus the kinetic energy at the start should equal the sum of that left when pure rolling has commenced and the work expended on friction. But the kinetic energy for our generalised body is, at the start, $bMr^2\omega_0^2/2c$, and when rolling it is $(b+c)Mr^2\omega^2/2c$. Also for the work done in slipping we have the product, frictional force into distance slipped. But the distance slipped is *half* initial speed of slip ($r\omega_0$) into the time of slip given in (8) and (8*a*).

We accordingly find the following expressions which furnish the desired check :—

$$\text{Final kinetic energy}+\text{work done}$$

$$=\tfrac{1}{2}\left(\frac{b+c}{c}Mr^2\right)\left(\frac{b}{b+c}\omega_0\right)^2+\mu Mg(\tfrac{1}{2}r\omega_0)\left(\frac{b}{b+c}\cdot\frac{\omega_0 r}{\mu g}\right)$$

$$= \frac{b^2}{2c(b+c)} Mr^2\omega_0^2 + \frac{b}{2(b+c)} Mr^2\omega_0^2$$

$$= \tfrac{1}{2}\left(\frac{b}{c} Mr^2\right)\omega_0^2 = \text{Initial kinetic energy} \quad . \quad . \quad . \quad . \quad . \quad (13).$$

**283. Motion on a Steep Plane with Initial Rotation.**—Let us now suppose a body with initial rotation about a horizontal axis to be gently placed on a plane whose steepness and roughness are such that the body if without initial motion would descend by combined rolling and slipping as in article 278. Or, in symbols, we have the condition (17) of article 277 violated. The problem presents two cases.

*Case I. Initial Slip is down the Incline.*—On referring to article 278 it will be seen that this initial rotation leaves the forces just as they were. Hence the accelerations remain as they were, and the motion presents simply this difference, that all through, the angular velocity now exceeds that in article 278 by its initial value, $\omega_0$ say. Thus the body fails to ascend the plane, although the slip is down and frictional force up, the linear acceleration being from the outset downwards.

*Case II. Initial Slip is up the Incline.*—Here the frictional force is initially downwards, and the linear acceleration during the first stage is down and numerically greater than in article 278. But obviously the moment of this force about the centre of the body is such as to retard the angular velocity.

Hence a point must be reached when the speed of the slip vanishes; it then changes sign, the frictional force reverses, and the accelerations become as in article 278. The whole motion might be represented by a graph as was done for a different case in Fig. 114. This is left as an exercise to the student.

**284. Motion up and down a Rough Plane slightly inclined.**—We now suppose the condition (17) of article 277 for pure rolling in free descent to be fulfilled and place our body with initial angular velocity $\omega_0$ so as to endeavour to roll up the plane at first. Just as on the level plane it will here, after an initial slip, reach the pure rolling stage; it may then ascend with diminishing speed, pause, and return, the accelerations for pure rolling being obviously those of articles 274-276. It is accordingly the initial stage that requires consideration. This differs from the pure rolling, *not* in a reversal of the frictional force but in its being increased if necessary to its *full limiting* value instead of having the perhaps smaller

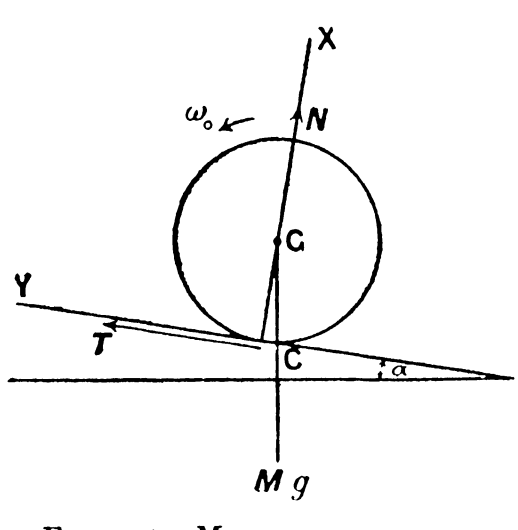



FIG. 115. MOTION UP OR DOWN ROUGH PLANE.

value of the same sign, which satisfies the geometrical condition of pure rolling.

But, during the initial stage of slip, the body might fail to ascend the plane but simply descend with a smaller acceleration; or, even remain without motion of its centre of mass, till the slip had ceased. This will appear more clearly from the analysis.

Let us take axes and forces as shown in Fig. 115. Then calling the linear and angular accelerations $b_1$ and $\gamma_1$, and distinguishing by subscripts 1 the values of accelerations or forces which apply to this first stage, we have the following equations for a general body :—

$$\left.\begin{aligned} N &= Mg\cos\alpha \\ T_1 &= \mu N \\ T_1 - Mg\sin\alpha &= Mb_1 \\ T_1(-r) &= \frac{b}{c}Mr^2\gamma_1 \end{aligned}\right\} \quad \ldots\ldots\ldots (14).$$

Whence
$$b_1 = g(\mu\cos\alpha - \sin\alpha),$$
or
$$b_1 = -\left(1 - \frac{\mu}{\tan\alpha}\right)g\sin\alpha \quad \ldots\ldots (15).$$

Thus, by the condition for pure rolling, (17) of article 277, $b_1$ may be either positive, or negative of value just reaching that found for $a$ in articles 274-276. We also have from (14)

$$\gamma_1 = -\frac{c\,\mu g\cos\alpha}{br} \quad \ldots\ldots\ldots (16).$$

Hence, whether $b_1$ is positive or negative, $\gamma_1$ is negative, showing the initial upward rotation will be destroyed.

If at time $t_1$ the linear and angular velocities are $v$ and $\omega$, the speed of slip is then $v - \omega r$. So on substitution of their values from (15) and (16) we have

$$\text{Speed of slip} = gt(\mu\cos\alpha - \sin\alpha) - \left(\omega_0 - \frac{c\,\mu g t\cos\alpha}{br}\right)r$$

$$= \left(\frac{b+c}{b}\mu\cos\alpha - \sin\alpha\right)gt - \omega_0 r \quad \ldots (17).$$

Hence the slip vanishes at the time

$$t = \frac{\omega_0 r}{g\left(\frac{b+c}{b}\mu\cos\alpha - \sin\alpha\right)}$$

$$= \frac{b\omega_0 r}{g(b+c)\left[\frac{\mu}{\tan\alpha} - \frac{b}{b+c}\right]\sin\alpha} \quad \ldots (18).$$

And, by the condition for pure rolling, the term in square brackets cannot be negative. Thus, the slip must vanish, and after this the accelerations must be as in articles 274-276.

**285. Rolling Oscillations.**—Let us now consider the pure rolling

of a cylinder in a cylinder, a sphere in a sphere, or other body of revolution inside a circular surface so that the motion is parallel to a given vertical plane. Then it is obvious that oscillations are possible about the lowest position as centre.

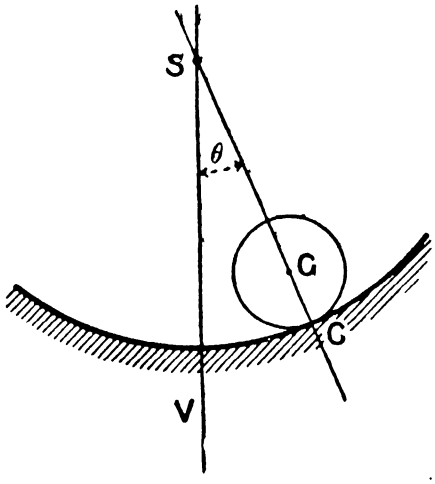



FIG. 116. ROLLING OSCILLATIONS.

Referring to Fig. 116, let the body have angular displacement $\theta$, its moment of inertia about the central axis perpendicular to the plane of the diagram being $bMr^2/c$, and the radius SC of the surface on which it rolls being $R$.

Then since $\theta$ is the inclination to the horizontal of the slope on which the body is now situated, it follows from articles 274-276 that the linear acceleration of its centre of mass G is approximately given by $-cg\theta/(b+c)$ if we restrict ourselves to small arcs for which $\sin\theta=\theta$ nearly. Thus, since the linear acceleration may be expressed by $(R-r)\ddot{\theta}$, we have for the equation of motion

$$(R-r)\ddot{\theta}=-\frac{c}{b+c}g\theta \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Whence, the motion is simply harmonic of period, given by

$$\tau=2\pi\sqrt{(b+c)(R-r)/cg} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus, for a cylinder in a cylinder this becomes

$$\tau_1=2\pi\sqrt{3(R-r)/2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

And for a sphere in a sphere we have

$$\tau_2=2\pi\sqrt{7(R-r)/5g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Assuming $g$ and observing $\tau$, $(R-r)$ may be determined by these relations. Of course, the condition for pure rolling must be fulfilled, but for small oscillations this is practically always the case.

## EXAMPLES—LII.

1. Obtain an expression for the linear acceleration of a body of revolution rolling down an incline, and show what condition must be fulfilled in order that sliding shall be absent.
2. Discuss the conditions and motion occurring when both sliding and rolling of a body on an incline are possible.
3. Investigate the behaviour of a solid wheel and axles rolling down a pair of parallel inclined bars on which the axles rest. What advantage in the determination of $g$ would this form of experiment possess over the rolling of a sphere on an inclined plane?
4. A sphere rolls down a V groove of uniform width and shape in an inclined plane. Determine the linear and angular accelerations.
5. 'Obtain the expression for the kinetic energy of a rigid body whose motion is parallel to one plane, in terms of its angular velocity and the velocity of its centre of mass.

'A thin spherical shell of negligible mass, quite smooth internally, is filled

with water, and allowed to roll down a length $l$ of a rough plane inclined to the horizon at the angle $i$; find the time taken.

'If the water freezes and becomes rigidly attached to the shell, what will the time be?'

(LOND. B.SC., PASS, APPLIED MATH., 1906, III. 4.)

6. 'Find the moment of inertia of a uniform solid sphere about a diameter.

'A sphere of radius $r$, rotating with angular velocity $\omega$ about a horizontal diameter, is gently placed on a horizontal table with which its coefficient of friction is $\mu$. Show that there will be slipping at the point of contact for a time $2\omega r/7\mu g$, and that then the sphere will roll with angular velocity $2\omega/7$.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, III. 10.)

7. 'Explain what is meant by the principle of energy, and show how it can be used to obtain the motion of any system having only one degree of freedom.

'Two rough solid cylindrical rollers of radius $a$ and mass $M$ are placed with their axes parallel and horizontal upon a fixed plane inclined at an angle $i$ to the horizontal. A log of mass $2M$ is laid across the rollers. If there be no slipping between either the log or plane and the rollers, find the acceleration with which the log moves.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, III. 1.)

# CHAPTER XIV

## SOLID RIGID KINETICS

**286. Motions of a Rigid Body with One Point fixed.**—In dealing with the kinetics of a rigid body we shall follow the order used in Chapter VIII. on the kinematics of the subject, taking first the motions when one point is fixed and afterwards those when no point is fixed. And, for the kinetics of the case when one point is fixed, there are two methods of attack open to us which are characterised by the use of fixed axes and by moving axes respectively.

Thus, *first*, we may find the angular momenta (of the form $\Sigma mrv$) about each of the three fixed axes at right angles with origin at the fixed point of the body, and then equate the rates of change of these momenta to the corresponding torques about these fixed axes.

Or, *second*, we may refer the motions to moving rectangular axes with origin at the fixed point, use expressions like those for angular accelerations (see equation (1), article 123), but, for the angular velocities on the right sides thereof, substitute *angular momenta*; the left sides then become the corresponding torques about these moving axes. But in this second plan we have to note that the angular momentum about any axis is *not necessarily* the product of moment of inertia and angular velocity about that axis. It is given fundamentally and always by an expression of the form $\Sigma mrv$, where $m$ is a particle of the body at a perpendicular distance $r$ from the axis about which the momentum is taken and $v$ its component of velocity perpendicular both to $r$ and to the axis. And this expression only reduces in specially simple cases to the product of moment of inertia and angular velocity, as we shall see presently.

As to choice between these methods, the first is often unnecessarily long, though for a simple case it is very instructive, and by those wanting only one or two simple examples may be preferable. The second method is more generally useful, and is often very expeditious when the necessary preliminaries have been discussed and the required expressions deduced.

We shall accordingly give here a single concrete example of the first method and then pass on to the details of the second.

**287. Maintenance of Rectangular Precession of Top.**—As an example, then, of the fixed axes method of treatment let us consider the case of a body of revolution rotating with uniform angular speed $\omega$ about its geometrical axis OA, which is horizontal, while this axis is

itself turning with uniform angular velocity $\Omega$ about OZ, which is vertical. We here make no use of the theory of moving axes nor do we assume that angular momentum is a vector, though incidentally that fact is illustrated.

Take, as shown in Fig. 117, OXYZ as fixed axes, OABC as moving axes, OC being coincident with OZ and vertical, the other four axes being horizontal; and consider the top when as illustrated the axis OA makes the angle $\phi$ with OX. Then we may write

$$\phi=\phi_0+\Omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

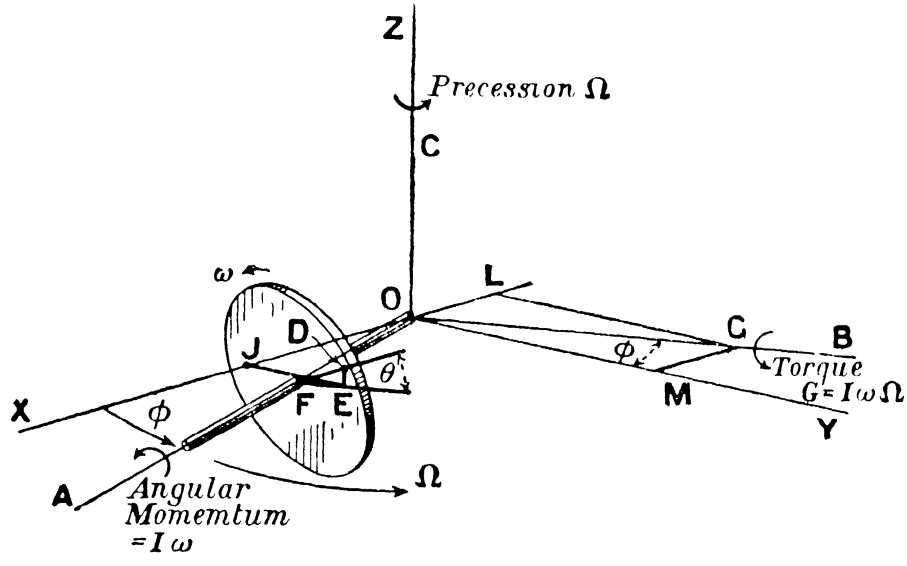



FIG. 117. GYROSCOPE.

Let us now take in the top a particle of mass $m$ situated at D of co-ordinates $x, y, z$ with respect to the fixed axes OXYZ, and co-ordinates $a, b, c$ with respect to the moving axes OABC. Further, let the radius FD have length $r$ and be inclined $\theta$ to the horizontal radius FE. Then we may write

$$\theta=\theta_0+\omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and using it we obtain

$$\left.\begin{aligned} b&=r\cos\theta,\ c=r\sin\theta \\ \dot b&=-\omega c \text{ and } \dot c=\omega b \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

We may further find relations between the two sets of co-ordinates. Thus, making EJ parallel to YO, we have

$$\text{OJ}=\text{OF}\cos\phi-\text{EF}\sin\phi \text{ and } \text{JE}=\text{OF}\sin\phi+\text{EF}\cos\phi,$$

or

$$x=a\cos\phi-b\sin\phi,\ y=a\sin\phi+b\cos\phi,\ z=c=\text{ED} \quad (4).$$

Then, differentiating (4), remembering $a$ is constant, and using (3), we obtain

$$\left.\begin{aligned} \dot x&=(-\Omega a+\omega c)\sin\phi-\Omega b\cos\phi \\ \dot y&=(\Omega a-\omega c)\cos\phi-\Omega b\sin\phi \\ \dot z&=\omega b \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Let the moments of inertia of the top about OA and OC be $I$ and $K$ respectively, and note that on account of the symmetry about OA all the products of inertia are zero. We thus have

$$I=\Sigma m(b^2+c^2),\ K=\Sigma m(a^2+b^2),\ \Sigma mbc=\Sigma mca=\Sigma mab=0 \quad . \quad (6).$$

Now let the angular momenta about the fixed axes OX, OY, and OZ be $P$, $Q$, and $R$ respectively.

Then, remembering that the moment of momentum of a particle is the algebraic sum of the moments of its component momenta, and using (4) and (5), we obtain

$$P=\Sigma m(\dot{z}y-\dot{y}z)$$
$$=\Sigma m(\omega ab\sin\phi+\omega b^2\cos\phi-\Omega ca\cos\phi+\omega c^2\cos\phi+\Omega bc\sin\phi).$$

Hence, omitting the products which are zero by (6), we have

$$P=\omega\cos\phi\,\Sigma m(b^2+c^2)=I\omega\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Similarly

$$Q=\Sigma m(\dot{x}z-\dot{z}x)$$
$$=\Sigma m(-\Omega ca\sin\phi+\omega c^2\sin\phi-\Omega bc\cos\phi-\omega ab\cos\phi+\omega b^2\sin\phi),$$

or $$Q=\omega\sin\phi\,\Sigma m(b^2+c^2)=I\omega\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

and $$\dot{R}=\Sigma m(\dot{y}x-\dot{x}y)$$
$$=\Sigma m\begin{cases}(\Omega a\cos\phi-\omega c\cos\phi-\Omega b\sin\phi)(a\cos\phi-b\sin\phi)\\ -(-\Omega a\sin\phi+\omega c\sin\phi-\Omega b\cos\phi)(a\sin\phi+b\cos\phi),\end{cases}$$

or $$R=\Omega\Sigma m(a^2+b^2)=K\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

**288.** Then, in accordance with the principle of the method, on differentiating $P$, $Q$, and $R$ we obtain the corresponding torques $L$, $M$, and $N$ about the fixed axes OX, OY, and OZ respectively.

We thus have

$$L=\dot{P}=-I\omega\Omega\sin\phi,\quad M=\dot{Q}=I\omega\Omega\cos\phi,\quad N=\dot{R}=0 \quad . \quad . \quad (10).$$

If we now choose the instant when the axes AOB coincide with XOY, then $\phi=0$ and equation (10) reduces to

$$L=0,\quad M=I\omega\Omega=G \text{ say},\quad N=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

That is, if we have a positive angular momentum $I\omega$ about OX, a positive torque $G=I\omega\Omega$ about OY will *maintain an already established positive precession* $\Omega$ about OZ. But it is only for the instant when the axis of the top OA coincides with OX that the torque $G$ is about an axis which coincides with OY. This can be seen more clearly if we go back to equation (10) and compound the values of $L$ and $M$ there expressed. For, as shown in Fig. 117, laying off OL and OM to represent $L$ and $M$ to scale as vectors, it is clear that their resultant is OG of magnitude

$$G=I\omega\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

and that its axis is coincident with OB at angle $\phi$ with OY, *i.e.* at right angles to OA. Consequently the torque axis, being OB, rotates about OC with angular velocity $\Omega$ just as OA does.[1]

**289.** And, in the case shown in the figure, if the top's spindle is supported at the origin O only, and the torque $G$ is due to gravity, we easily see that

$$G=g\Sigma ma=M_0g\bar{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

[1] Thus if a rigid solid of revolution has an angular momentum $I\omega$ about its geometrical axis which is also turning at the steady rate $\Omega$ radians per second about a perpendicular axis; then the body is under the action of a torque whose magnitude is $I\omega\Omega$, whose axis is always at right angles to the other two axes, and whose sense is such that the angular momentum is turning towards coincidence with the torque.

if $M_0$ is the mass of the top and $\bar{a}$ is the co-ordinate of its centre of mass; and that the torque axis automatically remains at right angles to the axis of the top. Hence the rate of precession $\Omega$ which, if started by other means, can be maintained by this torque due to gravity, is from (11), (12), and (13) given by

$$\Omega = G/I\omega = M_0 g\bar{a}/I\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Thus, if the top consists mainly of a disc of mass 72 lbs., 1 foot diameter, with $\bar{a}=8$ inches, and spins at 6000 revolutions per minute, we have $I=9$ lbs. ft.$^2$ and $\omega=200\,\pi$; whence $\Omega=0{\cdot}273$ radian per second, or one revolution in about $22\frac{1}{2}$ seconds.

Referring again to (12), we see that corresponding to a quicker precession we have a larger $G$; hence if the precession be *hurried* beyond the value in (14), while only the gravity torque is available, the axis OA will *rise* from the horizontal. Conversely, if the precession be slowed or prevented, the torque remaining constant, the axis OA will fall. The foregoing is perhaps the simplest way to account for these possible rises or falls. Another instructive way of regarding the matter is to treat the hurrying or retarding of the precession as a (positive or negative) torque about OC which corresponds to a precession about OB. Both views are useful, and the phenomena in question have important practical applications, as we shall see later.

### EXAMPLES—LIII.

1. What chief methods are open to us in the discussion and treatment of the motions of a rigid body with one point fixed? Indicate what you consider to be the relative advantages of the various methods.
2. Find the torques necessary to maintain a steady precession about one axis of a constant angular momentum about a perpendicular axis.
3. Give a numerical instance of question 2, in which the torque is supplied by gravity, using either *c.g.s.* units or British.
4. Enumerate several familiar illustrations of the above-mentioned precessional phenomena, accounting in each case for the sign of the precession which occurs.

**290. General Expressions for Angular Momenta.**—The length of the foregoing example of the first method of treatment, simple as the problem was, well illustrates the necessity for generally using the second method involving moving axes. We now take a step towards the second method by obtaining expressions for the angular momenta $h_1$, $h_2$, and $h_3$ about each of three rectangular axes OXYZ which we will at first suppose fixed in space, the simultaneous components of angular velocity about them being denoted by $\omega_x$, $\omega_y$, and $\omega_z$. Then we shall find that the angular momentum about any one axis depends, in the general case, upon all the three components of angular velocity, and is not necessarily the simple product of moment of inertia into angular velocity about the axis in question The fundamental expression for the angular momentum about the axis of $z$ say is $\Sigma m(\dot{y}x-\dot{x}y)$. And, although when there is rotation about OZ only this reduces to $C\omega_z$ where $C$ is the moment of inertia about the $z$ axis, this is by no means the case when there are simultaneous rotations about the other axes. For in the general case these other rotations modify the motions of many of the particles

in the body, and some of these changes of velocity have moments about the axis of $z$, and therefore modify the angular momentum about it.

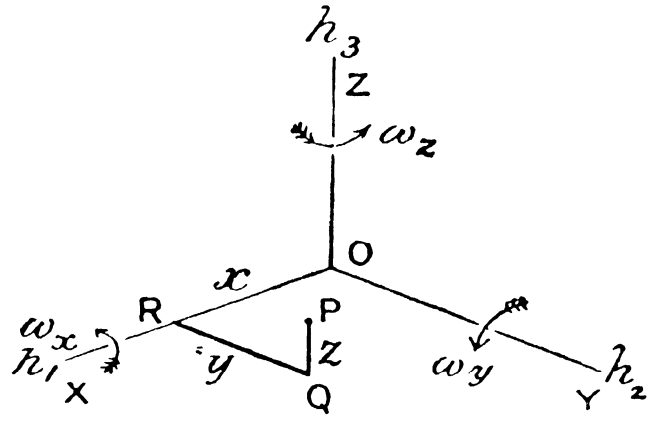



FIG. 118. ANGULAR MOMENTA.

Thus, consider the point P of co-ordinates $x$, $y$, $z$ in Fig. 118. Then from equation (1) of article 132, the origin being now at rest, we find

$$\left.\begin{aligned}\dot{x}&=\omega_y z-\omega_z y\\ \dot{y}&=\omega_z x-\omega_x z\\ \dot{z}&=\omega_x y-\omega_y x\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Putting these values in the expressions for angular momenta, we have

$$\begin{aligned}h_1&=\Sigma m(\dot{z}y-\dot{y}z)\\ &=\Sigma m(\omega_x y^2-\omega_y xy-\omega_z xz+\omega_x z^2)\\ &=\omega_x\Sigma m(y^2+z^2)-\omega_y\Sigma mxy-\omega_z\Sigma mzx.\\ h_2&=\Sigma m(\dot{x}z-\dot{z}x)\\ &=\Sigma m(\omega_y z^2-\omega_z yz-\omega_x yx+\omega_y x^2)\\ &=-\omega_x\Sigma mxy+\omega_y\Sigma m(z^2+x^2)-\omega_z\Sigma myz.\\ h_3&=\Sigma m(\dot{y}x-\dot{x}y)\\ &=\Sigma m(\omega_z x^2-\omega_x zx-\omega_y zy+\omega_z y^2)\\ &=-\omega_x\Sigma mzx-\omega_y\Sigma myz+\omega_z\Sigma m(x^2+y^2).\end{aligned}$$

If we now write for the moments and products of inertia of the body about these axes at the instant in question $A$, $B$, $C$ and $D$, $E$, $F$, we then have

$$\left.\begin{aligned}&A=\Sigma m(y^2+z^2),\ B=\Sigma m(z^2+x^2),\ C=\Sigma m(x^2+y^2)\\ &D=\Sigma myz,\ E=\Sigma mzx,\ F=\Sigma mxy\end{aligned}\right\} \quad . \quad (2).$$

Hence, on substituting these abbreviations, in the above expressions for the angular momenta, they become

$$\left.\begin{aligned}h_1&=+A\omega_x-F\omega_y-E\omega_z\\ h_2&=-F\omega_x+B\omega_y-D\omega_z\\ h_3&=-E\omega_x-D\omega_y+C\omega_z\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

These expressions are easily remembered by noting that the $A$, $B$, and $C$ appear in order as positive coefficients in a diagonal, the $D$, $E$, and $F$ occurring *twice* each as negative coefficients in order returning round from $C$ to $A$.

The above formulae (3) are quite general, and give the *instantaneous*

angular momenta whether the *axes are fixed or not.* For, if the axes of reference are moving, the motion of the body in an element of time is constructed by using the components of motion *as if the axes were instantaneously fixed.* And the above axes may be any whatever. Hence, if they are chosen to coincide for the instant with any given set of moving axes, the above formulae give the instantaneous angular momenta about them.

We may note here a few typical cases and the values to which the general expressions for angular momenta then reduce.

Thus, if the products of inertia vanish, the axes in question are called *principal axes* for the given origin, and we have

$$\left.\begin{aligned} &D=E=F=0 \\ &h_1=A\omega_1,\ h_2=B\omega_2,\ h_3=C\omega_3 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4),$$

the subscripts 1, 2, and 3 being now used for the $\omega$'s, since we have seen the results apply to moving as well as fixed axes.

If the body is a plane *lamina perpendicular to the third axis*, then two of the products of inertia vanish and one moment is the sum of the other two, we thus have

$$\left.\begin{aligned} &D=E=0,\ C=A+B \\ &h_1=A\omega_1-F\omega_2,\ h_2=-F\omega_1+B\omega_2,\ h_3=(A+B)\omega_3 \end{aligned}\right\} \quad (5).$$

For a body of *revolution about the third axis*, all the products of inertia vanish and two moments are equal. Thus we have

$$\left.\begin{aligned} &D=E=F=0,\ A=B \\ &h_1=A\omega_1,\ h_2=A\omega_2,\ h_3=C\omega_3 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6).$$

For a *homogeneous sphere* or a spherical shell or a sphere built up of concentric shells each of which is homogeneous with origin at centre, we have

$$\left.\begin{aligned} &D=E=F=0,\ A=B=C \\ &h_1=A\omega_1,\ h_2=A\omega_2,\ h_3=A\omega_3 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7).$$

**291. Angular Momentum is a Vector directed along its Axis.**— From the fundamental expressions for angular momenta, it may be seen that they are vectors directed along the axes about which the momenta are taken. Thus, for the value of $h_1$ the fundamental expression is $\Sigma m(\dot{z}y-\dot{y}z)$. So that, apart from mass and time, we have the directed quantities $y$ and $z$ which define the $yz$ plane characterised by its normal, the axis of $x$, which is therefore the direction of the vector $h_1$, the angular momentum about the axis of $x$. The same treatment of the like expressions for the other angular momenta, or for the general expression $\Sigma mrv$, shows that the angular momentum about any axis is a vector directed along that axis.

If, however, we now turn from these fundamental expressions in terms of velocities $v$ and arms $r$ to the general expressions in equation (3) of article 290, a doubt may be felt by some as to whether each term on the right side of any equation is a vector along the axis of the corresponding angular momentum. Indeed, at first sight, it might be imagined that each angular momentum was expressed as made up of three parts directed along the three rectangular axes. But this is

not the case. Each term on the right in any one equation has the same direction, namely, that of the axis indicated by the subscript to the $h$ on the left side. To establish this we may conveniently apply the method of examining the dimensions of the factors that constitute the terms on the right, but we must retain the idea of the *direction* of each one and not represent each length by $L$ simply. Thus making this dimensional or directional inspection of the terms for $h_1$, and indicating masses, direction, and time by $M$, $XYZ$, and $T$, we find

$$\left.\begin{aligned}
&\text{The dimensions of } A\omega_x \text{ are } M(Y^2+Z^2)\left(\frac{Z}{YT} \text{ or } \frac{Y}{ZT}\right) = \frac{MYZ}{T} \\
&\text{The dimensions of } F\omega_y \text{ are } (MXY)\frac{Z}{XT} = \frac{MYZ}{T} \\
&\text{The dimensions of } E\omega_z \text{ are } (MZX)\frac{Y}{XT} = \frac{MYZ}{T}
\end{aligned}\right\} \quad (8).$$

We thus see that each term in this general expression for $h_1$ is a vector directed along the axis of $x$, the normal to the plane defined by $YZ$. Similarly $h_2$ is expressed by three terms each of which is directed along the axis of $y$, and $h_3$ consists of three terms each directed along the axis of $z$. It therefore appears by this method that angular momentum about any axis is a vector directed along that axis.

**292. Axes of Resultant Velocity and Momentum.**—Thus, since angular momentum is a vector, if we have given the angular velocities of a body about each of the three axes, also the moments and products of inertia for these axes, we may find the three angular momenta, and can then determine both the resultant angular velocity and the resultant angular momentum, using vector addition in each case. At first sight it may seem startling to find that the directions or axes of these two resultants are not *necessarily coincident.* A numerical example of this is illustrated in Fig. 119, which shows angular velocities and momenta about the three axes OXYZ of the ellipsoid of semi-axes 3, 2, and 1, with centre at the origin and geometrical axes along the axes of co-ordinates. Thus its equation is

$$\frac{x^2}{3^2}+\frac{y^2}{2^2}+\frac{z^2}{1^2}=1.$$

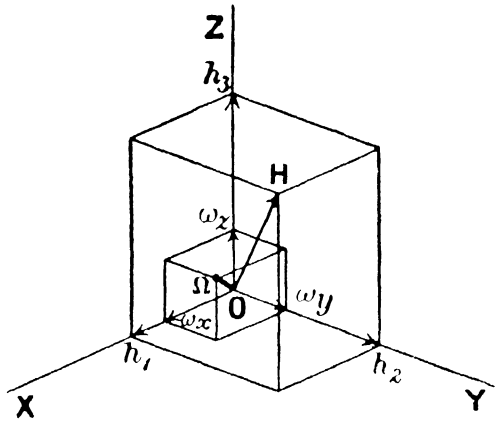



FIG. 119. RESULTANT ANGULAR VELOCITY AND MOMENTUM.

Its products of inertia are all zero on account of symmetry, and its moments of inertia are $A=1\frac{1}{4}$, $B=2\frac{1}{2}$, and $C=3\frac{1}{4}$ if the mass is $\frac{1}{4}$. Thus, if the component angular velocities are 60, 48, and 40, the corresponding component

angular momenta are 75, 120, and 130. Then, the resultant angular velocity $\Omega$, represented by O$\Omega$ on the figure, has magnitude

$$\Omega = \sqrt{60^2+48^2+40^2} = 86{\cdot}6 \ldots$$

and direction cosines 60/86·6, 48/86·6, and 40/86·6.

Whereas the resultant angular momentum, represented by OH in the figure, has magnitude

$$H = \sqrt{75^2+120^2+130^2} = 192{\cdot}1 \ldots$$

and direction cosines 75/192·1, 120/192·1, and 130/192·1.

Thus the angle between the directions of the two resultants is given by

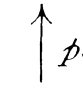
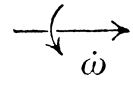
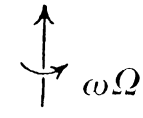
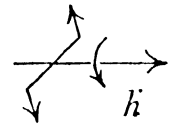
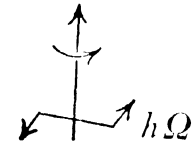
$$\cos \mathrm{HO}\Omega = \frac{60\times 75+48\times 120+40\times 130}{86{\cdot}6\times 192} = 0{\cdot}9298.$$

Whence the angle $\mathrm{HO}\Omega = 21^\circ\ 35'\ 30''$ nearly, as may be seen also by taking the logarithmic cosine.

**293. Analogous Relations between Mechanical Quantities.**— Looking again at Fig. 119, we easily see that, if any component angular velocity receives an increase, the point $\Omega$, which defines the resultant angular velocity, moves a corresponding distance and parallel to the axis of acceleration. Also, if any component angular momentum receives an increase, the point H, defining the resultant angular momentum, moves a corresponding distance and parallel to the axis about which the increase occurred. Also the rate of increase of angular momentum about any axis may be equated to the torque about that axis, which is therefore equal to the velocity of the point H in the diagram. (See *Notes on Dynamics*, by Sir G. Greenhill, p. 195, 1908.) Thus we may construct for *angular velocities and angular momenta loci analogous to the hodograph* of a point moving in any way.

We may also collect in a compact form the analogous relations as to linear velocities and momenta. These are shown together in a form easily remembered in Table XII., and are capable of numerous applications.

TABLE XII. ANALOGOUS CHANGES IN DIRECTED QUANTITIES.

| | Quantity. | Continuous change in *magnitude only* at rate $d/dt$ requires a *collinear* :— | Continuous change in *direction only* at rate $\Omega$ rad./sec. requires a *perpendicular* :— |
|---|---|---|---|
| LINEAR. | Velocity.  | Acceleration.  | Acceleration.  |
| | Momentum.  | Force.  | Force.  |
| ANGULAR. | Velocity.  | Angular Acceleration.  | Angular Acceleration.  |
| | Angular Momentum.  | Torque.  | Torque.  |

A continuous change of any quantity in both magnitude and direction requires a suitable combination of the above collinear and perpendicular components.

## EXAMPLES—LIV.

1. Obtain a set of general expressions for the angular momenta of a rigid body.
2. Prove that an angular momentum is a vector directed along its axis.
3. Show that the axes of resultant angular velocity and momentum are not necessarily in coincidence. Take a numerical illustration, and find the angle between the two axes.
4. Discuss linear and angular velocities and momenta and their changes, and show that a useful analogy may be traced. Exhibit your chief conclusions in tabular form.

**294. Torques about Moving Axes.**—If, in equations (1) of article 123, we now substitute the values of the angular momenta (from

equation (3) of article 290) or denote them by $h_1$, $h_2$, and $h_3$, we obtain expressions for the torques about the set of moving rectangular axes OA, OB, and OC. These are

$$\left.\begin{aligned} L &= \dot{h}_1 - h_2\theta_3 + h_3\theta_2 \\ M &= \dot{h}_2 - h_3\theta_1 + h_1\theta_3 \\ \text{and} \quad N &= \dot{h}_3 - h_1\theta_2 + h_2\theta_1 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1).$$

Substituting the general values for the $h$'s, we have for the torque about the moving axis OA

$$\left.\begin{aligned} L = \frac{d}{dt}(A\omega_1 - F\omega_2 - E\omega_3) \\ -(-F\omega_1 + B\omega_2 - D\omega_3)\theta_3 \\ +(-E\omega_1 - D\omega_2 + C\omega_3)\theta_2 \end{aligned}\right\} \quad \ldots\ldots \quad (2).$$

Similar equations give the values of $M$ and $N$. The values of $A$, $F$, and $E$, as well as those of the angular velocities, are subject to variation, as the moving axes may move with respect to the *body* as well as with respect to *space*. They are accordingly all kept under the sign of differentiation and must be held as subject to it. The method of article 291 applied to equations (1) or (2) of the present article would show each term on the right of any one equation to have the same direction. Thus, every term for $L$ is of the dimensions $MYZT^{-2}$.

Equation (2) is so long as to suggest the necessity for simplification. This can be effected in various ways, which we shall note in order. Thus, first, if the axes are at rest, the $\theta$'s all disappear, and (2) reduces to $L = \dot{h}_1$; see equation (3) of article 290.

**295. Moving Axes fixed in the Body.**—Let us next suppose the axes to be *moving* with respect to *space* but *fixed in the body*. Then we have

$$\omega_1 = \theta_1, \ \omega_2 = \theta_2, \text{ and } \omega_3 = \theta_3 \quad \ldots\ldots \quad (3).$$

Also, if at the instant of consideration the moving axes OABC coincide with the fixed axes OXYZ, we have

$$\omega_1 = \omega_x, \ \omega_2 = \omega_y, \text{ and } \omega_3 = \omega_z \quad \ldots\ldots \quad (4),$$

where the subscripts 1, 2, 3 refer to the angular velocities about the moving axes and the subscripts $x$, $y$, $z$ refer to those about the fixed axes. We have now to prove that the rates of increase of the above angular velocities are also equal each to each, as far as the first order of small quantities. Thus (following Routh) let OR, OR′ be the resultant axes of rotation of the body at the instants $t$ and $t+dt$, *i.e.* when (1) the moving and fixed axes coincide and (2) at the time $dt$ later. Let a rotation $\Omega dt$ about OR bring the body into the position in which OC is in coincidence with OZ at the time $t$. And let a further rotation $\Omega' dt$ about OR′ bring the body into some adjacent position at the time $t+dt$, while in the same interval $dt$, OC moves into the position OC′. Then, according to the definition of a differential coefficient, we have

$$\frac{d\omega_3}{dt} = \text{the limit of } \frac{\Omega' \cos \mathrm{R'C'} - \Omega \cos \mathrm{RC}}{dt}$$

$$\text{and} \qquad \frac{d\omega_z}{dt} = \text{the limit of } \frac{\Omega' \cos \mathrm{R'Z} - \Omega \cos \mathrm{RZ}}{dt}.$$

But the angles RC and RZ are equal by hypothesis because at time $t$, RO and OC coincide with RO and OZ. Further, since OC is fixed in the body, it makes a constant angle with OR′ as the body turns round OR′. Hence, the angles R′C′ and R′Z are also equal. Therefore the above two differential coefficients are equal. We can accordingly write

$$\dot{\omega}_1 = \dot{\omega}_x,\ \dot{\omega}_2 = \dot{\omega}_y,\ \text{and}\ \dot{\omega}_3 = \dot{\omega}_z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

for *moving axes fixed in the body at the instant of their coincidence with the fixed axes.* The great advantage of using *moving axes fixed in the body* lies in the obvious fact that the moments and products of inertia which appear in the equations are then *constant* quantities.

**296. Euler's Dynamical Equations.**—We now introduce a third simplification, namely, that the moving axes OABC fixed in the body are also the principal axes for the fixed point O. Then we have the products of inertia all zero, or

$$D = E = F = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Accordingly, the angular momenta reduce to

$$h_1 = A\omega_1,\ h_2 = B\omega_2,\ h_3 = C\omega_3 \quad . \quad . \quad . \quad . \quad (7).$$

Thus (3), (4), (5), and (7) in (1) yield for the torques about these moving principal axes fixed in the body

$$\left.\begin{aligned} L &= A\dot{\omega}_1 - (B-C)\omega_2\omega_3 \\ M &= B\dot{\omega}_2 - (C-A)\omega_3\omega_1 \\ N &= C\dot{\omega}_3 - (A-B)\omega_1\omega_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

These are the well-known *Dynamical Equations of Euler.* But instead of using them it is often found preferable to take a set of moving axes of which only one is *fixed* in the *body*, one *fixed* in *space*, the other *moving* with respect both to *space* and the *body*.

EXAMPLES—LV.

1. Obtain an expression for the torque about one of a set of moving axes for any specified angular velocities of a given body. From what drawback does this treatment suffer?
2. If the set of rectangular moving axes are chosen so as to be *fixed* in the body, establish the relations and state the simplifications which then follow.
3. Assuming the results of questions 1 and 2, establish Euler's equations.
4. Apply Euler's equations to prove that the maintenance of the steady precession $\Omega$ about a vertical axis of the angular momentum $A\omega$ about a horizontal axis requires the torque $A\omega\Omega$ about a perpendicular horizontal axis.

**297. Steady Precession of Top.**—Let us now apply the method of moving axes and the torques about them to the case of the steady precession of a body of revolution, with one point fixed. Let the moving rectangular axes OABC have OC inclined at an angle $\theta$ with the vertical, OA inclined $\theta$ with the horizontal, and OB horizontal. Let the fixed point of the top be at the origin of co-ordinates, the axis of the top coinciding with OC, about which its angular velocity is $\omega$ and

its moment of inertia $C$. Then the other two moments of inertia are alike, and the products of inertia all vanish since the body is one of revolution about OC. We may thus write, in the usual notation for moments and products of inertia,

$$A=B,\ D=E=F=0 \quad (1).$$

Further, let the axis OC move at constant angular velocity $\Omega$ round the vertical fixed axis OZ say. Then, by the kinematics of the case, worked in article 124, equations (2) to (4), we have for the angular velocities $\omega_1$, $\omega_2$, $\omega_3$ of the top about OA, OB, and OC and for the angular velocities $\theta_1$, $\theta_2$, $\theta_3$ of the axes about their instantaneous positions

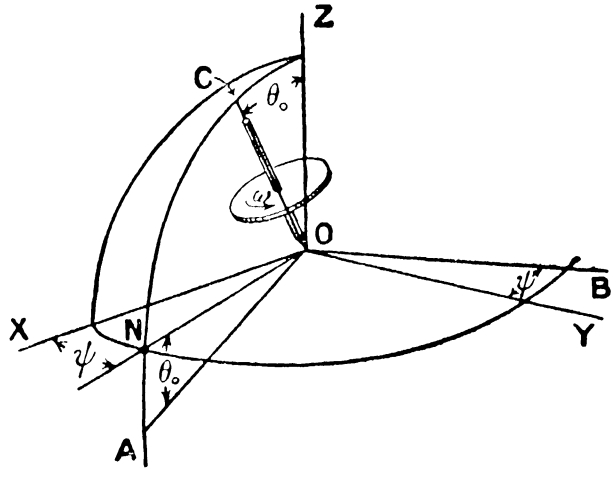



FIG. 119A. STEADY CONICAL PRECESSION OF TOP.

$$\omega_1=-\Omega\sin\theta,\ \omega_2=0,\ \omega_3=\omega \quad . \quad . \quad . \quad . \quad . \quad (2),$$
$$\theta_1=-\Omega\sin\theta,\ \theta_2=0,\ \theta_3=\Omega\cos\theta \quad . \quad . \quad . \quad (3),$$
$$\dot{\omega}_1=\dot{\omega}_2=\dot{\omega}_3=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Thus, by substitution of these values in the general expressions for the angular momenta $h_1$, $h_2$, and $h_3$, equation (3) of article 290, we obtain

$$\left.\begin{aligned} h_1&=-A\Omega\sin\theta \\ h_2&=0 \\ h_3&=C\omega \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

And then, using the expressions given in equation (1) of article 294 for the torques about the axes OA, OB, and OC, we find

$$\left.\begin{aligned} L&=0 \\ M&=C\omega\Omega\sin\theta-A\Omega^2\sin\theta\cos\theta \\ N&=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6).$$

These results should be compared with equations (5), (6), and (7) of article 124, which gave the corresponding angular accelerations.

We easily see that, if the torque $M$ is due to the weight $W$ of the top, its centre of mass being at a distance $h$ from O,

$$M=Wh\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Hence by combining (6) and (7)

$$Wh=C\omega\Omega-A\Omega^2\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Accordingly this is the condition for the *maintenance* of the existing motion by a torque due to gravity. This may be compared to equation (14) of article 288, to which it is equivalent on putting $\theta=\pi/2$ and $C=I$.

We may further note from (6) and (8) that since these are quadratics in $\Omega$, there are in general *two* rates of precession, either of which, if

established, could be maintained by a given torque. We see from (8) that these are given by

$$\Omega=\frac{C\omega\pm\sqrt{C^2\omega^2-4AWh\cos\theta}}{2A\cos\theta} \quad . \quad . \quad . \quad . \quad (9).$$

The two roots are, however, coincident when the quantity under the radical sign vanishes. And, for a smaller value of $\omega$, the rate of spinning, it is obvious that the rate of precession is imaginary. Hence we have as the *minimum* speed of spinning for the maintenance by gravity of the steady precession

$$\omega=\frac{2}{C}\sqrt{AWh\cos\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

**298. Conical Precession without Torque.**—On reference to equations (6) or (8) of article 297, we see that if

$$C\omega-A\Omega\cos\theta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

the torque usually required to maintain the precession vanishes.

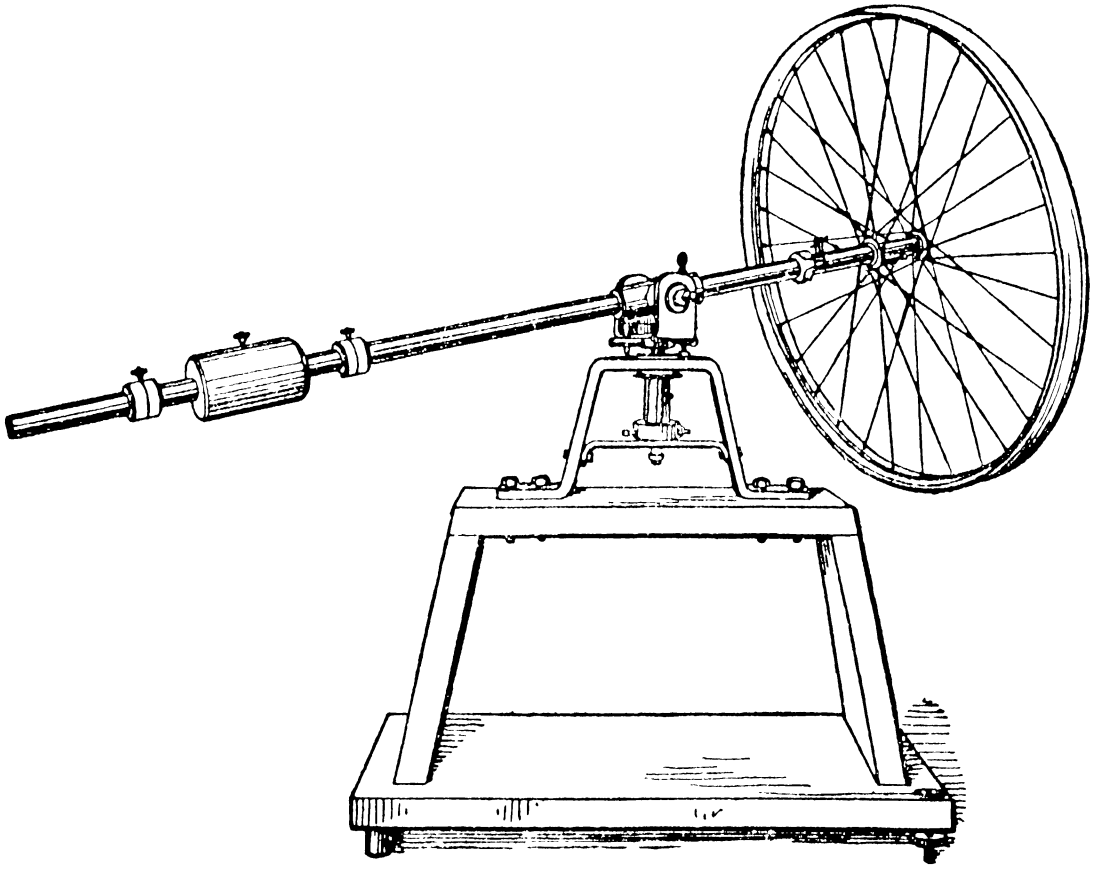


FIG. 120. AUTOMATIC PRECESSION.

Hence, the motion being once established, it continues without any impressed torque.

Sir G. Greenhill has shown that a loaded bicycle wheel is very useful for demonstrating various gyroscopic phenomena (*Notes on Dynamics*, p. 197, 1908).

One mounted as shown in Fig. 120 is in use at Nottingham. It shows very well this conical precession without torque, just referred to. For, when balanced carefully by sliding the balance weight along the tube at the side remote from the wheel, the wheel spun smartly by hand, and then the far end of the tube moved with the right direction and speed as found by a few trials, the motion is maintained in a circle when the tube is let go.

Many other gyroscopic properties can also be demonstrated with this apparatus more effectively than with the smaller gyroscopes sold as toys. For details of the various experiments the reader is referred to such works as Worthington's *Dynamics of Rotation*, or Crabtree's *Spinning Tops and Gyroscopic Motion.*

**299. General Expressions for Kinetic Energy of Rotations.**— Consider any body which has simultaneous angular velocities $\omega_x$, $\omega_y$, and $\omega_z$ about the axes of $x$, $y$, and $z$, the origin of co-ordinates being at rest. Then, from equation (1) of article 132, we have for the velocities of a point whose co-ordinates are $x$, $y$, and $z$ the following expressions :—

$$\left.\begin{aligned}\dot{x}&=\omega_y z-\omega_z y\\ \dot{y}&=\omega_z x-\omega_x z\\ \dot{z}&=\omega_x y-\omega_y x\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Thus, if we imagine there is a particle of mass $m$ at the point in question, the kinetic energy is given by

$$\begin{aligned}T&=\Sigma\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)\\ &=\tfrac{1}{2}\Sigma m\{(\omega_y z-\omega_z y)^2+(\omega_z x-\omega_x z)^2+(\omega_x y-\omega_y x)^2\}\\ &=\tfrac{1}{2}\omega_x^2\Sigma m(y^2+z^2)+\tfrac{1}{2}\omega_y^2\Sigma m(z^2+x^2)+\tfrac{1}{2}\omega_z^2\Sigma m(x^2+y^2)\\ &\quad-\omega_y\omega_z\Sigma myz-\omega_z\omega_x\Sigma mzx-\omega_x\omega_y\Sigma mxy.\end{aligned}$$

Or, with the usual notation for the moments and products of inertia, this becomes

$$T=\tfrac{1}{2}A\omega_x^2+\tfrac{1}{2}B\omega_y^2+\tfrac{1}{2}C\omega_z^2-D\omega_y\omega_z-E\omega_z\omega_x-F\omega_x\omega_y \quad . \quad . \quad (2).$$

Comparing this with equation (3) of article 290, which gives the general expressions for the angular momenta $h_1$, $h_2$, $h_3$, we see that

$$h_1=\frac{dT}{d\omega_x},\ h_2=\frac{dT}{d\omega_y},\ \text{and}\ h_3=\frac{dT}{d\omega_z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

the differentiations being partial. Further, we have

$$2T=h_1\omega_x+h_2\omega_y+h_3\omega_z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

When the axes in question are principal axes, the products of inertia are all zero, and we have the simplified expressions

$$T=\tfrac{1}{2}A\omega_x^2+\tfrac{1}{2}B\omega_y^2+\tfrac{1}{2}C\omega_z^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

These rectangular axes of rotation may be either fixed or the instantaneous positions of moving axes.

As a check upon the above relations we may reduce to a simple case. Thus, let the kinetic energy $T$ of rotation about a single axis be produced by the expenditure of work $W$ in the form of a steady torque $G$ exerted through an angle $\theta$, accordingly giving a final

angular velocity $\omega$ (or $\alpha t$) and angular momentum $H$ about this axis. Then, the moment of inertia in question being $I$, we have

$$T = W = G\theta = \tfrac{1}{2}Gt\omega = \tfrac{1}{2}H\omega = \tfrac{1}{2}I\omega^2 \quad \ldots\ldots\ldots \quad (6),$$

since $\theta = \frac{1}{2}\omega t$ and $H = I\omega$, the initial values $T_0$ and $H_0$ being each zero.

## EXAMPLES—LVI

1. Show that the maintenance of a steady precession $\Omega$ about OZ of the plane ZOC while an angular momentum $C\omega$ exists about OC requires about an axis OB, perpendicular to the plane ZOC, the torque
$$C\omega\Omega \sin\theta - A\Omega^2 \sin\theta\cos\theta,$$
where $A$ is the moment of inertia of the body about the axes OA or OB perpendicular to OC and $\theta = \text{ZOC}$.
2. From the result of question 1 prove that, if the torque is due to gravity, OZ being vertical, different values of $\theta$ will require different values of $\Omega$. Also find the minimum speed of spin for the maintenance of the precession by gravity.
3. Account for the fact that, if a body be thrown into the air spinning, its axis of spin is often seen to describe a cone about a line of fixed direction moving with the body. Do you know of any apparatus which illustrates the same phenomena?
4. Obtain general expressions for the kinetic energy of rotation of a body, and check this by considering some simple case.

**300. Starting Precession.**—We are now in a position to attack a simple example of the unsteady motion of a gyroscope in which the precession and inclination are each variable. We take the case of a body of revolution initially rotating at the speed $\omega_0$ about its geometrical axis OC, which is then inclined at the angle $\theta_0$ with the vertical OZ, the body being supported at the origin only and having no other motion than that specified, but free to acquire any other motions. This initial state of things is indicated in Fig. 121, in which OXYZ are the fixed rectangular axes, OABC the moving rectangular axes, $\psi$ being the angle between the planes ZOX and ZOC.

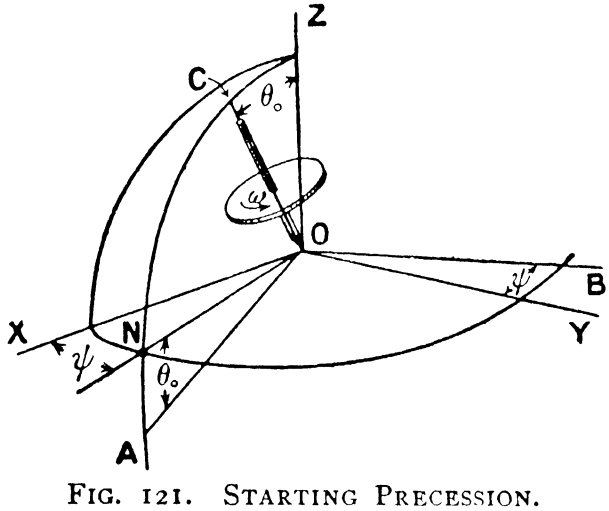


FIG. 121. STARTING PRECESSION.

Let the weight of the top be $W$, its centre of mass be distant $h$ from the fixed point of support O. Then, neglecting all frictional resistances, the sole torque acting upon the top is that due to gravity, and is about the *horizontal axis* OB, which is perpendicular to the plane ZOC. This gravity torque has accordingly no component torque

about either of the axes OC or OZ. Hence, the angular momentum about OC is constant, or in symbols,

$$\dot{\omega}=0 \text{ and } \omega=\omega_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

Further, the angular momentum $h_z$ about OZ is constant, or

$$\dot{h}_z=0 \text{ and } h_z=\text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

But, to solve the problem, we need the values at any time $t$ of the three angular velocities $\omega$ about OC, $\dot{\theta}$ about OB, and $\dot{\psi}$ about OZ. We accordingly need a *third* equation. This could be obtained by equating the torque about OB to the corresponding rate of change of angular momentum. It is, however, somewhat simpler to use the principle of conservation of energy. Thus, finding general expressions for the kinetic and potential energies, we may equate their sum to that of their initial values. This gives the third equation required, which we may now write as

$$T+V=T_0+V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

We have next to substitute the actual values in (2) and (3). Let $C$ be the moment of inertia of the top about OC, $B=A$ that about either OB or OA. And, since the body is a solid of revolution about OC, its three products of inertia, $D$, $E$, and $F$, all vanish. It is seen from the figure that the angular velocity $\dot{\psi}$ about OZ has a component $-\dot{\psi}\sin\theta$ about OA. We accordingly obtain for the angular momenta about OA, OB, and OC the values

$$h_1=A(-\dot{\psi}\sin\theta),\ h_2=B\dot{\theta}=A\dot{\theta}, \text{ and } h_3=C\omega.$$

Further, the angular momentum $h$ about OZ receives no component from OB, which is perpendicular to it, but only from those about OA and OC.

Thus, for the angular momentum about OZ, we have

$$\begin{aligned} h_z&=h_1(-\sin\theta)+h_3\cos\theta \\ &=A\dot{\psi}\sin^2\theta+C\omega\cos\theta. \end{aligned}$$

And, since by (2) this is constant, we may write

$$A\dot{\psi}\sin^2\theta+C\omega\cos\theta=C\omega\cos\theta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

as the equation of conservation of angular momentum about OZ.

We have still to form the expressions for the energy and substitute in (3). Thus, from article 299, we have

$$T=\tfrac{1}{2}A(-\dot{\psi}\sin\theta)^2+\tfrac{1}{2}B\dot{\theta}^2+\tfrac{1}{2}C\omega^2.$$

Also the potential energy, reckoned from the level of O, is obviously

$$V=Wh\cos\theta.$$

Thus, substituting in (3), and cancelling $C\omega^2$ from each side, we have

$$A(\dot{\psi}^2\sin^2\theta+\dot{\theta}^2)+2Wh\cos\theta=2Wh\cos\theta_0 \quad . \quad . \quad . \quad . \quad (5).$$

Equations (4) and 5 determine the whole subsequent motion of the top. From (4) we obtain the rate of precession in terms of $\theta$, viz.

$$\dot{\psi}=\frac{C\omega(\cos\theta_0-\cos\theta)}{A\sin^2\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Thus, at the start when $\theta=\theta_0$, we have for the initial value of the precession

$$\dot{\psi}_0=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

But (6) shows that for any larger value of $\theta$ than $\theta_0$, the precession $\dot{\psi}$ has a finite value. Thus, though the top does not immediately precess, it cannot fall without doing so. If now we differentiate (6), we find for the precessional acceleration

$$\ddot{\psi}=\frac{C\omega\dot{\theta}(1-2\cos\theta_0\cos\theta+\cos^2\theta)}{A\sin^3\theta} \quad . \quad . \quad . \quad (8).$$

Thus since at the start $\dot{\theta}=0$, the initial value of $\ddot{\psi}$ is zero also. Hence the first motion must be an increase of $\theta$ simply, that is, a falling of the axis OC from the vertical in the initial position of the plane ZOC. But, immediately $\dot{\theta}$ has a finite value, $\ddot{\psi}$ has a finite value also, and $\dot{\psi}$ accordingly grows; that is, the plane ZOC acquires a velocity about the vertical OZ. For, if $\dot{\theta}$ is finite while $\theta$ is still practically equal to $\theta_0$, (8) reduces to

$$\ddot{\psi}=C\omega\dot{\theta}/A\sin\theta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

**301. Nutations or Oscillations in the Azimuthal Plane.**—We have just seen that the starting of the precession is inseparable from a change in the value of $\theta$, that is, a fall or rise of the inclined axis OC of the top. We may next naturally ask whether this motion, which is expressed by $\dot{\theta}$, and which lies in the azimuthal plane ZOC, is initially a rise or a fall, whether it has any limits, and what is its general character.

To answer these questions we begin by eliminating $\dot{\psi}$ between equations (5) and (6) of article 300. Thus substituting from (6) in (5), we find

$$C^2\omega^2(\cos\theta_0-\cos\theta)^2+A^2\dot{\theta}^2\sin^2\theta=2AWh(\cos\theta_0-\cos\theta)\sin^2\theta.$$

The stationary values of $\theta$, if any occur, will be obtained from this by putting in it $\dot{\theta}=0$. We thus find

$$(\cos\theta_0-\cos\theta)\{2AWh\sin^2\theta-C^2\omega^2(\cos\theta_0-\cos\theta)\}=0.$$

Hence, the stationary values of $\theta$ are given by

$$\theta=\theta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

and the roots of

$$\sin^2\theta-2\lambda(\cos\theta_0-\cos\theta)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where $2\lambda=C^2\omega^2/2AWh$. Equation (11) may be written

$$\cos^2\theta-2\lambda\cos\theta+2\lambda\cos\theta_0-1=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

giving for cos $\theta$ the values

$$\cos\theta=\lambda\pm\sqrt{1-2\lambda\cos\theta_0+\lambda^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Since cos $\theta_0$ is less than unity, it follows that the quantity under the radical sign is greater than $(1-\lambda)^2$. Thus, if the radical has the value $\pm(1-\lambda+k)$, taking the upper sign would give for cos $\theta$ the value $1+k$; and this value, though real, gives no real cosine. We are therefore limited to the negative sign of the radical, and calling the corresponding value of the angle $\theta_1$, we have

$$\cos\theta_1=\lambda-\sqrt{1-2\lambda\cos\theta_0+\lambda^2}=2\lambda-1-k \text{ say} \quad . \quad . \quad . \quad (14).$$

Thus, as the azimuthal plane ZOC rotates about OZ with the variable velocity $\dot{\psi}$, the axis OC of the top also oscillates in the azimuthal plane between the positions defined by the values $\theta_0$ and $\theta_1$

of the angle ZOC, $\theta_0$ being the initial value when OC was at rest, $\theta_1$ the other limiting value reached at some later instant. This oscillation of the axis in the azimuthal plane is called *nutation.*

It is obvious that the inclination $\theta_1$ is below $\theta_0$, for the additional kinetic energy of the precessional motion $\dot{\psi}$, which we saw was associated with the change in $\theta$, could only be obtained at the expense of potential energy. Thus $\theta_0$ and $\theta_1$ are the minimum and maximum values of $\theta$ assumed by the axis OC. These limits are indicated by $C_0$ and $C_1$ in Fig. 122. The path described on the surface of a sphere by a point C on the axle is also shown by $C_0$, D, E, F.

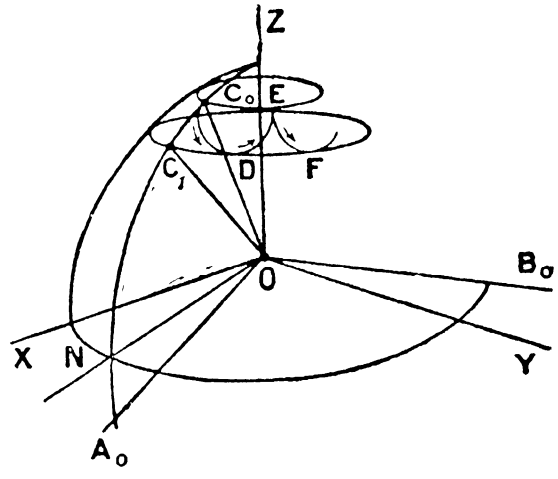



FIG. 122. NUTATION.

It is thus seen that the curve has a cusp for each time the upper limit is reached, as at $C_0$ and E. This motion may be easily observed if a gyroscope be spun slowly, placed in an inclined position with its point in the cup, and then suddenly let go. If, instead of letting go simply, the top be given a forward or backward impulse, the curve described will be a wavy curve or a looped one respectively, instead of the cusped one just dealt with.

This experiment for showing nutation is much more striking if shown with the bicycle wheel apparatus represented in Fig. 120, article 298.

**302. Precessional Velocities at Limiting Inclinations.**—Let us now obtain the values of $\dot{\psi}$ for the highest and lowest positions of the axis OC, that is, for the limiting values of $\theta$, viz. $\theta_0$ and $\theta_1$. Rewrite here equations (4) and (5) of article 300 in the form

$$A\dot{\psi}\sin^2\theta = C\omega(\cos\theta_0 - \cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (15),$$

and

$$A\dot{\psi}^2\sin^2\theta + A\dot{\theta}^2 = 2Wh(\cos\theta_0 - \cos\theta) \quad . \quad . \quad . \quad . \quad (16).$$

Then (15) multiplied by $\dot{\psi}$ gives

$$A\dot{\psi}^2\sin^2\theta = C\omega\dot{\psi}(\cos\theta_0 - \cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (17).$$

Hence, for the limiting inclinations, when $\dot{\theta}=0$, the left sides of (16) and (17) are equal. Thus, on equating their right sides, we have

$$(\cos\theta_0 - \cos\theta)(C\omega\dot{\psi} - 2Wh) = 0 \quad . \quad . \quad . \quad (18).$$

Accordingly, *either* (1) $\theta=\theta_0$, in which case we have from (15)

$$\dot{\psi}_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

*or* (2)

$$C\omega\dot{\psi} - 2Wh = 0,$$

that is,

$$\dot{\psi}_1 = 2Wh/C\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

The subscripts to the $\psi$'s here correspond to those of the $\theta$'s, to which positions they refer.

Equation (19) agrees with what was previously found in (7) of article 300, the result in (20) being new.

It may be noted here that this maximum rate of precession $\dot{\psi}_1$ at the limit $\theta_1$ of inclination exceeds the steady rate of precession which, if established, the given torque is able to maintain. Thus if $\theta_1$ is $\pi/2$, we reduce to the case of rectangular precession, in which, as shown in equations (14) of article 288 and (8) of 297, we have

$$\Omega = Wh/C\omega \quad \ldots\ldots\ldots \quad (21),$$

so that the steady $\Omega$ is in this case only half the maximum value of $\dot{\psi}$. On the other hand, the minimum value of $\dot{\psi}$ is zero.

**303. Tilting Velocity of Top.**—From equations (15) and (16) we may now find the value of $\dot{\theta}$, the angular velocity about OB, that is, the velocity of tilting in the azimuthal plane ZOC.

Thus, substituting in (16) the value of $\dot{\psi}$ from (15), we have

$$\frac{C^2\omega^2(\cos\theta_0 - \cos\theta)^2}{A\sin^2\theta} + A\dot{\theta}^2 = 2Wh(\cos\theta_0 - \cos\theta).$$

Whence $$\dot{\theta} = \frac{\sqrt{2AWh(\cos\theta_0 - \cos\theta)\sin^2\theta - C^2\omega^2(\cos\theta_0 - \cos\theta)^2}}{A\sin\theta} \quad (22).$$

And dividing this by the equation obtained from (15) expressing the precessional velocity $\dot{\psi}$, we obtain the ratio of the two velocities

$$\frac{\dot{\theta}}{\dot{\psi}} = (\sin\theta)\sqrt{\frac{\sin^2\theta}{2\lambda(\cos\theta_0 - \cos\theta)} - 1} \quad \ldots\ldots\ldots \quad (23),$$

where $2\lambda$ is used as before to denote $C^2\omega^2/2AWh$.

This equation gives, in terms of $\theta$, the direction of the path traced on a spherical surface by any point on the axis OC. Its projection on the horizontal plane $xy$ lies between two circles which correspond to the limiting inclinations $\theta_0$ and $\theta_1$ of the axis. Further, the path meets the inner circle with *cusps tangential to the radii*, because there $\dot{\theta}/\dot{\psi} = \infty$, as shown by (23), and *touches* the outer circle, because there $\dot{\theta}/\dot{\psi} = 0$, as shown by (6) of article 300 and the fact that $\theta$ is stationary at $\theta_1$.

Thus, before any precession is established, the top yields slightly to the tilting torque, but soon the precession exceeds the steady value which the torque could maintain, and the top rises to the next cusp.

**304. Minimal Velocities for Top to Spin and to 'Sleep.'**—The foregoing considerations lead us to note that if the value of $\theta_1$ is such that the body of the top catches the horizontal plane, it will cease to spin. Suppose the inclination when the top thus catches the plane is $\theta_2$, then for spinning we must have $\theta_2 > \theta_1$. And this may be shown to necessitate a minimum value of the angular velocity $\omega$ of spin. Thus (following the method of Crabtree) we may write from (15) of article 302

$$\dot{\psi} = \frac{C\omega(1 - \cos\theta - \kappa)}{A(1 - \cos^2\theta)} = \frac{C\omega}{A}\left(\frac{1}{1 + \cos\theta} - \frac{\kappa}{1 - \cos^2\theta}\right) \quad (24),$$

where $\kappa$ is the positive constant $1-\cos\theta_0$. Hence $\dot\psi$ increases with $\theta$. Thus if $\dot\psi_2$ corresponds to $\theta_2$, we have $\dot\psi_2>\dot\psi_1$. But, reverting again to (15), we have

$$\dot\psi_2=\frac{C\omega(\cos\theta_0-\cos\theta_2)}{A\sin^2\theta_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

And by (20) we have

$$\dot\psi_1=2\,Wh/C\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

So, on substituting the values of these two precessions in the inequality connecting them, we have

$$\frac{C\omega(\cos\theta_0-\cos\theta_2)}{A\sin^2\theta_2}>\frac{2\,Wh}{C\omega}.$$

Or

$$\omega^2>\frac{2A\,Wh\sin^2\theta_2}{C^2(\cos\theta_0-\cos\theta_2)} \quad . \quad (27).$$

For the top to *sleep*, that is, spin in a vertical position, put $\theta_0=0$. Then (27) gives for a top starting vertical and not falling so far as $\theta_2$

$$\omega^2>\frac{2A\,Wh(1+\cos\theta_2)}{C^2} \quad . \quad (28).$$

If we now limit $\theta_2$ also to zero, so that the top if disturbed from verticality returns to it, we have finally

$$\omega^2>4A\,Wh/C^2 \quad . \quad . \quad . \quad . \quad (29).$$

Of course, in the case of any actual top whose spinning is not maintained, friction will check the speed until it falls below the value in (29), and the top then begins to wobble, and finally falls.

**305. Examples of Gyroscopic Motion.**—Perhaps the most important example of gyroscopic motion is that of the earth itself, which we may regard as a top spinning about its polar axis and subject to fluctuating torques or couples due to the attractions between its equatorial protuberances and the moon or sun. Fig. 123 illustrates this effect of the sun or moon when in its most favourable position for producing it. For the attracting body B is shown in the plane containing the polar axis of the earth. Now since the earth is not spherical we may regard the attractions on it as separable into three components, viz. (1) the main one corresponding to the sphere on the polar diameter and acting at the centre; (2) a small force of like sign on the eastern equatorial protuberance; and (3) one of opposite sign on the western protuberance. These smaller forces are obviously due to the one protuberance being nearer to and the other farther from the attracting body than the earth's centre is. We have thus a counter-clockwise torque or couple acting on

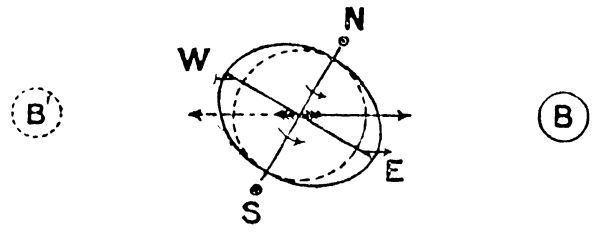



FIG. 123. CAUSE OF PRECESSION OF THE EQUINOXES.

the earth as shown in the diagram. Then by the phenomena of precession, as already dealt with, we see that since the earth is rotating from west to east, instead of simply yielding to this torque, it will turn about the third rectangular axis WE, and in such sense that N comes towards the spectator and S recedes as indicated by the conventional signs ⊙ and ⊕, representing respectively the point and the feathers of an arrow.

If now we transfer the attracting body to the same distance on the other side as indicated by the dotted circle B′, it is evident that though the main attraction is reversed, as shown by the dotted arrow, the couple represented by the small arrows remains unchanged. But, if B is brought to a position on the normal to the diagram through the centre of the earth, then the couple vanishes whether the attracting body is before or behind the plane represented. For the W and E protuberances shown are in those cases equidistant from the attracting body and the near and far protuberances (not shown in the figure) are exactly in the line of centres. Thus we see that as the moon goes round the earth the torque alternately waxes and wanes, vanishing twice in the period of the moon's orbit, but has *always the same sign.* This torque produces the lunar precession. The sun also produces the solar precession; and similar remarks apply to this as to waxing and waning and constancy of sign, the period being now that of the earth's orbit, or a year. The solar and lunar precessions are roughly in the ratio of 3 : 7. The combined effect is sometimes called the luni-solar precession. In consequence of this the pole P of the earth on the celestial sphere slowly describes a circle of radius 23° 27′ round K, the pole of the ecliptic on the celestial sphere. The average angle described in a year is about 50″·2, and the time of a complete revolution is of the order 25,800 years.

But, superposed upon this precession, there is also nutation; and this is due partly to the sun and partly to the moon. The lunar nutation is due to the change of the moon's nodes, and has a period of about 18 years 220 days, being that of a sidereal revolution of the moon's nodes. The solar nutation is due to the variation of the sun's declination, and has a period of half a tropical year (which year is the time between successive vernal equinoxes). The amplitudes of these lunar and solar nutations are of the order 9″ and 1″·2. Hence the pole P of the earth's equator traces a *wavy line* on the celestial sphere round the pole K of the ecliptic.

There are many other examples of gyroscopic motions, some of which are quite familiar. Thus, the turning of a hoop when rolling along and made to lean, or of a coin, are cases in point. The rearing or plunging of a very fast single-screw turbine steamboat, instead of answering its helm, is another.

Two cases in which gyroscopic actions have been applied may be mentioned. Thus Otto Schlick has arranged to steady a vessel at sea by means of gyroscopes, a rolling amplitude of 15° each side the vertical being thus reduced to an arc of rolling (out to out) of 1°.

Louis Brennan has applied gyroscopes to balance a special monorail car, both when going on the straight and round curves, the principle underlying his automatic devices being that of hurrying the precession, and thus causing the car to rise from a tilted position. It is beyond the scope of this book, however, to enter into details of these ingenious inventions or the many other problems of gyroscopic motion.

The interested reader may consult H. Crabtree's *Spinning Tops and Gyroscopic Motion*, A. M. Worthington's *Dynamics of Rotation*, and Sir G. Greenhill's *Notes on Dynamics*; also *Engineering*, vol. lxxxiii., 1907, pp. 442, 448, 623, and 794, and p. 797 of June 24, 1910, and *Nature*, Professor Perry's article, *The Use of Gyrostats*, vol. lxxvii. pp. 447-450, March 12, 1908; Professor H. Lamb's *Theory of Gyroscope*, *Roy. Soc. Edin.*, *Proc.* 35, pp. 153-161, 1914-1915.

### EXAMPLES—LVII.

1. A top is spinning about an inclined axis with its highest and lowest points supported, discuss what happens when the upper point is released and the axis is accordingly set free to move.
2. Explain what are meant by the terms *precession* and *nutation*, and give examples from an experiment and from the solar system.
3. In a typical case of the motion of a top under gravity, find an expression for the velocity of nutation or movement in the azimuthal plane.
4. Explain how the rate of precession changes when nutation is occurring, and obtain the limiting expressions for this rate.
5. Show that there are minimum velocities of spin for a top to clear the ground and for it to 'sleep.'
6. Give several examples of gyroscopic actions; some of them presenting difficulties to be overcome, others being useful applications of this action.

**306. Centrifugal Reactions and Torques.**—Referring now to Euler's equations, obtained in article 296, we note that the whole changes occurring in the values of the component angular velocities $\omega_1$, $\omega_2$, and $\omega_3$ are not entirely due to the direct action of the external torques $L$, $M$, and $N$, but are due in part to the centrifugal reactions which occur in virtue of the motions of the particles and the rigidity of the body. Thus, taking the last equation of (8) in the article referred to, it may be written

$$\frac{d\omega_3}{dt} = \frac{N}{C} + \frac{A-B}{C}\omega_1\omega_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Hence, of the increase $d\omega_3$ occurring in the time $dt$ in the velocity $\omega_3$, the part $Ndt/C$ is due to the direct action of the forces whose moment is $N$, and the part $(A-B)\omega_1\omega_2 dt/C$ is due to *centrifugal* reactions. Following Routh, this may be enunciated and proved thus:—

If a rigid body be rotating about an axis OI with an angular velocity $\omega$, then the moment of the centrifugal reactions for the whole body about the axis of $z$ is $(A-B)\omega_1\omega_2$, OZ being the instantaneous position of the moving axis OC, the component velocities being $\omega_1$, $\omega_2$, and $\omega_3$.

In Fig. 124, let P be the position of any particle $m$ of the body, its co-ordinates being $x$, $y$, and $z$, which are represented respectively by OR, RQ, and QP. Let PS$=r$ be the perpendicular

upon OI and let OS$=u$. Then, as the particle $m$ rotates round OI at angular velocity $\omega$, the centripetal force needed on it is $m\omega^2 r$ directed from P to S. Accordingly the *reaction* exerted by the particle is oppositely directed and of equal magnitude. Hence, *it is* $m\omega^2 r$ acting at P and directed *from* S. Since the vector SP$=r$ is equivalent to the vector sum of SO, OR, RQ, and QP, this force $m\omega^2 r$ is equivalent to the four forces $-m\omega^2 u$, $m\omega^2 x$, $m\omega^2 y$, and $m\omega^2 z$, all acting at P and parallel to $u$, $x$, $y$, and $z$ respectively. The moment of $m\omega^2 x$ round OZ is $-m\omega^2 xy$, while that of $m\omega^2 y$ is the same but with the positive sign. The moment of $m\omega^2 z$ round OZ is zero. Thus, three of these four components produce no moment about OZ. The remaining force, $-mu\omega^2$ acting at P parallel to OI, may in like manner be resolved into three forces parallel to the axes and of values $-mu\omega\omega_1$, $-mu\omega\omega_2$, and $-mu\omega\omega_3$ respectively, since the direction cosines of OI are $\omega_1/\omega$, $\omega_2/\omega$, and $\omega_3/\omega$. The moment of these forces round OZ is

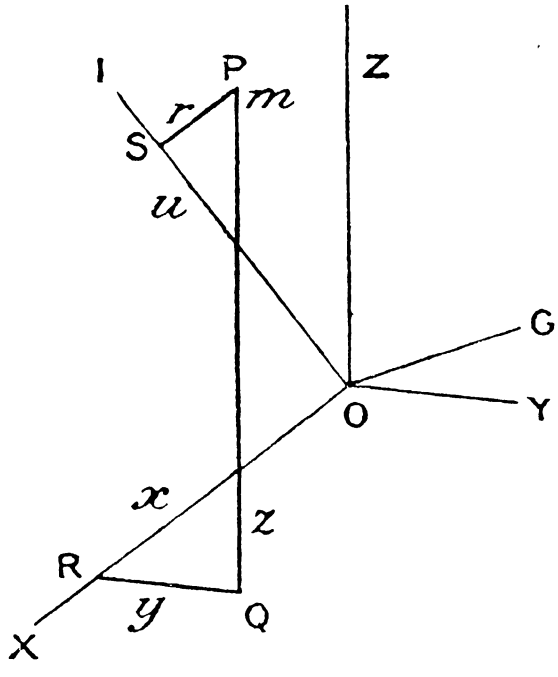


FIG. 124. CENTRIFUGAL TORQUE.

$$mu\omega(\omega_1 y-\omega_2 x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Also, by projecting upon OI the broken line OR, RQ, QP, PS, we have

$$u=\frac{\omega_1}{\omega}x+\frac{\omega_2}{\omega}y+\frac{\omega_3}{\omega}z \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Then, substituting (3) in (2), we find for the centrifugal torque round OZ

$$m(\omega_1 x+\omega_2 y+\omega_3 z)(\omega_1 y-\omega_2 x)$$
$$=m\{\omega_1\omega_2(y^2-x^2)+\omega_3\omega_1 yz-\omega_2\omega_3 zx+(\omega_1^2-\omega_2^2)xy\} \quad . \quad . \quad . \quad (4).$$

Let us now suppose that the moving axes OABC fixed in the body, which instantaneously coincide with OXYZ, are principal axes, so that all the products of inertia vanish. Then, on summing up the expression (4) for all particles in the body and omitting the sums which vanish, we obtain for the centrifugal torque about OZ

$$N'=\omega_1\omega_2\Sigma m(y^2-x^2)=\omega_1\omega_2\Sigma m(y^2+z^2-z^2-x^2) \quad . \quad . \quad (5).$$

We can now by symmetry write the three component centrifugal torques, thus:—

$$L'=(B-C)\omega_2\omega_3,\ M=(C-A)\omega_3\omega_1,\ N'=(A-B)\omega_1\omega_2 \quad (6).$$

Let the resultant *centrifugal torque* or *couple* be $G$, and let the angle between its axis OG and that of rotation OI be called $\theta$. Then, since

the direction cosines of these lines are respectively $L'/G$, $M'/G$, $N'/G$, and $\omega_1/\omega$, $\omega_2/\omega$, $\omega_3/\omega$, we have

$$\cos\theta = \frac{L'\omega_1 + M'\omega_2 + N\omega_3}{G\omega} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

because the numerator of the fraction vanishes in virtue of (6). Thus $IOG = \pi/2$, or *the axis of the centrifugal torque is at right angles to the instantaneous axis of rotation.*

Hence, returning to (1) and inserting the values from (6), we have for the three angular accelerations

$$A\dot{\omega}_1 = L + L',\ B\dot{\omega}_2 = M + M', \text{ and } C\dot{\omega}_3 = N + N' \quad . \quad . \quad (8).$$

**307. Independence of Translation of Centre of Mass and Rotation about it.**—We have hitherto in this chapter supposed one point of our rigid body to be fixed. Let this restriction be now removed. Then the subsequent work is often much simplified by the fact that the translation of the centre of mass and the rotation about any axis through it are quite independent of each other. That this is so might have been inferred from what was previously shown for coplanar motions. But it is more satisfactory to have an independent proof. We shall deal in turn with linear momenta, kinetic energy, angular momenta, and their rates of change.

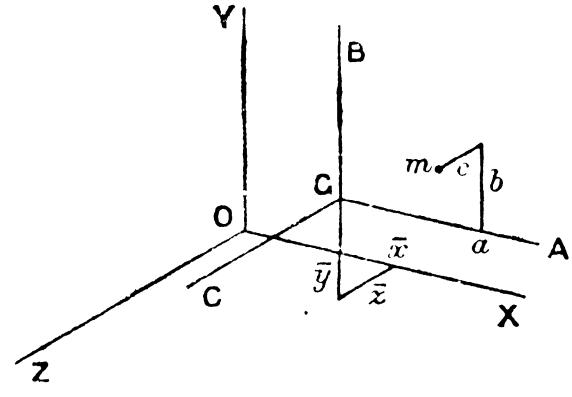



FIG. 125. INDEPENDENCE OF TRANSLATION AND ROTATION.

**Linear Momenta.**—Referring to Fig. 125, let OXYZ denote fixed axes and GABC axes whose origin G is at the centre of mass of the moving body and whose directions remain parallel to those of $x$, $y$, and $z$. Let a particle of mass $m$ of the body have co-ordinates $x$, $y$, $z$ with respect to the fixed axes and $a$, $b$, $c$ with respect to GABC. Further, let the moving point G have co-ordinates $\bar{x}$, $\bar{y}$, and $\bar{z}$ and velocity components $u$, $v$, and $w$, the body having angular velocity components $\omega_x$, $\omega_y$, and $\omega_z$. Then we have the following relations among the co-ordinates and velocities:—

$$\left.\begin{array}{lll} x = \bar{x} + a, & y = \bar{y} + b, & z = \bar{z} + c \\ \dot{a} = \omega_y c - \omega_z b, & \dot{b} = \omega_z a - \omega_x c, & \dot{c} = \omega_x b - \omega_y a \\ \dot{x} = u + \omega_y c - \omega_z b, & \dot{y} = v + \omega_z a - \omega_x c, & \dot{z} = w + \omega_x b - \omega_y a \end{array}\right\} \quad . \quad (1).$$

Let the linear momenta be $p_x$, $p_y$, and $p_z$, then we have

$$p_x = \Sigma m\dot{x} = u\Sigma m + \omega_y\Sigma mc - \omega_z\Sigma mb.$$

But since G, the origin of $a$, $b$, and $c$, is the centre of mass, $\Sigma mc = 0 = \Sigma mb = \Sigma ma$. Thus, writing $M$ for the total mass, we find

$$p_x = Mu,\ p_y = Mv, \text{ and } p_z = Mw \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

In other words, the linear momenta of the body are not affected by any rotations and equal those of the whole mass at G.

**308. Kinetic Energy of Rigid Body in General Motion.**—Let us now form the expression for the kinetic energy of a rigid body whose centre of mass has velocities $u$, $v$, and $w$, the body at the same time having rotations $\omega_x$, $\omega_y$, and $\omega_z$ about axes parallel to the fixed axes OXYZ. Then, using (1) above, we have for the kinetic energy

$$T=\tfrac{1}{2}\Sigma m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$
$$=\tfrac{1}{2}\Sigma m\{(u+\omega_y c-\omega_z b)^2+(v+\omega_z a-\omega_x c)^2+(w+\omega_x b-\omega_y a)^2\}.$$

Whence, squaring and omitting the terms which vanish in virtue of G being the centre of mass, we find

$$\left.\begin{aligned} T=&\tfrac{1}{2}M(u^2+v^2+w^2)+\tfrac{1}{2}A\omega_x^2+\tfrac{1}{2}B\omega_y^2+\tfrac{1}{2}C\omega_z^2\\ &-D\omega_y\omega_z-E\omega_z\omega_x-F\omega_x\omega_y \end{aligned}\right\} \quad . \quad . \quad . \quad (3),$$

the letters $A$, $B$, $C$ denoting the instantaneous moments of inertia and $D$, $E$, $F$ the corresponding products of inertia. It is easily seen that the first term on the right expresses the kinetic energy of the whole mass as if concentrated at G, while the other six terms give the kinetic energy of rotation about the axes through the centre of mass. (See equation (2) of article 299.) But it should be noted here that, since the *axes* GABC are *not fixed in the body*, but *only the point G*, the values of the moments and products of inertia in (3) may be continually changing.

**309. Angular Momenta of Rigid Body in General Motion.**—Using again Fig. 125 and equations (1) in article 307, we may form the expressions for the angular momenta $h_x$, $h_y$, and $h_z$ about the axes of $x$, $y$, and $z$.

Thus, for the first we have

$$h_x=\Sigma m(\dot{z}y-\dot{y}z)$$
$$=\Sigma m\{(w+\omega_x b-\omega_y a)(\bar{v}+b)-(v+\omega_z a-\omega_x c)(\bar{z}+c)\}.$$

And, on performing the multiplications, omitting as before the vanishing terms, we find

$$h_x=(w\bar{y}-v\bar{z})\Sigma m+\omega_x\Sigma m(b^2+c^2)-\omega_y\Sigma mab-\omega_z\Sigma mca \quad . \quad (4).$$

Hence, on substituting the usual symbols for the mass, moments, and products of inertia, and writing the other momenta by symmetry, we obtain

$$\left.\begin{aligned} h_x&=M(w\bar{y}-v\bar{z})+A\omega_x-F\omega_y-E\omega_z\\ h_y&=M(u\bar{z}-w\bar{x})-F\omega_x+B\omega_y-D\omega_z\\ h_z&=M(v\bar{x}-u\bar{y})-E\omega_x-D\omega_y+C\omega_z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

These again show that each of the angular momenta splits into two parts, one representing about the given axis the angular momentum of the whole mass concentrated at its centre G, and the other expressing the angular momentum of the actual body about a parallel axis through G. (See equations (3) of article 290.)

## EXAMPLES—LVIII.

1. In Euler's equations show that the increases of angular velocities are partly due to centrifugal reactions, and determine the resultant axis of these reactions.
2. In the case of a rigid body rotating under torques as expressed by Euler's equations, prove that the moment of the centrifugal reactions

about any principal axis is the continued product of the difference of moments of inertia about the other two axes and the two corresponding angular velocities.

3. Show that the linear momentum of a rigid body moving generally in solid space is that of the whole mass as if at its centre.

4. Prove that the kinetic energy of a rigid body in general motion in three dimensions splits into two terms, one expressing that of the whole mass at its centre of mass and the other that of the rotation of the actual body about an axis through the centre of mass as though it were at rest.

5. Show that the angular momentum of a rigid body in any motion is the sum of that of the whole mass at its centre and that of the actual body about a parallel axis through that centre.

**310. Equations of Motion for Axes of Fixed Directions.**—We have just shown that, as regards linear and angular momenta and kinetic energy, a rigid body moving in any way is replaceable by the total mass at its centre of mass together with the actual body in its actual motions relative to the centre of mass. It is clear without a formal proof that the same independence will hold for rates of change of angular momenta since it holds for the momenta themselves. Hence, adhering to our co-ordinate axes GABC moving parallel to themselves with their origin G always at the centre of mass of the body, we can now write general equations of motion of the rigid body. For each product, mass of a particle into its acceleration, corresponds to a force, and each change of linear momentum to an impulse. Thus, if the force components are represented by $X$, $Y$, and $Z$ acting at the point $x$, $y$, $z$, we may write, by the results of articles 307-309, the following equations:—

$$\Sigma X dt = M du,\ \Sigma Y dt = M dv,\ \Sigma Z dt = M dw \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$\Sigma(X dx + Y dy + Z dz) = dT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

$$\Sigma(Zy - Yz)dt = dh_x,\ \Sigma(Xz - Zx)dt = dh_y,\ \Sigma(Yx - Xy)dt = dh_z \quad (8),$$

$$\Sigma X = M\dot{u},\ \Sigma Y = M\dot{v},\ \Sigma Z = M\dot{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

$$\Sigma(Zy - Yz) = \dot{h}_x,\ \Sigma(Xz - Zx) = \dot{h}_y,\ \Sigma(Yx - Xy) = \dot{h}_z \quad . \quad . \quad . \quad (10),$$

$$\left.\begin{aligned} \Sigma(Zb - Yc) &= \frac{d}{dt}(A\omega_x - F\omega_y - E\omega_z) \\ \Sigma(Xc - Za) &= \frac{d}{dt}(-F\omega_x + B\omega_y - D\omega_z) \\ \Sigma(Ya - Xb) &= \frac{d}{dt}(-E\omega_x - D\omega_y + C\omega_z) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where $a$, $b$, and $c$ are the co-ordinates with respect to the axes GABC of the point of application of the force components $X$, $Y$, $Z$.

If we perform the differentiations indicated by $\dot{h}_x$ in (10), using the values from (5) of article 309, and remembering that $a$, $b$, and $c$ are variables as shown in (1) of article 307, we obtain

$$\left.\begin{aligned} \Sigma(Zy - Yz) = M(\dot{v}\bar{y} - \dot{v}\bar{z}) + A\dot{\omega}_x - F\dot{\omega}_y - E\dot{\omega}_z \\ - D(\omega_y^2 - \omega_z^2) - (B - C)\omega_y\omega_z + F\omega_z\omega_x - E\omega_x\omega_y \end{aligned}\right\} \quad (12).$$

The other two equations for $\dot{h}_y$ and $\dot{h}_z$ could then be written by symmetry. The cumbrousness of these expressions (due to the fact that

*A*, *B*, *C*, *D*, *E*, *F*, are varying) shows that it is usually desirable to adopt moving axes so that the above coefficients become constants.

Thus, since we have already seen that the translation of the centre of mass and the rotation about any axis through it are independent, we may use for the latter Euler's equations and for the former the equations for a particle. Another system is given in the next article.

**311. Hayward's Equations.**—We have just been discussing the equations of motion with respect to axes with origin fixed in the moving body and at its centre of mass but with directions always parallel to those of the axes fixed in space.

Let us now refer the general motion of a rigid body to rectangular axes whose *origin is fixed in space* but whose directions vary by *rotating* at angular velocities $\theta_1$, $\theta_2$, and $\theta_3$ about their *own instantaneous positions*, being, in fact, the system usually referred to as 'moving axes.'

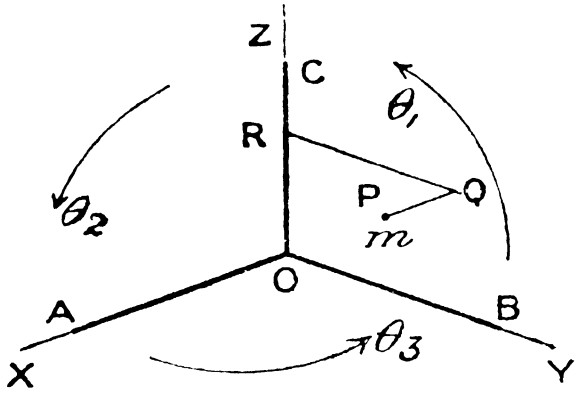



FIG. 126. LINEAR MOMENTA AND MOVING AXES.

We have already found (articles 290 and 294) the form assumed in this case by the expressions for the angular momenta and their rates of change. And it might be inferred that precisely similar expressions would hold for linear momenta. It is, however, desirable to give here an independent proof of this.

At the instant in question let the moving axes OABC coincide with the fixed axes OXYZ as shown in Fig. 126, and consider the point P of co-ordinates $x$, $y$, $z$ with respect to OABC and velocity components $u$, $v$, $w$ in the directions of the moving axes, but reckoned with respect to the fixed origin O. Then the velocity $u$ is the algebraic sum of

(1) that of P relative to Q;

(2) that of Q relative to R; and

(3) that of R relative to O. Thus, writing these values in from the figure and by symmetry the similar expressions for $v$ and $w$, we find

$$\left.\begin{aligned} u &= \dot{x} - y\theta_3 + z\theta_2 \\ v &= \dot{y} - z\theta_1 + x\theta_3 \\ w &= \dot{z} - x\theta_2 + y\theta_1 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1).$$

If we now wish to pass from the velocities to the corresponding linear accelerations $a$, $b$, and $c$, parallel to OABC but with respect to the fixed origin O, let OR, RQ, and QP in Fig. 126 now represent to some scale the velocities $w$, $v$, and $u$. Then, to the same scale, the component velocities of P will represent the accelerations required. Hence we have, by substitution in (1),

$$\left.\begin{aligned} a &= \dot{u} - v\theta_3 + w\theta_2 \\ b &= \dot{v} - w\theta_1 + u\theta_3 \\ c &= \dot{w} - u\theta_2 + v\theta_1 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2).$$

If now we suppose a particle of mass $m$ to have these accelerations, we may sum the products of mass into acceleration over the whole body and equate to the corresponding sum of force components $X$, $Y$, and $Z$. Thus

$$\Sigma X = \Sigma ma = \Sigma m\dot{u} - \Sigma mv\theta_3 + \Sigma mw\theta_2.$$

Or, if the sums of the force components are denoted by $U$, $V$, and $W$ and the linear momenta by $p_1$, $p_2$, and $p_3$, we have

$$\left.\begin{aligned} U &= \dot{p}_1 - p_2\theta_3 + p_3\theta_2 \\ V &= \dot{p}_2 - p_3\theta_1 + p_1\theta_3 \\ W &= \dot{p}_3 - p_1\theta_2 + p_2\theta_1 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3).$$

Let us now quote here the equations (1) from the beginning of article 294, expressing the relations between the torques $L$, $M$, and $N$ about moving axes and the corresponding angular momenta $h_1$, $h_2$, and $h_3$.

$$\left.\begin{aligned} L &= \dot{h}_1 - h_2\theta_3 + h_3\theta_2 \\ M &= \dot{h}_2 - h_3\theta_1 + h_1\theta_3 \\ N &= \dot{h}_3 - h_1\theta_2 + h_2\theta_1 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4).$$

It is now seen that these two sets of relations between forces and linear momenta in (3) and torques and angular momenta in (4) are precisely alike in form. Further, by the independence we have already seen to exist between the translations of the centre of mass of a rigid body and its rotations about any axis through the centre of mass, equations (4) still hold when the body has a motion of translation *provided the origin of these axes now coincides with the moving centre of mass of the body.*

The equations of motion in the forms shown by (3) and (4) were first given by R. B. Hayward, F.R.S. (see *Camb. Phil. Trans.*, Part I. vol. x., 1856).

Thus, in the preceding and present article, general equations have been obtained for the translations and rotations possible and referred in each article to different systems of moving co-ordinate axes. It is not, however, by any means necessary or desirable to use either system in its entirety for any given problem. On the contrary, we may often with advantage treat the translation of the centre of mass as though the whole body were concentrated there and the rotations by equations involving moving axes (say Euler's) as though the centre of mass were at rest. This is illustrated in the next article.

**312. Motions of a Quoit.**—Following the treatment of Tait, let us now consider the motions possible to a quoit when thrown. Let the moment of inertia about OA, the axis of figure, be $A$, and $B$ and $C$ those about any two rectangular axes OB and OC in the plane of the ring. Then obviously $A > B = C$, and by Euler's equations we find

$$A\dot{\omega}_1 = 0, \text{ or } \omega_1 \text{ is constant} \quad \ldots\ldots \quad (1),$$

$$B\dot{\omega}_2 - (B - A)\omega_3\omega_1 = 0 \quad \ldots\ldots \quad (2),$$

$$B\dot{\omega}_3 - (A - B)\omega_1\omega_2 = 0 \quad \ldots\ldots \quad (3)$$

For brevity write

$$\frac{A-B}{B}\omega_1 = n, \text{ a constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Then (2) and (3) become respectively

$$\dot{\omega}_2 + n\omega_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and

$$\dot{\omega}_3 - n\omega_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

From (5) we obtain

$$\omega_3 = -\dot{\omega}_2/n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

and this differentiated and substituted in (6) gives

$$\ddot{\omega}_2 + n^2\omega_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

But the solution of this may be written

$$\omega_2 = \omega_0 \cos(nt+\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

the corresponding angular acceleration being

$$\dot{\omega}_2 = -n\omega_0 \sin(nt+\phi) \quad . \quad . \quad . \quad . \quad (10),$$

in which $\omega_0$ and $\phi$ depend upon the initial conditions of the throw.

This value of $\dot{\omega}_2$ put in (7) gives for the other angular velocity

$$\omega_3 = \omega_0 \sin(nt+\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Thus, the resultant of $\omega_2$ and $\omega_3$ about the perpendicular axes in the plane of the quoit is an angular velocity $\omega_0$ about an axis OD in the plane of BOC and making at time $t$ the angle $(nt+\phi)$ with OB, as shown in Fig. 127.

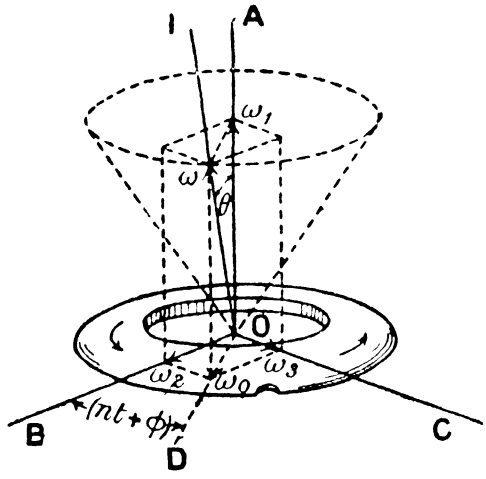



FIG. 127. MOTIONS OF A QUOIT.

Hence, compounding this angular velocity $\omega_0$ with $\omega_1$, we find the magnitude $\omega$ and axis OI of the instantaneous angular velocity to be given by

$$\omega^2 = \omega_1^2 + \omega_0^2 \quad . \quad . \quad . \quad (12),$$

and

$$\tan\theta = \omega_0/\omega_1 \quad . \quad . \quad . \quad (13),$$

where $\theta$ is the angle between OA and OI.

Thus the instantaneous axis of rotation describes with respect to the quoit a right circular cone of semi-vertical angle $\theta$ about the axis of figure OA. This is the body cone shown dotted in the figure. Further, as shown by equations (9) and (11), this cone is described with angular velocity $n$, in the same direction as that in which the body is rotating. In other words, the *body cone rolls externally upon the space cone*, their common vertex being the centre of mass, which is the origin of co-ordinates, and their line of contact being the instantaneous axis of rotation OI.

So far we have dealt with the rotations only as though translations were absent. But, as shown in article 131, the most general motion of a rigid body may be represented as the rolling of a cone, fixed relatively to the body, on a cone which *moves* in space, the vertices of the cones being common. Thus we have further to specify the motion of the common vertex. But in the present case this vertex is the centre of

mass of the quoit. Hence, apart from the resistance of the air, the trajectory of this point is easily seen to be a parabola whose elements depend upon the angle of elevation and the velocity at the instant of the throw.

We have further to specify the space cone on which the body cone rolls. In the present very simple problem, since $B=C$, it might be inferred from symmetry that the space cone is a right circular cone. But the full treatment of the space cone is quite beyond the scope of the present work; indeed its equation cannot, in general, be found.

In the case of the earth there is a shift of the instantaneous axis within it analogous to the description of the body cone just noticed for the quoit. This shift of the earth's instantaneous axis of rotation gives the phenomenon known as the change of latitude. The value of $(A-B)/B$ for the earth is of the order 0·00328, which would give a period of 305 days to this change of latitude. But, owing to imperfect rigidity of the earth, this period is about 428 days. (J. H. Jeans' *Theoretical Mechanics*, p. 310, Boston, 1907.)

When a body is thrown into the air spinning, the trajectory of its centre of mass is a parabola, and the motion of the body about an axis through that centre is as though that centre were fixed. This leads us to Poinsot's discussion of the motion of a rigid body *about a fixed point* under no forces, in which an ellipsoid is regarded as rolling in contact with a plane, the locus on the ellipsoid of this point of contact being called the *polhode*, the locus on the tangent plane of the same point being called the *herpolhode*.

When the body cone closes to a single line, it is evident that the space cone is reduced to a single line also. This illustrates the case with which the study of rotation naturally commences, namely, that in which the axis of rotation is fixed both in the body and in space.

Simple problems like the rolling of spheres on rough planes under the action of forces passing through the centre are easily solved by the principles already illustrated. For the more complicated cases of the motions of rigid bodies in three dimensions the reader is referred to the classic treatises by Routh, and to Professor A. G. Webster's recent *Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies* (Leipzig, 1904).

## EXAMPLES—LIX.

1. Obtain an equation of motion for a rigid body, using axes whose origin moves with it while the directions of the axes are unchanged. What drawback has this form of equation?
2. Derive the set of six relations for the motion of a rigid body known as Hayward's equations. What special advantages do they present?
3. A body of revolution, say a cylinder or disc, is thrown into the air with a spin about its geometrical axis; discuss its subsequent motion.
4. What do you understand by the terms *polhode* and *herpolhode*?

# PART IV.—STATICS

## CHAPTER XV

### STATICS OF PARTICLES

**313. Forces on Body devoid of Acceleration.**—When force is defined as the product of mass and acceleration, we have to inquire how the forces are to be estimated respecting a body devoid of acceleration, *i.e.* either at rest or in uniform motion. There is usually no difficulty in this, for it is generally easy to separate the conditions affecting a body into two or more divisions such that certain accelerations correspond to one or more of those divisions. Take, for example, the case of a body of mass $m$ resting on the top of a table. Then, relative to the earth, we have no acceleration of the body. But we can separate the conditions under which the body is placed into two divisions :— (1) the proximity of the earth to the body ; (2) the contact of the table top with the body. Now we know that, with the first set of conditions only, the acceleration of the body would be vertically downwards of value $g$ (about 981 cm./sec.$^2$ or 32·2 ft./sec.$^2$). If we now regard the equilibrium resting state of the body as the resultant of two accelerations opposite but numerically equal, we see that the reaction $R$ of the table must correspond to an acceleration $-g$, *i.e.* vertically upwards. We may accordingly write

$$R = -mg = -W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $W$ is the weight of the body.

Or, generally, if a body is at rest we can divide the conditions under which it rests into two sets or divisions corresponding to opposite accelerations $a$, $a'$ and forces $F$, $F'$. Then we have

$$a = -a' \text{ and } F = ma = -F' \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Or, in other words, one of the forces said to be acting upon the body at rest is gauged by *minus* the product, mass into the acceleration, which would occur in it, *were this force removed.*

Again, we may express the circumstances of the equilibrium in the above cases in the following obvious manner :—

$$W + R = 0 \text{ and } F + F' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and this is in accord with the general custom for statical problems.

It may be noted here that a body may be instantaneously at rest though not in equilibrium, or instantaneously in equilibrium without being at rest. On the other hand, it may be for a finite time at rest in equilibrium. A particle at the end and at the middle of rectilinear

simple harmonic vibrations illustrates the first two, and, when stopped, the last.

**314. Composition and Resolution of Forces.**—Since a force is the product of the *scalar* quantity mass and the *vector* quantity acceleration, it is itself a *vector* quantity of the *same* direction as the acceleration. Hence the composition of forces is simply an example of vector addition (see articles 14-16 and 23-25). Thus the parallelogram, triangle, and polygon of forces need no further proof. We may, however, note a few results of vectorial addition in typical cases.

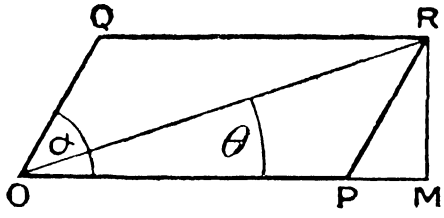



FIG. 128. PARALLELOGRAM OF FORCES.

If two forces $P$ and $Q$, represented in Fig. 128 by OP and OQ, act on a particle at the point O, their resultant is represented by the diagonal OR of the parallelogram OPRQ. It is easily shown that the magnitude and direction of the resultant are given by

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where $\alpha$ is the angle between $P$ and $Q$, $\theta$ that between $P$ and $R$. For $\alpha$ is the supplement of OPR, hence the cosines of these two angles differ in algebraic sign only; this gives (4). Equation (5) follows from $\tan\theta = \mathrm{MR} \div (\mathrm{OP} + \mathrm{PM})$. We may also note the useful though not independent relation

$$R = P\cos\theta + Q\cos(\alpha - \theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

If instead of compounding $P$ and $Q$ into their resultant $R$ we have to resolve $R$ into a pair of components, it is soon obvious there are an infinite number of ways of doing so. To make the problem of resolution definite we must have the angles fixed; thus if $\alpha$ and $\theta$ are given as well as $R$ we can easily find $P$ and $Q$, which are now determinate. The preceding relations are not, however, suitable for this inverse process. We may accordingly replace them by

$$\frac{P}{\sin(\alpha - \theta)} = \frac{Q}{\sin\theta} = \frac{R}{\sin\alpha} \quad . \quad . \quad . \quad . \quad . \quad (7),$$

obtained by applying to the triangle OPR the constancy of the ratio (side ÷ sine of opposite angle).

Of course, both composition and resolution may be performed graphically when preferred and if the utmost accuracy is not desired. In compounding graphically the forces applied at O, we may quite legitimately draw OP, PR, and OR instead of the whole parallelogram. But it should be noted that OR would need *shifting parallel* to itself to be the resultant of OP and PR, *acting along those lines* instead of both at O. See article 398.

It is seen by the equations (7) that, when a given force $R$ is re-

solved into two components $P$ and $Q$, the magnitude of each one depends upon *both* the angles which they make with $R$.

Thus, in order to give definiteness to the convenient phrase, 'component in a given direction,' we must have some convention as to the direction of the other component. The ordinary convention is this, that the two components are at right angles to one another when nothing is said to the contrary. Hence if the horizontal eastward component of a certain horizontal force is said to be four dynes, it is understood that the other component is along the north and south line. Or, generally, if the component parallel to the axis of $x$ is $X$, it is understood that the other of the two components is taken parallel to the axis of $y$, if the original force is in the plane of $xy$. This convention is adopted because usually simplicity results from resolving forces into directions parallel to rectangular co-ordinate axes. Hence, if $\theta$ is the angle between the original force $F$ and a component $P$, we have the relation

$$P = F\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

the other component being understood to make an angle $\pi/2 - \theta$ with the force $F$ and on the other side of it.

### EXAMPLES—LX.

1. Accepting the definition that force is the product of mass into acceleration, how is it possible to speak of a force or forces acting on a body at rest? Give examples, making your meaning quite clear.
2. Obtain analytical expressions for the magnitude and direction of the resultant of two specified forces.
3. Show how forces may be resolved each into two components. When speaking of one component of a force, what restriction is essential about the other component?
4. Find a pair of forces of fixed directions at the angle $a$, acting at a point and so related that as one force $P$ remains of constant value and the other $Q$ grows from zero, the resultant $R$ at first diminishes in magnitude, then reaches a minimum, and thereafter increases indefinitely.
5. Give a graphical construction for the composition of three or more forces, and write its analytical equivalent.

**315. Conditions of Equilibrium.**—If we have several forces acting on a particle, it is obvious that the condition of equilibrium is that their resultant vanishes. But the resultant is represented by the line which closes the polygon, whose sides represent the forces as if placed end to end. Hence for equilibrium this closing side must vanish, or the polygon of forces must be itself closed.

We may also give the same ideas in analytical form. Thus, let forces $F_1$, $F_2$ . . . be applied to the particles in the plane of $xy$, their components being $X_1$, $Y_1$, $X_2$, $Y_2$, etc. Then, for equilibrium we obviously have the conditions

$$\Sigma X = 0, \ \Sigma Y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

since the resultant $R$ is given by

$$R^2=(\Sigma X)^2+(\Sigma Y)^2,$$

and accordingly only vanishes when (1) is satisfied.

**316. Inclined Plane.**—Let us now consider the equilibrium of a particle on a rough inclined plane under the action of a force at an angle with the plane. Let the weight of the particle be $W$, the coefficient of friction between it and the plane $\mu=\tan\beta$, the inclination of the plane to the horizontal $\alpha$, the angle between the force $P$ and the plane $\theta$. Take the axis of $x$ down the plane, the origin at the particle, call the normal reaction of the plane $N$, and suppose the equilibrium to be such that the particle is just on the point of moving down the plane. Then the frictional reaction on the plane is upwards and equals its limiting value $\mu N$. The forces mentioned are shown in Fig. 129.

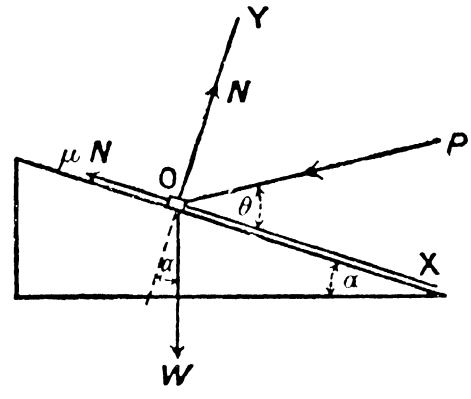



FIG. 129. EQUILIBRIUM ON ROUGH INCLINE.

Then on reference to the figure and using the conditions of equilibrium as expressed in (1), we have

$$\Sigma X=W\sin\alpha-\mu N-P\cos\theta=0 \quad . \quad . \quad . \quad . \quad . \quad (2),$$

$$\Sigma Y=-W\cos\alpha+N-P\sin\theta=0 \quad . \quad . \quad . \quad . \quad . \quad (3),$$

two equations from which to find $P$ and $N$.

Eliminating $N$ between the two equations we find

$$P=W\frac{\sin(\alpha-\beta)}{\cos(\theta-\beta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Then, substituting for $P$ in (2) and (3), we have

$$N=W\left(\cos\alpha+\frac{\sin\theta\sin(\alpha-\beta)}{\cos(\theta-\beta)}\right) \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Equation (4) shows that for a minimum *supporting* value of $P$ we must have $\theta-\beta=0$, when

$$P\min.=W\sin(\alpha-\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

If the particle is to be on the point of moving *up* the plane instead of down, we must reverse the sign of $\mu$ in (2). We then obtain the other limiting case in which the force, now distinguished by a dash, is

$$P'=W\frac{\sin(\alpha+\beta)}{\cos(\theta+\beta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

And now for the minimum value of $P'$, which is on the point of dragging the particle up, we find $\theta+\beta=0$, *i.e.* $P'$ is pushing at angle $\beta$ below OX or pulling at the angle $\beta$ above $\mu N$ in the figure, the corresponding value of the force being

$$P'\min.=W\sin(\alpha+\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

**317. The Wedge.**—For $P'$ horizontal put $\theta=\alpha$ in equation (7), and we obtain

$$P'=W\tan(\alpha+\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Further, on putting two such planes of inclination $\alpha$ base to base, we pass to a wedge of angle $2\alpha=\gamma$ say, the total horizontal force being

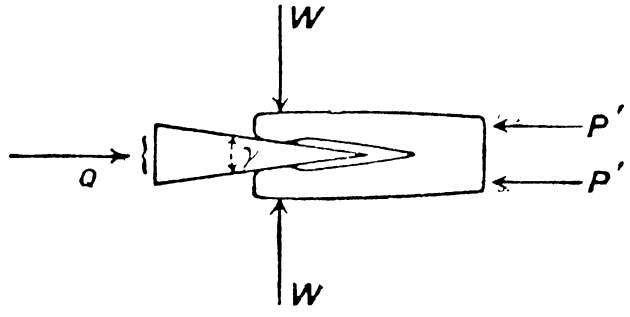



FIG. 130. THE WEDGE.

$2P'=Q$ say. We have then the state of things represented in Fig. 130. The relation between $Q$ and $W$ is obviously represented by

$$\left.\begin{aligned} P'&=W\tan(\alpha+\beta),\\ \text{or}\qquad Q&=2W\tan\left(\frac{\gamma}{2}+\beta\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Of course, if the friction is negligible, this reduces to

$$Q=2W\tan\gamma/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

or if the angle is small

$$Q=W\gamma \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

If the angle is not small, and the resistances are taken $R$, $R$ normally to the inclined forces of the wedge, we have instead of (11)

$$Q=2R\sin\gamma/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

as may be seen at once from the triangle of forces, which is a figure like the wedge shown but turned through a right angle, the base representing $Q$ and the sides the equal resistances $R$ and $R$.

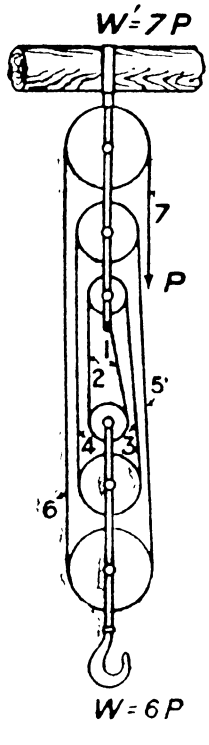



FIG. 131. TACKLE OF SINGLE MULTIPLIED CORD.

