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THE ESSENTIALS OF SCHOOL GEOMETRY

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THE ESSENTIALS
OF
SCHOOL GEOMETRY

BY

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MACMILLAN AND CO., LIMITED
ST. MARTIN'S STREET, LONDON
1933

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PRINTED IN GREAT BRITAIN

PREFACE

THOSE teachers and examiners who are in a position to compare the results obtained by the teaching of Geometry in schools to-day with those obtained before the dethronement of Euclid agree almost unanimously that there have been both gain and loss. On the one hand, almost all pupils to-day acquire much more power in applying and reasoning from the fundamental facts of Geometry than did their predecessors, but, on the other hand, their reasoning is often less rigorous, and the average pupil often fails lamentably to reproduce the standard proofs when called upon in examinations. Almost all would agree that the gain outweighs the loss; for the educational value of the subject lies far more in the former than in the latter accomplishment. There are many, however, who think that the loss need not accompany the gain, and one of the objects of this text-book is to help the average pupil by providing proofs of the standard theorems which, within limits, shall be accurate and rigorous. For this reason, in writing out proofs of theorems, great care has been taken to give adequate references *in words* to previous results and, in the interests of the average pupil, it has been thought better to err, if at all, on the side of prolixity rather than brevity. In particular, stress has been laid upon accurate references to the four congruence theorems. Those who examine School Certificate papers frequently find that these references are very slipshod and lead to much poor work not only in book-work but in riders. It is hoped that the stress laid in this book upon accurate references to previous theorems will not only help the pupils who use it to avoid a totally unnecessary loss of marks in examinations, but will also result in their learning to express themselves clearly and accurately in all their work.

It is now almost generally admitted that there should be three stages in the teaching of Geometry up to School Certificate standard :

(*a*) The experimental stage, in which the pupil is introduced to the common geometrical notions and figures, and in which through a large variety of drawing exercises proficiency in the use of instruments is acquired.

(*b*) The deductive stage, in which the pupil learns to prove theorems and riders and to write out proofs. In this stage many results will be established intuitively, the formal proof belonging to the later stage.

(*c*) The systematising stage, in which the knowledge obtained in the previous stages will be systematised, and in which the theorems will be arranged in a logical sequence depending on a comparatively small number of axioms. In this stage the reasoning should be rigorous, relatively to the mental age of the pupils.

This book presupposes a course in experimental Geometry, and no attempt has been made to include in the text the usual preliminary oral instruction which deals with the fundamental concepts. It contains, however, all that is necessary for revising such a course. No attempt has been made to indicate completely what propositions should be regarded as intuitive in the deductive stage ; most teachers will prefer to make their own choice. The theorems are therefore arranged in their final order, and the sequence is, with few exceptions, that recommended by the I.A.A.M. Committee. The only important exception is that the fundamental theorems concerning the properties of parallel lines have been placed after the first congruence theorem. This has been done not only to meet the requirements of certain important examining bodies, but also because the author thinks that it is both wrong and unnecessary to weaken whatever sense of logic the pupils may have by attempting to persuade them, *at the systematising stage*, that the " Parallel " theorems may be *proved* by an appeal to the idea of direction as axiomatic. He

believes that most pupils, at this stage, are capable of appreciating the logical development of the subject, while being thankful that few examining bodies expect the reproduction of the proofs of these theorems in their examinations.

Euclid I. 27 has been proved by means of I. 4, thus allowing I. 16 to be treated as a corollary of I. 32 and removing a constant source of confusion. The first two locus theorems have, with some reluctance, been placed where recommended by the I.A.A.M. Committee. Many teachers prefer to take them after the Intercept Theorem, and there is no reason why this should not be done, provided that a small number of the examples, which involve a knowledge of area, are postponed.

Certain theorems and sections have been marked with an asterisk. It is recommended that ordinary pupils should not be required to reproduce these proofs unless they have to prepare them for a public examination.

In considering what proof of the fundamental congruence theorem ought to be inserted the writer of a text-book has to choose between a proof by superposition, which is "condemned by all competent authorities," and a proof such as that given in the report issued by the Mathematical Association, the adoption of which, in the opinion of many experienced teachers, would "condemn the great majority of pupils to learn by heart a series of phrases and sentences which they cannot be expected to understand." A "proof" by superposition has therefore been inserted as the lesser of two evils, but the other congruence theorems have been deduced from the first without the further use of superposition. It is very much to be regretted that *all* examining bodies have not agreed to treat the fundamental congruence theorem as an axiom.

Throughout the book a number of alternative proofs of theorems have been provided. It is hoped that few teachers will desire to alter the proofs offered, but a number of blank pages have been inserted so that if a teacher desires to replace a proof in the book an alternative proof can be copied into it.

It is hoped that the treatment of Constructions and Loci will be found more complete than is usual in school text-books. *For convenience in revision* most results concerning loci have been stated as corollaries of the appropriate propositions. A complete list of loci is given, for reference, at the end of the book.

No use has been made of the method of limits in proving propositions; experience has shown that the average pupil is confused rather than helped by such proofs.

The examples are numerous, and have been arranged with great care. After each proposition or small group of related propositions there is a set of examples containing, where possible, both easy numerical examples and easy riders. There is also a set of riders of moderate difficulty (i.e. approximately School Certificate standard) after each group of propositions. These are called "Further Examples," and should not, as a rule, be worked during the first reading of the book. A pupil who is able to do these should have no difficulty with a School Certificate paper; but for the benefit of the abler pupils a set of Miscellaneous Examples, containing harder questions, has been provided. The more important examples have been printed in thick type.

No attempt has been made to include a formal course of Solid Geometry, it being thought that most teachers will prefer to establish the main facts intuitively. Wherever possible, however, three-dimensional examples have been provided, to be used at the discretion of the teacher either at the deductive or at the systematising stage. In dealing with loci, the three-dimensional locus has, in most instances, been given in the list at the end of the book.

Acknowledgments are due to Mr. T. Grantley Powell, M.A., of the Cowley School, St. Helens, to Mr. Jacob Morgan, M.A., Headmaster of the Boys' County School, Brecon, and to my colleagues, Mr. C. Kingsley Dove, M.A., and Mr. T. Marsden, M.A., B.Sc., for most valuable assistance and suggestions at all stages of the work. Thanks are also due to the Senate of the

University of London, the Cambridge Local Examinations Syndicate, the Delegates of the Oxford Local Examinations, the Joint Matriculation Board, the Oxford and Cambridge Schools Examination Board, and the Central Welsh Board for permission to use questions from the printed papers set in their examinations.

January 1933.

CONTENTS

PART I

	PAGE
INTRODUCTORY	1
ANGLES AT A POINT	11
CONGRUENT TRIANGLES	16
PARALLELS	22
ANGLES OF A TRIANGLE AND POLYGON	28
CONGRUENT TRIANGLES (*Continued*)	35
ISOSCELES TRIANGLES	38
CONGRUENT TRIANGLES (*Continued*)	42
THE USE OF INSTRUMENTS	53
ELEMENTARY CONSTRUCTIONS	58
INEQUALITIES CONNECTED WITH THE TRIANGLE	70
CONSTRUCTION OF TRIANGLES FROM GIVEN DATA	78
PARALLELOGRAMS	82
INTERCEPT THEOREMS	93
DIVISION OF A LINE INTO EQUAL PARTS	99
CONSTRUCTION OF TRIANGLES, QUADRILATERALS AND POLYGONS FROM GIVEN DATA	101
CONCURRENCY PROPERTIES OF THE TRIANGLE	105
INEQUALITIES (*Continued*)	114

PART II

PAGE

AREA OF RECTANGLE - - - - - - - - 118

AREA OF PARALLELOGRAMS AND TRIANGLES - - - 127

AREA CONSTRUCTIONS - - - - - - - - 134

AREA OF QUADRILATERAL AND TRAPEZIUM - - - 135

AREA CONSTRUCTIONS (*Continued*) - - - - - 136

PYTHAGORAS' THEOREM - - - - - - 144

LOCI - - - - - - - - - - - 157

PART III

CHORDS OF A CIRCLE - - - - - - - 174

ANGLE PROPERTIES OF A CIRCLE - - - - - 184

ANGLES SUBTENDED BY EQUAL ARCS AND CHORDS - 204

TANGENT PROPERTIES - - - - - - - 218

CONSTRUCTION OF TANGENTS AND COMMON TANGENTS - 226

ANGLE IN ALTERNATE SEGMENT - - - - - 233

CONSTRUCTION OF CIRCLES FROM VARIOUS DATA - - 242

INSCRIBED AND CIRCUMSCRIBED FIGURES - - - 250

CIRCLES CONNECTED WITH THE TRIANGLE - - - 256

LOCI CONNECTED WITH THE CIRCLE - - - - 264

PART IV

THE APPLICATION OF ALGEBRA TO GEOMETRY - - 273

GEOMETRICAL EQUIVALENTS OF ALGEBRAIC IDENTITIES - 275

EXTENSIONS OF PYTHAGORAS' THEOREM - - - 284

APOLLONIUS' THEOREM - - - - - - - 288

THE AREA OF A TRIANGLE WITH GIVEN SIDES - - 294

PAGE

THE RECTANGLES CONTAINED BY SEGMENTS OF CHORDS OF A CIRCLE - - - - - - - - - 296

VISIBLE HORIZON - - - - - - - - - 304

FURTHER CONSTRUCTIONS - - - - - - - 309

PART V

RATIO AND PROPORTION - - - - - - - - 317

PROPORTIONAL DIVISION OF LINES - - - - - - 322

THE BISECTOR OF THE VERTICAL ANGLE OF A TRIANGLE 326

SIMILAR FIGURES - - - - - - - - - 332

SIMILAR TRIANGLES - - - - - - - - 333

AREAS OF SIMILAR TRIANGLES - - - - - - 352

FURTHER CONSTRUCTIONS AND LOCI - - - - - 360

THE CIRCLE OF APOLLONIUS - - - - - - - 370

HOMOTHETIC FIGURES - - - - - - - - 372

THE RADICAL AXIS - - - - - - - - 375

NOTE ON INCOMMENSURABLE MAGNITUDES - - - - 378

TABLE OF STANDARD LOCI - - - - - - - 395

INDEX - - - - - - - - - - - - 399

ABBREVIATIONS

THE following abbreviations are commonly used in Geometry. In this book they are sparingly used in the text, but they have been freely used to reduce the space taken up by the examples.

$\therefore$	therefore.
$\because$	because.
$=$	is (or are) equal to.
$\neq$	is (or are) not equal to.
$\equiv$	is (or are) congruent to.
$+$	plus or together with.
$-$	minus or less than.
$\angle$(s)	angle(s) or the angle(s).
par^{m}	parallelogram.
sq.	square.
$\parallel$	is (or are) parallel (to). parallel (adjective). parallel (noun).
rt.	right.
$>$	is (or are) greater than.
$\ngtr$	is (or are) not greater than.
$<$	is (or are) less than.
$\nless$	is (or are) not less than.
Δ	triangle or the triangle.
quad.	quadrilateral.
rect.	rectangle.
$\perp$	is (or are) perpendicular (to). perpendicular (adjective). perpendicular (noun).
pt.	point.
st.	straight.
opp.	opposite.
vert.	vertically.
adj	adjacent.
diag.	diagonal.
$\bigcirc$	circle.
$\bigcirc^{ce}$	circumference.
hyp.	hypothesis.
hypot.	hypotenuse.
alt.	alternate.
corres.	corresponding.
def.	definition.
cor.	corollary.
diam.	diameter.
ext.	exterior.
int.	interior.

The word ' line ' is often used to denote " straight line " when this may be done without ambiguity.

PART I

ANGLES, CONGRUENCES, PARALLELS, INEQUALITIES, AND PARALLELOGRAMS

INTRODUCTORY

Geometry is the science of space, and consists of the study of the shapes, sizes and positions of objects which occupy space.

All reasoning is founded on certain simple principles, the truth of which is admitted without proof. These self-evident truths are called **Axioms**. The following is a list of axioms which apply to magnitudes of all kinds. Certain other axioms relating to geometrical magnitudes only will be stated later as they are required.

1. Things which are equal to the same thing are equal to one another.

2. If equals are added to equals, the sums are equal.

3. If equals are taken from equals, the remainders are equal.

4. If equals are added to unequals, the sums are unequal.

5. If equals are taken from unequals, the remainders are unequal.

6. Things which are the same multiples of equals are equal to one another, e.g. Doubles of equals are equal to one another.

7. Things which are the same parts of equals are equal to one another, e.g. Quarters of equals are equal to one another.

8. The whole is greater than its part.

DEFINITIONS

Surface. The space occupied by any object (say a penny or a football) is limited by boundaries which separate it from surrounding space. These boundaries are called **surfaces.**

A **surface** has length and breadth, but no thickness.

Surfaces intersect (or meet) in **lines.**

A **line** has length, but no breadth. Lines intersect in **points.**

A **point** has position, but no magnitude.

In practice a line is represented by the mark traced by a thin pencil-point on a sheet of paper, and a point is represented either by a small dot on the paper or, preferably, by the intersection of two small lines, e.g. X.

Straight lines. A line may be straight or curved. Everyone knows what is meant by a straight line, and it is almost impossible to add to this knowledge by a formal definition. The following is the definition usually given :

A straight line is that which lies evenly between its extreme points.

Axiom. Two straight lines cannot enclose a space.

It follows that two straight lines cannot intersect in more than one point. For, if they met in two points, they would enclose a space.

POSTULATES

In order to draw the figures required for the study of geometry certain instruments are required. These are (i) a straight ruler, (ii) a pair of compasses. It is also necessary to assume the possibility of performing certain simple operations by using these instruments.

A geometrical construction which we can take for granted that we can perform is called a **postulate.** The following postulates concern a straight line :

Postulate 1. Let it be granted that a straight line may be drawn from one point to any other point.

Postulate 2. Let it be granted that a straight line may be produced to any length in a straight line.

One and only one straight line can be drawn between two given points. When we draw this straight line, we are said to **join** the points.

The Plane. A **plane** is a surface such that the straight line joining any two points on the surface lies wholly in that surface. If it is desired to find out whether a surface is plane or not, a straight edge, e.g. the edge of an ordinary ruler, should be taken and laid across it in many positions. If the surface is plane, the ruler will always lie flat against it.

Angles. Two straight lines which meet at a point are said to form an **angle.** The straight lines are called the **arms** of the angle; the point at which they meet is called the **vertex** of the angle.

The angle contained by two straight lines XA and XB is called "the angle AXB" or "the angle BXA," or, if there is only one angle at the point X, "the angle X." It may also be denoted by a small letter, e.g. α, which is usually marked in the figure, as shown.

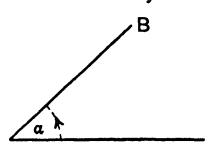


FIG. 1.

An angle may be considered as being made by the turning of a line from the position of one arm, say XA, into the position of the other arm, XB; the size of the angle AXB is measured by the amount of turning involved.

It should be noticed that the size of an angle does not depend upon the lengths of its arms.

Two angles (BXC, CXD) which have a common vertex and lie on either side of a common arm are said to be **adjacent.**

The angle BXD is the sum of the angles BXC, CXD.

The angle CXD is the difference of the angles BXD, BXC.

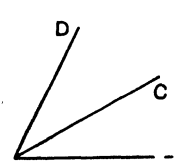


FIG. 2.

When two straight lines such as AB, CD cross one another at X, the angles CXA, BXD are said to be **vertically opposite**; so also are the angles AXD, CXB.

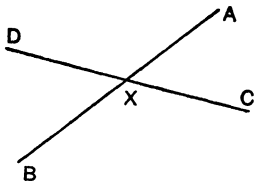


FIG. 3.

When one straight line, standing on another straight line, makes the adjacent angles equal, each of these angles is called a **right angle**; and each line is said to be **perpendicular** to the other.

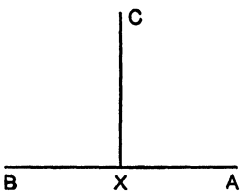


FIG. 4.

Axiom. All right angles are equal.

A right angle is, for convenience, divided into 90 equal parts called **degrees** (°); each degree into 60 equal parts called **minutes** (′); each minute into 60 equal parts called **seconds** (″).

In the above figure, if a straight line revolves about X from the position XA into the position XB, it turns through two right angles or 180 degrees. The angle AXB may also be called a straight angle. If the line makes a complete revolution about X, starting from XA and returning to its original position, it turns through four right angles or two straight angles or 360 degrees.

An angle less than one right angle is called an **acute** angle.

An angle which is greater than one right angle, but less than two right angles, is called an **obtuse** angle.

An angle which is greater than two right angles, but less than four right angles, is said to be **reflex**.

Thus an acute angle lies between 0° and 90°, a right angle equals 90°, an obtuse angle lies between 90° and 180°, a straight angle equals 180°, a reflex angle lies between 180° and 360°.

It should be noticed that when two straight lines meet, two angles are formed, one greater and one less than two right

angles. The first is obtained by supposing XB to have revolved from the position XA the longer way round, marked (i) in the figure; the other by supposing XB to have revolved the shorter way round, marked (ii). Unless otherwise stated, the angle AXB will be considered to be that which is less than two right angles.

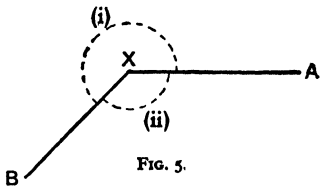


FIG. 5.

Any portion of a plane surface bounded by one or more lines is called a **plane figure.**

A **circle** is a plane figure contained by a line traced out by a point which moves so that its distance from a certain fixed point is always the same.

In the figure the point P moves so that its distance from the fixed point O is always the same.

The fixed point is called the **centre**; the bounding line is called the **circumference.**

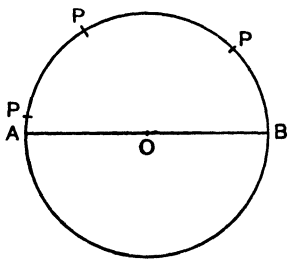


FIG. 6.

A **radius** of a circle is a straight line drawn from the centre to the circumference. All radii of a circle are, by definition, equal.

A **diameter** of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter. In the figure above AB divides the circle into two semi-circles.

A **chord** of a circle is a straight line joining any two points on the circumference.

An **arc** of a circle is any part of the circumference.

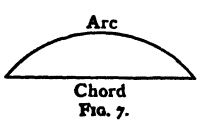


FIG. 7.

A **segment** of a circle is the figure bounded by a chord of the circle and one of the arcs into which it divides the circumference.

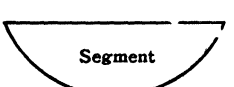


FIG. 8.

A **sector** of a circle is the figure bounded by two radii and the arc intercepted between them.

Circles that have the same centre are said to be **concentric.**

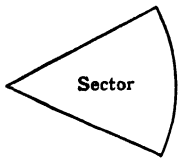


FIG. 9.

Note. When no ambiguity arises as a result, the word circle is sometimes used to denote the circumference of the circle.

Postulate. Let it be granted that a circle may be drawn with any point as centre and with a radius of any length. It follows that from the greater of two given straight lines a part can be cut off equal to the less.

Axiom. (i) If a point X moves from A to B along the straight line AB, it must pass through one position in which it divides AB into two equal parts. When X is in this position it is said to **bisect** AB or to be the **mid-point** of AB.

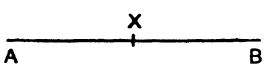


FIG. 10.

(ii) If a straight line XC, revolving about X, turns from XA to XB, it must pass through one position in which it divides the angle AXB into two equal parts. When XC is in this position it is said to bisect the angle AXB.

In other words, (i) every finite straight line has a point of bisection and (ii) every angle has a line of bisection.

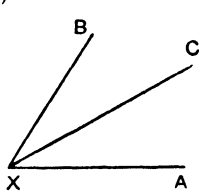


FIG. 11.

HYPOTHETICAL CONSTRUCTIONS

From the preceding, it follows that we may suppose :

(i) A straight line to be drawn perpendicular to a given straight line from any point in it.

(ii) A finite straight line to be bisected at a point.

(iii) An angle to be bisected by a straight line.

SUPERPOSITION AND TESTS OF EQUALITY

Axiom. Magnitudes which can be made to coincide with one another are equal.

This axiom implies that any line, angle or figure, may be taken up from its position, and without change of shape or size, laid down upon another line, angle or figure, for the purpose of comparison. Two such magnitudes are equal when one can be exactly placed over the other. This process is called **superposition**, and the first magnitude is said to be applied to the other.

Thus **equal straight lines** are those which can be made to coincide. Such lines are said to have the same **length.**

The **distance between two points** is the length of the straight line joining them.

If X is a point in a straight line AB, lengths measured from X towards B are said to be measured in the opposite **sense** to those measured from X towards A. If AB is produced beyond B, we are said to **produce** AB; if beyond A, we are said to **produce** BA.

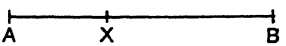


FIG. 12.

Any part of a straight line is called a **segment** of that straight line.

Plane Geometry deals with the properties of lines and points which are in the same plane. The subject is divided into **propositions,** which are either **theorems** or **constructions.** A **theorem** is a proposition in which some statement has to be proved. A **construction** is a proposition in which it is required to draw some particular figure.

A proposition consists of the following parts :

(i) The **general enunciation,** which is a preliminary statement, describing what has to be proved or constructed.

(ii) The **particular enunciation,** which repeats the statement already made with special reference to a particular diagram.

(iii) The **construction,** which directs the drawing of such lines and circles as may be necessary.

(iv) The **proof,** which shows that the property stated in a theorem is true, or that the figure required has been constructed.

The enunciation of a theorem consists of two parts. The first part, called the **hypothesis,** states what is assumed. The second part, called the **conclusion,** states what has to be proved.

If two theorems are such that the hypothesis of each is the conclusion of the other, then either of the theorems is said to be the **converse** of the other. It should be particularly noted that **the converse of a theorem is not necessarily true.**

It sometimes happens that, when a theorem has been proved, it is found that other important results follow so easily that they hardly require further proof. Such results are called **Corollaries.**

In writing out Theorems, an accurate figure drawn with instruments is not expected. A neat freehand drawing is quite sufficient.

TRIANGLES

A **plane figure** is any part of a plane surface bounded by one or more lines. The sum of the bounding lines is called the **perimeter** of the figure. The amount of surface enclosed by the perimeter is called the **area** of the figure.

A **rectilineal** (or **rectilinear**) **figure** is one which is bounded by straight lines.

A **triangle** is a plane figure bounded by three straight lines. The straight lines YZ, ZX, XY which bound a triangle XYZ are called its **sides** and the points X, Y, Z its **angular points** or **vertices.** It is usual for one angular point to be called **the vertex**; the opposite side is then called **the base.** In the figure, if X is the vertex, YZ is the base.

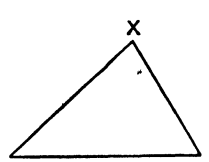


FIG. 13.

A **quadrilateral** is a plane figure bounded by four straight lines.

A **polygon** is a plane figure bounded by more than four straight lines.

A polygon with 5 sides is called a **pentagon.**
,, ,, 6 ,, ,, **hexagon.**
,, ,, 7 ,, ,, **heptagon.**
,, ,, 8 ,, ,, **octagon.**
,, ,, 10 ,, ,, **decagon.**

A rectilineal figure is said to be **convex** if each of its angles is less than two right angles. If a boy walks round a rectilineal figure, and each side is produced in the direction in which he walks along it, the sides are said to be produced **in order.**

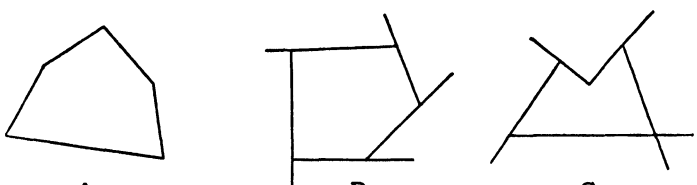
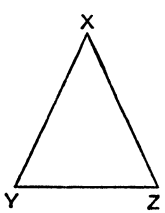


FIG. 14.

Thus the figures A and B are convex, the figure C is not convex. The sides of figure B are produced in order; the sides of figure C are not produced in order.

A rectilineal figure is said to be **equilateral,** when all the sides are equal; **regular,** when all its sides are equal and all its angles are equal.

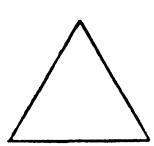
An **isosceles triangle** is a triangle which has two equal sides. The side which is unequal to the others is always called the **base,** and the angular point opposite to the base is called the **vertex.** Thus, in the figure, XY=XZ; YZ is the base, and X is the vertex.



Isosceles Triangle

FIG. 15.

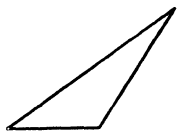
An **equilateral** triangle is a triangle whose three sides are equal.



Equilateral Triangle

FIG. 16.

A **scalene** triangle is a triangle whose three sides are all unequal.



Scalene Triangle

FIG. 17.

In a triangle ABC, the letters A, B, C often denote the magnitude of the several angles (as measured in degrees); and the letters a, b, c the lengths of the opposite sides (as measured in some convenient unit of length). The three sides and the three angles are sometimes called the **parts** of the triangle.

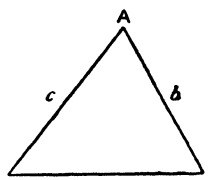
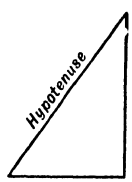


FIG. 18.

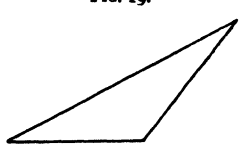
A **right-angled triangle** is a triangle one of whose angles is a right angle. The side opposite the right angle is called the **hypotenuse**.



Right-angled Triangle

FIG. 19.

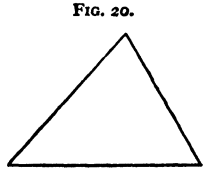
An **obtuse-angled triangle** is a triangle which has one obtuse angle.



Obtuse-angled Triangle

FIG. 20.

An **acute-angled triangle** is a triangle which has three acute angles.



Acute-angled Triangle

FIG. 21.

In any triangle the straight line joining a vertex to the middle point of the opposite side is called a **median.**

THE COMPARISON OF TWO TRIANGLES

Two figures which can be made to coincide are called **congruent.** Congruent figures are said to be **equal in all respects.** In two congruent figures, the sides and angles which coincide, when one figure is applied to the other, are said to **correspond.** They are also called **corresponding sides** and **corresponding angles** respectively.

In two congruent triangles corresponding sides are opposite equal angles, and corresponding angles are opposite equal sides.

It should be noted that, in order to make two triangles coincide, it may be necessary to turn one of them over.

* THEOREM 1

If a straight line stands on another straight line, the sum of the adjacent angles so formed is equal to two right angles.

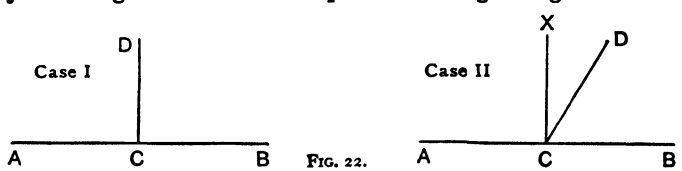


FIG. 22.

Let the straight line CD stand on the straight line AB. It is required to prove that the angles ACD, DCB are together equal to two right angles.

Case I. If the angles ACD, DCB are equal, each is by definition a right angle and they are together equal to two right angles.

Case II. If the angles ACD, DCB are unequal, let CX represent a line drawn through C perpendicular to AB.

Proof. $\angle ACD + \angle DCB$
$= \angle ACX + \angle XCD + \angle DCB$
$= \angle ACX + \angle XCB$
$= 2$ right angles, by construction.

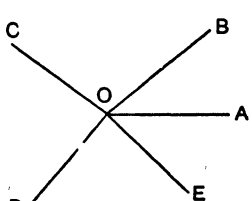


FIG. 23.

Corollary. If any number of straight lines are drawn from a given point, the sum of the consecutive angles so formed is equal to four right angles, i.e. in the figure

$\angle AOB + \angle BOC + \angle COD + \angle DOE + \angle EOA = 4$ right angles.

DEFINITIONS

Two angles whose sum is two right angles are said to be **supplementary**; and each angle is called the **supplement** of the other.

Thus in each of the figures of Theorem 1 the angles ACD, DCB are supplementary. Also the angle 105° is the supplement of the angle 75°.

Two angles whose sum is one right angle are said to be **complementary**; and each angle is called the **complement** of the other.

Thus in the figure for Case II of Theorem 1 the angles XCD, DCB are complementary. Also the angle 18° is the complement of the angle 72°.

It should be noted that (i) supplements of the same angles are equal, (ii) complements of the same angles are equal.

* THEOREM 2

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines are in one and the same straight line.

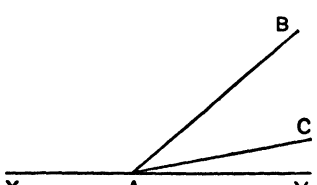


FIG. 24.

At the point A in the straight line AB, let the straight lines AX, AC, on opposite sides of it, make the adjacent angles XAB, CAC together equal to two right angles.

It is required to prove that XA, AC are in the same straight line.

Construction. Produce XA beyond A to any point Y.

Proof. Since by construction XAY is a straight line,

$$\therefore \angle XAB + \angle BAY = 2 \text{ right angles.}$$

But, it is given that $\angle XAB + \angle BAC = 2$ right angles;

$$\therefore \angle XAB + \angle BAC = \angle XAB + \angle BAY.$$

From each of these equals, take the angle XAB;

$$\therefore \angle BAC = \angle BAY;$$

$\therefore$ AC and AY are the same straight line.

But, by construction, AY is in the same straight line with XA;

$\therefore$ AC is in the same straight line with XA.

* THEOREM 3

If two straight lines intersect, the vertically opposite angles are equal.

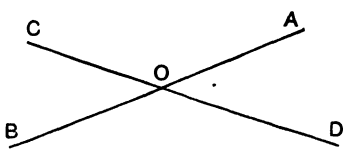


FIG. 25.

Let the straight lines AB, CD cut one another at the point O. It is required to prove that $\angle AOC = \angle BOD$ and that $\angle AOD = \angle BOC$.

Proof. Because OC stands on the straight line AOB,

$$\therefore \angle AOC + \angle COB = 2 \text{ right angles;}$$

and because OB stands on the straight line COD,

$$\therefore \angle BOD + \angle COB = 2 \text{ right angles;}$$

$$\therefore \angle AOC + \angle COB = \angle BOD + \angle COB.$$

From each of these equals take the angle COB;

$$\therefore \angle AOC = \angle BOD.$$

Similarly it can be shown that $\angle AOD = \angle BOC$.

EXERCISE 1 [Theorems 1-3]

1. Write down the supplements of $\frac{3}{4}$ rt. $\angle$, 43°, 118°, 93° 35′, 36° 43′, 161° 9′, $\frac{7}{8}$ rt. $\angle$.

2. Write down the complements of $\frac{2}{7}$ rt. $\angle$, 27°, 32° 26′, $41\frac{2}{3}°$, 79° 33′, 49° 37′ 12″.

3. If in the figure of Theorem 1, Case II, $\angle XCD = 27°$, how large are the $\angle$s ACD, DCB?

4. If in the figure of the corollary of Theorem 1,

$$\angle EOA = \angle AOB = 38°, \ \angle BOC = \angle EOD = 112° 30',$$

what is $\angle DOC$?

5. If in the figure of Theorem 3, $\angle AOC = 126°$, what are the other angles in the figure?

6. If in the figure of Theorem 3, $\angle AOD = 36° 17'$, what are the other angles in the figure?

7. If in the figure of Theorem 1, Case II, $\angle XCD = x°$, how large are the $\angle$s ACD, DCB?

8. If in the figure of Theorem 3, $\angle AOC = x°$, what are the other angles in the figure?

9. What is the angle in degrees between the hands of a watch at (i) 1 o'clock, (ii) 4 o'clock, (iii) 5 o'clock, (iv) x o'clock, x being a whole number?

10. Through what angles does (i) the minute hand, (ii) the hour hand of a clock turn in (*a*) 15 minutes, (*b*) 40 minutes, (*c*) x minutes, x being a whole number less than 60?

11. What is the angle, in degrees, between the hands of a clock at (i) 12.20 p.m., (ii) 2.40 a.m.?

12. Is any of the information given in Question 11 irrelevant?

13. A clock is started at noon and keeps accurate time: through what angles will the hour hand have turned by (i) 2.45 p.m. the same day, (ii) 6.10 p.m. the same day? What will be the time when it has turned through 135°?

14. A clock is started at noon and loses one minute every hour. Through what angles will the hour hand have turned when the true time is (i) 6 p.m. the same day, (ii) 10.15 p.m. the same day?

---

15. In the figure of Theorem 3, prove that the bisectors of the $\angle$s AOC, BOC are at right angles.

16. In the figure of Theorem 3, prove that the bisector of the $\angle AOC$, when produced, bisects the $\angle BOD$.

17. XYZ is a $\triangle$ having $\angle Y = \angle Z$; XY is produced to P, XZ to Q. Prove that $\angle ZYP = \angle YZQ$.

18. XYZ is a $\triangle$ having $\angle Y = \angle Z$; YZ is produced to M and ZY is produced to L. Prove that $\angle XYL = \angle XZM$.

19. If in the figure $\angle AOB = \angle COD$, prove that $\angle AOC = \angle BOD$.

20 If in the figure $\angle AOC = \angle BOD$, prove that $\angle AOB = \angle COD$.

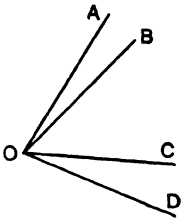


FIG. 26.

21 XP, XQ, XR, XS are straight lines drawn so that

$$\angle PXQ = \angle RXS \text{ and } \angle RXQ = \angle SXP.$$

Prove that PX, XR and also QX, XS are in the same straight line.

22. XOY and OZ are two straight lines and $\angle XOP = \angle YOZ$, P, Z being on opposite sides of the straight line XOY. Prove that ZO, OP are in the same straight line.

23. If two straight lines RS and PQ intersect at O, prove that the bisectors of the $\angle$s POR, SOQ are in the same straight line.

24. OA, OB, OC are three straight lines such that OB bisects the angle between the other two. OP is any other straight line through O. Prove that the $\angle BOP$ may be either half the sum or half the difference of the $\angle$s POC, POA.

25 OX bisects the $\angle POQ$, and XO is produced to Y; prove that $\angle POY = \angle QOY$.

* THEOREM 4

Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other.

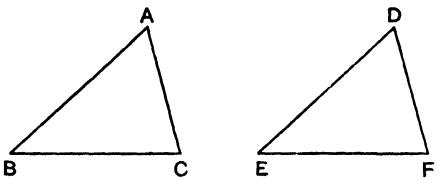


FIG. 27.

Let ABC, DEF be two triangles in which

$$AB = DE,$$
$$AC = DF,$$

and the included angle BAC = the included angle EDF.

It is required to prove that the triangles are congruent.

Proof. Apply the triangle ABC to the triangle DEF so that the point A falls on the point D, the straight line AB falls along the straight line DE, and the point C falls on the same side of DE as the point F.

Then because AB = DE (*given*),

$\therefore$ the point B must fall on the point E.

Also, since AB falls along DE and it is given that $\angle BAC = \angle EDF$,

$\therefore$ AC must fall along DF;

and since it is given that AC = DF,

$\therefore$ the point C must fall on the point F.

Then since B coincides with E, and C with F,

$\therefore$ the side BC must coincide with the side EF.

Hence the triangle ABC coincides with the triangle DEF,

$\therefore$ the triangles ABC, DEF are congruent.

When this theorem is quoted the following abbreviation may be used: "2 *sides*, **included** $\angle$." The insertion of the word "**included**" is **essential**; it will be proved later (Theorem 14A) that triangles are not necessarily congruent if two sides and a

non-included angle of one triangle are respectively equal to two sides and a non-included angle of the other.

In applying this theorem (and other congruence theorems) **the work should be written out as in the following example.**

Ex. The equal sides ZX, YX of an isosceles $\triangle XYZ$ are produced to P, Q so that $XP = XQ$; prove that $YP = ZQ$.

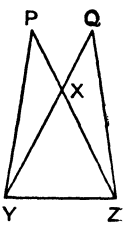


FIG. 28.

[We try to prove that a triangle having YP as one side is congruent to a triangle having ZQ as a corresponding side. In this case try the $\triangle$s XPY, XQZ.]

In the $\triangle$s XPY, XQZ,

$XY = XZ$ (*given*),

$XP = XQ$ (*given*),

$\angle PXY =$ the vert. opp. $\angle QXZ$,

and these are $\angle$s included by the equal sides;

$\therefore$ the $\triangle$s $\begin{matrix} XPY \\ XQZ \end{matrix}$ are congruent † (2 *sides*, **included** $\angle$),

$\therefore\ YP = ZQ$.

† Note carefully the order of the letters. If trouble is taken at this stage to write the corresponding letters each above the other the remaining parts which are equal can be written down without even looking at the figure. The pupil can easily test the correctness of his work by seeing whether each of the three equal parts named in proving the congruence appears above the part to which it is equal. Thus in the above case he will look to see

(1) whether XY is above XZ (or YX above ZX),
(2) whether XP is above XQ (or PX above QX),
(3) whether X (the middle letter of the $\angle PXY$) is above X (the middle letter of the $\angle QXZ$).

Only if all three are correct has the right order been written down. When the right order has been found all the corresponding parts are then one above the other, and the remaining results can be written down at once. Thus in the above case we have

(1) $YP = ZQ$, (2) $\angle XPY = \angle XQZ$, (3) $\angle XYP = \angle XZQ$.

The beginner should always write down all three results, and then pick out what is required to answer the question. It is especially important to do this whenever there is a second part to the question, for the results needed for the solution of the second part of the question may not be the same as those required for the first part. The beginner is apt to overlook them, unless he makes a habit of writing down all the information he can obtain from any pair of congruent triangles.

Note also that the parts of the $\triangle XPY$ are written on the left and the parts of the $\triangle XQZ$ are written on the right. The pupil should never write some parts of one triangle on the right and others on the left.

EXERCISE 2 [Theorem 4]

In the following examples it may be assumed

(i) that all the sides of a square are equal,
(ii) that all the angles of a square are right angles,
(iii) that all the angles of a rectangle are right angles,
(iv) that the angles of an equilateral triangle are each equal to 60°.

1. A st. line XY is bisected at Z, and through Z a st. line ZO is drawn ⊥ to XY. Prove that OX = OY.

2. XYZ is an equilateral △ ; E, F are the mid-points of XY, XZ. Prove that EZ = FY.

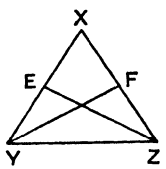


FIG. 29.

3. ABCD is a quad. in which BD bisects AC at right angles ; show that BD bisects the opposite angles ABC and ADC.

4. ABCD is a square. Prove that the diagonals AC, BD are equal.

5. PQRS is a rectangle ; X is the mid-point of PQ. Y, Z are points on PS, QR respectively such that PY = QZ. Prove that XY = XZ.

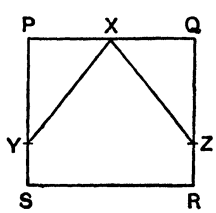


FIG. 30.

6. A hexagon has six equal sides and the bisectors of the angles meet in a point O. Prove that O is equidistant from the mid-points of the sides of the hexagon.

7. In the figure, E, F are the mid-points of AB, AC.

CE, BF are produced to X, Y so that EX = CE and FY = BF. Prove that the △s AEX, BEC are congruent, and write down the remaining pairs of equal parts.

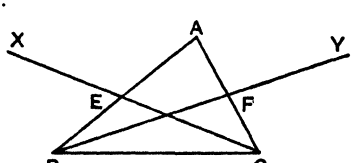


FIG. 31.

8. In the figure of Question 7 find a △ congruent with the △BFC, and prove that AX = AY.

9. In the $\triangle PQR$, the sides QP, QR are bisected at X, Z. Through X and Z $\perp$s are drawn to QP, QR respectively to meet at O. Prove that $OP = OQ = OR$.

10. ABCD is a square, and P, Q, R are the mid points of AB, BC, and CD respectively. Prove that

(i) $PQ = QR$, (ii) $AQ = DQ$, (iii) $AQ = AR$, (iv) $BR = DQ$.

[Draw a separate figure in each case.]

11. Upon the sides BA, AC of a $\triangle ABC$ are described equilateral $\triangle$s ABD, ACE, with their vertices D and E outside the $\triangle ABC$. Prove that $BE = DC$.

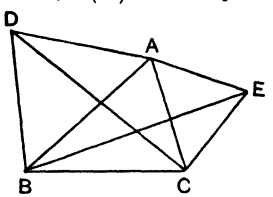


FIG. 32.

12. ABCD is a rectangle. On AB, BC equilateral $\triangle$s APB, BQC are described external to the rectangle. Prove that $AQ = PC$.

13. ABC is an equilateral $\triangle$; BC, CA, AB are produced for equal lengths to A′, B′, C′ respectively. Prove that the $\triangle A'B'C'$ is also equilateral.

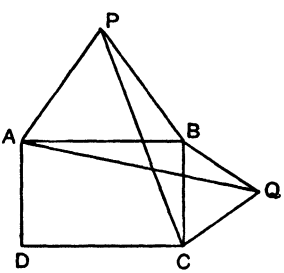


FIG. 33.

14. ABC is an equilateral $\triangle$; D is any point inside it. On DC an equilateral $\triangle$ DEC is drawn, as in the figure. Prove that the $\triangle$s ACE, BCD are congruent.

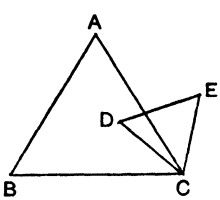


FIG. 34.

15. ABC is an equilateral $\triangle$. D, E, F are points on BC, CA, AB respectively such that $BD = CE = AF$. Prove that the $\triangle$ DEF is also equilateral.

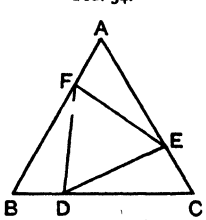


FIG. 35.

16. If two st. lines PQ, XY bisect each other at right angles at O, prove that PX = XQ = QY = YP.

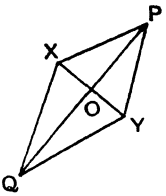


FIG. 36.

17. PQR is a △ in which PQ = PR. S, T are points on PQ, PR respectively such that PS = PT.

Prove that (i) SR = TQ,
(ii) ∠PSR = ∠PTQ,
(iii) ∠QSR = ∠RTQ,
(iv) ∠SQR = ∠TRQ.

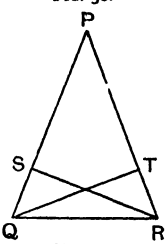


FIG. 37.

18. In the figure,

OP = OR and OQ = OS ;

prove that (i) QR = SP,
(ii) ∠OQP = ∠OSR,
(iii) ∠PQR = ∠RSP.

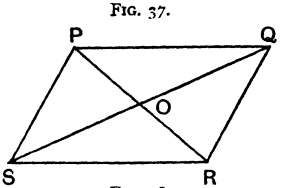


FIG. 38.

19. In the figure,

OA = OB, OC = OD, ∠AOB = ∠COD ;

prove that AC = BD.

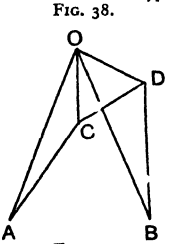


FIG. 39.

20. ABCD is a square. On its sides are taken the points L, M, N, P, so that AL = BM = CN = DP.

Prove that all the sides of LMNP are equal.

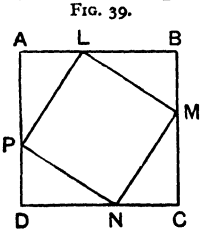


FIG. 40.

Note. The teacher may think it profitable to point out that some of the above examples, e.g. No. 11, may be solved by rotating one triangle about a vertex until it coincides with another triangle.

The pupil should work the following questions with a model in front of him.

21. The figure represents an ordinary rectangular block.

The edges AG, BH, CE, DF are equal.
The edges AD, BC, GF, HE are equal.
The edges AB, DC, GH, FE are equal.
The edges AG, AD, AB are unequal.

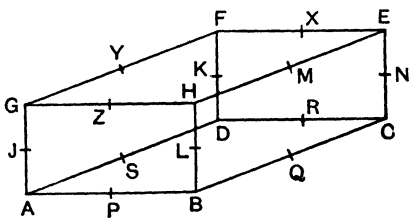


FIG. 41.

The middle points of the edges are marked in the figure.

Find as many triangles as possible congruent with (i) △ASP, (ii) △ADR, (iii) △ADC, (iv) △ADH, (v) △AGE.

22. In the figure of Question 21, prove that (i) AR = HX, (ii) HC = AF, (iii) YX = QP.

23. BCD is an equilateral △. A is a point outside the plane BCD such that the △s ABC, ACD, ABD are equilateral. The middle points of the edges are marked in the figure.

Find as many triangles as possible congruent with (i) △BXQ, (ii) △BAY.

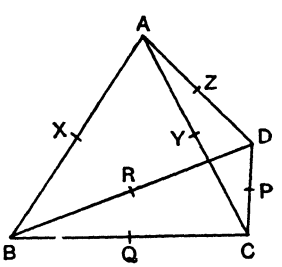


FIG. 42.

24. In the figure of Question 23, prove that (i) XQ = YZ, (ii) CX = AP.

25. OP is a vertical pole standing on a level piece of ground. O, A, B, C are points on the ground such that OA = OB = OC. Prove that FA = PB = PC.

PARALLELS

Straight lines, which are in the **same plane** and which do not meet, however far they are produced in either direction, are called **parallel lines.**

If two straight lines lie in the same plane they are said to be **co-planar.**

Playfair's Axiom. Two intersecting straight lines cannot both be parallel to the same straight line.

Postulate. Let it be granted that a straight line can be drawn, through a given point, parallel to a given straight line.

It follows from Playfair's Axiom that only one such straight line can be drawn.

Any straight line drawn to cut two or more given straight lines is called a **transversal.**

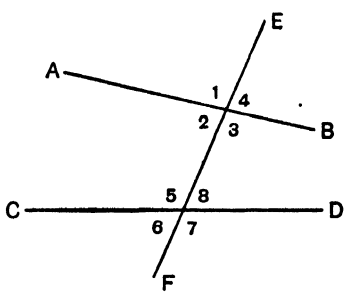


FIG. 43.

If a transversal makes with two straight lines angles marked as in the figure, then

1, 4, 6, 7 are called **exterior** angles :

2, 3, 5, 8 are called **interior** angles :

2 and 8 are called **alternate** angles ; so also are 3 and 5 :

1 and 5 are called **corresponding** angles ; so also are 4 and 8 ; 2 and 6 ; 3 and 7.

* THEOREM 5

If a straight line cuts two other straight lines so as to make

(i) the alternate angles equal,

OR **(ii) the corresponding angles equal,**

OR **(iii) the interior angles on the same side of the cutting line supplementary,**

then in each case the two straight lines are parallel.

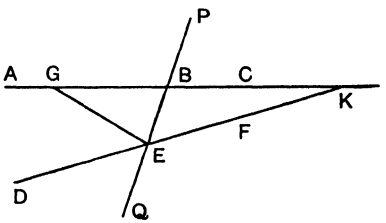


FIG. 44.

(i) Let the straight line PQ cut the two straight lines AC, DF at B and E so as to make the alternate angles ABE, FEB equal to one another.

It is required to prove that AC and DF are parallel.

Proof. If AC and DF are not parallel, they will meet, when produced, either towards C and F, or towards A and D. If possible, let AC and DF, when produced, meet towards C and F, at the point K.

Let G be the point on CB produced such that GB = KE.

Join GE.

Then in the Δs GBE, KEB,

$$GB = KE \text{ (\textit{construction})},$$

$$EB = BE \text{ (\textit{common})},$$

$$\angle GBE = \angle KEB \text{ (\textit{given})};$$

$\therefore$ the Δs $\begin{matrix} GBE \\ KEB \end{matrix}$ are congruent (2 *sides*, **included** *angle*);

$$\therefore \angle GEB = \angle KBE.$$

But $\angle KEB = \angle GBE$ (*given*),

$\therefore \angle GER + \angle KEB = \angle KBE + \angle GBE = 2$ right angles, since G, B, K lie on a straight line ;

$\therefore$ G, E, K lie on a straight line,

i.e. two straight lines intersect at G and K, which is impossible ;

$\therefore$ our assumption is wrong, and it is not possible for AC and DF to meet when produced towards C and F.

It may similarly be proved that AC and DF cannot meet when produced towards A and D ;

$\therefore$ AC and DF are parallel.

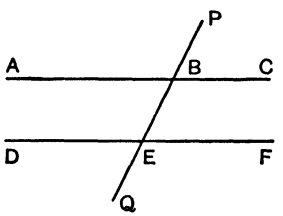


FIG. 45.

(ii) Let the corresponding angles PBC, BEF be equal.

It is required to prove that AC and DF are parallel.

Proof. $\angle BEF = \angle PBC$ (*given*),

$= \angle ABE$ (*vertically opposite*).

But these are alternate angles ;

$\therefore$ AC and DF are parallel.

(iii) Let the two interior angles CBE, BEF be supplementary.

It is required to prove that AC and DF are parallel.

Proof. $\angle BEF =$ the supplement of $\angle CBE$ (*given*)

$= \angle ABE$ (*since* C, B, A *lie on a straight line*).

But these are alternate angles ;

$\therefore$ AC and DF are parallel.

Note. In this theorem we have used an indirect method of proof of very frequent use in Geometry. It consists in showing that, if the theorem were untrue, we should be led to some impossible conclusion This form of proof is known as **Reductio ad Absurdum**. It is most frequently used to demonstrate the converse of some previous theorem.

* THEOREM 6

If a straight line cuts two parallel lines, it makes

(i) **the alternate angles equal to one another ;**

(ii) **the corresponding angles equal to one another ;**

(iii) **the two interior angles on the same side of the cutting line supplementary.**

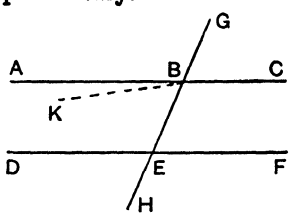


FIG. 46.

Let the straight lines AC, DF be parallel, and let the straight line GBEH cut them at B, E, respectively.

It is required to prove that

(i) $\angle ABE = \angle BEF$,

(ii) $\angle GBC = \angle BEF$,

(iii) $\angle CBE + \angle BEF =$ two right angles.

Proof. (i) Suppose that $\angle ABE$ is not equal to $\angle BEF$, and let the straight line BK be drawn, making $\angle EBK$ equal and alternate to $\angle BEF$.

Then BK is parallel to DF.

But it is given that AC is parallel to DF ;

$\therefore$ AC and BK are both parallel to DF ; but, by Playfair's Axiom, this is impossible, for AC and BK intersect ;

$\therefore$ it is wrong to suppose that $\angle$s ABE, BEF are unequal ;

$$\therefore \angle ABE = \angle BEF.$$

(ii) $\angle GBC = \angle ABE$ (*vertically opposite*)
$= \angle BEF$ (*proved above*).

(iii) $\angle EBC =$ the supplement of the angle ABE (*since* A, B, C *lie on a straight line*)
= the supplement of the angle BEF (*since the* $\angle ABE$ *has been proved equal to the* $\angle BEF$).

Note. When using this theorem to prove the equality of two angles it is not sufficient to state that the angles are equal because they are alternate or corresponding angles. In general, alternate or corresponding angles are not equal; the reason for the equality of the angles is that certain lines are parallel, and it is therefore essential that these lines should be named, e.g.

$\angle ABE = \angle BEF$ (AB, EF *parallel, alternate angles*).

* THEOREM 7

Co-planar straight lines which are parallel to the same straight line are parallel to one another.

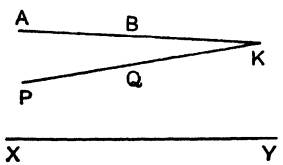


FIG. 47.

Let AB, PQ be co-planar straight lines which are each parallel to XY.

It is required to prove that AB and PQ are parallel.

Proof. If possible, suppose that AB, PQ (produced if necessary) meet at K. Then through K there are two straight lines KA, KP, both of which are parallel to XY.

But this is impossible by Playfair's Axiom;

$\therefore$ our supposition that AB, PQ meet at K is wrong;

$\therefore$ AB, PQ cannot meet, and since they are co-planar, therefore they must be parallel.

EXERCISE 3 [Theorems 5-7]

1. In the given figure AB is parallel to CD.

(i) If $\angle b = 53°$, find all the other angles in the figure.

(ii) If $\angle e = 114°$, find all the other angles in the figure.

In each case give your reasons clearly.

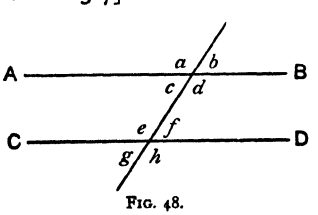


FIG. 48.

2. In the given figure, AB, CD are parallel, as also are EF, GH.

If $\angle b = 117°$, and $\angle p = 93°$, find all the other angles in the figure.

Give your reasons clearly.

[The teacher can give as much practice as is needed by varying the two angles given.]

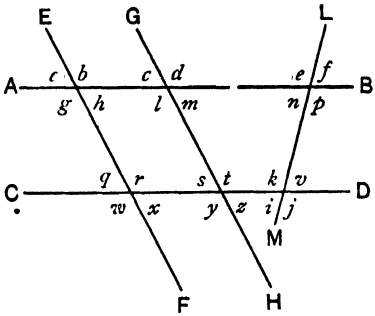


FIG. 49.

3. Prove that co-planar st. lines which are $\perp$ to the same st. line are parallel.

4. AB, CD are two parallel st. lines. XY is $\perp$ to AB. Prove that it is $\perp$ to CD.

5. ABCD is a quad. with both pairs of opposite sides parallel. Prove that both pairs of opposite angles are equal.

6. Prove that the angle between any two intersecting st. lines is either equal or supplementary to the angle between any parallels to these lines.

7. Two st. lines PQ, XY bisect each other at O. Prove that PY, XQ are parallel, and also PX, YQ.

8. AB, PQ are equal and parallel st. lines. Prove that the st. lines AP, BQ are also equal and parallel.

DEFINITION

If the side BC of a triangle is produced to D, the $\angle$ ACD is called an **exterior angle** of the triangle, and the $\angle$s ABC, BAC are said to be **interior opposite angles.** The bisector of the $\angle$ ACD is called the **external bisector of the $\angle$C of the triangle.**

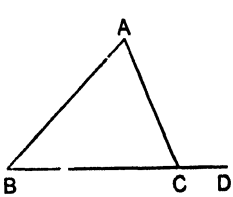


FIG. 50.

THEOREM 8

The sum of the three angles of a triangle is equal to two right angles.

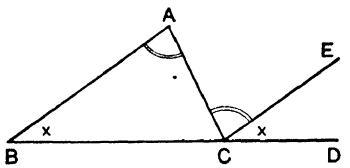


FIG. 51.

Let ABC be a triangle.

It is required to prove that the three ∠s ABC, BCA, CAB together = two right angles.

Construction. Produce BC to any point D ; and suppose CE to be the line through C parallel to BA.

Proof. Because BA and CE are parallel and AC meets them,

∴ ∠ACE = the alternate ∠CAB.

Again, because BA and CE are parallel and BD meets them,

∴ ∠ECD = the corresponding ∠ABC ;

∴ **the whole exterior ∠ACD = the sum of the two interior opposite angles,** CAB, ABC.

To each of these equals add the ∠BCA ; then the ∠s BCA, ACD together = the three ∠s BCA, CAB, ABC.

But, by construction, the ∠s BCA, ACD together = two right angles ;

∴ the ∠s BCA, CAB, ABC together = two right angles.

Corollary 1. If a side of a triangle is produced the exterior angle so formed is equal to the sum of the two interior opposite angles.

Corollary 2. If a side of a triangle is produced the exterior angle so formed is greater than either of the interior opposite angles.
This result is frequently of considerable use.

Corollary 3. If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angles are equal.

Corollary 4. In a right-angled triangle, (i) the right angle is the greatest angle ; (ii) the sum of the remaining angles is equal to a right angle and each of these angles is an acute angle.