**318. The Multiplied Cord or Tackle.**—The combination of several plies of a cord, rope, or chain with pulley blocks is called a tackle or purchase. The pulleys are of great practical importance in certain cases as they lessen friction at the places where the directions of the cord change considerably, but the essential efficacy of the contrivance lies in the reduplication of the cord or its disposition in repeated plies or parts side by side, and the pulleys are in some cases omitted with advantage. Many arrangements and combinations may be made and are in use. It will suffice here to consider two illustrative types.

Take first the system represented in Fig. 131. In

this tackle, since only one cord is used and pulleys are provided at the bends, the tension will be practically the same throughout if we regard friction and stiffness of the cord as negligible. Hence, the resistance $W$, which may be balanced by the applied force $P$ at the free end of the hauling part of the rope, is given by the expression

$$W = nP \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $n$ is the number of plies supporting the *lower* or running block. Thus in the figure $W = 6P$. If we write $W'$ for the load put by the upper or fixed block on the beam supporting it, obviously

$$W' = W + P = n'P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where $n'$ is the number of plies hanging from the fixed block, in this case seven. The $W$ is, of course, the total resistance which the plies of the cord balance, and includes that of the running block itself. Thus if this block has the weight $w$, only an additional load $W - w$ could be supported by the force $P$.

Consider now a system with several cords and several running or movable pulleys. This is shown in Fig. 132, from which it is seen that the first movable pulley is supported by two plies of the cord to which the force $P$ is applied. Hence the cord from it has a tension $2P$. But this cord has two plies which support the next movable pulley, which can accordingly exert on its cord a tension $4P$. Finally, as shown, two plies of this rope support the weight $W$, which is accordingly $8P$. Thus, as we take for the number of separate cords and movable pulleys 1, 2, 3, . . . $n$, we see that the ratios, weights which can be supported to the force $P$ applied, are given by $2^1, 2^2, 2^3, \ldots 2^n$. Hence the general relation for this system of plies and pulleys

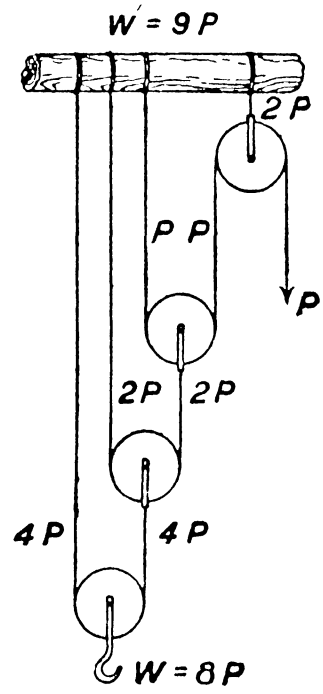



FIG. 132. TACKLE OF SEVERAL MULTIPLIED CORDS.

$$W = P.2^n \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where $n$ is the number of separate cords and of movable pulleys. If here, again, $W'$ is the weight put on the beam by the upper or fixed block, we have

$$W' = W + P = P(2^n + 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

which is the relation required if the system is inverted.

Of course, if the weights of the pulleys themselves are comparable with the tensions in the plies they must be taken into consideration. Thus if $w_1$ is the weight of the pulley supported by the first cord of tension $P$, the tension of the second cord is $2P - w_1$. If the pulley supported by this second cord has weight $w_2$, the tension of the third cord is $2(2P - w_1) - w_2$, and double this value is available to support the weight $w_3$ of the third pulley and the load hung on it.

EXAMPLES—LXI.

1. Express analytically and graphically the conditions for the equilibrium of a particle under a number of coplanar forces.
2. Find the force required to support a body on a rough incline and plot force against inclination, thus confirming the analytic result that the minimum result is $W \sin(\alpha - \beta)$, where $W$ is the weight of the body, $\alpha$ the inclination of the plane, and $\tan\beta$ the coefficient of friction.
3. Obtain the ratio of resistances to force in the case of a wedge (i) when the resistances are in opposite directions along the same line, and (ii) when they are each normal to the respective faces of the wedge.
4. Explain why a wedge when driven into wood does not slip out again. Give a numerical instance, and work it out to support your explanation.
5. Sketch two arrangements of 'tackle' or 'purchase,' and find in each case the relation of load to applied force, allowances being made for the weights of the pulleys.

**319. Work for given Force and Displacement.**—Consider the work done on a particle when it has a displacement OD under the action of a force $P$ inclined at an angle $\phi$ with OD as shown in Fig. 133. From $P$ let fall PQ perpendicular to OD produced, and from D the perpendicular DE on OP. Then, if OP represents to scale the force $P$, OQ will represent to scale the component force $Q$ along OD, the actual displacement. Also OE represents the component displacement in the direction of the force $P$. Now, by a definition in article 212, work is the product force into displacement in the direction of the force.

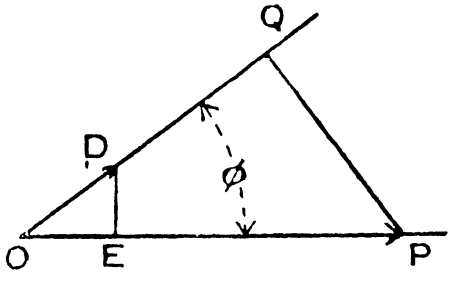



FIG. 133. WORK FOR GIVEN FORCE AND DISPLACEMENT.

Hence we have three forms for the work $W$ in question, viz.

$$W = P.OE = Q.OD = P.OD \cos\phi \quad . \quad . \quad . \quad . \quad . \quad (1).$$

These are easily seen to be identical, for $OE = OD \cos\phi$ and $Q = P\cos\phi$. If the component displacement in the direction of either force be denoted by the corresponding small letter, we may state the above results still more compactly thus:—

$$W = Pp = Qq = Pq\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

**320. Resultant Work is Algebraic Sum of Components.**—Let now a particle have a displacement OS while under the action of forces, say $A$, $B$, and $C$, whose resultant is $R$. Estimate the works of each of these forces as in the last article. Then it may be readily seen that the work of the resultant force $R$ is the algebraic sum of the works of the component forces. Thus, let the forces $A$, $B$, and $C$ be

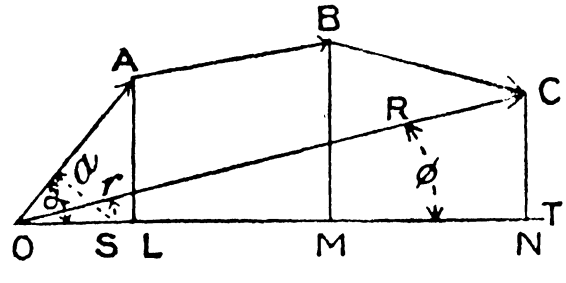



FIG. 134. RESULTANT WORK IS SUM OF COMPONENTS.

represented to scale, end to end, by OA, AB, and BC, their resultant by $OC=R$, and the displacement by OS along OT. Let fall from A, B, and C the perpendiculars AL, BM, and CN on OT, and denote by $\alpha$, $\beta$, $\gamma$, and $\phi$ the angles which the forces $A$, $B$, $C$, and $R$ make with OT. Then, by the geometry of the figure, we have

$$OL+LM+MN=ON,$$

or
$$A\cos\alpha+B\cos\beta+C\cos\gamma=R\cos\phi.$$

Multiply throughout by $OS=s$ say, giving

$$A(s\cos\alpha)+B(s\cos\beta)+C(s\cos\gamma)=R(s\cos\phi).$$

But the quantities in brackets are the component displacements in the directions of the various forces. Hence, denoting these by the corresponding small letters, we may write

$$Aa+Bb+Cc=Rr \quad . \quad . \quad . \quad . \quad (3).$$

Or, in a general case, where the forces are $P_1$, $P_2$, etc., and the component displacements $p_1$, $p_2$, etc., we have

$$P_1p_1+P_2p_2+\ldots=\Sigma Pp=Rr \quad . \quad . \quad . \quad . \quad (4),$$

for what holds for these forces is obviously valid for any number.

It may be noted that the forces $A$, $B$, $C$ or $P_1$, $P_2$, etc., are not restricted to a plane but may be distributed in any way in solid space. Further, for these results to hold each displacement must be small enough for the forces to be practically constant in magnitude and direction throughout it.

**321. Virtual Work is Zero for Equilibrium.**—Suppose a single free particle at the point O is in equilibrium under the action of given forces $P_1$, $P_2$, etc., and let it be *imagined* to have an infinitesimal displacement $ds$, whose components or resolved portions in the directions of the forces are respectively $dp_1$, $dp_2$, etc.

Then $ds$, $dp_1$, $dp_2$, etc., are called *virtual displacements* of the particle in the given directions; also, the products $P_1dp_1$, $P_2dp_2$, etc., are called the *virtual works* of the corresponding forces.

(Sometimes the phrases virtual *velocities* and virtual *moments* are used instead of the above.)

If, for an instant, we imagined the forces to have a resultant $R$, along whose direction the virtual displacement was $dr$, we should have from (4)

$$P_1dp_1+P_2dp_2+\ldots=\Sigma Pdp=Rdr \quad . \quad . \quad . \quad . \quad (5).$$

But, since for *equilibrium* $R$ must be *zero*, it follows that the *total virtual work is then zero* for *any* virtual displacement. Or, in symbols,

$$\Sigma Pdp=0 \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Conversely, if the total virtual work vanishes for any virtual displacement, the particle is in equilibrium.

This dual statement expresses what is called the *Principle of Virtual Work* as applied to a single free particle. The principle also applies to a particle on a curve or surface (as we shall presently see), and even to rigid bodies, as we shall note in a later chapter.

Thus, for a particle in equilibrium on a smooth curve or a smooth

surface, there would be the reaction $Q$ of the curve or surface in addition to the other impressed forces $P_1$, $P_2$, etc. Hence by (6) we should have for any small displacement

$$\Sigma Pdp + Qdq = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

If, however, we take $ds$ along the curve or in the surface, $dq$ vanishes, because $Q$ and $ds$ are at right angles, and (7) accordingly reduces to (6).

Conversely, consider the particle constrained to move along a curve, the virtual work for a tangential displacement being zero. Then, the resolved part in that direction of the resultant of the impressed forces is zero also. Hence the particle is in equilibrium.

Also, for a particle constrained to move on a smooth surface, if the virtual works for any *two* displacements, *not* in the same straight line, are *each* zero, the particle is in equilibrium. Because in that case obviously the resultant force had no component in either of those directions, and accordingly vanished.

If the curve or surface were rough, the frictional forces would need taking into account. And usually the principle of virtual work, now under discussion, is not then of any service, it being as hard to find those frictional forces as to solve the whole problem by some other method.

A more general statement of the principle of virtual work is the following, quoted from Todhunter's *Statics*:—

'If any system of particles is in equilibrium, and we conceive a displacement of all the particles which is consistent with the conditions to which they are subject, the sum of the virtual moments (*i.e.* works) of all the forces is zero, whatever be the displacement. And conversely, if this relation hold for all the virtual displacements, the system is in equilibrium.'

It is easily seen that this form applies to all systems of multiplied cords or tackles, and gives the results already obtained.

## EXAMPLES—LXII.

1. When the displacement of a point is inclined to a force acting upon it, as in the case of a canal boat towed by a horse, find several equivalent expressions for the work done.
2. Prove that the resultant work is the algebraic sum of the component works in the case of a particle displaced under the action of various forces.
3. State clearly the principle of *Virtual Work*, and establish it for a particle.
4. 'Enunciate the principle of Virtual Work. If a material system is in equilibrium under the action of gravity and smooth constraints, under what condition will it rest in all positions into which it can be placed without violating the constraints? Apply this to the following case:—$A$ is a fixed smooth pulley; a light flexible cord passing over the pulley has a freely hanging mass of weight $P$ attached to one end and a ring of weight $W$ to the other; if this ring is constrained to move along a smooth fixed wire in the form of a certain conic having $A$ for focus, the system will rest in all positions.'

(LOND. B.SC., PASS, APPLIED MATH., 1905, I. 6.)

**322. Equilibrium of Bent Cord.**—If between two points P and P′ on a stretched cord there are no forces applied (except the tensions), it is almost obvious that this portion of the cord would remain straight and of constant tension throughout. This has already been assumed in dealing with the multiplied cords and tackles. If, on the other hand, isolated forces occur at various points P, P′, etc., of a stretched cord, it is evident that the cord will change its direction or the magnitude of its tension at each of these points, remaining straight and of constant tension between them. The equilibrium at each of these points may obviously be dealt with by the polygon of forces.

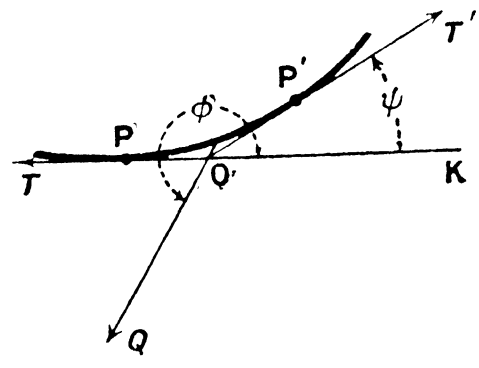



FIG. 135. EQUILIBRIUM OF BENT CORD.

If, however, a fine inextensible flexible cord is subject to forces $Q$ *per unit length* between the points P and P′, it is obvious that the tensions $T$ and $T'$ at the points must differ in magnitude or direction or both. And the precise mode of these differences requires examination.

Thus, referring to Fig. 135, and resolving along and perpendicular to OK, we have

$$\left.\begin{aligned} T'\cos\psi - T + \mathrm{PP'}.\Sigma Q\cos\phi = 0 \\ T'\sin\psi + \mathrm{PP'}.\Sigma Q\sin\phi = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1),$$

the sign of summation being introduced before $Q$ to cover the cases where various forces occur, of which $Q$ is the type and all are stated *per unit length*. The above equations are written in the form which applies when the angle $\psi$ between the tangents at P and P′ is finite. Now let P′ and P approach and the angle become the infinitesimal one $d\psi$, then $\cos d\psi$ is unity and $\sin d\psi$ becomes $d\psi$. Also $T'-T$ is $dT$ and PP′ is the infinitesimal $ds$. Thus equations (1) transform into

$$\left.\begin{aligned} \frac{dT}{ds} + \Sigma Q\cos\phi = 0 \\ \frac{d\psi}{ds} + \frac{\Sigma Q\sin\phi}{T} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2).$$

These are the general differential equations for a cord in equilibrium under coplanar forces. They may be expressed in words thus:—

*Space rate of change of tension* equals tangential component of forces.

*Curvature* equals quotient of normal component of forces divided by tension. It should be remembered that the forces are estimated *per unit length*. We easily see from (2) that when $Q$ vanishes $dT/ds=0$, and $d\psi/ds=0$, or $\rho=\infty$, that is, the cord is of constant tension and straight.

**323. Cords wrapped on Curves.**—Let us now suppose the forces $Q$

to be due to the reaction of a curved surface on which the stretched cord is wrapped. At first suppose the surface to be perfectly smooth so that the angle $\phi$ becomes $3\pi/2$, and the force $Q$ may now be called $N$ since it is normal to the surface.

Thus equations (2) reduce to the following:—

$$\frac{dT}{ds}=0 \text{ and } \frac{d\psi}{ds}=\frac{N}{T}=\frac{1}{\rho} \quad \ldots\ldots \quad (3).$$

These show that round a smooth curve the tension is constant and that the normal reaction $N$ is $T/\rho$, where $\rho$ is the radius of curvature.

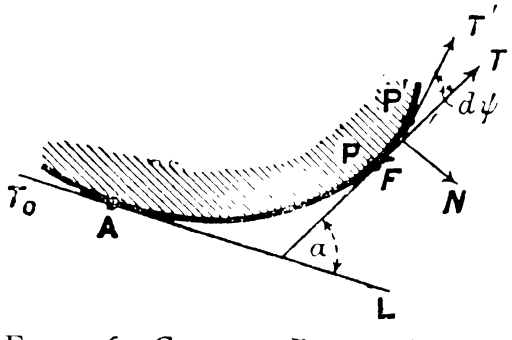



FIG. 136. CORD ON ROUGH CURVE.

Consider now a rough surface, and denote the normal reaction by $N$ and the tangential frictional force by $F$ as shown in Fig. 136. Suppose the cord to be on the point of slipping in the direction of $T'$, the tension applied at P'. Then $F=\mu N$ numerically and its $\phi=\pi$, the $\phi$ for $N$ being still $3\pi/2$ as for the smooth curve. Hence for the rough surface equations (2) give

$$\frac{dT}{ds}=\mu N \text{ and } \frac{d\psi}{ds}=\frac{N}{T}=\frac{1}{\rho} \quad \ldots\ldots \quad (4).$$

Eliminating $ds$ between the two equations of (4) we find

$$dT/d\psi=\mu T. \quad \ldots\ldots \quad (5).$$

Thus, if the tension is $T_0$ at $A$ and $T$ at $P$, the angle between these directions being $\alpha$, we have by integration of (5) between these limits

$$\int_{T_0}^{T}\frac{dT}{T}=\mu\int_0^{\alpha}d\psi.$$

Whence $\log_e T-\log_e T_0=\mu\alpha=\log_e(T/T_0)$.

Thus, on raising $e$ to the powers indicated by the two sides of this equation, we have

$$\left.\begin{aligned} e^{\mu\alpha}&=T/T_0, \\ T&=T_0e^{\mu\alpha} \end{aligned}\right\} \quad \ldots\ldots \quad (6).$$

or

It is noteworthy that the ratio of the tensions is independent of the *form* of the curved surface, but depends only on the coefficient of friction and the *total* angle involved.

Also, owing to the exponential form, the ratio of the tensions increases very quickly with the angle $\alpha$. Thus a turn and a half of a cord round a cylinder will give a threefold tension if the coefficient of friction is slightly over one-tenth (say 0·1166). And three turns would give a ninefold tension. This explains the possibility of hauling against a great resistance $T$ with a rope which has a couple of turns round a capstan, the slack of the rope being taken off with a small but

indispensable tension $T_0$. Thus, with a rope round a wooden post, a single complete turn may yield a twenty-fold tension.

**324. Uniform Cord under Gravity.**—Let us now consider the equilibrium of a uniform, inextensible, and flexible cord under gravity. If attached at one point only, it is clear that its position of equilibrium is that of the vertical line below that point, the tension at each section being the total weight of the portion of cord below that level.

If, however, the cord is attached at two points not in the same vertical line, the cord will hang in a curve which needs investigating. There are various problems to be solved and different ways of attacking them. We will first apply the methods already used and then pass to others which are desirable for the sake of the further light they throw upon the subject.

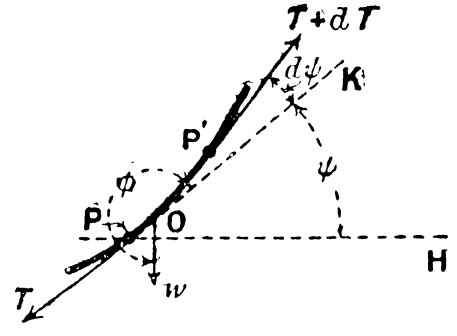



FIG. 137. UNIFORM CORD UNDER GRAVITY.

Let the cord have weight $w$ per unit length, and consider an element of length $PP'=ds$ inclined at an angle $\psi$ with the horizontal, the tensions at its ends being $T$ and $T+dT$, as shown in Fig. 137. Then we see that, in the notation of article 322, the angle $\phi$ between POK and the weight $Ow$ is $3\pi/2-\psi$; thus $\cos\phi=-\sin\psi$ and $\sin\phi=-\cos\psi$. Hence equations (2) of that article reduce to

$$\left.\begin{aligned}\frac{dT}{ds}&=w\sin\psi\\ \frac{d\psi}{ds}&=\frac{w\cos\psi}{T}\end{aligned}\right\} \quad \text{and} \quad \ldots\ldots\ldots (7).$$

These may be put into the rather more useful form

$$\left.\begin{aligned}dT&=wds\sin\psi=wdy\\ T&=\rho w\cos\psi\end{aligned}\right\} \quad \text{and} \quad \ldots\ldots (8),$$

where $y$ and $\rho$ are the vertical ordinate and radius of curvature respectively at the point P to which $T$ applies. From the second of these it is clear that for $\psi=0$, *i.e.* the bottom point of the cord, we have for the tension there $cw$, the letter $c$ denoting the radius of curvature at this point. But from the first of (8) we see that the tension increases proportionally to the vertical height; hence if we chose the origin of $y$ so as to make the constant of integration zero, we have

$$\left.\begin{aligned}T&=wy\\ T_0&=wc\end{aligned}\right\} \quad \text{and} \quad \ldots\ldots\ldots (9),$$

showing that the horizontal axis of $x$ must be taken a depth $c$ below the lowest point of the cord at which $\psi=0$. The tension at any point P is then the weight of cord whose length equals the vertical ordinate of P, reckoned from this axis of $x$. The curve assumed by a uniform cord under gravity is called the *common catenary*.

**325. Elementary Relations for Catenary.**—We began treating the problem of a cord by taking a finite length, which was then indefinitely reduced. Another method is to pass from the finite portion to the infinitesimal element by differentiation, which will be convenient now in deriving further elementary relations from the uniform cord at rest under gravity.

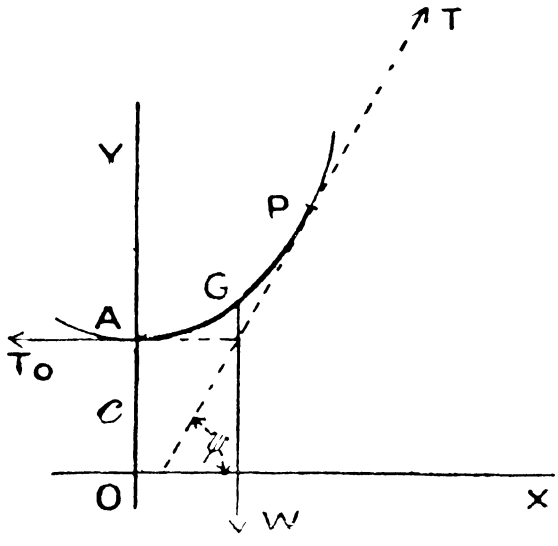



FIG. 138. ELEMENTARY RELATIONS FOR CATENARY.

Thus, referring to Fig. 138, let us begin by considering the equilibrium of the finite portion $AP=s$ of the cord from its lowest point A, under the forces $T_0$ at A horizontally to the left, $T$ at P upwards to the right at inclination $\psi$, and its weight $W$ vertically downwards through its centre of mass G. The origin is taken at the distance $c$ vertically below A, so that $T_0=cw$ and $W=sw$ where $w$ is the weight of the cord per unit length; we may also for convenience write $T=tw$.

Resolving the forces parallel to the axes of $x$ and $y$ and equating the sums to zero gives

$$cw=tw\cos\psi \text{ and } sw=tw\sin\psi \quad . \quad . \quad . \quad . \quad (10).$$

Whence

$$\tan\psi=s/c \text{ and } t^2=c^2+s^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

It is convenient to represent these relations graphically also, as in Fig. 139, which obviously forms the triangle of forces for the equilibrium of AP.

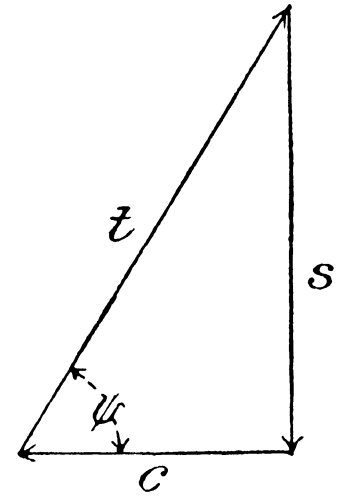



FIG. 139. TRIANGLE OF FORCES FOR EQUILIBRIUM OF CORD.

This diagram and the equations (10) express all that can be stated on the basis of the mechanical conditions alone. We now introduce a geometrical relation and combine it with (11). Thus

$$\frac{dy}{dx}=\tan\psi=\frac{s}{c} \quad . \quad . \quad . \quad . \quad . \quad (12).$$

To eliminate the $x$ we use another geometrical relation and combine with (12), so obtaining

$$\left(\frac{dy}{ds}\right)^2=\frac{dy^2}{dx^2+dy^2}=\frac{s^2}{c^2+s^2}.$$

Then, taking the root and integrating from A to P, we have

$$\int_c^y dy=\int_0^s \frac{sds}{\sqrt{c^2+s^2}}.$$

And, on evaluating, we find

$$y-c=\sqrt{c^2+s^2}-c,$$

or
$$y^2=c^2+s^2 \quad \ldots\ldots\ldots\ldots\ldots (13).$$

Referring to the second equation of (11), or to Fig. 139, we see that

$$t=y \text{ and } T=wy \quad \ldots\ldots\ldots\ldots (14),$$

thus confirming equation (9) of the previous article. We may now see from Fig. 139 or equations (11) that

$$y=t=c\sec\psi \quad \ldots\ldots\ldots\ldots (15),$$

the tension $T$ being $w$ times this.

Many simple problems on the common catenary can be solved from the above elementary relations. It is, however, important also to derive the cartesian equation of the curve, which we proceed to do in article 326.

### EXAMPLES—LXIII.

1. In the case of a flexible cord in equilibrium, prove (i) that the space rate of change of tension equals the tangential components of the forces, and (ii) that the curvature multiplied by the tension equals the normal component of the forces, these forces being reckoned per unit length.
2. 'Obtain a formula to show the variation of tension along an inextensible weightless string which is wound tightly round an imperfectly rough post and is just on the point of slipping.

'How many turns of a rope round a post (coefficient of friction = 0·2) would enable a tension of 1 lb. to support a weight of 1000 lbs.? It may be assumed that $\log_e 10 = 2{\cdot}3$.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, I. 8.)

3. 'Investigate the catenary formulas

$$s=a\tan\psi,\ y=a(\sec\psi-1),$$

connecting the arc $s$ and the height $y$ with the slope $\psi$.

'Prove that in flying a kite at a maximum height of 1 mile at the end of 2 miles of steel thread 0·03 inch in diameter, the tension at the lowest point is the weight of $1\frac{1}{2}$ miles of thread, or nearly 20 lbs., and the pull of the kite is the weight of $2\frac{1}{2}$ miles of thread, or 33 lbs., acting at about 37° with the vertical.'

(LOND. B.SC., PASS, MIXED MATH., 1902, II. 2.)

**326. Equations of Common Catenary.**—To obtain the equation of the catenary in terms of $x$ and $y$ we eliminate $s$ from equations (12) and (13), thus obtaining

$$\frac{dy}{dx}=\frac{\sqrt{y^2-c^2}}{c}.$$

Transforming this so as to separate the variables and integrating from A to P gives

$$\int_0^x dx=\int_c^y \frac{c\,dy}{\sqrt{y^2-c^2}}=c\Big[\log_e(y+\sqrt{y^2-c^2})\Big]_c^y.$$

Thus
$$x=c\log_e\left(\frac{y+\sqrt{y^2-c^2}}{c}\right).$$

Hence, on taking exponentials, we obtain

$$ce^{x/c}=y+\sqrt{y^2-c^2}.$$

Transposing and squaring gives

$$c^2e^{2x/c}-2yce^{x/c}+y^2=y^2-c^2.$$

Whence $$y=\frac{c^2e^{2x/c}+c^2}{2ce^{x/c}}=c\frac{e^{x/c}+e^{-x/c}}{2}$$

or $$y=c\cosh x/c \quad \ldots\ldots (16),$$

which is the cartesian equation required, and shows that the common catenary is a 'cosh' graph.

Putting this value of $y$ in (13) we find

$$s^2=c^2(\cosh^2 x/c-1),$$

or $$s=c\frac{e^{x/c}-e^{-x/c}}{2}=c\sinh x/c \quad \ldots\ldots (17).$$

By addition and subtraction of (16) and (17) we have also

$$\left.\begin{aligned} y+s&=ce^{x/c}\\ y-s&=ce^{-x/c}\end{aligned}\right\} \quad \ldots\ldots (18),$$

relations which are sometimes useful.

**327. Alternative Method for Equations of Catenary.**—We may now very briefly indicate a method for obtaining the equations of a catenary in which the relation between $s$ and $x$ is first obtained and from it that between $x$ and $y$.

The mechanical equations are

$$t\sin\psi=s,\ t\cos\psi=c.$$

Combining with them the geometrical relation we have

$$\frac{dy}{dx}=\tan\psi=\frac{s}{c} \quad \ldots\ldots (19).$$

The expression for $ds$ then gives

$$\left(\frac{ds}{dx}\right)^2=\frac{dy^2+dx^2}{dx^2}=\frac{s^2+c^2}{c^2}.$$

Thus $$\int_c^s\frac{cds}{\sqrt{s^2+c^2}}=\int_0^x dx,$$

and $$c\sinh^{-1}s/c=x,$$

or $$s/c=\sinh x/c \quad \ldots\ldots (20),$$

which agrees with (17). Returning to (19), we now have

$$\frac{dy}{dx}=\frac{s}{c}=\sinh\frac{x}{c}.$$

Whence $$\int_c^y dy=\int_0^x \sinh(x/c)dx,$$

and $$y=c\cosh(x/c) \quad \ldots\ldots (21),$$

agreeing with (16).

**328. Elementary Properties of the Common Catenary.**—Let us now note some of the definitions and elementary properties of the common catenary. Thus, on reference to Fig. 140, the lowest point A

of the curve is called the *vertex*, the vertical line OY through it is the *axis*, the length $c$ represented by AO and equal to the radius of curvature at the vertex is called the parameter, the horizontal line OX at this depth below the vertex is called the *directrix*.

Take any point P on the curve and from it draw the tangent, normal and ordinate, cutting the directrix at Z, H, and N respectively. Also from N let fall NL perpendicular to the tangent and meeting at the point L. Then, calling the inclination of the tangent to the horizontal $\psi$, we see that the angles PNL and NPH are each equal to $\psi$. Referring to Fig. 139 and equation (14) we see that NP represents $t$ as well as $y$, and that in consequence NL$=c$ and LP$=s$. Further, from equations (8) and (14) we have

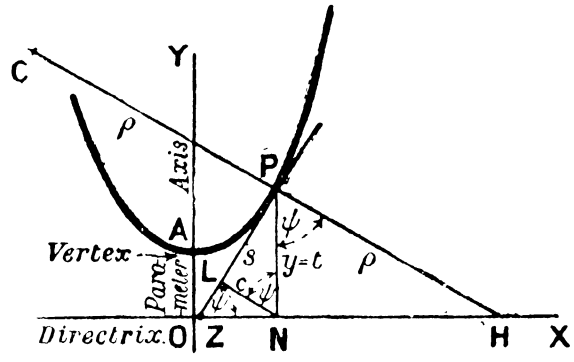



FIG. 140. ELEMENTARY PROPERTIES OF THE COMMON CATENARY.

$$y=t=\rho\cos\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Thus the normal PH on Fig. 140 represents by its length the radius of curvature $\rho$ for the point P, the corresponding centre of curvature C being, of course, on the concave side of the curve on HP produced where PC$=$PH. This is seen to agree with the parameter OA$=c$ being the radius of curvature at the vertex.

It is easily seen that the common catenary is a curve of the class having only a single parameter each. Thus different catenaries differ in size only and not in shape. Familiar examples of this class are the circle, parabola, and cycloid, defined respectively by radius, latus rectum, and radius of generating circle.

For some further geometrical properties of the catenary and methods of constructing the curve the student is referred to Minchin and Dale's *Mathematical Drawing*.

## EXAMPLES—LXIV.

1. 'The ends of a uniform chain are attached to two fixed points, and the chain hangs freely under the action of gravity; deduce the equation of the curve in which it hangs.

'A uniform chain has a mass of 2 lbs. per foot length, and the catenary in which it hangs has for equation $y=10\cosh(x/10)$; find, by using the tables, the tension and the vertical component of tension at the point for which $x=25$ feet.' (LOND. B.SC., PASS, APPLIED MATH., 1907, I. 9.)

2. 'If a string hangs in equilibrium in the form of a catenary, show that the tension at any point is equal to the weight of a length of the string equal to the height of the point above the directrix of the catenary. A kite is flown with 600 feet of string from the hand to the kite, and a spring balance held in the hand shows a pull equal to the weight of 100 feet of the string inclined at an angle of 30° to the horizon; find the vertical height of the kite above the hand.'

(LOND. B.SC., PASS, APPLIED MATH., 1908, I. 9.)

3. 'Show that the length of an endless chain which will hang over a circular pulley of radius $a$ so as to be in contact with two-thirds of the circumference of the pulley is

$$a\left\{\frac{3}{\log_e(2+\sqrt{3})}+\frac{4\pi}{3}\right\}.'$$

(LOND. B.SC., PASS, APPLIED MATH, 1909, I. 9.)

4. 'Prove that the difference between the tensions at two points of a uniform chain hanging under gravity is equal to the weight of a portion of the chain whose length is the vertical distances between the points.

'A uniform chain is stretched between a point on the ground and a point 100 feet above the ground. The tension at the ground is the weight of 3100 feet of chain, and the inclination there of the tangent to the horizon is $\cos^{-1}(16/31)$.

'Find the length of the chain to the nearest inch.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, I. 9.)

5. Draw carefully a catenary, indicating its vertex, axis, and directrix. Obtain graphically the length of the curve from the vertex to any point P, also the radius of curvature of the catenary at P.

6. 'A uniform chain 100 feet long is to be suspended from two points in the same horizontal line with such a span that the tension at the ends is to be three times that at the middle.

'Find (with the help of tables) the required span to the nearest inch.'

(LOND. B.SC., PASS, APPLIED MATH., 1905, I. 9.)

**329. Approximations to the Catenary.**—When $x$ is so *small* that its fourth power is negligible, we may see from equations (16) or (21), on expanding the exponentials, that the approximate value of the ordinate is

$$y=\frac{c}{2}\left\{1+\frac{x}{c}+\frac{x^2}{2c^2}+\frac{x^3}{6c^3}+\ldots\right.$$
$$\left.+1-\frac{x}{c}+\frac{x^2}{2c^2}-\frac{x^3}{6c^3}+\ldots\right\}$$
$$=\frac{c}{2}\left\{2+\frac{2x^2}{2c^2}+\ldots\right\}$$

Thus $y=c+x^2/2c$,

or $x^2=2c(y-c)$ nearly . . . . . (23),

which is the parabola of latus rectum $2c$ with vertex at $(0, c)$ and axis vertically upwards, which approximates to the catenary at its lower portion.

On the other hand, when $x$ is large the negative exponential $e^{-x/c}$ vanishes in comparison with $e^{x/c}$. Thus, for the higher parts of the catenary the curve approximates to the positive exponential only. Or,

$$y=\frac{c}{2}e^{x/c} \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

There is an intermediate portion of the catenary for which neither (23) nor (24) furnishes a close approximation.

**330. Sag and Excess Length in Telegraph Wires.**—We may now find the sag and extra length due to it in the case of wires tightly

stretched between points on the same horizontal level and at a distance $p$ apart, as in the case of telegraph or telephone wires. Clearly we are here concerned with a relatively small portion at the bottom of a very large catenary. Hence we may commence with the parabola approximation of (23). Thus putting $x=p/2$ and denoting by $q$ the sag or dip $(y-c)$ in the middle, we have

$$q=p^2/8c \quad \ldots\ldots\ldots\ldots\ldots\ldots (25).$$

For the unknown $c$ on the right side we may use the approximate value $t=T/w$, the tension $T$ at the ends in terms of the weight $w$ per unit length being supposed known. We accordingly write as the working approximation

$$q=p^2/8t \quad \ldots\ldots\ldots\ldots\ldots\ldots (26).$$

For the excess length we may conveniently calculate $s$ from its true expression as in (17) and (20). Thus, expanding the exponentials,

$$s=\frac{c}{2}\left\{1+\frac{x}{c}+\frac{x^2}{2c^2}+\frac{x^3}{6c^3}+\ldots\right.$$

$$\left.-1+\frac{x}{c}-\frac{x^2}{2c^2}+\frac{x^3}{6c^3}-\ldots\right\}.$$

Hence, omitting powers of $x/c$ beyond the cube, we find

$$s=x+x^3/6c^2.$$

Accordingly, for $x=p/2$, we have for the excess length in a single span

$$2s-p=p^3/24c^2 \text{ nearly} \quad \ldots\ldots\ldots\ldots (27).$$

For the $c$ on the right we may here use $c=y=t$ nearly, or the closer approximation

$$c=y-q=t-q \quad \ldots\ldots\ldots\ldots\ldots (28),$$

$q$ being approximately known from (26).

The case in question may be more conveniently illustrated by a numerical example than by a diagram. Thus, let the poles be 88 yards apart and the tension of the order, weight of a mile of wire. Then, by equation (26), the sag is given by

$$q=88^2/8\times 1760=11/20 \text{ yard}=19{\cdot}8 \text{ inches}.$$

Again, by (27) and $c=t$ nearly, the excess length due to this sag may be written

$$2s-p=88^3/24t^2=11/1200 \text{ yard}=0{\cdot}33 \text{ inch}!$$

By (28) we note that $c$ is 1759·45 yards, hence the directrix of the catenary is practically a mile below the wires.

**331. Parameter of Catenary.**—Where the foregoing approximations are not sufficiently close for the purpose in view, it may be necessary to determine the parameter $c$ of the catenary from the data and then

use the true curve. This may be done graphically as follows:—Suppose the $x$ and $s$ for a given point are known; then, using (17) or (20),

$$s/c = \sinh x/c.$$

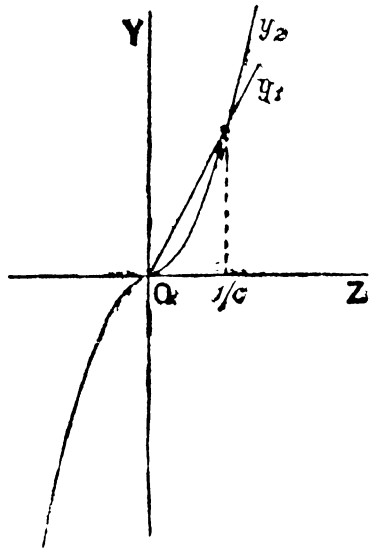



FIG. 141. GRAPHIC SOLUTION FOR PARAMETER OF CATENARY.

Transform this by writing $c = 1/z$ and $x = b$. It thus becomes

$$sz = \sinh bz \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

Now plot the two graphs

$$\left.\begin{aligned} y_1 &= sz \\ \text{and} \quad y_2 &= \sinh bz \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (30),$$

as shown in Fig. 141.

Their point of intersection, I, gives by its abscissae $z$ the required root $z$ of (29). And the reciprocal of this is the value sought for the parameter $c$.

**332. Parabolic Cord requires Uniform Horizontal Load.**—Let us now suppose a cord to hang in the form of a parabola, and let us find the law of distribution of the load per unit *horizontal* length for this to be the position of equilibrium. It is, of course, indifferent for our purpose whether the load is due to the weight of the cord or to masses supported by it. The equation of the parabola may be written

$$x^2 = 4ay,$$

giving

$$\tan\psi = \frac{dy}{dx} = \frac{x}{2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

The mechanical conditions for equilibriums are

$$T\cos\psi = T_0 \text{ and } T\sin\psi = W,$$

giving

$$\tan\psi = \frac{W}{T_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

where $W$ is the total weight from the lowest point of tension $T_0$ to that at inclination $\psi$ and tension $T$. Thus, equating the right sides of (31) and (32), we find

$$W = xT_0/2a \text{ and } dW/dx = T_0/2a = h \text{ say} \quad . \quad . \quad (33).$$

We accordingly see that the parabolic curve requires the total weight to be proportional to $x$ or the law of load per unit *horizontal* length to be one of uniformity, for $h$ is evidently a constant.

If we are only dealing with a small portion of the curve near its lowest point, it is clear that the uniformity of load horizontally is practically a uniformity of load along the curve, which is here so near horizontal. Thus, if the $2a$ of the parabola equals the $c$ of the catenary, we have $h = w$, as should be the case.

The importance of the parabola for the form of a hanging cord or chain lies in the fact that it is the curve assumed by the chains of a suspension bridge, the load due to the roadway being practically uniform

per unit horizontal length. We now see from another point of view how well the approximation of article 330 was justified in the numerical example given in illustration at the end of that article. For the parabola, to which the catenary was there assimilated, is rigorously exact for a uniform horizontal load, and the excess distance in a 88 yard span was less than one-third of an inch!

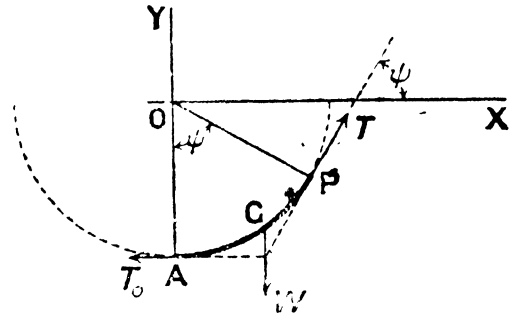



FIG. 142. CORD HANGING IN A CIRCLE.

**333. Laws of Load for a Cord to hang in a Circle.**—Let us now find the distribution of load both along the curve and along the horizontal when the cord hangs in a circle. Let its equation be

$$x^2+y^2=a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

Then, for the inclination of the tangent at any point P, see Fig. 142, we have

$$\tan\psi=\frac{dy}{dx}=-\frac{x}{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

But, from the mechanical conditions of equilibrium of the portion of weight $W$ from the lowest point A to P, viz.

$$T\cos\psi=T_0 \text{ and } T\sin\psi=W,$$

we find

$$W=T_0\tan\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

Consider first the law of load per unit length of curve $s=a\psi$. Then, using (34) and (36), we have

$$\frac{dW}{ds}=\frac{dW}{ad\psi}=\frac{T_0}{a}\sec^2\psi.$$

But, by (35) and (34), since P is on the circle, we see that

$$\sec^2\psi=1+x^2/y^2=a^2/y^2.$$

Hence

$$\frac{dW}{ds}=\frac{aT_0}{(-y)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

The negative sign is here inserted before the $y$ as we are concerned only with the lower half of the circle for which the ordinates are negative.

Again, for the law of load per unit horizontal length we substitute (35) in (36) and differentiate. Thus

$$\frac{dW}{dx}=T_0\frac{d}{dx}\left(-\frac{x}{y}\right)=T_0\frac{(-y-x^2/y)}{y^2}.$$

Then, simplifying by (34), we have

$$\frac{dW}{dx}=\frac{a^2T_0}{(-y)^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

We accordingly see from (37) and (38) that at the ends of the horizontal diameter where $y$ vanishes, the circular form of the cord

would require an infinite load per unit length whether estimated along the curve or on the horizontal.

It may be noticed here that since AP is in equilibrium under the three inclined forces $T_0$, $T$, and $W$, they must intersect. Hence the centre of mass G of AP is vertically over this point as shown.

## EXAMPLES—LXV.

1. 'Prove that a chain loaded so that the weight of any portion is proportional to its projection on the horizontal will hang in the form of a parabola.

'Also prove that the horizontal tension is equal to $wl$, where $w$ is the weight per foot at the bottom and $l$ is the semi-latus rectum.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, I. 9.)

2. 'Démontrer qu'une chaîne homogène en équilibre sous l'influence de son propre poids prendra la forme de la courbe

$$y = a \cosh (x/a).$$

'Si l'on construit une parabole ayant même axe, même sommet et même courbure au sommet, que la chaînette, déterminer si cette parabole passe au dessus ou au dessous de la chaînette.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, I. 8.)

3. 'If any heavy flexible chain, whether uniform or not, is suspended vertically from two fixed points, show that the tension has everywhere a constant horizontal component. A uniform chain of length $2l$ and weight $W$ is suspended from two points, $A$, $B$, in the same horizontal line. A load $P$ is now suspended from the middle point of the chain, and the depth of this point below $AB$ is found to be $h$. Prove that each terminal tension is now

$$\tfrac{1}{2}\left[P\frac{l}{h} + W\frac{h^2+l^2}{2hl}\right].'$$

(LOND. B.SC., PASS, MIXED MATH., 1904, II. 4.)

4. Show that the lower part of a catenary is approximately a parabola, and express the latus rectum of that parabola in terms of the parameter of the catenary. Prove also that the two curves have the same curvature at their common vertex.

5. Prove that to hang in a form of a circular arc a flexible cord must have at every point a linear density inversely as the square of its depth below the centre of the circle.

6. Find the law of load per unit horizontal length which, attached to a flexible cord of negligible weight, bends that cord into a circular arc.

# CHAPTER XVI

## ATTRACTIONS AND POTENTIAL

**334. Attractions of Two Particles.**—Having given any law of attraction between particles, it is an important problem to deduce the resultant mutual attraction of any two given bodies in specified relative positions. Such problems may be based upon any assumed law of attraction, but as we are so specially concerned with the law of gravitation, varying inversely as the distance squared (see article 209), we shall here practically limit ourselves to this law of paramount importance, which is often referred to as the *natural* law.

Consider first the case of two particles of masses $m$ and $m'$ at P and Q respectively, their distance apart, $PQ=r$, being very great compared with the size of either particle, and $m'$ very small compared with $m$, as shown in Fig. 143. Then, according to Newton's law of universal gravitation, we have the attraction proportional to $mm'/r^2$. Or, introducing the factor $\gamma$ to convert the proportionality into an equality, we may write for the attractive forces in question

$$F=\gamma mm'/r^2 \quad . \quad (1),$$

where the force on $m$ is in the direction PQ and that on $m'$ is in the direction QP.

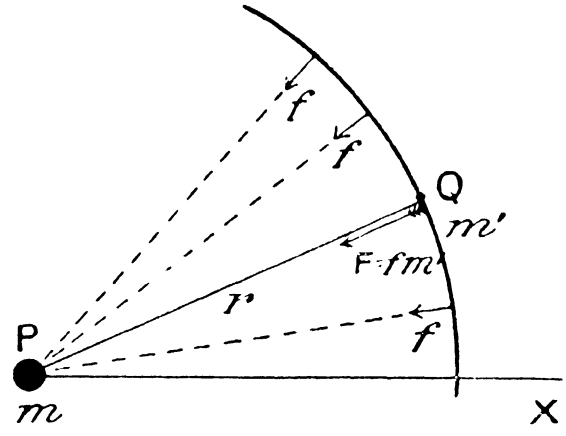



FIG. 143. ATTRACTION AND FIELD.

The factor $\gamma$ is called the Newtonian constant of gravitation. Its numerical value obviously varies with the units in use, and in the *c.g.s.* system it is approximately

$$\gamma=6{\cdot}7\times10^{-8}\ c.g.s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The method of deducing this value may be dealt with later. See article 350.

**335. Gravitational Field.**—Referring again to equation (1), we may note that the force of attraction experienced by the mass $m'$ is the product of its mass and the factor $\gamma m/r^2$, which accordingly expresses in some way the magnitude of the influence due to the existence of the mass $m$ at a distance $r$ away. And, if for $m'$ were substituted any other mass, still the attractive force would be found as the product of the

factor just referred to and the mass of our new particle. We are thus led to connect this expression $\gamma m/r^2$ with the presence of the particle of mass $m$ at P, a distance PQ$=r$ away from Q. And we may regard the point Q as temporarily impressed with a certain property to a certain quantitative degree in virtue of the existence of the particle of mass $m$ at P. There may not be any gross matter at Q, nor any force applied there. But, owing to the particle at P, there is such a specialised state of things at Q that gross matter if there will inevitably experience the force as defined by equation (1). This conception is concisely expressed by stating that there exists at Q a *gravitational field* of magnitude $\gamma m/r^2$ in the direction QP. Thus, *field* is a vector quantity like force, for it is the quotient *force per unit mass*. We may accordingly write for the field $f$ due to $m$

$$f=\gamma m/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

while

$$f'=\gamma m'/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

similarly expresses the field $f'$ due to $m'$, the $r$ in either case measuring from the mass in question. It is evident that we may now write (1) for the force $F$ in the new forms

$$F=fm'=f'm \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and (3) for the fields in the new forms

$$f=F/m',\ f'=F/m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Equations (3) and (3$a$) expressed the fields due to given particles, whereas (5) leads us to a more general point of view. For whatever particles, known or unknown, may be producing the field $f$ at a point Q, if a particle $m'$ placed there experiences a force $F$ in virtue of this field, then that field at Q is measured by the quotient $F/m'$.

Referring again to equations (1) and (2), we see that the numerical value of $\gamma$ in any given system of units expresses in that system the attractive force between unit masses condensed into points at unit distance apart. But $\gamma$ is not itself a force, being of different dimensions. Thus, from (1) we have the dimensional equation, in terms of mass $M$, length $L$, and time $T$,

$$MLT^{-2}=[\gamma]M^2L^{-2}.$$

Whence the dimensions of $\gamma$ are given by

$$[\gamma]=M^{-1}L^3T^{-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Accordingly the Newtonian constant of gravitation is approximately $6{\cdot}7\times10^{-8}$ cc. per gm. per sec. per sec.

Again, from (4) or (5), we see that the dimensions of gravitational field are given by

$$[f]=MLT^{-2}\div M=LT^{-2} \quad . \quad . \quad (7).$$

In other words, a gravitational field has the dimensions of an acceleration, as could also have been seen from the definitions of field and force. Hence we may rightly speak of the acceleration $g$ of the gravitational field in which we live on the earth's surface.

Also, in the case of the field $f$ at Q due to the particle of mass $m$ at P, a distance $r$ away, we see that $f=\gamma m/r^2$ expresses the value of the acceleration which would be experienced by a free particle of any mass

placed there, the frame of reference having its origin at the centre of mass of the two particles. Or, if the mass $m$ at P is very large in comparison with $m'$ at Q, we may then without sensible error take the origin of the co-ordinate axes at P itself.

Thus, for the acceleration of a stone falling near the earth's surface, we may take our origin at the centre of mass of earth and stone, or at that of the earth simply. If the masses of the two attracting bodies are comparable, we must take $\gamma m/r^2$ to be the acceleration of $m'$ towards the centre of mass of the two. The mutual or relative acceleration of the two bodies is the sum of those separate accelerations, and is accordingly given strictly by

$$f=\gamma(m+m')/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Thus, ignoring $m'$ in comparison with $m$ is like taking the frame of reference fixed in the larger body of mass $m$. Hence (8) would be reduced to the usual approximate relation

$$f=\gamma m/r^2=\mu/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a),$$

where the $\mu$ is the factor used in central accelerations (articles 83-91 of Chapter v.). It is now obvious that we may think of the $f$ either as the field or as the acceleration of any point defined by the $r$. As the magnitude of the field is independent of the direction of $r$, we see that it must be a radial one having the same value at any point on a sphere of given radius $r$ about the position P as centre.

We may note from (4) that if the mass $m'$ at Q is unity, then $F$ is equal numerically to $f$. And it is customary to speak of the attraction at a point on a unit particle and to denote it by $F$ and its cartesian components by $X$, $Y$, and $Z$. This convenient practice will be followed here, but it should be borne in mind that we may sometimes be thinking of a force $F$ and sometimes of a field or acceleration of the same numerical value and represented by the same symbol, yet the natures and dimensions of the two quantities are different.

**336. Filament and Particle: Axial Case.**—Let us now consider the attraction between a straight filament and a particle of unit mass at a point either in the direction of the filament or off to one side.

Thus, for the axial case which is specially simple, let the particle be at P in the line of the filament AB as shown in Fig. 144.

Let the linear density of the filament be $\lambda$ and its ends A and B be distant from P by $a$ and $b$ respectively, the gravitation constant being denoted by $\gamma$.

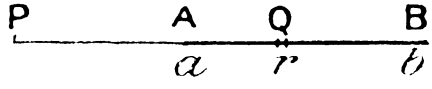



FIG. 144. AXIAL ATTRACTION OF FILAMENT.

Then, for an infinitesimal element of the filament at Q distant $r$ from P, the attraction is given by $dF=\gamma\lambda dr/r^2$, whence

$$F=\gamma\lambda\int_a^b\frac{dr}{r^2}=\gamma\lambda\left(\frac{1}{a}-\frac{1}{b}\right)=\gamma\frac{M}{ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $M$ is the mass of the filament. This initial result is important since it shows that, in respect of this attraction, the filament is replace-

able *not* by a particle at its centre of mass, but by a particle at a distance from P which is the *geometric* mean of PA and PB.

We easily see that if, while A is stationary, the filament is lengthened by moving B to infinity, the attraction becomes

$$F'=\gamma\lambda/a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

**337. Filament and Particle: General Case.**—We pass now to the second case, in which the particle is at a point P off the line of the filament AB and at a perpendicular distance, $p$ say, as shown by PL in Fig. 145, the point P being at the origin of co-ordinates and the filament lying along the line $x=p$. Join PA, PB and denote their inclinations with the axis of $x$ by $\alpha$ and $\beta$. Also take any line PQ at inclination $\theta$, another PM making the angle $d\theta$ with PQ, and let fall QN perpendicularly on PM. Then the angle MQN is $\theta$ also. Let the filament as before have the linear density $\lambda$, and consider the attraction $dF$ on unit mass at P due to the infinitesimal element QM. Then we have from the figure

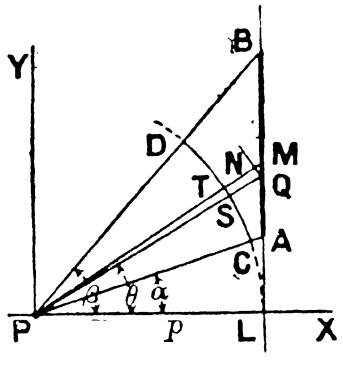



FIG. 145. ATTRACTION OF FILAMENT OFF ITS AXIS.

$$dF=\gamma\lambda\frac{\mathrm{QM}}{\mathrm{PQ}^2}=\gamma\lambda\frac{(\mathrm{QN}\sec\theta)}{\mathrm{PQ}.p\sec\theta}=\gamma\lambda\frac{(\mathrm{PQ}d\theta)}{\mathrm{PQ}p}=\frac{\gamma\lambda}{p}d\theta \quad . \quad . \quad (3).$$

Taking now the $X$ and $Y$ components of this attraction of the element QM, we have

$$dX=dF\cos\theta \text{ and } dY=dF\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Hence, on substituting from (3) and integrating, we find

$$X=\frac{\gamma\lambda}{p}\int_\alpha^\beta\cos\theta\,d\theta=\frac{\gamma\lambda}{p}(\sin\beta-\sin\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and

$$Y=\frac{\gamma\lambda}{p}\int_\alpha^\beta\sin\theta\,d\theta=\frac{\gamma\lambda}{p}(\cos\alpha-\cos\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Let us now denote the resultant attraction by $R$ at an angle $\psi$ with PX. Then from (5) and (6) we obtain

$$R=\sqrt{X^2+Y^2}=\frac{\gamma\lambda}{p}\sqrt{2-2\cos(\beta-\alpha)}=\frac{\gamma\lambda}{p}2\sin\frac{\beta-\alpha}{2} \quad (7),$$

and

$$\tan\psi=Y/X=\frac{\cos\alpha-\cos\beta}{\sin\beta-\sin\alpha}=\tan\frac{\alpha+\beta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Thus $R$ is directed along the bisector of the angle APB.

We see from (6) and the figure that the attraction component parallel to the filament may be written

$$Y=\gamma\lambda\left(\frac{1}{\mathrm{PA}}-\frac{1}{\mathrm{PB}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

which may be compared with (1) for the axial attraction.

From (7) and (8) we may write at once the values for an infinite filament extending one or both ways from A or L to infinity.

Thus, if it extends infinitely both ways from L, $\alpha=-\pi/2$ and $\beta=+\pi/2$. Hence the resultant attraction is

$$R'=2\gamma\lambda/p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

**338. Particle and Circular Filament.**—Let a circle be described about P with radius PL, cutting the lines PA and PB at C and D, and let a filament of linear density $\lambda$ be supposed to occupy the arc CD. Then it is easy to see that the attraction at P of the infinitesimal element ST of this filament lying between PQ and PM is given by

$$dF=\gamma\lambda\frac{pd\theta}{p^2}=\frac{\gamma\lambda d\theta}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

But this agrees with (3), giving the attraction of the corresponding element QM of the straight element. Hence we see that the attractions of corresponding finite lengths of the straight and circular filaments are precisely alike.

**Particle and Two Straight Filaments.**—For the attraction of two or more straight filaments on a unit mass at any point, it is evident that we could find the resultant attraction of each filament and then compound them by vectorial addition for the final resultant.

**339. Mutual Attraction of Collinear Filaments.**—Suppose now that we have two straight filaments lying along the same straight line, and let it be required to find their mutual attraction. We may simplify the work a little by assuming the attraction of one filament for a particle along its axis, as found in article 336; the particle in the present case is then taken as an element in the other filament. Thus, let the filaments be AL and BM of lengths $a$ and $b$ and of linear densities $\lambda$ and $\mu$, the distance AB between their near ends being $c$; all as shown in Fig. 146. Take a point P in BM distant $s$ from A and $(s+a)$ from L, and consider the attraction of the filament AL on the element $ds$ at P. Then, by (1) of article 336, we may write for this attraction

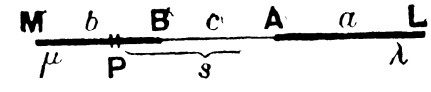



FIG. 146. ATTRACTION OF COLLINEAR FILAMENTS.

$$dF=\gamma a\lambda\frac{\mu ds}{s(s+a)}.$$

Hence, integrating between the prescribed limits, we find

$$F=\gamma a\lambda\mu\int_c^{b+c}\left(\frac{1}{as}-\frac{1}{a(s+a)}\right)ds=\gamma\lambda\mu\log_e\frac{(b+c)(c+a)}{(a+b+c)c} \quad (11a),$$

which is seen to be a symmetrical expression of the right dimensions.

If either of the filaments becomes infinite by the addition of matter at the end *remote* from the other, the attraction is still finite. Thus for

**25a. Moments.** DEFINITION.—The *moment* of a vector with respect to a point is the product of the vector into the perpendicular from that point upon the line in which the vector is localised, and is reckoned positive when the direction of the vector about the point is counter-clockwise.

*Theorem.*—If two vectors are localised at a point O, then the algebraic sum of their moments with respect to any point P in their plane is equal to the moment of their resultant about P.

In Fig. 6A, let OA, OB represent the component vectors, OC their resultant, found by the parallelogram law, and all localised in O; also let P, in the plane of OABC, be the point about which the moments are to be taken.

*Proof.*—Then, by definition, the moment of any vector is represented to scale on the figure by twice the area of the triangle

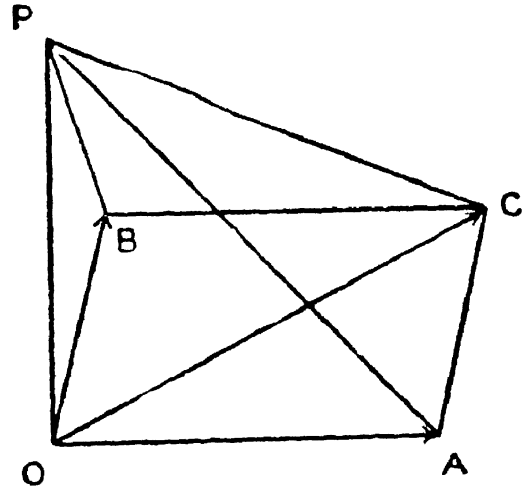



FIG. 6A. THEOREM OF MOMENTS.

whose base in the vector and whose vertex is the point P about which its moment is taken. Thus, half the moment of OC

$$\begin{aligned} &= \text{area of } \triangle OCP = \triangle OBP + \triangle OCB + \triangle BCP \\ &\qquad = \triangle OBP + \triangle OAP \\ &\qquad = \text{half-sum of moments of OB and OA.} \end{aligned}$$

This accordingly establishes that case of the proposition represented in the figure. When P is differently placed the needed demonstration follows in like manner.

**25b. Composition of Angular Velocities.**—Measuring angles in radians, we naturally measure *angular velocities* in radians per second. Thus, if a point P moves with an angular velocity $\omega = \theta/t$ about a given axis from which it is distant by the radius $r$, we have the relations

$$\omega = \theta/t = s/rt \text{ or } s = \omega rt,$$

where $s$ is the arc described by the point in time $t$. Hence in a given time the displacement $s$ is proportional to the product $\omega r$. This

suggests an analogy with the moments of a vector with respect to a point. Thus, referring to Fig. 6A, consider the point P in the plane of the diagram as having an angular velocity about OA and proportional to the length of OA $=\omega_1$ say. Then in time $dt$, P would suffer a displacement perpendicular to the plane of the diagram of amount $ds_1=\omega_1 dt\times$ perpendicular from P upon OA $=dt\times$ double area of OAP. Similarly if the point P had a coexistent angular velocity $\omega_2$ proportional to OB about OB, its displacement in virtue of that in time $dt$ would be given by $ds_2=\omega_2 dt\times$ perpendicular from P upon OB. Hence the sum of these displacements would be represented to a certain scale on the diagram by the sum of the areas OAP and OBP. But, as we have just seen, this sum equals the area of OCP, which consequently represents to the same scale the displacement $ds$ which P would experience in time $dt$ under the influence of an angular velocity about OC of amount $\omega$ proportional to the length OC. Thus, so far as a point P in the plane of the diagram is concerned, the resultant of the coexistent angular velocities whose axes are OA and OB and magnitudes OA and OB respectively is an angular velocity whose axis is OC and whose magnitude is OC.

Suppose now a point P′ is taken out of the plane of the diagram such that PP′$=p$ is perpendicular to that plane. Then the displacements we have just considered for P would also be the displacements perpendicular to that plane for P′. The displacements parallel to the plane for P′ would obviously be $\omega_1 pdt$, which would appear in the diagram as perpendicular to OA, $\omega_2 pdt$ appearing perpendicular to OB, and $\omega pdt$ appearing perpendicular to OC. Thus these displacements would form in the diagram a parallelogram of the same shape as OACB, but of different size, and with sides perpendicular to the original parallelogram. Hence these component displacements due to $\omega_1$ and $\omega_2$ about OA and OB would give that due to the resultant angular velocity $\omega$ about OC. This, which has been proved for any one point, is obviously true for any assemblage of points.

We have accordingly seen that if two coexistent angular velocities are respectively represented in magnitude and axis by two lines meeting in a point, then the line found as the resultant by the addition of vectors represents in magnitude and axis the resultant angular velocity. In other words, an angular velocity is a vector. It should be observed that associated with a certain direction of drawing or describing the line representing an angular velocity must be associated a certain direction of rotation about that line as axis. We shall adopt the convention that the direction of rotation and corresponding drawing of the axis are those of rotation and advance of a right-handed screw in a stationary nut, as for example in driving an ordinary screw into wood. Thus, in Fig. 6A, OA, OB, and OC each denote an angular velocity which would bring P out from the plane of the diagram towards the reader.

It must be distinctly noted that this relation or theorem as to the composition of *coexistent* angular *velocities* in no wise applies to *successive finite rotations* or *angular displacements*.

**25c. Curvature.**—The *curvature* of a plane curve is defincon-
rate of change of its direction per unit length measured along ling

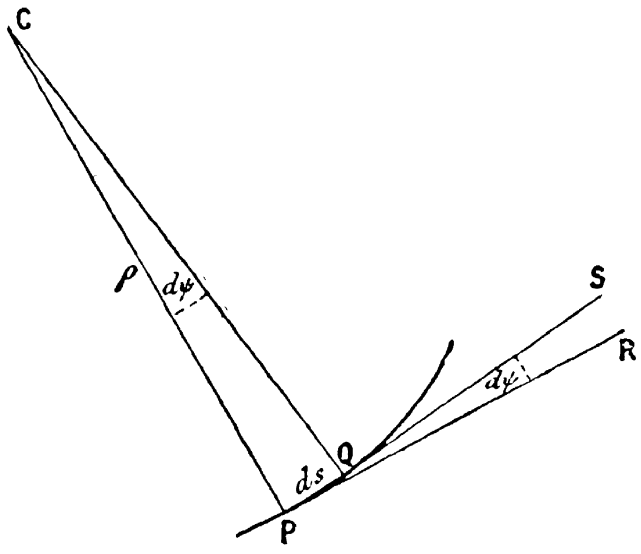



FIG. 6B. CURVATURE.

Thus take two points P and Q, Fig. 6B, an infinitesimal distance $ds$ apart on a plane curve, and let tangents PR, QS and normals PC, QC be drawn through P and Q making angles $d\psi$ with each other, and the normals meeting at C where $CP=CQ=\rho$.

Then, regarding the tangents, we see by the definition that the curvature at PQ is $d\psi/ds$. But, regarding the normals, we see that $d\psi=ds/\rho$. Hence we obtain

$$\text{curvature}=d\psi/ds=1/\rho.$$

Or, the curvature is also measured by the reciprocal of $\rho$, a length which is called the *radius of curvature*. The point C where the consecutive normals meet is called the *centre of curvature*.

**25d. Centroids.**—It is often desirable in the various sections of Mechanics to find a centre or point of mean position for a system of points, a line, a surface, a volume, or for some other distributed quantity. The formal development of this subject naturally occurs in Chapter XVII., since its chief mechanical application is to centres of mass, forming part of the section of Statics there treated. But the idea itself is purely geometrical and independent of any such particular application, so must be introduced here. Moreover, this early notice is a convenience, since the conception is required in Chapter XIII. when dealing with the Kinetics of Rigid Bodies.

Obviously the centre of two points is the point of bisection of the straight line joining them. Or, if the abscissae of the points were $x_1$ and $x_2$ and that of their centre $\bar{x}$, we could write $\bar{x}=\frac{1}{2}(x_1+x_2)$. Thus, for any number $n$ of points each having different abscissae, $\bar{x}=\frac{1}{n}(x_1+x_2+\ldots x_n)$. But some of the points might have abscissae of the same value. Thus, let $m_1$ points have the abscissa $x_1$, and $m_2$ points the abscissa $x_2$, and so on; the same notation with $y$'s holding for their ordinates. Then the co-ordinates of their centre would be

$$\bar{x}=\Sigma(mx)/\Sigma m \text{ and } \bar{y}=\Sigma(my)/\Sigma m \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).^1$$

We may extend the application by supposing the $m$'s to refer to the

[1] We may apply this to the resultant $R$ of $n$ vectors acting at a point O. For $R=\sqrt{\{(\Sigma x)^2+(\Sigma y)^2\}}$, and the centroid G of $n$ unit particles at the ends $(x, y)$ of the vectors is given by $OG=\sqrt{(\bar{x}^2+\bar{y}^2)}$. Thus, by (1) in the text, $R=n$ OG.

magnitudes of elements of length, area, volume, or other scalar quantity situated at the points defined by the corresponding $x$ and $y$. The point $(\bar{x}, \bar{y})$ so determined is called the *centroid* of the system, figure, or body in question.

If the co-ordinates of points with respect to the centroid are $x'$, $y'$, we have

$$x=\bar{x}+x' \text{ and } y=\bar{y}+y' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Hence $\Sigma mx=\bar{x}\Sigma m+\Sigma mx'$ and $\Sigma my=\bar{y}\Sigma m+\Sigma my'$.

But, by (1), these reduce to

$$0=\Sigma mx'=\Sigma my' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Thus, if equations (1) are taken as defining the centroid, we may regard (3) as giving some of its properties. It is, however, often convenient to take (3) as giving the definition of the centroid and equations (1) as forming the working rules for the determination of its position.

For quantities distributed throughout space of three dimensions we have simply to add the corresponding equations, involving $z$ by analogy with (1), (2), and (3), thus giving the following additional relations:—

$$\left.\begin{aligned} \bar{z}&=\Sigma mz/\Sigma m \\ z&=\bar{z}+z' \\ \text{and}\quad 0&=\Sigma mz' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

**26. Constraints and Degrees of Freedom.**—If a point is not constrained in any way, it is said to have *three degrees of freedom*, since it is obviously free to move parallel to the three co-ordinate axes of solid space. If the point is constrained to remain on a plane surface, say that of $xy$, it has then lost one degree of freedom, namely, the motion parallel to the axis of $z$, and retains two degrees of freedom only, namely, those along the axes of $x$ and $y$. Similarly, if the point is constrained to remain in a line, say the axis of $x$, it is evident that it has lost two degrees of freedom and retains only one. With any actual small particle these two cases may be represented by floating on still water and confinement in a straight tube respectively.

Let us now contemplate an extended body whose parts are debarred from any relative motion; this is called a *rigid body*. Then obviously, if it has no constraints whatever, it has what may be called *six degrees of freedom*, viz. translation parallel to each of the three co-ordinate axes and rotation about each of them. It should be noted that what is styled a single degree of freedom of a rigid body may (namely, if it is a rotation) involve the two-dimensional motion of its points. And if with this rotation we combine a translation along the axis of rotation, we have a three-dimensional motion of its points. Whereas two translations and a rotation about the axis perpendicular to both of them involves only a two-dimensional motion, for all the motions occurring are parallel to the plane containing the two translations. Thus the classification of motions according to their constraints and degrees of freedom, though very useful and needing to be borne in mind, differs materially from that according to dimensions in

ternal particle the same attraction as though its whole mass were concentrated at its centre O.

*Case II. Particle Inside.*—For the particle inside, the inferior limit of integration changes sign. We thus have

$$F=-\frac{\gamma\pi a\sigma}{c^2}\int_{a-c}^{a+c}\left(1+\frac{c^2-a^2}{r^2}\right)dr=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

That is, *no attraction* is experienced by a particle within a uniform spherical shell of material attracting according to the law of the inverse square of the distance.

*Case III. Particle on Shell.*—We consider finally the case of a particle on the outer surface of the shell. Here $c=a$, and the limits of integration are o and $2a$; but if we attempt to integrate (4) with these inserted, a difficulty is experienced with respect to the term $(c^2-a^2)/r^2$. It appears at first sight to be zero. But this is not the case where $r$ is very small near the lower limit of integration. Perhaps the simplest method is to write first for $c$ the sum $a+h$, and then finally make $h$ vanish. The limits of integration are accordingly $h$ and $2a+h$, and we have from (4)

$$\begin{aligned}F&=-\frac{\gamma\pi a\sigma}{c^2}\int_h^{2a+h}\left(1+\frac{(a+h)^2-a^2}{r^2}\right)dr\\&=-\frac{\gamma\pi a\sigma}{c^2}\left[r-\frac{2ah}{r}\right]_h^{2a+h}\\&=-\gamma 4\pi a^2\sigma/c^2 \text{ when } h \text{ vanishes.}\end{aligned}$$

Thus $$F=-\gamma\frac{M}{c^2}=-\frac{\gamma M}{a^2}=-4\pi\gamma\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Or, the law for the outside region holds good up to and including the surface of the shell itself.

In article 359 will be found an alternative method for obtaining these fields from the potential. This, though apparently lacking directness, is quicker for this particular problem.

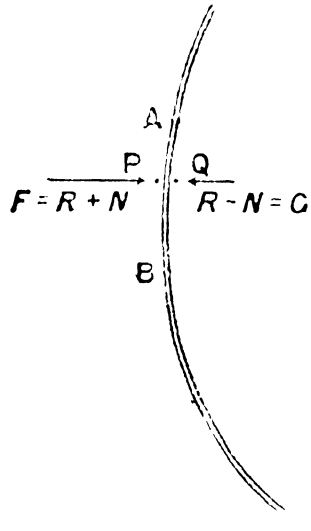



FIG. 149. SUDDEN CHANGE OF FIELD THROUGH SHELL.

**346. Sudden Change of Field through a Shell is $4\pi\gamma\sigma$.**—In article 340, equation (16), we saw that the attraction on unit mass near the centre of a large disc is $2\pi\gamma\sigma$, where $\sigma$ is the surface density of attracting matter in the disc. Hence the sudden change in the field on passing through the disc is from plus to minus the above, *i.e.* the change is $4\pi\gamma\sigma$. Again, in article 345 we saw by equations (6) and (7) that the attraction is zero inside a spherical shell and $4\pi\gamma\sigma$ just on it or infinitely near it outside. Thus in this case also the sudden change on passing through the shell is, as before, $4\pi\gamma\sigma$. These are only special examples of a general case, as may be seen, thus:—

Consider a portion of a very thin shell of gravitating matter and the fields $F$ and $G$ at very near points P and Q (Fig. 149) just opposite each other to the left and right of the shell. Take in the shell a disc-like portion AB whose diameter is *large*, compared with the distances of P and Q from its centre, but *small* compared with the rest of the shell. Then each of the fields at P and Q may be regarded as made up of the two terms, $R$ due to the portion of the shell beyond the disc AB, and $N$ due to the disc AB. The former term $R$ is practically the same in magnitude and direction for P and Q. Whereas the latter $N$, if called positive at P, is negative at Q, its magnitude remaining unchanged while its direction is reversed on passing through the disc. But $N$, as seen in article 340, equation (6), is $2\pi\gamma\sigma$. Hence the change of field on passing through the shell from one to the other of the infinitely near points P and Q on opposite sides of it is

$$F-G=R+2\pi\gamma\sigma-(R-2\pi\gamma\sigma)=4\pi\gamma\sigma \quad . \ . \ . \quad (8).$$

## EXAMPLES—LXVII.

1. Obtain the attraction of a thin circular disc for a particle on the axis. What does this become for an infinite diameter?
2. Pass from the attraction of a disc for a particle to the cases of
   (i) a disc and an axial filament, and
   (ii) a cylinder and an axial particle.
3. Prove that the attraction of a thin uniform spherical shell for an external particle is equal to that of the shell condensed to its centre.
4. Show that on passing through a thin shell of surface density $\sigma$ there is a sudden change in the field of $4\pi\gamma\sigma$, where $\gamma$ is the gravitation constant.
5. 'Show by any method that the attraction of an infinite uniform plane stratum of matter is the same at all points external to it.
   'How could this result have been predicted from the theory of "dimensions"?'
   (LOND. B.SC., PASS; APPLIED MATH., 1905, II. 8.)
6. 'Prove that the attraction of a thin uniform circular disc upon a particle of unit mass situated on its axis at a distance $z$ from the disc is $2\pi\gamma\rho[1-z/\sqrt{a^2+z^2}]$, where $a$ is the radius of the disc, $\rho$ its mass per unit of area, and $\gamma$ is the gravitation constant.'
   (LOND. B.SC., PASS, APPLIED MATH., 1906, II. 10, 1st part.)
7. 'Find the attraction at any point, internal or external, due to an infinite uniform layer of matter of thickness $2a$ and density $\rho$, bounded by parallel plane faces.
   'Show that if $N$ be the normal attraction of such a distribution, and $z$ be measured perpendicular to the layer, $dN/dz$ is discontinuous at the interface of this layer.
   'Show that, in the case of an infinitely thin plane layer of uniform surface density, $N$ itself is discontinuous as we pass through this layer.'
   (LOND. B.SC., PASS, APPLIED MATH., 1910, II. 7.)
8. 'A hole, bounded by the circle $r=a$, is cut in a uniform lamina of surface density $\sigma$ whose edge is the curve $r=be^{\cos\theta}$; prove that the attraction at the origin is $\pi\gamma\sigma$, where $\gamma$ is the constant of gravitation.
   'Prove that if the edge of the lamina is $r=f(\theta)$, and the edge of the hole $r=cf(\theta)$, where $c$ is a constant, the attraction at the origin is zero.'
   (LOND. B SC., PASS, APPLIED MATH., 1908, II 10.)

**347. Solid Sphere and Thick Shells.**[1]—We may easily pass from the attraction of a thin shell to that of a thick one bounded by concentric spheres, for unit mass at points outside, in the substance of the shell itself, or within the cavity. Let the external and internal radii of the shell be $a$ and $b$ respectively and the distance of the particle of unit mass from the shell's centre be $c$, also let the shell have density $\rho$ and total mass $M$.

*Case I. Particle Outside.*—Then when $c>a$, the particle being at the outside of all the thin concentric shells of which we may regard the thick shell built up, each shell is replaceable by its mass at the centre (equation (5), article 345). Hence the field is given by

$$F=-\gamma M/c^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

which clearly holds also for a *solid sphere.*

*Case II. Particle in Cavity.*—Here $c<b$, and the particle is inside every one of the thin shells which build up the thick one. Hence the field is zero, for each such shell produces no field in its cavity (equation (6), article 345), or

$$F=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

*Case III. Particle in Substance of Shell.*—Here we have $a>c>b$. We must accordingly divide the thin shells into two sets:—(i) those whose radii exceed $c$, which clearly contribute nothing to the field, and (ii) those whose radii do not exceed $c$, and which may be replaced by their masses at the centre (equation (5), article 345).

Thus we find, for the field in question,

$$F=-\tfrac{4}{3}\pi\gamma(c^3-b^3)\rho/c^2 \quad . \quad . \quad . \quad . \quad . \quad (11).$$

*Case IV. Field in Solid Sphere.*—On putting $b=0$ and $c=r$ in (11), we pass to the field at any point in the substance of a solid sphere, which is accordingly

$$F=-\tfrac{4}{3}\pi\gamma\rho r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

or $F\propto r$. To make this applicable to the earth, supposed solid and homogeneous, on the surface of which at radius $R$ the field is $-g$, we have

$$g=\tfrac{4}{3}\pi\gamma\rho R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Hence (12) may be written

$$F=-gr/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Of course, $g$, $r$, and $R$ must be expressed in terms of the same system of units to give $F$ correctly in any system.

**348. Graphical Representation of Fields.**—It is often very desirable to represent fields graphically, their intensities being plotted as ordinates and the distances of the points as abscissae. In other words, the distance of the point P from the centre of shell or sphere is plotted as $x$ and the corresponding value of the field as $y$. Figs. 150 and 151 show the fields in this way for the thin spherical shell and solid sphere. Since the field is to the left when the distance is to the right, the ordinates are always negative when the abscissae are positive. The student may with advantage draw other fields for himself, preferably

[1] Art. 352 and the principle of symmetry afford a useful alternative method.

using squared paper on which to plot to scale. Fig. 150 gives an interesting academic example of a discontinuity in the field. This discontinuity would not, however, occur in nature, for the shell to have

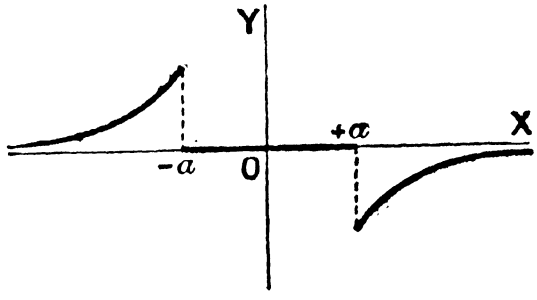



FIG. 150. FIELD OF THIN SHELL.

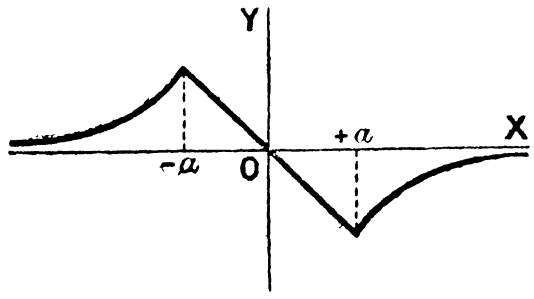



FIG. 151. FIELD OF SOLID SPHERE.

a sensible value of the surface density $\sigma$ would really need an appreciable thickness also. And in this thickness the change of field $4\pi\gamma\sigma$ would occur without actual discontinuity.

**349. Field in Eccentric Spherical Cavity.**—Consider now a homogeneous sphere of radius $a$ and density $\rho$ with an eccentric spherical cavity of radius $b$, and let it be required to find the field at any point in the cavity. Let the centre of the sphere be S, that of the cavity C, and consider the field at P (Fig. 152). Regard the eccentric shell as built up of a complete sphere of radius $a$ and density $\rho$, and a second sphere of radius $b$ and density *minus* $\rho$, their centres being at S and C respectively. Then the field at P has components in the directions PS and CP due to these component spheres and expressed by

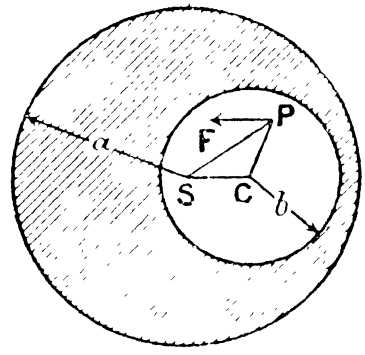



FIG. 152. FIELD IN ECCENTRIC SPHERICAL CAVITY.

$$-\tfrac{4}{3}\pi\gamma\rho\mathrm{SP} \text{ and } -\tfrac{4}{3}\pi\gamma(-\rho)\mathrm{CP}.$$

Hence these components may be represented to scale by PS and CP and their resultant $F$ by PF parallel and equal to CS. Thus we have for the field at P

$$F = -\tfrac{4}{3}\pi\gamma\rho\mathrm{SC} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

And since this value is independent of the co-ordinates of P, it applies to any point in the cavity whose field is consequently uniform throughout.

**350. Newtonian Constant of Gravitation.**—For detailed accounts of the determination of the Newtonian constant of gravitation the reader is referred to physical text-books. It may, however, be remarked here that many of the methods consist essentially in finding the attraction between given bodies at a specified distance apart in terms of the weight of a standard body or unit. For this determination Professor C. V. Boys used a special torsion balance, Professor J. H. Poynting

used a special form of the ordinary beam balance, as did also Drs. F. Richarz and Krigar-Menzel. Suppose by any method it is found that the force of attraction between masses $M$ and $m$ a distance $r$ apart equals the weight of a mass $n$. Then we have

$$F=\gamma Mm/r^2=ng \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But, if the earth be taken as a nearly homogeneous sphere of radius $R$ and mean density $\Delta$, we have (by equation (13) of article 347)

$$g=\tfrac{4}{3}\pi\gamma\Delta R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus, by division, we find the density in terms of measurable quantities:—

$$\Delta=\frac{3Mm}{4\pi nr^2R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

The value of the Newtonian constant may be found from (1) or (2) provided $g$ is known from experiments with the pendulum or by other means. Thus from (1) or (2)

$$\gamma=\frac{ngr^2}{Mm} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

The values found for $\gamma$ and $\Delta$ by the investigators referred to are given in Table XIII.

TABLE XIII. GRAVITATION CONSTANT AND EARTH'S DENSITY.

| INVESTIGATOR. | NEWTONIAN CONSTANT. $\gamma$ in *c.g.s.* units. | EARTH'S MEAN DENSITY. $\Delta$ in gm /cc. |
|---|---|---|
| C. V. Boys . . . | $6{\cdot}6576\times10^{-8}$ | 5·527 |
| J. H. Poynting . | $6{\cdot}6984\times10^{-8}$ | 5·4934 |
| Richarz and Krigar-Menzel | $6{\cdot}685\times10^{-8}$ | 5·505 |

## EXAMPLES—LXVIII.

1. Find the fields inside and outside a solid sphere of homogeneous material, and plot them as graphs.
2. Show that the field is uniform in an eccentric spherical cavity of a homogeneous sphere.
3. 'Prove that the attraction exerted at any external point by a homogeneous solid sphere of gravitating matter is the same as that which would be exerted by a particle of equal mass situated at the centre. Find the law of attraction in the interior of the sphere.

'Show that, if a smooth straight tunnel could be cut between any two places on the earth's surface, it would be traversed by a particle, starting from rest and influenced only by gravity, in about 42½ minutes.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, II. 8.)

4. 'Prove that the gravitational force within a uniform solid sphere varies directly as the distance from the centre,

'A uniform solid sphere of mass $M$ is cut in two by a diametral plane; show that the resultant attraction between the halves is

$$\tfrac{3}{16}\gamma M^2 a^{-2},$$

where $a$ is the radius of the sphere and $\gamma$ the constant of gravitation.'

(LOND. B.SC., PASS, APPLIED MATH., 1908, II. 9.)

5. 'If the volume density of a sphere varies uniformly from zero at the centre to $2\rho$ at the surface, show that the intensity of its attraction at a point of its surface is half as great again as that of a sphere of the same radius whose volume density is uniformly equal to $\rho$.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, II. 7.)

**351. Solid Angles, Lines and Tubes of Force.**—Let ABCDE (Fig. 153) be any closed figure, curved or polygonal, plane or not, and let lines be drawn through each point of the closed figure to any other point O. Then these lines form a cone, and the closed figure is said to subtend at O a *solid* or *conical* angle, which we will denote by $\omega$. To define it quantitatively, describe about O as centre a sphere of radius $r$ intersecting the conical surface at *abcde*, and let the area of the spherical surface thus enclosed be $S$. Then

$$\omega = S/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the relation which defines the magnitude of the solid angle. It is easily seen, as in the case of a plane angle, that $\omega$ is independent of $r$ for a given cone.

If there is a particle of gross matter of mass $m$ at O in Fig. 153, it is obvious that the generating lines of the cone in question, or of any cone with vertex at O, would be directions along which are directed the forces of attraction of $m$ for any other particles lying on those generating lines. And even if the other particles are not present to experience that attraction, we can still think of the *field* round $m$, and describe it by drawing these lines and specifying the intensity of the field $\gamma m/r^2$ at a number of points. Accordingly the lines in question might be suitably called *lines of the field*. They are, however, usually called, with less appropriateness, *lines of force*.

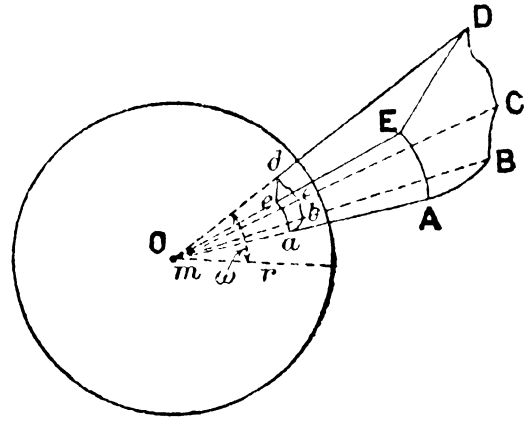



FIG. 153. SOLID ANGLE.

If we take an infinite number of such lines side by side, say those which practically constitute the conical surface between ABCDE and *abcde* (Fig. 153), we then have what is usually called a *tube of force*. Here again *tube of the field* would be a better phrase.

Now each line of the field has at each point the direction of the attraction experienced by a small particle placed there, and the wall of the tube is composed of these lines. Hence no attraction crosses the wall of such a tube from inside to outside or *vice versa*.

Consider now the field $F$ at a distance $r$ from a particle of mass $m$. It is given by

$$F=\gamma m/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

which accordingly expresses the field at the spherical surface in Fig. 153. Let us now take the product, field into area, for a portion of the spherical surface, say that contained by the cone. Then this product is $FS$; or, substituting from (1) and (2) the values of $S$ and $F$, we have

$$FS=\gamma m\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

which is a constant for the given cone of solid angle $\omega$ with the mass $m$ at its vertex O, independent of the distance $r$ from O. Thus, if we describe another sphere about O with radius $r'$, the field there being $F'$ and surface enclosed by the cone $S'$, we should have the product of the same numerical value as before. But if we were taking the product, field into area, positive when *entering* this frustum of a cone, we should have to give the two products for its bases opposite signs. And we have seen before that no field crosses the sides of a tube of force. Accordingly for the whole product entering such a tube of force we have

$$FS+F'S'=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

If we drew the bases of the frustum *oblique*, making an angle $\theta$ say with the spherical surface, then an equation like (4) would still hold if we took the *component of the field normal* to the new surface. For obviously this component would be the true field *multiplied by cos* $\theta$, and the new surface would be the corresponding portion of spherical surface *divided by cos* $\theta$. Hence (4) holds for any surfaces provided the $F$'s always mean components *normal* to those surfaces.

It is easy to give to these relations a graphical aspect. For, since the product $FS$ remains constant for the cross section of a tube of force, it is evident that we may represent the intensity of a field at any section by the *number per unit area of continuous lines* drawn through it. Thus the product $FS$ will then be the *total number* of such lines in the tube, which by their continuity remains throughout a constant quantity, as required by equation (3). Hence, these lines may pass continuously from a given mass *without annihilation or creation* in space *devoid* of ponderable matter.

Another and perhaps better plan of graphic representation is to imagine the tube of force divided into a number of smaller tubes lying side by side, such that the cross section of any such small tube shall have its value of $FS$ equal to unity. Such tubes may be called *unit tubes.*

If, for our conical tube of force round a mass $m$, we take the whole external space, we must write for $\omega$ the value $4\pi$. Hence, for all the lines or unit tubes from a mass $m$, we have by (3)

$$FS=4\pi\gamma m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

If our field is due to two or more particles the lines of force and tubes of force will usually be curved, but the above relations still hold, as will be seen presently.

**352. Gauss' Theorem.**—The important theorem given by Gauss in 1839 may be stated as follows (see Routh's *Statics*, ii. p. 52, 1902):—

'Let $S$ be any closed surface, and let $M$ be the sum of the attracting masses which lie within the surface, $M'$ the masses outside. Let $dS$ be any element of area of this surface, $F$ the normal component at this element of the attraction of the whole mass both internal and external. Then shall

$$\int F dS = \pm 4\pi M,$$

where the integration extends over the whole surface of $S$ and the upper or lower sign is taken according as $F$ is estimated positive or negative when the normal force acts inwards.'

In the above enunciation the units of force, length, and mass are supposed to be such as make the Newtonian constant of gravitation unity. Thus $\gamma$ does not appear. If we write the equation on the understanding that ordinary units are employed, then $\gamma$ appears on the right side. If also we take $F$ positive inwards, the above equation becomes

$$\int F dS = 4\pi\gamma M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

This theorem may be easily established by reference to Fig. 154 and use of the equations (3), (4), and (5).

For consider first at O' outside the surface $S$ any particle $m'$ of the total $M'$ outside. And draw from O' a small cone cutting the surface $S$ at A'B'C'D'. Then by (3) and (4) the products $FdS$ at A' and B' annul each other, as do also the other pair of values at C' and D'. And it is evident that any cone from an outside point O' will intersect any such closed surface in an *even* number of places. Hence no such cone with vertex outside $S$ can contribute anything to the final sum expressed by the left side of (6).

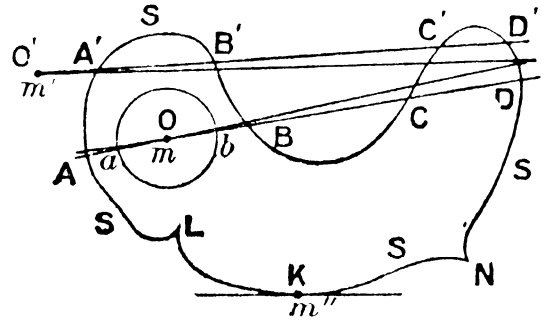



FIG. 154. GAUSS' THEOREM.

Second, consider a mass $m$ at the point O inside the closed surface $S$, and draw from it a small cone cutting the surface at ABCD. Also describe round O and inside the surface $S$ a sphere cutting this cone at $a$ and $b$. Then, here again, we see by (4) that the surfaces CD will contribute nothing to the sum required, for they give products of equal numerical value and opposite signs. But the surfaces at A and B will contribute something to the integral sought, for they each give positive products of the same value as expressed by (3). And, when we have taken all possible cones with vertices at O so as to embrace the whole closed surface $S$, we shall also have included the whole of the sphere $ab$ subtending at O the solid angle $4\pi$. Hence, by (3) or (5), we see

that the particle $m$ at O contributes to the required integral the product, which we may write

$$\int f dS = 4\pi\gamma m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where $f$ refers to the field due to $m$ only. Thus, when we sum $m$ to obtain the total internal mass $M$, $f$ becomes $F$, and we have

$$\int F dS = 4\pi\gamma M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

as was required to be shown.

From this result it follows that for any tube of force, straight or curved, containing no masses, the sum or normal flux $\int F dS$ is zero for the whole surface of the tube. But since there is no $F$ through the sides of the tube, we have for the ends $FS + F'S' = 0$, as was obtained simply in (4).

We may further notice that, if a mass $m''$ is situated at an ordinary point on the surface itself (say at K, Fig. 153), then the whole surface $S$ subtends at this point the solid angle $2\pi$. Hence for such a particle the total value of the integral would be $2\pi\gamma m''$. Thus for a mass $M''$ consisting of particles on the surface at ordinary points, we should have generally

$$\int F dS = 2\pi\gamma M'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

This applies only to particles at points like $K$ which may be touched by single tangent planes. At internal or external conoidal cusps resembling dimples or spikes, as L and N, the above value would obviously be modified, and might range anywhere between the extreme values of $4\pi\gamma m$ and zero.

Calling the left sides of (6), (7), (8), and (9) the Gauss integral, we may summarise the results of this article thus :—

The Gauss integral for a closed surface $S$ receives no contribution from masses outside it, and has the value $4\pi\gamma$ times the total mass inside it. Particles on the surface itself contribute to the integral the value $2\pi\gamma$ times their mass if at points of the surface each of which may be touched by a single tangent plane, but a larger or smaller value if situated at the vertex of an internal or external conoidal cusp. We sometimes for brevity use the term *flux* instead of Gauss' integral.

**353. Potential Introduced.**[1]—As we have already seen, the gravitational field in any region may be specified by stating its magnitude and direction at a sufficient number of points. It may also be represented graphically by unit lines or unit tubes, the closeness of these lines or tubes per unit area being directly as the intensity of the field.

Further, if at a certain point P the field has a magnitude $F$ and a given direction, then the component of the field in any other direction inclined $\theta$ to the lines of the field at P is clearly given by $F \cos\theta$, since the field is a vector quantity. Hence, the field being specified by its

[1] For symbolic definitions of potential, if desired at the outset, see Equations (11) and (12) of art. 355.

magnitude and direction, we can find the components of the field in any other directions whatever.

All the foregoing may be illustrated by a possible method of indicating the slopes of a hilly district on a map. Thus we might state, for each of a number of points on the map, the direction of the steepest slope there and its gradient (analogous to the direction of the field and its magnitude). And from these supposed details on the map we could then deduce the gradient in any other direction at any one of these points.

But, instead of carrying out on maps this idea (like our specification of a field), an entirely different method is generally taken to present quantitatively the features of the hills and valleys. Thus, we usually have the height above sea-level stated for a number of points, and often series of lines are shown, each point of any one line denoting the same height above sea-level. These are called *contour lines*, and are given on many ordnance and tourist maps, sometimes with the intervening bands in special colours. Then, the heights being known, the gradient in any direction is inferred as the *rate of change of height* per unit horizontal distance in that direction. Here, then, there is substituted a *scalar* quantity, *height* above sea-level, in place of the *vector* quantity, *steepest gradient*. A distinct advantage in simplicity results from this method of describing or specifying the region, and very little disadvantage, if any, is entailed in consequence. For each point requires only the magnitude of the scalar quantity to be given, whereas both magnitude and direction are needed for a vector quantity. And, as to the readiness of using the map, it is practically as easy to read the gradients in any direction from the heights and their distances apart as it would be from the steepest gradients if shown.

A like advantage is often obtained in the theory of gravitational fields, if instead of specifying them as heretofore we state for each point a scalar quantity called the *potential*, whose *rate of change* in any direction gives the corresponding field component. The *maximum* rate of change of the potential at any given point gives by its direction and magnitude the lines and intensity of *the field* at that point.

This potential, from which the gravitational field is derived, is often called the Newtonian potential to distinguish it from the electric or magnetic potentials from which the corresponding fields are in like manner derived.

**354. Potential and Field.**—Let $V$ denote the potential at any point P, and let $X$, $Y$, and $Z$ be the field components parallel to the axes of co-ordinates. Then, by the relations between them, we have

$$X=\frac{dV}{dx},\quad Y=\frac{dV}{dy},\quad Z=\frac{dV}{dz} \quad \ldots\ldots \quad (1).$$

In any direction defined by the co-ordinate $s$ (see Fig. 155) the field $F$ is given by

$$F=\frac{dV}{ds} \quad \ldots\ldots\ldots\ldots \quad (2).$$

The resultant $R$ of $X$, $Y$, and $Z$ gives *the* field, or the direction and

magnitude of *maximum* space rate of change of $V$. Thus, we have the field given by

$$R^2=X^2+Y^2+Z^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

stating its magnitude, and the following set of direction cosines

$$X/R,\ Y/R,\ Z/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

defining its direction.

If we denote by $r$ the co-ordinate along the direction of $R$, we may write

$$R=\frac{dV}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

If we suppose the direction cosines of the co-ordinate $s$ to be

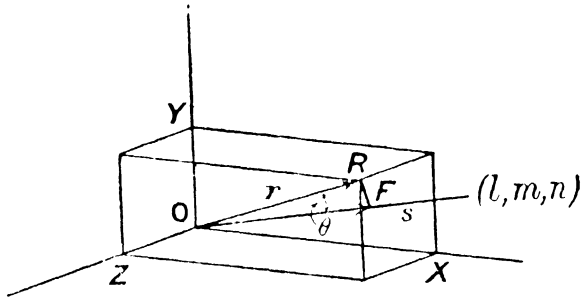



FIG. 155. POTENTIAL AND FIELDS.

$l$, $m$, $n$ referred to the axes of $x$, $y$, $z$, and the angle it makes with that of $r$ to be $\theta$, we have by projecting $R$ upon $s$

$$F=R\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

But also by projecting upon $s$ the $X$, $Y$, $Z$ components of $R$, we obtain

$$F=Xl+Ym+Zn \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Hence the right sides of (6) and (7) should agree. This is easily seen to be the case if we write the usual expression for the cosine of ROF. Thus, from (4),

$$\cos\theta=\frac{X}{R}l+\frac{Y}{R}m+\frac{Z}{R}n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

so that the two expressions for $F$ in (6) and (7) are seen to be identical.

**355. Potential and Work.**—We have hitherto written for the field $F$, as derived from the potential $dV/ds$ simply, without regard as to whether the positive or negative sign was appropriate. This question must now be examined. We see from (1) that change of potential equals field multiplied by distance. If we place a particle in the field it experiences a force. So the passage of a particle from one potential to another involves the product force multiplied by distance. Accordingly such a passage involves work. Thus, if we have the signs rightly arranged, a body must have work done on it to pass to a *higher potential*, and then possesses the *exact equivalent of that work in its potential energy*. To secure this convention of signs, the potential and potential energy must increase when the motion is against the field. That is, when the force exerted on the body is $-Rm$, where $R$ is the

field and $m$ the mass of the body moved in it. Thus, we must have for the increment of work done on the body

$$dW = mdV = (-Rm)dr$$

or

$$R = -dV/dr \quad \ldots\ldots\ldots (9).$$

And, if the field $R$ is due to a mass $M$ at distance $r$, we have

$$R = -\gamma M/r^2 \quad \ldots\ldots\ldots (10).$$

Thus, by (10) in (9),

$$dV = \gamma \frac{Mdr}{r^2} = -\frac{dW}{m} \quad \ldots\ldots\ldots (11),$$

and we may well note here that the equality of the first and last quantities in (11) forms the basis of the definition of potential in elementary electrical theory.

Now let the *zero* value of $V$ be arbitrarily chosen to correspond to the infinite distance from the mass $M$ to which it is due. Then we may integrate (11) between the limits as follows:—

$$\int_0^V dV = \gamma M \int_\infty^r \frac{dr}{r^2}.$$

Whence

$$V = -\gamma M/r \quad \ldots\ldots\ldots (12).$$

Thus we see by (9) and (12) that, with these conventions for mutually attracting material, the field is the rate of *decrease* of the potential, and that the potential is itself *negative* for all finite values of the radius $r$.

But as mutual repulsions also have to be dealt with in electrostatics and magnetism, it is customary in mechanics to calculate the numerical values of field and potential without inserting the negative signs in either of the relations shown in (9) and (12). Hence, in the following working of the potentials and fields for various cases, we shall usually regard them as expressed by

$$V = \gamma M/r \quad \ldots\ldots\ldots (13),$$

and

$$F = dV/ds \quad \ldots\ldots\ldots (14),$$

the appropriate signs being afterwards inserted if required.

But, in plotting curves for $V$ and $F$, the correct relations are observed in this work (see Figs. 160 and 161 of article 361).

It may be noted here that, if the velocities of two attracting bodies are derived from their loss of potential energy in approaching each other, then the velocities of *each* body with respect to the *centre of mass* of the two bodies will be so determined. The kinetic energy of either body as found from its velocity relative to the other will not necessarily equal the loss of potential energy of the system by the mutual approach of its parts.

The forms of (13) and (14) suggest some other expressions for potential and change of potential. Thus, from (13), we see that if the potential at a point P is due to various masses $m_1$, $m_2$, etc., distant $r_1$, $r_2$, etc., from P, then we may write

$$V = \gamma \Sigma \frac{m}{r} \quad \ldots\ldots\ldots (15).$$

For obviously, each element of the potential contributed by any quotient $m/r$, being a scalar, they all add arithmetically. If, however,

the mass is continuously distributed with a density $\rho$ at the distance $r$ from P, then

$$V=\gamma\iiint\frac{\rho}{r}dxdydz \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

gives the resulting potential, the integration extending over the whole volume occupied by the matter in question. One of the forms, (15) and (16), is the basis of any calculation of the potential produced by specified masses.

Another view of potential difference, viz. the *line integral of the field*, is suggested by (14). Thus, transforming it, integrating between $s'$ and $s$ along the path taken in the field, and calling the two potentials $V'$ and $V$, we find

$$V-V'=\int_{V'}^{V}dV=\int_{s'}^{s}Fds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

In this equation $F$ is the field *component along the curve* $s$.

**356. Equipotential Surfaces.**—We have already seen that the resultant field $R$ has the direction and magnitude of the maximum rate of change (or *gradient*) of the potential. And further, that the value $F$ of the field inclined at an angle $\theta$ to $R$ is $R\cos\theta$. Hence at right angles to $R$ the field vanishes. Thus, since the potential difference is the line integral of the field, there can be no potential difference if we move in any direction at right angles to the resultant field at the place. By moving in such a manner we should describe an *equipotential surface*. And it is evident that a region in which gravitational or other attractions are experienced may be graphically represented by plotting the field lines or tubes and the sections of the equipotential surfaces. Thus, for a single particle, the field lines or lines of force are radial and equally distributed, the tubes of force are of equal conical form with a common vertex at the particle, and the equipotential surfaces are concentric spheres. They may be drawn to represent a common difference of potential by use of equation (13).[1]

## EXAMPLES—LXIX.

1. Explain what you mean by lines and tubes of force, and how a field may be specified in terms of them.
2. Defining *potential* as that function of the co-ordinates whose space rate of change is the field, prove that the difference of potential at two points is the quotient $W/m$, where $W$ is the work done on a particle of mass $m$ as it passes from one of those points to the other.

[1] *Potential of Straight Filament.*—For the uniform filament SS′, of length $l$ and linear density $\lambda$, let it be required to find the potential at any point P distant $r$ and $r'$ respectively from the ends S and S′. Then, by art. 337, the direction of the *field* at P bisects the angle SPS′, so it is the normal at P to the ellipse through P whose foci are S and S′. Hence, for the filament SS′, this ellipse is the equipotential line through P for the plane SPS′. Let S′S produced meet this ellipse at A. Then the potential at P equals that at A, and this last is readily found by use of (15) of art. 355. Thus, $V_P=V_A=\gamma\lambda\int_{AS}^{AS'}\frac{dx}{x}=\gamma\lambda\log\frac{r+r'+l}{r+r'-l}$. It is obvious that the equipotential *surfaces* are ellipsoids derived from the ellipses by revolution about SS′.

3. 'Define the terms *tube of force* and *equipotential surface.* Prove that in a field of force due to gravitating matter systems of tubes of force and of equipotential surfaces can be drawn, such that at any point of free space the force varies inversely as the cross section of the tubes, and also inversely as the distance between consecutive surfaces.
'Will these properties be affected if the attraction does not follow the Newtonian law?'
(LOND. B.SC., PASS, APPLIED MATH., 1905, II. 7.)

4. 'Show that, if the law of attraction be that of the inverse square, the surface integral of normal attraction over any closed surface $=4\pi\gamma$ (mass inside), where $\gamma$ is the constant of gravitation.'
(LOND. B.SC., PASS, APPLIED MATH., 1907, II. 10.)

5. 'Define equipotential surfaces, lines of force, tube of force. Show that if $F$ be the force intensity along any tube of force, $\sigma$ the cross section of the tube, then $F\sigma =$ constant along the tube of force.'
(LOND. B.SC., PASS, APPLIED MATH., 1908, II. 8.)

6. 'Define the potential of a given distribution of matter at a point external to the attracting masses. Show how from the value of the potential to deduce the attraction in any direction at a given point.'
(LOND. B.SC., PASS, APPLIED MATH., 1906, II. 9.)

7. 'Define gravitation potential, and show that the component of attraction in any direction is the corresponding space gradient of potential.'
(LOND. B.SC., PASS, APPLIED MATH., 1907, II. 7.)

**357. Laplace's and Poisson's Theorems.**—These theorems enable us from a knowledge of the potential at and near a point to determine the density of any of the matter there.

To establish them simply by use of Gauss' theorem (article 352), consider an infinitesimal parallelepiped at the origin of co-ordinates and of edges $dx$, $dy$, $dz$ as shown in Fig. 156, and take for its bounding surface the total flux, or sum of products, normal field into area. Let $V_1$ be the mean potential at the face in the $yz$ plane. Then that at the opposite face may be written $V_1+(dV/dx)dx$. Then, differentiating these expressions with respect to $x$ to obtain the normal fields, and taking their difference since one is in and the other out, we have

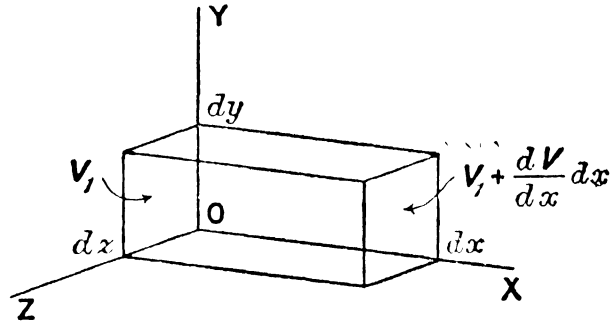



FIG. 156. LAPLACE'S AND POISSON'S THEOREMS.

$$\frac{dV_1}{dx}+\frac{d}{dx}\left(\frac{dV}{dx}\right)dx-\frac{dV_1}{dx}=\frac{d^2V}{dx^2}dx \quad \ldots\ldots (18).$$

Thus, multiplying by $dy$, $dz$, the area of either of the pair of faces at right angles to the $x$ axis, we see that they contribute to the required flux the quantity

$$\frac{d^2V}{dx^2}dxdydz.$$

The other two pairs of faces contribute corresponding terms, and so give for the total normal flux required, all over the parallelepiped, the symmetrical expression

$$FdS=\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dxdydz \quad . \quad . \quad . \quad . \quad (19).$$

But, by Gauss' theorem, we already know that this flux may be expressed in terms of the matter in the volume enclosed by the surface $S$ (see equation (8) of article 352).

*First*, suppose that this infinitesimal parallelepiped contains none of the gross matter to which this potential is due. The above sum is accordingly zero. This leads to *Laplace's Theorem*, which is then expressed by

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

*Second*, suppose that there is gross matter of density $\rho$ within the parallelepiped and to which the potential is wholly or partly due. Its mass is therefore $\rho dxdydz$, and, by Gauss' theorem, equation (19), then gives

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=4\pi\gamma\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

which expresses *Poisson's Theorem*.

In accordance with the convention of signs here adopted, the right side of (21) is positive. In using the theorem for an electric field, we have $F=-dV/ds$, and the right side is negative.

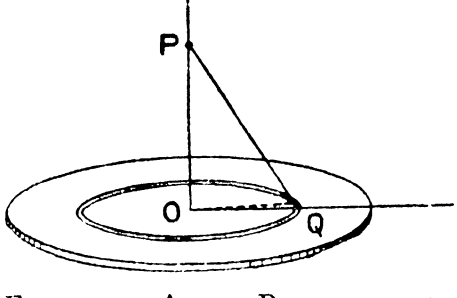



FIG. 157. AXIAL POTENTIAL OF DISC.

**358. Axial Potential of Disc.**—Consider now the potential $V$ at a point P distant $z$ along the axis from the centre of a disc of radius $a$ and surface density $\sigma$. Take, as the infinitesimal element, a ring of radius $r$ and width $dr$ passing through Q as shown in Fig. 157. Now, for each part of the ring, as that near Q and subtending the angle $d\theta$ at O, the contribution to the potential at P is simply (mass ÷ PQ), and all such parts add arithmetically. Hence for the whole ring element we may write the mathematical expression for this and integrate between the appropriate limits.

Thus $$\int_0^V dV=2\pi\gamma\sigma\int_0^a \frac{rdr}{\sqrt{r^2+z^2}}.$$

Whence $$V=2\pi\gamma\sigma(\sqrt{a^2+z^2}-z) \quad . \quad . \quad . \quad . \quad (1).$$

From this, by differentiating with respect to $z$, we obtain the $Z$ component of the field, *i.e.* the field along the axis of the disc.

$$Z=\frac{dV}{dz}=-2\pi\gamma\sigma\left(1-\frac{z}{\sqrt{z^2+a^2}}\right) \quad . \quad (2).$$

It may be seen that this agrees with the value found directly for the field and given in equation (15) of article 340.

Having the potential of a disc at *any* point on its axis, we could apply this to the potential at a *fixed* point P contributed by any thin slice of a cylinder of finite length, and so pass to the potential for the whole cylinder and then to its field. But for this particular problem the field is obtained more readily by the direct method, whereas in the case of the spherical shell it is rather quicker to obtain the potential first and then derive the field.

**359. Potential of Spherical Shell.**[1]—Let us now calculate the potential of any point P due to a thin spherical shell of radius $a$ and surface density $\sigma$. Let the centre of the shell be distant $c$ from P, and consider the infinitesimal ring, element QML, where PQ$=r$ and the angle POQ$=\theta$, all as shown in Fig. 158.

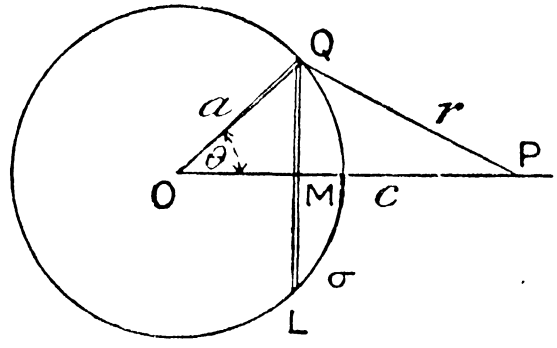



FIG. 158. POTENTIAL OF SPHERICAL SHELL.

Then the radius of this ring is $a \sin\theta$, its width $ad\theta$, and the distance of each part of it from P is $r$ simply. Thus its contribution to the potential at P is expressed by

$$dV = \gamma 2\pi(a\sin\theta)(ad\theta)\sigma/r = 2\pi\gamma a^2\sigma \sin\theta\, d\theta/r \qquad (3).$$

But we have here two variables $\theta$ and $r$, one of which we must eliminate by the properties of the triangle POQ. Thus

$$r^2 = c^2 + a^2 - 2ca\cos\theta,$$

which, on differentiating, yields

$$rdr = ca\sin\theta\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Using this, (3) reduces to

$$dV = 2\pi\gamma a\sigma dr/c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The treatment now falls under three heads according to the position of P.

*Case I. Point outside Shell.*—Integrating (5) between the appropriate limits we have

$$V = \frac{2\pi\gamma a\sigma}{c}\int_{c-a}^{c+a} dr = \frac{4\pi\gamma a^2\sigma}{c} = \frac{\gamma M}{c} \quad . \quad . \quad . \quad . \quad . \quad (6),$$

where $M$ is the mass of the shell and $c$=OP. Thus the potential is the same as for the given mass collected at its centre O.

[1] The principle of symmetry and the Theorem of Gauss [Eqn. (6) art. 352] furnish a useful alternative method. Thus, for any external point at distance $r$ from the centre of a spherical shell or solid sphere of mass $M$, the field $F$ is given by $\int FdS = F4\pi r^2 = 4\pi\gamma M$, or $F = \gamma M/r^2$. But, by (14) of art. 355, $F = dV/dr$; whence $V = \gamma M/r$ numerically.

*Case II. Point on Shell.*—Again integrating (5) with the limits required, we find

$$V=\frac{2\pi\gamma a\sigma}{a}\int_0^{2a}dr=\frac{4\pi\gamma a^2\sigma}{a}=\frac{\gamma M}{a} \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where $M$ is the mass of the shell and $a$ its radius.

*Case III. Point inside Shell.*—Using again the correct limits in (5) and integrating, we have

$$V=\frac{2\pi\gamma a\sigma}{c}\int_{a-c}^{a+c}dr=\frac{4\pi\gamma a\sigma c}{c}=\frac{\gamma M}{a} \quad . \quad . \quad . \quad . \quad . \quad (8),$$

showing that the potential is constant throughout the interior.

*Field outside Shell.*—Writing in (6) $x$ for $c$, and differentiating with respect to $x$, we find for the corresponding radial field

$$X=\frac{dV}{dx}=-\frac{\gamma M}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

as in (5) of article 341.

*The Field inside the Shell* is seen from (8) to be zero, because the potential shows no variation inside the shell.

**360. Potential of Solid Sphere.**—Let it be required to find the potential at any point of a solid sphere of radius $a$ and uniform volume density $\rho$.

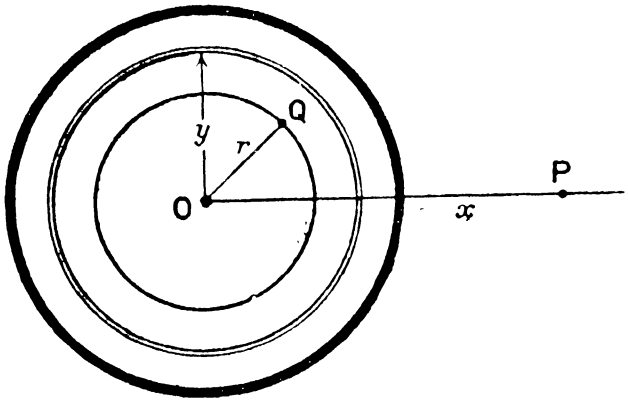



FIG. 159. POTENTIAL OF SOLID SPHERE.

*Case I. Point outside Sphere or on Surface.*—First, let the point be P outside the sphere as shown in Fig. 159, and denote OP by $x$.

Then since each thin shell of which we may regard the sphere as composed is inside P, its contribution to the potential at P is as though the mass of the shell were concentrated at the centre O of the sphere. Thus the potential at P is given by

$$V=\gamma\tfrac{4}{3}\pi a^3\rho/x=\gamma M/x \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

The corresponding radial field *outside* is

$$X=\frac{dV}{dx}=-\frac{\gamma M}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

And the above relations obviously hold when P moves up to and coincides with the outer surface of the sphere.

*Case II. Point inside Sphere.*—Now take the point Q inside the sphere at distance $r$ from its centre O. Then all the shells whose radii do not exceed $r$ may be replaced by their masses at the centre O, thus giving a contribution to the potential at Q expressed by $\gamma\frac{4}{3}\pi r^3\rho/r$. We have next to consider the shells external to Q of radius

$y$ say. Each of these gives to the potential at Q a contribution expressed by (mass $\div y$). Hence for the entire potential at Q we find

$$V=\tfrac{4}{3}\pi\gamma r^2\rho+\gamma 4\pi\rho\int_r^a y\,dy=\tfrac{4}{3}\pi\gamma\rho\left(r^2+\frac{3a^2-3r^2}{2}\right),$$

or
$$V=\tfrac{4}{3}\pi\gamma\rho\frac{3a^2-r^2}{2}=\gamma M\frac{3a^2-r^2}{2a^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Thus, on differentiating, we find for the radial field *inside*

$$R=\frac{dV}{dr}=-\tfrac{4}{3}\pi\gamma\rho r=-\frac{\gamma Mr}{a^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Further, we may note from (12) that the potential at the surface of the sphere is to that at the centre as 2 : 3, the values being $\gamma M/a$ and $3\gamma M/2a$ respectively.

**361. Graphic Representation of Fields and Potentials.** — We now give as illustrations of the relation between field, potential, and distance their graphic representation for the two important cases of a spherical shell and a solid sphere in Figs. 160 and 161 respectively. The distances from the centre are plotted as abscissae, the potential and field as ordinates. The curves for the field are in full lines marked $F$ and the potentials in broken lines marked $V$. The potential and field are each plotted of the appropriate sign which make potential correspond to potential energy, the zero value of potential being at infinity.

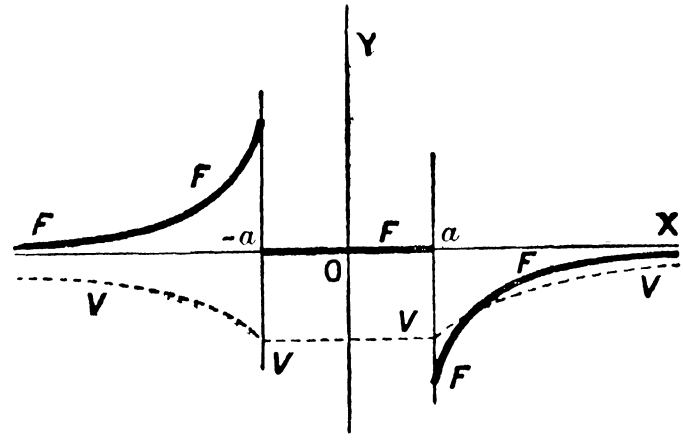



FIG. 160. POTENTIAL AND FIELD FOR SPHERICAL SHELL.

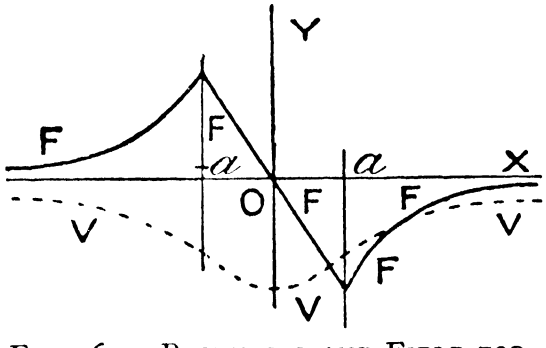



FIG. 161. POTENTIAL AND FIELD FOR SOLID SPHERE.

The student may make similar diagrams for other cases, say a thick spherical shell. The work of plotting is reduced if squared paper is used.

The discontinuity shown in the field for the spherical shell is only an ideal case, and would not occur with any real material shell, but would be there replaced by a line connecting the maximum field just outside the shell to the zero field inside.

## EXAMPLES—LXX.

1. Establish Laplace's and Poisson's theorems as to the space rates of change of field at places unoccupied and occupied by gravitating matter.
2. Find the axial potential of a thin circular disc, and from it derive the field.
3. Find both for external and internal points the potential due to uniform distribution of gravitating matter in an infinitely thin layer on a spherical surface.
4. 'Show that the mean value of the potential over a spherical surface, due to matter outside the sphere, is equal to the potential of this matter at the centre of the sphere.

   'How is the mean value affected by matter inside the sphere?'

   (LOND. B.SC., PASS, APPLIED MATH., 1905, II. 10.)
5. 'State Gauss' theorem of the surface integral of normal force.

   'Find the attraction of an infinite solid circular cylinder at an external point.'

   (LOND. B.SC., PASS, APPLIED MATH., 1909, II. 9.)
6. 'Find the potential of a uniform circular ring at a point on its axis.

   '$A$, $B$ are the ends of the axis of a straight uniform thin tube of circular section and mass $m$ per unit length : $\alpha$, $\beta$ are points on the corresponding edges ; $P$ is a point on $BA$ produced.

   'Prove that the potential at $P$ is

   $$\gamma m \log \{(PB + P\beta)/(PA + P\alpha)\} ;$$

   and show that if a particle, starting from rest at $A$, moves under the attraction of the tube, it will arrive at the middle point $O$ of $AB$ with velocity equal to

   $$\sqrt{[2\gamma m \log \{(OA + O\alpha)^2/A\alpha(AB + A\beta)\}]}.'$$

   (LOND. B.SC., PASS, APPLIED MATH., 1908, II. 11.)
7. 'Two similar spheres of radius $a$ are placed with their centres at a distance $4a$ apart ; find the time they take to come into contact under their mutual attraction.'

   (LOND. B.SC., PASS, APPLIED MATH., 1909, II. 8.)

# CHAPTER XVII

## PLANE STATICS OF RIGID BODIES

**362. Resultant of Parallel Forces.**—Suppose we have two parallel forces $P$ and $Q$ applied at the points A and B in a rigid body, and let it be required to find their resultant. Since the body is supposed rigid we can without change apply to the system at A and B two opposite forces each equal to $F$ and acting along AB as shown in Fig. 162. Then, compounding $P$ and the $F$ acting at A, also $Q$ and the $F$ acting at B, we obtain AP′ and BQ′ respectively. These we may produce to meet in a point O. Then at O we may resolve AP′ into $OF_1$ parallel to the $F$ applied at B and a force equal to $P$ and parallel to its original direction, viz. along OC. Similarly, BQ′ may be resolved at O into $OF_2$, equal and opposite to $OF_1$, and a force $Q$ along OC. Hence $OF_1$ and $OF_2$ annul each other, and we are left with a force equal to $P+Q$ along OC. We may accordingly write for the magnitude of the resultant

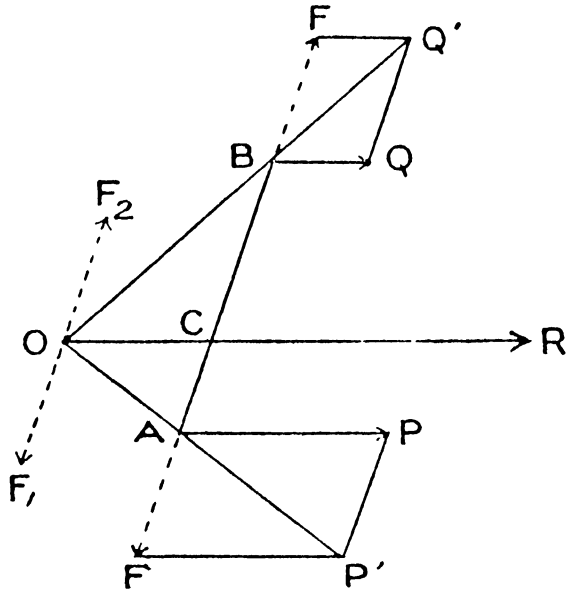



FIG. 162. RESULTANT OF PARALLEL FORCES.

$$R=P+Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

For its point of application $C$ we have by the geometry of the figure

$$CA/OC=F/P \text{ and } BC/OC=F/Q.$$

Thus $$P.CA=F.OC=Q.BC,$$

or $$\frac{P}{BC}=\frac{Q}{CA}=\frac{R}{AB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Since the body to which $P$ and $Q$ are applied is supposed to be rigid, the resultant $R$ may be regarded as applied at any point of the body in the line OCR.

The figure has been drawn for $P$ and $Q$ in the same direction, that is, each may be reckoned positive. If they are in opposite directions

one may be reckoned positive and the other negative. Then $R$ has the magnitude of their algebraic sum or numerical difference, and one of the segments BC or CA exceeds the length of the line AB, thus showing that C falls on AB produced.

To picture easily the resultant in any case, it is well to recall the set of three parallel forces *in equilibrium* as shown by $P$, $Q$, and $S$ in Fig. 163, and related in magnitude and position like $P$, $Q$, and $R$ just dealt with. Then, the resultant of *any two of these three* is obtained by *reversing the direction of the third*, leaving its magnitude and line of action unchanged. Thus, the resultant of AP and CS is BQ'.

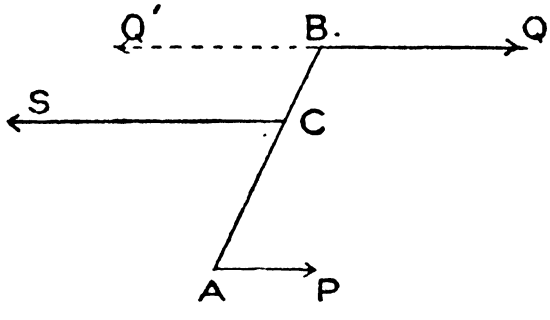



FIG. 163. PARALLEL FORCES IN EQUILIBRIUM.

Having seen how to determine the resultant of two parallel forces it is clear that, by repetition of the process, that of any number could be found. Thus, for forces $P_1$, $P_2$, $P_3$, etc., the resultant has magnitude

$$R=\Sigma P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

And if moments be taken about any point from which the perpendiculars on the forces and resultant are respectively $p_1$, $p_2$, $p_3$, etc., and $r$, we may find from (2) that

$$Rr=\Sigma Pp. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

**363. Couples.**—Suppose we consider two parallel forces of equal magnitude acting in opposite directions. By equation (1) the magnitude of their resultant disappears, and by (2) BC and CA are each infinite but of undetermined sign. In other words, we have here a system which refuses to reduce to a single force as its resultant, and the system in question must be treated as an entity in analysis which may be defined and described as follows:—

DEFINITIONS.—A pair of forces numerically equal and acting in opposite directions along parallel lines is called a *couple*.

The plane containing these localised forces is the *plane* of the couple.

Any line perpendicular to this plane may be regarded as the *axis* of the couple.

The product, either force into perpendicular distance between their lines of action, is called the *moment* of the couple.

This product, since its factors are perpendicular vectors, is itself a vector perpendicular to both component vectors, and hence parallel to, or along, the axis of the couple. For the only direction characterising the plane containing the factors of the moment is the normal to that plane, along which the vector representing the moment accordingly lies.

The moment of a couple may thus be represented graphically to

scale by a suitable length measured in the right direction along its axis. We shall here follow the right-handed system and take the *positive direction along the axis* related to the positive direction of rotation due to the couple as that of advance to rotation in a right-handed screw. Thus, if a couple tend to produce a counter-clockwise rotation in the plane of this paper, it would be represented by a line of suitable length drawn perpendicularly to the paper *towards* the reader.

The reason for defining as above the moment of a couple lies in the fact that this is the value of the algebraic sum of the moments of the two forces about any axis perpendicular to their plane.

Thus, if the forces are $P$ and $-P$, their perpendicular distance apart $p$, and we consider their moments about any point O, whose perpendicular distance from the nearer force $-P$ is $r$, then the sum of the moments about O is

$$-Pr+P(r+p)=Pp \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

This accordingly is the magnitude of the vector to be drawn along the axis in the direction as described to represent the couple. We can then compound any couples by the addition of vectors in the usual way.

**364. Resultant of Coplanar Forces.**—Let us now find the resultant of a system of any number of coplanar forces acting in different directions at various points of a rigid body.

Let the forces $P_1$, $P_2$, etc., act at points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, etc., and have components parallel to the co-ordinate axes denoted by $X$ and $Y$, with subscripts corresponding to the forces. Then, taking the component $X_1$, we may, as shown in Fig. 164, introduce two other opposite forces $X_1$ and $X'$ at the origin parallel to the original component and each of the same magnitude as $X_1$. This is legitimate, because the body is supposed rigid. Then we have the force $OX_1$ at the origin and also the couple formed by $A_1X_1$ and $OX'$, which is of moment $(-X_1y_1)$.

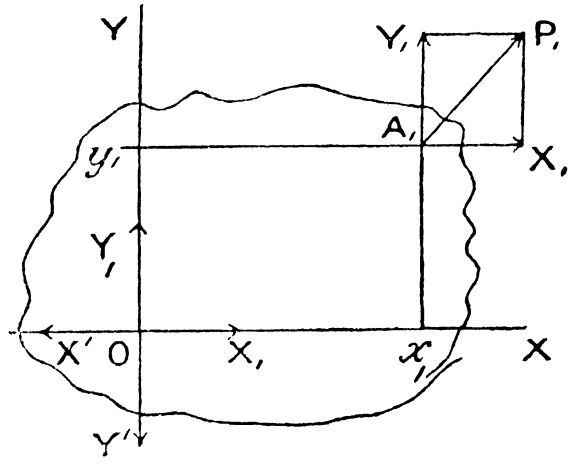



FIG. 164. REDUCTION OF COPLANAR FORCES.

Similarly, we may introduce forces $Y_1$ and $Y'$ at the origin in opposite directions, but each of the magnitude of $Y_1$ at $A_1$, and parallel to its line of action. Then we obtain the force $Y_1$ at the origin and the couple formed of $A_1Y_1$ and $OY'$, which is of moment $(+Y_1x_1)$.

Proceeding in like manner with the remaining forces, and summing by algebraic addition the three sets of vectors so obtained, we find for the whole system :—

Forces along OX $= \Sigma X = U$ say . . . . . (1).
Forces along OY $= \Sigma Y = V$ say . . . . . (2).
Moments in the plane XOY $= \Sigma(Yx - Xy) = G$ say (3).

The moments of the couples are reckoned positive when they tend to turn OX towards OY, and they are then denoted as vectors by the positive direction along OZ.

The forces evidently yield a resultant $R$ applied at O and acting at an angle $\theta$ with OX, which may be expressed by

$$R^2 = U^2 + V^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and

$$\tan\theta = V/U \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

If the forces $P_1$, $P_2$, etc., act at angles $\alpha_1$, $\alpha_2$, etc., with OX, we evidently have

$$X_1 = P_1 \cos\alpha_1 \text{ and } Y_1 = P_1 \sin\alpha_1, \text{ etc. } . \quad . \quad . \quad (6).$$

In the case where $U = V = 0$, then $R$ vanishes, and the resultant is the couple $G$, whose moment is, of course, the same about any point in the plane XOY. On the other hand, $G$ may vanish, and the resultant of the system reduce to $R$ simply.

But when both $R$ and $G$ are finite a further reduction to a single force is possible.

Thus, referring to Fig. 165, let $G = Rr$, and draw for the two $R$'s representing the couple $G$ the forces $OR_1$ and $O'R'$, where $OR_1$ is opposite to OR, and therefore cancels it. There is thus left as the final resultant of the system O'R', equal and parallel to OR, their perpendicular distance apart being given by

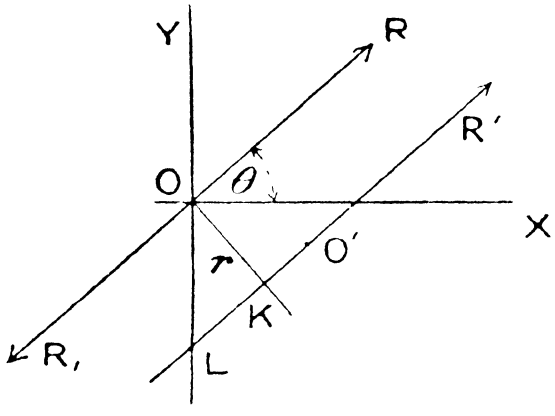



FIG. 165. SYSTEM REDUCED TO SINGLE FORCE.

$$OK = r = G/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

**365. Change of Axes.**—Suppose the axes of co-ordinates to move parallel to themselves till the new origin is at $(a, b)$, then by (1) to (5) we see that $R$ is unchanged, but that $G$ changes to $G'$ where

$$G' = \Sigma\{Y(x-a) - X(y-b)\} = G - (Va - Ub) \quad . \quad . \quad (8).$$

If, on the other hand, the origin is retained while the axes are rotated through any angle, $\phi$ say, then neither $R$ nor $G$ suffers any change.

This is easily seen on reference to Fig. 164. For this shows that $A_1P_1$ is replaced by the force $P_1$ applied at the origin together with moments about the origin which are equivalent to $P_1p_1$, where $p_1$ is the perpendicular from the origin on to $A_1P_1$ produced if necessary. Hence neither the force nor the moment in question is altered by any rotation of the axes. And since this holds for the first typical force, it holds for every other and for their combined effects. This mode of

regarding the matter has also the advantage of bringing into view another aspect of the resultant force and couple $R$ and $G$. For evidently $R$ is the *vector* sum of the $P$'s and $G$ is the *algebraic* sum of the products $Pp$. Or

$$\left.\begin{aligned} R &= P_1 \hat{+} P_2 \hat{+} P_3 \hat{+} \ldots \\ G &= P_1p_1 + P_2p_2 + P_3p_3 + \ldots = Rr \end{aligned}\right\} \quad \ldots\ldots (9).$$

Let us now revert to the final reduction of $R$ and $G$, when both are finite, to a parallel $R$ at distance $r$ from its original position. Then we see from (8) that for a new origin making $G'=0$, we have

$$G = Va - Ub \quad \ldots\ldots\ldots\ldots (10),$$

in which equations (1) to (7) hold. Hence, substituting for $a$ and $b$ the current co-ordinates $x$ and $y$ and dividing through by $U$, we find as the equation of the line on which the origin must lie to make the couple vanish

$$y = x \tan\theta - r \sec\theta \quad \ldots\ldots\ldots\ldots (11).$$

But this is the line KO'R' of Fig. 165, thus leading to the single resultant O'R' already found, which is thus definite except for the point of application O' on the line. This may be anywhere in the body since it is considered rigid.

**366. Poinsot's Analogy between Statics and Kinematics.**— Several analogies between statics and kinematics have been pointed out by Poinsot. The following interesting one may be noted here. Referring to article 94, we see that

(1) *In Plane Kinematics*, a linear displacement $ds$ of a point A in a rigid body and an angular displacement $d\theta$ about A are together equivalent to an equal angular displacement $d\theta$ alone, but about a point distant $ds/d\theta$ from A in a direction perpendicular to both $ds$ and the axis of $d\theta$.

Referring now to articles 364-365, we have

(2) *In Plane Statics*, a force $R$ and a couple $G$ are together equivalent to an equal force $R$ alone, but shifted a distance $G/R$ from its original line of action in a direction perpendicular both to $R$ and to the axis of $G$.

It is particularly noteworthy that, in the kinematical case, the combination of linear and angular displacements reduce to an equal *angular* resultant shifted parallel to itself. Whereas in the statical analogue the combination of force and couple concerned respectively with linear and angular accelerations reduce to an equal *force* shifted parallel to itself. Thus the analogy presents a cross connection which is somewhat startling.

Yet it is easy by a single dynamical illustration to link up these apparently conflicting kinematical and statical propositions.

For, let the motion, first considered kinematically, be that of a rigid body of appreciable mass.

Then, by the theorem of the independence of translation and rotation (article 269, etc.), we may fitly reduce the whole coplanar motion to the linear velocity $v$ of the centre of mass and the angular velocity

$\omega$ of the body about an axis through the centre of mass. But, for the instant, these further reduce to an equal angular velocity $\omega$ alone about a parallel axis shifted through $v/\omega$. Again, the linear and angular momenta corresponding to $v$ and $\omega$ may be equated to the impulse and impulsive couple competent to produce them, each to each. And these linear and angular impulses have as factors a force and couple respectively which we may denote by $R$ and $G$ and then further reduce to an equal force $R$ alone shifted a distance $G/R$. Hence this new force corresponds to an impulse which could produce the whole motion originally contemplated. Thus the entire momentum of the body may be represented by $P$ (that of the whole mass at the centre of mass and moving at its speed) together with $H_0$, the angular momentum of the actual body about an axis through the centre of mass. But, for some purposes, the entire momentum may be further reduced to $P$ alone shifted parallel to itself through a distance $H_0/P$.

Further, the shifted or instantaneous axis of rotation in the kinematical case and the shifted line of action of linear momentum in the dynamical case are, for the body in question, an axis of rotation and the corresponding line of percussion.

**367. Conditions of Equilibrium.**—If we consider the state of a rigid body capable of any motions parallel to a given plane, XOY say, it is easy to see that the possible motions are three in number, viz. the translations parallel to OX and OY and the rotation in the plane XOY.

Hence the conditions of equilibrium of such a body given initially at rest are, no accelerations in any of the above ways. Thus, considering that force equals mass into linear acceleration and couple equals moment of inertia into angular acceleration, we have as the required

Conditions of Equilibrium

$$\Sigma X=0,\ \Sigma Y=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and

$$\Sigma(Yx-Xy)=0=\Sigma Pp \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Equations (1) may be called the *equations of resolution* and (2) the *equation of moments*.

A little consideration will show that in the equations (1) the axes need not be at right angles. For in order that there should be equilibrium both $R$ and $G$ must vanish since they cannot balance each other. But if one of the axes, $x$ say, were taken by chance perpendicular to $R$, so that $\Sigma X$ vanishes though $R$ were finite, then any direction of $y$ not parallel to $x$ would furnish a finite value for $\Sigma Y$.

Thus, the additional vanishing of $\Sigma Y$ oblique to the axis of $x$ would show that $R$ was zero.

As to the couple $G$, though its value varies for different positions of the origin when $R$ does not vanish; its value is the same for any origin when $R$ does vanish. Hence the vanishing of $G$ with respect to *any* origin is sufficient.

We may thus put the conditions of equilibrium of a rigid body under forces in one plane as follows:—

The sums of the component forces parallel to each of any two

inclined axes in the plane of the forces must vanish, and also the algebraic sum of the moments of all the forces about any one point must vanish.

We easily see from the foregoing that if three forces in a plane maintain a rigid body in equilibrium, their directions either meet in a point or are all parallel. For if two of their directions meet in a point, then the moments of these two about that point vanish. Hence for $G=0$ with this point as origin the other force must pass through the same point.

**368. Alternative Conditions for Equilibrium.**—Suppose now it is known that for three different points $(a_1, b_1)$, $(a_2, b_2)$, and $(a_3, b_3)$ the algebraic sums of the moments of the coplanar forces vanish. Then by equation (10) we have

$$\left.\begin{aligned} G &= Va_1 - Ub_1, \\ G &= Va_2 - Ub_2, \\ \text{and} \quad G &= Va_3 - Ub_3 \end{aligned}\right\} \quad \ldots\ldots \quad (3),$$

where $G$ is the moment about the origin.

Whence, on eliminating $G$, we may write

$$\left.\begin{aligned} V(a_1 - a_2) &= U(b_1 - b_2) \\ \text{and} \quad V(a_2 - a_3) &= U(b_2 - b_3) \end{aligned}\right\} \quad \ldots\ldots \quad (4).$$

So that either

$$\frac{a_1 - a_2}{a_2 - a_3} = \frac{b_1 - b_2}{b_2 - b_3} \quad \ldots\ldots \quad (5),$$

or,

$$U = V = 0 \quad \ldots\ldots \quad (6).$$

But (5) expresses the condition that the three points $(a_1b_1)$, $(a_2b_2)$, and $(a_3b_3)$ shall lie on one straight line. So, unless this is fulfilled, (6) must be satisfied, in which case, by (3), $G$ also vanishes; or

$$U = V = G = 0 \quad \ldots\ldots \quad (7).$$

Thus, if the algebraic sums of the moments of a system of coplanar forces on a rigid body vanish with respect to three points in the plane *not* in a *straight* line, that system is in equilibrium.

## EXAMPLES—LXXI.

1. Sketch a set of three parallel forces in equilibrium, indicating the relation between their magnitudes and positions and showing also what construction gives the resultant of any pair of the forces.
2. Explain fully what you understand by a *couple*, and show how it may be represented by a line, and thus facilitate the composition of coplanar couples.
3. Reduce any set of coplanar forces to a single force and a single couple. What may the system reduce to in certain cases?
4. How are the resultant force and couple of question 3 affected by a change of axes?
5. Enunciate one of Poinsot's analogies between statics and kinematics.
6. Give two forms of the conditions for equilibrium of a rigid body in circumstances allowing motions in a given plane, and establish one of these forms.
7. Show that any system of coplanar forces may be reduced to two forces if not to one, and indicate the conditions of each case.

8. 'Show that a system of forces acting in one plane on a rigid body can be reduced to a single force or to a couple.

'$ABC$ is an equilateral triangle and $P$ is the foot of the perpendicular from $C$ on $AB$. Find in magnitude and line of action the resultant of the following forces :—

'10 acting from $A$ to $B$, 8 from $B$ to $C$, 12 from $A$ to $C$, and 6 from $C$ to $P$.'

(LOND. B.SC., PASS, APPLIED MATH., 1905, I. 1.)

9. 'Show that any number of couples acting simultaneously on a rigid body can be replaced by a single couple.

'(The proof may be confined to the case of couples all acting in the same plane.)

'$ABC$ is a triangular plate ; $A'$, $B'$, $C'$ are respectively the middle points of $BC$, $CA$, $AB$ ; forces represented in magnitude and sense by $k.AB$, $k.BC$, $k.CA$, $\lambda.B'A'$, $\lambda.A'C'$, and $\lambda.C'B'$ keep the plate in equilibrium ; what is the relation between $k$ and $\lambda$ ?'

(LOND. B.SC., PASS, APPLIED MATH., 1907, I. 1.)

10. 'Show that a system of coplanar forces may be reduced to two components along chosen axes at right angles to one another together with a couple in the plane of the axes.

'The algebraical sums of the moments of a system of coplanar forces about points whose co-ordinates are (1, 0), (0, 2), and (2, 3) referred to rectangular axes are $G_1$, $G_2$, $G_3$ respectively ; find the tangent of the angle the direction of the resultant forces makes with the axis of $x$.'

(LOND. B.SC., PASS, APPLIED MATH , 1908, I. 1.)

11. 'Show that a system of forces in one plane is in equilibrium if its moment, about three points of the plane not lying in a straight line, is zero.

'The moments of a system of forces about two points $A$, $B$ are $P$ and $Q$ respectively. Construct a point in the line of action of their resultant.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, I. 1.)

12. '$ABCD$ is a square lamina which is acted upon by forces of 5 units along $BA$, 3 units along $BD$, 7 units along $DC$, and by a couple of moment $8a$ in the sense $ABCD$, where $2a$ is the length of a side of the square. Find the equation of the line of action of the resultant of the system referred to AB, AD as axes of co-ordinates.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, I. 1.)

**369. Determination of Centroids.**—The conception of a *centroid* was introduced in article 25*d*, was used a little in Chapter XIII., but must now be more fully dealt with and its positions calculated for some typical cases.

As before stated, the term centroid may be applied to the centre of a number of points, to the centre of given lines, surfaces, or volumes, or to the centre of any scalar quantity distributed discretely or continuously in space.

We are here particularly concerned with the centroid in its use or application as the *centre of mass* or centre of inertia ; often referred to as the centre of *gravity*. But it should be noted that every distribution of mass has a single definite centre of mass fixed relatively to that distribution ; whereas, strictly speaking, it is the exception for bodies to have a true, single, fixed centre of gravity, when they are large enough with respect to the distance of the attracting body to make the attractions of their separate particles sensibly inclined.

Bodies that have this exceptional property are called *centrobaric*.

Homogeneous spheres afford an example of this class, as we saw in the sixteenth chapter on attractions.

Of course, any small body, commonly used in experiments on the earth, has its centre of gravity then practically fixed at its centre of mass. But the two conceptions are entirely distinct. See, *e.g.*, equation (1) of article 336 and the remark following it.

It is the centroid that we usually find in what follows, whether of particles supposed condensed at points, or material condensed into lines or surfaces, or material distributed in solid space. If we abstract the idea of the associated matter, we have the centroid of the given points, lines, surfaces, or volumes. If we keep the notion of the matter in the foreground, we may then fitly speak of the centre of mass. If the body is small, the forces of gravity on its particles, though *vectors* instead of *scalars*, may be regarded as *parallel*, and hence the single invariable point found as the centroid may be taken as representing with sufficient accuracy the really movable point which is the true centre of gravity.

The working rule for finding the centroid may be quoted here from article 25*d*.

$$\bar{x}=\frac{\Sigma mx}{\Sigma m},\ \bar{y}=\frac{\Sigma my}{\Sigma m},\ \bar{z}=\frac{\Sigma mz}{\Sigma m} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

From this it may easily be seen that if the centroids of certain portions of the system are known, then, for the centroid of the whole each such portion may be replaced by its magnitude at its centroid. Hence, we may sometimes easily find the centroid of the whole from those of its parts. To formally prove this, let the $x$'s have the subscripts 1 and 2 to distinguish two parts of the system. Then finding the centroid for the whole from its two parts, we have

$$\bar{x}=\frac{\bar{x}_1\Sigma m_1+\bar{x}_2\Sigma m_2}{(\Sigma m_1+\Sigma m_2)}=\frac{\Sigma m_1x_1+\Sigma m_2x_2}{\Sigma(m_1+m_2)} \quad . \quad . \quad (2),$$

or $\bar{x}=\dfrac{\Sigma mx}{\Sigma m}$, as found from the direct treatment of the whole system and given in equation (1).

Before passing to the use of integration for centroids we notice a few simple cases which may be solved more quickly by other methods.

### 370. Elementary Examples of Centroids.

*Three Equal Masses at the Corners of a Triangle.*—Take the origin of co-ordinates at one corner and the axis of $y$ parallel to the opposite side. Then the co-ordinates of the masses may be written (0, 0), ($a$, $b$), and ($a$, $c$), and each mass denoted by $m$. Hence by (1) we have

$$\bar{x}=\tfrac{2}{3}a,\ \bar{y}=\frac{b+c}{3}.$$

Thus the centroid is *two-thirds* along any median from the vertex, which agrees with the result found by taking first the centroid of two masses and supposing the doubled mass to be concentrated there. Hence we see that the three medians intersect at one point.

*Four Equal Masses at the Corners of a Tetrahedron.*—Following the analogy of the previous case, we see that the centroid of the four masses must be at *three-fourths* from any vertex along the line to the centroid of the corresponding base. Another way to regard the matter is to take the centroid of one pair of masses at the middle of the edge joining them, then similarly the centroid of the other pair at the middle of the opposite edge. Thus the centroid required will be found by bisecting the line joining these two middle points of edges. But as the tetrahedron has six edges, three such lines may be drawn. Obviously therefore they bisect one another, which is another geometrical property derived from the conception of the centroid.

*Sides of Triangle*.—Consider the sides of a triangle as though they were very thin uniform wires or filaments, and let it be required to find the centroid of this ideal triangular framework. Clearly we may replace each side by a mass proportional to its length and placed at the centre of the side. The co-ordinates of the centroid may then be readily found by the usual relations or by simple geometrical considerations. Take the latter method first. Hence, in the triangle ABC shown in Fig. 166, we first replace each side by a particle at its middle point and of equivalent mass, *i.e.* proportional to the length of that side. We thus obtain particles at D, E, and F of masses proportional to $a$, $b$, $c$, the sides of ABC, but these are proportional to the sides EF, FD, and DE. Hence, to find on DE say the centroid $g_3$ of the particles at D and E, we must divide it inversely as the masses of those particles, which is directly as the sides FD and FE. Hence, according to the well-known theorem, we must bisect the angle DFE in $FGg_3$. Similarly to find $g_1$ on EF, we must bisect the angle FDE by $DGg_1$. Thus, the intersection G of these two bisectors is the centroid required. In other words, it is the centre of the circle inscribed in DEF, the triangle whose corners are the middle points of the sides of the original triangle.

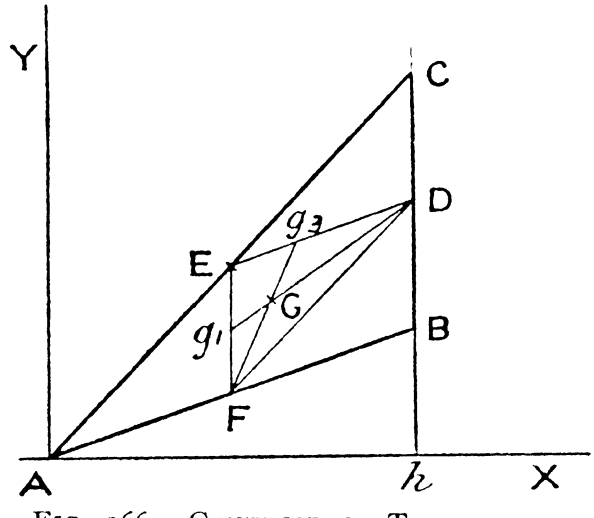



FIG. 166. CENTROID OF TRIANGULAR FRAME.

Taking now the analytical method, let us denote the points A, B, and C by the co-ordinates (o, o), $(h, k)$, and $(h, k+a)$ in accordance with Fig. 166. Then we easily find for the co-ordinates $\bar{x}, \bar{y}$ of the centroid G

$$\left.\begin{aligned} 4s\bar{x} &= h(2a+b+c) \\ 4s\bar{y} &= k(2a+b+c)+a(a+b) \end{aligned}\right\} \quad \cdots \quad (3),$$

where $2s = a+b+c$, the sum of the sides.

**371. Surface of a Triangle.**—Take any triangle, as ABC in Fig. 167, and draw two medians AD and BE intersecting at G. Then G is the required centroid, for each median bisects all elements of the triangle parallel to the corresponding base, and therefore contains the centroid of the whole surface. By construction and similar triangles, we have

$$\frac{CD}{CB}=\frac{CE}{CA}=\frac{ED}{AB}=\frac{GD}{AG}=\tfrac{1}{2}.$$

Whence $AG=\frac{2}{3}$ of AD . . . . . (4).

*Volume of a Tetrahedron.*—Let ABCD in Fig. 168 represent the tetrahedron, and take in it the plane CDE through the edge CD and the middle point E of the opposite edge AB. Then by symmetry this plane must contain the centroid sought. Take in CE and DE the centroids F and G of the faces ABC and DBA; join CG and DF intersecting at H. Then since by symmetry CG passes through the centroids of all slices parallel to ABD, it contains the centroid of the tetrahedron. Similarly, so does DF. Accordingly their intersection H is the centroid required. Join FG, then we have, by construction and similar triangles, the following equal ratios:—

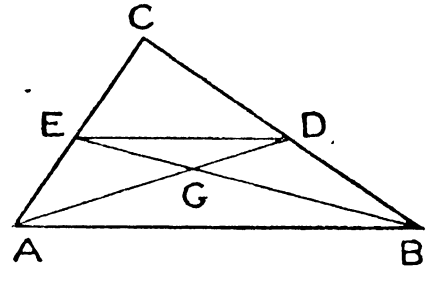



FIG. 167. CENTROID OF A TRIANGLE.

$$\frac{EF}{EC}=\frac{EG}{ED}=\frac{FG}{CD}=\frac{FH}{HD}=\tfrac{1}{3}.$$

Whence $DH=\frac{3}{4}DF$. . . . . (5).

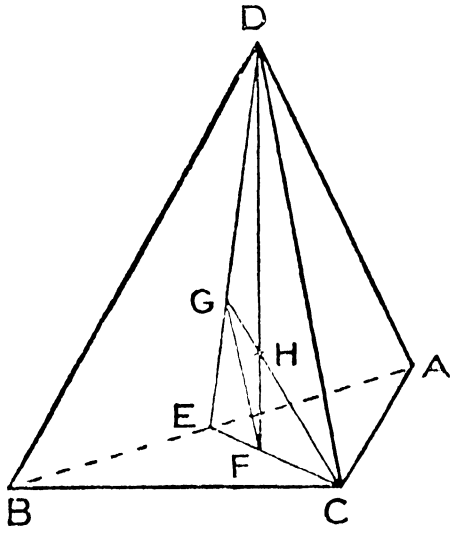



FIG. 168. CENTROID OF A TETRAHEDRON.

*Cone or Pyramid.*—We thus see that for a tetrahedron, a cone, or any pyramid, if we draw the line DF from the apex D to the centroid F of the base, then the centroid of the whole volume is H where $DH=\frac{3}{4}DF$.

**372. Frustum of a Pyramid.**—As shown in Fig. 169, let the vertex of the *completed* pyramid be O, A and B being the centroid of larger and smaller ends respectively. Let F and G be respectively the centroids of the whole completed pyramid and of the small pyramid needed for the completion. Let $OA=a$ and $OB=b$ and $OH=\bar{z}$ where H is the required centroid of the frustum. Then the volumes of the large whole pyramid and small completing one are

to each other as $a^3$ and $b^3$. Thus by the relation (2) of article 369 we may write

$$OF.a^3 = OG.b^3 + OH(a^3 - b^3),$$

or
$$\tfrac{3}{4}a^4 = \tfrac{3}{4}b^4 + \bar{z}(a^3 - b^3).$$

Whence
$$\bar{z} = \tfrac{3}{4}\frac{a^4 - b^4}{a^3 - b^3} = OH \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Of course, this may be cancelled down somewhat or other expressions substituted if desired, but this has the advantage of compactness when reckoning from the apex of the completed pyramid as origin.

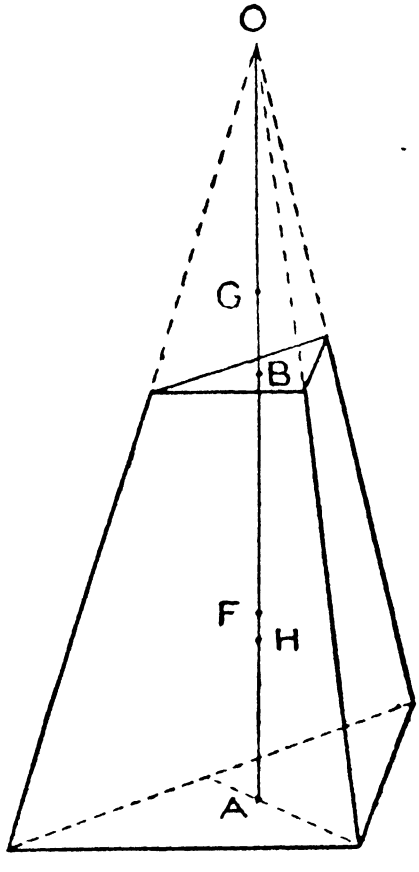



FIG. 169. CENTROID OF A FRUSTUM.

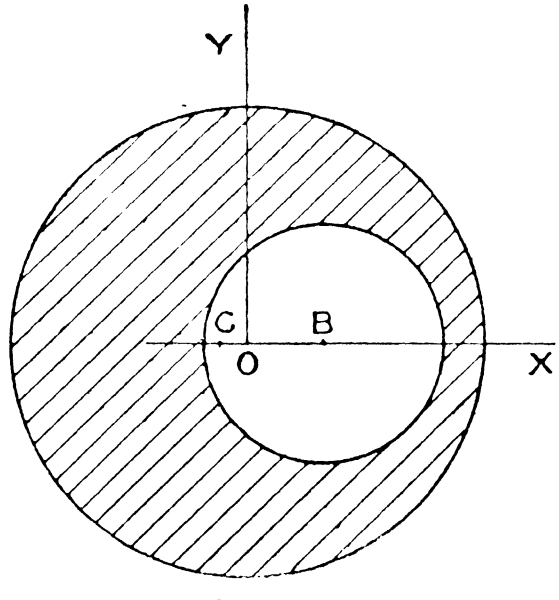



FIG. 170. CENTROID OF PIERCED CIRCLE.

**373. Difference of two Simple Figures.**—As a further illustration of the principle just used, let us now determine the centroid of the difference of any two simple figures. Thus take the figure left when a circle is cut from a larger circle as shown in Fig. 170.

We may write the equations of the circles $x^2 + y^2 = a^2$ and $(x-b)^2 + y^2 = c^2$, where $a < (b+c)$. Then writing $\bar{x}$ for the abscissa of G, the centroid of the remaining surface of the pierced circle, we have

$$\pi c^2 b + \pi(a^2 - c^2)\bar{x} = 0.$$

Thus
$$\bar{x} = \frac{-bc^2}{a^2 - c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Of course, $\bar{y} = 0$.

Obviously the same method may be applied to any figures formed from ellipses, squares, triangles, or other combinations of simple figures.

## EXAMPLES—LXXII.

1. Distinguish between *centroid*, *centre of mass*, and *centre of gravity*, giving as illustrations figures or bodies in which these three points are not all coincident.
2. Calculate the position of the centre of mass of four equal masses at the corners of a tetrahedron, and from this establish a geometrical property of the figure.
3. Establish the positions of the centroids of the surface of a triangle and the volume of a pyramid.
4. Obtain the centroid of a frustum of a pyramid in any form and reduce it to the form, distance from centroid of larger base of area $A$ towards the centroid of the smaller base of area $B$ is

$$\frac{h}{4} \cdot \frac{A+2\sqrt{AB}+3B}{A+\sqrt{AB}+B},$$

where $h$ is the height of the frustum. Show also that the distance of the centroid from the smaller base is the above fraction altered by the transference of the coefficient 3 from the $B$ to the $A$ in the numerator.

5. A homogeneous cube has a pyramid cut off by a plane passing through the three corners of the cube adjacent to that original corner of the cube which forms the vertex of the pyramid. Show that the centroid of the remaining portion of the cube is on a diagonal and one-twentieth of its length from its centre.
6. 'A uniform square plate is cut in two along a straight line joining a corner to the middle point of a side. Prove that the mass centre of the larger portion coincides with that of four particles of masses 4, 6, 5, 3 situated at the corners of the original square.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, I. 1.)

7. 'A triangular plate $ABC$ of uniform thickness rests horizontally on three vertical props at $A$, $B$, and $C$; show that the pressures on the props are equal.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, I. 2.)

8. 'A square board $ABCD$ rests with its plane perpendicular to the plane of a smooth vertical wall, one corner $A$ of the board being in contact with the wall, and another corner $B$ tied by a string, equal in length to a side of the square, to a point in the wall. Draw carefully a diagram showing the position of equilibrium of the board, and show that the distances of the corners $B$, $C$, $D$ from the wall are in the ratio 1 : 4 : 3.'

(LOND. B.SC., PASS, APPLIED MATH., 1908, I. 2.)

9. 'An equilateral triangular lamina $ABC$ rests in equilibrium in a vertical plane with its sides $AB$, $AC$ in contact with two smooth pegs in the same horizontal line at a distance $a$ apart. Prove that if $AD$, the perpendicular from $A$ on $BC$, is not vertical it must make an angle $\theta$ with the vertical where $\cos\theta = h\sqrt{3}/6a$ and $h$ is the length of $AD$.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, I. 2.)

**374. Centroids by Integration.**—We commence this section of the subject by the treatment of lines, passing afterwards to surfaces and then to volumes.

**Circular Arc.**—Consider first a circular arc AB of radius $a$ and subtending at the centre O of the circle the angle $2\beta$. Take the axis of $x$ through $C$, the middle of the arc, the origin being at O, as shown

in Fig. 171. Let the infinitesimal element PQ, subtending at O the angle $d\theta$, be defined by the co-ordinates $x, y$ of P and the angle $COP=\theta$. Then for P, $x=a\cos\theta$, and $m$ is represented by $PQ=ad\theta$. Thus, applying the usual expression for the centroid G, we have

$$\bar{x}=\Sigma mx/\Sigma m,$$

or
$$\bar{x}=a^2\int_{-\beta}^{+\beta}\cos\theta\,d\theta\div a\int_{-\beta}^{+\beta}d\theta=\frac{a\sin\beta}{\beta}=\frac{\text{chord}}{\text{arc}}\text{ radius} \quad . \quad (1).$$

That this may be put in the convenient form added in words is

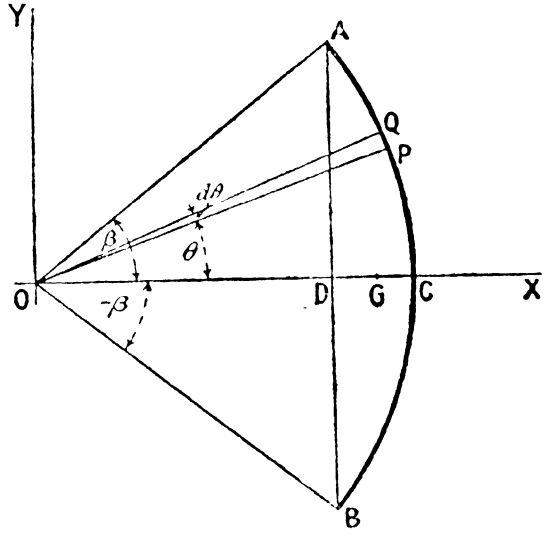



FIG. 171. CENTROID OF CIRCULAR ARC.

easily seen. It is also clear that G is rather nearer to C than to D, the intersection of the chord with OC.

Obviously $\bar{y}=0$.

**375. The Catenary.**—Taking the directrix as the axis of $x$ and the axis of the curve as that of $y$, the catenary is expressed by

$$y=c\cosh(x/c) \text{ and } s=c\sinh(x/c) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Thus
$$ds=\cosh(x/c)dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Hence for a portion $s$ the working rule gives for the abscissa

$$\bar{x}=\int_0^s x\,ds\div\int_0^s ds,$$

or
$$s\bar{x}=\int_0^x x\cosh(x/c)dx=c\int_0^x x\,d\{\sinh(x/c)\}$$
$$=[cx\sinh(x/c)-c^2\cosh(x/c)]_0^x$$
$$=xc\sinh(x/c)-c^2\cosh(x/c)+c^2.$$

Hence
$$s\bar{x}=xs-cy+c^2$$

and
$$\bar{x}=x-c(y-c)/s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

For the ordinate, the rule gives similarly

$$s\bar{y}=\int_0^s y\,ds=c\int_0^x\cosh^2(x/c)dx$$

$$= \frac{c}{4}\int_0^x (e^{2x/c} + 2 + e^{-2x/c})dx$$

$$= \frac{c}{4}\left[\frac{c}{2}e^{2x/c} + 2x - \frac{c}{2}e^{-2x/c}\right]_0^x$$

$$= \frac{c^2}{8}\left(e^{2x/c} - e^{-2x/c}\right) + \frac{cx}{2}.$$

Thus $s\bar{y} = \frac{ys}{2} + \frac{cx}{2}$,

or, $\bar{y} = \frac{1}{2}(y + cx/s)$ . . . . . (4).

For any other curve the difficulty is only that of finding the expressions in terms of $x$ for $ds$, the element of length.

**376. Centroids of Surfaces.**—Passing now to the centroids of surfaces, we begin with plane surfaces, taking first of all a

**Circular Sector.**—Let the origin of co-ordinates be at the centre of the circle of radius $a$, the axis of $x$ bisecting the sector of angle $2\beta$, so that $\bar{y} = 0$ as shown in Fig. 172. Take as element the concentric

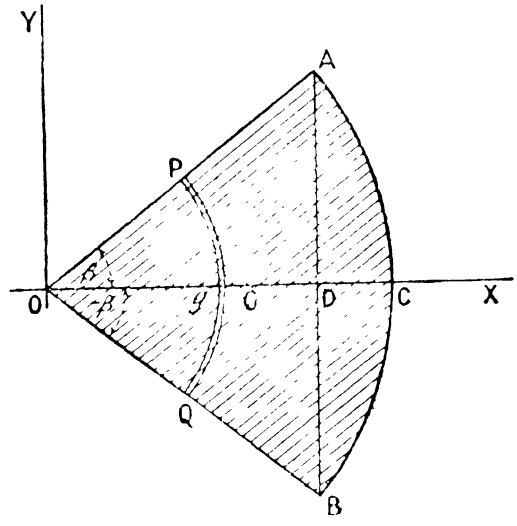



FIG. 172. CENTROID OF SECTOR.

circular strip PQ of radius $x$ and width $dx$. Then its centroid $g$ is distant $(x \sin\beta)/\beta$ from O and its area is $2\beta x dx$. Thus by the ordinary working rule we have for the abscissa of the centroid of the sector

$$\bar{x} = \int_0^a \frac{x \sin\beta}{\beta} 2\beta x dx \div \int_0^a 2\beta x dx,$$

or $$\bar{x} = \frac{\sin\beta}{\beta}\int_0^a x^2 dx \div \int_0^a x dx.$$

Whence $$\bar{x} = \frac{\sin\beta}{\beta}\frac{a^3}{3} \div \frac{a^2}{2} = \frac{2}{3}a\frac{\sin\beta}{\beta}$$ . . . . . (1).

Thus $$OG = \frac{2}{3} \cdot \frac{\text{chord}}{\text{arc}} \times \text{radius}.$$

Another mode of arriving at the same result is to take as elements

triangles with their common vertices at O and their bases infinitesimal portions of the arc ACB. Then their centroids lie along a concentric arc between OA and OB and of radius two-thirds that of the circle. Hence the centroid of the sector is that of this circular arc, and so is (chord/arc)$\times\frac{2}{3}$ radius, agreeing with that stated above.

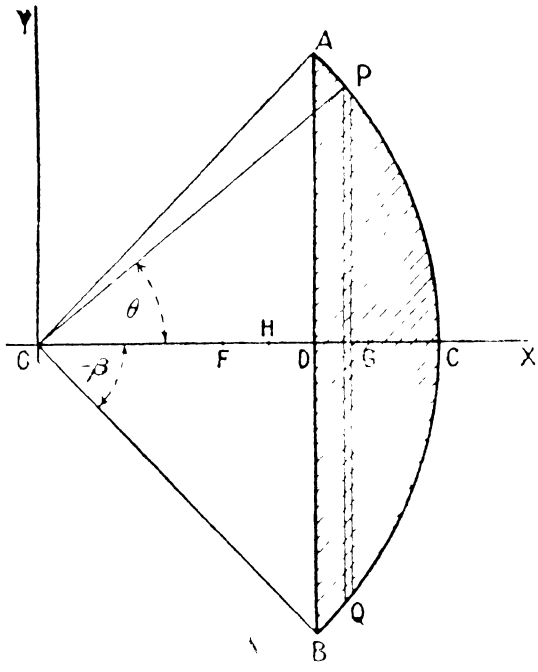



FIG. 173. CENTROID OF SEGMENT OF A CIRCLE.

**377. Segment of a Circle.**— Let us now find the centroid G of the segment ACBD subtending an angle $2\beta$ of the circle of radius $a$, as shown in Fig. 173.

In the segment take the element PQ parallel to the base ADB, and let the axis of $x$ bisect the segment in DC so that $\bar{y}=0$. Let the abscissa of P and Q be $x$, the width of the strip $dx$, and denote by $\theta$ the angle XOP. Then we have $x=a\cos\theta$, $dx=-a\sin\theta\,d\theta$, and $PQ=2a\sin\theta$, the limits of integration for the segment being $\theta=\beta$ and $\theta=0$. Hence, applying the working rule, we find

$$\bar{x}=-2a^3\int_\beta^0 \sin^2\theta\cos\theta\,d\theta\div -2a^2\int_\beta^0 \sin^2\theta\,d\theta$$

$$=a\int_\beta^0 \sin^2\theta\,d(\sin\theta)\div\int_\beta^0 \frac{1-\cos 2\theta}{2}\,d\theta$$

$$=a\left[\frac{\sin^3\theta}{3}\right]_\beta^0\div\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_\beta^0.$$

So, finally, we obtain as the abscissa of the centroid

$$\bar{x}=\tfrac{2}{3}a\frac{\sin^3\beta}{\beta-\sin\beta\cos\beta}=OG \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

As an alternative method we may derive the above value by considering the sector OACB as made up of the triangle OAB and the segment ADBC. Thus for the respective areas and abscissae of centroids we have

| | *Sector.* | *Triangle.* | *Segment.* |
|---|---|---|---|
| Areas . | $a^2\beta$. | $a^2\sin\beta\cos\beta$. | $a^2(\beta-\sin\beta\cos\beta)$. |
| Abscissae | $OH=\frac{2}{3}a\frac{\sin\beta}{\beta}$. | $OF=\frac{2}{3}a\cos\beta$. | $OG=\bar{x}$. |

Thus, applying the relation for the area and abscissae of parts and the whole, we have

$$a^2\beta\tfrac{2}{3}a\frac{\sin\beta}{\beta}=a^2\sin\beta\cos\beta\,\tfrac{2}{3}a\cos\beta+a^2(\beta-\sin\beta\cos\beta)\bar{x}.$$

Whence
$$\bar{x}=\frac{\frac{2}{3}a^3\sin\beta(1-\cos^2\beta)}{a^2(\beta-\sin\beta\cos\beta)}=\tfrac{2}{3}a\frac{\sin^3\beta}{(\beta-\sin\beta\cos\beta)} \quad (2a),$$

as found before in (2).

As checks on the results of the present and preceding article we may note that for a semicircle, which may be regarded either as a sector or a segment, for which $\beta=\pi/2$, both (1) and (2) reduce to $\bar{x}=4a/3\pi$. For $\beta=0$ or very small we easily find for the sector $\bar{x}=\frac{2}{3}a$, as should be the case. But for the very small segment where obviously $\bar{x}$ approaches $a$ in value, the right side of (2) becomes indeterminate of he form 0/0. So here, applying the method of the differential calculus, we must repeatedly differentiate the numerator and denominator, after each such differentiation putting $\beta=0$ to test if the quotient is then determinate. Thus, if the trigonometrical part of (2) is written

$$\frac{f(\beta)}{\phi(\beta)}=\frac{\sin^3\beta}{\beta-\sin\beta\cos\beta},$$

we find
$$\frac{f'''(\beta)}{\phi'''(\beta)}=\frac{6\cos^3\beta-21\sin^2\beta\cos\beta}{4\cos^2\beta-4\sin^2\beta}.$$

So that

$$\frac{f'''(0)}{\phi'''(0)}=\tfrac{3}{2}.$$

Then on inserting this value on the right side of (2) we find that for $\beta=0$, $\bar{x}=a$, as should be the case.

Another way of evaluating the indeterminate form is, of course, available by expanding the functions in terms of $\beta$. Thus we may write

$$\frac{\sin^3\beta}{\beta-\frac{\sin 2\beta}{2}}=\frac{\beta^3-\frac{\beta^5}{2}\cdots}{\beta-\frac{1}{2}\left(2\beta-\frac{4\beta^3}{3}\cdots\right)}=\frac{\beta^3\cdots}{\frac{2}{3}\beta^3\cdots}=\tfrac{3}{2},$$

as found before.

**378. Parallel Portion of any Plane Area.**—Let us now derive general formulae for the centroid of a portion of a plane area cut off by the axis of $x$ and two parallel ordinates, the fourth boundary being any curve expressed by $y=f(x)$ say, as shown by AB*ba* in Fig. 174.

Take in the area any infinitesimal element PQ*qp* bounded by ordinates at $x$ and $x+dx$. Then the area of this strip is $ydx$, and its centroid is, in the limit, given by $x$ and $y/2$, where $y$ is the value of $p$P found from the equation of the curve AB. Hence, considering the area in question as made up of these strips, and applying the usual relation,

we find for the co-ordinates of the centroid the following formulae, in which $a$ and $b$ are the limiting abscissae $Oa$ and $Ob$ respectively :—

$$\bar{x} = \int_b^a xy\,dx \div \int_b^a y\,dx \quad \ldots\ldots\ldots\ldots \quad (1),$$

$$\bar{y} = \tfrac{1}{2}\int_b^a y^2\,dx \div \int_b^a y\,dx \quad \ldots\ldots\ldots\ldots \quad (2).$$

Of course, the values of $y$ in terms of $x$ must be inserted before the integrals can be evaluated.

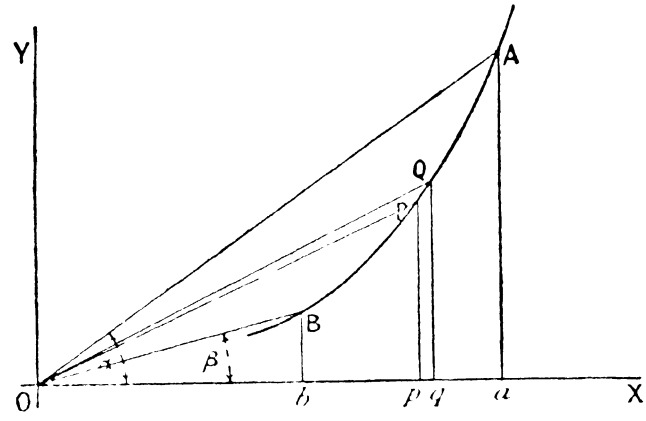



FIG. 174. CENTROID OF PORTION OF PLANE AREA.

**379. General Sectorial Area.**—Referring again to Fig. 174, let us now take the sectorial area AOB bounded by the same curve AB. It will now be convenient to use polar co-ordinates for part of the work. We shall accordingly regard AB as given by $r = F(\theta)$, the limiting values XOA and XOB of $\theta$ being respectively $\alpha$ and $\beta$. We now take as infinitesimal element POQ, where $OP = r$, $XOP = \theta$, and $POQ = d\theta$. Thus the area of the element is $\frac{1}{2}r^2 d\theta$, and the cartesian co-ordinates of its centroid are $(\frac{2}{3}r\cos\theta, \frac{2}{3}r\sin\theta)$. Hence, applying the usual rule, we find the following formulae for centroid in cartesian co-ordinates :—

$$\bar{x} = \int_\beta^\alpha (\tfrac{2}{3}r\cos\theta)\tfrac{1}{2}r^2 d\theta \div \int_\beta^\alpha \tfrac{1}{2}r^2 d\theta \quad \ldots\ldots \quad (3),$$

$$\bar{y} = \int_\beta^\alpha (\tfrac{2}{3}r\sin\theta)\tfrac{1}{2}r^2 d\theta \div \int_\beta^\alpha \tfrac{1}{2}r^2 d\theta \quad \ldots\ldots \quad (4).$$

Here the values of the $r$'s must be inserted in terms of $\theta$ from the equation of the curve AB before the integrals can be evaluated.

**380. Plane Areas by Double Integration.**—Let us now consider a plane area bounded by any two ordinates $x = a$ and $x = b$ and any two curves $y = \phi(x)$ and $y = \psi(x)$, as shown by ABCD in Fig. 175. This might be treated by single integrals as in article 378, but, to illustrate another method sometimes preferable, we will here adopt double integration. We now divide the area into elementary strips parallel to both axes of co-ordinates. Thus the strip $Ppq$Q is bounded by the abscissae $x$ and $x + dx$. The strip R$rs$S is similarly bounded by the ordinates $y$ and $y + dy$. Hence the infinitesimal rectangle $uv$, common

to both strips, and which now forms an element of the area ABCD, has area $dxdy$ and, in the limit, its centroid has co-ordinates $x$ and $y$. Thus, applying the usual rule and inserting the limits of each integration, we have the following formulae :—

$$\bar{x} = \int_b^a \int_{\psi(x)}^{\phi(x)} x\,dydx \div \int_b^a \int_{\psi(x)}^{\phi(x)} dydx \quad \ldots\ldots\ (5),$$

$$\bar{y} = \int_b^a \int_{\psi(x)}^{\phi(x)} y\,dydx \div \int_b^a \int_{\psi(x)}^{\phi(x)} dydx \quad \ldots\ldots\ (6).$$

Since the limits of $y$ involve $x$, it should be noted that the integration of $y$ must be taken first. After this integration is effected we have the expressions that might have been obtained in the single integral method. The advantage of the double integral lies in its power to deal with a lamina occupying the area in question and varying in surface density with distance from the co-ordinate axes. Thus, if the density were $\sigma xy$, we should simply have $\sigma xy$ introduced in each of the above four integrals in (5) and (6).

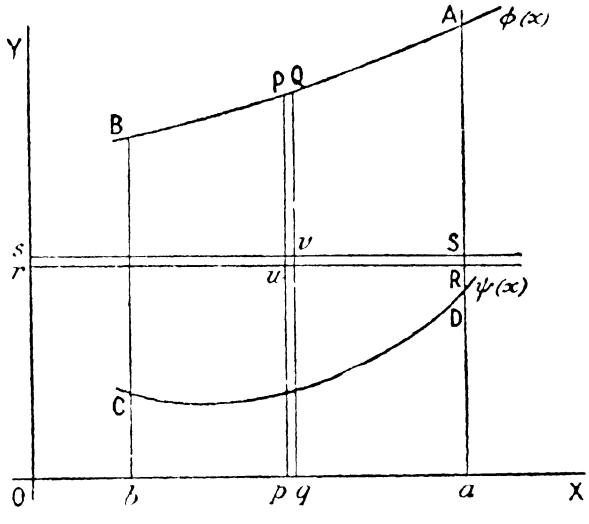



FIG. 175. CENTROID OF PLANE AREA BY DOUBLE INTEGRATION.

**381. Plane Areas by Double Integration in Polar Co-ordinates.**— Let us now consider an area bounded by any radii at angles $\alpha$ and $\beta$ and the curves whose equations are $r=\phi(\theta)$ and $r=\psi(\theta)$, as shown by ABCD in Fig. 176. We take as the infinitesimal element of the double integration the figure $st$ bounded by the radii at angles $\theta$ and $\theta+d\theta$ and by the concentric circles of radii $r$ and $r+dr$. Then the element has area $rd\theta dr$ and, in the limit, its centroid has co-ordinates ($r\cos\theta$, $r\sin\theta$). Thus, applying the usual rule and inserting the limits of integration, we derive the following formulae for the centroid of the area :—

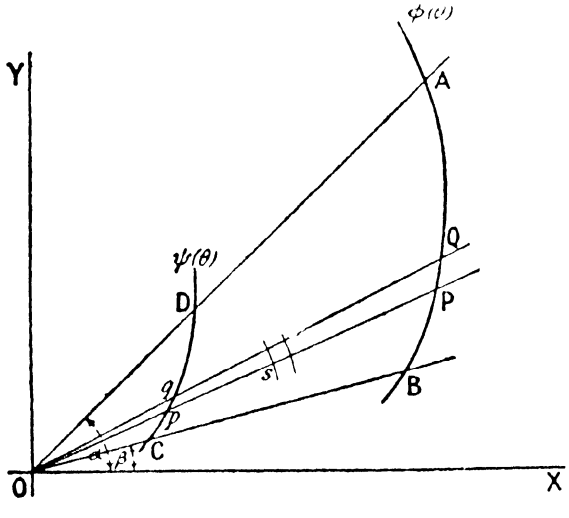



FIG. 176. PLANE AREAS BY DOUBLE POLAR INTEGRATION.

$$\bar{x}=\int_{\beta}^{\alpha}\int_{\psi(\theta)}^{\phi(\theta)} r^2\cos\theta\, dr d\theta \div \int_{\beta}^{\alpha}\int_{\psi(\theta)}^{\phi(\theta)} r dr d\theta \quad . \quad . \quad . \quad (7),$$

$$\bar{y}=\int_{\beta}^{\alpha}\int_{\psi(\theta)}^{\phi(\theta)} r^2\sin\theta\, dr d\theta \div \int_{\beta}^{\alpha}\int_{\psi(\theta)}^{\phi(\theta)} r dr d\theta \quad . \quad . \quad . \quad . \quad (8).$$

As before, the inner integral (in $r$) must be evaluated first, as it involves $\theta$ in the limits.

The advantage of this method would be most felt in the case of a lamina with surface density a function of $r$, or of $\theta$, or of both.

## EXAMPLES—LXXIII.

1. Find by integration an expression for the centroid of a circular arc (or uniform wire of that shape). Hence show that the centroids of a quadrantal and of a semicircular wire are distant from the centre by $2a\sqrt{2}/\pi$ and $2a/\pi$ respectively where $a$ is the radius.
2. Determine by any method the centroid of a plane sector of a circle, and confirm the result by another method.
3. Calculate by integration the position of the centroid of the plane area of a segment of a circle, and check it by reference to a segment which is also a sector.
4. Derive formulae for the centroid of a parallel portion of any plane area. Apply them to the quarter of an ellipse between its axes, and check it by reference to the quadrantal sector of a circle.
5. Obtain formulae for the centre of mass of a plane lamina in which the mass per unit area was proportional to the distance from the origin. Apply these to show that for a semicircular lamina of the foregoing distribution of mass the centre of mass is distant from the centre $3/2\pi$ of the radius.
6. 'Prove that the centre of mass of a uniform semicircular disc of radius $a$ is at a distance $4a/3\pi$ from the centre.

   'A uniform solid semicircular cylinder is placed with its axis horizontal and its curved surface in contact with an imperfectly rough plane (of coefficient $\mu$) which is inclined to the horizon at an angle $\alpha$. Prove that, provided

   $$\tan\alpha<\mu \text{ and } \sin\alpha<4/3\pi,$$

   there is a position of equilibrium in which the inclination of the plane face to the horizontal is

   $$\sin^{-1}(\tfrac{3}{4}\pi\sin\alpha).'$$

   (LOND. B.SC., PASS, APPLIED MATH., 1908, I. 3.)
7. '$AOB$ is a diameter of a circle whose centre is $O$. On $AO$, $OB$ as diameters semicircular arcs are described, on opposite sides. If $G$, $G'$ be the mass centres of the two equal portions into which the area of the circle is divided by these arcs, prove that the inclination of $GG'$ to $AB$ is $\tan^{-1}(4/\pi)$.'

   (LOND. B.SC., PASS, APPLIED MATH., 1906, I. 5.)

**382. Surfaces of Revolution.**—Let it now be required to determine the centroid of the surfaces generated by revolution, about either of the co-ordinate axes $x$ and $y$, of any plane curve defined by $y=f(x)$, lying between the limits $x=a$ and $x=b$, as shown by AQPB in Fig. 177.

In the curve AB take the infinitesimal element PQ of length $ds$, P having the co-ordinates $(x, y)$. Then, by revolution about OX, PQ

will describe the ring element of area $2\pi y ds$ and, in the limit, its centroid will have co-ordinates $(x, 0)$. Hence, by the working rule, we find for the co-ordinates of the centroid the following expressions:—

$$\bar{x} = \int_b^a 2\pi y x \frac{ds}{dx} dx \div \int_b^a 2\pi y \frac{ds}{dx} dx \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and $\bar{y} = 0$.

The values of $y$ and $ds/dx$ must be inserted from the equation to the curve before the integrals can be evaluated.

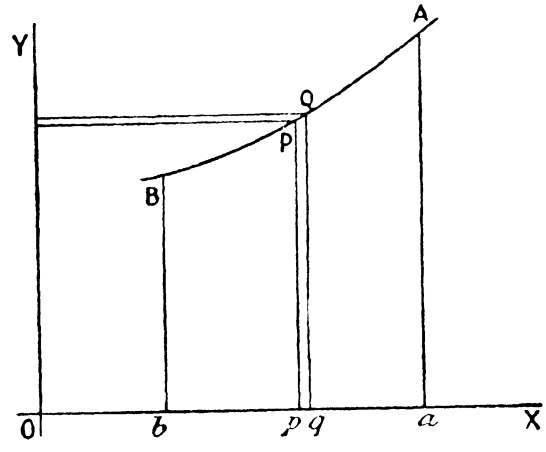



FIG. 177. CENTROIDS OF SURFACES OF REVOLUTION.

For the surface generated by revolution of the same curve AB about OY, the abscissae of the centroid are obviously

$$\bar{y} = \int_b^a xy \frac{ds}{dx} dx \div \int_b^a x \frac{ds}{dx} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

and $\bar{x} = 0$.

CASE I.—For a *cylinder* with its axis as the axis of $x$, the $y$ is the constant radius and cancels out, and $ds/dx = 1$, so that from (1) we have

$$\bar{x} = \frac{\frac{1}{2}(a^2 - b^2)}{a - b} = \frac{a+b}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

as evidently should be the case.

CASE II.—For a *sphere*, let the equation be $x^2 + y^2 = c^2$. Then $dy/dx = -x/y$ and $yds = cdx$, *i.e.* the surface of each ring element has the same area as that of the corresponding ring of the circumscribing cylinder. Thus here also, for a spherical zone or cap, we have, as for the cylinder,

$$\bar{x} = \frac{a+b}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

CASE III.—For a *cone* of semi-vertical angle $\alpha$ and vertex at the origin we have $y = x \tan \alpha$ and $ds = \sec \alpha . dx$. Hence (1) reduces to

$$\bar{x} = \int_b^a x^2 dx \div \int_b^a x dx = \frac{\frac{2}{3}a^3 - b^3}{a^2 - b^2} \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Or, for the *complete* cone,

$$\bar{x} = \tfrac{2}{3}a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

**383. Conical Surface by Projection.**—Let us now illustrate the method of projection by applying it to find the centroid of a portion of the surface of a right cone on a circular base, as shown by ABC in Fig. 178.

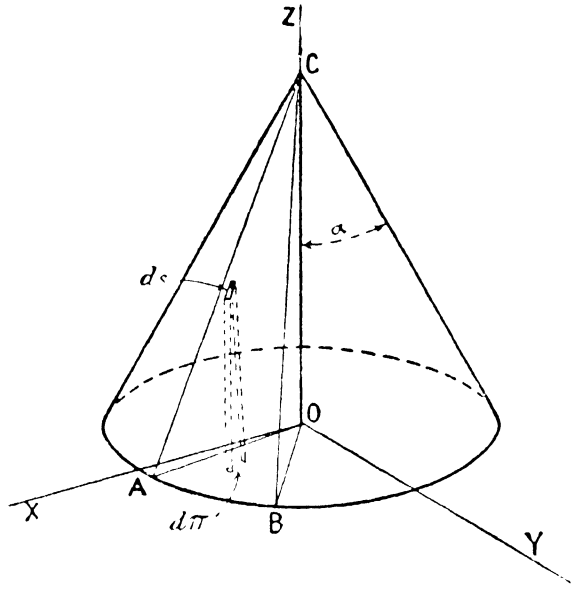



FIG. 178. CENTROID OF CONICAL SURFACE.

The angle between an element $dS$ of the conical surface and $d\Pi$, its projection on the base, is that between any generator and the base. Hence it is the complement of $\alpha$ if $\alpha$ is the semi-vertical angle of the cone. Thus $d\Pi/dS = \sin\alpha$. And the axis of the cone being that of $z$, it follows from the orthogonal projection on the base that $dS$ and $d\Pi$ have the same co-ordinates in $x$ and $y$, and accordingly the like equalities hold for the centroids of the conical surface $S$ and its projection $\Pi$. Or, in other words, the *projection of the centroid* of any portion of the surface of a right circular cone on a plane perpendicular to the axis is the *centroid of the projection* of that surface.

For the $z$ co-ordinate of the centroid of $S$ we have, by application of the usual rule and the relations between $S$ and $\Pi$, the following expressions:—

$$\bar{z} = \frac{\int z dS}{\int dS} = \frac{\int z d\Pi}{\int d\Pi} = \frac{V}{\Pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where $V$ is volume of the prism included between the conical surface $S$, its projection $\Pi$, and the lines of projection parallel to the axis.

Where, as in the figure, $\Pi$ is the triangle AOB and $S$ is the corresponding conical triangle ACB, it is easily seen that

$$\bar{z} = \tfrac{1}{3}\mathrm{OC} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

which result holds also for the whole conical surface.

Projections of spherical surfaces on a diametral plane or on the circumscribing cylinder are sometimes useful.

**384. Centroid of a Solid of Revolution.**—Suppose a curve AB to rotate about the axis of $x$, and let it be required to find the centroid of the *volume* thus generated. Take in the curve AB, Fig. 179, the point P of co-ordinates $(x, y)$ and the adjacent point Q of abscissa $x + dx$. Then, as the curve revolves about OX, the element PQ will sweep out a volume $\pi y^2 dx$ whose centroid, in the limit, has the abscissa $x$ simply. Hence, by the rule, we find for the abscissa of the centroid

$$\bar{x}=\int_b^a \pi x y^2 dx \div \int_b^a \pi y^2 dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Also by symmetry we obviously have

$$\bar{y}=\bar{z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

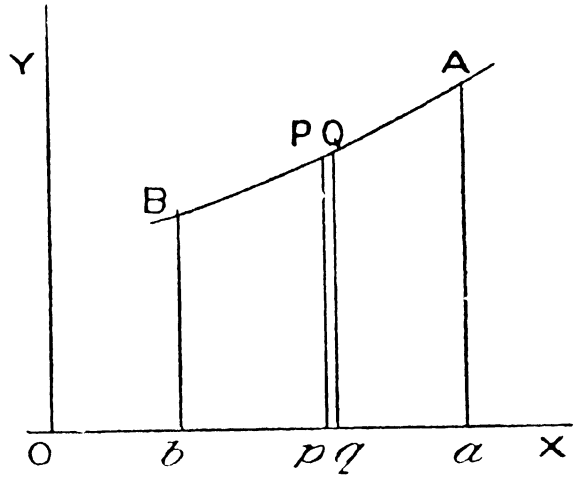



FIG. 179. CENTROID OF SOLID OF REVOLUTION.

Thus, for a *parallel slice of a sphere* of radius $c$, we have $y^2=c^2-x^2$, and (1) becomes

$$\left.\begin{aligned}\bar{x}&=\frac{\frac{1}{2}(a^2-b^2)c^2-\frac{1}{4}(a^4-b^4)}{(a-b)c^2-\frac{1}{3}(a^3-b^3)}\\&=\tfrac{3}{4}(a+b)\frac{2c^2-(a^2+b^2)}{3c^2-(a^2+ab+b^2)}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

For a *hemisphere*, $a=c$ and $b=0$, and this reduces to

$$\bar{x}=\tfrac{3}{8}c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

For a *solid spherical sector* of semi-vertical angle $\beta$ we might use the above general method, but each integral in (1) would split into two. Thus from $x=0$ to $c\cos\beta$ we should have $y=x\tan\beta$, while from $x=c\cos\beta$ to $x=c$ we should have $y^2=c^2-x^2$. Hence on this plan the work would be somewhat long. We may therefore with advantage adopt a device founded on the known centroids of the pyramid and spherical cap. For we may consider the spherical sector as composed of a number of pyramids with their common vertices meeting at the centre of the sphere and their bases making up the spherical surface of the sector. Then, since the centroid of each such pyramid is at $\frac{3}{4}c$ for the centre of the sphere, the whole sector is replaceable, for our purpose, by a spherical cap of radius $\frac{3}{4}c$ and semi-vertical angle $\beta$ like that of the sector. But, by what has been found for a spherical zonal surface or cap, the centroid of this cap is distant from the centre by the arithmetic mean of the limiting distances of the zone or cap. These distances are respectively $\frac{3}{4}c$ and $\frac{3}{4}c\cos\beta$. Hence for the abscissa of the centroid we have

$$\bar{x}=\tfrac{3}{8}c(1+\cos\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

as would be found by the general method.

If now we regard a hemisphere as a solid sector, for which the

semi-vertical angle $\beta$ is $\pi/2$, we see that by this expression we find, as before in (4), $\bar{x}=\frac{3}{8}c$.

Finally, if the sector is of vanishingly small base, $\beta$ is very small, and $\cos\beta=1$ nearly. Thus for this figure, which is practically a right circular cone, we have $\bar{x}=3c/4$, as should be the case.

**385. Centres of Mass for Uniform or Variable Densities.**—The positions of the centroids hitherto found for lines, surfaces, or solid figures of course apply immediately to those of the centres of mass of bodies of uniform density and approximating to those lines or surfaces or occupying those volumes.

If the density varies in any way with the co-ordinates either cartesian or polar, this fact must be introduced as a factor in each of the integrals whose quotient gives the corresponding co-ordinate of the centre of mass.

We may use single, double, or triple integrals, and sometimes shorten the work by a device. Thus, take the following example:—

**Solid Spherical Sector with Density directly as Radius.**—Using the device which led to equation (5) of article 384, we have now only to find the new position for the centre of mass of the elementary pyramids with their common vertices at the centre of the sphere. Let the very small solid angle of one such pyramid be $\omega$, the density at any point distant $r$ from the centre be $r\rho_0$, and consider the slice at $r$ of thickness $dr$. Its area is $\omega r^2$, its mass $\omega r^2 dr.r\rho_0$, and the moment of this about the centre $r$ times the latter expression. Thus for the abscissa of the centre of mass of the pyramid we have

$$\bar{x}=\omega\rho_0\int_0^c r^4dr\div\omega\rho_0\int_0^c r^3dr=\tfrac{4}{5}c \quad . \quad . \quad . \quad . \quad (6)$$

if the radius of the sphere is $c$. Hence the solid sector is replaceable by a spherical cap of radius $\frac{4}{5}c$ and of semi-vertical angle, $\beta$ say, that of the sector. Accordingly the centroid is at the arithmetic mean of the limiting distances of this cap. Thus we have

$$\bar{x}=\tfrac{2}{5}c(1+\cos\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for the solid sector of density proportional to the radius.

For a hemisphere with this law of density we thus find

$$\bar{x}=\tfrac{2}{5}c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

**386. Pappus' Theorems.**—The two following theorems, due to Pappus, are useful in connection with centroids:—

*Enunciation.*—Let any plane area $A$ revolve through any angle about an axis in its own plane, then

(1) The area $S$ of the surface generated by the perimeter of $A$ is equal to the product of the perimeter into the length of the path described by the centroid of the perimeter.

(2) The volume $V$ of the solid generated by the area $A$ is equal to the product of the area $A$ into the length of the path described by the centroid of $A$.

In both these theorems the axis is supposed not to intersect the plane area or perimeter.

*Proof.*—Take the plane of $A$ as that of $xy$ and the axis of rotation as that of $x$. Denote by $s$ the length of the perimeter BCD of $A$ (Fig. 180), and let the distances from OX of the centroids of $A$ and of $s$ be respectively $\bar{a}$ and $\bar{s}$. In $s$ take the element $PQ=ds$ of ordinate $y$; then, in the rotation through the angle $\theta$ about OX, PQ will generate the area given by

$$dS=\theta y ds.$$

Hence, for the whole perimeter, we have

$$S=\theta\int y ds=\theta\bar{s}\int ds=(\theta\bar{s})s \quad . \quad . \quad . \quad . \quad . \quad (1),$$

which establishes the first theorem.

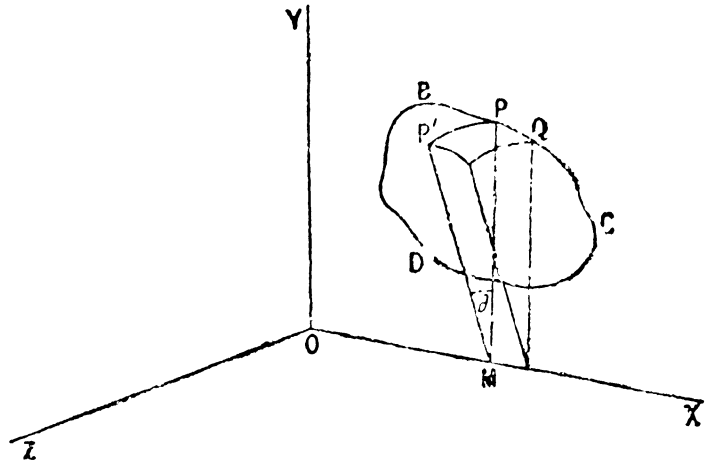



FIG. 180. PAPPUS' THEOREMS.

Again, for the volume $dV$ generated by a small element $dA$ situated at $y$ in the plane area $A$ on its revolution through $\theta$ about OX, we have $dV=\theta y dA$. So, for the whole volume, we find

$$V=\theta\int y dA=\theta\bar{a}\int dA=(\theta\bar{a})A \quad . \quad . \quad . \quad . \quad (2),$$

which is the symbolic expression of the second theorem.

## EXAMPLES—LXXIV.

1. Find the centroid of the whole curved surface of a right circular cone and of any part of it.
2. Show that the centroid of any parallel zone of a spherical surface is the same as that of the corresponding zone of the circumscribing cylindrical surface.
3. Find expressions for the centroids of a segment and of a sector of a homogeneous solid sphere. Check them by showing that for a hemisphere each formula gives 3/8 of the radius as the distance of the centroid from the centre of the sphere.
4. 'A uniform solid hemisphere rests with its curved surface in contact with a rough plane, which is gradually tilted. Find at what inclination the hemisphere will be on the point of toppling over. If the inclination be less than this, is the equilibrium stable?'

   (LOND. B.SC., PASS, APPLIED MATH., 1906, I. 7).
5. 'Find the position of the centre of gravity of a uniform solid hemisphere.
   'A uniform solid hemisphere of weight $W$ rests with its curved surface on a smooth horizontal plane. A body of weight $P$ is suspended from the rim of the hemisphere; find the inclination of the base of the hemi-

sphere to the horizon in the position of equilibrium. Is the result modified if the horizontal plane is rough? (Give the reason of your answer.)' (LOND. B.SC., PASS, APPLIED MATH., 1907, I. 7.)

6. 'Show that the centre of gravity of the smaller portion of a solid sphere of radius $a$ cut off by a plane distant $b$ from the centre is at a distance

$$\tfrac{3}{4}\frac{(a+b)^2}{2a+b}$$

from its centre.

'A spherical cap is cut off from a solid homogeneous sphere by a plane whose distance from the centre is half the radius of the sphere, and the remainder of the sphere is placed with the plane boundary in contact with a perfectly rough inclined plane. Find the greatest inclination of the plane if the truncated sphere is not to topple over.'

(LOND. B.SC, PASS, APPLIED MATH., 1909, I. 8.)

7. 'A frustum of a right circular cone of axial length 6 feet, and radii of ends 1 foot and 4 feet, rests with its curved surface in contact with the ground. If the weight of the frustum is 3 tons, find the number of foot-tons of work that must be expended to raise the frustum into such a position that it can fall over into a situation of equilibrium with its larger end on the ground.'

(LOND. B.SC., PASS, APPLIED MATH, 1910, I. 3.)

**387. The Principle of Virtual Work for Rigid Bodies.**—In article 321 the principle of virtual work was seen to hold for forces on a particle. Let us now, following the treatment of Routh, consider the principle more fully, not restricting the system to a particle, but supposing it to be a rigid body or system of rigid and pliable bodies in two (or three) dimensions.

*Enunciation.*—Let any number of forces $P_1$, $P_2$, etc, act at the points $A_1$, $A_2$, etc., of a system of bodies. These bodies may be connected together in any way so as to allow or exclude relative motion: they may accordingly exert on each other mutual actions and reactions. Let the system be slightly displaced so that the points $A_1$, $A_2$, etc., assume neighbouring positions, the projections upon the directions of the forces of these displacements of the corresponding points being respectively $dp_1$, $dp_2$, etc. Also write

$$dW = P_1 dp_1 + P_2 dp_2 + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Then the system is in equilibrium if

$$dW = 0 = \Sigma P dp \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for all displacements consistent with the geometrical connections between the bodies of the system.

Also the system is not in equilibrium if one or more displacements can be found for which $dW$ is not equal to zero.

Routh points out that, in strictness, we should say, not that $dW$ is zero, but that it is a small quantity of the second order.

It is to be understood that these displacements called 'virtual' are 'imaginary motions which the system might, but does not necessarily, take. The principle of virtual work supplies a test, whether a given position of the system is one of equilibrium or not. We first consider what are the possible ways in which the system could begin to move

out of the given position. If for any one of these the sum $\Sigma Pdp$ is zero, then the system will not begin to move in that mode of displacement. In this way all the possible displacements are examined, and if $\Sigma Pdp$ is zero for each and every one, the given position is one of equilibrium.'

**388. Concrete Example of Virtual Work.**—The principle of virtual work was made by Lagrange the basis of his *Mécanique Analytique*, and he made a brilliant attempt to give a general proof of it. Routh, however, states that 'no satisfactory method has yet been found by which the principle for a *system of bodies* can be deduced directly from the elementary axioms of statics.' The full treatment of the subject is accordingly regarded as beyond the scope of the present text-book, but the following simple example of a rigid bar resting on smooth inclines at its ends will serve to illustrate the principle, and afford an insight into its meaning and application. In Fig. 181 the bar touches *smooth* inclined surfaces at $A_1$ and $A_2$, its centre of mass being G. We accordingly have forces $P_1$, $P_2$ acting *normally* to the inclines at $A_1$ and $A_2$, also the weight $P_3 = Mg$ say acting vertically downwards at G. Now, if any motion of the bar is supposed to occur with the restriction that it remains in contact with the surfaces, it is clear that, on account of the smoothness, $dp_1 = 0 = dp_2$. If we take the axis of $z$ vertically downwards we may denote $dp_3$ by $dz$. We accordingly have, by (1) of article 387,

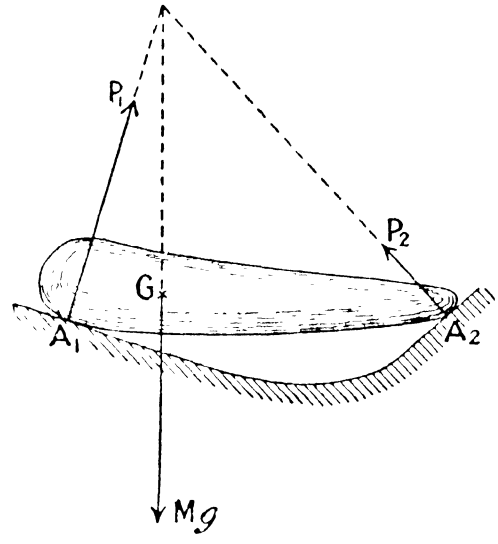



FIG. 181. VIRTUAL WORK FOR A RIGID BODY.

$$dW = Mgdz = Mg\frac{dz}{ds}ds \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where $ds$ is an element of the path $s$ described by G when the contacts $A_1$ and $A_2$ move on their surfaces. Hence by (2) the condition for equilibrium here becomes

$$\frac{dz}{ds} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Or, in words, subject to contact between the rigid bar and its smooth supports, the centre of mass G of the bar can describe a certain surface, $S$ say. Then, for equilibrium of the bar, the point G must occupy in the surface $S$ a point at which the tangential plane to $S$ is horizontal.

Many examples on equilibrium occurring presently and later may be dealt with by virtual work; but some are done quicker without it, especially if friction is present. Discretion must be exercised as to which method should be adopted for each case. The principle of virtual work is specially useful where there are *pairs* of equal and

opposite forces with the same virtual displacements as at the junction or contact of parts of the system or forces normal to all possible displacements, for these contribute nothing to $dW$, and can therefore be omitted.

**389. Lever: Wheel and Axle.**—Let us now examine those simple machines which include some rigid parts, taking first the lever which consists of a rigid bar, straight or bent, movable about a fixed axis called the *fulcrum*. It is acted on by at least two forces, $P$ and $W$ say, besides that of the fulcrum. The parts of the lever between the fulcrum and the points of application of the other two forces are called the *arms* of the lever, which we may denote by $a$ and $b$ (Fig. 182). Let the force $P$ act at an angle $\alpha$ with the arm $CA=a$ and $W$ at an angle $\beta$ with the arm $CB=b$. Then, for equilibrium, we have by moments

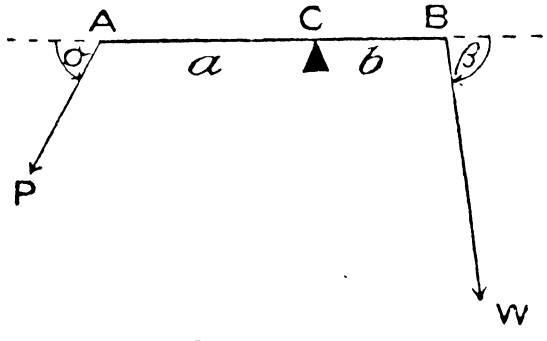



FIG. 182. THE LEVER.

$$Pa\sin\alpha = Wb\sin\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

By virtual work we should obtain for equilibrium

$$Pdp + Wdw = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

But if the displacements are derived from a rotation $d\theta$ of the lever, we see that

$$dp = ad\theta\sin\alpha \text{ and } dw = -bd\theta\sin\beta \quad (3).$$

Hence by use of (3) equation (2) reduces to (1).

Of course, for $\alpha=\pi/2=\beta$, equation (1) reduces to

$$Pa = Wb \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

*The Wheel and Axle* is only a modification of the lever in a form allowing of continuous motion through large angles, since the arms $a$ and $b$ of the lever are replaced by the radii of the wheel and axle respectively.

Other modifications of the lever occur by placing B (Fig. 182) between C and A or by placing A between B and C. The ratio of $W$ to $P$, called the *mechanical advantage*, for any such lever is easily seen to be always

$$\frac{W}{P} = \frac{a\sin\alpha}{b\sin\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

Or, in words, the magnitudes of the forces are inversely as the perpendiculars from the fulcrum on their lines of action.

*The Differential Wheel and Axle* winds the end on a thick part of the axle of radius $b$, while it unwinds it from a thin part of the axle of radius $c$, the loop of cord passing round a pulley supporting the weight. Hence $Pa = W\frac{1}{2}(b-c)$ where $a$ is the radius of the wheel.

*Weston's Pulley Blocks* is a compact tackle on the above principle, in which $a=b$.

**390. Efficiency of a Simple Machine.**—In dealing with the lever we have supposed that the only resistance which opposes its motion is the force $W$, which it is used to overcome by exertion of the force $P$, which may be called the effort. In that case, as we have noticed, the work done against the resistance is exactly equal to that done by the effort. Indeed, it was this equality which, on the principle of virtual work, gave the relation between $W$ and $P$. But in any actual lever or other simple machine (notably in the case of screws) there is always some *frictional* resistance to be overcome in addition to the main resistance for which the machine is used. And, since the total work done against all resistances can but equal that done by the effort which drives the machine, the *useful* work done will now fall short of that *total*. The proper fraction expressing this ratio is called the *efficiency*.

Thus, in any actual displacement of the machine (whether lever or other machine), let the effort $P$, the resistance $Q$, and the frictional or other wasteful resistance $R$ have displacements in their directions of $dp$, $-dq$, and $-dr$ respectively. Then, by the principle of virtual work, and taking this actual displacement as the virtual one, we find

$$\left.\begin{aligned} dW = Pdp + Q(-dq) + R(-dr) = 0, \\ \text{or} \qquad Pdp = Qdq + Rdr \end{aligned}\right\} \quad \ldots \quad (1).$$

Hence the efficiency $\eta$ is given by

$$\eta = \frac{Qdq}{Pdp} = 1 - \frac{Rdr}{Pdp} \quad \ldots \quad (2).$$

Or, in words, the *efficiency* of a machine is the ratio of the useful work done by it to that done by the effort on it when the machine receives any small actual displacement.

**391. The Screw.**—A screw thread may be cut in a cylinder by a tool moving parallel to the axis with uniform speed while the cylinder rotates uniformly. If the edges of the tool are respectively parallel and perpendicular to the axis, the screw is said to be *square-threaded*; if inclined, it is said to be *V-threaded*. For simplicity's sake we shall here confine attention to square-threaded screws.

Consider such a screw mounted so as to be capable of rotation about its axis, while any motion of the screw parallel to the axis is prevented. Also, on this screw, suppose there is a *nut* (or piece with internal screw fitting on the other) capable of motion parallel to the axis while rotation of the nut is prevented.

Let a torque act on the screw of magnitude $G$, which we may suppose due to a force $P$ acting perpendicular to an arm $a$ so that $G = Pa$.

Let the radius of the screw be $b$ and its *pitch*, or advance of the thread axially per revolution, be $q$. Then if we develop the thread by unrolling it from the cylinder into a plane we see that the angle between the thread and the base of the cylinder is given by $\alpha$ where $\tan\alpha = q/2\pi b$. We may note here that although $b$ and $\alpha$ are each variable quantities, differing from point to point along a radius, the pitch $q$ is perfectly

**definite.** Hence if $a$ and $b$ are involved in any equation they must be interpreted as the mean values of angle and radius, the expression involving them being approximate only.

**From** the symmetry of the screw it is obvious that we may develop the cylindrical thread into an incline of angle $\alpha$, and treat the problem as one of two dimensions. The horizontal force $Pa/b$ acting at the radius of the screw thus produces the vertical force $Q$ on the nut, as shown in Fig. 183.

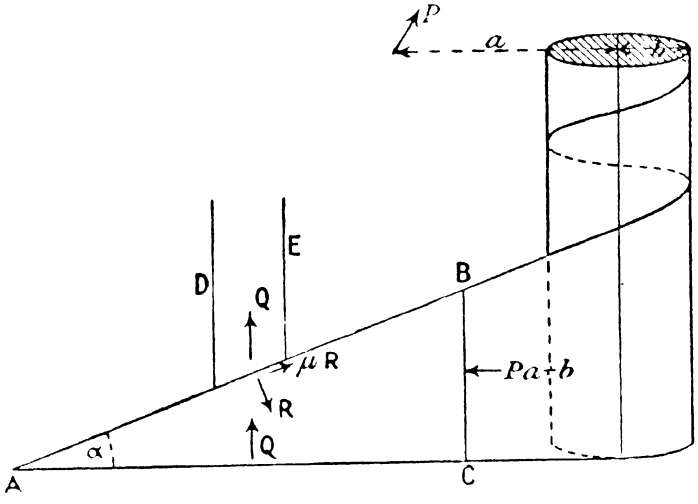



FIG. 183. SCREW DEVELOPED INTO A PLANE.

Let the normal reaction between the screw and nut threads be $R$ and the coefficient of friction between them be $\mu=\tan\beta$, then the tangential reaction may be anything up to $\pm\mu R$. If we suppose the torque acting on the screw is on the point of prevailing, then the frictional force acts upwards on the portion of screw thread AB, as shown in the figure.

Thus, for equilibrium, we have from the diagram by resolving horizontally and vertically,

$$Pa/b=\mu R\cos\alpha+R\sin\alpha,$$

and $$Q=R\cos\alpha-\mu R\sin\alpha.$$

Whence $$Pa=Qb\tan(\alpha+\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $\beta$ is the angle of friction.

Hence the ratio of $Q$ to $P$, usually called the mechanical advantage, is

$$\frac{Q}{P}=\frac{a}{b}\cdot\frac{1}{\tan(\alpha+\beta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

But, since it is the essential property of a screw to produce or overcome an axial resistance, the effort being a *torque* about that axis, it would seem preferable to quote as the mechanical advantage the ratio of $Q$ to $G$. We then have

$$\frac{Q}{G}=\frac{1}{b\tan(\alpha+\beta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

The efficiency of the screw pair (*i.e.* nut and screw) is $Qb\theta\tan\alpha/Pa\theta$, or

$$\eta=\frac{Qb\tan\alpha}{Pa}=\frac{\tan\alpha}{\tan(\alpha+\beta)} \quad . \quad . \quad . \quad . \quad . \quad (4),$$

when $P$ is on the point of prevailing. If $Q$ is on the point of prevailing the sign of $\beta$ must be reversed. We then find $\eta=\tan(\alpha-\beta)/\tan\alpha$.

If the friction is negligible $\beta$ vanishes, the efficiency is, of course, unity, and the mechanical advantage is given by

$$\left.\begin{aligned} Q/P &= 2\pi a/q, \\ Q/G &= 2\pi/q \end{aligned}\right\} \quad \text{or} \quad \ldots\ldots\ldots \quad (5),$$

as would be obtained at once by the principle of virtual work.

If for a given $\beta$, $\alpha$ is at our disposal, we may choose it so as to make $\eta$ a maximum.

Thus, differentiating $\eta$ to $\alpha$ by (4), we find that, for a maximum efficiency, $\tan 2\alpha = \cot\beta$, or $\alpha = \frac{\pi}{4} - \frac{\beta}{2}$ . . . . . (6).

As to the efficiency of any actual mechanism involving a screw and nut, it should be noted that it will always be lower than the expression (4), for that allows for friction only at the surfaces of the screw pair itself (*i.e.* the helical surfaces of the screw threads), but there must be friction also at the devices which prevent axial motion of the screw and rotation of the nut. Hence equation (4) is to be regarded as giving an ideal efficiency which is approached when the friction of parts other than the screw threads are almost negligible.

To take these other frictions into account, referring to Fig. 183, we should need to introduce a horizontal resistance increasing $P$ along AC, and also vertical resistances along the side of DE, thus reducing the available portion of $Q$. Both these new terms would reduce both the mechanical advantage and the efficiency. For approximate allowances for these quantities and tables of efficiency when $\mu = 0{\cdot}15$, see Goodman's *Mechanics Applied to Engineerng*, pp. 239-240 (London, 1908).

## EXAMPLES.—LXXV.

1. State in words and by equations the principle of virtual work as applied to rigid bodies.
2. What do you mean by the terms *mechanical advantage* and *efficiency* when applied to simple machines? Give an actual illustration, and find for it the value of each of the ratios mentioned.
3. 'Determine the mechanical advantage of a screw lifting jack, neglecting friction.

   'Calculate the tension in a stay which is tightened up by a force of P pounds acting on a lever $a$ inches long, which turns a double screw, composed of a right-handed screw of $m$ threads to the inch and a left-handed screw of $n$ threads to the inch.'

   (LOND. B.A. AND B.SC., PASS, MIXED MATH., 1902, I. 4.)
4. 'Enunciate the principle of virtual velocities, and mention the class of mechanical problem to which it is applicable.

   'Prove that if a horse is assimilated to an articulated parallelogram, the horizontal tractive force is

   $$1 \div (\tan\alpha - \tan\theta)$$

   of his weight, when the legs make an angle $\alpha$ with a horizontal road and the traces slope downwards at angle $\theta$.'

   (LOND. B.SC., PASS, MIXED MATH., 1902, II. 1.)
5. 'State the laws of friction between solid bodies, and describe the experimental verification.

'Prove that in lifting a body of weight $W$ with tongs of weight $W'$, they must if vertical be grasped with a force

$$\frac{W+W'}{2\mu'}, \text{ at a distance } \frac{W}{W+W'}\cdot\frac{\mu'}{\mu}$$

of their length from the hinge, $\mu$ denoting the coefficient of friction of the body and $\mu'$ of the hand on the surface of the tongs, both surfaces of contact being on the point of slipping.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1903, I. 2.)

6. 'Mention the class of statical problem to which the principle of virtual velocities is suitable for application.

'Prove that the virtual work of a couple is the product of its moment and the angle in radians through which it works, and prove that the balancing couples on the rods $AC$, $BD$ of a jointed quadrilateral $ACDB$ pivoted at $A$ and $B$ are as

$$AC/CE \text{ to } BD/DE,$$

where $E$ is the point of intersection of $AC$ and $BD$.'

(LOND. B.SC., PASS, MIXED MATH., 1903, II. 1.)

7. 'Enunciate the principle of virtual work for the equilibrium of any given system of connected bodies.

'$ABCD$ is a square formed by four equal uniform bars, each of weight $W$, freely jointed together at $A$, $B$, $C$, $D$; a strut of negligible weight connects the joints $B$ and $D$, so as to preserve the square figure when the system is suspended vertically from $A$. Show by virtual work that the pressure in the strut $= 2W$.'

(LOND. B.SC., PASS, MIXED MATH., 1904, II. 1.)

8. 'A ladder $AB$ rests on the ground at $A$ and against a vertical wall at $B$. If $AB$ is inclined to the vertical at an angle less than the angle of friction between the ladder and the ground, show geometrically that no load, however great, suspended from any point on the ladder will cause it to slip.' (LOND. B.SC., PASS, APPLIED MATH., 1905, I. 3.)

9. 'An effort $P$ is applied at the end of an arm of length $a$ to overcome a load $W$ placed on top of a rough screw press. The radius of the cylinder is $r$, the inclination of the thread of the screw to the horizon is $i$, and $\lambda$ is the angle of friction. Prove that

$$P = W\frac{r}{a}\tan(i+\lambda).$$

'If the radius of the cylinder is 2 inches, the effort arm 12 inches, the coefficient of friction 0·1, and there are 8 threads to the inch, find $P$. Will this screw reverse if $P$ is withdrawn?'

(LOND. B.SC., PASS, APPLIED MATH., 1905, I. 4.)

10. 'Prove that under a certain condition the altitude of the centre of gravity of a system of bodies in equilibrium is stationary for small displacements.

'Illustrate this by the case of a bar (not uniform) resting with its extremities on two smooth inclined planes which face one another.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, I. 6.)

11. 'A system of forces $(X_1, Y_1)$, $(X_2, Y_2)$, ... act at the points $(x_1, y_1)$, $(x_2, y_2)$, ... respectively of a plane lamina. If the lamina receives a small displacement such that the component displacements of the origin are $(\alpha, \beta)$ and the angle of rotation of the lamina is $\omega$, prove analytically that the total work of the forces is

$$\alpha.\Sigma(X)+\beta.\Sigma(Y)+\omega.\Sigma(xY-yX).$$

'Deduce the principle of virtual work.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, I. 10.)

12. 'State the principle of work as applied to a machine working uniformly against resistance.

'Apply it in the case of a screw press which is (1) smooth, (2) rough, assuming all requisite data.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, I. 5.)

13. 'A rod of weight $w$ and length $2l$ can revolve freely in a vertical plane about one end which is fixed. At a vertical height $h$ above the fixed end is a smooth peg over which passes a string, one end of which is attached to a smooth light ring which slides freely along the rod while the other end carries a weight $P$. Apply the principle of virtual work to find the position of equilibrium of the rod, explaining fully the argument on which the equation used is based.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, I. 5.)

**392. Stability of Equilibrium.**—Let us suppose a body to be in equilibrium in any position A under the action of any forces. Let the body be successively placed at rest in each of any two adjacent positions B and C on opposite sides of A. Then the type of the equilibrium at A may be defined as follows :—

(1) Let the body remain at rest at B and at C; its equilibrium at A is then said to be *neutral.*

(2) Let the body start moving towards A from both B and C; its equilibrium at A is then said to be *stable.*

(3) Let the body when at B start moving away from A and when at C start moving either from or to A; its equilibrium at A is then said to be *unstable*, for in either case it will finally move away from A.

Let us now find the analytical conditions to which these various types of equilibrium correspond. Suppose the body to be under the action of forces like gravity or the elastic reaction of a frictionless spring of constant properties. We then have the relations

$$T+V=\text{constant}, \quad dT+dV=0 \quad . \quad . \quad . \quad (1),$$

where $T$ and $V$ denote respectively the kinetic and potential energies of the body. But when a body starts to move from rest or increases any speed it has, $T$ is increasing; thus, by (1), $V$ must be decreasing. In other words, a body moves spontaneously so as to *diminish its potential energy.*

Thus for *neutral* equilibrium, since no motion occurs after a displacement, we have

$$dT=0=dV \text{ or } V=\text{a constant, } V_0 \text{ say} \quad . \quad (2).$$

For *stable* equilibrium, since motion towards A occurs from either side, the potential energy is a minimum there, $V_0$ say, and its increase in the neighbourhood is expressed by an *even* power of the displacement ($x$ say) multiplied by a *positive* coefficient. Or, in symbols,

$$V=V_0+a_2x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

For *unstable* equilibrium, if the motion after either displacement is always *from* A, we have by similar reasoning an even power of $x$ associated with a negative coefficient, or

$$V=V_0-a_2x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

While for *unstable* equilibrium with motion from A at one side, to A from the other and then through A and finally away, we have an odd power of $x$ involved, or

$$V = V_0 \pm a_3 x^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

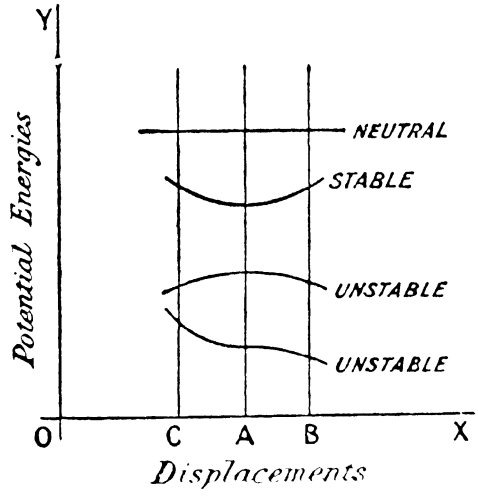



FIG. 184. POTENTIAL ENERGY GRAPHS, SHOWING STABILITY OF EQUILIBRIUM.

These results are collected in Table XIV. and illustrated by graphs of the potential energy in Fig. 184.

TABLE XIV. STABILITY OF EQUILIBRIUM.

| TYPES OF EQUILIBRIUM AT POSITION A. | INCREASE OF POTENTIAL ENERGY AT $x$ FROM A $= V - V_0 =$ |
|---|---|
| Neutral. | Zero. |
| Stable. | $+a_2x^2$. |
| Unstable. | $-a_2x^2$, or $\pm a_3x^3$. |

We may note here that the virtual work for any imagined frictionless displacement is numerically equal to the corresponding change of potential energy. Hence the principle of virtual work as a criterion of equilibrium is equivalent to the statement that for *equilibrium of any type* the first power of the displacement must vanish in the expression for $dW = \Sigma P dp$, the higher powers being regarded as negligibly small. We now see that the *determination of the type* of equilibrium requires retention of those higher powers and an examination of their indices and coefficients.

Or, looking at the graphs for the potential energy $V$ in Fig. 184, we may say that equilibrium of *any* type at A requires the *slope to be zero* there, whereas the *type* of equilibrium at A depends upon the *curvature* there or its rate of change.

But the slope, curvature, etc., near A depend respectively on the

first, second, and higher powers of $x$ in the expansion for $V$, which may be written generally

$$V = V_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \quad \ldots \quad (6).$$

Thus the various views of the subject are harmonised.

Referring again to the graphs for the potential energy $V$, it is seen that the positions of stable and unstable equilibrium may occur at minima and maxima; thus, in the absence of cusps and points of inflection in the curve, these two types may be expected to occur alternately in the equilibrium of a body. Simple examples of such alternation occur in the case of a loaded sphere rolling on a table and in that of a rod revolving about a horizontal or inclined axis near one end.

By bearing in mind the graphs of Fig. 184 we can often assert immediately whether the equilibrium of a body or system is stable or unstable.

**393. The Balance.**—The ordinary balance affords a good example of the lever, of equilibrium, of stability, and also of *sensitiveness* or ratio of inclination to difference of loads. Leaving to practical treatises the details of construction and manipulation, the essentials of a delicate balance for our purpose are indicated in Fig. 185.

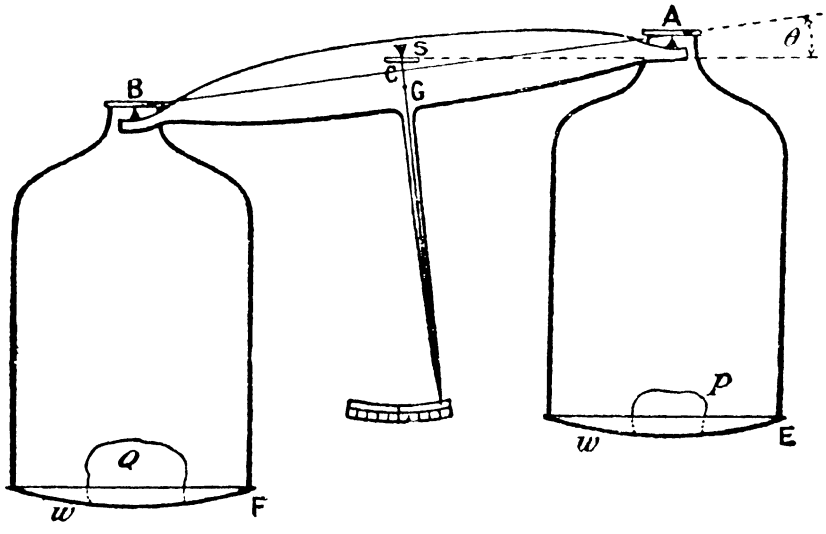



FIG. 185. THE BALANCE.

In this diagram the beam is suspended by a knife edge at S and, from other knife edges at A and B, hang the similar scale pans E and F.

Let the points A and B be joined by a straight line, and upon it from S let fall the perpendicular SC, and produce to G the centre of mass of the beam. Denote the arms of the balance CA and CB by $a$ and $b$ respectively. Let $SC = c$ and $SG = h$. Also let the beam have weight $W$, the scale pans each have weight $w$, and when weights $P$ and $Q$ are in the pans E and F, let the balance be in equilibrium with A Bat inclination $\theta$ to the horizontal as shown.

Then, since the forces at A and B due to scale pans and contents

are vertical, their arms are the horizontal projections of SA and SB or of $SC\hat{+}CA$ and $SC\hat{+}CB$ respectively. Thus, taking moments about S, we easily obtain the equation

$$(P+w)(a\cos\theta+c\sin\theta)+Wh\sin\theta=(Q+w)(b\cos\theta-c\sin\theta) \quad (1).$$

Whence
$$\tan\theta=\frac{(Q+w)b-(P+w)a}{(P+Q+2w)c+Wh} \quad . \quad (2).$$

A good balance in correct adjustment possesses three important properties, viz. the true zero position for equality of loads $P$ and $Q$, stability, and sensitiveness, which we will notice in this order.

*True Zero.*—When the loads in the pans are equal the line AB should be horizontal and the pointer indicate the centre of the scale. For this we must have $\theta=0$ for $Q=P$. On reference to (2) we see that this is obtained by making $b=a$, which is aimed at in the adjustment of the balance.

*Stability* of the equilibrium is obviously assured by pro viding that G is below S, for then G describes round S the lower part of a circle, and we have equation (3) of article 392 fulfilled. When the zero position of the balance is disturbed, the loads being equal, it is obvious that oscillations will occur. And, by the methods of Chapter XIII., article 258, we may write for the period of the oscillations (if $c=0$)

$$\tau=2\pi\sqrt{K/Wh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where $K$ is the moment of inertia of the beam, pans, and load; these being suspended at A and B, but *without* rotation, being reckoned as *particles* of same masses placed at A and B. Hence to reduce the period so as to facilitate quick working, we should have to diminish $K$, and therefore the arms $a$, but increase $h$. It is no use increasing $W$, for that equally increases $K$.

*Sensitiveness.*—Reverting to equation (2), we see that on putting $b=a$ as needed for the true zero, this may then be written

$$\frac{\tan\theta}{Q-P}=\frac{a}{(P+Q+2w)c+Wh} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

which then expresses the sensitiveness which may be measured by the ratio of $\theta$ or $\tan\theta$ to the difference of the loads. If we choose, this may be put in the form of a differential coefficient. Thus writing $Q=P+dQ$, and considering the increment $d(\tan\theta)$ to correspond to it, we have for this increment $\sec^2\theta\,d\theta=d\theta$ nearly. Hence (4) becomes

$$\frac{d\theta}{dQ}=\frac{a}{2(P+w)c+Wh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Thus by either (4) or (5) the sensitiveness for a given load may be increased by

(i) increasing $a$;
(ii) decreasing $c$, *or*, making it *negative*;
(iii) decreasing $h$;
(iv) decreasing $W$ and $w$.

It will be noticed that some of the ways of securing great sensitiveness clash with some of those for securing a small period. Hence any actual good balance is one presenting the kind of *compromise* most suitable for a certain purpose in view. By making $c$ negative, that is, bringing the point of suspension S *below* the line AB of the beam, the sensitiveness may be very greatly increased.

It is seen that the sensitiveness as expressed by (4) and (5) varies with the load $P+Q$ or $2P$, although $c$ and $h$ are there supposed constant. But in any actual balance the beam is not perfectly rigid. Accordingly these very small quantities $c$ and $h$ may appreciably change with the load, and thus cause an additional change in the sensitiveness. To approximately allow for this we may write these quantities as linear functions of the load. Thus let

$$c=c_0+c'P \text{ and } h=h_0+h'P \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Then, substituting in (5), we have as a first approximation for the sensitiveness of a balance with an elastic beam the equation

$$\frac{d\theta}{dQ}=\frac{a}{2(P+w)(c_0+c'P)+W(h_0+h'P)} \quad . \quad . \quad . \quad (7).$$

EXAMPLES—LXXVI.

1. Discuss the various types of equilibrium as to their stability, and obtain curves and equations applicable to the various possibilities.
2. Obtain expressions for the sensitiveness and period of oscillation of a delicate balance, and discuss the design to be preferred where both accuracy and quick working are needed.
3. 'Prove that the sum of the moments of a system of coplanar parallel forces about any point is equal to the moment of their resultant.

'In a precision balance the addition of 0·1 gram in one scale pan makes the pointer move over 6 mm. of its scale; find the depth of the C.G. of the beam below the plane of the three knife edges, having given that the distance of the extreme edges is 25 cm., the length of the pointer 18 cm., and the weight of the beam alone 200 grams.'

(LOND. B.A., PASS, APPLIED MATH., 1906, I. 2.)