Corollary 5. In an obtuse-angled triangle, (i) the obtuse angle is the greatest angle ; (ii) the sum of the remaining angles is less than a right angle and each of these angles is an acute angle.

Corollary 6. If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.

Corollary 7. Any two angles of a triangle are together less than two right angles, and every triangle must have at least two acute angles.

Corollary 8. Only one perpendicular can be drawn to a straight line from a given point outside it.

For, if two perpendiculars PX, PY could be drawn to AB from P, we should have a triangle PXY in which each of the angles PXY, PYX would be a right angle, which is impossible.

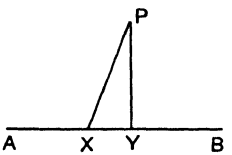


FIG. 52.

EXERCISE 4 [Theorem 8]

1. In a right-angled $\triangle$, one angle $= 36°$; find the third angle.

2. In the $\triangle PQR$, $\angle P = 102°$, $\angle Q = 23°$; find the $\angle R$.

3. In the $\triangle XYZ$, $\angle X = 26°$, the exterior angle at $Z = 98°$; find the $\angle Y$.

4. Calculate the angles of a triangle if they are in the ratio $2 : 3 : 4$.

5. The bisector BX of the vertical angle of a $\triangle ABC$ meets the base AC at X. If $\angle ABC = 68°$ and $\angle BXC = 86°$, find, by calculation, the $\angle$s A and C.

6. ABC is a $\triangle$ in which $A = 60°$, $B = 52°$, and the side BC is produced to D. The bisectors of the $\angle$s ABC and ACD meet at X. Calculate the $\angle BXC$

7. A line AB makes angles of $65°$ with each of two lines OP, OQ. What is the angle between these lines ? A fourth line makes an angle of $35°$ with OP. At what angles can it be inclined (i) to AB, (ii) to OQ ?

8. The internal bisector of the $\angle B$ of a $\triangle ABC$ meets AC at P, and the internal bisector of the $\angle C$ meets AB at Q. The lines BP, CQ meet at O, the $\angle ABC$ is $60°$ and the $\angle POQ$ is $110°$. Calculate the $\angle$s AQC and ACB.

9. In the $\triangle ABC$, $\angle B = 63°$, $\angle C = 38°$. Find, by calculation, the acute $\angle$ between the bisector of $\angle BAC$ and the $\perp$ from B to AC.

10. The $\triangle ABC$ is right-angled at A. Prove that the sum of the exterior angles at B and C is three times the exterior angle at A.

11. ABC is a $\triangle$. The bisectors of the $\angle$s B and C meet at O, and the $\angle BOC = 126°$. Calculate the $\angle A$.

12. If the external bisector of the vertical angle of a $\triangle$ is inclined at an angle of 20° to the base, show that the difference of the base angles is equal to 40°.

13. The sum of the $\angle$s B and C of a $\triangle ABC$ is 105° and the $\angle A$ exceeds the $\angle C$ by 40°. Calculate the $\angle$s A, B, C.

14. ABC is a $\triangle$ having one of its angles equal to five times the least angle and the other angle equal to three times the least angle. Calculate the angles.

15. PQR is a $\triangle$ with QR produced to S. The $\angle PRS = 108°$ and $\angle QPR = 3\angle PQR$. Calculate the angles of the $\triangle PQR$.

16. The greatest angle of a $\triangle$ is twice the least, while the other is 12° greater than the least. Find the angles of the $\triangle$.

17. PQR is a $\triangle$. QX is $\perp$ to PR and RY is $\perp$ to PQ. Prove that $\angle PQX = \angle PRY$.

18. PQR is a $\triangle$ in which $\angle Q = \angle R$. RS is drawn $\perp$ to QR to meet QP produced at S. Prove that $\angle PRS = \angle PSR$.

19. Two st. lines AX, BY are cut by a third st. line at A, B; prove that if the bisectors of the $\angle$s BAX and ABY be $\perp$ to one another, then AX and BY are parallel.

20. In any $\triangle ABC$, prove that the internal bisector of the angle B and the external bisector of the angle C cannot be parallel, and also prove that their point of intersection lies on the same side of BC as A.

21. If P is a point within a $\triangle ABC$, and PB PC be joined, prove that the excess of the $\angle BPC$ over the $\angle BAC$ is equal to the sum of the $\angle$s ABP, PCA.

22. Prove that if the lines which bisect the opposite angles A and C of a quad. ABCD are parallel to one another, the $\angle$s B and D are equal.

23. AB, CD are two parallel st. lines. XY is a st. line cutting them at P, Q respectively. Prove that the bisector of the $\angle APQ$ is $\parallel$ to the bisector of the $\angle PQD$ and $\perp$ to the bisector of the $\angle CQP$.

THEOREM 9a

All the interior angles of a polygon, together with four right angles, are equal to twice as many right angles as the figure has sides.

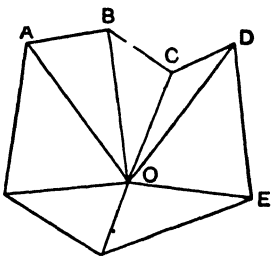


FIG. 53

Let ABCDE... be a polygon of n sides.

It is required to prove that all the interior angles + 4 right angles = $2n$ right angles.

Construction. If possible choose a point O inside the polygon, such that all the straight lines joining O to the vertices lie entirely within the polygon.

Proof. The polygon has been divided into n triangles by these lines. The sum of the three angles of each triangle equals two right angles;

$\therefore$ the sum of all the angles of all the triangles equals $2n$ right angles.

But all the angles of all the triangles make up all the interior angles of the polygon, together with the angles at O, which = 4 right angles;

$\therefore$ all the interior angles of the polygon + 4 right angles = $2n$ right angles.

Note. In the figure chosen the angle of the polygon at C is "*reflex*," the polygon is "*non-convex*" or "*re-entrant.*" Theorem 9a is true for such polygons as well as for convex polygons. There may be no point O inside the polygon such that the straight lines joining O to the vertices lie within the polygon. The above proof then fails, but the theorem is still

true. The reader will find it instructive to verify this statement in a few simple cases, e.g. :

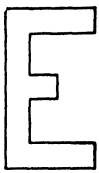


FIG. 54.

The following theorem is only true for convex polygons :

THEOREM 9b

If the sides of a convex polygon are produced in order, the sum of the exterior angles so formed is equal to four right angles.

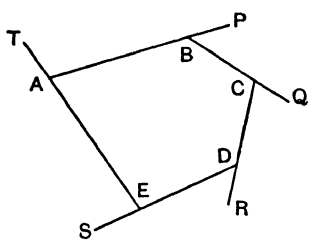


FIG. 55.

Let ABCD... be a convex polygon of n sides. AB, BC, CD, DE... are produced to P, Q, R, S... respectively.

It is required to prove that the sum of the angles PBQ, QCR, RDE, SEA... is equal to four right angles.

Proof. At *each* vertex the interior angle + the exterior angle = 2 right angles ;

and there are n vertices,

∴ the sum of all the interior angles + the sum of all the exterior angles = $2n$ right angles.

But we have just proved that

the sum of all the interior angles + 4 right angles = $2n$ right angles,

∴ the sum of all the exterior angles = 4 right angles.

Note. This theorem is usually regarded as a corollary of Theorem 9a. If it is set in an examination the candidate is advised not to assume the result of Theorem 9a. He should first prove Theorem 9a and then deduce Theorem 9b as above.

EXERCISE 5 [Theorems 9a and 9b]

1. In a quadrilateral ABCD, $\angle A = 74°$, $\angle B = 85°$, $\angle C = 103°$; find $\angle D$.

2. In a quadrilateral ABCD, $\angle A = 39°$, $\angle B = 114°$, and $\angle C = 2\angle D$; find $\angle C$ and $\angle D$.

3. What is the size of each exterior angle of a regular hexagon? Hence find the size of each interior angle.

4. What are the exterior angles of regular polygons of 8, 10, 12, 20, 5, n sides? Hence find the size of each interior angle of these polygons.

5. The exterior angle of a regular polygon is 40°; how many sides has it?

6. The interior angle of a regular polygon is 160°; how many sides has it?

7. How many sides have the regular polygons whose exterior angles are (i) 12°, (ii) $7\frac{1}{2}°$?

8. Is it possible to have regular polygons whose exterior angles are (i) 18°, (ii) 11°, (iii) 13°, (iv) $2\frac{1}{2}°$?

9. Is it possible to have regular polygons whose interior angles are (i) 144°, (ii) 154°, (iii) 175°, (iv) 117°? In the cases which are possible, find the number of sides.

10. In the quad. ABCD the angles are in the ratio 1 : 3 : 7 : 9. Find them.

11. If AC, CB are two sides of a regular polygon of $2n$ sides, express $\angle ACB$ as a fraction of a right angle.

12. The interior $\angle$s of a quad. ABCD are 40°, 100°, 64° and 156°. The bisectors of the exterior $\angle$s form a new quad. Find the interior $\angle$s of this new quad.

13. In a certain pentagon one of the angles is four-thirds of a right angle; of the other four angles, one is double, another three times, and another four times the smallest angle; find the number of degrees in each angle.

14. A certain polygon has seven sides; one of its angles is 25° greater than each of the others, which are equal. Find the number of degrees in each angle.

15. Two angles of a certain octagon are right angles, two others are each one and a half right angles; of the remaining angles one is 10° greater and the other is 20° less than each of the other two angles. Find the number of degrees in each unknown angle.

16. One of the angles of a pentagon is a right angle; the other four angles are equal to each other. How many degrees are there in each?

17. The $\angle$s A, B, C of a quad. ABCD are 96°, 140°, 76° respectively, and the bisectors of the four exterior $\angle$s of the quad. are drawn, forming a quad. PQRS; find the $\angle$s of this quad.

18. In the hexagon ABCDEF, the $\angle$s at A, C, E are equal to one another, as also are the $\angle$s at B, D, F, the former being double the latter. Find the angles of the hexagon.

19. In a pentagon ABCDE, the $\angle$s A, B are each 120°, and the $\angle$s C, E are each 95°. Alternate sides are produced to meet, forming a star-shaped figure. Calculate the angles of the star.

20. The $\triangle$ABC has $\angle B = 40°$ and $\angle C = 80°$; points P, Q, R are taken on BC, CA, AB respectively so that $\angle APC = x°$, $\angle BQA = y°$, $\angle CRB = z°$. Calculate the $\angle$s of the $\triangle$ formed by AP, BQ, CR, and hence find the values of x, y, z if this $\triangle$ is equilateral and BQ is $\perp$ to AC.

21. In a pentagon ABCDE the $\angle$D is double the $\angle$B, and the $\angle$s A, C, E are each half the sum of the $\angle$s B and D. Find all the $\angle$s of the pentagon.

22. A nine-sided figure has one angle a right angle, and the other angles are equal to each other. Find their size in degrees.

23. Two angles of a convex polygon are right angles, and each of the other angles is 120°. How many sides has the polygon?

24. ABCDEF is a hexagon in which $\angle FAB = 110°$, $\angle ABC = 115°$, CD is $\perp$ to AB and DE to AF, and $\angle DEF = \angle EFA$. Find the remaining $\angle$s of the hexagon.

25. Show that if all the $\angle$s of a hexagon are equal, the opposite sides must be parallel.

26. The sides AB, DC of a quad. meet when produced at P, and the sides BC, AD meet when produced at Q. The lines bisecting the $\angle$s P and Q meet at O. Prove that the $\angle$ POQ or its supplement is equal to half the sum of BAD and BCD.

27. If the $\angle$s of a quad. taken in order are in the ratio of 1 4 : 6 : 9, prove that two of its sides are parallel.

28. ABCD is a quad. Prove that the $\angle$ between the bisectors of the $\angle$s BAD, ADC $= \frac{1}{2}(\angle ABC + \angle BCD)$.

* THEOREM 10

Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the other.

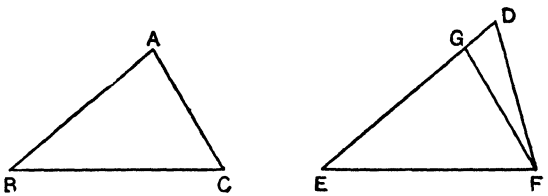


FIG. 56.

Let ABC, DEF be two triangles in which BC = EF, $\angle B = \angle E$, $\angle A = \angle D$.

Since the three angles of each triangle are together equal to two right angles, $\therefore$ the remaining $\angle C$ = the remaining $\angle F$.

Proof. Suppose that BA is not equal to ED. Then there is a point G on ED (or on ED produced) such that EG = BA. Join FG.

Then in the $\triangle$s ABC, GEF,

BA = EG (*construction*),

BC = EF (*given*),

$\angle ABC = \angle GEF$ (*given*) ;

$\therefore$ the $\triangle$s $\begin{matrix} ABC \\ GEF \end{matrix}$ are congruent (2 *sides*, **included** $\angle$),

$\therefore$ $\angle BCA = \angle EFG$, but $\angle BCA = \angle EFD$ (*given*),

$\therefore$ $\angle EFG = \angle EFD$, i.e. a part is equal to the whole, which is absurd,

$\therefore$ our supposition is wrong ;

$\therefore$ BA = ED.

Then in the $\triangle$s ABC, DEF,

BA = ED (*proved*),

BC = EF (*given*),

$\angle ABC = \angle DEF$ (*given*) ;

$\therefore$ the $\triangle$s $\begin{matrix} ABC \\ DEF \end{matrix}$ are congruent (2 *sides*, **included** $\angle$).

Note. When this theorem is quoted the following abbreviation may be used: "2 ∠s, **corresponding** side." The insertion of the word "**corresponding**" is **essential**. Two triangles are not necessarily congruent if two angles and a side of one triangle are respectively equal to two angles and a side of the other. This is clear from a consideration of the figure below:

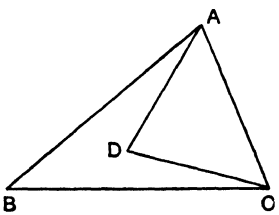


FIG 57.

Let $\angle ABC = \angle ACD = 40°$,

$\angle ACB = \angle DAC = 60°$.

Then, in the △s ABC, ADC,

$AC = AC$,

$\angle ABC = \angle ACD$,

$\angle ACB = \angle DAC$;

i.e. two angles and a side of one triangle are respectively equal to two angles and a side of the other. But the triangles are clearly not congruent.

EXERCISE 6 [Theorem 10]

[See note at the head of Ex. 2.]

1. AD is the bisector of the ∠ between AB and AC; P is any point on AD. Prove that the ⊥s drawn from P to AB and AC are equal.

2. In any △XYZ, the bisectors of the ∠s Y and Z meet at O; from O, OP, OQ, OR are drawn ⊥ to YZ, ZX, XY respectively. Prove that $OP = OQ = OR$.

3. In the △ABC, the ⊥ AD from A to the bisector BD of the ∠ABC meets the base BC (produced if necessary) at E. Prove that AD = DE.

4. Prove that if the line bisecting the vertical angle of a $\triangle$ is $\perp$ to the base, the $\triangle$ is isosceles.

5. In any $\triangle$ PQR, PQ and PR are produced, and the bisectors of the exterior angles at Q and R meet at O. OX, OY, OZ are drawn $\perp$ to the three sides of the $\triangle$. Prove that OX = OY = OZ.

6. ABC is a $\triangle$ in which AB = AC. Prove that the $\perp$ from B to AC is equal to the $\perp$ from C to AB.

7. PQR is a $\triangle$ in which $\angle Q = \angle R$; S is the mid-point of QR. Prove that the $\perp$s from S to PQ, PR are equal.

8. AD is the bisector of the angle between AB and AC; P is any point on AD. Through P is drawn a line $\perp$ to AD to meet AB and AC at X, Y respectively. Prove that AX = AY.

9. ABC is an acute-angled $\triangle$. The $\perp$ from B to AC is equal to the $\perp$ from C to AB. Prove that AB = AC.

10. If CA, CB are radii of a circle at right angles to each other, and AM, BN are drawn $\perp$ to any diameter meeting it at M and N, prove that AM = CN and BN = CM.

11. On the sides AB, BC of a $\triangle$ABC, which is right-angled at A, squares ARSB, BPQC are drawn outside the $\triangle$. PB produced meets SR (produced if necessary) at T. Prove that the $\triangle$s TSB and CAB are congruent.

12. ABCD is a rectangle, M the middle point of BC. AM is drawn, and produced to meet DC produced at E. Prove that CE = AB. Prove also that, if B and E are joined, BE is equal to one of the diagonals of the rectangle.

13. A $\triangle$ABC has the $\angle C = 90°$ and AC > BC. AP is cut off on AC equal to BC, and PN, CM are the $\perp$s to AB from P and from C; prove that the $\triangle$s BCM, PAN are congruent.

14. K is a point inside a square ABCD such that $\angle BKC = 90°$ and CK > BK. The $\perp$ drawn from D to CK meets CK at N. Prove that DN = CK.

15. If the diagonal of a quad. bisects the angles which it joins, prove that it bisects the other diagonal at right angles.

THEOREM 11

If two sides of a triangle are equal, the angles opposite those sides are equal.

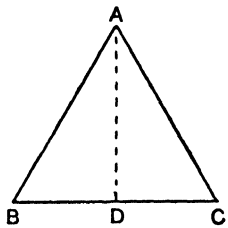


FIG. 58.

Let ABC be a triangle in which AB = AC.

It is required to prove that $\angle B = \angle C$.

Construction. Let AD represent the bisector of the angle BAC, and let it meet BC at D.

Proof. In the $\triangle$s ABD, ACD,

$$AB = AC \ (\textit{given}),$$

$$AD = AD \ (\textit{common}),$$

$$\angle BAD = \angle CAD \ (\textit{construction});$$

$\therefore$ the $\triangle$s $\begin{matrix} ABD \\ ACD \end{matrix}$ are congruent (2 *sides*, **included** *angle*);

$$\therefore \ \angle B = \angle C.$$

Corollary 1. If the equal sides AB, AC of an isosceles $\triangle$ are produced, the exterior angles EBC, FCB are equal; for they are the supplements of the equal angles at the base.

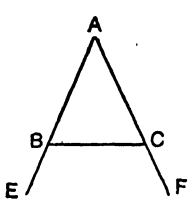


FIG. 59.

Corollary 2. If a triangle is equilateral each of its angles is 60°.

A figure is said to be **symmetrical about a line when,** on being folded about that line, the parts of the figure on each side of it can be made to coincide. The straight line is called an **axis of symmetry.**

Corollary 3. An isosceles triangle is symmetrical about the bisector of its vertical angle. An equilateral triangle is symmetrical about the bisector of any one of its angles.

Corollary 4. The bisector of the vertical angle of an isosceles triangle (i) **bisects the base** ; (ii) **is perpendicular to the base.**

For it has been shown that the triangles $\begin{matrix}ABD\\ACD\end{matrix}$ are congruent ; $\therefore$ BD = CD and $\angle$ ADB = $\angle$ ADC, and each angle is therefore by definition a right angle ; $\therefore$ AD is perpendicular to BC.

Note. The teacher should explain why the construction given is the only valid construction. Candidates in examinations often let AD represent the median or the perpendicular from A to BC, and then proceed to prove congruence by means of Theorems 13 or 14. It cannot be too strongly emphasised that this is " argument in a circle."

THEOREM 12

If two angles of a triangle are equal, the sides opposite those angles are equal.

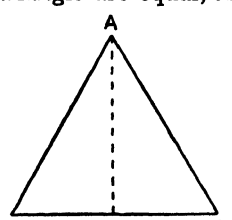


FIG. 60.

Let ABC be a triangle in which $\angle$ B = $\angle$ C.

It is required to prove that AB = AC.

Construction. Let AD represent the bisector of the angle BAC, and let it meet BC at D.

Proof. In the $\triangle$s ABD, ACD,

$\angle$ B = $\angle$ C (*given*),

$\angle$ BAD = $\angle$ CAD (*construction*),

AD = AD (*common*),

$\therefore$ the $\triangle$s $\begin{matrix}ABD\\ACD\end{matrix}$ are congruent (*2 angles*, **corresponding** *side*) ;

$\therefore$ AB = AC.

EXERCISE 7 [Theorems 11 12]

1. In a $\triangle PQR$, $PQ = PR$; find the $\angle$s of the $\triangle$ in the following cases: (i) $\angle Q = 68°$, (ii) $\angle P = 34°$, (iii) $\angle P = 128° 34'$, (iv) $\angle R = 13° 33'$, (v) $\angle Q = 37° 30'$, (vi) $\angle P = 98° 9'$.

2. Find the $\angle$s of an isosceles $\triangle$ in which each of the base angles is (i) three-quarters, (ii) four times the vertical angle.

3. PQR, SQR are two isosceles $\triangle$s on opposite sides of the same base QR. Prove that $\angle PQS = \angle PRS$.

4. PQR, SQR are two unequal isosceles $\triangle$s on the same side of the same base QR. Prove that $\angle PQS = \angle PRS$.

5. ABCDE is a regular pentagon. Calculate the $\angle$s ABC and ACD.

6. ABCDE is a regular pentagon. On AB the square ABPQ is described lying entirely within the pentagon. Calculate the $\angle$s CBP, PBD.

7. If ABCDE is a regular pentagon, and AC, AD be joined, calculate the number of degrees in each of the three angles at A.

8. ABC is a $\triangle$ with $AB = BC = 2''$, and $\angle ABC = 60°$. CB is produced to E making $BE = 2''$. Calculate the $\angle CAE$.

9. ABC is a $\triangle$; a line DE $\parallel$ to AB cuts AC at D and BC at E. If $\angle BAC = 35°$, $\angle BDC = 70°$, $\angle DEC = 98°$, find all the other angles in the figure, and prove that $AD = DB$.

10. ABC is a $\triangle$ in which $\angle CAB = 35°$, $\angle ABC = 100°$. BD is drawn $\perp$ to AC. Prove that $BD = DC$.

11. The $\angle$s A, B of a $\triangle ABC$ are 70° and 60° respectively. Lines AD, AE are drawn within the $\angle BAC$ to cut BC at D and E, such that $\angle BAD = \angle ACB$, and $\angle CAE = \angle ABC$. Prove that $AD = AE$.

12. Two angles of a $\triangle$ are 40° and 120° respectively. Show that the $\triangle$ may be divided into two isosceles $\triangle$s in two different ways.

13. The side BC of the $\triangle ABC$ is produced to D, and BD is made equal to BA, the $\angle ACD$ is 80° and the $\angle BAC$ is 24°. Calculate the $\angle$s ABC, ACB, CAD, CDA.

14. A $\triangle ABC$ has $\angle B = 42°$, $\angle C = 104°$. E is a point on a line through A drawn $\parallel$ to BC, such that $AE = AB$ and the $\angle EAB$ is obtuse. Join BE and calculate the $\angle AEB$.

15. ABCDE, ABPQRS are respectively a regular pentagon and a regular hexagon described on the same side of a given line AB. Join PC. Calculate the $\angle$s QPC, PCD.

16. A square ABCD, and a regular pentagon ABPQR are described on opposite sides of a line AB. Calculate the ∠s RAD, PBD.

17. ABCDEFG is a regular heptagon ; calculate to the nearest minute the angles of the figure, of the △ACF, and of the △ formed by AC, DF and the perpendicular at G to FG.

18. A, B, C, D, E are the vertices of a regular pentagon. AC, CE, EB, BD, DA are drawn. Find every angle in the figure.

19. ABC is a △ in which AN is drawn ⊥ to BC. AB < AC. A circle, centre A and radius AB, cuts BC at P. If ∠BAN = 19° and AP = PC, calculate the ∠CAN.

20. ABC is a △ in which AB = AC = 5 cm. and BC = 2 cm. BC is produced both ways a distance 3 cm. to P and Q so that BP = 3 cm and CQ = 3 cm. Join AP, AQ. Point out in the figure a pair of congruent isosceles △s.

If ∠ABC = $x°$, prove that ∠BAC = $(180 - 2x)°$ and ∠BAQ = $\left(90 - \frac{x}{2}\right)°$, and deduce that ∠PAQ = ∠ABC.

21. In a △ABC the line AD bisects ∠BAC and cuts BC at D. If AD = DB, and ∠ACB = 66°, calculate ∠ABC.

22. ABC is an isosceles △. A straight line || to the base BC meets the other sides at X, Y. Prove that △AXY is isosceles.

23. ABC is an isosceles △ ; the bisectors of the equal ∠s B, C meet at O. Prove that △OBC is isosceles.

24. ABC is an isosceles △ ; P, Q are points on the base BC such that BQ = CP. Prove that AP = AQ.

25. XYZ is a △, whose sides YX, ZX are produced to P, Q, so that XP = XZ, XQ = XY. Prove that ZP, YQ are ||.

26. ABC is a △ and BA is produced to E ; AD bisects the ∠CAE. CX is drawn through C || to DA to cut AB, or AB produced, at X. Prove that AX = AC.

27. XYZ is a △ whose ∠X = 90°. On YZ points P, Q are taken such that ZP = ZX and YQ = YX. Prove that ∠PXQ = 45°.

28. Prove that the △ formed by joining the mid-points of the 3 sides of an isosceles △ is isosceles.

29. XYZ is a △ in which XY = XZ. YQ, ZR are drawn ⊥ to XZ, XY respectively, meeting at S. Prove that △SYZ is isosceles.

30. AXBY is a quad. in which AX = AY and ∠X = ∠Y. Prove that XB = YB.

31. In the base YZ of a △XYZ points R, S are taken such that ∠YXR = ∠ZXS ; if XR = XS, prove that △XYZ is isosceles

32. X is a point in the hypotenuse BC of a right-angled $\triangle ABC$, such that $\angle BAX = \angle ABX$. Prove that $AX = BX = CX$.

33. Prove that the bisectors of the base angles of an isosceles $\triangle$ are equal.

34. PQRS is a quad. in which PQ, SR are $\parallel$ and $\angle P = \angle Q$. Prove that $PS = QR$.

35. PQR is a $\triangle$ in which $PQ = PR$. The bisector of the $\angle Q$ meets PR at L and through L is drawn a line $\parallel$ to RQ to meet PQ at M. Prove that $QM = ML = LR$.

36. Through any point in the bisector of an angle a line is drawn parallel to either of the arms of the angle. Prove that the $\triangle$ so formed is isosceles.

* THEOREM 13

Two triangles are congruent if the three sides of one triangle are respectively equal to the three sides of the other.

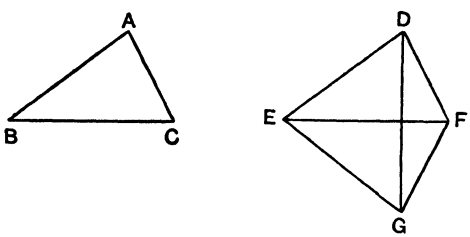


FIG. 61.

Let ABC, DEF be two triangles in which $AB = DE$, $AC = DF$, $BC = EF$.

It is required to prove that the triangles are congruent.

Construction. † Let BC, EF be the sides which are not shorter than the other sides.

Consider a $\triangle GEF$ on the opposite side of EF to D, having $\angle GEF = \angle ABC$ and $\angle EFG = \angle BCA$. Join DG.

Proof. In the $\triangle$s ABC, GEF,

$\angle ABC = GEF$ (*construction*),

$\angle BCA = \angle EFG$ (*construction*),

$BC = EF$ (*given*);

$\therefore$ the $\triangle$s $\begin{matrix} ABC \\ GEF \end{matrix}$ are congruent (2 *angles*, **corresponding** *side*);

$\therefore$ $AB = GE$, $AC = GF$, $\angle BAC = \angle EGF$

But $DE = AB$, $DF = AC$ (*given*),

$\therefore$ $DE = GE$ and $DF = GF$,

$\therefore$ EDG and FDG are isosceles Δs,

$\therefore$ $\angle EDG = \angle EGD$ and $\angle GDF = \angle DGF$ (*base angles of an isosceles* Δ);

$\therefore$ $\angle EDG + \angle GDF = \angle EGD + \angle DGF$,

$\therefore$ $\angle EDF = \angle EGF$.

But $\angle EGF = \angle BAC$ (*proved*),

$\therefore$ $\angle EDF = \angle BAC$.

Then, in the Δs ABC, DEF,

$AB = DE$ (*given*),

$AC = DF$ (*given*),

$\angle BAC = \angle EDF$ (*proved*);

$\therefore$ the Δs $\begin{matrix} ABC \\ DEF \end{matrix}$ are congruent (2 *sides*, **included** *angle*).

When this theorem is quoted the following abbreviation may be used: "3 *sides*."

† This line is inserted to avoid the necessity of considering more than one figure. Had it been left out, it would have been necessary to modify the above proof for the figures given below.

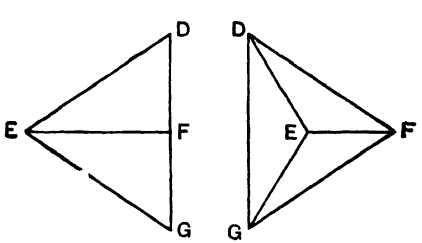


FIG. 62.

EXERCISE 8 [Theorem 13]

1. The four sides of a quad. are equal; prove that its angles are bisected by the diagonals.

2. The opposite sides of a quad. are equal; prove that its opposite sides are parallel.

3 ABC is an isosceles △ ; D is the mid-point of the base BC. Prove that AD bisects the vertical angle of the △ABC and is also ⊥ to BC.

4. ABCD is a square. E, F, G are the mid-points of AB, BC, CD respectively. Prove that the △s AEF, DGF are congruent.

5. ABCD is a quad. of which no two sides are parallel, but it is given that AB = CD. The lines drawn through the mid-points of AD, BC ⊥ to AD, BC respectively meet at O. Prove that the △s OAB, ODC are congruent.

6. If ABC, DBC are two △s on opposite sides of the common base BC, having AB = AC, and DB = DC, prove that AD bisects the ∠s BAC, BDC and also bisects BC at right angles.

7. ABCD is a quad. ; the diagonals AC, BD intersect at X. If AC = BD and DX = CX, prove that the triangles ADB and BCA are congruent and that AB, DC are ||.

8. In the quads. ABCD, PQRS, it is given that AB = PQ, BC = QR, CD = RS, DA = SP, and AC = PR. Prove that ∠BAD = ∠QPS and that BD = QS.

9. ABCD is a quad. in which AB = CD. Also there is a point O inside the quad. such that OA = OD and OB = OC. Prove that BC is || to AD.

10. In the figure of Ex. 2, No. 21, find as many △s as possible congruent with (i) △ASR, (ii) △APR, (iii) △SRQ, (iv) △ARQ, (v) △AXM, (vi) △AYE, (vii) △AYM, (viii) △AKN, (ix) △AFE.

11. In the figure of Ex. 2, No. 23, find as many △s as possible congruent with (i) △XYQ, (ii) △BXY, (iii) △BYZ, (iv) △XPZ.

THEOREM 14

Two right-angled triangles are congruent if the hypotenuse and a side of one are respectively equal to the hypotenuse and a side of the other.

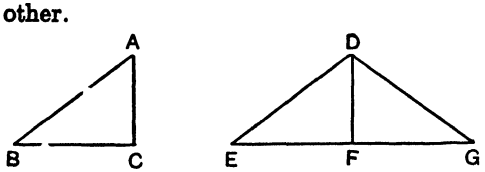


FIG. 63.

Let ABC, DEF be two triangles in which AB = DE, AC = DF, ∠ACB = ∠DFE = a right angle.

It is required to prove that the triangles are congruent.

Construction. Produce EF to G, making FG = CB. Join DG.

Proof. EFG is a straight line and ∠EFD = a right angle (*given*),

$\therefore$ ∠DFG = a right angle.

Then, in the △s ABC, DGF,

AC = DF (*given*),

BC = GF (*construction*),

∠ACB = ∠DFG (*both right angles*) ;

$\therefore$ the △s $\begin{matrix}\text{ABC}\\\text{DGF}\end{matrix}$ are congruent (2 *sides*, **included** *angle*),

$\therefore$ ∠ABC = ∠DGF and AB = DG.

But AB = DE (*given*), $\therefore$ DE = DG,

$\therefore$ DEG is an isosceles △,

$\therefore$ the base angles DEF, DGF are equal.

But ∠DGF = ∠ABC (*proved*),

$\therefore$ ∠DEF = ∠ABC.

Then, in the △s ABC, DEF,

AC = DF (*given*),

∠ACB = ∠DFE (*given*),

∠ABC = ∠DEF (*proved*) ;

$\therefore$ the △s $\begin{matrix}\text{ABC}\\\text{DEF}\end{matrix}$ are congruent (2 *angles*, **corresponding** *side*).

Note. By combining this theorem with Theorem 4 it will be seen that two triangles are congruent if two sides and a right angle of one triangle are respectively equal to two sides and a corresponding right angle of the other.

When this theorem is quoted, either of the following abbreviations may be used :

(1) " rt. ∠, hypot., side " ;

or (2) " 2 sides, **corresponding** rt. angle."

* THEOREM 14a

If two triangles have two sides of the one equal to two sides of the other, each to each, and also have the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary, and in the former case the triangles are congruent.

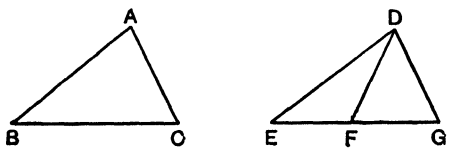


FIG. 64.

Let ABC, DEF be two triangles in which

$$AB = DE,\ AC = DF,\ \angle ABC = \angle DEF.$$

It is required to prove that

either $\angle ACB = \angle DFE$, and $\triangle ABC \equiv \triangle DEF$,

or $\angle ACB + \angle DFE = 2$ right angles.

Construction. Let G be a point on EF or EF produced such that EG = BC.

Proof. *Either* the point G coincides with the point F *or* it does not.

If G coincides with F, BC = EF. In this case the $\frac{ABC}{DEF}$ are congruent (2 *sides*, **included** *angle*),

$$\therefore\ \angle ACB = \angle DFE.$$

If G does not coincide with F, join DG.

Then, as above, the $\frac{ABC}{DEG}$ are congruent and $\angle ACB = \angle DGE$ and AC = DG. But AC = DF (*given*), $\therefore$ DF = DG;

$\therefore$ the $\triangle$DFG is isosceles and the base angles DFG, DGF are equal.

Also EFG is a straight line,

$$\therefore\ \angle DFE + \angle DFG = 2 \text{ right angles};$$

$$\therefore\ \angle DFE + \angle DGF = 2 \text{ right angles}.$$

But $\angle DGF = \angle ACB$ (*proved*),

$$\therefore\ \angle DFE + \angle ACB = 2 \text{ right angles}.$$

NOTE ON THE IDENTICAL EQUALITY OF TWO TRIANGLES

Four cases of the congruence of triangles have now been dealt with in Theorems 4, 10, 13, 14, the results of which may be summarised as follows :

Two triangles are congruent in the following cases :

(1) If two sides and the included angle of the one are respectively equal to two sides and the included angle of the other. (Theorem 4.)

(2) If two angles and a side are respectively equal to two angles and the corresponding side of the other. (Theorem 10.)

(3) If the three sides of one are respectively equal to the three sides of the other. (Theorem 13.)

(4) If the triangles are right-angled and the hypotenuse and a side of one are respectively equal to the hypotenuse and a side of the other. (Theorem 14.)

It should be particularly noted that two triangles are **not necessarily congruent** in the following cases :

(1) If the three angles of one are respectively equal to the three angles of the other. For example, in the figure, $\angle$s A, B, C

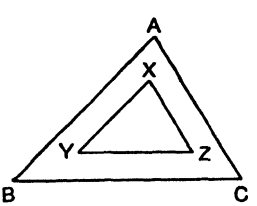


FIG. 65.

are respectively equal to $\angle$s X, Y, Z, but the triangles are not congruent.

(2) If two sides and an angle of one triangle are respectively equal to two sides and an angle of the other, the equal angles being opposite equal sides in each triangle. (See Theorem 14a.)

(3) If two angles and a side of one triangle are respectively equal to two angles and a side of the other. (See note on Theorem 15.)

The pupil should, on this account, exercise the greatest care when quoting Theorems 4, 10, 14.

When quoting Theorem 4 it should be emphasised that the equal angles must be the **included** angles.

When quoting Theorem 10 it should be emphasised that the equal sides must be **corresponding** sides.

When quoting Theorem 14 it should be emphasised that congruence follows **because the angles are right angles and not because they are equal.**

EXERCISE 9 [Chiefly on Theorems 14 and 14a]

1. ABC is a $\triangle$ in which AB = AC; AD is drawn $\perp$ to BC meeting it at D. Use Theorem 14 to prove that BD = DC.

2. ABCD is a square; P is the mid-point of AB. A circle, centre P, cuts AD, BC at Q, R respectively. Prove that

$$\angle AQP = \angle BRP.$$

3. P is a point within the arm of the $\angle$BAC such that $\perp$s from P to AB, AC are equal. Prove that AP bisects the $\angle$BAC.

4. PQRS is a square. QM, SN are drawn $\perp$ to any line through P, which lies entirely without the square, meeting it at M, N. Prove that PM = SN.

5. ABCD is a square. From BD a length BX is cut off equal to BA. From X a st. line XY is drawn $\perp$ to BD to meet AD at Y. Prove (i) that DX = XY; and (ii) by congruent $\triangle$s that XY = AY.

6. In a $\triangle$ABC, the $\perp$ from A to BC is equal to the $\perp$ from C to AB; prove that AB = CB.

7. In the acute-angled $\triangle$s ABC, DEF, AB = DE, AC = DF, and the $\perp$ from A to BC is equal to the $\perp$ from D to EF; prove that the $\triangle$s are congruent. Prove that the result is also true if both $\triangle$s are obtuse-angled at C and F.

8. ABCD is a square. A point P is taken on BC and a point Q on CD such that AP = BQ. Prove that AP and BQ intersect at right angles.

9. ABC is a △ ; D is the mid-point of BC. The ⊥s from D to AB AC are equal. Prove that the △ABC is isosceles.

The following examples depend upon Th. 14a. It is recommended that they should be omitted during the first reading of the book.

10. In the △ABC the bisector of the ∠BAC also bisects the side BC. Show that the △ is isosceles.

11. ABCD is a square, AXD an equilateral △ drawn inside it. BX is joined and produced to meet the diagonal AC at Y. Calculate the ∠s of the △BAX and hence deduce that ∠DXY = 45°. Then use Th. 14a to prove that XY = CY.

State and prove the corresponding results when the equilateral △AXD is drawn outside the square.

12. Prove Ex. 7, No. 30, by means of Th. 14a.

13. ABC is a △ in which AB = AC. X is a point within the △ such that ∠ABX = ∠ACX. Prove that AX is the bisector of the ∠BAC. Is the theorem true if X is outside the △ ?

14. On the base BC of a △ABC a right-angled isosceles △BDC is drawn on the opposite of BC to A. If AD bisects the ∠BAC, prove that the △ABC must be either right-angled or isosceles.

15. ABC is a △ such that the isosceles right-angled △BDC, which is drawn on the base BC, lies entirely within the △ABC. If AD bisects the ∠BAC, prove that the △ABC must be isosceles.

16. If two △s ABC, PQR have the sides AB, AC equal to the sides PQ, PR respectively and the ∠s at C and R unequal and supplementary, prove that the ∠s at B and Q are equal and that the sum of the ∠s at A and P is twice the complement of either B or Q.

[HINT : Take S on QR such that PS = PR.]

EXERCISE 10

Further Examples, chiefly on Theorems 4, 10, 13, 14

1. ABCD, CEFG are two squares, so placed that BCE is a st. line, G lies in CD, and the squares do not overlap. From BC the length BP is cut off equal to CE. Prove (i) that AP = PF, (ii) that ∠APF = 90°.

2. E and F are the mid-points of the sides AB, AC of a △. CE, BF are produced to X, Y respectively, so that EX = CE and FY = BF. Prove that XAY is a st. line.

3. The equal sides QP, RP of an isosceles $\triangle PQR$ are produced to S, T, so that $PS = PT$; QT, RS meet at V. Prove that $\triangle QVR$ is isosceles.

4. ABCD is a square; points P, Q, R, S are taken in the sides AB, BC, CD, DA respectively, such that $AP = BQ = CR = DS$. Prove that PQRS is a square.

5. ABCD is a square; AL, BM are drawn $\perp$ to any line through D. AN is drawn $\perp$ to BM or BM produced. Prove that ALMN is a square.

6. A line drawn $\parallel$ to the base BC of an isosceles $\triangle ABC$ meets the equal sides AB, AC at E, F respectively. Lines through E, F $\parallel$ to AC, AB meet at G. Show that AG, produced if necessary, will bisect BC.

7. A quad. ABCD has the adjacent sides AB, AD equal, and also the opposite $\angle$s B, D equal. Prove that AC is $\perp$ to BD.

8. From the mid-points of the sides of a rectangle st. lines are drawn outwards $\perp$ to the sides and equal to half of the respective sides. Prove that the ends of these st. lines form the vertices of a square, whose sides each pass through an angular point of the rectangle.

9. ABXU is a square and C a point inside it such that the $\angle BCA = 90°$ and $BC > CA$. On BC a square BCYZ is described on the side of BC on which X lies. Show that X lies on the line YZ.

10. ABCD is a square. Points P, Q are taken in the sides BA, BC respectively such that $BP = BQ$. AQ, CP cut at O. Prove that $AO = CO$.

11. O is the intersection of the diagonals AC, BD of a square ABCD. POR, QOS are any two lines $\perp$ to each other, cutting AB, CD at P and R respectively, and BC, AD at Q and S respectively. Prove that $OP = OQ$, and that PQRS is a square.

12. ABC and SCB are two congruent $\triangle$s standing on the same side of the common base BC. Prove that if the corresponding sides, produced if necessary, meet at O and P, OP produced will bisect BC at right angles.

13. A, B, C, D are four points in order on a st. line with $CD = AB$; on one side of this line squares ABEF, BDGH are drawn, and the points F, C, G joined. Prove that $\triangle FCG$ is isosceles and right-angled.

14. If the sides AB, AC of a $\triangle$ are at right angles, and squares ABDE, ACFG are described outwards on them, and if DE, FG are produced to meet at X, prove that XA is $\perp$ to BC and equal to it.

15. A point D is taken in a side BC of an equilateral $\triangle$ ABC and an equilateral $\triangle$ CDE is described on CD, the vertices A and E being on opposite sides of BC. Prove that AD = BE ; and if AD produced meets BE at F, prove that $\angle$ BFD = 60°.

16. ABC is a $\triangle$ right-angled at A, BCDE is the square on BC described outside the $\triangle$, EG is drawn $\parallel$ to BA to meet AC (produced if necessary) at G, and BF is drawn $\parallel$ to AC to meet EG at F. Prove that ABFG is a square.

17. ABC is an equilateral $\triangle$ and D, E, F are points on BC, CA, AB respectively which are such that the $\triangle$ DEF is also equilateral. Prove that BD = CE.

18. ABCD is a square ; any two $\parallel$ straight lines APS, CRQ are drawn through A, C respectively, so that P, Q, R, S all lie within the square and the straight lines BQP, DSR are $\perp$ to APS and CRQ. Prove that PQRS is a square.

19. On the sides AB, AD of a parallelogram ABCD squares ABXY, ADVZ are described. Show that YZ is equal to one or other of the diagonals of the parallelogram according as the squares are described on the one side or the other of AB, AD.

20. ABC is a $\triangle$ in which AB = AC ; the side BC is produced in both directions to D and E respectively, so that BD = CE ; F is any point inside the $\triangle$ and on the bisector of the $\angle$ BAC. Prove that DF = EF. Prove also that DF divides AB into parts which are equal respectively to the parts into which EF divides AC.

EXERCISE 11

Further Examples, chiefly on Theorems 5-9, 11, 12

1. The sides AB, BC, CD, DE, EA of a pentagon inscribed in a circle subtend angles 40°, 50°, 60°, 100° and 110° respectively at the centre of the circle. Find the $\angle$s of the pentagon.

2. A chord QP of a circle is produced to N so that PN is equal to the radius of the circle. N is joined to O, the centre of the circle, and NO is produced to T. Prove that $\angle$ QOT = 3 $\angle$ QNT.

3. AB, AC are two lines meeting at an $\angle$ of 60°, and AC = 2AB. Prove that $\angle$ ABC = 90°.

4. OX, OY are two st. lines at right angles. S is any point in their plane. Show how to find a point P in OY such that its distance from S = its distance from O.

5. O is the mid-point of the hypotenuse AB of a right-angled $\triangle ABC$, and OC is drawn. Denoting the $\angle BAC$ by A, express the other $\angle$s in the figure in terms of A. AC is produced both ways, and points E beyond A and F beyond C are taken such that $AE = AB$, $CF = CO$. FO produced meets EB at K. Prove that $KE = KF$.

6. ABC is a $\triangle$ in which $\angle ACB > \angle ABC$. From AB cut off $AX = AC$ and join XC. Prove that the $\angle XCB$ is one half the difference between the $\angle$s ACB and ABC.

7. ABC is a $\triangle$ in which $AB = AC$. A point X is taken in AB and AC is produced to S so that $CS = XB$. The line XS cuts BC at O. Prove that $XO = OS$. [Draw a line through X $\parallel$ to AC.]

8. In a $\triangle ABC$, $AB > AC$. An arc of a circle is drawn with centre A and radius AB to cut AC produced at D. Prove that the $\angle ADB$ is equal to half the sum of the angles B and C of the $\triangle ABC$.

9. Two isosceles $\triangle$s AOB, COD ($AO = OB$, $CO = OD$) have a common vertex O and $\angle OAB = \angle OCD$. Prove that *either* $AC = BD$, *or* $AD = BC$.

10. ABC is an isosceles $\triangle$, and equilateral $\triangle$s A'BC, B'CA, C'AB are described externally upon the sides. Prove that A'B'C' is isosceles.

11. ABC is a $\triangle$, in which $\angle A = 90°$ and $BC = 2AC$. CA is produced to D so that $CA = AD$. Prove that $\angle BDC = 60°$, and that $\angle BCA = 60°$.

12. ABCD is a quad. with its opposite sides AD, BC $\parallel$. The side AB is produced to E so that $BE = BC$, and BA to F so that $AF = AD$; show that EC and FD produced meet at right angles.

13. The bisectors of the interior $\angle$s at B and C of the $\triangle ABC$ meet the bisector of the exterior $\angle$ at A in the points P, Q respectively, and BP, CQ meet at O. Prove that $\angle OPQ = \angle ACO$, $\angle OQP = \angle ABO$, and $\angle BOC = 90° + \frac{1}{2} \angle BAC$.

14. The bisectors of the exterior $\angle$s at B and C of the $\triangle ABC$ meet at X. Prove that $\angle BXC = 90° - \frac{1}{2} \angle BAC$.

15. XYZ is a $\triangle$ right-angled at X. YS is drawn $\perp$ to YZ to meet at S the bisector of the $\angle XZY$. Prove that $YP = YS$, P being the point of intersection of XY and ZS.

16. ABCD is a parallelogram; show that the internal bisectors of the $\angle$s at A, B, C, D together form a rectangle.

17. From the vertex A of a right-angled $\triangle$ a perpendicular is drawn to the hypotenuse BC. The bisector of the angle B meets this perpendicular at P and meets AC at Q. Prove that the $\triangle APQ$ is isosceles.

18. From any point P in the base BC of an isosceles $\triangle ABC$, straight lines are drawn parallel to the sides meeting them at D and E. Prove that the sum of PD and PE is the same for all positions of P.

19. ABCDE is a regular pentagon. AD, BE, CE are joined ; BE, CE cut AD at S, T respectively. Prove that AT = DS.

20. In a $\triangle ABC$, BC is produced beyond C to D, so that CD = AC ; CB is produced beyond B to E, so that BE = AB. Show that EAD is an obtuse $\angle$. If AE = AD, prove that $\triangle ABC$ is isosceles.

21. P, O, Q are three points in this order on a st. line. An acute $\angle POA$ is drawn on one side of PQ, and an $\angle QOB = 3$ times $\angle POA$ is drawn on the other side of PQ, OB being equal to OA. If AB cuts PQ at C, prove that OC = CA.

22. BC is the longest side of a $\triangle ABC$. From BC, BP is cut off equal to BA ; and from CB, CQ is cut off equal to CA. If AP = AQ prove that CP = BQ.

23. ABCD is a par^{m}. BC is produced to P, making CP = CD, and BA is produced to Q, making AQ = AD. Prove that P, D and Q are in one st. line.

24. PQR is a $\triangle$, Z is the mid-point of PQ, and the line through Z $\parallel$ to QR meets the bisector of the $\angle Q$ at Y. Prove that

$$\angle PYQ = 90°.$$

THE USE OF INSTRUMENTS

The following instruments are required :

1. A hard pencil.

2. A ruler about 6″ long, one edge being graduated in inches and tenths of an inch, and the other in centimetres and millimetres.

3. Two set squares, one with angles of 45°, and the other with angles of 60° and 30°.

4. A pair of compasses with a hard pencil-point.

5. A pair of dividers.

6. A protractor.

This book presupposes a course in Practical Geometry, and it has not been thought necessary to describe the use of the ordinary instruments.

Points of the Compass. The directions, North, East, South, West, at any point of the earth's surface, are called the **Cardinal Points** of the Compass.

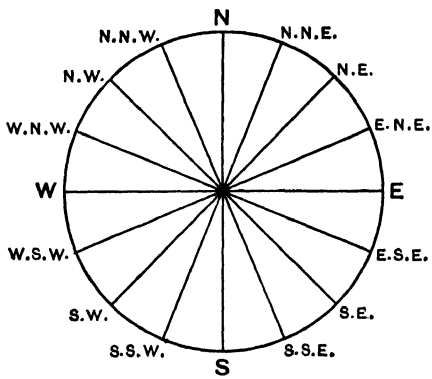


FIG. 66.

The chief points of the Compass are shown in the figure. Note the method of lettering. Thus N.W. denotes the point bisecting the angle between the points N. and W.; N.N.E. denotes the point bisecting the angle between the points N. and N.E.

The **Bearing** of an object Y as seen from a point X is the point of the compass towards which a man at X would be facing if he were looking in the direction of Y.

By "S. 15° E." or "15° E. of S." is meant the direction in which you would be looking if you first faced due South and then turned through an angle of 15° towards the East.

Horizontal and Vertical. The student is probably already familiar with the use of these words. Any line along the floor of a room is horizontal; so also is the floor. A spirit level is used to test whether a surface is horizontal. The vertical is the direction of a plumb line. Any plane which contains a vertical line is called a vertical plane.

If, looking upwards from a point A, I see an object at C; and AB is the horizontal line in the same vertical plane as AC, the angle BAC is called the **angle of elevation** of C. It is also called the **altitude** or **elevation** of C.

If C is observed by looking downwards from A, the angle BAC is called the **angle of depression** of C.

Notice that each of these angles is the angle between the line of sight AC and the **horizontal** line in the same vertical plane as AC.

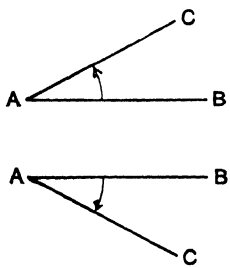
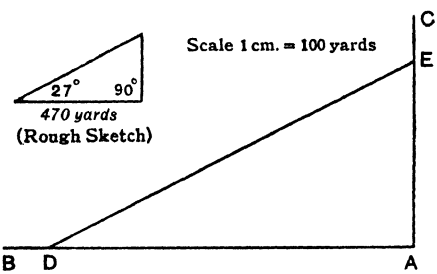


FIG. 67.

Diagrams drawn to Scale. By means of a scale, a plan of any large figure, such as a field, can easily be drawn. In order to do this the unit of the scale is taken to represent some greater length connected with the object.

Example. The angle of elevation of the top of a tower on level ground is found to be 27°. If the point of observation is on the ground 470 yards from the foot of the tower, find the height of the tower.

Let 100 yards be represented by 1 cm.; ∴ 470 yards will be represented by 4·7 cm.



F.G. 68.

Draw two straight lines AB, AC at right angles. Along AB set off AD = 4·7 cm. At D draw a straight line making an angle of 27° with AB and meeting AC at E. Then AE represents the height of the tower.

By measurement AE = 2·4 cm. approximately,

∴ The height of the tower is approximately 240 yards.

Note. The scale must always be indicated on the diagram. It is always advisable to draw a rough sketch and mark on it

the dimensions given. In deciding what scale to use, remember that the larger the diagram the more accurate is the result likely to be. The diagram should therefore be as large as your paper will allow.

EXERCISE 12

[Diagrams to Scale]

1. The distance from Cambridge to Huntingdon by road is 16 miles. Draw a line to represent this distance, if 1″ represents 5 miles.

2. A man walks 50 yd. West, then 31 yd. North, then 43 yd. East; find by drawing a diagram (1″ = 10 yd.) his distance from the starting point.

3. Draw a plan of a room measuring 28 ft. by 18 ft.; find the distances between opposite corners. [Scale 1 cm. = 2 ft.]

4. The legs of a pair of compasses are 8 cm. long. I open them to an angle of 55°. What is the distance between the compass points?

5. XY is a ladder, with its foot X on the ground, and Y against a vertical wall, Z being the foot of the wall. Draw a plan (scale 1 cm. = 10 ft.) in the following cases:

(i) XY = 75 ft., YZ = 72 ft.; find XZ.
(ii) XY = 38 ft., XZ = 11 ft.; find YZ.

6. A is 3 miles E. of B, and C is 4·7 miles S. of A. What is the distance from B to C? What is the bearing of C from B?

7. B is 42 miles N. of C, and A is 22 miles N. 30° E. of C. Find the distance from B to A, and the bearing of B from A.

8. The range of a gun is 8·5 miles, and its distance from a straight road is 4·7 miles. Find by a diagram what length of road is dangerous.

9. A gun is 6 miles from a straight road, and it can shell a length of 15 miles of road. What is the gun's range?

10. A ship observes a lighthouse bearing N. 47° W. at a distance of 8 miles. She then steams due S. 4 miles, and again observes the light. What is the bearing and distance of the light at the second observation?

11. A is 500 yards due W. of B. C is a place N.W. of A, and its bearing from B is N. 58° W. Find the distance from A to C.

12. B and C are two places 10 miles distant from A, and, as viewed from A, they are in the directions N. 34° W. and N. 20° W. respectively. Find the distance and bearing of B from C.

13. Two ships sail from a port, one due S. at 12 miles an hour, the other W.S.W.; at the end of half an hour they are in line with a lighthouse which is 9 miles due W. of the port. At what rate does the second ship sail ?

14. X is 26 kilometres S. 22° E. of Z and Y is 23 kilometres S. 65° E. of Z. Find the distance and bearing of X from Y.

15. An observer on a cliff 350 ft. above the sea finds that the angle of depression of the boat is 32°. What is the boat's distance (*a*) from the observer, (*b*) from the foot of the cliff ?

16. The angle of elevation of the top of a church spire at a point 690 feet from its base was found to be 15°. What is the height of the spire ?

17. A vertical flagstaff 45 feet high stands on a horizontal plane. Find the angle of elevation of the top of the flagstaff from a point on the horizontal plane 20 yards from the foot of the flagstaff.

18. A man is walking along a straight road which runs due East. He notices a monument in front of him, whose direction is N. 68° E.; after walking 2 miles he finds that its bearing is N. 52° E. How far is the monument from him when he makes the first observation ?

19. A man facing East notices an aeroplane at a height of 2000 feet and an elevation of 30°. Another man, facing West, notices that the elevation is 38°. What is the distance between the two men ?

20. A man on a cliff 300 feet high notices two buoys out at sea; the angle of depression of one is 28°, of the other 52°. How far apart are they, if they are in the same vertical plane as the man ?

21. The angle of elevation of the top of a tower from a point on the ground, 60 feet from the base of the tower, is 47°. From the same point the angle of elevation of the top of a vertical flagstaff standing on the tower is 58°; find the height of the tower and of the flagstaff.

22. The shadow of a tree is 12 yards long when the sun's altitude is 35°. Find the height of the tree.

23. A is 5 miles due W. of B. From A, C bears N. 26° E. and D bears N. 35° E. From B, C bears N. 42° W. and D bears N. 18° W. Find the distance and bearing of D from C.

24. A is 5 miles from B and its bearing from B is S. 78° W. From A, C bears N. 24° E. and D bears N. 42° E. From B, C bears N.W. and D bears due N. Find the distance and bearing of C from D.