**394. Graphical Statics.**—The vectorial addition of forces has already been noticed, both for their composition and as a criterion of equilibrium (articles 314-315). In some cases it matters little whether this vectorial addition is performed analytically or graphically. In many simple cases the analysis is quicker, especially if drawing materials are not at hand. But consider the case of a rigid frame in equilibrium under the action of various forces applied at different points. Here we have a number of points in equilibrium, viz. each place where the bars or members of the frame meet. Now, to treat any one such point graphically requires a single force polygon. But to treat a second adjacent point some one or more of the lines already drawn will form part of the second polygon then needed. Moreover, in dealing with roofs, girders, and other structural work approximate

values, easily obtained by drawing, are often accurate enough for the purpose, as ample strength is always provided in each member. Further, engineering and architectural calculators have the necessary drawing equipment and instruments at hand, and are expert in their use. Consequently the geometrical method of dealing with such problems has been highly developed, and now forms the very important branch of mechanics called *graphical statics*, a branch which no one studying statics can afford to neglect.

**395. Reciprocal Figures.**—The formal treatment of graphical statics naturally begins with some notice of the properties of those pairs of figures with parallel sides first completely pointed out by Maxwell 'On Reciprocal Figures and Diagrams of Forces' (*Phil. Mag.*, 1864; *Edin. Trans.*, vol. xxvi., 1870; *Scientific Papers*, vol. i. pp. 514-525, Cambridge, 1890). Following Routh, we note the fundamental definitions and properties thus:—

Two plane rectilinear figures are said to be reciprocal when

(i) they consist of an equal number of straight lines or edges such that *corresponding edges are parallel*, and

(ii) the edges which *meet in a point or corner of either figure* correspond to lines which form a *closed polygon or face in the other figure.*

It is owing to this second property that the term *reciprocal* has been given to these figures.

Any figure being given, it cannot have a reciprocal unless

(iii) every corner has at least *three edges* meeting at it, and

(iv) the figure can be resolved into faces such that each edge forms a base for two faces and for two only.

Since a closed polygon must have at least three sides, it is evident that to satisfy (ii) and (iv) we must have

(v) *at least three edges* meeting at each corner of each figure.

The edges of a figure can sometimes be combined in various ways so as to form different polygons. Only those polygons are to be regarded as faces in one figure which correspond to corners in the reciprocal figure. The figure is then said to be resolved into its faces.

Any side of any face in one figure corresponds to a parallel edge terminated at the corresponding corner of the reciprocal figure. Since an edge can only have two ends, and each represents a face in the other figure, it follows that

(vi) two faces and two only intersect in each edge.

**396. Frame and its Force Polygon.**—To illustrate the above properties, let us now take a very simple example of reciprocal figures in which one figure is a loaded frame whose bars are supposed rigid and freely jointed; and the reciprocal figure is the corresponding force polygon, which thus serves to graphically determine the forces in the bars or members of the frame. For, since the bars are supposed

freely jointed, the forces in each bar are along its length. These are shown by Figs. 186 and 187 respectively.

Since parallel lines in the two figures correspond, one obvious method of lettering to express this correspondence would be to place the same single letter on the middle of the parallel lines in each figure, say capitals in one and small in the other (see article 398). But this method suffers from the objection that lines may partly coincide. Thus in Fig. 187 $P$ and $Q$ do not coincide with $R$, and so may be separately lettered. But if in Fig. 186 $P$, $Q$, and $R$ had been given all vertical, it is evident that in Fig. 187 $P$ and $Q$ would have been end to end and $R$ coincident with $P+Q$. Again, if we indicate a line by letters at each end in one figure, it is impossible to indicate

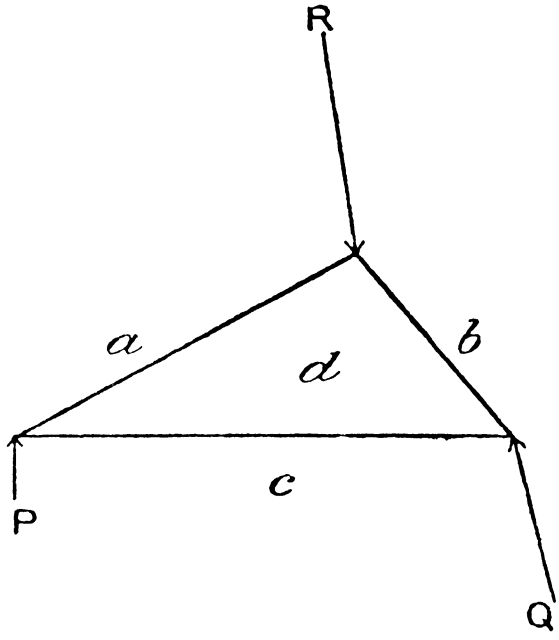



FIG. 186. FRAME DIAGRAM.

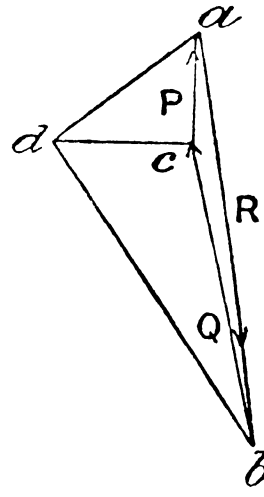



FIG. 187. FORCE POLYGONS.

the corresponding line in the reciprocal figure by the same letters at each end on account of the method of construction, which brings different lines requiring different letters to meet at any given point.

But since by condition (ii) of article 395 the *faces* of one figure correspond to the *points* in its reciprocal, these *faces* in one figure and corresponding *points* in the reciprocal may bear the *same letters*. This constitutes *Bow's notation* (1873), which is adopted in the small letters *abcd* of Figs. 186 and 187, and will often be used in what follows for a frame diagram and its force polygon.

Thus the bars of the frame in Fig. 186 may be called *ad*, *bd*, and *cd* respectively. And the forces in them are represented to scale by the lengths in Fig. 187 of the lines *ad*, *bd*, and *cd*. In the force polygon the letters here stand at the ends of any line called by them, whereas in the frame diagram the letters stand on the faces divided by the

line in question. But no difficulty arises from this, the line intended being in each case quite definite. Again, a point in the frame diagram is called by the letters of the faces meeting there, *acd* say; and, in the force polygon, the same letters occur at the corners of the polygon, which represent the forces *ca*, *ad*, and *dc* by which that point *acd* is in equilibrium. If desired, for further distinction, the letters may be capitals in one figure and small in the other.

**397. Construction of Force Polygon and its Interpretation.**— Referring still to Figs. 186 and 187, let us suppose we have given (i) the frame, (ii) the magnitude and direction of the force $R$ applied at *abd*, and (iii) the fact that the point *acd* rests on a roller so that the force $P$ must be vertical. Further, let it be required to find the forces $P$ and $Q$ and the forces exerted by the members or bars of the frame at their points of junction or joints.

We cannot determine $P$ and $Q$ directly by the force polygon of Fig. 187, for it is evident that $P$ may be drawn vertically of any magnitude we please, and $Q$ then follows accordingly. We may therefore, by reference to Fig. 186, take moments of $R$ and of $P$ about the point *bcd*, and equate their magnitudes. This determines $P$, so $Q$ follows as the vector which joins $R$ and $P$ in Fig. 187. Then, by the method of lettering which we have adopted, the vector $Q$ in Fig. 187 is to be lettered *bc*, since in Fig. 186 $Q$ divides the faces *b* and *c*; next, $P$ must be lettered *ca*; then $R$ is already lettered *ab* as it should be. We next draw through the corners *a*, *b*, *c* of this triangle, in Fig. 187, lines parallel to the bars *ad*, *bd*, and *cd* in Fig. 186. At first these parallel lines may be produced both ways from the corners till it is seen where they are likely to meet in the point *d*. The figures may then be looked over and checked to make sure all is right. It may be noted here that in the force polygon the *point c* stands at the *junction of P and Q*, just as in the frame diagram the *face c intervenes between P and Q*, and so on all round. In Fig. 187 arrow heads are placed along the lines representing $P$, $Q$, and $R$ just as they are in Fig. 186; this is done because these directions are all quite definite. But along the lines *ad*, *bd*, and *cd* in Fig. 187 no arrow heads are shown, because each of these lines represents oppositely directed but numerically equal forces, when in turn it forms a side of a different polygon. Thus, if we consider the force polygon *acd*, the clue to the way round is afforded by the force $P$, and we have the forces *ca*, *ad*, and *dc* respectively. But these are the forces which maintain equilibrium at the point *acd* of Fig. 186. Hence the member *cd* is pulling to the right at *acd*; it is consequently in tension, or is a *tie*, which fact is shown in Fig. 186 by leaving it as a *thin* line. The bar *ad*, on the other hand, is seen to be pushing at the point *acd*; it is therefore in compression, and is called a *strut*, and to express this in the diagram it is shown by a *thick* line.

Thus, as the investigation of the forces in the frame proceeds by the construction and interpretation of the force diagram (or *stress* diagram as it is often called), the nature of the opposite forces (or

*stresses*) exerted by each bar upon the joints at its ends is shown by this thickening of the lines, where necessary, to denote struts in the frame. This double use of the lines in the force polygons for opposite forces is seen if we now take the figure *bcd*. Here the force $Q$ gives the clue as to direction, and it follows that we are to write the forces *bc*, *cd*, and *db*. But this shows that the member *cd* is pulling to the left at the point *bcd*, whereas the same member was before found to be pulling to the right at the point *acd*. This shows, as before concluded, that it is a *tie*, or in tension.

It is also desirable to point out that the external forces applied to the frame should be denoted on the frame diagram, as in Fig. 186, by *external lines so as not to cut into the spaces in the frame itself.*

We have thus, in this very simple case, found the supports or reactions $P$, $Q$ applied to the frame under a given load $R$, and also the magnitudes and natures of the stresses in each member of the frame.

## EXAMPLES—LXXVII.

1. 'A light framework of freely jointed rods in the form of a right-angled isosceles triangle is suspended from the right angle. Weights $w$ and $2w$ are suspended from the other two joints. Determine the stresses in the rods.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, I. 3.)

2. 'Points $A$, $B$, $C$, $D$ are taken on a straight line, such that $AB=\frac{1}{2}BC=CD$. On $AB$, $BC$, $CD$ and on the same side of these are described equilateral triangles $AEB$, $BFC$, $CGD$. $EF$ and $FG$ are joined. The completed figure represents a system of freely jointed light rods in a vertical plane with $AD$ horizontal and lowest. Supports are placed at $A$ and $D$, and weights of 4 and 6 tons are hung on at $B$ and $C$ respectively. Draw a force diagram for the system. Thence determine the stresses in the rods which meet at $F$, indicating in each case which are tensions and which are thrusts.'

(LOND. B.SC, PASS, APPLIED MATH., 1907, I. 6.)

3. '$AD$ is a straight line trisected at $B$ and $C$, and $BCEF$ is a square. Let $AF$, $AB$, $BF$, $FE$, $BE$, $BD$, $ED$ be rods forming a freely jointed framework, and let this framework be supported at $A$ and $D$ so that $AD$ is horizontal and $BF$ vertically upwards. Find, by graphical construction, the thrusts or pulls in all the rods due to a weight $W$ placed at $E$.'

(LOND. B.SC., PASS, APPLIED MATH., 1908, I. 4.)

4. 'Three equal light rods $AB$, $AC$, $AD$, each of length $a$ loosely jointed at $A$, have their other ends joined by three strings each of length $b$, and rest with $B$, $C$, $D$ on a smooth horizontal plane so as to form a tripod. From $A$ is suspended a load $W$. Show how to find the tensions in the strings by a graphical construction or otherwise.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, I. 5.)

5. '$ABCD$, $PBCQ$ are squares on opposite sides of $BC$, and $E$ is the centre of the latter square. A framework is made of weightless rods $AB$, $AD$, $DC$, $BC$, $DB$, $BE$, $CE$ freely jointed to one another. $A$ is freely pivoted to a smooth vertical wall, and $D$ presses against the wall below $A$. A weight $W$ is hung from $E$. Calculate the stresses in all the rods.' (LOND. B.SC., PASS, APPLIED MATH., 1910, I. 7.)

**398. Funicular Polygon for Resultant of Coplanar Forces.**—In the example dealt with in articles 396-397 we had only a single force $R$ applied above the frame, and found by calculation the corresponding values of the reactions or supports. But usually there are a number of forces or loads applied on the upper side of the frame, and in this and other cases it is often desirable to determine *graphically* the magnitude, direction, and *line of action* of the resultant of such a system of coplanar forces. This may be done by what is termed the *funicular* polygon or *link* polygon.

The method of drawing such a polygon and the proof of its pro-

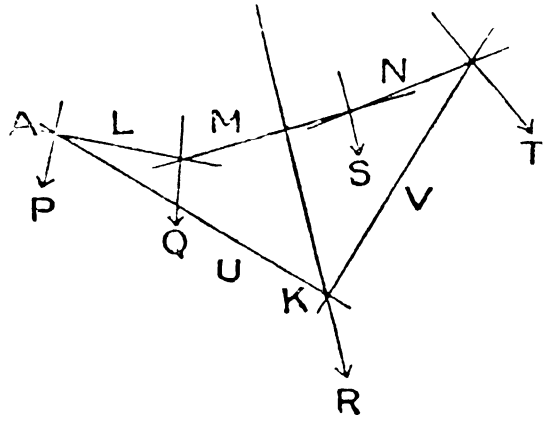



FIG. 188. SET OF COPLANAR FORCES AND FUNICULAR POLYGON.

perties may be seen from the following example:—Let the set of forces $P$, $Q$, $S$, $T$ be given as shown in Fig. 188 and their resultant required. The force polygon $pqst$ in Fig. 189 determines the magnitude and direction of the resultant $R$ which are equal to those of $r$, but the line of action of $R$ is still to be found.

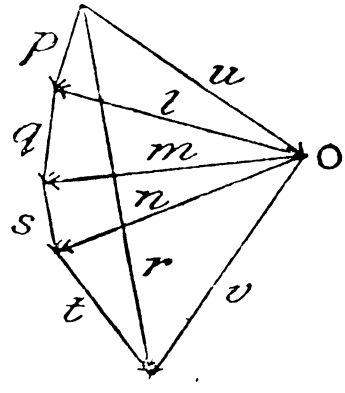



FIG. 189. FORCE POLYGONS FROM WHICH TO DERIVE FUNICULAR POLYGON.

*Construction.*—To determine this proceed as follows:—

*First*, in the diagram, Fig. 189, take any convenient point O, called the *pole*, and join it to the corners of the force polygon by the lines $u$, $l$, $m$, $n$, $v$.

*Second*, in the line of action of $P$ in Fig. 188 take any point A, and from it draw the lines $U$ and $L$ parallel to $u$ and $l$ in Fig. 189. From the intersection of $L$ with $Q$ draw $M$ parallel to $m$. *Continue* in this way, drawing next $N$ and *finally* $V$ intersecting $U$ at K, thus completing the funicular polygon whose sides are *ULMNV*. Finally, through K, in Fig. 188, draw $R$ equal in magnitude in direction to $r$, in Fig. 189. Then K shall be a point on the line of action of the resultant, which is thus truly represented by $R$.

*Proof.*—Referring to Fig. 189, we see that $P$ is represented by $p$, and is equivalent to the components $u$ and $l$ taken in the directions of the arrows. Thus $P$ in Fig. 188 may be replaced by forces of these magnitudes along $U$ and $L$ *meeting at* A *on the line of action of* $P$. Again, $Q$ is represented by $q$ or its components $-l$ and $m$, or by forces of these magnitudes along $L$ and $M$ meeting on the line of action of $Q$. Thus the force along $L$ is cancelled, being taken in turn positively and negatively. In like manner, as we proceed along the *funicular polygon ULMNV*, all the intermediate forces $l$, $m$, and $n$ are cancelled, and only those along $U$ and $V$ equal to $u$ and $v$ are left. But their resultant is $r=R$, which must act through their intersection K as shown.

This line of thought may be expressed symbolically as follows, it being understood that when the small letters are used the corresponding forces are represented as to magnitude and direction only, the large letters denoting in addition the correct line of action. The sign $\hat{}$ over the $+$ shows that the addition is vectorial.

$$P\hat{+}Q\hat{+}S\hat{+}T=p\hat{+}q\hat{+}s\hat{+}t$$
$$=(u\hat{+}l)\hat{+}(-l\hat{+}m)\hat{+}(-m\hat{+}n)\hat{+}(-n\hat{+}v)$$
$$=u\hat{+}v=r$$

$=(u$ along $U)\hat{+}(v$ along $V)$

$=R$ through K.

It may easily be seen that, by shifting the point A, we change to a second funicular polygon with sides respectively parallel to those of the first. Whereas, if we shift O, we change to a second funicular polygon with sides in general not parallel to those of the first. But, in either case, K remains on the same line of action of $R$, as shown in Fig. 189A.

**398a. Link Polygons with Different Poles.**—We may now establish the following important *theorem* on link polygons:—

ENUNCIATIONS.—*The corresponding sides of any two link polygons for a given system of forces intersect on a straight line, which is parallel to that joining the pole of the two funiculars.*

Referring to Figs. 189A and 189B, the system of forces is denoted in the former by $PQST$ with resultant $R$, and in the latter by the force polygon $p$, $q$, $s$, $t$ closed by $r$.

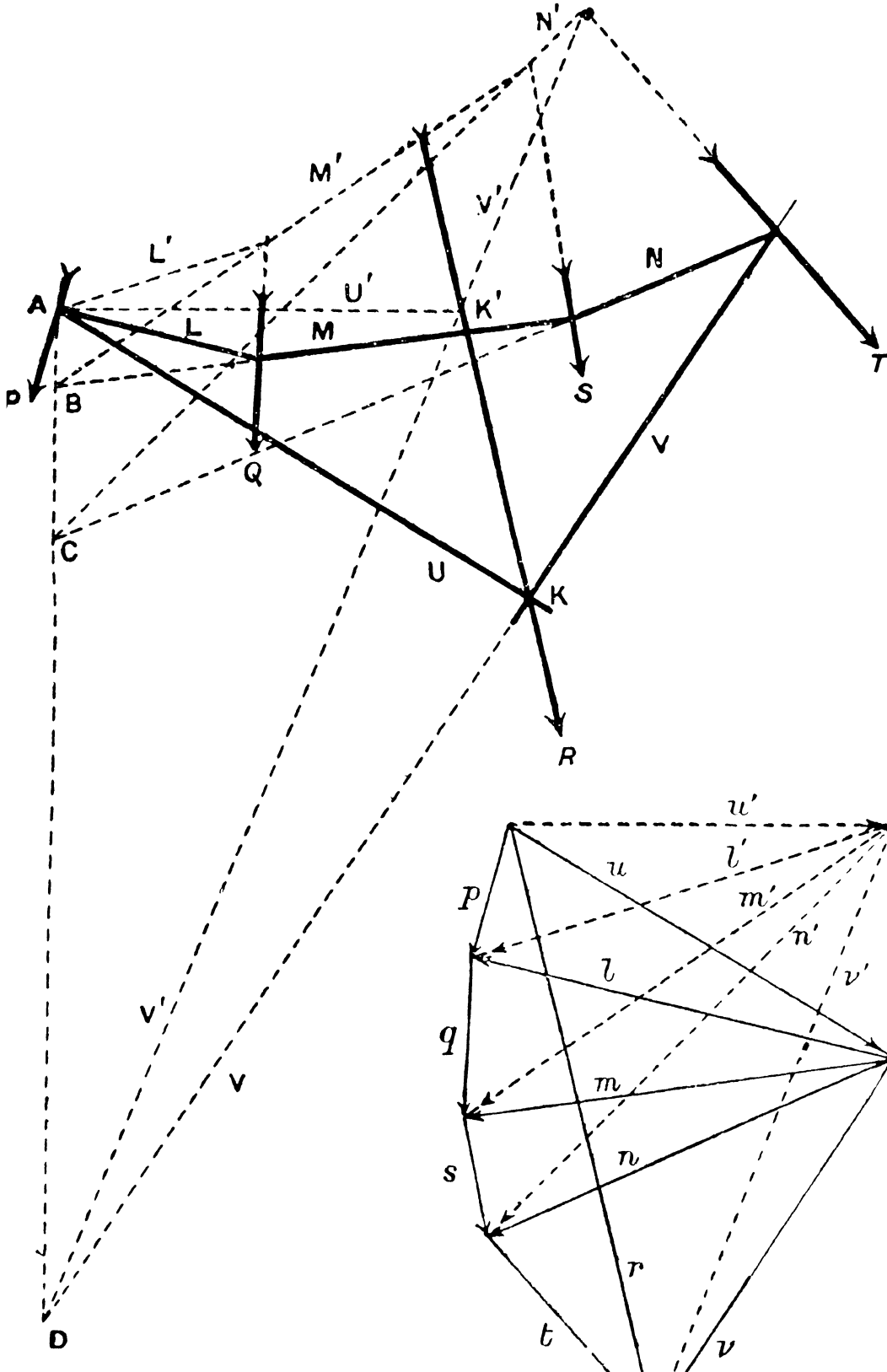



FIG. 189A. SET OF FORCES AND TWO LINK POLYGONS.

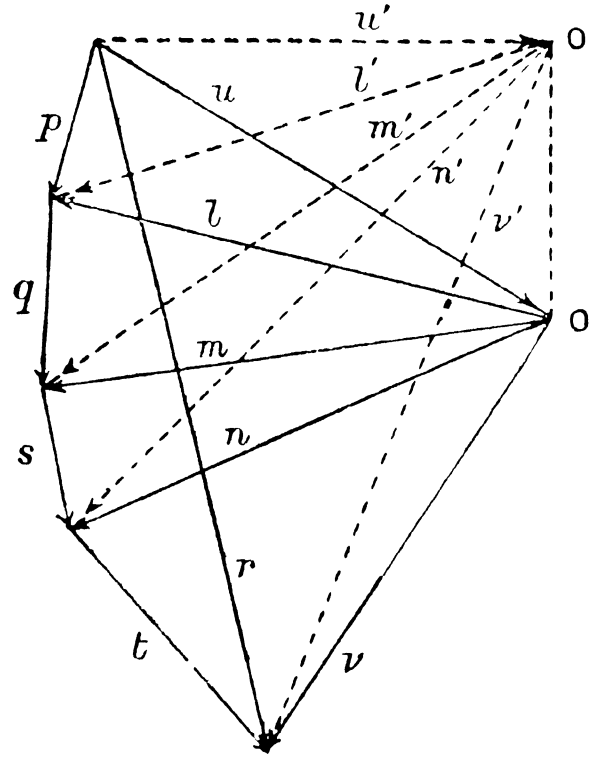



FIG. 189B. FORCE POLYGON AND TWO POLES FOR LINK POLYGONS.

In Fig. 189B the poles O and O′ are taken, and the two sets of rays $u$, $l$, $m$, $n$, $v$ and $u'$, $l'$, $m'$, $n'$, $v'$ drawn. Corresponding to these we have in Fig. 189A the two funicular polygons $U$, $L$, $M$, $N$, $V$ and $U'$, $L'$, $M'$, $N'$, $V'$. Let the intersections of the sides $UU'$, $LL'$, $MM'$, $NN'$, and $VV'$ be A, A, B, C, and D respectively. Then shall ABCD in Fig. 189A be a right line parallel to OO′ (in Fig. 189B).

*Proof by a Statical Method.*—At the intersection of $L$ and $M$ in Fig. 189A, let the force $Q$ act as shown. Also at the intersection of $L'$ and $M'$ let the force $-Q$ act. Hence, with the component forces along $L$, $M$, $L'$, and $M'$, we shall have two sets of forces in equilibrium, three in each set, giving in all six forces in equilibrium, viz.

$$\left.\begin{array}{l} +Q,\ -l \text{ along } L,\ +m \text{ along } M \\ -Q,\ +l' \text{ along } L',\ -m' \text{ along } M' \end{array}\right\} \quad \ldots \quad (1).$$

But since the two $Q$'s are equal and opposite along the same line, they are in equilibrium, and consequently the other four are in equilibrium. Hence any pair of these four will equilibrate the remaining pair. Thus

$-l$ along $L$ and $+l'$ along $L'$, each applied at A,

will equilibrate $-m'$ along $M'$ and $+m$ along $M$, each applied at B. Accordingly each pair must represent a force along AB.

But by Fig. 189B

$-l$ and $+l'$ have a resultant of magnitude and direction O′O,

whereas $-m'$ and $+m$ have a resultant of magnitude and direction OO′.

Therefore AB is parallel to OO′. And in the same way the like relation may be established for ACD.

**399. Graphical Conditions of Equilibrium.**—We may now enunciate from the graphical standpoint the conditions of equilibrium of a rigid body under the action of coplanar forces. We recall that, as stated analytically in (1) and (2) of article 367, these conditions are equivalent to

(i) Resultants of forces parallel to $x$ or $y$ each equals zero.

(ii) Resultant torque in plane of $xy$ equals zero.

Referring to Figs. 188 and 189 of article 398, we see that if five forces were given, viz. $P$, $Q$, $S$, $T$, and one of the magnitude of $R$ but opposite in direction, then the force polygon would show that the resultant force is zero, thus fulfilling the first condition of equilibrium. But if this reversed force $R'$ say did not pass through the point K in Fig. 188, but was parallel to the $R$ there shown and at a perpendicular distance $r'$ from it, the whole system would be equivalent to a couple of magnitude $R'r'$. Moreover, on the line of action of this force $R'$, the sides $U$ and $V$ of the funicular polygon would not intersect, but there would be two points where these sides $U$ and $V$ would respectively meet the line of action of $R'$. This is described by stating that the funicular polygon would not be closed. But to make the resultant couple zero *it must be closed*; that is, $U$

and $V$ must meet in the same point K on the line of action of the reversed $R$.

Thus the formal graphical conditions for equilibrium of a rigid body or system under coplanar forces are the following two:—

(i) *The force polygon must be closed; and*

(ii) *The funicular polygon must be closed.*

And these clearly correspond to the equations (1) and (2) of article 367 each to each.

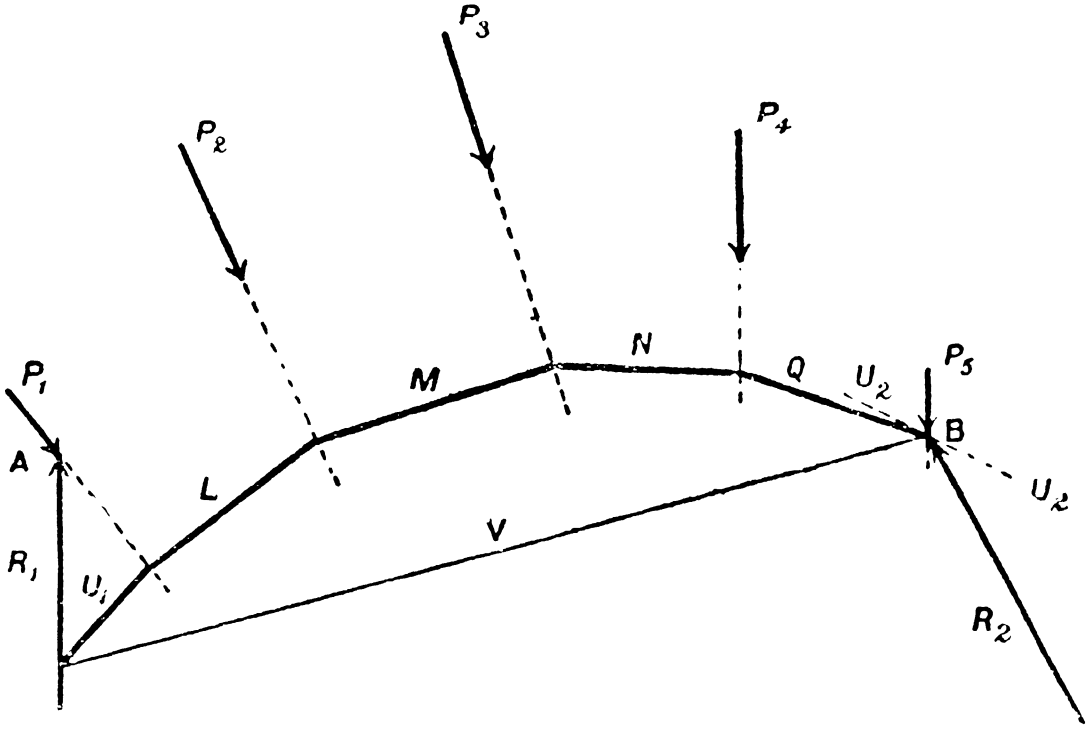



FIG. 190. COPLANAR FORCES, REACTIONS, AND FUNICULAR POLYGON.

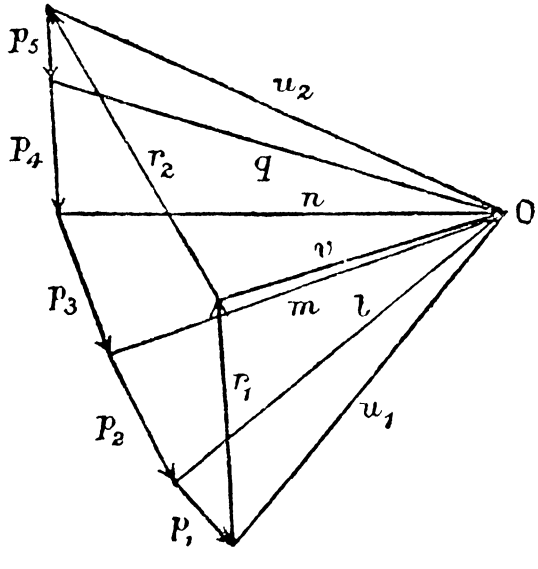



FIG. 191. FORCE POLYGONS FOR FUNICULAR.

**400. Reactions determined by Funicular or Link Polygon.** —We have seen that the single resultant of a system of coplanar forces may be completely determined by the use of the funicular polygon, and that the equilibrium of a rigid body requires that the force polygon and the funicular polygon should each be closed. We now pass to a third application of the funicular polygon, in which it is used to determine each of the two reactions at the supports of a rigid bar or frame which is loaded by any set of coplanar forces. The case in question is illustrated by Fig 190, where $P_1$, $P_2$, $P_3$,

$P_4$, $P_5$ denote the forces which constitute the load on some frame (not shown in the diagram); the conditions as to the reactions at the supports being that they occur at the *points* A *and* B, and that the *reaction at* A *is vertical* because the frame rests on a roller there.

For the graphical determination of this problem we may proceed as follows:—Of the force polygon draw the sides $p_5$, $p_4$, $p_3$, $p_2$, $p_1$ (Fig. 191), and join the corners to a convenient point O chosen as pole by the lines $u_2$, $q$, $n$, $m$, $l$, $u_1$. We have now to find a suitable point in Fig. 190 at which we may begin drawing the sides of the funicular polygon parallel to the lines meeting at O. We cannot begin at A, because that would leave it impossible to decide where a given side of the funicular *not passing through* B would intersect the reaction $R_2$, since its direction is not at first known. We therefore commence the funicular polygon at B, the only point known in the reaction $R_2$. We then draw through B the lines $U_2$ and $Q$ parallel respectively to $u_2$ and $q$ in Fig. 191. From the intersection of $Q$ with $P_4$, produced if necessary, we draw $N$ parallel to $n$, and in like manner $M$, $L$, and $U$, meeting the known line of action of the reaction $R_1$ of the support at A. We can then join this point of intersection to B by the line $V$, and parallel to $V$ we draw $v$ through O in Fig. 191; this cuts the vertical $r_1$ and enables us to complete the force polygon by drawing $r_2$.

Then *the reactions sought are represented by* $R_1$ *and* $R_2$ *acting at* A *and* B in Fig. 190 and parallel and equal to $r_1$ and $r_2$ in Fig. 191 each to each.

It may be noticed that though $U_2$ has been drawn in Fig. 190 it is not required, since the forces it should join intersect at B.

**401. Roof with Asymmetrical Load.**—Let us now suppose the set of coplanar forces just dealt with to be the asymmetrical load due to wind on a specified roof principal (or roof truss) as shown in Fig. 192. We may then accept the reactions at the supports as found by the funicular polygon in article 400, and proceed to find the stresses in the members of the frame by the force polygons of Fig. 193 on the principles explained in articles 396 and 397. We may conveniently deal with the points of the frame in the order of the numbers shown in Fig. 192.

And the various applied forces or stresses in the bars at each point may be taken in the order indicated by the letters after each number in the following scheme:—

1. *badc*,
2. *cdfe*,
3. *fdag*,
4. *efghk*,
5. *khml*,
6. *hgam*,
7. *nlma*.

Thus it is well to begin at either 1 (or 7), because as there are only

two unknown stresses here, the polygon is determinate. The point 2 is taken next in preference to 3, because the former has only two unknown stresses, while the latter has three until that in *df* has been determined by dealing with point 2.

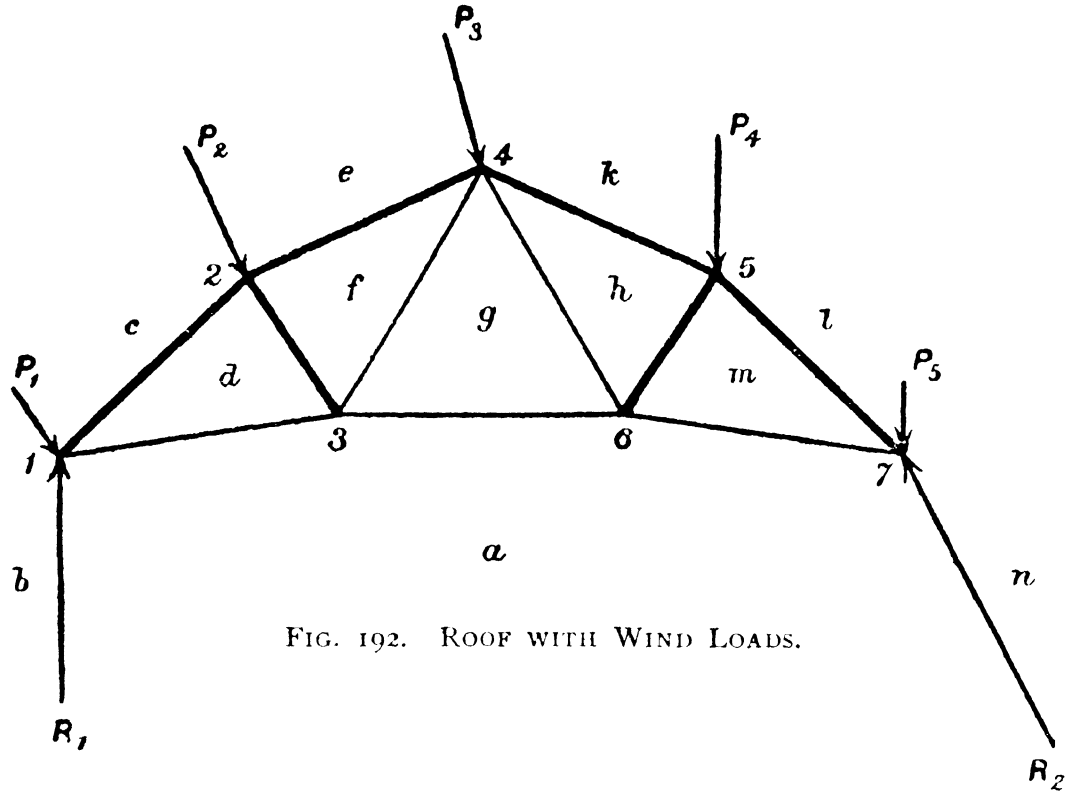



FIG. 192. ROOF WITH WIND LOADS.

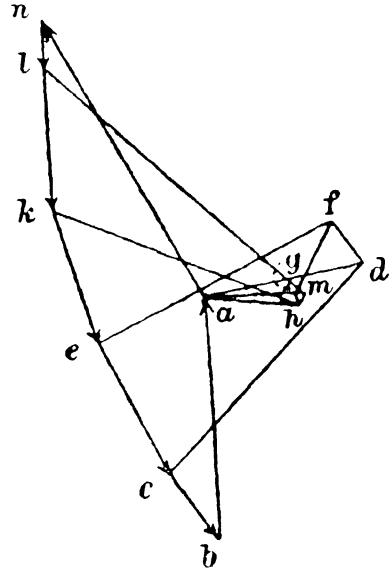



FIG. 193. FORCE POLYGONS FOR ROOF

In the above scheme, where a *pair* of letters is underlined it denotes that it has just been found that the bar in question is a *strut*, *i.e.* is under compression. In carrying out the work, the student at this stage may well thicken the line which represents the strut in the frame diagram.

**402. Evaluation of Stresses apparently Indeterminate.**—Let us now consider a frame, of a form often adopted for roofs, in which at a

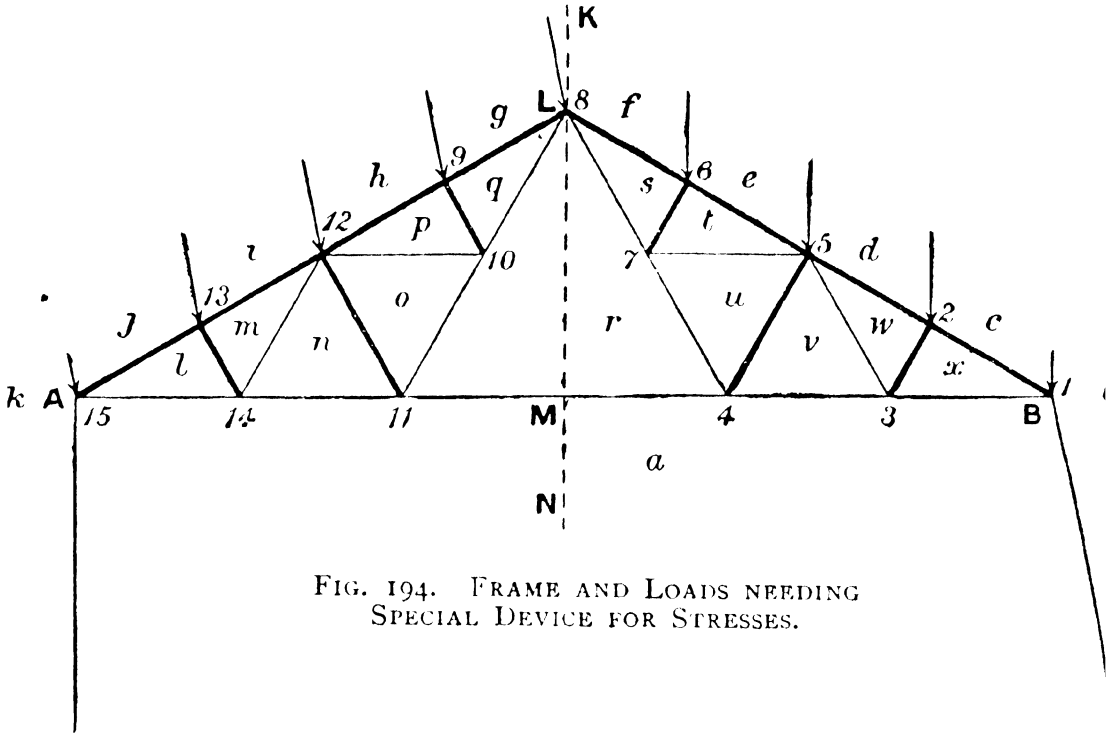



FIG. 194. FRAME AND LOADS NEEDING SPECIAL DEVICE FOR STRESSES.

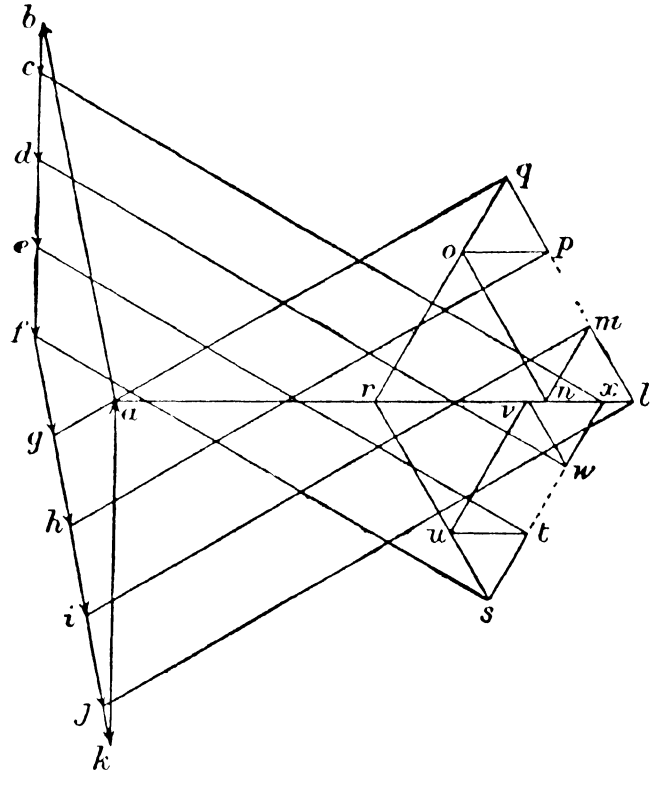



FIG. 195. FORCE POLYGONS FOR ABOVE FRAME.

certain stage the methods hitherto adopted for the stresses needs supplementing by some other device. Thus, let the frame AMBL of

Fig. 194 be given with the loads there shown, and let it be required to find the reactions and the stresses in every member of the frame.

We attack this problem as in articles 400 and 401, finding the reactions by the funicular polygon and then proceeding with the stresses at each point. Suppose we begin at B (marked also 1 in Fig. 194), taking next in order the points 2 and 3. There is no difficulty so far, as each point had only two unknown forces, so each of the corresponding force polygons was determinate. But a difficulty arises as to the next point to be taken. For point 4 has the *three* unknowns *vu*, *ur*, *ra*, if taken before 5. While point 5 has the *three* unknowns *et*, *tu*, *uv*, if taken before 4.

There are various ways of proceeding at this juncture. Perhaps for our purpose the most convenient is that known as the *Method of Sections*. Goodman, in his *Mechanics Applied to Engineering*, p. 506 (London, 1908), states that this method, usually ascribed to Ritter, is really due to Rankine.

Adopting it in the present case, we take a *section* of the frame along, say, the central vertical line KLMN (Fig. 194), and consider the equilibrium of the right half of the frame under (i) the loads applied to that half, (ii) the reaction at B, and (iii) the stress in the member *ra* at M. Now only the last-named of these is unknown. Hence, taking moments about L and equating to zero their algebraic sum, we determine by computation the magnitude and nature of the stress in *ra*. There are then only two unknowns at point 4, which is accordingly dealt with in the usual way. Thus, *vu* being found, point 5 has only two unknowns, and is therefore determinate. The other points may then follow in the order as numbered, and give no further difficulty.

The whole procedure may now be summarised as follows, the references being to the drawing of the force polygons in Fig. 195:—

1. *abcxa*,
2. *cdwxc*,
3. *axwva*.

By method of sections find *ra*,

4. *ravur*,
5. *detuvwd*,
6. *efste*,
7. *tsrut*,
8. *fgqrsf*,
9. *ghpqg*,
10. *qporq*,
11. *arona*,
12. *phimnop*,
13. *mijlm*,
14. *mlanm*,
15. *ljkal*.

As before, the pairs of letters in this scheme underlined are those referring to members just found to be in compression. At each step

they should accordingly be thickened in Fig. 194 to indicate a strut.

It may be mentioned here that the loads shown in the roofs hitherto as applied *at the joints* are supposed to be usually derived from loads really *distributed between those joints.* They are replaced by equivalent forces at the joints, because we are not here concerned with the bending of the members which are imagined rigid.

In the examples already noticed the reactions were usually needed as a preliminary to the determination of the stresses, because there were more than two unknowns where the loads were applied. But, suppose the load is practically a single force, as in the case of some forms of crane for lifting heavy weights. And let it be applied at a point in which only two members meet. We may then begin at that point in the evaluation of the stresses, the reactions at the supports being determined by the force polygons simply, without any recourse to the funicular polygon.

### EXAMPLES—LXXVIII.

1. 'Show how to find the magnitude and line of action of the resultant of a number of coplanar forces by means of a vector and a link polygon.
'If two different poles $O$ and $O'$ be taken for the link polygon, prove that the intersections of corresponding links all lie on a straight line parallel to $OO'$.'
(LOND. B.SC., PASS, APPLIED MATH., 1909, I. 2.)
2. 'Translate and explain the following passage :—Die Grösse und Richtung der Resultante eines ebenen Kräftesystemes ergiebt sich durch die Schlusslinie des Kräftepolygones. Es genügt aber zur Festlegung der Resultante schon die Kentniss eines einzigen Punktes auf ihrer Aktionslinie. Zur graphischen Bestimmung eines solchen wird ein Seilpolygon gezeichnet. Somit setzt sich die Resultante des ganzen Kräftesystemes aus den beiden Kraften zusammen, welche sich in den äussersten Seiten des Seilpolygones ergeben, und geht durch deren Schnittpunkt hindurch.'
(LOND. B.SC., PASS, APPLIED MATH., 1908, I. 5.)
3. 'Three concurrent forces are in equilibrium ; show *ab initio* that any funicular triangle of the system is closed.
'Also prove that if two funiculars be drawn, the intersections of corresponding sides are collinear.'
(LOND. B.SC., PASS, APPLIED MATH., 1906, I. 3.)
4. Draw a set of coplanar forces acting on a frame or beam, and find by the stress and link polygons the reactions.
5. For a French truss roof under asymmetrical loads find the reactions at the ends and stresses in all the members. (See Fig. 194.)

**403. Reactions at Joints.**—In a jointed arrangement of rigid bars, or bars and cords, it is often a matter of interest and importance to find the reaction at some of the joints.

In attacking such problems the work may frequently be shortened by keeping clearly in mind two principles pointed out by Routh as to the direction of these reactions. These may be stated as follows :—

(i) Let the body be hinged at two points A and B, and let it be acted on by no other forces except the reactions at A and B. Since

the body is in equilibrium under these two reactions, they must act *along the straight line joining the hinges* and be equal and opposite.

(ii) Let the body and the external forces be both symmetrical about some straight line *through the hinge*. Then the action and reaction between the two bars must be symmetrically situated. But they are also equal and opposite. Hence, to fulfil both conditions, the action and reaction must each be *perpendicular to the line of symmetry*.

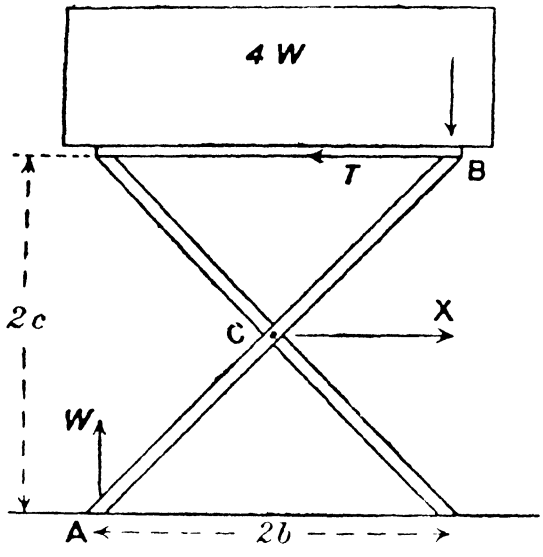



FIG. 196. REACTION AT JOINT OF CAMP STOOL.

The first of these principles as to direction of reactions has already been illustrated in dealing with roofs by graphical methods, and will be useful again presently. Let us now illustrate the second principle by two examples of symmetrical problems.

Take first the case of a camp stool on a smooth floor loaded by a smooth box of weight $4W$ as shown in Fig. 196. Each side of the stool consists of two inclined bars as illustrated, jointed at their centre C, whose action and reaction are sought. Let the tension in the cross straps at the top be $T$. Then the bars and forces being all symmetrical about a *vertical line through the hinge*, the reactions there are *horizontal*, and may accordingly be denoted by $X$ simply, no vertical component $Y$ being needed. Call the height of the frame $2c$ and its width $2b$, and consider a single inclined bar AB. Then the load at B and the reaction at A are each vertical and of magnitude $W$. Hence, taking moments about B, we find

$$W2b = Xc, \text{ or } X = 2Wb/c \quad . \quad (1).$$

Then, resolving horizontally,

$$T = X = 2Wb/c \quad . \quad . \quad . \quad (2).$$

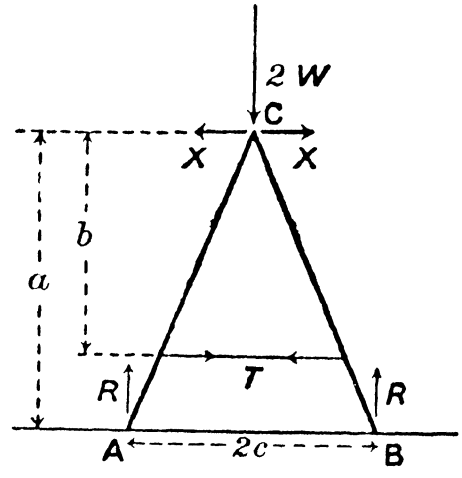



FIG. 197. REACTION AT TOP OF STEPS.

As a second example of a frame and forces all symmetrical consider now the set of steps which in its simplified form makes the A shown in Fig. 197, the floor being supposed smooth.

Let the load at the top be $2W$, and let the reaction there and the tension of the cord below be required. Denote by $2c$ the spread AB of the legs, let the top C be at a height $a$ above the ground and $b$ above

the cord, whose tension is $T$. Then, by the principle of symmetry, the reaction at C is horizontal, and will be denoted by $X$. Also, because the floor is smooth, the reactions at A and B are each vertical of value $R$ say. Resolving vertically shows that $R = W$; resolving horizontally for either leg (AC say) gives $T = X$. Then finally, taking moments about A, we have $Xa = Wc + T(a - b)$.

Whence $X = Wc/b = T$ . . . . . (3),

thus giving the reactions sought.

**404. Separation of Bars.**—To find the reactions at joints it is sometimes a convenience to suppose a bar separated from the others, and then represent the reactions it bears, and so determine them. This method will be illustrated by the framework of seven members shown in Fig. 198, one of the side pieces AG being considered detached to show the forces more clearly, as represented on a larger scale in Fig. 199.

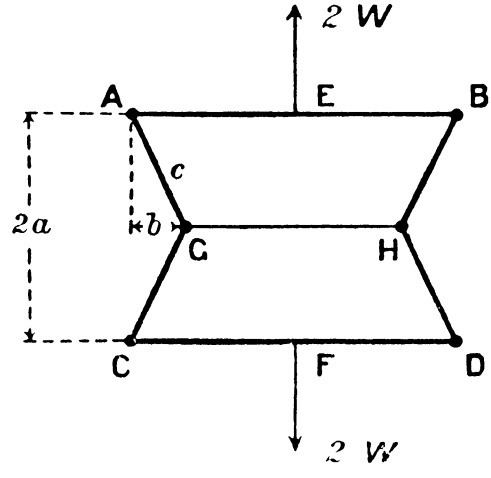



FIG. 198. REACTIONS AT JOINTS A AND G.

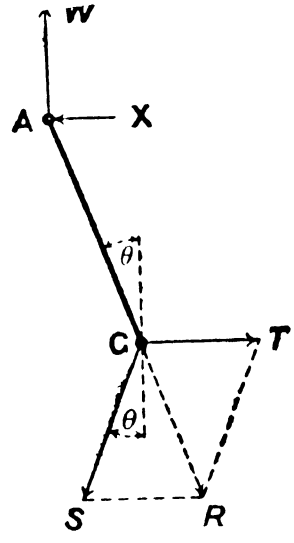



FIG. 199. BAR AG SEPARATED FROM FRAME.

The frame AEBCFDGH is suspended at E, the middle point of AB, and has a weight $2W$ attached at F, the middle point of CD. Let it be required to find the reactions on the ends of one of the inclined bars, AG say. We cannot treat this by the symmetrical principle, because the line of symmetry does not pass through either of our points. We may, however, apply the other principle of article 403 to show that the action $S$ exerted at G by the bar GC must be along GC, since this bar has forces at its ends only. The same principle shows us that at A the action due to the bar AB is not solely along it, for there is the force $2W$ at E which is not a joint, AB and CD being stiff bars with joints at their ends only. We thus obtain the view of the forces on AG which is represented in Fig. 199, viz. that A has the force $W$ vertically upwards and the thrust $X$ horizontally to the left, while G has the tension $T$ horizontally to the right and $S$ obliquely

downwards in the direction of the bar GC in Fig. 198, making an angle, $\theta$ say, with the vertical. Let $AC=2a$, $AB-GH=2b$, the lengths of each of the four inclined bars being $c$. Then $\sin\theta=b/c$ and $\cos\theta=a/c$.

We may now consider the equilibrium of the bar AG in the usual way. Thus, resolving vertically gives $W=S\cos\theta=Sa/c$; resolving horizontally, $X+Sb/c=T$; and, taking moments about G, $Xa=Wb$.

Hence we have

$$S=Wc/a,\ X=Wb/a, \text{ and } T=2Wb/a \quad . \quad . \quad . \quad (4),$$

showing that the resultant of $W$ and $X$ is along GA as should be.

Compounding $S$ and $T$ to give the resultant $R$, we find that

$$R=S, \text{ and is along AG} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

as obviously should be the case, and so provides another check.

These relations might also have been obtained graphically.

**405. Reactions inside Bodies.**—We now pass to the consideration of the reactions between two adjacent portions of the same continuous body by which the equilibrium of each of those portions is maintained in spite of certain impressed forces or systems of forces. We here confine our attention to the determination of these forces of reaction and their variation with the circumstances of the case, because at present we are dealing only with bodies supposed rigid.

The discussion of any changes in size or shape of the body consequent upon the operation of these forces forms part of the theory of elasticity and is accordingly deferred to Chapter XXI., which is devoted to bodies exhibiting elastic properties.

We start with the following simple example:—Let a horizontal beam be subject to a vertical force $P_1$ at $x_1$, and consider the reactions $\Sigma X$ and $\Sigma Y$ which must act horizontally and vertically at $x$ to maintain in equilibrium the portion of the beam between these two points (see Fig. 200). We suppose the beam to be held in equilibrium by some forces applied at or beyond $x$, but with the exact nature of these forces we are not now concerned.

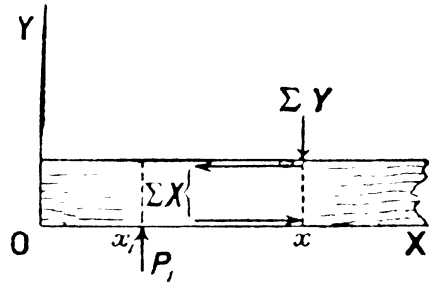



FIG. 200. REACTIONS IN A BEAM.

Then, for the equilibrium of the part of the beam under consideration, resolving vertically and horizontally, and taking moments about the point of co-ordinates $(x, 0)$, we find

$$\Sigma Y=-P_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$\Sigma X=0 \text{ and } \Sigma(-Xy)=P_1(x-x_1) \quad . \quad . \quad . \quad . \quad (2).$$

The signs of $\Sigma X$ and $\Sigma Y$ must be reversed if we wish to express the actions of this portion of the beam upon that portion beyond $x$.

The vertical components of this action and reaction, acting parallel to the vertical section at $x$, constitute what is called the *shearing stress* there, its magnitude being *force per unit area*. We are here concerned with the *pair of total forces* of this stress, and shall denote them by $S$, which refers to the magnitude of either force, the positive sign being

used for the present case, in which the positive force is experienced on the positive side of the section in question.

The horizontal components have a distribution as yet undetermined, but have a definite moment as given by (2), and their resultant is equivalent to a pure couple, the possible force being zero. They constitute what is called the *bending moment* at the section under consideration. This moment will be denoted by $M$, the positive sign being used when the parts near the section in question tend to become concave towards the positive direction of $y$.

**406. Two or more Forces.**—Suppose that to the same beam, in addition to $P_1$, other forces $P_2$, $P_3$, etc., are now applied at $x_2$, $x_3$, etc., each such abscissa being less than $x$; and let it be required to express the reactions at $x$.

Then, by the same reasoning which led to the former equations for the single force, we find

$$P_1+P_2+P_3+\ldots=\Sigma P=S \quad \ldots\ldots\ldots\ldots \quad (3),$$

and $$\Sigma(-Xy)=P_1(x-x_1)+P_2(x-x_2)+P_3(x-x_3)+\text{etc.}=M \quad (4).$$

As before, $\Sigma X=0$. These expressions for the shearing forces and the bending moments show that there is a simple relation between them. For, on differentiating (4) with respect to $x$, we have

$$\frac{dM}{dx}=P_1+P_2+P_3+\ldots=\Sigma P=S \quad \ldots\ldots \quad (5).$$

Or, in words, though the absolute value of the bending moment depends upon the positions as well as the magnitudes of the forces, its rate of change along the beam at any place depends only on the sum of all the forces up to that place, which is also the shearing force at the place in question.

**407. Distributed Load.**—It is obvious from (3), and the reasoning which led to it, that, if the forces instead of being few and finite are very many and correspondingly small so as to become practically a continuous or distributed load, still the sum of all the forces up to a point expresses the shearing forces at that point, and we also see that this stress will then be a continuous function of the abscissa. Hence, for this case, we may again differentiate to $x$, operating upon equation (5). We thus find

$$\frac{d^2M}{dx^2}=\frac{dS}{dx}=Q \text{ say} \quad \ldots\ldots\ldots\ldots \quad (6),$$

where $Q$ clearly denotes the total impressed force per unit length at $x$.

**408. Diagrams for Bending Moments and Shearing Forces in Beams.**—It is now desirable to consider how these reactions in beams may be graphically represented. For the diagrams so formed illustrate the relations just developed, and also lend themselves to the solution of problems.

Take first the case of a horizontal beam held at or beyond $x$ and with vertical forces $P_1$, $P_2$, $P_3$ acting at $x_1$, $x_2$, $x_3$ respectively. And

let us draw diagrams in which the abscissae represent lengths along the beam, the ordinates representing in one diagram the bending moments $M$ and in the other the shearing forces $S$. The beam and these diagrams are shown in Figs. 201, 202, and 203.

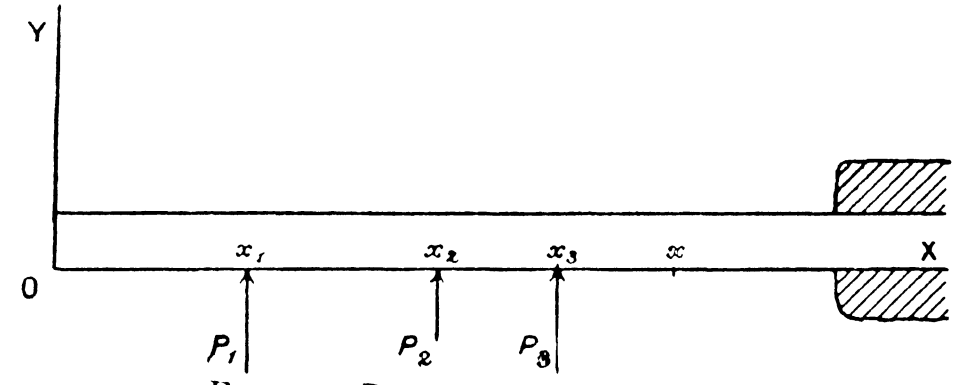



FIG. 201. BEAM WITH THREE FORCES.

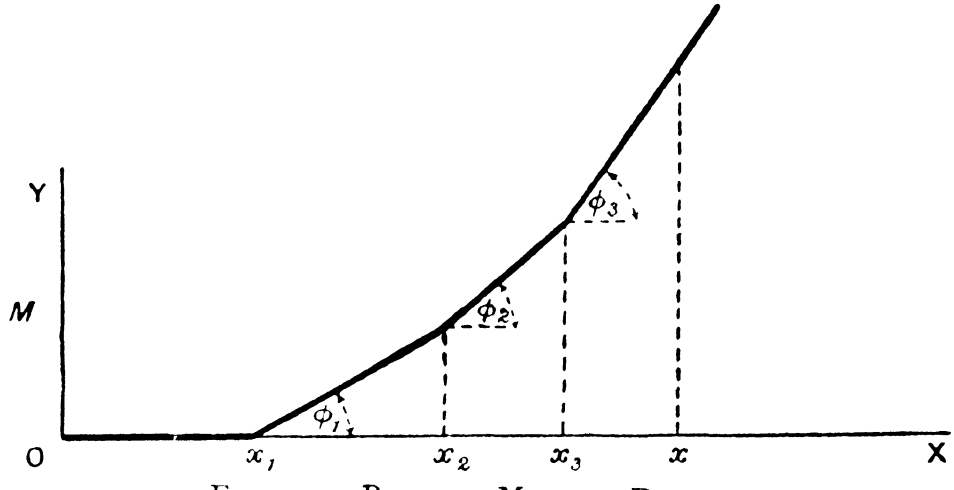



FIG. 202. BENDING MOMENT DIAGRAM.

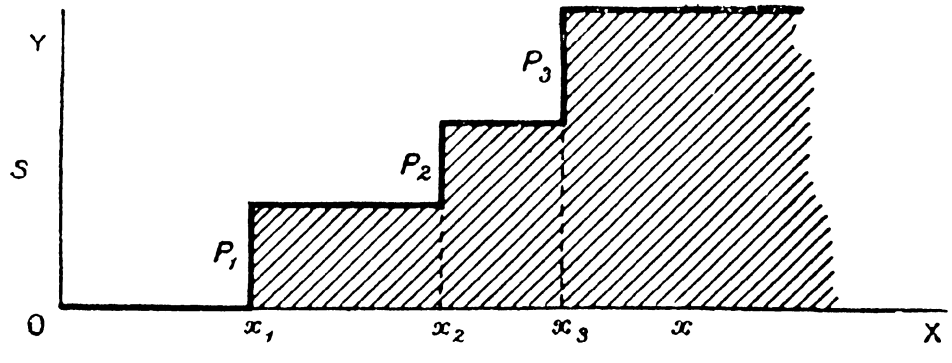



FIG. 203. SHEARING FORCE DIAGRAM.

In the bending moment diagram the inclinations of the line are denoted by $\phi_1$, $\phi_2$, and $\phi_3$ after the applications of the forces $P_1$, $P_2$, and $P_3$ respectively. And, since at the various parts in this diagram the values of $dM/dx$ are the tangents of the corresponding angles, we have the general relations

$$\tan\phi = dM/dx = \Sigma P = S \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

We may also represent the increase, $M - M_1$, of $M$ between two

points, $x_1$ and $x$ say, as the areas of the shear diagram over that range. For from (7) we have

$$M - M_1 = \int_{x_1}^{x} S dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

But, in using either (7) or (8), care is necessary as to the scales used in the horizontal and vertical directions of the diagrams. Thus, if the horizontal scale of the bending moment diagram were 1 inch to the foot, and the vertical scale 1 inch to 10 foot-tons weight, a line inclined at 45° would have a tangent 10 when properly interpreted instead of unity, as in ordinary geometry, where the scales are equal in all directions.

The comparison of Figs. 202 and 203 shows instructively what is expressed by equations (6) and (7), namely, that the slope of the bending moment diagram always equals the shearing force, and therefore any change in this slope equals the increase in the ordinates of the shear diagram at the place in question. These changes in slope occur abruptly because the loads are discrete forces. The next example better illustrates equation (6).

**409. Uniformly loaded Cantilever and Beam.**—Let us now take the case of a cantilever (or beam projecting from a wall into which it is built) loaded uniformly along its length $l$, the forces being downwards

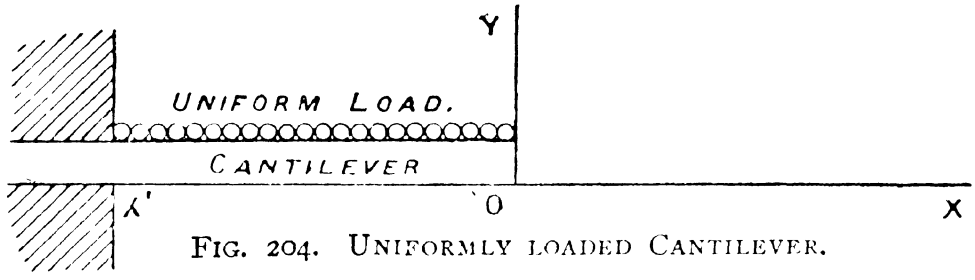



FIG. 204. UNIFORMLY LOADED CANTILEVER.

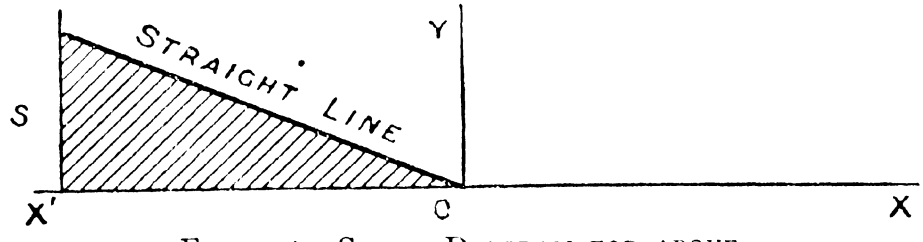



FIG. 205. SHEAR DIAGRAM FOR ABOVE.

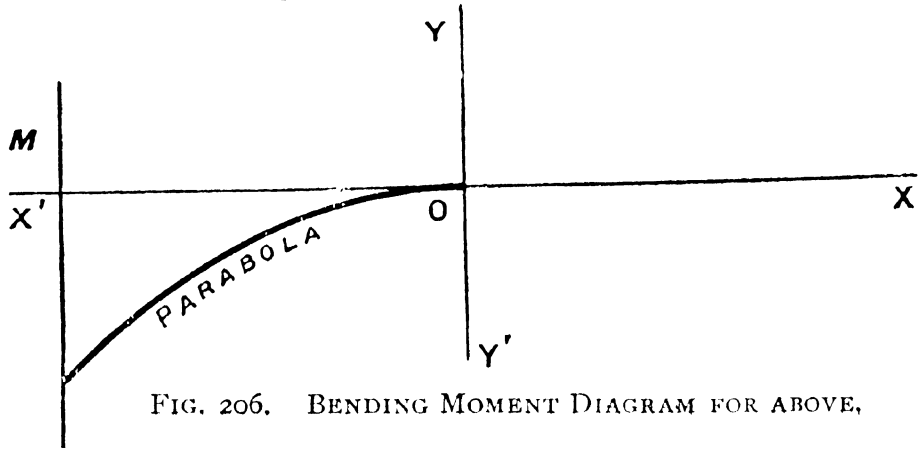



FIG. 206. BENDING MOMENT DIAGRAM FOR ABOVE.

and of distribution whose numerical value is 7 per unit length. Let us draw the cantilever projecting to the right and take the origin of co-ordinates at its free end with $y$ upwards, as shown in Figs. 204, 205, and 206, giving the beam itself and the two diagrams.

Then $Q = -7$, and by (6) $S = -7x$ . . . (9).

But by (5) $dM = Sdx = -7xdx$.

Whence $$M = -7\int_0^x xdx = -7x^2/2 = Qx^2/2 \quad . \quad . \quad (10).$$

Thus, since our abscissae are all negative $S$ is by (9) positive, as shown in Fig. 205; and $M$ is by (10) negative for any values of $x$, and is accordingly shown negative in Fig. 206. We see from (10) that the bending moment diagram is part of the parabola

$$x^2 = -\tfrac{2}{7}y = \frac{2}{Q}y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Hence its vertex is at the origin, its axis is vertical, and its branches extend downwards.

If the length of the cantilever is $l$, we see that the maximum shearing force at the root, where $x = -l$, is given by $-Ql = +7l$. Also the bending moment at the root, by (10) or (11), on substitution of $-l$ for $x$, is seen to be $-7l^2/2$.

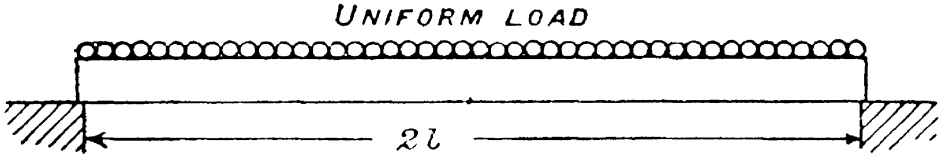



FIG. 207. UNIFORMLY LOADED BEAM

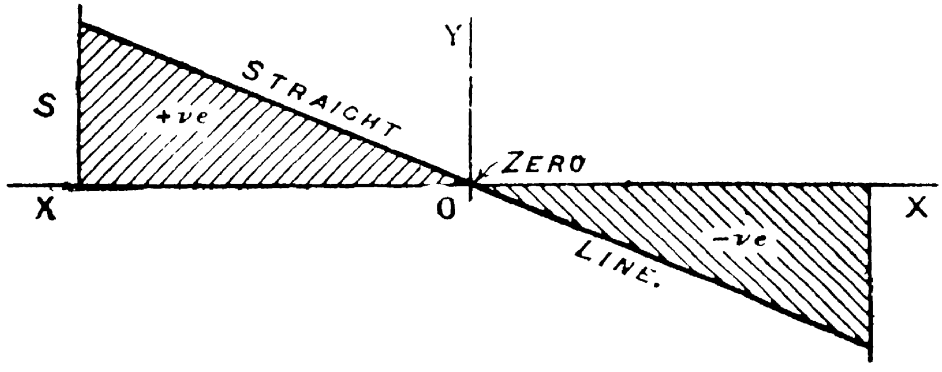



FIG. 208. SHEAR DIAGRAM OF SAME.

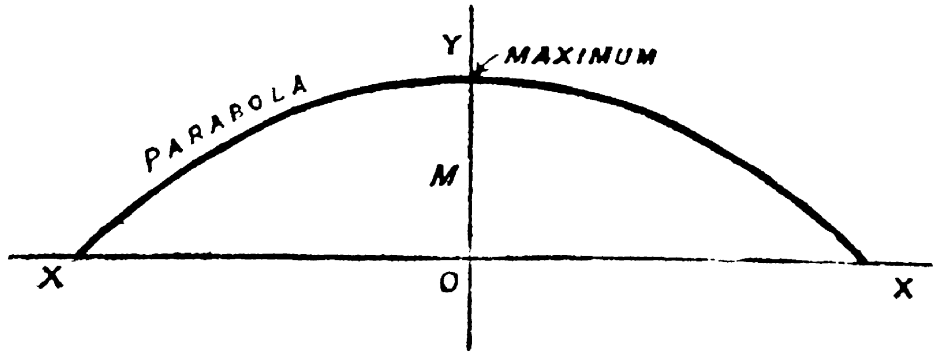



FIG. 209. BENDING MOMENT DIAGRAM.

We may now very simply pass to the case of a beam supported at each end with clear span of $2l$ and a downward load of 7 per unit length, *i.e.* as before $Q=-7$. The diagrams for this case are given in Figs. 207-209, and need little if any further explanation. It may be noted that although the shear and bending moment diagrams seem simply extended to twice the width by producing their lines, still in the case of the latter (Fig. 209) the ordinates are all different, the maximum moment being now at the place of zero shear instead of both vanishing together as before.

In choosing the scale of ordinates for the bending moment diagram it is neither necessary nor in general convenient to represent by the same height a moment $M=S\times 1$, as was used in the shear diagram to represent the shearing force $S$.

To avoid making the ordinates in the bending moment diagram unduly large, $S\times$ one-third or half the length of the beam may be represented by the same ordinate as $S$ itself in the shear diagram. See also the next article.

**410. Bending Moment Diagram a Particular Link Polygon.**—It is now desirable to note that the bending moment diagram as hitherto drawn on a horizontal base is a particular form of the link polygon for vertical forces.

To illustrate this, consider the case of a beam supported at the ends and loaded at three points as shown in Fig. 210. From the data there shown a force diagram is drawn as in Fig. 211, in which the polygon closes to a single line BAC since all the forces are vertical. Now let some point O be taken as pole and the *rays* OB, $OW_1$, $OW_2$, $OW_3$ be drawn, and consider their slopes. Obviously, as we pass from ray to ray, the change of tangent of the inclination is, on the scales chosen, equal to the corresponding forces interposed between those rays. Hence, if any one ray has the right slope for the corresponding part of the bending moment diagram, they all have the right slopes. Suppose $R_1$ is the reaction at the support at the origin in Fig. 210, and in Fig. 211 take $AB=R_1$, and let AO be perpendicular to AB. Then, if the bending moment diagram begins with a slope parallel to OB in Fig. 211, it is evidently right, on the understanding that the scale of ordinates is such that a bending moment equal to $(OA\times AB)$ is represented by AB simply. Then this first slope being right, the others will be right if drawn parallel to $OW_1$, $OW_2$, and $OW_3$ respectively, the junctions of the slopes corresponding to the points of application of the loads $W_1$, $W_2$, and $W_3$.

Thus $R_1$ being found, by calculation or a preliminary link polygon with any pole O′, and the pole O being taken as shown, the shear diagram may be appropriately drawn, as in Fig. 212, level with the force polygon, the bending moment diagram being at foot as in Fig. 213.

If $R_1$ is determined by a link polygon with any pole O′, that link polygon may be retained as the bending moment diagram. But if O′ were *not* level with A in Fig. 211, as O is, then the bending moment diagram so obtained would be on a *sloping base* parallel to O′A. Ordinates could still be measured on it, as on the other, but it would

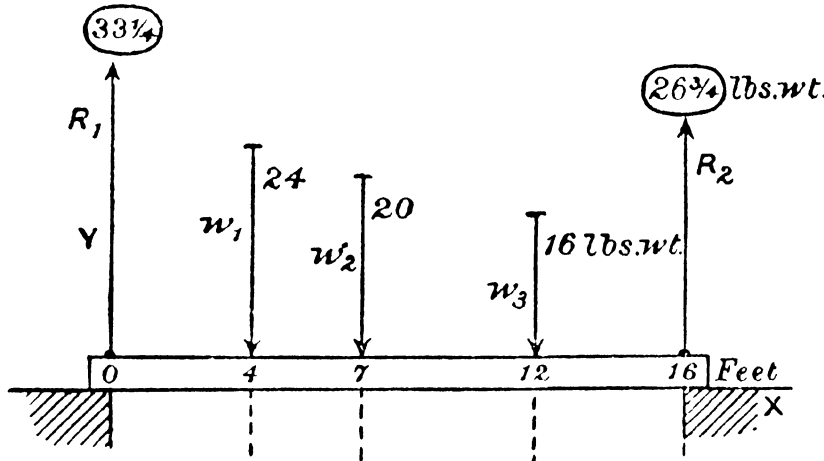



FIG. 210. BEAM WITH IRREGULAR LOADS.

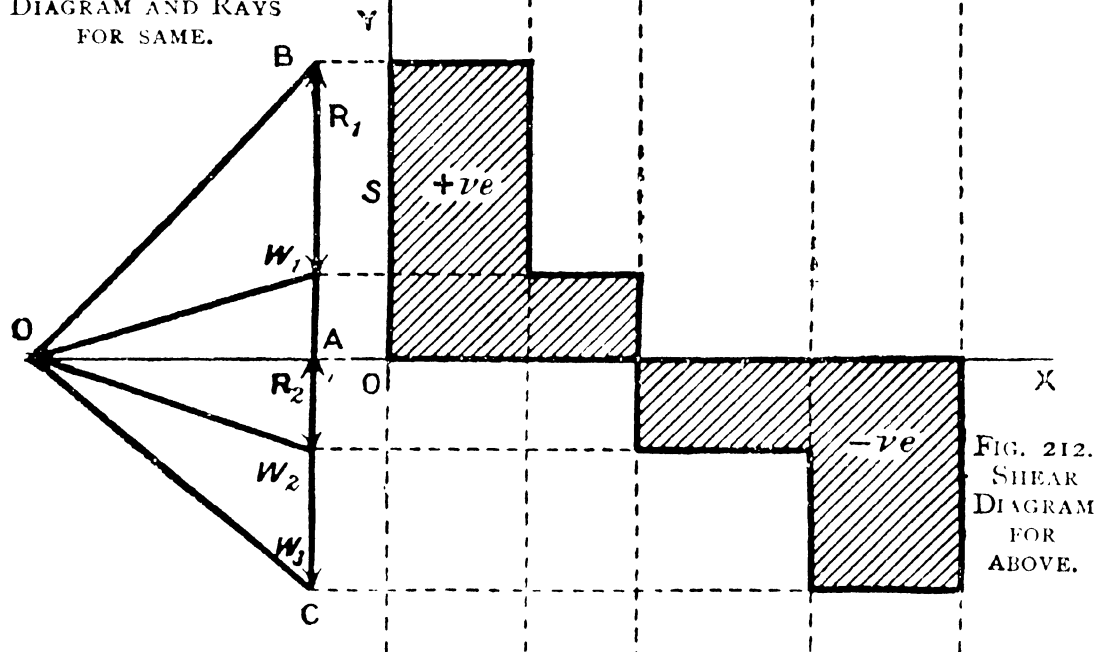



FIG. 211. FORCE DIAGRAM AND RAYS FOR SAME.

FIG. 212. SHEAR DIAGRAM FOR ABOVE.

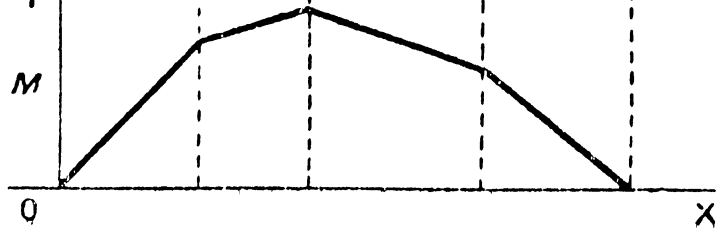



FIG. 213. BENDING MOMENT DIAGRAM AS LINK POLYGON.

not be possible to apply to it the equations written for rectangular co-ordinates and suppose them to be still valid as rectangular co-ordinate equations.

The relation of the scales of the ordinates in the shear and bending moment diagrams has been shown to depend upon the pole distance OA. A full statement as to all the scales may be put as follows :—

In all diagrams let 1 inch of abscissae represent $a$ feet, and in the shear diagram let 1 inch of ordinate represent a force of $b$ lbs. wt.; then, in the bending moment diagram, 1 inch of ordinate shall represent a bending moment of $abc$ ft. lbs. wt. where $c$ is the pole distance OA, and the slopes of the bending moment diagram are parallel to the corresponding rays from O.

For further information on these beam diagrams, such works as Professor A. Morley's *Strength of Materials* or D. A. Low's *Applied Mechanics* may be consulted.

## EXAMPLES—LXXIX.

1. State how to find the reactions at joints of a frame, illustrating your answer by a numerical example.
2. Explain the nature of the reaction between the two parts of a beam, divided by an imaginary cross section, when one or more forces are applied at places beyond this section.
3. Explain the terms *shearing forces* and *bending moment* as applied to a beam under load, and obtain relations between the above quantities and the load per unit length.
4. Make shear and bending moment diagrams for a beam fixed in a wall and loaded at two or more points.
5. Discuss the stresses in a beam resting on two supports and loaded uniformly throughout its length. Draw the diagrams for the shear and for the bending moments, and obtain the equations of the curves.
6. 'A bar $AB$ is in equilibrium under the action of given forces; show how to find the resultant action exerted over the cross section at any point $P$ of the bar by the portion $BP$ on the portion $PA$.

   'In particular, if $AB$ is a uniform bar resting on two supports at $A$ and $B$ in a horizontal line, what is the action between the two halves of the bar? Explain the result.'

   (LOND. B.SC., PASS, APPLIED MATH., 1905, I. 2.)
7. 'A framework $ABCD$ is formed of four similar uniform heavy rods freely jointed at their extremities. The rod $AB$ is of length $2a$, the rod $CD$ of length $3a$, and the rods $AD$, $BC$ each of length $a$. If the framework is suspended at the middle point of the rod $AB$, show that the ratio of the reaction at an upper joint to that at a lower joint is $\sqrt{91} : \sqrt{43}$.'

   (LOND. B.SC., PASS, APPLIED MATH., 1909, I. 4.)
8. 'Explain the usual specification of stress across a section of a thin rod subject to forces. How is the specification simplified in the case of a flexible string?

   'A girder 25 feet long (whose weight may, for the present purpose, be neglected), rests horizontally on smooth supports at its ends, and carries loads of 4 tons, 10 tons, and 4 tons at distances of 7, 12, and 20 feet respectively from one end. Make link and vector polygons for the forces on the girder, indicate how the bending moment varies from point to point of the girder, and find its greatest value.'

   (LOND. B.SC., PASS, APPLIED MATH., 1910, I. 6.)

# CHAPTER XVIII

## SOLID STATICS OF RIGID BODIES

**411. Resultants of any System of Forces acting on a Rigid Body. Poinsot's Method.**—Let a typical force acting at $(x_1, y_1, z_1)$ be denoted by $P_1$ and have rectangular components $X_1$, $Y_1$, and $Z_1$, as shown in Fig. 214.

Introduce at the origin the opposite forces $OX_1$ and $OX'_1$, and at B the opposite forces $BX_1$, $BX'_1$, all parallel to the axis of $x$ and numerically equal to the $X$ component of $P_1$. We have then five forces which may be regarded as a single one at the origin and two couples. For the one at the origin we must take $OX_1$ precisely like the $X$ component of $P_1$, except for its point of application. For the first couple take the pair of forces $CX_1$ and $BX'_1$, which form the couple of moment $+X_1z_1$ in a plane parallel to ZOX, so we may regard it as having the axis OY. We have now left the pair of forces $BX_1$ and $OX'_1$, which form the couple of moment $-X_1y_1$ in the plane of $xy$ or about the axis OZ.

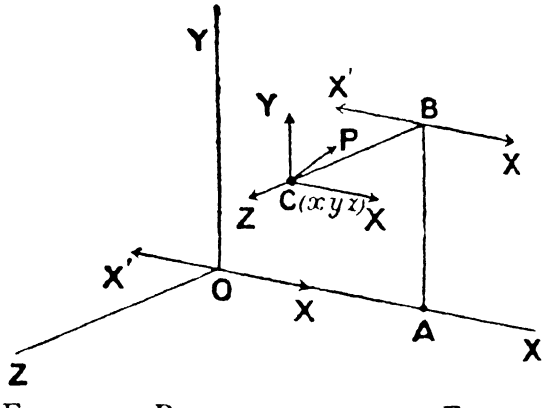



FIG. 214. RESULTANTS OF ANY FORCES ON RIGID BODY.

Thus any one component, applied away from all the axes, yields an equal force at the origin along the axis to which it is parallel and couples about the other two axes.

By introducing other pairs of forces equal to the $Y$ component of $P_1$, we can deal similarly with it, and then in like manner for the $Z$ component. Or, we may write these other values from those obtained by symmetry alone.

We thus find that the $Y$ component yields the force $Y_1$ at the origin, and the couples $Y_1x_1$ about OZ and $-Y_1z_1$ about OX. Similarly it is found that the $Z$ component yields the force $Z_1$ at the origin, and the couples $Z_1y_1$ about OX and $-Z_1x_1$ about OY.

Hence, summing up for the three components, we find that the force $P_1$ yields forces $X_1$, $Y_1$, and $Z_1$ at the origin along the axes OX, OY, and OZ, together with the couples $(Z_1y_1 - Y_1z_1)$, $(X_1z_1 - Z_1x_1)$, and $(Y_1x_1 - X_1y_1)$ about the axes OX, OY, and OZ respectively. The same would apply to the next force $P_2$ with the necessary change of the subscripts, and so on till all the forces of the system were dealt with. We

should accordingly have forces along the co-ordinate axes represented by

$$\left.\begin{aligned} X_1+X_2+X_3+\ldots &= \Sigma X = U \\ Y_1+Y_2+Y_3+\ldots &= \Sigma Y = V \\ \text{and} \quad Z_1+Z_2+Z_3+\ldots &= \Sigma Z = W \end{aligned}\right\} \quad \ldots \quad (1).$$

The couples along the corresponding axes would have moments

$$\left.\begin{aligned} \Sigma(Zy-Yz) &= L \\ \Sigma(Xz-Zx) &= M \\ \Sigma(Yx-Xy) &= N \end{aligned}\right\} \quad \ldots \quad (2).$$

The above three forces and three couples are called the *six components of the forces.*

We may then compound the forces $U$, $V$, and $W$ into a single force $R$ acting at the origin and defined in magnitude by

$$R^2 = U^2+V^2+W^2 \quad \ldots \quad (3)$$

and in direction by

$$\frac{l}{U} = \frac{m}{V} = \frac{n}{W} = \frac{1}{R} \quad \ldots \quad (4),$$

where $l$, $m$, and $n$ are the direction cosines of $R$.

Similarly the couples $L$, $M$, and $N$ may be compounded into the single couple $G$, given in magnitude and axis by

$$G^2 = L^2+M^2+N^2 \quad \ldots \quad (5)$$

and

$$\frac{\lambda}{L} = \frac{\mu}{M} = \frac{\nu}{N} = \frac{1}{G} \quad \ldots \quad (6),$$

where $\lambda$, $\mu$, and $\nu$ are the direction cosines of the axis of $G$.

The processes of the foregoing reduction may be compactly summarised as shown in Table XV. This at once forms an aid to the memory, and offers a guide which it may be well to follow when dealing with numerical examples.

TABLE XV. REDUCTION OF FORCES ON A RIGID BODY.

| Forces dealt with. | Components of Forces transferred to Origin acting along | | | Co-ordinates of Points of Application. | | | Moments of Couples about | | |
|---|---|---|---|---|---|---|---|---|---|
| | OX | OY | OZ | $x$ | $y$ | $z$ | OX | OY | OZ |
| $P_1$ | $X_1$ | $Y_1$ | $Z_1$ | $x_1$ | $y_1$ | $z_1$ | $0$<br>$-Y_1z_1$<br>$Z_1y_1$ | $X_1z_1$<br>$0$<br>$-Z_1x_1$ | $-X_1y_1$<br>$Y_1x_1$<br>$0$ |
| $P_1$<br>$P_2$<br>... | $X_1$<br>$X_2$<br>... | $Y_1$<br>$Y_2$<br>... | $Z_1$<br>$Z_2$<br>... | $x_1$<br>$x_2$<br>... | $y_1$<br>$y_2$<br>... | $z_1$<br>$z_2$<br>... | $Z_1y_1-Y_1z_1$<br>$Z_2y_2-Y_2z_2$<br>... ... | $X_1z_1-Z_1x_1$<br>$X_2z_2-Z_2x_2$<br>... ... | $Y_1x_1-X_1y_1$<br>$Y_2x_2-X_2y_2$<br>... ... |
| $\Sigma P$ | $\Sigma X = U$ | $\Sigma Y = V$ | $\Sigma Z = W$ | | | | $\Sigma(Zy-Yz) = L$ | $\Sigma(Xz-Zx) = M$ | $\Sigma(Yx-Xy) = N$ |
| | $R$ | | | | | | $G$ | | |
| Resultants. | | | | | | | | | |

The force $R$ is called the *principal force* at O, and the couple $G$ is called the *principal couple* at O. The components $L$, $M$, $N$ of $G$ are called the moments of the forces about the axes. The legitimacy of the term may seem almost obvious from the figure and the definition of moments, but is further dealt with in article 414.

**411a. Change of Base.**—The base of reference O to which the forces have been transferred has thus far been the origin of co-ordinates. Suppose we wish to make the transference to some other point O′ of co-ordinates $a$, $b$, and $c$.

We must then replace $x$, $y$, and $z$ in the previous expressions by $(x-a)$, $(y-b)$, and $(z-c)$ respectively. But the expressions for the components of $R$ do not contain $x$, $y$, and $z$. Hence the *principal force* $R$ is the *same* in magnitude and direction *whatever base is chosen.*

Let the components of the couple $G'$ for the new base be $L'$, $M'$, and $N'$. We then find

$$\left.\begin{aligned} L' &= \Sigma\{Z(y-b)-Y(z-c)\} = L - Wb + Vc \\ M' &= \Sigma\{X(z-c)-Z(x-a)\} = M - Uc + Wa \\ N' &= \Sigma\{Y(x-a)-X(y-b)\} = N - Va + Ub \end{aligned}\right\} \quad . \quad . \quad (7).$$

We accordingly see that the magnitude and axis of the principal couple are in general different for different bases.

**412. Conditions of Equilibrium.**—For equilibrium it is evident that the resultants of the system of forces must vanish, *i.e.* for the reduction already effected

$$R = 0 \text{ and } G = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

But these involve the vanishing of each of the six components of the forces, giving as the conditions

$$U = 0,\ V = 0,\ W = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

and

$$L = 0,\ M = 0,\ N = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Suppose the base is shifted to O′, the principal force and couple being now $R$ and $G'$. We then have (9) as before, but (10) replaced by

$$L' = 0,\ M' = 0, \text{ and } N' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

But by (7) we see that (11) reduces to (10) when (9) is fulfilled. Hence, for any base whatever, the conditions of (9) and (10) are sufficient. It is not, however, necessary that the axes should be at right angles. It may easily be seen that any oblique axes that are non-coplanar will serve.

It may be noted here that the six conditions of equilibrium correspond to the absence of linear and angular accelerations of the six possible modes open to a rigid body or in the six degrees of freedom possessed by it.

**413. Components of a Force.**—As seen from Table xv. of article 410, it is sometimes convenient, *for the sake of the possible addition*, to consider as the *six components of a force* $P$ the expressions

$$X,\ Y,\ Z,\ Zy - Yz,\ Xz - Zx, \text{ and } Yx - Xy \quad . \quad . \quad . \quad (12),$$

instead of regarding, as usual, its magnitude and the equations of its line of action.

To represent on this plan the line of action of the force apart from its magnitude, we may temporarily write for the magnitude *unity*. Then, if $(x, y, z)$ are the co-ordinates of a point on the line and $(l, m, n)$ are its direction cosines, the six components of this unit force (or co-ordinates of the line) are

$$l, m, n; \lambda = ny - mz, \mu = lz - nx, \nu = mx - ly \quad . \quad . \quad . \quad . \quad (13).$$

From which it is evident we have the relation

$$l\lambda + m\mu + n\nu = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

If the force along this line be $P$ instead of unity, it follows that its six components are

$$Pl, Pm, Pn; P\lambda, P\mu, P\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

To compound several such forces we obviously have, using the former notation,

$$U = \Sigma(Pl), \quad V = \Sigma(Pm), W = \Sigma(Pn) \quad . \quad . \quad . \quad . \quad (16).$$

$$L = \Sigma(P\lambda), M = \Sigma(P\mu), \quad N = \Sigma(P\nu) \quad . \quad . \quad . \quad . \quad (17).$$

But it is clear that the relation

$$LU + MV + NW = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

corresponding to (14), is *not now necessarily* true. It may be shown later that *when* (18) holds then either (i) $R = 0$, (ii) $G = 0$, or (iii) the couple can be made zero by shifting $R$. (See ends of articles 415 and 420.)

**414. Moment of a Force about a Line.**—It was stated at the end of article 411 that $L$, $M$, $N$ are called the moments of the forces about the axes. Let us now examine how this term agrees with the definition of the moment of a vector with respect to a point. The latter was defined as the product of the vector and the perpendicular upon it from the point (article 25*a*). The same definition can obviously be extended to the moment of a vector about a line, if that line passes through the point previously named and is *perpendicular to the plane* containing the point and the vector. Thus in dealing with coplanar vectors we might speak indifferently of their moments about given *points* in their common plane, or about *axes* through those points and *perpendicular* to this common plane.

But, when the vectors and the axes are inclined, further definition is needed. Thus, referring to Fig. 214 of article 411, what is the moment about OZ of the force $P$ of components $X$, $Y$, and $Z$ applied at the point $(x, y, z)$? Let the moment of $P$ about OZ be the algebraic sum of the moments of its components about the same axis, and consider as *zero* the moment of a vector about a *parallel line*. Thus the moment of $Z$ about OZ vanishes, and the moments of $Y$ and $X$ are seen to be $+Yx$ and $-Xy$ respectively. For these components lie on the plane parallel to XOY and distant $z$ from it, and this plane is intersected at right angles at $(0, 0, z)$ by the axis OZ. Let two inclined vectors $P_1$ and $P_2$ be compounded and the moments be taken about OZ of the separate vectors and of their resultant. Then in each of the three cases the $Z$ components give no moment and, as already seen for the plane case, the moment of the resultant equals

the algebraic sum of the moments of the components. Hence adding the moments on this plan will correctly give that of their resultant. And what holds for the axis OZ will apply to any other axis.

Consider again the single force $P$, and let it make the angle $\phi$ with OZ; then the resultant of $X$ and $Y$ will be the component of $P$ parallel to the $xy$ plane, and will be $P\sin\phi$.

Produce this line if necessary, and let fall upon it a perpendicular of length $p$ say, from the point (o, o, $z$) where the plane of the components $X$ and $Y$ cuts OZ. Then obviously the moment of $P$ about OZ may be denoted by $(P\sin\phi)p$; hence

$$Yx - Xy = Pp\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

We may thus enunciate generally as follows:—

DEFINITION.—The product $Pp\sin\phi$ is the *moment about any straight line* AB of the vector $P$ localised in a line inclined at the angle $\phi$ to AB, the shortest distance between the lines being $p$.

The usual relation between rotation and translation is to be observed in fixing the sign of the above product.

If the line in which the vector is localised is called CD, it is obvious that the quantity $p\sin\phi$ has reference to those lines only, and that the numerical value of the moment is the same however $P$ acts along one line, the other being the axis of the moment. Thus the product $p\sin\phi$ may be called *the moment of either line about the other*, or *their mutual moment.*

Thus the principal couple $G$, of a system of forces, may be seen to be the algebraic sum of the moments of all those forces about the line which is the axis of $G$.

A more formal proof of this is as follows:—If we keep to the same origin, $R$ is independent of the direction of the axes, for it is the resultant of all the forces as if transferred to the origin. But if, by a new set of axes with the same origin, we could obtain a different couple $G''$ say, then $R$ and $G''$ would be equivalent to $R$ and $G$, and therefore $R$, $G$, $-R$, and $-G''$ would form a system in equilibrium. But this is impossible unless $G''=G$ and their axes were coincident or parallel. That is, both in magnitude and direction $G$ is independent of the direction of the axes if the origin remains fixed. Thus, since the direction of the axes is arbitrary, we may let the axis of $x$ coincide with that of $G$; then $M=0$, $N=0$, and $G$ and $L$ are identical. Hence $G$ is the algebraic sum of the moments of all the forces with respect to the straight line which is the axis of $G$.

## EXAMPLES—LXXX.

1. Show how to reduce any system of forces acting on a rigid body to a single force and single couple.
2. If, in the reduction of the former question, the base is changed, show that the principal force is unaltered but that the principal couple usually is altered.
3. State what is meant by the six components of a system of forces, and express them in a second form.

4. Explain carefully what is meant by the moment of a force about a line, and show that the principal couple of a set of forces is the algebraic sum of the moments of all the forces about that line which is the axis of the couple.

**415. Conditions for a Single Resultant.**—Having seen how to reduce any system of forces acting on a rigid body to a force and a couple, let us now examine the conditions for these to give a single resultant, one force only.

Obviously one set of conditions would be

$$\left.\begin{array}{l} R \neq 0 \text{ and } G=0 \\ \text{or} \quad U,\ V, \text{ and } W \text{ not all zero, but } L=M=N=0 \end{array}\right\} \quad \ldots \quad (20).$$

But a second set of conditions may be found. For, if *both G and R are finite but at right angles*, as in article 364, they may be reduced to an equal force shifted parallel to itself, as there shown.

The condition that $G$ and $R$ are at right angles is obviously that the cosine of the angle $\theta$ between them shall vanish. But

$$\cos\theta = l\lambda + m\mu + n\nu = \frac{U}{R}\cdot\frac{L}{G} + \frac{V}{R}\cdot\frac{M}{G} + \frac{W}{R}\cdot\frac{N}{G}.$$

Thus, the second set of conditions is analytically expressed by

$$\left.\begin{array}{l} LU+MV+NW=0 \\ \text{where} \quad U,\ V, \text{ and } W \text{ do not all vanish} \end{array}\right\} \quad \ldots \quad (21).$$

When (21) is satisfied the reduction to a single force proceeds as in article 364. Thus the plane of the couple $G$ is made to contain $R$ and to consist of a force opposite but numerically equal to $R$ and applied at the same point, and another force equal to $R$ in magnitude and direction but distant from it by the arm $G/R$. Hence the couple is absorbed, $R$ at the original point is annulled, and we have left only the equal force $R$ transferred parallel to itself through the distance $G/R$.

We have now arrived at the proof alluded to after equation (18) at the end of article 413. For, in (18) or the identical equation of (21), (i) if we have $U=V=W=0$, then $R=0$; (ii) if $L=M=N=0$, then $G=0$; and (iii) if neither (i) nor (ii) is fulfilled, $R$ and $G$ are each finite but at right angles, and $G$ may be absorbed by the shifting of $R$, as just seen.

**416. Line of Action of Single Resultant.**—Supposing there is a single resultant for a system of forces acting on a rigid body, let it be required to determine the equations of its line of action. We may conveniently find these by shifting the origin to O', whose co-ordinates are $(a, b, c)$, writing the values for the corresponding moments $L'$, $M'$, and $N'$, and then equating them to zero. For the single resultant force acts through that new origin for which the couple vanishes.

Thus, quoting (7) in article 411*a*, we have

$$L' = L - Wb + Vc = 0 \quad \ldots \quad (22).$$
$$M' = M - Uc + Wa = 0 \quad \ldots \quad (23).$$
$$N' = N - Va + Ub = 0 \quad \ldots \quad (24).$$

But, since there is a single resultant, we also have, as in (21),

$$LU+MV+NW=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

And this shows that the former three equations are not all independent. For, if we eliminate $c$ between (22) and (23), we obtain

$$LU+MV+W(Va-Ub)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

and (25) and (26) give (24). We are thus confined to two of the three equations of (22) to (24), say (22) and (23), and these give two relations for the three unknowns $a$, $b$, $c$, which accordingly may have an infinite number of values subject to these conditions. That is, (22) and (23) do not determine some one definite point O′ of fixed co-ordinates $(a, b, c)$, but define a locus of O′. Hence, writing the current co-ordinates $x$, $y$, and $z$ instead of $a$, $b$, and $c$, we have

$$\left.\begin{aligned} L-Wy+Vz&=0 \\ M-Uz+Wx&=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27),$$

which are the equations of a straight line any point in which is an origin O′ for which the couple $G$ vanishes. In other words, these equations are those of the line of action of the single resultant force $R$ to which the system was reducible. They may be thrown into the form

$$\frac{x+M/W}{U/R}=\frac{y-L/W}{V/R}=\frac{z}{W/R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

showing that the line has direction cosines $U/R$, $V/R$, and $W/R$, as we know should be the case, and also that it intersects the $xy$ plane at the point whose co-ordinates are $(-M/W, L/W)$.

**417. Reduction to Two Forces.**—We have already seen that any system of forces acting on a rigid body may be reduced to a force and a couple, which, under certain conditions, may be a couple only, or even a single force only. But we may now notice that though these further reductions are particular, we may in *all other cases* reduce the system to *two forces*. We may also make one of them act at an assigned point and give to the other an assigned value.

Thus, suppose the system has been reduced to the force $R$ and couple $G$ for the base or origin O. We may then consider the couple to be composed of any two unlike parallel forces $F$ and $-F$ distant $G/F$ apart, $-F$ acting at O, and both in the plane through O perpendicular to the axis of $G$. These two forces may, of course, have any orientation we choose provided they are in the above plane. We may then compound $R$ and $-F$ at the origin, giving the resultant $S$ say.

Thus the system has been reduced to two forces $S$ and $F$. Further, since the origin may be anywhere we choose, one of the forces $S$ may be made to pass through any assigned point.

Also, since the magnitude of $F$ was arbitrary, we may choose it to have any assigned value, though, of course, this choice modifies the magnitude and direction of the $S$ thereafter determined.

**418. Equilibrium of a Body with One or Two Points Fixed.**—Take first the case of a rigid body acted on by any applied forces and with *one point fixed*, which we will choose as the origin of co-ordinates.

Then these forces will produce on the fixed point a pressure whose components may be denoted by $X'$, $Y'$, and $Z'$. Hence, using the former notation, we have

$$U-X'=0,\ V-Y'=0,\ W-Z'=0 \quad . \quad . \quad . \quad . \quad (1).$$
$$L=0,\ M=0,\ N=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus, (1) gives the pressure components at the fixed point, and (2) gives the three conditions of equilibrium, viz. that the moments of the applied forces shall vanish with respect to any three rectangular axes meeting in the fixed point. It is easily seen that these three conditions correspond to the three degrees of freedom possessed by the body.

Take now the case in which *two points are fixed*. Let the axis of $z$ pass through both points, their distances from the origin being $z'$ and $z''$, and the components of pressures being $X'$, $Y'$, $Z'$, and $X''$, $Y''$, $Z''$. Then we have

$$U-X'-X''=0,\ V-Y'-Y''=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$
$$W-Z'-Z''=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$
$$L+Y'z'+Y''z''=0,\ M-X'z'-X''z''=0 \quad . \quad . \quad . \quad . \quad . \quad (5).$$
$$N=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Hence the four equations (3) and (5) determine the components $X'$, $Y'$, $X''$, $Y''$ of the pressures on the fixed points, while (4) gives the sum of $Z'$ and $Z''$, it being impossible to discriminate between them for an ideally rigid body.

Finally, (6) gives the sole condition of equilibrium, viz. the vanishing of the moment of the applied forces about the line through the two fixed points, which obviously corresponds to the sole degree of freedom left to the body.

**419. Equilibrium of a Body with Three Points on a Smooth Plane.**—Let the plane be that of $xy$, the co-ordinates of the points of contact being $(x', y', 0)$, $(x'', y'', 0)$, $(x''', y''', 0)$, the corresponding pressures of the body on the points being $Z'$, $Z''$, and $Z'''$. Then the equations are

$$U=0,\ V=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$
$$W-Z'-Z''-Z'''=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$
$$\left.\begin{aligned} L-Z'y'-Z''y''-Z'''y'''=0 \\ M+Z'x'+Z''x''+Z'''x'''=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$
$$N=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Hence, of these six equations, the three contained in (8) and (9) determine the pressures $Z'$, $Z''$, $Z'''$ exerted by the body on the plane; while the other three numbered (7) and (10) form the conditions of equilibrium corresponding to the three degrees of freedom still left to the body.

## EXAMPLES—LXXXI.

1. If a set of forces applied to a rigid body have the components $U$, $V$, and $W$ parallel to the co-ordinate axes and the moments $L$, $M$, and $N$ about them, show that they all reduce to a single force if $LU+MV+NW=0$ and $L$, $M$, and $N$ are not all zero.
2. When a general system of forces may be reduced to a single resultant force, find the equation to the line of action of that resultant.
3. Show that any system of forces in three dimensional space may be reduced to two forces if not to one.
4. State the conditions of equilibrium of a rigid body with no points fixed, and show to what these reduce if two points are fixed.
5. If a rigid body is constrained to have three points in contact with a plane, obtain the pressures on these points under any set of applied forces, and find also the conditions of equilibrium of the body.

**420. Reduction of Forces to a Wrench.**—We have already seen (article 411) how to reduce a set of forces to their six components $L$, $M$, $N$, $U$, $V$, $W$ and thence to a couple $G$ and force $R$. It is now desirable to note how they may be still further reduced to a single force and a couple in a plane perpendicular to (*i.e.* with axis parallel to) the direction of the force. Such a combination is known as a *wrench*, this use of the word being due to Sir R. Ball.

Let the axis of $G$ be inclined at the angle $\theta$ to the direction of $R$ as shown in Fig. 215. Resolve $G$ into its rectangular components $OQ=G\cos\theta$ and $OS=G\sin\theta$, with axes along and perpendicular to the direction of $R$. Further, let the component couple OS be represented by forces $-R$ at O and $R'$ at O', each parallel and numerically equal to $R$. Then OO' is obviously perpendicular to the plane of $R$ and the axis of $G$, and its length is $(G\sin\theta)/R$. Further, $R$ and $-R$ each applied to O annul each other, so we are left with $R$ at O', equal and parallel to the original $R$, together with the couple $OQ=G\cos\theta$, with its axis parallel to the original $R$.

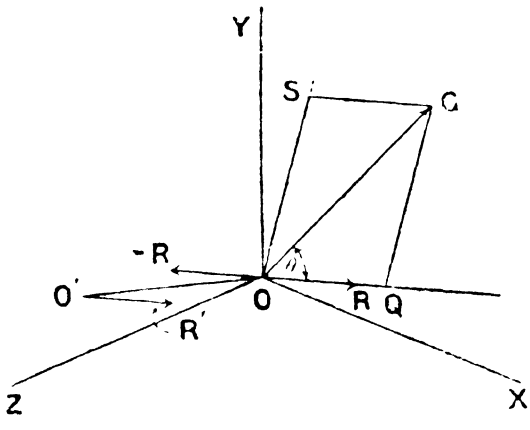


FIG. 215. REDUCTION TO WRENCH.

Since the axis of a couple may be regarded as shifted to any parallel line, we have thus a single force $R$ at O' and a couple, $G\cos\theta=\Gamma$ say, with axis coincident with the line of action of this force.

We have accordingly reduced the forces to a *wrench*, as was desired.

The axis O'R', to which the force $R$ has been transferred, is called *Poinsot's Central Axis*. It is obviously constructed as the line through O' parallel to $R$, where OO' is perpendicular both to $R$ and to the axis of $G$, of length $OO'=(G\sin\theta)/R$, and of direction such that the couple $G\sin\theta$ would carry O' in the direction of $R'$.

And the wrench, to which we have reduced all the forces, is the force $R$ along this central axis, together with the couple $\Gamma = G\cos\theta$ about the central axis.

Since the cosine of the angle between two lines is the sum of the products of their corresponding direction cosines, we have for the angle $\theta$ between $R$ and $G$

$$R\Gamma = RG\cos\theta = LU + MV + NW = I \quad . \quad . \quad . \quad . \quad (1),$$

where $I$ is an *invariant* for the given forces. For, however the base is changed, $U$, $V$, and $W$ remain unaltered. And if $L$, $M$, and $N$ are changed to $L'$, $M'$, and $N'$ by shifting the base, it may be seen (from equation (7) of article 411*a*) that the above sum of three products is not changed thereby.

Thus for $I = 0$ we have either $R = 0$ or $\Gamma = 0$, which is another proof of the statement at the end of article 413.

To obtain the equations of the central axis, take any point $(x, y, z)$ on it and form the expressions $L'$, $M'$, $N'$ for the moments of the forces about parallel axes through this new origin (see equation (7) of article 411*a*). Then, since the central axis is parallel to $R$, these moments are proportional to $U$, $V$, $W$, the components of $R$. The symbolic expression of these relations gives the equations desired for the central axis, viz.

$$\frac{L - Wy + Vz}{U} = \frac{M - Uz + Wx}{V} = \frac{N - Vx + Uy}{W} = \frac{I}{R^2} \quad . \quad (2).$$

The final expression on the right is obtained by multiplying the three on the left by $U^2$, $V^2$, and $W^2$ respectively, adding, and remembering (1).

If $U$, $V$, and $W$ all vanish, (2) fails to give the equations sought. But in this case $R$ is zero, and any straight line parallel to the axis of $G$ is the central axis.

We may now show that the couple of the wrench has the *minimum* value for the principal couple.

Let any base be chosen, and denote by $G'$ and $L'$, $M'$, $N'$ the corresponding principal couple and its components. Then we have already seen that

$$L'U + M'V + N'W = LU + MV + NW = I \quad . \quad . \quad (3).$$

We may also write

$$R^2G'^2 = (U^2 + V^2 + W^2)(L'^2 + M'^2 + N'^2).$$

And, by using (3), this may be thrown into the form

$$R^2G'^2 = (N'V - M'W)^2 + (L'W - N'U)^2 + (M'U - L'V)^2 + I^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

But, as both $R$ and $I$ are invariants, the minimum value of $G'$ obviously corresponds to the vanishing of each of the three bracketed squares of (4).

Hence by (1)

$$R^2G'^2_{\text{min.}} = I^2 = R^2\Gamma^2, \text{ or } G'_{\text{min.}} = \Gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

**421. Tripod by Virtual Work.**—Let us now apply the principle of virtual work to the solution of a few problems of equilibrium of bodies or systems in three dimensions.

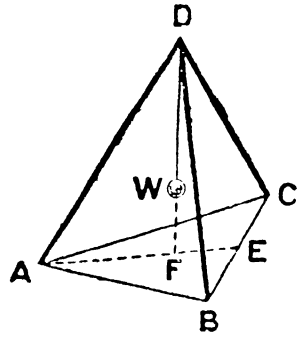



FIG. 216. TRIPOD BY VIRTUAL WORK.

Take first the case of a weight $W$ hanging from the freely jointed summit of a tripod ABCD, Fig. 216, whose legs have each the length $a$, and whose feet touch a *smooth* horizontal plane at the corners of an equilateral triangle of side $s$, and are there bound by a string whose tension $T$ is to be found.

All being symmetrical, it is evident that the vertical from D meets the triangle ABC at the point F, where AF is $2/3$ of AE and E is the middle point of BC. Thus $AE = s\sqrt{3}/2$ and $AF = s/\sqrt{3}$. Denote by $h$ the height FD, then we have

$$h^2 + s^2/3 = a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Hence, on differentiating, we obtain

$$3hdh + sds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

But, by the principle of virtual work, we have

$$Wdh + T(3ds) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Thus the comparison of coefficients in (2) and (3) gives

$$W/3h = 3T/s,$$

or

$$\frac{T}{W} = \frac{s}{9h} = \frac{s}{9\sqrt{a^2 - s^2/3}} \quad . \quad . \quad . \quad (4).$$

**Pile of Four Equal Spheres.**—Let us now imagine three equal *smooth* spheres in contact on a *smooth* plane and encompassed by a fine thread in contact with them in the horizontal plane through their centres, the thread having *no tension until* a fourth equal *smooth* sphere of weight $W$ is placed on the other three and in contact with them all. It is required to determine the tension $T'$ of the thread.

This problem, so different in the form of the bodies composing the system, is easily seen to be essentially the same as the tripod just treated, except that now $a = s$.

Thus (4) yields for this case

$$\frac{T'}{W} = \frac{1}{3\sqrt{6}}.$$

**422. Bifilar Suspension by Virtual Work.**—Let us now consider the relation between the couple $G$ about a vertical axis and the angle $\theta$ of twist which it can maintain in a *bifilar suspension*, whose equal threads have length $a$, the distance between their fixed points of support being $2b$, and the mass they carry being $M$. The bifilar suspension is shown in Figs. 217-219, the full lines representing the displaced position and the dotted lines the equilibrium position.

When the bar of the bifilar is turned through the angle $\theta$ about a vertical axis, let its depth below the fixed points CD be decreased from

$a$ to $h$. Next imagine the virtual displacements $d\theta$ and $dh$, then we have the relation

$$Gd\theta + Mgdh = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

which states for this case the principle of virtual work. It is accordingly the only mechanical relation needed, the others required being

FIG. 217. SIDE ELEVATION.

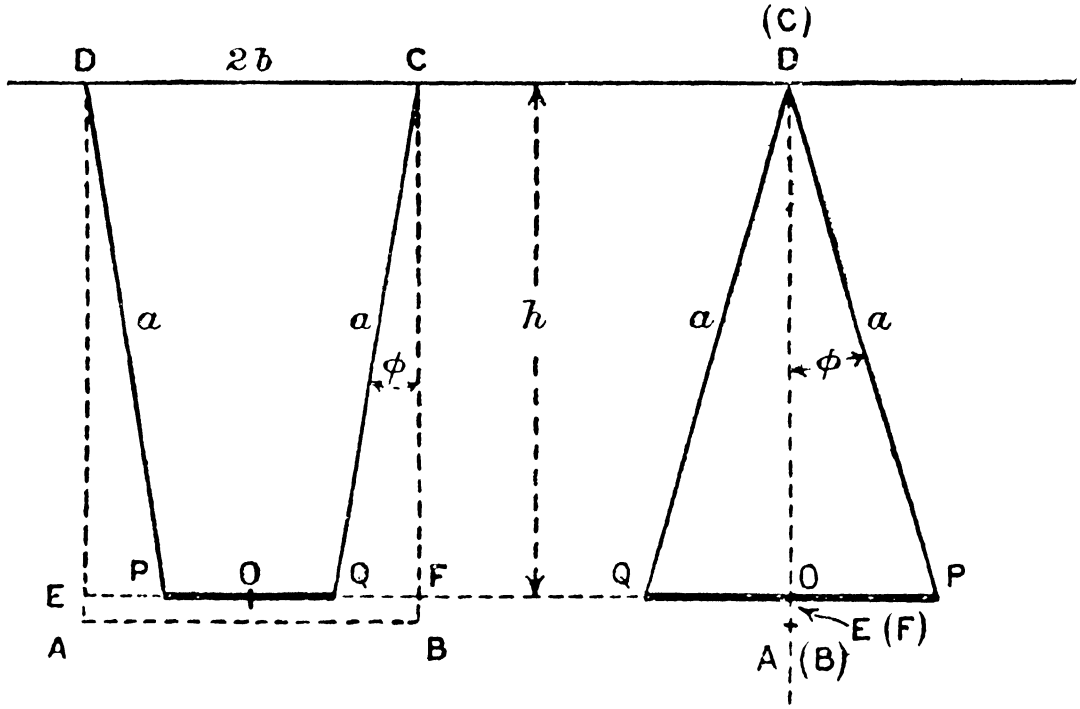
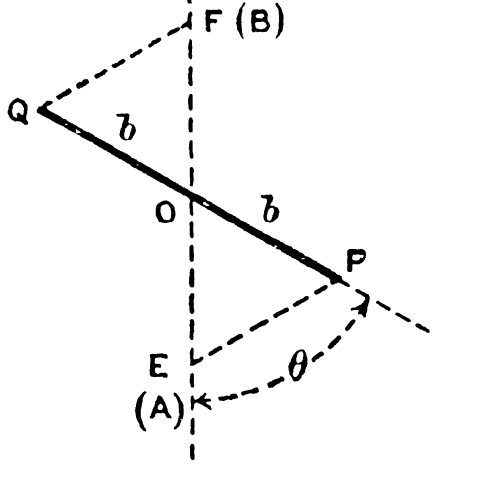



FIG. 218. EDGE ELEVATION.