25. At two points on opposite sides of a tree the angles of elevation of its top are 37° and 33°. If the height of the tree is 52 ft., find the distance between the two points.

ELEMENTARY CONSTRUCTIONS

The following constructions are **purely geometrical constructions.** In such constructions ruler and compasses only must be used. The ruler is used only for joining points and producing straight lines.

The compasses may be used

(i) to draw a circle with a given centre and radius ;

(ii) from the greater of two given straight lines, to cut off a part equal to the less.

The ruler and compasses may not be used for any other purpose.

The scale and protractor may only be used in numerical applications.

In writing out constructions an accurate figure must be drawn. All lines required for the construction must be clearly shown ; they should not be rubbed out when the construction has been completed.

Any additional lines, which are inserted only for purposes of proof, should be dotted or drawn very lightly, to distinguish them from lines necessary to the construction.

CONSTRUCTION 1

To bisect a given angle.

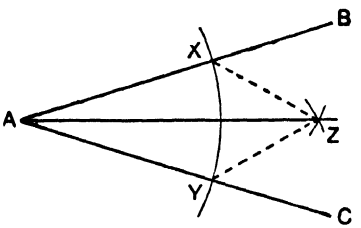


FIG. 69.

Let BAC be the given angle to be bisected.

Construction. With centre A, and any radius, draw an arc of a circle, cutting AB, AC at X and Y respectively.

With centres X and Y and any radius greater than $\frac{1}{2}$XY (the same for both circles), draw two arcs cutting at Z.

Join AZ.

Then AZ bisects the ∠BAC.

Proof. Join XZ, YZ.

In the △s XAZ, YAZ,

XA = YA (*radii of a circle*),

XZ = YZ (*radii of equal circles*),

AZ = AZ (*common*) ;

∴ the △s $\begin{matrix}\text{XAZ}\\\text{YAZ}\end{matrix}$ are congruent (3 *sides*),

∴ ∠XAZ = ∠YAZ ;

∴ AZ bisects the ∠BAC.

CONSTRUCTION 2

To bisect a given finite straight line.

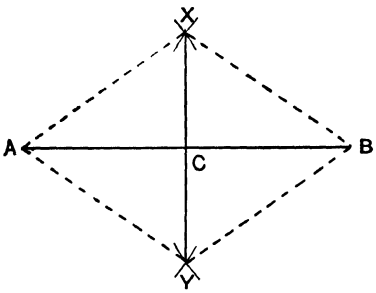


FIG. 70.

Let AB be the finite straight line which is to be bisected.

Construction. With centre A and any radius greater than $\frac{1}{2}$AB, draw two arcs, one on each side of AB.

With centre B and the same radius, draw two arcs, one on each side of AB, cutting the first arcs at X and Y.

Join XY, cutting AB at C.

Then AB is bisected at C.

Proof. Join AX, BX, AY, BY.

In the $\triangle$s AXY, BXY,

AX = BX (*radii of equal circles*),

AY = BY (*radii of equal circles*),

XY = XY (*common*) ;

$\therefore$ the $\triangle$s $\begin{matrix}\text{AXY}\\ \text{BXY}\end{matrix}$ are congruent (3 *sides*),

$\therefore$ $\angle$AXY (i.e. $\angle$AXC) = $\angle$BXY (i.e. $\angle$BXC).

Again, in the $\triangle$s AXC, BXC,

AX = BX (*radii of equal circles*),

$\angle$AXC = $\angle$BXC (*proved*),

XC = XC (*common*) ;

$\therefore$ the $\triangle$s $\begin{matrix}\text{AXC}\\ \text{BXC}\end{matrix}$ are congruent (2 *sides*, **included** $\angle$),

$\therefore$ AC = BC ; i.e. AB is bisected at C.

Note. From the congruence of the Δs AXC, BXC it follows that $\angle ACX = \angle BCX$, and each angle is therefore by definition a right angle;

$\therefore \angle ACX = 90°$, i.e. XY bisects AB at right angles.

This construction, therefore, gives the **perpendicular** bisector of AB.

CONSTRUCTION 3

To draw a straight line perpendicular to a given straight line at a given point in it.

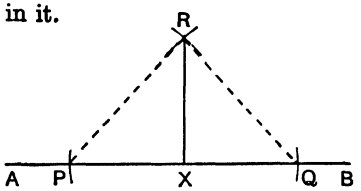


FIG. 71.

Let AB be the given straight line and X the given point in it.

It is required to draw a perpendicular to AB at X.

Construction. Along XA and XB, cut off two equal lengths, XP and XQ.

With centres P and Q and any radius greater than $\frac{1}{2}$PQ (the same for both circles), draw two arcs cutting at R.

Join XR.

Then XR is $\perp$ to AB.

Proof. Join RP, RQ.

In the Δs RPX, RQX,

RP = RQ (*radii of equal circles*),

PX = QX (*construction*),

RX = RX (*common*);

$\therefore$ the Δs $\begin{matrix}\text{RPX}\\ \text{RQX}\end{matrix}$ are congruent (3 *sides*),

$\therefore \angle RXP = \angle RXQ$,

and each angle is therefore by definition a right angle; i.e. RX is $\perp$ to AB.

Note. If the point X is near one end of AB, the above construction is unsuitable, and one of the following constructions should be used instead

CONSTRUCTION 3. SECOND METHOD

Construction. With centre X and any radius draw an arc of a circle CDE cutting XB at C.

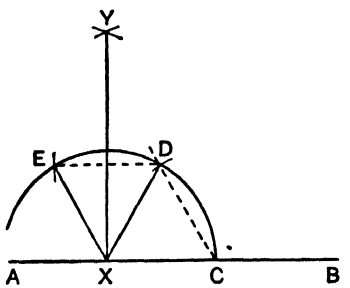


FIG. 72.

With centre C and the same radius as before draw an arc cutting the arc CDE at D.

With centre D and the same radius as before draw an arc cutting the arc CDE at E. Join XD, XE.

Bisect the ∠DXE by XY. (*Construction* 1.)

Then XY is ⊥ to AB.

Proof. Join CD, DE.

By construction, CX, CD, DE, XD, XE are radii of equal circles;

∴ all these lines are equal;

∴ the △s CXD, DXE are equilateral;

∴ ∠CXD = ∠DXE = 60°.

But, by construction, XY bisects the ∠DXE;

∴ ∠DXY = 30°;

∴ ∠BXY = ∠CXD + ∠DXY = 90°;

∴ XY is ⊥ to AB.

CONSTRUCTION 3. THIRD METHOD

Construction. Take any point C outside AB.

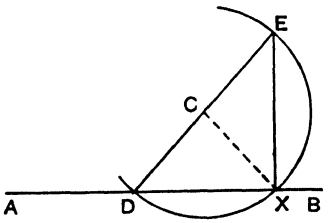


FIG. 73.

With centre C, and radius CX, draw a circle cutting AB again at D.

Join DC, and produce it to meet the circle again at E.

Join XE.

Then XE is $\perp$ to AB.

Proof. Join CX.

CE = CX, being radii of a circle; $\therefore$ $\triangle$CEX is isosceles;

$$\therefore \angle CEX = \angle CXE.$$

CD = CX, being radii of a circle; $\therefore$ $\triangle$CDX is isosceles;

$$\therefore \angle CDX = \angle CXD,$$

$$\therefore \angle CEX + \angle CDX = \text{the whole } \angle DXE.$$

But $$\angle CEX + \angle CDX + \angle DXE = 180^\circ,$$

$$\therefore \angle DXE = 90^\circ,$$

$\therefore$ XE is $\perp$ to AB.

CONSTRUCTION 4

To draw a straight line perpendicular to a given straight line from a given point outside it.

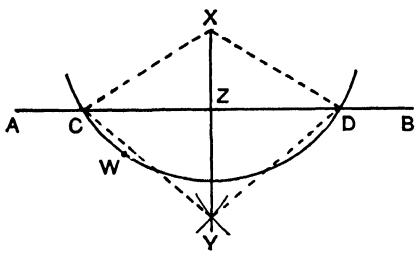


FIG. 74.

Let AB be the given straight line and X the given point outside it.

It is required to draw a perpendicular from X to AB.

Construction. Take any point W on the side of AB remote from X.

With centre X and radius XW, draw an arc of a circle to cut AB at C and D.

With centres C and D and any radius greater than $\frac{1}{2}$CD (the same for both circles), draw two arcs cutting at Y.

Join XY cutting AB at Z.

Then XZ is $\perp$ to AB.

Proof. Join CX, CY, DX, DY.

In the $\triangle$s XCY, XDY,

XC = XD (*radii of a circle*),

CY = DY (*radii of equal circles*),

XY = XY (*common*);

$\therefore$ the $\triangle$s $\begin{matrix}\text{XCY}\\ \text{XDY}\end{matrix}$ are congruent (3 *sides*),

$\therefore$ $\angle$CXY (i.e $\angle$CXZ) = $\angle$DXY (i.e. $\angle$DXZ).

Again, in the $\triangle$s XCZ, XDZ,

$$XC = XD \text{ (\textit{radii of a circle})},$$

$$\angle CXZ = \angle DXZ \text{ (\textit{proved})},$$

$$XZ = XZ \text{ (\textit{common})};$$

$$\therefore \text{ the } \triangle\text{s } \begin{matrix} XCZ \\ XDZ \end{matrix} \text{ are congruent (2 \textit{sides}, \textbf{included} } \angle),$$

$$\therefore \angle XZC = \angle XZD,$$

and each angle is therefore by definition a right angle; i.e. XZ is $\perp$ to AB.

Note. If the point X is nearly opposite one end of AB, the above construction is unsuitable, and one of the following constructions should be used instead.

CONSTRUCTION 4. SECOND METHOD

Construction. Take any point D in AB. Join DX.

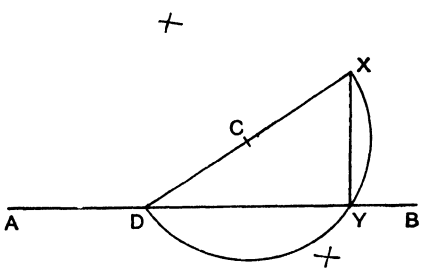


FIG. 75.

Bisect DX at C. (*Construction* 2.)

With centre C, and radius CD, draw a circle cutting AB again at Y. Join XY.

Then XY is $\perp$ to AB.

For, as in Construction 3, Third Method, the $\angle XYD = 90°$.

CONSTRUCTION 4. THIRD METHOD

Construction. Take any two points D and E in AB.

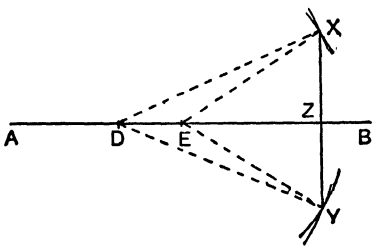


FIG. 76.

With centre D and radius DX, draw an arc of a circle, on the side of AB remote from X.

With centre E and radius EX, draw another arc cutting the former at Y.

Join XY, cutting AB at Z.

Then XZ is $\perp$ to AB.

Proof. Join DX, DY, EX, EY.

In the $\triangle$s XDE, YDE,

XD = YD (*radii of a circle*),

XE = YE (*radii of a circle*),

DE = DE (*common*) ;

$\therefore$ the $\triangle$s $\begin{matrix}\text{XDE}\\ \text{YDE}\end{matrix}$ are congruent (3 *sides*),

$\therefore$ $\angle$XDE (i.e. $\angle$XDZ) = $\angle$YDE (i.e. $\angle$YDZ).

Again, in the $\triangle$s XDZ, YDZ,

XD = YD (*radii of a circle*),

$\angle$XDZ = $\angle$YDZ (*proved*),

DZ = DZ (*common*) ;

$\therefore$ the $\triangle$s $\begin{matrix}\text{XDZ}\\ \text{YDZ}\end{matrix}$ are congruent (2 *sides*, **included** $\angle$),

$\therefore$ $\angle$XZD = $\angle$YZD,

and each angle is therefore by definition a right angle: i.e. XZ is $\perp$ to AB.

CONSTRUCTION 5

At a given point in a given straight line to make an angle equal to a given angle.

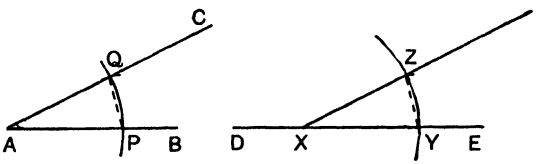


FIG. 77.

Let BAC be the given angle, and DE the given straight line; and let X be the point at which an angle is to be made equal to the $\angle$BAC.

Construction. With centre A and any radius, draw an arc of a circle, cutting AB, AC at P, Q, respectively. Join PQ.

With centre X and the same radius, draw an arc of a circle to cut XE at Y.

With centre Y and radius equal to the straight line PQ, draw an arc of a circle to cut the former arc at Z.

Join XZ, YZ.

Then XZ is the required line.

Proof. In the $\triangle$s APQ, XYZ,

AP = XY (*radii of equal circles*),

AQ = XZ (*radii of equal circles*),

PQ = YZ (*construction*);

$\therefore$ the $\triangle$s $\begin{matrix}\text{APQ}\\ \text{XYZ}\end{matrix}$ are congruent (3 *sides*),

$\therefore$ $\angle$PAQ (i.e. $\angle$BAC) = $\angle$YXZ;

$\therefore$ XZ is the required line.

CONSTRUCTION 6

Through a given point to draw a straight line parallel to a given straight line.

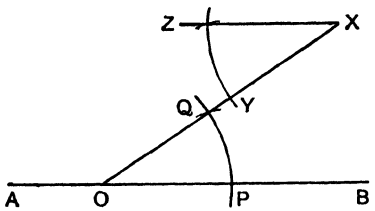


FIG. 78.

Let AB be the given straight line, and X the given point.

It is required to draw a parallel to AB through X.

Construction. In AB take any point O, and join XO.

Using Construction 5, at the point X in the line OX make an $\angle OXZ$ equal to the $\angle XOB$ and alternate to it.

Then XZ is parallel to AB.

Proof. OX, a transversal cutting the lines XZ and AB, makes the alternate $\angle$s OXZ, XOB equal;

$\therefore$ XZ is parallel to AB.

Note. Constructions 3, 4, 6 are more easily performed by the aid of set squares, and the methods given above are not usually followed in practical drawing. They must, however, be used if strict geometrical constructions are required.

EXERCISE 13 [Constructions 1-6]

[These exercises should be done as far as possible without using the protractor or set-square.]

1. On a given st. line as base construct an equilateral Δ.

2. Construct an equilateral Δ of given height.

3. Construct angles of 60°, 30°, 45°, 15°, 22½°, 7½°, 105°.

4. Draw a st. line AB 5·3″ long and take a point C in it 0·5″ from A. Erect a $\perp$ to AB at C (i) by the second method, (ii) by the third method of Construction 3.

5. Draw a st. line AB 3·2″ long and take a point C in it 1·4″ from B. Erect a $\perp$ to AB at C by the first method of Construction 3.

6. Draw any $\triangle$XYZ. Bisect the $\angle$s X, Y, Z and verify that the bisectors meet in a point.

7. Draw any $\triangle$XYZ. Draw the $\perp$s from X, Y, Z, to the opposite sides, and verify that these lines meet in a point.

8. Draw any $\triangle$XYZ. Draw the $\perp$ bisectors of YZ, ZX, XY and verify that these lines meet in a point. If P, Q, R are the mid-points of YZ, ZX, XY, verify that XP, YQ, ZR meet in a point.

9. A lies 7 miles N. 30° W. of B; C is 4 miles S. 67½° E. of A. Construct the $\perp$ from C to AB.

10. A lies 10 miles N. 22½° W. of B; D lies on AB, 2½ miles from A. Construct the $\perp$ to AB at D and find two points on it, E, F, one on each side of D and 5 miles from D. Find the distance and bearing from B of E and F.

11. Draw any $\triangle$XYZ. Through X draw a st. line $\parallel$ to YZ. Draw the bisector of the $\angle$XYZ to meet the $\parallel$ to YZ at S. Verify that XY = XS.

12. A, B are two ships at sea. B is 12 miles due S. of A. A landmark X bears N. 30° W. from A and N. 15° W. from B. A buoy is observed to be in the line of sight from B to X and also to be on the bisector of the $\angle$BAX. Find the distance and the bearing of the buoy from X.

13. Through a given point C, draw a st. line which shall meet a given st. line AB at a given angle.

14. Through a given point A draw a st. line such that the $\perp$s drawn to it from two points X and Y may be equal. When is it impossible to do this?

INEQUALITIES CONNECTED WITH THE TRIANGLE

THEOREM 15

If two sides of a triangle are unequal, the greater side has the greater angle opposite it.

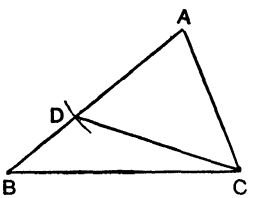


FIG. 79.

Let ABC be a triangle in which $AB > AC$.

It is required to prove that $\angle ACB > \angle ABC$.

Construction. With centre A and radius AC, describe a circle. Since $AB > AC$, this circle must cut AB at a point D **between A and B.** Join CD.

Proof. The $\triangle ADC$ is, by construction, an isosceles $\triangle$;

$\therefore$ the base $\angle$s ADC, ACD are equal.

But the exterior $\angle ADC$ of the $\triangle BDC$ is greater than the interior opposite $\angle DBC$;

$\therefore \angle ACD > \angle DBC.$

But $\angle ACB > \angle ACD,$

$\therefore \angle ACB > \angle DBC,$ i.e. $\angle ABC.$

Note. The teacher should point out the importance of stressing the fact that D lies between A and B. The pupil should realise that in placing D between A and B he is making use of the data, and that nowhere else in the proof are the data used. If anyone is inclined to doubt the necessity of this let him criticise the following " proof " and expose its unsoundness.

Let D be a point on AB such that BD = DC.

[Or, Let the ⊥ bisector of BC meet AB at D ; then BD = DC.]

Then ∠ACB > ∠DCB.

But ∠DCB = ∠DBC (base ∠s of the isosceles △DBC),

i.e. ∠DCB = ∠ABC,

∴ ∠ACB > ∠ABC.

THEOREM 16

If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

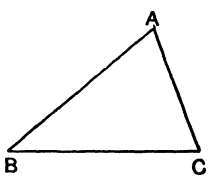


FIG. 80.

Let ABC be a triangle in which ∠ACB > ∠ABC.

It is required to prove that AB > AC.

First Proof. *Either* AB = AC *or* AB < AC *or* AB > AC.

If AB were equal to AC, △ABC would be isosceles and the base angles ABC, ACB would be equal, which is contrary to the hypothesis.

If AB were less than AC, the angle opposite the greater side AC would be greater than the angle opposite the smaller side AB, i.e. ∠ABC would be greater than ∠ACB, which is contrary to the hypothesis ;

∴ AB is neither equal to nor less than AC ;

∴ AB is greater than AC.

Second Proof.

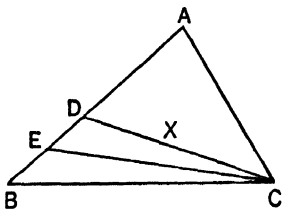


FIG. 81.

Construction. Since $\angle ACB > \angle ABC$, there is a line CX within the $\angle ACB$ such that $\angle ACX = \angle ABC$. Let CX meet AB at D and let the bisector of $\angle BCD$ meet AB at E.

Then D lies between A and B and E lies between D and B, as in the figure.

Proof. $\angle ACE = \angle ACD + \angle DCE$. But, by construction,

$$\angle ACD = \angle ABC, \quad \text{and} \quad \angle DCE = \angle ECB;$$

$$\therefore \angle ACE = \angle ABC + \angle ECB.$$

But the exterior $\angle AEC$ of the $\triangle EBC$ is equal to the sum of the interior opposite angles ABC, ECB,

$$\therefore \angle ACE = \angle AEC,$$

$$\therefore \text{the } \triangle AEC \text{ is isosceles and } AE = AC;$$

$$\therefore AB > AE > AC.$$

Note. The first proof given above is an example of **Proof by Exhaustion.** It is applicable in cases when **one** of certain suppositions must be true; and it consists in proving that each of these suppositions is false **with one exception only**; hence the truth of the remaining supposition is established.

* THEOREM 16a

Any two sides of a triangle are together greater than the third side.

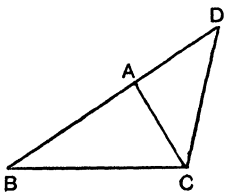


FIG. 82.

Let ABC be a triangle.

It is required to prove that any two of its sides are together greater than the third side.

Let BC be a side which is not less than AB or AC.

Then it is sufficient to prove that $AB + AC > BC$.

Construction. Produce BA to D, making $AD = AC$. Join DC.

Proof. Because $AD = AC$, the $\triangle ADC$ is isosceles,

$$\therefore \angle ADC = \angle ACD.$$

But $$\angle BCD > \angle ACD,$$

$$\therefore BCD > \angle ADC, \text{ i.e. } \angle BDC;$$

$\therefore$ In the $\triangle BDC$, the side BD opposite the greater $\angle BCD$ is greater than the side BC opposite to the smaller $\angle BDC$.

But $$BD = BA + AD = BA + AC \ (\textit{construction}),$$

$$\therefore BA + AC > BC.$$

Note. The truth of this theorem is usually considered to be self-evident. The proof given above is, however, useful as an exercise, and the figure should be carefully studied, as it will frequently be of use in later work.

THEOREM 17

Of all straight lines which can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.

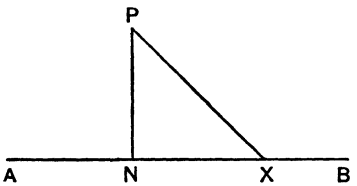


FIG. 83.

Let AB be a straight line, P a given point outside it, PN the perpendicular from P to AB meeting it at N, and X any other point in AB. Join PX.

It is required to prove that PN $<$ PX.

Proof. In the $\triangle$PNX, $\angle$PNX is a right angle (*given*),

$\therefore$ $\angle$PXN is less than a right angle

(for the 3 angles of a triangle make up two right angles);

$\therefore$ $\angle$PXN $<$ $\angle$PNX;

$\therefore$ In the $\triangle$PNX the side PN opposite the smaller $\angle$PXN is less than the side PX opposite the greater $\angle$PNX.

Corollary. Since there can only be one perpendicular from P to AB, it follows that **if PC is the shortest straight line from P to AB, then PC is perpendicular to AB.**

EXERCISE 14 [Theorems 15-17]

1. In a $\triangle$ABC, $\angle$A $=20°$, $\angle$C $=40°$. The bisector of the $\angle$B meets AC at X. Prove that (i) AX $>$ BX, (ii) CX $>$ BX, (iii) BC $>$ XC.

[HINT. First calculate all the angles in the figure.]

2. In a $\triangle$ABC, $\angle$A $=75°$, $\angle$B $=62°$. AD is drawn $\perp$ to BC, BE is drawn $\perp$ to CA; AD, BE meet at O. Prove that (i) AO $<$ BO, (ii) AE $<$ OE.

3. ABC is a $\triangle$ in which $\angle A = 76°$, $\angle B = 38°$. BC is produced to D and CE is drawn to bisect $\angle ACD$ and to meet at K the bisector of the $\angle ABC$. Prove that $CK < BC$.

4. ABC is a $\triangle$ in which $\angle A = 76°$, $\angle B = 38°$. The bisectors of the exterior $\angle$s at B and C meet at L. Prove that $CL > BL$.

5. ABC is an isosceles $\triangle$, with $AB = AC$ and $\angle A = 26°$. P is a point in AC such that $PA = PB$. Prove that $BC > PC$.

6. In a $\triangle ABC$, $\angle A = 90°$. Prove that BC is the greatest side of the $\triangle$.

7. In a $\triangle ABC$, BC is the greatest side. Prove that the $\angle$s B and C are both acute and that $\angle A$ is greater than 60°.

8. ABC is a $\triangle$, in which OB, OC bisect the $\angle$s ABC, ACB respectively. If $AB < AC$, prove that $OB < OC$.

9. AD bisects the $\angle BAC$ of a $\triangle$ and meets the base BC at D. Prove that (i) $AB > BD$, (ii) $AC > DC$.

10. ABCD is a quadrilateral with its opposite sides AB, DC parallel. $\angle A$ is an obtuse angle. The bisectors of the $\angle$s A and D meet at E. Prove that $DE > AE$.

11. X is a point inside an equilateral $\triangle PQR$, such that $XQ > XR$; prove that $\angle XQP > \angle XRP$.

12. The sides XY, XZ of a $\triangle XYZ$ are equal; A is a point in XY. Prove that $AY < AZ$.

13. ABC is a $\triangle$, AD is $\perp$ to BC. If $AC < AB$, prove that
$$\angle BAD > \angle CAD.$$

14. PQRS is a quadrilateral, the $\angle$s Q, S being right angles. If $PQ > PS$, prove that $RS > RQ$.

15. In an acute-angled $\triangle ABC$, in which $AB > AC$, the lines drawn from B, C $\perp$ to the opposite sides intersect at O. Prove that $OB > OC$.

16. In a $\triangle ABC$, $AB > AC$, the bisectors of the exterior $\angle$s at B and C meet at I. Prove that $IC > IB$.

17. The base YZ of an isosceles $\triangle XYZ$ is produced to any point S. Prove that $XS > XY$ or XZ.

18. XZY is a st. line, ZS its $\perp$ bisector. P is any point on the same side of ZS as Y. Prove that $PX > PY$.

[HINT. Let PX meet ZS at T. Join YT.]

19. D is a point on the base EC of a $\triangle ABC$. Prove that either AB or AC is greater than AD.

[HINT. Consider separately the cases in which $\angle ADB$ is acute, right and obtuse.]

20. ABCD is a quadrilateral, in which AB > BC > CD > DA. Prove that ∠CDA > ∠CBA.

21. ABCD is a quadrilateral, in which AB is the greatest and CD the least side. Prove that (i) ∠CDA > ∠CBA, (ii) ∠DCB > ∠DAB.

22. ABCD is a quadrilateral whose diagonals AC, BD intersect at O.

(*a*) If ∠ABO > ∠DCO, prove that ∠CDO > ∠OAB.

(*b*) If OC > OD > OB > OA, prove that ∠BAD is obtuse and ∠BCD is acute.

23. Find whether it is possible to make △s whose sides are :

(i) 3·5″, 4·5″, 5·5″ ; (ii) 0·5″, 1·5″, 2·5″ ;

(iii) 5 cm., 6 cm., 11 cm. ; (iv) 6 cm., 7 cm., 8 cm.

24. A ⊥, OP, is drawn to a given line from a point O outside it, and Q is any point on the line. Prove that if R is any point on the line between P and Q, OR < OQ.

25. Prove that the perimeter of a quadrilateral is greater than the sum of its diagonals.

26. Prove that the sum of the distances of a point P within a quadrilateral from the angular points is greater than half the sum of its sides.

27. ABC is an isosceles △ in which AB = BC. E is any point on AB produced and D is any point on the bisector of the ∠ABC. Prove that CB + BE < CD + DE.

28. Show that the sum of the distances of any point from the angular points of a closed polygon is greater than half the perimeter of the polygon.

29. Show that the perimeter of a polygon is always greater than that of a polygon of which the successive corners lie in the sides of the first polygon taken in order.

EXERCISE 15 [Further Examples on Theorems 15-17]

1. In a △ABC, AB > AC ; the ∠BAC is bisected by a line meeting BC at D. Show that the ∠ADB is obtuse.

2. The mid-point M of the side BC of a △ABC is joined to A. Prove that (i) if AM < BM, the ∠BAC is obtuse ; (ii) if AM > BM, the ∠BAC is acute ; (iii) if AM = BM, the ∠BAC = 90°.

3. In a △ABC, ∠B = 2∠C and is acute. Prove that AC < 2AB.

[HINT. Take N in BC so that AN = AB.]

4. XYZ is an obtuse-angled Δ, $\angle XYZ$ being the obtuse angle Prove that it is not possible to construct a Δ with sides $\frac{1}{2}YZ$, ZX, $\frac{1}{2}XY$. Under what conditions is the result true if XYZ is an acute-angled Δ?

5. ABCX is a trapezium, the parallel sides BC, AX each being $\perp$ to CX; $AB = BC$ and $BC > AX$. YBZ is the bisector of the $\angle ABC$, Y being outside the trapezium and Z being inside it. Prove that AZ is greater than the $\perp$ from Z to CX, and that AY is greater than the $\perp$ from Y to XC produced.

6. If P be any point within a ΔABC, prove that $\angle BPC > \angle BAC$.

[HINT. Join AP and produce to Q.]

7. If P be any point within a ΔABC, prove that

$$BP + PC < BA + AC.$$

[HINT. Produce BP to meet AC at D. Prove that

$$BP + PD < BA + AD$$

and $PC < PD + DC$. Then add these results.]

8. ABC is a Δ; Q is any point on the external bisector of the $\angle A$. Prove that $BQ + QC > BA + AC$.

[HINT. Produce BA to D, making $AD = AC$. Join DQ.]

9. Prove that if any point P be taken within the ΔABC,

$$BC + CA + AB > PA + PB + PC > \tfrac{1}{2}(BC + CA + AB).$$

10. AM is a median of the ΔABC. Prove that $AB + AC > 2AM$.

[HINT. Produce AM to X, making $MX = AM$. Join XC.]

11. Prove that the sum of the three medians of a Δ is less than the perimeter of the Δ.

12. XY is a straight line; A, B are two given points on the same side of it; find a point R in XY such that, if S is any other point in XY, then $AR + RB < AS + SB$.

[Draw AN $\perp$ to XY and produce it to T, so that $NT = AN$. Join BT meeting XY at R.] Prove also that AR, RB are equally inclined to XY.

Note. T is called the **image** of A in XY.

13. APB is an arc of a circle less than a semi-circle. The $\perp$ OM from the centre O on the chord AB is produced to meet the arc at Q; P is any other point on the arc, and OP meets the chord AB at N. Prove that $QM > PN$.

14. If ABC be a Δ and BPQR ... C any convex polygon having BC for one side, and its other vertices P, Q, R, etc., lying inside ABC, show that the perimeter of ABC is greater than the perimeter of BPQR ... C.

THE CONSTRUCTION OF TRIANGLES

CONSTRUCTION 7

To draw a triangle, given the lengths of the three sides.

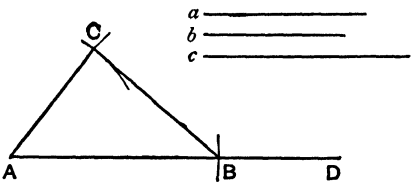


FIG. 84.

Let a, b, c be the given lengths.

Construction. Draw any straight line AD, and cut off from it a part AB equal to c.

With centre A and radius b, draw an arc of a circle.

With centre B and radius a, draw an arc of a circle.

Let these arcs cut at C.

Then ABC is the required triangle.

Note. In order that the construction may be possible it is necessary that any two of the given sides should be together greater than the third side (Theorem 16a); for otherwise the arcs drawn with centres A and B would not cut.

CONSTRUCTION 8

To draw a triangle, given two sides and the included angle.

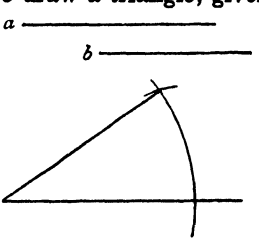


FIG. 85.

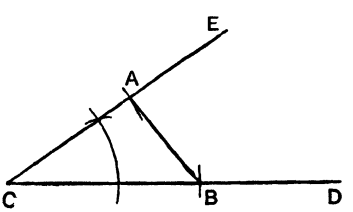


FIG. 86.

Let a, b be the given sides and X the given angle.

Construction.

Draw any straight line CD, and cut off from it a part CB equal to a.

At C make the $\angle DCE$ equal to the given $\angle X$ (Construction 5).

From CE cut off a part CA equal to b.

Join AB.

Then ABC is the required triangle.

CONSTRUCTION 9

To draw a triangle, given a side and two angles.

(i) Given two angles X, Y and the side a opposite the given angle X.

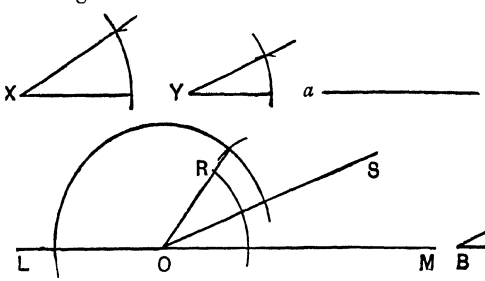


FIG. 87.

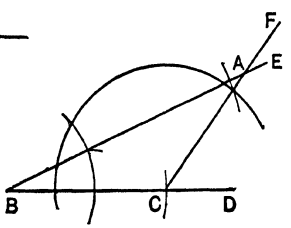


FIG. 88.

Construction. Draw a straight line LOM. At O make angles MOS, SOR equal to Y, X respectively. Then the $\angle ROL$ is equal to the third angle of the required triangle, since the $\angle$s MOS, SOR, ROL are together equal to two right angles.

Draw a straight line BD, and cut off from it a length BC equal to a.

At B make an $\angle DBE$ equal to $\angle Y$ (Construction 5).

At C make an $\angle BCF$ equal to $\angle ROL$ (Construction 5).

Let BE, CF intersect at A.

Then ABC is the required triangle.

(ii) Given two angles X, Y and the side a adjacent to these angles.

The construction is the same as (i), with the omission of the first step.

CONSTRUCTION 10

To draw a triangle, given two sides and an angle opposite to one of them.

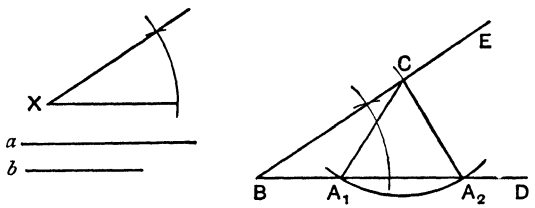


FIG. 89.

Let a, b be the given sides and X the given angle which is to be opposite b.

Construction. Draw a straight line BD and at B make an $\angle$DBE equal to X (Construction 5).

From BE cut off a length BC equal to a.

With centre C and radius equal to b, draw an arc of a circle.

Several cases are possible.

(1) If, as in the figure, the circle cuts BD, or BD produced, at two points A_1, A_2, there are two $\triangle$s satisfying the given conditions, namely the $\triangle$s CBA_1, CBA_2.

(2) If the circle cuts BD or BD produced at F and DB produced at F′, there is only one solution, namely the $\triangle$CBF.

(3) If the circle meets BD or BD produced at only one point G, there is only one solution, namely the $\triangle$CBG.

(4) If the circle does not meet BD or BD produced, there is no solution. No triangle exists with the given parts.

The first case is called the **Ambiguous Case.**

The pupil should draw figures to illustrate all four cases.

Case 1 arises when b is less than a, but greater than the $\perp$ from C to BD.

Case 2 arises when $b > a$.

Case 3 arises when b is equal to the $\perp$ from C to BD.

Case 4 arises when b is less than this perpendicular.

CONSTRUCTION 11

To draw a right-angled triangle, given the hypotenuse and one side.

This is a special case of Construction 10

NOTE ON THE CONSTRUCTION OF TRIANGLES

The constructions given above are purely geometrical constructions, but, in general, the pupil is not called upon to perform these. In most cases, when triangles have to be constructed, the actual dimensions are given, and the ruler and protractor may be used to measure lengths and angles respectively.

EXERCISE 16 [Constructions 7-11]

In Examples 1-18, if possible, construct $\triangle$s to the given measurements.

1. BC $=5{\cdot}2''$, CA $=3{\cdot}8''$, AB $=3{\cdot}3''$.
2. XY $=6{\cdot}8$ cm., YZ $=7{\cdot}2$ cm., ZX $=5{\cdot}7$ cm.
3. PQ $=4$ cm., QR $=5$ cm., RP $=9$ cm.
4. DE $=3{\cdot}2''$, EF $=2{\cdot}7''$, FD $=1{\cdot}4''$.
5. XY $=3{\cdot}1''$, YZ $=0{\cdot}9''$, ZX $=1{\cdot}9''$.
6. $\angle$BAC $=47°$, AB $=2{\cdot}3''$, AC $=2{\cdot}7''$.
7. $\angle$XYZ $=83°$, XY $=4$ cm., YZ $=3{\cdot}3$ cm.
8. $\angle$ABC $=65°$, $\angle$ACB $=37°$, BC $=2{\cdot}1''$.
9. $\angle$XYZ $=21°$, $\angle$XZY $=118°$, YZ $=3{\cdot}2$ cm.
10. $\angle$DEF $=102°$, $\angle$FDE $=36°$, EF $=1{\cdot}8''$.
11. $\angle$PQR $=77°$, $\angle$QRP $=54°$, PQ $=3{\cdot}8$ cm.
12. AB $=5$ cm., $\angle$BAC $=29°$, BC $=3{\cdot}1$ cm.
13. AB $=5{\cdot}2$ cm., $\angle$BAC $=73°$, BC $=6{\cdot}3$ cm.
14. PQ $=3{\cdot}8''$, $\angle$QPR $=30°$, QR $=1{\cdot}9''$.
15. PQ $=3{\cdot}4''$, $\angle$QPR $=42°$, QR $=2{\cdot}7''$.
16. XY $=5{\cdot}4$ cm., $\angle$XYZ $=127^c$, XZ $=8{\cdot}3$ cm.
17. XY $=3{\cdot}6$ cm., $\angle$XYZ $=82°$, XZ $=1{\cdot}3$ cm.
18. XZ $=7$ cm., $\angle$XYZ $=90°$, XY $=5{\cdot}3$ cm.

PARALLELOGRAMS

A **quadrilateral** is a plane figure bounded by four straight lines.

The straight lines which join opposite angular points are called the **diagonals** of the quadrilateral. Thus, in the figure, PR, SQ are the diagonals of the quadrilateral PQRS.

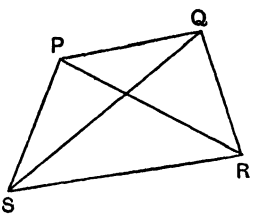
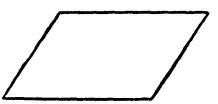


FIG. 90.

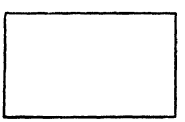
A **parallelogram** is a quadrilateral which has both pairs of opposite sides parallel.



Parallelogram

FIG. 91.

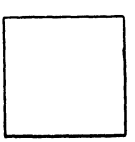
A **rectangle** is a parallelogram which has one of its angles a right angle.



Rectangle

FIG. 92.

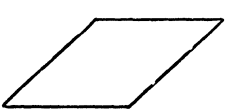
A **square** is a rectangle which has two adjacent sides equal.



Square

FIG. 93.

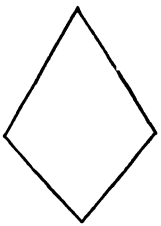
A **rhombus** is a parallelogram, having two adjacent sides equal, but none of its angles right angles.



Rhombus

FIG. 94.

A **kite** is a quadrilateral with two pairs of adjacent sides equal.



Kite

FIG. 95.

A **trapezium** is a quadrilateral which has one pair of sides parallel.

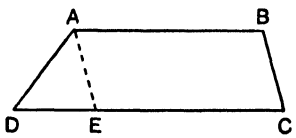


FIG. 96.

Note. A trapezium can be divided into a triangle and a parallelogram. Thus, if ABCD is a trapezium with AB, DC parallel, we can, by drawing AE parallel to BC, divide the trapezium into the triangle ADE and the parallelogram AECB. It will be seen later that nearly all the properties of a trapezium may be deduced from this fact.

Note on the preceding definitions. In defining anything, the definition should say **just enough** to define the thing exactly **and no more.**

Starting with the definitions given above, and using the properties of parallel straight lines, we shall now proceed to prove that

(1) The opposite sides and angles of a parallelogram are equal,

(2) All the angles of a rectangle or of a square are right angles,

(3) All the sides of a square or of a rhombus are equal.

But these **additional facts must not be mentioned** in the definitions.

THEOREM 18

1. The opposite angles of a parallelogram are equal.

2. A quadrilateral is a parallelogram if both pairs of opposite angles are equal.

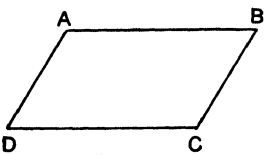


FIG. 97.

1. Let ABCD be a parallelogram.

It is required to prove that $\angle A = \angle C$ and $\angle B = \angle D$.

Proof. AB, DC are parallel lines cut by the transversal AD,

$\therefore \angle A + \angle D = 2$ right angles ;

AD, BC are parallel lines cut by the transversal DC,

$\therefore \angle D + \angle C = 2$ right angles ;

$\therefore \angle A + \angle D = \angle D + \angle C$.

From each of these equals take the $\angle D$,

$\therefore \angle A = \angle C$.

Similarly it may be proved that $\angle B = \angle D$.

2. Let ABCD be a quadrilateral in which

$\angle A = \angle C$ and $\angle B = \angle D$.

It is required to prove that ABCD is a parallelogram.

Proof. The 4 angles of a quadrilateral are together equal to 4 right angles,

$\therefore \angle A + \angle B + \angle C + \angle D = 4$ right angles.

But $\angle A = \angle C$ and $\angle B = \angle D$,

$\therefore 2\angle A + 2\angle B = 4$ right angles,

$\therefore \angle A + \angle B = 2$ right angles.

But these are interior angles on the same side of the transversal AB, $\therefore$ AD, BC are parallel.

Also, since $\angle A + \angle B = 2$ right angles and $\angle B = \angle D$,

$\therefore \angle A + \angle D = 2$ right angles.

But these are interior angles on the same side of the transversal AD, $\therefore$ AB, DC are parallel ;

$\because$ ABCD has both pairs of opposite sides parallel,

$\therefore$ it is a parallelogram.

THEOREM 19

1. The opposite sides of a parallelogram are equal, and each diagonal bisects the area of the parallelogram.

2. A quadrilateral is a parallelogram if both pairs of opposite sides are equal.

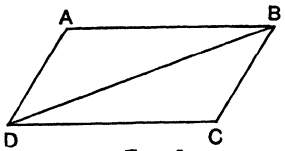


FIG. 98.

1. Let ABCD be a parallelogram, of which BD is a diagonal. It is required to prove that

(i) AB = CD and AD = CB,

(ii) the △ABD = the △CDB in area.

Proof. In the △s ABD, CDB,

∠ABD = ∠CDB (*alternate angles*, AB, DC ||),

∠ADB = ∠CBD (*alternate angles*, AD, BC ||),

BD = DB (*common*);

∴ the △s $\begin{matrix}\text{ABD}\\\text{CDB}\end{matrix}$ are congruent (2 *angles*, **corresponding** *side*),

∴ AB = CD, AD = CB and the triangles are equal in area.

Note. Theorem 18 (1) may also be proved in this way.

2. Let ABCD be a quadrilateral in which AB = CD and AD = CB. It is required to prove that ABCD is a parallelogram.

Construction. Join BD.

Proof. In the △s ABD, CDB,

AB = CD (*given*),

AD = CB (*given*),

BD = DB (*common*);

∴ the △s $\begin{matrix}\text{ABD}\\\text{CDB}\end{matrix}$ are congruent (3 *sides*);

∴ ∠ABD = ∠CDB and these are alternate angles;

∴ AB, DC are parallel;

and ∠ADB = ∠CBD and these are alternate angles;

∴ AD, BC are parallel;

∴ ABCD has both pairs of opposite sides parallel

∴ it is a parallelogram.

THEOREM 20

1. The diagonals of a parallelogram bisect one another.

2. A quadrilateral is a parallelogram if its diagonals bisect one another.

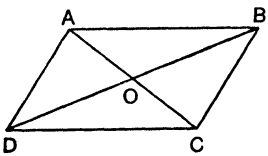


FIG. 99.

1. Let ABCD be a parallelogram, and let the diagonals AC, BD meet at O.

It is required to prove that AO = CO and BO = DO.

Proof. In the △s ABO, CDO,

AB = CD (*opposite sides of a parallelogram*),

∠OAB = ∠OCD (*alternate angles*, AB, DC ||),

∠OBA = ∠ODC (*alternate angles*, AB, DC ||);

∴ the △s $\frac{\text{ABO}}{\text{CDO}}$ are congruent (*2 angles*, **corresponding** *side*);

∴ AO = CO and BO = DO.

2. Let ABCD be a quadrilateral in which the diagonals AC, BD bisect one another at O, so that AO = CO and BO = DO.

It is required to prove that ABCD is a parallelogram.

Proof. In the △s ABO, CDO,

AO = CO (*given*),

BO = DO (*given*),

∠AOB = ∠COD (*vertically opposite*);

∴ the △s $\frac{\text{ABO}}{\text{CDO}}$ are congruent (*2 sides*, **included** *angle*);

∴ ∠ABO = ∠CDO and these are alternate angles,

∴ AB, DC are parallel.

Similarly, by considering the △s ADO, CBO, it may be proved that ∠ADO = ∠CBO and, therefore, that AD, BC are parallel;

∴ ABCD has both pairs of opposite sides parallel,

∴ it is a parallelogram.

THEOREM 21

A quadrilateral is a parallelogram if one pair of opposite sides are both equal and parallel.

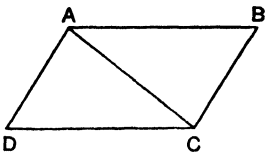


FIG. 100.

Let ABCD be a quadrilateral in which AB, DC are both equal and parallel.

Construction. Join AC.

Proof. In the $\triangle$s ABC, CDA,

$AB = CD$ (*given*),

$AC = CA$ (*common*),

$\angle BAC = \angle DCA$ (*alternate angles*, AB, DC $\|$);

$\therefore$ the $\triangle$s $\begin{matrix} ABC \\ CDA \end{matrix}$ are congruent (2 *sides*, **included** *angle*);

$\therefore$ $\angle ACB = \angle CAD$, and these are alternate angles;

$\therefore$ AD, BC are parallel.

But AB, DC are also parallel (*given*);

$\therefore$ ABCD has both pairs of opposite sides parallel,

$\therefore$ it is a parallelogram.

Note. The enunciation of this theorem is sometimes given as follows:

"The straight lines which join the ends of two equal and parallel straight lines towards the same parts are themselves equal and parallel."

Corollary 1. If P moves in a plane so that its distance from a fixed straight line AB is constant, the locus of P consists of two straight lines parallel to AB, one on either side of it, at a fixed distance from it.

Corollary 2. If P moves in a plane so that its distance (measured parallel to a given straight line) from a fixed straight line AB is constant, the locus of P consists of two straight lines parallel to AB, one on either side of it, at a fixed distance from it.

Corollary 3. If P moves in a plane so that it is equidistant from two fixed parallel straight lines AB, CD, the locus of P is a straight line parallel to AB, CD and mid-way between them.

SUMMARY OF THEOREMS 18-21

Properties of a Parallelogram. In any parallelogram,

(*a*) the opposite angles are equal,
(*b*) the opposite sides are equal,
(*c*) each diagonal bisects the parallelogram,
(*d*) the diagonals bisect each other.

Conditions for Parallelograms. A quadrilateral is a parallelogram if

(*a*) both pairs of opposite angles are equal,
(*b*) both pairs of opposite sides are equal,
(*c*) its diagonals bisect each other,
(*d*) one pair of opposite sides are both equal and parallel.

EXERCISE 17 [Theorems 18-21]

1. Prove that, if the $\angle$A of a parallelogram ABCD is a right angle, the $\angle$s B, C, D are also right angles.

2. Prove that if the diagonals of a parallelogram are equal, it is a rectangle.

3. PQRS is a parallelogram ; PX, QY are the $\perp$s from P, Q to SR or SR produced. Prove that PX = QY.

4. ABCD is a parallelogram, AB is produced to E so that BE = AB. Prove that ED bisects BC.

5. ABC is a $\triangle$. D is any point on AC, E any point on AB. Can BD, CE bisect each other ?

6. ABCD, ABEF are parallelograms on the same base AB, and between the same parallels AB, DE, and AF passes through the mid-point of BC. Prove that DE is trisected at C and F.

7. In a quad. ABCD, $\angle C = \angle D = 2\angle A = 2\angle B$. Calculate the angles of the figure. If, in addition, BC = CD and AB = 2″, find the length of DA.

[HINT. Through D draw a line || to CB to meet AB at E.]

8. ABCD is a parallelogram. The diagonals meet at O. Prove that any st. line through O, which is terminated by a pair of opposite sides of the parallelogram, is bisected at O.

9. ABCD is a parallelogram. Prove that the ⊥s from B, D, to AC are equal.

10. ABXY, XYPQ are two parallelograms with a common side XY; prove that ABQP is a parallelogram. Point out the exceptional case.

11. ABCD is a parallelogram. P, Q are the mid-points of AB, DC respectively. Prove that APCQ is a parallelogram.

12. PQRS is a parallelogram. A st. line is drawn through R parallel to QS to meet PS produced at X and PQ produced at Y. Prove that (i) PS = SX, (ii) PQ = QY.

13. If a parallelogram is not a rectangle, prove that the diagonals are unequal.

14. ABCD is a parallelogram; E, F, G, H are points in AB, BC, CD, DA respectively. AE = CG and BF = DH. Prove that EFGH is a parallelogram.

15. ABCDE is a convex pentagon with the side BC equal and parallel to AE, and the three sides AB, CD, DE all equal to one another. If the angle at A is 70°, find each of the other angles of the pentagon. [Join EC.]

16. On two adjacent sides AB, AD of a parallelogram ABCD, squares ABEF, ADGH are described, outside the parallelogram. Prove that HF is equal to the diagonal AC of the parallelogram.

17. The bisectors of the opposite angles A and C of a parallelogram ABCD meet the diagonal BD at E and F respectively; prove that the △s ABE, CDF are congruent.

18. In a parallelogram ABCD the side AB is greater than the adjacent side AD; prove that the point E where the bisector of the angle A cuts the side DC is between C and D. If AE is produced to meet BC produced at F, show that CE = CF.

19. ABCD is a parallelogram whose diagonals meet at O. Any equal lengths AE, CF are marked off along AP, CD respectively. Show that EOF is a st. line.

20. If A is a point on a given line, and B is a point on a parallel line, and if a st. line is drawn through the mid-point of AB to meet the two parallel lines in C, D, prove that ACBD is a parallelogram.

21. The side BC of a rhombus ABCD is produced through C to a point E so that CE is greater than BC. The line ED is drawn and produced to cut CA produced at F, and F is joined to B. Prove that the angles BFA, EFA are equal.

22. ABCD is a parallelogram. Equal st. lines CF, DF are drawn in opposite directions (outside the parallelogram) $\perp$ to BC and DA respectively. Prove that EF bisects CD.

23. ABCD is a quad. in which $AB = CD$, and $\angle ABC = \angle BCD$. Prove that AD is $\parallel$ to BC.

24. Squares ABDE, ACFG are described outwardly on the sides AB, AC of a right-angled $\triangle$ whose right angle is at A. Perpendiculars to BC at B and C respectively meet DE and FG (produced if necessary) at H and L. Prove that BHLC is a square.

25. ABCD is a parallelogram. A point E is taken in the side CD between C and D. AE, BD are joined, meeting at L. Prove that $BL > LD$.

26. ABCD is a quad. in which $AB = CD$ and AD is $\parallel$ to BC. If AB and CD are not $\parallel$, prove that $\angle ABC = \angle BCD$.

EXERCISE 18

[Further Examples on Theorems 18-21]

1. The sides AB, AC of a $\triangle ABC$ are bisected at N and M respectively. CN, BM meet at O. ON is produced to Q so that $NQ = ON$, and OM is produced to P so that $MP = OM$. Prove that the $\triangle$s QAP and BOC have their corresponding sides parallel and are congruent.

2. ABCDEF is a hexagon such that AB, ED are $\parallel$ to the diagonal FC, BC, EF are $\parallel$ to the diagonal AD, and CD, AF are $\parallel$ to the diagonal BE. Prove that the sum of the diagonals equals the sum of the sides.

3. ABCD is a parallelogram with AC and BD diagonals intersecting at E ; prove that the sum of AB and AD is greater than the sum of AE and EB.

4. E is any point in the side BC of a rectangle ABCD. EF is drawn || to the diagonal CA to meet AB at F, and EG || to BD to meet CD at G. Show that EF and EG together make up the length of a diagonal.

5. ABCD is a parallelogram with all its sides indefinitely produced: a point P is taken anywhere in the plane. Show, with proof, how to draw through P a st. line such that the part of it intercepted between one pair of parallel sides shall be equal to the part intercepted between the other two sides of the parallelogram.

6. The $\angle$ BAC of a $\triangle$ ABC is bisected and a line through C || to AB meets the bisecting line at D. If AB is not equal to AC, show that BD is not || to AC.

7. P is any point on a given circle whose centre is O. From P a fixed length PQ is drawn in a fixed direction. Find the locus of Q.

8. From a point O in a diagonal of a parallelogram, lines FOG, LOM are drawn || to its sides, meeting the other sides at F, G, and L, M respectively; prove that two of the parallelograms thus formed are equal in area. Prove also that the quad. FLGM is of the same area whatever point be chosen for O.

9. ABCD is a rectangle whose diagonal AC is twice the side AB. CD is produced to P making DP = CD. If O is the mid-point of AC, prove that OP is $\perp$ to AC and equal to AD.

10. ABCD is a square, and points P, Q, R, S are taken upon the sides AB, BC, CD and DA respectively, such that AP = CR and BQ = DS. Show that PR and QS meet at the centre of the square.

11. O is any point inside the $\triangle$ ABC; the parallelograms BOCD, COAE, AOBF are completed; show that the $\triangle$s ABC, DEF are congruent.

12. Points P and Q are taken within a parallelogram ABCD so that AP and QC and also BP and QD are ||. Prove that AQ is equal and parallel to PC.

13. From the diagonal BD of a parallelogram ABCD two equal portions, BE, DF, are measured off. Prove that the figure AECF is a parallelogram.

14. ABCDEF is an irregular hexagon, with its three pairs of opposite sides respectively || Also AB, BC are respectively equal to their opposite sides DE, EF. Prove that the diagonals AD, BE, CF meet in a point, and that the remaining sides CD, FA are equal to one another.

15. ABCD is a quad. If AB = CD and if the angles at A and B are supplementary, prove that the figure is either a parallelogram or a trapezium with $\angle A = \angle D$ and $\angle B = \angle C$.

16. If the angles of a parallelogram are not all equal, prove that the diagonal which joins the acute angles is longer than the other.

17. On opposite sides AB, CD of a square ABCD equilateral $\triangle$s ABE, CDF are described outside the square. Prove that ED and BF are ||.

18. On the sides of a parallelogram ABCD and outside the parallelogram equilateral $\triangle$s are described whose other vertices are E, F, G, H. Prove that EFGH is a parallelogram.

19. Prove that if two pairs of tram-lines of the same gauge cross, the figure formed by their intersections is a rhombus.

20. O is the point of intersection of the diagonals AOC, BOD of a parallelogram ABCD ; EOF is a st. line drawn through O meeting AB at E and CD at F. Show that the $\triangle$s BOE, DOF are congruent ; and hence that EOF divides the area of the parallelogram into two equal parts.

21. In a trapezium ABCD the longest side AD is double each of the other sides (which are equal). If O is the mid-point of AD, show that the $\triangle$OBC is equilateral.

22. AB, BC, CD are rigid rods, freely jointed at B and C. AB = DC and BC = AD. AB and CD can turn freely about A and D so that A, B, C, D remain in one plane. PB and PC are two other rods jointed to B and C. Prove that, as the rods AB and DC turn about A and D, P traces out a circle.

THEOREM 22

The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.

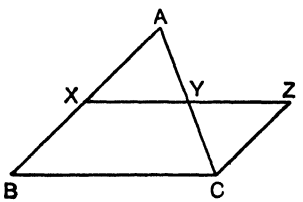


FIG. 101.

Let ABC be a triangle, and let AB be bisected at X.

Through X let a straight line be drawn, parallel to BC, to meet AC at Y.

It is required to prove that AY = CY.

Construction. Through C draw a straight line parallel to BA to meet XY produced at Z.

Proof. It is given that XZ is parallel to BC, and, by construction, CZ is parallel to BA, i.e. BX ;

$\therefore$ BCZX is a parallelogram ;

$\therefore$ its opposite sides BX, CZ are equal.