FIG. 219. PLAN OF THE BAR.




THREE VIEWS OF BIFILAR SUSPENSION.

obtainable from the geometrical conditions. In deriving these it is convenient to use the angle $\phi$ with the vertical assumed by the threads in their displaced positions. It should be noted that this angle is not seen at its true value in either of the elevations, for in neither case is its plane coincident with that of the diagram. In the figures, AB denotes the equilibrium position of the bar, EF its position if raised $(a-h)$ parallel to itself, PQ its position in the actual displacement in which it turns through the angle $\theta$ about the vertical through its centre and at the same time rises vertically by $(a-h)$, the threads swinging till at the angle $\phi$ with the vertical in oblique planes.

Then by the plan, Fig. 219, we have

$$EP=FQ=2b\sin\theta/2,$$

and by it and the elevations we see that

$$EP=FQ=a\sin\phi.$$

Hence
$$a\sin\phi=2b\sin\theta/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Again, by the elevations, we have

$$h=a\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

which completes the required geometrical equations. We must now eliminate $\phi$ between (2) and (3), differentiate the equation so obtained, and use the result in (1).

We thus find in turn

$$h^2=a^2-a^2\sin^2\phi=a^2-4b^2\sin^2\theta/2,$$

and
$$2hdh=-4b^2\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,(\tfrac{1}{2}d\theta),$$

or
$$dh=-\frac{b^2\sin\theta}{\sqrt{a^2-4b^2\sin^2\frac{\theta}{2}}}d\theta \quad . \quad . \quad . \quad (4).$$

And this in (1) gives the *exact* result

$$G=\frac{Mgb^2\sin\theta}{\sqrt{a^2-4b^2\sin^2\frac{\theta}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Whence, when $2b\sin\frac{\theta}{2}$ is small compared with $a$, we have the *approximate* result

$$G=\frac{Mgb^2\sin\theta}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

For the work done in twisting the bifilar through the angle $\beta$ from the equilibrium position, we have from (6) the approximate value

$$W=\int_0^\beta Gd\theta=\frac{Mgb^2}{a}(1-\cos\beta). \quad . \quad . \quad (7).$$

## EXAMPLES—LXXXII.

1. Reduce any system of forces to a *wrench*, and show that the couple then involved is the minimum for the system in question.
2. Apply the principle of Virtual Work to obtain the couple required to maintain any specified angular displacement of a bifilar suspension.
3. A horizontal bar of radius of gyration $k$ about a central vertical axis is suspended by two parallel threads each of length $a$ and at a distance $2b$ apart.

   Show that if slightly disturbed the bar will oscillate in the period given by

$$\tau = \frac{k}{b} 2\pi \sqrt{\frac{a}{g}}.$$

4. '(i) A bar $AB$, of weight $W$, is guided by rings at its ends so that $A$ can move on a smooth horizontal rail $OX$, and $B$ on a smooth vertical rail $OY$. Employ the principle of Virtual Work to evaluate the horizontal force at $A$ necessary to maintain equilibrium when the angle $OAB$ is $\theta$.

   '(ii) A nut of weight $W$ is mounted on a fixed smooth screw of pitch $p$, whose axis is inclined to the vertical at an angle $\alpha$; what couple is required to keep the nut from moving?'

   (LOND. B.SC., PASS, APPLIED MATH., 1908, I. 7.)

5. 'Three equal spheres are lying in contact on a horizontal plane and are held together by a string. A cube of weight $W$ is placed with one diagonal vertical so that its lower faces touch the spheres, and the cube is supported in this position by the spheres; show that the tension in the string is

$$\tfrac{1}{3}\sqrt{\tfrac{2}{3}}W.$$

   (LOND. B.SC., PASS, APPLIED MATH., 1908, I. 8.)

**422*a*. Alternative Proof for Bifilar Suspension.**—For threads of length $a$ sustaining a weight $W$ and at distances apart $2b$ at top and $2c$ at bottom (or vice versa) let the inclinations of the threads to the vertical be very small for the angle of twist $\theta$. Also let the bottoms of the threads be thereby displaced horizontally a distance $r$ each, and let $p$ be the perpendicular let fall from the centre of the cross-bar upon $r$. Then the couple $G$ may be written as follows:—

$$G = 2 \text{ (Horizontal component of thread tension) effective arm,}$$
$$= 2\left(\frac{W}{2}\ \frac{r}{a}\right)p = \frac{W}{a}bc \sin\theta, \quad \ldots\ldots\ldots \quad (8)$$

since $pr$ and $bc \sin\theta$ are alternative expressions for twice the area of a triangle seen in the plan of the bifilar.

# PART V.—HYDROMECHANICS

## CHAPTER XIX

### HYDROSTATICS

**423. Natures of Fluids, Liquids, and Gases.**—In this, the fifth part of the present work, we consider the rest and motion of fluids, forms of matter offering only very *small resistances* to changes of *shape* however large, provided only that *time enough* is allowed in which those changes may occur. This distinguishes them from *rigid* bodies which are supposed to retain their exact shape under all circumstances, and from *elastic* solids each of which exhibits a certain nearly constant ratio between its almost instantaneous changes of shape and the external actions to which those changes are ascribed.

If the resistance offered by a fluid to even a sudden change of shape is quite negligible, the fluid is said to be very mobile or of negligible *viscosity*. This last term denotes a property really possessed by all fluids and which, in the case of sufficiently sluggish fluids, needs taking into account according to a quantitative definition. It is obvious that in the present chapter on the statics of fluids we are not concerned with viscosity, for we examine the equilibrium state after sufficient time has been allowed for all motions to cease. And in the next chapter on the motions of fluids we shall, for simplicity's sake, exclude all notions of viscosity (as being negligible) except where it is expressly introduced.

We may now subdivide fluids, discriminating between liquids and gases. *Liquids* are fluids whose volume per unit mass are practically independent of the pressures to which they are subjected; in particular, the specific volume is finite when the pressure is almost at zero. In other words, the density of a given liquid is practically constant and its specific volume always finite. *Gases* are fluids whose volumes per unit mass may become as large as we please by our suitably diminishing the pressure to which they are subjected. In other words, the density of any gas is a fairly simple function of the pressure such that its specific volume has no finite limit. We may accordingly sum up the chief points of distinction by the semi-popular remark that a solid body has both size and shape, a given mass of liquid has size only, while for a given mass of gas there is neither shape nor size. Hence a solid body requires no vessel to hold it, a liquid requires no lid to the vessel, but a gas needs both vessel and lid, or it would expand to fill all the exterior space open to it.

In the case of gases near the change of state called liquefaction, the relations between pressure and volume are often much more complicated than the ideal simple forms which ordinarily represent them with sufficient accuracy. The fluid is then called a *vapour*. But with the details of these complexities we are not here concerned. They must be studied in physical rather than in mechanical text-books.

**424. Hydrostatic Pressure Independent of Direction.**—In a fluid at rest take a small triangular pyramid OABC bounded by the three rectangular co-ordinate planes and a base ABC of area $\Delta$ whose perpendicular distance from the origin has length $h$ and direction cosines $l$, $m$, $n$. And consider the equilibrium of this pyramid under the forces $X$, $Y$, $Z$ per unit mass parallel to the axes, the *normal* pressures $P$, $Q$, $R$ on the mutually rectangular faces meeting at O, and $N$ on the oblique base, as shown in Fig. 220. It should be noticed that whether the fluid has appreciable viscosity or not, these pressures must be normal since the fluid is at rest.

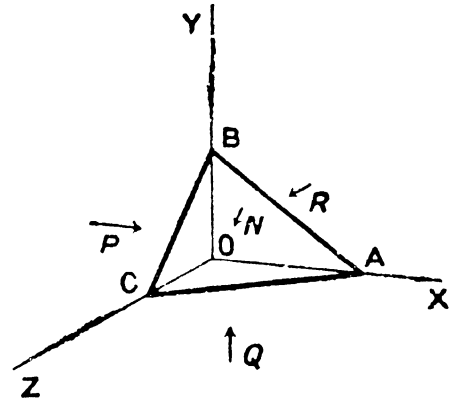



FIG. 220. HYDROSTATIC PRESSURE INDEPENDENT OF DIRECTION.

Then, by the geometry of the figure, the areas of the three mutually rectangular faces are respectively $l\Delta$, $m\Delta$, $n\Delta$ (since their inclinations with the base are those between the axes and the normal $h$); also the mass of fluid in the pyramid is $\frac{1}{3}\rho h\Delta$ where $\rho$ denotes the density. Hence, taking components parallel to the axes, we have

$$Pl\Delta - Nl\Delta + \tfrac{1}{3}\rho hX\Delta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and two similar equations.

But when, to make the pressures $P$, $Q$, $R$, and $N$ all act at the same point O, the pyramid is indefinitely reduced in size, its volume $\frac{1}{3}h\Delta$, being of the third order of small linear quantities, vanishes in comparison with $\Delta$, which is of the second order only. Hence, each equation reduces to its first two terms, and the set simplifies to

$$\left.\begin{aligned} P &= N \\ Q &= N \\ R &= N \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Or, in words, the *hydrostatic pressure* of a fluid in equilibrium on any small surface through a given point *acts normally* to that surface but is otherwise *independent of its aspect.* Or again, the normal of the surface is the direction of the pressure, its *magnitude remaining the same* for all orientations of that surface.

**425. Fundamental Equations.**—Consider a small parallelepiped of edges $\alpha$, $\beta$, and $\gamma$ parallel to the co-ordinate axes, and let it be occupied by fluid of density $\rho$ in equilibrium under the pressures and the forces $X$, $Y$, and $Z$ per unit mass. Let $+p$ be the positive pressure

on the $\beta\gamma$ face at the negative end of the element and $-p'$ the corresponding pressure on the opposite face. Then, taking all components parallel to the $x$ axis, we have the condition of equilibrium

$$p\beta\gamma - p'\beta\gamma + \rho\alpha\beta\gamma X = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

But $p' = p + \alpha dp/dx$, and accordingly $p - p' = -\alpha dp/dx$. Thus (3) and the two similar relations for the other axes yield the set

$$\left.\begin{aligned} \rho X - \frac{dp}{dx} &= 0 \\ \rho Y - \frac{dp}{dy} &= 0 \\ \rho Z - \frac{dp}{dz} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

These are the fundamental equations of hydrostatics, and show that the space rate of increase of pressure in any direction equals the applied force per unit volume in the same direction.

**426. Pressure and Depth in a Liquid.**—For the case of a liquid of constant density $\rho$ in equilibrium under gravity only let us take the $z$ axis vertically upwards. Then we have

$$X = Y = 0, \; Z = -g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

And these, put in (4) of the last article, give

$$\frac{dp}{dx} = \frac{dp}{dy} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

and

$$\frac{dp}{dz} = -\rho g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Equation (6) shows that the pressure is constant at all points in any horizontal plane. It therefore agrees with the common statement that 'water finds its own level'; or the familiar experiment of Pascal's vases, in which the liquid reaches the same level in any limbs of a complicated vessel of communicating parts of bulbous and other forms.

Since the density $\rho$ is supposed constant for our liquid under all moderate pressures, equation (7) yields

$$\int_{p_0}^{p} dp = -\rho g \int_{z_0}^{z} dz,$$

or

$$p - p_0 = \rho g(z_0 - z) = \rho g h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

where $p_0$ is the pressure at some standard height $z_0$ (say the free surface of the liquid), $h$ being the *depth of $z$ below $z_0$*.

If either $\rho$ or $g$ vary, or both, (8) would need modifying by keeping these variables under the sign of integration and dealing with them as functions of $z$.

Since the pressure, or force per unit area, is quite independent of the area, the ratio between forces on given movable surfaces in contact with the same liquid may be magnified as much as we please by correspondingly magnifying the ratio of the areas of those surfaces.

Thus if the same pressure $p$ is exerted by a liquid on the plunger of a pump of radius $a$ and on the ram of radius $b$ in a hydraulic press, the corresponding forces being $P$ and $Q$, we have

$$P/Q = pa^2/pb^2 = a^2/b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Hence with $a/b$ one hundredth, we have $P/Q$ one ten thousandth. Or one pound weight on the plunger gives a force of 10,000 lbs. wt. (or nearly four and a half tons weight) on the ram. The fact that, by means of an intervening fluid, a small force can be made to balance a much larger one is often referred to as the 'hydrostatic paradox.'

The above expression given in (9) is, of course, easily obtained on the principle of virtual work by the kinematical relation between the two corresponding displacements possible to the plunger and ram connected by the liquid, supposed incompressible.

The principle of this article as contained in equation (8) has many applications scientific, technical, and familiar. Thus the method of determination of densities of liquids by balancing one against another in a U-tube, the use of siphons and pumps, may be mentioned here, but call for no detailed treatment.

**427. Resultant Force on a Submerged Plane Area.**—Take the axes of $x$ and $y$ in the upper surface of the liquid and that of $z$ vertically downwards as shown in Fig. 221, ABCD being the area inclined at an angle $\theta$ with the vertical.

Take a point E in the plane at a depth FE$=z$ below the free surface of the liquid, and through E take the horizontal line BD in the plane. Take also in the plane a parallel line at a depth $z+dz$ so as to cut off a slice of the plane of width $dz/\cos\theta$. Then, if the length of BD is denoted by $l$, we have, as the force due to the liquid pressure on one side of this element, the expressions

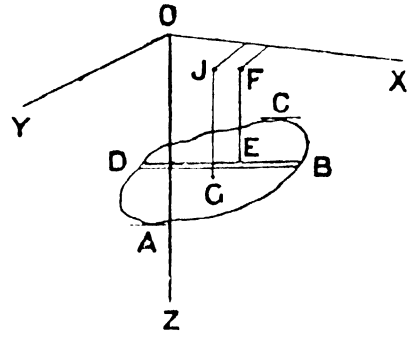



FIG. 221. FORCES ON A PLANE AREA.

$$dR = p\frac{ldz}{\cos\theta} = (p_0+\rho gz)\frac{ldz}{\cos\theta} \quad . \quad . \quad . \quad . \quad . \quad (1),$$

the pressure $p_0$ being that on the free surface of the liquid. To obtain the resultant force on the whole surface due to the liquid we have simply to integrate (1), because since the area in question is plane, all the forces of the liquid pressures on it are parallel. We thus find

$$R = \int_c^a \left(p_0+\rho gz\right)\frac{ldz}{\cos\theta} = p_0S+\rho gSh \quad . \quad . \quad . \quad (2),$$

where $c$ and $a$ are the limiting depths of the plane ABCD, $S$ its area, and $h$ the depth, JG, of its centroid G below the level of the free surface of the liquid of density $\rho$.

Where $p_0$ is zero or negligible the expression for $R$ obviously

reduces to its last term, which gives the pressure due to the liquid only. It is worth noting that the *magnitude* of this term expresses the weight of the column of liquid that would stand vertically on the area ABCD if *rotated into the horizontal plane through its centroid*; the *direction* of $R$ is, however, normal to the plane in its actual position as shown.

The *vertical* component of $R$ is evidently

$$R\sin\theta = p_0 S\sin\theta + \rho g S h\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a),$$

showing its value to be that of the weight of the vertical columns of fluid standing on the *area* in question in its actual *inclined* position.

EXAMPLES—LXXXIII.

1. Discuss carefully the distinction between solids and fluids and that between liquids and gases.
2. Show that the hydrostatic pressure of a fluid at a point is independent of the orientation of the surface on which it presses.
3. Obtain the fundamental equations of a fluid in equilibrium under specified forces, and apply them to the state of a fluid at rest under the action of gravity only.
4. Find the relation between pressure and depth from the free surface of a heavy liquid.
5. Obtain an expression for the resultant force on a plane area submerged in a heavy liquid and also one for the vertical component of this force.
6. 'Prove that the average thrust per unit area of a liquid on a plane area immersed vertically is equal to the pressure intensity at the centroid of the area.

   'The water upon one side of a dock gate 15 feet wide is 10 feet deep and upon the other side is 20 feet. Taking the gate as rectangular, find the resultant thrust of the water and its line of action.'

   (LOND. B.SC., PASS, APPLIED MATH., 1905, III. 9.)
7. 'If at any point $P$ in a perfect fluid a small plane area is imagined as separating fluid on one side from fluid on the other side of the area, and if the *direction* of the force exerted over this area by the one part of the fluid on the other is always normal to the area, whatever be the aspect of the area at $P$, prove that the *magnitude* of the force is constant for all positions of the area.

   'A canal lock gate is 12 feet broad, and the depths of the water at opposite sides of the gate are 16 and 10 feet; find, in tons weight, the magnitude of the resultant water pressure on the gate, assuming that a cubic foot of water has a mass of 62·5 lbs.'

   (LOND. B.A., PASS, APPLIED MATH., 1906, II. 1.)

**423. Centre of Pressure.**—We have just found the resultant force of the liquid pressures on a plane area and know its direction. We have now to find a point on the line of its action. This point, if taken in the plane itself, is called the *centre of pressure*. Obviously its position sideways, or in the horizontal direction, is simply that of the centroid of a lamina of the same shape as the plane area in question, but of surface density proportional to the depth below the free surface of the liquid. We are therefore concerned now simply with the depth $z$ of the centre of pressure below the liquid surface. To find this we consider moments of the horizontal components of the forces about the surface of the liquid, which we still take as the $xy$ plane, referring

again to Fig. 221. Draw any vertical plane whose intersection with ABCD is a horizontal line, then $S\cos\theta$ is the area of the projection of ABCD on this vertical plane, and our horizontal components will be perpendicular to this vertical plane. We shall also need symbols for the radii of gyration of this projection about horizontal lines in the plane of projection. When this line or axis is in the free surface of the liquid, or $xy$ plane, let $K$ be the radius of gyration, $k$ denoting the corresponding value for the parallel axis through the centroid of the projection.

Then, writing $H$ for the horizontal component of pressure on the whole surface and $M$ for its moment about the $xy$ plane, we find

$$H=\rho g\int_c^a zl\,dz=\rho gSh\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and

$$M=\rho g\int_c^a z^2l\,dz=\rho gSK^2\cos\theta \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Thus, for the depth of the centre of pressure, we have

$$\bar{z}=\frac{M}{H}=\frac{K^2}{h}=\frac{h^2+k^2}{h}=h+\frac{k^2}{h} \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The pressure, if any, on the free surface of the liquid is here supposed negligible.

We accordingly find that the depths of the centroid and of the centre of pressure of a plane area below the free surface of the liquid are related to each other like the corresponding distances of the centroid and centre of oscillation of a physical pendulum from the axis of suspension (article 258).

**429. Resultant Force on a Closed Surface.**—Consider any closed surface $S$ described in a liquid at rest in equilibrium as shown in Fig. 222, in which the axes of $x$ and $y$ are taken horizontally in the free surface of the liquid and that of $z$ vertically downwards. Take, in the volume enclosed by $S$, any prism AB with axis horizontal and of an infinitesimal cross section. Let the pressure at this level be $p$, and denote by $dS$ the area of the intersection of the prism with the surface $S$ at A. Then the force on this base of the prism due to the liquid pressure is $pdS$, and, if its normal is inclined $\theta$ with the horizontal, the horizontal component of this force is $pdS\cos\theta$. But $dS\cos\theta$ is the area, $d\sigma$ say, of the *normal* cross section of the prism, so the horizontal component of *either* end force is the symmetrical expression $pd\sigma$. Hence the horizontal component of the inward forces on $S$ at A is opposed by one of equal magnitude at B, so that the resultant horizontal force on this prism is zero. And as this applies to any *horizontal* prism parallel to the axes of $x$ or $y$, or

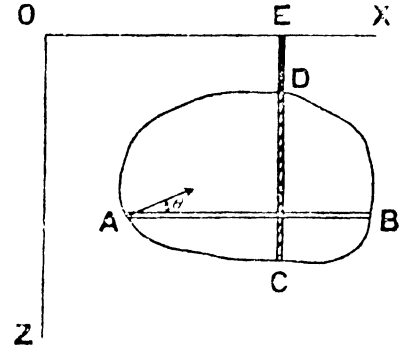


FIG. 222. RESULTANT ON A CLOSED SURFACE.

oblique to them, and at any level, it is clear that the horizontal component of the resultant force on the entire closed surface $S$ vanishes.

Consider now, in the closed surface $S$, any vertical prism CD, and produce it to E in the free surface of the liquid.

Then, by what we found at the end of article 427, the vertical components of the forces due to the liquid at C and D are each equal to the weights of the columns of liquid which could stand on C and D. Hence, taking these forces inwards on the closed surface $S$, we see that they are in opposite senses, and have for their resultant a vertically upward force equal to the weight of the column of the liquid CD extending between the lower and upper limits of $S$ along the vertical prism in question. And this is true whether there is one liquid only, or two liquids, or more round the surface $S$.

Accordingly, the resultant of all the inward forces on the closed surface $S$, due to the pressure of the surrounding liquid (or liquids) at rest in equilibrium, is a *vertically upward force, equal to the weight of the liquid (or liquids) occupying the interior of S, and acting through the centre of gravity of the liquid (or liquids) so contained.* The centre of gravity of the liquid (or liquids) thus contained by the closed surface $S$ is called *the centre of buoyancy* of $S$ under the circumstances in question.

*Corollary* 1.—If the closed surface $S$ be that of a solid body introduced into the liquid, it is evident that the body will be subject to the resultant force just described, which is now the weight of liquid (or liquids) displaced by the body. Usually some other force is required to keep the solid in equilibrium beside that of the liquid pressures. This may be supplied by a thread attached above and from which the body hangs, if it is denser than the liquid If, on the other hand, the liquid is denser than the body, then the latter may be tethered down by a thread attached below, or the body may be held by a sinker or cage of denser material suspended from above.

Such devices are used when finding the densities of bodies by the hydrostatic balance Thus, a body first in air and then in water is balanced each time by weights in air, and the difference in grams gives its volume in c.c. (nearly). Then the density is found as the quotient, mass divided by volume.

Further, by taking into account the weight of the air displaced by bodies and by the weights of the balance, the weighings may be *reduced to vacuo* when extreme accuracy is desired.

*Corollary* 2.– If the closed surface $S$ were at a great depth in an incompressible liquid of small density, we might have a considerable pressure, but differing only very slightly at the upper and lower limits of $S$, the weight of the liquid displaced by $S$ being correspondingly small. Hence, if the depth of $S$ below the free surface were continually increased while the density of the liquid were proportionally diminished, we might maintain the pressure finite while the weight of liquid displaced by $S$ and the difference of pressures at top and bottom of $S$ each vanished. We may accordingly state that *the resultant of any uniform normal pressure on any closed surface is zero.*

for a plane surface we can deduce that for an unclosed surface, if closable by a plane; *e.g.* the curved surface of a cone.

**430. Floating Bodies.**—Let a body float in equilibrium partly immersed in one liquid, the upper portion of the body being in another liquid or fluid, all in equilibrium as in Fig. 223.

Then the body is in equilibrium under the action of two resultant forces:—(1) its own weight $W$ acting vertically downwards through G, the centre of gravity of the body, and (2) the force $R$ equal to the weight of fluids displaced[1] and acting vertically upwards through the centre of buoyancy H. Hence we have

$$R = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

also

G and H are in the same vertical line (2),

as the conditions for equilibrium; for these ensure that final resultant forces and torques are each zero.

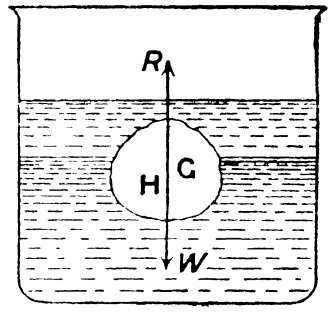



FIG. 223. FLOATING BODY.

If the upper fluid be air and the lower any liquid, it is often near enough for practical requirements to neglect the part of $R$ contributed by the air. We then obtain the special forms of (1) and (2) known as the Principle of Archimedes.[2]

The *hydrometers* of fixed or variable immersion are scientific devices for obtaining the densities of solid and liquids by application of this principle.

**431. Stability of Floating Bodies: Metacentre.**—Having seen what are the conditions for the equilibrium of floating bodies, it is now desirable to examine the stability of that equilibrium and the circumstances on which it depends. Any shift of a floating body can be regarded as made up of translations and rotations. The only translation with which we can be concerned here is a vertical one, and for this the equilibrium is obviously stable. We turn therefore to the question of the stability of a floating body when slightly tilted but without change of the volume $V$ of liquid displaced, which may be called its *displacement*.

Let this tilt occur in the plane of the diagram Fig. 224, which may be regarded as a cross-sectional elevation of a boat. Fig. 225 shows a sectional plan of the same boat at the water-line, AKBL being called the *surface of flotation*, whose area we shall denote by $S$.

Instead of drawing the boat twice, in the equilibrium and tilted positions, it is shown once only, viz. upright, the water-level being shown horizontal by a full line AB, corresponding to the boat's equilibrium position, and again by the broken line A′B′, inclined $d\theta$, corresponding to the boat's tilted position. Thus, since the volume displaced by the boat is to be the same in each position, the wedge of

[1] That is, required to *replace the body* if removed and yet preserve the interface of the fluids unshifted.

[2] For further details see *The Works of Archimedes*, etc., T. L. Heath (Camb.), 1897; or *Mechanics of Fluids*, pp. 95-96 (London), 1915.

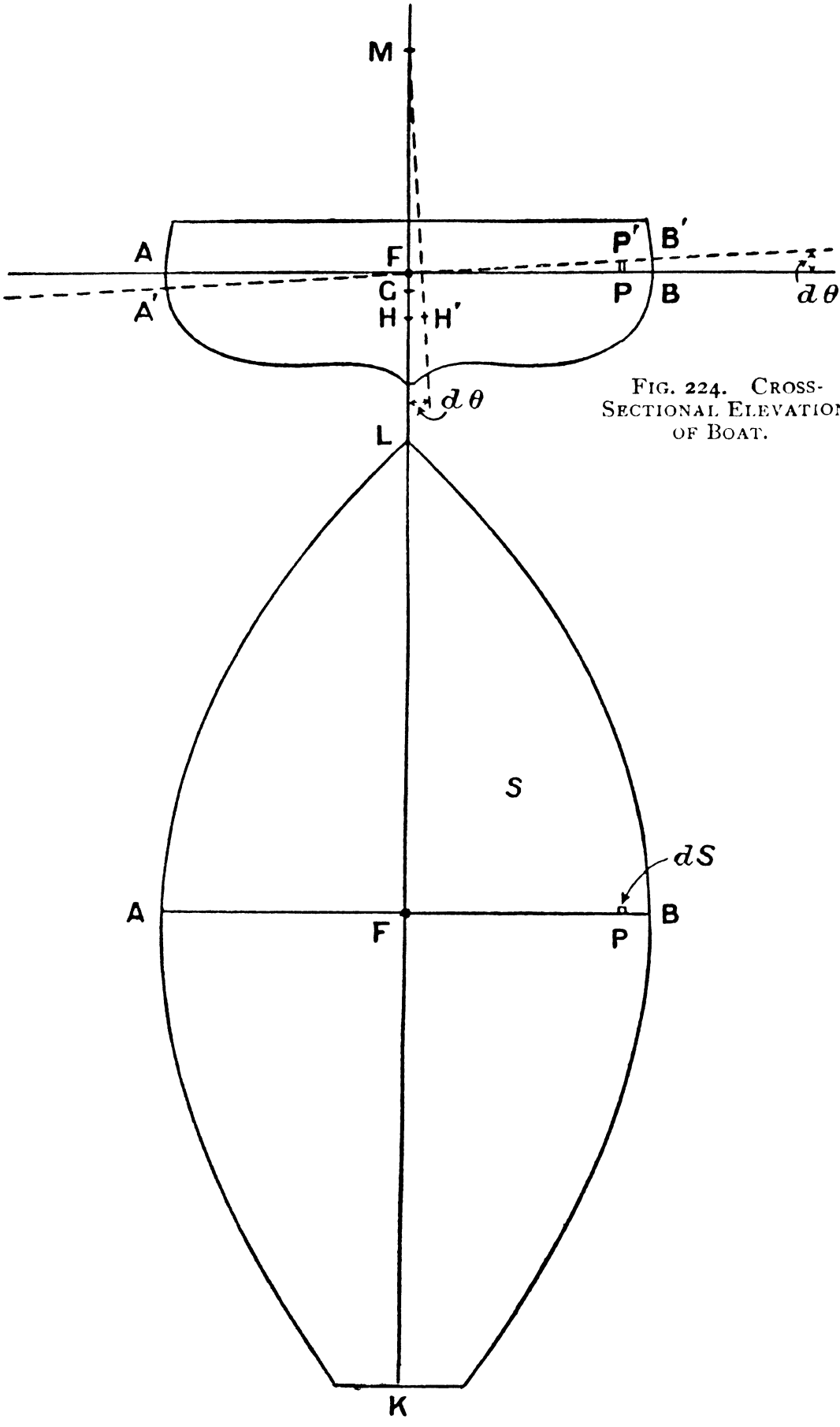



FIG. 224. CROSS-SECTIONAL ELEVATION OF BOAT.

immersion FBB′ must be equal in volume to the wedge of emersion FAA′. This circumstance serves to define the position of F where the wedges meet. For, take the origin at F, and let the positive direction of $x$ be from F towards B, and at the distance FP$=x$ take a small vertical prism of horizontal cross section $dS$ and extending from P to P′, the height ($x$ tan $d\theta$) of the wedge of immersion at the place. Then the volume of this prism will be tan $d\theta.xdS$. And, when we pass to the wedge of emersion, $x$ will be negative and make the volume negative. Hence, by integrating the above expression over the whole surface of flotation (AKBL, Fig. 225) we obtain the addition to the displaced volume $V$ caused by the tilting. But this addition is zero, accordingly we have

$$0 = \tan d\theta \int_A^B x dS = \tan d\theta . S\bar{x} \quad \ldots\ldots\ldots \quad (1),$$

where $\bar{x}$ is the distance of the centroid of $S$ from F. We thus see that

$$\bar{x} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

or the centroid of $S$ is on the line through F perpendicular to the plane of Fig. 224. In other words, the wedges of immersion and emersion meet on a line passing through the centroid of the surfaces of flotation. We shall let F represent this centroid in each figure.

Let H and H′ be the respective centres of buoyancy in the equilibrium and slightly tilted positions, and let the verticals through them in each case meet in M, then M is called the *metacentre* of the body for the type of tilt in question, *i.e.*, in the present case, for *rolling*.

The metacentre must be located to determine the behaviour of the body for the tilts under consideration, and its position depends only on the form of that part of the body which is immersed. But to determine the stability of the body, when loaded so as to sink to the given mark, a knowledge of the position of G, the centre of gravity of the floating body, is also required. For, in the tilted position, we evidently have a force equal to the weight $W$ of the body acting down through G parallel to MH′, and an equal force $V\rho g$ acting up through H′ along H′M, and due to the displacement of a volume $V$ of liquid of density $\rho$. These forces form the restoring or *righting* couple, if G is *below* M; the couple vanishes if G *coincides with* M, while the couple tilts the body farther if G is *above* M.

To locate M we may conveniently find HM. To do this consider the body in the tilted position and, about the axis of tilt through F, take the moments of the buoyancy due to the liquid displaced. We may write two expressions for this moment, one regarding the total force $V\rho g$ acting through H′, and another regarding the volume displaced in the tilted position as made up of (i) the *original* volume in the equilibrium position, (ii) *minus* the wedge of *emersion* FAA′, (iii) *plus* the wedge of *immersion* FBB′. But the wedge of emersion has a negative moment about F, so taking this wedge away *adds* a term to the moments, just as adding the wedge of immersion does (since its

moment is positive). We thus obtain the following equation of moments :—

$$-V\rho g\mathrm{HF}\sin d\theta+\rho g\tan d\theta\int_{\mathrm{A}}^{\mathrm{F}}x^2dS+\rho g\tan d\theta\int_{\mathrm{F}}^{\mathrm{B}}x^2dS=V\rho g(\mathrm{HM}-\mathrm{HF})\sin d\theta$$

Whole displacement in equilibrium position. Wedge of emersion. Wedge of immersion. Whole displacement. in tilted position.

or
$$\tan d\theta\int_{\mathrm{A}}^{\mathrm{B}}x^2dS=V\,\mathrm{HM}\sin d\theta\ .\ .\ .\ .\ .\ .\ (3).$$

Let us now denote by $I$ the moment of inertia of the surface of flotation AKBL about the axis of tilt KFL. Then

$$I=\int_{\mathrm{A}}^{\mathrm{B}}x^2dS\ .\ .\ .\ .\ .\ .\ .\ (4).$$

Hence using this in (3), and in the limit sinking the distinction between tan $d\theta$ and sin $d\theta$, we obtain

$$\mathrm{HM}=I/V\ .\ .\ .\ .\ .\ .\ .\ .\ (5).$$

This distance HM is sometimes called the metacentric height, but it is to be noted that GM is the height upon which the stability depends. So it is perhaps safer to avoid the vague phrase metacentric height where any confusion might occur.

If HG is known to be less than the value given by (5) for HM, or if HG can be adjusted so as to be less than HM, then G falls below M, and stability is attained.

Since for any initial infinitesimal tilt H cannot vary in height, it is evident that M is the intersection of consecutive normals to the locus of H, which locus is called the *surface of buoyancy*. In other words, the *metacentre* for any plane of tilt is the *centre of curvature* of the corresponding section at H of the surface of buoyancy.

**432. Practical Determination of Metacentric Height.**—On ships the height GM may be found conveniently by shifting a weight across the deck, or, what is practically the same thing, alternately filling with water two boats at opposite sides of the deck. The consequent inclination is found by observing the shift of a plumb bob on a string of known length. With the movable weight at one side, let the floating body be at rest in the symmetrical position; then its centroid is at G, Fig. 224. Call the total weight $W$, and let the inclination $d\theta$ be produced by shifting the weight $dW$ across through a distance $a$. Then the centroid of the whole body floating must have shifted from G to some point G′ in MH′, in order that the weight acting at G′ and the equal buoyancy at H′ shall act along the same straight line. Hence the *change* of moment of the floating body's weight may be regarded in either of two ways :—

(i) $dW.a\cos d\theta$; (ii) $W.\mathrm{GM}.\sin d\theta$.

Hence, remembering that the cosine of a small angle may be written unity and its sine assimilated to its circular measure, we have

$$\mathrm{GM}=\frac{a\,dW}{W\,d\theta}\ .\ .\ .\ .\ .\ .\ .\ .\ (6).$$

As an illustration of this method, we may take the following numerical example :—

Let $W = 5000$ tons weight,
$dW = 20$ ,, ,,
$a = 50$ feet,
$d\theta = 1/20$.

Then $$GM = \frac{50 \times 20}{5000 \times 1/20} = 4 \text{ feet.}$$

It should be borne in mind that the theoretical treatment obtains the height HM of the metacentre above the *centre of buoyancy*, whereas the practical method determines the height GM of the metacentre above the *centre of gravity* of the ship as *then loaded*.

Thus the former result holds for the given floating body every time it is sunk to the place in question, while the latter result varies also according to the arrangement of the loading, but is the vital criterion of stability for that loading.

## EXAMPLES—LXXXIV.

1. Define *centre of pressure* of a plane area submerged in a heavy liquid, and obtain an expression which locates this point.
2. Show that if a closed surface be described in a liquid at rest under gravity, the resultant of the inward pressures on this surface is numerically equal to the weight of the liquid inside that surface and acts upwards along the same line as that weight.
   If such a surface is that of a solid, what follows?
3. Obtain an expression for the height of the metacentre above the centre of buoyancy of a floating body.
4. 'State and prove the principle of buoyancy.
   'A solid cone is floating in water with its axis vertical and vertex downwards. To cause it to sink until 3/4 of its axis is immersed requires a load of 50 grams on its base; and to cause 4/5 of the axis to be immersed requires a load of 96 grams. Show that the specific gravity of the body is very nearly 0·324.'
   (LOND. B.A., PASS, APPLIED MATH., 1906, II. 3.)
5. 'Two liquids which have different densities and do not mix are poured into a vessel. Prove that their surface of separation is a horizontal plane.
   'A solid cylinder of specific gravity 0·7 floats with its axis vertical in a vessel containing two liquids whose specific gravities are 0·6 and 0·9, the cylinder being completely submerged; how much of its axis is in the upper fluid?
   'Explain how the upper fluid contributes towards the *upward* force on the body.' (LOND. B.A., PASS, APPLIED MATH., 1906, II. 4.)
6. 'Show how the depth of the centre of pressure on any given plane area in a liquid is calculated.
   'A circular area of radius $r$ whose plane is vertical has its highest point in the surface of water, and its centre of pressure is at a depth $r/4$ below the centre of the circle. Prove this by considering the separate equilibrium of the hemisphere of water standing on the given circular area, having given that the centre of gravity of a homogeneous hemisphere is $3r/8$ from the centre.'
   (LOND. B.SC., PASS, APPLIED MATH., 1906, III. 8.)

7. 'Define a *metacentre*, and establish the formula for its position in the case of a solid of revolution floating with its axis vertical.
'A solid right circular cone, of specific gravity $s$, floats in water with its axis vertical and vertex downwards. If $r$ is the radius of the base and $h$ the height, find the condition for stability.'
(LOND. B.SC., PASS, APPLIED MATH., 1907, III. 7.)

8. 'Prove that the resultant of the pressure on a body immersed in a fluid is an upward force equal to the weight of the fluid displaced.
'A thin rectangular board of specific gravity $\sigma$ is hinged along one of its shorter edges to the flat bottom of a tank. Find the position assumed by the board when water is poured in to a depth $h$, and prove that the vertical position is attained when $h$ is $\sqrt{\sigma}$ times the length of the longer edge of the board.'
(LOND. B.SC., PASS, APPLIED MATH., 1908, III. 9.)

9. 'Prove that if a ship be displaced through a small angle about a longitudinal axis in the section of flotation, then, approximately, Righting Couple = Displacement × metacentric height × sin (angle of heel).
'A weight of 10 tons is shifted through 22 feet across the deck of a ship of 7000 tons. The bob of a pendulum suspended from a height of 70 feet above the deck is found to move $5\frac{1}{2}$ inches across the deck at the same time. Calculate the metacentric height of the ship.'
(LOND. B.SC., PASS, APPLIED MATH., 1909, III. 8.)

10. 'Find formulae giving the position of the centre of pressure on a rectangular lamina, submerged with two sides horizontal.
'$ABCD$ is a square of side $2a$, and $P$, $Q$, $R$ are the middle points of $DA$, $AB$, $BC$. A lamina in the form of the pentagon $PQRCD$ is submerged in water so that $Q$ is in the surface, $DC$ is horizontal, and the plane of the lamina makes an angle $\theta$ with the vertical. Compare the total pressures on the portions $PQR$ and $PRCD$; find the positions of the corresponding centres of pressure. (*N.B.*—The atmospheric pressure is to be neglected).'
(LOND. B.SC., PASS, APPLIED MATH., 1910, III. 6.)

11. 'Discuss the relation between the stability of a floating body and its metacentric height.
'The section of a barge by the plane of flotation is a rectangle of 40 feet by 10 feet. Compare the couples required to produce a given small angular displacement about the fore and aft line and about a perpendicular horizontal line amidships.'
(LOND. B.SC., PASS, APPLIED MATH., 1910, III. 8.)

**433. Heights by Barometer.**—In article 426 we treated the relation between pressure and height in a liquid of constant density. We now turn to the problem of the form that the relation assumes for a fluid whose density is proportional to the pressure.

Taking as before the axes of $x$ and $y$ horizontal and that of $z$ vertically upwards, we may quote equations (6) and (7) from article 426.

$$\frac{dp}{dx}=\frac{dp}{dy}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and

$$\frac{dp}{dz}=-\rho g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

For a fluid like air we may now write the approximate relation expressing the behaviour of an ideal gas, viz.

$$p/\rho=R\theta, \text{ or } \rho=p/R\theta \quad . \quad . \quad . \quad . \quad (3)$$

where $R$ is a constant for the gas in question and $\theta$ is its absolute temperature.

Then, putting (3) in (2), and integrating from the height $z_0$ at which the pressure is $p_0$, we have

$$\int_{p_0}^{p} \frac{dp}{p} = -\int_{z_0}^{z} \frac{g}{R\theta} dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Hence, provided $g$ and $\theta$ can be treated as constants, we find

$$\log_e p - \log_e p_0 = -\frac{g}{R\theta}(z - z_0),$$

or

$$z - z_0 = \frac{R\theta}{g} \log_e \left(\frac{p_0}{p}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Now using (3), and inserting numerical values for air at 0° C. and standard pressure, we have

$$R = \frac{p}{\rho\theta} = \frac{76 \times 13{\cdot}6 \times g}{0{\cdot}001293 \times 273} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Hence, by (6) in (5), and transforming to common logarithms, we obtain finally

$$z - z_0 = \frac{76 \times 13{\cdot}6 \times \theta}{0{\cdot}001293 \times 273} \times 2{\cdot}3026 \times \log_{10}\left(\frac{p_0}{p}\right) \quad (7).$$

Since the height of the standard mercury barometer has been written 76 cm. and the density of mercury has been put at 13·6 gm./c.c., etc., etc., the difference of heights $z - z_0$ will be expressed in centimetres. It may be noted that for the $p_0$ and $p$ in (7), since they form a pure ratio, any units may be used provided they are the same for each. Indeed, the values of $\rho$ at each level could be put in (5) or (7) instead of the $p$'s, since the $\rho$'s are proportional to the $p$'s.

The fall of the mercury on ascending is roughly of the order 1 cm. in 11,000 cm. or 1 inch per 1000 feet.

If the temperature varies considerably in a given ascent, equation (7) may be applied to separate sections of the ascent, the mean temperature being used for each such section.

Hence a barometer and thermometer give the data for a determination of the heights ascended on a mountain or in the air.[1]

For other refinements as to convective equilibrium and variation of $g$ the student may consult Webster's *Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies*, p. 466 (Leipzig, 1904).

**434. Pressure on Curved Membrane of Uniform Tension.**—As a preliminary to a brief treatment of surface tension or capillary phenomena, it will be convenient to find here the relation between the difference of pressures on the two sides of a membrane or surface, its radii of curvature, and its tension or force per unit length, supposed the same in every direction.

The method followed is that used in the writer's *Text-Book of*

[1] The height of the atmosphere, *if throughout of the density* at the earth's surface, is called the *height of the homogeneous atmosphere*. It is seen to be given by $p/\rho g = R\theta/g$. It is of the order 8,000 metres.

*Sound* (pp. 262-263, 1908), and consists of equating the work done by the pressures for a small imaginary displacement of an element of the surface to that done against the tensions. It is, in fact, an application of the principle of Virtual Work.

The first expression for the work is obviously the normal force into normal displacement or excess of normal pressure on concave side into volume described by the element of surface. The second expression for the work is the tension of the surface into the increment of area acquired by the element in its normal displacement. Let the plane of $xy$ be tangential to the surface. And, at the point of contact, take as the element an infinitesimal rectangle of sides $\alpha$ and $\beta$ parallel to the axes of $x$ and $y$, the corresponding radii of curvatures of the surface being $r_1$ and $r_2$. Then as we shall suppose these radii large compared with the sides of our element, its normal displacement at each point may be written $dz$. Hence the work considered as pressure into volume is

$$dW = p\alpha\beta dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $p$ is the *excess* of the pressure on the concave side over that on the convex side.

To obtain the increment of the area of the surface in consequence of the normal displacement, we need an expression for the increase of each side of the rectangle. Thus

$$\frac{\alpha}{r_1} = \frac{\alpha + d\alpha}{r_1 + dz} = \frac{d\alpha}{dz},$$

since each of these expressions is the circular measure of the angle subtended by the side of length $\alpha$ at the centre of its curvature. Hence

$$d\alpha = \frac{\alpha}{r_1} dz \text{ and similarly } d\beta = \frac{\beta}{r_2} dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus, if $T$ is the uniform tension of the membrane, or whatever occupies the surface, we have as our second expression for the work

$$dW = Td(\alpha\beta) = T(\beta d\alpha + \alpha d\beta) = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\alpha\beta dz \quad . \quad . \quad . \quad (3).$$

Accordingly, equating the right sides of (1) and (3), we obtain as the relation sought

$$p = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

**435. Soap Bubbles and Films.**—It is known from experiments that liquids behave as though their bounding surface in contact with air, etc., were a membrane under a tension which is constant for the given materials meeting at the interface and for a given temperature. For a water-air surface this tension is of the order 74·32 dynes per cm. (see P. O. Pederson's 'Surface Tension by Jet Vibration,' *Roy. Soc. Phil. Trans.*, A. 207, pp. 341-392, December 20, 1907; *Science Abstracts*, No. 385, p. 138, March 1908). For the surface of a soap solution the tension is of the order 27 dynes per cm.

Now, thin as a soap film may be when blowing a bubble or drawing it out between wires, etc., it has under ordinary circumstances two of

the specialised portions or *skins*, each of which corresponds to the surface tension usually denoted by $T$. Hence for a spherical bubble of radius $r$ cm. we have from (4) of last article

$$p = 2T\left(\frac{2}{r}\right) = 4\frac{T}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

where $p$ is the *excess* pressure of the interior in dynes per sq. cm. and $T = 27$ dynes per cm. nearly.

If a cylindrical soap film of circular section and radius $r$ be produced by separating two wire rings dipped in the solution, one of the radii becomes infinite and its reciprocal disappears. Hence, again using (4), we find for the excess of the interior pressure

$$p = 2T/r \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6).$$

Hence this state of things is only possible when the ends of the cylinder are closed to preserve this difference of pressures. If the ends are soap films of tension $T$ and their radii $r'$, we have by (5) and (6)

$$\left.\begin{aligned} p &= 2T/r = 4T/r'. \\ \text{Whence} \quad r' &= 2r \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

Owing to the extreme thinness of the films in the above cases the effect of gravity on the mass of liquid is neglected. For any curved plates whose weights are negligible in comparison to the pressures the above formulae may be applied, the $T$ or $2T$ being replaced by tension of plate.

**436. Capillary Ascent.**—Let us now consider cases in which a considerable body of liquid is held up or depressed by surface tension phenomena. First consider the case of a vertical tube of small circular bore of radius $r$ dipping in a liquid of density $\rho$ and surface tension $T$ (*i.e.* $T$ is the tension of the liquid air interface). Suppose now the liquid spontaneously ascends a height $h$ in the tube, the inclination of the liquid surface to the wall of the tube being $\phi$, called *the angle of contact*. It is required to find the relation between $h$ and $r$ in terms of the constants involved.

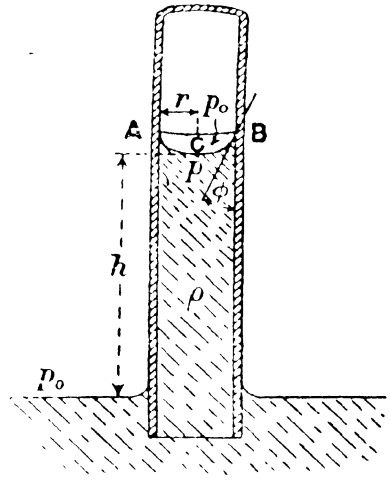



FIG. 226. CAPILLARY ASCENT.

By corollary 2 of article 429 we may replace the resultant of the normal forces on the curved surface or *meniscus* ACB, Fig. 226, by that of the same forces per unit area on the plane AB. Hence, writing $p_0$ and $p$ for the pressures above and below the meniscus and resolving vertically, we have

$$(p_0 - p)\pi r^2 = 2\pi r T \cos\phi \quad \ldots\ldots\ldots\ldots \quad (8).$$

But by hydrostatic considerations (see equation (8) of article 426) we have

$$p_0 - p = \rho g h \quad \ldots\ldots\ldots\ldots \quad (9).$$

Thus (9) in (8) gives

$$2T\cos\phi=\rho ghr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

This shows that for a given liquid $hr$ is a constant, or $h\propto 1/r$. It also expresses the product $T\cos\phi$ in terms of readably observable quantities.

The value of $\phi$, the angle of contact, varies with the liquid and solid concerned, and must be experimentally determined for any given pair of substances. Since water *wets* glass, the value of $\phi$ is zero for water in a glass tube. Thus, considering also $\rho$ practically unity for water, (10) simplifies to the approximate equation for water in a glass tube.

$$2T=ghr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

It should be noticed that in dealing with the meniscus we were concerned only with the radius of the tube at that place. Further, equation (9), giving the relation between pressures and difference of levels, is independent of the radius altogether. Hence the tube may be of any section we please provided it is circular and of radius $r$ at the meniscus ABC and the equations (8), (9), and (10) are still valid. It may, however, be observed that if the tube have a bulbous portion just below ABC, the liquid would not spontaneously ascend past that bulb but would need forcing or sucking up to ABC, and would then remain in equilibrium there.

It is usually near enough to measure the height $h$ to the middle point of the meniscus as shown in the figure.

If a liquid like mercury is used which does not wet glass and whose meniscus is convex upwards in equilibrium, the values of $\cos\phi$ and of $h$ become negative. In other words, the ascent is changed into a depression.

**437. Ascent between Plates.**—If we use parallel vertical plates dipping into the liquid a distance $a$ apart instead of a tube, equation (8) of article 436 is replaced by

$$(p_0-p)la=2lT\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11a),$$

a horizontal length $l$ along the faces of the plates being considered.

Our previous equation (9) still holds, viz.

$$p_0-p=\rho gh.$$

Hence, we obtain

$$2T\cos\phi=\rho gha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

showing that $ha$ is constant, or $h\propto 1/a$. Thus, if two vertical plates are used very nearly touching at one vertical edge and quite so at the other, the form of the upper surface of the liquid is an equilateral hyperbola with the free liquid surface and the line of contact of the plates as asymptotes.

## EXAMPLES—LXXXV.

1. Establish the law that in an atmosphere of uniform temperature the pressures diminish in a geometrical progression as the corresponding heights increase in an arithmetical progression.
2. If the tabular density of air is taken as 0·00129 gm. per c.c., show that at

15° C. a fall of the barometer from 76 to 75 cm. corresponds to an ascent of 11,196 cm., and a fall from 30 to 29 inches to an ascent of 940 feet. (Use seven figure logs.)

3. Show that the resultant force of uniform normal pressures $P$ all over the convex surface of a hemisphere of radius $a$ is $\pi a^2 P$ perpendicular to the base.
4. Find the relation between the excess pressure on the concave side of a membrane, film, or other flexible sheet, and its curvature and tension.
5. A long cylindrical glass tube of 1 cm. internal bore has an excess pressure inside of 50 atmospheres, show that the circumferential tension is $25 \times 10^6$ dynes per cm. nearly.
6. Obtain formulae for the surface tension of a soap bubble of given size and excess internal pressure, also for the tension per square inch in a spherical steel shell subjected to high internal pressure.
7. Derive formulae for the capillary ascent of liquids in tubes and between plates.
8. A rectangular frame of opposite wires of length $c$ connected at their ends by threads is dipped in soap solution, and one wire is fixed horizontally the other supporting a weight $W$; the vertical height between the wires is then found to be $b$, and the distance apart of the threads midway between the wires is $a$. Show that the surface tension $T$ of the soap solution is given by

$$T = W \frac{c - a}{c^2 + b^2 - a^2}.$$

**438. Equilibrium Form of Large Drop.**—Let us now consider the form of a large drop of a liquid upon a horizontal plate which it does not wet, say mercury on clean glass. Take the origin of co-ordinates at the centre of its upper surface, the axis of $z$ being vertically downwards and that of $x$ in the plane of the diagram, see Fig. 227, which shows a central vertical section of the drop resting on the surface $c$C. Let the pressure outside the drop be $p_0$, practically the same at all parts of it, the pressure inside it at the level $z$ being $p$. Then, for the point P on the surface at this level, we have from article 426, equation (8), and article 434, equation (4), the following expressions—

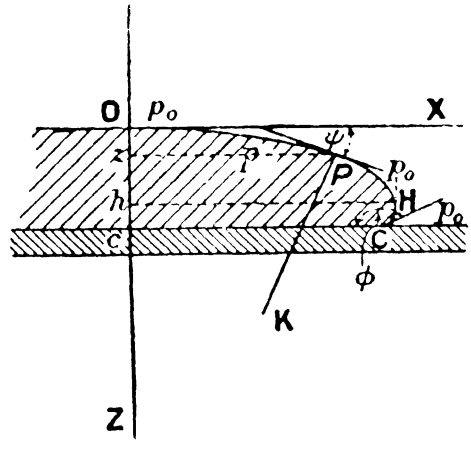



FIG. 227. EQUILIBRIUM OF LARGE DROP.

$$p - p_0 = \rho g z \quad \ldots\ldots\ldots\ldots \quad (1),$$

$$p - p_0 = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad \ldots\ldots\ldots \quad (2).$$

Whence

$$\rho g z = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad \ldots\ldots\ldots \quad (3),$$

$T$ being the surface tension of the liquid air interface and $r_1$, $r_2$ the radii of curvature at P. Let $r_2$ denote the radius on a plane perpendicular to that of the diagram, and passing (of course) through PK, the

normal at P, to the surface of the drop. Then, as the drop is supposed large, we see that $r_2$ will be large, and therefore its reciprocal may be neglected in comparison to that of $r_1$, which refers to the plane of the diagram. Also, if $\psi$ denote the angle between OX and the tangent at P, and $s$ is distance along the curve, we have

$$\frac{1}{r_1}=\frac{d\psi}{ds}=\frac{d\psi}{dx}\cdot\frac{dx}{ds}=\cos\psi\frac{d\psi}{dx} \quad . \quad . \quad (4).$$

But, in (3) and (4), we have still three variables, $z$, $\psi$, and $x$. Let us therefore eliminate the latter by use of the relation

$$\frac{dz}{dx}=\tan\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Then (4) and (5) in (3) give

$$\rho g z\frac{dz}{dx}=T\cos\psi\frac{d\psi}{dx}\tan\psi,$$

or

$$\rho g\int_0^s z\,dz=T\int_0^\psi \sin\psi\,d\psi.$$

Whence, on integrating,

$$\rho g z^2=2T(1-\cos\psi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

At the point H, where the vertical is tangential to the curve, let the depth below the summit be $h$. Then, since $\psi=\pi/2$ at H, equation (6) gives

$$\rho g h^2=2T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

a formula which may be used in the experimental determination of $T$ without the angle $\phi$ of contact being known.

Let the total thickness of the drop be $c$; then, as $\psi$ at the depth $c$ is equal to $\phi$, equation (6) gives

$$\rho g c^2=2T(1-\cos\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

And from this $\phi$ may be experimentally determined when $T$ is known from (7).

What is here only approximate for a drop of finite size (and therefore of finite curvature in plan) would be rigidly true for the shape of the liquid surface near a plane plate dipped into the liquid.

As shown in Fig. 227, it would be right for a liquid not wetting the plate, the surface, convex upwards, being accordingly depressed near the plate, as mercury is near glass. If the curve of Fig. 227 were inverted by rotation through 180° about OX, it would then correctly represent the liquid surface, concave upwards, and raised above the general free surface OX, because it was near a plate which that liquid wets, as for water and glass. In either case the curve would be valid from O to P, H, etc., according to the inclination of the plate dipping into the liquid and the angle of contact of that liquid with it.

Further, the investigation made and shown for a drop on the upper surface of a plate applies also to an *air bubble* blown in liquid on the *under surface* of a plate.

**439. Relative Equilibrium of a Liquid in an Accelerated Vessel.**—The problems of liquids in motion belong, of course, to the next chapter, but the relative equilibrium of liquids in vessels whose acceleration is uniform or follows a very simple law may be noticed here.

We have seen in articles 226-229 how gravity appears to be changed in magnitude and direction by the acceleration of the chamber in which a plumb bob or pendulum is hung. And, since the free equilibrium surface of a liquid is at right angles to gravity, we may at once deduce the form of a liquid surface in relative equilibrium in an accelerated vessel. The relative or disturbed value of gravity $g'$ at an angle $\psi$ say with the vertical is a normal to the surface of relative equilibrium of the liquid in this chamber or vessel. This surface is accordingly inclined at the angle $\psi$ with the horizontal, the vertical plane which cuts it at the steepest angle being that which contains the acceleration of the vessel.

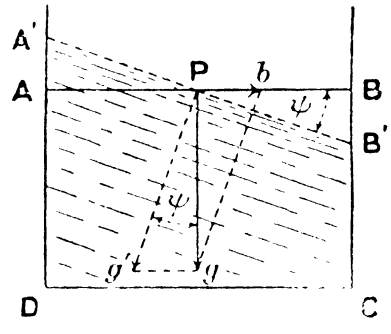



FIG. 228. LIQUID IN ACCELERATED VESSEL.

Thus, in Fig. 228, let the vessel ABCD have horizontal acceleration $Pb$ in the plane of the diagram. Then we may regard this acceleration as *vectorially subtracted* from the acceleration $Pg$ due to gravity, thus leaving the effective or relative gravity $Pg'$ at an angle $\psi$ with the vertical and in the plane of the diagram. Thus the relative equilibrium surface of the liquid is A'PB' at the angle $\psi$ with the horizontal in the plane of the diagram.

The quantities concerned are obviously connected by the relations

$$\tan\psi = -b/g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$g'^2 = g^2 + b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

If there is liquid in a chamber moving with acceleration down an incline we should have horizontal and vertical accelerations, say $b$ and $a$ respectively. Then they could be both vectorially subtracted from the vertical $g$, leaving the effective $g'$, which is a normal to the surface of the liquid. (See articles 226-228.)

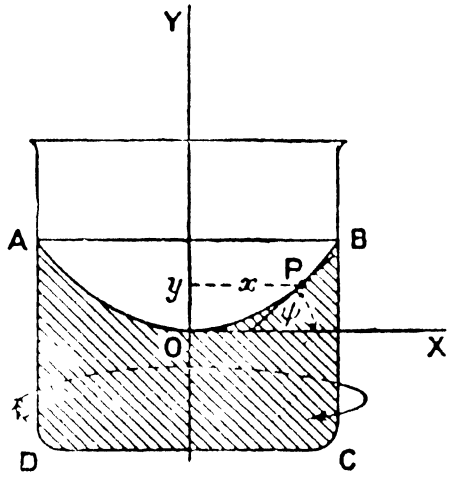



FIG. 229. SURFACE OF ROTATING LIQUID.

**440. Uniform Rotation about a Vertical Axis: Liquid Surface a Parabola.**—Let the vessel ABCD rotate uniformly at speed $\omega$ about the vertical axis OY, Fig. 229; and let it be required to find the form of the liquid surface when it has settled to a steady state and is rotating like a rigid solid. Let AOPB represent this form, and consider the point P of co-

ordinates $x$ and $y$. At this point the acceleration of the liquid is obviously expressed by

$$b = -\omega^2 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Thus, if the inclination of the surface to the horizontal is here $\psi$, we have by (1) of article 439

$$\tan\psi = \omega^2 x/g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

But, by the geometry of the figure,

$$\tan\psi = dy/dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Hence, equating and integrating, we have

$$\int_0^y dy = \frac{\omega^2}{g}\int_0^x dx.$$

Whence

$$x^2 = \frac{2g}{\omega^2}y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

showing that the central vertical section of the surface is a parabola of latus rectum $2g/\omega^2$, with vertex at the origin and axis vertically upwards.

This problem is referred to later and dealt with in a more general manner. (See article 447.)

## EXAMPLES—LXXXVI.

1. Explain how the surface tension and angle of contact may be obtained for a liquid like mercury by using a very large drop on a glass plate. Derive any formulae that are needed.
2. Show how the surface of a liquid at rest relative to the containing vessel depends upon the acceleration components of that vessel.
3. Prove that the free surface of a liquid in a vessel rotating about a vertical axis assumes the form of a paraboloid of revolution.
4. A tray containing liquid is placed upon a long plane inclined $\alpha$ to the horizontal, the coefficient of friction between the tray and the plane being $\tan\beta$, where $\beta$ is less than $\alpha$.

   Show that as the tray slides down the plane under gravity the inclination of the free surface of the liquid (when steady) with the horizontal is $(\alpha - \beta)$, or, with the plane, $\beta$ simply.

# CHAPTER XX

## HYDROKINETICS

**441. Equation of Continuity.**—The general equations which apply to fluids in motion are of three kinds, viz.

(1) the so-called *equation of continuity* ;

(2) the three *equations of motion* ;

(3) the relation between *density and pressure*. We shall discuss them in the above order.

The first equation is based on a property of matter so fundamental as to be usually tacitly assumed as underlying all our notions of matter and as being a conception so deeply ingrained as to need no formal definition. The property referred to is the continuous existence and constant physical value of a given portion of matter, or the utter absence of its disappearance and reappearance under any circumstances to which physical equations apply. In other words, the quantity and the inertia of the matter *in a given volume can change only* by the algebraic sum of the like quantities passing into and out from that volume *through the closed surface which forms its boundary*.

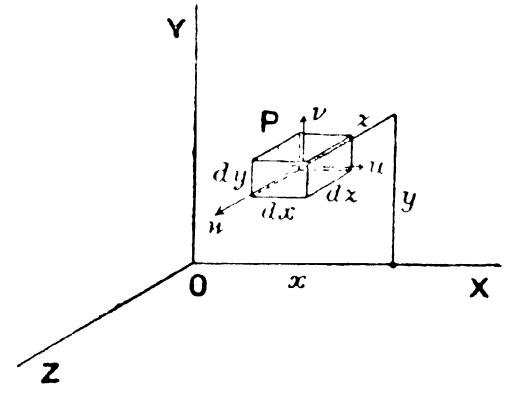



FIG. 230. CONTINUITY EQUATION.

We now proceed to give this principle mathematical expression. Consider the fluid of density $\rho$ occupying the parallelepiped of edges $dx$, $dy$, and $dz$ with centre at $P$, $(x, y, z)$, the velocity components there at time $t$ being $u$, $v$, and $w$ respectively (see Fig. 230).

Then the rate of change of quantity of matter in the volume $dxdydz$ is

$$\frac{d\rho}{dt}dxdydz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

For the flow through the boundary, take first the pair of opposite faces parallel to the $yz$ plane. Then the normal flows there are

$$\left[\rho u \mp \frac{d(\rho u)}{dx} \cdot \frac{dx}{2}\right] dydz.$$

Hence, the net flow inwards through this pair of faces is the algebraic sum of the above expressions, taking the first (with the upper

sign) as it stands and the second with the lower sign reversed. Th gives

$$-\frac{d(\rho u)}{dx}dxdydz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

As to any variation of $u$ over the $dydz$ faces, it should be note that $u$ means volume passing per unit time per unit area. Thus, our volume is infinitesimal with centre at $x$, $y$, $z$, the meaning of there is the limit of the quotient (flow/area). Hence $udydz$ expres the volume passing per unit time through the area in question at t centre of the parallelepiped.

It is obvious that for the whole net inward flow of fluid we nee two other expressions similar to (2) but for the other two pairs faces. Writing these, adding the three, equating the sum to (1), an cancelling out the common factors, we obtain

$$\frac{d\rho}{dt}+\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad (3),$$

which is the *equation of continuity* in its *general* form.

To apply it to a liquid of practically constant density, the first ter disappears and $\rho$ cancels out from the others. The equation accor ingly reduces to

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad (3a).$$

which is the equation of continuity for *an incompressible liquid.*

**442. Equations of Motion.**—Still referring to Fig. 230, let $p$ l the pressure at the centre of the parallelepiped. Then the pressures o the opposite $dydz$ faces are

$$p\mp\frac{dp}{dx}\cdot\frac{dx}{2}.$$

Hence the resultant force due to pressures acting on the fluid i the parallelepiped in the positive direction of the axis of $x$ is

$$-\frac{dp}{dx}dxdydz.$$

Let the external forces have cartesian components $X$, $Y$, and $Z$ *pe unit mass.*

Then, for the total force components on the fluid in the parallel piped, we shall have

$$\left(X\rho-\frac{dp}{dx}\right)dxdydz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and two corresponding expressions for $Y$ and $Z$.

But each such force may be equated to the product of the mas concerned and its corresponding acceleration. For the $x$ direction w may write this product in the form

$$\rho dxdydz\frac{Du}{Dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where $D/Dt$ denotes *particle* differentiation; *i.e.* we are to follow i

imagination an *individual* particle or set of particles in their course and note *their* rate of increase of speed. Another and more usual method is to regard the velocity components $u$, $v$, and $w$ each as a function of $x$, $y$, $z$, and $t$ and take *partial* differentiations. In other words, instead of following *any one particle* in its course, we note the speeds at various places at a particular instant of *whatever particles* may be there then, and find the changes of those speeds with place and time.

Since the change of speed of an individual particle is made up of the four possible changes due to change of $t$ and of $x$, $y$, and $z$, we may exhibit the relation of the two styles of differentiation as in the following equation :—

$$Du=\frac{du}{dt}\,dt+\frac{du}{dx}\,dx+\frac{du}{dy}\,dy+\frac{du}{dz}\,dz.$$

Passing from the particle (or total) *differential* to the particle *differentiation*, the $dx$, $dy$, and $dz$ on the right side become the corresponding velocity components.

Hence we obtain

$$\frac{Du}{Dt}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Thus, using (6) in (5), equating to (4), and writing the two similar equations for $y$ and $z$, we have

$$\left.\begin{aligned}\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}&=X-\frac{1}{\rho}\frac{dp}{dx}\\ \frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}&=Y-\frac{1}{\rho}\frac{dp}{dy}\\ \frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}&=Z-\frac{1}{\rho}\frac{dp}{dz}\end{aligned}\right\} \quad . \quad . \quad . \quad (7).$$

These are the *equations of motion* of a fluid.

The equation of continuity (equation (3) of article 441), together with these equations of motion, are called *Euler's fundamental equations of hydrodynamics.*

It must be always remembered in using the above that the differentiations are partial. Thus, $du/dt$ in (7) means the rate of increase with time $t$ of the speed $u$ of whatever particles are passing the place in question. Again, $du/dx$ means the rate of increase with co-ordinate $x$ of the speed $u$ of whatever particles are passing at these places at the instant in question. In other words, we are dealing with the procession or flow of particles at certain points or instants and are not following individual particles.

**443. Relations between Pressure and Density.**—In order to express the five functions $u$, $v$, $w$, $p$, and $\rho$ in terms of the four independent variables $x$, $y$, $z$, and $t$, we need five equations. Equation (3) is a kinematical one, the three of (7) are dynamical ones, the fifth now required is a physical one expressing the relation between pressure and density as experimentally found for different fluids or types of fluids.

Thus, for liquids under ordinary pressures, we may write the approximation

$$\rho = \rho_0 = \text{a constant nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a).$$

We should need to depart from this in the case of great depths in the ocean, but it would be near enough for all ordinary cases of the flow of liquids on the earth.

For gases well above their points of liquefaction, the ideal gaseous law or *characteristic equation* of gases is a near approximation, viz.

$$p/\rho = R\theta,$$

where $R$ is a constant for the gas in question and $\theta$ is the absolute temperature, which is about $(273 + t^\circ \text{ C.})$.

Hence, for temperature constant, the above equation yields the expression of Boyle's law.

$$\rho \propto p, \text{ or } \rho = p/R\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8b).$$

Again, if there is no communication of heat when expansion or compression occurs, it is found that the temperature varies in such a way that

$$\left.\begin{aligned} p/\rho^\gamma &= \text{constant}, \\ \text{or} \quad \rho &\propto p^{1/\gamma} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8c),$$

where $\gamma$ is the ratio of the specific heats of the gas at constant pressure and constant volume respectively.

From the characteristic equation we easily see that the constant $R$ may be expressed by

$$R = \frac{p}{\rho\theta} = \frac{p_s}{\rho_0 273} = \frac{76 \times 13{\cdot}6 \times 981}{\rho_0 \times 273} \text{ c.g.s. units.} \quad . \quad . \quad (9),$$

where $p_s$ is the standard atmosphere and $\rho_0$ the tabular density under it at $0^\circ$ C.

Thus equations (3), (7), and the form of (8) appropriate to the case give the five fundamental equations required for the motion of fluids.

**444. Steady Motion under Gravity, etc. Bernoulli's Theorem.**—Let us now consider a fairly simple case of motion of great importance and obtain integrals of the corresponding equations.

Suppose that the fluid is in what is called *steady* motion, *i.e.* the velocity components at each point remain of the same constant values. And further, let us suppose that the external forces are due to gravity, or other causes, such that they are derivable from a *potential.*

Then we may write

$$\frac{du}{dt} = \frac{dv}{dt} = \frac{dw}{dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

and

$$X = -\frac{dV}{dx}, \; Y = -\frac{dV}{dy}, \; Z = -\frac{dV}{dz} \quad . \quad . \quad . \quad . \quad (11).$$

In steady motion it is clear that the lines of motion coincide with the paths of the particles moving. We can accordingly draw curves, called *stream lines*, such that a particle at any point Q has its resultant velocity $q$ along the tangent to the curve at that point of co-ordinate $s$.

We may thus write

$$q^2=u^2+v^2+w^2 \quad \ldots\ldots\ldots\ldots (12),$$

and

$$u=q\frac{dx}{ds},\ v=q\frac{dy}{ds},\ w=q\frac{dz}{ds} \quad \ldots\ldots\ldots (13).$$

We must now apply these new relations to the simplification of our equations of motion (7) of article 442. Hence, introducing into them (10) and (13), we obtain on the left side of the first

$$q\frac{du}{dx}\cdot\frac{dx}{ds}+q\frac{du}{dy}\cdot\frac{dy}{ds}+q\frac{du}{dz}\cdot\frac{dz}{ds}=q\frac{du}{ds} \quad \ldots\ldots\ldots (14).$$

So that, on using (11) also, we have for the complete set

$$\left.\begin{aligned} q\frac{du}{ds}&=-\frac{dV}{dx}-\frac{1}{\rho}\frac{dp}{dx}\\ q\frac{dv}{ds}&=-\frac{dV}{dy}-\frac{1}{\rho}\frac{dp}{dy}\\ q\frac{dw}{ds}&=-\frac{dV}{dz}-\frac{1}{\rho}\frac{dp}{dz}\end{aligned}\right\} \quad \ldots\ldots\ldots (15).$$

Multiplying these three in order by $dx/ds$, $dy/ds$, and $dz/ds$ and adding, we obtain

$$u\frac{du}{ds}+v\frac{dv}{ds}+w\frac{dw}{ds}=-\frac{dV}{ds}-\frac{1}{\rho}\frac{dp}{ds} \quad \ldots\ldots\ldots (16).$$

Then, using (12) in the left side and putting $S$ for the force per unit mass along a stream line on the right, we have

$$q\frac{dq}{ds}=S-\frac{1}{\rho}\frac{dp}{ds} \quad \ldots\ldots\ldots (17).$$

Now multiply either equation (16) or (17) by $ds$ and integrate, this integration being accordingly *along a stream line.*

We then find

$$V+\int\frac{dp}{\rho ds}ds+\tfrac{1}{2}q^2=C \quad \ldots\ldots\ldots\ldots (18),$$

where $C$ is the integration constant for the *particular stream line in question* but may be *different* for any other line. There are certain circumstances possible in which the constant is the same for all the lines.

If now the motion is occurring under the action of gravity simply and we take the axis of $z$ vertically upwards, we may write

$$V=gz \quad \ldots\ldots\ldots (19).$$

Also if the pressures are all moderate and the fluid concerned a liquid, we have approximately

$$\rho=\rho_0, \text{ a constant} \quad \ldots (20).$$

Then, substituting (19) and (20) in (18), we obtain the important relation or theorem due to Daniel Bernoulli, viz.

$$gz+\frac{p}{\rho_0}+\tfrac{1}{2}q^2=E \quad \ldots\ldots\ldots (21),$$

in which $E$ is now written for the new constant of integration. It may

be noted that it, like every other term of the equation, is of the nature *energy per unit mass.* Thus the physical interpretation of the formula is that the energy per unit mass of liquid along a stream line under gravity is constant and may consist of three parts :—

(1) $gz$, its potential energy in the gravitational field ;

(2) $p/\rho_0$, its potential energy in the *pressure field* (or the energy required to pump it in against the pressure, but without raising it or imparting velocity to it) ;

(3) $\frac{1}{2}q^2$, its kinetic energy of translation. Indeed, it was from this standpoint and the principle of energy that Bernoulli's theorem was originally derived. See Lamb's *Hydrodynamics* (p. 23, 1895).

We may get another set of most useful conceptions, in connection with this steady gravitational flow, by dividing the above equations through by $g$. Thus

$$z+\frac{p}{g\rho_0}+\frac{q^2}{2g}=H \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

We have here written $H$ for $E/g$ ; it is therefore a constant for the particular stream line in question. The interpretation of the terms is now that each is a *height.*

And the meaning of the whole equation is that for any one stream line there is an associated height $H$, which is constant for all points in that line. Further, this height is made up of three parts :—

(1) $z$, the *actual height* of the point in question above the origin ;

(2) $p/g\rho_0$, the height or head corresponding to the pressure or to which the pressure might be considered due, and called the *pressure head* ;

(3) $q^2/2g$, the height or head corresponding to the velocity or to which the velocity may be considered due, and called the *velocity head.*

Hence if from any point Q in the given stream line we raise a vertical whose length is the sum of the pressure head and velocity head at the point Q in question, we reach the same *horizontal plane* at a height $H$ above the origin.

From either (21) or (22) we see that along any one horizontal stream line the places of greater velocity have smaller pressure, and *vice versa.* This is often at first sight surprising, but a moment's reflection shows that, apart from gravity, the liquid can only increase its speed when passing to a place of lower pressure.

Hence by causing water to flow in a jet of diminishing cross section, a lowering of pressure is produced or a partial vacuum. And this may be utilised to draw other fluids into this space, as is done in jet pumps and aspirators.

**445. Torricelli's Theorem of Outflow Velocity.**—Let us now consider the velocity of flow of liquid from a small hole in the horizontal base of a large vessel in which the liquid stands at a height $h$, practically constant for the short time in question.

Take the $z$ axis vertically upwards with the origin at the hole, then (21) of article 444 will apply.

At a certain height $z'$ from the base of the vessel let $w=0$ practically, then the value of the pressure there is $p_0+(h-z')\rho_0 g$, where $p_0$ is the atmospheric pressure. Hence we have, from (21) for $z=z'$,

$$gz'+\frac{p_0+(h-z')\rho_0 g}{\rho_0}=E=\frac{p_0}{\rho_0}+gh \quad . \quad . \quad . \quad (23).$$

Again, for $z=0$ we have the following *approximate* relations:— $p=p_0$, $u=v=0$, and $q=w$. Thus the velocity $w$ of outflow is given by putting these values in (21) and substituting for $E$ from (23), viz.

$$\frac{p_0}{\rho_0}+\tfrac{1}{2}w^2=\frac{p_0}{\rho_0}+gh,$$

or

$$w^2=2gh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24),$$

which is the approximate formula known as *Torricelli's Theorem.*

It is seen that the velocity thus determined is the same as that of a body falling from the level of the free liquid surface. The same result might have been obtained by considerations of energy.

**446. Vena Contracta.**—We must now examine the approximate relations used in obtaining Torricelli's theorem and note how, for strictness, they need modification.

In the first place, the stream lines above the orifice must be of a converging character. Further, this convergence continues for a short distance outside the orifice, as may easily be observed. Hence, although at the *surface* of the issuing jet the pressure of the liquid may be atmospheric only, *within the jet itself* the pressure is rather greater than atmospheric. Accordingly, the velocity here is less than that applicable to the surface as given by Torricelli's theorem, which would in consequence fail to give a correct estimate of the quantity of liquid issuing from a given orifice.

The exact behaviour of the liquid near the orifice presents great theoretic difficulties which have been only partially overcome. But experiments have shown that at a little distance from a simple orifice the stream lines have all become parallel, and at this place, called the *vena contracta*, the cross section of the jet has a minimum value. Thus, over the cross section of the vena contracta, we may take the velocity as uniform and the pressure practically atmospheric. Hence, if we know experimentally the position and size of the vena contracta for a given orifice, we may make an amended calculation of the total outflow.

For a circular hole in a thin plate the ratio of the area $S'$ of the vena contracta to that $S$ of the orifice is approximately

$$S'/S=0{\cdot}62 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

This ratio is called the *coefficient of contraction.*

The distance of the vena contracta from the circular orifice may be taken as between 0·39 and 0·5 of its diameter.

For practical details relating to a variety of openings the student may refer to Goodman's *Mechanics Applied to Engineering* (pp. 548-563, 1908).

**447. Liquid Rotating as Rigid Solid.**—Suppose a mass of liquid of constant density to rotate as a rigid solid with velocity $\omega$ about the axis of $z$, which is taken vertically upwards.

Then we have

$$\left.\begin{aligned} u=-\omega y,\ v=+\omega x,\ w=0 \\ X=0,\quad Y=0,\quad Z=-g \end{aligned}\right\} \quad . \quad (26).$$

The equation of continuity is then satisfied, and these equations (26) put in (7) of article 442 give

$$\left.\begin{aligned} -\omega^2 x &= -\frac{1}{\rho}\cdot\frac{dp}{dx} \\ -\omega^2 y &= -\frac{1}{\rho}\cdot\frac{dp}{dy} \\ 0 &= -g-\frac{1}{\rho}\frac{dp}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (27).$$

Multiplying these in order by $dx$, $dy$, and $dz$, adding, and integrating, we find

$$\frac{\omega^2}{2}(x^2+y^2)-gz=\frac{p}{\rho}+C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

Taking the origin at the surface, where the pressure is $p_0$, the constant of integration is

$$C=-p_0/\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

Hence the equation of the surface is expressed by

$$\omega^2(x^2+y^2)=2gz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

which is a paraboloid of revolution with latus rectum $2g/\omega^2$, as found by other considerations in article 440.

### EXAMPLES—LXXXVII.

1. Derive the equation of continuity for a compressible and for an incompressible fluid.
2. Obtain the equations of motion for a fluid.
3. State the various relations which hold under different circumstances between the pressure and density of a fluid.
4. Assuming Euler's fundamental hydrodynamical equations, obtain the equation of steady motion along a stream line in the form

$$q\frac{dq}{ds}=S-\frac{1}{\rho}\frac{dp}{ds}.$$

5. Obtain an equation which expresses Bernoulli's theorem, and interpret each term. Can the equation be put in another useful form?
6. Show that, with the assumption of certain approximate relations, the outflow velocity of a liquid is that due to a free fall through the *pressure head.*
7. How is the result of question 6 affected by the phenomenon known as the *vena contracta*?

**448. Angular Velocities of Elements of a Fluid.**—We have hitherto expressed the motion of a fluid in terms of its linear velocity components. Let us now introduce expressions for the angular velocities of its elements and determine the relation between the two.

Take the point $P(x, y, z)$ as one corner of a parallelepiped of edges $dx$, $dy$, $dz$, and consider its change of shape parallel to the $xy$ plane in consequence of the linear velocity components $u$, $v$, $w$ of the fluid occupying it (see Fig. 231).

Thus in time $dt$ let the face PACB change to the size and shape PA'C'B', in which the corner originally at P is shown as if brought back there to simplify the figure.

Let the angles APA' and BPB' be $d\alpha$ and $d\beta$. Then the mean angular displacement about OZ of the element of fluid in time $dt$ may be represented by $\frac{1}{2}(d\alpha - d\beta)$.

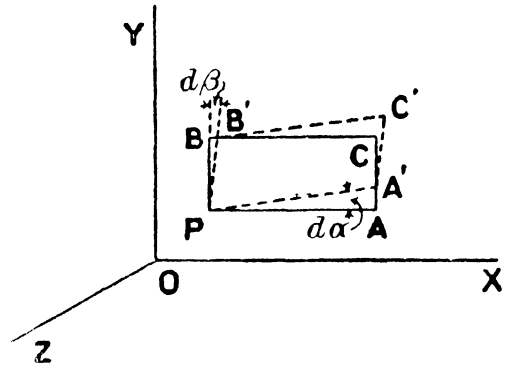



FIG. 231. ANGULAR VELOCITIES OF A FLUID.

But $$d\alpha dx = AA' = (v' - v)dt = \frac{dv}{dx}dx dt,$$

where $v'$ is the velocity parallel to the $y$ axis at A.

We thus have $$d\alpha = \frac{dv}{dx}dt,$$

and similarly $$d\beta = \frac{du}{dy}dt.$$

Hence, for the angle of mean rotation in time $dt$ about the $z$ axis, we have

$$\frac{d\alpha - d\beta}{2} = \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)dt = \zeta dt \text{ say.}$$

Thus, denoting by $\xi$ and $\eta$ the angular velocities about the axes of $x$ and $y$ respectively, and obtaining the corresponding expressions by symmetry, we find

$$\left.\begin{aligned} 2\xi &= \frac{dw}{dy} - \frac{dv}{dz} \\ 2\eta &= \frac{du}{dz} - \frac{dw}{dx} \\ 2\zeta &= \frac{dv}{dx} - \frac{du}{dy} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1),$$

expressions which give the component angular velocities in terms of the linear velocities as required.

**449. Velocity Potential.**—When the three angular velocity components all vanish, the fluid is *devoid of rotation of its elements*, and its linear velocity components are then derivable as the space differentials of a function $\phi$ of the space co-ordinates and time, and called the *velocity potential.*

We then have $$u = -\frac{d\phi}{dx},\ v = -\frac{d\phi}{dy},\ w = -\frac{d\phi}{dz} \quad \ldots \quad (2),$$

and $$0 = \xi = \eta = \zeta \quad \ldots\ldots\ldots \quad (3).$$

Thus the velocity potential stands in the same relation to the linear velocity as the gravitational potential does to the field or attraction. The introduction of the velocity potential accordingly effects a similar simplification in certain cases.

**450. Angular Accelerations of Elements of a Liquid.**—To obtain the angular accelerations of the elements of a liquid, we must introduce the expressions for the angular velocities into the equations of motion. We may conveniently do this as follows :—By differentiation of the square of the resultant linear velocity and its components, we have

$$\tfrac{1}{2}\frac{dq^2}{dx}=\tfrac{1}{2}\frac{d}{dx}(u^2+v^2+w^2)=u\frac{du}{dx}+v\frac{dv}{dx}+w\frac{dw}{dx} \quad . \quad . \quad (4).$$

Now take the right side of (4) from the left side of the first of the equations of motion (7) of article 442, and we obtain

$$\frac{du}{dt}+v\left(\frac{du}{dy}-\frac{dv}{dx}\right)+w\left(\frac{du}{dz}-\frac{dw}{dx}\right)=X-\frac{1}{\rho}\frac{dp}{dx}-\tfrac{1}{2}\frac{dq^2}{dx} \quad . \quad . \quad (5).$$

Then, introducing in this the angular velocities from (1) of article 448, and writing the other two equations by symmetry, we find

$$\left.\begin{aligned}\frac{du}{dt}+2(w\eta-v\zeta)&=X-\frac{1}{\rho}\frac{dp}{dx}-\tfrac{1}{2}\frac{dq^2}{dx}\\ \frac{dv}{dt}+2(u\zeta-w\xi)&=Y-\frac{1}{\rho}\frac{dp}{dy}-\tfrac{1}{2}\frac{dq^2}{dy}\\ \frac{dw}{dt}+2(v\xi-u\eta)&=Z-\frac{1}{\rho}\frac{dp}{dz}-\tfrac{1}{2}\frac{dq^2}{dz}\end{aligned}\right\} \quad . \quad . \quad (6).$$

Now differentiate with respect to $y$ the third of equations (6), and from the result subtract the second differentiated with respect to $z$.

Then the last two terms on the right disappear for constant $\rho$, and the first on the left, interpreted by (1), gives $2d\xi/dt$. We accordingly have

$$\frac{d\xi}{dt}+\frac{d}{dy}(v\xi-u\eta)-\frac{d}{dz}(u\zeta-w\xi)=\tfrac{1}{2}\left(\frac{dZ}{dy}-\frac{dY}{dz}\right) \quad . \quad . \quad (7).$$

Now, considering the liquid to be incompressible, we may apply the special equation of continuity, (3$a$) of article 441, viz.

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

But using (1), which defines the rotations, we find by differentiation that

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Then, by aid of (8) and (9), we may transform (7) into the following :—

$$\frac{d\xi}{dt}+u\frac{d\xi}{dx}+v\frac{d\xi}{dy}+w\frac{d\xi}{dz}-\xi\frac{du}{dx}-\eta\frac{du}{dy}-\zeta\frac{du}{dz}=\tfrac{1}{2}\left(\frac{dZ}{dy}-\frac{dY}{dz}\right) \quad . \quad . \quad . \quad (10).$$

We have hitherto used $\xi$, $\eta$, and $\zeta$ to apply to whatever portions of liquid occupied a given position at a specified instant. Let us now change to some particular portion of liquid which at time $t$ is at $(x, y, z)$

and is followed in imagination in its motion. Then, using as before $D/Dt$ as the symbol of particle differentiation, we have

$$\frac{D\xi}{Dt}=\frac{d\xi}{dt}+u\frac{d\xi}{dx}+v\frac{d\xi}{dy}+w\frac{d\xi}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

and similar equations for $\eta$ and $\zeta$ just as we had for $u$, $v$, and $w$. See (6) of article 442.

Hence, using (11) in (10), it becomes the first of the following set, the others being written by symmetry :—

$$\left.\begin{aligned}\frac{D\xi}{Dt}&=\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz}+\tfrac{1}{2}\left(\frac{dZ}{dy}-\frac{dY}{dz}\right)\\ \frac{D\eta}{Dt}&=\xi\frac{dv}{dx}+\eta\frac{dv}{dy}+\zeta\frac{dv}{dz}+\tfrac{1}{2}\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)\\ \frac{D\zeta}{Dt}&=\xi\frac{dw}{dx}+\eta\frac{dw}{dy}+\zeta\frac{dw}{dz}+\tfrac{1}{2}\left(\frac{dY}{dx}-\frac{dX}{dy}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad (12).$$

Now let the forces be derivable from a potential $V$ so that

$$X=-\frac{dV}{dx},\ Y=-\frac{dV}{dy},\ Z=-\frac{dV}{dz} \quad . \quad . \quad . \quad . \quad . \quad (13),$$

then the component torques due to these forces are all zero, as seen by the terms in round brackets at the right in (12).

If, in addition, we have also at any instant

$$\xi=\eta=\zeta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

then the other angular accelerative terms on the right of (12) vanish.

Hence, for the case under consideration, *no rotation being present in our ideal liquid, it appears that none can be acquired.*[1]

It should be noted that viscosity is supposed quite absent here. Its presence would change this conclusion.

**451. Coaxial Circular Currents.**—Let us now consider the case of the flow of an incompressible liquid in coaxial circles round the axis of $z$, which is taken vertically upwards. Within the cylinder of radius $r_0$, let the value of the angular velocity of the elements be given by $\zeta=\zeta_0$, a constant. But, outside this cylinder of radius $r_0$, let $\zeta=0$. It is required to find the velocities of the particles and the velocity potential, if any.

If $\omega$ is the angular velocity about OZ at $r$, the radius vector, we have for the velocity and force components

$$\left.\begin{aligned}u&=-\omega y,\ v=\omega x,\ w=0\\ X&=0,\quad Y=0,\quad Z=-g\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (15).$$

Then, remembering that $r^2=x^2+y^2$ and $dr/dy=y/r$, we find by partial differentiation

$$\left.\begin{aligned}\frac{du}{dy}&=-\omega-y\frac{d\omega}{dr}\cdot\frac{dr}{dy}=-\omega-\frac{y^2}{r}\cdot\frac{d\omega}{dr}\\ \text{and}\quad \frac{dv}{dx}&=\omega+x\frac{d\omega}{dr}\cdot\frac{dr}{dx}=\omega+\frac{x^2}{r}\cdot\frac{d\omega}{dr}\end{aligned}\right\} \quad . \quad . \quad (16).$$

[1] The proof in the text seems the easiest for elementary students. Those desiring a fuller investigation (with a better claim to rigour) should refer to modern classical treatises.

Whence, by definition, we have

$$\zeta=\tfrac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right)=\omega+\frac{r}{2}\frac{d\omega}{dr} \quad . \quad . \quad . \quad . \quad (17).$$

Thus, within the cylinder $r=r_0$ we have

$$\omega-\zeta_0+\frac{rd\omega}{2dr}=0,$$

or

$$\int\frac{d\omega}{\omega-\zeta_0}+\int\frac{2dr}{r}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

And, on integrating,

$$\log(\omega-\zeta_0)+\log r^2=\log A,$$

or

$$\omega=\zeta_0+\frac{A}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (19),$$

where $A$ is the integration constant.

But, to avoid an infinite value of $\omega$ at the axis where $r=0$, we see that the integration constant $A$ must be zero.

Hence, for the whole interior of the cylinder of radius $r_0$, we have

$$\omega=\zeta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

which means that this rotates as a *rigid solid* with angular velocity $\zeta_0$, and that each part of it is therefore rotating at the same angular velocity as specified at the outset.

We now deal with the outer part, for which the rotation of the individual parts is zero. Then, from (17), we have

$$\omega+\frac{r}{2}\frac{d\omega}{dr}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

Whence, proceeding as before, we find

$$\omega=\frac{B}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

To avoid a discontinuity of velocities, we will make the $\omega$'s for the inner cylinder and beyond it agree for the value $r=r_0$. Thus

$$\zeta_0=\frac{B}{r_0^2}, \text{ or } B=\zeta_0 r_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

Accordingly the outer linear velocities in the circles are given by

$$\omega r=\frac{\zeta_0 r_0^2}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

And this expresses the *grading* of the velocities of flow with radius, which correspond to *the absence of rotation of the parts*. We see that in this region the angular momentum about the axis of $z$ per unit mass is constant, for it is given by

$$\omega r^2=\zeta_0 r_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Let us now find the velocity potential $\phi$ of this outer region. We evidently have from (24)

$$\frac{d\phi}{dr}=0 \text{ and } -\frac{d\phi}{rd\theta}=\omega r=\frac{\zeta_0 r_0^2}{r} \quad . \quad . \quad . \quad (26).$$

Thus, by the first of these $\phi$ does not contain $r$, and by the second we have

$$\phi=-\zeta_0 r_0^2\theta+C=C-\zeta_0 r_0^2\tan^{-1}(y/x) \quad (27).$$

For the inner cylinder of radius $r_0$ there is no velocity potential, since the condition for it, that $\xi$, $\eta$, $\zeta$ should all vanish, is not fulfilled.

Or, we may note direct from (15) that no velocity potential can be assigned to this motion.

The free surfaces of a liquid rotating like this under gravity are easily found. For the inner cylinder it is, of course, the paraboloid we have dealt with before in (30) of article 447, viz. from $r=0$ to $r=r_0$.

$$2gz=\zeta_0^2(r^2-r_0^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

the origin being now at the level of the liquid at $r=r_0$.

While for the outer part, using (24) for $\omega r$, whose square is proportional to the change of level, we find the hyperboloid, for $r>r_0$,

$$2gz=\zeta_0^2(r_0^2-r_0^4/r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

Thus, the depression of the centre below the surface for $r=\infty$ is expressed by

$$c=\zeta_0^2 r_0^2/g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

the level at $r=r_0$ being equidistant from the levels at the centre and at infinite radius.

**452. Steady Flow past Cylinder.**—Let us now consider a plane problem of the steady flow of an incompressible liquid parallel to the circular base of a right cylinder of infinite length, which forms the obstacle in the path of the stream, whose velocity far away from this disturbance is $u_0$, parallel to the $x$ axis, that of $z$ being the axis of the cylinder of radius $c$.

Then everywhere we have $w=0$, and far away from the cylinder $v=0$ also, but near the cylinder $v$ is finite, and both it and $u$ are variables, being functions of $r$, where $r^2=x^2+y^2$.

Since the liquid is incompressible the simple form of the equation of continuity applies, and, using the velocity potential $\phi$, we may put (3$a$) of article 441 in the form

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But for our case, since $w=0$, this reduces to

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

This equation, together with the condition as to the undisturbed velocity, is satisfied by

$$\phi=\frac{Ax}{r^2}-u_0x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

as may be seen by differentiation, remembering that $dr/dx=x/r$ and $dr/dy=y/r$.

We have yet to introduce the boundary condition of no normal velocity at the surface of the cylinder. This gives

$$\left(\frac{d\phi}{dr}\right)_{r=c}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Let us write $\cos\alpha = x/r$, then (3) becomes

$$\phi = \frac{A}{r}\cos\alpha - ru_0\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

And, by use of (4), we see that the constant $A$ is given by

$$A = -u_0c^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Hence, putting this in (3), we have

$$\phi = -u_0x(1 + c^2/r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Thus, differentiating for $u$ and $v$, we obtain

$$u = -\frac{d\phi}{dx} = u_0\left(1 + \frac{c^2}{r^2}\right) - \frac{2u_0c^2x^2}{r^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

and

$$v = -\frac{d\phi}{dy} = -\frac{2u_0c^2xy}{r^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Hence, for the resultant velocity $q$ *at the surface* of the cylinder, we have

$$q^2 = u_c^2 + v_c^2 = \left(\frac{2u_0y^2}{c^2}\right)^2 + \left(\frac{-2u_0xy}{c^2}\right)^2 = \frac{4u_0^2y^2}{c^2} \quad . \quad (10).$$

Then, introducing this value of $q$ in equation (21) of article 444, we find for the normal pressure on the cylindrical surface

$$p = \rho_0\left(E - gz - \frac{2u_0y^2}{c^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

The pressure is accordingly just as great on the hinder part of the cylinder, where the liquid is flowing away from it, as on the fore part, where the liquid is meeting it. Hence the cylinder has no resultant force from this *ideal* liquid flowing past it. Or, if the liquid is conceived as at rest at all parts far from the cylinder, which is moving through it at speed $u_0$, then the cylinder would, under the ideal conditions assumed, experience no resistance from the liquid.

The differences between this state of things and those obtaining in any actual experiment are due to the presence of viscosity in all actual liquids and of friction between the solid and the liquid, called *skin friction*. Further, one cannot in an experiment obtain the ideal geometric relations here supposed.

**453. Water Waves.**—Of the many kinds of waves possible in liquids, theory recognises three chief classes:—

(1) *Long waves*, or *tidal* waves, in which the motions of the particles are chiefly horizontal, and are equally great on the surface and below, where the bottom is level.

(2) *Ripples*, or very small disturbances on the surface, in which the restoring forces called into play are due to the surface tension, the liquid skin tending to flatten itself and assume the form of minimum area.

(3) The commonest class of all, and those most noticeable, which are variously called *oscillatory* waves, *surface* waves, and *gravity* waves. Their oscillatory character they have in common with the ripples, from which, however, they are distinguished by the term gravity, which refers to the nature of their restoring forces. The term surface dis-

tinguishes them from the tidal waves, in that, unlike the tidal waves, these are produced by surface disturbances and are confined to a region near the surface, the amplitude of the disturbance diminishing rapidly as we descend.

In the present article we now confine attention to this commonest class of water waves, which we shall suppose to be propagated horizontally along the axis of $x$, that of $z$ being taken vertically upwards from the level bottom of the vessel. We have accordingly no motion parallel to the $y$ axis, hence the equation of continuity reduces to the two terms

$$\left.\begin{aligned} \frac{du}{dx}+\frac{dw}{dz}&=0 \\ \text{or}\qquad \frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dz^2}&=0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1).$$

We next suppose the wave to be of the simplest harmonic form and test the legitimacy of that assumption. Thus let

$$\phi=F\cos k(x-at) \quad \ldots\ldots\ldots \quad (2),$$

where $F$ is a function of $z$ only. Then, differentiating (2) and putting in (1), we obtain

$$\frac{d^2\phi}{dx^2}=-k^2F\cos k(x-at),$$

$$\frac{d^2\phi}{dz^2}=\frac{d^2F}{dz^2}\cos k(x-at),$$

and
$$\frac{d^2F}{dz^2}=k^2F \quad \ldots\ldots\ldots \quad (3).$$

Whence
$$F=Ae^{kz}+Be^{-kz} \quad \ldots\ldots\ldots \quad (4).$$

Now for the bottom of the liquid, where $z=0$, we must have $w=0=d\phi/dz$. But, by (2) and (4),

$$\frac{d\phi}{dz}=k(Ae^{kz}-Be^{-kz})\cos k(x-at) \quad \ldots\ldots \quad (5).$$

Hence for $z=0$ one of two alternatives must be chosen. Thus (i) we may write $B=A$ in (5) and (4), which reduces (2) to the form

$$\phi=A(e^{kz}+e^{-kz})\cos k(x-at) \quad \ldots\ldots \quad (6).$$

Or (ii) if the depth is great we might write $B=0$ and put $A$ small, so that $A$ into the factor $e^{kz}$ is considerable at the surface, although practically zero long before the bottom is reached.

To obtain $a$, the velocity of propagation of the waves, we must use the equations of motion (7) of article 442. In these, if we suppose the amplitudes we deal with are small, we may neglect the products $udu/dx$, etc., which reduces the left side of each equation to its first term.

Hence, introducing the velocity potential, these become

$$-\frac{d}{dx}\left(\frac{d\phi}{dt}\right)=-\frac{dV}{dx}-\frac{1}{\rho}\frac{dp}{dx},$$

and two similar ones for $y$ and $z$. Hence multiplying them by

$dx$, $dy$, $dz$ respectively, adding, remembering the differentiations are partial, and integrating, we obtain

$$-\frac{d\phi}{dt}+gz+\frac{p}{\rho_0}=C \quad \ldots\ldots\ldots \quad (7),$$

$gz$ being written for $V$, since it is supposed due to gravity only.

Further, since we suppose the amplitudes to be small, we may consider the pressure everywhere independent of the time. Thus differentiating (7), we find

$$-\frac{d^2\phi}{dt^2}+g\frac{dz}{dt}=0 \quad \ldots\ldots\ldots \quad (8).$$

But

$$\frac{dz}{dt}=w=-\frac{d\phi}{dz} \quad \ldots\ldots\ldots \quad (9).$$

Hence (8) becomes

$$\frac{d^2\phi}{dt^2}+g\frac{d\phi}{dz}=0 \quad \ldots\ldots\ldots \quad (10).$$

We have to obtain these terms from (6) and substitute them in (10). Thus

$$\frac{d^2\phi}{dt^2}=-k^2a^2A(e^{kz}+e^{-kz})\cos k(x-at) \quad \ldots \quad (11),$$

and

$$g\frac{d\phi}{dz}=gkA(e^{kz}-e^{-kz})\cos k(x-at). \quad \ldots \quad (12).$$

Whence, equating the sum to zero, we have

$$ka^2(e^{kz}+e^{-kz})=g(e^{kz}-e^{-kz}) \quad \ldots\ldots\ldots \quad (13).$$

Thus to obtain the velocity of propagation of the waves at any height above the bottom of the liquid, we put the corresponding value of $z=h$ say in (13) and find

$$a^2=\frac{g}{k}\cdot\frac{e^{kh}-e^{-kh}}{e^{kh}+e^{-kh}}=\frac{g}{k}\tanh(kh). \quad \ldots \quad (14).$$

We may now fitly introduce the wave length $\lambda$, and so transform (14) by aid of the relation derived from (6), etc., viz.

$$k\lambda=2\pi, \text{ or } k=2\pi/\lambda \quad \ldots\ldots\ldots \quad (15).$$

Thus (14) may be written

$$a^2=\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi h}{\lambda}\right). \quad \ldots\ldots\ldots \quad (16).$$

If $h$ is very great compared to $\lambda$, then the exponential fraction in (14), involving $e^{2\pi h/\lambda}$, reduces to unity, and we have

$$a^2=\frac{g}{k}=\frac{g\lambda}{2\pi} \text{ nearly} \quad \ldots\ldots\ldots \quad (17).$$

We thus see that the velocity of propagation in each case depends on the wave length, varying directly as its square root; thus there is in water waves an analogy to the phenomena of optical dispersion.

Further, again referring to the exponential fraction in (14), we see that for small values of $h$ a reduction in $h$ involves a reduction in $a$. That is, the waves advance more slowly in shallower water.

If the depth of the water is great, we may now take the alternative

to (6), viz. that B must be zero. We then write for the velocity potential

$$\phi = Ae^{kz}\cos k(x-at) \quad . \quad . \quad : \quad . \quad . \quad . \quad . \quad (18).$$

Thus for the velocities at $(x, z)$ we have

$$\left.\begin{aligned} u &= -\frac{d\phi}{dx} = kAe^{kz}\sin k(x-at) \\ \text{and} \quad v &= -\frac{d\phi}{dz} = -kAe^{kz}\cos k(x-at) \end{aligned}\right\} \quad . \quad . \quad . \quad (19).$$

Hence the path of the particles at $(x, z)$ is a circle, described with angular velocity $ak$ and of radius $r$, such that

$$\left.\begin{aligned} rak &= q = kAe^{kz}, \\ \text{or} \quad r &= \frac{A}{a}e^{kz} = \frac{A}{a}e^{2\pi z/\lambda} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Thus the radii of these circles diminish in geometrical progression as depth below the surface increases in arithmetical progression. And, at the depth of one wave length only, the ratio of diminution is $e^{2\pi}$, or 535 : 1 nearly. Thus, as Tait mentions, an Atlantic roller of 40 feet high from trough to crest (if such occur) and 300 feet long would produce a disturbance from the mean position of only half an inch or less at a depth of 300 feet.

For aerial waves, works on physics may be consulted, as, *e.g.*, the writer's *Text-Book on Sound* (Macmillan, 1908), Chapter IV., articles 119-122, 125-127, 130-139, 145-146, and 149-155.

**454. Steady Flow of Viscous Liquid through a Narrow Cylinder.**—We have hitherto usually ignored viscosity but will now in conclusion deal with one very simple case in which viscosity is paramount.

It is the problem of the steady flow under pressure and gravity (or the former only) of a viscous liquid through a narrow cylindrical tube. We suppose the velocity of the liquid to be zero at the wall of the tube and to increase to a maximum which is reached only at the centre. Thus, the liquid is sliding in concentric cylindrical layers, each inner one urging the outer one along, which in turn is retarded by the next outer layer, which moves slower than itself. The magnitude of this effect for a given liquid is expressed by the *coefficient of viscosity* $\eta$, which may be defined by

$$dA = \pm\eta\frac{du}{dy}dzdx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $dA$ is the force parallel to the $x$ axis on the elementary area $dzdx$, when the velocity $u$ in that direction changes as we pass along the axis of $y$.

To fix our ideas, let us consider the arrangement shown in Fig. 232, in which liquid, kept at a constant depth $c$ in the upper vessel, flows through the fine tube of radius $a$ and length $l$ set obliquely so that the difference of levels of its ends is $h$.

In the tube, consider an elementary cylindrical layer of liquid, of radii $r$ and $r+dr$, in steady flow, the velocity $u$ parallel to the axis

and down the slope being a function of $r$ only and the velocity extremely small. Then the motion is executed and maintained con-

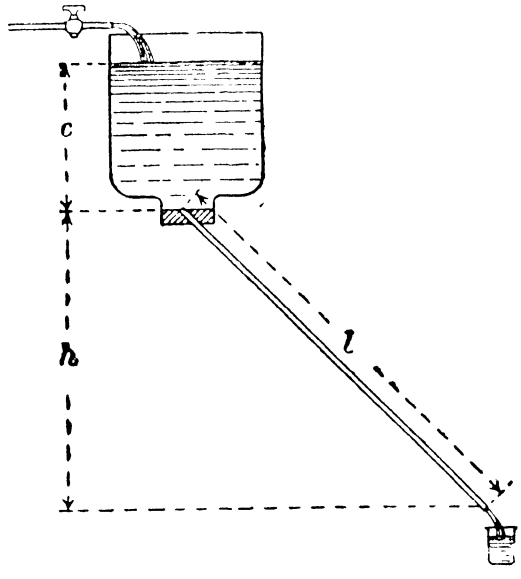



FIG. 232. STEADY VISCOUS FLOW.

stant by a zero force, which is the resultant of five separate forces falling into three groups, viz.

(1) That due to gravity, which is

$$\frac{gh}{l}\cdot 2\pi r dr l\rho = 2\pi\rho g h r dr \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

(2) That due to pressures at ends

$$\{(p_0+c\rho g)-p_0\}2\pi r dr = 2\pi\rho g c r dr \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where $p_0$ is the atmospheric pressure.

(3) That due to the tangential viscous forces on its inner and outer surfaces.

The former is $\left(-\eta\frac{du}{dr}2\pi r l\right)$, and acts downwards, that is, positively, $du/dr$ being negative. The latter is numerically greater by the differential of the former, whose variable part is $r\,du/dr$. Hence their difference, the former minus the latter, is given by

$$2\pi l\eta\frac{d}{dr}\left(r\frac{du}{dr}\right)dr \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

and acts upwards. Now, equating the sum of these expressions to zero, we find

$$2\pi\rho g(c+h)r dr + 2\pi l\eta d\left(r\frac{du}{dr}\right) = 0,$$

or

$$f r dr + \eta d\left(r\frac{du}{dr}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where

$$= \rho g(c+h)/l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Integrating (5) we obtain

$$\frac{fr^2}{2}+\eta r\frac{du}{dr}=C,$$

or

$$\frac{frdr}{2}+\eta du=\frac{C}{r}dr \quad \ldots\ldots\ldots \quad (7).$$

Hence, integrating again, we have

$$\frac{fr^2}{4}+\eta u=C\log r+D \quad \ldots\ldots\ldots \quad (8).$$

Now for $r=0$, if $u$ is not infinite, we have $C=0$. Again for $r=a$, $u=0$ by hypothesis, so $D=fa^2/4$.

Thus (8) becomes

$$u=f(a^2-r^2)/4\eta \quad \ldots\ldots\ldots \quad (9).$$

To find the volume $Q$ of liquid discharged per second, we have

$$dQ=2\pi urdr=\frac{\pi f(a^2-r^2)rdr}{2\eta}.$$

Hence integrating we obtain

$$Q=\int_0^a dQ=\frac{\pi fa^4}{8\eta} \quad \ldots\ldots\ldots \quad (10);$$

or, putting in from (6) the value of $f$, we find

$$Q=\frac{\pi\rho g(c+h)a^4}{8l\eta} \quad \ldots\ldots\ldots \quad (11).$$

These relations may be used for a practical determination of the viscosity coefficient $\eta$ at any given temperature. Of course, if the tube be vertical $h$ becomes $l$, while if it be placed horizontally $h$ vanishes.

In the present chapter much help has been derived from Professor G. Jäger's excellent brief treatise on *Theoretical Physics* (Sammlung Göschen, Leipzig, 1906).

## EXAMPLES—LXXXVIII.

1. Obtain expressions for the angular velocities of the elements of a fluid. If the angular velocity for any element is zero, does that element necessarily behave like a rigid solid? Take some actual example of motion, and indicate how the element behaves although rotation is absent.
2. Assuming the fundamental equations, derive expressions for the angular accelerations of the elements of a liquid. If at any instant, in an ideal fluid under forces derived from a potential, the angular velocities of the elements are all absent, what follows? Prove your assertion.
3. All the liquid in a certain region is in coaxial circular motion about a vertical axis, the portions inside a certain coaxial cylinder having linear velocities directly as the radii and the portions outside that cylinder having linear velocities inversely as the radii, there being no discontinuity of velocity anywhere.

   Show that the free surface is a paraboloid within the cylinder mentioned and a hyperboloid outside it.

Also show that the depression of the centre is double that of the boundary of the cylinder, both being reckoned from the free surface at infinity.

Prove that the liquid outside the cylinder has no rotation of its elements, but is not like a rigid body, while that within is like a rigid body and has rotation of its elements.

4. Obtain the flow of an ideal liquid past an infinitely long right circular cylinder in planes perpendicular to its axis. Plot curves showing the stream lines, and mark the speeds at several points on one line.

5. Derive the differential equation of motion for gravity waves, and solve it, showing that the motion of the particles is in circles which diminish as we descend.

6. Obtain an expression for the velocity of propagation of gravity waves, and point out how it varies with depth and wave length.

7. Derive a general formula for the steady flow of a viscous liquid through a capillary tube, and adapt it to the cases where this tube is (i) horizontal and (ii) vertical.

8. For very small vibratory disturbances of a compressible fluid of negligible weight obtain the equations of motion in the form

$$\frac{d^2s}{dt^2} = a^2\left(\frac{d^2s}{dx^2} + \frac{d^2s}{dy^2} + \frac{d^2s}{dz^2}\right),$$

where $s$ is defined by $\rho = \rho_0(1+s)$.

# PART VI.—ELASTICITY

## CHAPTER XXI

### STATICS OF ELASTIC SOLIDS

**455. Nature of Elastic Bodies.**—Elasticity may be regarded as that property of matter in virtue of which a body (i) resists forces tending to change its bulk or form or both; (ii) requires the continuance of those forces for the unimpaired maintenance of those changes; and (iii) recovers its original bulk and form when those forces are removed. Adopting the word *strain* to express the change in volume or form or both combined, and the word *stress* to express that combination of forces associated with the strain, we may say that the theory of elasticity is that branch of mechanics which discusses the mutual relations of stress and strain.

But before proceeding to this theory, even to that elementary degree compatible with the plan of this work, we must note the general significance of certain other terms applied to bodies in this connection, and also give more formal and quantitative definitions of stress and elasticity.

If, on the removal of the stress, the strain entirely disappears, the body is said to be *perfectly elastic*. If, however, on the removal of the stress some part of the strain remains, that part is called the *permanent set*, and the body in question is said to be *imperfectly elastic* for such stresses. The condition at which a marked permanent set occurs is called the *elastic limit* of a material. Very small stresses and strains are found to be practically proportional. In other words, they satisfy Hooke's law: *ut tensio sic vis*. The place at which this simple proportionality ceases to hold is called the *proportionality point* on the graph of stresses and strains. It is usually near the elastic limit. On the removal of a stress, if practically none of the strain disappears, the body is said to be *plastic*; and if only a small part of the strain disappears, it is said to be *ductile*.

If a stress, maintained constant, causes in a body a strain which increases continually with the time, that material is said to be *viscous*. When a continuous alteration in form is produced only by stresses *exceeding a certain value*, the material is called a *solid*, *however soft* it may be. But if the *very smallest stresses*, when continued *long enough*, cause a constantly increasing change of form, the material must be regarded as a *viscous fluid*, *however hard* it may be. (Maxwell's *Heat*, p. 303, London, 1894.)

'A body is called *homogeneous* when any two equal, similar parts of it, with corresponding lines parallel and turned towards the same parts, are undistinguishable from one another by any difference in quality' (Kelvin and Tait's *Natural Philosophy*, Part II. p. 216, Cambridge, 1890). If we push our scrutiny to the utmost conceivable limit perhaps no material would survive the test and be held as homogeneous. But, in the theory of elasticity, glass, continuous crystals, india-rubber, and fluids are usually considered as homogeneous.

'The substance of a homogeneous solid is called isotropic when a spherical portion of it, tested by any physical agency, exhibits no difference in quality however it is turned. Or, which amounts to the same, a cubical portion cut from any position in an isotropic body exhibits the same qualities relatively to each pair of parallel faces' (*ibid.* p. 217).

In what follows we shall restrict our attention to materials which are both homogeneous and isotropic.

**456. Stress and its Relation to Strain.**—When the terms stress and strain were introduced into the theory of elasticity in 1854 by Rankine, he used stress to denote the equilibrating set of forces which represents the elastic reaction of a strained body.

In the following year Kelvin adopted the term stress, but used it for the numerically equal but opposite set of forces, defining as follows:—

'*Definition.*—A stress is an equilibrating application of force to a body.

'*Corollary.*—The stress on any part of a body in equilibrium will thus signify the force which it experiences from the matter touching that part all round, whether entirely homogeneous with itself, or only so across a part of its bounding surface.

'*Definition.*—A strain is any definite alteration of form or dimensions experienced by a solid' (*Encyclopædia Britannica*, ninth edition, vol. vii. p. 819).

It is well to note here that Kelvin's use of the term stress brought the mechanics of elasticity into line with that previously developed for particles and rigid bodies. For just as

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

and $$\text{Torque} = \text{Moment of Inertia} \times \text{Angular Acceleration},$$

so $$\textit{Stress} = \textit{Elasticity} \times \textit{Strain}.$$

Or, in symbols, the three analogous relations may be written

$$\left.\begin{aligned} F &= Ma \\ G &= I\alpha \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

and $$S = Ks \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

or $$K = S/s \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

where $S$ denotes stress in force per unit area, $s$ strain as fractional change, and $K$ the elastic constant or *elasticity* concerned, it being understood that the stresses and strains are kept below the elastic limit. Thus (2) and (3) each give the symbolic quantitative definition of elasticity.

But although the meaning attached by Kelvin to the word stress

was probably the one whose need to be named was then most keenly felt, experience showed that a word was required for another closely allied conception, and for this also the same word stress became generally adopted. We must accordingly be prepared to meet the word in its more modern usage, which is indicated in the following passages or quasi-definitions.

When there is no tendency to relative motion between the parts of a body, so that if cuts are made in it there is neither opening, closing, nor sliding anywhere, the body is said to be in a state of ease. When, however, on that test being made, it is found that there is tendency to relative motion as shown by opening, closing, or sliding, the body is said to have been in *a state of stress*.

*Stress* is *the pair of forces* constituting the mutual interaction in or across a plane, or the entire action and reaction, of which force is one-half.

*Stress* is *force per unit area* in or across a plane.

Some writers reserve the word stress for the *interior* of a body and use *load and reactions* for the weight and supports applied externally. Whereas by Kelvin's definition the load and reactions would constitute the stress on the whole body, and the force per unit area in any direction at any place in it would be strictly a stress component on an element there.

**457.** We may further illustrate the different uses of the word stress by diagrams. Thus, in Fig. 233, eight forces are shown, being four pairs at the planes 1 to 4.