But it is given that BX = AX ;

$\therefore$ CZ = AX ,

$\therefore$ In the $\triangle$s AXY, CZY,

AX = CZ (*proved*),

$\angle$XAY = $\angle$ZCY (*alternate angles* ; XA, CZ $\|$),

$\angle$AXY = $\angle$CZY (*alternate angles* ; XA, CZ $\|$) ;

$\therefore$ the $\triangle$s $\begin{matrix}\text{AXY}\\ \text{CZY}\end{matrix}$ are congruent (*2 angles*, **corresponding** *side*),

$\therefore$ AY = CY.

THEOREM 23

The straight line joining the middle points of two sides of a triangle is parallel to the third side, and equal to one-half of it.

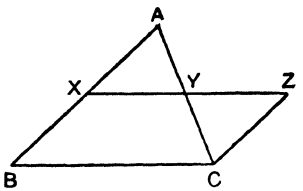


FIG. 102.

Let ABC be a triangle, and let AB be bisected at X and AC at Y.

It is required to prove that XY is parallel to BC and equal to half BC.

Construction. Through C draw a straight line parallel to BA to meet XY produced at Z.

Proof. In the △s AXY, CZY,

$$AY = CY \text{ (\textit{given})},$$

$$\angle XAY = \angle ZCY \text{ (\textit{alternate angles} ; CZ, XA } \|),$$

$$\angle AXY = \angle CZY \text{ (\textit{alternate angles} ; CZ, XA } \|) ;$$

$\therefore$ the $\triangle s \begin{matrix} AXY \\ CZY \end{matrix}$ are congruent (*2 angles*, **corresponding** side),

$$\therefore AX = CZ \text{ and } XY = ZY.$$

But it is given that $BX = AX$;

$\therefore BX = CZ$, and, by construction, BX is parallel to CZ,

$\therefore$ BXZC is a parallelogram,

$\therefore$ XZ (i.e. XY) is parallel to BC.

Also $XZ = BC$ (*opposite sides of the parallelogram* BXZC).

But $XY = YZ = \frac{1}{2}XZ$ (*proved*),

$$\therefore XY = \tfrac{1}{2}BC,$$

i.e. XY is parallel to BC, and equal to one-half of it.

THEOREM 24

If three or more parallel straight lines make equal intercepts on any transversal, they make equal intercepts on any other transversal.

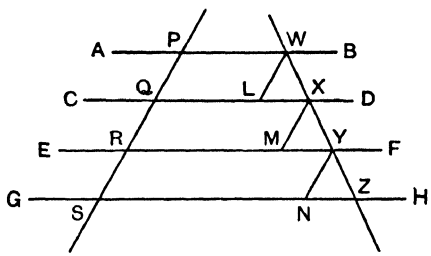


FIG. 103.

Let tne parallels AB, CD, EF, GH cut off equal intercepts PQ, QR, RS from the transversal PQRS ; and let WX, XY, YZ be the corresponding intercepts cut off from any other transversal WXYZ.

It is required to prove that WX = XY = YZ.

Construction. Through W, X, Y let straight lines be drawn parallel to PQRS to meet QX, RY, SZ at L, M, N respectively.

Proof. Since WL, XM, YN are each drawn parallel to PQRS, they are parallel to each other.

Also the figures PWLQ, QXMR, RYNS are, by construction, parallelograms,

$\therefore$ WL = PQ, XM = QR, YN = RS (*opposite sides of a parallelogram*).

But, it is given that PQ = QR = RS,

$\therefore$ WL = XM = YN ;

$\therefore$ in the $\triangle$s WLX, XMY, YNZ,

WL = XM = YN (*proved*),

$\angle$LWX = $\angle$MXY = $\angle$NYZ (*corresponding* $\angle$s ; WL, XM, YN $\|$),

$\angle$WXL = $\angle$XYM = $\angle$YZN (*corresponding* $\angle$s ; LX, MY, NZ $\|$);

$\therefore$ the $\triangle$s $\begin{matrix}\text{WLX}\\ \text{XMY}\\ \text{YNZ}\end{matrix}$ are congruent (2 *angles*, **corresponding** *side*) ;

$\therefore$ WX = XY = YZ.

Note 1. If, in the above figure, PQ = RS and QR is not equal to them, it is proved above that WX = YZ.

Note 2. This theorem may be quoted briefly as "The Intercept Theorem."

Corollary. In a △ABC, if a set of lines PX, QY, RZ, ... drawn ‖ to the base, divide one side AB into equal parts, they also divide the other side AC into equal parts. Also

$$PX = \frac{1}{n+1}BC,\ QY = \frac{2}{n+1}BC,\ RZ = \frac{3}{n+1}BC,\ \ldots,$$

where n is the number of lines.

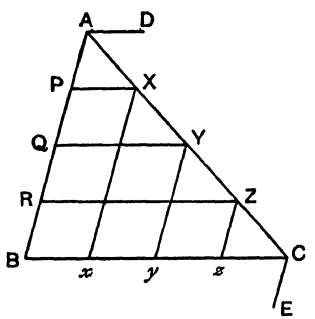


FIG. 104.

Construction. Suppose that there are 3 lines. Through X, Y, Z draw straight lines ‖ to AB, meeting BC at x, y, z. Draw AD ‖ BC and CE ‖ AB.

Proof. Since AD, PX, QY, RZ are, by construction, ‖ to BC;

∴ AD, PX, QY, RZ, BC are ‖ to one another.

But, by construction, these parallels make equal intercepts on AB; ∴ they also make equal intercepts on AC;

∴ AX = XY = YZ = ZC.

Also, since Xx, Yy, Zz, CE are, by construction, ‖ to AB;

∴ AB, Xx, Yy, Zz, CE are ‖ to one another.

But it has been proved that these parallels make equal intercepts on AC; ∴ they also make equal intercepts on BC;

∴ Bx = xy = yz = zC.

But PXxB, QYyB, RZzB are, by construction, parallelograms ;

$\therefore$ PX = the opposite side Bx = $\frac{1}{4}$BC.

Similarly QY = By = $\frac{2}{4}$BC.

RZ = Bz = $\frac{3}{4}$BC.

EXERCISE 19 [Theorems 22-24]

1. A, B are two points respectively 1″ and 1½″ from the st. line PQ. Find the distance of the mid-point of AB from PQ, (i) if A and B are on the same side of PQ, (ii) if they are on opposite sides of it.

2. ABCD is a parallelogram and through B a st. line XY is drawn outside the figure. The ⊥s from A and from the mid-point of DB to XY are 1½″ and 2¼″ respectively. Calculate the ⊥s from D and C to XY.

3. ABC is a △, F is the mid-point of AB, E the mid-point of AC. X is any point on BC. Prove that AX is bisected by EF.

4. D is the mid-point of the side BC of a △ABC, and E the mid-point of AD ; BE produced meets AC at F, and DG, drawn ‖ to BE, meets AC at G. Prove that AC is divided into three equal parts at F and G.

5. The base BC of a △ABC is bisected at D. A st. line through A lies outside the △, and L, M, N are the feet of the ⊥s drawn to this line from B, C, D respectively. Prove that DL = DM.

6. ABCD is a parallelogram, and E, F are the mid-points of the opposite sides AD, BC. Prove that BE and DF trisect the diagonal AC.

7. Prove that the quadrilateral formed by joining the mid-points of the sides of a rectangle is a rhombus.

8. D, E, F are the mid-points of the sides BC, CA, AB of a △ABC. Prove that AD bisects EF.

9. Prove that, if the diagonals of a quadrilateral are equal, then the figure formed by joining the mid-points of its adjacent sides is a rhombus.

10. ABCD is a parallelogram with AB bisected at E ; BF ‖ to ED meets AD produced at F. Prove that BC = DF.

11. O is the middle point of a st. line AB, and C is any point in AB produced. Prove that the ⊥ from O on any line through C is half the sum of the ⊥s from A and B on the same line.

12. Through C, the mid-point of the base AB of a △OAB, a st. line PCQ is drawn, meeting OA at P and OB produced at Q. Prove that CQ > CP.

13. ABC is a Δ ; D, E are points on AB, AC such that $AD = \frac{1}{4}AB$, $AE = \frac{1}{4}AC$. Prove that $DE = \frac{1}{4}BC$.

14. ABC, DBC are two Δs on the same base BC and on the same side of it, AD not being $\parallel$ to BC. P, Q, R, S are the mid-points of AB, AC, BD, CD respectively. Prove that PS, QR bisect each other.

EXERCISE 20

[Further Examples on Theorems 22-24]

1. A, B, C, D are the angular points of a square, whose centre is O. Perpendiculars OL, AP, BQ, CR are drawn from O, A, B, C upon any line through D, not lying inside the square. Prove that $AP + BQ + CR = 4OL$.

2. ABCD is a rhombus, and the side BA is produced to E so that $AE = BA$. EC is joined. Prove that EC bisects AD and divides the diagonal BD into two parts, one of which is double the other.

3. AD is a median of a ΔABC ; PQ is any line $\perp$ to AD cutting lines through B, C parallel to AD at P and Q respectively. Prove that $\angle PAQ$ is bisected by AD.

4. Prove that the mid-points of the sides of the quadrilateral ABCD are the corners of a parallelogram. Show that the result is true whether A, B, C, D are co-planar or not.

5. D and E are the mid-points of the sides AB and AC of the ΔABC. F is a point on BC produced, such that $CF = \frac{1}{2}BC$; DF cuts AC at G. Prove that $AG = 3GC$.

6. Show that the st. lines which join the mid-points of opposite sides of a quad. ABCD, bisect one another. Is the result true if A, B, C, D are not co-planar ?

7. ABCD is a tetrahedron, and L, M, N, P are the mid-points of the edges AB, BD, CD, AC respectively. Prove that the figure LMNP is a parallelogram.

8. In a given ΔABC, A is joined to the mid-point M of BC, and B is joined to a point N in AC such that $AN = \frac{1}{4}AC$. AM, BN meet at O. Prove, by drawing a st. line through N $\parallel$ to AM, that $ON = \frac{1}{5}BN$.

9. ABCD is a trapezium of which the non-parallel sides AD and BC are bisected respectively at E and F. Prove that EF is $\parallel$ to AB and DC and equal to half their sum.

10. On the side AB of a parallelogram ABCD any point P is taken, on the side CD a point Q, and on the diagonal AC a point R, such that $BP = CQ = CR$. Show (i) that the locus of the mid-point M of QR is the bisector of the $\angle ACD$; (ii) that the locus of the mid-point N of PR is the line $\parallel$ to CM drawn through the mid-point of BC.

11. D, E, F are the mid-points of the sides BC, CA, AB of the $\triangle ABC$; BE, DF meet at P; CF, DE meet at Q. Prove that (i) $PQ = \frac{1}{4}BC$, (ii) PQ is $\parallel$ to BC.

12. ABCD is a quad. having AB = CD. E, F, G, H are the mid-points of AD, AC, BD, BC; prove that EH is $\perp$ to FG.

CONSTRUCTION 12

To divide a given straight line into a given number of equal parts.

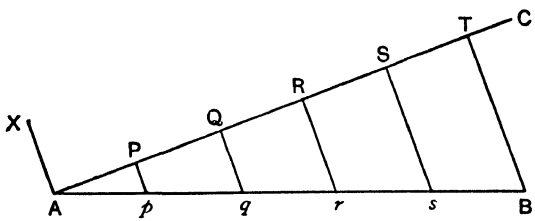


FIG. 105.

Let AB be the given straight line.

It is required to divide it into a given number of equal parts (say five).

Construction. From A draw AC, a straight line of unlimited length, making any convenient angle with AB.

From AC mark off five equal lengths, AP, PQ, QR, RS, ST.

Join TB.

Through P, Q, R, S draw straight lines $\parallel$ to TB, meeting AB at p, q, r, s.

Then Ap, pq, qr, rs, sB are the required parts.

Proof. Through A draw AX $\parallel$ to BT.

Then, by construction, XA, Pp, Qq, Rr, Ss, are $\parallel$ to TB;

$\therefore$ XA, Pp, Qq, Rr, Ss, TB are $\parallel$ to one another.

But, by construction, these parallels make equal intercepts on AC;

$\therefore$ they also make equal intercepts on AB;

$\therefore$ AB is divided into five equal parts at p, q, r, s.

The Diagonal Scale. The figure below is a diagonal scale showing inches, tenths and hundredths of an inch.

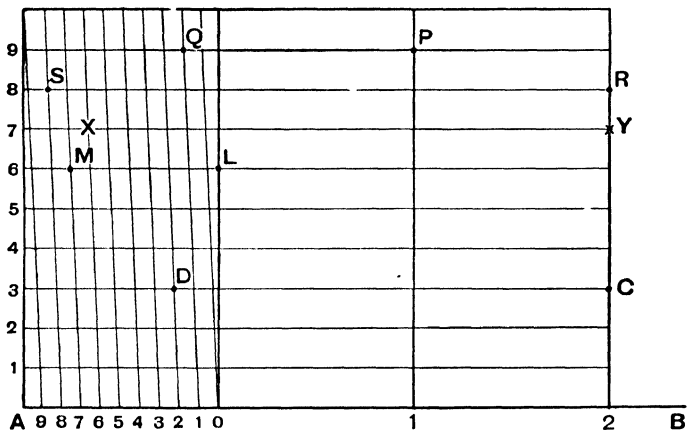


FIG. 106

A straight line AB is divided from A into inches, and the points of division are marked 0, 1, 2, ... The first division A0 is divided into tenths of an inch and the points of division are numbered (along A0) 9, 8, 7, 6, 5, 4, 3, 2, 1. It is now possible to read on AB inches and tenths of an inch.

In order to read hundredths of an inch, ten lines are taken at any equal intervals || to AB; and perpendiculars are drawn through A, 0, 1, 2 ... .

The first division on the tenth parallel is then divided into ten equal parts; and diagonal lines are drawn, as in the figure. The scale is now complete.

Example. Obtain from the scale a length of 2·67″. Place one point of the dividers at 2 in AB. Move this point along the perpendicular to AB through 2 until it reaches the seventh parallel. Extend the dividers until the other point reaches the diagonal which starts from 6 in A0. The distance XY between the points of the dividers is then 2·67″.

The proof depends upon the corollary to Theorem 24.

EXERCISE 21

[Construction 12 and the Diagonal Scale]

1. Draw a line 2·6″ long and divide it into seven equal parts.

2. Read off the lengths from C to D, L to M, P to Q, R to S, points marked on the diagram of the diagonal scale.

3. Draw a line 3·67″ long and divide it into five equal parts.

4. A distance of 100 miles is represented on a map by 1″. Draw lines to represent 234 miles and 447 miles.

CONSTRUCTION OF QUADRILATERALS AND POLYGONS

The construction of polygons usually depends upon the construction of triangles. In all cases the pupil should draw a neat freehand sketch and insert in it the magnitudes given in the question. It will usually be seen that a start can be made by constructing a triangle, after which it is easy to complete the figure. In constructing parallelograms it is frequently necessary to make use of the properties of parallelograms. In constructing a trapezium it is usually necessary to regard the trapezium as consisting of a triangle and a parallelogram.

Example 1. Construct a quadrilateral ABCD, having BC = 2·5″, CD = 2·1″, DA = 1·4″, BD = 2·3″, AC = 2·6″.

Draw the rough sketch.

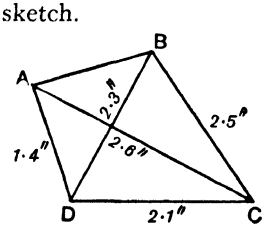


FIG. 107

It will then be seen that the 3 sides of the △s ADC, BDC are known. Construct these △s on the base DC = 2·1″ [as in Construction 7].

Join AB, and the quadrilateral is drawn.

Example 2. Construct a parallelogram ABCD, having CD = 3″, CA = 4″, DB = 5″.

Draw the rough sketch and let the diagonals AC, BD meet at O.

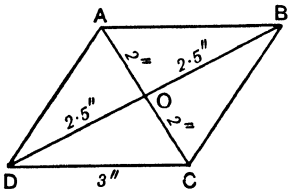


FIG 108

Since the diagonals of a parallelogram bisect one another, CO = 2″, DO = 2·5″;

∴ The sides of the △DOC are known.

Construct the △DOC [as in Construction 7]. Produce DO to B, making DB = 5″; produce CO to A, making CA = 4″.

Join DA, AB, BC.

Example 3. Construct a trapezium ABCD, having the parallel sides AB, DC equal to 4″ and 5·2″ respectively, AD = 2·8″, BC = 2·2″.

Draw the rough sketch, and draw AE || to BC to meet DC at E.

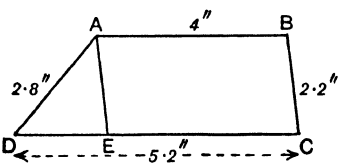


FIG. 109.

Then ABCE is a parallelogram;

∴ AE = 2·2″, EC = 4″.

∴ DE = 1·2″.

∴ △ADE can be constructed (Construction 7).

The remainder of the construction is obvious.

EXERCISE 22

[Constructions of Quadrilaterals and Polygons]

In Examples 1-24 construct quadrilaterals to the given measurements ; the order of the letters is in each case A, B, C, D.

1. AB = 2·4″, BC = 2·2″, CD = 3·5″, DA = 1·7″, BD = 3·1″.

2. AB = 6·2 cm., CD = 6·4 cm., BC = 3·6 cm., AC = 4·2 cm., AD = 3·4 cm.

3. AB = CD = 2·6 cm., BC = AD = 1·6 cm., BD = 3·6 cm.

4. AC = 2·9″, BD = 2·5″, AB = 1·9″, BC = 2·75″, CD = 2·25″.

5. AC = 5·6 cm., BD = 5·4 cm., AB = 3·3 cm., AD = 4·7 cm., CD = 3·8 cm.

6. AC = 3 cm., BD = 2·9 cm., AB = 1·7 cm., AD = 2·6 cm., BC = 2·1 cm.

7. AB = 3·7″, BC = 2·8″, CD = 2·2″, DA = 4·7″, ∠A = 63°.

8. AB = 5·3 cm., BC = 6·2 cm., CD = 3·3 cm. DA = 4·1 cm., ∠C = 117°.

9. AB = 5·3 cm., ∠B = 85°, BC = 7·2 cm., ∠C = 93°, CD = 6·5 cm.

10. AB = 1·7″, BC = 1·1″, AD = 1·55″, ∠A = 72°, ∠B = 84°.

11. AB = 2·6′, BC = 3·1″, CD = 2·3″, ∠B = 71°, ∠C = 54°.

12. AC = 4·2″, BD = 4·5″, AB = 3·3″, ∠A = 88°, ∠B = 93°.

13. AC = 5·9 cm., BD = 7·2 cm., CD = 5·1 cm., ∠C = 117°, ∠D = 67°.

14. AC = 2·97″, BD = 4·14″, BC = 2·73″, ∠B = 71°, ∠C = 109°.

15. ∠B = 114°, BC = 1·3″, ∠C = 97°, CD = 1·8″, ∠D = 93°.

16. ∠A = 42°, ∠B = 123°, ∠D = 54°, AB = 2·7″, AD = 4·3″.

17. ∠A = 123°, ∠B = 127°, ∠C = 63°, AB = 3·1 cm., BC = 2·8 cm.

18. AD = 6 cm., ∠D = 116°, ∠DAC = 28°, ∠BAC = 37°, AB = 4·7 cm.

19. AC = 2·8″, ∠BAC = 56°, ∠DAC = 69°, ∠BCA = 56°, ∠DCA = 42°.

20. AB = 2·8″, BD = 2·6″, CD = 2·1″. ∠ABD = 109°, ∠BDC = 21°.

21. AB = 3·7 cm., BC = 6·5 cm., AD = 5·3 cm., ∠C = 94°, ∠DBC = 26°.

22. BC = 3·8 cm., CD = 4·2 cm., BD = 2·3 cm., ∠ABD = 63°, ∠ADB = 58°.

23. AB = 2·7″, ∠BAC = 45°, ∠A = 85°, ∠B = 100°, ∠ABD = 35°.

24. AB = 2·9″, BC = 3″, CD = 1·57″, DA = 1·68″, AC = 2·67″.

In Examples 25-34 construct trapezia to the given measurements; the order of the letters is in each case A, B, C, D and AB, DC are the parallel sides.

25. AB = 1·8″, BC = 1·5″, CD = 2·5″, DA = 1·3″.

26. AB = 9 cm., CD = 5 cm., ∠DAB = 77°, ∠ABC = 56°.

27. ∠BAD = 135°, ∠ACD = 60°, DC = 2″, DB = 2·5″.

28. AB = 4·8 cm., CD = 7·6 cm., ∠ADC = 70°, ∠DCB = 30°.

29. AB = 10 cm., BC = 4 cm., CD = 7 cm., DA = 6 cm.

30. AB = 6·4 cm., BC = 2·9 cm., CD = 9·3 cm., ∠BCD = 62°.

31. AB = 6·4 cm., BC = 2·9 cm., CD = 2·3 cm., ∠BDC = 62°.

32. AB = 6·4 cm., BC = 9 cm., CD = 9·3 cm., ∠BDC = 62°.

33. AB = 3·2″, ∠A = 117°, ∠B = 108°, AC = 4·9″.

34. AB = 1·75″, BD = 2·34″, DC = 2·39″, CA = 2·47″

[Draw CE parallel to DB to meet AB produced at E. First construct △CAE.]

In Examples 35-40 construct parallelograms to the given measurements; the order of the letters is in each case A, B, C, D.

35. AD = 2·1″, DC = 2·9″, ∠D = 43°.

36. AB = 4·3 cm., BC = 3·2 cm., ∠C = 112°.

37. AD = 2·1″, DC = 2·9″, AC = 1·7″.

38. AD = 2·1″, DC = 2·8″, DB = 3·5″.

39. AC = 3·4 , DB = 3·8″, AD = 3·2″.

40. AC = 4·2 cm., DB = 5·8 cm., AB = 4·6 cm.

41. Construct a square on a side 2·32″ long.

42. Construct a rhombus ABCD, having AB = 2·3″, AC = 3·7″.

43. Construct pentagons ABCDE to the following measurements:

(i) AB = 1·2″, BC = 1·4″, CD = 2·6″, DE = 1·9″, EA = 1·3″, ∠ABC = 140°, ∠BAE = 124°.

(ii) AB = 3 cm., BC = 3·2 cm., CD = 1·9 cm., DE = 4·2 cm., EA = 2·4 cm., AC = 5·7 cm., BE = 3·8 cm.

(iii) AB = 2·5″, BC = 1·85″, CD = 1·8″, DE = 2·15″, EA = 1·9″, AC = 3·2″, AD = 3·35″.

(iv) AC = BD = CE = DA = EB = 2″; AB = CD = 1″.

44. ABC is an isosceles △ in which AB = AC; XY is drawn ‖ to the base BC, at a distance 1″ from it, cutting AB at X and AC at Y. Given that BC = 4″ and XY = 2·5″, construct the △ABC.

45. Draw a line AC 3·75″ long, and divide it into seven equal parts by a geometrical construction. Construct a quadrilateral ABCD of which the above line is one diagonal, from the following data. BD cuts AC at O so that AO is three-sevenths of AC. AB = 2·2″, ∠AOB = 49°, ∠BCD = 57°.

46. Construct a quadrilateral ABCD in which AB = 2·2″, ∠A = 30°, ∠C = 80°, ∠ADB = 65°, ∠BDC = 15°.

47. Construct a trapezium in which AB = BC = DA = 1″ and DC = 2″.

48. Construct the quadrilateral ABCD from the following data: AB = 7·0 cm., BC = 9·4 cm., AC = 11·0 cm., ∠ACD = 47°, AD = 9·1 cm. Show that there are two solutions, and measure the length of the diagonal BD in each quadrilateral. If the first four data are as before, but AD is made as short as possible, find by measurement this shortest length and also the length of BD for this case.

49. Draw a quadrilateral ABCD, symmetrical about the diagonal BD, which is to be 3″ long. Make

$$\angle ABC = 140°, \qquad \angle A = \angle C = 65°.$$

50. In a circle, centre O, radius 2″, inscribe a quadrilateral ABCD having BC = CD, the ∠ABC = 100°, and the ∠AOB = 50°.

EXERCISE 23

[Constructions of Triangles and Quadrilaterals. In each example give a purely geometrical construction, if possible.]

[These examples should be postponed until a late stage in the School Certificate course.]

1. Construct an isosceles Δ, given the base and the altitude.

2. Construct an isosceles Δ, given the base and the vertical angle.

3. Construct an isosceles Δ, given the altitude and the vertical angle.

4. Construct an isosceles Δ, given the altitude and the base angles.

5. Construct a ΔABC, given AB, AC and the $\perp$ from A to BC.

6. Construct a ΔABC, given AB, BC and the $\perp$ from A to BC.

7. Construct a ΔABC, given $\angle$s ABC, ACB and the $\perp$ from A to BC.

8. Construct a ΔABC, given $\angle$s ABC, BAC and the $\perp$ from A to BC.

9. AB, CD are two parallel st. lines. O is any point. Through O draw a st. line OPQ meeting AB, CD at P, Q, so that PQ shall be equal to a given st. line. When is the construction impossible ?

10. ABC is a Δ. Construct a rhombus APQR, P being on AB, Q on BC, and R on CA.

11. Draw a line parallel to a given st. line at any given distance.

12. (i) P is a point within the arm of the $\angle AOB$. Through P draw a st. line terminated by OA, OB and bisected at P. (ii) OA, OB, OC are three st. lines meeting at O. Draw a st. line terminated by OA and OC, and bisected by OB.

13. Construct a parallelogram, given the lengths of two adjacent sides and the $\perp$ distance between one pair of opposite sides.

14. Construct a parallelogram, given one angle and the $\perp$ distances between pairs of opposite sides.

15. Is it always possible to construct a trapezium whose parallel sides are a, b, and whose other sides are c, d ?

16. Prove the correctness of the following alternative to Construction 12:

"Let AB be the given st. line, which is to be divided into five equal parts. From A draw AC at any angle with AB, and on it mark off four equal parts AP, PQ, QR, RS. From B draw BD || to CA, and on it mark off BE, EF, FG, GH, each equal to the parts marked on AC. Join ES, FR, GQ, HP, meeting AB at X, Y, Z, W. Then AB is divided into five equal parts at these points."

17. Construct a △ABC, given ∠B, ∠C, and the perimeter.

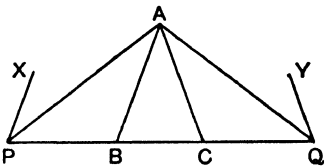


FIG. 110

[Draw PQ equal to the perimeter. At P make ∠QPX = ∠B. At Q make ∠PQY = ∠C; X, Y, being on the same side of PQ. Draw the bisectors of the ∠s QPX, PQY, meeting at A. Through A draw AB || XP and AC || YQ to meet PQ at B, C. Then ABC is the required △. Supply the proof.]

18. Construct a △ABC, given the perimeter, the altitude, and one base angle.

19. Construct a △ABC, given ∠A, AB, CA + CB.

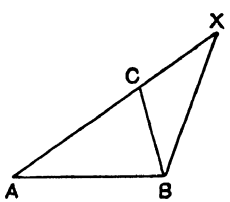


FIG. 111.

[Draw the given base AB. At A make ∠BAX = ∠A and

$$AX = CA + CB.$$

Join BX. Let the ⊥ bisector of BX meet AX at C. Then ABC is the required △. Supply the proof.]

20. Construct a △ABC, given ∠A, AB, CA − CB.

21. Construct a $\triangle ABC$, given $\angle A$, BC, CA + AB.

[In the figure of Th. 16a, $\angle BDC = \frac{1}{2}A$; $\therefore$ $\triangle BDC$ can be constructed. Also A lies on the $\perp$ bisector of CD. Supply the proof.]

22. Construct a $\triangle ABC$, given $\angle A$, BC, AB − AC.

[In the figure of Th. 15, $\angle BDC = 90 + \frac{A}{2}$; $\therefore$ $\triangle BDC$ can be constructed, etc.]

23. Construct a $\triangle ABC$, given BC, AB − AC, $\angle C - \angle B$.

[In the figure of Th. 15, $BD = AB - AC$; $\angle BCD = \frac{1}{2}(\angle C - \angle B)$; $\therefore$ $\triangle BDC$ can be constructed, etc.]

24. Construct a $\triangle ABC$, given BC, AB − AC, $\angle B + \angle C$.

[In the figure of Th. 15, $BD = AB - AC$, $\angle ADC = \frac{1}{2}(C + B)$; $\therefore$ $\angle BDC$ is known; $\therefore$ $\triangle BDC$ can be constructed, etc.]

25. Construct a $\triangle ABC$, given BC, AC + AB, $\angle C - \angle B$.

[In the figure of Th. 16a, $\angle BCD = 90° + \frac{1}{2}(\angle C - \angle B)$; $\therefore$ the $\triangle BDC$ can be constructed, etc.]

26. Construct a $\triangle ABC$, given BC, $\angle A$, , $\angle C - \angle B$.

[In the figure of Th. 15, $\angle BCD = \frac{1}{2}(\angle C - \angle B)$ and

$$\angle BDC = 90° + \tfrac{1}{2}\angle A;$$

$\therefore$ $\triangle BDC$ can be constructed, etc.]

27. P, Q, are two fixed points on the same side of a straight line AB; find a point X in AB, such that PX, QX are equally inclined to AB. [See Ex. 15, No. 12.]

28. Construct the quadrilateral ABCD, having given the lengths of the sides and the angle between AB and CD.

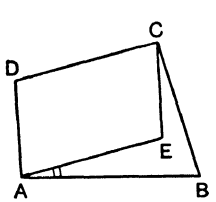


FIG. 12

[Draw the parallelogram ADCE. Join BE. $\triangle BAE$ can be constructed. Then $\triangle BCE$, etc.]

29. Construct the quadrilateral ABCD, having given the lengths of BD, AC, AB, CD and the angle between AB and CD.

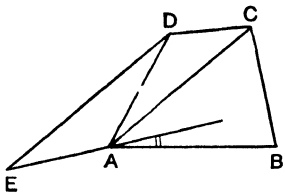


FIG. 113.

[Draw the parallelogram ACDE and proceed as before, starting from $\triangle BAE$.]

30. Construct the quadrilateral ABCD, having given BC, AD, AC, BD and the angle between AC and BD.

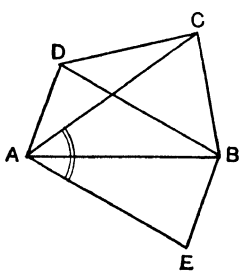


FIG. 114.

[Draw the parallelogram ADBE and proceed as before, starting from the $\triangle ACE$.]

THE CONCURRENCE OF CERTAIN STRAIGHT LINES IN A TRIANGLE

[This section should be omitted in a first reading of the book.]

THEOREM A

The perpendicular bisectors of the three sides of a triangle meet at a point.

Let ABC be a $\triangle$, and let XO, YO be the $\perp$ bisectors of BC, CA, meeting at O.

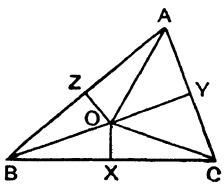


FIG. 115.

Let Z be the mid-point of AB. Join OZ. It is required to prove that OZ is $\perp$ to AB.

Construction. Join AO, BO, CO.

Proof. In the $\triangle$s BXO, CXO,

$$BX = CX \text{ (\textit{given})},$$
$$XO = XO \text{ (\textit{common})},$$
$$\angle BXO = \angle CXO \text{ (\textit{given}, \textit{rt.} } \angle \text{s)};$$
$$\therefore \text{ the } \triangle \text{s } \begin{matrix} BXO \\ CXO \end{matrix} \text{ are congruent (2 \textit{sides}, \textbf{included} } \angle),$$
$$\therefore BO = CO.$$

Similarly by comparing the $\triangle$s CYO, AYO, it may be proved that

$$CO = AO;$$
$$\therefore BO = AO.$$

Hence, in the $\triangle$s BZO, AZO,

$$BZ = AZ \text{ (\textit{given})},$$
$$ZO = ZO \text{ (\textit{common})},$$
$$BO = AO \text{ (\textit{proved})};$$
$$\therefore \text{ the } \triangle \text{s } \begin{matrix} BZO \\ AZO \end{matrix} \text{ are congruent (3 \textit{sides})},$$

$\therefore \angle BZO = \angle AZO$, and each angle is therefore by definition a right angle; $\therefore$ OZ is $\perp$ to AB.

Definition. Three or more straight lines are said to be **concurrent** when they meet at the same point.

Definition. Three or more points are said to be **collinear** when they are in the same straight line.

THEOREM B

The bisectors of the three angles of a triangle are concurrent.

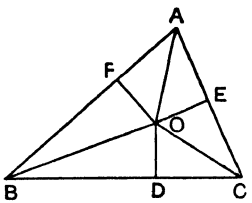


FIG. 116.

Let ABC be a $\triangle$, and let BO, CO be the bisectors of the $\angle$s ABC, ACB, meeting at O. Join OA. It is required to prove that OA bisects the $\angle$BAC.

Construction. Draw OD, OE, OF $\perp$ to BC, CA, AB respectively.

Proof. In the $\triangle$s BOD, BOF,

$$BO = BO \ (\textit{common}),$$

$$\angle OBD = \angle OBF \ (\textit{given}),$$

$$\angle ODB = \angle OFB \ (\textit{construction});$$

$\therefore$ the $\triangle$s $\begin{matrix} BOD \\ BOF \end{matrix}$ are congruent (2 $\angle$s, **corresponding** *side*),

$$\therefore \ OD = OF.$$

Similarly, by comparing the $\triangle$s COD, COE, it may be proved that

$$OD = OE,$$

$$\therefore \ OE = OF.$$

Hence, in the $\triangle$s AOE, AOF,

$$OE = OF \ (\textit{proved}),$$

$$AO = AO \ (\textit{common}),$$

$$\angle AEO = \angle AFO \ (\textit{rt. } \angle s \textit{ by construction});$$

$\therefore$ the $\triangle$s $\begin{matrix} AOE \\ AOF \end{matrix}$ are congruent (*rt.* $\angle$, *hypot.*, *side*),

$$\therefore \ \angle OAE = \angle OAF,$$

$\therefore$ OA bisects the $\angle$BAC.

THEOREM C

The three medians of a triangle are concurrent.

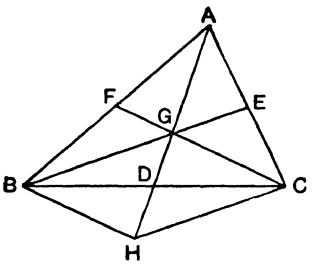


FIG. 117.

Let ABC be a △, and let E, F be the mid-points of AC, AB. Let the straight lines BE, CF meet at G.

Join AG and produce it to cut BC at D.

It is required to prove that BD = DC.

Construction. Produce AG to H, making GH = AG. Join BH, CH.

Proof. In the △ ABH, FG is, by construction, the line joining the mid-points of the sides AB, AH ;

∴ it is || to the base BH ; ∴ FGC is || to BH.

Similarly, in the △AHC, the line EGB may be proved to be || to CH,

∴ Both pairs of opposite sides of the figure BGCH are parallel ;

∴ BGCH is a parallelogram.

But the diagonals of a parallelogram bisect one another ;

∴ BD = DC.

Definition. The point of intersection of the medians is called the **centroid** of the triangle.

Corollary. The centroid is a third of the way up the median DA, measured from D.

For, in the above figure, AG = GH, by construction.

But, since the diagonals of a parallelogram bisect one another ;

∴ GD = DH, i.e. GD = $\frac{1}{2}$GH,

∴ GD = $\frac{1}{2}$AG, i.e. GD = $\frac{1}{3}$AD.

Similarly, it may be proved that GE = $\frac{1}{3}$BE, GF = $\frac{1}{3}$CF.

THEOREM D

The perpendiculars, drawn from the vertices of a triangle to the opposite sides, are concurrent.

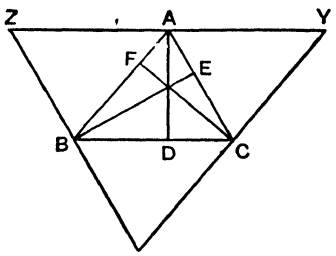


FIG. 118.

Let ABC be a $\triangle$, and let AD, BE, CF be the $\perp$s from A, B, C to BC, CA, AB respectively.

It is required to prove that AD, BE, CF are concurrent.

Construction. Through A, B, C draw straight lines $\parallel$ to BC, CA, AB respectively, forming the $\triangle XYZ$.

Proof. By construction, XBAC is a parallelogram,

$\therefore$ the opposite sides are equal ; $\therefore$ BX = AC ;

Also, by construction, ZACB is a parallelogram,

$\therefore$ the opposite sides are equal ; $\therefore$ ZB = AC ;

$\therefore$ ZB = BX.

Also AC is $\parallel$ to ZX and BE is $\perp$ to AC ;

$\therefore$ BE is $\perp$ to ZX,

$\therefore$ BE is the $\perp$ bisector of ZX.

Similarly AD, CF are the $\perp$ bisectors of YZ, XY.

But the $\perp$ bisectors of the sides of the $\triangle XYZ$ are concurrent,

AD, BE, CF are concurrent.

Definition. The point of concurrence of the perpendiculars, drawn from the vertices of a triangle to the opposite sides, is called the **orthocentre** of the triangle.

The following theorems are still occasionally set in some examinations.

* THEOREM E

If two triangles have two sides of the one respectively equal to two sides of the other but the included angles unequal, then the triangle with the greater included angle has also the greater side.

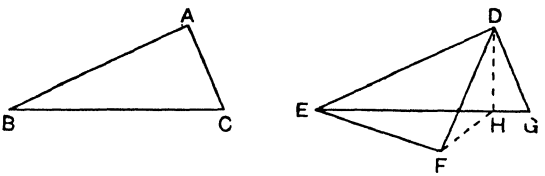


FIG. 119.

Let ABC, DEF be two triangles in which AB = DE, AC = DF, but ∠BAC is greater than ∠EDF.

It is required to prove that BC is greater than EF.

Proof. Construct the △DEG, on the same side of ED as F, having DG = AC and ∠EDG = ∠BAC.

Then, since ∠EDG > ∠EDF, DF falls within the ∠EDG.

Also the △s ABC, DEG are congruent (2 *sides*, **included** ∠),

$$\therefore \text{ BC} = \text{EG}.$$

Let the bisector of the ∠FDG meet EG at H. Join FH.

Then, in the △s DFH, DGH,

$$\text{DF} = \text{DG} \ (\textit{both equal to } \text{AC}),$$

$$\text{DH} = \text{DH} \ (\textit{common}),$$

$$\angle\text{FDH} = \angle\text{GDH} \ (\textit{construction});$$

$$\therefore \text{ the } \triangle\text{s } \begin{matrix}\text{DFH}\\ \text{DGH}\end{matrix} \text{ are congruent (2 } \textit{sides}, \textbf{ included } \angle),$$

$$\therefore \text{ FH} = \text{GH}.$$

But BC = EG, ∴ BC = EG = EH + HG = EH + HF.

But EH + HF > EF, for (i) if F is not on EG, EH and HF are two sides of the △EHF and are therefore together greater than the third side EF; (ii) if F is on EG, EH alone is greater than EF, for the order of the points must be E, F, H, G;

$$\therefore \text{ BC is greater than EF}.$$

* THEOREM F

If two triangles have two sides of the one respectively equal to two sides of the other, but their bases are unequal, then the triangle with the greater base has also the greater included angle.

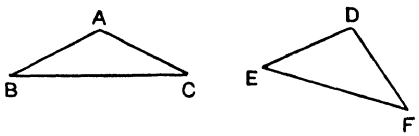


FIG. 120.

Let ABC, DEF be two triangles in which AB = DE, AC = DF, but BC is greater than EF. It is required to prove that ∠BAC is greater than ∠EDF.

Proof. ∠BAC is either less than, equal to, or greater than ∠EDF,

(i) if ∠BAC is less than ∠EDF, BC is less than EF (Theorem E), which is contrary to hypothesis ;

(ii) if ∠BAC is equal to ∠EDF, BC is equal to EF

[for then $\triangle s \frac{BAC}{EDF}$ are congruent (2 *sides*, **included** ∠)],

which is contrary to hypothesis ;

∴ ∠BAC must be greater than ∠EDF.

EXERCISE 24 [Theorems A-F]

1. The sides CA, CB of a △ABC are produced. Prove that the bisectors of the interior ∠ at C and the exterior ∠s at A and B are concurrent.

2. Construct a △, having given the mid-points of the sides.

3. Construct a △, having given the centroid and two of the mid-points of the sides.

4. Construct a △, having given the lengths of two sides and of the median which bisects the third side.

5. Construct a Δ, having given the lengths of one side and of the medians which bisect the other two sides.

6. Construct a Δ, having given the lengths of the three medians.

7. Prove that any two medians are together greater than the third.

8. In the figure of Theorem C, prove that (i) $AB + AC > 2AD$, (ii) $AD + BE > \frac{3}{2}AB$.

9. In the figure of Theorem C, prove that, if $BE = CF$, then $AB = AC$.

10. In any Δ prove that the perimeter $>$ the sum of the medians $>$ three-quarters of the perimeter.

11. Prove that the medians of a Δ are also the medians of the Δ formed by joining the mid-points of the sides.

12. The medians of a ΔABC meet at G, and the parallelograms CBGH, CAGK are completed. Prove that C is the point of intersection of the medians of the ΔGHK.

13. If within a ΔPQR a point S can be found such that $PS = PQ$, prove that $QR > SR$.

14. In the ΔPQR, S is the mid-point of PQ and $PR > QR$. Prove (i) that the $\angle PSR$ is obtuse, and (ii) that if T is any point on RS, then $PT > QT$.

15. In a quadrilateral PQRS, the opposite sides PQ, RS are equal and $\angle PQR > \angle QRS$. Prove that $\angle RSP > \angle SPQ$.

16. AC is the greatest side of a ΔABC, BC the least. On AC a rectangle ACDE is described having CD equal to BC, and on AB a rectangle ABFG having BF equal to BC, both rectangles falling outside the ΔABC. Prove that $CF > BD$. (Consider separately the cases when the $\angle ABC$ is (i) acute, (ii) obtuse, (iii) right.)

PART II

AREAS AND LOCI

AREAS

1. The **area** of a figure is the amount of surface contained within its bounding lines.

2. The **unit of area** is the area of a square on a side of unit length. Thus, a **square inch** is the area of a square drawn on a side one inch in length.

* THEOREM 25

The area of a rectangle is measured by the product of the measures of its sides.

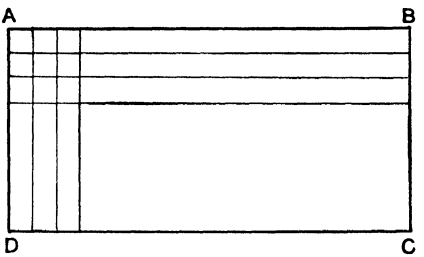


FIG. 121.

Let ABCD be a rectangle in which AB is a units long and AD is b units long, the unit being chosen so that a and b are whole numbers.†

It is required to prove that the area of ABCD is ab square units.

Proof. Let AB be divided into a equal parts and through the points of division let straight lines be drawn parallel to AD.

Then ABCD will be divided into a equal rectangles.

Let AD be divided into b equal parts and through the points of division let straight lines be drawn parallel to AB. Then each of the a equal rectangles will be divided into b squares, each of which is 1 unit of area;

$\therefore$ ABCD contains ab units of area.

Consider the application of the above theorem to a rectangle $2\frac{2}{3}$ inches long, $1\frac{1}{2}$ inches wide. In order that a and b should be whole numbers the unit of length may be taken as $\frac{1}{6}$ inch. The length of the rectangle is then 16 units, and the breadth of the rectangle is 9 units.

$\therefore$ the area of the rectangle $= 16 \times 9$ units of area.

But a unit of area is a square of side $\frac{1}{6}$ inch.

† This is not possible if a and b are incommensurable numbers. See note on p. 378.

It is easily seen that a square of side 1 inch can be divided into 6×6 small squares, each of which is a square of side $\frac{1}{6}$ inch ;

$\therefore$ 6×6 units of area $= 1$ square inch,

$\therefore$ the area of the rectangle $= \frac{16 \times 9}{6 \times 6}$ square inches

$= \frac{16}{6} \times \frac{9}{6}$ square inches

$= 2\frac{2}{3} \times 1\frac{1}{2}$ square inches.

Thus the area of a rectangle is measured by the product of the measures of its sides, whether these measures are integers or fractions.

DEFINITIONS

1. If any side of a triangle is taken as its base, the perpendicular to that side from the opposite corner is called the **altitude** or **height** of the triangle.

2. If any side of a parallelogram is taken as its base, the perpendicular distance between that side and the side parallel to it is called the **altitude** or **height** of the parallelogram.

It is clear that a triangle has three distinct altitudes (AD, BE, CF in the figures), and that a parallelogram has two distinct altitudes (XY, TW in the figure).

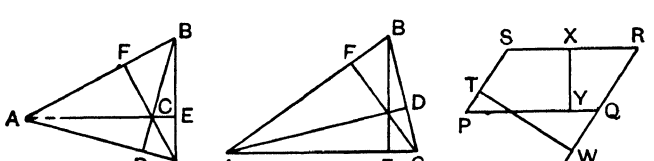


FIG. 122.

3. Parallelograms and triangles, situated as in the figures below, are said to be **between the same parallels.**

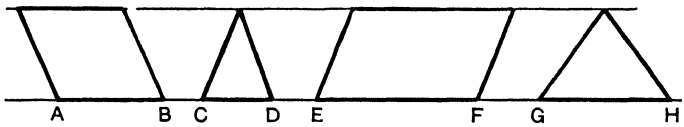


FIG. 123.

Since the perpendicular distance between two parallel straight lines is everywhere the same, these figures have the same height if AB, CD, EF, GH are taken as the bases.

4. If two figures are of equal area they are said to be **equivalent** or **equal.** It should be noticed that equal $\triangle$s are equal in area but are not necessarily congruent.

5. The rectangle ABCD is said to be **contained** by AB, AD. It may be called the figure AC or the figure BD if this is not ambiguous.

The following abbreviations are also used :

The rectangle whose adjacent sides are AB, AD is denoted by *rect.* AB, AD, or AB $\times$ AD, or AB . AD.

A square drawn on the side AB is denoted by *sq. on* AB, or AB^2.

There is so much disagreement about the treatment of the following group of theorems that it has been thought best to give two distinct sets of proofs. Either of these arrangements is generally accepted as satisfactory, and the teacher should decide which is to be used. It is strongly recommended that one or the other should be adopted **as a whole.**

THEOREM 26. FIRST METHOD

Parallelograms on the same base and between the same parallels are equal in area.

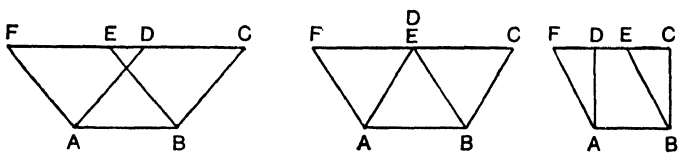


FIG. 124.

Let ABCD, ABEF be parallelograms on the same base AB and between the same parallels AB, FC.

It is required to prove that ABCD, ABEF are equal in area.

Proof. In the $\triangle$s FDA, ECB,

FA = EB (*opposite sides of a parallelogram*),

$\angle$ AFD = $\angle$ BEC (*corresponding angles* ; AF, BE $\|$),

$\angle$ ADF = $\angle$ BCE (*corresponding angles* ; AD, BC $\|$) ;

$\therefore$ the $\triangle$s $\begin{matrix}\text{FDA}\\ \text{ECB}\end{matrix}$ are congruent and therefore equal in area (2 *angles*, **corresponding** *side*).

Subtract each triangle in turn from the whole figure FABC,

$\therefore$ the remaining figures ABCD, ABEF are equal in area.

Note. It is essential to prove the equality of the parallelograms by subtracting the equal triangles from the whole figure. If we attempt to add pieces together to form the parallelograms it will be necessary to give different proofs for the different figures.

Corollary 1. The area of a parallelogram is equal to that of a rectangle whose adjacent sides are equal to the base and altitude of the parallelogram, respectively. The area of a parallelogram is therefore measured by the product of the measures of its base and its altitude.

Corollary 2. Parallelograms on equal bases and between the same parallels are equal in area.

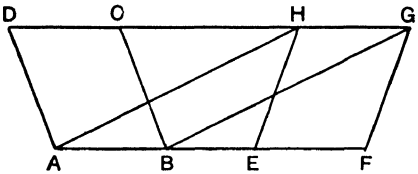


FIG. 125.

Let ABCD, EFGH be two parallelograms on equal bases AB, EF and between the same parallels DG, AF.

It is required to prove that ABCD, EFGH are equal in area.

Construction. Join AH, BG.

Proof. HG = EF (*opposite sides of a parallelogram*),

EF = AB (*given*),

$\therefore$ HG = AB.

But HG, AB are parallel,

$\therefore$ The quadrilateral ABGH has one pair of opposite sides both equal and parallel,

$\therefore$ it is a parallelogram,

$\therefore$ ABCD = ABGH (*parallelograms on the same base* AB *and between the same parallels* AB, DG),

and ABGH = EFGH (*parallelograms on the same base* HG *and between the same parallels* AF, HG),

$\therefore$ ABCD = EFGH.

Note. Theorem 26 appears in some schedules in the following form: "**Parallelograms on the same (or equal) bases and of the same altitude are equal in area.**"

If confronted with this in an examination, the candidate should first point out that the parallelograms can be placed so as to be on the same (or equal) bases and between the same parallels, and then proceed as above.

THEOREM 27. FIRST METHOD

1. Triangles on equal bases (or on the same base) and between the same parallels are equal in area.

2. Triangles of equal area which are on equal bases in the same straight line (or on the same base) and on the same side of it are between the same parallels.

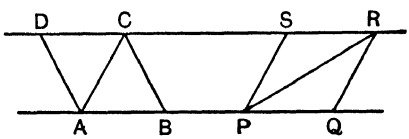


FIG. 126.

1. Let ABC, PQR be triangles on equal bases AB, PQ and between the same parallels AQ, CR.

It is required to prove that ABC, PQR are equal in area.

Construction. Through A draw a straight line parallel to BC to meet RC produced at D, and through P draw a straight line parallel to QR to meet CR at S.

Proof. Each of the figures ABCD, PQRS is, by construction, a parallelogram, and they are on equal bases AB, PQ and between the same parallels AQ, DR;

$$\therefore \text{ par}^{m} \text{ ABCD} = \text{par}^{m} \text{ PQRS};$$

But each parallelogram is bisected by a diagonal;

$$\therefore \triangle \text{ABC} = \tfrac{1}{2} \text{ par}^{m} \text{ ABCD}$$

and

$$\triangle \text{PQR} = \tfrac{1}{2} \text{ par}^{m} \text{ PQRS};$$

$$\therefore \triangle \text{ABC} = \triangle \text{PQR}.$$

In the same way it can be shown that triangles on the same base and between the same parallels are equal in area.

2. Let the triangles ABC, DEF be equal in area, and on the equal bases BC, EF, the bases being in the same straight line, and the triangles on the same side of it.

It is required to prove that AD is parallel to BF.

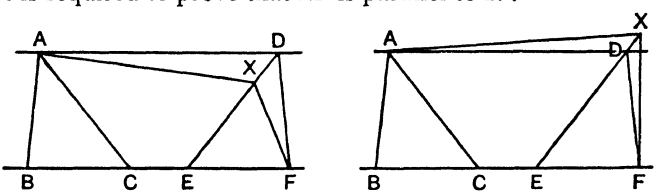


FIG 127.

Proof. Suppose that AD is not parallel to BF, and through A let a straight line be drawn parallel to BF to meet ED or ED produced at X. Join XF.

It is given that the triangles ABC, XEF are on equal bases BC, EF; and, by construction, they are between the same parallels BF, AX;

$$\therefore \triangle ABC = \triangle XEF.$$

But it is given that

$$\triangle ABC = \triangle DEF;$$

$$\therefore \triangle XEF = \triangle DEF.$$

But this is impossible, for a part cannot be equal to the whole,

$\therefore$ Our supposition that AD is not parallel to BF is wrong;

$\therefore$ AD is parallel to BF.

In the same way it can be shown that triangles of equal area which are on the same base and on the same side of it are between the same parallels.

Corollary 1. Triangles on the same or equal bases and of the same altitude are equal in area.

For they can be placed so as to be on the same base and between the same parallels.

Corollary 2. If triangles (or parallelograms) of the same area have the same or equal bases, their altitudes are equal.

For they can be placed so as to be on the same base, or on equal bases in the same straight line and on the same side of it.

Corollary 3. If AB is a fixed straight line and P moves so that the area of the △APB is constant, the locus of P consists of two straight lines parallel to AB, one on either side of it, at a fixed distance from it.

THEOREM 28. FIRST METHOD

If a parallelogram and a triangle stand on the same base, and between the same parallels, the area of the parallelogram is double that of the triangle.

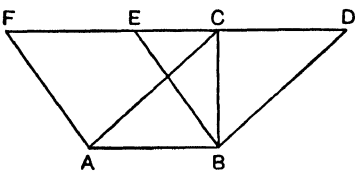


FIG. 128.

Let the parallelogram ABEF and the triangle ABC be on the same base AB and between the same parallels AB, FC.

It is required to prove that the area of ABEF is double that of ABC.

Construction. Draw a straight line through B parallel to AC to meet FC produced at D.

Proof. It is given that AB is parallel to CD, and, by construction, BD is parallel to AC ;

$\therefore$ ABDC is a parallelogram.

But BC is a diagonal, and therefore bisects the parallelogram,

$\therefore$ par^{m} ABDC $= 2\triangle$ ABC.

Also the parallelograms ABDC, ABEF are on the same base AB and between the same parallels AB, FD ;

$\therefore$ par^{m} ABDC $=$ par^{m} ABEF ;

$\therefore$ par^{m} ABEF $= 2\triangle$ ABC.

Corollary 1. If a parallelogram and a triangle are on equal bases and of the same altitude, the area of the parallelogram is double that of the triangle.

For they can be placed so as to be on the same base and between the same parallels.

Corollary 2. The area of a triangle is equal to half that of a rectangle whose adjacent sides are respectively equal to the base and altitude of the triangle. The area of a triangle is therefore measured by half the product of the measures of its base and its altitude.

THEOREM 26. SECOND METHOD

The area of a parallelogram is equal to the area of a rectangle on the same base and between the same parallels.

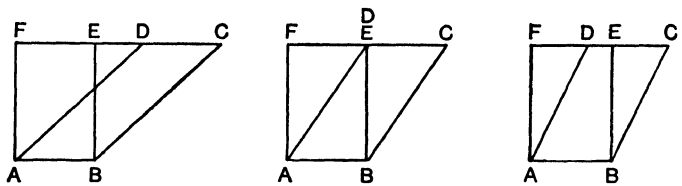


FIG. 129.

Let ABEF be a rectangle and ABCD a parallelogram on the same base AB and between the same parallels AB, FC.

It is required to prove that ABCD, ABEF are equal in area.

Proof. In the $\triangle$s FDA, ECB,

$FA = EB$ (*opposite sides of a rectangle*),

$\angle AFD = \angle BEC$ (*corresponding angles*; AF, BE $\|$),

$\angle ADF = \angle BCE$ (*corresponding angles*; AD, BC $\|$);

$\therefore$ the $\triangle$s $\begin{matrix}FDA\\ECB\end{matrix}$ are congruent, and therefore equal in area (2 *angles*, corresponding *side*).

Subtract each triangle in turn from the whole figure FABC,

$\therefore$ the remaining figures ABCD, ABEF are equal in area.

Note. It is essential to prove the equality of the parallelogram and the rectangle by subtracting the equal triangles from the whole figure. If we attempt to add pieces together to form the parallelogram and the rectangle it will be necessary to give different proofs for the different figures.

Corollary 1. The area of a parallelogram is measured by the product of the measures of its base and altitude.

For the area of

ABCD = the area of ABEF = AB × AF = base × altitude.

Corollary 2. Parallelograms on the same or equal bases and of the same altitude are equal in area.

Corollary 3. If parallelograms of the same area have the same or equal bases their altitudes are equal and they can be placed so as to be on the same base and between the same parallels.

THEOREM 27. SECOND METHOD

The area of a triangle is equal to one-half of the area of a rectangle on the same base and between the same parallels.

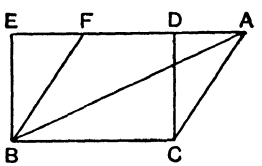


FIG. 130.

Let ABC be a triangle and BCDE a rectangle on the same base BC and between the same parallels BC, EA.

It is required to prove that the area of ABC is half that of BCDE.

Construction. Through B draw a straight line parallel to CA to meet EA at F.

Proof. It is given that FA is parallel to BC, and, by construction, BF is parallel to CA ;

$\therefore$ BCAF is a parallelogram.

But BA is a diagonal and therefore bisects the parallelogram,

$\therefore$ $\triangle ABC = \frac{1}{2}$ par^{m} BCAF.

Also the par^{m} BCAF and the rectangle BCDE are on the same base BC and between the same parallels BC, EA ;

$\therefore$ par^{m} BCAF = rectangle BCDE,

$\therefore$ $\triangle ABC = \frac{1}{2}$ rectangle BCDE.

Corollary 1. The area of a triangle is measured by one-half the product of the measures of its base and its altitude.

For area ABC $= \frac{1}{2}$ area BCDE $= \frac{1}{2}$BC . CD.

But the perpendicular from A to BC is equal to CD, being opposite sides of a rectangle,

$\therefore$ area ABC $= \frac{1}{2}$ base $\times$ altitude.

Corollary 2. Triangles on the same or equal bases and of the same altitude are equal in area.

Corollary 3. If triangles of the same area have the same or equal bases their altitudes are equal.

Corollary 4. Two triangles of equal area which are on equal bases in the same straight line (or on the same base) and on the same side of it are between the same parallels.

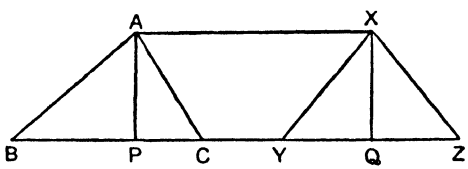


FIG. 131.

Let ABC, XYZ be the triangles, of equal area, with equal bases BC, YZ, in the same straight line.

Then by Corollary 3 the altitudes AP, XQ are equal.

But AP, XQ are parallel since each is perpendicular to BQ, i.e. AP, XQ are both equal and parallel ;

$\therefore$ AXQP is a parallelogram,

i.e. AX is parallel to PQ or BZ.

Corollary 5. If AB is a fixed straight line and P moves so that the area of the $\triangle$ APB is constant, the locus of P consists of two straight lines parallel to AB, one on either side of it, at a fixed distance from it.

THECREM 28. SECOND METHOD

If a parallelogram and a triangle stand on the same base and between the same parallels, the area of the parallelogram is double that of the triangle.

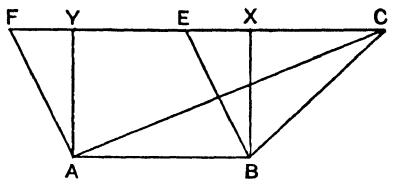


FIG. 132.

Let the parallelogram ABEF and the triangle ABC be on the same base AB and between the same parallels AB, FC.

It is required to prove that the area of ABEF is double that of ABC.

Construction. Through A, B draw straight lines perpendicular to FC to meet it at Y, X respectively.

Proof. It is given that AB is parallel to YX, and, by construction, AY, BX are parallel and the $\angle AYX$ = a right angle ;

$\therefore$ ABXY is a rectangle ;

$\therefore$ ABXY is a rectangle and ABEF a parallelogram on the same base AB and between the same parallels AB, FX ;

$\therefore$ rectangle ABXY = par^{m} ABEF.

Also the rectangle ABXY and the triangle ABC are on the same base AB and between the same parallels AB, FC ;

$\therefore$ rectangle ABXY $= 2\triangle$ ABC,

$\therefore$ par^{m} ABEF $= 2\triangle$ ABC.

Corollary. If a parallelogram and a triangle are on equal bases and of the same altitude, the area of the parallelogram is double that of the triangle.

For they can be placed so as to be on the same base and between the same parallels.

EXERCISE 25 [Theorems 25-28]

NUMERICAL [Calculation of Areas]

In Nos. 1-12 draw a rough sketch of the figure named, and find the area by calculation. In Nos. 13-18 draw the figure to a convenient scale, and find the areas by measurement.

1. A rectangle $2{\cdot}5''$ by $4''$.

2. A rectangle 3·2 cm. by 4·6 cm.

3. A right-angled $\triangle$, the sides containing the rt. $\angle$ being $2{\cdot}5''$ and $6''$.

4. A $\triangle ABC$ in which $\angle B = \angle C = 45°$, $AB = 1{\cdot}8$ cm.

5. A parallelogram, having the base $3''$, and the height $1{\cdot}8''$.

6. A parallelogram, having the base 4·8 cm., and the height 3 cm.

7. A $\triangle$ having the base $2''$, and the altitude $1{\cdot}6''$.

8. A rhombus having the diagonals $4{\cdot}8''$ and $1{\cdot}2''$.

9. A trapezium having the parallel sides $3''$ and $4''$, and the perpendicular distance between them $1{\cdot}4''$.

10. A parallelogram having two adjacent sides 6 cm. and 4·8 cm. and the included angle 30°.

11. The area of a parallelogram is 4·2 sq. cm. and the base is 1·4 cm. Find the height.

12. The area of a $\triangle$ is 3·6 sq. in.; the height is $2{\cdot}4''$. Find the base.

13. A rectangle, having one side $2{\cdot}2''$ and a diagonal $2{\cdot}9''$.

14. A rectangle ABCD, having $AB = 4{\cdot}2$ cm. and $\angle ABD = 39°$.

15. A $\triangle ABC$, having $\angle A = 52°$, $AB = 5{\cdot}3$ cm., $AC = 3{\cdot}2$ cm.

16. A $\triangle ABC$, having $\angle A = 47°$, $\angle B = 72°$, $BC = 1{\cdot}8''$.

17. A parallelogram ABCD, having $AB = 4{\cdot}2$ cm., $\angle ABC = 112°$, $AC = 7{\cdot}4$ cm.

18. A parallelogram ABCD, having

$$AB = 2{\cdot}2'', \quad AD = 1{\cdot}4'', \quad \angle BCD = 58°.$$

[If further practice is needed the areas of some of the figures in Exercise 22 may be found by drawing and measurement.]

19. ABCD is a square. P is a point in AB such that $AP = \frac{1}{5}AB$; Q is a point in PC such that $PQ = \frac{1}{5}PC$. Prove that the area of APQD is $\frac{1}{5}$ of the area of ABCD.

20. AD, BC are the parallel sides of a trapezium ABCD, and the diagonals AC, BD cut at O. The areas of the $\triangle$s AOB, BOC are 3 sq. in. and 2 sq. in. respectively. Calculate the areas of the $\triangle$s COD, AOD.

21. M, A, N are points in a st. line; $MA = 7''$, $AN = 5''$. MB, NC are drawn $\perp$ to MN and on the same side of it such that $MB = 5''$, $NC = 9''$. Find the area of the $\triangle ABC$.

22. A rectangle PBCQ and an acute-angled $\triangle ABC$ are on the same base BC and on the same side of it, the height of the $\triangle$ being twice that of the rectangle. The line PQ cuts AB at X and AC at Y. Express the area of the $\triangle AXY$ as a fraction (*a*) of the area of the $\triangle ABC$, (*b*) of the area of the rectangle.

23. ABCD is a quadrilateral in which the diagonal AC is bisected at right angles by the diagonal BD. If P, Q are the mid-points of the sides AB, AD, prove that the area of the $\triangle CPQ$ is three-eighths of the area of the whole quadrilateral.

24. A triangular area ABC is represented on a map, the scale of which is 2 inches to the mile, by a $\triangle A'B'C'$. $A'B' = \frac{1}{2}''$, $A'C' = \frac{3}{4}''$, $\angle A = 30°$. Find the number of sq. yards in the area ABC.

25. The diagonals AC, BD of a parallelogram meet at O. From M, the mid-point of AD, a parallel is drawn to DB meeting AO at H, and a parallel to AC meeting DO at K. Express the area MHOK as a fraction of that of ABCD.

26. The perpendicular distances from the vertices of a $\triangle$ to the opposite sides are in the ratios 10 : 15 : 24, and the longest side of the $\triangle$ is $60''$. Find the shortest.

27. The sides BC, CA, AB, of a $\triangle$ are produced to D, E, F, so that $CD = BC$, $AE = CA$, $BF = AB$. Prove that the area of $\triangle DEF$ is seven times the area of $\triangle ABC$.

28. ABCD is a parallelogram; BC is bisected at E, and CD at F; AE and DC are produced to meet at G, and AF and BC are produced to meet at H. Prove that the area of the $\triangle AGH$ is equal to one and a half times the area of the parallelogram.

29. ABCD is a rectangle in which $AB = 6''$, $BC = 3\frac{1}{2}''$. X and Y are points on DC between D and C such that $DX = 3''$ and $CY = 1''$. What are the areas of (i) the $\triangle ADX$; (ii) the figure ABYX?

30. A point P is taken on the side CA of a $\triangle ABC$ such that $CP = \frac{1}{3}CA$. BP is produced to Q so that AQ is $\parallel$ to BC. Express the area of the $\triangle CPQ$ as a fraction of the area of the $\triangle ABC$.

EXERCISE 26 [Theorems 25-28]

THEORETICAL

1. Prove that a median of a $\triangle$ divides it into two parts of equal area.

2. ABC is a $\triangle$ and PQ is drawn $\parallel$ to BC, cutting AB, AC at P, Q respectively. BQ, CP meet at R. Prove that

(i) $\triangle PBC = \triangle QBC$, (ii) $\triangle BRP = \triangle CRQ$,

(iii) $\triangle PQB = \triangle PQC$, (iv) $\triangle ABQ = \triangle ACP$.

3. PQRS is a parallelogram ; X, Y the mid-points of PQ, PS. What fraction of the area of the parallelogram is (i) $\triangle PXY$, (ii) $\triangle QYR$, (iii) $\triangle YXR$?

4. The base BC of a $\triangle ABC$ is divided at D so that $BD = \frac{5}{11}BC$, prove that $\triangle ABD = \frac{5}{8}\triangle ACD$.

5. Two $\triangle$s of equal area stand on the same base. Prove that the st. line joining their vertices is either bisected by the base (produced if necessary) or is $\parallel$ to it.

6. X is the mid-point of the diagonal AC of a quad. ABCD. Prove that the quads. ABXD, CBXD are equal in area.

7. Prove that a parallelogram is divided by its diagonals into four $\triangle$s of equal area.

8. P is any point on the median AD of the $\triangle ABC$. Prove that $\triangle ABP = \triangle ACP$.

9. ABCD is a parallelogram ; X is any point in the diagonal AC. Prove that $\triangle ABX = \triangle ADX$.

10. ABCD is a parallelogram ; P is any point on AB or AB produced and Q is any point on BC or BC produced. Prove that $\triangle PCD = \triangle QAD$.

11. ABC is a $\triangle$. If AB, AC are fixed lengths, prove that the area of the $\triangle$ is greatest when $\angle BAC = 90°$.

12. ABC is a $\triangle$; D is the mid-point of AB. DE is drawn $\parallel$ to BC to meet AC at E. Prove that

(i) $\triangle ADE = \triangle DBE$, (ii) $\triangle DEC = \triangle ADE$.

Deduce an alternative proof of Theorem 22.

13. ABC is a $\triangle$; D, E are the mid-points of AB, AC respectively. Prove that (i) $\triangle ADE = \triangle BDE$, (ii) $\triangle BDE = \triangle CDE$. Deduce an alternative proof of Theorem 23.

14. ABCD is a quad. The diagonals AC, BD meet at X. If $\triangle ABX = \triangle CDX$, prove that AD is $\parallel$ BC.

15. ABCD, ABEF are two parallelograms in the same plane, with a common side AB. If CE, DF be joined, show that the figure CDFE is a parallelogram, and that its area is either the sum or the difference of the areas of the two given parallelograms.

16. ABCD is a parallelogram, E any point in CD; AE, BC produced meet at F; show that

(i) $\triangle ADF = \triangle ABE$, (ii) $\triangle DEF = \triangle BEC$.

17. POQ is a right angle, P and Q being any two points on the arms. QR is drawn $\perp$ to PQ to meet PO produced at R, and PS drawn $\perp$ to QP meets QO produced at S. Prove that

$$OR \cdot OS = OP \cdot OQ.$$

18. ABCD, AEFG are two parallelograms having a common point at A, and having the vertex E on BC, and the vertex D on FG. Prove that the parallelograms are equal in area.

19. Prove that if the diagonals of a quad. divide it into four $\triangle$s equal in area, the quad. must be a parallelogram.

20. DEFGH is an irregular pentagon in which DG is $\parallel$ to EF and GH to ED. Prove that $\triangle HDE = \triangle GDF$.

21. P is a point in the side AB of a $\triangle ABC$. PC is joined, and BK, $\parallel$ to PC, meets AC produced at K. AK is bisected at Q, and PQ joined. Show that $\triangle APQ = \frac{1}{2}\triangle ABC$.

AREA CONSTRUCTIONS

CONSTRUCTION 13

To construct a rectilineal figure equal in area to a given rectilineal figure and having fewer sides by one than the given figure.

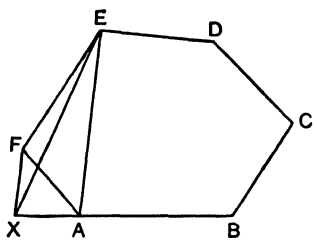


FIG. 133.

Let ABCDEF be a rectilineal figure with (say) six sides.

It is required to construct a rectilineal figure equal to ABCDEF, with five sides.

Construction. Join AE. Draw FX parallel to EA to meet BA produced at X. Join EX.

Then XBCDE is the required figure.

Proof. The △s AFE, AXE are on the same base EA and between the same parallels FX, EA ;

$$\therefore \triangle AXE = \triangle AFE.$$

To each add the figure ABCDE ;

$$\therefore \text{ the figure } XBCDE = \text{the figure } ABCDEF.$$

Note 1. By a succession of steps similar to the above we can construct a triangle equal in area to any given rectilineal figure.

Note 2. The area of any rectilineal figure may be found by dividing the figure into triangles, by joining any vertex to the remaining vertices ; the area of each triangle may then be found and the sum of these areas is the area of the required figure.

This method is not, however, of much practical use. The more usual method is to reduce the figure to a single triangle by successive applications of the above construction. There are, however, two special cases of some importance.

(1) Let ABCD be a quadrilateral, in which $\angle A = \angle C = 90°$. The quadrilateral may be divided into two right-angled triangles by joining BD. Its area is easily seen to be

$$\tfrac{1}{2}(AB \cdot AD + CB \cdot CD).$$

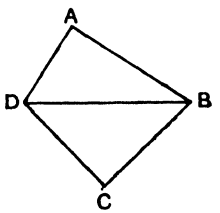


FIG. 134.

(2) Let ABCD be a trapezium with AB parallel to DC.

Draw AE $\perp$ to DC, CF $\perp$ to AB produced. Join AC.

Then $\triangle ADC = \tfrac{1}{2}DC \cdot AE$,

$\triangle ABC = \tfrac{1}{2}AB \cdot CF$.

But since AECF is, by construction, a rectangle, $\therefore AE = CF$;

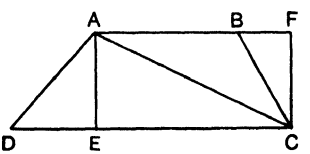


FIG. 135.

$\therefore$ Area of ABCD $= \triangle ADC + \triangle ABC = \tfrac{1}{2}DC \cdot AE + \tfrac{1}{2}AB \cdot AE$

$= \tfrac{1}{2}(DC + AB)AE.$

Thus the area of a trapezium is the product of half the sum of the parallel sides and the perpendicular distance between them.

EXERCISE 27 [Construction 13]

1. ABCD is a quad.; AB = 2″, AD = BC = 2·5″, CD = 3″, BD = 4″. Construct the quad. and also a $\triangle$ equal to it in area. Find this area.

2. Construct a quad. ABCD in which AB = 2″, BC = 3·5″, CD = 3″, DA = 2·8″, and the diagonal AC = 4″. Find the area of the quad. by first reducing it to a $\triangle$ and then finding the area of the $\triangle$.

3. ABCD is a quad.; AB = 5 cm., BC = 7 cm., CD = 8 cm., DA = 9 cm., AC = 10 cm. Draw a rectangle, with a side 10 cm., whose area is twice the area of the quad. Hence find the area of the quad.

4. Construct a quad. ABCD, having AB = 2″, BC = 3″, and the ∠s at A, B and D equal respectively to 125°, 70° and 95°. Construct a △, having BC as one of its sides, equal in area to the quad.

5. Construct a pentagon ABCDE in which AB = 3·1″, BC = 2″, CD = 1·65″, DE = 1·3″, EA = 2·9″, BD = 3·22″, AD = 4·1″. Find, by construction, a △ equal in area to the pentagon and thence calculate the area.

6. Construct a quad. ABCD having AB = 2·6 cm., AC = 5 cm., ∠ABC = 100°, ∠BAD = 130°, ∠CDA = 90°.

Construct a △ equal in area to this quad., having AB as one side and the opposite vertex on BC produced. Hence calculate its area.

7. Two △s ABC, DBC are on the same side of BC, and AB = 1″, BC = 2″, CD = 3″, DB = 3·5″, CA = 2·5″. Find a point M in BD such that △MBC = △DBC − △ABC.

8. Bisect a given △ABC by a st. line through A.

9. Divide a given △ABC into seven equal parts by st. lines through A.

10. Bisect a given △ABC by a st. line through a given point P in the side AB.

[Let F be the mid-point of AB. Join CF, CP. Through F draw a st. line ‖ to PC to meet BC at Q. Join PQ. Then PQ bisects the △.

For APQC = △APC + △PCQ
= △APC + △PFC (why ?)
= △AFC = ½△ABC (why ?).]

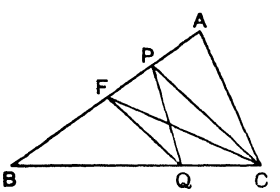


FIG. 136.

11. Trisect a given △ABC by st. lines drawn from a given point P in the side AB.

[Trisect AB at D, E. Join CD, CP, CE. Through D, E draw st. lines ‖ to PC to meet AC, BC at H, K.

Join PH, PK. Prove that these are the required lines.]

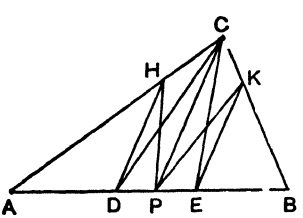


FIG. 137.

12. Bisect a quadrilateral by a st. line drawn through an angular point.

[Obtain a △ equal in area to the quad. and draw the median of this △.]

CONSTRUCTION 14

To construct a parallelogram equal in area to a given triangle and having one of its angles equal to a given angle.

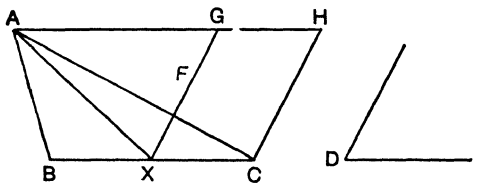


FIG. 138.

Let ABC be the given △, and D the given angle.

It is required to construct a parallelogram equal in area to the △ABC, and having one of its angles equal to the angle D.

Construction. Bisect BC at X. (*Construction* 2.)

At X in BC, make ∠CXF equal to ∠D. (*Construction* 5.)

Through A draw a straight line parallel to BC. (*Construction* 6.)

Let this straight line (produced if necessary) meet XF produced at G.

Through C draw a straight line parallel to XF to meet AG produced at H

Then XCHG is the required parallelogram.

Proof. Join AX.

The △s ABX, AXC are on equal bases BX, XC, and between the same parallels BC, AG,

$$\therefore \triangle ABX = \triangle AXC,$$

$$\therefore \triangle ABC = 2\triangle AXC.$$

But the figure XCHG is, by construction, a parallelogram.

Also, since it is on the same base XC, and between the same parallels XC, GH, as the △AXC ;

$$\therefore \text{ the par}^{m} \text{ XCHG} = 2\triangle AXC ;$$

$$\therefore \text{ the par}^{m} \text{ XCHG} = \triangle ABC ;$$

and one of its angles, GXC, is by construction equal to ∠D,

$\therefore$ XCHG is the required parallelogram.

Note. A particular case of the above is to construct a rectangle equal in area to a given triangle. **By Constructions 13 and 14 we can therefore draw a rectangle equal to any rectilineal figure. By these constructions we can also draw a parallelogram equal in area to a given rectilineal figure, and having an angle equal to a given angle.**

CONSTRUCTION 15

To construct a parallelogram equiangular to a given parallelogram, of equal area, and with one of its sides equal to a given straight line.

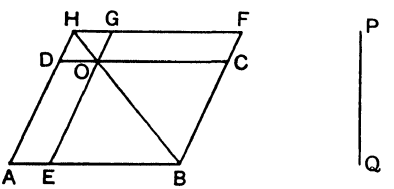


FIG. 139.

Let ABCD be the required parallelogram, and PQ the given straight line.

It is required to construct a parallelogram equiangular to the parallelogram ABCD, of equal area, and with one of its sides equal to PQ.

Construction. From BA (produced if necessary) cut off BE equal to PQ.

Through E draw a straight line parallel to BC to meet CD (produced if necessary) at O.

Join BO. Let the intersection of BO and AD be H.

[In the above figure BO and AD must both be produced to meet. If E is on BA produced, BO and AD meet without being produced.]

Through H draw a straight line parallel to AB to meet EO, BC (produced if necessary) at G, F, respectively.

Then EBFG is the required parallelogram.

Proof. ABFH, DHGO, EBCO are, by construction, parallelograms;

∴ Since a parallelogram is bisected by a diagonal,

$$\triangle AHB = \triangle FHB,$$

$$\triangle DHO = \triangle GHO,$$

$$\triangle OEB = \triangle OCB;$$

$$\therefore \triangle AHB - \triangle DHO - \triangle OEB = \triangle FHB - \triangle GHO - \triangle OCB;$$

∴ the figure ADOE = the figure OCFG.

To each add the figure EBCO,

∴ the figure ABCD = the figure EBFG.

But EBFG is clearly a parallelogram, by construction, and its angles are equal to the angles of the parallelogram ABCD.

Definition. In a parallelogram ABCD, if through any point O in the diagonal AC parallels EG, FH are drawn to the sides, meeting them at E, F, G, H, as shown in the figure, then the figures EOHA, OFCG are called **parallelograms about** AC, and the figures EOFB, OGDH are said to be their complements.

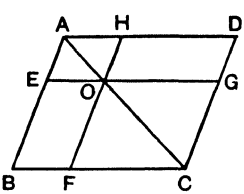


FIG. 140.

Note. It should be noticed that in the proof of Construction 15 we have incidentally proved that, in that figure, ADOE = OCFG; i.e. **the complements of the parallelograms which are about the diagonal of a parallelogram are equal.**

EXERCISE 28 [Constructions 14, 15]

1 Draw a △ABC, in which AB = 3·2″, BC = 2·3″, ∠ABC = 74°. On AB as base construct a parallelogram ABDE whose area is twice that of the △ABC and in which ∠EAB = 57°.

2. Draw, full size, a parallelogram ABCD having AB = 3·6″, BC = 2·4″, and ∠ABC = 42°. On AB as base construct a △OAB of area equal to ABCD, and having one of its base angles equal to 65°. Also construct a parallelogram ABRS of area equal to ABCD having a diagonal AR equal to AB

3. Construct a square with each side 2″ in length, and on a diagonal of the square as base, construct (i) an isosceles △ whose area is equal to the area of the square, (ii) a right-angled △ whose area is equal to the area of the square.

4. Show how to construct the △DBC which has the same base BC as a given △ABC, but has its area double that of ABC and the ∠DBC half the ∠ABC.

5. Construct a rhombus equal in area to a given parallelogram, having one side equal to a given st. line. Is the construction always possible ?

6. A rectangle ABCD has AB = 2·5″, BC = 1·4″. Construct geometrically a rectangle ACPQ of equal area.

7. Construct an isosceles △ equal in area to a given △ and having the same base.

8. On a line AB, 2·5 cm. long, draw a square ABCD. On the diagonal AC as base construct a △ACE equal to the square in area and having the ∠ACE = 125°.

9. Construct, without performing any calculation, a rhombus of side 2·5″ equal in area to a square of side 2″.

10. ABC is a given △. Construct an equivalent △ with its vertex on a given st. line XY. Is the construction always possible ?

11. On a base AB 4 cm. long, describe a △ having ∠ABC = 35°, ∠BAC = 55°. Construct a rectangle equal in area to this △ and having one side 3 cm. long.

12. Construct a △ABC, having BC = 2″, CA = 1·7″, AB = 1·2″ ; then construct a △BLM of equal area, having the same angle at B, but having BL = 0·8″.

13. Without making any calculations, construct a parallelogram equal in area to an equilateral △ each of whose sides is 2″ long. The parallelogram must also have one angle 45° and one side 1·4″ long.

14. Draw a △ with sides of length 2·5″, 3″ and 4·3″ ; then construct a parallelogram equal in area, having an angle 60° and one side of length 4″.

15. Draw, without calculation, a rectangle, one side of which is 3·2″ long, equal in area to a square of side 2·7″.

16. Construct the △ABC, given BC, ∠B, and that the area is equal to that of a given △XYZ.

EXERCISE 29.

[Further Examples on Theorems 25-28, etc.]

1. Two sides of a $\triangle$ are $1{\cdot}7''$ and $2{\cdot}4''$, and its area is $1{\cdot}2$ sq. in. Construct any $\triangle$s which satisfy these conditions. How many distinct $\triangle$s are there ?

2. A $\triangle$ has one side $3{\cdot}2''$ long, its area is $5{\cdot}2$ sq. in., and the radius of its circumcircle is $2{\cdot}3''$. Construct the $\triangle$.

3. Construct a quad. ABCD, given that $AB = 2''$, $AC = 3''$, $BC = 1{\cdot}6''$, $\angle CAD = 43^\circ$, and the area of the $\triangle ACD$ is 2 sq. in.

4. A quad. ABCD is such that its diagonal AC bisects it. Draw the quad. when $AB = 9{\cdot}6$ cm., $BC = 4{\cdot}2$ cm., $CA = 8{\cdot}8$ cm., $\angle ACD = 68^\circ$.

5. Construct the greatest possible parallelogram whose diagonals are $4{\cdot}9$ cm., $5{\cdot}8$ cm.

6. ABCD is a parallelogram and P is any point within it : prove that the sum of the $\triangle$s PAB, PCD = the sum of the $\triangle$s PBC, PAD.

7. Prove that, if the diagonal of a quad. bisects the other diagonal, it will bisect the area.

8. Prove that, if a diagonal of a quad. bisects its area, it will bisect the other diagonal.

9. X is a point in the side AB of the $\triangle ABC$. Find a point Y in BC so that $\triangle YAX = \triangle CAY$.

10. In a $\triangle ABC$, the $\perp$ from A to BC = the $\perp$ from C to AB. Prove that $AB = BC$.

11. AB, AC are the equal sides of an isosceles $\triangle ABC$. At B a line BD is drawn $\perp$ to BC to meet CA produced at D ; prove that A is the mid-point of CD. A line is drawn $\parallel$ to BD to meet CB, produced beyond B, at E and to meet a $\parallel$ to CB through D at F. If AB trisects the area of the trapezium FDCE, prove that $BC = 4BE$.

12. The side BC of a $\triangle ABC$ is produced to X so that $CX = BC$, and Y is the mid-point of AB. Prove that the $\triangle$s ABC, XBY are equal.

13. AB, XY, CD are three $\parallel$ lines, X being on AC and Y on BD. Prove that the quads. BXCY, AXDY are equal in area.

14. ABCD is a parallelogram whose diagonals AC, BD meet at O. M and N are the mid-points of OB and OD respectively. Prove that AMCN is a parallelogram whose area is half that of ABCD.

15. AD and BCEF are two || st. lines with BC = EF. Another st. line is drawn || to the given st. lines and between them, meeting AB at G and DE at H. Prove that $\triangle AGC = \triangle DHF$.

16. On the side AB of a $\triangle ABC$ the square ABST is described. If the area of the $\triangle SBC$ is equal to one quarter of that of the square, prove that the $\triangle CAB$ is isosceles.

17. In the $\triangle ABC$, B is joined to E, the mid-point of AC, and CK is drawn || to EB, to cut AB produced at K; prove that

$$\triangle AKC = 2\triangle ABC.$$

18. AP, BQ, CR are || st. lines; two lines are drawn through C, viz. CB meeting BQ at B, and CP meeting AP at P; two lines are drawn through R, viz. RA || to CB meeting AP at A, and RQ || to CP meeting BQ at Q; show that the quads. ABCR, PQRC are equal in area.

19. ABCD is a quad. R is any point of the side DC. The || to AR through D intersects the || to BR through C in the point S. Prove that the area of the $\triangle ABS$ is equal to that of the quad ABCD.

20. The sides AB, AC of a $\triangle ABC$ are bisected at D, E. CD and BE intersect at F. Prove that the $\triangle BFC$ is equal to the quad. ADFE in area.

21. A point P is taken on the base BC of an isosceles $\triangle ABC$, and PM, PN are drawn $\perp$ to AB, AC respectively, meeting them at M, N. Prove that the sum of PM and PN is the same for all positions of P on BC. If P is taken on BC produced, what modification of the theorem is required?

22. If AD bisects the side BC of a $\triangle ABC$, and if P is any point on AD, prove that the areas of the $\triangle$s PAB, PAC are equal. Prove also that if P is any point on the line through A || to BC, the $\triangle$s PAB, PAC are equal.

If E, F are any given points on the sides AB, AC, respectively of a $\triangle ABC$, find two positions of a point P on BC (produced either way if necessary) such that the areas of the $\triangle$s PEA, PFA are equal.

23. ABC is a given $\triangle$ and BA, CA are produced to D and E, making AD equal to AB and AE equal to AC. If P is any point in BC, show that the sum of the $\triangle$s PAD and PAE is constant.

24. Prove that if the diagonals of a parallelogram are given in length, its area is greatest when it is a rhombus.

25. AFE is a $\triangle$ with B the mid-point of AF and C the mid-point of BE, AC produced meets EF at D. Prove that the $\triangle$s FBD, ADB, ADE are equal in area.

26. From a point X in the diagonal AC of a parallelogram ABCD lines are drawn || to the sides, meeting AB at F, BC at G, CD at H, and DA at K. Prove that (i) $\triangle XFG = \triangle XHK$, (ii) GH, FK are parallel.

27. Taking the figure of the previous example, prove (i) that GH, DB are parallel (consider $\triangle$s DGH, BGH), (ii) that

$$\triangle CHG + \triangle CKF = \tfrac{1}{2} \text{ parm } ABCD.$$

28. AB, DC are the parallel sides of the trapezium ABCD. X, Y are the mid-points of AD, BC, respectively. Prove that XY is || to DC.

29. Prove that a parallelogram is bisected by any st. line which passes through the intersection of its diagonals. Hence show how a parallelogram PQRS may be bisected by a st. line drawn (i) through a given point O, (ii) parallel to a given line XY.

30. ABCD is a trapezium in which AB is || to DC. DF drawn || to CB meets BA, produced if necessary, at F. E is the mid-point of AF. Prove that the area of $\triangle$EBC is half that of the trapezium. Consider both cases.

31. ABCD is any ordinary convex quad. A || to AB drawn through C meets a || to AC drawn through D at E. Prove that the area of the trapezium ABCE is equal to that of the quad. ABCD.

32. Using the two previous results, show how to find a point F in the side AB (or AB produced) of an ordinary quad. ABCD such that the area of the $\triangle$FBC is half that of the quad. Show also how, given a point P in AB, to find a point Q in CD, such that PQ divides the quad. into two quads. of equal area.

33. ABE is any $\triangle$; DC drawn || to the base BE cuts AB, AE in D, C respectively. Show that the $\triangle$s ABC, ADE have equal areas. Hence show how to draw a $\triangle$ of the same area as a given $\triangle$ having one of its angles the same as one of the angles of the given $\triangle$, and having one of its sides containing that angle of a given length.

THE THEOREM OF PYTHAGORAS

This theorem is of great interest and importance. The discovery of the theorem has always been ascribed to Pythagoras, the famous Greek philosopher who lived in the latter part of the sixth century B.C., but there is reason to believe that the truth of the theorem was known, at any rate for the simpler cases, e.g. 3, 4, 5; 5, 12, 13, at a very much earlier date. The proof which follows is due to Euclid, of Alexandria (about 300 B.C.), who wrote a text-book of Geometry which remained the standard introduction to the subject for more than two thousand years, and which has only been displaced in this country during the last fifty years. The proof is long, but each part of it is quite simple, and it is suggested that the teacher should make the class familiar with the separate steps before attempting to take the theorem as a whole. The following method has been found useful. Draw the figure of Theorem 29 on the blackboard, and by the side of it draw a figure in which the squares are described on the sides of an acute-angled triangle. Questions such as the following may be discussed orally:

(1) Find as many equal angles (other than right angles) as possible in each figure.

(2) Pick out sets of 3 points which lie on a straight line in the first figure, but not in the second.

(3) Prove that the $\triangle$s GBC, ABH are congruent in both figures.

(4) How many $\triangle$s are there with base BG and area equal to half BAFG? Discuss each figure separately.

(5) How many $\triangle$s are there with base BH and area equal to half BHLJ? Discuss each figure separately.

If the pupil has gained familiarity with the figure in this way the formal proof will be found quite easy.

Some teachers may like to show that the squares on AB, AC may be cut in pieces and fitted together to make the square on BC. The best-known method of dissection is due to Périgal, but many others are known. It should, of course, be emphasised that dissection and piecing together again does not constitute proof, and the fallacy in trusting too much to appearances may be illustrated by showing that a rectangle 13×5 can be made to appear to be made up of the same pieces as a square 8×8.

The following is Perigal's dissection.

Draw on thin cardboard a right-angled $\triangle ABC$ and the squares on the three sides. Choose one of the smaller squares, say the square on AC, and cut it up in the following way.

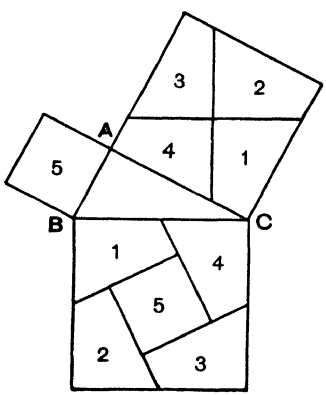


FIG. 141.

Through the centre of the square (i.e. the intersection of the diagonals) draw straight lines parallel and $\perp$ to the hypotenuse BC.

Cut along these lines, getting the four pieces marked 1, 2, 3, 4.

It will be found that these, together with the square marked 5, can be fitted together, as shown, to make the square on BC.

THEOREM 29. [Pythagoras' Theorem]

In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

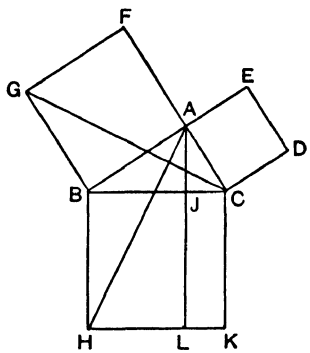


FIG. 142

Let ABC be a triangle with the angle A a right angle.

It is required to prove that the square on BC = the square on CA + the square on AB.

Construction. On BC, CA, AB and outside the triangle draw the squares BCKH, CAED, ABGF.

Through A draw a straight line parallel to BH and therefore to CK to meet BC at J and HK at L.

Join CG, AH.

Proof. The adjacent angles BAC, BAF are each right angles and their sum = 2 right angles, ∴ CA, FA are in the same straight line.

Similarly BA, AE are in the same straight line.

Also the ∠s GBA, HBC, being angles of a square, are each right angles.

To each of these equals add $\angle ABC$,

$$\therefore \angle GBC = \angle ABH ;$$

$\therefore$ In the $\triangle$s GBC, ABH,

$$GB = AB \text{ (}\textit{sides of a square}\text{)},$$

$$BC = BH \text{ (}\textit{sides of a square}\text{)},$$

$$\angle GBC = \angle ABH \text{ (}\textit{proved}\text{)} ;$$

$\therefore$ the $\triangle$s $\begin{matrix} GBC \\ ABH \end{matrix}$ are congruent and therefore equal in area (*2 sides*, **included** *angle*).

But the square ABGF and the $\triangle GBC$ are on the same base GB, and between the same parallels GB, FC ;

$$\therefore \text{ sq. } ABGF = 2\triangle GBC.$$

Also the rectangle BHLJ and the $\triangle ABH$ are on the same base BH, and between the same parallels BH, AL ;

$$\therefore \text{ rect. } BHLJ = 2\triangle ABH.$$

But $$\triangle GBC = \triangle ABH \text{ (}\textit{proved}\text{)},$$

$$\therefore \text{ sq. } ABGF = \text{rect. } BHLJ.$$

Similarly, by joining AK, BD, it can be shown that

$$\text{sq. } CAED = \text{rect. } CKLJ ;$$

$$\therefore \text{ sq. } ABGF + \text{sq. } CAED = \text{rect. } BHLJ + \text{rect. } CKLJ$$

$$= \text{sq. } BCKH.$$

THE APPLICATION OF PYTHAGORAS' THEOREM

Although the theorem states a fact about the squares on the sides of a right-angled triangle it is very rarely necessary to draw these squares when applying the theorem. Most of the applications are numerical, and the chief use of the theorem is to enable us to calculate the length of one side of a right-angled triangle,

when the other two sides are known. The following example illustrates the method:

Ex. The foot of a ladder rests on level ground 10′ from a wall, and the top of the ladder rests against the wall at a height of 24′ from the ground. Find the length of the ladder.

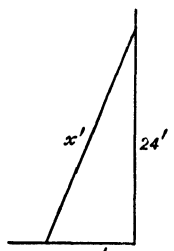


FIG. 143.

Let the ladder be x ft. long.

Then, by Pythagoras' Theorem,

x^2 sq. ft. $= 24^2$ sq. ft. $+ 10^2$ sq. ft.,†

$\therefore\ x^2 = 24^2 + 10^2$

$= 576 + 100$

$= 676$;

$\therefore\ x = 26$,

$\therefore$ the ladder is 26′ long.

THEOREM 30

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, then the angle contained by these sides is a right angle.

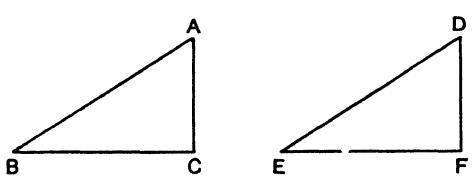


FIG. 144.

Let ABC be a triangle in which the square on AB = the sum of the squares on AC, CB.

It is required to prove that the angle C is a right angle.

Construction. Draw a triangle DEF, in which DF = AC, ∠DFE = a right angle and FE = CB.

† This step may be omitted as soon as the pupil is familiar with the method.

Proof. Since $\angle DFE =$ a right angle,

$$\therefore\ DE^2 = DF^2 + FE^2 \ (\textit{Pythagoras'}).$$

But by construction $DF = AC$ and $FE = CB$;

$$\therefore\ DE^2 = AC^2 + CB^2.$$

But it is given that $AB^2 = AC^2 + CB^2$,

$$\therefore\ DE^2 = AB^2,$$

$$\therefore\ DE = AB\ ;$$

$\therefore$ in the $\triangle$s ABC, DEF,

$$AB = DE \ (\textit{proved}),$$

$$AC = DF \ (\textit{construction}),$$

$$CB = FE \ (\textit{construction})\ ;$$

$$\therefore \text{ the } \triangle\text{s } \begin{matrix} ABC \\ DEF \end{matrix} \text{ are congruent } (3 \textit{ sides}),$$

$$\therefore\ \angle ACB = \angle DFE.$$

But $\angle DFE = 90°$ (*construction*) ;

$$\therefore\ \angle ACB = 90°.$$

EXERCISE 30 [Theorems 29, 30]

NUMERICAL

1. Find the hypotenuses of a right-angled $\triangle$ when the sides containing the right angle are (i) $5''$, $12''$; (ii) $2{\cdot}5''$, $6''$; (iii) 8 cm., 15 cm. ; (iv) $5{\cdot}04''$, $6{\cdot}72''$; (v) $4{\cdot}2$ miles, $5{\cdot}7$ miles.

2. Find the remaining side of a right-angled $\triangle$, given the hypotenuse and one side as follows : (i) hyp. $= 1{\cdot}7''$, side $= 0{\cdot}8''$; (ii) hyp. $= 4{\cdot}5''$, side $= 3{\cdot}6''$; (iii) hyp. $= 16{\cdot}9$ cm., side $= 6{\cdot}5$ cm. ; (iv) hyp. $= 180$ yards, side $= 100$ yards.

3. Find the diagonal of a rectangle, whose sides are $3''$ and $2{\cdot}7''$.

4. Find the remaining side of a rectangle, given diagonal $= 14{\cdot}8$ cm. and one side $= 11$ cm.

5. In a $\triangle$ABC, AN is drawn $\perp$ to BC, the point N being between B and C. AN $= 3''$, AB $= 4''$, AC $= 5''$. Calculate the length of BC.

6. A ladder whose foot is $8'$ from the front of a house reaches to a window-sill $22'$ above the ground. Calculate the length of the ladder.

7. A ladder 29′ long just reaches the top of a wall 21′ high. How far from the foot of the wall is the foot of the ladder ?

8. A ship sails 56 miles due North. It then sails due East until it is 65 miles from its starting point. How far East does it sail ?

9. The diagonals of a rhombus are 5·6 cm. and 3·3 cm. Find the length of a side.

10. A side of a rhombus is 1·7 cm., one diagonal is 3 cm. Find the length of the other diagonal.

11. Find the altitude of an isosceles $\triangle$, given base $= 1{\cdot}6''$, side $= 1{\cdot}7''$.

12. Which of the $\triangle$s whose sides are given below are right-angled ?

(i) $4''$, $5''$, $6''$; (ii) $1{\cdot}2''$, $3{\cdot}7''$, $3{\cdot}5''$; (iii) 4 cm., 9·6 cm., 10·4 cm.; (iv) $2{\cdot}2''$, $3{\cdot}3''$, $4{\cdot}4''$; (v) $2{\cdot}6''$, $2{\cdot}4''$, $1''$.

13. The altitude of a $\triangle$ is $6''$ long and divides the base internally into two parts, $4''$ and $9''$ long; prove that the $\triangle$ is right-angled.

14. The diagonals of a parallelogram are of lengths 12 cm. and 5 cm., and each of one pair of parallel sides is of length 6·5 cm. Prove that the parallelogram is a rhombus and calculate its area.

15. The shortest distance AP from a point A to a st. line QR is 12 cm., and Q, R are respectively 15 cm. and 20 cm. distant from A on opposite sides of AP. Prove that $\angle QAR = 90°$.

16. A lawn in the shape of a quad. ABCD has the following measurements : $AB = 60'$, $BC = 25'$, $\angle ABC = 90°$, $CD = 39'$, $DA = 52'$. Calculate the area of the lawn.

17. ABCD is a quad. in which $AB = 9$ cm., $BC = DA = 6$ cm., $CD = 3$ cm., and $\angle CDA = 90°$. Prove that $\angle ACB = 90°$.

18. ABC is an isosceles $\triangle$ of which the base $BC = 18''$; the $\perp$ from A to $BC = 12''$. A point E is taken on BC produced such that $AE = 20''$. Prove that $\angle BAE = 90°$.

19. A ladder 26′ long is placed with its lower end 10′ from the foot of a vertical wall. Find by calculation where the top of the ladder touches the wall, and also the distance from the lower end of the ladder of the point on it which is nearest the foot of the wall.

20. A man walks from A to B by going 2 miles due East, 3 miles North-East, and finally 1 mile due East. Calculate the direct distance from A to B.

21. A ship sails 17 miles due West, then 16 miles due North, and finally 47 miles due East ; how far is it now from its starting point ?

22. AD is the $\perp$ from a point A on a line BC. If AB = 7″, AC = 5″, BC = 6″, find BD, DC, AD and the area of the △ABC.

23. A ship steams 4·2 miles due North, then 3·7 miles due East, then 1·9 miles due South. Calculate (i) its final distance from its starting point, and (ii) the greatest distance from its starting point that is reached during the journey.

24. The upper end of a ladder rests against a vertical wall at a point 26′ above the ground, which is level, and the end of the ladder on the ground is 6·9′ from the wall. Find the length of the ladder. If the lower end is moved away from the wall through a distance of 1′, find the distance through which the upper end moves downwards.

25. It is found that a rope 50′ long attached to the top of a wall can be stretched to a point on the ground 17′ from the foot of the wall. Find, by calculation, to what distance from the foot of the wall a rope 65′ long can be similarly stretched.

26. Prove that a △ the length of whose sides are $x^2 - 1$, $2x$, $x^2 + 1$, has a right angle, whatever x may be.

27. A is 3″ distant from O, the centre of a circle of radius 5″. From A a st. line AB 5″ long is drawn $\perp$ to OA. Find, by calculation, how far B is outside the circle (i) measured along AB, (ii) measured straight towards the centre of the circle.

28. Find the slant side of a cone of (i) height 6″, base-radius 4″; (ii) height 11 cm., base-radius 4·6 cm.

29. Find the base-radius of a cone of (i) slant side 9′, height 6′; (ii) slant side 22·6 cm., height 11·4 cm.

30. Find the height of a cone of (i) slant side 11″, base-radius 8″; (ii) slant side 2·9 cm., base-radius 2·75 cm.

31. A box measures 36″ by 9″ by 12″. Find the length of the diagonal joining one corner to the opposite corner.

32. A box measures 2″ by 2″ by 3·5″. Find the length of its diagonal.

33. Calculate to the nearest inch the length of the longest straight stick that can be packed inside a packing-case of inside measurements 3′ by 4′ by 6′.

34. ABC is a △ having AB = BC = 15 cm., AC = 18 cm. Along the bisector of the ∠ABC a point O is taken such that BO = 8 cm. At O a line OV, 15 cm. long, is drawn $\perp$ to the plane of the △ABC. Calculate the lengths VA, VB, VC. Also find these lengths by drawing.

35. A symmetrical pyramid stands on a square base ABCD of side 8″, and each of the slant edges is 7″. V is its vertex and O the centre of the square base. Calculate its altitude VO which is ⊥ to the base ABCD. Also find VO by drawing.

36. The height of a pyramid standing on a base 3 inches square = 3″. Find the length of one of the slanting edges.

37. A pyramid is to have a rectangular base, the sides being 8 cm. and 4 cm., and the length of each slant edge is to be 6 cm. Draw a figure suitable for cutting out to make a model of the pyramid, and find out what the height of the pyramid would be.

38. The bottom face of a rectangular block is a square of side 18 cm. The height of the block is 6 cm. Find the distance between the middle point of the front edge of the top face and the middle point of a side edge of the bottom face.

39. The edges AB, AD, AE of a rectangular block are 4·5″, 6″, 10″ respectively. What is the length of the st. line joining the mid-point of AB to the mid-point of the edge opposite to AE ?

40. On each side of a square PQRS, side 6″, an equilateral △ is drawn outside the square. These △s are folded about the sides of the square to form a pyramid, OPQRS, vertex O. Calculate the height of the pyramid, and prove that △POR is right-angled.

41. A cube has an edge of 2″. A △ is formed by joining the mid-point of one face of the cube to the ends of one of the edges of the opposite face. Calculate the lengths of the sides and the area of this △.

42. A wall ABCD is 20′ long and 9′ high, AD being the top of the wall. O is a point on the level ground opposite B (i.e. so placed that ∠CBO = 90°), OA slopes upward at 3 vertical to 5 along the slope. Prove that OD slopes upward at 9 vertical to 25 along the slope.

EXERCISE 31 [Theorems 29, 30]

THEORETICAL

1. PX is the altitude of a △PQR. Prove that

$$PQ^2 - PR^2 = QX^2 - RX^2.$$

2. The diagonals of a quad. intersect at right angles. Prove that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair.

3. OX, OY are two ⊥ st. lines. Z is any point on OY. Prove that $OY^2 + XZ^2 = OZ^2 + XY^2$.

4. OX, OY are two $\perp$ st. lines. P is any point on OX, Q any point on OY. Prove that $XY^2 + PQ^2 = QX^2 + PY^2$.

5. Prove that the sum of the squares on the diagonals of a rectangle is equal to the sum of the squares on its sides.

6. Prove that the sum of the squares on the diagonals of a rhombus is four times the square on one of its sides.

7. The line joining the points P, Q is $\perp$ to the line joining two other points A, B. P is further than Q from AB. Prove that

$$PA^2 - QA^2 = PB^2 - QB^2.$$

8. ABC is a $\triangle$ in which $\angle B = \angle C = 30°$. Prove that $BC^2 = 12AD^2$, D being the mid-point of BC.

9. In the $\triangle ABC$, $\angle C = 90°$. Squares ACDE, BCFG are described so as to lie wholly outside the $\triangle ABC$. Prove that

$$EB^2 - AG^2 = AC^2 - BC^2.$$

10. ABC is an equilateral $\triangle$, D is the mid-point of BC and M is the mid-point of AD. Prove that $4BM = \sqrt{7}\,BC$.

11. From a point O inside the $\triangle ABC$ $\perp$s OD, OE, OF are drawn to the sides BC, CA, AB respectively ; prove that

$$BD^2 + CE^2 + AF^2 = BF^2 + AE^2 + CD^2.$$

12. In the quad. ABCD it is given that the diagonals are $\perp$, and also that $AB^2 + BC^2 = AD^2 + DC^2$. Prove that AC bisects BD.

13. Show that if three right-angled isosceles $\triangle$s are described on the three sides of a right-angled $\triangle$ as hypotenuses, the largest of them is equal to the sum of the other two.

14. A convex quad. ABCD has the $\angle DAB = 90°$ and the diagonal BD is $\perp$ to one of the sides. Prove that the square described on the longest side of the quad. is equal to the sum of the squares described on the other three sides.

15. ABC is any $\triangle$; D is the foot of the $\perp$ from A to BC. BH is drawn $\perp$ to AB and equal to CD, and CK is drawn $\perp$ to AC and equal to BD. Prove that AH = AK.

16. In the figure of Th. 29, prove that

(i) G, A, D are in a st. line,
(ii) AH is $\perp$ to CG,
(iii) BF is $\parallel$ to CE,
(iv) $\triangle CDK$ is equivalent to $\triangle ABC$.

17. A $\triangle$ is drawn with sides equal in length to BF, CE, CH (in the figure of Th. 29). Prove that this $\triangle$ is right-angled.

18. PQRS is a rectangle whose diagonals meet at O. PQ subtends an angle of 120° at O. X is taken on QR, between Q and R, such that $RX = \frac{1}{4}RQ$. Prove that $RQ = \frac{1}{4}PX$.

CONSTRUCTION 16

To draw straight lines respectively $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, ... units long.

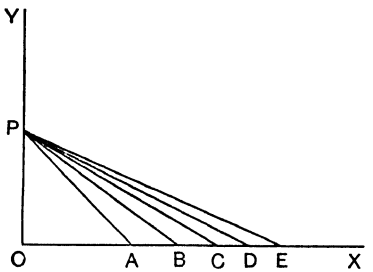


FIG. 145.

Take OX, OY at right angles to one another, and from them mark off OA, OP, each one unit of length. Join PA.

Then, by Pythagoras', $PA^2 = OA^2 + OP^2 = 1 + 1 = 2$,

$\therefore$ PA $= \sqrt{2}$ units of length.

From OX mark off OB equal to PA, and join PB.

Then $PB^2 = OB^2 + OP^2 = 2 + 1 = 3$, $\therefore$ PB $= \sqrt{3}$ units of length.

From OX mark off OC equal to PB, and join PC.

Then $PC^2 = OC^2 + OP^2 = 3 + 1 = 4$, $\therefore$ PC $= \sqrt{4}$ units of length.

From OX mark off OD equal to PC, and join PD.

Then $PD^2 = OD^2 + OP^2 = 4 + 1 = 5$, $\therefore$ PD $= \sqrt{5}$ units of length.

Similarly, OE $= \sqrt{6}$ units of length, and by continuing the process we may find the square root of any number desired.

Note. This process is, in practice, very clumsy, and it is usually better to proceed as follows. By making use of the identity $a^2 - b^2 = (a+b)(a-b)$, any number may be expressed as the difference of two squares. Thus $47 = 47.1$, so that we must take $a + b = 47$, $a - b = 1$; $\therefore$ $a = 24$, $b = 23$.

$46 = 46 \cdot 1$, so that $a + b = 46$, $a - b = 1$; $\therefore$ $a = 23 \cdot 5$, $b = 22 \cdot 5$.

Ex. To construct a straight line $\sqrt{46}$ units long.

Take OX, OY at right angles to one another, and from OY mark off OA = 22·5 units. With centre A, radius 23·5 units, describe an arc cutting OX at B. Then OB is the required line. Join AB.

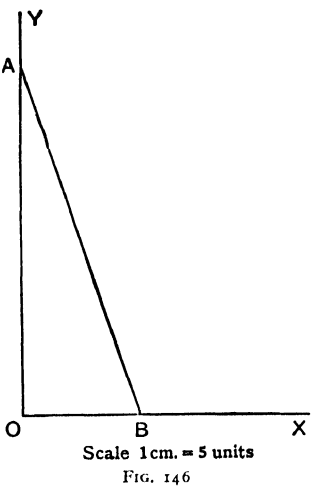


FIG. 146

Then in the right-angled △OAB, by Pythagoras',

$$OB^2 + OA^2 = AB^2,$$

$$\therefore\ OB^2 = AB^2 - OA^2$$

$$= 23{\cdot}5^2 - 22{\cdot}5^2$$

$$= (23{\cdot}5 + 22{\cdot}5)(23{\cdot}5 - 22{\cdot}5)$$

$$= 46{\cdot}1 = 46.$$

$\therefore$ OB = $\sqrt{46}$ units of length.

This method is quite general, but large numbers are involved, and it is usually necessary to take a rather small scale. A better construction will be given later in the book.

EXERCISE 32 [Construction 16]

1. Express the following numbers as the sum of two squares: 10, 13, 17, 45, 73. Hence construct st. lines of lengths

$$\sqrt{10},\ \sqrt{13},\ \sqrt{17},\ \sqrt{45},\ \sqrt{73}.$$

2. Express the following numbers as the difference of two squares: 27, 32, 33, 18·6, 14·8. Hence construct st. lines of lengths $\sqrt{27}$, $\sqrt{32}$, $\sqrt{33}$, $\sqrt{18{\cdot}6}$, $\sqrt{14{\cdot}8}$.

3. Construct a square 6 square inches in area.

4. Construct a square equal to half a given square.

5. Construct a square equal to (i) the sum of two given squares, (ii) the difference of two given squares, (iii) the sum of n given squares.

6. Prove that $\sqrt{a^2 - b^2} > a - b$.

7. Construct geometrically the side of a square which is equal to three times the square on a given line AB, 2·3″ long.

8. A square is divided into two unequal rectangles ; show how to obtain, by a geometrical construction, two squares whose areas are respectively equal to those rectangles.

[Consider the figure of Th. 29.]

9. A, B are two fixed points ; P is any variable point on the st. line AB, between A and B. PQ is drawn $\perp$ to AB, on one side of it, and equal to PB. Prove that the locus of Q is a segment of a st. line passing through B. Deduce a construction for finding the position of P which satisfies the condition $AP^2+PB^2=c^2$, where c is a given length. Is the construction always possible ?

10. AB is a st. line 4·4″ long. Find, by a geometrical construction, a point P between A and B such that $AP^2-PB^2=4\cdot8^2-3\cdot7^2$.

[If X is any point on the $\perp$ to AB at P, $XA^2-XB^2=AP^2-PB^2$.]

Deduce a general construction for finding a point P between A and B such that $AP^2-PB^2=c^2$, where c is a given length.

EXERCISE 33 [Further Examples on Theorems 29, 30]

1. A ladder 27 ft. long rests against a wall with 5 feet projecting over the top ; when the foot of the ladder is pushed 3 feet nearer the wall, 7 feet project over the top. Find the height of the wall.

2. ABC is a $\triangle$, right-angled at B, with AB=2BC. CB is produced to D, so that CD=CA. From BA, BE is cut off equal to BD. Show that $AB\,.\,AE=BE^2$. [Let AB=2a units.]