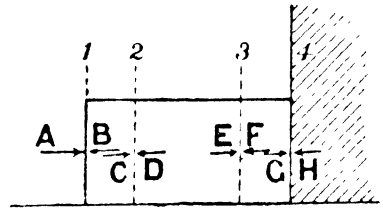



FIG. 233. COMPRESSIVE STRESSES.

Then on Kelvin's original definition and strict usage A and H constitute the stress on the whole body included between the planes 1 and 4, A and D the stress on the part 1 to 2, and so forth. On the more modern method, A and H would be called the load and reaction, C and D, *or either alone*, the stress at the plane 2, and so on. In either case all the stresses are called compressive, because they are normal and tend to compress or crush the substance. Also, in either usage the value of the stress is the quotient force per unit area.

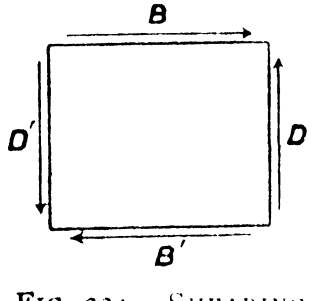



FIG. 234. SHEARING STRESS.

Again, in Fig. 234 is shown a cube about to be subjected to a shearing stress, *i.e.* one which produces the strain called a simple shear.

Then, in the modern usage, any one of the four tangential forces $B$, $B'$, $D$, or $D'$, reckoned per unit area, would be called a shearing stress, because it is a tangential force or a force in the plane and not normal to it or inclined. On Kelvin's definition the whole set of four forces is required to constitute an 'equilibrating application

of force'; and so nothing less is, by that definition, a shearing stress, the separate forces being stress components merely.

It should be noted also that, in this respect, Kelvin's definition is strictly logical. For any one force only, say $B$, would produce linear acceleration simply. A pair of forces, on opposite faces, as $B$ and $B'$, constitute a couple, and would accordingly, by themselves, produce angular acceleration. There is no necessary production of strain until we have completed the set of forces $B$, $B'$, $D$, $D'$, which, on a *rigid* body, would be an equilibrating set, the strain then follows if the body is elastic.

But however the word stress is used qualitatively to apply to one force, two, or four, both usages are in accord quantitatively, as the various forces in question are equal and any one is the gauge of the rest. We may thus without confusion use the term stress in the various recognised senses, as found convenient, generally leaving the context to show the exact meaning intended in each case.

**458. General Homogeneous Stress and its Components.**—Let us now consider a general application of forces to a body, simplify it till it forms a homogeneous equilibrating system or stress, and then specify that stress by its rectangular components. Further, let us take as our element which is the subject of this stress a cube, which may be the whole of the body or may be in the interior of a larger body from which it is separated in imagination only.

Whatever the forces applied to the faces may be, we can denote them by their rectangular components, which are respectively normal to the face and parallel to its edges.

We have thus three possible components for each face, and therefore eighteen in all for the six faces of the cube. But to reduce these to an equilibrating system we first make the normal components on opposite faces numerically equal. This reduces the total number of differing values from eighteen to fifteen.

We next make the parallel tangential components on opposite faces equal and opposite so as to form a couple. This reduces the fifteen different components to nine, as there are two tangential forces along each of the three pairs of opposite faces.

But there is a still further reduction of these nine different components to six; for, referring to Fig. 234, not only must $B'=B$ and $D'=D$ as already just provided for, but also $B$ must equal $D$. For $B=B'=D=D'$ measures the simple shearing stress corresponding to a simple shear in the plane of that diagram.

Hence the most general application of forces to the faces of a cube, when reduced to an equilibrating system or stress, become a pair of equal and opposite normal forces on a pair of opposite faces, a set of four equal tangential forces parallel to the edges of these faces and applied on the other four faces as in Fig. 234; and then a similar set for each of the other two pairs of opposite faces. Thus to specify a stress we require only six different components in all, three normal and three tangential.

The subject may also be treated analytically. Thus, take the stage at which the parallel forces on opposite faces have been made equal, so that we have three forces to specify on each of three adjacent faces, or nine in all. Then we may denote them by the following scheme of symbols:—

$$\left.\begin{matrix} X_x,\ X_y,\ X_z \\ Y_x,\ Y_y,\ Y_z \\ Z_x,\ Z_y,\ Z_z \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots \quad (4),$$

in which the large letters denote the direction of the forces and the subscripts give the normals to the faces on which those forces act.

Following now a notation analogous to that used for strains in Table v. of article 179, we may express these nine components by the first letters of the alphabet:—

$$\left.\begin{matrix} A,\ B,\ C \\ D,\ E,\ F \\ G,\ H,\ I \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots \quad (5).$$

Comparing these two schemes, we see that the normal components are denoted by $A$, $E$, $I$, the first three vowels, corresponding to the three elongations, which were previously denoted by $a$, $e$, $i$.

We also see that our reduction from the nine to the final six components will consist in the equations

$$Y_x = X_y,\ Z_x = X_z,\ Z_y = Y_z \quad \ldots\ldots \quad (6),$$

or the equivalent set

$$D = B,\ G = C,\ H = F \quad \ldots\ldots \quad (7).$$

Hence our set of six components which specify the stress are

$$\left.\begin{matrix} A,\ B,\ C \\ B,\ E,\ F \\ C,\ F,\ I \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots \quad (8),$$

the full nine being retained to show the equalities. These correspond to the general pure strain shown in Table v. of article 179.

The relation of these six components of a general stress will perhaps be best understood by reference to a diagram such as that in Fig. 235. One force of each pair of equal ones is shown on the near face of the cube, the equal and opposite one on the hidden face being understood. All the forces must be imagined distributed equally over the face to which it is applied, the capital letters must be taken as denoting the values of the quotients *force per unit area*, and the positive directions of the components on the cube are those indicated in the figure.

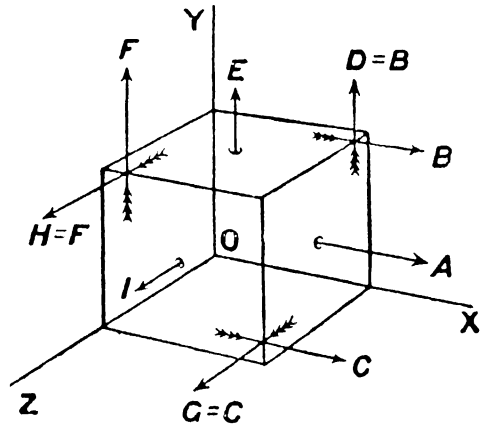



FIG. 235. SIX COMPONENTS OF GENERAL STRESS.

## EXAMPLES—LXXXIX.

1. Give an account of the nature of elastic bodies, carefully distinguishing between the various special terms used in this connection.
2. Explain carefully the various meanings which have been assigned to the word *stress*, indicating how we may still use the word quantitatively without fear of confusion.
   Illustrate your answer by concrete examples and sketches.
3. Consider the most general set of forces acting on the faces of a cube, and show how, to represent the most general homogeneous stress, the eighteen components reduce to six.
4. With the notation of article 458, specify a uniform hydrostatic stress and two shearing stresses, showing by a diagram how the forces act. Also prove that six components are sufficient to specify the most general homogeneous stress.
   If the application of force were such as to require more than six quantities to specify it, what would happen to the body?

**459. Stress across any Plane.** Let the stress to which a material is subjected be specified by its six components $A, E, I, F, C, B$, referred to the co-ordinate planes, and take any plane cutting the axes at U, V, W, the direction cosines of its normal being $l, m, n$. It is required to find, on UVW, the stress $N$ with direction cosines $\lambda$, $\mu$, $\nu$ and components $P$, $Q$, and $R$ parallel to the axes (see Fig. 236).

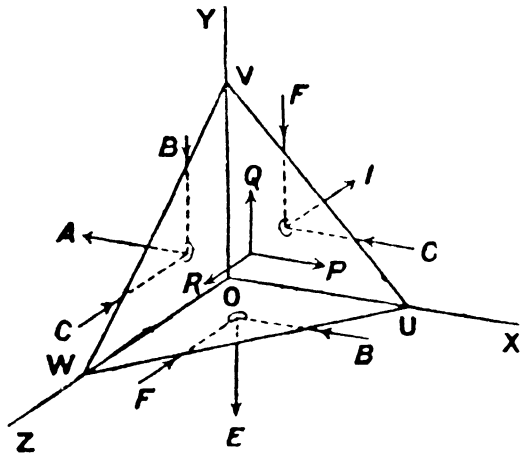



FIG. 236. STRESS AT GIVEN PLANE.

Let the area of the plane triangle UVW be $\Delta$, then those of its projections OVW, OWU, and OUV will be respectively $l\Delta$, $m\Delta$, and $n\Delta$. Then, using the conditions of equilibrium for the portion of material OUVW, we have by resolving parallel to the axes

$$P\Delta = Al\Delta + Bm\Delta + Cn\Delta,$$

and two similar equations. Thus, cancelling out the $\Delta$'s all through, we find for the components and resultant

$$\left.\begin{aligned} P &= Al + Bm + Cn = N\lambda \\ Q &= Bl + Em + Fn = N\mu \\ R &= Cl + Fm + In = N\nu \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1),$$

also $$N^2 = P^2 + Q^2 + R^2$$

and $$\left.\begin{aligned} \frac{\lambda}{P} = \frac{\mu}{Q} = \frac{\nu}{R} = \frac{1}{N} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (2).$$

To embody this result in a geometrical form consider the ellipsoid

$$S = Ax^2 + Ey^2 + Iz^2 + 2(Fyz + Czx + Bxy) = K \quad \ldots \quad (3),$$

and take on its surface a point defined by the co-ordinates

$$x = lr,\ y = mr,\ z = nr \quad \ldots\ldots\ldots\ldots \quad (4),$$

then the line of length $r$ to this point from the origin has the direction of the normal to the plane UVW, whose stress is in question.

Now, differentiating (3), we find

$$\left.\begin{aligned}\frac{dS}{dx}&=2(Ax+By+Cz)\\ \frac{dS}{dy}&=2(Bx+Ey+Fz)\\ \frac{dS}{dz}&=2(Cx+Fy+Iz)\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(5).$$

Thus, remembering (4) and (1), we have

$$\frac{dS}{dx}:\frac{dS}{dy}:\frac{dS}{dz}=\lambda:\mu:\nu\quad\ldots\ldots\ldots\quad(6),^{1}$$

which shows that the perpendicular (of length $p$ say) from the origin to the tangent plane at $x$, $y$, $z$ has the same direction cosines as the resultant stress $N$ across the plane UVW, whose normal has the same direction as the radius vector $r$ to the point just named.

For, from (5) and (1) we may write

$$\left.\begin{aligned}Nr\lambda&=Ax+By+Cz\\ Nr\mu&=Bx+Ey+Fz\\ \text{and}\quad Nr\nu&=Cx+Fy+Iz\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(7).$$

Multiplying these equations by $x$, $y$, and $z$ respectively, adding, and using (3) and (6), we obtain

$$Nr(\lambda x+\mu y+\nu z)=K=Nrp\quad\ldots\ldots\ldots\quad(8),$$

or

$$N=K/pr\quad\ldots\ldots\ldots\quad(9).$$

Thus, 'For any fully specified state of stress in a solid, a quadric surface may always be determined, which shall represent the stress graphically in the following manner:—

'To find the direction and the amount per unit area of the force acting across any plane in the solid, draw a radius perpendicular to this plane from the centre of the quadric to its surface. The required force will be equal' (or *proportional, unless* $K=1$) 'to the reciprocal of the product of the length of this radius into the perpendicular from the centre to the tangent plane at the extremity of the radius, and will be perpendicular to this tangent plane.

'From this it follows that for any stress whatever there are three determinate planes at right angles to one another such that the force acting in the solid across each of them is precisely perpendicular to it. These planes are called the *principal* or *normal planes* of the stress; the forces upon them, per unit area, its *principal* or *normal tractions*; and the lines perpendicular to them its *principal* or *normal axes*, or simply its *axes*. The three principal semi-diameters of the quadric surface are equal (or proportional) to the reciprocals of the square roots of the principal tractions. If, however, in any case each of the three principal tractions is negative, it will be convenient to reckon

[1] The student not familiar with this may easily verify the corresponding relations for

them rather as *pressures*; the reciprocals of the square roots of which will be the semi-axes of a real stress ellipsoid representing the distribution of force in the manner explained above, with pressure substituted throughout for traction' (Kelvin and Tait's *Natural Philosophy*, Part II. pp. 207-208, Cambridge, 1890).

**460. Composition of Stresses.**—The composition of stresses may, of course, be effected by the algebraic addition of like components. Thus, if several stresses are defined by $A_1$, $E_1$, $I_1$, $F_1$, $C_1$, $B_1$, and the same with subscripts 2, 3, etc., the resultant stress has components

$$\left.\begin{aligned} \Sigma A &= A_1 + A_2 + \dots \\ \Sigma E &= E_1 + E_2 + \dots \\ \Sigma I &= I_1 + I_2 + \dots \\ \Sigma F &= F_1 + F_2 + \dots \\ \Sigma C &= C_1 + C_2 + \dots \\ \Sigma B &= B_1 + B_2 + \dots \end{aligned}\right\} \quad \dots\dots\dots (10).$$

If all the stresses to be compounded have their principal axes coincident, each one reduces to its three principal tractions $A$, $E$, and $I$, and the resultant is, of course, given by the three terms $\Sigma A$, $\Sigma E$, and $\Sigma I$.

Finally, if the strains to be compounded are each a simple traction, and all at right angles, we have them represented by $A$, 0, 0; 0, $E$, 0; and 0, 0, $I$, and their resultant by $A$, $E$, $I$.

**461. Two Aspects of Shearing Stress.**—In the chapter on strains we considered a simple shear at first (article 165) as a combination of an elongation and an equal rectangular contraction. We afterwards saw (articles 170-174) that it could be viewed as a sliding of planes at 45° to the above elongation and contraction, the amount of the shear, or slides per unit distance apart, being double the elongation.

In referring to shearing stresses in the present chapter we have taken first the tangential-force aspect of the matter which corresponds to the sliding of the planes, which in the strain was considered second. We now note the conversion of the other form into this, and shall find that here *no doubling nor halving occurs in passing from one aspect to the other.*

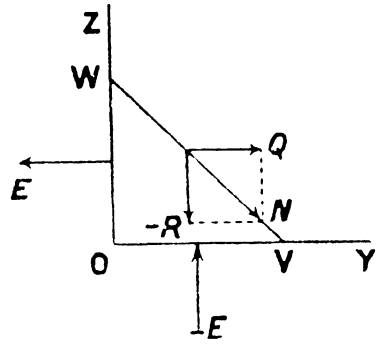



FIG. 237. TWO ASPECTS OF SHEARING STRESS.

Thus let the shearing stress be specified by its principal tractions 0, $E$, $-E$, and take a plane at angles of 45° with the axes of $y$ and $z$, so that its normal lies in their plane and between their positive directions, as shown in Fig. 237.

Then, using the results in (1) and (2) of article 459 for the stress on this inclined plane, we have

$$\left.\begin{aligned} P &= 0 \quad\quad = N\lambda \\ Q &= E/\sqrt{2} \quad = N\mu \\ R &= -E/\sqrt{2} = N\nu \end{aligned}\right\} \quad \dots\dots (11).$$

Whence $N=E,\ \lambda=0,\ \mu=1/\sqrt{2}=-\nu$ . . . (12).

Thus, showing that the stress on the plane inclined at 45° to the axes of traction and pressure has a tangential force of *equal value per unit area* and directed as shown in the figure.

This result can also be found easily by elementary considerations without the general formulae of article 459.

**462. Elasticities and their Relations.**—Let us now take symbols for the chief elasticities, find expressions for them, and establish relations between them.

Let the traction $P$ along the axis of $x$ produce in an isotropic material the strain $(a, -i, -i)$, or $(a, -\sigma a, -\sigma a)$ where $\sigma$ is called *Poisson's ratio.*

The elasticity involving change of size only is called the *volume elasticity* or *bulk modulus.* It is measured by the quotient *hydrostatic pressure* divided by *fractional diminution of volume*, or uniform normal tractions divided by fractional increase of volume. Thus denoting this elasticity by $k$, and using equations (2) of article (164) and (11) of 169, we find

$$k=\frac{P}{\delta}=\frac{P}{3d}=\frac{P}{3a(1-2\sigma)} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The elasticity involving change of shape only is usually called *rigidity*, and will be denoted by $n$. It is measured by the quotient of any one of the *tangential forces per unit area* of the shearing stress divided by the *amount* of the corresponding *shear.*

Hence, using the result of article 172 and equation (12) of 169, we obtain

$$n=\frac{P}{\chi}=\frac{P}{2e}=\frac{P}{2a(1+\sigma)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

From these two elasticities the behaviour of an isotropic material is calculable since from the corresponding strains any homogeneous strain may be built up as already seen in Chapter X.

But another so-called elasticity is in general use, namely, *Young's modulus*, which is the quotient *traction* divided by *fractional elongation*, the lateral contraction being left free to occur but not taken into account.

Denoting this constant by $q$, we may write, immediately from the definition,

$$q=\frac{P}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Let us now express $q$ in terms of the pure elasticities $k$ and $n$. Thus, putting (3) in (1) and (2), we find

$$\left.\begin{aligned} 1-2\sigma&=\frac{q}{3k}\\ \text{and}\qquad 2(1+\sigma)&=\frac{q}{n}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Whence, by addition, we have the well-known relation

$$q=\frac{9kn}{3k+n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

For a given wire or rod, the product, Young's modulus into area of cross section, is sometimes called the *modulus* for the rod.

Again, by division of the equations (4), we eliminate $q$, and find

$$\sigma=\frac{3k-2n}{6k+2n} \quad \ldots\ldots\ldots\ldots \quad (6),$$

thus expressing Poisson's ratio in terms of $k$ and $n$. This shows that when $n$ is negligibly small (as in some jellies) $\sigma$ approaches the limiting value $1/2$. If it exceeded this value, hydrostatic pressure would cause a dilatation, as seen by equation (1).

The other limit to the value of $\sigma$ is zero, which (as Kelvin and Tait point out) is practically reached by cork.

We may next take the case of the elasticity in which a simple elongation occurs under traction, lateral contractions being prevented by other tractions. But just as the lateral contractions were omitted in estimating the strain for Young's modulus, we may now omit reference to these lateral tractions in estimating the stress for this simple *elongational elasticity*, and measure it by the quotient *traction* $P$ divided by corresponding fractional *elongation* $e$. Thus, denoting it by $j$, and using equation (10) of article 169, we have

$$j=\frac{P}{e}=\frac{P(1-\sigma)}{a(1+\sigma)(1-2\sigma)}=\frac{q(1-\sigma)}{(1+\sigma)(1-2\sigma)} \quad (7).$$

We can throw this expression into another convenient form by using equations (6), (5), and (4). We thus find

$$\left.\begin{aligned} 1-\sigma&=\frac{3k+4n}{6k+2n}\\ 1+\sigma&=\frac{q}{2n}\\ 1-2\sigma&=\frac{3n}{3k+n}\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8).$$

Introducing these values in (7) we obtain

$$j=k+\tfrac{4}{3}n \quad \ldots\ldots\ldots\ldots \quad (9).$$

On referring again to equation (10) of article 169, we may write the lateral traction $Q$ in terms of $P$ by making them proportional to the corresponding axial strains as there shown.

Hence

$$\frac{Q}{a_1}=\frac{P}{a_2},\quad Q=P\frac{a_1}{a_2},$$

or

$$Q=P\frac{\sigma}{1-\sigma} \quad \ldots\ldots\ldots\ldots \quad (10).$$

Thus for $\sigma=0$, $Q=0$, but for a moderate value of $\sigma=1/4$ say, $Q=P/3$.

The chief results of this paragraph are collected in Table XVI. (For a larger table on a similar plan see page 125 of the writer's *Text-Book on Sound*, London, 1908.)

TABLE XVI. RELATIONS BETWEEN ELASTIC CONSTANTS.

| ELASTIC CONSTANTS. | | EXPRESSED IN TERMS OF | | |
|---|---|---|---|---|
| Names. | Symbols. | Stress and Strain. | $q$ and $\sigma$. | $k$ and $n$. |
| Volume Elasticity. | $k$ | $\frac{P}{\delta}$ | $\frac{q}{3-6\sigma}$ | $k$ |
| Rigidity. | $n$ | $\frac{P}{\chi}$ | $\frac{q}{2+2\sigma}$ | $n$ |
| Young's Modulus. | $q$ | $\frac{P}{a}$ | $q$ | $\frac{9kn}{3k+n}$ |
| Poisson's Ratio. | $\sigma$ | $\frac{-i}{a}$ | $\sigma$ | $\frac{3k-2n}{6k+2n}$ |
| Elongational Elasticity. | $j$ | $\frac{P}{e}$ | $\frac{q(1-\sigma)}{(1+\sigma)(1-2\sigma)}$ | $k+\frac{4}{3}n$ |

EXAMPLES—XC.

1. From the general components of a homogeneous stress find the direction and magnitude of the stress across any given plane in the substance.
2. State and prove the relation between the stress inside a solid, the components of the stress outside the body, and a certain ellipsoidal surface.
3. How may stresses be compounded? Using the symbols $A$, $E$, $I$ to denote the normal stresses, and $F$, $C$, $B$ for the tangential ones, which set refers to shearing stresses? Can a shearing stress be denoted by any combination from the other set; if so, how do the two shearing stresses differ from each other?
4. Define *volume elasticity*, *rigidity*, and *Young's modulus*, and obtain expressions for each.
5. Obtain expressions for Poisson's ratio and Young's modulus in terms of volume elasticity and rigidity.
6. Express the simple elongational elasticity in terms of volume elasticity and rigidity. How could you find Poisson's ratio from this elongational elasticity and Young's modulus?

**463. The Work of Stress and Strain.**—Take a unit cube and let the infinitesimal strain ($da$, $de$, $di$, $df$, $dc$, $db$) occur under the practically constant stress ($A$, $E$, $I$, $F$, $C$, $B$). Then the traction $A$ for the elongation $da$ does work $Ada$, and similar expressions hold for the other normal tractions $E$ and $I$.

Again, the tangential force $F$ with the shear $df$ does work $Fdf$. And this is so, whether we regard the shear as a progressive sliding parallel to $y$ of the $xy$ planes, or as a sliding parallel to $z$ of the $zx$ planes. The simultaneous occurrence of both slidings simply obviates shift or rotation of the whole, and does not introduce any additional shear or

work, for each slide involves the other as regards the strain itself. The other tangential forces $C$ and $B$ involve the corresponding expressions.

Thus the element of work on the unit cube is given by

$$dW_1 = Ada + Ede + Idi + Fdf + Cdc + Bdb \quad . \quad . \quad . \quad (1).$$

If the side of the cube is $s$ instead of unity, the displacements are affected by the factor $s$ and the corresponding forces by $s^2$. Thus the increment of work $W$ is given by

$$dW = s^3 dW_1 \text{ or } dW_1 = dW/s^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

showing that the expression $dW_1$ is the *increment of work per unit volume.*

Let us now consider a cube with unit edges when free from stress, and let it be acted upon by a general stress increasing from zero till it has the value $(A, E, I, F, C, B)$, the strain then being $(a, e, i, f, c, b)$. We can then write for the variable traction $A$ the product $ra$ where $r$ is a constant and $a$ the instantaneous value of the corresponding strain component. Thus the element of work for $A$ and $a$ would be given by

$$\int_0^a Ada = r\int_0^a ada = \tfrac{1}{2}ra^2 = \tfrac{1}{2}Aa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Hence for the unit cube we have

$$W_1 = \frac{W}{s^3} = \tfrac{1}{2}(Aa + Ee + Ii + Ff + Cc + Bb) \quad . \quad . \quad (4),$$

the capital letters in the brackets now denoting the *final* values of the stress components, their average values being only half.

We have hitherto confined our attention to cases of homogeneous strain. We now notice a few very simple problems beyond this limitation.

**464. Bent Bar.**—Let a bar, which is straight when in a state of ease, be slightly bent in the plane of the diagram, Fig. 238. It is required to determine the bending moment.

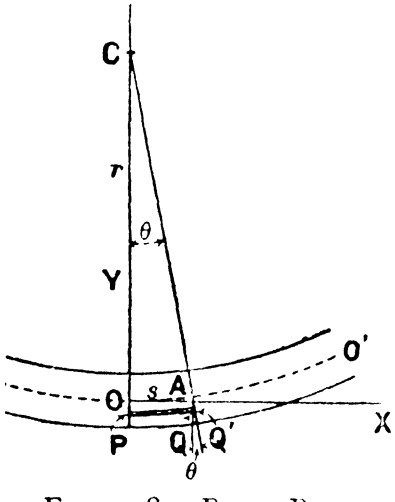



FIG. 238. BENT BAR.

Clearly the outer parts of the bar are slightly elongated and the inner ones compressed. We may accordingly assume that there is an intermediate part, as shown by the broken line OAO', and called the *neutral surface*, which is neither lengthened nor shortened by the bending.

Consider a small portion of the bar OA of length $s$, and let the radius of curvature there be OC$=r$, $s$ subtending at C the angle $\theta$. Thus a small filament PQ of ordinate $-y$, which had an initial length $s$, is lengthened by the bending to PQ'.

Hence $$\frac{QQ'}{AQ} = \theta = \frac{s}{r} = \frac{QQ'}{y}.$$

Thus the fractional elongation of the filament is

$$\frac{QQ'}{PQ}=\frac{QQ'}{s}=\frac{y}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let the filament PQ have cross-sectional area $dS$, and the bar be of material for which the Young's modulus is $q$. Then the force $dF$ on the filament satisfies the relation

$$q=\frac{dF}{dS}\cdot\frac{r}{y}, \text{ or } dF=q\frac{y}{r}dS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The moment of this force about O is therefore

$$ydF=\frac{q}{r}y^2dS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Hence the entire bending moment needed for the whole cross section of area $S$ is given by

$$N=\frac{q}{r}\int y^2dS=\frac{q}{r}K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where $K$ is the moment of inertia of the cross section about its intersection with the neutral surface.

But, as we are here only concerned with slight curvatures, we may write the approximate relation

$$\frac{1}{r}=\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

so that (4) then becomes

$$N=qK\frac{d^2y}{dx^2} \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

This gives the solution of the preliminary problem stated at the outset.

Now let us determine the form of a bar under the following simple stress:—The bar is clamped with a length $l$ projecting, and the free

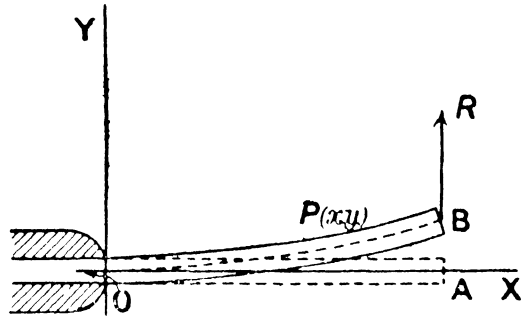



FIG. 239. BAR WITH TERMINAL FORCE.

end has a perpendicular force $R$ as shown in Fig. 239, the weight of the bar being negligible.

Then, at any point P of co-ordinates $(x, y)$, the bending being so small that $x$ remains practically unaltered, the bending moment is given by

$$N=R(l-x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Thus, equating (6) and (7), we have

$$R(l-x)=qK\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Hence, multiplying by $dx$ and integrating from the origin to P, we find

$$R\left(lx-\frac{x^2}{2}\right)=qK\frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

which gives the slope of the bar at any point $x$ and may be utilised in the experimental determination of $q$.

Again multiplying by $dx$ and integrating between O and P, we obtain

$$R\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)=qKy \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

which is the equation of the curve assumed by the bar under the stress in question.

At the free end, where $x=l$, let $y=b$, then equation (10) becomes

$$Rl^3=3qKb,$$

or

$$q=\frac{Rl^3}{3Kb} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

a formula which is often used in the laboratory exercise of finding $q$ for wood and metals. For symmetrical cross sections it is clear that the neutral surface is central.

Passing from the case of a clamped bar to that of one drooping $b$ at the middle under a central load $W$, the distance between the supports being $L$, we see that

$$l=L/2 \text{ and } R=W/2 \quad \ldots\ldots\ldots\ldots (12).$$

Thus writing these values in (11), it transforms into

$$q=\frac{WL^3}{48Kb} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

this arrangement being often more convenient than the former, to which (11) applies.

**465. Twisted Cylinder.**—Let us now consider a right circular cylinder of radius $a$ and length $l$ with axis along the axis of $z$, the base in the $xy$ plane held still while the opposite end is rotated through the angle $\theta$ in its own plane, $\theta/l$ being called the *twist* and denoted by $\tau$. It is required to determine the stress needed to maintain this twist.

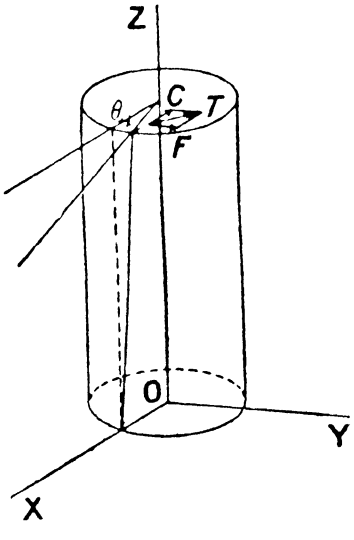



FIG. 240. TWISTED CYLINDER.

It is almost self-evident that the stress in question will consist of equal and opposite couples in the planes of the ends of the cylinder respectively. And further, it is obvious that the strain will consist of a progressive angular displacement of planes parallel to $xy$, increasing uniformly with $z$ from zero at the base to $\theta$ at the other end where $z=l$, none of these planes being themselves distorted.

A little reflection shows that the mutual interaction of any elements of the material meeting at these planes is the

undiminished handing on of a force parallel to the $xy$ plane and perpendicular to the radius, such interaction being imposed by the application at the ends of the cylinder of the forces constituting the stress, which is a pair of equal but opposite couples about the axis.

Hence, referring to Fig. 240, we may write the strain as follows:—

$$\left.\begin{aligned} x'-x &= -\tau yz \\ y'-y &= +\tau xz \\ z'-z &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1).$$

Whence, with our ordinary notation, we may write

$$\left.\begin{aligned} a=e=i&=0 \\ f=\tau x,\ c=-\tau y,\ b&=0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2).$$

Thus, for the stress components, we have

$$\left.\begin{aligned} A=E=I&=0 \\ F=n\tau x,\ C=-n\tau y,\ B&=0 \end{aligned}\right\} \quad \ldots\ldots \quad (3),$$

where $n$ is the rigidity of the material.

All these equations show that the strain with which we have here to do is not of the type called homogeneous, for the displacements now involve the *products* of the variables instead of being a linear function of them.

The components $F$ and $C$ give a resultant (refer, if necessary, to Fig. 235, at end of article 458) tangential to circles about the axis of $z$ and of value expressed by

$$T=n\tau r \quad \ldots\ldots\ldots\ldots \quad (4),$$

as shown at the top of the cylinder in Fig. 240.

Hence for any ring of radii $r$ and $r+dr$ the force would be $2\pi r(n\tau r)dr$, and its moment about the axis of $z$ would be $r$ times that value. Thus, for the whole area, the torque $G$ is given by

$$G=2\pi n\tau\int_0^a r^3 dr=\frac{\pi n\tau a^4}{2} \quad \ldots\ldots \quad (5),$$

or by

$$\frac{G}{\theta}=\frac{\pi n a^4}{2l} \quad \ldots\ldots\ldots\ldots \quad (6).$$

An alternative method of obtaining this relation is given in the writer's *Text-Book of Sound*, pp. 128-129, and is there followed by a description of some methods for the determination of the rigidity of a material.

If the rod is variable in radius we might write $r$ (as a function of $z$) instead of $a$. In this case the twist, angle per unit length, will also vary along the axis from point to point. We may accordingly write $d\theta/dz$ in place of $\tau$. Making these substitutions in (5), we find

$$d\theta=\frac{2G}{\pi n}\cdot\frac{dz}{r^4} \quad \ldots\ldots\ldots\ldots \quad (6a).$$

**466. Elongation of Helical Spring.**—To deal rigorously with the problems of a helical spring is beyond the scope of this work, but by the simple device used in Perry's *Applied Mechanics*, pp. 628-629 (London, 1898), we may calculate approximately the small elongations of a close helical spring of circular wire.

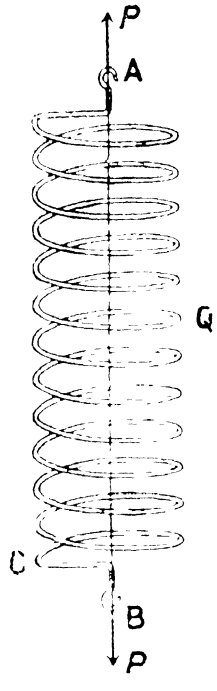



FIG. 241. ELONGATION OF HELICAL SPRING.

In Fig. 241 is shown the spring, made of wire of radius $r$ bent into coils of radius $R$ to the centre of the wire. It is fixed at A, and at the lower end B in the axis a vertical force $P$ is applied depressing that point by the amount $p$ say. Let the weight of the spring be negligible in comparison with $P$.

At any point Q in the spring take a section by a plane through the axis AB. Then, since the spring is a close one, this section is practically a normal section of the wire. Consider the equilibrium of the portion BQ. The force $P$ down the axis is balanced by some forces at the section at Q. These must be equivalent to a force $P$ vertically upwards and a torque of magnitude $PR$ acting on the upper end Q of the portion BQ of the spring. And since Q is any point, this force and torque remain constant throughout the purely helical portions of the spring.

The vertical force $P$ is a shear component of the stress, and its effect may be neglected in comparison with the torsion.

The torque $PR$ applied throughout the spring will produce the corresponding uniform twist as found in equation (6) of article 465. And this twist will give the depression $p$ of the point B proper to its final angle and the arm $BC = R$.

Thus, writing $L$ for the length of the wire bent into the helical form and $\theta$ for the total angle due to the twist, we have from (6)

$$\frac{PR}{\theta} = \frac{\pi n r^4}{2L} \quad \ldots\ldots\ldots \quad (7),$$

and

$$p = R\theta \quad \ldots\ldots\ldots \quad (8).$$

Whence, eliminating $\theta$ between these equations,

$$\frac{P}{p} = \frac{\pi n r^4}{2LR^2} = \frac{nr^4}{4NR^2} \quad \ldots\ldots \quad (9),$$

where $N$ is the number of turns in the helix, its slope being accounted negligible.

For rigorous theories of helical springs the interested student may consult Perry (*ibid.* pp. 633-638) and Gray, *Physics*, vol. i. pp. 600-609 (London, 1901).

## EXAMPLES—XCI.

1. Obtain expressions for the work per unit volume when an elastic body is strained from a state of ease to a specified finite strain.
   What does this general expression reduce to in the case of a wire stretched by an amount $f$, the final force being $F$?
2. Prove that to bend a bar to radius $r$ needs the application of bending moments of the value $qK/r$, where $q$ is the Young's modulus of the bar and $K$ the moment of inertia of the cross section about its intersection with the neutral plane.
3. Show that if a straight uniform bar is fixed at one end and very slightly bent by a perpendicular force at the other, its equation may be written
   $$y = Ax^2 - Bx^3,$$
   and state the values of the constants $A$ and $B$.
4. Find the inclinations at the ends of a beam loaded in the middle and supported at the ends, also the droop in the middle.
5. A beam of clear length $L$ between its supports has a load $W$ uniformly distributed along it; show that the droop in the middle is $5WL^3/384qK$, where $q$ is the Young's modulus of the material and $K$ the moment of inertia of the cross section of the beam about the line where the neutral surface cuts it.
6. Treat the problem of the pure torsion of a right circular cylinder of radius $a$ and length $l$, and show that the couple per radian of total angular displacement of one end relatively to the other is $\pi na^4/2l$, where $n$ is the rigidity of the material.
7. Assuming the result of the previous example, show that the torsional oscillations of a cylinder or other body, of moment of inertia $I$ about a central vertical axis, when suspended by a wire fixed at its top, may be expressed by $\tau = 2\pi\sqrt{2lI/\pi na^4}$, whence the rigidity of the wire is given by $n = 8\pi lI/\tau^2 a^4$.
8. Investigate the pure torsion of a frustum of a cone whose bases have the slightly differing radii $a$ and $b$, the length of the frustum being $l$, and show that the total angle $\theta$ through which one base is turned relatively to the other by the opposite couples of magnitude $G$ is given by
   $$\theta = \frac{2Gl}{3\pi n} \cdot \frac{a^2 + ab + b^2}{a^3 b^3}.$$
9. A filament of circular cross section fluctuates several times between the radii $a$ and $b$, all parts of it being conical without any intervening cylindrical portions; show that, if one end is fixed, the couple per radian displacement of the other end is expressed by
   $$\frac{G}{\theta} = \frac{\pi^2 n a^3 b^3}{2V},$$
   where $V$ is the volume of the filament.
10. Show that for a close helical spring of radius $R$, made of circular wire of length $L$ and of radius $r$, the relation between load and axial elongation is that between load and displacement when the load is applied to an arm of length $R$ on the end of the straight wire of length $L$ and radius $r$ of the same material as the spring.
11. A uniform beam is placed horizontally on two end supports, and the intervening portion then bears a uniformly distributed load. Taking the origin at centre of the beam, the axis of $x$ horizontally parallel to the original position of the beam before loading, and the axis of $y$ vertically upwards, show that the *shearing force*, the *bending moment*, the *slope* of the beam, and its *ordinate* are proportional respectively to the *first*, *second*, *third*, and *fourth* powers of the abscissa $x$.

## EXAMPLES—XCII. : CHIEFLY KINEMATICS.

1. 'Show how to connect linear, angular, and areal velocity, period, and revolutions per second in uniform motion in a circle; and calculate the angular velocity of the hands of a clock.

'Determine the revolutions per second of a bullet fired with velocity 2000 feet per second from a rifle, the grooves of the rifling making one turn in 10 inches.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1902, I. 5.)

2. 'Investigate the simple harmonic vibration of a weight suspended by a vertical spiral spring, and determine the length of the simple pendulum which will synchronise.

'Prove that a train on a perfectly straight railway, not curved to the radius of the earth, will oscillate if unresisted on each side of the position of equilibrium in the same period as a grazing satellite, about one-seventeenth of a day, or 1 h. 25 m.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1902, I. 9.)

3. 'Write down formulas connecting angular velocity, linear velocity in feet/second, revolutions/minute, and period of revolution in seconds.

'Prove that a bicycle geared up to $D$ inches requires $336\,S/D$ revolutions/minute of the pedals for a speed of $S$ miles/hour.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1903, I. 5.)

4. 'Explain the theory, units, and notation of the formulas

$$\text{(i) } Pt=\frac{Wv}{g}\text{; (ii) } Ps=\frac{Wv^2}{2g}\text{; (iii) } v=2\frac{s}{t}.$$

'Supposing that one in $m$ is the steepest incline a train can crawl up with uniform velocity, and one in $n$ is the steepest incline on which the brakes can hold the train, prove that the quickest run up an incline of one in $p$ from one station to stop at the next, a distance of $a$ feet, can be made in

$$\sqrt{\left\{\frac{\left(\frac{1}{m}+\frac{1}{n}\right)}{\left(\frac{1}{m}-\frac{1}{p}\right)\left(\frac{1}{n}+\frac{1}{p}\right)}\cdot\frac{2a}{g}\right\}}\text{ seconds.}$$

'Calculate for $m=50$, $n=5$, $p=100$, $a=5280$.'

(LOND. B.A AND B.SC., PASS, MIXED MATH., 1903, I. 6.)

5. 'Calculate the velocity at any point in a centrifugal railway and the thrust on the rails of a car, where its C. G. describes a vertical circle of radius $a$ feet, due to entering at the lowest point from an incline with a fall of $h$ feet vertical.

'Prove that $h$ should not fall short of $5a/2$, and the car should be strong enough to sustain the weight sixfold.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1903, I. 8.)

6. 'Show that an angular velocity $\omega$ about any axis is equivalent to an equal angular velocity about a parallel axis, together with a velocity of translation $\omega a$ in a direction at right angles to the plane containing the axes, the distance between which is $a$.

'$A$ and $C$ are given points in a plane, in which a bar $AB$ is turning about $A$ with angular velocity $\omega$. $AB$ is jointed at $B$ to a bar $BD$, which is constrained to pass through $C$. In any position of the linkwork, draw $AE$ to meet $BD$ at right angles in $E$; draw $EF$ parallel to $AB$ to meet $AC$ in $F$; let $v$ be the velocity of the point in $BD$ which is passing through $C$, and let $\omega'$ be the angular velocity of $BD$.

'Show that

$$v=\omega.AE,\ \omega'=\frac{AF}{AC}\omega.'$$

(LOND. B.SC., PASS, APPLIED MATH., 1905, II. 6.)

7. 'One end $B$ of a rod $AB$ describes a circle, while the other end $A$ is constrained to move along a line which passes through the centre $C$ of the circle. Prove that the ratio of the speeds of $A$ and $B$ is equal to the ratio of $AM$ to $AN$, where $BN$ is the perpendicular from $B$ upon $AC$ and $AM$ is that from $A$ upon $CB$.

'If $AB$ (or $AB$ produced) meets the circle again in $B'$, show that the rate of increase of $AB'$ is to the speed of $A$ as $2AC$ is to $AB$.'

(LOND. B.Sc., PASS, APPLIED MATH., 1906, II. 6.)

8. 'If a particle is projected from $O$, under the action of gravity, at an elevation $\alpha$, with a velocity due to a height $h$, show that the equation of the parabola described with reference to horizontal and vertical axes at $O$ is

$$y = mx - \frac{x^2}{4h}(1 + m^2),$$

where $m = \tan\alpha$. Find the greatest height attained.'

(LOND. B.A., PASS, APPLIED MATH., 1907, I. 6.)

9. 'A particle describes an ellipse under a centre of force in a focus which produces an acceleration $\mu/r^2$; prove the formula for the velocity

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right),$$

where $a$ is the major semi-axis of the ellipse.

'The maximum velocity of the earth in its orbit is 30,000 metres per second, and the minimum velocity is 29,200 metres per second; deduce the eccentricity of the earth's orbit.'

(LOND. B.A., PASS, APPLIED MATH., 1907, I. 9.)

10. 'Obtain a formula for the velocity at any point of an elliptic orbit described under a central force to one of the foci.

'The greatest and the least velocities in such an orbit being 110 ft. per sec. and 90 ft. per sec. respectively, and the periodic time being 20 min., calculate the eccentricity and (approximately) the length of the major axis.'

(LOND. B.Sc., PASS, APPLIED MATH., 1907, II. 5.)

11. 'A particle of unit mass moves in a straight line from rest under a constant accelerating force $g$ and a retarding force $kv^2$. Show that after describing a distance $x$ its velocity is given by

$$v^2 = \frac{g}{k}(1 - e^{-2kx}).'$$

(LOND. B.Sc., PASS, APPLIED MATH., 1908, II. 7.)

12. 'The arms $AC$, $CB$ of a wire bent at right angles slide upon two fixed circles in a plane. Show that the locus of the instantaneous centre in space is a circle, and that its locus in the body is a circle of double the radius of the space centrode.'

(LOND. B.Sc., PASS, APPLIED MATH., 1908, III. 2.)

13. 'A particle is projected at right angles to the line joining it to a centre of force attracting according to the law of the inverse square with a velocity $\sqrt{3}V/2$, $V$ being the velocity from infinity. Find the eccentricity of the orbit described, and show that the periodic time is $2\pi T$, $T$ being the time taken to describe the major axis of the orbit with uniform velocity $V$.'

(LOND. B.Sc., PASS, APPLIED MATH., 1908, III. 3.)

14. 'Explain the principle of relative velocities.

'Two points $P_1$, $P_2$ describe coplanar concentric circles of radii $a_1$ and $a_2$ with velocities $v_1$ and $v_2$ respectively; prove that the velocity of $P_1$ relative to $P_2$ is in the direction $P_1P_2$ when $P_1P_2$ subtends an angle $A$ at the centre of the circles such that

$$v_2(a_2 - a_1\cos A) + v_1(a_1 - a_2\cos A) = 0.'$$

(LOND. B.Sc., PASS, APPLIED MATH., 1910, II. 2.)

15. 'If $(r, \theta)$ be the polar co-ordinates of a point $P$ moving in any manner in a plane, show that the accelerations of $P$ along and perpendicular to the radius vector are

$$\ddot{r} - r\dot{\theta}^2, \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$

'A smooth horizontal tube $OA$ of length $a$ is movable about a vertical axis $OB$ through the extremity $O$. A particle placed at the extremity $A$ is suddenly projected towards $O$ with velocity $a\omega$, while at the same time the tube is made to revolve about $OB$ with angular velocity $\omega$. Show that the particle will have travelled half-way down the tube after a time $\frac{1}{\omega}\log_e 2$, and will not reach $O$ in any finite time.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, II. 3.)

16. 'Translate :—

'Les organes qui permettent de faire passer le mouvement, de la pièce menante à la pièce menée, se nomment *mécanismes*. Les mécanismes peuvent être classés en deux catégories ; la première catégorie comprend les systèmes articulés dans lesquels les angles varient, les distances des articulations restant constants. La deuxième catégorie comprend les systèmes constitués par une pièce $P$ animée d'un mouvement $M$ qui, par contact continu, imprime un mouvement $M'$ à une autre pièce $P'$.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, II. 10.)

17. 'Prove that the orbit described by a particle under a force which tends to a fixed centre, and varies inversely as the square of the distance from the centre, is a conic.

'Prove that, if $P$ is the particle, $S$ the centre of force, and $N$ the foot of the perpendicular from $P$ on the axis of the conic, the velocity of $N$ is greatest when it coincides with $S$.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, III. 1.)

18. 'Investigate the motion of a heavy particle allowed to fall from rest in a medium which offers a resistance proportional to the velocity.

'Prove that the velocity constantly increases, but tends to a finite limit.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, III. 4.)

19. 'Translate the following passage :—

'Das Planetensystem erleidet allerdings im Laufe der Jahrtausende bedeutende Umgestaltungen. Doch treffen sie diejenigen beiden Elemente nicht, welche man mit vollstem Recht als die hauptsächlichsten bezeichnen kann, nämlich die mittleren Entfernungen der Planeten von der Sonne und ihre Umlaufszeiten um dieselbe. Hat man die letzteren als Mittel von hunderten oder tausenden von beobachteten Umläufen bestimmt, so ist man sicher, dass dieses Mittel die unveränderliche Umlaufszeit genau darstellt.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, III. 10.)

20. 'Prove that, at any point of the path of a moving particle, the normal component of the acceleration is $v^2R$, where $v$ is the velocity and $R$ the curvature at the point. Deduce its expression in terms of $x$, $y$, and $t$.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 47.)

21. 'What is meant by the motion of a lamina in its own plane?

'Show that a lamina can be moved in its own plane from any one position to another by a rotation round a point in that plane. Point out any case that presents an apparent exception.

'The motion of a body at any instant can be represented by two angular velocities round parallel axes ; find a simpler mode of representing the motion.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 48.)

22. 'Show that two equal and opposite rotations, effected successively round two parallel axes $A$ and $B$, are equivalent to a single motion of translation.

'Illustrate your answer with reference to a carefully drawn diagram of the following case:—Suppose that $AB$ is a side of a square and that the square is made to turn in its own plane round $A$ through 60°, and then, in the opposite direction, round $B$ through an angle of 60°; also, show the direction of the translation.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 46.)

23. 'A motor starts from rest to go a distance $S$, and the acceleration of its velocity at time $t$ is $k(1-\frac{t}{\tau})$, $\tau$ being the time taken to acquire the maximum velocity, which after that time is maintained.

'If $T$ be the whole time for the distance, and $T_1$ that of the same motor with a flying start, prove that

$$\tau = 3(T - T_1),$$

$$k = \tfrac{2}{3} \cdot \frac{S}{T_1(T - T_1)}.'$$

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 47.)

24. 'Suppose that a square $ABCD$ undergoes a very small elongation parallel to the side $AD$ caused by a uniform tension $T$; also that it undergoes a very small compression parallel to the side $AB$ caused by a uniform pressure $T$. Show that the square is under a shear in the direction $AC$, and find its magnitude.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1909, 62.)

25. 'Explain what is meant by a rotation and by an axis of rotation. Consider a body which turns round an axis, and a straight line in the body which is in a plane at right angles to the axis of rotation. Show that, at the end of the motion, the line makes with its initial position an angle equal to that through which the body has been turned.

'$AB$, $AC$, $AP$ are three lines forming a solid angle at $A$, and $AP$ is such that a rod coinciding with $AB$ can be brought into the position $AC$ by a rotation round $AP$. Supposing that $AB$ and $AC$ are fixed, find the locus of $AP$. Find also the relation between the angle of rotation and the angle $BAC$.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1910, 48.)

## EXAMPLES—XCIII. · CHIEFLY PARTICLE KINETICS.

1. 'Prove that a jet of water of delivery $P$ lb. per second, and velocity $v$ feet per second, impinging on a plane pallet fixed perpendicular to its direction, will exert a thrust $Pv/g$ pounds.

'If such a series of pallets are mounted on the circumference of a wheel moving with velocity $u$, the horse-power given out will be

$$\frac{P}{550g}\left\{\frac{v^2}{4} - \left(u - \frac{v}{2}\right)^2\right\},$$

a maximum $Pv^2/2200g$ when $u = v/2$; and half the energy of the jet is then utilised by the wheel.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1902, I. 7.)

2. 'A train of 200 tons acquires a speed of 20 miles an hour from rest in a mile and a quarter. Find (in tons) the excess of tractive force over resistance, assumed constant.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1904, I. 2.)

3. 'A particle $m$ describes in succession the sides of a regular polygon, each with the same constant velocity $v$. Prove that in order that it may do this an impulse of amount $mv^2t/r$ must be applied to it at each corner, where $t$ is the time of describing a side and $r$ is the radius of the circle circumscribing the polygon.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1904, I. 5.)

4. 'A force whose magnitude is at each instant completely given acts always in a given right line; show how to draw a diagram representing

($a$) the work done by the force in a given interval;

($b$) the whole impulse of the force in this interval.

'A ball whose mass is 4 ounces strikes normally a fixed plane with a velocity of 50 f.s.; the coefficient of restitution is 2/5, and the time of contact is 1/256 second; taking $g=32$, what is the mean pressure between the ball and the plane in lbs. weight?

'What is, approximately, the *greatest* pressure?'

(LOND. B.SC., PASS, MIXED MATH., 1904, II. 3.)

5. 'At the top, $B$, of a rough plane inclined to the horizon tan $^{-1}3/4$ is fixed a pulley; a uniform chain having a mass of 5 lbs. per foot lies on the plane along the line of greatest slope and passes over the pulley at $B$. If the coefficient of friction is 1/2, and 20 feet of chain lie on the plane, find the amount of work done against friction when the free end of the chain is pulled until

($a$) 10 feet of chain have come over;

($b$) 20 " " " ."

(LOND. B.SC., PASS, APPLIED MATH., 1905, I. 7.)

6. 'A particle makes small abrasions on a horizontal straight line under the influence of a spring of negligible mass attached to a fixed point. Discuss the motion, and prove that for different particles the time of oscillation varies as the square root of the mass attached.'

(LOND. B.SC., PASS, APPLIED MATH., 1905, III. 1.)

7. 'Assuming that the attraction of the earth is directed towards the centre, and is constant at all points of its surface, prove that the deviation of the direction of gravity from the radius due to the rotation of the earth at a place in latitude $\lambda$ varies as $\sin 2\lambda$, and that its maximum value is 6′ approximately.'

(LOND. B.SC, PASS, APPLIED MATH., 1905, III. 2.)

8. 'Prove that the work done by an impulse on a particle in the direction of its line of motion is measured by the product of the impulse and the mean of the initial and final velocities of the particle.

'Hence find the loss of kinetic energy in the direct impact of two given inelastic spheres.'

(LOND. B.SC., PASS, APPLIED MATH., 1905, III. 3.)

9. 'Discuss the motion of a heavy particle making complete revolutions within a smooth circular tube which is fixed in a vertical plane.

'If the speed at the lowest point is $n$ times the speed at the highest point, prove that the pressure of the particle on the tube, when the particle is moving vertically, bears to the pressure at the lowest point the ratio of

$$2(n^2+1) : 5n^2 - 1.'$$

(LOND. B SC., PASS, APPLIED MATH., 1905, III. 4.)

10. 'A mass $M$ strikes a mass $m$ which is at rest with velocity $v$; prove that the velocity communicated to $m$ is always less than $2v$.

'A hammer whose head weighs 3 lbs. strikes a steel ball weighing 5 ozs. with a velocity of 50 f.s.; find the velocity communicated to the latter, assuming that there is no loss of energy in the impact.'

(LOND. B A., PASS, APPLIED MATH., 1906, I. 6.)

11. 'A number of equal particles of mass $m$ are connected by strings, each of length $a$, so as to form a regular polygon of $n$ sides, and the whole revolves about the centre, the velocity of each particle being $v$. Find the tension in each string.'

(Lond. B.A., Pass, Applied Math., 1906, I. 8.)

12. 'Define *simple harmonic motion*, and find the velocity in any given phase, in terms of the period and the amplitude.

'A mass of 10 lbs. is executing a S. H. motion of amplitude 3 ins., with a frequency of $2\frac{1}{2}$ complete vibrations per second. Find (in foot-lbs.) its maximum kinetic energy.'

(Lond. B.A., Pass, Applied Math., 1906, I. 9.)

13. 'Find in gravitation units the force which must act along the normal on a particle of weight $w$ which is moving in a curved path. In what *sense* along the normal must this force act?

'A uniform chain $AB$ of length $l$ and weight $w$ per unit length rests on a smooth horizontal table. If the chain revolves freely round $A$, which is fixed, with uniform angular velocity $\omega$, find the tension at a point distant $x$ from $A$.'

(Lond. B.Sc., Pass, Applied Math., 1906, III. 1.)

14. 'What is meant by a work diagram? A force acts in a fixed right line $OA$, its point of application being $P$; the magnitude of the force is directly proportional to $OP$, and has the value $F$ when $OP = a$; draw the diagram representing the work done by the force in displacing $P$ from the given position $B$ to the position $C$ along $OA$.

'A mass of 500 lbs. moves down 100 feet of a rough plane inclined at $\sin^{-1}0{\cdot}05$ to the horizon, frictional resistance being 15 lbs. weight. By a direct application of the principle of work and energy, find the velocity of the body when it reaches the foot of the incline.'

(Lond. B.A., Pass, Applied Math., 1907, I. 3.)

15. 'If a particle of weight $w$ is moving in a plane curved path whose radius of curvature at a given point is $\rho$, prove that there must act on the particle a force equal to $wv^2/g\rho$ along the normal towards the concave side of the path, $v$ being the velocity of the particle at the point.

'If a particle of weight $w$ is suspended from the roof of a railway carriage which is moving with a constant speed $v$, prove that in the position in which the particle is at rest relatively to the carriage the tension of the cord is

$$w(1 + v^4/g^2\rho^2)^{1/2}.'$$

(Lond. B.Sc., Pass, Applied Math., 1907, III. 1.)

16. 'A particle revolves in a circle about a spherical body attracting according to the law of the inverse square. Show that, if $T$ be the time of a complete revolution, the mass of the attracting body is given in astronomical units by $4\pi^2 r^3/T^2$, $r$ being the radius of the orbit described.

'Given that the time of revolution of the Earth about the Sun is approximately 13 times that of the Moon about the Earth, and the Sun's distance is 400 times the Moon's distance, compare the masses of the Sun and Earth.'

(Lond. B.Sc., Pass, Applied Math., 1908, II. 5.)

17. 'A particle acted upon by gravity is projected with velocity $v$, and at an inclination $\alpha$ in a uniform medium of which the resistance varies as the velocity; find the altitude of the particle at a given time, and show it is a maximum at time

$$-\frac{1}{k}\log\left(1 + \frac{k}{g}v\sin\alpha\right),$$

where $k$ is the resistance per unit mass experienced by the body when it is moving with unit velocity.'

(Board of Education, Theo. Mech., Solids, Stage 3, 1909, 45.)

18. 'A heavy particle hangs from a point O by a string of length $a$. It is projected horizontally with velocity $v$ such that

$$v^2 = (2 + \sqrt{3})ga.$$

'Show that the string becomes slack when it has described an angle

$$\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right),$$

and that the subsequent path of the particle passes through O.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 48.)

19. 'A heavy particle is placed at the highest point of a smooth vertical circular disc; it is connected by an inextensible string with an equally heavy particle which is at the extremity of a horizontal radius. If motion be allowed to ensue, prove that the upper particle will leave the disc when at an angular distance from the highest point given by the equation $2\cos\theta = \theta + 1$.

'At that instant find the tension of the string and the velocity of the system. Find also an approximate value of $\theta$.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1910, 41.)

20. 'Two particles, whose masses are given, are connected by an inextensible string, and are projected in any way, but so as to move in a vertical plane. Define their motion, and find the tension of the string at any instant during the motion.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1910, 65.)

## EXAMPLES—XCIV.: CHIEFLY RIGID DYNAMICS.

1. 'Write down the radius of gyration of a homogeneous billiard ball, and prove that it will roll down a plane slope $\alpha$ with acceleration $\frac{5}{7}g\sin\alpha$.

'Prove that if rolled horizontally on the plane with velocity $V$, the ball will proceed to describe a parabola with latus rectum $\frac{14}{5}V^2/g\sin\alpha$. (Galileo's experiment.)'

(LOND. B.SC., PASS, MIXED MATH., 1902, II. 9.)

2. 'Investigate the harmonic vibration of the balance wheel of a chronometer of moment of inertia $I$ (lb.-ft.$^2$), and prove that if the wheel swings through $N^\circ$ from rest to rest in $T$ seconds, the maximum couple exerted by the balance spring must be

$$\pi^3 IN/360gT^2 \text{ (lb.-ft.)}.'$$

(LOND. B.SC., PASS, MIXED MATH., 1902, II. 10.)

3. 'Define the centre of oscillation of a compound pendulum, and show that it is convertible with the centre of suspension.

'A uniform solid rectangular parallelepiped has its edges 6, 9, and 12 inches long; what is the least time in which it can oscillate about a horizontal axis, and how must the axis be fixed in the body?'

(LOND. B.SC., PASS, MIXED MATH., 1904, II. 9.)

4. 'A uniform rod turns in a vertical plane about a point distant one-third of its length from one end. Find the centre of percussion.

'Determine also the impulse on the axis when the rod receives a given horizontal impulse at a point distant $c$ from the lower end.'

(LOND. B.SC., PASS, APPLIED MATH., 1905, III. 7.)

5. 'A fly-wheel is movable in smooth bearings, about a horizontal axis, and is set in motion by a descending mass of weight[1] $P$, which hangs from one end of a cord coiled round the axle of radius $r$. This mass is found to descend through $h$ feet in $n$ seconds; prove that the moment of inertia of the wheel about its axis is

$$\left(\frac{gn^2}{2h} - 1\right)P.r^2.'$$

(LOND. B.SC., PASS, APPLIED MATH., 1906, III. 3.)

[1] If $P$ is in *lbs. weight*, the answer is in lbs.-ft.$^2$

6. 'A uniform rod of length $l$ and weight $w$ is held at an inclination $a$ to the vertical with its lower end in contact with a smooth horizontal plane, and is then let fall. Find its angular velocity and its pressure on the plane, when it is inclined at $\theta$ to the vertical.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, III. 9.)

7. 'Obtain a formula for the period of the oscillations of a compound pendulum.

'A uniform rod of length $2a$ is swinging as a pendulum about one of its ends, its greatest angular deviation from the downward vertical being $a$. At an instant when the rod is vertical its fixed end is suddenly released; find how far the centre of the rod descends before the rod is again vertical.'

(LOND. B.SC., PASS, APPLIED MATH., 1908, III. 5.)

8. 'Define the principal axes of a rigid body.

'If the body be a plane lamina, explain why one of the principal axes at any point must be at right angles to the plane.

'If the lamina be an equilateral triangle of uniform density, find the principal axes at one of the angular points. Find also the principal moments of inertia at that point.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 49.)

9. 'Find the period of a complete double oscillation of a compound pendulum when the angular swing is small. How must the pendulum be suspended to make the period a minimum?'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 50.)

10. 'Find an expression for the kinetic energy of a body moving, in any given way, in one plane.

'$AB$ is an inclined plane and $BC$ is the horizontal plane, through the lowest point $B$, and both planes are smooth. A uniform rod is placed on $AB$ with one end at $B$, and is allowed to slide down. Find its angular velocity just before its upper end leaves the inclined plane.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 51.)

11. 'A uniform rod can turn freely round one end. It is held in its highest position, and is then allowed to fall. On reaching its lowest position it encounters a fixed obstacle at its lower end. There is no rebound.

'Find the impact on the obstacle and that on the point of suspension.

'The rod is 12 ft. long and weighs 10 lbs.; compare the impacts with the momentum of a body which weighs 5 lbs. and has a velocity of 8 ft. a second.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 49.)

12. 'A uniform beam $AB$ of length $2a$ hangs vertically from the end $A$, which lies in a smooth horizontal groove.

'If the end $A$ be projected with velocity $u$ along the groove, show that the middle point of the beam will move with a velocity whose horizontal component is

$$\frac{k^2}{a^2 + k^2} u,$$

where $k$ is the radius of gyration of the beam about its middle point.

'Find the equation which determines the angular velocity of the beam at any time.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 51.)

13. 'Discuss the sensibility of a balance with equal arms.

'Show how to find the period of a small oscillation and, for a given load, prove that it varies directly as the square root of the sensibility.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1908, 62.)

14. 'A uniform rod $AB$ lies on a smooth horizontal plane; $C$ is a fixed point vertically over $B$; a thread carries a weight $P$, and after passing over $C$ is fastened to $A$; the end $B$ can move freely in a vertical guide or groove coinciding with the vertical line $BC$. The weights of $P$ and of the rod are equal. If $P$ is allowed to fall, find ($a$) the angular velocity of the rod in any assigned position, ($b$) the direction of the motion of the centre of gravity, ($c$) the reaction of the groove on the end $B$.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1908, 68.)

15. 'A beam of mass $M$ and length $a$ rotates about one extremity on a smooth horizontal plane, there being no forces except the resistance of the atmosphere. If the retarding effect of the resistance on a small element of the beam be equal to $\lambda$ times the square of its velocity, show that in time $t$ the angular velocity will be reduced from $\Omega$ to $\omega$, where

$$\omega^{-1} = \Omega^{-1} + \frac{\lambda a^4}{4Mk^2}t.'$$

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1909, 66.)

16. 'A uniform rod is placed very nearly upright on a smooth horizontal floor and against a smooth vertical wall, and it slides down in a plane at right angles to both wall and floor. Find its position at the instant of its leaving the wall; find also how it is moving at that instant, and the pressure on the floor.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1909, 67.)

17. 'A cone 8 inches high, radius of base 4 inches, weighs 5 lbs. Determine its moment of inertia about an axis through its centre of gravity parallel to its base.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1910, 47.)

18. 'In a compound pendulum form the equations of motion to determine the horizontal and vertical components, $X$ and $Y$, of the force acting at the point of suspension at any instant.

'Let $W$ be the weight, $r$ the distance of the centre of gravity from the point of suspension, and $k$ the radius of gyration about the same point. Suppose that the pendulum is allowed to fall from the position in which the centre of gravity and the point of suspension are in the same horizontal line; show that, when the pendulum is inclined at an angle of 45° to the vertical,

$$X - Y = \left(1 - \frac{r^2}{k^2}\right)W.'$$

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1910, 50.)

19. 'Find, in the motion of a ballistic pendulum, the relation between the centre of percussion and the axis of spontaneous rotation.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1910, 51.)

20. 'Show how to find the moment of inertia of a hemisphere (of uniform density) about an axis drawn through its centre of gravity and parallel to its base, assuming that the moment of inertia of a sphere about a diameter is $2mr^2/5$.

'A hemisphere rests with its curved surface in contact with a rough horizontal plane. It is slightly disturbed from its position of equilibrium; find the time of a small oscillation.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1910, 52.)

## EXAMPLES—XCV. : CHIEFLY STATICS.

1. 'Define the angle of friction $\phi$, and by means of it determine graphically where jamming begins when a drawer is pulled out by a handle to one side.

'Prove that a sash window of height $a$, counterbalanced by weights, cannot be raised or lowered by a vertical force unless it is applied within a middle distance $a \cot \phi$; and prove that if the cord of a counterbalance breaks, the window will fall unless the width is greater than $a \cot \phi$.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1902, I. 2.)

2. 'Discuss the conditions of equilibrium of three forces. Determine graphically the stress in each bar of a jointed triangular framework, strained by three forces acting at the angles.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1903, I. 1.)

3. 'Determine the force to be exerted by the hand at the end of each arm $a$ feet long of a copying press to set up a thrust of $P$ pounds, the screw being smooth and cut with $n$ threads to the foot.

'Sketch in plan the forces which act on the press and the man to maintain equilibrium.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1903, I. 4.)

4. 'A uniform bar is bent into the shape of a V with equal arms, and hangs freely from one end. Prove that a plumbline suspended from this end will cut the lower arm at a distance of one-third its length from the angle.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1904, I. 9.)

5. '$ABCD$ is a rectangle; $AB=12$ inches, $AD=8$; at $A$, $B$, $C$, $D$ are placed particles whose masses are proportional to 8, 10, 6, 16 respectively. Find the position of the centre of mass

($a$) by the theorem of mass moments;

($b$) by means of funicular polygons.'

(LOND. B.SC., PASS, APPLIED MATH., 1905, I. 5.)

6. 'A uniform square plate is divided into two portions by a straight line joining a corner to the middle point of a side. Prove that the line joining the mass centres of the two portions is perpendicular to the dividing line.'

(LOND. B.A., PASS, APPLIED MATH., 1906, I. 1.)

7. 'Four equal light bars are jointed freely so as to form a rhombus $ABCD$, and the corners $A$, $C$ are connected by a light chain. The whole hangs from $A$, which is uppermost; and two equal weights $W$ are suspended from $B$ and $D$. Find (graphically or otherwise) the tension in the chain.'

(LOND. B.A., PASS, APPLIED MATH., 1906, I. 4.)

8. 'A uniform ladder, of length $l$ and weight $W$, rests with its foot on the ground (rough) and its upper end against a smooth wall, the inclination to the vertical being $a$. A force $P$ is applied horizontally to the ladder at a point distant $c$ from the foot so as to make the foot approach the wall. Prove that $P$ must exceed

$$W \frac{l}{l-c}(\mu + \tfrac{1}{2}\tan a),$$

where $\mu$ is the coefficient of friction at the foot.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, I. 2.)

9. 'Assuming the position of the centre of gravity of a triangular pyramid, deduce the position of the centre of gravity of a homogeneous solid right circular cone.

'The radii of the bases of a frustum of such a cone are 6 feet and 3 feet, and the thickness of the frustum is 9 feet; find the position of its centre of gravity.'

(LOND. B.A., PASS, APPLIED MATH., 1907, I. 7.)

10. 'A uniform bar, $AB$, rests with its end $A$ on a rough horizontal plane, for which the angle of friction is $\lambda$; the bar is to be kept at a given inclination to the horizon by means of a cord attached to $B$. Exhibit in the figure the extreme directions of the cord which will allow of equilibrium.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, I. 4.)

11. 'Es ist zu beweisen dass die an einem starren Körper angreifenden Krafte, falls sie in einer Ebene liegen, im allgemeinen einer einzigen resultierenden Kraft statisch aquivalent sind, welche auch in ein Poinsot'sches Kraftepaar ubergeben kann. Wie setzt man die Krafte graphisch zusammen?'

(LOND. B SC., PASS, APPLIED MATH., 1907, I. 10.)

12. 'Two equally rough pegs $A$ and $B$ are a distance $2a$ apart in a straight line inclined at an angle $\theta$ to the vertical. A rod passes over the peg $A$ and under the peg $B$, and is just kept from sliding down by friction at the pegs. Prove that the centre of gravity of the rod must be at a distance from the upper peg $A$ equal to

$$a(\cot\theta/\mu - 1),$$

where $\mu$ is the coefficient of friction between the rod and the pegs.'

(LOND. B.SC., PASS, APPLIED MATH., 1908, I. 6.)

13. 'Show how to reduce a system of coplanar forces to a single force and a couple.

'Forces of magnitudes 1, 2, 3, 4, 5, 6 act in the order named round the sides of a regular hexagon, the senses of the forces in adjacent sides being either both towards, or both away from, the corresponding vertex. Find the single force at the centre of the hexagon and the couple to which the system reduces.'

(LOND. B SC., PASS, APPLIED MATH., 1909, I. 3.)

14. 'A heavy uniform bar rests with one end on the ground and the other end against the vertical face of a rectangular block, which also rests on the ground. The vertical plane through the bar is perpendicular to the given face, and passes through the centre of gravity of the block. The coefficient of friction for the contact of the bar, both with the ground and with the block, is $\mu$, and the weight of the block is four times that of the bar. Show that if the bar is on the point of slipping and the block on the point of overturning, the ratio of the length of the bar to the width of the block is $(1+\mu^2)(2+3\mu^2)/\mu(1-\mu^2)$.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, I. 4.)

15. A sphere is composed of a solid homogeneous hemisphere and a very thin hemispherical shell of equal mass.

Show that the composite body cannot rest in equilibrium upon a rough plane if its inclination with the horizontal exceeds the angle whose sine is 0·0625.

16. '$ABCDE$ is a frame of five equal bars, kept in the form of a regular pentagon by two bars $AC$, $AD$. The frame is hung up by the point $A$, and carries equal weights ($W$) at $B$ and $E$. Find the stresses in the bars, putting out of the question the weight of the bars and the friction of the joints.

'Explain how the results would be altered if the weights were hung at $C$ and $D$ instead of at $B$ and $E$.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 41.)

17. 'Show that any system of forces, acting on a rigid body, can be replaced by a single resulting force acting at any chosen point and a couple.
'Give further information concerning this force.
'Define Poinsot's central axis, and show how to construct it for any given system of forces.'
(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 43.)

18. 'A particle of weight $w$ rests against the circumference of a circular plate, whose plane is vertical; a cord attached to the particle passes over a pulley placed vertically above the highest point of the circle at a distance from the circle equal to the radius and carries a weight $p$; show that the particle will rest at an angular distance $\theta$ from the highest point of the circle where $\cos\theta = 5/4 - p^2/w^2$).
'Prove also that the pressure on the circle is independent of the magnitude of $p$.'
(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1908, 61.)

19. 'The end of a cylinder is pressed against a rough plane by a force which is equally distributed over the area of contact. The cylinder could move freely round its axis were it not for the friction. Find the force applied along a tangent of the base which will just make the cylinder turn.'
(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 44.)

20. 'A system of concurrent forces in space are given in magnitude and line of action.
'Show how from the plans and elevations of these forces to draw the plan and elevation of the resultant force.
'A horizontal triangle has sides 10′, 10′, and 5′ respectively; from the three corners hang three ropes, each 6′ long; they are joined at their extremities and support a weight of 1 cwt.; find the pull on each rope.'
(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1910, 61.)

## EXAMPLES—XCVI.: CHIEFLY ATTRACTIONS.

1. 'Find the attraction at any point in the substance of a solid sphere of given uniform density.
'A thick shell of uniform density is bounded by spherical surfaces which are not concentric; prove that the attraction in the internal cavity is uniform in magnitude and direction.'
(LOND. B.SC., PASS, APPLIED MATH., 1905, II. 9.)

2. 'Démontrer que lorsqu'on passe au travers d'une couche de densité superficielle $\sigma$ l'intensité de la force d'attraction dans la direction perpendiculaire à la surface reçoit un accroissement subit de $4\pi\gamma\sigma$.'
(LOND. B.SC., PASS, APPLIED MATH., 1907, II. 9.)

3. 'Show that the attraction of a solid homogeneous sphere at any point outside it is the same as if its mass were collected at the centre.
'Prove that if the earth were homogeneous throughout, the decrease in gravitational attraction as one rose through a certain height in a balloon would be approximately twice the decrease as one descended an equal depth into a mine.'
(LOND. B.SC., PASS, APPLIED MATH., 1910, II. 6.)

4. In a homogeneous sphere, of radius $a$ and density $\rho$, is drilled to the centre a cylindrical hole of very small radius $b$. Show that the attraction of the sphere on a plug of same density filling the hole is $\frac{2}{3}\gamma\pi^2\rho^2a^2b^2$.

5. 'Investigate the attraction of a thin homogeneous circular plate of radius

$a$ at a point which is at a perpendicular distance $c$ from the centre of the plate.

'Remark upon the cases :—

(1) $a$ infinite, $c$ finite ;

(2) $a$ finite, $c$ infinitesimal.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 44.)

6. 'Define the potential of a system of attracting or repelling masses at any point.

'Mass, attracting according to the law of nature, is uniformly distributed on the circumference of a circle. Prove that the chord of contact of tangents drawn from an external point divides the mass into two parts having equal potentials at the point.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 46.)

7. 'Define the potential of a single particle and of a given distribution of matter at an assigned point, and state what is its physical meaning in the case of gravity.

'Find the potential of a spherical shell of uniform density at an assigned external point.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1909, 61.)

8. 'Consider a cylindrical surface of given length and radius and of uniform surface density ; find the attraction at one end, $P$, of the axis.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1910, 42.)

9. 'Find the attraction of a plane, of uniform surface density and of indefinite extent, at a point outside it.

'Let $AB$ be a diameter of a spherical surface of uniform surface density. Show how to draw a plane at right angles to $AB$, which will divide the surface into two parts, such that their attractions at $A$ shall be equal.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1910, 63.)

## EXAMPLES—XCVII. : CHIEFLY HYDROSTATICS.

1. 'Show how to determine the specific gravity (*S. G.*) of a solid or liquid by the Hydrostatic Balance, and investigate the formula.

'Prove that a brass pound weight made to equilibrate the standard pound of platinum in air is too great by the fraction

$$\left(\frac{A}{B}-\frac{A}{P}\right)\Big/\left(1-\frac{A}{B}\right),$$

where $A$, $B$, $P$ denote the *S. G.* of air, brass, and platinum.'

(LOND. B.SC., PASS, MIXED MATH., 1902, II. 3.)

2. 'State the principle of Pascal for the transmission of fluid pressure.

'A rectangular area is immersed vertically in water with one side horizontal at a depth of 9 feet and the opposite side at a depth of 15 feet ; show that the centre of pressure is 3 inches below the middle point of the rectangle.'

(LOND. B.SC., PASS, MIXED MATH., 1904, II. 7.)

3. '$ABC$ is a triangular area immersed vertically in water with $C$ in the surface and $AB$ horizontal ; show how to divide the area by a horizontal line $PQ$ into two portions on which the pressures are equal, $P$ and $Q$ being points in $AC$ and $BC$ respectively.

'If $h$ is the length of the perpendicular from $C$ on $AB$, prove that the height above $AB$ of the centre of pressure on the area $APQB$ in the above case is

$$\frac{h}{8}(3\times 4^{1/3}-4).'$$

(LOND. B.A., PASS, APPLIED MATH., 1906, II. 2.)

4. 'Obtain an expression for the total pressure exerted by water on a plane area occupying any position.

'A rectangular area $ABCD$ has the side $AB$ in the surface of water, the side $AD$ (10 feet long) being vertical and submerged; divide the area by horizontal lines into three parts on each of which the pressure is the same.'

(LOND. B.SC., PASS, APPLIED MATH., 1905, III. 7.)

5. 'A hollow cone consisting of a curved surface closed by a circular base, both made of thin sheet metal, is 12 inches high and has a radius of 5 inches. The cone is to rest completely submerged in water with its vertex fixed and its axis horizontal; find the necessary weight of metal per square inch of surface, assuming that a cubic foot of water has a mass of 1000 ounces.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, III. 9.)

6. 'A rectangular tank is divided into two compartments by a vertical diaphragm. The two compartments are then filled to heights $h$, $k$ with liquids of densities $\rho$, $\sigma$ respectively. Show how to choose $h$ and $k$ so that the resultant of the pressures on the diaphragm shall reduce to a couple, and find the magnitude of this couple per unit breadth of the tank.'

(LOND. B.A., PASS, APPLIED MATH., 1905, II. 1.)

7. 'State the conditions for equilibrium and for stability of a body floating freely in water.

'A solid sphere of radius $r$, weight $W$, and specific gravity $s$ lies in the bottom of a cylindrical vessel of radius $a$ and height $h$, which contains water just up to the top. Prove that the work required to raise the sphere just clear of the water is

$$W\left\{h-\frac{4r^3}{3a^2}-\frac{1}{s}\left(h-r-\frac{2r^3}{3a^2}\right)\right\}.'$$

(LOND. B.SC., PASS, APPLIED MATH., 1907, III. 5.)

8. 'A rectangular area is immersed vertically in water with one side horizontal, and at a depth $x$, the opposite side being at a depth $h+x$; show that the distance of the centre of pressure from the upper side is

$$\frac{h}{3}\cdot\frac{3x+2h}{2x+h}.$$

'Show that if the area is divided by a horizontal line into two parts on which the water pressures are the same, the depth of this line below the surface of the water must be $(x^2+hx+\frac{1}{2}h^2)^{1/2}$.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, III. 6.)

9. 'A thin hollow vessel in the shape of a paraboloid of revolution floats in water with its axis vertical and vertex downwards. If the weight of the vessel itself be neglected, find approximately to what height it must be filled with mercury (specific gravity 13·6) in order that its vertex may be 18 inches below the free surface of the water.'

(LOND. B.SC., PASS, APPLIED MATH., 1908, III. 10.)

10. 'State and prove the principle of Archimedes.

'A vessel contains two liquids that do not mix, and a cylinder floating with axis vertical; the lighter liquid is 5 in. deep, and its sp. gr. is 0·8; the sp. gr. of the heavier liquid is 1·15, and $1\frac{1}{2}$ in. of the height of the cylinder is above the upper liquid. If the sp. gr. of the cylinder is 0·75, what is its height?'

(LOND. B.SC., PASS, APPLIED MATH., 1909, III. 7.)

11. 'A uniform lead pipe, 16 feet long and closed at one end, is bent so as to form three quarters of the circumference of a circle, and is held in a vertical plane, so that the closed end is the highest point of the circle. Liquid is then poured in, and, when it is just on the point of overflowing,

the imprisoned air is found to occupy half the length of the pipe. Find the specific gravity of the liquid, assuming the water barometer to stand at 33 feet.'

(LOND. B.Sc., PASS, APPLIED MATH., 1909, III. 9.)

12. 'A circular hole, of radius $a$, in the plane vertical side of a cistern is exactly filled by a solid wooden sphere of specific gravity $\sigma$, which is free to turn about a fixed smooth horizontal axle which is diametral to both the circle and the sphere. The cistern is filled with water of density $\rho$ to a height $h$, greater than $a$, above the centre of the sphere. Evaluate the action between the sphere and the axle.

'What couple, if any, is required to keep the sphere from rotating?'

(LOND. B.Sc., PASS, APPLIED MATH., 1910, III. 7.)

13. A casting is to be made by running molten metal (of density 1/4 lb. to the cubic inch) into a sand mould consisting of top and bottom parts. The casting is a rectangular table 12 feet by 5 feet with legs 3 feet high. The pattern is moulded face down and legs up, the joint between top and bottom parts being at the junction of the legs and the rectangular face or table. Show that, to hold the top part of the mould down at the instant of casting, a distributed load of nearly 35 tons is needed, with almost an additional ton for every inch of accidental excess height in the legs when pouring the metal in.

14. 'Explain a general method for determining the position of the centre of pressure of a plane surface immersed in any manner in a fluid.

'A plate in the form of a quadrant of a circle is immersed vertically in water with one edge in the surface. Find the distance of the centre of pressure from the horizontal and vertical edges respectively.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 43.)

15. 'A cylindrical vessel, radius $a$, contains water, and a cylindrical body (of the same height), whose radius is $b$, is lowered into the vessel until it stands upright on the bottom; none of the water is spilt. Show that the ratio of the increase of the potential energy of the water to its original potential energy is $b^2/(a^2-b^2)$.

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 47.)

16. 'Define the capillary curve. Find an expression for its radius of curvature at any point of its length.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 52.)

17. 'Find the centre of pressure in the case of an equilateral triangle, completely immersed with one edge horizontal, and its plane inclined at a given angle to the horizon.

'A vessel has the form of a regular tetrahedron. It is filled with water and made watertight. It is held with one face horizontal. Find the resultant pressures on the several faces (*a*) when the vertex is below the horizontal face, (*b*) when it is above that face.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1909, 44.)

18. 'Find the least value of the specific gravity of a cube of uniform density that the conditions of equilibrium may be satisfied when it has no more than one angular point out of water. Also show that, when its specific gravity exceeds that value, the parts above water of the edges meeting in that point are equal.

'Consider the case when the specific gravity is $47 \div 48$.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1909, 70.)

19. 'Investigate the height to which a liquid rises in a capillary tube of any cross section.

'Show that it rises as high in a cylindrical tube of diameter $d$ as in one having for cross section a square of side $d$.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1910, 50.)

20. 'If $\rho$ be the density of air at the earth's surface, show that at height $h$ the density is $\rho e^{-gh/k}$.

'What assumptions have been made in obtaining this result? What is $k$? Determine it in foot-second units from the data

height of barometer = 30 in.,
specific gravity of mercury = 13·596,
$g = 32{\cdot}2$,
specific gravity of air = 0·0013.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1910, 52.)

## EXAMPLES—XCVIII.: CHIEFLY HYDROKINETICS.

1. 'Discuss the flow of a fluid in a tube of non-uniform cross section, showing how the pressure varies.

'Will the fluid tend to push the tube before it?'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1908, 72.)

2. 'If water be flowing steadily through an inclined pipe of varying section, show that

$$\frac{u^2}{2g} + \frac{p}{w} + z = \text{constant},$$

where at any section

$u$ = velocity in feet per second,
$p$ = pressure in lb. per sq. ft.,
$w$ = weight of a cubic foot of water,
$z$ = height of section in feet above a fixed horizontal plane.

'Show that for steady motion $u$ cannot exceed a certain limiting value.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1909, 72.)

3. 'State the physical fact expressed by the "equation of continuity" in the motion of fluids.

'Establish that equation in the form

$$\frac{d\kappa\rho}{dt} + \frac{d\kappa\rho v}{dx} = 0,$$

and show how to express it in terms of rectangular co-ordinates.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1910, 69.)

4. 'A vessel of water discharges through a large pipe of variable section. If the flow be steady, investigate the relation between velocity, pressure, and height above datum level at any section of the pipe.

'If the vessel supply a stream running in a channel of any size, either closed or open, and is undisturbed by frictional resistances, show that the *total* energy of all parts of the water is the same.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1910, 70.)

5. 'A vessel symmetrical about a vertical axis contains a gas. If it be rotated uniformly about the axis, investigate the pressure at any point of the gas, assuming it to move in relative equilibrium with the vessel.

'Apply to the case in which the vessel, a cylinder of radius $a$ and height $h$, contains a weight $W$ of gas.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1910, 71.)

## EXAMPLES—XCIX.: CHIEFLY ELASTICITY.

1. 'State the law (Hooke's) connecting the tension of an elastic cord with its extension.

'A uniform india-rubber cord has a length of 26 inches under a tension of 2½ pounds weight, and a length of 20 under a tension of 1 pound

weight; calculate the amount of work done in stretching it from its natural length to a length of 30 inches, and draw a work diagram.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, I. 8.)

2. 'Prove that the potential energy stored up in a stretched elastic string is half the product of the tension and the extension.

'A uniform bar 6 feet long, weighing 20 lbs., lies on perfectly rough horizontal ground. Evaluate the work done in raising one end from the ground to a height of 3 feet, by means of a vertical string attached to that end, (i) when the string is inelastic; (ii) when the string is elastic, 2 feet long, and such that its length would be doubled by a pull of 30 lbs. weight.'

(LOND. B.SC., PASS, APPLIED MATH., 1908, I. 10.)

3. 'A beam length $a$, width $b$, depth $d$, coefficient of elasticity $E$, is held by one end in a horizontal position so as to be bent simply by its own weight, which is $w$ per unit of length; show that the curvature at a distance $x$ from the fixed end is

$$\frac{6w(a-x)^2}{Ebd^3}.$$

'Find also the deflection of the other end of the beam.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1908, 65.)

4. 'A rod, naturally straight, is slightly bent by forces at right angles to its length in one plane. Show that the radius of curvature at any point of its length is given by the formula

$$EAK^2 \div (\text{Bending Moment}),$$

and explain the notation.

'A rod of uniform cross section is supported horizontally on three points, viz. one at each end and one in the middle, so that there is no droop at the middle. Show that the greatest droops are very nearly at one-fifth of the length from each end. $(\sqrt{33} = 5{\cdot}7446.)$'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1908, 66.)

5. 'An iron bar, 2 inches in diameter, for which Young's modulus is 29,000,000, is bent into an arc of a circle 400 feet in diameter. Find the maximum stress at any point of the transverse section.

'Show further that if the stress be limited to 4 tons per square inch, the diameter of the circle must not be less than 540 feet.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1909, 63.)

6. 'A uniform horizontal beam is supported in any manner. Establish the equation

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

of the deflection curve.

'If the beam be supported at the ends and centre, show that there is no bending moment at points which are at a distance from an end equal to 3/8 of the length of the beam.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1909, 64.)

7. 'A uniformly loaded beam is supported at the ends, the supports being in the same horizontal line and propped in the middle.

'If the centre prop is at the same level as the supports, find the points of zero bending moment.

'If now the centre prop be raised a distance equal to 1/4 of the deflection of the centre when the prop is wholly removed, prove that it sustains a pressure equal to 25/32 of the whole load.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1910, 62.)

## EXAMPLES—C.: MISCELLANEOUS.

1. State an analogy, pointed out by Poinsot, between kinematics and statics, and give a dynamical illustration which links up the apparently conflicting elements in the analogy.

2. 'Determine the position of equilibrium of a body movable about a fixed point, having one or more spherical cavities in which a sphere or some liquid can roll about.

'Prove that a tilting basin, of thin metal in the form of a segment of a spherical surface, movable about a diameter of the base, will not upset when water is poured into it, if the weight of the basin exceeds

(radius of base/height of basin)$^2$ − 1

times the weight of water.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1902, I. 3.)

3. 'Prove that if $W$ tons is conveyed $s$ feet in $t$ seconds, being moved from rest by a force of $P_1$ tons up to velocity $v$ feet per second, and then brought to rest by a force of $P_2$ tons,

$$(1)\ \frac{Wv^2}{2g} = \frac{P_1P_2s}{P_1+P_2};\quad (2)\ \frac{Wv}{g} = \frac{P_1P_2t}{P_1+P_2};$$

$$(3)\ t = \sqrt{\frac{2s}{g}\left(\frac{W}{P_1}+\frac{W}{P_2}\right)}.$$

'With a coefficient of adhesion $\mu$, a motor car actuated and braked on the hind axle can get up a speed $v$ in $x$ feet, or be brought to rest again in $y$ feet, given by

$$x = \frac{v^2}{g}\left(\frac{1}{\mu}-\frac{h}{a}\right),\quad y = \frac{v^2}{g}\left(\frac{1}{\mu}+\frac{h}{a}\right),$$

$a$ denoting the distance between the axles and $h$ the height of the C. G. (midway between the wheels) from the ground.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1902, I. 6.)

4. 'Prove that the line joining the centre of gravity and the centre of buoyancy of a floating body is vertical in a position of equilibrium, and normal to the curve or surface of buoyancy; and show how the stability is determined.

'Find where a cylindrical wooden pile of given S. G. will begin to leave the vertical position when lowered into water by a chain fastened to the top.'

(LOND. B.SC., PASS, MIXED MATH., 1902, II. 4.)

5. 'On the experimental law that the resistance of similar steamers is proportional to the wetted surface and the square of the velocity, prove that if a 6-foot model of 0·01 ton displacement run at a speed of 2 knots in an experimental tank experiences a resistance of 0·2 pound, a similar 600-foot 10,000-ton steamer at 20 knots would experience a resistance equivalent to an incline of one in 112, and require over 12,000 effective horse-power.

'Show that for the Atlantic passage an increase of 1% in speed requires 6% increase in displacement tonnage and 7% increase in horse-power.'

(LOND. B.SC., PASS, MIXED MATH., 1902, II. 5.)

6. 'Prove that the bursting rim velocity of a circular wire ring is $\sqrt{gh}$, where $h$ is the breaking length of the wire when straight and hung up vertically.

'Determine the greatest velocity in miles an hour possible with wheel tires of density 1 cwt./ft.$^3$ and tensile strength 2 cwt./in.$^2$'

(LOND. B.SC., PASS, MIXED MATH., 1902, II. 7.)

7. 'Prove that as the C. G. of a part of a body of weight $P$ is moved about in a curve, the C. G. of the whole body of weight $W$ will describe a similar curve, reduced in the linear scale of

$$P \text{ to } W.$$

'A rectangular block of stone is slung by two equal parallel chains fastened to points in its upper face, and suspended from the ends of a uniform beam supported by a fulcrum. Prove that the block can be tilted from the horizontal to any desired inclination $\theta$ by moving the fulcrum support through a distance

$$(1 - P/W)h \tan\theta,$$

where $h$ is the depth of the C. G. of the block below the upper face, $W$ the total weight, and $P$ that of the beam and chains.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1903, I. 3.)

8. 'Prove that if a body is resisted by a force proportional to its displacement, the body is in a position of stable equilibrium, about which it will perform harmonic oscillation in an invariable period; and mention some familiar illustrations.

'Prove that the free oscillation of the mercury of a barometer in a U tube of uniform section will synchronise with a pendulum of half the length of the mercury column.'

(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1903, I. 7.)

9. 'Prove that if a chain $ABC$ fastened at $A$ is led over a pulley $B$, so as to rest on a smooth inclined plane $BC$, the part $AB$ will assume a catenary curve in which the tension at any point $P$ will be the same as at $Q$ at the same level on $BC$; and deduce the analytical properties of the catenary.

'Prove that if the plane is rough the length of the chain $BC$ may be altered without affecting $AB$, within the limits

$$BC \sin\alpha \cos\phi \operatorname{cosec}(\alpha \pm \phi),$$

$\alpha$ denoting the slope of $BC$ and $\phi$ its limiting angle of friction.'

(LOND. B.SC., PASS, MIXED MATH., 1903, II. 2.)

10. 'Define the metacentre; and prove that the metacentric height of a ship of $W$ tons displacement is $mbP/W$, if a moment of $bP$ foot-tons obtained by moving $P$ tons transversely $b$ feet gives the ship a heel of one in $m$.

'Assuming the experimental laws that the normal pressure of the wind and the tangential friction of the water per square foot are proportional to the square of the velocity, prove that a ship and its model are heeled over to the same angle by a wind proportional to the square root of the length or the sixth root of the tonnage, and move through the water at a proportional speed.'

(LOND. B.SC., PASS, MIXED MATH., 1903, II. 3.)

11. 'On the experimental law of the last question, prove that steamers geometrically similar, propelled at speed proportional to the sixth root of the tonnage, will experience the same resistance expressed in pounds per ton or equivalent incline, and will burn the same coal per ton-mile; but the horse-power (H.P.) per ton and the period of revolution of the screw will be proportional to the speed, and the steam pressure to the square of the speed.

'Taking the H.P./ton as 1/16 (speed in knots), prove that this implies a resistance of about 20 lbs. per ton, or an equivalent incline of one in 112; and with a coal consumption of 2 lbs. per H.P.-hour, the coal capacity required for a voyage of 3000 miles is about one-sixth of the tonnage.

'Calculate for a steamer of 26,000 tons at a speed of 23·5 knots.'

(LOND. B.SC., PASS, MIXED MATH. 1903, II. 4.)

12. 'Determine the motion of a circular hoop of radius $a$ feet, whirling in a vertical plane on a round horizontal stick, if released when the centre is moving with velocity $V$ f./s. at an angle $\alpha$ with the horizon; and prove that it will make $2\pi\alpha/V$ revs./sec. in the air.
'Prove that the tension in the hoop will be the weight of a length $V^2/g$ feet of the rim (the tension length).'
(LOND. B.SC., PASS, MIXED MATH., 1903, II. 6.)

13. 'Prove that about the diameter of a homogeneous sphere
(radius of gyration)$^2$ = 0·1(diameter)$^2$;
and show how this may be verified experimentally by allowing the sphere to roll down a slit of uniform breadth in an inclined plane.
'Explain why the cushion of a billiard table is made to receive the impact at a height 0·7 of the diameter of the billiard ball.'
(LOND. B.SC., PASS, MIXED MATH., 1903, II. 7.)

14. 'Investigate the torsional vibration of a body suspended by a vertical wire in which the restoring couple is proportional to the angle turned through from the position of equilibrium.
'Prove that the angular velocity is $2\pi$/(period) times the G. M. of the angular distance in radians from the two ends of the swing.'
(LOND. B.SC., PASS, MIXED MATH., 1903, II. 8.)

15. 'A rectangular tray with vertical sides is made of thin sheet metal. If its length be 3 feet, its breadth 2 feet, and its depth 4 inches, find the height of the centre of gravity above its base.'
'Also find approximately how much the centre of gravity will be raised if the tray is filled with water; the density of the metal being eight times that of water, and its thickness one-sixteenth of an inch.'
(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1904, I. 8.)

16. 'A light bar $AB$ can turn freely about the end $A$, which is fixed, and is supported in a horizontal position by a string $CB$, $C$ being a fixed point vertically above $A$. If a weight $W$ be suspended from any point $P$ of the bar, find geometrically the direction and magnitude of the reaction at $A$; also the tension of the string. Work out numerically the tension of the string when $W$ = 10 lbs., $AB$ = 18 in., $AP$ = 12 in., $AC$ = 9 in.'
(LOND. B.A. AND B.SC., PASS, MIXED MATH., 1904, I. 10.)

17. 'If two systems of mass lying in a plane have the same centre of mass and the same radii of gyration about three different lines in the plane, prove that they have the same radii of gyration about every line.
'Hence prove that when calculating the radius of gyration of a triangular area (or uniform plate), we may replace the triangle by three equal particles at the middle points of the sides.'
(LOND. B.SC., PASS, MIXED MATH., 1904, II. 10.)

18. 'An unclosed curved surface of given shape is immersed in a given position in water; show how to find the vertical component of the pressure exerted over one side of the surface.
'A hollow cone of inner radius 4 feet and inner height 10 feet, and not closed by a base, is placed with its rim on a horizontal plane; the cone is filled with water through a small hole at the vertex and the water does not flow out. Find the force, in tons weight, with which the water tends to lift the cone, assuming the mass of water per cubic foot to be 62·5 lbs.'
(LOND. B.SC., PASS, MIXED MATH., 1904, II. 6.)

19. '$AB$, $BC$, and $CD$ are three bars in a horizontal plane, freely jointed at $B$ and $C$, and movable about fixed pins at $A$ and $D$. At a given point, $P$, in $BC$ is applied perpendicularly to $BC$ a given force $F$; and the bars are to be kept in a given configuration by means of a couple applied to the bar $AB$. Assuming all necessary data, calculate the magnitude of

this couple, and exhibit the lines of action of the stresses at the points $A$, $B$, $C$, $D$.'

(LOND. B Sc., PASS, APPLIED MATH., 1905, I. 8.)

20. 'A rod of length $l$ and weight $W$ rests against a rough horizontal rail with its lower end on a smooth horizontal plane. The height of the rail above this plane is $h$, and the angle of friction between the rail and the rod is $\lambda$. Find the greatest possible inclination of the rod to the vertical, and the corresponding pressure on the horizontal plane. [Assume that $h > \frac{1}{2}l \sin \lambda$.]'

(LOND. B.A., PASS, APPLIED MATH., 1906, I. 3.)

## EXAMPLES—CI.: MISCELLANEOUS.

1. 'Show that a uniform rod, mass $m$, is kinetically equivalent to three particles rigidly connected and situated one at each end of the rod and one at its middle point, the masses of the particles being $\frac{1}{6}m$, $\frac{1}{6}m$, $\frac{2}{3}m$.

'A rod consists of two parts of equal length which are uniform but of different densities. Find three particles which situated respectively at the ends and the middle point of the rod form a system kinetically equivalent to the rod. If $M$ and $M'$ are the masses of the two portions of the rod, prove that this is always possible with actual particles, if $M/M'$ lies between 5 and 1/5.'

(LOND. B Sc., PASS, APPLIED MATH., 1905, III. 5.)

2. 'Prove that the moment of momentum of a rigid body moving in two dimensions about an axis through the mass centre perpendicular to the plane of motion is $Mk^2\omega$.

'A uniform heavy sphere, whose mass is 1 lb. and radius 3 inches, is suspended by a wire from a fixed point, and the torsion couple of the wire is proportional to the angle through which the sphere is turned from the position of equilibrium. If the period of an oscillation is 2 secs., find the couple which will hold the sphere in equilibrium in a position in which it is turned through four right angles from the equilibrium position.'

(LOND. B Sc., PASS, APPLIED MATH., 1905, III. 8.)

3. 'Eine zylindrische Taucherglocke von der Hohe $a$ wird in Wasser getaucht, bis ihre oberste Spitze in einer Tiefe $b$ unter der Wasserflache ist. Bestimmen Sie wie weit die Luft komprimirt wird, wenn das Wasserbarometer auf $h$ steht.

'Wenn die Glocke mit einer gleichmässigen Geschwindigkeit $v$ sinkt, so bestimmen Sie die Geschwindigkeit, mit welcher man Luft mit atmospharischem Druck in die Glocke pumpen muss, um kein Wasser in dieselbe zu bekommen.'

(LOND. B Sc., PASS, APPLIED MATH., 1905, III. 10.)

4. 'Four bars are loosely jointed so as to form a plane quadrilateral $ABCD$; and it is in equilibrium under the action of four forces $P$, $Q$, $R$, $S$, applied to the joints $A$, $B$, $C$, $D$ respectively. The lines of action of $P$, $Q$ meet in $E$, and those of $R$, $S$ meet in $F$. Prove that $EF$ produced will pass through the intersection of $AD$, $BC$.'

(LOND. B.Sc., PASS, APPLIED MATH., 1906, I. 4.)

5. 'If a particle is acted upon by a force always directed towards a fixed point and varying inversely as the square of the distance, obtain the conditions to be satisfied when the particle is initially projected so that the path of the particle may be (1) a circle, (2) a parabola.'

(LOND. B.Sc., PASS, APPLIED MATH., 1906, III. 3.)

6. 'Write down the general equation for the transformation of a given mass of gas whose volume, temperature, and intensity of pressure are altered.

'Calculate the height to which the water will rise in a cylindrical diving bell, 12 feet high, when its top is lowered to a depth of 60 feet, being given that the temperature of air at the surface is 80° F., height of water barometer at surface 33 feet, and temperature of water 40° F.'

(LOND. B.SC., PASS, APPLIED MATH., 1906, III. 10.)

7. 'State the necessary and sufficient conditions of equilibrium to be satisfied by a system of forces acting in one plane on a rigid body.

'$AB$ is a uniform ladder 37 feet long resting at $A$ on the ground, where the coefficient of friction is 1/2, and at $B$ against a vertical wall, where the coefficient is 5/12; the distance of $A$ from the wall is 12 feet; a horizontal force is applied at $A$ to move the ladder towards the wall; find the magnitude of the requisite force in terms of the weight, $W$, of the ladder.'

(LOND. B.A., PASS, APPLIED MATH., 1907, I. 1.)

8. 'Two uniform bars, $AB$ and $AC$, are rigidly attached together at $A$, so that the angle $BAC$ is 120°, and are freely movable in a vertical plane about $A$, which is fixed; find the inclination of $AB$ to the horizon in the position of equilibrium, being given length of $AB=6$ feet, mass of $AB=10$ lbs., length of $AC=5$, mass of $AC=8$.'

(LOND. B.A., PASS, APPLIED MATH., 1907, I. 2.)

9. 'What is meant by unavailable energy? A mass $P$ placed on a smooth horizontal table is connected by a light slack cord with a mass $Q$ lying on the ground. Show that, if $P$ is projected along the table with a velocity $V$, the energy available for raising $Q$ is

$$\frac{P}{P+Q}\cdot\frac{PV^2}{2g},$$

and write down the energy which has become unavailable.'

(LOND. B.A., PASS, APPLIED MATH., 1907, I. 4.)

10. 'Give a proof that the time of a small semi-oscillation of a simple pendulum is $\pi\sqrt{l/g}$.

'If the bob of the pendulum is drawn out from the vertical to a deviation $\alpha$ such that the tension in the vertical position exceeds the weight of the bob by 1/10 of the weight, show that $\alpha$ must be 18° 11'.'

(LOND. B.A., PASS, APPLIED MATH., 1907, I. 5.)

11. 'Two perfectly elastic particles impinge directly on each other; prove that the product of the sum of their masses and the amount of energy transferred from one to the other is equal to the product of their total momentum and the momentum transferred.

'Find the ratio of the momentum transferred to the total momentum in the case of two imperfectly elastic spheres of masses 6 and 10, with coefficient of restitution 3/4, when the smaller one moving with velocity 8 impinges directly on the larger one moving with velocity 3 in the same sense.'

(LOND. B.A., PASS, APPLIED MATH., 1907, I. 8.)

12. 'To one end of the cord of an Atwood's machine a mass of 110 grammes is attached; to the other end is attached a smooth pulley whose mass is 40 grammes, and over which passes a cord with masses of 35 grammes and 25 grammes hanging at its ends. Find the acceleration of each mass.'

(LOND. B.A., PASS, APPLIED MATH., 1907, I. 10.)

13. An ordinary gauge consists of a U tube of uniform bore and open to the atmosphere at the outer limb, the increase of pressure in the inner

limb being read by the depression there of the surface of the liquid of density $\Delta$ occupying the lower portions of each limb.

A special gauge has the upper parts, of each limb of the U tube, enlarged in cross-sectional area to $n$ times that of the lower parts, and is charged with two liquids as follows :—A liquid of density $\rho$ meets the atmosphere in the large upper portion of the outer limb, and extends round the bend of the U, finishing at some level P in the thin tube of the inner limb. Above P in the inner limb there extends a second liquid of density $\sigma$, not mixing with the former, but reaching to the upper enlarged portion of the tube where it is exposed to the pressure to be determined. The gauge is read by the position of the interface P of the two liquids.

Show that the ratio of the sensibility of this special gauge to that of the ordinary one first mentioned may be expressed by

$$\frac{2n\Delta}{\rho(n+1)-\sigma(n-1)}.$$

14. 'Two reservoirs, 10,000 square feet in area, with vertical walls, contain the one salt and the other fresh water. They are both filled to the same absolute level. If the reservoirs are connected by a pipe 50 feet below the free surface of either, find how much salt water has passed into the fresh water reservoir before equilibrium is established, the specific gravity of the salt water being 1·026, assuming that any salt water which enters the fresh water reservoir sinks below the level of the pipe.'

(LOND. B.A., PASS, APPLIED MATH., 1907, II. 2.)

15. 'State the conditions of equilibrium of bodies wholly or partly immersed in a fluid.

'A cylinder of radius $r$ floats in liquid of density $\sigma$ inside a cylindrical vessel of radius $a$. Show that if a mass $M$ be placed on the floating cylinder, it will sink by an amount

$$\frac{M}{\pi\sigma}\left(\frac{1}{r^2}-\frac{1}{a^2}\right).'$$

(LOND. B.A., PASS, APPLIED MATH., 1907, II. 4.)

16. 'A thin rod of length $2a$ and density $\sigma$ floats in a sloping position in a liquid of density $\rho$, the end out of the liquid being suspended by a string. Find the proportion of the length of the rod immersed, and show that it is independent of the inclination of the rod. Find also the tension of the string.'

(LOND. B.A., PASS, APPLIED MATH., 1907. II. 5.)

17. 'Prove that, if $r$, $\theta$ be the polar co-ordinates of a point moving in a plane, the radial component of acceleration is $\ddot{r}-r\dot{\theta}^2$.

'A particle $P$ of unit mass, free to slide on a straight smooth wire, is attracted towards a point $O$ of the wire with a force $\mu.OP$. If the wire be made to revolve about $O$ in a horizontal plane with constant angular velocity $\omega$, show that the motion of $P$ on the wire is a simple harmonic motion of period $2\pi/\sqrt{\mu-\omega^2}$, and that when $\omega^2=\mu/2$ the path of $P$ in space is a circle.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, II. 4.)

18. 'Two equal particles are connected by an inextensible string of length $a+b$, which passes through a hole in a smooth table, so that one particle hangs freely and the other is on the table. At an instant when the system is at rest, and a length $a$ of the string is horizontal, the particle on the table is suddenly made to move perpendicularly to the string with velocity $\sqrt{4gc}$. Employ the principles of energy and angular

momentum to show that the hanging particle will not rise to the level of the table unless

$$b<c-a+\sqrt{c(2a+c)};$$

and find with what velocity the particle arrives at the hole if this inequality is satisfied.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, II. 6.)

19. 'A uniform bar, $AB$, is oscillating in a vertical plane about a smooth horizontal axis fixed at $A$; find in any position of the bar the magnitudes of the forces on the axis in the directions along and perpendicular to the bar.

'If at any point $P$ of the bar we consider a transverse section of it, the stress on this section consists of transverse and longitudinal forces together with a bending moment. Indicate the method of calculating these.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, III. 3.)

20. 'Eine zylindrische Glasrohre, welche 125 cm. lang und mit zwei Hahnen versehen ist, steht senkrecht. Der untere Hahn wird geschlossen, in die Rohre eine Wassersaule von 90 cm. Hohe und uber dieselbe eine Lage Öl von 20 cm. Hohe gebracht. Das specifische Gewicht des Öls ist 0·75. Der ubrige Teil der Rohre ist mit Luft angefullt unter dem atmospharischen Druck 75 cm. Nun wird der obere Hahn geschlossen und der untere teilweise geöffnet, so dass das Wasser tropfenweise ausfliessen kann, bis Gleichgewicht eintritt. Um wie viel muss die Oberflache des Öls sinken? Spec. Gew. Quecksilvers 13·6.'

(LOND. B.SC., PASS, APPLIED MATH., 1907, III. 8.)

## EXAMPLES—CII.: MISCELLANEOUS.

1. 'A bead is free to slide on a smooth circular wire of radius $a$, which is made to rotate about a vertical diameter with angular velocity $\omega$. Show that, if $a\omega^2<g$, the bead will be in stable equilibrium at the lowest point of the wire, and will, if disturbed, perform small oscillations about this position of equilibrium in a period

$$2\pi\sqrt{\frac{a}{g-a\omega^2}}.'$$

(LOND. B.SC., PASS, APPLIED MATH., 1908, II. 6.)

2. 'Explain how the effect of a very large force acting for a very short time is estimated.

'A sleigh weighing 5 tons is travelling along a horizontal road. As it passes under a bridge a load of rubbish weighing 1 ton is shot into it from a height of 9 feet. If the sleigh was moving at 4 miles an hour at the time, find its speed immediately afterwards, the coefficient of impulsive friction between the ground and sleigh being 1/5.'

(LOND. B.SC., PASS, APPLIED MATH., 1908, III. 1.)

3. 'State the laws of Boyle and Charles.

'Bubbles of air rise through water from a depth of 18 feet. In what ratio will the diameter of a bubble have altered when it arrives at the surface? (Assume that the height of the mercury barometer is 30 inches and that the specific gravity of mercury is 13·6.)'

(LOND. B.SC., PASS, APPLIED MATH., 1908, III. 11.)

4. 'A rectangular thin board $ABCD$ is hinged along a horizontal axis $AB$, lying in a fixed plane inclined at an angle $\alpha$ to the horizon. A smooth heavy sphere of radius $r$ is introduced between the inclined plane and the board above the hinge. If $l$ be the length of the side $BC$ of the

board, and the weight of the sphere be half the weight of the board, find what condition must be satisfied if the sphere is not to be forced out.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, I. 6.)

5. 'Translate :—

'Les Anglais enseignent la mécanique comme une science expérimentale ; sur le continent on l'expose toujours plus ou moins comme une science déductive et *à priori*. Ce sont les Anglais qui ont raison, cela va sans dire ; mais comment a-t-on pu persévérer si longtemps dans d'autres errements? Pourquoi les savants continentaux qui ont cherché à échapper aux habitudes de leurs devanciers, n'ont-ils pas pu le plus souvent s'en affranchir complètement?

'D'autre part, si les principes de la mécanique n'ont d'autre source que l'expérience, ne sont-ils donc qu'approchés et provisoires? Des expériences nouvelles ne pourront-elles un jour nous conduire à les modifier ou même a les abandonner?'

(LOND. B.SC., PASS, APPLIED MATH., 1909, I. 10.)

6. 'Translate :—

'Ueberhaupt findet zwischen der Statik und der Geometrie ein sehr inniger Zusammenhang statt, indem nicht allein erstere Wissenschaft der Hulfe der letztern unumganglich bedarf, sondern weil auch umgekehrt, gleichsam zum Lohne fur die geleistete Hulfe die Statik der Geometrie neue Satze zufuhrt, Sätze die nicht selten wiederum zum Vortheile der Statik verwendet werden können.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, II. 10.)

7. 'A light elastic string carries a mass $m$ attached to a point of trisection. It is stretched between two points $A$ and $B$ of a smooth fixed table, the distance $AB$ being $1\frac{1}{2}$ times the natural length $l$ of the string. If the weight of a mass $5m$ be required to stretch the string to twice its natural length, find the period of small oscillations when the mass $m$ is displaced longitudinally.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, III. 2.)

8. 'A body describing a parabolic orbit, about a centre of force attracting according to the inverse square law, is at one extremity of the latus rectum of its orbit when it collides with a body of equal mass, describing a circle about the same centre of force, but revolving about this centre in the opposite sense. The two bodies coalesce after collision. Show that they must ultimately fall into the centre of force.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, III. 4.)

9. 'An open cubical cistern, 2-foot edge, has one of its sides hinged to the bottom and kept from falling outwards by a rope connecting the middle point of its upper edge to the corresponding point of the opposite side. Find the tension in this rope when the cistern is half filled ; and find how the tension is altered by tilting the cistern through 45° about the edge containing the hinges. It may be assumed that 1 cubic foot of water weighs $62\frac{1}{2}$ lbs., and that the weight of a side of the cistern is 10 lbs.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, III. 6.)

10. 'Translate the following passage :—

'Die Thatsache, dass Luft eine Flussigkeit ist, die auf andere Körper durch Druck wirkt, scheint zuerst von *Torricelli* und *Otto von Guericke* bemerkt worden zu sein. Die Beziehung zwischen dem Druck in einer Luftmasse und dem Volumen, das sie einnimmt, wurde zuerst von *Boyle* untersucht. Die Idee, dass die Abnahme der Barometerhöhe beim Emporsteigen uber die Erdoberflache zur Messung der Berghöhen benutzt werden könnote, verdankt man *Pascal*.'

(LOND. B.SC., PASS, APPLIED MATH., 1909, III. 10.)

11. 'A simple pendulum is swinging in a vertical plane about a fixed point $O$. Prove that if the pendulum just makes complete revolutions when the bob is attached to $O$ by a light rigid rod, then if the rod be replaced by a string and the velocity at the lowest point be kept the same, the string will become slack when the bob has risen through 5/6 of the diameter of the circle it describes.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, II. 1.)

12. 'Translate the following passage :—

'Im Hinblick auf diese grossen Erfolge waren Newtons Nachfolger bestrebt, die übrigen Naturerscheinungen ganz nach der Methode Newtons lediglich unter passenden Modifikationen und Erweiterungen zu erklaren. Unter Benutzung einer alten, schon von Demokrit herruhrenden Hypothese dachten sie sich die Korper als Aggregate sehr zahlreicher materieller Punkte, der Atome. Zwischen je zweien derselben sollte ausser der Newtonschen Anziehung noch eine Krafte wirken, welche man sich in gewissen Entfernungen abstossend, in anderen anziehend dachte, wie es eben zur Erklärung der Erscheinungen am geeignetsten schien.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, II. 10.)

13. 'A heavy bead slides along a fixed rough horizontal circular wire ; initially the bead has a velocity $V$, and it comes to rest after describing an arc of length $l$. Prove that

$$V^2 = ag \sinh(2\mu l/a),$$

where $a$ is the radius of the circle and $\mu$ the coefficient of friction.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, II. 5.)

14. 'Three equal particles of mass $m$ are placed at the corners of an equilateral triangle of side $a$ and allowed to move from rest under their own gravitation. Show that they will come together after a time

$$\pi a^{3/2}/2\sqrt{6\gamma m},$$

$\gamma$ being the constant of gravitation.'

(LOND. B.SC., PASS, APPLIED MATH., 1910, II. 8.)

15. 'The ends of a light elastic string of natural length $a$ and of modulus $\lambda$ are fastened to the ends of a rod $AB$ of the same length $a$ and of mass $m$. A mass $M$ is attached to the middle point $C$ of the string. The system lies on a smooth horizontal table, and $AB$ is held fast while $C$ is drawn away from $AB$ till $ACB$ is an equilateral triangle. Prove that, if the rod and the mass $M$ are now simultaneously released, their relative velocity when $M$ strikes the rod will be

$$\left\{a\lambda\left(\frac{1}{M}+\frac{1}{m}\right)\right\}^{1/2}.$$

(LOND. B.SC., PASS, APPLIED MATH., 1910, III. 3.)

16. 'Draw an isosceles triangle $ABC$, and from $C$ draw $CD$ at right angles to the base $BC$; also draw a circle to touch $AC$ and $CD$. Suppose that $ABC$ represents a cross section of a beam lying on the ground, and that the circle represents a cross section of a cylinder resting between the beam and a wall $CD$. Taking account of the friction between the beam and the ground, but not of the friction between the cylinder and the beam or the wall, find the relation between the weights of the beam and the cylinder when the beam is just beginning to slide out.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 42.)