3. With the corners of a square of side 2a as centres, circles of radius b (less than a) are drawn, determining 8 points on the sides of the square. If these points form a regular octagon, calculate $\frac{b}{a}$ to 3 places of decimals.

4. A square PQRS, of area 58 sq. cm., is placed in a square, ABCD, of area 100 sq. cm., with its vertices P, Q, R, S on AB, BC, CD, DA respectively. Calculate the lengths of AP and AS.

5. ABC is a $\triangle$ and AN, the $\perp$ from A to BC, falls within the $\triangle$. If $BN\,.\,NC=AN^2$, prove that $\angle BAC=90°$.

6. A quad. ABCD has its diagonals $\perp$ to each other. A $\triangle$PQR is drawn having PQ=AB, QR=CD and $\angle PQR=90°$. Prove that if another $\triangle$ is drawn with sides respectively equal to BC, DA, PR, this $\triangle$ will also contain a right angle.

7. ABCD is a rectangle. M, N are the feet of the ⊥s drawn from A, C to the diagonal BD. Prove that $BM^2+BN^2=DM^2+DN^2$.

8. A, B, C, D are four points in order on a st. line, with CD=AB; on one side of this line squares ABEF, BDGH are drawn, and the points F, C, G are joined. Prove that △FCG is equal to half the sum of the squares.

9. Two circles intersect orthogonally, i.e. at each point of intersection their tangents are ⊥. The radii of the circles are 2″ and 2·1″. Find the distance between their centres.

10. ABCD is a square. E, F, G, H are points in the sides AB, BC, CD, DA respectively, such that AE=BF=CG=DH; prove that $FH^2=2(AE^2+BE^2)$.

11. ABC is a △ with ∠A=90°; on the sides of AB and AC, remote from BC, squares ABDE and ACFG are described. Prove that DAF is a st. line and also that $DF^2=2BC^2+4BC\,.\,AM$, where M is the foot of the ⊥ from A upon BC.

12. If in Qu. 10 the area of EFGH is five-ninths of the area of ABCD, calculate the ratio AE : AB.

13. The base of a right circular cone has a radius of 3″ and the height of the cone when standing on its base is 4″. The cone is placed on a horizontal table with its curved surface lying on the table. Find the height above the table of the highest point of the cone.

14. ABCD is a tetrahedron in which BC=CA=AB=6′, and DA=DB=DC=4′. DP is the ⊥ from D upon the plane ABC. Show that $AP=2\sqrt{3}$ feet, and that DP=2 feet.

15. AO, AB, AC are concurrent edges of a rectangular block, and P, Q, R are the mid-points of OA, AB, BC respectively. Prove that △PQR is right-angled and that $4PR^2=OA^2+BC^2$.

LOCI

Definition. If a point which moves in accordance with some given law describes a certain path, and if every point on this path satisfies the given law, then the path is called the **locus of the point.** The locus of a point may be (i) a line (straight or curved) or part of a line, or (ii) several lines (straight or curved) or parts of lines, or (iii) a surface (or part of a surface). In

considering motion in three dimensions it is sometimes necessary to consider the locus of a line which moves in accordance with some given law. The locus is usually one or more surfaces or parts of one or more surfaces.

Most of the loci considered here are two-dimensional loci, but at a suitable stage the pupil should extend the results to three dimensions. In a large number of cases the three-dimensional locus may easily be obtained by rotating the plane locus about a suitable straight line in the plane. In the list of loci, which is placed for reference at the end of the book, the extension to three dimensions is given in most cases.

To investigate the locus of a point satisfying a given law, we must first obtain several points on the locus, and then try to guess what the locus is; afterwards, we must test our locus to see whether our guess is a correct one. The beginner will find the first process the harder; it is a comparatively simple matter, in many cases, to prove that a certain path is the locus, when we have realised that it is the locus. It is suggested that no attempt should be made to give a sound theoretical proof of any locus question until the pupil has had plenty of practice in guessing what the locus is. In attempting to do this two courses are open to him; either he may attempt to visualise the actual path by following the actual motion of the point, e.g. he will easily describe the locus of the tip of the pendulum of a clock during its motion; or he may construct, either with or without the help of his instruments, a number of points satisfying the given conditions, e.g. he will in this way more easily guess the locus of a point which moves so as to be equidistant from two fixed points.

It is suggested that the greater part of the following exercise should be discussed orally in class.

EXERCISE 34

The pupil should try to see the loci in Nos. 1-17 without actually plotting points. In Nos. 18-28 sufficient points should be plotted either roughly or with the use of instruments.

1. What is the locus of the tip of the minute hand of a clock which is keeping true time ? How many times is the locus described in 24 hours ?

2. What is the locus of a point marked on a door as the door is opened ?

3. What is the locus of the bell of a bicycle as a man rides it in a st. line on a level road ?

4. What is the locus of the tip of a man's nose if he stands perfectly still in a lift which descends vertically ?

5. What is the locus of the tip of the pendulum of a clock during its motion ?

6. What is the locus of the centre of a half-crown which is rolled along a st. line ?

7. What is the locus of the centre of a halfpenny which is rolled around a florin, in contact with it the whole time ?

8. A point is marked on the pane of a sash-window. What is the locus of the point as the window is closed ?

9. A point is marked on the pane of a window. What is the locus of the point as the window is closed, (*a*) if the window moves about a horizontal axis, (*b*) if the window moves about a vertical axis ?

10. What is the locus of the tassel of a blind, as the blind is drawn down ?

11. Mark a point on the hypotenuse of your set-square. What is the locus of this point as the set-square is rotated about one of the shorter sides as axis ?

12. Mark a point on one of the shorter sides of your set-square. What is the locus of this point as the set-square is rotated about the hypotenuse as axis ?

13. Mark a point on the circumference of your semicircular protractor. What is the locus of this point as the set-square is rotated about the diameter as axis ?

14. Mark a point on the edge of your rectangular protractor. What is the locus of this point as the set square is rotated about one of the other edges as axis ?

15. A pencil rolls straight along a horizontal table. What is the locus of the pencil point ?

16. A ball rolls along a straight groove on the top of a desk. What is the locus of its centre ?

17. A wooden match-box has edges AB=4″, AD=3″, AE=2″, and stands on a horizontal table. It is turned about the edge AB as axis until the face ABFE lies on the table. Find the locus of (*a*) E, (*b*) H, (*c*) the mid-point of HG, (*d*) any fixed point P on the wood which you care to pick out for yourself.

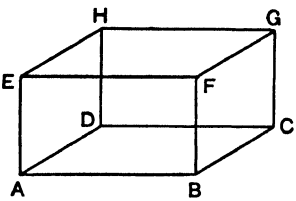


FIG. 147.

18. Take two points A and B, 3 cm. apart. Mark about 12 points which are equidistant from A and B, taking points on both sides of the line AB. What does the locus of points equidistant from A and B appear to be ?

19. Draw in your exercise book the locus of a point which is

(i) 3″ from the right-hand edge of the paper ;
(ii) equidistant from the top and bottom edges of the paper ;
(iii) one inch from a diagonal of the paper ;
(iv) equidistant from the bottom and left-hand edge of the paper.

20. Draw a st. line XY across the middle of your page. Mark about 12 points which are distant 1″ from XY, taking points on both sides of the line XY. What does the locus of points 1″ from XY appear to be ?

21. Mark a point A on your paper. Take a number of points distant 2 cm. from A. What does the locus of points 2 cm. from A appear to be ?

22. OX, OY are two ⊥ st. lines. P is a point within the angle XOY, and M is the foot of the ⊥ from P to OX.

By plotting sufficient points satisfying the given law, find the locus of P,

(i) if PM=½OM,
(ii) if PM+OM=3,
(iii) if 3PM+2OM=6.

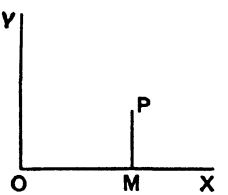


FIG. 148.

[The exercise may be extended by giving *a*, *b*, *c* different values in the equation $a\text{PM} \pm b\text{OM} = c$.]

23. A, B are two fixed points 3″ apart. By plotting sufficient points satisfying the given law, find the locus of P,

(i) if AP+PB=5″, (ii) if AP−PB=2″, (iii) if AP∼PB=2″.

[Take points on both sides of the line AB.]

24. Take any st. line XY, and any point S not on it. By plotting sufficient points satisfying the given law, plot the locus of P,

(i) if SP = the distance of P from XY ;

(ii) if SP = ½ the distance of P from XY.

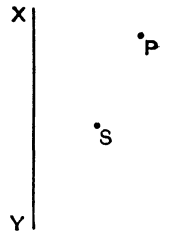


FIG. 149.

25. A, B are two fixed points in a plane, and a variable point P moves in the plane, so that ∠APB = 30°. Find, experimentally, the locus of P. [Stick two pins into your paper at A and B, perpendicular to the paper. Take your set-square, with angles 90°, 60° and 30°, and slide it between the pins so that the arms of the angle 30° of the set-square pass through A, B. Mark on the paper the various positions of the vertex of the angle. Be careful to obtain points on both sides of the line AB.]

Repeat this exercise, if desired, with ∠APB = 45°, 60° or 90°.

26. Take two points, A, B, 1·8″ apart. By plotting sufficient points satisfying the given law, plot the locus of P, if 3AP = 2BP.

[Take points on both sides of the line AB.]

27. Draw a st. line AB, and take any fixed point P outside it. If X is a variable point on AB, plot the locus of the mid-point of PX.

28. Draw any circle, and take any fixed point P outside it. If X is a variable point on the circle, plot the locus of the mid-point of PX.

We must now proceed to consider a typical example, in which the locus must first be found and then proved to be correct. There are in general three stages. In the first stage the locus must be found ; in the second stage it must be proved that any point which obeys the given law lies on the specified locus ; in the third stage it must be proved that if a point lies on the specified locus, it satisfies the given law. The omission of the third stage is a common error which must be guarded against ; it is often, but not always, a simple piece of work (see Theorems G, I, on pp. 371, 374). It is most important that the pupil should realise that the formal proof of a locus always involves the proof of a theorem and its converse.

In searching for the locus it is always useful to notice whether there are any fixed points which satisfy the given law. It is usually sufficient to plot about six points satisfying the given law, but in cases of doubt it may be necessary to plot a larger number, e.g. if the locus consists of the arc of a circle together with a straight line, the pupil would be unlikely to guess the complete locus unless considerably more than six points were plotted.

Two of the most important loci are found in the following theorems:

THEOREM 31

To find the locus of a point which is equidistant from two given points.

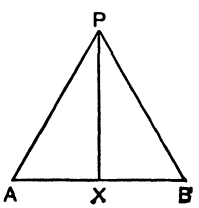


FIG. 150.

Let A, B be two fixed points.

[Analysis. Let X be the mid-point of AB. X is clearly one position of the moving point. Mark a number of points, on both sides of AB, which are equidistant from A and B. All these points appear to lie on the perpendicular bisector of the straight line AB. We therefore try to prove that this is correct.]

1. Let P be **any** position of a point which moves so that PA = PB.

It is required to prove that P lies on the perpendicular bisector of AB.

Construction. Let X be the mid-point of AB. Join PX.

Proof. In the Δs PXA, PXB,

$$PA = PB \text{ (\textit{given})},$$

$$AX = BX \text{ (\textit{construction})},$$

$$PX = PX \text{ (\textit{common})};$$

$$\therefore \text{ the } \Delta\text{s } \begin{matrix} PXA \\ PXB \end{matrix} \text{ are congruent (3 \textit{sides})},$$

$$\therefore \angle AXP = \angle BXP,$$

and each angle is therefore by definition a right angle; $\therefore$ PX is $\perp$ to AB and bisects it,

$\therefore$ P lies on the perpendicular bisector of AB.

2. Let P be a point on the perpendicular bisector of AB.

It is required to prove that PA = PB.

Construction. Let X be the mid-point of AB. Join PA, PB.

Proof. In the Δs PXA, PXB,

$$PX = PX \text{ (\textit{common})},$$

$$AX = BX \text{ (\textit{given})},$$

$$\angle PXA = \angle PXB \text{ (\textit{rt. } $\angle$\textit{s, given})};$$

$$\therefore \text{ the } \Delta\text{s } \begin{matrix} PXA \\ PXB \end{matrix} \text{ are congruent (2 \textit{sides}, \textbf{included} } \angle),$$

$$\therefore PA = PB.$$

Note. It is usual, in examinations, to ask candidates to " prove that the locus of points equidistant from two given points is the perpendicular bisector of the straight line joining the two given points."

Note. In writing out Theorems 31 and 32 the pupil may omit the analysis given in square brackets.

THEOREM 32

To find the locus of points equidistant from two given intersecting straight lines.

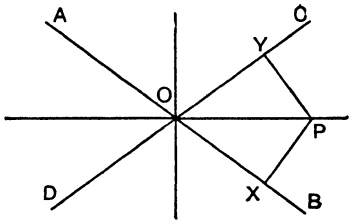


FIG 151.

Let AOB, COD be the given intersecting straight lines.

[Analysis. O is clearly one position of the moving point. Mark a number of points, in each of the four angles formed by the intersecting straight lines, which are equidistant from the straight lines. These points appear to lie on one or other of the bisectors of the angles between these straight lines. We therefore try to prove that this is correct.]

(1) Let P be a position of a point which moves so that the $\perp$s PX, PY, drawn from P to the given lines, are equal.

It is required to prove that P lies on one of the two straight lines bisecting the angles between the given lines.

Consider first a point P situated in the angle BOC.

Construction. Join PO.

Proof. In the $\triangle$s PXO, PYO,

$$PX = PY \text{ (\textit{given})},$$

$$PO = PO \text{ (\textit{common})},$$

$$\angle PXO = \angle PYO \text{ (\textit{rt. } $\angle$\textit{s, given})};$$

$\therefore$ the $\triangle$s $\begin{matrix} PXO \\ PYO \end{matrix}$ are congruent (*rt.* $\angle$, *hypot.*, *side*),

$$\therefore \angle POX = \angle POY,$$

$\therefore$ P lies on the line bisecting the angle BOC.

In the same way, if P is situated in either of the angles COA, AOD, DOB, it lies on the bisector of those angles.

But since BOC, AOD are vertically opposite angles, the bisector of $\angle BOC$ is the same straight line as the bisector of $\angle AOD$. For the same reason the bisector of $\angle COA$ is the same straight line as the bisector of $\angle DOB$, $\therefore$ P lies on one of the **two** straight lines bisecting the angles between the given lines.

(2) Let P be any point on the bisector of one of the angles between the given lines.

It is required to prove that the perpendiculars PX, PY, drawn from P to the given lines, are equal.

Construction. Join PO.

Proof. In the $\triangle$s PXO, PYO,

$$PO = PO \ (\textit{common}),$$

$$\angle PXO = \angle PYO \ (\textit{rt. } \angle s, \textit{ given}),$$

$$\angle POX = \angle POY \ (\textit{given}) ;$$

$\therefore$ the $\triangle$s $\begin{matrix} PXO \\ PYO \end{matrix}$ are congruent (2 $\angle s$, **corresponding** *side*),

$$\therefore PX = PY.$$

Note. It is usual, in examinations, to ask candidates to "prove that the locus of points equidistant from two given intersecting straight lines is the pair of straight lines which bisect the angles between the two given lines."

Note. These theorems, together with the loci found in the next exercise, are most important, and should be mastered by the pupil before the rest of this section is attempted.

EXERCISE 35

1. Find the locus (a) in a plane, (b) in space of a point P which moves so that its distance from a fixed point O is constant.

2. Find the locus (a) in a plane, (b) in space of a point P which moves so that its distance from a fixed straight line AB is constant.

3. Find the locus of a point P which moves in a plane so that its distance (measured parallel to a given straight line) from a fixed straight line AB is constant.

4. Find the locus (a) in a plane, (b) in space of a point P which moves so that it is equidistant from two fixed parallel straight lines.

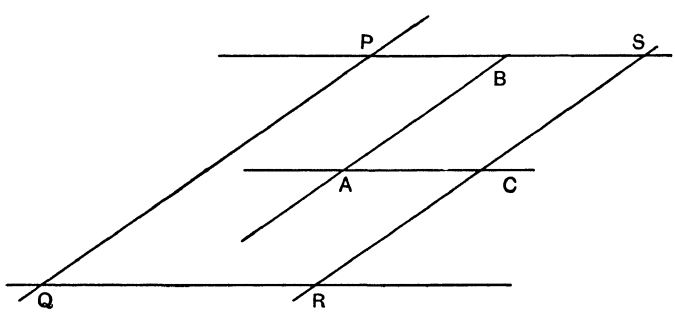
5. Extend Theorem 31 to three dimensions.

6. Extend Theorem 32 to three dimensions.

INTERSECTION OF LOCI

The problem of finding a point which satisfies two conditions may be solved as in the following examples.

Ex. 1. Given two straight lines AB, AC, find as many points as possible which are distant 1″ from AB and $1\frac{1}{2}$″ from AC.



Scale 1 cm. = 1 inch

FIG. 152.

The locus of a point which moves so as to be always distant 1″ from AB is two straight lines || to AB, the distance between AB and each line being 1″. Similarly, the locus of a point which

moves so as to be always distant $1\frac{1}{2}$" from AC is two straight lines || to AC, the distance between AC and each line being $1\frac{1}{2}$". Draw these two loci. The points of intersection of the loci, i.e. P, Q, R, S, are the points which satisfy both conditions.

Ex. 2. Find the points which are equidistant from two fixed points A, B, and which are distant 2·4 cm. from another fixed point C.

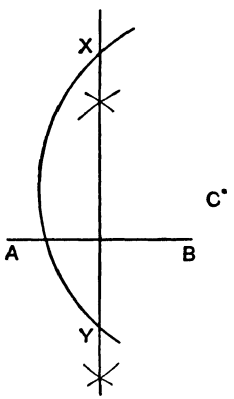


FIG. 153.

The locus of a point which is equidistant from A and B is the perpendicular bisector of AB.

The locus of a point which is distant 2·4 cm. from C is a circle with centre C and radius 2·4 cm.

Draw these two loci. The points of intersection of the loci, i.e. X, Y, are the points which satisfy both conditions.

The following example should be omitted in a first reading of the book.

The beginner is apt to apply the standard locus theorems to a particular case without making careful reference to the figure. In so doing there is a risk (i) that points not on the locus may be included, (ii) that points on the locus may be omitted. Careful study of the following example will show the need of exercising caution in the application of these theorems.

Ex. A point P moves so that its distance from the line AOB exceeds its distance from the line COD by 2 cms. What is the locus of P ?

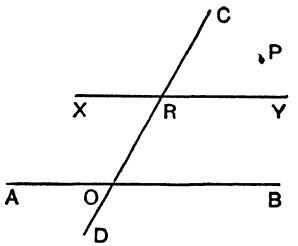


FIG 154.

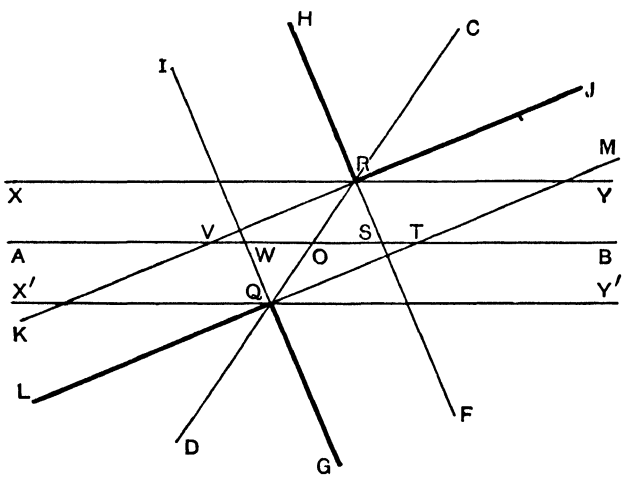


FIG. 155.

The beginner might possibly reason as follows. " If a line XRY is drawn || to AOB, 2 cm. from it, meeting CO at R, P must be equidistant from this line and from CD ; ∴ the locus of P is the bisectors of the angles between these lines." It will be seen from the large figure that this is not so.

Consider the bisectors of the angles between CD and XY, JRVK and HRSF. It is clear that any point on RO or RH satisfies

the required condition, but points between R and V and between R and S do not. For such points the *sum* of the distances from AOB and COD is 2 cms. Similarly for points on SF or VK the distance from AOB is *less than* its distance from COD by 2 cms. Even so, the portions RJ and RH do not constitute the complete locus, for, by drawing X'Y', on the other side of AOB, || to it and 2 cms. from it, and considering the bisectors of the angles between CD and X'Y', it will be seen that the portions QL, QG of these bisectors also form part of the locus. The complete locus is therefore four segments of four straight lines.

The pupil should verify, by considering the lines drawn || to COD at a distance of 2 cms., that no more points can be obtained on the locus.

Note. In the above work the length of the perpendicular from a point to a line has been taken to mean the actual measured distance, without consideration of sign. At a suitable stage the teacher may use this example to illustrate how the introduction of a convention of sign modifies the above discussion.

EXERCISE 36

1. In a given line a point A is taken, and B is a point outside the line. Find a point in the line equidistant from A and B. Can such a point always be found ?

2. Find a point, or points, equally distant from AB, AC, and also equally distant from the points B and C.

3. Draw a st. line, XY, roughly diagonally across the page, and fix a point O in it, about the middle of the page. Then find two points A and B, on the same side of XY, to satisfy the following conditions : B is to be $1\frac{1}{2}''$ from A and $1''$ from the line XY. A is to be $2\frac{1}{2}''$ from O and $2''$ from the line XY.

4. A tree is equidistant from two straight paths AB, AC which intersect at an angle of 30°, and is 70 yards distant from a third straight path which crosses the other two at B and C. AB is 500 yards and AC 250 yards. Find all the possible positions of the tree and their distances from A.

5. Draw a $\triangle ABC$ in which $AB=3''$, $BC=2{\cdot}4''$, $\angle ABC=47^\circ$. Find a point X such that X lies on the median through A and $BX=AX$.

6. Find a point which is at a given distance from a given point, and equidistant from two intersecting st. lines. Discuss the number of solutions possible in various cases.

7. In a st. line XY find a point equidistant from two given st. lines AB, AC. Is it always possible to do this ?

8. In a st. line XY find a point equidistant from two given parallel st. lines AB, CD. When is it impossible to do this ?

9. (i) Draw the bisector of the angle between two st. lines which meet off the paper ; (ii) OX, OY are two given st. lines. A is a fixed point. Through A draw a line to make equal angles with OX, OY. How many solutions are there ?

10. Find a point equidistant from three given points X, Y, Z, which are not in the same st. line.

11. Construct a $\triangle$, having given the base, the altitude, and the length of the median which bisects the base.

12. PQ, XY are two given st. lines. Find points 2 cm. distant from PQ and 2·5 cm. from XY. When is it impossible to do this ?

13. Find a point which is at a given distance from a given point, and equidistant from two given parallel st. lines. Discuss the three possible cases.

14. Find a point equidistant from the three sides of a $\triangle$.

15. Draw two $\perp$ st. lines OAB, OCD, making $OA=2{\cdot}5''$ and $AB=OC=CD=1''$. Find a point P so that the $\triangle$s PAB and PDC may be congruent.

16. A, B, C, D are four points such that $AC=AD$ and $BC=BD$. Prove that any point on the st. line through A and B is equidistant from C and D.

17. Draw a $\triangle$ with $AB=3''$, $BC=6''$, $CA=5''$. Find points P, Q in the side BC such that (i) P is equidistant from the sides CA, AB ; (ii) Q is one inch nearer to the side CA than to the side AB. Measure the distances of P and Q from the sides CA, AB.

18. A, B are two points $3{\cdot}4''$ apart and BC is a line such that $\angle ABC=55°$. By geometrical constructions and measurement find the distances from A of (1) a point P in BC which is equidistant from A and B ; and (2) a point Q in BC which is $1{\cdot}2''$ nearer B than A.

19. AB, AC are two $\perp$ lines ; $AB=2''$, $AC=3''$. Find, on BA produced, a point P such that PB is $1''$ longer than PC. Measure PB, PC.

20. Two st. lines OX, OY are $\perp$, and from P, a point within the angle XOY, $\perp$s PM, PN are drawn to OX, OY respectively. Plot the locus of P when $PM+PN=3''$. Then find, by geometrical reasoning, the locus of P. How would your answers be modified if P is not confined to the arm of the angle XOY ?

21. AB, CD are two || st. lines. O is a fixed point. Through O is drawn a st. line meeting AB, CD at P, Q. Find the locus of the mid-point of PQ.

22. A, B, C are three fixed points, in order, upon a st. line. P is a variable point such that $\angle PBC = 2\angle PAB$. Prove that the locus of P is a circle.

23. Construct the $\triangle ABC$, given

(i) base $BC = 7{\cdot}7$ cm., height $= 4{\cdot}3$ cm., $\angle C = 42°$;

(ii) base $BC = 3{\cdot}8''$, $\angle B = 72°$, median $AX = 2{\cdot}4''$;

(iii) base $BC = 3{\cdot}8''$, $\angle B = 72°$, median $CZ = 4{\cdot}3''$;

(iv) base $BC = 8{\cdot}2$ cm., height $AD = 3{\cdot}2$ cm., median $AX = 5{\cdot}3$ cm. ;

(v) base $BC = 5$ cm., area $= 10$ sq. cm., median $AX = 4{\cdot}3$ cm.

24. OA, OB are two fixed $\perp$ rods. A st. rod PQ of given length slides between them, P being always on OA and Q being always on OB. The rectangle OPSQ is drawn. Find the locus of S.

25. The sides AB, BC of a $\triangle$ are $11{\cdot}2$ cm. and 5 cm. long respectively, and the length of the $\perp$ from C on AB is 3 cm. Construct the $\triangle$ and measure CA.

26. Draw two st. lines AB, AC containing an angle of $45°$. Find a point P on AB and a point Q on AC such that $\angle APQ = 50°$ and $PQ = 1''$.

27. P is a point outside a circle, centre O, and OA is any st. line drawn through O. Show how to find a point Q in OA such that PQ is bisected by the arc of the circle. Show that the number of solutions may be 2, 1 or 0 according to the position of P.

28. A, B are fixed points, and $\triangle APB$ is a $\triangle$ of constant area ; what is the locus of P ?

29. The area of a $\triangle ABC$ is 3 sq. ins., and the base BC is $3''$, this being the longest side, while AC is the shortest side. Prove that the locus of the vertex consists of segments of two st. lines.

30. The $\triangle PQR$ is of constant area on a fixed base QR ; if X is the mid-point of QR, find the locus of the mid-point of PX.

31. Construct a quad. ABCD, given diagonal $AC = 5{\cdot}3''$, diagonal $BD = 4{\cdot}8''$, distances of A, C from BD $2{\cdot}5''$ and $2''$ respectively, side $AB = 3{\cdot}5''$.

32. Construct a trapezium ABCD, in which AB is || to CD, given

(i) $AB = 4{\cdot}2''$, distance between the || sides $= 2{\cdot}7''$, $AD = 3{\cdot}1''$, $BC = 2{\cdot}9''$;

(ii) $AB = 8$ cm., distance between the || sides $= 5{\cdot}4$ cm. diagonal $AC = 9{\cdot}2$ cm., diagonal $BD = 8{\cdot}9$

How many solutions are there ?

33. ABCD is a parallelogram with fixed base AB and of constant area. Find the locus of the intersection of the diagonals.

34. ABC is a $\triangle$ with fixed base BC and of constant area. Find the locus of the centroid.

35. $\triangle$s ABC of the same area are drawn on the same side of BC. Find the locus of the foot of the $\perp$ from A to the bisector BD of the $\angle$ ABC.

36. Show clearly in a diagram where a point P must lie within a $\triangle$ABC in order that each of the $\triangle$s PBC, PCA, PAB may be less in area than half the $\triangle$ABC.

37. ABDC is a quad. whose sides AB, AC are fixed and the fourth angular point D moves in such a way that the sum of the areas of the $\triangle$s ABD, ACD remains constant. Find the locus of D. Can any part of the locus lie outside the $\angle$ BAC ?

38. XYZ is a fixed $\triangle$. Find the locus of P, given that $\triangle PXY = \triangle PXZ$. [See Ex. 29, No. 22.]

39. AOC, BOD are diagonals of a parallelogram ABCD. P is a variable point such that $\triangle POA + \triangle POB =$ constant. Prove that the locus of P consists of segments of four straight lines.

[Consider separately positions of P in each of the angles AOB, BOC, COD, DOA and notice that $\triangle POA = \triangle POC$, $\triangle POB = \triangle POD$. The area of a $\triangle$ is taken to be represented by a positive quantity in each case. If the convention of sign is introduced, the above result must be modified.]

PART III

THE CIRCLE

THE chief definitions in connection with the circle will be found on p. 5. Other definitions will be introduced as required.

The following properties of a circle are easily seen to be true.

(1) A circle is a closed curve, so that if any point inside the circumference of a circle be joined by a straight line to any point outside it, the straight line will meet the circumference of the circle.

(2) Circles with equal radii are congruent figures.

(3) A point is outside, on, or inside the circumference of a circle, according as its distance from the centre is greater than, equal to, or less than the radius.

(4) Concentric circles, with equal radii, coincide entirely.

(5) Concentric circles, with unequal radii, cannot intersect.

THEOREM 33

(1) The straight line which joins the centre of a circle to the mid-point of a chord (which is not a diameter) is perpendicular to the chord.

(2) The straight line drawn from the centre of a circle perpendicular to a chord bisects it.

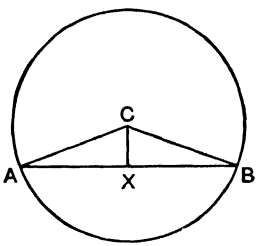


FIG. 156.

Let C be the centre and AB a chord of a circle.

(1) Let X be the middle point of AB. Join CX.

It is required to prove that CX is perpendicular to AB.

Construction. Join CA, CB.

Proof. In the $\triangle$s CAX, CBX,

$CA = CB$ (*radii of the same circle*),

$CX = CX$ (*common*),

$AX = BX$ (*given*) ;

$\therefore$ the $\triangle$s $\begin{matrix} CAX \\ CBX \end{matrix}$ are congruent (3 *sides*),

$\therefore \angle CXA = \angle CXB$,

and each angle is therefore by definition a right angle;

$\therefore$ CX is perpendicular to AB.

(2) Let CX be perpendicular to AB.

It is required to prove that $AX = BX$.

Construction. Join CA, CB

Proof. In the Δs CAX, CBX,

$$CA = CB \text{ (\textit{radii of equal circles})},$$
$$CX = CX \text{ (\textit{common})},$$
$$\angle CXA = \angle CXB \text{ (\textit{rt. angles, given})};$$

$\therefore$ the Δs $\begin{matrix} CAX \\ CBX \end{matrix}$ are congruent (\textit{rt.} $\angle$, \textit{hypot.}, \textit{side}),

$$\therefore AX = BX.$$

Corollary 1. The locus of mid-points of parallel chords of a circle is a diameter of the circle, $\perp$ to the chords.

Corollary 2. The perpendicular bisector of a chord of a circle passes through the centre.

SYMMETRY

The definitions of symmetry about a line and axis of symmetry are given on p. 38. It should be noted that if AB is the axis of symmetry, and if P′ is the point with which P can be made to coincide, then PP′ is bisected at right angles by AB. The point P′ is said to be the **image** of P in the axis. The image of P′ in the axis is easily seen to be P.

A figure may also be symmetrical about a point O.

Definition. A figure is said to be **symmetrical about a point** O, if, for every point P, there is a corresponding point P′ such that O bisects the straight line PP′. O is called the **centre of symmetry.**

Examples of both kinds of symmetry may be found by considering many common figures and objects. The pupil will most easily grasp the idea of symmetry if the following questions are discussed orally:

1. What types of triangle have (*a*) symmetry about a line, (*b*) symmetry about a point?

2. What types of quadrilateral have (*a*) symmetry about a line, (*b*) symmetry about a point?

3. Consider the capital letters of the alphabet. How many of these have symmetry of either kind?

4. Prove that a circle is symmetrical about any diameter.

5. Prove that a circle is symmetrical about its centre.

The teacher who wishes to do so may now point out that many properties of circles, which are usually proved by the use of congruent triangles, may easily be proved by considerations of symmetry.

THEOREM 34

There is one circle and only one circle which passes through three given points not in the same straight line.

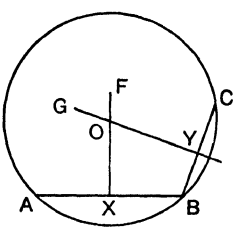


FIG. 157

Let A, B, C be three points not in the same straight line.

It is required to prove that one circle, and only one, can pass through A, B, C.

Construction. Join AB, BC. Draw XF, YG, the perpendicular bisectors of AB, BC respectively.

Then, since AB, BC are not in the same straight line, XF and YG are not parallel. Let XF, YG meet at O.

Proof. Because XF bisects AB at right angles, $\therefore$ XF is the locus of points which are equidistant from A and B.

Similarly YG is the locus of points which are equidistant from B and C;

$\therefore$ all points equidistant from A, B, C must lie both on XF and on YG ;

$\therefore$ O is a point, and the only point, which is equidistant from A, B, C,

$\therefore$ the circle with centre O and radius OA passes through A, B, C, and there is no other circle which passes through these three points.

Note. In the Construction above we have shown how to draw a circle passing through the vertices of any triangle.

Corollary 1. The size and position of a circle are completely determined if it is known to pass through three given points.

For then the position of the centre and the length of the radius can be uniquely determined.

Corollary 2. Two circles cannot cut one another in more than two points without coinciding entirely.

For if they have a third common point they coincide.

Corollary 3. If O is a point within a circle from which three equal straight lines OA, OB, OC can be drawn to the circumference, O is the centre of the circle.

EXERCISE 37 [Theorems 33-34]

1. The diameter of a circle is 5 cm. long, a chord is 4·8 cm. long. Calculate its distance from the centre.

2. A chord of a circle is 6 cm. long and 1·6 cm. from the centre. Calculate the radius of the circle.

3. Calculate the length of a chord which is 2·5″ distant from the centre of a circle whose radius is 6·5″.

4. In a circle of radius 6″ a chord is placed equal in length to the radius. Calculate its distance from the centre.

5. A chord distant 2 cm. from the centre of a circle is 18 cm. long. Calculate the length of a chord of the same circle which is 6 cm. distant from the centre.

6. Two chords of a circle of diameter 18″ are of lengths 16″ and 14″ and intersect at right angles. Calculate the distance of their point of intersection from the centre.

7. Two parallel and equal chords of a circle whose radius is 6″ are 2″ apart. Find the length of each chord.

8. AB, CD are parallel chords of a circle on the same side of the centre. AB = 3″, CD = 1·4″, and the distance between the chords is 0·4″. Find the radius of the circle.

9. Repeat Ex. 8, taking the distance between the chords to be 4·4″ and the chords on opposite sides of the centre.

10. Find the locus of mid-points of parallel chords of a circle.

11. If two circles, centres A, B, cut at X, Y, prove that (i) AB is ⊥ to XY, (ii) AB bisects XY.

12. AB, AC are equal chords of a circle, centre O. Prove that $\angle OAB = \angle OAC$.

13. Find the locus of centres of circles passing through two fixed points A, B.

14. AB, AC are two chords of a circle which make equal angles with the line joining A to the centre of the circle. Prove that the chords are equal.

15. If two chords of a circle bisect one another, prove that their point of intersection must be the centre of the circle. [Use Th. 33, Cor. 2.]

16. ABCD is a parallelogram inscribed in a circle. Prove that the centre of the circle must be at the intersection of the diagonals of the parallelogram. Deduce that the parallelogram must be a rectangle.

17. A circular field is crossed by two parallel paths, 66′ apart; the lengths of the paths are 144′ and 120′. Calculate the length of the diameter of the field.

18. A st. line cuts two concentric circles, one at A and B, the other at C and D. Prove that AC = DB. Prove also that the mid-points of AC, DB are equidistant from the centre of the circle.

19. A line is drawn through a point of intersection of two circles ∥ to the line of centres and terminated by the circumferences. Prove that its length is twice that of the line joining the centres.

20. Construct the chord of a given circle which is bisected at a given point P within the circle.

21. A right circular cylinder is 5″ long and its ends are circles of radius 7·5″. A chord XY, 9″ long, is drawn on one end of the cylinder, and two chords PQ, RS, equal and ‖ to XY, on the other end. Find the lengths of PX, RX.

22. AB is a diameter of a sphere of radius 2·5″, and a plane ⊥ to AB cuts it 1″ from A. Prove that the section of the sphere by this plane is a circle, and find its radius.

23. A rectangular piece of cardboard, 4″ by 3″, is fitted into a hemispherical bowl of diameter 13″. Find the distance of the intersection of the diagonals from the centre of the bowl.

24. AB, CD are two given finite st. lines in a plane. Show how to draw two separate concentric circles, one having AB for a chord, the other having CD for a chord. When is this impossible ?

THEOREM 35

(1) **Equal chords of a circle are equidistant from the centre.**

(2) **Chords of a circle which are equidistant from the centre are equal.**

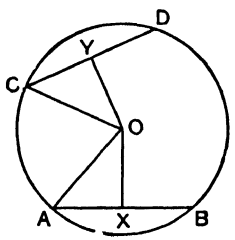


FIG. 158.

(1) Let AB, CD be equal chords of a circle of which O is the centre.

It is required to prove that the perpendiculars OX, OY to these chords are equal.

Construction. Join OA, OC.

Proof. Since the perpendicular drawn from the centre of a circle to a chord bisects it,

$$\therefore\ AX = \tfrac{1}{2}AB,\ CY = \tfrac{1}{2}CD.$$

But $AB = CD$ (*given*),

$$\therefore\ AX = CY;$$

$\therefore$ in the $\triangle$s AXO, CYO,

$AO = CO$ (*radii of the same circle*),

$AX = CY$ (*proved*),

$\angle OXA = \angle OYC$ (*rt. angles, given*);

$\therefore$ the $\triangle$s $\begin{matrix}AXO\\CYC\end{matrix}$ are congruent (*rt.* $\angle$, *hypot., side*),

$$\therefore\ OX = OY.$$

(2) Let AB, CD be chords of a circle of which O is the centre. Let the perpendiculars OX, OY to these chords be equal. It is required to prove that AB = CD.

Construction. Join OA, OC.

Proof. In the △s AXO, CYO,

$$AO = CO \text{ (\textit{radii of the same circle})},$$

$$OX = OY \text{ (\textit{given})},$$

$$\angle OXA = \angle OYC \text{ (\textit{rt. angles, given})};$$

$$\therefore \text{ the } \triangle s \begin{matrix} AXO \\ CYO \end{matrix} \text{ are congruent (\textit{rt.} } \angle, \textit{ hypot., side}),$$

$$\therefore AX = CY.$$

But the perpendicular drawn from the centre of a circle to a chord bisects it,

$$\therefore 2AX = AB, \ 2CY = CD;$$

$$\therefore AB = CD.$$

Corollary 1. The results of this theorem are also true for chords of equal circles. The proof is almost identically the same.

Corollary 2. The locus of mid-points of equal chords of a circle is a concentric circle. It will be seen later (Th. 46) that this circle touches all the chords.

* THEOREM 36

(1) If two chords of a circle be unequal, the greater is nearer the centre.

(2) If two chords of a circle be at different distances from the centre, the one nearer the centre is the greater.

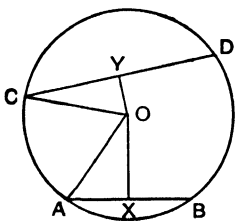


FIG. 159

Let AB, CD be chords of a circle of which the centre is O. Let OX, OY be the perpendiculars from O to AB, CD respectively.

It is required to prove that

(1) If CD > AB, then OY < OX.

(2) If OY < OX, then CD > AB.

Construction. Join OA, OC.

Proof. Since the perpendiculars drawn from the centre of a circle to a chord bisects it,

$$\therefore \quad AX = \tfrac{1}{2}AB, \quad CY = \tfrac{1}{2}CD.$$

But since the angles OXA, OYC are right angles (*given*), we have

$$OX^2 + AX^2 = OA^2 \text{ (\textit{Pythagoras'})}$$
$$= OC^2 \text{ (OA = OC, \textit{radii of the same circle})}$$
$$= OY^2 + CY^2 \text{ (\textit{Pythagoras'})};$$

$\therefore$ (1) If CD > AB, then $\tfrac{1}{2}CD > \tfrac{1}{2}AB$, i.e. CY > AX;

$\therefore CY^2 > AX^2$; $\therefore OY^2 < OX^2$; $\therefore$ OY < OX.

(2) If OY < OX, then $OY^2 < OX^2$;

$\therefore CY^2 > AX^2$; $\therefore$ CY > AX, $\therefore \tfrac{1}{2}CD > \tfrac{1}{2}AB$; $\therefore$ CD > AB.

Corollary 1. The greatest chord in a circle is a diameter.

Corollary 2. The results of this theorem are also true for chords of equal circles. The proof is almost identically the same.

Note. The above proof may be adapted so as to provide an alternative proof for the results of Theorem 35.

EXERCISE 38 [Theorems 35-36]

1. Find the locus of the mid-points of equal chords of a circle.

2. AB and CD are two equal chords of a circle with N and M their mid-points. Prove that NM makes equal angles with AB and CD.

3. APQ, AXY are secants drawn from an external point A to a circle, centre O, and the chords PQ, XY are equal. Prove that the secants are equally inclined to AO, and that AP=AX.

4. APQ, AXY are secants drawn from an external point A to a circle, centre O, and the secants are equally inclined to AO. Prove that the chords PQ, XY are equal.

5. AB, AC are two st. lines, and circles are drawn so that the intercepts DE and FG which any one of them makes upon AB and AC are equal. Find the locus of the centres of the circles.

6. AXB, CXD are equal chords of a circle intersecting at a point X within the circle. Prove that AX is equal either to CX or DX.

7. What is (i) the greatest, (ii) the least possible chord which can be drawn through a given point within a circle ?

8. PQ is a fixed chord of a circle, centre O. C is a point on PQ. Draw the chord AB which is bisected at C. Describe the variation of the length of the chord AB as C moves along PQ from P to Q.

9. ABC is a $\triangle$. Construct a circle passing through B and C and cutting off equal chords on AB and AC (produced if necessary).

EXERCISE 39 [Further examples on Theorems 32-36]

1. A circle has radius 2·9″, and chords of length 1·8″ are placed in it ; on these chords as bases are drawn isosceles $\triangle$s of side 2·3″ on the sides of the chords remote from the centre of the circle. Find the locus of the vertices of these $\triangle$s.

2. Prove that the feet of the $\perp$s from the ends of a diameter of a circle to any chord (produced if necessary) are equally distant from the ends of the chord.

3. If two circles which intersect are cut by a straight line || to the common chord, show that the parts of it intercepted between the circumferences are equal.

4. A st. line is drawn meeting each of three given circles. How must the circles be situated if the mid-points of the three chords so formed coincide ?

5. Two circles have radii 1·2″, 2·3″ and their centres are 0·75″ apart. Calculate to the nearest tenth of an inch the lengths of the greatest and least chords of the outer circle which can be drawn to touch the inner.

6. If two chords of a circle are equal, prove that they are equidistant from any point on the longest chord drawn through their point of intersection.

7. Two circles, whose centres are X and Y, intersect at R and S, and O is the mid-point of XY. A chord is drawn through R ⊥ to OR, meeting the circles again at A and B. Prove that AR=RB.

8. A st. line is drawn cutting two unequal circles, || to the line joining their centres. Prove that the chord intercepted by the smaller circle is shorter than the chord intercepted by the greater.

9. Two unequal circles, centres X, Y, meet at P, Q. Through P is drawn a st. line meeting the circles again at A and B. Prove that the chord APB is always less than 2XY, unless it is || to XY. What is the greatest possible value of the length AB ?

10. Two chords AB, CD of a circle with centre S cut at right angles at X, within the circle. Prove that $AB^2+CD^2+4SX^2=8SA^2$.

11. From two points X, Y, in the diameter of a semi-circle, equidistant from the centre, parallel straight lines are drawn to meet the circumference at A, B. Show that, if A, B be joined, ∠XAB=∠YBA=90°.

THEOREM 37

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Let BC be an arc of a circle of which O is the centre, and let A be any point on the remaining part of the circumference.

It is required to prove that ∠BOC=2∠BAC.

Construction. Join AO and produce it to X. (In Fig. 3 X is on OC.)

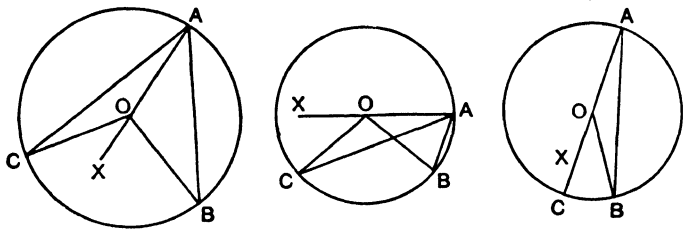


Fig. 1 Fig. 2 Fig. 3

FIG. 160.

Proof. The exterior angle BOX of the $\triangle AOB$ equals the sum of the interior opposite angles OAB, OBA.

But since $OA = OB$ (*radii of the same circle*),

$\therefore \angle OAB = \angle OBA$ (*base angles of an isosceles triangle*),

$\therefore \angle BOX = 2\angle OAB$. (In Fig. 3 this completes the proof.)

Similarly it may be proved that $\angle COX = 2\angle OAC$,

$\therefore$ In Fig. 1 $\angle BOX + \angle COX = 2\angle OAB + 2\angle OAC$,

$\therefore \angle BOC = 2\angle BAC$.

In Fig. 2 $\angle BOX - \angle COX = 2\angle OAB - 2\angle OAC$,

$\therefore \angle BOC = 2\angle BAC$.

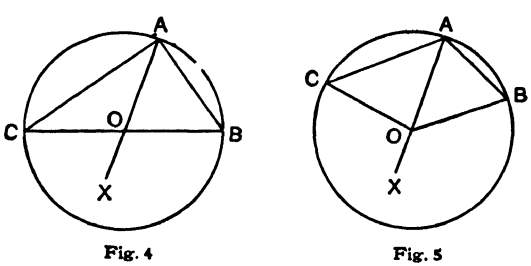


Fig. 4 Fig. 5

FIG. 161.

Note. The proof for Fig. 1 applies without change to Figs. 4 and 5, so that the theorem is true whether the given arc is less than, equal to, or greater than half the circumference of the circle.

DEFINITIONS

If a chord of a circle does not pass through the centre of the circle it divides the circumference into two unequal arcs; of these, the greater is called the **major arc**, and the less the **minor arc.** The major arc is greater, and the minor arc less than half the circumference. The major and minor arcs, into which a circumference is divided by a chord, are said to be **conjugate** to one another. The segment bounded by the chord and the major arc is called the **major segment**, that bounded by the chord and the minor arc, the **minor segment.**

The angle subtended at a point P by two points A and B (or by the straight line AB) is the $\angle APB$.

An **angle in a segment** of a circle is an angle subtended by the chord of the segment at a point on the arc of the segment. The segment is said to **contain** the angle.

Similar segments of circles are those which contain equal angles.

A rectilineal figure which is such that all its vertices lie on the circumference of a circle is said to be **inscribed in** the circle.

The circle is said to be **described about** the figure or **circumscribed to** it.

The circle which passes through the vertices of a triangle is called the **circum-circle** of the triangle; its radius is called the **circum-radius** and its centre the **circum-centre.**

A rectilineal figure which is such that a circle can be described about it is said to be **cyclic.**

Four or more points such that a circle can be drawn through them are said to be **concyclic.**

THEOREM 38

Angles in the same segment of a circle are equal to one another.

Let ADCB be a segment of a circle ADCBX, centre O, and let ADB, ACB be any angles in the segment.

It is required to prove that $\angle ADB = \angle ACB$.

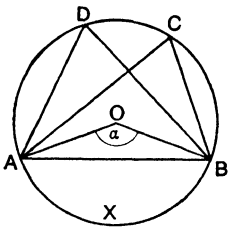


Fig. 1

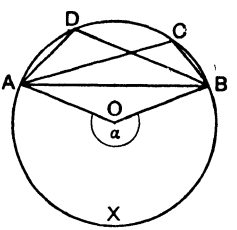


Fig. 2

FIG. 162.

Construction. Join OA, OB.

Proof. The arc AXB subtends at the centre of the circle an angle AOB which is double that which it subtends at any point on the remaining part of the circumference ;

$$\therefore \quad \angle AOB(\alpha) = 2\angle ADB,$$

and

$$\angle AOB(\alpha) = 2\angle ACB ;$$

$$\therefore \quad 2\angle ADB = 2\angle ACB,$$

$$\therefore \quad \angle ADB = \angle ACB.$$

Note. The given segment may be greater than a semi-circle, as in Fig. 1, or less than a semi-circle, as in Fig. 2 : in the latter case the angle AOB will be reflex. But, since Theorem 37 was proved for all possible cases, the above proof applies to both figures.

Corollary 1. Angles in a major segment are acute ; angles in a minor segment are obtuse.

In Fig. 1 the angles ADB, ACB are in a major segment, so that $\angle AOB$ is less than two right angles ;

$\therefore$ $\angle ADB$ which equals $\frac{1}{2}\angle AOB$ is less than one right angle, i.e. $\angle ADB$ is acute.

In Fig. 2 the angles ADB, ACB are in a minor segment, so that $\angle AOB$ is reflex, i.e. greater than two right angles ; $\therefore$ $\angle ADB$ which equals $\frac{1}{2}\angle AOB$ is greater than one right angle, i.e. $\angle ADB$ is obtuse.

EXERCISE 40 [Theorems 37, 38]

1. AXB, CXD are two chords of a circle, meeting at X. If $\angle XAC = 32°$ and $\angle XCA = 40°$, find $\angle$s XBD, XDB.

2. X is a point outside a circle. XAB, XCD are two st. lines cutting the circle at A, B, C, D. $\angle ABC = 27°$, $\angle CXB = 37°$, find $\angle$s ADC, XAD.

3. A, B, C, D are four points in order on a circle. $\angle BAD = 62°$, $\angle CDA = 57°$, $\angle ABD = 82°$. Find $\angle$s ACD, BDA, ACB, DBC.

4. ABCD is a cyclic quadrilateral. $\angle ADB = 22°$, $\angle BDC = 43°$, $\angle ACD = 58°$. Find $\angle$s CAB, BCA, DAB.

5. B, C, D are three points on a circle, O the centre. $\angle BOC = 38°$, $\angle OCD = 24°$. Find $\angle$s DBC, BCD.

6. AXB, CXD are two chords of a circle. If $\angle AXC = 82°$, $\angle ADX = 37°$, find $\angle BCD$.

7. AXB, CXD are two chords of a circle, centre O. If $\angle AXC = 72°$, $\angle ADX = 23°$, find $\angle BOD$.

8. A, B, C, D are four points in order on the circumference of a circle: and the arcs AB, BC, CD subtend at the centre $\angle$s of 60°, 90° and 100° respectively. Join the points and find the $\angle$s of the quad. ABCD and the $\angle$s between the diagonals AC, BD.

9. B, A, C are points on a circle whose centre is O and $\angle BAC = 115°$. Calculate in degrees $\angle$s BOC, OBC.

10. Two chords AB, CD of a circle intersect within the circle; AC subtends an $\angle$ of 20° at the centre, and BD subtends an $\angle$ of 30° at the centre. Find the acute $\angle$ between the chords.

11. P is any point on the circumference of a circle, centre M; Q is any point on the circumference of a circle, centre N. The circles meet at A and B; $\angle AMB = 110°$, $\angle ANB = 150°$. Find all possible values for the sum of the $\angle$s APB, AQB.

12. Two radii OA, OB contain an angle of 130°, and a point C is taken on the minor arc AB such that $\angle OAC = 45°$. CB is joined. Calculate the remaining $\angle$s of the quad. OACB.

13. ABCD are four points in that order on a circle. $\angle DAC = 25°$, $\angle ADB = 33°$, $\angle ABD = 41°$. Find each of the $\angle$s of the quad. ABCD.

14. ABCD is a cyclic quad.; AC, BD intersect at O. $\angle ABC = 70°$, $\angle BAD = 95°$, $\angle BOC = 85°$. Calculate $\angle BAC$, and prove that AB = BC.

15. The $\triangle ABC$ is inscribed in a circle, centre O. If AOCB is a parallelogram, prove that $\angle AOC = 120°$

16. OP, OQ are two radii of a circle inclined at an angle 48°. The diameter SR of the circle is drawn || to PQ. Calculate the acute ∠ between the diagonals of the quad. PQRS.

17. ABCD is a cyclic quad. AB, DC produced meet at X and AC, BD meet at Y. Prove that the following pairs of △s are equiangular : (i) ABY, DCY ; (ii) AYD, BYC ; (iii) XAC, XDB.

18. ABC is an acute-angled △ and O is the centre of the circle ABC ; prove that ∠OCB+∠BAC=90°.

19. If X is any point on the arc of a segment of which AB is the chord, prove that ∠XAB+∠XBA is the same for all positions of X.

20. Two circles intersect at X, Y. Through X two st. lines AXB, CXD are drawn meeting the first circle at A, C, and the second circle at B, D. Prove that ∠AYC=∠BYD.

21. RST is an angle in the minor segment of a circle whose centre is O. Prove that ∠RST−∠ORT=90°.

22. APB, CPD are two ⊥ chords of a circle which cut at P, within the circle, and O is the centre of the circle. Prove that the ∠s AOD, BOC are supplementary.

23. ABC is a △, obtuse-angled at B, with AB=CB. A circle, passing through A and B, cuts AC at D and CB produced at E. Prove that DC=DE.

24. ABCD is a cyclic quad. ; AN is || to BC, meeting BD, not produced, at N. Prove that ∠ADN=∠NAC.

25. The bisectors of the angles of a △ABC meet the circumscribing circle at A′, B′, C′. Find the magnitudes of the ∠s A′, B′, C′ of the △A′B′C′ in terms of A, B, C.

26. O is the intersection of the ⊥s drawn from the angular points of a △ABC to the opposite sides ; AO meets BC at D and the circle ABC at E. Prove that ∠EBD=∠DBO, and that ED=DO.

27. AB is the chord of a given segment of a circle. C is the point on the segment which is equidistant from A and B. With centre C and radius CA, describe a circle. APQ is a st. line through A meeting the first arc at P and the other circle at Q. Prove that PQ=PB.

28. An equilateral △ABC is inscribed in a circle whose centre is O ; OD is the radius ⊥ to BC. Prove that OBD, OCD are equal equilateral △s.

29. Two circles intersect at A and B. Through A any st. line XAY is drawn terminated by the circumferences. Prove that ∠XBY is the same for all positions of XY.

30. The internal bisectors of the ∠s B, C of a △ABC meet the ⊙ ABC at Q, R respectively. Prove that, if BR is ‖ to CQ,

$$\angle BAC = 60°.$$

31. P is a point on a semi-circle whose diameter is AB and centre O ; AP is produced to T ; if PT=OP, prove that

$$\angle POT = \tfrac{1}{3}\angle TOB.$$

32. Two segments of circles are described on the same side of a st. line AB. Any point P is taken on the outer segment, and PA meets the inner segment at Q. Prove that ∠PBQ is the same for all positions of the point P.

33. Prove that the bisector of the angle A of a △ABC is equally inclined to the ⊥ from A to BC and to the line joining A to the centre of the ⊙ABC.

34. Construct a △ABC, given ∠A=64°, ∠AOB=112°, AB=1·5″, where O is the centre of the ⊙ABC.

THEOREM 39

If the straight line joining two points subtends equal angles at two other points on the same side of it, then the four points are concyclic.

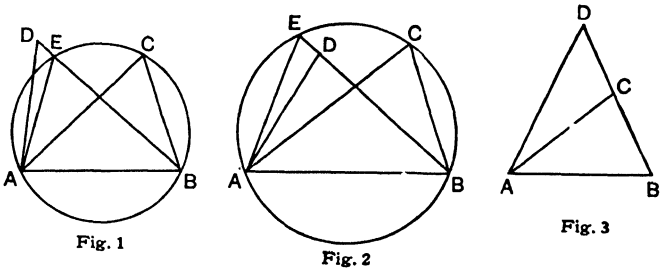


FIG. 163.

Let AB be a straight line subtending equal angles at two points C, D on the same side of AB.