17. 'A spherical shell of uniform density attracts an external particle according to the law of gravity ; find the resultant attraction.

'A sphere of uniform density attracts a particle, the mass of which is 1 lb., at a distance of 4000 miles from its centre, with a force of

32 poundals. Find the force with which it would attract an equal particle ($P$) placed at a distance of $60 \times 4000$ miles.

'If $P$ describes a circle round the centre of the sphere, find the periodic time. ($N.B.$ — $\sqrt{116} = 10{\cdot}4881$.).'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 45.)

18. 'A heavy rod is constrained to slide in a vertical line with its lower end on the curved surface of an equally heavy smooth hemisphere, the hemisphere sliding on a smooth horizontal plane. Determine the motion.

'Solve the equations of motion for the case in which initially the rod is very nearly in its highest position.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1908, 52.)

19. '$D$ is any point in $BC$, a side of a triangular lamina $ABC$. Show that the moment of inertia of $ABC$ about $AD$ equals

$$\tfrac{1}{6}(\text{mass}) \times (BD^2 - BD.DC + DC^2) \sin^2 ADB.'$$

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 41.)

20. 'Examine minutely the following statement:—

'Since the specific gravity of brass is greater than that of a diamond and less than that of gold, if diamonds and gold are sold by brass weights a purchaser would find it to his advantage to buy diamonds in fine weather and gold in bad weather.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 42.)

## EXAMPLES—CIII.: MISCELLANEOUS.

1. 'A lamina of uniform density is in shape an equilateral triangle; it is entirely submerged with its centre of gravity fixed. When it is placed in any position with its plane vertical, find the co-ordinates of its centre of pressure.

'If now it is made to turn in the vertical plane, show that the locus of its centre of pressure is a circle, and explain how to draw the circle.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 44.)

2. 'Under what circumstances will a floating body, if slightly disturbed, make small vertical oscillations?

'Show how to find the time of such an oscillation.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 45.)

3. 'A U-shaped glass tube of uniform section contains liquid, to a height $a$ in each leg, which can move without friction in the tube; if the liquid be slightly disturbed, show that the small oscillations are synchronous with those of a simple pendulum of length $a$.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 46.)

4. 'A litre of dry air at zero centigrade and 760 mm. pressure weighs 1·2932 grammes; find the weight of $v$ cubic decimetres when the temperature is $T°$ C. absolute temperature and the pressure $p$ mm.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 49.)

5. 'When a thin plate of any substance is in a state of tension, how is the tension measured?

'A vessel made of a thin material has the shape of a prism, whose base is an equilateral triangle. It stands upright and is filled with water, the weight of which is put out of the question. The water is then put under a pressure of $p$ lbs. per square inch. Show that the sides of the vessel are under a tension equal to $pa \div 2\sqrt{3}$ per inch of the length of a vertical edge, where $a$ denotes a side of the equilateral triangle.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 48.)

6. 'What is the effect of inserting a small quantity of air in the Torricellian

vacuum of a cylindrical barometer tube? If the height of the barometer be 30 inches, the length of the Torricellian vacuum 5 inches, and the cross section of the tube 1 square inch, calculate the effect, on the column of mercury, of the insertion of 1/4 cubic inch of air into the vacuum, the absolute temperature having changed during the experiment in the ratio of 48 to 49.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 50.)

7. 'State the laws of the ascent and depression of liquid in capillary tubes.

'Show in a diagram the state of equilibrium when a capillary tube is immersed vertically in a liquid

($a$) when the liquid wets the tube;

($b$) when it does not do so.

'In each case, show by lines with arrow heads, supplemented by explanations, how the forces act which maintain the liquid inside the tube in equilibrium.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1908, 51.)

8. 'In a suspension bridge the roadway, weighing $w$ lb. per foot of the length, is upheld by a uniform chain suspended from two points in a horizontal line. Neglecting the weight of the chain in comparison with that of the roadway, prove that the tension at a point is equal to $Nw$ where $N$ is the length of the normal at the point between the curve and the axis of symmetry.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1908, 63.)

9. 'A uniform string hangs from two points under the action of gravity. Prove that $s = c \tan\psi$, $y^2 = s^2 + c^2$. The sag at the middle point of a uniform string, suspended from two points in the same horizontal line, is $(1/n)^{\text{th}}$ of the length of the string. Prove that the ratio of the tension at either end to the weight of the string is $(n^2+4)/8n$.

(LONDON B SC., 1918, I. 10.)

10. 'Find the centre of gravity of a circular arc of uniform density.

'A uniform flexible rope is wrapped round a cylinder, whose axis is horizontal, and the length of the rope equals the circumference of the cylinder. Its free end is at the end of a horizontal diameter. The cylinder is turned through a right angle, so that the free end falls through a distance equal to a quarter of the circumference. Find the work done by gravity.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1908, 67.)

11. 'Draw a triangle $ABC$ with $BC$ vertical, and suppose it to represent a frame of three weightless bars. A weight $W$ is hung from $A$, and $BC$ is kept vertical by being fastened to a wall at two given points, $X$ and $Y$. Find the stresses in $AB$ and $AC$, and the forces at $X$ and $Y$ caused by $W$.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 41.)

12. 'Show that a force, acting on a rigid body in an assigned direction along a given line, can be replaced by an equal force, acting in the same direction along a parallel line, and a couple.

'Let $AB$, $AC$, $AD$ be three edges of a cube, and consider $AD$ and two other edges parallel to $AB$ and $AC$ respectively, which neither intersect $AD$ nor one another. Suppose that equal forces, $P$, act along these edges respectively in the directions $AB$, $AC$, $AD$. Show how to reduce them to a resultant force and a couple. Also, show how to draw the line along which the resultant acts when the plane of the couple is at right angles to the direction of the resultant.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 42.)

13. 'A ladder, whose centre of gravity is half-way between its ends, rests

upon a pavement and against a wall in the usual manner, but the wall slopes inwards and the pavement downwards each at an angle $a$.

'If $\phi$ be the limiting angle of friction both at the wall and at the pavement, show that the ladder, when in limiting equilibrium, makes an angle $\theta$ with the pavement such that

$$\theta = \tfrac{1}{2}\pi + a - 2\phi.'$$

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 43.)

14. 'A heavy isosceles right-angled triangle hangs from its vertex; determine the time of a small oscillation about an axis in its plane parallel to its hypothenuse.

'What would be the result if the axis of rotation were at right angles to its plane?'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 50.)

15. 'State the principle of the Conservation of Energy.

'A spherical shell, the interior radius of which is one-half of the exterior, is filled with fluid of the same density with itself; show that it will run down a perfectly rough plane of inclination $a$ with acceleration

$$\tfrac{80}{111} g \sin a.'$$

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1909, 52.)

16. 'The base of a right prism is a right-angled triangle; find the moment of inertia about the edge which passes through the right angle.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1909, 41.)

17. 'Investigate the position of the centre of pressure on a plane area immersed vertically in liquid at a given depth.

'Find also the effect upon the centre of pressure

(i) of turning the area about the line in which its plane cuts the surface of the liquid;

(ii) of rotating the area in its own plane about its centre of gravity.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1909, 43.)

18. 'Liquid in a vessel increases in density uniformly from $\rho_1$ at the surface to $\rho_2$ at a depth $h$; a small body of density $\sigma (< \rho_1)$ is held at a depth $h$ and then released; show that it will reach the surface with the velocity due to the height

$$\left( \frac{\rho_1 + \rho_2}{2\sigma} - 1 \right) h.'$$

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1909, 45.)

19. 'The height of a cylinder equals a diameter of its base, and its specific gravity is 0·5; show that it is in unstable equilibrium when placed in water with half its height covered.

'If one end of a thread were fastened to the centre of the base, and the other end to the floor at the bottom of the water, find how the length must be adjusted that the cylinder may float with its axis vertical. Find also the tension of the thread. If the thread were cut, how would the cylinder move?'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1909, 46.)

20. 'What is the "Reserve of Buoyancy" of a floating body?

'If a homogeneous body of density $\sigma$ floats partly in a liquid of density $\rho$ and partly in one of density $\rho'$ with a certain section, $A$, of itself in the horizontal plane of separation of the liquids, show that it can float inverted with the same section, $A$, in the plane of separation if its density be changed to $\sigma'$, where

$$\sigma' = \rho + \rho' - \sigma.'$$

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1909, 47.)

## EXAMPLES CIV.: MISCELLANEOUS.

1. 'An elastic fluid, having a constant temperature, is at rest under the action of gravity; show that the surfaces of equal pressure are horizontal planes.

'If $P$ is the pressure at a given point, and $p$ the pressure at a height $z$ above that point, show that

$$p = Pe^{-gz/k}.$$

'Explain what is denoted by the symbol $k$, and find its value from the following approximate data, viz.:—air weighing 540 grains per cubic foot exerts a pressure of 15 lbs. per square inch.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1909, 50.)

2. 'A stream of water impinges on a fixed surface. Explain the mechanical principles by which the pressure exerted on the surface can be determined.

'Find the pressure when the stream has a section of 4 square inches and a velocity of 48 feet a second, on the supposition that the water is simply stopped.

'How would the result be affected if there were a coefficient of restitution equal to 0·25?'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1909, 52.)

3. It has been calculated that during a century the earth loses eight seconds of time, if judged by an ideally perfect clock rated by the earth at the beginning of that century. If this retardation were due to tangential resistances uniformly distributed over an equatorial belt two miles wide and acting with constant value throughout the century, show that this resistance would exceed 1000 tons weight per square mile. Take the earth to be a homogeneous sphere of radius 4000 miles and density 5·5 gms. per c.c.

4. 'A particle acted upon by gravity descends from a point $O$ down a curve $OA$, which it presses at each point of its descent with a force varying as the square of its distance below the horizontal line through $O$. Taking the axes $Ox$, $Oy$ horizontal and vertical, show that $OA$ is the elastic curve

$$\frac{dy}{dx} = \frac{(a^4 - y^4)^{1/2}}{y^2}.$$

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1909, 65.)

5. 'A rod, $AB$, in an inclined position has the end, $B$, on a horizontal plane; it is kept in its position by two pegs, $P$ and $Q$, on opposite sides of the rod. If all the surfaces are smooth, find the pressures on the plane and on the pegs.

'If we suppose the plane to be removed and the pegs to be rough, so that the rod is still supported, find what will now be the pressure on the peg and the frictions that the pegs exert.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1910, 41.)

6. 'A square, $ABCD$, is formed by four equal weightless rods, loosely jointed at the angular points, and kept in shape by a diagonal tie, $AC$. It is hung up by $A$, and carries equal weights at $B$ and $D$. Find, by virtual work, the tension of the tie.

'Verify your result by resolution of forces, and find the stresses in the rods which form the square.

'Point out the rods which would be liable to bend if the weights were large.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1910, 43.)

7. 'A body turning about a fixed axis is symmetrical with respect to a plane

at right angles to the axis and passing through the centre of gravity. Show that the centrifugal force is the same as if its mass were concentrated at its centre of gravity.

'A hoop, with a radius of 3 feet and weighing 20 lbs., turns in its own plane about its centre ten times a second. Show that the rotation sets up a tension in the hoop of more than half a ton weight.'

(BOARD OF EDUCATION, THEO. MECH., SOLIDS, STAGE 3, 1910, 49.)

8. '$ABCD$ is a rectangular lamina of uniform density. Find the moment of inertia, about the side $AB$, of each of the triangles into which it is divided by the diagonal $BD$.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1910, 41.)

9. 'Find the position of the centre of pressure of a plane area immersed in water, supposing the plane to be vertical.

'Apply the method to the case in which the area is the part of a parabola cut off by the focal chord at right angles to the axis, the chord being vertical and just immersed.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1910, 43.)

10. 'Prove that if a volume be cut off a solid body by a plane section and an equal volume be *supposed* cut off by any other plane section, making a small angle with the first plane, the two planes will intersect in a line passing through the centre of gravity of the first plane.

'Explain the importance of this theorem in the question of the stability of a floating body.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1910, 45.)

11. 'A body floating in stable equilibrium is slightly disturbed, and makes small vertical oscillations. Find the time of an oscillation.

'Apply the method to the case of a cone of given height, whose vertical angle is 60° and specific gravity 5/8. (*N.B.* — $\sqrt[3]{5} = 1{\cdot}71$. It may be assumed that the cone will float in stable equilibrium with its vertex downwards.)'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1910, 46.)

12. 'Describe the hydrometer of variable immersion, and explain how the specific gravity of a liquid may be found by means of it.

'A hydrometer floats with 8 inches of the stem above the surface of a liquid whose specific gravity is 0·95; also it floats with 2 inches of the stem above the surface of a liquid whose specific gravity is 0·75. Find the specific gravity of the liquid in which it is found to float with 5·5 inches of the stem above the surface.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1910, 47.)

13. 'A spherical surface of radius $a$ and of small thickness $c$ contains gas at a given pressure $p$. Investigate the magnitude of the tension per unit area of the section of the material at any point.

'A thin india-rubber ball contains air. Show that the tension per unit area of the material varies as the absolute temperature.'

(BOARD OF EDUCATION, THEO. MECH., FLUIDS, STAGE 3, 1910, 51.)

14. 'A body is in motion about a fixed point $O$, and three rectangular axes, fixed in the body, are drawn through $O$. The angular velocities of the body about these axes respectively are known. Find equations for determining the motion of the body with reference to three rectangular axes drawn through $O$ and fixed in space.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1910, 64.)

15. 'In the motion of a rigid body in space of two dimensions establish the independence of the motions of the centre of gravity of the body and of the body relative to the centre of gravity.

'Examine the motion of a heavy rod sliding between a smooth vertical

plane and a smooth horizontal plane in a plane perpendicular to both.

'When will the rod leave the vertical plane?'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1910, 66.)

16. 'State the mechanical principle called the conservation of energy.

'Explain the application of the principle to the following case :—

'A thin but heavy cylindrical shell is fitted with a solid cylinder or core, supposed to be perfectly smooth. The compound body is placed on a rough inclined plane, and is allowed to roll down it. Find how the body is moving when it has rolled down a given length of the plane.

'Also compare the results that would be obtained : —(1) If the core had not been put in. (2) If the core had adhered firmly to the shell, the masses of shell and core being equal.'

(BOARD OF EDUCATION, THEO. MECH., HONOURS, 1910, 67.)

## EXAMPLES—CV. : MISCELLANEOUS.

1. 'Define the centre of gravity of a body and establish the formula $(\Sigma mz) \div \Sigma m$ for the distance from a plane of the centre of gravity of a number of particles.

'Find the position of the centre of gravity of the part of a solid sphere which lies between a diametral plane and a parallel plane whose distance from the former is half the radius of the sphere.'

(LOND. B.SC., APP. MATH., SUBSIDIARY TO HONS. PHYSICS, 1909, I. 2.)

2. 'State the principle of Virtual Work. In what respects is the method of Virtual Work preferable to that of resolving and taking moments as a means of solving problems?

'A circular cylinder of radius $a$ is fixed with its axis horizontal and its curved surface in contact with a vertical wall. A disc of radius $b$ $(b>a)$ rests on the cylinder and against the wall, the plane of the disc being parallel to the axis of the cylinder. If in the position of equilibrium the plane of the disc makes an angle $\alpha$ with the horizon, prove that

$$a = b\cos\alpha(1+\sin\alpha).'$$

(LOND. B.SC., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, I. 4.)

3. 'Define (i) *potential*, (ii) *tube of force*, and prove the fundamental property of a tube of force.

'If the lines of force are arcs of circles whose centres are at a fixed point $O$ and whose planes pass through a fixed line $OA$, compare the intensity of the force at two points in free space which lie on the same line of force.'

(LOND. B.SC., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, I. 6.)

4. 'Find the attraction of a thin uniform circular plate at any point of the line through its centre at right angles to its plane.

'Find the force of attraction exerted by a solid uniform hemisphere of density $\rho$ and radius $a$ upon a particle of mass $m$ placed at the centre of the plane face.'

(LOND. B.SC., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, I. 7.)

5. 'Determine the potential of a straight uniform rod at any point, and deduce the law of attraction of an infinite rod extending to infinity in both directions.

'Prove that if two such infinite rods intersect, their plane is cut by the equipotential surfaces in hyperbolas.'

(LOND. B.SC., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, I. 8.)

6. 'Find by any method the centre of pressure of a circular area, wholly immersed in water.

'Prove that, if $a$ is the radius of the circle and $h$ is the height of the water barometer, the centre of pressure lies on or within a concentric circle of radius

$$\tfrac{1}{4}a^2 \div (a+h).'$$

(LOND. B.Sc., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, I. 9.)

7. 'If a particle is subjected to two simple harmonic motions of the same period in perpendicular directions, show that the resulting motion is in general elliptic.

'If the motions have the same period and amplitude, differ in phase by one quarter of a period, and are in directions inclined to one another at 60°, find the resulting motion.'

(LOND. B.Sc., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, II. 1.)

8. 'Find expressions for the tangential and normal accelerations of a particle moving in a plane curve. If the motion is such that the tangential and normal accelerations are always equal, and $v_1$, $v_2$ are the velocities at any two points of the path, show that

$$v_2 = v_1 e^{\theta},$$

where $\theta$ is the angle turned through between the points.'

(LOND. B.Sc., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, II. 2.)

9. 'A particle of unit mass moves in a straight line under the action of a force $kx$ directed towards a fixed point in the straight line, where $x$ is the distance of the particle at any time from the fixed point and $k$ is constant. The resistance to motion at any time is $l$ times the velocity of the particle where $l$ is constant. Write down the equation of motion of the particle, and show that if $l^2 < 4k$, it is satisfied by an expression of the form

$$x = Ae^{-pt}\sin(nt+a),$$

where $A$ and $a$ are arbitrary constants.

'Find also the values of $p$ and $n$ in terms of $k$ and $l$.'

(LOND. B Sc., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, II. 3.)

10. 'In a wheel and axle a mass $P$ is suspended from the rope round the wheel, which is of radius $a$, and a mass $W$ is suspended from the rope round the axle, which is of radius $b$; show that the angular acceleration of the system is

$$\frac{(Pa \sim Wb)g}{Pa^2 + Wb^2 + (M)k^2},'$$

where $M$ is the mass and $k$ is the radius of gyration of the wheel and axle about the axis of turning.'

(LOND. B.Sc., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, II. 4.)

11. 'If an orbit is described under a central attraction $\mu(\text{distance})^{-2}$, show that the velocity at any point at a distance $r$ from the centre of attraction is given by

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right).$$

'Show also that the periodic time in the orbit is $2\pi(a^3/\mu)^{1/2}$.'

(LOND. B Sc., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, II. 5.)

12. 'If $k$ is the radius of gyration of a body about an axis through the centre of gravity, find the time of a small oscillation of the body about a parallel axis at a distance $h$ from the centre of gravity.

'Find the position of the axis in an elliptic disc about which the time of a small oscillation of the disc is the least possible.'

(LOND. B.Sc., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, II. 7.)

13. 'Show that in a given time a uniform solid circular cylinder will slide from rest under gravity down a given smooth inclined plane half as far again as it would roll down the same plane if perfectly rough.'

(LOND. B.Sc., APP. MATH., SUBSY. TO HONS. PHYSICS, 1909, II. 9.)

## EXAMPLES—CVI.: ESSAY SUBJECTS.

*Three Hours allowed for each Essay.*

1. The limitations and subdivisions of mechanics.
2. The composition and resolutions of vectors, localised and unlocalised.
3. The small vibrations of a particle with one degree of freedom.
4. The motion of projectiles.
5. Planetary motions.
6. The possible motions of a rigid body, (i) parallel to one plane, and (ii) with one point fixed.
7. The most general motion of a rigid body.
8. Quadric linkages.
9. Homogeneous strains.
10. Ancient and modern views on the foundations of mechanics, and their enunciation in laws, axioms, and definitions.
11. Moments of inertia.
12. Gyroscopic motions, including nutation.
13. Attractions between straight filaments and their combinations.
14. Attractions between a cylinder and a coaxial sphere.
15. Potentials and fields of attraction.
16. Graphical statics.
17. Stability of floating bodies.
18. Steady flow of liquids under gravity.
19. The chief elasticities and their relations.
20. Screws and wrenches.
21. The equilibrium under gravity of inextensible cords, uniform or not.
22. Small and large oscillations of rigid bodies.
23. Gravity waves in liquids.
24. Surface tension phenomena.

## EXAMPLES—CVII.: MISCELLANEOUS.

1. 'Prove that a system of coplanar forces can be reduced to a force at a given point and a couple. Find expressions for the magnitude of the force and the moment of the couple.

'$AB$, $A'B'$ are two equal lines in the same plane, $C$ and $C'$ their middle points. Prove that forces represented by $AA'$ and $BB'$ are equivalent to a force represented by $2CC'$ and a couple whose moment is $2AC^2$ multiplied by the sine of the angle between $AB$ and $AB'$.'

(CALCUTTA B.A. AND B.SC., HONOURS, MATH., 1909, V. 1.)

2. 'State the principle of virtual work, and prove it for a system of coplanar forces acting on a rigid body.

'A freely jointed framework is formed of five equal uniform rods each of weight $W$. The framework is suspended from one corner, which is also joined to the middle point of the opposite side by an inextensible string; if the two upper and the two lower rods make angles $\theta$ and $\phi$ respectively with the vertical, prove that the tension of the string is to the weight of a rod as

$$4\sin\theta + 2\sin\phi : \sin\theta + \sin\phi.'$$

(CALCUTTA B.A. AND B.SC., HONOURS, MAHT., 1909, V. 2.)

3. 'State the laws of friction.

'A uniform rectangular lamina of sides $2a$, $2b$ rests with one corner on a rough horizontal plane and another corner against an equally rough vertical wall, the plane of the lamina being vertical and perpendicular

to the wall Prove that when friction is limiting at each corner, the inclination to the horizon of the side of length $2a$ joining the two corners in contact is

$$\tan^{-1}\left(\frac{a\cos 2\epsilon}{b+a\tan\epsilon}\right),$$

where $\tan\epsilon$ is the coefficient of friction.'

(CALCUTTA B.A. AND B.SC., HONOURS, MATH., 1909, V. 3.)

4. 'Prove that the centroid of the projection of any plane area on any plane is the projection of the centroid of the area.

'A portion of a circular cone is cut off by a plane which makes an angle $\theta$ with the axis and is at a perpendicular distance $p$ from the vertex. Prove that the centre of gravity of the curved surface is at a distance $\frac{1}{3}p$ cosec $\theta$ from the centre of the plane section.'

(CALCUTTA B.A. AND B.SC., HONOURS, MATH., 1909, V. 4.)

5. 'Assuming the general equations of equilibrium for a flexible inextensible string deduce the equation of the curve formed by a string hanging under the action of gravity, its extremities being attached to fixed points in the same horizontal plane.

'Show that the tension at any point of the catenary is equal to the weight of a portion of the string whose length is equal to the ordinate of the point.'

(CALCUTTA B.A. AND B.SC., HONOURS, MATH., 1909, V. 5.)

6. 'Define momentum, kinetic energy, potential energy, power.

'An engine of 350 horse-power, whose weight is 20 tons, is attached to a train weighing 130 tons, and pulls it up an incline of 1 in 300 at a rate of 40 miles an hour. Find the resistance per ton due to friction, etc.'

(CALCUTTA B.A. AND B.SC., HONOURS, MATH., 1909, V. 6.)

7. 'A point is moving in a curve with velocity $v$; show that its acceleration inwards along the normal is $v^2/\rho$, where $\rho$ is the radius of curvature of the curve at the point.

'A particle is projected, from the lowest point inside a smooth elliptic cylinder whose major axis is vertical, in a direction perpendicular to the generators. Show that the particle will go completely round the cylinder if the velocity of projection is greater than $\sqrt{\{(5-e^2)ag\}}$, where $2a$ is the major axis and $e$ is the eccentricity of the cylinder.'

(CALCUTTA B.A. AND B.SC., HONOURS, MATH., 1909, V. 7.)

8. 'Define the hodograph of the path of a particle, and show that the velocity in the hodograph is equal to the acceleration in the path. By means of the hodograph determine the acceleration of a point describing a circle with constant speed.

'A heavy particle slides down a smooth cycloid with its axis vertical and its vertex uppermost. If the particle start from rest at the vertex, prove that the hodograph consists of the quadrant of a circle and a straight line touching the circle.'

(CALCUTTA B.A. AND B.SC., HONOURS, MATH., 1909, V. 8.)

9. 'Discuss the motion of a particle of mass $m$, attached by a weightless elastic string of length $l$ to a fixed point, which has been allowed to fall from a point vertically below the fixed point and at a distance $h$ from it ($h<l$). Determine the maximum tension, $\rho$ being Young's modulus for the material of the string.'

(CALCUTTA B.A. AND B.SC., HONOURS, MATH., 1909, V. 9.)

10. 'Prove the relation $F=\frac{h^2}{p^3}\cdot\frac{dp}{dr}$ in the case of a particle moving in a central orbit. Show that if $F=kr$ the particle will move in an ellipse whose centre coincides with the centre of force, and that its velocity at

any moment will be proportional to the length of the diameter conjugate to $r$.'

(CALCUTTA B.A. AND B.SC., HONOURS, MATH., 1909, V. 10.)

11. 'Two smooth imperfectly elastic spheres moving in a given manner impinge on one another obliquely; find equations to determine their motion after impact.

'If the masses of the spheres be $m_1$, $m_2$, and their velocities $u_1$, $u_2$ in directions at right angles to one another, making angles $\alpha$ and $90^\circ + \alpha$ with the line of centres at the instant of impact, prove that their directions after impact will also be at right angles to one another, if the coefficient of restitution be

$$\frac{m_1^2 u_1 + m_2^2 u_2 \tan \alpha}{m_1 m_2 (u_1 + u_2 \tan \alpha)}.$$

(CALCUTTA B.A. AND B.SC., HONOURS, MATH., 1909, V. 11.)

## EXAMPLES—CVIII.: MISCELLANEOUS.

1. 'Show that for a beam supported horizontally and loaded in any manner the ordinates of the funicular polygon represent to some scale the Bending Moment for the beam, at any point, and show how to find this scale.

'Draw to scale the Shearing Force and Bending Moment diagram for a uniform beam $AF$, 20 feet long, under the following conditions:—It is supported at one end $A$ and an intermediate point $B$, where $AB$ is 15 feet, while weights of 10, 15, and 20 cwts. are attached at points $C$, $D$, and $E$ where $AC$, $AD$, and $AE$ equal 10, 12, and 16 feet respectively; in addition, there is a uniform load of half a cwt. per foot run over $EF$. The weight of the beam may be neglected.'

(CALCUTTA B.E., STATICS AND DYNAMICS, 1909, 4.)

2. 'A mass $M$ draws up another $M'$ on the wheel and axle; if $a$ is the radius of the wheel and $a'$ that of the axle, find the motion and the tensions of the strings, assuming $\mu$ to be the mass of the revolving body and $k$ its radius of gyration about axis of revolution.'

(CALCUTTA B.E., STATICS AND DYNAMICS, 1909, 5.)

3. 'Show that, assuming the earth to be a homogeneous sphere, it would be necessary for the rotatory motion to be about seventeen times as fast as it is at present, if the centrifugal force was to exactly neutralise the action of gravity. The earth's radius equals 20,880,000 feet, and the acceleration due to gravity is 32·2 feet per sec. per sec.'

(CALCUTTA B.E., STATICS AND DYNAMICS, 1909, 6.)

4. 'A particle of mass $m$ is acted on by a force varying inversely as the square of the distance of the centre of mass from a fixed point; show that the particle will describe an ellipse round the fixed point, that equal areas will be traced out about the centre of force in equal times, and that the square of the periodic time of revolution bears a constant ratio to the cube of major axis of the ellipse described.'

(CALCUTTA B.E., STATICS AND DYNAMICS, 1909, 7.)

5. 'Show that for a particle moving in a plane the radial and transversal accelerations with reference to any axes fixed in the plane can be expressed in the form

$$\ddot{r} - r\dot{\theta}^2 \text{ and } 2\dot{r}\dot{\theta} + r\ddot{\theta}.$$

'Use this result to investigate the motion of two particles of masses $m$ and $m'$, connected together by a light string of length $l$ and revolving in a smooth tube in a horizontal plane at constant angular velocity.'

(CALCUTTA B.E., STATICS AND DYNAMICS, 1909, 8.)

6. 'State the fundamental notion that distinguishes "solids" from "fluids," and show that in a fluid the pressure at any point is the same in all directions.
'Differentiate between "Total Pressure," "Centre of Pressure," and "Pressure at a Point."'

(CALCUTTA B.E., HYDROSTATICS, 1909, 2.)

7. 'Investigate the necessary conditions of equilibrium for a body floating freely in a liquid, pointing out the use and meaning of the terms "metacentric height," "plane of flotation," and "surface of buoyancy."
'A rectangular block of wood of given volume and square in section floats in a homogeneous liquid. Find the least ratio of breadth to height that the block may just float upright.'

(CALCUTTA B.E., HYDROSTATICS, 1909, 3.)

## EXAMPLES—CIX.: KINEMATICS.

1. 'With the usual notation show that the acceleration of a point which is describing a central orbit is $\frac{h^2}{p^3}\frac{dp}{dr}$.
'If the point is describing a parabola about a centre of force in the focus, show that the velocity $v$ and the acceleration $f$ at any point are connected by the relation $v^2 = 2fr$.'

(LOND. B.Sc., 1911, II. 5.)

2. 'When the coordinates $x, y$ of a moving point are given functions of the time $t$, how can we calculate its velocity and acceleration?
'If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ and $\theta$ increases at a uniform rate $\omega$, prove that the velocity of the point is $a\theta\omega$; and find the inclination of the velocity to the axis of $x$.'

(LOND. B.Sc., 1912, II. 1.)

3. 'Define (*a*) the angular velocity of a moving point about a fixed point, (*b*) the angular velocity of a lamina which moves in its own plane.
'Prove that, if $\Omega$ be the angular velocity of a lamina, the angular velocity of a point $P$ of the lamina about a fixed point $O$ is $\Omega(PN/PO)$, where $N$ is the foot of the perpendicular from the instantaneous centre upon $PO$.'

(LOND. B.Sc., 1912, II. 2.)

4. 'A particle is projected horizontally with velocity $\sqrt{(ngq)}$ from a point on the surface of the Earth (supposed a sphere of radius $a$): prove that, if $n$ is between 1 and 2, the orbit will be an ellipse of eccentricity $(n-1)$ which lies entirely outside the Earth.'

(LOND. B.Sc., 1912, III. 1.)

5. 'A vector quantity has components $\xi$, $\eta$ in two directions at right angles, which are rotating at a rate $\omega$. Show that the components in those directions of the rate of change of the vector are

$$\dot{\xi} - \omega\eta \text{ and } \dot{\eta} + \omega\xi.$$

'Obtain the formulæ for the component velocities and accelerations in polar coordinates $r$, $\theta$.'

(LOND. B.Sc., 1913, II. 1.)

6. 'Prove that a particle moving under a central attraction varying as the distance will describe an ellipse with its centre at the centre of force.
'Show that, if two particles move in the same ellipse in opposite directions under the same force to the centre, the line joining them moves parallel to itself.'

(LOND. B.Sc., 1913, II. 4.)

7. 'The coordinates $(x, y)$ of a moving point are given functions of the time; show how to determine the magnitude and direction of its acceleration.

'The ends of a straight rod move in two straight grooves intersecting at right angles, and one end is constrained to describe its groove with uniform velocity. Show that the acceleration of any definite particle of the rod is perpendicular to this groove and inversely proportional to the cube of the distance from it'

(LOND. B.Sc., 1914, II. 1.)

8. 'The component accelerations of a particle are $g$, $-\mu y^{-3}$. It is set free from rest at a point of ordinate $k$, where these components are numerically equal. Prove that it reaches the axis of $x$ in time $(k/g)^{\frac{1}{2}}$, having described the arc of a parabola from an end of the latus-rectum to the vertex.

(LOND. B.Sc., 1915, II. 3.)

9. 'Two equal uniform rods $AB$, $BC$ are rigidly connected at $B$ at right angles to one another. If the rods rest on a smooth horizontal plane and an impulse $P$ is applied along $AB$, prove that the angular velocity acquired by the rods is $3P/10ma$, when $2a$ is the length and $m$ is the mass of a rod.'

(LOND. B.Sc., 1915, III. 6.)

10. 'If a lamina move in its own plane, show that at any instant it is rotating about a definite point; and construct that point when the velocity of any one point of the lamina is given together with the angular velocity of the lamina.

'One end, $P$, of an ordinary connecting rod moves uniformly in a circle, and the other, $Q$, moves in a straight line through the centre, $C$, of the circle. Show that the velocity of $Q$ is proportional to the intercept made by $QP$ on the line through $C$ perpendicular to $QC$.'

(LOND. B.Sc., 1916, II. 3.)

11. 'The coordinates $x$, $y$ of a moving point being given functions of the time, show how to calculate its velocity and acceleration.

'A straight line of constant length moves with its ends on two rectangular axes $Ox$, $Oy$, and $P$ is the foot of the perpendicular from $O$ on the straight line. Show that the velocity of $P$ perpendicular to $OP$ is $\omega . OP$ and along $OP$ is $2\omega . CP$, where $C$ is the middle point of the line, and $\omega$ is the angular velocity of $C$ about $O$.'

(LOND. B.Sc., 1917, II. 1.)

12. 'A particle moves in an orbit under a central acceleration $\mu r^{-2}$ along the radius vector; obtain the equations of energy and angular momentum.

'The particle is projected with velocity $u$ at right angles to the radius at distance $c$ from the origin; prove that $c$ is its least distance from the origin if $u^2 > \mu/c$.'

(LOND. B.Sc., 1917, II. 5.)

13. 'Having given the velocities of two points, show how to find the angular velocity of the line joining them.

'$ABC$ is a rigid equilateral triangular lamina, centre $G$. It is moving so that the velocities of $A$ and $B$ are each $u$ in the directions $AG$, $GB$ respectively. Find the angular velocity of the lamina, and the velocity of the corner $C$ in magnitude and direction.'

(LOND. B.Sc., 1917, II. 7.)

14. 'A man throws a stone with velocity $u$ at elevation $\alpha$, and $t$ seconds later he throws another stone with velocity $v$ at elevation $\beta$, so as to hit the first stone. Prove that

$$2uv \sin(\alpha - \beta) = (u \cos \alpha + v \cos \beta) gt.'$$

(LOND. B.Sc., 1918, II. 1.)

15. 'A particle describes an orbit under the action of a central force; prove that its acceleration is represented by

$$\frac{h^2}{p^3}\frac{dp}{dr}$$

'If the acceleration at distance $r$ be $\mu r^{-5}$ and the particle be projected at a distance $a$ from the centre of force with velocity $(\mu/2a^4)^{\frac{1}{2}}$, prove that the orbit is a circle, through the centre of force, of diameter $a$ cosec $\alpha$, where $\alpha$ is the inclination of the direction of projection to the radius vector.'

(LOND. B.Sc., 1918, II. 7.)

16. 'A rod $AB$ moves with its ends on two fixed lines $OA$, $OB$; show that, if the rod turns with uniform angular velocity $\omega$, the velocity of any point $P$ of the rod is equal to $\omega . IP$ perpendicular to $IP$, where $IA$, $IB$ are drawn perpendicular to the fixed lines.

'Prove also that the acceleration of $P$ is equal to $\omega^2 . OP$ towards $O$.'

(LOND. B.Sc., 1919, II. 1.)

17. 'Having given the positions and the linear velocities of two particles moving in the same plane show how to find their relative angular velocity.

'$P$ is a fixed point on a circle of radius $a$ which rolls with angular velocity $\omega$ on the outside of an equal circle whose centre is $O$; prove that the angular velocity of $OP$ is $3\omega\,(1 - a^2/OP^2)/4$.'

(LOND. B.Sc., 1920, II. 1.)

18. 'A particle is describing an ellipse of eccentricity $e$ about a centre of force in a focus. Prove that, with the usual notation,

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \text{ and } h^2 = \mu a\,(1 - e^2).$$

'If when the particle is at an extremity of the minor axis its velocity is suddenly doubled, show that it will proceed to describe a hyperbola and find the eccentricity of the new path.'

(LOND. B.Sc., 1920, II. 6.)

19. 'A circle rolls on a fixed circle. Find in terms of the radii the ratio of the angular velocities of the rolling circle and the line joining their centres.

'If $A$ is the centre of the rolling circle and $a$ its radius, and $B$ the centre of the fixed circle, prove that, if the angular velocity of the rolling circle is constant and equal to $\omega$, the acceleration of any point $P$ connected with the rolling circle is completely represented by $\omega^2 . PC$, where $C$ is a point on $AB$ such that $AC . AB = a^2$.'

(LOND. B.Sc., 1921, II. 1.)

20. 'Prove the formula $v^2 - \frac{2\mu}{r} = \text{constant}$, for a planetary orbit.

'Prove that the angular velocity of the centre of the planet about the focus which does not contain the centre of force is a minimum when the centre of the planet is at an extremity of the minor axis.'

(LOND. B.Sc., 1922, II. 4.)

## EXAMPLES—CX.: PARTICLE KINETICS

1. 'The tractive force exerted by an engine is $\frac{1}{100}$th of the weight of the whole train, and the maximum brake force capable of being exerted $\frac{1}{10}$th of the weight of the train. Find the time in which the train travels from rest to rest up 2800 yards of a slope which is inclined to the horizon at $\sin^{-1}(1/200)$, the brakes being applied at the instant when the steam is shut off.'

(LOND. B.Sc., 1911, II. 1.)

2. 'If $(r, \theta)$ are the polar coordinates of a moving point, find expressions for the component acceleration of the point along and perpendicular to the radius vector.

'A point moves in a circular path of radius $a$ so that its angular velocity about a fixed point in the circumference of the circle is constant and equal to $\omega$; show that the resultant acceleration of the point at every point of the path is of constant magnitude $4a\omega^2$.'

(LOND. B.Sc., 1911, II. 2.)

3. 'A particle is projected from the surface of the earth, with velocity $u$, along the radius; prove that if $u^2<2g\alpha$, the particle will come to rest at a distance $x$ from the centre of the earth, where

$$\frac{a}{x}=1-\frac{u^2}{2g\alpha}.$$

'The earth is regarded as a uniform gravitating sphere of radius $a$, and $g$ is the value of gravity at the surface.'

(LOND. B.Sc., 1912, II. 4.)

4. 'A light elastic string $OP$ of natural length $a$ and modulus of elasticity $E$ lies in a straight line upon a smooth table: the end $O$ is fixed to the table, and $P$ is attached to a mass $m$. The mass is now projected with velocity $V$ at right angles to the string. Prove that the greatest length $na$ to which the string is stretched in the ensuing motion satisfies the equation

$$\frac{n^2(n-1)}{n+1}=\frac{mV^2}{aE}.$$

(LOND. B.Sc., 1912, II. 6.)

5. 'A boy can throw a ball vertically upwards to a height $h$ feet. Show that he cannot throw the ball so as to clear a wall, $a$ feet away and $b$ feet above the level at which he throws, if $2h<b+\sqrt{a^2+b^2}$.'

(LOND. B.Sc., 1913, II. 2.)

6. 'A particle of mass $m$ moves under a central attraction $mf$; show that the orbit is given by the equation

$$\frac{d^2u}{d\theta^2}+u=\frac{f}{h^2u^2}.$$

'If $f=\mu u^2$, and the particle is projected at a distance $R$ with velocity $V$ in any direction, show that the path is a conic, and that its eccentricity $e$ is given by the equation

$$\mu^2(1-e^2)=h^2(2\mu/R-V^2).'$$

(LOND. B.Sc., 1913, II. 5.)

7. 'Obtain the expressions $\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2$, and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$ for the component accelerations in polar coordinates.

'A particle starts from rest under the action of a force which always acts at right angles to the radius vector from a fixed point, and is inversely proportional to that radius vector; show that the area which the radius sweeps out in any time is proportional to the square of the time, and that the kinetic energy acquired is proportional to the angle described about the fixed point.'

(LOND. B.Sc., 1914, II. 2.)

8. 'Prove that in polar coordinates $r$, $\theta$, the radial and transverse components of the acceleration of a moving point are

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2 \text{ and } \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).'$$

'A particle of mass $m$ describes the curve

$$r^2=a^2\cos^2\theta+b^2\sin^2\theta$$

under an attraction to the origin. Prove that the attraction at distance

$r$ is $mh^2\{2(a^2+b^2)r^2-3a^2b^2\}r^{-7}$, where $mh$ is the moment of momentum of the particle about the origin.'

(LOND. B.Sc., 1915, II. 4.)

9. 'A train travels from rest to rest between two stations 5 miles apart. The total mass moved is 200 tons; there is a constant road resistance of 2400 lbs. weight, and the engine exerts a uniform pull of 5 tons weight until the maximum speed of 30 miles per hour is reached. This speed is maintained until, steam being shut off, an *additional* resistance equal to ·075 the weight of the train is applied to bring the train to rest. Find the time between the stations. (Take $g=32$.)'

(LOND. B.Sc., 1915, II. 5.)

10. 'A heavy particle is supported at rest by a light vertical string of length $a$. It is projected horizontally with velocity $V$, where $2V^2=7ag$. Find where the string becomes slack, and show that, when the string becomes taut again, it is vertical.'

(LOND. B.Sc., 1916, II. 1.)

11. 'Prove that the components of acceleration of a particle moving in a plane curve are $vdv/ds$ along the tangent and $v^2/\rho$ along the normal.

'A particle is projected from the lowest point along the inside of a smooth elliptic ring, the minor axis of the ellipse being vertical. Find the pressure between the particle and the ring in any position, and show that the least velocity of projection that will carry the particle completely round the ring without losing contact is given by $u^2=g(a^2+4b^2)/b$, where $a$, $b$ are the semi-axes.'

(LOND. B.Sc., 1917, II. 6.

12. 'Find expressions for the tangential and normal accelerations of a particle moving in a plane curve.

'A smooth elliptical wire of axes $a$, $b$ rotates with constant angular velocity $\omega$ about the axis $a$ which is vertical. Show that a bead free to slide on the wire will remain in equilibrium relative to the wire at a depth $a^2g/b^2\omega^2$ below the centre.'

(LOND. B Sc., 1918, II. 4.)

13. 'Three particles of masses $m_1$, $m_2$, $m_3$ at the vertices $A$, $B$, $C$ of a triangle are connected by inextensible strings of lengths $a$, $b$, $c$. They are projected under gravity in the same vertical plane, so that the strings are all stretched. Prove that in the ensuing motion the angular velocity $\omega$ of the particles about their centroid is constant and that the tension in the string $BC$ is equal to

$$am_2m_3\omega^2/(m_1+m_2+m_3).'$$

(LOND. B.Sc., 1919, II. 3.)

14. 'Two small balls of masses $m$ and $m'$ impinge directly; prove that the loss of kinetic energy due to the impact is

$$\tfrac{1}{2}(1-e^2)mm'u^2/(m+m'),$$

where $u$ is their velocity of approach and $e$ is the coefficient of restitution.

'The balls are hung from a point by two strings, each of length $l$, and the ball of mass $m$ is then raised so that its string is fully extended horizontally; it is then released from rest. Prove that, if $m'/m=1+2/e$, the second impact occurs at the lowest point and find the loss of kinetic energy at this impact.'

(LOND. B.Sc., 1919, II. 4.)

15. 'A ball is projected from the ground at an angle $a$ to the horizontal and rebounds from a smooth vertical wall to the point of projection. If the line joining the point of projection to the point of impact makes an angle $\theta$ with the horizontal, prove that $(1+e)\tan\theta=\tan a$, where $e$ is the coefficient of restitution.'

(LOND. B Sc., 1920, II. 2.)

16. 'Two masses $m_1$, $m_2$ are in equilibrium on a wheel and axle. If the masses be interchanged, prove that the acceleration of the descending mass $m_1$ is

$$m_1(m_1 - m_2)g/(m_1^2 - m_1m_2 + m_2^2)$$

the friction and inertia of the machine being neglected.'

(LOND. B.Sc., 1920, II. 5.)

17. 'Two smooth spheres moving with given velocities collide obliquely. Obtain equations which give their velocities after impact.

'A smooth sphere of mass $m$ strikes a sphere of mass $nm$, which is at rest, so that immediately before impact the velocity of the former sphere makes an angle $\theta$ with the line joining their centres. Prove that the final velocity of this sphere makes a right angle with its first velocity, if

$$\tan^2\theta = (en - 1)/(n + 1),$$

where $e$ is the coefficient of restitution.'

(LOND. B.Sc., 1921, II. 5.)

18. 'Show that the work done in stretching a uniform elastic string is the product of the extension and the arithmetic mean of the initial and final tensions.

'A light elastic string of natural length $a$ hangs vertically with its upper end fixed and a weight attached to its lower end. Its length is then $l$. If the weight be depressed through a further distance $h$ and then released, show that it will just rise to the level of the top of the string if $h^2 = l^2 - a^2$.'

(LOND. B.Sc., 1921, II. 7.)

19. 'A particle which moves in a straight line, and is acted on by a force which works at a constant rate, changes its velocity from $u$ to $v$ in passing over a distance $x$. Prove that the time taken is

$$3(u + v)x/2(u^2 + uv + v^2).'$$

(LOND. B.Sc., 1922, II. 1.)

20. 'A packet is let fall from an airship momentarily at rest, and is acted on by gravity together with a resistance $Av^2$ per unit of mass, where $A$ is a constant. Find the velocity after falling a distance $x$ and show that the velocity tends towards the value $\sqrt{g/A}$, and that as the distance dropped increases in arithmetical progression the acceleration diminishes in geometric progression.'

(LOND. B.Sc., 1922, II. 2.)

## EXAMPLES—CXI.: RIGID DYNAMICS

1. 'A fly-wheel has a light cord coiled round its axle, and the cord is pulled with a constant force of $m$ pounds weight, until a length of $l$ feet has unwound, when the cord slackens and comes off. It is found that the wheel is then rotating $n$ times a second; prove that its moment of inertia in ft. lb. units is equal to

$$\tfrac{1}{2}ml/g/(n^2\pi^2).$$

'If a constant frictional force is now applied at a distance of $a$ feet from the axis equal to the weight of $m'$ pounds, show that the wheel will stop after

$$\frac{m}{m'}\,\frac{l}{a}\,\frac{1}{n\pi} \text{ seconds.}'$$

(LOND. B.Sc., 1911, III. 5.)

2. 'A lamina lying on a horizontal plane is acted on by simultaneous impulses in the plane. State (without proof) sufficient equations to determine the manner in which the lamina begins to move.

'A uniform square lamina $ABCD$ lies on a horizontal plane and is free to turn about the corner $A$, which is fixed. A blow is struck at $C$ along the edge $CD$. Prove that there is an impulse at $A$ which bears to the blow the ratio $\sqrt{(10)} : 4$.

(LOND. B.Sc., 1911, III. 3.)

3. 'Prove the theorem concerning the moments of inertia of a body about parallel axes.

'Prove that the moment of inertia of a thin spherical shell of mass $M$ and radius $a$ about a diameter is $\frac{2}{3}Ma^2$; and find the moments of inertia of a thin hemispherical bowl (i) about the radius through the centre of gravity, (ii) about a perpendicular line through the centre of gravity.'

(LOND. B.Sc., 1912, III. 2.)

4 'Obtain the equations of motion of a plane lamina acted on by impulsive forces in its plane.

'A uniform circular hoop lying on a smooth table receives a blow at a point $P$, the direction of the blow lying in the plane of the hoop and making an angle $\alpha$ with the radius through $P$. Show that $P$ begins to move in a direction inclined to the radius through $P$ at an angle $\tan^{-1}(2\tan\alpha)$.'

(LOND. B Sc., 1913, III. 4.)

5. 'Find the moment of inertia of a uniform circular disc, of mass $M$ and radius $a$, about a straight line through its centre at right angles to its plane.

'Such a disc can rotate with its plane vertical about a fixed horizontal axis through a point on its circumference. If it starts from rest in a position in which the radius through this point makes an angle $\alpha$ with the downward-drawn vertical, find its angular velocity when this radius makes an angle $\theta$ with the vertical.

'Find also, in this position, the components of the pressure on the axis, along and perpendicular to the radius through the point.

(LOND. B.Sc., 1914, III. 1.)

6. 'The mass of a flywheel is 100 lb. and a mass of 10 lb. hangs by a string wrapped round the axle, which is horizontal and has a radius of 2 in. If the mass of 10 lb. falls through 20 feet from rest in 16 seconds, show that the radius of gyration of the flywheel is a little over 9 inches.'

(LOND. B.Sc., 1914, III. 3.)

7. 'Prove that, with the usual notation, the kinetic energy of a body moving in two dimensions can be expressed in the form

$$\tfrac{1}{2}m(u^2+v^2+k^2\omega^2).$$

'A system consisting of a uniform isosceles triangular lamina and a particle, of mass one-third that of the lamina, attached to its vertex, is free to swing about the base of the triangle as a horizontal axis. If the system is started from rest in its position of unstable equilibrium, find its angular velocity in its lowest position.'

(LOND. B.Sc., 1915, III. 3.)

8. 'State and prove the principle of the conservation of linear momentum.

'Two uniform planks, of the same length $a$, but of weights in the ratio 3 : 1, are freely hinged together at one end and stand (like a common step-ladder) on a smooth horizontal plane, being kept from falling by a cord. If the cord is cut, show that the velocity of the hinge, when it reaches the ground, is $\sqrt{(3ga\cos\alpha)}$, where $2\alpha$ is the original angle between the planks. Show also that the hinge will have moved a horizontal distance $\frac{1}{4}a(1-\sin\alpha)$.'

(LOND. B.Sc., 1916, III. 4.)

9. 'An impulse is applied at any point of a lamina, which is moving in any manner in its own plane ; prove that the angular momentum about the point is unchanged.

'An impulse is applied at a point $P$ in the rim of a uniform circular disc in such a manner that $P$ starts to move along the tangent to the rim at $P$. Prove that the initial velocity of the centre is one-third of that of $P$.'

(LOND. B.Sc., 1916, III. 6.)

10. 'A thin circular cylindrical shell is placed at rest with its axis horizontal on a rough inclined plane. Show that it will begin to slide or roll down the plane as the inclination of the plane to the horizon is greater or less than $\tan^{-1} 2\mu$, where $\mu$ is the coefficient of friction between the cylinder and the plane.'

(LOND. B.Sc., 1917, III. 5.)

11. 'The moment of inertia of a body about any line being given, show how to find the moment about any parallel line.

'Find the moment of inertia of a rectangle about a line through its centre parallel to either pair of opposite sides. Find also the moments of inertia about the sides and about the diagonals.'

(LOND. B.Sc., 1918, III. 1.)

12. 'An equilateral triangular lamina is rotating in its own plane about its centroid, and one corner is suddenly fixed. Prove that the angular velocity is immediately reduced to one-fifth of its original amount, and find the loss of kinetic energy.'

(LOND. B.Sc., 1918, III. 5.)

13. 'Find the product and moments of inertia of a right-angled triangle at the right angle.

'The sides of a right-angled triangle are 5, 12 and 13. Find the directions of the principal axes of inertia at the right angle.'

(LOND. B.Sc., 1919, III. 3.)

14. 'Prove that if the moments of inertia of a lamina about two lines in its plane are equal, the principal axes of inertia at their intersection will bisect the angles between the lines.

'If $P$ and $Q$ are two points taken on the minor axis of an uniform elliptical lamina at equal distances from the centre $C$ and such that $2PC = 2QC = FC$ where $F$ is a focus of the ellipse, prove that the principal axes of inertia at any point $R$ in the plane of the ellipse bisect the angles between the lines $RP$, $RQ$.'

(LOND. B.Sc., 1920, III. 1.)

15. 'A lamina at rest on a horizontal table receives a horizontal blow at a given point. Prove that, whatever the magnitude and direction of the blow, the instantaneous centre of rotation lies upon a fixed straight line.'

(LOND. B.Sc., 1920, III. 4.)

16. 'A uniform solid sphere of radius $a$ is placed at rest on a rough plane inclined to the horizontal at an angle $\alpha$. Show that it will begin to roll down the plane without sliding unless $\mu$, the coefficient of limiting friction between the sphere and the plane, is less than $(2 \tan \alpha)/7$.'

(LOND. B.Sc., 1921, III. 3.)

17. 'Obtain the equations of motion of a system acted on by impulsive forces in two dimensions.

'Two uniform rods $AB$ and $BC$, alike in all respects, are freely jointed at $B$ and lie on a smooth horizontal plane with $AB$ and $BC$ in one straight line. $AB$ receives a horizontal blow at its middle point at right angles to its length. Show that the two rods begin to move with the same angular velocity.'

(LOND. B.Sc., 1921, III. 5.)

18. 'Given the moment of inertia of a lamina about any axis in its plane, state without proof how to find its moment of inertia about any parallel axis in its plane.

'Find the radius of gyration of a uniform circular arc, of radius $a$ and subtending an angle $2\alpha$ at its centre, about an axis perpendicular to its plane through the mid-point of the arc.'

(LOND. B.Sc., 1922, III. 1.)

19. 'Explain what is meant by the conservation of angular momentum.

'A uniform disc of mass $M$ and radius $a$ can turn freely in a horizontal plane about an axis at right angles to its plane through its centre. It is set rotating with angular velocity $\Omega$ and an insect of mass $nM$ crawls along a radius with uniform velocity $V$ relative to the disc, starting close to its centre. Show that when the insect reaches the edge of the disc, the disc has turned through an angle

$$(a\Omega \tan^{-1} \sqrt{2n})/(V\sqrt{2n}).'$$

(LOND. B.Sc., 1922, III. 5.)

20. 'Obtain the equations of motion of a system in two dimensions acted on by impulsive forces.

'Two uniform rods $AB$ and $BC$, alike in all respects, and freely jointed at $B$, rest on a smooth horizontal table with the angle $ABC$ a right angle. $AB$ is struck a blow horizontally at its middle point at right angles to its length. Show that the velocity of the centre of gravity of $AB$ is four times that of $BC$.'

(LOND. B.Sc., 1922, III. 7.)

## EXAMPLES—CXII.: STATICS

1. 'Show that a system of coplanar forces may be reduced to a single force through a chosen point in the plane together with a couple.

'The sums of the moments of a system of coplanar forces about three points whose coordinates referred to rectangular axes are $(a_1, b_1)$ $(a_2, b_2)$ $(a_3, b_3)$ are $M_1$, $M_2$, $M_3$ respectively; show that the resultant of the system is a single force through the origin of coordinates if

$$\begin{vmatrix} a_1 & b_1 & M_1 \\ a_2 & b_2 & M_2 \\ a_3 & b_3 & M_3 \end{vmatrix} = 0.'$$

(LOND. B.Sc., 1911, I. 1.)

2. 'From the principle of virtual work deduce the equations of equilibrium for a system of coplanar forces.

'Four equal uniform rods, each of weight $W$, are freely jointed at their extremities, forming a rhombus $ABCD$. The system is suspended from the joint $A$ and is kept in shape, so that the rods $AB$, $AD$ are inclined at an angle $\alpha$ to the vertical, by a *light* rod jointed to $B$ and $D$. Show that the thrust in $BD$ is $2W \tan \alpha$ and the reaction at the joint $C$ is $\frac{1}{2}W \tan \alpha$.'

(LOND. B.Sc., 1911, I. 8.)

3. 'Define a *couple*. How is it measured? And what effect does it tend to have upon a body?

'Why can a ship, that has twin propellers, turn in a shorter distance than it could if it had only a single propeller?'

(LOND. B.Sc., 1912, I. 3.)

4. 'Prove that a set of coplanar forces will be in equilibrium if, and only if, the algebraic sum of their moments vanishes about each of the vertices of a triangle in their plane.

'Given a triangle $ABC$ and a force $F$ in its plane, show that $F$ can be

resolved in one, and only one, way into three component forces acting along the sides of the triangle.

'If $p$, $q$, $r$ are the distances of $A$, $B$, $C$ from the line of action of $F$, find the values of these components.'

(LOND. B.Sc., 1913, I. 1.)

5. 'A uniform plank, inclined at the angle $\tan^{-1}\frac{3}{4}$ to the vertical, is placed with one end on the ground and the other against a vertical wall. The coefficient of friction between the plank and both the ground and the wall is $\mu$. Prove that equilibrium is impossible, if $\mu$ is less than $\frac{1}{3}$.

'If $\mu$ is greater than $\frac{1}{3}$, prove that the reactions are indeterminate, and that if $\mu$ is $\frac{3}{5}$, the friction between the plank and the wall cannot exceed $\frac{9}{32}$ of the weight of the plank.'

(LOND. B.Sc., 1913, I. 6.)

6. 'Prove that the resultant of two forces $l \cdot OA$ and $m \cdot OB$ is $(l+m)\,OC$, where $C$ divides $AB$ in the ratio $m : l$.

'Two rods $AB$, $CD$ are joined by strings $AC$, $BD$; the strings are tightened by equal and opposite forces $P$ applied to the rods at their mid-points $M$, $N$. Prove that either the rods or the strings must be parallel in equilibrium; and that in the former case the tensions $T_1$, $T_2$ in $AC$ $BD$ are given by

$$\frac{T_1}{AC} = \frac{T_2}{BD} = \frac{P}{2MN}.$$

(LOND. B.Sc., 1914, I. 4.)

7. 'State the laws of friction, limiting and non-limiting; and explain what is meant by the angle of friction.

'A uniform heavy plank rests in a horizontal position with its ends on two rough planes, which are inclined at equal angles ($\alpha$) to the horizontal, the vertical plane through the plank being perpendicular to the (horizontal) line of intersection of the planes, and it is in limiting equilibrium when a weight equal to its own is placed on one of its ends. If the coefficient of friction ($\tan\lambda$) is the same between the plank and each plane, prove that $\sin 2\lambda = \frac{1}{2}\sin 2\alpha$.'

(LOND. B Sc., 1915, I. 6.)

8. 'Obtain the equations of the common catenary.

$$y = c\cosh\frac{x}{c}. \quad s = c\sinh\frac{x}{c}.$$

'A uniform string of length $2l$ passes over two smooth pegs fixed at the same level at a distance $2a$ apart, and carries at each end a weight equal to a length $k$ of the string, the portion between the pegs hanging in a catenary of parameter $c$. Show that $c$ is given by the equation $ce^{a/c} = l + k$.'

(LOND. B.Sc., 1915, I. 10.)

9. 'Find the centre of gravity of a uniform triangular lamina.

'Deduce, or otherwise find the centre of gravity of a solid sphere, whose density at any point varies inversely as the distance from a fixed tangent plane.'

(LOND. B.Sc., 1916, I. 3.)

10. 'Prove that the moment of the resultant of any number of coplanar forces about a point in their plane is equal to the sum of their moments about the same point.

'Six weights in geometrical progression of common ratio $k$ are fixed at intervals of 60° round the rim of a wheel free to turn about a horizontal axis. Prove that the radius to the first weight is inclined at an angle $\theta$ to the horizontal where $k\sqrt{3}\tan\theta = 2 - k$.'

(LOND. B.Sc., 1917, I. 4.)

11. 'A heavy uniform sphere is held at rest on a rough inclined plane making an angle $\alpha$ with the horizontal by means of a string fastened to a point on the sphere and kept parallel to the lines of greatest slope. If $\mu > \frac{1}{2}\tan\alpha$ show that the sphere will remain at rest provided that the distance between the string and the plane exceeds $a/(1+\mu\cot\alpha)$, where $a$ is the radius of the sphere.'

(LOND. B.Sc., 1917, I. 9.)

12. 'Find the position of the centre of gravity of a sector of a circle.
'A portion of the annular area included between two circles of radii $a$, $b$ is cut off by radii inclined at 60° to each other. Find the position of the centre of gravity of this portion.'

(LOND. B.Sc., 1918, I. 7.)

13. 'Explain, with proof, the method of constructing the resultant of a system of forces by means of the funicular polygon.
'If a funicular polygon of four forces in equilibrium is a rectangle, prove that the sum of the squares of one pair of forces is equal to the sum of the squares of the other pair.'

(LOND. B.Sc., 1919, I. 3.)

14. 'A smooth parabolic wire is fixed with its axis vertical and its vertex downwards and the ends of a uniform rod of length greater than the latus rectum rest upon the wire. Prove that there is a position of unstable equilibrium in which the rod passes through the focus.'

(LOND. B.Sc., 1919, I. 8.)

15. 'Define a couple and find the resultant of any number of coplanar couples acting on a rigid body.
'$ABCD$ is a square and $E$ is the middle point of $CD$; forces $P$, $2P$ and $\sqrt{5}P$ acting along $AB$, $BC$ and $AE$ respectively together with a force $Q$ acting at the middle point of $AD$ are equivalent to a couple. Find the magnitude of $Q$ and show that its line of action bisects $DE$.'

(LOND. B.Sc., 1920, I. 2.)

16. 'State the law connecting the tension of an elastic string with its extension.
'When the lengths of the string are $a_1$ and $a_2$, the tensions are $T_1$ and $T_2$ respectively; find expressions for (i) the natural length of the string, (ii) its modulus of elasticity, (iii) the work done in stretching the string from its natural length to a length $b$.'

(LOND. B.Sc., 1920, I. 8.)

17. 'Prove that necessary and sufficient conditions for the equilibrium of a system of coplanar forces acting on a rigid body are that the moments of the system about each of three non-collinear points in the plane should vanish.
'Prove also that, if the system is not in equilibrium and its moments about three collinear points $A$, $B$, $C$ are $L$, $M$, $N$ respectively,

$$L.BC + M.CA + N.AB = 0.'$$

(LOND. B.Sc., 1921, I. 2.)

18. '$ABC$ is a plane figure formed by the axis $AB$, the ordinate $BC$, and the arc $AC$ of a parabola. A uniform solid of revolution is formed by the revolution of the area about the ordinate $BC$. Prove that the centre of gravity of this solid is at a point $D$ on $BC$ such that $BD : DC = 5 : 11$.'

(LOND. B.Sc., 1921, I. 7)

19. 'A uniform hemisphere rests with its curved surface in contact with an inclined plane rough enough to prevent slipping. Prove that, if the inclination of the plane to the horizon is less than $\sin^{-1}\frac{8}{9}$, there is a position of stable equilibrium.'

(LOND. B.Sc., 1922, I. 8.)

20. 'A uniform chain hangs from two fixed points at the same horizontal level. Obtain the relation between the horizontal span and the angle which the end tensions make with the horizon.'

'A uniform chain weighing 1 lb./ft. hangs from two fixed points at each of which the horizontal and vertical components are 30 lb. and 40 lb. respectively. Prove that the horizontal span is 65·9 ft. approximately.'

(LOND. B.Sc., 1922, I. 10.)

## EXAMPLES CXIII.: ATTRACTIONS

1. '$AB$ is a uniform thin rod, the mass of which per unit length is $m$. Show that the component parallel to the rod of the attraction of the rod on a particle of unit mass placed at a point $P$ is $\gamma m \{1/PA - 1/PB\}$.'

(LOND. B.Sc., 1911, II. 6.)

2. 'Find the potential of a solid homogeneous sphere of mass $M$ and radius $a$ at an external point and also at an internal point.

'If the mass of the sphere were unaltered but its density, instead of being uniform, were proportional at any point to the distance of that point from the centre, show that the potential at all external points would be unaltered, but the potential at an internal point distant $r$ from the centre would be decreased by the quantity

$$\gamma M(a-r)^2\,(a+2r)/6a^3.'$$

(LOND. B.Sc., 1911, II. 8.)

3. 'Prove that the direction of the attraction of a uniform rod $AB$ upon a particle $P$ bisects the angle $APB$; and deduce the form of the equipotential surfaces of the rod.'

(LOND. B.Sc., 1912, II. 7.)

4. 'Show that the potential of a circular disc, of mass $M$ and small uniform thickness, at any point on its axis is $2\gamma M/(r_1 + r_2)$, where $r_1$ and $r_2$ are respectively the distances of the point from the rim and the centre of the disc.

'If the thickness is small but not uniform, being proportional at any point to the square of the distance of that point from the centre of the disc, show that the potential is

$$\frac{4\gamma M}{3}\,\frac{(r_1+2r_2)}{(r_1+r_2)^2},$$

(LOND. B.Sc., 1913, II. 9.)

5. 'Show that a thin uniform spherical shell attracts all external points as if concentrated at its centre and exerts no attraction at any internal point.

'Deduce formulae for the potential at internal and external points respectively.'

(LOND. B.Sc., 1913, II. 10.)

6. 'Define a *tube of force*, and show that the intensity of attraction at any point in empty space of a tube of small cross-section varies inversely as the area of the cross-section of the tube at that point.

'Prove that the intensity of attraction due to an infinitely long homogeneous circular cylinder, at an external point, whose distance from its axis is $r$, is $2\gamma M/r$, where $M$ is the mass per unit length of the cylinder.'

(LOND. B.Sc., 1914, II. 9.)

7. 'Define gravitation potential, and prove that the intensity of attraction in any direction is equal to the rate of variation of the potential in that direction.

'Show that at any point inside a thin spherical shell the potential is constant; and hence determine its value.'

(LOND. B.Sc., 1914, II. 10.)

8. 'Define gravitational potential, and prove that the potential at the vertex of a thin solid cone of mass $M$ and length $l$ is $\frac{3}{2}\gamma M/l$.
'A shell is bounded by concentric spheres of radii $r_1$, $r_2$. Prove that the potential at the centre of any portion of this shell cut out by a conical surface with its vertex at the centre is $\frac{3}{2}\gamma M(r_1+r_2)/(r_1^2+r_1r_2+r_2^2)$, where $M$ is the mass cut out.'

(LOND. B.Sc., 1915, II. 9.)

9. 'Prove that the attraction of a homogeneous spherical shell upon an external body is the same as if its mass were collected at the centre, and that the attraction on an internal body is zero.
'A small smooth cylindrical hole is drilled through a solid uniform sphere. Show that a particle set free at the surface will oscillate in a period independent of the direction of the hole.'

(LOND. B.Sc., 1916, II. 8.)

10. 'Show that the attraction of a uniform thin shell in the form of a paraboloid of revolution, extending to infinity, upon a particle at its focus is $8\gamma\pi\sigma/3$, where $\sigma$ is the density per unit area and $\gamma$ the constant of gravitation.'

(LOND. B.Sc., 1916, II. 9.)

11. 'Define the potential of a system of attracting particles in terms of work. Deduce the value $\gamma\Sigma(m/r)$, and show that the component in any direction of the attraction of the system at any point is the space-rate of increase of the potential in that direction.'

(LOND. B.Sc., 1917, II. 9.)

12. 'Prove that the potential of a thin uniform spherical shell of mass $M$ and radius $a$ is $\gamma M/r$ at any external point at distance $r$ from the centre, and $\gamma M/a$ at any internal point.
'Show that two uniform spherical shells, external to one another, attract each other as if concentrated at their centres.'

(LOND. B.Sc., 1917, II. 10.)

13. 'Prove that the potential of a uniform solid sphere of radius $a$ and density $\rho$ at a point distant $r$ ($<a$) from the centre is $2\pi\gamma\rho(3a^2-r^2)/3$.
'If the density is proportional to the distance from the centre, show that the potential is $\frac{1}{3}\pi\gamma\rho'(4a^2-r^3/a)$, where $\rho'$ is the density at the surface.'

(LOND. B.Sc., 1918, II. 9.)

14. 'Prove that the component attraction at $P$ due to a uniform rod $AB$, resolved parallel to the rod, is equal to $\gamma m(1/AP-1/BP)$, where $m$ is the mass per unit length of the rod.
'When the length of the rod is very great compared with $AP$, show that the resultant attraction is given by the following construction:—Take $G$ on the rod so that $AG=AP$; then the attraction is along $PG$ and is equal to $2\gamma m/PG$.'

(LOND. B.Sc., 1919, II. 7.)

15. 'Find the attraction of a uniform thin circular plate at any point of its axis.
'Prove that the attraction of a homogeneous hemisphere of radius $a$ at the centre of its base is equal to $\pi\gamma\rho a$.'

(LOND. B.Sc., 1919, II. 8.)

16. Find the magnitude and direction of the attraction of a thin uniform straight rod at an external point.
'Prove that the attraction of a uniform lamina in the form of a segment of a circle at a point $P$ on the remaining arc of the circle acts through the middle point of the arc of the segment.'

(LOND. B.Sc., 1920, II. 8.)

17. 'Find the attraction of a uniform thin circular disc of radius $a$ at a point of its axis at a distance $h$ from the disc.

'A uniform solid circular cone of height $h$ is divided into two parts by a plane perpendicular to its axis at a distance $h'$ from the vertex; find the ratio of the attractions of the two parts at the vertex.'

(LOND. B.Sc., 1920, II. 9.)

18. 'Prove that the direction of the resultant attraction of a uniform straight rod $AB$ at a point $P$ bisects the angle $APB$.
'If the end $B$ is infinitely distant so that the rod is semi-infinite, and if the direction of the resultant attraction at $P$ meets the rod in $C$, prove that the magnitude of the resultant attraction is $2\gamma m/PC$ where $m$ is the mass per unit length of the rod.'

(LOND. B.Sc., 1921, II. 8.)

19. 'Find the potential of a uniform solid sphere at an internal point.
'A uniform solid sphere has an eccentric spherical cavity. Prove that the force at any point inside the cavity is constant in magnitude and direction.'

(LOND. B.Sc., 1922, II. 9.)

20. 'Find the attraction of a uniform circular disc at a point on its axis.
'Prove that the attraction of a uniform cylinder of density $\rho$, radius $r$ and length $h$, at the centre of one of its faces is

$$2\pi\gamma\rho\{h + r - \sqrt{h^2 + r^2}\}.'$$

(LOND. B.Sc., 1922, II. 10.)

## EXAMPLES—CXIV.: HYDROSTATICS

1. 'Prove that the centre of pressure of a triangle immersed in water with its base in the surface is at half the depth of the vertex, atmospheric pressure being neglected.
'If the atmospheric pressure is taken into account, and the depth of the vertex is $1/n^{\text{th}}$ of the height of the water-barometer, prove that the depth of the centre of pressure is to the depth of the vertex as

$$2n + 1 : 6n + 2.'$$

(LOND. B.Sc., 1911, III. 6.)

2. 'What is meant by the metacentre of a floating body?
'It is found that when 10 tons of water are transferred from one side to the other of a vessel of 8000 tons displacement the bob of a 10 ft. pendulum swings through 6 inches. Show that the height of the metacentre above the centre of gravity is $13\frac{1}{2}$ inches, given that the breadth of the deck is 45 feet.'

(LOND. B.Sc., 1911, III. 8.)

3. 'A cylindrical block of wood, of weight $W$, the area of whose section is $A$, is floating in a cylinder of water whose section is $B$: it is slowly pushed further under water. Show that the work done is equal to

$$\tfrac{1}{2} Pl(1 - A/B),$$

where $l$ is the distance between the first and last positions of the water-line on the block and $P$ is the pressure applied to the block in the final position.
'If the block is lifted from the floating position just clear of the water, calculate the work done in terms of $A$, $B$, $W$ and $w$, the weight of unit volume of water.'

(LOND. B.Sc., 1912, III. 9.)

4. 'A solid immersed in fluid has a plane face and a curved surface; show how to find the pressure on the curved surface.

'A right circular solid cone (of specific gravity greater than unity), in which the height is double the radius of the base, is suspended by a string attached to a point on its base and rests completely immersed in water, with the point of attachment of the string in the surface of the water. Show that the horizontal and vertical components of the resultant fluid pressure on its curved surface are respectively $\frac{6}{5}W$ and $\frac{8}{5}W$, where $W$ is the weight of the water displaced by the cone.'

(LOND. B.Sc., 1913, III. 6.)

5. 'A plane area is completely immersed in water and is in a vertical plane. If horizontal and vertical axes of reference are taken in the area, the origin being at the centroid, express the coordinates of the centre of pressure of the area in terms of the product and moments of inertia about these axes, and of the depth of the centroid.

'If the area is a square of side $2a$, and $h$ is the depth of its centre, show that the centre of pressure is on the vertical through the centre at a depth $\frac{a^2}{3h}$ below it, whatever the inclination of the sides of the square to the vertical.'

(LOND. B.Sc., 1913, III. 7.)

6. 'Investigate the conditions of equilibrium of a body floating freely, partly immersed in a liquid.

'The cross-section of a uniform log is an isosceles triangle $ABC$, with a right angle at $C$. The log floats in water with the edge $C$ immersed, the edge $A$ in the surface, and half the face $BC$ under water, equilibrium being maintained by a load fastened at the middle point of the edge $A$. Show that the weight of this load is one-ninth of the weight of the log, and that the specific gravity of the log is '45.'

(LOND. B.Sc., 1914, III. 7.)

7. 'Enunciate Boyle's law connecting the pressure and volume of a gas at a given temperature.

'A heavy air-tight piston can slide freely in a vertical cylinder of length $h$. Initially the air below it is at atmospheric pressure, and the piston is allowed to descend. Show that, if the air below the piston is assumed to obey Boyle's law, the piston will come to instantaneous rest at a distance $z$ from the bottom, given by the equation

$$\log_e (h/z) = (h - z)/k,$$

where $k$ is the distance from the bottom at which the piston would rest in equilibrium.'

(LOND. B.Sc., 1914, III. 9.)

8. 'A body is floating freely in a homogeneous fluid; prove the formula $HM = AK^2/V$ for the height of its metacentre above the centre of buoyancy.

'A log of length $l$ whose cross-section is a square of side $a$ floats in liquid of twice its density with a diagonal of each end horizontal. Prove that its equilibrium is stable for small angular displacements about its axis, but is only stable for small angular displacements about the horizontal diagonal of the central cross-section if $l > a$.'

(LOND. B.Sc., 1915, III. 9.)

9. 'Find the centre of pressure of a triangle, which has one corner in the free surface of a heavy fluid, and the opposite side parallel to the surface.

'A right circular cone of specific gravity $\frac{1}{2}$ floats in water with its axis in the free surface. Prove that the resultant fluid pressure on the curved surface immersed passes through the centre of the base, if the vertical angle of the cone is 60°.'

(LOND. B.Sc., 1916, III. 7.)

10. 'Prove that, if the atmosphere be in isothermal equilibrium, the density at a point at height $z$ above the point where the density is $\rho_0$, is equal to $\rho_0 e^{-z/H}$, where $H$ is the height of the homogeneous atmosphere of density $\rho_0$.
'If the specific gravity of mercury is 13·59, and that of air at the pressure of 76 cm. of mercury is ·00129, find the value of $H$.'
(LOND. B.Sc., 1916, III. 10.)

11. 'Show how to calculate the horizontal and vertical components of pressure on a surface immersed in a liquid.
'A cone, whose vertical angle is $2\alpha$, has its lowest generator horizontal and is filled with liquid; prove that the resultant pressure on the curved surface is $\sqrt{(1+15\sin^2\alpha)}$ times the weight of the liquid.'
(LOND. B.Sc., 1917, III. 7.)

12. 'A closed thin cylindrical vessel of height $a$ contains air at atmospheric pressure, and floats in water with its axis vertical and length $b$ immersed. If there is a small hole in the bottom of the vessel, show that water will leak in until there is a depth $ab/(h+b)$ inside, $h$ being the height of the water barometer.'
(LOND. B.Sc., 1917, III. 10.)

13. 'A thin straight cylindrical tube with its lower end closed floats in water: its length is $a$, its weight $W$ and the weight of the water it can hold is $aw$. Show that the smallest depth of water that must be poured into the tube in order that it may float upright is $\frac{1}{2}(aw-W)/w$. The thickness of the tube and the mass of the closed end may be neglected.'
(LOND. B.Sc., 1918, III. 8.)

14. 'Prove that when a plane lamina is immersed in water the depth of the centre of pressure below the centroid is inversely proportional to the depth of the centroid below the surface.
'A rectangular trap-door is in the side of a water-tank; it is hinged along the upper edge, which is horizontal. Calculate what force, applied at right angles to the lower edge, is required to keep the door shut.'
(LOND. B.Sc., 1919, III. 7.)

15. 'Explain what is meant by the metacentre of a floating body.
'If a small weight $w$ is transferred from one side to the other of a ship, a hanging pendulum is found to swing through a small angle $\theta$; prove that the height of the metacentre above the centre of gravity is equal to $wa/W\sin\theta$, where $a$ is the breadth of the deck and $W$ is the weight of the ship.'
(LOND. B.Sc., 1919, III. 9.)

16. 'Find the centre of pressure of a circular area immersed to any depth in a uniform liquid.
'A cone of vertical angle 60° is held immersed in a uniform liquid with its highest generator in the surface. Prove that the resultant thrust of the liquid on the curved surface passes through the centre of the base.'
(LOND. B.Sc., 1920, III. 7.)

17. 'Prove that the pressure at any point of a liquid rotating in a state of relative equilibrium is $g\rho h$, where $\rho$ is the density and $h$ the vertical depth below the free surface.
'A sphere floats half immersed in a rotating liquid with its centre on the axis of rotation. If the density of the sphere is one-third that of the liquid and its radius is 1 ft., prove that the angular velocity is $5\frac{1}{3}$ radians per second. [Take $g=32$ ft./sec.$^2$.]'
(LOND. B.Sc., 1920, III. 8.)

18. 'State and prove the conditions for the stability of a body floating freely.

'A uniform cube floats in water in stable equilibrium with two of its faces horizontal. Obtain limits for the value of its specific gravity.'

(LOND. B.Sc., 1921, III., 10.)

19. 'Define the metacentre of a solid of revolution floating in a homogeneous liquid and determine its height above the centre of buoyancy.
'A uniform right circular cone of vertical semi angle $\alpha$ and specific gravity $\sigma$ floats in water with its axis vertical and vertex downwards. Show that the equilibrium is stable if $\sigma > \cos^6\alpha$.'

(LOND. B.Sc., 1922, III. 9.)

20. 'Show that, if the temperature of the atmosphere be regarded as constant, the pressure at a height $z$ above the Earth's surface is given by

$$p = p_0 e^{-z/H}$$

where $p_0$ is the pressure at the surface and $H$ would be the height of the atmosphere, if homogeneous.
'Show further that, if the variation of gravity with height be taken into account, this relation becomes

$$p = p_0 e^{-\frac{az}{H(a+z)}}$$

where $a$ is the Earth's radius.'

(LOND. B.Sc., 1922, III. 10.)

## EXAMPLES—CXV.: PASSAGES FOR TRANSLATION

1. 'Translate the following passage :—
'Ein Korper, welcher keiner Kraft unterworfen ist kann nur eine geradlinige und gleichformige Bewegung haben.
'Liegt darin eine Wahrheit, welche sich dem Verstande *a priori* aufdrangt? Wenn dem so ware, wie konnten die Griechen sie dann verkennen? Wie hatten sie glauben konnen, dass die Bewegung in dem Augenblicke anhalt, in dem die Ursache, welche die Bewegung entstehen liess, aufhort? Oder wie konnten sie sogar glauben, dass jeder Körper, sobald ihm nichts in den Weg kommt, eine kreisformige Bewegung machen wurde, welche die angeblich vornehmste aller Bewegungen sein sollte?

(LOND. B.Sc., 1911, III. 10.)

2. 'Translate :—
'Si, par un procédé mécanique quelconque, on assujettit un point matériel à décrire une trajectoire donnée, on admet, comme fait d'expérience, qu'il exerce sur la matière avec laquelle il est en contact une action normale à la trajectoire, et qu'il est soumis à une autre action, égale et contraire à celle-là. Par exemple, un point matériel pesant, suspendu à l'extrémité d'un fil dont l'autre extrémité est fixe, et abandonné à lui-même, exerce à chaque instant sur le fil une action qui a pour effet de le tendre suivant une ligne droite, et reçoit, de la part du fil ainsi tendu, une réaction qui le maintient sur la trajectoire. Un point matériel qui se meut dans ces conditions se nomme un pendule simple.'

(LOND. B.Sc., 1911, II. 10.)

3. 'Translate the following :—
'Maupertuis hat einen auf das Gleichgewicht bezuglichen interessanten Satz gefunden, dem er den Namen 'Loi de repos' gegeben hat. Wenn wir an einen System unendlich kleine Verschiebungen vornehmen, so entspricht denselben eine Summe virtuellen Momenta $Pp + P'p' + P''p'' + \ldots$, welche nur im Gleichgewichtsfalle $= 0$ ist. Diese Summe ist

die den Verschiebungen entsprechende Arbeit. Wenn wir von einer gewissen Anfangsconformation des Systems ausgehen, und bis zu einer beliebigen Endconformation übergehen, so entspricht diese Procedur eine gewisse geleistete Arbeit.'

(LOND. B.Sc., 1911, I. 10.)

4. 'Translate :—

'C'est un fait ordinaire des mécaniques élémentaires de dire que le pendule simple est irréalisable. Il est beaucoup plus exact et plus intéressant de montrer qu'une balle métallique homogène de rayon $r$, suspendue au bout d'un fil très léger de longueur $l - r$, réalise un pendule simple de longueur $l$ avec une très grande approximation, à la seule condition que $l/r$ soit assez grand.

'L'erreur relative faite sur la période en assimilant le pendule composé à un pendule simple de longueur $l$ est $\frac{1}{5}(r/l)^2$.

'C'est avec un tel pendule que *Borda* fit à la fin du XVIII$^e$ siècle ses mémorables expériences sur la mesure de $g$.'

(LOND. B.Sc., 1912, II. 10.)

5. 'Translate :—

'Der Satz des Kräftenparallelogramms stellt sich in der Form, in welcher derselbe von Newton und Varignon gegeben wird, deutlich als ein Erfahrungssatz dar. Ein von zwei Kraften ergriffener Punkt fuhrt zwei von einander unabhängige Bewegungen mit den Kräften proportionalen Beschleunigungen aus. Darauf gründet sich die Parallelogrammkonstruktion. Daniel Bernoulli war nun der Meinung, dass der Satz des Kraftenparallelogramms eine geometrische (von physikalischen Erfahrungen unabhängige) Wahrheit sei.'

(LOND. B.Sc., 1912, I. 10.)

6. 'Translate :—

'Der grundlegende Begriff in der Hydrostatik ist der Begriff des Flussigkeitsdruckes. Archimedes scheint zu seiner Einfuhrung in die Mechanik durch die allgemeine Erfahrung veranlasst zu sein. Er stellte den Grundsatz auf, dass jeder Theil einer Flussigkeit von anderen Theilen gedruckt wird, und benutzte die so erhaltene Vorstellung vom Flussigkeitsdruck zum Aufbau einer Theorie schwimmender Körper.'

(LOND. B.Sc., 1912, III. 10.)

7. 'Translate :—

'Les planètes, dans leurs mouvements autour du soleil, obéissent aux lois suivantes, que le génie de Kepler a fait jaillir des observations de Tycho-Brahé ;

'(i) Les planètes se meuvent dans les courbes planes et leurs rayons vecteurs décrivent des aires proportionelles aux temps :

'(ii) Les orbites des planètes sont des ellipses dont le soleil occupe un foyer :

'(iii) Les carrés des durées des révolutions sidérales des planètes autour du soleil sont entre eux comme les cubes des grands axes de leurs orbites.'

(LOND. B.Sc., 1913, II. 11.)

8. 'Translate :—

'Der Begriff der Kraft ist uns anschaulich durch das Gefühl des Druckes, den ein in die Höhe gehobener Körper auf die Hand ausubt. Dieser Druck kann sehr verschieden an Stärke sein je nach Grösse und Materie des Korpers. Da er immer nach dem Erdmittelpunkt gerichtet ist, so schreiben wir ihn der Einwirkung der Erde auf die Korper zu, und sprechen daher von einer Kraft, die die Erde auf die Körper ausubt. Daher sagen wir :—Die Erde ubt auf verschiedene Korper verschieden

starke Kräfte aus, die wir die Gewichte der Körper nennen. Das Gewicht ist eine durch sinnliche Beobachtung erkannte Eigenschaft eines Körpers.'

(LOND. B.Sc., 1913, I. '10.)

9. 'Translate :—

'Stevin fand auf seinem eigenen Wege die wichtigsten Sätze der Hydrostatik und deren Ableitungen wieder. Es sind hauptsachlich zwei Gedanken, aus welchen Stevin seine fruchtbaren Folgerungen schopft. Der ein Gedanke ist ganz ahnlich demjenigen betreffend die geschlossene Kette. Der andere besteht in der Annahme, dass die Erstarrung der im Gleichgewicht befindlichen Flussigkeit das Gleichgewicht nicht stört.'

(LOND. B.Sc., 1913, III. 10.)

10. 'Translate :—

'Ein Korper, der einen andern druckt oder zieht, wird nach Newton von dem andern ebenso viel gedruckt oder gezogen. Druck und Gegendruck, Kraft und Gegenkraft sind einander stets gleich. Da Newton in der Zeiteinheit erzeugte Bewegungsgrösse (Masse × Geschwindigkeit), als Kraftmaass definirt, so folgt, das aufeinander wirkende Körper sich in gleichen Zeiten gleiche entgegengesetzte Bewegungsgrossen ertheilen, oder entgegengesetzte ihren Massen umgekehrt proportionirte Geschwindigkeiten annehmen.'

(LOND. B.Sc., 1914, I. 10.)

11. 'Translate :—

'Le pendule, oscillant sous l'action de la pesanteur, permet de donner une démonstration rigoureuse de la première loi de la chute des corps, à savoir que la pesanteur agit sur tous également. Il suffit de constater que la durée de l'oscillation est absolument indépendante de la nature du corps constituant le pendule.

'On assure même que ce fut l'observation de l'égalité de durée des oscillations de deux pendules également longs mais differemment pesants, qui conduisit Galilée à la découverte de cette loi fondamentale de la pesanteur. Bessel fit des mesures précises ; opérant avec un pendule de Borda, dont la boule était tour à tour constituée par les corps les plus différents (métaux, ivoire, pierres, etc.), il trouva que les durées de l'oscillation différaient à peine de ·00001 d'un corps à l'autre.'

(LOND. B.Sc., 1914, II. 11.)

12. 'Translate :—

'Das Experiment ist die einzige Quelle der Wahrheit : dieses allein kann uns etwas Neues lehren ; dieses allein kann uns Gewissheit geben. Das sind zwei Punkte, die durch nichts bestritten werden können

'Wenn aber das Experiment alles ist, welcher Platz bleibt dann fur die mathematische Physik ubrig? Was hat die Experimental-Physik mit einem solchen Hilfsmittel zu schaffen das unnutz und wohl gar gefahrlich zu sein scheint?

'Und dennoch existiert die mathematische Physik ; sie hat unleugbare Dienste geleistet : darin liegt eine Tatsache, die notwendigenweise erklart werden muss.'

(LOND. B.Sc., 1914, III. 10.)

13. 'Translate :—

'Stevin (1548-1620) untersuchte zuerst die mechanischen Eigenschaften der schiefen Ebene und zwar auf eine ganz originelle Weise. Liegt ein Gewicht auf einem horizontalen Tisch, so sieht man, weil der Druck senkrecht gegen die Ebene des Tisches ist, nach dem bereits mehrfach verwendeten Symmetrieprincip das Bestehen des Gleichgewichts sofort ein. An einer verticalen Wand hingegen wird ein Gewicht an seinen

Fallbewegung gar nicht gehindert. Die schiefe Ebene wird also einen Mittelfall zwischen die beiden Grenzfallen darbieten. Das Gleichgewicht wird nicht von selbst bestehen, wie auf die horizontale Unterlage, es wird aber durch ein geringeres Gegengewicht zu erhalten sein, als an der verticalen Wand.'

(LOND. B.Sc., 1915, I. II.)

14. 'Translate :—

'Ein anderes Beispiel, welches zeigt, welche Vereinfachung die Beschreibung naturlicher Bewegungen durch die Einfuhrung des Begriffs der Kraft erfahrt, ist die Bewegung der Planeten um die Sonne. Es ist diese mit einem gewissen Grade der Genauigkeit beschrieben durch die sogenannten Kepler'schen Gesetze ; es wird uns gelingen diese zusammen zu fassen in einen Satz von grosser Einfachheit.

'Nach dem ersten Kepler'schen Gesetze bewegt ein Planet sich so, dass sein von der Sonne gezogener radius vector in gleichen Zeiten gleiche Flachenraume beschreibt ; nach dem zweiten ist die Bahn eines Planeten eine Ellipse, in deren einen Brennpunkte die Sonne steht.'

(LOND. B.Sc., 1916, II. II.)

15. 'Translate :—

'In der Kinematik ist es weder notwendig noch zweckdienlich, auf eine Definition des Massenbegriffs einzugehen. Dies uberlasst man besser der Dynamik. Es ist vollstandig hinreichend, sich auf den Standpunkt Kirchhoffs zu stellen, der in der Masse, welche einem Punkte beizulegen ist, nichts weiteres sieht, als einen Zahlfaktor des Geschwindigkeits- oder Beschleunigungsvektors, der die Beschreibung der Bewegungserscheinungen auf bequeme Weise gestattet. Jedenfalls ist diese Auffassung eine sehr nuchterne und zugleich dem geschichtlichen Entwicklungsgange der Mechanik nicht widersprechende.'

(LOND. B.Sc., 1917, II. II.)

16. 'Translate :—

'Wir bermerken noch, dass die Auszeichnung des einen Systems vor dem anderen, welche in den Worten "beweglich" und "fest" liegt, vom Standpunkte der reinen Kinematik eigentlich ungerechtfertigt ist und dass es richtiger wäre, etwa von einem ersten und zweiten System zu sprechen. Geometrisch ist nämlich jede Bewegung mit ihrer umgekehrten, bei welcher die Rolle des beweglichen und des festen Systemes vertauscht ist, gleichberechtigt. In der Kinematik handelt es sich also immer nur um Relativbewegungen. Anders in der Kinetik : die zur Erzeugung einer Bewegung erforderlichen Krafte andern sich durchaus wenn wir das bewegliche System mit dem festen vertauschen.'

(LOND. B.Sc., 1919, II. II.)

17. 'Translate :—

'Als Newton entdeckte dass die Bewegungen der Himmelskorper mit einer ausserordentlichen Genauigkeit vorausberechnet werden können unter der Annahme, dass sie sich nach seinem beruhmten Gesetz gegenseitig anziehen, betrachteten weder er noch seine Zeitgenossen die Frage als erschöpft. Trotz ihrer grossen Einfachheit hatte diese Erklarung der Bewegungen der Himmelskorper durch die Einfuhrung einer Kraft, die ohne Zwischenmedium in der Ferne, oder, was beinahe auf dasselbe hinauskommt, momentan wirkt, fur jene Männer etwas höchst unwahrscheinliches und abstossendes'

(LOND. B.Sc., 1920, I. II.)

18. 'Translate :—

'La Mécanique repose sur un petit nombre de principes qu'il est impossible de vérifier directement et auxquels on a été conduit par une

longue suite d'inductions ; les conséquences qu'on en déduit sont vérifiées par l'observation. La première idée de ces principes remonte à Galilée qui, dans l'étude des lois de la chute des corps (plan incliné, pendule, mouvement parabolique), a introduit les notions d'inertie, d'accélération, de composition des mouvements. Huygens fut le continuateur de Galilée dans la théorie du mouvement d'un point : il étudia le premier le mouvement d'un système matériel ; enfin Newton étendit le champ de la Mécanique par la découverte de la loi d'attraction universelle.'

(LOND. B.Sc., 1920, II. 11.)

19. 'Translate :—

'Fur alle Fälle berechnet man den Bodendruck wenn man den Flächeninhalt des Bodens multipliziert mit der Flussigkeitshöhe und mit dem specifischen Gewichte der Flüssigkeit. Fur das cylindrische Gefass ist dies leicht begreiflich, weil nach dieser Berechnung das Gewicht der Flüssigkeitssäule herauskommt, die auf dem horizontalen Boden recht. Folgt es aus dem hydrostatischen Gesetz dass in einem Gefass mit Flussigkeit der Druck auf ein Flächeneinheit nur von der Tiefe des betreffenden Flächenstückes unter der Oberfläche und dem specifischen Gewicht der Flüssigkeit, nicht aber von der Gestalt der Gefasswande abhangt.'

(LOND. B.Sc., 1921, III. 11.)

20. 'Translate :—

'Der Einfachheit halber werden wir die Kernmässe als unendlich in Verhältniss von Masse ansehen, und ferner auch von den kleinen Änderungen in der Bewegung absehen welche die von der Relativitatstheorie geforderte Veranderlichkeit der Masse mit seiner Geschwindigkeit bedingt.

'Unter diesen Annahmen wird das Electron nach der gewöhnlichen mechanischen Vorstellungen eine geschlossene elliptische Bahn beschreiben mit dem Kerne in dem einen Brennpunkte und die Umlaufzahl und die grosse Achse dieser Bahn werden durch einfachen Formeln, die den Keplerschen Gesetzen Ausdruck geben, mit der Energie des Systems verknupft sein.'

(LOND. B.Sc., 1922, II. 11.)

## EXAMPLES—CXVI. : MISCELLANEOUS

1 'Find the position of the centre of gravity of the area of a segment of a circle.

'A circular lamina of radius $a$ is divided into two parts by a chord which subtends an angle of 60° at the centre of the circle ; show that the distance of the centre of gravity of the greater part from the centre of the circle is $a/(10\pi + 3\sqrt{3})$.'

(LOND. B.Sc., 1911, I. 3.)

2. 'A rod rests in a vertical plane within a hemispherical bowl whose radius is equal to the length of the rod. If $\mu$ is the coefficient of friction between the rod and the bowl show that in limiting equilibrium the rod makes an angle with the horizon equal to $\tan^{-1}\{4\mu/(3-\mu^2)\}$.'

(LOND. B.Sc., 1911, I. 4.)

3. 'Find a formula for the length of the simple equivalent pendulum when a heavy rigid body oscillates about a fixed horizontal axis.

'The centre of gravity of a bicycle wheel is in its axis. When a small valve of mass $m$ is fixed to the rim at a distance $h$ from the axis, and the axis is held horizontally, the wheel oscillates in the same period as

a simple pendulum of length $l$. What is the moment of inertia of the wheel (apart from the valve) about the axis?'

(LOND. B.Sc., 1911, III. 2.)

4. 'A tram-car of mass $M$ has four wheels each of radius $a$, and $I$ is the moment of inertia of each wheel about its axis; the brakes are applied on each wheel with pressure $P$ and coefficient of friction $\mu$. Prove that the retardation of the car is equal to

$$4\mu P/(M + 4I/a^2),$$

no matter at what point of the rim the brake is applied to the wheel, if axle-friction is neglected and the wheels do not skid.'

(LOND. B.Sc., 1911, III. 4.)

5. 'If a particle moves in a straight line towards a centre of force which attracts according to the inverse square of the distance, starting from rest at a distance $2a$ from the centre, show that the time of motion from the distance $2a$ to the distance $a$ is to the time from the distance $a$ to the centre in the ratio $\pi + 2 : \pi - 2$.'

(LOND. B.Sc., 1911, II. 3.)

6. 'Find from mechanical considerations the equation $s = c \tan \psi$ for the common catenary and deduce the equation $y = c \sec \psi$.

'A uniform heavy chain hangs in the form of a catenary with its extremities attached to two pegs, $P$ and $Q$, which are at the same level. $P'$ and $Q'$ are two points of the catenary where the tangents make with the vertical acute angles which are twice the acute angles made with the vertical by the tangents at $P$ and $Q$. Show that the chain could hang over two smooth pegs at $P'$ and $Q'$ with its extremities hanging freely and vertically and the portion between the pegs hanging in the same catenary as before. Show also that the resultant pressures on the pegs at $P'$ and $Q'$ would be the same as those on the pegs at $P$ and $Q$.'

(LOND. B.Sc., 1911, I. 9.)

7. 'Show that a system of forces represented in direction, magnitude, and *position* by the sides of a closed polygon is equivalent to a couple.

'$ABCD$ is a quadrilateral in which the sides $BC$ and $AD$ are parallel. If forces $l_1AB$, $l_2BC$, $l_3CD$, $l_4DA$ acting along $AB$, $BC$, $CD$, $DA$ respectively are equivalent to a couple, show that $l_1 = l_3$ and $(l_1 - l_4)AD = (l_1 - l_2)BC$.'

(LOND. B.Sc., 1911, I. 7.)

8. 'Prove that the form of the free surface of liquid rotating about a vertical axis is a paraboloid of revolution.

'A cylinder whose axis is vertical and whose top is closed is partly filled with water and is made to rotate about the axis. Prove that for speeds at which the liquid partly covers the top of the cylinder and wholly covers the bottom, the angular velocity is directly proportional to the distance of the vertex of the paraboloid below the top of the cylinder.'

(LOND. B.Sc., 1911, III. 7.)

9. 'Determine the direction of projection which (for a given velocity) produces the greatest range on a horizontal plane.

'A cricketer throws the ball 100 yards; prove that (neglecting air-resistance) the time taken is about $4\frac{1}{3}$ secs. if he throws in the direction determined above. Prove that the necessary velocity of projection is about 67 miles an hour.'

(LOND. B.Sc., 1912, II. 3.)

10. 'Prove the formula for the velocity at any point of an elliptic orbit described under the action of a force which varies inversely as the square of the distance.

'Regarding the earth's orbit as circular, prove that the eccentricity of the orbit of a comet, whose period is 27 years, and whose smallest distance from the sun is half the radius of the earth's orbit, is 17/18. Compare the greatest and least velocities of the comet with the velocity of the earth.'

(LOND. B.Sc., 1912, II. 5.)

11. 'A uniform square lamina is rotating in its own plane about the middle point of one of its sides; one of the ends of that side is then seized and held fixed, the middle point of the side being released. Prove that the angular velocity is reduced in the ratio 8 : 5.'

(LOND. B.Sc., 1912, III. 3.)

12. 'An Atwood's machine has a pulley whose moment of inertia is $I$, and whose radius is $a$; the masses attached at the ends of the string are each $M$, and the rider has mass $m$. Prove that the acceleration of the masses is $f$, where

$$\frac{g}{f} = 1 + 2\frac{M}{m} + \frac{I}{ma^2},$$

assuming that the string does not slip on the pulley and neglecting axle-friction.'

(LOND. B.Sc., 1912, III. 4.)

13. 'Show that the resultant of forces represented by $m.OA$ and $n.OB$ is $(m+n)\,OC$, where $C$ is the centre of mean position of masses proportional to $m$ and $n$ placed at $A$ and $B$ respectively.

'A uniform triangular lamina $ABC$ is suspended from a point $O$ by two strings $OA$ and $OB$ attached to the corners $A$ and $B$. If the tensions in these strings are proportional to their lengths, show that the straight line joining $O$ to $C$ must pass through the middle point of $AB$.'

(LOND. B.Sc., 1912, I. 1.)

14. 'Find the centre of gravity of the area inside a parabola bounded by a chord perpendicular to the axis.

'Tangents are drawn to a parabola, of vertex $A$, from a point $T$ in the axis. Prove that the centre of gravity of the area, between the parabola and the two tangents, divides $TA$ in the ratio 4 : 1.'

(LOND. B.Sc., 1912, I. 6.)

15. 'State the "Principle of Virtual Work" and prove it for the case of a system of forces in one plane acting at a point.

'A uniform solid prism, the cross section of which is a rectangle, rests with two edges, but not an entire face, in contact, one with each of two smooth fixed inclined planes which have a common horizontal straight line and make equal angles with the vertical. Show that, whether the two edges are adjacent or opposite, the plane which contains them must be horizontal.'

(LOND. B.Sc., 1912, I. 8.)

16. 'Find the equations of the common catenary,

$$y = c\cosh\frac{x}{c} \text{ and } s = c\sinh\frac{x}{c}.$$

'A uniform wire of length $2l$ is stretched between two points at the same height, distant $2a$ apart, the difference between $l$ and $a$ being a small fraction of either of them. Show that the least tension in the wire is approximately the weight of a length of it equal to $\sqrt{a^3/6(l-a)}$, and that the ratio of the greatest tension to the least is approximately $(3l-2a)/a$.'

(LOND. B.Sc., 1912, I. 9.)

17. 'Prove that, in homogeneous liquid at rest under gravity, the pressure increases uniformly with the depth below the surface.
'A door in the vertical side of a cistern can open by turning about a horizontal axis through its centre of gravity; it is prevented from doing so by a bolt at a point of its perimeter. Prove that (whatever be the shape of the door), when the door is entirely below the surface of the water, the force on the bolt is independent of the depth of the water.
'If the door is a square of side 9″, two of whose sides are horizontal, and the bolt is in the lowest side, prove that the force on the bolt is nearly $4\frac{1}{2}$ lbs. weight.'*

(LOND. B.Sc., 1912, III. 7.)

18. 'Determine the vertical pressure on any area immersed in homogeneous liquid.
'A wine glass when filled holds one-fifth of a pint;* its rim is a circle of diameter $2\frac{1}{4}$ inches, and the greatest depth of the contained liquid is $2\frac{2}{3}$ inches. The glass, filled with water, is inverted on a horizontal plane. Prove that equilibrium is impossible in this position if the weight of the glass when empty is less than 2·13 oz. [Take $\pi = 3{\cdot}1416$].'

(LOND. B.Sc., 1912, III. 8.)

19. 'A particle of mass $m$ attached to a string of length $a$, the other end of which is fixed, describes a circle in a vertical plane. Show that, if $V$ be its velocity when at the level of the centre, the tension in the string when the particle is at height $z$ above the centre is

$$m(V^2 - 3gz)/a.'$$

(LOND. B.Sc., 1913, II. 3.)

20. 'A projectile discharged with an initial velocity $V$ is subject to a retardation $kv^m$. Show how to find its velocity at the end of any time, and the space traversed, considering the motion to be in a horizontal straight line.
'If the velocity of projection is 3600 ft. 1 sec. show that it will be reduced to 2600 ft. 1 sec. in approximately one second, the values of $m$ and $k$ for this range of velocities being given as $m = 1{\cdot}55$, $\log k = 3{\cdot}60905$.'

(LOND. B.Sc., 1913, II. 6.)

## EXAMPLES—CXVII.: MISCELLANEOUS

1. 'Show how to find the product and moments of inertia of a plane lamina about two straight lines in its plane, when those about parallel axes through the centre of gravity of the body are known.
'Find the product and moments of inertia of a uniform rectangular lamina, of mass $M$, and sides $2a$ and $2b$, about two adjacent sides.'

(LOND. B.Sc., 1913, III. 1.)

2. 'Show that the rate of change of momentum of a material system in a fixed direction is equal to the component in that direction of the external force acting on the system.
'A uniform solid circular cylinder, of mass $M$, can rotate freely about its axis, which is fixed in a horizontal position; a light inextensible string is coiled round the cylinder and carries at its free end a particle of mass $m$. If the system is allowed to move, show that the particle will descend with uniform acceleration $2mg/(M + 2m)$.'

(LOND. B.Sc., 1913, III. 2.)

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* A cubic foot of water weighs 1000 oz.; a pint of water weighs 20 oz.

3. 'Prove that when a body of mass $M$ is moving in two dimensions, its kinetic energy is equal to $\frac{1}{2}(Mu^2 + I\omega^2)$, where $u$ is the velocity of the centre of inertia, $\omega$ is the angular velocity of the body, and $I$ is the moment of inertia about an axis through the centre of inertia perpendicular to the plane of motion.

'A uniform thin hollow cylinder, of radius $a$, has a particle of equal mass attached to it at a point in its inner surface. The cylinder is placed on a perfectly rough horizontal plane in its position of unstable equilibrium and is slightly disturbed. If $V$ is the velocity of the axis of the cylinder when it has turned through an angle $\theta$, show that

$$V^2(2 + \cos\theta) = ag(1 - \cos\theta).'$$

(LOND. B.Sc., 1913, III. 5.)

4. 'A force whose components parallel to the rectangular axes of $x$ and $y$ are $X$, $Y$, acts at a point $x$, $y$. Show how to replace it by forces along the axes and a couple.

'A set of coplanar forces is in equilibrium. Each force is now turned in the same sense about its point of application through a right angle. Prove that the new system is in general equivalent to a couple, and find the moment of this couple.'

(LOND. B.Sc., 1913, I. 3.)

5. 'Find the centre of gravity of a sector of a circle.

'Deduce by orthogonal projection, or obtain otherwise, the position of the centre of gravity of a sector of an ellipse cut off by two conjugate semi-diameters.'

(LOND. B.Sc., 1913, I. 4.)

6. 'A body with a spherical base of radius $a$ is balanced on the top of a fixed rough sphere of radius $b$: prove that the equilibrium will be stable if $\frac{1}{h} > \frac{1}{a} + \frac{1}{b}$ where $h$ is the height of the centre of gravity above the point of contact.

'If the body is a uniform hemisphere of radius $a$, show that the equilibrium will be stable provided that $3b > 5a$.'

(LOND. B.Sc., 1913, I 8.)

7. 'Obtain the intrinsic equation $s = c\tan\phi$ for the uniform catenary, and deduce the expressions for $x$ and $y$ in terms of $\phi$.

'A uniform wire of length $l$ hangs in a nearly flat catenary between two points distant $a$ apart on the same level; show that approximately

$$\frac{l}{a} - 1 = \frac{1}{24}\left(\frac{a}{c}\right)^2.$$

'The horizontal tension in a telegraph wire is not to exceed the weight of 1500 yards of the wire, and the wire is to be carried, in spans of 100 yards each, over a distance of 20 miles. Prove that the least length of the wire is nearly 20 feet more than the 20 miles.'

(LOND. B.Sc., 1913, I. 9.)

8. 'Explain the terms 'potential' and 'equipotential surface.'

'Show that the direction of the force of attraction at a point due to an attracting body is normal to the equipotential surface through the point.'

(LOND. B.Sc., 1913, II. 7.)

9. 'Show that the resultant attraction at any point $P$ due to a uniform rod $AB$, of mass $M$, bisects the angle $APB$.

'Deduce that the equipotential surfaces of the rod are confocal prolate spheroids, and that the potential at any point $P$ is

$$\frac{\gamma M}{AB}\log\left(\frac{AP + PB + AB}{AP + PB - AB}\right).'$$

(LOND. B.Sc., 1913, II. 8.)

10. 'Define the term metacentre in connection with floating bodies.
'A hollow buoy of sheet-metal of uniform thickness is in the form of a hemisphere with a conical top. Show that, if it floats in stable equilibrium, with part of the spherical surface immersed, the ratio of the height of the cone to the radius of its base must not exceed 8 : 5 approximately.'

(LOND. B.Sc., 1913, III. 8.)

11. 'Show that, on the hypothesis of constant temperature, the density in the atmosphere at height $z$ above the earth's surface is $\rho_0 e^{-z/h}$, where $\rho_0$ is the density at the surface, and $h$ a constant length.
'Taking the height of the water barometer as 33·8 feet, and the specific gravity of air at the earth's surface as ·0013, show that $h = 26{,}000$ feet.'

(LOND. B.Sc., 1913, III. 9.)

12. 'A heavy particle of mass $m$ falls from rest in a medium in which the resistance is $mk$ (velocity); show that the velocity acquired and the space fallen in time $t$ are given by the equations

$$v = \frac{g}{k}(1 - e^{-kt}) \text{ and } x = \frac{gt}{k} - \frac{g}{k^2}(1 - e^{-kt}).'$$

(LOND. B.Sc., 1914, II. 3.)

13. 'A bullet is fired, with velocity due to a height $h$, from a point in a plain at a tower standing in the plain. Prove that it will not reach the tower if the distance of the latter from the firing point is greater than $2h$; and that, in order to strike the tower at a point as far above the ground as possible, the bullet must be aimed at a point in the vertical line of the tower whose height above the plain is $2h$.'

(LOND. B.Sc., 1914, II. 4.)

14. 'A heavy particle is attached to an elastic string, whose other end is fixed, and hangs freely. Show that as long as the string is extended, the motion in the vertical is simple harmonic about the position of equilibrium.
'The modulus of elasticity of the string is one half the weight of the particle, and its natural length is $a$. The particle is let fall from rest at the fixed point to which the string is attached. Show that the string attains its greatest length,

$$a(3 + 2\sqrt{2}), \text{ in time } \frac{3\pi + 4}{4}\sqrt{\frac{2a}{g}}.'$$

(LOND. B.Sc., 1914, II. 5.)

15. 'Prove that the hodograph of a particle, which is describing an ellipse under an attraction to the centre, is a similar ellipse.
'Two particles $P$ and $Q$, each of unit mass, describe the same ellipse under the same attraction to the centre $C$, in such a way as to be always at the extremities of conjugate diameters of the ellipse. Show that, if $\mu$ is the attraction at unit distance, the relative velocity of the particles is $2\mu^{\frac{1}{2}}CR$, where $R$ is the point midway between $P$ and $Q$.'

(LOND. B.Sc., 1914, II. 6.)

16. 'Obtain the formula, $v^2 = \frac{2\mu}{r} - \frac{\mu}{a}$, for the velocity of a particle describing an ellipse under an acceleration $\frac{\mu}{r^2}$ to a focus.
'If, when the particle is at an extremity of the minor axis, the intensity of attraction is doubled, prove that the axes of the new orbit are $\frac{4}{3}a, \frac{2}{\sqrt{3}}b$, where $2a$, $2b$ are the old axes.'

(LOND. B.Sc., 1914, II. 7.)

17. 'A uniform solid circular cylinder is placed at rest with its axis horizontal on a rough plane inclined at an angle $\alpha$ to the horizon. Show that it will begin to roll down the plane without sliding, if the coefficient of friction between the cylinder and the plane is not less than $\frac{1}{3}\tan\alpha$.'

(LOND. B.Sc., 1914, III. 2.)

18. 'Prove that necessary and sufficient conditions for the equilibrium of a system of coplanar forces are that the sums of their moments about three non-collinear points are zero.

'Forces $p.BC$, $q.CA$, $r.AB$ act along the sides of a triangle $ABC$ taken in order; show that their resultant passes through the centroid of the triangle, if $p+q+r=0$.'

(LOND. B.Sc., 1914, I. 1.)

19. 'Prove that two couples in the same plane are equivalent if their moments are equal.

'The ends of a uniform rod, 10 inches long, are attached to two fixed points, 24 inches apart, by two equal elastic strings, each of natural length $6\frac{1}{2}$ inches. In the position of equilibrium the tension of each string is $T$ poundals, the rod and strings being in the same straight line. Show that the moment of the couple necessary to hold the rod at rest, when turned through a right angle about its centre, is $10\,T$ in ft. poundals.'

(LOND. B.Sc., 1914, I. 2.)

20. 'A smooth hemispherical bowl of radius $a$ contains a rod of $n$ times its weight, and is suspended from a point of its rim. Show that, if in the position of equilibrium the rod lies entirely within the bowl, it must be horizontal. Prove also that the inclination of the plane of the rim to the horizontal is $\tan^{-1}2(n+1)$.

'[The centroid of a hemispherical bowl bisects the radius to its vertex.]'

(LOND. B.Sc., 1914, I. 3.)

## EXAMPLES—CXVIII.: MISCELLANEOUS

1. 'Find by direct integration the moment of inertia of a triangle about a side.

'Prove that a triangle of mass $m$ is equimomental with three equal particles of mass $\frac{1}{12}m$ at the corners and a particle of mass $\frac{9}{4}m$ at the centroid.'

(LOND. B.Sc., 1915, III. 2.)

2. 'Prove that, if the forces acting on a material system have no component in a given direction, the component of momentum of the system in that direction is constant.

'A uniform smooth circular tube of mass $M$, containing within it a particle of mass $m$, lies on a smooth horizontal plane. The tube is projected along the plane with velocity $V$ in a direction at right angles to the diameter through the particle; prove that the tube does not rotate, and that when the particle has described a quadrant of the tube, the component velocities of the centre of the tube are $MV/(M+m)$, $mV/(M+m)$, parallel to and perpendicular to its original direction of motion.'

(LOND. B.Sc., 1915, III. 5.)

3. 'Find the position of the centre of gravity of a uniform semicircular lamina.

'The cross-section of a homogeneous prism is a semicircle of radius $a$, with an isosceles triangle of height $h$ described on its bounding

diameter as base. Show that the prism can rest in stable equilibrium with its curved surface in contact with a horizontal plane, if $h^2 < 2a^2$.'

(LOND. B.Sc., 1915, I. 7.)

4. 'Explain the terms *shearing stress*, *bending moment* as applied to a loaded beam.

'Draw diagrams to represent the shearing stress and the bending moment for a light horizontal beam, 12 feet long, supported by a prop 1 foot from one end and a prop 2 feet from the other end, and loaded with 36 lbs. at its middle point.'

(LOND. B.Sc., 1915, I. 8.)

5. 'State and explain the principle of virtual work.

'$AB$, $BC$ are two uniform heavy beams of equal weight and of lengths $a$ and $b$, smoothly jointed at $B$. They are placed with $B$ on a smooth horizontal plane and with their other ends $A$ against one and $C$ against another of two parallel smooth vertical walls, whose distance apart is $c$. Prove that $y$, the height of their centre of gravity, is given by the equation $16y^2 = a^2 + b^2 + 2ab\cos(\alpha - \beta) - c^2$, where $\alpha$, $\beta$ are the inclinations of the beams to the horizontal.

'Prove that there is a position of equilibrium in which $\alpha = \beta = \cos^{-1}\{c/(a+b)\}$, and that it is unstable.'

(LOND. B.Sc., 1915, I. 9.)

6. 'Investigate the conditions of equilibrium of a body floating completely immersed in two liquids which do not mix.

'If a prism of triangular section floats with one edge in the common surface of two liquids which do not mix, show that the opposite face of the prism must be vertical, the prism being wholly immersed.'

(LOND. B.Sc., 1915, III. 8.)

7. 'Prove that the sum of the moments about any point of two intersecting forces is equal to the moment of their resultant about that point.

'Four forces act along the sides of a quadrilateral, and are in equilibrium. Prove that each force is inversely proportional to the perpendicular on it from the intersection of the diagonals, and state the directions of the forces necessary for equilibrium.'

(LOND. B.Sc., 1916, I. 2.)

8. 'Show that any force which does not lie in the plane of the triangle $ABC$ can be uniquely represented by three forces parallel to the given force and acting at $A$, $B$, $C$, respectively.

'A table is supported by three vertical legs placed at the corners of a triangle $ABC$. Show that, if a weight is placed on the table at the orthocentre of the triangle $ABC$, the additional thrusts on the legs are as $\tan A : \tan B : \tan C$.'

(LOND. B.Sc., 1916, I. 4.)

9. 'Find necessary and sufficient conditions of equilibrium of a system of coplanar forces.

'Four rods smoothly jointed at their extremities form a cyclic quadrilateral $ABCD$, the opposite corners being joined by strings in a state of tension. Prove that the tensions in $AC$ and $BD$ are as $\sin A : \sin B$.'

(LOND. B.Sc., 1916, I. 5.)

10. 'Find the angular velocity in any position of a compound pendulum, which is swinging about a smooth horizontal axis.

'A uniform rod, of length $2a$, hinged at one end to a fixed point $O$, is let fall from the horizontal position; when it becomes vertical the hinge breaks. Prove that, when the rod is next horizontal, the horizontal and the vertical distances of its middle point from $O$ are $\pi a/2$ and $a + \pi^2 a/12$ respectively.'

(LOND. B.Sc., 1916, III. 3.)

11. 'A light string connecting two equal particles passes through a smooth ring fixed at the highest point of a smooth circular wire in a vertical plane. The length of the string is greater than the diameter of the circle, and one particle hangs freely, while the other is constrained to move on the circle. Show that the position of equilibrium in which this particle is at the lowest point of the circle is stable, and the position in which it is at angular distance 60° from the highest point is unstable.'

(LOND. B.Sc., 1916, I. 9.)

12. 'A circular hoop of weight $W$, free to turn about a point on its circumference, rests on a rough horizontal plane on which it presses evenly. Show that the least force, applied to the hoop, that will move it horizontally is $2\mu W/\pi$, where $\mu$ is the coefficient of friction.'

(LOND. B.Sc., 1916, I. 7.)

13. 'State and explain the principle of virtual work.
'Four equal uniform rods $AB$, $BC$, $CD$, $DE$, each of weight $W$, are suspended from fixed points $A$ and $E$ in the same horizontal. A string connects the middle points of $BC$ and $CD$. Show that, if $\alpha$ and $\beta$ are the angles made by $AB$ and $BC$ with the horizontal, the tension of the string is $W(3\cot\alpha - \cot\beta)$.'

(LOND. B.Sc., 1916, I. 8.)

14. 'Prove the formula, $HM = Ak^2/V$, for the height of the metacentre above the centre of buoyancy of a solid of revolution floating, with its axis vertical, in homogeneous fluid.
'Show that a uniform circular cylinder, of sp. gr. 1/3, cannot be in stable equilibrium, when floating upright in water, if its length exceeds three-quarters of its diameter.'

(LOND. B.Sc., 1916, III. 9.)

15. 'A particle is projected from the origin along the axis of $x$ with velocity $u$, and has an acceleration $g - \mu y$ parallel to the axis of $y$. Show that it describes a sine curve, whose convolutions touch the $x$-axis.'

(LOND. B.Sc., 1917, II. 4.)

16. 'Find the principal moments of inertia at the centroid of a homogeneous rectangular lamina of sides $2a$, $2b$.
'Prove that, if $a = 2b$, two of the principal moments of inertia at the middle point of either longer side are equal.'

(LOND. B.Sc., 1917, III. 2.)

17. 'Show that any system of coplanar forces not in equilibrium is equivalent to a single force or a single couple.
'Four forces are represented completely by the sides $AD$, $BC$ and the diagonals $CA$, $DB$ of a quadrilateral $ABCD$. Show that they are equivalent to a couple of moment $AB \cdot CD \sin\theta$ where $\theta$ is the angle between $AB$ and $CD$.'

(LOND. B.Sc., 1917, I. 1.)

18. 'A heavy uniform beam 16 feet long is supported at points distant one foot from each end. Draw curves showing the bending moment and shearing stress at all points of the beam.'

(LOND. B.Sc., 1917, I. 3.)

19. 'Find the position of the centre of gravity of the area of a segment of a circle.
'The area of a circle of radius $a$ is divided in the ratio 2 : 1 by a chord. Show that the distance between the centres of gravity of the two parts is $3(a/\pi)\sin^3\alpha$ where $3(\alpha - \sin\alpha\cos\alpha) = \pi$.'

(LOND. B.Sc., 1917, I. 5.)

20. 'State the principle of virtual work.
'A rhombus, of four equal light rods of length $a$ smoothly jointed has its opposite corners joined by elastic strings of the same material and of natural lengths $a\sqrt{2}$ and $\frac{1}{3}a\sqrt{2}$ respectively, and rests on a horizontal plane. Show that in equilibrium an angle of the rhombus is 30°.'
(LOND. B.Sc., 1917, I. 6.)

## EXAMPLES—CXIX.: MISCELLANEOUS

1. 'Obtain the expression $Ak^2/V$ for the height of the metacentre of a floating body above the centre of buoyancy.
'A uniform circular cylinder, whose radius is two-thirds of its height, floats in water with its axis vertical. Prove that the equilibrium cannot be stable, if the specific gravity of the cylinder lies between $\frac{1}{3}$ and $\frac{2}{3}$.'
(LOND. B.Sc., 1917, III. 9.)

2. 'A mass of 100 lb. hangs freely from the end of a rope. The mass is hauled up vertically from rest by winding up the rope. The pull starts at 150 lb. weight and diminishes uniformly at the rate of 1 lb. weight for every foot wound up. Find the velocity after 50 ft. have been wound up, neglecting the weight of the rope.'
(LOND. B.Sc., 1918, II. 5.)

3. 'Three light strings meet at a point $P$ where they are knotted together. They pass over three pulleys $A$, $B$, $C$ and each supports a weight $W$. Show that the knot can only be in equilibrium within the triangle $ABC$ if none of the angles of this triangle exceeds 120°. If there were four strings and four coplanar pulleys at the corners of a convex quadrilateral where would the knot rest?'
(LOND. B.Sc., 1918, I. 1.)

4. 'Show that a system of coplanar forces is in equilibrium if the algebraic sum of their moments about each of three points not in a straight line vanishes.
'Three equal forces $P$ act along the sides of a triangle $ABC$ taken in order; show that they are equivalent to forces $P\sin\frac{A}{2}$, $P\sin\frac{B}{2}$, $P\sin\frac{C}{2}$ acting along the external bisectors of the angles $A$, $B$, $C$ respectively.'
(LOND. B.Sc., 1918, I. 3.)

5. 'Establish formulae for determining the position of the centre of gravity of a system of particles of known masses and positions in a plane.
'Prove that if two particles of the system are interchanged the position of the centre of gravity of the whole system is unchanged only if the particles have equal masses.'
(LOND. B.Sc., 1918, I. 5.)

6. 'A uniform rod of weight $W$ can turn about one extremity on the rough horizontal plane on which it lies. A force $P$ is applied at one end, in a fixed direction, initially at right angles to the rod which is at rest. Show that the rod will not begin to move unless $P > \frac{1}{2}\mu W$, where $\mu$ is the coefficient of friction; and find in what position the rod will subsequently come to rest.'
(LOND. B.Sc., 1918, III. 6.)

7. 'State and prove the conditions of equilibrium of a floating body.
'Prove that a cubical block of specific gravity ·75 can float in water with one edge in the surface.'
(LOND. B.Sc., 1918, III. 7.)

8. 'A vertical semicircle of radius $a$ is immersed in water with its diameter in the surface. Show that the depth of the centre of pressure is $3\pi a/16$. The semicircle being a plane face of a spherical quadrant, find the resultant pressure on the curved surface.
'[The centre of gravity of a semicircle is at a distance $4a/3\pi$ from its centre, and of a hemisphere $3a/8$ from its centre.]'

(LOND. B Sc., 1918, III. 9.)

9. 'Two points describe coplanar concentric circles of radii $a$ and $b$ $(a<b)$ with uniform angular velocities $\omega$ and $\omega'$ respectively $(\omega>\omega')$; initially they are collinear with the centre and on the same side of it. Prove that the angular velocity of either point relative to the other is zero after a time which is equal to

$$\frac{1}{(\omega-\omega')}\cos^{-1}\frac{a^2\omega+b^2\omega'}{ab(\omega+\omega')}.$$

(LOND. B.Sc., 1919, II. 2.)

10. 'The position of a point in a plane is determined by its polar co-ordinates $(r, \theta)$; find expressions for its component accelerations along and perpendicular to the radius vector.
'A particle is acted on by an attractive force $\mu r^{-n}$ $(n<3)$ to a fixed point and its orbit is nearly circular; prove that the apsidal angle is equal to

$$\pi/(3-n)^{\frac{1}{2}}.'$$

(LOND. B.Sc., 1919, II. 6.)

11. 'Explain how any uniplanar motion of a rigid body can be obtained by the rolling of a curve fixed in the body on a curve fixed in space.
'A rod passes through a fixed point and one extremity moves along a fixed line. Prove that the motion can be obtained by the rolling of a curve, of the type $r\cos^2\theta=\text{const.}$, upon a certain parabola.'

(LOND. B.Sc., 1919, III. 1.)

12. 'A particle is projected in a medium in which the resistance varies as the cube of the velocity, and the effect of other forces may be neglected; the time $t_0$ is observed which the particle takes in travelling from $P$ to $Q$, prove that the velocity at the mid-point of $PQ$ is equal to $s_0/t_0$, where $s_0$ is the length of $PQ$.'

(LOND. B.Sc., 1919, III. 2.)

13. 'Find a formula for the length of the simple equivalent pendulum when a uniform bar of length $l$ swings about a point distant $c$ from its centre.
'Show that for a bar equal in length to the seconds pendulum the least time of a beat (from rest to rest), for different points of suspension, is about $\frac{3}{4}$ second.'

(LOND. B.Sc., 1919, III. 4.)

14. 'A truck has a total mass $M$; it has two pairs of wheels and axles, each pair having a moment of inertia $I$ and a radius $a$. Prove that if the truck when moving with velocity $V$ is brought to rest by striking a fixed obstacle, the total impulse is equal to $(M+2I/a^2)V$.'

(LOND. B.Sc., 1919, III. 5.)

15. 'A uniform thin cylindrical shell with axis horizontal is placed in contact with the highest generator of a fixed equal cylinder with axis horizontal and coefficient of friction $\frac{1}{2}$. Prove that the motion changes from rolling to sliding when the plane through the axes of the cylinders makes an angle $\tan^{-1}\frac{3}{4}$ with the vertical, the shell starting from rest.'

(LOND. B.Sc., 1919, III. 6.)

16. 'Prove that a system of coplanar forces can, in an infinite number of ways, be replaced by two forces each acting through a given point.

'If $A$ and $B$ are the points and $AA'$ and $BB'$ represent such a pair of forces, prove that the loci of $A'$ and $B'$ are straight lines and that $A'B'$ always passes through a fixed point.'

(LOND. B.Sc., 1919, I. 1.)

17. 'Prove that two coplanar couples of equal and opposite moments are in equilibrium.

'$ABCD$ and $A'B'C'D'$ are two coplanar parallelograms; prove that forces acting along and proportional to $AA'$, $CC'$, $B'B$, $D'D$ are equivalent to a couple.'

(LOND. B.Sc., 1919, I. 2.)

18. 'Explain what is meant by the angle of friction.

'One end of a bar rests on a rough plane which is inclined at an angle $a$ to the horizon and the other end is supported by a vertical force. Show that the angle of friction is not less than $a$ and that the bar can be supported in this manner at any inclination to the vertical.'

(LOND. B.Sc., 1919, I. 6.)

19. 'Determine the position of the centre of gravity of a hollow triangular prism with open ends formed from three rectangular sheets of metal of uniform thickness, the ends of the prism being perpendicular to the edges.

'The cross section of such a prism is a triangle $ABC$ with an obtuse angle at $C$ and the prism is placed with the face $BC$ in contact with a horizontal plane; prove that it will topple over if

$$2a^2 + a(c - b) < c^2 - b^2.'$$

(LOND. B.Sc., 1919, I. 9.)

20. 'Prove that, with the usual notation, the equations

$$s = c \tan \psi, \quad y^2 = s^2 + c^2$$

are satisfied by the common catenary.

'A uniform chain of length $l$ lies in a straight line on a rough horizontal table, the coefficient of friction being $\mu$. One end of the chain is slowly raised in a vertical plane through the chain until the chain is on the point of slipping along the table. Prove that, if this end is at a height $z$ above the table, the length of the portion in contact with the table is

$$l + \mu z - \{(\mu^2 + 1)z^2 + 2\mu l z\}^{\frac{1}{2}}.'$$

(LOND. B.Sc., 1919, I. 10.)

## EXAMPLES—CXX.: MISCELLANEOUS

1. 'A sphere of weight $W$ and radius $a$ floats half-immersed in a large expanse of water. Find an expression for the work done in (i) lifting it just completely out of the water and (ii) just immersing it completely.'

(LOND. B.Sc., 1919, III. 8.)

2. 'Determine the potential at any point within a sphere of uniform density $\rho$.

'Within a sphere (centre $A$), a spherical cavity is scooped out (centre $B$); show that inside the cavity the equipotential surfaces are planes perpendicular to $AB$, and that on the plane bisecting $AB$ at right angles and passing through the cavity the value of the potential is $2\pi\gamma\rho\ (a^2 - b^2)$, where $a$, $b$ are the radii of the spheres.'

(LOND. B.Sc., 1919, II. 9.)

3. 'A particle of unit mass falls from rest under gravity in a medium whose resistance is $k$ times the velocity; prove that the distance fallen through in time $t$ is

$$g(kt - 1 + e^{-kt})/k^2.$$

'If the particle is projected initially horizontally with a velocity $u$, find the horizontal and vertical velocities after a time $t$ and hence prove that the hodograph of its path is a straight line.'

(LOND. B.Sc., 1920, II. 4.)

4. 'A uniform square lamina of mass $M$ lying on a horizontal table has a particle of mass $m$ attached to one corner by an inextensible string. The particle is projected with velocity $V$ along the table in a direction along one of the sides through the point of attachment. Prove that the loss of energy of the system when the string becomes taut is

$$\tfrac{1}{2}mV^2 \Big/ \left(1 + \frac{5}{2}\cdot\frac{m}{M}\right).'$$

(LOND. B.Sc., 1920, III. 3.)

5. 'Prove that the kinetic energy of a two-dimensional body of mass $M$ moving in its own plane can be expressed in the form $\frac{1}{2}M(u^2+k^2\omega^2)$, where $u$ is the velocity of its centre of gravity and $\omega$ its angular velocity.

'Find the kinetic energy of a rhombus of freely-jointed uniform rods in terms of the velocity of the centre of the rhombus and the angular velocities of its sides.'

(LOND. B.Sc., 1920, III. 6.)

6. 'Prove that in general any system of coplanar forces acting on a rigid body can be reduced to a single force and mention the exceptions.

'Forces $P$, $Q$, $P+Q$ act along the sides $BC$, $CA$, $BA$ respectively of a triangle $ABC$; prove that, however $P$ and $Q$ vary, the resultant of the three forces passes through a fixed point.'

(LOND. B.Sc., 1920, I. 1.)

7. 'Any number of coplanar forces have fixed points of application. If all the forces are turned round their points of application in their plane through the same angle and in the same sense, prove that their resultant will always pass through a fixed point.

'Find the coordinates of this point for two perpendicular forces $X$ and $Y$ through the points $(a, 0)$ and $(0, b)$ respectively.'

(LOND. B.Sc., 1920, I. 3.)

8. 'Show how to find the position of the centre of gravity of a number of coplanar masses when their distances from any two intersecting lines in their plane are known.

'Four small bags of shot are placed at fixed points $A$, $B$, $C$, $D$ of a plane, and a weight $W$ of shot is transferred from $A$ to $B$ and an equal weight $W$ from $C$ to $D$. Show that in general, as $W$ varies, the centre of gravity of the four bags of shot describes part of a straight line.'

(LOND. B.Sc., 1920, I. 7.)

9. 'A uniform solid paraboloid of revolution with a plane base perpendicular to its axis has a circular cylindrical hole drilled symmetrically through it along its axis. Prove that the centre of gravity of the remaining solid is at a distance $\frac{1}{3}l$ from the base of the paraboloid, where $l$ is the length of the hole.'

(LOND. B.Sc., 1920, I. 9.)

10. 'Prove the formula $x = c \log(\sec\phi + \tan\phi)$ for the common catenary, where $\phi$ is the angle made with the horizon by the tangent at any point.

'A chain 100 ft. long hangs in a common catenary, the directions of the tensions at the ends making angles of 45° with the horizon. Calculate the span of the chain and the sag of its lowest point.'

(LOND. B.Sc., 1920, I. 10.)

11. 'A cylindrical diving-bell is lowered with uniform velocity $v$ ft. per second, air being pumped into it as it descends. Prove that in order that the level of water in the bell may remain unaltered, air at the rate per second of $vV/H$ cubic feet at atmospheric pressure must be pumped in, where $V$ is the volume of the bell occupied by air and $H$ the height of the water barometer.'

(LOND. B.Sc., 1920, III. 9.)

12. 'A particle describes a hyperbola about a centre of force attracting as the inverse square of the distance. Show that, with the usual notation,

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right).$$

'A particle approaches such a centre of force, its velocity at infinity being $V$ along an asymptote whose perpendicular distance from the centre of force is $p$. Find the angle between the asymptotes of the orbit, and show that, when $V$ is large, this is approximately $\frac{2\mu}{pV^2}$.'

(LOND. B.Sc., 1921, II. 4.)

13. 'Prove that, if $Mk^2$ is the moment of inertia of a body of mass $M$ about an axis through its centre of gravity, its moment of inertia about a parallel axis distant $a$ from the other axis is $M(k^2 + a^2)$.

'Find the moments of inertia of a uniform semicircular lamina of mass $M$ and radius $a$ about an axis perpendicular to its plane and passing through (i) its centre, (ii) a point on its circumference.'

(LOND. B.Sc., 1921, III. 1.)

14. 'A disc of mass $m$ is moving in any manner in a plane when a point $P$ of the disc is suddenly fixed. Show how to determine the subsequent motion, and show that the loss of kinetic energy is $\frac{1}{2}mv^2\left(1 - \frac{p^2}{k^2}\right)$, where $v$ is the velocity of $P$ immediately before being fixed, $p$ the perpendicular from the centre of gravity on the direction of $v$ and $k$ the radius of gyration about an axis through $P$ perpendicular to the plane of the disc.'

(LOND. B.Sc., 1921, III. 6.)

15. 'Three equal uniform rods each weighing 3 lbs. and 5 feet in length are jointed at one end to form a tripod and stand on a smooth horizontal table. The middle points of the rods are joined by 3 light strings, each 18 ins. long; find the tension in each string when a weight of 10 lbs. is placed on the tripod.'

(LOND. B.Sc., 1921, I. 3.)

16. 'Find formulae to determine the position of the centre of gravity of a system of coplanar particles whose weights and positions are given.

'A uniform wire is bent to form a plane rectilineal figure $ABCD$, where $AB = 3$ ins., $BC = 2$ ins., $CD = 1$ in., the angles at $B$ and $C$ are right angles and $A$, $D$ are on the same side of $BC$. If the wire is hung freely from the corner $B$, prove that the side $AB$ makes with the vertical an angle $\tan^{-1}(\frac{4}{5})$.'

(LOND. B.Sc., 1921, I. 5.)

17. 'Prove the formula $y = c\cosh x/c$ for the common catenary.

'A wire rope weighing 2 lb. per foot has a horizontal span of 150 ft. and the sag in the middle is 5 ft. Prove that the horizontal tension throughout is 1125 lb. wt. approximately.'

(LOND. B.Sc., 1921, I. 10.)

18. 'Find the potential of a uniform gravitating sphere at an internal point.

'Show that the work done in raising a given mass from a mine of depth $h$ to the Earth's surface is less than that calculated from the value

of $g$ at the surface in the ratio $1 : 1 - h/d$, where $d$ is the diameter of the Earth, which is assumed to be uniform.'

(LOND. B.Sc., 1921, II. 9.)

19. 'Find the attraction of a uniform circular disc at a point on its axis.
'A particle is constrained to move on a smooth wire which coincides with the axis of a disc in the shape of a uniform annulus of mass $M$ and external and internal radii $a$, $b$. Show that, if displaced from its position of equilibrium at the centre of the disc, it will execute small oscillations of period

$$\pi\left\{\frac{2ab(a+b)}{\gamma M}\right\}^{\frac{1}{2}},$$

(LOND. B.Sc., 1921, II. 10.)

20. 'A particle is describing an ellipse under a force to its centre. Find expressions for its acceleration along the principal axes of the ellipse in terms of the eccentric angle $\phi$ of its position, and the derivatives of $\phi$. Hence show that $\frac{d\phi}{dt}$ is constant and that the force must vary as the direct distance.'

(LOND. B.Sc., 1922, II. 3.)

## EXAMPLES—CXXI.: MISCELLANEOUS

1. 'Show that a uniform thin spherical shell exerts no attraction on a particle placed at any point inside it.
'If the shell is divided into two unequal portions along the circumference of a circle, show that the attraction of either portion on a particle placed at the centre of the shell is proportional to the area of the circle.'

(LOND. B.Sc., 1911, II. 7.)

2. 'Find the attraction of a uniform thin circular plate at a point on its axis; and show that the attraction is discontinuous as the point passes from one side of the plate to the other, the difference between the attractions at the two sides being $4\pi\gamma\sigma$, where $\sigma$ is the density of the plate.'

(LOND. B.Sc., 1911, II. 9.)

3. 'Prove that the surface integral of the normal attraction of any material system taken over a closed surface is equal to $4\pi\gamma M$, where $M$ is the mass of the part of the system which lies inside the surface, and $\gamma$ is the constant of attraction.
'Deduce the attraction at any point exerted by a thin homogeneous spherical shell bounded by two concentric spheres.'

(LOND. B.Sc., 1912, II. 8.)

4. 'Prove that the attraction at points within a uniform sphere is $gr/a$, where $r$ is the distance from the centre, $a$ is the radius of the sphere, and $g$ is the value of gravity at the surface.
'Show that if a smooth fine tunnel is made in the sphere, the motion of a particle along the tunnel will be simple harmonic: and that the period of the motion is the same for all positions of the tunnel.'

(LOND. B.Sc., 1912, II. 9.)

5. 'A smooth trough is formed of two planes each making an angle $\alpha$ with the horizontal. Three equal cylinders are laid symmetrically in the trough with the two lowest cylinders each touching one of the planes, and the uppermost resting on the other two. Show that (in the absence of any friction between the cylinders) the two lower cylinders will not separate if

$$\tan\alpha > 1/(3\sqrt{3}).$$

'[The axes of the cylinders are horizontal and parallel to the edge of the trough.]'

(LOND. B.Sc., 1914, I. 5.)

6. 'The radii of the circular ends of a frustum of a uniform solid right circular cone are $a$ and $b$, and its axial thickness is $h$; show that its centroid is at a distance from the middle point of the axis equal to

$$\frac{h}{4}\,\frac{a^2-b^2}{a^2+ab+b^2},$$

measured along the axis towards the end of larger radius $a$.'

(LOND. B.Sc., 1914, I. 6.)

7. 'State the principle of virtual work.

'Show that, if a step-ladder consists of two equal halves, each of weight $W$, and a load $W'$ is placed on the top of the ladder, the tension in the cord is equal to $(W+W')b^2/2ah$, where $2a$ is the length of the cord, $h$ is the height of the ladder, and $2b$ the distance between its ends.

'[The cord is supposed to be fastened to the two halves at equal distances from the top.]'

(LOND. B.Sc., 1914, I. 7.)

8. 'A uniform plank of weight $W$ and length $2a$ is supported at its ends and loaded (i) with a weight $2W$ at a distance $\frac{1}{2}a$ from one end (ii) with two weights each equal to $W$ at distances $\frac{1}{2}a$ from the two ends.

'Draw graphs to show the variation of the bending-moment in the plank, in each of the two cases. Prove that in case (i) the maximum value is $\frac{5}{16}Wa$ and that in case (ii) it is $\frac{3}{4}Wa$.'

(LOND. B.Sc., 1914, I. 8.)

9. 'Prove the equations $s=c\tan\phi$, $y=c\sec\phi$ for the common catenary, and show that if a point at arcual distance $s$ from the vertex be at height $h$ above it, $(s^2-h^2)/h$ is constant.

'A chain of length $2l$ is suspended from two points in the same horizontal and has a sag $h$ in the middle; show that the span is

$$\frac{l^2-h^2}{h}\log_e\left(\frac{l+h}{l-h}\right).'$$

(LOND. B.Sc., 1914, I. 9.)

10. 'A particle is projected from a point $P$ to pass through another point $Q$, at a horizontal distance 150 ft. from $P$ and 200 ft. vertically *below* $P$. The velocity of projection is that due to falling freely under gravity through $81\frac{1}{4}$ ft. Find the two possible angles of projection; and taking $g=32$ f.s.s. show that the two times of flight are $2\frac{1}{2}$ sec. and $6\frac{1}{4}$ sec.'

(LOND. B.Sc., 1915, II. 1.)

11. 'Find the tangential and normal components of the acceleration of a particle describing a plane curve.

'A smooth circular wire of radius $a$ is fixed in a horizontal position. A small ring, attached to a point of the wire by an elastic string of natural length $a$, is placed at the point of the wire opposite to the point of attachment and slightly disturbed. Show that the horizontal component of the pressure of the wire on the ring will be outwards until the extension of the string is $2a/3$, and will afterwards be inwards.'

(LOND. B.Sc., 1920, II. 7.)

12. 'A particle moves so that its velocity resolved parallel to a fixed line is proportional to its distance from that line, and its velocity resolved parallel to a fixed perpendicular line is proportional to its distance from that perpendicular line. Show that its path is a conic and that its acceleration is always directed towards the intersection of the fixed lines.'

(LOND. B.Sc., 1921, II. 2.)

13. 'The frustum of a uniform solid right cone is placed with its base on a rough inclined plane, the radii of its upper face and base being respectively $r$ and $R$. If the inclination of the plane to the horizon is gradually increased, prove that equilibrium will be broken by tumbling if $\mu > 4R(R^2+Rr+r^2)/h(R^2+2Rr+3r^2)$, where $h$ is the height of the frustum.'

(LOND. B.Sc., 1922, I. 6.)

14. Prove that any system of coplanar forces can in general be reduced to a force acting at an arbitrary point of the plane together with a couple.
'Show that such a system may be replaced in an infinite number of ways by two forces acting one through each of two fixed points, $A$ and $B$, in the plane; and that if $AP$ is drawn through $A$ to represent on any fixed scale the force through $A$, then the locus of $P$ is a straight line parallel to $AB$.'

(LOND. B.Sc., 1922, I. 1.)

15. 'A uniform parabolic plate, the focus of which is $F$, is bounded by the finite portion of the curve and by a double ordinate $AB$. The triangular portion $AFB$ is cut out. Prove that the centre of gravity of the remainder is at $F$ if the distance of $AB$ from the vertex is $\sqrt{5}a$, $4a$ being the latus rectum of the parabola.'

(LOND. B.Sc., 1922, I. 7.)

16. 'Explain what is meant by the principle of virtual work.
'Four equal uniform rods, each of weight $W$ and length $2a$, are freely jointed so as to form a rhombus $ABCD$. The rhombus is suspended from the joint $A$ and prevented from collapsing by a strut of negligible weight and length $2b$ joining the middle points of $CB$ and $CD$. Show that the stress in this strut is $4Wb/(a^2-b^2)^{\frac{1}{2}}$.'

(LOND. B.Sc., 1922, I. 9.)

17. 'A uniform circular cylinder of radius $a$ is projected transversely along a rough horizontal plane with a velocity $V$ and a spin $\Omega$ in the sense which would tend to make it roll backwards. Show that it will again pass the point of projection if $a\Omega > 2V$, and that it will do so before slipping ceases if $a\Omega > 5V$.'

(LOND. B.Sc., 1922, III. 6.)

18. 'Obtain an expression for the resultant attraction of a uniform straight rod $AB$ at any external point $P$.
'If the resultant attraction be resolved along and perpendicular to $AP$, prove that the former component is $\gamma M/(AP.BP)$, where $\gamma$ is the constant of gravitation and $M$ the mass of the rod.'

(LOND. B.Sc., 1922, II. 8.)

19. 'The radii of the plane ends of a frustum of a right circular cone are $a$ and $b$; the length of the slant edge is $l$. Show that the centroid of the total surface (plane and convex) divides the line joining the centres of the circular ends in the ratio

$$3b^2+(2b+a)l : 3a^2+(2a+b)l.'$$

(LOND. B.Sc., 1923, I. 3.)

20. 'A particle is projected from a point on a plane inclined at an angle $a$ to the horizon with velocity $V$ in a vertical plane through a line of greatest slope. Show that the range is a maximum if the direction of projection bisects the angle between the vertical and a line of greatest slope.
'If the particle is smooth and perfectly elastic, and if it is projected as above described, show that the distance up the plane between the first and second impacts is

$$\frac{1-2\sin a}{1+\sin a}\cdot\frac{V^2}{g}.$$

(LOND. B.Sc., 1923, II. 1.)

## EXAMPLES—CXXII. : MISCELLANEOUS

1. 'Find expressions for the components of acceleration in polar coordinates $r$, $\theta$.
'A straight smooth tube is constrained to rotate in a horizontal plane with constant angular velocity $\omega$ about a vertical axis through a point of its length. The tube contains a particle which initially was at rest relatively to the tube at a distance $a$ from the axis. Show that at any subsequent time $t$ the reaction between the particle and the tube is proportional to $\sinh \omega t$.'

(LOND. B.Sc., 1923, II. 2.)

2. 'Show how to find the moments and product of inertia of a plane lamina about a pair of rectangular axes in the lamina, when those about parallel axes through its centre of inertia are known.
'Show that a principal axis of the lamina at its centre of inertia is a principal axis at any point of its length.'

(LOND. B.Sc., 1923, III. 1.)

3. 'Obtain the conditions of equilibrium of a system of forces acting in one plane on a rigid body.
'Two equal uniform ladders, each of length $l$ and weight $w$, are hinged together at the upper end of each, and tied by a light string attached to them at points distant $c$ from the lower ends. The system rests on smooth ground, with the ladders inclined to the vertical at the same angle $\alpha$, and a weight $W$ is suspended from the middle of one ladder. Prove that the tension of the string is

$$\frac{l \tan \alpha}{4(l-c)}(W + 2w).'$$

(LOND. B.Sc., 1923, I. 4.)

4. 'Prove the formula $v^2 = \mu(2/r - 1/a)$ for elliptic motion about a focus.
'A particle is projected in a direction making an angle $\alpha$ with the line joining it to a fixed centre of force attracting as the inverse square of the distance. The speed of projection is $n$ times that which the particle would acquire in moving from rest at infinity to the point of projection ($n < 1$). Show that the eccentricity of the orbit is

$$\{1 - 4n^2(1 - n^2)\sin^2\alpha\}^{\frac{1}{2}}.'$$

(LOND. B.Sc., 1923, II. 3.)

5. 'A diving bell is in the form of a circular cylinder of diameter $h$ and height $h$, surmounted by a hemisphere of equal diameter. If it is lowered into water and no air is pumped in, find the depth of the bell when the water just fills the cylinder. Show that to expel the water from the cylinder the volume of air at atmospheric pressure that must be pumped in is $(3 + h/H)V$, where $V$ is the volume of the bell, $H$ the height of the water barometer.'

(LOND. B.Sc., 1923, III. 10.)

6. 'State the *laws of friction.*
'A uniform rod $AB$ rests with one end $A$ on rough ground, and is supported in limiting equilibrium at an inclination $\alpha$ to the horizontal by a string $BC$, attached to the other end $B$. Prove that, if the angle $ABC$ is $\pi - \beta$, and $\alpha + \beta > \frac{1}{2}\pi$, the coefficient of friction is

$$\frac{2\cos\alpha\cos(\pi - \alpha - \beta)}{3\sin\beta - \sin(2\alpha + \beta)}.$$

(LOND. B.Sc., 1923, I. 5.)

7. 'Explain what is meant by the "terminal velocity" of a particle moving under gravity in a resisting medium.

'A particle falls from rest in a medium whose resistance is proportional to the velocity. Show that the velocity after a time $t$ is $U(1-e^{-gt/U})$, where $U$ is the terminal velocity.'

(LOND. B.Sc., 1923, II. 4.)

8. 'A ring of mass $M$ and radius $a$ can turn freely in its own plane, which is vertical, about a point $O$ of its circumference. A particle of the same mass $M$ is rigidly attached to the ring at the point diametrically opposite to $O$. Prove that, when the system makes small oscillations about $O$, the length of the equivalent simple pendulum is $2a$.

'If the system is making finite oscillations in which the diameter through $O$ makes in its extreme positions an angle $\alpha$ with the downward-drawn vertical, find the reaction at $O$, when the diameter through $O$ is vertical.'

(LOND. B.Sc., 1923, III. 2.)

9. 'A uniform sphere of weight $W$ has a particle of weight $w$ attached to its surface. Show that there are two positions in which it can rest on a rough inclined plane, provided the angle of inclination of the plane $< \sin^{-1}\left(\frac{w}{W+w}\right)$.

'Prove that the position in which the particle is below the centre is the stable position.'

(LOND. B.Sc, 1923, I. 7.)

10. 'Find an expression for the potential energy of a stretched light elastic string.

'A smooth wire in the form of a parabola of latus-rectum $4a$, is fixed with its plane horizontal. A small ring of mass $m$ can slide on the wire and is attached to the focus of the parabola by a light elastic string of natural length $a$ and modulus of elasticity $\lambda$. If the ring is held at an extremity of the latus-rectum of the parabola and released, show that its distance from the axis of the parabola at any subsequent time $t$ is

$$2a\cos\left(t\sqrt{\frac{\lambda}{4ma}}\right).'$$

(LOND B.Sc., 1923, II. 7.)

11. 'A uniform solid sphere is placed at rest on a rough plane inclined to the horizontal at an angle $\alpha$, and $\mu$, the coefficient of limiting friction between the sphere and the plane, is greater than $(2\tan\alpha)/7$. Prove that the sphere will roll down the plane without sliding.

'If the sphere is projected directly up the plane with velocity $V$ without rotation, show that it will begin to roll at the end of time

$$2V \div g(2\sin\alpha + 7\mu\cos\alpha).'$$

(LOND. B.Sc., 1923, III. 4.)

12. 'Show how the *Principle of Virtual Work* leads to the conditions of equilibrium of a rigid body subject to forces in one plane.

'A uniform rod $AB$ of length $a$ and weight $W$, free to turn about one end $A$, is supported, in a position in which $B$ is below the level of $A$, and $AB$ makes an angle $\theta$ with the horizontal, by a light string attached to $B$, passing over a smooth peg, at the same level as $A$ and distant $b$ $(>a)$ from it, and carrying a weight $W'$. Prove that

$$2bW'\tan\theta = W\sqrt{(a^2+b^2-2ab\cos\theta)}.'$$

(LOND. B.Sc., 1923, I. 8)

13. 'Explain the term equipotential surface, and prove that the direction of the resultant force due to an attracting body on a particle placed at any point is normal to the equipotential surface through the point.

'Find the potential of a uniform straight rod at any point and show that

the equipotential surfaces are ellipsoids of revolution with their foci at the extremities of the rod.'

(LOND. B.Sc., 1923, II. 8.)

14. 'State the principle of the conservation of linear momentum.
'A smooth wedge, of mass $M$ and angle $\alpha$, is held with one face in contact with a smooth horizontal plane and a particle of mass $m$ is placed on the other face at a distance $a$ from the plane, measured along a line of greatest slope. If the system is allowed to move freely, show that the distance that the wedge has moved when the particle reaches the plane and the time taken by the particle to reach the plane are respectively

$$\frac{ma\cos\alpha}{M+m} \text{ and } \left\{\frac{2(M+m\sin^2\alpha)a}{(M+m)g\sin\alpha}\right\}^{\frac{1}{2}}.$$

(LOND. B.Sc., 1923, III. 3.)

15. 'Obtain the equations of a catenary in the form $s=c\tan\psi$, $y=c\sec\psi$, $x=c\log(\sec\psi+\tan\psi)$, $T=wy$.
'A uniform chain of length $2l$ is suspended from two points in the same horizontal and has a sag $h$ at the centre. Show that the span is

$$\frac{l^2-h^2}{h}\log_e\left(\frac{l+h}{l-h}\right).$$

'The length of the chain is 100 ft., the sag is 10 ft. Prove that the tension at each end is the weight of 130 ft. of chain, and, by expanding the logarithm, show that the span is approximately 97·3 ft.'

(LOND. B.Sc., 1923, I. 10.)

16. 'Prove that the integral of the normal component of attraction over any closed surface is equal to $-4\pi\gamma$ times the mass enclosed by the surface.
'An infinite hollow cylindrical shell whose cross section consists of two concentric circles of radii $a$, $b$ $(a>b)$ has a density which is inversely proportional to the distance from the axis. Show that the attraction at a point within the material of the shell at a distance $r$ from the axis is

$$\frac{2\gamma M(r-b)}{r(a-b)}$$

where $M$ is the total mass per unit length of the shell.'

(LOND. B.Sc., 1923, II. 9.)

17. 'Find the conditions of equilibrium of a body floating freely in water.
'The cross-section of a prism is an isosceles right-angled triangle $ABC$, $A$ being the right angle. Show that, if the specific gravity of the prism is 13/16, it can float freely in water with the edge through $A$ above the surface, and with 1/4th and 3/4ths of the sides immersed.'

(LOND. B.Sc., 1923, III. 8.)

18. 'Show that in homogeneous fluid at rest under gravity, the pressure increases uniformly with the depth below the surface.
'A sphere is immersed in fluid, the depth of the centre below the surface being equal to the diameter of the sphere. Prove that the resultant pressure on the lower half of its surface is twice the weight of the fluid displaced by the sphere.'

(LOND. B.Sc., 1923, III. 7.)

19. 'Find the centre of pressure of a triangle having one side in the free surface.
'A triangle $ABC$ has the vertex $A$ in the surface and rotates in its own plane about $A$. Show that as long as the triangle is completely immersed the centre of pressure lies in $E'F'$, where $E'$, $F'$ are the mid-points of the sides $DE$, $DF$ of the median triangle.'

(LOND. B.Sc., 1923, III. 9.)

20. 'Prove that two couples in the same plane are equivalent if they have the same moment.

'A uniform rod $AB$ of length $l$ and weight $W$ is partly supported by a light string $OA$ attached to a fixed point $O$ and to the end $A$, and partly by a couple of moment $G$ applied to it in the vertical plane containing it. Prove that in equilibrium $OA$ must be vertical, and find $G$ if the rod is horizontal.'

(LOND. B.Sc., 1923, I. 1.)

# ANSWERS TO EXAMPLES

## III. PAGE 24.

4. Acceleration = 32 units of length per sec. per sec.

7. Acceleration $= 2\dfrac{a(t_2 - t_3) - b(t_1 - t_2)}{(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)}$;

Velocity at $P_2 = \dfrac{a(t_2 - t_3)^2 + b(t_1 - t_2)^2}{(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)}$.

## IV. PAGE 33.

1. (*a*) 13·8 miles per hour in direction 5°54′ E. of N.,
   (*b*) 16·8 m./hr. 2°36′ W. of N.,
   (*c*) 15·1 m./hr. 7°36′ W. of N.

4. Resultant Velocity = 968 cm. per sec. nearly, making with the northerly, easterly and upward directions respectively the angles whose cosines are 707/968, 662/968, and 5/968.

8. $\omega \sin\lambda$ about axis through Greenwich, and $a\cos\lambda$ about the perpendicular axis. Or, taking $\lambda = 51°29'$ and $\omega = 2\pi$ per day, the angular velocities are 4·91615 and 3·912815 radians per day respectively; *i.e.*, one complete turn in 1·278075 and in 1·6058 days respectively.

10. 3/4 of a mile.

12. (*a*) Two blunt spikes in a horizontal groove and a third spike touching a horizontal plane.
   (*b*) As beam of balance, art. 26.
   (*c*) Two blunt spikes in a circular groove on a horizontal plate and a third spike touching the horizontal plate.

## V. PAGE 35.

1. 32 ft. per sec. per sec.
2. + 1/18 and − 1/9 mile per min. per min.
3. 40 ft. per sec. per sec.; 37·4 ft. per sec.

## VI. PAGE 37.

3. OM = $a \sin(4t + \epsilon)$, where $a = 5\sqrt{5}/2$ and $\tan t = 2$.

4. (*a*) $\pm 3/2$ cm. per sec.,
   (*b*) $\pm 3\sqrt{3}/4$ cm. per sec.,
   (*c*) $\pm 3/4$ cm. per sec.$^2$,
   (*d*) − 1/4 per sec.$^2$.

## VII. PAGE 40.

2. $y = 4{\cdot}96 \sin(wt + 54°)$.

4. Resultant amplitude = 18·7 and
   Phase angle = 23°50′ nearly.

## VIII. PAGE 43.

3. 2·106 miles per sec. (or, 2·096 m./sec., by the approximation).
4. 5 min. 5·8 sec.
5. 6·98 miles per sec.

## IX. PAGE 51.

2. Time to halve $V$ is $\frac{V}{\lambda}\log_e 2$.
3. See Equations (2) and (5) of art. 41.
4. $V = k(1 + k^2/U^2)^{-1/2}$.

## X. PAGE 55.

2. If the original motion were $y = \sin pt$ and the retardation introduced were $2\kappa$ per unit velocity, then the amplitude would be affected by the factor $e^{-\kappa t}$ and the period by the factor $p/q$ where $q^2 = p^2 - \kappa^2$.
   But if the period were 1/100 sec. and the damping such as to reduce the amplitude to $(1/e)$th in 100 vibrations, we should have $\kappa = 1$ per sec., and $p = 200\pi$ per sec.; hence
   $$q^2 = p^2 - \kappa^2 = 40{,}000\pi^2 - 1 = p^2 \text{ nearly.}$$
   Thus the effect on the period is only about 1 in 800,000, though the amplitude fell to $1/e$th in a second.
3. $y = ae^{-5t}\sin(12t + \epsilon)$, where $a$ and $\epsilon$ are arbitrary constants depending on the initial conditions, $\tau = \pi/6$, the logarithmic decrement is $5\pi/12$ per half period or $5\pi/6$ per period.

## XI. PAGE 58.

3. Angle of elevation = 45° + half that of incline; or, in other words, the direction of projection bisects the angle between the vertical and the incline.
6. $261\frac{4}{11}$ feet from point at which velocity was 1200 ft. per sec., and $238\frac{7}{11}$ ft. from point at which velocity was 1000 ft. per sec.
8. Speed of projection = 120·5 ft./sec. nearly, if we take $g = 32$ ft. per sec.$^2$, angle of elevation = 41°38′.

## XII. PAGE 62.

3. (*a*) $s = al\cos\{\sqrt{(g/l)}t\}$ (*b*) $s = \frac{v}{\sqrt{(g/l)}}\sin\{\sqrt{(g/l)}t\}$.
4. Period = $2{\cdot}03\pi(l/g)^{1/2}$ nearly.
5. Velocity = 1·7 ft./sec. nearly.

## XIII. PAGE 67.

1. Base = $2\pi a$, Curve = $8a$, where $a$ is radius of circle.
2. $s = 4a\sin\phi$ (with the notation of art. 58).
6. The constant angular velocity is $\pm\sqrt{(g/4a)}$.

## XV. PAGE 72.

1. (*a*) Circle. (*b*) Two coincident straight lines.
3. Co-ordinates of ends of arc are $(-a, \pm a)$; Velocity = $na$ and is perpendicular to axis.

## XVII. PAGE 78.

3. Radial acceleration is $-g \sin\theta$; Transversal acceleration is $-g\cos\theta$; where $\theta$ is the angle between $r$ and the horizontal.

4. Angular velocity $= \dfrac{bu - av}{a^2 + b^2 - 2(au + bv)t + (u^2 + v^2)t^2}$.

## XX. PAGE 91.

4. For Parabola $v^2 = 2\omega/r$; for Hyperbola $v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$.

5. Point where the velocity is checked becomes an end of the major axis of the new elliptic orbit, the sun being at the distant focus. The periodic time is diminished according to the expression $\tau_0^2/\tau^2 = (2 - v^2/v_0^2)^3$, in which the letters with subscripts refer to the original period and velocity.

## XXI. PAGE 94.

1. (1) 13 radians per sec., (2) 40 radians, (3) $8\pi$ radians$^2$ per sec.$^2$.
2. 5 radians per sec. per sec.
3. 4 (or $-7$) seconds.
4. $\theta = \frac{\pi}{6}\cos 3t$.

## XXII. PAGE 101.

1. $C$ is the midpoint between the initial and final positions of A.
4. The component accelerations of the highest point are $2f$ horizontally forwards and $v^2/a$ vertically downwards, the acceleration of the lowest point is $v^2/a$ vertically upwards.

## XXIII. PAGE 105.

3. OP is $x = y = z$; OQ is $\frac{x}{4} = \frac{y}{3} = \frac{z}{2}$;

   PQ is $\frac{x-1}{3} = \frac{y-1}{2} = \frac{z-1}{1}$.

4. $\cos\theta = 9/\sqrt{87}$.

## XXIV. PAGE 108.

4. Velocities are zero, $va(a^2+p^2)^{-1/2}$, $vp(a^2+p^2)^{-1/2}$;
   Accelerations are $-av^2(a^2+p^2)^{-1}$, zero, zero.
5. Vertical acceleration is downwards and is 1·8856 cm./sec.$^2$; Radial acceleration is inwards to the centre of the circle and is 24·1 cm./sec.$^2$.

## XXVII. PAGE 125.

2. The plane of vibration would appear to rotate at the rate of $(2\pi \sin\lambda)$ radians per day, or one complete revolution in $(\operatorname{cosec}\lambda)$ days.
4. An angular acceleration is required about an axis at right angles to both the axes referred to in the question, of magnitude equal to the product of the two angular velocities, and of sense such that the axis of angular velocity is turning towards coincidence with that of the angular acceleration.
7. See equations (5)-(9) of art. 124.

## XXVIII. PAGE 130.

3. (1) The central axis QS is parallel to the horizontal axis PR (see Fig. 53) and at a distance of 5 cm. from it and from a parallel axis through P′ the final position of P. (2) 5 cm. (3) 60°.

## XXXIII. PAGE 152.

1. (i) Yes. (ii) Yes.

## XXXIV. PAGE 157.

2. The speed $v$ of the crosshand and piston is connected with $u$, that of the crank pin, by the equation

$$v = u(\sin\theta)\left(1 + \frac{\cos\theta}{4}\right).$$

For a very long coupler, this becomes

$$v = u\sin\theta \text{ nearly.}$$

## XXXV. PAGE 165.

4. The elongations along the three axes are as follows, each being accompanied by lateral contractions of one-fifth of those values.

$$(a)\ \frac{0{\cdot}01}{3},\ \frac{0{\cdot}01}{3},\ \frac{0{\cdot}01}{3}.$$

$$(b)\ 0{\cdot}0025,\ 0,\ -0{\cdot}0025.$$

$$(c)\ \frac{0{\cdot}005}{18},\ \frac{0{\cdot}020}{18},\ \frac{0{\cdot}005}{18}.$$

## XXXVII. PAGE 179.

1. $(a)\ x^2+y^2+z^2=(1{\cdot}001)^2$; $(b)\ \dfrac{x^2}{(1{\cdot}002)^2}+y^2+\dfrac{z^2}{(0{\cdot}998)^2}=1$;

$(c)\ \dfrac{x^2}{(1{\cdot}008)^2}+\dfrac{y^2}{(0{\cdot}998)^2}+\dfrac{z^2}{(0{\cdot}998)^2}=1.$

6. No; successive pure strains involving only *elongations* along the *same* co-ordinate axes cannot result in a rotation.

## XLI. PAGE 205.

5. Distance of penetration $s$ against uniform resistance $r$ varies as (1) mass $m$ and (2) as the square of the velocity $v$. This is because $\frac{1}{2}mv^2 = rs$. Velocity of emergence from board one inch thick is

$$\tfrac{1000}{3}\sqrt{34} = 1943{\cdot}65 \text{ ft./sec.}$$

6. When the thread is at an angle $\theta$ with the radius which is drawn vertically upward from the centre, the tension is given by

$$T = m\left(\frac{v^2}{r} - g\cos\theta\right),$$

where $m$ is the mass of the body, $v$ its linear speed, and $r$ the radius of the circle.

## XLII. PAGE 210.

3. 464/17 (or 27·294 . . .) and 600/17 (or 35·294 . . .) ft./sec.
4. −832/17 (or −48·9412) and +120/17 (or +7·0588 . . .) ft /sec.

## XLIV. PAGE 217.

1. During the acceleration $a$, the equilibrium position of the thread is in the vertical plane containing the acceleration and at an angle $\beta$ with the vertical where $\tan\beta = -a/g$. Oscillations may occur of period $\tau_1 = 2\pi\sqrt{(l/g_1)}$ where $g_1^2 = a^2 + g^2$. When the carriage is on the curved part of the line, the equilibrium position will be in the vertical plane containing $\rho$ and at the angle $\gamma$ with the vertical where $\tan\gamma = -v^2/\rho g$ and possible oscillations are of the period

$$\tau_2 = 2\pi\sqrt{(l/g_2)}, \text{ where } g_2^2 = g^2 + v^4/\rho^2.$$

3. The elevation of the outer rail is 2·55 inches.

## XLVII. PAGE 233.

1. $87\frac{3}{4}$, 270, and $303\frac{3}{4}$ lbs. inches$^2$ about the 9 inch, $4\frac{1}{2}$ inch, and 3 inch edges respectively.

3. $\frac{2}{3}Ma^2$ and $\frac{5}{3}Ma^2$ for shell;
$\frac{2}{5}Ma^2$ and $\frac{7}{5}Ma^2$ for solid sphere.

5. $\frac{3}{20}M(4a^2 + b^2)$, where $a$ is axis and $b$ base radius.

6. $\frac{M}{20}(2a^2 + 3b^2)$.

## XLIX. PAGE 243.

1. 12·88 ft./sec.$^2$, 25·76 radians/sec.$^2$, 193·2 poundals or 6 lbs. wt.

4. Error would be $2g/105$. Spurious determination would be $20g/21$.

## L. PAGE 249.

1. Periods are (i) $2\pi\sqrt{(2h/g)}$ and (ii) $2\pi\sqrt{(3h/2g)}$.

2. (i) Two-thirds total length from point of suspension.
(ii) A quarter of the radius below the centre.

7. Minimum period for bar is $2\pi\sqrt{(2a/g\sqrt{3})}$.
Minimum period for plate is $2\pi\sqrt{(s/g\sqrt{3})}$, where $s$ is side.
Greatest period approaches infinity as axis approaches centre of gravity.

8. No; because the thread would not remain straight, since each particle (if alone on the thread) would have its own special period of oscillation.

10. Length of equivalent simple pendulum $= h + a^2/4h$, where $a$ is the radius of the disc and $h$ is the distance of the axis from its centre.

## LI. PAGE 259.

3. Impulse = 15 lbs. ft./sec; angle $= \cos^{-1}\frac{23}{25}\frac{1}{6} = 25^\circ\ 19'$ nearly.

5. $P$ is $a/\sqrt{3}$ from middle of bar of length $2a$.

9. The linear acceleration of the centre of mass of the lamina of mass $M$ is $X/M$, and is parallel to the direction of the force $X$. The angular acceleration of the lamina is $-(Xr\sin\theta)/Mk^2$, where $r$ is the radius from the centre of mass to the point of application of the force $X$ and $\theta$ is the angle between $r$ and $X$, finally $k$ is the radius of gyration of the lamina about an axis through its centre of mass and perpendicular to its plane.

## LII. PAGE 270.

3. It would 'dilute' $g$ more, *i.e.*, make the linear acceleration smaller.
4. If the incline is at $\alpha$ with the horizontal and the groove has the semi-vertical angle $\theta$, then the linear acceleration of the sphere is $(5 \sin^2\theta\, g \sin\alpha)/(2 + 5\sin^2\theta)$, and the angular acceleration is $(5 \sin\theta\, g \sin\alpha)/R(2 + 5\sin^2\theta)$ where $R$ is the radius of the sphere.
5. Time with liquid $= (2l/g\sin i)^{1/2}$; time with ice $= (14l/5g\sin i)^{1/2}$.
7. Acceleration is $\frac{12}{11} g \sin i$.

## LIII. PAGE 275.

2. The magnitude of the torque equals the product of angular momentum and rate of precession and occurs about an axis perpendicular to both the other axes.

## LVI. PAGE 286.

2. Minimum speed of spin is $\frac{2}{C}\sqrt{(AWh\cos\theta)}$, the notation being that of article 297.

## LIX. PAGE 301.

1. The drawback lies in the fact that the moments and products of inertia in use may be all varying.
2. They present in a very compact, comprehensive, and harmonious form the basis of particle and rigid dynamics.

## LX. PAGE 304.

4. $270° > \alpha > 90°$; for minimum $R$ we have $Q + P\cos\alpha = 0$, the minimum $R$ being $\sqrt{(P^2 - Q^2)}$.

## LXIII. PAGE 315.

2. 5·49 turns nearly.

## LXIV. PAGE 317.

1. Tension 122·68 lbs. wt. Vertical component 121·04 lbs. wt.
2. 555·74 ft.
4. 116 ft. 2 inches.
6. 62 ft. 4 inches.

## LXV. PAGE 322.

2. The parabola is *below* the catenary. The equation of the parabola may be written $y = c\left(1 + \frac{x^2}{2c^2}\right)$ and that of the catenary

$$y = c\left(1 + \frac{x^2}{2c^2} + \frac{x^4}{24c^4} + \ldots\right).$$

6. If $W$ is the weight on the cord from lowest point to where ordinate is $y$ (of negative value), then the law of load per unit horizontal length is

$$\frac{dW}{dx} = \frac{a^2 T_0}{(-y)^3},$$

where $a$ is the radius of the circle and $T_0$ is the horizontal tension at lowest point.

## LXVI. PAGE 328.

4. For wire from o to $\theta$, the components of attraction are

$$X=\frac{\rho}{a}\int_0^\theta \cos\theta d\theta \text{ and } Y=\frac{\rho}{a}\int_0^\theta \sin\theta d\theta.$$

The conditions for equilibrium are

$$\int_0^{2\pi} f(\theta)\cos\theta d\theta = 0 \text{ and } \int_0^{2\pi} f(\theta)\sin\theta d\theta = 0.$$

5. Resultant intensity of attraction acts at $(a, a)$, is directed to the origin and has magnitude $(2-\sqrt{2})\gamma m/a$.

## LXVII. PAGE 333.

5. Gravitative field has dimensions $\gamma M/L^2$, and so here involves $\gamma$ and $\sigma$, and since $\sigma = M/L^2$ dimensionally it can involve nothing more except numerics and $\pi$. Also we see that $\gamma$ and $\sigma$ must each occur raised to the power plus one.

## LXIX. PAGE 344.

3. Yes; it is only for the inverse square law that these tubes and equipotential surfaces have the properties named.

## LXX. PAGE 349.

4. Let $\overline{V}$ be the mean potential of the spherical surface of centre O and radius $a$ due to matter $m$ external to the sphere, of which an element $dm$ is at P distant $r$ from a point Q on the spherical surface at which the potential is $V$ and an element of surface $dS$ is taken. Then we may write

$$4\pi a^2\overline{V} = \iint V dS = \iiint \frac{dm}{r} dS = \iiint \frac{dS}{r} dm$$

$$= \frac{4\pi a^2}{OP}\int dm = 4\pi a^2 \int \frac{dm}{OP}$$

$$= 4\pi a^2 [\text{Potential at O of matter external to spherical surface}],$$

or, in words, the mean potential $\overline{V}$ equals that at O due to the same matter, as was to be shown.

5. If the cylinder is of density $\rho$, of finite radius $a$ and extends in length from $-\infty$ to $+\infty$, then the *radial* field at distance $r$ from the axis is

$$R = -2\pi\gamma a^2\rho/r.$$

If the same cylinder *has one end accessible* and the *axial* field is required at a distance $c$ from that end, the other end being at $\infty$, we have

$$Z = -2\pi\gamma\rho\{\sqrt{(c^2+a^2)} - c\}.$$

7. Time is $(\pi+2)(a^3/\gamma M)^{1/2}$ sec. where $M$ is the mass of each sphere.

## LXXI. PAGE 357.

8. For P as origin and PB and PC as axes of $x$ and $y$, we have

$$R = \sqrt{(480 - 120\sqrt{3})} = 16{\cdot}5,$$

and its line of action is $12y - (10\sqrt{3}-6)x - AB\sqrt{3} = 0$.

9. $\lambda = 4k$.

10. The tangent of the angle is $\dfrac{G_1 - 3G_2 + 2G_3}{G_3 - 2G_1 + G_2}$.

12. $3x + (3-2\sqrt{2})y + 6a(\sqrt{2}-1) = 0$.

## LXXIV. PAGE 375.

4. At inclination $\sin^{-1} 3/8$. Yes, it is stable for smaller inclinations; but for inclinations approaching the critical value the stability has very small range on one side.[1]
5. $\mathrm{Tan}^{-1}(8P/3W)$. No, friction would not modify it but adhesion might.
6. $\mathrm{Tan}^{-1}(4\sqrt{3}/5) = 54^\circ 11'$.
7. Work is 5·176 ft. tons wt.

## LXXV. PAGE 381.

3. Tension is $2\pi P a m n/(m+n)$ lbs. wt.
9. $P = (0{\cdot}183)W$. No, for $i < \lambda$.
13. Angle rod makes with vertical is $\tan^{-1}(Ph/wl)$.

## LXXVI. PAGE 387.

3. Depth is 3/16 cm.

## LXXVII. PAGE 391.

1. Hypotenuse has compression $4w/\sqrt{10}$. The equal sides have tensions $6w/\sqrt{5}$ and $3w/\sqrt{5}$, the greater tension being on the side connecting $2w$ and point of support.
2. $BF$ has 2 tons tension, $CF$ has 3·75 tons tension, $EF$ has 4·5 tons compression, and $GF$ has 5·5 tons compression.
3. The tensions are $AB = BF = W/3$, $BD = 2W/3$. The compressions are $AF = BE = W\sqrt{2}/3$, $DE = W2\sqrt{2}/3$, $EF = W/3$.
4. Tensions are each $= Wb/\{9\sqrt{(a^2 - b^2/3)}\}$.
5. The tensions are $AB = 3W/2$, $BC = W/2$, $DA = W$, $BE = W/\sqrt{2}$. The compressions are $CD = W/2$, $BD = CE = W/\sqrt{2}$.

## LXXIX. PAGE 411.

5. With origin in the middle of the beam and length $2l$, equation for shear is $y = Qx$ (where $Q$ is negative), and equation for bending moment is $x^2 = \frac{2}{Q}(y + Ql^2/2)$.
6. The shear is zero and the bending moment is the product (half weight of bar into a quarter of its length). (See Figs. 207-209).
8. The stress in a flexible string is tension only. Greatest value of bending moment is 86·56 ft. tons wt., and occurs where the 10 ton load is.

## LXXXII. PAGE 425.

4. (i) Horizontal force is $\frac{1}{2}W\cot\theta$.
   (ii) Couple is $(Wp\cos\alpha)/2\pi$.

[1] Or, in more detail, for angles of inclination less than $\tan^{-1}3/8$ ($20^\circ 33'$) there is *one* position of equilibrium which is *stable*.

For inclinations between $\tan^{-1}3/8$ ($20^\circ 33'$) and $\sin^{-1}3/8$ ($22^\circ 1'$) there are *two* positions of equilibrium, *one stable and one unstable*. For inclinations greater than $\sin^{-1}3/8$ there are no positions of equilibrium.

## LXXXIII. PAGE 430.

6. Resultant thrust acts horizontally from the high level side to the low level side, normally to the dock gate; at a point in the median vertical line $7\frac{7}{9}$ feet above the bottom common to the water on both sides the gate. For fresh water of density 62·3 lbs. per c. ft. the magnitude of this resultant is 140,175 lbs. wt., for sea water of specific gravity 1·03 (compared with that of fresh water) the magnitude is 144,380 lbs. wt.
7. Resultant thrust acts horizontally from the high level side to the low level side, normally to the lock gate at a point in the median vertical line $6\frac{8}{13}$ feet above the bottom common to the water on both sides the gate. The magnitude of the resultant is 26·116 tons wt.

## LXXXIV. PAGE 437.

5. Two-thirds of the axis is in the upper liquid. The upper liquid contributes to the upward force on the body by pressing down on the lower liquid, and so increasing the pressure which it exerts at any given level and in any direction normal to a surface exposed to it.
7. $s \propto \cos^6\theta$, where $\tan\theta = r/h$.
8. Let $l$ be the longer edge of the board and $\theta$ its angle with the vertical, then $\sigma l^2\cos^2\theta = h^2$. Hence for $\theta = 0$, $h = l\sqrt{\sigma}$.
9. 4·8 feet or 4 ft. 9·6 inches.
10. Total pressures on $PQR$ and $PRCD$ are as 2 : 9.
Centres of pressure are in the median line through $Q$ and at vertical depths $\frac{3}{4}a\cos\theta$ and $\frac{14}{9}a\cos\theta$ for $PQR$ and $PRCD$ respectively.
11. Required couples are as 1 : 16, if the centre of gravity coincides with the centre of buoyancy.

## XCII. PAGE 484.

1. Minute hand $\pi/1800$ radians per sec., hour hand $\pi/21{,}600$ radians per sec., bullet 2400 revolutions per sec.
2. Length of equivalent simple pendulum is $W/w$, where $W$ is suspended weight and $w$ is weight per unit extension of spring.
4. Quickest run in the numerical case is made in 3 min. 5·9 sec.
(Take $g = 32$ ft. per sec.$^2$).
8. Greatest height attained is $\frac{hm^2}{1+m^2}$ or $h\sin^2\alpha$.
9. The eccentricity is 1/74 or 0·0135.
10. The eccentricity is one-tenth, and the major axis $\frac{36{,}000\sqrt{11}}{\pi}$ feet or $37{,}994\frac{8}{11}$ feet.
13. The eccentricity is one-half.
25. The locus of $AP$ is a plane at right angles to $BAC$ and bisecting the angle $BAC$, which equals $2\theta$ say. Then, if the angle of rotation about $AP$ is $2\phi$ and $\psi$ is the angle between $AP$ and the normal to $BAC$, we have $\tan\phi = \tan\theta\sec\psi$.

## XCIII. PAGE 487.

2. For $g=32$ ft./sec.$^2$, the excess force is 11/27 tons wt.

4. Mean pressure is 140 lbs. wt.
The value found for the greatest force depends on the assumptions made in the absence of exact knowledge. Let us suppose throughout that during approach while in contact the force increases almost uniformly with time and that it decreases uniformly with time during recession. There are still other assumptions which may be taken successively. Thus, during approach and recession, let there be either

(i) *Equal mean forces*,
(ii) *Equal spaces* traversed by the centre of the ball, or
(iii) *Equal times occupied.*

Then we have as the approximate values of the greatest pressure in each case the following :—(i) 280 lbs. wt., (ii) 700 lbs. wt., and (iii) 400 lbs. wt.

5. (*a*) 300 ft. lbs. wt. (*b*) 400 ft. lbs. wt.

10. 90·566 ft./sec.

11. The tension is $mv^2/2r \sin(\pi/n) = mv^2/a$.

12. The maximum kinetic energy is $125\pi^2/16g = 2{\cdot}41$ ft. lbs. wt.

13. The tension at $x$ from $A$ is $w\omega^2(l^2-x^2)/2g$ lbs. wt., if $w/g$ is in lbs. and $l$ in feet.

14. The velocity is 11·31 ft./sec. taking $g=32$ ft./sec.$^2$, or 11·34 ft./sec., taking $g=32{\cdot}2$ ft./sec.$^2$.

16. Ratio of masses of Sun and Earth is $64\times10^6 \div 169 = 378{,}698$.

17. Altitude $y$ at time $t$ is given by

$$y=\left(\frac{v\sin\alpha}{k}+\frac{g}{k^2}\right)(1-e^{-kt})-\frac{gt}{k}.$$

19. If $m$ is the mass of each particle, $T$ the tension of the thread, and $a$ the radius of the disc, we have generally $T=\frac{mg}{2}(1-\sin\theta)$ and, at the critical position, $v^2=ga\cos\theta$. We then find from the given equation $\theta=35°44'$, whence at the critical instant $T=0{\cdot}208\,mg$ and $v=(0{\cdot}901)\sqrt{(ga)}$.

20. If the masses are $M$ and $m$ and their distances are $R$ and $r$ from their centre of gravity G, then the point G will describe the usual parabolic trajectory of a projectile, while the masses will revolve round G at the initial rate of rotation ($\omega$ say). The tension of the thread is given by either of the equivalent expressions $MR\omega^2 = mr\omega^2$.

## XCIV. PAGE 490.

3. Axis for minimum period must be parallel to the 12 inch edges and be at a distance of $\sqrt{(13/192)}=0{\cdot}26021$ ft. from the centre, the direction of this distance being immaterial. The value of the minimum period is $2\pi\sqrt{(0{\cdot}52042/g)}=0{\cdot}799$ sec.

4. The centre of percussion is the end remote from the axis. The impulse $Q$ on axis is given by $-3cP/4a$, where $P$ is impulse at lower end and $2a$ is the length of the rod.

6. The angular velocity is $\left\{\frac{12g}{l}\cdot\frac{\cos\alpha-\cos\theta}{1+3\sin^2\theta}\right\}^{1/2}$, and the reaction on the plane is $w\,(4-6\cos\alpha\cos\theta+3\cos^2\theta)/(1+3\sin^2\theta)^2$.

7. The smallest vertical distance of descent is $\pi^2 a/6\sin^2(\alpha/2)$, the rod is then upside down. At four times this descent the rod will be again vertical and (for the first time) again erect.

8. See article 290.

10. Critical angular velocity is $\left\{\frac{3g\sin\alpha}{2a(1+3\cot^2\alpha)}\right\}^{1/2}$, where $2a$ is the length of the rod.

11. If $P$ is blow on obstacle and $Y$ that on point of suspension we have $P=M\sqrt{(2ga/3)}$ and $Y=M\sqrt{(ga/6)}$, where $M$ is the mass of the rod and $2a$ its length. For $g=32$ ft /sec.$^2$ we have $P/40\doteq 2\sqrt{2}$ and $Y/40=\sqrt{2}$.

12. If the inclination of the beam with the vertical is $\theta$, then we have

$$\dot\theta^2\,(k^2+a^2\sin^2\theta)=\frac{a^2k^2u^2}{(a^2+k^2)^2}-2ga(1-\cos\theta).$$

14. Let the rod $AB$ have length $2a$ and mass $M$, then when it reaches an inclination $\theta$ with the horizontal, we have

(*a*) its angular velocity given by $\dot\theta^2=\dfrac{3g\sin\theta}{2a(1+3\cos^2\theta)}$,

(*b*) the direction of motion of its centre of gravity is at $\theta$ with the vertical, and

(*c*) the reaction of the groove on end $B$ is

$$9Mg\,\frac{\sin 2\theta(3+\cos^2\theta)}{8(1+3\cos^2\theta)^2}.$$

16. At the instant of leaving the wall the rod (of length $2a$) makes with the vertical the angle $\cos^{-1}2/3$ and is turning at the rate $\sqrt{(g/2a)}$ radians per second, its centre of gravity has the component speeds $\sqrt{(2ga)}/3$ horizontally from the wall and $\sqrt{(5ga)}/3\sqrt{2}$ vertically downwards, and its weight on the floor is one quarter that of the rod.

17. The moment of inertia is $\frac{3}{20}$ mass (height$^2$ + 4 radius$^2$) = 1/6 lb. ft.$^2$.

19. If $P$ is the centre of percussion for the centre of suspension $S$, then $S$ is also the centre of spontaneous rotation for $P$. (See art. 270).

20. Moment of inertia required for hemisphere is $83mr^2/320$. Period of small oscillations is $2\pi\sqrt{(26r/15g)}$ sec.

## XCV. PAGE 493.

5. Centre of mass is 4·8 inches from $AD$ and 4·4 inches from $AB$.

7. Tension in chain $= W$.

9. The centre of gravity of the frustum is $3\frac{15}{28}$ feet or 3·5357 from the centre of the larger base.

10. The extreme positions of the cord are in the vertical plane through the rod and make the angles $\tan^{-1}\lambda$ with the vertical.

13. Resultant force is $2\sqrt{3}$ in direction at angle of 30° with force 1 reversed. Couple is $3a$, where $2a$ is the distance between parallel sides, or, couple is $3s\sqrt{3}$, where the side is $2s$.

16. With weights $W$ at $B$ and $E$, the stresses are as follows :—
In $AB$ and $AE$ tensions of 0·325 $W$, in $AC$ and $AD$ tensions of 0·851 $W$, in $BC$ and $DE$, compressions of 0·851 $W$, and in $CD$ a compression of 0·586 $W$.
With weights $W$ at $C$ and $D$, the stresses are as follows :—
In $AC$ and $AD$ tensions of 1·052 $W$ and in $CD$ a compression of 0·325 $W$. The members $AB$, $BC$, $AE$, and $DE$ are theoretically without stress. (The above results have been obtained from four-figure tables ; when solving by small scale drawings on the graphical method, an accuracy of about one per cent. is all that can be expected.)

19. If the pressure is $P$ units of force per unit area, then the force to be applied along the tangent is $2\mu\ P\pi a^2/3$, where $a$ is the radius of the cylinder.

20. The rope from the vertex of the triangle has a pull of 0·916 cwt. or 102·65 lbs. wt., the ropes from the ends of the base have each a pull of 0·524 cwt. or 58·66 lbs. wt.

## XCVI. PAGE 495.

8. The attraction is $2\pi\gamma\sigma\left\{1 - \frac{a}{\sqrt{(a^2+b^2)}}\right\}$, or $\frac{\gamma M}{ab}\left\{1 - \frac{a}{\sqrt{(a^2+b^2)}}\right\}$; where $\gamma$ is the constant of gravitation, $\sigma$ is the surface density, $a$ is the radius, $b$ is the length, and $M$ is the total mass of the cylinder.

9. If the plane required cuts $AB$ at $C$, then the distance $AC$ approaches zero. (See art. 346.)

## XCVII. PAGE 496.

4. If the lines pass through $E$ and $F$ on $AD$, then $AE = 10\sqrt{3}/3$ feet and $AF = 10\sqrt{6}/3$ feet.

5. Weight of metal required is 0·635 oz. per sq. inch.

6. Condition for couple is $\rho h^2 = \sigma k^2$; the couple is then $(\rho h^3 \sim \sigma k^3)/6 = \rho h^2 (h \sim k)/6$.

9. The height to which it must be filled with mercury is 4·881 inches.

10. Height of cylinder is $8\frac{1}{16}$ inches.

11. If the bore of the pipe is so small that the first portion of liquid immediately fills it and imprisons the air, the specific gravity of the liquid is 13·745. But, if the bore is such that the first portion of the liquid runs down to the bottom of the pipe and closes it there, then the specific gravity is only 4·582.

12. Upward vertical force on axle is $\frac{2}{3}\pi a^3 wg(1 - 2\sigma)$, outward horizontal force on axle is $\pi a^2 wgh$. No couple is required to keep the sphere from rotating, for the weight of the sphere passes through the axle and so do all the liquid pressures since they are normal to the surface and therefore act along radii of the sphere. (The vertical components of the liquid pressures give a resultant upward through the centre of gravity of the immersed hemisphere, the horizontal components of the liquid pressures give a resultant acting outwards through the centre of pressure of the vertical section of the sphere; but these two forces have equal and opposite moments about the axle.)

14. Let the radius be $a$, axis $OX$ in the surface and axis $OY$ vertically downwards; then the centre of pressure has co-ordinates $\bar{x} = 3a/8$ and $\bar{y} = 3\pi a/16$.

17. (*a*) Pressure on each of the three inclined faces is perpendicular to that face, has magnitude $2\sqrt{2}a^3wg/3$ (where $2a$ is the length of the edge and $w$ the density of the water), and acts at the middle point of the median of that face which passes through the vertex.

(*b*) Pressure on each of the three slant faces acts perpendicularly to that face through a point in its median line through the vertex and at one quarter of the median from the bottom horizontal edge, its magnitude is $4\sqrt{2}a^3wg/3$. The pressure on the bottom face acts vertically down through its centroid and has magnitude $2\sqrt{2}\ a^3wg$

18. The least value of the specific gravity is 5/6.

20. It has been assumed that the temperature is constant throughout. The quantity denoted by the symbol $k = p/\rho$ where $p$ is the pressure; in other words, $k$ is $g$-times the height of the homogeneous atmosphere. $k = 841{,}906$ ft. sec. units.

## XCVIII. PAGE 499.

1. The fluid would tend to push the tube before it if the tube were smaller at the outflow end than at the inflow, and vice versa. If the two ends had equal cross sections there would be no resultant longitudinal force on the tube.

5. From equation (27) of art. 447, by addition but without integrating, we have

$$dp = \rho\{\omega^2 x\,dx + \omega^2 y\,dy - g\,dz\}.$$

Then, assuming temperature constant and writing $p = k\rho$, and integrating we find

$$k \log \rho = \omega^2 \frac{x^2 + y^2}{2} - gz + C, \text{ whence } \rho = e^{C/k} \cdot e^{(\omega^2 r^2 - 2gz)/2k},$$

where $C$, the integration constant, is yet to be determined. In the question let the total *mass* be $M$, and express it by the double integration of a ring element of radius $r$ and height $z$. Thus following Besant and Ramsay (*Hydromechanics*, Part I.), we write $M = 2\pi \int_0^h \int_0^a \rho\, r\,dr\,dz$.

On substituting for $\rho$ and integrating, we have for the total *weight*,

$$W = Mg = \frac{2\pi k^2}{\omega^2} e^{C/k} \left(e^{\omega^2 a^2/2k} - 1\right)\left(1 - e^{-gh/k}\right),$$

from which $C$ may be determined.

## XCIX. PAGE 499.

1. Work is $2\frac{1}{4}$ ft. lbs. wt.

2. Work is (i) 30 ft. lbs. wt. and (ii) $33\frac{1}{3}$ ft. lbs. wt.

3. Deflection at end of beam is $-5wa^4/2Ebd^3$. Equation of beam's centre line is $$y = \frac{wx}{2Ebd^3}(x^3 - 4ax^2 + 6a^2x - 8a^3).$$

5. Maximum stress is at outer and inner margins of bent rod and is $12{,}083\frac{1}{3}$ lbs. wt. per sq. inch. This is reduced to 10 lbs. wt. per sq. inch less than 4 tons per sq. inch for a diameter of 540 ft.

## C. PAGE 501.

4. Let the radius of the pile be $r$, its total length $l$, the immersed length $i$, and its specific gravity be $s$; then it begins to leave the vertical position when $i/l = 1 - \sqrt{(1 - s - r^2/l^2)}$.

6. 67·1882 miles per hour.

11. Horse power is 38,187·5, resistance = $232\frac{1}{2}$ tons wt., coal consumption is 34·096 tons per hour or $4{,}333\frac{1}{3}$ tons per trip of 3000 miles.

15. Height of centre of gravity of tray is 5/7 inch. The centre of gravity of the whole system is raised 1·08 inch on filling with water.

16. In the numerical case the tension is 14·91 lbs. wt. (or $20\sqrt{5/3}$).

18. Close the surface in the simplest way then equate the resultant force on the closed surface to the components on the unclosed and closing surfaces respectively. If, for example, the unclosed surface were the curved part of a hemisphere, close it by the base.
Upward force in the numerical case is 9·35 tons wt.

20. Greatest deviation of rod from vertical is $\frac{\pi}{2} - \lambda$ and the pressure on the horizontal plane is then $W\left(1 - \frac{l \sin \lambda}{2h}\right)$.

## CI. PAGE 504.

1. The masses of the particles in order, beginning at the end of the rod of mass $M$, are $(5M - M')/12$, $2(M + M')/3$, and $(5M' - M)/12$. Hence real particles are possible (*i.e.* their masses are positive) when the ratio $M/M'$ lies between 5 and 1/5.

2. The couple required is $\pi^3/20 = 1{\cdot}55$ foot poundals.

3. The air in the diving bell is compressed from height $a$ to a height $ah/(a+b+h)$ nearly, when this height is negligible in comparison with $(a+b+h)$.
If the rate of introduction of air at atmospheric pressure be measured by $V = f/t$, where $f$ is the fraction of the height $a$ of the bell which it would fill at atmospheric pressure in time $t$; then $V = v/h$.

5. (1) To describe a circle the velocity must have magnitude $\sqrt{(\mu/c)}$, and must be directed at right angles to the initial distance $c$ of the particle from the centre of force whose proportionality factor is $\mu$.
(2) To describe a parabola the velocity must have magnitude $\sqrt{(2\mu/c)}$, but may have any direction.

6. The water will rise 8·25 feet, leaving only 3·75 ft. for the air.

7. When $A$ is on the point of moving toward the wall the horizontal force applied there is $89W/120 = 0{\cdot}7417\ W$.

8. Inclination of $AB$ is $\tan^{-1} 2/\sqrt{3} = 49^\circ\ 6'\ 24''$.

9. Energy which has become mechanically unavailable is
$$\frac{PQV^2}{(P+Q)2g}.$$

11. (See equation (11) on page 202 and note just after it). Momentum transferred is to total momentum as 175 : 416.

12. Since it is stated that the pulley of mass 40 gm. is smooth we must suppose it does not turn; then the accelerations are as follows :—
The 110 gm. has $7g/125$ down (and the 40 gm. pulley the same up); the 35 gm. has $3g/25$ down, and the 25 gm. has $29g/125$ up.

14. 12,670·75 cubic feet of salt water must pass into the fresh water tank to establish equilibrium.

16. Fraction of total length which is immersed is given by
$$f = 1 - \sqrt{(1 - \sigma/\rho)}.$$
The tension on the string is then given by
$$T = W(1 - f\rho/\sigma),$$
where $W$ is the weight of the rod.

20. The oil will sink 1·7213 cm.

## CII. PAGE 507.

2. Taking $g=32$ ft./sec.$^2$, we find speed of sleigh reduced to $2\frac{26}{33}$ miles per hour; for $g=32{\cdot}2$ ft./sec.$^2$ final speed is 2·786 miles per hour.

3. Ratio of diameter at surface to that below is $\sqrt[3]{(26/17)}$ or 1·15215.

4. The condition required is $\tan\alpha < \dfrac{2lr(l^2-r^2)}{(r^2+l^2)^2+(2lr)^2}$.

7. Period is $2\pi\sqrt{(2l/45g)}$. This cannot be numerically evaluated until we know the units of $l$.

9. First tension is 10·417 lbs. wt. New tension is 62·461 lbs. wt.

16. If vertical angle of beam is $2\alpha$, its coefficient of friction on the ground $\mu$, its weight $W$, and that of the cylinder $w$; then when the beam is about to slip, we have

$$\frac{w}{W}=\frac{\mu\sin\alpha}{\cos\alpha-\mu\sin\alpha}.$$

17. Attraction at sixty-fold distance is 2/225 or 0·00888889 poundals. The period at this distance is $\pi\sqrt{(2\times225\times60\times4000\times5280)}$ sec., or 2,372,358 sec., or 27·45785 days.

18. When the rod touches the hemisphere at the end of a radius at $\theta$ with the vertical, let the rod's speed down be $v$ and the hemisphere's speed horizontally be $u$, then $v=u\tan\theta$. Also if the system started from rest at $\theta=\alpha$, we have as the energy equation $u^2+v^2=2ga(\cos\alpha-\cos\theta)$, where $a$ is the radius of the sphere. Whence, $u^2=2ga(\cos\alpha-\cos\theta)\cos^2\theta$ and $v^2=2ga(\cos\alpha-\cos\theta)\sin^2\theta$. For the special case of $\alpha$ practically zero, these become

$$u=2\sin\frac{\theta}{2}\cos\theta\sqrt{(ga)} \text{ and } v=2\sin\frac{\theta}{2}\sin\theta\sqrt{(ga)}.$$

## CIII. PAGE 510.

4. The weight is $\frac{pv}{T}$(0·464531) gms. wt.

6. New reading of barometer is 28·77 inches (on the assumption that the *mercury* has *not* changed in temperature).

10. The work is $a^2w(\pi^2-8)/8$, where $a$ is the radius of the rope's centre line and $w$ its weight per unit length.

14. When the axis is parallel to the plane of the triangle the period of oscillation is $2\pi\sqrt{(3p/4g)}$, where $p$ is the length of the perpendicular on the hypotenuse from the right angle. When the axis of rotation is at right angles to the plane, the period is $2\pi\sqrt{(p/g)}$.

16. If the prism has mass $M$ and equal sides of its base have length $s$, the moment of inertia required is $Ms^2/3$.

17. See arts. 427 and 428.

19. Assuming the thread to be vertical it must be adjusted so that $\sqrt{(3/8)}$ of the height of the cylinder is immersed. The tension of the thread is then $\left(\frac{\sqrt{6}}{2}-1\right)$ of the weight of the cylinder. If the thread were cut the cylinder would rise vertically till less than $(2+\sqrt{2})/4$ (or 0·85355) of its height were immersed, when it would become unstable and settle with its axis horizontal.

## CIV. PAGE 513.

1. The symbol $k$ denotes $p/\rho$ where $\rho$ is the density; or, it denotes $g$-times the height of the homogeneous atmosphere. The value of $k$ is 901,600 ft.$^2$/sec.$^2$.

2. For a cubic ft. of water 62·5 lbs. and $g = 32$ ft./sec.$^2$., the pressures are 4,500 and 5,625 lbs. wt. per sq. foot. With the values 62·3 and 32·2 for the same quantities, we find for the pressures 4,457·74 and 5,572·18 lbs. wt. per sq. ft. The total forces exerted over the area of four square inches would then be
   (i) 125 and 156·25 lbs. wt., or (ii) 123·82 and 154·78 lbs. wt.

5. Let the rod have length $2l$ and the pegs be at $P$ and $Q$ distant respectively $a$ and $b$ from $B$; then,
   (i) when *all are smooth*, the pressure on the plane $= W$, the weight of the rod, and the pressures on the pegs are each $Wl \sin\theta/(a-b)$, where $\theta$ is the angle of the rod to the vertical.
   (ii) For the case where the plane is removed and the pegs become rough let the coefficient of friction be $\mu$.

   Then the condition for possible equilibrium is
   $$\mu \tan\theta = (a-b) \div (2l - a - b).$$
   Normal pressure from rod on peg at $P$ is $W\dfrac{l-b}{a-b}\sin\theta$.

   Normal pressures from rod on peg at $Q$ is $W\dfrac{l-a}{a-b}\sin\theta$.

   Tangential frictional forces at pegs are $\mu$-times the above.

6. Let the equal weights be each $W$, then the tension of the tie is $W$, and the stress in the rods are all of magnitude $W/\sqrt{2}$. The upper pair are in tension and the lower pair in compression, so that the lower pair would be liable to bend if $W$ were large.

9. Let $a$ be the distance of focus from vertex, then the centre of pressure is at a depth of $12a/5$ below the surface, and is distant $2a/5$ horizontally from the focal chord.

11. Time of a vertical oscillation is $2\pi\sqrt{(V/Ag)}$, where $V$ is volume of liquid displaced and $A$ is area of plane of flotation.
   Time for the cone if height is $h$ is $2\pi\sqrt{(1\,71h/6g)} = 3{\cdot}35431\sqrt{(h/g)}$.

12. Specific gravity of third liquid is 0·855.

15. The rod will leave the vertical plane when the end in contact has fallen to a height 2/3 of its original height. Or, if its original inclination to the horizontal were $a$, it will leave the vertical plane when inclined to the horizontal at $\theta$, where $\sin\theta = \frac{2}{3}\sin a$.

16. Let the plane be inclined $\theta$ to the horizontal. Then after rolling a length $s$ of the plane from rest, the linear velocity of the centre is given by $v^2 = 2(\frac{5}{7}g\sin\theta)s$, when the core of same mass as shell is loose and does not rotate:
   (1) $v^2 = 2(\frac{1}{2}g\sin\theta)s$, when there is no core; and
   (2) $v^2 = 2(\frac{1}{3}g\sin\theta)s$ when the core adheres firmly to shell.
   The factors in brackets in each case are the linear acceleration of the centre of the shell.

## CV. PAGE 515.

1. The position of the centre of gravity is 21/88 of the radius from the centre of the diametral plane and on the radius perpendicular to that plane.

4. The attraction is $\pi\gamma\rho am$, where $\gamma$ is the gravitation constant.

7. The motion is elliptic, its equation referred to the axes inclined at 60° being $x^2+y^2=a^2$, where $a$ is the common value of the amplitude along the axes of $x$ and $y$.

12. The axis for minimum period is parallel to the major axis and distant $b/2$ from it, where $b$ is the semi-axis minor. The minimum period is $2\pi\sqrt{(b/g)}$.

## CVII. PAGE 517.

6. The resistance is $14\frac{49}{120}=14{\cdot}408$ lbs. wt. per ton.

9. The greatest stretching force attained is

$$mg\left[1+\sqrt{\left\{1-\frac{2\rho\sigma(l-h)}{mgl}\right\}}\right],$$

where $\sigma$ is the cross-sectional area of the string. The motion of the mass $m$ is thenceforth simple harmonic of period $\tau=2\pi\sqrt{(lm/\rho\sigma)}$.

## CVIII. PAGE 519.

2. The angular acceleration is $(Ma-M'a')g/D$, and the tensions of the strings on the wheel and on the axle respectively are

$$\{M'a'(a+a')+\mu k^2\}Mg/D \text{ and}$$
$$\{Ma(a+a')+\mu k^2\}M'g/D\text{ ; where}$$
$$D \text{ is } Ma^2+M'a'^2+\mu k^2.$$

7. Required ratio of breadth to height is $\sqrt{\{6s(1-s)\}}$, where $s$ is the specific gravity of the solid in terms of the liquid in which it floats.

## CIX. PAGE 520.

2. The inclination of the velocity to the axis of $x$ is $\theta$.

13. Angular velocity is $u/AB$. Velocity of $C$ is $u\sqrt{3}$ and is perpendicular to $GC$.

18. The eccentricity of the new hyperbolic path is $\sqrt{(9-8e^2)}$, where $e$ is the eccentricity of the old elliptical path.

## CX. PAGE 522.

1. Taking $g=32{\cdot}2$ ft./sec.$^2$ the time from rest to rest is 5 min. 48·911 secs.

10. When it has swung through the angle 120° and the particle has risen $3a/2$.

## CXI. PAGE 525.

1. The moment of inertia about axis is $Ma^2/2$. The angular velocity is $\omega$, where $\omega^2=4g(\cos\theta-\cos\alpha)/3a$. Radial force on axis is $\frac{1}{3}Mg(7\cos\theta-4\cos\alpha)$, and the transverse force is $\frac{1}{3}Mg\sin\theta$.

7. Angular velocity required is $4\sqrt{(g/3p)}$.

12. Loss is $\frac{1}{8}$ of original energy.

13. Moments of inertia in numerical case are $25M/6$ about side 12 and $24M$ about side 5. The product of inertia about the same side is $5M$, where $M$ is throughout the mass of the triangle. Principal axes of inertia at the right angle make the angles $\frac{1}{2}\tan^{-1}(60/119)$ with the sides containing the right angle.

## CXII. PAGE 528.

9. The centre of gravity is at 2/3 the radius from the tangent plane and on the radius perpendicular to that plane.

12. The centre of gravity is on the radius of symmetry and at a distance $\frac{2}{\pi}\frac{a^2+ab+b^2}{a+b}$ from the centre.

15. $Q=2P\sqrt{5}$ and acts along the line joining the middle points of $DE$ and $DA$.

16. (i) The natural length is $a = \dfrac{T_1a_2 - T_2a_1}{T_1 - T_2}$,

(ii) The modulus for the string is $\dfrac{T_1a_2 - T_2a_1}{a_1 - a_2}$,

(iii) The work done from $a$ to $b$ is $\frac{1}{2}(b-a)^2\dfrac{T_1 - T_2}{a_1 - a_2}$.

20. Let the horizontal span be $2x$. the inclination of the end tensions to the horizontal $\psi$, then we have $\cosh(x/c) = \sec\psi$, or $x = c\log_e(\sec\psi + \tan\psi)$; where $c$ is the parameter of the catenary.

## CXIII. PAGE 531.

17. Ratio of attraction of frustum to that of small cone is $(h - h')/h'$.

## CXIV. PAGE 533.

3. Work is $\dfrac{B - A}{2ABw}W^2$.

10. 8006·5 metres.

14. Force required is $\frac{2}{3}abw(4a + 3h)$, where $2a$ denotes the vertical and $2b$ the horizontal sides of the door, $h$ is the depth of its hinges, and $w$ the weight of water per unit volume.

18. The limits for the value of its specific gravity are

$$\frac{1}{2} \pm \frac{\sqrt{3}}{6}.$$

## CXVI. PAGE 540.

3. The moment of inertia of the wheel alone is $mh(l - h)$.

10. $\sqrt{35}/3$ and $1/3\sqrt{35}$.

## CXIX. PAGE 549.

2. $5\sqrt{g}$.

3. At the intersection of the diagonals.

6. The rod comes to rest when $P$ makes with the rod produced the angle $\sin^{-1}(\mu W/2P)$.

8. Pressure has horizontal component $2a^3w/3$ through centre of pressure and vertical component $\pi a^3w/3$ through centre of gravity of quadrant.

## CXX. PAGE 551.

1. Work of lifting clear is $5Wa/8$, work of submerging completely is also $5Wa/8$.

5. The total kinetic energy of the rhombus is

$$2m[u^2 + \tfrac{2}{3}a^2(\omega^2 + \omega'^2)],$$

where $m$ is the mass and $2a$ the length of each rod, $u$ is the linear velocity of the centre of the rhombus, $\omega$ and $\omega'$ are the angular velocities of opposite pairs of parallel sides.

7. The co-ordinates are $\left\{\dfrac{X(Xa + Yb)}{X^2 + Y^2}, \quad \dfrac{Y(Xa + Yb)}{X^2 + Y^2}\right\}$.

10. Span is 88·14 ft., sag is 20·71 ft.

13. (i) $\frac{1}{2}Ma^2$, (ii) $\frac{3}{2}Ma^2$.

15. Tension is 2·06 lbs. wt.

## CXXII. PAGE 557.

8. Reaction is vertical of value $Mg(5 - 3\cos\alpha)$.

20. $G = W/2$

# MATHEMATICAL TABLES

(REPRODUCED HERE BY KIND PERMISSION OF THE CONTROLLER OF H.M. STATIONERY OFFICE.)

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*(A copy of these Tables will be supplied to each candidate at the Examinations in Practical Plane and Solid Geometry, Machine Construction and Drawing (Stage 3 and Honours), Building Construction (Stage 3 and Honours), Naval Architecture, Practical Mathematics, Applied Mechanics, and Heat Engines.)*

## USEFUL CONSTANTS.

1 Inch = 25·40 millimetres.

1 Gallon = ·1604 cubic foot = 10 lb. of water at 62° F.

1 Knot = 6080 feet per hour = 1 Nautical mile per hour.

Weight of 1 lb. in London = 445,000 dynes.

One pound avoirdupois = 7000 grains = 453·6 grammes.

1 Cubic foot of water weighs 62·3 lb.

1 Cubic foot of air at 0° C. and 1 atmosphere, weighs ·0807 lb.

1 Cubic foot of Hydrogen at 0° C. and 1 atmosphere, weighs ·00559 lb.

1 Foot-pound = $1·3562 \times 10^7$ ergs.

1 Horse-power-hour = $33000 \times 60$ foot-pounds.

1 Electrical unit = 1000 watt-hours.

Joule's Equivalent to suit Regnault's H, is $\begin{cases} 774 \text{ ft.-lb.} = 1 \text{ Fah. unit.} \\ 1393 \text{ ft.-lb.} = 1 \text{ Cent. } \text{,,} \end{cases}$

1 Horse-power = 33000 foot-pounds per minute = 746 watts.

Volts × ampères = watts.

1 Atmosphere = 14·7 lb. per square inch = 2116 lb. per square foot = 760 mm. of mercury = $10^6$ dynes per sq. cm. nearly.

A column of water 2·3 feet high corresponds to a pressure of 1 lb. per sq. inch.

Absolute temp., $t = \theta°$ C. + 273° or $\theta°$ F. + 460°.

Regnault's H = 606·5 + ·305 $\theta°$ C. = 1082 + ·305 $\theta°$ F.

$pu^{1·0646} = 479$.

$$\log_{10} p = 6·1007 - \frac{B}{t} - \frac{C}{t^2},$$

where $\log_{10} B = 3·1812$, $\log_{10} C = 5·0882$.

$p$ is in pounds per sq. inch, $t$ is absolute temperature Centigrade.

$u$ is the volume in cubic feet per pound of steam.

$\pi = 3·1416$.

One radian = 57·30 degrees.

To convert common into Napierian logarithms, multiply by 2·3026.

The base of the Napierian logarithms is $e = 2·7183$.

The value of $g$ at London = 32·182 feet per sec. per sec.

## LOGARITHMS.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4<br>4 | 9<br>8 | 13<br>12 | 17<br>16 | 21<br>20 | 26<br>24 | 30<br>28 | 34<br>32 | 38<br>37 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4<br>4 | 8<br>7 | 12<br>11 | 15<br>15 | 19<br>19 | 23<br>22 | 27<br>26 | 31<br>30 | 35<br>33 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 | 3<br>3 | 7<br>7 | 11<br>10 | 14<br>14 | 18<br>17 | 21<br>20 | 25<br>24 | 28<br>27 | 32<br>31 |
| 13 | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3<br>3 | 7<br>7 | 10<br>10 | 13<br>12 | 16<br>16 | 20<br>19 | 23<br>22 | 26<br>25 | 30<br>29 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3<br>3 | 6<br>6 | 9<br>9 | 12<br>12 | 15<br>15 | 18<br>17 | 21<br>20 | 24<br>23 | 28<br>26 |
| 15 | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3<br>3 | 6<br>5 | 9<br>8 | 11<br>11 | 14<br>14 | 17<br>16 | 20<br>19 | 23<br>22 | 26<br>25 |
| 16 | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3<br>3 | 5<br>5 | 8<br>8 | 11<br>10 | 14<br>13 | 16<br>15 | 19<br>18 | 22<br>21 | 24<br>23 |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 | 3<br>2 | 5<br>5 | 8<br>7 | 10<br>10 | 13<br>12 | 15<br>15 | 18<br>17 | 20<br>19 | 23<br>22 |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2<br>2 | 5<br>5 | 7<br>7 | 9<br>9 | 12<br>11 | 14<br>14 | 16<br>16 | 19<br>18 | 21<br>21 |
| 19 | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2<br>2 | 4<br>4 | 7<br>6 | 9<br>8 | 11<br>11 | 13<br>13 | 16<br>15 | 18<br>17 | 20<br>19 |
| 20 | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| 25 | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2 | 3 | 5 | 7 | 8 | 10 | 11 | 13 | 15 |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 30 | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 12 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 35 | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 38 | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| 40 | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 |
| 41 | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 43 | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 44 | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 45 | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 46 | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 47 | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 50 | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |

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## LOGARITHMS.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 51 | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 52 | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| 53 | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 54 | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 55 | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 59 | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 60 | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 61 | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 65 | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 70 | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 71 | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 73 | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 74 | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 75 | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 76 | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 77 | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 80 | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 81 | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 82 | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 83 | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 84 | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 85 | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 86 | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 87 | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 88 | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 89 | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 90 | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 91 | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 92 | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 93 | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 94 | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 95 | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 96 | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 97 | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 98 | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 99 | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |

## ANTILOGARITHMS.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·00 | 1000 | 1002 | 1005 | 1007 | 1009 | 1012 | 1014 | 1016 | 1019 | 1021 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·01 | 1023 | 1026 | 1028 | 1030 | 1033 | 1035 | 1038 | 1040 | 1042 | 1045 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·02 | 1047 | 1050 | 1052 | 1054 | 1057 | 1059 | 1062 | 1064 | 1067 | 1069 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·03 | 1072 | 1074 | 1076 | 1079 | 1081 | 1084 | 1086 | 1089 | 1091 | 1094 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·04 | 1096 | 1099 | 1102 | 1104 | 1107 | 1109 | 1112 | 1114 | 1117 | 1119 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·05 | 1122 | 1125 | 1127 | 1130 | 1132 | 1135 | 1138 | 1140 | 1143 | 1146 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·06 | 1148 | 1151 | 1153 | 1156 | 1159 | 1161 | 1164 | 1167 | 1169 | 1172 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·07 | 1175 | 1178 | 1180 | 1183 | 1186 | 1189 | 1191 | 1194 | 1197 | 1199 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·08 | 1202 | 1205 | 1208 | 1211 | 1213 | 1216 | 1219 | 1222 | 1225 | 1227 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| ·09 | 1230 | 1233 | 1236 | 1239 | 1242 | 1245 | 1247 | 1250 | 1253 | 1256 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| ·10 | 1259 | 1262 | 1265 | 1268 | 1271 | 1274 | 1276 | 1279 | 1282 | 1285 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| ·11 | 1288 | 1291 | 1294 | 1297 | 1300 | 1303 | 1306 | 1309 | 1312 | 1315 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| ·12 | 1318 | 1321 | 1324 | 1327 | 1330 | 1334 | 1337 | 1340 | 1343 | 1346 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| ·13 | 1349 | 1352 | 1355 | 1358 | 1361 | 1365 | 1368 | 1371 | 1374 | 1377 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·14 | 1380 | 1384 | 1387 | 1390 | 1393 | 1396 | 1400 | 1403 | 1406 | 1409 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·15 | 1413 | 1416 | 1419 | 1422 | 1426 | 1429 | 1432 | 1435 | 1439 | 1442 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·16 | 1445 | 1449 | 1452 | 1455 | 1459 | 1462 | 1466 | 1469 | 1472 | 1476 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·17 | 1479 | 1483 | 1486 | 1489 | 1493 | 1496 | 1500 | 1503 | 1507 | 1510 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·18 | 1514 | 1517 | 1521 | 1524 | 1528 | 1531 | 1535 | 1538 | 1542 | 1545 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·19 | 1549 | 1552 | 1556 | 1560 | 1563 | 1567 | 1570 | 1574 | 1578 | 1581 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| ·20 | 1585 | 1589 | 1592 | 1596 | 1600 | 1603 | 1607 | 1611 | 1614 | 1618 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| ·21 | 1622 | 1626 | 1629 | 1633 | 1637 | 1641 | 1644 | 1648 | 1652 | 1656 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| ·22 | 1660 | 1663 | 1667 | 1671 | 1675 | 1679 | 1683 | 1687 | 1690 | 1694 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| ·23 | 1698 | 1702 | 1706 | 1710 | 1714 | 1718 | 1722 | 1726 | 1730 | 1734 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| ·24 | 1738 | 1742 | 1746 | 1750 | 1754 | 1758 | 1762 | 1766 | 1770 | 1774 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| ·25 | 1778 | 1782 | 1786 | 1791 | 1795 | 1799 | 1803 | 1807 | 1811 | 1816 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| ·26 | 1820 | 1824 | 1828 | 1832 | 1837 | 1841 | 1845 | 1849 | 1854 | 1858 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| ·27 | 1862 | 1866 | 1871 | 1875 | 1879 | 1884 | 1888 | 1892 | 1897 | 1901 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| ·28 | 1905 | 1910 | 1914 | 1919 | 1923 | 1928 | 1932 | 1936 | 1941 | 1945 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·29 | 1950 | 1954 | 1959 | 1963 | 1968 | 1972 | 1977 | 1982 | 1986 | 1991 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·30 | 1995 | 2000 | 2004 | 2009 | 2014 | 2018 | 2023 | 2028 | 2032 | 2037 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·31 | 2042 | 2046 | 2051 | 2056 | 2061 | 2065 | 2070 | 2075 | 2080 | 2084 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·32 | 2089 | 2094 | 2099 | 2104 | 2109 | 2113 | 2118 | 2123 | 2128 | 2133 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·33 | 2138 | 2143 | 2148 | 2153 | 2158 | 2163 | 2168 | 2173 | 2178 | 2183 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·34 | 2188 | 2193 | 2198 | 2203 | 2208 | 2213 | 2218 | 2223 | 2228 | 2234 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·35 | 2239 | 2244 | 2249 | 2254 | 2259 | 2265 | 2270 | 2275 | 2280 | 2286 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·36 | 2291 | 2296 | 2301 | 2307 | 2312 | 2317 | 2323 | 2328 | 2333 | 2339 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·37 | 2344 | 2350 | 2355 | 2360 | 2366 | 2371 | 2377 | 2382 | 2388 | 2393 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·38 | 2399 | 2404 | 2410 | 2415 | 2421 | 2427 | 2432 | 2438 | 2443 | 2449 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·39 | 2455 | 2460 | 2466 | 2472 | 2477 | 2483 | 2489 | 2495 | 2500 | 2506 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| ·40 | 2512 | 2518 | 2523 | 2529 | 2535 | 2541 | 2547 | 2553 | 2559 | 2564 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| ·41 | 2570 | 2576 | 2582 | 2588 | 2594 | 2600 | 2606 | 2612 | 2618 | 2624 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| ·42 | 2630 | 2636 | 2642 | 2649 | 2655 | 2661 | 2667 | 2673 | 2679 | 2685 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| ·43 | 2692 | 2698 | 2704 | 2710 | 2716 | 2723 | 2729 | 2735 | 2742 | 2748 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| ·44 | 2754 | 2761 | 2767 | 2773 | 2780 | 2786 | 2793 | 2799 | 2805 | 2812 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| ·45 | 2818 | 2825 | 2831 | 2838 | 2844 | 2851 | 2858 | 2864 | 2871 | 2877 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| ·46 | 2884 | 2891 | 2897 | 2904 | 2911 | 2917 | 2924 | 2931 | 2938 | 2944 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| ·47 | 2951 | 2958 | 2965 | 2972 | 2979 | 2985 | 2992 | 2999 | 3006 | 3013 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| ·48 | 3020 | 3027 | 3034 | 3041 | 3048 | 3055 | 3062 | 3069 | 3076 | 3083 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| ·49 | 3090 | 3097 | 3105 | 3112 | 3119 | 3126 | 3133 | 3141 | 3148 | 3155 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |

## ANTILOGARITHMS.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·50 | 3162 | 3170 | 3177 | 3184 | 3192 | 3199 | 3206 | 3214 | 3221 | 3228 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| ·51 | 3236 | 3243 | 3251 | 3258 | 3266 | 3273 | 3281 | 3289 | 3296 | 3304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| ·52 | 3311 | 3319 | 3327 | 3334 | 3342 | 3350 | 3357 | 3365 | 3373 | 3381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| ·53 | 3388 | 3396 | 3404 | 3412 | 3420 | 3428 | 3436 | 3443 | 3451 | 3459 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| ·54 | 3467 | 3475 | 3483 | 3491 | 3499 | 3508 | 3516 | 3524 | 3532 | 3540 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| ·55 | 3548 | 3556 | 3565 | 3573 | 3581 | 3589 | 3597 | 3606 | 3614 | 3622 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| ·56 | 3631 | 3639 | 3648 | 3656 | 3664 | 3673 | 3681 | 3690 | 3698 | 3707 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| ·57 | 3715 | 3724 | 3733 | 3741 | 3750 | 3758 | 3767 | 3776 | 3784 | 3793 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| ·58 | 3802 | 3811 | 3819 | 3828 | 3837 | 3846 | 3855 | 3864 | 3873 | 3882 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| ·59 | 3890 | 3899 | 3908 | 3917 | 3926 | 3936 | 3945 | 3954 | 3963 | 3972 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| ·60 | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| ·61 | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·62 | 4169 | 4178 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·63 | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·64 | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·65 | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·66 | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| ·67 | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| ·68 | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| ·69 | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| ·70 | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| ·71 | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| ·72 | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| ·73 | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| ·74 | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5585 | 5598 | 5610 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| ·75 | 5623 | 5636 | 5649 | 5662 | 5675 | 5689 | 5702 | 5715 | 5728 | 5741 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| ·76 | 5754 | 5768 | 5781 | 5794 | 5808 | 5821 | 5834 | 5848 | 5861 | 5875 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| ·77 | 5888 | 5902 | 5916 | 5929 | 5943 | 5957 | 5970 | 5984 | 5998 | 6012 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| ·78 | 6026 | 6039 | 6053 | 6067 | 6081 | 6095 | 6109 | 6124 | 6138 | 6152 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| ·79 | 6166 | 6180 | 6194 | 6209 | 6223 | 6237 | 6252 | 6266 | 6281 | 6295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| ·80 | 6310 | 6324 | 6339 | 6353 | 6368 | 6383 | 6307 | 6412 | 6427 | 6442 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| ·81 | 6457 | 6471 | 6486 | 6501 | 6516 | 6531 | 6546 | 6561 | 6577 | 6592 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| ·82 | 6607 | 6622 | 6637 | 6653 | 6668 | 6683 | 6699 | 6714 | 6730 | 6745 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| ·83 | 6761 | 6776 | 6792 | 6808 | 6823 | 6839 | 6855 | 6871 | 6887 | 6902 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| ·84 | 6918 | 6934 | 6950 | 6966 | 6982 | 6998 | 7015 | 7031 | 7047 | 7063 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 15 |
| ·85 | 7079 | 7096 | 7112 | 7129 | 7145 | 7161 | 7178 | 7194 | 7211 | 7228 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| ·86 | 7244 | 7261 | 7278 | 7295 | 7311 | 7328 | 7345 | 7362 | 7379 | 7396 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| ·87 | 7413 | 7430 | 7447 | 7464 | 7482 | 7499 | 7516 | 7534 | 7551 | 7568 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| ·88 | 7586 | 7603 | 7621 | 7638 | 7656 | 7674 | 7691 | 7709 | 7727 | 7745 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| ·89 | 7762 | 7780 | 7798 | 7816 | 7834 | 7852 | 7870 | 7889 | 7907 | 7925 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| ·90 | 7943 | 7962 | 7980 | 7998 | 8017 | 8035 | 8054 | 8072 | 8091 | 8110 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| ·91 | 8128 | 8147 | 8166 | 8185 | 8204 | 8222 | 8241 | 8260 | 8279 | 8299 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| ·92 | 8318 | 8337 | 8356 | 8375 | 8395 | 8414 | 8433 | 8453 | 8472 | 8492 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| ·93 | 8511 | 8531 | 8551 | 8570 | 8590 | 8610 | 8630 | 8650 | 8670 | 8690 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| ·94 | 8710 | 8730 | 8750 | 8770 | 8790 | 8810 | 8831 | 8851 | 8872 | 8892 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| ·95 | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 17 | 19 |
| ·96 | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| ·97 | 9333 | 9354 | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| ·98 | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| ·99 | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 |

| Angle. | | Chord. | Sine. | Tangent. | Co-tangent. | Cosine. | | | |
|---|---|---|---|---|---|---|---|---|---|
| Degrees. | Radians. | | | | | | | | |
| 0° | 0 | 0 | 0 | 0 | ∞ | 1 | 1·414 | 1·5708 | 90° |
| 1 | ·0175 | ·017 | ·0175 | ·0175 | 57·2900 | ·9998 | 1·402 | 1·5533 | 89 |
| 2 | ·0349 | ·035 | ·0349 | ·0349 | 28·6363 | ·9994 | 1·389 | 1·5359 | 88 |
| 3 | ·0524 | ·052 | ·0523 | ·0524 | 19·0811 | ·9986 | 1·377 | 1·5184 | 87 |
| 4 | ·0698 | ·070 | ·0698 | ·0699 | 14·3007 | ·9976 | 1·364 | 1·5010 | 86 |
| 5 | ·0873 | ·087 | ·0872 | ·0875 | 11·4301 | ·9962 | 1·351 | 1·4835 | 85 |
| 6 | ·1047 | ·105 | ·1045 | ·1051 | 9·5144 | ·9945 | 1·338 | 1·4661 | 84 |
| 7 | ·1222 | ·122 | ·1219 | ·1228 | 8·1443 | ·9925 | 1·325 | 1·4486 | 83 |
| 8 | ·1396 | ·140 | ·1392 | ·1405 | 7·1154 | ·9903 | 1·312 | 1·4312 | 82 |
| 9 | ·1571 | ·157 | ·1564 | ·1584 | 6·3138 | ·9877 | 1·299 | 1·4137 | 81 |
| 10 | ·1745 | ·174 | ·1736 | ·1763 | 5·6713 | ·9848 | 1·286 | 1·3963 | 80 |
| 11 | ·1920 | ·192 | ·1908 | ·1944 | 5·1446 | ·9816 | 1·272 | 1·3788 | 79 |
| 12 | ·2094 | ·209 | ·2079 | ·2126 | 4·7046 | ·9781 | 1·259 | 1·3614 | 78 |
| 13 | ·2269 | ·226 | ·2250 | ·2309 | 4·3315 | ·9744 | 1·245 | 1·3439 | 77 |
| 14 | ·2443 | ·244 | ·2419 | ·2493 | 4·0108 | ·9703 | 1·231 | 1·3265 | 76 |
| 15 | ·2618 | ·261 | ·2588 | ·2679 | 3·7321 | ·9659 | 1·218 | 1·3090 | 75 |
| 16 | ·2793 | ·278 | ·2756 | ·2867 | 3·4874 | ·9613 | 1·204 | 1·2915 | 74 |
| 17 | ·2967 | ·296 | ·2924 | ·3057 | 3·2709 | ·9563 | 1·190 | 1·2741 | 73 |
| 18 | ·3142 | ·313 | ·3090 | ·3249 | 3·0777 | ·9511 | 1·176 | 1·2566 | 72 |
| 19 | ·3316 | ·330 | ·3256 | ·3443 | 2·9042 | ·9455 | 1·161 | 1·2392 | 71 |
| 20 | ·3491 | ·347 | ·3420 | ·3640 | 2·7475 | ·9397 | 1·147 | 1·2217 | 70 |
| 21 | ·3665 | ·364 | ·3584 | ·3839 | 2·6051 | ·9336 | 1·133 | 1·2043 | 69 |
| 22 | ·3840 | ·382 | ·3746 | ·4040 | 2·4751 | ·9272 | 1·118 | 1·1868 | 68 |
| 23 | ·4014 | ·399 | ·3907 | ·4245 | 2·3559 | ·9205 | 1·104 | 1·1694 | 67 |
| 24 | ·4189 | ·416 | ·4067 | ·4452 | 2·2460 | ·9135 | 1·089 | 1·1519 | 66 |
| 25 | ·4363 | ·433 | ·4226 | ·4663 | 2·1445 | ·9063 | 1·075 | 1·1345 | 65 |
| 26 | ·4538 | ·450 | ·4384 | ·4877 | 2·0503 | ·8988 | 1·060 | 1·1170 | 64 |
| 27 | ·4712 | ·467 | ·4540 | ·5095 | 1·9626 | ·8910 | 1·045 | 1·0996 | 63 |
| 28 | ·4887 | ·484 | ·4695 | ·5317 | 1·8807 | ·8829 | 1·030 | 1·0821 | 62 |
| 29 | ·5061 | ·501 | ·4848 | ·5543 | 1·8040 | ·8746 | 1·015 | 1·0647 | 61 |
| 30 | ·5236 | ·518 | ·5000 | ·5774 | 1·7321 | ·8660 | 1·000 | 1·0472 | 60 |
| 31 | ·5411 | ·534 | ·5150 | ·6009 | 1·6643 | ·8572 | ·985 | 1·0297 | 59 |
| 32 | ·5585 | ·551 | ·5299 | ·6249 | 1·6003 | ·8480 | ·970 | 1·0123 | 58 |
| 33 | ·5760 | ·568 | ·5446 | ·6494 | 1·5399 | ·8387 | ·954 | ·9948 | 57 |
| 34 | ·5934 | ·585 | ·5592 | ·6745 | 1·4826 | ·8290 | ·939 | ·9774 | 56 |
| 35 | ·6109 | ·601 | ·5736 | ·7002 | 1·4281 | ·8192 | ·923 | ·9599 | 55 |
| 36 | ·6283 | ·618 | ·5878 | ·7265 | 1·3764 | ·8090 | ·908 | ·9425 | 54 |
| 37 | ·6458 | ·635 | ·6018 | ·7536 | 1·3270 | ·7986 | ·892 | ·9250 | 53 |
| 38 | ·6632 | ·651 | ·6157 | ·7813 | 1·2799 | ·7880 | ·877 | ·9076 | 52 |
| 39 | ·6807 | ·668 | ·6293 | ·8098 | 1·2349 | ·7771 | ·861 | ·8901 | 51 |
| 40 | ·6981 | ·684 | ·6428 | ·8391 | 1·1918 | ·7660 | ·854 | ·8727 | 50 |
| 41 | ·7156 | ·700 | ·6561 | ·8693 | 1·1504 | ·7547 | ·829 | ·8552 | 49 |
| 42 | ·7330 | ·717 | ·6691 | ·9004 | 1·1106 | ·7431 | ·813 | ·8378 | 48 |
| 43 | ·7505 | ·733 | ·6820 | ·9325 | 1·0724 | ·7314 | ·797 | ·8203 | 47 |
| 44 | ·7679 | ·749 | ·6947 | ·9657 | 1·0355 | ·7193 | ·781 | ·8029 | 46 |
| 45° | ·7854 | ·765 | ·7071 | 1·0000 | 1·0000 | ·7071 | ·765 | ·7854 | 45° |
| | | | Cosine. | Co-tangent. | Tangent. | Sine. | Chord. | Radians. | Degrees. |
| | | | | | | | | Angle. | |

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