It is required to prove that A, B, C, D lie on a circle.

Proof. It is not possible for BC to coincide with BD, for in that case (see Fig. 3) we should have the exterior angle ACB of

the $\triangle ACD$ equal to the interior opposite angle ADB, which is impossible;

$\therefore$ one of the angles ABC, ABD must be greater than the other.

Suppose that $\angle ABC$ is the greater, so that BD lies in the angle ABC.

Draw a circle through ABC and suppose that it does not pass through D.

Since BD lies in the angle ABC, the circle ABC must cut BD or BD produced. Let E be the point of intersection (see Figs. 1 and 2).

Then $\angle AEB = \angle ACB$ (*angles in the same segment*).

But $\angle ACB = \angle ADB$ (*given*),

$\therefore$ $\angle AEB = \angle ADB$.

But this is impossible because one of these is the exterior angle and the other is the interior opposite angle of the $\triangle ADE$;

$\therefore$ our supposition that the circle does not pass through D is wrong;

$\therefore$ A, B, C, D lie on a circle.

Corollary. If a fixed straight line AB subtends a constant angle at a variable point P, the locus of P consists of a segment of a circle standing on the chord AB, containing the given angle, together with an equal segment standing on the chord AB and on the opposite side of it.

THEOREM 40

The angle in a semi-circle is a right angle.

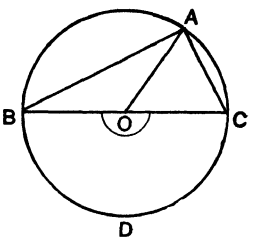


FIG. 164.

Let ABDC be a circle of which O is the centre and BOC a diameter.

It is required to prove that ∠BAC is a right angle.

1st Proof. The ∠BAC subtended at the circumference by the arc BDC is half the ∠BOC subtended by it at the centre.

But ∠BOC = 2 right angles,

∴ ∠BAC = 1 right angle.

2nd Proof. Join OA.

Then OA = OB (*radii of the same circle*),

∴ ∠OAB = ∠OBA (*base angles of an isosceles* △) ;

Also OA = OC (*radii of the same circle*),

∴ ∠OAC = ∠OCA (*base angles of an isosceles* △) ;

∴ The whole ∠BAC = ∠OAB + ∠OAC

= ∠OBA + ∠OCA.

But the 3 angles of the △ABC together = 2 right angles,

∴ ∠BAC = one-half of 2 right angles,

= one right angle.

THEOREM 41

The circle described on the hypotenuse of a right-angled triangle as diameter passes through the vertex.

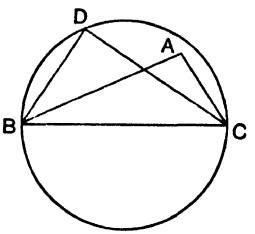


FIG. 165.

Let ABC be a triangle with $\angle BAC$ a right angle.

It is required to prove that the circle on BC as diameter passes through A.

1st Proof. Describe the circle on BC as diameter.

On its circumference, on the same side of BC as A, take a point D. Join BD, CD.

Then $\angle BDC$, being the angle in a semi-circle, is a right angle ;

$$\therefore \angle BDC = \angle BAC,$$

i.e. BC subtends equal angles at two other points D, A, on the same side of it,

$$\therefore \text{B, C, D, A lie on a circle,}$$

i.e. A lies on the semi-circle on BC as diameter.

2nd Proof.

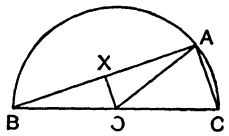


FIG. 166.

Let O be the mid-point of BC, and X the mid-point of AB.

Join OX, OA.

Since the line joining the middle points of two sides of a triangle is parallel to the third, ∴ OX is parallel to CA.

But CA is perpendicular to AB (*given*),

∴ OX is perpendicular to AB,

∴ OX is the perpendicular bisector of AB and the locus of points equidistant from A and B,

∴ OA = OB.

But OB = OC (*given*),

∴ OA = OB = OC ;

∴ The circle, centre O and radius OB, passes through C and A, and this is the circle on BC as diameter.

Corollary. If a fixed straight line AB subtends a right angle at a variable point P, the locus of P is a circle on AB as diameter.

THEOREM 42

1. The opposite angles of a cyclic quadrilateral are supplementary.

2. If one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

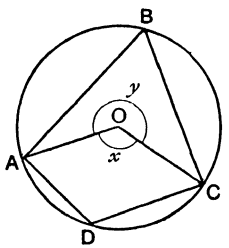


FIG. 167.

1. Let ABCD be a quadrilateral inscribed in a circle of which O is the centre.

It is required to prove that the angles B and D are supplementary and that the angles A and C are supplementary.

1st Proof. Join AO, CO.

Let the arc ADC subtend the angle x at the centre, and let the arc ABC subtend the reflex angle y at the centre.

Then, because the angle which an arc subtends at the centre is double that which it subtends at the circumference,

$$\angle x = 2\angle B \text{ and reflex } \angle y = 2\angle D ;$$

$$\therefore \angle x + \angle y = 2\angle B + 2\angle D.$$

But $$\angle x + \angle y = 4 \text{ right angles},$$

$$\therefore \angle B + \angle D = 2 \text{ right angles}.$$

But the four angles of a quadrilateral are together equal to 4 right angles,

$$\therefore \angle A + \angle C = 2 \text{ right angles}.$$

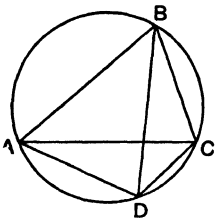


FIG. 168.

2nd Proof. Join AC, BD.

Then $\angle ABD = \angle ACD$ (*$\angle$s in the same segment*),

and $\angle CBD = \angle CAD$ (*$\angle$s in the same segment*);

$\therefore \angle ABC + \angle ADC = \angle ABD + \angle CBD + \angle ADC$

$= \angle ACD + \angle CAD + \angle ADC = 2$ right $\angle$s,

since the three angles of a $\triangle = 2$ right $\angle$s.

But the four angles of a quadrilateral are together equal to 4 right $\angle$s,

$\therefore \angle BAD + \angle BCD = 2$ right $\angle$s.

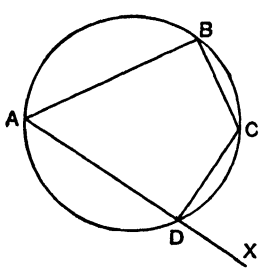


FIG. 169.

2. Let ABCD be a cyclic quadrilateral, and let the side AD be produced to X.

It is required to prove that the exterior angle CDX is equal to the interior opposite angle ABC.

Proof.

$\angle ADC + \angle CDX = 2$ right $\angle$s (*adjacent* $\angle$*s*, ADX *a st. line*),

$\angle ADC + \angle ABC = 2$ right $\angle$s (*opp.* $\angle$*s of a cyclic quad.*),

$\therefore \angle ADC + \angle CDX = \angle ADC + \angle ABC.$

From each of these equals take $\angle ADC$,

$\therefore \angle CDX = \angle ABC.$

This result is frequently of great use in solving riders.

Note 1. It is very desirable that the pupil should use a single small letter to denote an angle, as has been done in the 1st proof above. The working is thus made shorter and easier to read.

Note 2. If the first method of proof is adopted, it is important to draw a figure in which one of the $\angle$s AOC at the centre is clearly a reflex angle. It is also most important to state clearly which of the angles at the centre is twice $\angle B$ and which is twice $\angle D$. In examinations weak candidates often fail to do this and, for this reason, many may prefer to adopt the second proof.

EXERCISE 41 [Theorems 40-42]

1. AB, DC are two chords of a circle which, when produced meet at X outside the circle. If $\angle DAB = 72°$ and $\angle AXD = 31°$, and B, C are joined, calculate all the other angles in the figure. Repeat the calculations, taking $\angle DAB = x°$.

Prove, generally, that the $\triangle$s XBC, XDA are equiangular.

2. A, B, C, D are four points in order on a st. line such that $AB = 2''$, $BC = 1''$, $CD = 3$. Find a point P such that $CP = 2''$ and $\angle BPD = 90°$. Calculate AP.

3. At a certain instant a submarine bears 50° East of North from a point A and 40° West of North from another point B, and B is 70° East of North from A. What is the bearing of the submarine from the mid-point of AB ?

4. In the first figure of Theorem 42, $\angle ABC = 80°$, $\angle OAB = 19°$, $\angle ACD = 13°$; find $\angle$s OCB, DAB.

5. In the first figure of Theorem 42, $\angle OAC = 22°$, find $\angle$s ADC, ABC.

6. In a cyclic quad. ABCD the $\angle$s CAD, BDA, BDC are 15°, 65°, 35° respectively. Find the angles of the quad.

7. ABCD is a cyclic quad. having its diagonals at right angles. $\angle A = 105°$, $\angle B = 93°$. Write down $\angle C$ and $\angle D$. If $\angle ACB = x°$, express the $\angle$s ACD and CDB in terms of x, and calculate the value of x.

8. PQRS is a quad. inscribed in a circle; $\angle QPS = 110°$, $\angle PQS = 30°$, $\angle SQR = 30°$. PR, QS intersect at T. Find the $\angle$s QRS, PRS, QTR.

9. ABCDE is a pentagon inscribed in a circle. If the sides BC, CD, DE subtend $\angle$s of 30°, 40° and 38° respectively at A, while AB subtends an $\angle$ of 35° at C, find all the $\angle$s of the pentagon.

10. ABCD is a cyclic quad. and AD is produced to X. If $\angle CDX = 102°$, $\angle ABD = 68°$, $\angle BAC = 22°$, find $\angle BCD$.

11. The $\angle$s A, B of a cyclic quad. ABCD are 110°, 85° respectively, and AB subtends an angle of 65° at the point of intersection of the diagonals. Find the $\angle$s which the sides subtend at the centre of the circle. [Let $\angle AED = x°$.]

12. ABCD is a quad. inscribed in a circle; the sides AB, DC produced meet at E, and the sides AD, BC produced meet at F; if $\angle E = 30°$ and $\angle F = 40°$, find the $\angle$s of the quad. ABCD.

13. ABCDEF is a hexagon inscribed in a circle. $\angle ABC = 132°$, $\angle CDE = 108°$, find $\angle EFA$. [Join CF.]

14. ABCDEF is a hexagon inscribed in a circle. Prove that $\angle ABC + \angle CDE + \angle EFA = 360°$.

15. ABCD is a cyclic quad.; $\angle A = 68°$, CB = CD. Find $\angle CBD$.

16. AB, CD are parallel chords of a circle. AD, BC meet at O within the circle. Prove that (i) OA = OB, (ii) OC = OD, (iii) AC = BD.

17. AB, CD are parallel chords of a circle. CA, DB produced meet at X outside the circle. Prove that (i) XA = XB, (ii) XC = XD, (iii) AD = BC.

18. Two unequal circles meet at A, B. AP, AQ are diameters of these circles. Prove that P, B, Q are collinear.

19. AB is a chord of a circle, centre O. Prove that the circle on OB as diameter bisects the chord AB.

20. If a parallelogram is inscribed in a circle, it must be a rectangle.

21. ABC is an isosceles $\triangle$ in which AB = AC. Prove that the circles on AB, AC as diameter meet again at the mid-point of BC.

22. ABC is any $\triangle$. Prove that the circles on AB, AC as diameter meet again at a point on BC.

23. Find the locus of the middle points of all chords of a given circle drawn through a given point (a) inside, (b) outside, (c) on, the given circle. [Use Theorems 23 and 41.]

24. A, B are given points and LM a given st. line. Find a point in LM at which AB subtends a right angle. Discuss the number of solutions possible in the various cases.

25. P is a point 3″ from the centre O of a circle, radius 2″. PO produced meets the circle again at C. Construct a secant PAB of the circle, such that M, the mid-point of AB, is equidistant from P and C. [Use No. 23, above.]

26. Draw a circle of radius 2·7″ and take a point P which is 2·2″ from the centre O. Construct a chord of the circle passing through P, and having its middle point 1·4″ from P.

27. AB is a chord 3″ long of a circle, radius 1·8″. P is a point inside the circle on the opposite side of AB to the centre of the circle and such that AP=1·3″, BP=1·9″. Draw through P two chords of the circle which are bisected by AB.

28. In a quad. ABCD the ∠s A and C are each 90°. AB=6 cm., AD=8 cm., BC=2·8 cm. Construct the quad. Measure CD, and also find its length by calculation.

29. Find the locus of the intersection of the diagonals of a rhombus, one of whose sides is fixed in magnitude and position.

30. ABC is a circle. Show how to find a point P on BC such that if AP produced meets the circle at Q, AP=PQ.

31. A st. rod of given length slides between two fixed st. rods placed ⊥ to one another ; find the locus of its mid-point

32. Two chords PQ, RS of a circle intersect at X. Prove that, if SX=SQ, PX=PR.

33. A, B, C are three points on a circle such that ABC is an acute-angled △ ; BQ and CR are diameters of the circle, and AQ, AR are joined. Prove that ∠BAR=∠CAQ.

34. An equilateral △ABC is inscribed in a circle ; BD is the diameter through B. Prove that AD=the radius of the circle.

35. The circle, described upon a side AB of a △ABC as diameter, bisects the side BC. Prove that two angles of the △ are equal.

36. A diameter AB of a circle bisects a chord CD ; if BC is ‖ to DA, prove that CD is also a diameter.

37. ABCD is a quad. inscribed in a circle. Join AC, and prove that the ∠BAC is the difference between the ∠s ADC, ACB.

38. ABCD is a cyclic quad. ; DE is a chord bisecting the angle between CD produced and BD. Prove that AE (produced if necessary) bisects the angle between BA produced and AC.

39. A quad. ABCD is inscribed in a circle. The side BC is produced to any point E ; the angle at A is bisected by a line which cuts the circle again at F. Prove that CF bisects the angle DCE.

40. A, B, C are three points on the circumference of a circle ; a point P is taken on the side of AB remote from C so that the sum of the ∠s APB, ACB is greater than 180°. Prove that P lies within the circle.

41. With centre A, a point on the circumference of a circle, a second circle is drawn to intersect the first circle at P and Q. Prove that PB=QB, if B is the point on the first circle diametrically opposite to A.

42. In a △ABC, if AD, CF are drawn ⊥ to the opposite sides, cutting each other at H, and if BK is a diameter of the circumcircle of ABC, and CK, KA are joined, prove that AHCK is a parallelogram.

THEOREM 43

If a pair of opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.

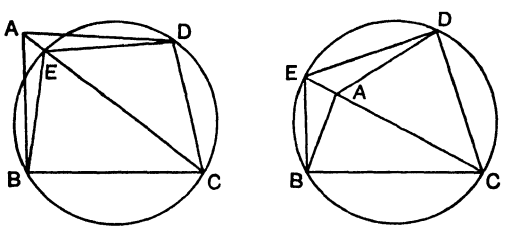


FIG. 170.

1st Proof. Let ABCD be a quadrilateral in which

$$\angle ABC + \angle ADC = 2 \text{ right angles.}$$

It is required to prove that A, B, C, D lie on a circle.

Proof. Draw the circle through B, C, D and suppose that it does not pass through A.

Then, since CA lies within the angle BCD, the circle BCD must cut CA or CA produced. Let E be the point of intersection. Join EB, ED.

Then since EBCD is a cyclic quadrilateral, its opposite angles are supplementary,

$$\therefore \angle EBC + \angle EDC = 2 \text{ right angles.}$$

But $\angle ABC + \angle ADC = 2$ right angles (*given*),

$\therefore \angle EBC + \angle EDC = \angle ABC + \angle ADC$,

i.e. the part is equal to the whole, which is absurd ;

$\therefore$ our supposition that the circle BCD does not pass through A is wrong,

$\therefore$ A, B, C, D lie on a circle.

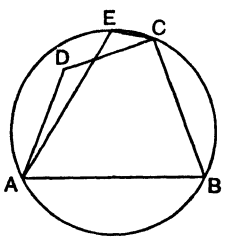


FIG. 171.

2nd Proof.

Let ABCD be a quadrilateral in which

$\angle ABC + \angle ADC = 2$ right angles.

It is required to prove that A, B, C, D lie on a circle.

Proof. Draw the circle ABC.

Let E be any point on the circumference on the same side of the chord AC as D. Join EA, EC.

Then, since ECBA is a cyclic quadrilateral, its opposite angles are supplementary,

$\therefore \angle AEC =$ the supplement of the $\angle ABC$

$= \angle ADC$ (*given*),

i.e. AC subtends equal angles at two other points E, D on the same side of it,

$\therefore$ A, C, E, D lie on a circle,

i.e. D lies on the circle AEC.

But the circle AEC is the same as the circle through A, B, C,

$\therefore$ A, B, C, D lie on a circle.

EXERCISE 42 [Theorems 39, 43]

1. The $\angle$s A, C of a quad. ABCD are 127° and 53° respectively, and $\angle ABD = 31°$. Find $\angle ACD$.

2. ABCD is a quad.; AC, BD meet at X. If $\angle ADB = 32°$, $\angle ACD = 24°$, $\angle ACB = 32°$, $\angle DXA = 45°$, find $\angle$s DAB, $\angle ABC$.

3. ABCD is a quad.; AC, BD meet at X. If $\angle ADB = 36°$, $\angle BCD = 82°$, $\angle ACD = 46°$, $\angle BXC = 62°$, find $\angle$s DAB, ABC.

4. ABC, ABD are two $\triangle$s on the same base AB, and on the same side of it. The base angles of the first $\triangle$ are 53° and 75°, of the second $\triangle$ are 84° and 44°. Prove that A, B, C, D lie on a circle.

5. ABC, ABD are two $\triangle$s on the same base AB, and on opposite sides of it. The base angles of the first $\triangle$ are 52° and 36°, of the second $\triangle$ are 48° and 54°. Are the four points A, B, C, D concyclic?

6. ABC is a given $\triangle$ and XY is a given st. line. Find a point P in XY such that $\angle APB = \angle ACB$. Draw figures showing that the number of solutions may be 0, 1, 2, 3 or 4.

7. Looking out to sea through a window 5′ wide I can just see 45 degrees of the horizon. What will my path be if I move so as to change my view but always to keep the same extent of horizon in view?

8. AD, BE, CF are the $\perp$s drawn from A, B, C to the opposite sides of an acute-angled $\triangle ABC$ and H is their point of intersection. Prove that the following sets of points are concyclic: (i) B, F, E, C; (ii) A, F, H, E. Pick out two more sets which are concyclic for the same reason as (i) the first set, (ii) the second set.

9. ABC is an isosceles $\triangle$, and PQ is drawn $\parallel$ to BC cutting the sides AB, AC at P, Q respectively. Prove that P, Q, C, B are concyclic.

10. Through two fixed points A, B pass two st. lines AC, BC which intersect at an angle of constant magnitude at C. If AD, BD be drawn $\perp$ to AC, BC respectively, find the locus of D.

11. PQRS is a parallelogram; any circle through P, S cuts PQ at X and SR at Y. Prove that QXYR is cyclic.

12. ABC is an acute-angled $\triangle$. The circle on AB as diameter cuts BC, CA at D, E respectively; AD, BE meet at P. Prove that $\angle APB = \angle ABC + \angle BAC$.

13. ABC is a $\triangle$. P, Q are points on AB, AC such that the st. line PQ makes with AB, AC angles equal to those which BC makes with AC, AB respectively; show that a circle can be described through B, C, P, Q. Is any part of the data redundant?

14. Two circles PAB, QAB intersect at A and B; their centres are C and D; P is a point on the circle PAB and lies outside the circle QAB; Q is a point on the circle QAB and lies outside the circle PAB. Prove that if $\angle APB + \angle AQB = 90°$, the points A, B, C, D are concyclic.

15. ABC is an acute-angled $\triangle$ in which $AB > AC$. P is a point in BC such that $AP = AC$. AP produced meets the circle ABC at Q. Prove that $BP = BQ$, and that if AM, BM be drawn $\parallel$ respectively to CB and PA to meet at M, then M lies on the circle.

16. Equilateral $\triangle$s are described outwardly on the sides of a given acute-angled $\triangle$. Prove that the circles which circumscribe these equilateral $\triangle$s meet at a common point.

17. If points P, Q, R are taken in the sides BC, CA, AB of a $\triangle$ respectively, prove that the circles circumscribing the $\triangle$s AQR, BRP, CPQ meet at a point.

18. The vertex A of an equilateral $\triangle$ is joined to any point D in the base BC produced and an equilateral $\triangle$ADE is described on AD, E and C being on opposite sides of AD. Prove that the $\perp$ bisectors of AC, AD and AE are concurrent.

19. A point A is taken on a circle whose centre is O and X, S are the mid-points of two chords AP, AQ. Prove that A, O, X, S lie on a circle.

20. XYZ is a $\triangle$ inscribed in a circle, centre O. The bisector of the angle XOZ is produced to meet YZ at R. Prove that XORY is a cyclic quad.

21. The base AB of a $\triangle$ is $2{\cdot}5''$, the area is 4 sq. in., and the vertical angle is greater than $30°$. Where must the vertex lie?

* THEOREM 44

1. In equal circles (or in the same circle), if two arcs subtend equal angles at the centres, or at the circumferences, they are equal.

2. In equal circles (or in the same circle), if two arcs are equal, they subtend equal angles at the centres and at the circumferences.

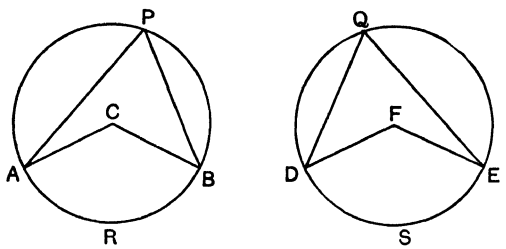


FIG. 172.

1. Let ARB, DSE be arcs of equal circles, subtending the angles ACB, DFE at the centres and angles APB, DQE at the circumferences.

If $\angle ACB = \angle DFE$, OR if $\angle APB = \angle DQE$, it is required to prove that the arc ARB = the arc DSE.

Proof. Since the arcs ARB, DSE subtend angles at the centres which are respectively double the angles which they subtend at the circumferences, it follows that if $\angle APB = \angle DQE$, then

$$\angle ACB = \angle DFE.$$

It is, therefore, only necessary to consider the case in which it is given that $\angle ACB = \angle DFE$.

Apply the circle APB to the circle DQE so that C falls on F and CA falls along FD.

Then because $\angle ACB = \angle DFE$, CB can be placed along FE.

And because the circles are equal, their radii are equal;

$\therefore$ A falls on D, B falls on E, and the circumferences of the circles will coincide entirely;

$\therefore$ the arc ARB must coincide with the arc DSE;

$\therefore$ the arc APB = the arc DSE.

2. Let ARB, DSE be equal arcs of equal circles, of which the centres are C and F.

It is required to prove that $\angle ACB = \angle DFE$ and that

$$\angle APB = \angle DQE.$$

Proof. Apply the circle APB to the circle DQE, so that C falls on F and CA falls along FD.

Then, because the circles are equal, their radii are equal ;

$\therefore$ A falls on D and the circumferences of the circles will coincide entirely.

But it is given that the arc ARB = the arc DSE,

$\therefore$ it is possible to make the arc ARB coincide with the arc DSE, in which case B falls on E ;

$\therefore$ CB falls along FE,

$\therefore \angle ACB = \angle DFE.$

But the angle which an arc subtends at the circumference is half that which it subtends at the centre,

$$\therefore \angle APB = \tfrac{1}{2}\angle ACB \text{ and } \angle DQE = \tfrac{1}{2}\angle DFE ;$$

$$\therefore \angle APB = \angle DQE.$$

Note. It is clear that any theorem relating to arcs, angles and chords in *equal* circles must also be true of the same circle.

Corollary. If the arcs ARB, DSE of two equal circles are equal, the sectors ACB, DFE which stand on these arcs are equal.

*THEOREM 45

1. In equal circles (or in the same circle), if two chords are equal, the arcs which they cut off are equal, the major arc to the major arc, and the minor to the minor.

2. In equal circles (or in the same circle), if two arcs are equal, the chords of the arcs are equal.

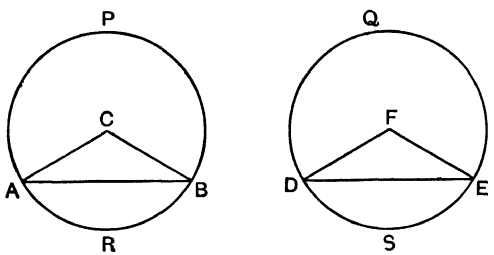


FIG. 173.

1. Let APB, DQE be equal circles of which the centres are C and F; and let the chord AB = the chord DE.

It is required to prove that the major arc APB = the major arc DQE and that the minor arc ARB = the minor arc DSE.

Construction. Join CA, CB, FD, FE.

Proof. In the △s ACB, DFE,

AB = DE (*given*),

AC = DF (*radii of equal circles*),

CB = FE (*radii of equal circles*);

∴ the △s $\begin{matrix}\text{ACB}\\\text{DFE}\end{matrix}$ are congruent (3 *sides*),

∴ ∠ACB = ∠DFE;

∴ the arcs ARB, DSE, of equal circles, subtend equal angles at the centres, ∴ they are equal.

But the circumferences of the circles are equal, ∴ the remaining arc APB = the remaining arc DQE.

2. Let ARB, DSE be equal arcs of equal circles, of which the centres are C and F.

It is required to prove that the chord AB = the chord DE.

Construction. Join CA, CB, FD, FE.

Proof. ARB, DSE are equal arcs of equal circles,

$\therefore$ they subtend equal angles at the centre,

i.e. $\angle ACB = \angle DFE$;

$\therefore$ in the $\triangle$s ACB, DFE,

$AC = DF$ (*radii of equal circles*),

$CB = FE$ (*radii of equal circles*),

$\angle ACB = \angle DFE$ (*proved*);

$\therefore$ the $\triangle$s $\begin{matrix} ACB \\ DFE \end{matrix}$ are congruent (2 *sides*, **included** *angle*),

$\therefore$ the chord AB = the chord DE.

IMPORTANT APPLICATIONS OF THEOREMS 44 AND 45

1. By combining the results of Theorems 44 and 45 (or by using congruent triangles), we obtain the following results :

(*a*) **Equal chords of equal circles (or of the same circle) subtend at the centres angles which are equal and at the circumferences angles which are either equal or supplementary.**

(*b*) **In equal circles (or in the same circle) chords are equal if they subtend angles at the centres which are equal** OR **angles at the circumferences which are either equal or supplementary.**

EXAMPLE. *Two equal circles intersect at* A *and* B. *A straight line through* A *meets the circles again at* P *and* Q. *Prove that the triangle* BPQ *is isosceles.*

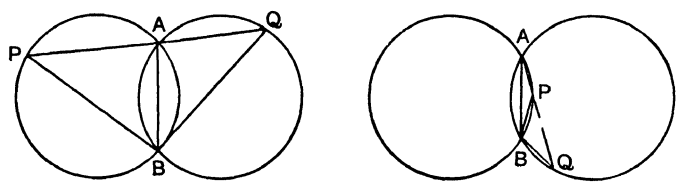


Fig. 1 Fig. 2

FIG. 174.

Join PB, AB, QB.

In the equal circles PAB, QAB the chord AB = the chord AB, ∴ the angles which AB subtends at the circumferences are either equal or supplementary, i.e. ∠APB is either equal to ∠AQB or supplementary to it.

(i) In Fig. 1 ∠APB and ∠AQB are two angles of the same triangle, ∴ they cannot be supplementary,

∴ they are equal,

∴ the triangle BPQ is isosceles.

(ii) In Fig. 2 $\angle APB$ is the exterior angle of the triangle BPQ, and is therefore greater than the interior opposite $\angle AQB$,

$\therefore$ in this case $\angle APB$ is supplementary to the $\angle AQB$.

But $\angle APB$ is supplementary to the $\angle BPQ$, since APQ is a straight line,

$$\therefore \angle BPQ = \angle AQB,$$

$\therefore$ the triangle BPQ is isosceles.

2. Lengths of arcs and areas of sectors.

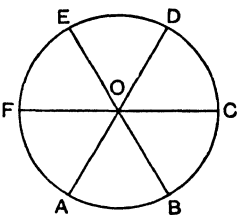


Fig. 1

FIG. 175A.

The length of any arc is easily obtained as a fraction of the length of the whole circumference from the consideration of the angle it subtends at the centre.

Thus in Fig. 1, if $\angle AOB = 60°$, it is clear that the circumference may be divided into six portions each equal to the arc AB ;

$\therefore$ the length of the arc $AB = \frac{1}{6}$ of the length of the whole circumference.

Also the area of the sector $AOB = \frac{1}{6}$ of the whole area of the circle.

In Fig. 2, if $\angle XOY = 37°$, the circumference may be divided into 360 equal portions each of which subtends an angle of 1° at the centre. XY contains 37 of these portions,

$\therefore$ the length of the arc $XY = \frac{37}{360}$ of the length of the whole circumference, and the area of the sector $XOY = \frac{37}{360}$ of the whole area of the circle.

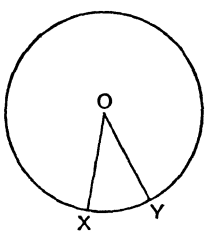


Fig. 2

FIG. 175B

Similarly, if $\angle XOY = 37\frac{7}{12}°$, the circumference may be divided into 4320 (360×12) equal portions, each of which subtends an angle of $\frac{1}{12}°$ at the centre. XY contains 451 of these portions,

$\therefore$ the length of the arc $XY = \frac{451}{4320}$ $\left(\text{or } \frac{37\frac{7}{12}}{360}\right)$ of the length of the whole circumference and the area of the sector XOY $= \frac{37\frac{7}{12}}{360}$ of the whole area of the circle.

More generally, if $\angle XOY = x°$, the length of the arc XY

$= \frac{x}{360}$ of the length of the whole circumference,

and the area of the sector XOY

$= \frac{x}{360}$ of the whole area of the circle.

This result is sometimes briefly written :

$$\frac{\text{arc XY}}{\text{whole circumference}} = \frac{\angle XOY}{\text{whole angle at centre}}$$

$$= \frac{\text{sector XOY}}{\text{whole area of circle}}.$$

Note 1. The above method is only applicable if the angle XOY can be expressed as an exact fraction of 360°, i.e. if the length of the arc XY can be expressed as an exact fraction of the length of the whole circumference.

If the length of the arc XY and the length of the whole circumference are incommensurable magnitudes, the proof of the above result is beyond the scope of this book.

Note 2. The Length of the Circumference and the Area of the Circle. It is well known that the ratio of the length of the circumference of a circle to the length of its diameter (d) is the same for all circles. This ratio is denoted by the Greek letter π, and the result is usually written: circumference $= \pi d$, or $2\pi r$, if r denotes the radius of the circle.

The area of the circle is known to be πr^2.

The proof of these statements is beyond the scope of this book.

π is an incommensurable number, and its value is approximately 3·1416. The student should note that the popular approximation $3\frac{1}{7}$ is not usually sufficiently accurate.

Note 3.

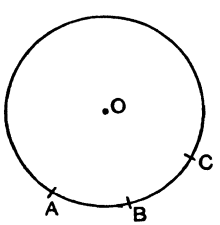


FIG. 176.

If AB, BC be equal arcs of the circle, centre O, then

$$\angle AOC = 2\angle BOC ;$$

but it should be noticed that the length of the chord AC is NOT twice that of the chord BC.

EXERCISE 43 [Theorems 44, 45]

1. ABC are three points on a circle, and $\angle ABC = 38°$, and $\angle ACB = 68°$. P, Q are, respectively, the mid-points of the minor arcs AC, AB. Calculate the $\angle$s BCP, CPQ.

2. ABCDE is a regular pentagon inscribed in a circle. Find the acute angle between BE and AD.

3. AB and AC are, respectively, sides of a square and a regular hexagon inscribed in the same circle. Calculate, in degrees, the $\angle BAC$. Consider both cases.

4. ABC is a $\triangle$ inscribed in a circle, the $\angle$s at B and C being 40° and 60° respectively ; D is the mid-point of the arc subtended by BC on the same side as A. Find the $\angle DCA$.

5. AB, CD are chords of a circle. If the arcs AC, CB, BD, DA are as 1 : 3 : 7 : 4, calculate $\angle CBD$. [Consider first the angles at the centre.]

6. AB, CD are chords of a circle. If the arcs AC, CB, BD, DA are as 2 : 5 : 4 : 7, calculate $\angle CBD$. If CE is drawn $\perp$ to AB to meet the circle at E, what fraction of the whole circumference is the minor arc BE ?

7. Find the acute angle between the st. lines joining (i) I to IX and V to X, (ii) II to XI and I to X, (iii) VI to IX and III to X, on the face of a clock.

8. Find the angle between the st. lines joining II to V and III to X on the face of a clock.

9. ABCD is a square and AXY an equilateral $\triangle$ inscribed in the same circle, X being on the minor arc BC. Calculate (i) $\angle BAX$, (ii) $\angle XDC$.

10. O is the centre of a circle and AB is an arc equal to one-ninth of the circumference. AC is drawn, outside the $\triangle OAB$, so that $\angle OAC = 30°$, and C is on the circumference. CD is drawn $\parallel$ to OB, cutting the circle again at D. What fraction of the circumference is the minor arc BD ?

11. Bisect a given arc of a circle by a purely geometrical construction.

12. AB, CD are two parallel chords of a circle. Prove that arc AC = arc BD.

13. AB, CD are two parallel chords of a circle. A, C are fixed points and B, D are variable points. Find the locus of the mid-point of BD [Use No 12.]

14. AB, DC are equal chords of a circle which, when produced, intersect outside the circle. Prove that $\angle BAD = \angle ADC$.

15. ABCDEFG is a regular heptagon inscribed in a circle, centre O. Prove that one of the angles between CE and DF is equal to the angle subtended at O by a side of the heptagon.

16. XYZ is a $\triangle$ inscribed in a circle. P, Q are the mid-points of the arcs XY, XZ, respectively. PQ meets XY, XZ at R, S, respectively. Prove that $\angle XRS = \frac{1}{2}(\angle XYZ + \angle XZY)$. Deduce that XR=XS.

17. The sum of the arcs cut off by two $\perp$ chords of a circle is equal to half the circumference.

18. AB, DC are two chords of a circle, centre O, which, when produced, meet at X outside the circle. Prove that

$$\angle AXD = \tfrac{1}{2}(\angle AOD - \angle BOC).$$

19. AC, BD are two chords of a circle, centre O, which meet at Y inside the circle. Prove that $\angle AYD = \frac{1}{2}(\angle AOD + \angle BOC)$.

20. If one pair of opposite sides of a cyclic quad. are equal, prove that the other pair are parallel.

21. Two st. lines AOB, COD, making with each other an angle of constant magnitude, turn about a fixed point O within a circle and intercept arcs AC, BD on the circumference. Prove that the sum of the arcs is constant.

22. AB is a fixed chord of a circle and P is any variable point on either the minor or the major arc cut off by AB. Prove that the bisector of the $\angle APB$ meets the circle again in the same point for all positions of P.

23. ABC is a $\triangle$; the bisector of the $\angle BAC$ meets the circle ABC again at K. KL, KM are drawn $\perp$ to AB, AC (produced if necessary). Prove that LB=MC.

24. The $\perp$ bisector of the side BC of a $\triangle ABC$ meets the circumference of the circle ABC at D on the side of BC remote from A. Prove that AD bisects the $\angle BAC$.

25. The $\perp$s from A, B to the opposite sides of the $\triangle ABC$ meet the circle ABC at P, Q. Prove that C is the mid-point of the arc PQ.

26. XZ is the diameter of a semi-circle XYZ, centre O. Prove that the st. line through O $\parallel$ to ZY bisects the arc XY.

EXERCISE 44 [Further Examples on Theorems 37-45]

1. The opposite $\angle$s B and D of a cyclic quad. ABCD are bisected ; prove that the acute $\angle$ between the bisecting lines is equal to $\frac{1}{2}(\angle BAD \sim \angle BCD)$.

2. ABCD is a cyclic quad. and AB, DC when produced meet at E. The internal bisector of $\angle BEC$ meets BC at F and AD at G. Prove that $\angle AGF = \angle BFG$.

3. ABCD is a quad. inscribed in a circle in which AB=AC ; the side CD is produced to E ; prove that AD bisects the $\angle BDE$.

4. Through A and B, the points of intersection of two circles ABEC, ABFD, two parallel st. lines are drawn, cutting one circle at C, E and the other at D, F. Prove that CD=EF.

5. D is the mid-point of the hypotenuse BC of a right-angled $\triangle$d ABC ; from any other point E on BC the line EP is drawn $\perp$ to AB meeting it at P, and the line EQ is drawn $\perp$ to AC meeting it at Q and cutting AD (produced if necessary) at R. Prove that the $\triangle$s BPE, RQA are congruent.

6. AOB, COD are two $\perp$ chords of a circle. Q is the mid-point of AD. Prove that QO produced is $\perp$ to BC.

7. ABC is a $\triangle$. I, I′, are the centres of the inscribed circle and the escribed circle opposite A, respectively. Let the st. line AII′ meet the circumcircle of the $\triangle$ABC at L. Prove that (i) L is the mid-point of the arc BC of the circumcircle, (ii) LI=LB=LI′.

8. ABCD is a quad. inscribed in a circle such that the centre O lies within the quad. and such that AB and CD are not parallel. P and Q are the mid-points of the minor arcs AB and CD. Prove that the chord PQ makes equal angles with the chords AB, CD. [Join OP, OQ.]

9. ABC is a $\triangle$ inscribed in a circle whose centre is O. D, E, F are the mid-points of the arcs BC, CA, AB respectively, and P is the other end of the diameter through D. Express the $\angle$s EOF, BOP in terms of the $\angle$s of the $\triangle$ABC, and prove that EF=BP.

10. If the bisectors of the $\angle$s B, C of a $\triangle$ABC meet the opposite sides at E, F respectively, and intersect at the point I, prove that, if AEIF is a cyclic quad., A=60°. Prove also that, in this case, EI=IF=the radius of the circle AEIF.

11. Two unequal circles intersect at A, B ; a st. line is drawn through A meeting one circle again at P and the other at Q, and a st. line is drawn through B meeting the first circle again at R and the second at S. Prove that PR is $\parallel$ to QS.

12. The circles ABC, ABDE meet at A and B ; from any point C on the circle ABC the st. lines CA, CB are drawn cutting the circle ABED at D and E ; another st. line is drawn through C cutting the circle ABC at P and the st. line DE at Q. Prove that A, P, Q, D are concyclic.

13. Two circles PAB, QAB intersect at A, B ; and any st. line through A cuts the circles at P and Q respectively. The st. lines joining a point O to P and Q cut the circles PAB, QAB at R and S respectively. Prove that O, R, B, S are concyclic.

14. ABCD is a rectangle ; X is any point in AB ; DY is drawn $\perp$ to DX cutting BC produced at Y ; V is the mid-point of XY. Prove that VB=VD. What is the locus of V as the position of X in AB changes ?

15. A $\triangle$ABC is inscribed in a circle whose centre is O ; E, F are the mid-points of AC, AB ; the st. lines EO, FO meet AB, AC (produced if necessary) at X, Y respectively. Prove that B, C, Y, X are concyclic.

16. A, C, B are three points in a st. line such that C lies between A and B and is nearer to B than to A. The circle on AB as diameter is drawn and also the circle centre C and radius CA. The side AS of the equilateral $\triangle$ASB cuts the first circle at N and the second at M. Prove that $NM=\frac{1}{2}(AC-CB)$.

17. ABC is an equilateral $\triangle$ inscribed in a circle. P is a point on the smaller arc AC. AP meets BC produced at R, BP meets AC at O. Prove that the $\triangle$s ABR and OAB are equiangular.

18. AB, AC are equal chords of a circle. The bisector of the $\angle$ABC meets the circle at D and AD produced meets BC produced at E. Prove that CE=CA.

19. ABC is a $\triangle$ inscribed in a circle and the bisectors of the $\angle$s A, B, C meet the circle at X, Y, Z respectively ; if the opposite $\angle$s of the figure AZBXCY are equal, prove that the $\triangle$ABC is equilateral.

20. AB and CD are two chords of a circle intersecting inside the circle at right angles. M is the mid-point of AD, and N the mid-point of BC. O is the centre of the circle. Prove that the $\triangle$s OAM, CON are congruent.

21. O is the point of intersection of two $\perp$ chords AB, CD of a circle. In OC, produced if necessary, a point P is taken so that OP=OD. Prove that AP, BC are $\perp$.

22. EF is a diameter of a circle, and P, Q are any two points on the circle ; EP, EQ cut the tangent at F in X, Y. Prove that P, Q, Y, X are concyclic.

23. A and B are the points of intersection of two circles. A st. line cuts one of the circles at P and Q, and the other at R and S. The parallel to AP through S cuts AB (produced if necessary) at X. Prove that Q, B X, S are concyclic.

24. ABC is an equilateral △. Any circle through B and C cuts AB, AC at D, E respectively. F is a point on this circle such that EF = ED. Prove that BF = BC.

25. Equilateral △s ABD, ACE are described on the sides AB, AC of a △ABC, so that D and E fall outside the △ ; BE, CD intersect at H. Prove that A, E, C, H lie on a circle.

26. Prove that, if a square be described externally on the hypotenuse of a right-angled △, and the right angle be joined to the centre of the square, the joining line will bisect the right angle.

27. Given the base BC and the vertical angle A of a △ABC, the angle A being acute, prove that if BM is drawn ⊥ to AC, cutting it at M, and if CN is drawn ⊥ to AB cutting it at N, the line MN is of constant length. [Consider the circle BCMN.]

28. ABCD is a quad. inscribed in a circle, and any st. line ‖ to AB meets AC and BD at P and Q respectively. Prove that a circle can be drawn through P, Q, C and D.

[Consider positions of PQ on either side of the intersection of AC, BD.]

29. AB, AC are equal chords of a circle. P, Q are any two points in BC, and AP, AQ meet the circle again at R and S. Prove that a circle can be described about PQSR.

30. A hexagon ABCXYZ whose sides are unequal to one another is inscribed in a circle. AB is parallel to XY, and BC to YZ. Prove that CX is ‖ to AZ.

31. Through two fixed points A, B on the circumference of a circle any two ‖ lines are drawn, cutting the circle at P, Q. Prove that the st. line PQ in all its positions touches a second circle concentric with the given circle. Draw a diagram, and construct the second circle.

32. Straight lines are drawn on the dial of a watch joining the positions I and V, II and X, IIII and VIII, VII and XI ; prove that the figure enclosed by the lines is a square.

33. AC and BD are two fixed diameters of a given circle, P is any point on the circumference, and M, N are the feet of the ⊥s from P on AC and BD respectively. Prove that the length of MN remains constant as P moves round the circle.

34. AOB is a diameter and AS a chord of a circle whose centre is O. The chord AP bisects the $\angle$BAS. AS is produced to Q so that PQ=PA. Prove that (i) OP is $\parallel$ to AQ; (ii) $\angle$APQ=$\angle$BPS; (iii) $\angle$SPQ=90°; (iv) $\triangle$s APB, QPS are congruent.

35. ABC is a $\triangle$ right-angled at B, P any point on the hypotenuse AC, O the mid-point of AC. A point Q is taken on BP produced such that OQ=OB. Prove that $\angle$ACB=$\angle$AQB.

36. ABC is a $\triangle$ right-angled at A and AD is the $\perp$ from A to BC; E and F are the feet of the $\perp$s from D to CA and AB respectively. Prove that $4FE^2=BC^2-4MD^2$, where M is the mid-point of the side BC.

37. Two st. lines AB, PQ are produced to meet at E. The circles EAQ and EBP meet again at O. Prove that the $\triangle$s OAB and OQP are equiangular.

38. If ABCD is a cyclic quad. and the bisectors of the $\angle$s BAD and BCD meet the circumference at E and F, prove that the $\angle$EAF=90°.

39. Prove that if a polygon having an even number of sides whose lengths are not necessarily equal, can be inscribed in a circle, the sum of one set of alternate angles must be equal to the sum of the other set.

40. ABCD is a quad. such that C is inside the circle through A, B, D. Prove that A is inside the circle through B, C, D, but that D is outside the circle through A, B, C.

41. D, E, F are respectively the feet of the $\perp$s from the vertices A, B, C of an acute-angled $\triangle$ to the opposite sides, and O is the point of intersection of the $\perp$s. Prove that (i) $\angle$FDO=90°−$\angle$BAC, (ii) if S is the centre of the circumscribed circle of the $\triangle$AFE, then D, F, S, E lie on a circle.

42. ABCD is a quad. such that AB+CD=AD+BC. Points P, Q, R, S are taken on AB, BC, CD, DA respectively such that AP=AS, BP=BQ, and CQ=CR. Prove that P, Q, R, S lie on a circle.

43. OA is $\perp$ to OB, and ON is $\perp$ to AB. C, D are taken in OA, OB respectively so that CD is bisected by ON. Prove that ABDC is a cyclic quad.

44. ABCD is a quad. inscribed in a circle. Prove that the internal bisector of the $\angle$BAD meets the external bisector of the $\angle$BCD at a point on the circle.

45. In a convex quad. ABCD, AB and DC when produced meet at E, DA and CB when produced meet at F, and AC and BD meet at G. Show that, if it were possible for GAFB and EBGC to be cyclic quads., the sum of the $\angle$s AFB and BEC would be two right angles. Hence show that GAFB and EBGC cannot both be cyclic quads.

TANGENCY

DEFINITIONS

1. A **secant** of a circle is a straight line of indefinite length which meets the circumference at two points.

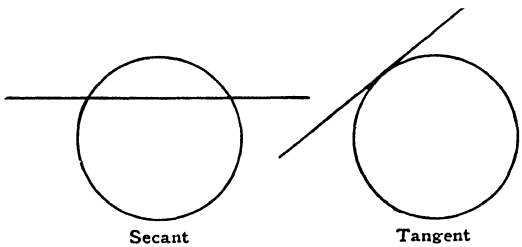


FIG. 177.

2. A **tangent** to a circle is a straight line which meets the circumference at one point and does not meet it again, however far it may be produced in either direction.

The point at which the tangent meets the circle is called the **point of contact**, and the tangent is said to **touch** the circle at this point.

It may be shown that every point on the tangent, except the point of contact, is outside the circle.

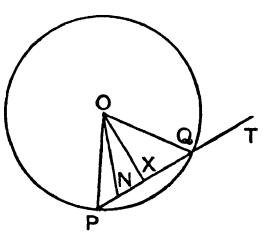


FIG. 178.

For let PT be a tangent to a circle whose centre is O, and, if possible, suppose that there is a point N on PT such that N lies within the circle. Join ON, OP.

Then, since N lies within the circle, ON < OP,

$$\therefore \angle OPN < \angle ONP,$$

$$\therefore \angle OPN \text{ is an acute angle ;}$$

$\therefore$ the perpendicular from O to PT meets it at a point X on the same side of P as N.

Let Q be the point on PX produced such that XQ = PX.

Join OQ.

Then in the $\triangle$s OPX, OQX,

$$PX = QX \ (\textit{given}),$$

$$OX = OX \ (\textit{common}),$$

$$\angle OXP = \angle OXQ \ (\textit{rt. } \angle s, \textit{ given}) ;$$

$$\therefore \text{ the } \triangle s \begin{matrix} OPX \\ OQX \end{matrix} \text{ are congruent } (2 \textit{ sides}, \textbf{included } \textit{angle}),$$

$$\therefore OP = OQ,$$

$\therefore$ Q lies on the circle, which is impossible, since a tangent cannot meet a circle at two points ;

$\therefore$ our supposition that there is a point N on the tangent within the circle is wrong ;

$\therefore$ every point on the tangent, except the point of contact, is outside the circle.

Note. This proof is somewhat beyond the range intended to be covered by this book, and it is not suggested that the pupil should be expected to reproduce it. It has been inserted because some teachers might otherwise hesitate to use the first proof of Theorem 46 (2). It is strongly recommended that the first proofs of Theorem 46 should be adopted as the proofs which the pupil is expected to learn, many experienced teachers having found that their pupils both understand and reproduce these proofs better than the alternatives available. The second proofs are inserted for the sake of their elegance and general interest.

Some teachers may prefer to avoid the difficulty by adopting the following definition of a tangent :

A **tangent** to a circle is a straight line which meets the circumference at one point and is such that **any** other point on it is outside the circle.

3. Two circles are said to **touch one another** when they meet at a point and have a common tangent at this point.

Circles are said to touch **externally**, when they are on opposite sides of the common tangent, and **internally**, when they are on the same side of the common tangent.

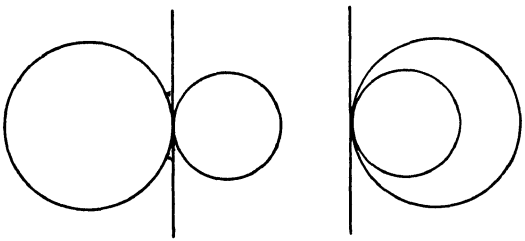


FIG. 179.

4. If all the sides of a rectilineal figure touch a circle, the figure is said to be **described about** the circle or **circumscribed to it**, and the circle is said to be **inscribed in** the figure.

5. A circle which touches the three sides of a triangle is called the **inscribed** circle, and circles which touch one side and the other two sides produced are called **escribed** circles. The centres of these circles are called respectively the **in-centre** and **ex-centres.**

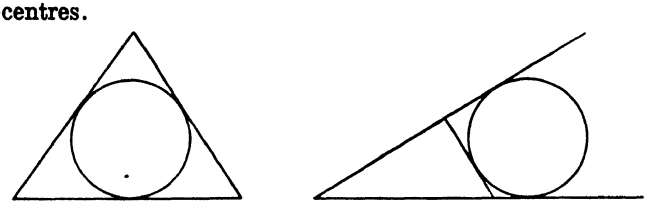


FIG. 180.

6. If a straight line cuts a circle at a point A, **the angle at which it cuts the circle** is the angle which the line makes with the tangent at A.

7. If two circles cut at a point A, **the angle at which they cut** is the angle between the tangents at A to the two circles.

THEOREM 46

(1) The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

(2) A tangent to a circle is perpendicular to the radius drawn through the point of contact.

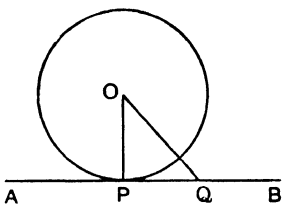


FIG. 181.

1st Proof.

(1) Let O be the centre and OP a radius of a circle.

Let APB be a straight line perpendicular to OP.

It is required to prove that APB is a tangent to the circle.

Construction. Let Q be **any** point on BC. Join OQ.

Proof. Since $\angle OPQ$ = a right angle (*given*),

$\therefore$ each other angle of the $\triangle OPQ$ is less than a right angle,

$$\therefore \angle OQP < \angle OPQ,$$

$\therefore$ In the $\triangle OPQ$ the side OP opposite the smaller $\angle OQP$ is less than the side OQ opposite the greater $\angle OPQ$.

But OP is a radius, $\therefore$ OQ is greater than a radius, $\therefore$ Q lies outside the circle.

Similarly **every** point on ABC except P lies outside the circle,

$\therefore$ AB is a tangent to the circle at P.

(2) Let PB be a tangent at the point P to a circle whose centre is O.

It is required to prove that PB is $\perp$ to the radius OP.

Construction. Take **any** point Q on PB. Join OQ.

Proof. Since PB is a tangent, every point on it except P is outside the circle,

$\therefore$ OQ is greater than the radius OP.

And this is true for **every** point Q on PB,

$\therefore$ OP is the shortest distance from O to PB,

$\therefore$ OP is $\perp$ to PB.

Corollary 1. At every point on a circle one and only one tangent to the circle can be drawn.

Corollary 2. The perpendicular to a tangent at its point of contact passes through the centre of the circle.

Corollary 3. The locus of the centre of a circle which touches a given straight line AB at a given point P is the straight line through P ⊥ to AB.

Corollary 4. The locus of the centre of a circle which touches a given straight line AB and has a given radius consists of two straight lines, parallel to the given straight line, one on each side of it, and at a distance from it equal to the given radius. (See Theorem 21, Cor. 1.)

Corollary 5. The locus of the centre of a circle which touches two given parallel straight lines is a straight line parallel to the given lines and mid-way between them. (See Theorem 21, Cor. 2.)

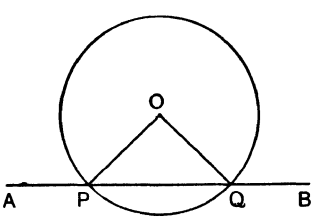


FIG. 182.

2nd Proof.

Let O be the centre and OP a radius of a circle.

Let APB be a straight line ⊥ to OP.

It is required to prove that APB is a tangent to the circle.

Construction. Suppose, if possible, that APB meets the circle again at Q.

Join OQ.

Proof. Then

OP = OQ (*radii of the same circle*),

∴ ∠OPQ = ∠OQP (*base angles of an isosceles △ are equal*).

But ∠OPQ is a right angle (*given*),

∴ the triangle OPQ has two angles each a right angle, which is impossible ;

∴ our supposition that APB meets the circle again is wrong,

∴ APB does not meet the circle except at P,

∴ APB is a tangent to the circle.

(2)

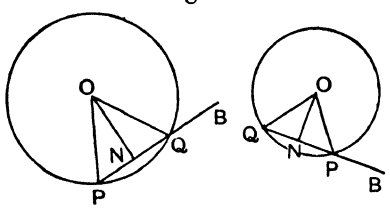


FIG. 183.

Let PB be a tangent at the point P to a circle whose centre is O.

It is required to prove that PB is perpendicular to the radius OP.

Construction. Suppose, if possible, that ∠OPB is not a right angle.

Draw the perpendicular from O to PB or BP produced ; this perpendicular will meet PB or BP produced at a point N, distinct from P.

Produce PN beyond N to Q, making NQ = PN. Join OQ.

Proof. In the △s OPN, OQN,

ON = ON (*common*),

PN = QN (*construction*),

∠ONP = ∠ONQ (*rt. ∠s, construction*) ;

∴ the △s $\begin{matrix}\text{OPN}\\\text{OQN}\end{matrix}$ are congruent (*2 sides*, **included** *angle*),

∴ OP = OQ,

∴ Q is on the circle, which, by hypothesis, is impossible, since PB is a tangent ;

∴ our supposition that ∠OPB is not a right angle is wrong,

∴ ∠OPB is a right angle.

Note. To construct the tangent to a circle at a point P on it.

Join OP. Draw PT ⊥ to OP at P (Construction 3, Second Method).

Then PT is the required tangent

THEOREM 47

If two tangents are drawn to a circle from an external point,

(1) the tangents are equal ;

(2) they subtend equal angles at the centre of the circle ;

(3) they make equal angles with the straight line joining the given point to the centre.

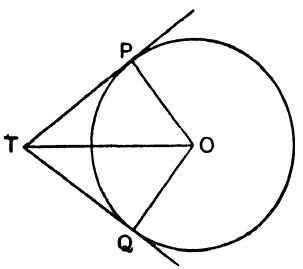


FIG. 184.

Let TP, TQ be tangents drawn to a circle whose centre is O from an external point T. Join TO, PO, QO.

It is required to prove that (1) TP = TQ, (2) ∠TOP = ∠TOQ, (3) ∠PTO = ∠QTO.

Proof. In the △s TPO, TQO, since a tangent to a circle is perpendicular to the radius drawn from the point of contact,

∠TPO = ∠TQO = a right angle,

PO = QO (*radii of the same circle*),

TO = TO (*common*) ;

∴ the △s TPO, TQO are congruent (*rt.* ∠, *hypot.*, *side*);

∴ (1) TP = TQ,

(2) ∠TOP = ∠TOQ,

(3) ∠PTO = ∠QTO.

THEOREM 48

If two circles touch one another, the straight line joining their centres (produced if necessary) passes through the point of contact.

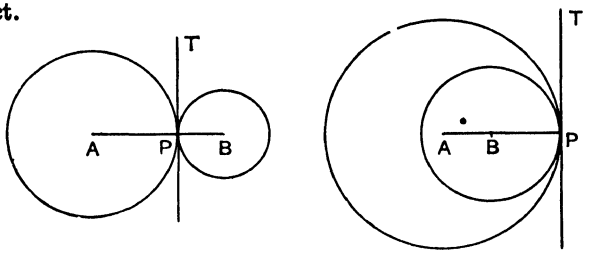


FIG. 185.

Let two circles with centres A and B touch one another, and let P be the point of contact.

It is required to prove that the points A, P, B are in the same straight line.

Proof. Since the circles touch at P, by definition, they have a common tangent at P. Let PT be the common tangent.

Then, since a tangent to a circle is perpendicular to the radius drawn from the point of contact, $\angle$s APT and BPT are both right angles;

$\therefore$ A and B both lie on the straight line through P, perpendicular to PT; $\therefore$ A, P, B lie on a straight line.

Corollary 1. If two circles touch externally the distance between their centres is equal to the sum of their radii.

Corollary 2. If two circles touch internally the distance between their centres is equal to the difference of their radii.

Corollary 3. The locus of the centre of a circle which touches a given circle, centre O, at a given point P is the straight line OP, produced indefinitely either way.

Corollary 4. The locus of the centre of a circle of given radius which touches a given circle, centre O, consists of two circles, each with centre O, one having its radius equal to the sum of the radii and the other having its radius equal to the difference of the radii of the two circles.

CONSTRUCTION 17

To draw a tangent to a circle from an external point.

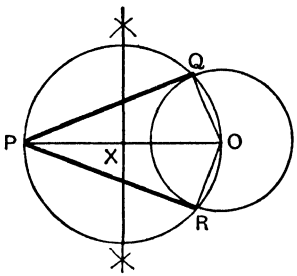


FIG. 186.

Let P be the given point and O the centre of the given circle.

It is required to construct a tangent from P to the circle.

Construction. Join PO and bisect it at X (Construction 2).

With centre X and radius XP, draw a circle, cutting the given circle at Q and R. Join PQ.

Then PQ is a tangent to the given circle.

Proof. Join OQ.

Since OX = XP, the circle, centre X, radius XP, passes through O, and OP is a diameter;

$\therefore$ $\angle$OQP, being by construction the angle in a semi-circle, is a right angle;

$\therefore$ PQ is $\perp$ to the radius through Q;

$\therefore$ PQ is a tangent at Q to the circle.

Note. It may similarly be shown that the straight line PR is a tangent to the circle at R.

Definition. If a straight line touches one circle at a point P and another circle at a point Q, it is said to be a **common tangent** of the two circles. If the circles are on the same side of the straight line PQ, the common tangent is said to be **direct** or **external** or **exterior.** If the circles are on opposite sides of the straight line PQ, the common tangent is said to be **transverse** or **internal** or **interior.**

CONSTRUCTION 18

To draw the direct common tangents to two given circles.

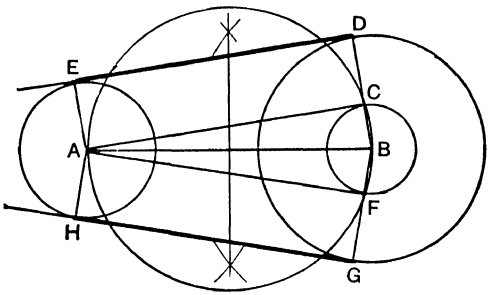


FIG. 127.

Let A, B be the centres and a, b the radii of the circles, and suppose that $b > a$.

It is required to draw the direct common tangents to the two circles.

Construction. Join AB.

With centre B and radius $b - a$, draw a circle.

Draw tangents AC, AF to this circle. [Construction 17.]

Join BC, BF and produce to cut the largest circle at D, G.

Draw AE, AH parallel to BC, BF to cut the circle, centre A, at E, H.

Join DE, HG.

Then DE, HG are the required common tangents.

Proof. By construction $BC = b - a$. Also $BD = b$.

$\therefore$ $CD = a = AE$;

$\therefore$ CD and AE are equal and parallel ;

$\therefore$ CDEA is a parallelogram.

But, by construction, AC is a tangent at C ;

$\therefore$ $\angle ACD = 90°$; $\therefore$ CDEA is a rectangle ;

$\therefore$ the $\angle$s AED, CDE are right angles ;

$\therefore$ DE is $\perp$ to the radii through C, D ;

$\therefore$ DE touches each of the given circles.

Similarly, it may be shown that HG touches each of the given circles.

CONSTRUCTION 19

To draw the transverse common tangents to two given non-intersecting circles.

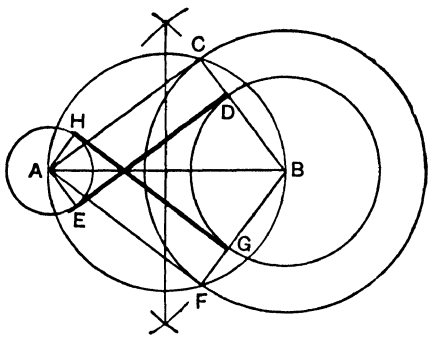


FIG. 188.

The construction is similar to the preceding construction—the only difference being that the circle described with centre B is of radius $b+a$. The required common tangents are DE, HG.

The proof is also almost identical; for BC by construction equals $b+a$, and $BD = b$; $\therefore CD = BC - BD = a = AE$. The rest of the proof is the same.

EXERCISE 45

[Theorems 46, 47, 48. Constructions 17-19]

1. Find the length of a tangent drawn to a circle of radius 21″ from a point 29″ distant from the centre of the circle.

2. The tangents to a circle from a point P are 24 cm. long and the radius of the circle is 7 cm. Calculate the distance of P from the centre of the circle.

3. Two concentric circles are drawn whose radii are 13″ and 5″. Calculate the length of any chord of the larger circle which touches the smaller.

4. ABC is a $\triangle$ in which $\angle A = 64°$, $\angle B = 70°$. The inscribed circle touches BC, CA, AB at D, E, F respectively. Find the $\angle$s of the $\triangle$ DEF.

5. Draw a circle of 1″ radius, and draw two radii OP, OQ at right angles. Draw the tangents at P and Q, meeting at A, and produce AP to B making PB=2″. From B draw another tangent and produce it to meet AQ produced at C. Calculate the lengths of AC and BC.

6. The inscribed circle of a △ABC touches BC, CA, AB at D, E, F respectively. BD=0·8″, BC=1·5″ AE=0·9″. Calculate the lengths of AB and AC.

7. Prove that tangents at the extremities of the diameter of a circle are ||.

8. If two || tangents are drawn to a circle, prove that their points of contact lie on a diameter of the circle.

9. Prove that a diameter XY of a circle bisects all chords || to the tangent at Y.

10. A number of concentric circles are drawn, and to each of them tangents are drawn from an external point. Prove that the points of contact lie on another circle.

11. AB, AC are tangents to a circle whose centre is O, B and C being the points of contact. If OB produced meets the ⊥ from A to AC at I, prove that OI=AI.

12. Two circles touch at P, and Q is any point on the common tangent at P. Prove that the other tangents to the circles from Q are equal.

13. Prove that the st. line joining the point P to the centre of a circle bisects at right angles the chord joining the points of contact of the two tangents from P to the circle.

14. A quad. ABCD is circumscribed to a circle. Prove that

$$AB+CD=BC+AD.$$

15. A quad. ABCD is such that AB+CD=BC+AD. Prove that a circle can be drawn to touch the sides of the quad.

[Draw a circle to touch AB, BC, CD. Draw the second tangent from D to this circle and use " reductio ad absurdum."]

16. If the four sides of a parallelogram touch a circle, prove that all four sides are equal.

17. A st. line AB touches two intersecting circles at A and B respectively ; a second st. line touches them at C and D respectively. Prove that AC and BD are ||.

18. P is a point outside a given circle. The ⊥ from P on a diameter of the circle meets that diameter at a point Q outside the circle. Prove that the difference of the squares of the tangents from P and Q to the circle is equal to the square on PQ.

19. Any tangent to a circle, centre O, cuts two fixed tangents (not produced) at A and B. Prove that $\angle AOB$ is of constant magnitude.

[If the points of contact of the fixed tangents are X, Y,

$$\angle AOB = \tfrac{1}{2}\angle XOY.]$$

20. Two parallel tangents are drawn to a circle, and any tangent cuts them at the points A and B. Prove that AB subtends a right angle at the centre of the circle.

21. Prove that opposite sides of a quad. circumscribed to a circle subtend supplementary angles at the centre.

22. Two unequal circles touch externally at P and a common tangent is drawn touching one circle at X and the other at Y. Prove that $\angle XPY = 90°$.

23. The tangents at A and B to a circle meet at O, and a third tangent is drawn to the circle at a point C on the same side of AB as O, meeting OA at P and OB at Q. Prove that

$$OP + PQ + QO = 2OA.$$

24. A circle is touched internally at P by another circle whose diameter is half that of the first circle. PQR is any chord through P cutting the circles again at Q and R. Prove that PQ = QR.

25. Find the locus of the centre of a circle which touches two equal fixed circles.

26. A, B are the centres of two unequal circles which touch each other at C, and any st. line through C cuts the circles again at P, Q respectively. Prove that AP is $\parallel$ to BQ. Consider both cases.

27. A circle, radius R, is drawn and inside it are described two circles touching each other externally and the other circle internally. Prove that the perimeter of the $\triangle$ formed by the three centres is equal to 2R.

28. A number of circles are drawn touching each other at O. A st. line is drawn through O meeting the circles again at A, B, C ... . Prove that the tangents at A, B, C ... are all parallel. [Draw the common tangent at O to meet the tangents at A, B, C ... in A′, B′, C′... .]

MOSTLY CONSTRUCTIONS

29. Draw two circles with radii 2·1″ and 1·4″, with their centres 2·5″ apart; draw a direct common tangent, and calculate its length.

30. Draw two circles with radii 1·5″ and 0·9″, with their centres 2·5″ apart; draw a transverse common tangent, and calculate its length.

31. (i) Draw two circles with radii 2·2″ and 1″, with their centres 2″ apart. Draw their common tangents, and calculate their lengths. (ii) **Draw two equal circles with radii 1·2″, with their centres 3″ apart. Draw their common tangents.**

32. Draw all the common tangents to two circles whose centres are 5·5 cm. apart and whose radii are 1·8 cm. and 1·5 cm. respectively. Calculate their lengths.

33. If circles have radii a and b and the distance between their centres is d $(d>a+b)$, prove that the lengths of the direct and transverse common tangents are $\sqrt{d^2-(a\sim b)^2}$ and $\sqrt{d^2-(a+b)^2}$ respectively.

34. A, B are two fixed points 3″ apart. Construct a st. line through A, such that the ⊥ from B to it may be 1·7″ long.

35. A, B are two fixed points 5·6 cm. apart. Construct a st. line such that the ⊥s to it from A and B may be 1·7 cm. and 2·3 cm. respectively.

36. A line AB is 5·9 cm. long; construct a circle with centre A so that the tangent to it from B is 2·7 cm. long.

37. Show how to draw two tangents to a circle so as to contain a given angle.

38. Show how to draw a tangent to a circle which shall be (i) ||, (ii) ⊥, to a given st. line.

39. Draw a circle of radius 3″ and construct two || chords of the circle AB and CD, 3·3″ and 4·7″ long respectively, on opposite sides of the centre.

40. Construct a tangent to an arc of a circle at a point A, when the centre of the arc is not on the paper.

41. Prove that all equal chords of a circle are tangents to a fixed concentric circle, the points of contact being the mid-points of the chords.

42. P is a point 7 cm. distant from the centre of a circle of radius 3 cm. Construct a chord of the circle which shall be 4 cm. long, and which, when produced, shall pass through P.

[Draw the tangents from P to the circle which is touched by all chords of length 4 cm.]

43. In a circle of radius 6·5 cm. draw a chord 12 cm. long. Calculate the shortest distance of this chord from the centre of the circle. Take a point P on this chord 3 cm. from one end, and draw an equal chord so as to pass through P.

44. Through a point P, 3″ from the centre of a circle radius 2″, draw a st. line so that the chord intercepted by the circle shall be $2\frac{1}{2}$″ long.

45. Draw, full size, two circles with radii 1·5″ and 1″, and centres 1·8″ apart. Through one of the points of intersection of the circles draw that chord of the larger circle (i) which is bisected by the circumference of the smaller circle, (ii) which has a given length. Is this always possible ?

46. Given a circle and a point within it, show how to draw through the point a chord of given length. Find the region of the circle within which the point must lie in order that the construction may be possible.

47. Given a circle, show how to draw a chord of given length which shall be (*a*) $\perp$, (*b*) $\parallel$ to a given st. line.

48. Two circles of radii 1·6″ and 1″ are drawn with their centres 3″ apart. Draw a st. line so that the chord intercepted by the larger circle shall be 2·4″ long and that by the smaller 1·2″ long. [See No. 41.]

49. The length of the tangent from P to a fixed circle, centre O, is given. Prove that the locus of P is a circle, centre O.

50. A circle of radius 2·5 cm. has its centre 4 cm. from a given st. line. Show how to draw tangents 6 cm. long to the circle from points on this line.

51. Draw a circle, centre A, radius 1″. Take a point B such that AB = 2″. Construct the tangents from B to the circle. Take another point C 2″ from A and 3″ from B. Describe a second circle, centre C, so that the tangents to the two circles from B are equal.

52. Draw a circle of radius 4 cm., and in it place a chord AB of length 6·8 cm. Then find all the points whose perpendicular distance from this chord (produced if necessary) is 2·6 cm., while the length of the tangent from each to the circle is 3·6 cm.

53. A circle and a st. line are drawn and the centre O of the circle is joined to any point A in the st. line. Explain how to find a point P in the st. line such that the tangent from P to the circle may equal OA.

54. In a circle of radius 3 cm. draw a chord AB of length 5 cm. By a geometrical construction, which must be clearly stated, find a point P in AB produced such that the tangent from P to the circle is equal to the chord AB. Measure PB.

55. Draw two circles P, Q of radii 1·8″ and 1·1″ respectively, the distance between their centres being 3·2″. Then construct the point from which the tangent to P is 1″ long, while the tangent from it to Q has the greatest value consistent with this condition.

56. A point P, inside a triangular field ABC, is 30 yards from the side AB, 50 yards from the side AC, 60 yards from the side BC, 100 yards from the corner A, and 150 yards from the corner B. Draw accurately the plan of the field on a scale of 50 yards to the inch.

THEOREM 49

If a straight line touches a circle and, from the point of contact, a chord is drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments of the circle.

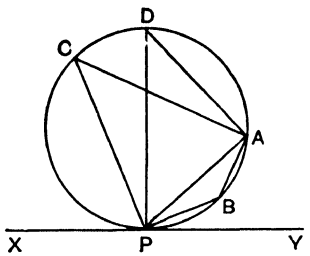


FIG. 189.

First Proof.

Let the straight line XY touch the circle at P. Let PA be a chord of the circle and let C, B be **any** points on the circumference of the circle on opposite sides of PA. Join CP, CA, BP, BA.

It is required to prove that

(1) the $\angle APY$ = the angle in the alternate segment, i.e. $\angle PCA$;

(2) the $\angle APX$ = the angle in the alternate segment, i.e. $\angle PBA$.

Construction. Draw the diameter PD. Join DA.

Proof. Since PY is a tangent to the circle and PD is a diameter drawn from the point of contact, $\therefore$ $\angle DPY$ is a right angle.

Since PD is a diameter, $\angle PAD$ is a right angle, being the angle in a semi-circle.

But the sum of the 3 angles of the $\triangle APD$ is two right angles,

$\therefore$ $\angle APD + \angle PDA$ = a right angle ;

$\therefore$ $\angle APD + \angle PDA = \angle DPY$ or $\angle APD + \angle APY$.

From each of these equals subtract the $\angle APD$,

$\therefore$ $\angle PDA = \angle APY$.

But $\angle PDA = \angle PCA$ (*angles in the same segment*),

$\therefore$ $\angle APY = \angle PCA$. .....(1)

Also $\angle PBA$ = the supplement of the $\angle PCA$ (*opp. $\angle$s of a cyclic quad.*).

But $\angle APY = \angle PCA$ (*proved*),

$\therefore$ $\angle PBA$ = the supplement of the $\angle APY$

= $\angle XPA$ (*adjacent angles on the st. line XPY*)....(2)

Note. The disadvantage of this proof is that the proof for the equality of the obtuse angles depends upon that for the equality of the acute angles. For this reason the following proof may be preferred.

Second Proof.

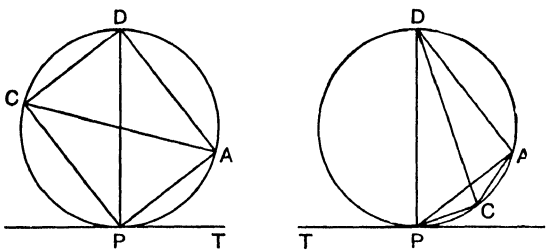


FIG. 190.

Let the straight line PT touch the circle at P. Let PA be a chord of the circle and let C be **any** point on the circumference of the circle such that C and T are on opposite sides of PA. Join CP, CA.

In each figure it is required to prove that the $\angle APT$ = the angle in the alternate segment, i.e. $\angle PCA$.

Construction. Draw the diameter PD. Join DA, DC.

Proof. Since PT is a tangent to the circle and PD is a diameter drawn from the point of contact, $\therefore \angle DPT$ is a right angle.

Since PD is a diameter $\angle PCD$ is a right angle, being the angle in a semi-circle.

Also $\angle DCA = \angle DPA$ (*angles in the same segment*),

$\therefore$ In Fig. 1 $\angle DPT - \angle DPA = \angle PCD - \angle DCA$,

i.e. $\angle APT = \angle PCA$. .....(1)

In Fig. 2 $\angle DPT + \angle DPA = \angle PCD + \angle DCA$,

i.e. $\angle APT = \angle PCA$. .....(2)

Note 1. If ∠APT is a right angle, the alternate segment is a semi-circle, and the angles in a semi-circle are right angles, so that the theorem is true in this case.

Note 2. The pupil should notice that the phrase "**alternate segment**" means the segment which is **not** on the same side of the chord AP as the angle TPA, referring to the figures used in the Second Proof.

THEOREM 49a

If through an extremity of a chord of a circle a straight line is drawn, making with the chord an angle equal to the angle in the alternate segment, then the straight line touches the circle.

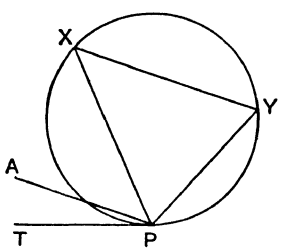


FIG. 191.

Let PX be the chord and PA a straight line through P such that ∠APX = ∠PYX in the alternate segment.

It is required to prove that PA touches the circle at P.

Proof. If possible, suppose that PA does not touch the circle at P and let PT be the tangent at P.

Then ∠TPX = ∠PYX in the alternate segment.

But ∠APX = ∠PYX (*given*),

∴ ∠TPX = ∠APX,

i.e. the whole is equal to the part, which is impossible ;

∴ our supposition that PA does not touch the circle is wrong,

∴ PA touches the circle at P.

EXERCISE 46 [Theorem 49]

1. A chord AB of a circle subtends an angle of 104° at the centre of the circle. Calculate the acute angle between the tangents at A and B.

2. A, B are two points on a circle, and the tangents at A, B meet at P, making $\angle APB = 44°$. AC is the chord of a circle drawn through A || to PB. Calculate each of the $\angle$s of the $\triangle ABC$.

3. ABC is a $\triangle$ in which $\angle A = 52°$, $\angle B = 62°$, $\angle C = 66°$. The tangents to the circle ABC at A, B, C form a new $\triangle$. Find the $\angle$s of this $\triangle$.

4. If two tangents PA, PB touch a circle at A, B and $\angle APB = 58°$, find in degrees the angle at the circumference of the circle standing on the larger arc AB.

5. PA, PB are two tangents to a circle which include an angle of 44°. C is a point on the circle on the opposite side of AB from P. Calculate the $\angle$ ACB.

6. Find the $\angle$s subtended at the centre of the circle by chords which make $\angle$s (i) of 23°, and (ii) of 140° with the tangents at their extremities.

7. A chord 1″ long of a circle makes an angle of 30° with the tangent at one end of the chord. Calculate the radius of the circle.

8. PTO is a tangent to a circle, T being the point of contact. PQR is a st. line meeting the circle at Q, R and such that $\angle TPQ = 50°$ and $\angle RTO = 100°$. Prove that (i) PQ = QT, (ii) QR is equal to the radius of the circle.

9. The sides PQ, QR, RS, SP of a quad. touch a circle at the points A, B, C, D respectively. The $\angle QPS = 96°$, and the $\angle QRS = 120°$. The st. lines AB, DC meet at E. Calculate the $\angle AED$.

10. Two circles cut at A and B and through A and B are drawn two st. lines PAQ and RBQ, cutting one circle at P and R and the other circle at Q. Prove that PR is || to the tangent at Q.

11. Two circles intersect at A and B. The tangent at A to the first circle meets the second circle at P and the tangent at A to the second circle meets the first circle at Q. Prove that the $\triangle$s PBA, ABQ are equiangular. Prove also that, if PBQ is a st. line, then $\angle PAQ = 90°$.

12. P is any point on the side BC of a $\triangle ABC$, and the circles are circumscribed to the $\triangle$s ABP, APC. Prove that one of the angles between the tangents at A to the two circles is equal to the $\angle BAC$.

13. ABC is a $\triangle$ and a circle is drawn to touch BA at A and to pass through C. If AP is the chord of this circle which is $\parallel$ to BC and if BP cuts this circle again at Q, prove that the $\angle$s QAB, QBC, QCA are equal.

14. ABCD is a cyclic quad.; prove that, if the circle passing through D and touching AB at A meets CD again at E, then AE is $\parallel$ to BC.

15. A, P, B are three points on a circle such that the chords AP, PB are equal: prove that the $\perp$s from P to the chord AB and the tangents at A, B are all equal.

16. Two circles intersect at the points P, Q, and a common tangent touches the circles at the points A, B. Prove that the $\angle$s APB, AQB are supplementary.

17. A regular pentagon ABCDE is inscribed in a circle and the tangent at A meets CB produced at F. Prove that AF=BC.
[Calculate the $\angle$s of the $\triangle$s AFB, ABC.]

18. If AB, AC are two equal chords of a circle, prove that BC is $\parallel$ to the tangent at A to the circle.

19. Two circles touch at a point O. AB is a chord of one circle and CD a chord of the other circle such that AC and BD meet at O. Prove that AB is $\parallel$ to CD. Consider both kinds of touching.

20. ABC is a $\triangle$; a circle is drawn through A to touch BC at C. AD is the chord of this circle which is $\parallel$ to BC. Prove that $\triangle$ACD is isosceles.

21. ABCD is a cyclic quad.; AN is $\parallel$ to BC, meeting BD, not produced, at N. Prove that AC touches at A the circle ADN.

22. With centre B, a point on the internal bisector of an angle ACD, a circle is drawn to pass through C and to cut CA at A. Prove that the circle circumscribing the $\triangle$ABC will touch CD at C.

23. PQRS is a parallelogram whose diagonals PR, QS meet at X. Prove that the circles PQX, RSX touch each other at X.

24. Two equal circles ACP, ACQ cut at A and C, and the tangent at A to the first circle cuts the other again at B. If AP be a chord of the first cutting the other again at Q, prove that AP=BQ.

25. A circle passes through the three points A, P, B, and O is its centre. O and P lie on the same side of AB and O does not lie between the arms of the $\angle$APB. If AO is a tangent to the circle through O, P, B, prove that AP bisects the $\angle$OPB.

26. ABCD is a cyclic quad., and a circle which passes through C and touches AB at B cuts BD at O. Prove that the $\triangle$s OCD, BCA are equiangular.

27. ABCD is a quad. such that the diagonal CA bisects the $\angle$ BCD. If AD touches the circle ABC, prove that AB touches the circle ACD.

28. Tangents are drawn from a point T to a circle whose centre is O, touching the circumference at A and B, and OT meets the circumference at P. Prove that AP bisects the $\angle$TAB.

29. From a point P a tangent PA is drawn to a circle, and a secant PBC cutting the circle at B and C. The bisector AQ of the $\angle$BAC cuts BC at Q. Prove that PQ=PA.

30. AB, AC are the equal sides of an isosceles $\triangle$ABC. The circle which passes through A and touches BC at B meets CA, produced if necessary, at E. Prove that BE=BC.

EXERCISE 47 [Further Examples on Theorems 46-49]

1. Two circles intersect at O and the tangents at O to the circles are at right angles. The tangent to the first circle cuts the second circle at P and the tangent to the second circle cuts the first circle at Q: show that PQ passes through the other point of intersection of the circles.

2 P and Q are the points of contact of tangents OP, OQ drawn to a circle whose centre is C; PC and OQ are produced to meet at L; QC and OP are produced to meet at M. Prove that LM is $\parallel$ to PQ.

3. If one of the internal common tangents to two non-intersecting circles makes an angle of 30° with the line joining the centres of the circles, prove that the distance between the centres is equal to the sum of the diameters of the circles.

4. Give a construction for finding points on the circumference of a given circle, such that tangents drawn from them to another given circle are of given length. How many solutions are there? When is this impossible?

5. If a quad. can be divided by a st. line crossing it so that circles can be inscribed in each of the two quads. into which it is divided, show that the length of the st. line is half the excess of one pair of opposite sides of the quad. over the sum of the other pair.

6. Two unequal circles touch externally at P; two points B and D are taken on opposite sides of P on the common tangent at P (and *not* on the part of it which lies between the direct common tangents of the two circles) and tangents are drawn from B and D to both circles. If the tangents from B and D to the first circle meet at A and those to the second circle meet at C, prove that AB+CD=BC+DA.

7. ABC is a $\triangle$ inscribed in a circle and the tangents at B, C meet at P; (i) if the $\angle$s BAC and BPC are equal, prove that $\angle$BAC=60°; (ii) if the $\angle$s ABC and BPC are equal, prove that the $\triangle$ABC is isosceles.

8. TA, TB are tangents to a circle whose centre is C. The circle with centre T and radius TA cuts CT at P, P being between C and T. Prove that AP bisects the $\angle$CAB.

9. ABC is a $\triangle$ inscribed in a circle, AD is drawn $\perp$ to BC and CE $\perp$ to the tangent at A. Prove that DE is $\parallel$ to BA.

10. The internal bisector of the $\angle$A of a $\triangle$ABC cuts the side BC at D and the circumscribing circle of the $\triangle$ at E. Prove that EC is a tangent to the circle through A, D, C.

11. From an external point P tangents PQ, PR are drawn to a circle of which RS is a diameter; SQ produced meets RP produced at T. Prove that $\angle$QPR = $\angle$ 2QTR.

12. ABCDEF are six points on a circle such that AB is $\parallel$ to DE. Prove that if DE meets AF produced at G, BD touches the circle through D, G, F.

13. Two circles touch externally at a point O. One of the common tangents touches them at A and B, another at C and D respectively. Prove that the circles OAB and OCD touch one another.

14. Two circles with centres A and B touch one another externally at P, and a common tangent touches them at Q and R respectively. Prove that AQ is a tangent to the circle through P, Q and R.

15. Two circles touch externally. If r_1 and r_2 are the radii of the circles, show that the length of a direct common tangent is $2\sqrt{r_1 r_2}$. A third circle touches these circles and a direct common tangent. Show that, if r is the radius of this circle,

$$r = \frac{r_1 r_2}{r_1 + r_2 + 2\sqrt{r_1 r_2}}.$$

16. A point O is taken on the side BC of a $\triangle$ABC, and the tangent at B to the circle AOB meets the tangent at C to the circle AOC at the point X. Prove that (i) A, B, X, C lie on a circle, (ii) $\triangle$s ABO, AXC are equiangular.

17. AB is a diameter of a circle, centre C, and upon AC, CB as diameters circles are described, and a fourth circle is drawn touching the first circle internally and the other two externally. Find the radius of the fourth circle, that of the first being of length a.

18. ABCD is a parallelogram and X, Y are the mid-points of the opposite sides AB and DC. If the circle with centre X and radius XA touches the circle with centre Y and radius YC, prove that ABCD is a rhombus.

19. BC is a common external tangent to two circles which touch one another externally at A ; show that BA, CA, produced, pass through the other ends of the diameters through C, B.

20. ABCD is a rectangle in which AD > 2AB ; two equal circles are drawn, one of which touches AD, BC and AB, while the other touches AD, BC and CD. One of the two transverse common tangents to these circles cuts AD at P and BC at Q. Prove that PQ = AD − AB.

21. Two unequal circles touch externally at A. A point P is taken on the common tangent at A, and from P two tangents are drawn, one to each circle, touching the circles at B and S respectively. The st. line BS cuts the circles again at C and R respectively. Prove (*a*) that BC and RS subtend equal angles at A ; (*b*) that the tangents at B, R, C and S form a parallelogram.

22. A st. line is drawn ‖ to the side BC of a △ABC meeting the sides AB, AC at X and Y respectively. Prove that the circles AXY, ABC have a common tangent at A.

23. A △ABC with AB = AC, is inscribed in a circle. The bisector of the ∠ABC meets the tangent at A to the circle at D. Prove that AD = AB.

24. AB and BC are two equal chords of a circle. Circles, centre A, radius AC and centre B, radius BC, intersect at C and T. Prove that AT touches the circle ABC at A.

25. A quad. ABCD is inscribed in a circle of which AB is a diameter. The diagonals AC, BD meet at O. Prove that the tangent at C to the circle through C, O, D passes through the centre of the circle ABCD.

26. The tangents at A and B to a circle whose centre is O meet at C ; a st. line through O cuts CA produced at P and CB produced at Q ; and the other tangents that can be drawn from P and Q meet at T. Prove that OT = OC.

27. TP, TQ are tangents to a circle, and R is any other point on the circle. The tangent at R is drawn, and a st. line through T, ‖ to this tangent, meets RP, RQ at X and Y. Prove that TX, TP, TQ, TY are all equal.

28. Construct a △XYZ, in which YZ = 3″, the ⊥ from Y to XZ = 2″, the ⊥ from Z to XY = 2·3″. How many solutions are there ?

29. Two circles touch internally at O ; from O the st. line OAB is drawn cutting the circles at A and B. From any point T on the common tangent, secants TAP, TBQ are drawn cutting one circle at A and P, the other at B and Q. Prove that O, P, Q and T are concyclic.

30. A, B, C, D are four points on a circle such that the chords AC, BD intersect at right angles inside the circle. Prove that the tangents to the circle at A, B, C, D form a quad. which can be inscribed in a circle.

31. Two segments of circles have their bases AB, BC in the same st. line ABC ; they lie on opposite sides of these bases, and the angle in one segment is the same as the angle in the other. Show that the arcs have a common tangent at B, and that the tangents at A and C are ||.

32. ABC is an isosceles △, having AB=AC. Any chord AP (produced if necessary) of the circle ABC meets BC (produced if necessary) at Q. Prove that AC touches the circle CPQ.

33. LMN is an isosceles △ in which the sides LM, LN are equal. The circle which touches MN at N and passes through L cuts LM at a point P between L and M, so that LP=MN. Prove that the ∠s LMN and LNM are each double the ∠MLN.

34. AB, AC are two chords of a circle ABC. BL, drawn || to the tangent at A, cuts AC (produced if necessary) at L. Prove that (i) the circle through B, L, C touches AB at B ; (ii) if CA=CB, then the circle BLC is equal to the circle ABC.

35. From a point P outside a circle a tangent PT is drawn touching the circle at T and a secant cutting it at Q and R ; on PT as diameter a circle is drawn cutting RT, produced if necessary, at Y and TQ, produced if necessary, at X. Prove that XY is ⊥ to PR.

36. Two circles touch internally at P, and a st. line cuts the inner circle at A, B, and the outer circle at C, D. PA, PB produced cut the outer circle at E and F respectively. Prove that EF is || to AB, and that ∠EPC=∠FPD.

37. ABC is a △ ; find the locus of a point P such that the ∠s PAB, PCA are equal, P lying inside the △.

38. ABC is a △ inscribed in a circle. Tangents are drawn to the circle at A, B, and C, those at B and C meeting at P. Through P a st. line is drawn || to the tangent at A meeting AB, AC produced at R and S respectively. Prove that RS is bisected at P.

39. O is the centre of a circle, B any point on the radius OA. The ⊥ bisector of AB meets the circle at P, and the tangent at P meets OA produced at T. Prove that ∠OPB+3∠APT=90°.

40. A, B, C, D are four points on a st. line in that order. X is a point at which AB, CD subtend equal angles. Prove that the circles XBC, XAD have a common tangent at X.

THE CONSTRUCTION OF CIRCLES TO SATISFY GIVEN CONDITIONS

In order to draw a circle the given conditions must be sufficient to determine (i) the position of the centre, (ii) the length of the radius. It is usually possible to find the centre as the intersection of two loci, each of which is obtained by using one of the given conditions, so that two conditions are needed to fix the centre. When the centre has been found, either the radius must be given or we must find it by being given some point through which the circle must pass. It follows that, in order to construct a circle, three independent conditions are needed. We shall first consider the straightforward constructions, in which the centre is always obtained as the intersection of two loci, and then proceed to consider a number of constructions in which other methods have to be used.

Before attempting the constructions of the next Exercise the pupil must be able to apply the following loci:

1. The locus of centres of circles which pass through a fixed point X and have a given radius. [See definition of a circle.]

2. The locus of centres of circles which pass through two fixed points. [See Theorem 31.]

3. The locus of centres of circles which touch a given straight line at a given point. [See Theorem 46, Cor. 3.]

4. The locus of centres of circles which touch a given straight line and have a given radius. [See Theorem 46, Cor. 4.]

5. The locus of centres of circles which touch a given circle at a given point. [See Theorem 48, Cor. 3.]

6. The locus of centres of circles which touch a given circle and have a given radius. [See Theorem 48, Cor. 4.]

7. The locus of centres of circles which touch two given intersecting straight lines. [See Theorem 32.]

8. The locus of centres of circles which touch two given parallel straight lines. [See Theorem 46, Cor. 5.]

It is suggested that, before the following Exercise is attempted, the teacher should discuss orally the application of the above loci to the solution of the following problems. In all cases the number of solutions possible should be carefully considered:

1. Construct a circle to pass through three points, not lying in the same straight line.

2. Construct a circle to touch three straight lines which form a triangle.

3. Construct a circle to touch two parallel straight lines and a transversal.

4. Construct a circle to pass through a point C and touch the straight line AB at B.

5. Construct a circle to touch a given straight line at a given point and also a second given straight line [There are two cases.]

6. Construct a circle to pass through a point C and touch a given circle at a given point B.

7. Construct a circle to touch a given circle at a given point B, and also a given straight line AC. [Notice that the required circle must touch the tangent at B to the given circle.]

8. Construct a circle of given radius to pass through two fixed points.

9. Construct a circle of given radius to pass through a fixed point and touch a given straight line.

10. Construct a circle of given radius to pass through a fixed point and touch a given circle.

11. Construct a circle of given radius to touch two intersecting straight lines.

12. Construct a circle to pass through a given point and touch two parallel straight lines.

13. Construct a circle of given radius to touch a given straight line and a given circle.

14. Construct a circle of given radius to touch two given circles. [Notice that the number of solutions may be any number up to eight.]

15. Construct a circle to touch the arc and bounding radii of a sector of a circle. [Notice that the required circle must touch the tangent to the sector at the mid-point of its arc.]

EXERCISE 48

(Construction of circles to satisfy given conditions.)

1. Draw a $\triangle ABC$ in which $AB=5$ cm., $BC=6$ cm., $CA=7$ cm., and draw a circle to pass through A, B and C.

2. Draw a $\triangle ABC$ in which $AB=3''$, $BC=4''$, $CA=6''$, and draw a circle to pass through A, B and C.

3. Draw a $\triangle ABC$ in which $AB=5{\cdot}6$ cm., $BC=5{\cdot}2$ cm., $CA=4{\cdot}6$ cm., and draw a circle to pass through A, B and C. Construct a second $\triangle ABD$ so that $\angle ADB=\angle ACB$ and $AD=3$ cm.

4. Construct a $\triangle$ with sides 4·2, 6·1, 7·3 cm.; describe a circle touching the 4·2 cm. side and also touching the 6·1 and 7·3 cm. sides produced.

5. Construct a $\triangle$ with sides $2{\cdot}1''$, $2{\cdot}3''$, $3{\cdot}7''$. Describe a circle touching the three sides.

6. Describe a circle touching each of three given st. lines. Discuss all the possible cases. When is the construction impossible ?

7. AB, CD are two $\parallel$ lines $1{\cdot}9''$ apart. XY is a transversal inclined to them at an angle of 47°. Construct two circles to touch AB, CD, XY.

8. ABC is an equilateral $\triangle$, side $1{\cdot}8''$. Describe a circle touching BC at B and passing through A.

9. Draw two straight lines OA, OB containing an angle of 49°, and make $OA=2{\cdot}3''$, $OB=1{\cdot}3''$. Describe a circle touching OA at A and passing through B. Produce OB to meet the circle again at C. Find two points D, E on the circle which are equidistant from AB (or BA produced) and AC.

10. Draw a $\triangle APC$ having $AB=2{\cdot}5''$, $BC=3{\cdot}25''$, $CA=2''$. Construct a circle touching AB at B and also touching a st. line through A $\parallel$ to BC.

11. Draw a $\triangle ABC$ having $AB=2{\cdot}5''$, $BC=3{\cdot}25''$, $CA=2''$. Construct a circle touching AB at B and also touching a st. line through C $\parallel$ to AB.

12. Draw a st. line AB of length 3·1″; find a point D distant 2·4″ from A and 1·4″ from B. Construct a △ABC whose inscribed circle touches BC at D.

13. Draw a circle, centre O, radius 2·4″, and mark a point A on it. Take a point B such that OB = 1·8″ and ∠AOB = 32°. Construct a circle to touch the first circle at A and pass through B.

14. Draw a circle, centre O, radius 2·6″, and mark a point A on it. Take a point B such that OB = 4·1″ and ∠AOB = 28°. Construct a circle to touch the first circle at A and pass through B.

15. Draw a circle, centre O, radius 1·8″, and mark a point A on it. Draw a st. line 2·7″ distant from O. Construct a circle to touch this line and also the first circle at A.

16. Draw a circle, centre O, radius 1·8″, and mark a point A on it. Draw a st. line 1″ distant from O. Construct a circle to touch this line and also the first circle at A.

17. Take two points A, B, 3 cm. apart. Draw a circle of radius 7 cm. to pass through A and B.

18. Take a point O and a st. line XY, 3″ distant from O. Construct a circle of radius 1·8″ to touch XY and pass through O.

19. Take a point O and a st. line PQ, 2·4 cm. distant from O. Construct a circle of radius 4·6 cm. to touch XY and pass through O.

20. Two points X, Y are taken 4·6 cm. apart; a circle is described with centre X and radius 2 cm. Describe a circle with radius 3 cm. to touch the first circle and to pass through Y. How many solutions are there?

21. Two points X, Y are taken 3·6 cm. apart; a circle is described with centre X and radius 2 cm. Describe a circle with radius 3 cm. to touch the first circle and to pass through Y. How many solutions are there?

22. P is a point 4·1″ distant from the centre of a circle of radius 2·2″. Through P construct two circles of radius 1·4″ to touch the original circle.

23. Draw a circle of 7 cm. radius, and take a point X in its plane distant 4·5 cm. from the centre. Draw a circle of radius 5 cm. to touch the first circle and to pass through X.

24. Draw two st. lines OX, OY intersecting at an angle of 65° and construct a circle of radius 2·3 cm. to touch OX and OY.

25. Draw two st. lines intersecting at an angle of 42°. Construct all the circles of 1·2″ radius which touch the two st. lines.

26. Construct a circle to touch the arms of a given angle and be equal to a given circle.

27. Draw two parallel st. lines 1·7″ apart and take a point 1·1″ from one of them and 0·6″ from the other; construct a circle touching both lines and passing through the point.

28. Construct, by a purely geometrical construction, a circle to touch two given parallel st. lines and to pass through a given point between them.

29. Draw a circle, radius 1·3″, and also a st. line at a distance 2″ from its centre. Construct a circle, radius 1·4″, to touch the circle and the st. line.

30. Draw a circle, 2·1″ radius, with a chord 1·1″ from the centre. Describe another circle, radius 1·3″, in the larger segment to touch the first circle and the chord.

31. A circle of 0·9″ radius rolls inside a circle of 2·1″ radius. Find the locus of its centre. Construct a circle of 1″ radius to touch internally the circumference and diameter of a semi-circle of 3″ radius.

32. Draw a circle of radius 2″ and a st. line distant 1″ from the centre of the circle. Then construct all the circles of radius 0·5″ which touch both the st. line and the circle. How many solutions are there? Can you modify the given lengths in such a way as to obtain four solutions?

33. Draw two circles, radii 4·1 cm. and 5·1 cm., with their centres 2 cm. apart. Then draw all the possible circles of radius 2·1 cm. which touch the given circles. How many circles are there?

34. Describe a circle of 2·5″ radius. Inside this describe a circle of radius 1″ touching it. Draw as many circles as possible of radius 0·6″ touching these circles. How many circles are there?

35. Draw two circles, radii 2·5 cm. and 2·4 cm., having their centres 6 cm. apart. How many circles of radius 7 cm. can be drawn to touch these circles? Construct a circle to touch the first circle internally and the second externally.

36. XZ is a st. line 3·4″ long and Y is its mid-point. On the same side of XYZ describe semi-circles on XY, XZ as diameters. Construct a circle of radius 0·8″ to touch the larger semi-circle internally and the smaller semi-circle externally.

37. Construct a circle of radius d″ to touch two circles, radii a″ and b″, whose centres are c″ apart. By giving a, b, c, d suitable values show that the number of possible solutions to this problem may be 0, 1, 2, 3, 4, 5, 6, 7 or 8.

38. O is the centre of a circle of radius 4 cm., and P, Q are two points on the circumference such that $\angle POQ = 47°$. Construct a circle to touch OP, OQ and the arc PQ.

39. Draw a sector of a circle of radius 3″, in which the arc of the sector subtends an angle of 60° at the centre. In this sector draw a circle touching the arc and both radii. Calculate the radius of this circle.

40. Draw a circle of radius 4·2 cm., and two radii OP, OQ, at an angle 43°. Construct a circle to touch the first circle externally, and also OP produced beyond P and OQ produced beyond Q.

41. In a circle ABC of radius 3″ place two chords AB, BC of lengths 2·8″ and 5·2″, and draw a chord BD ‖ to AC. Construct two circles, each of which touches the circle ABC internally and also the two ‖ chords AC, BD.

42. Each of three points A, B, C is 5 cm. distant from a fourth point D in the same plane. If $\angle ABC = 24°$ and $\angle BAC = 46°$, construct a possible figure.

43. Show how to draw two concentric circles, one of which passes through two given points A, B, while the other touches a given circle at a point P. In what case is the construction impossible ?

44. The $\triangle ABC$ has $AB = AC = 4·4$ cm., $BC = 2·9$ cm., and a circle is drawn with centre A and radius 2·4 cm. Construct the two circles which pass through B and C and touch this circle.

45. Draw a circle so as to pass through two given points and have its centre on a given circle. Give figures to show the different cases that may arise.

46. Two lines OX, OY meet at an angle of 32°; along OX two points P and Q are taken so that $OP = 1·4''$ and $OQ = 2·5''$. Construct a circle that shall pass through P and Q and have its centre 1·2″ from OY. How many such circles are there ?

47. Two adjacent sides of a quad. inscribed in a circle of radius 2·4″ subtend angles of 82° and 48° at the centre. The remaining two sides are equal. Construct the quad. and calculate its four angles.

48. Construct a circle, of radius 1″, touching a given st. line, so that the tangents to the circle from a given point may be 2″ in length. If the given point is at a distance of x inches from the given st. line, how do the number of solutions depend upon the value of x ?

CONSTRUCTION 20

On a given straight line to construct a segment of a circle containing a given angle.

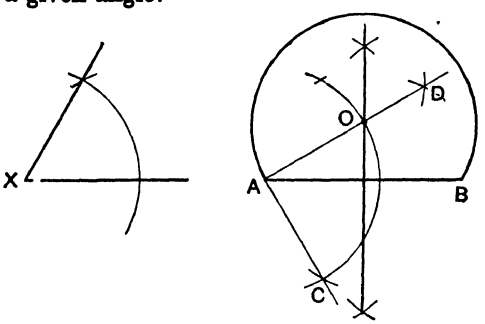


FIG. 192.

Let AB be the given straight line and X the given angle.

It is required to construct on AB a segment of a circle, containing an angle equal to the angle X.

Construction. At A make an $\angle BAC$ equal to the $\angle X$. (Construction 5.)

Draw AD $\perp$ to AC. (Construction 3, Method 2.)

Draw the $\perp$ bisector of AB, meeting AD at O. (Construction 2.)

With centre O, and radius OA, draw a circle.

Then the segment of this circle on the chord AB, alternate to the $\angle BAC$, is the required segment.

Proof. Since O lies on the $\perp$ bisector of AB, $\therefore$ OA = OB ;

$\therefore$ the circle drawn with centre O, and radius OA, passes through B.

But since, by construction, the radius OA is $\perp$ to AC, AC is a tangent to the circle ;

$\therefore$ $\angle BAC$ (*i.e.* $\angle X$) = the angle in the alternate segment, which is therefore the required segment.

Note. The above construction is a purely geometrical construction. If the use of the protractor is allowed it is easier to proceed as follows :

To construct, on AB, a segment of a circle containing an angle of $74°$.

Suppose the segment (centre O) drawn.

Then $\angle AOB$, being the angle subtended by AB at the centre, = twice the angle subtended by AB at the circumference $= 148°$. But $\triangle OAB$ is an isosceles $\triangle$, $\therefore$ each base $\angle = \frac{1}{2}(180° - 148°) = 16°$. We can therefore find the centre O of the required segment by constructing, on AB as base, an isosceles $\triangle$ having its base angles each $16°$. The vertex of this $\triangle$ is the required centre.

CONSTRUCTION 21

In a given circle to inscribe a triangle equiangular to a given triangle.

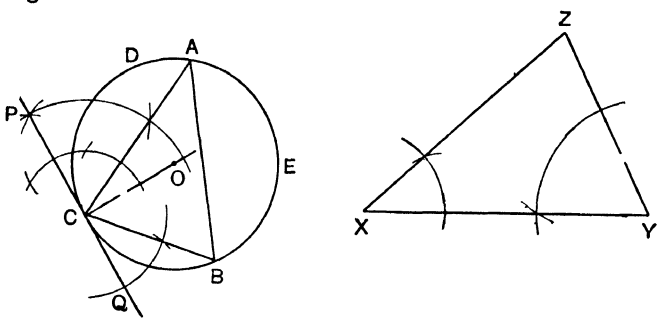


FIG. 193.

Let XYZ be the given △ and CDE the given circle.

It is required to inscribe in the circle CDE a △ equiangular to the △XYZ.

Construction. Join C to the centre O of the circle CDE.

Draw PCQ ⊥ to CO at C. (Construction 3, Method 2.)

Then PCQ is a tangent to the circle, being ⊥ to the radius through C.

Draw chords CB, CA, making the ∠s QCB, PCA equal to ∠s YXZ. XYZ respectively. (Construction 5.)

Join A, B.

Then ABC is the required triangle.

Proof. Because PCQ is a tangent and CA a chord of the circle,

∴ ∠PCA = ∠CBA in the alternate segment.

But ∠PCA = ∠XYZ (*construction*);

∴ ∠CBA = ∠XYZ.

Similarly ∠CAB = ∠YXZ;

∴ the remaining ∠ACB = ∠XZY.

CONSTRUCTION 22

About a given circle to describe a triangle equiangular to a given triangle.

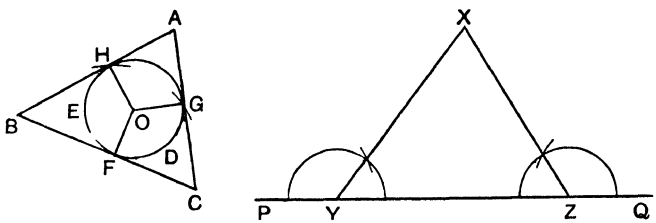


FIG. 194

Let XYZ be the given △ and EFD the given circle.

It is required to describe about the circle EFD a triangle equiangular to XYZ.

Construction. Join F to the centre O of the circle EFD.

Draw radii OH, OG, making the ∠s FOH, FOG equal to the supplements of the ∠s XYZ, XZY respectively. (*Construction* 5.)

Draw the tangents at F, G, H, forming the △ ABC. (Theorem 46, Note.)

Then ABC is the required △.

Proof. Since HB, FB are the tangents at H, F, they are ⊥ to the radii OH, OF respectively ;

$$\therefore \angle OHB = \angle OFB = \text{a right angle ;}$$

∴ since the angles of a quadrilateral are together equal to four right angles, ∠HBF = the supplement of ∠FOH.

But ∠FOH = the supplement of ∠XYZ (*construction*) ;

$$\therefore \angle HBF, \text{ i.e. } \angle ABC = \angle XYZ.$$

Similarly $\angle ACB = \angle XZY$;

$$\therefore \text{ the remaining } \angle BAC = \angle ZXY.$$

CONSTRUCTION 23

To inscribe a regular hexagon in a given circle.

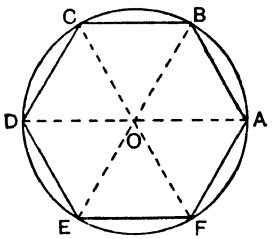


FIG. 195.

Let O be the centre of the circle in which it is required to inscribe a regular hexagon.

Construction. Take any point A on the circumference.

In the circle place five chords AB, BC, CD, DE, EF, each equal to the radius of the circle. Join AF.

Then ABCDEF is a regular hexagon.

Proof. Join O to A, B, C, D, E, F.

Then the $\triangle$s AOB, BOC, COD, DOE, EOF are all equilateral $\triangle$s, since each side of each $\triangle$ is, by construction, equal to the radius of the circle ;

$\therefore$ $\angle$s AOB, BOC, COD, DOE, EOF are each equal to 60°

But the sum of the angles at O is 360° ;

$\therefore$ $\angle AOF = 60°$.

But the sum of the angles of the $\triangle AOF$ is 180° ;

$\therefore$ $\angle OAF + \angle OFA = 120°$.

But OA = OF, being radii of the circle,

$\therefore$ $\angle OAF = \angle OFA = 60°$,

$\therefore$ the $\triangle OAF$ is equilateral ;

$\therefore$ AF = the radius OA ;

$\therefore$ the figure ABCDEF has all its sides equal.

Also each angle of the figure is equal to 120° ;

$\therefore$ ABCDEF is a regular hexagon.

Note 1. A regular polygon of n sides may be inscribed in a given circle, centre O, by drawing radii OA, OB, OC, OD, ... such that the $\angle$s AOB, BOC, COD, ... are each equal to $\frac{360^\circ}{n}$. If AB, BC, CD, ... be drawn, ABCD ... is the polygon required.

Note 2. A regular polygon of n sides may be described about a given circle, centre O, by drawing radii OA, OB, OC, OD ... such that the $\angle$s AOB, BOC, COD, ... are each equal to $\frac{360^\circ}{n}$. The tangents to the circle at A, B, C, D, ... are the sides of the required polygon.

EXERCISE 49 [Constructions 20-23]

Nos. 1-4 should be discussed orally.

Consider the number of solutions possible in each case.

1. From a given circle cut off a segment containing a given angle.

2. Construct a $\triangle$ having given the base, the vertical angle, and (i) another side, (ii) the altitude, (iii) the area, (iv) the median which bisects the base, (v) the foot of the perpendicular from the vertex to the base.

3. Construct a $\triangle$ having given the base, the vertical angle, and that the vertex lies on a given line, straight or curved.

4. Construct a $\triangle$ having given the base, the vertical angle, and the point where the bisector of the vertical angle meets the base.

[Notice that, if the bisector is produced, it will bisect the arc of the segment of the circumscribing circle which does not contain the vertex.]

5. Construct a $\triangle$PQR, in which PQ=4 cm., $\angle$R=47°, and whose area is 4·8 sq. cm.

6. X is a point on a given st. line PQ. Construct a circle which touches PQ at X and is such that every chord of length 2·3 cm. subtends an angle of 47° at the circumference.

7. Construct a parallelogram PQRS in which QR=5 cm., $\angle$PQR=32° and $\angle$RPS=42°. Construct also a quad. PQPT such that $\angle$PTR=32° and PT=QR, but T is a different point from S.

8. Solve the following problem by accurate drawing and measurement. A, B, C are three points such that AB is 5 miles, BC is 10 miles and CA is 7 miles. P is a point in the plane of ABC such that $\angle BPA$ is 30°, $\angle CPA$ is 30° and $\angle BPC$ is 60°. Find PA, PB, PC.

9. Inscribe a $\triangle$ in a circle of radius 2″ such that the two segments into which the bisector of the vertical angle divides the base are of lengths 1·3″ and 1·6″.

10. Construct a quad. PQRS in a circle, radius 2·7″, such that PQ subtends an angle 80° at the centre of the circle, $\angle QSR = 15°$, and the tangent at S makes an angle 68° with SR.

11. Construct a $\triangle XYZ$ given that YZ = 3″, $\angle YXZ = 64°$, and XW = 2·3″, where W is the mid-point of YZ.

12. Construct a quad. PQRS with diagonals PR, QS intersecting at O, such that PQ = 2″, QR = 2·4″, PR = 3·8″, PS = 2·9″, $\angle POQ = 103°$.

13 Construct a quad. PQRS such that PQ = 2 cm., QR = 4 cm., RS = 6 cm., $\angle QPR = \angle QRS = 39°$.

14. PQ is a st. line of length 2·5″. Find two points A, B, distant 3″ from P such that $\angle PAQ = \angle PBQ = 40°$.

15. Construct a $\triangle PQR$, in which PQ = 3·6 cm., QR = 5·2 cm., $\angle PQR = 128°$. On the side QR and on the same side of it as P, construct a segment of a circle containing an angle of 52°. Find also the position of a point X within the $\angle PQR$ such that the area of the $\triangle XPQ = 10·8$ sq. cm. and the $\angle QXR = 52°$.

16. A $\triangle$ has its base 4″ long, and its vertical angle 98°. Find, by drawing, measurement and calculation, the largest area it can possibly have. Draw the $\triangle$ when its area is two-fifths of this maximum.

17. On the same side of a st. line AB, of length 6 cm., construct $\triangle$s ACB, ADB such that $\angle ACB = \angle ADB = 75°$, AC = CB and the area of $\triangle ADB$ is one-third that of $\triangle ACB$.

18. X is a given point outside the circumference of a given circle, centre O. Show how to draw through X a st. line which will cut off from the circle a segment containing an angle equal to a given angle.

[Let XYZ be the required line meeting the circle at Y and Z. Consider the $\angle XYO$.]

19. Prove that (i) the bisectors of the angles of a regular polygon meet at a point, (ii) this point is the centre of the circle circumscribing the polygon, (iii) this point is the centre of the circle inscribed in the polygon.

20. Describe a circle of radius 4 cm. In it inscribe a $\triangle$ having angles 42°, 64°, 74°

21. Describe a circle of radius 2 cm. About it describe a $\triangle$ having angles 42°, 64°, 74°.

22. Show how to draw a $\triangle$ having angles 40°, 60°, 80°, so that a given circle touches the shortest side and the other two sides produced.

23. Show how to inscribe an equilateral $\triangle$ in a given circle, without using a protractor.

24. ABCDEF is a regular hexagon. Prove that the area of the $\triangle$BDF is half the area of the hexagon.

25. Prove the following construction for drawing a circle to touch a given circle and also a given st. line PQ at a fixed point A :

Let O be the centre of the given circle. Through A draw AZ $\perp$ to PQ; through O draw a st. line $\perp$ to PQ meeting the circle at B and C. Join AB, AC and let these lines, produced if necessary, meet the circle at D, E. Join OD, OE and let these lines, produced if necessary, meet AZ at X, Y. Then the circle with centre X and radius XA, and the circle with centre Y and radius YA are the required circles.

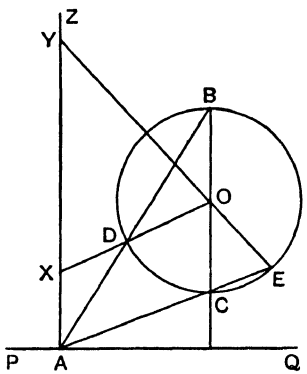


FIG. 196.

26. Draw a circle to touch a given circle at a fixed point D and also a given st. line PQ. [Adapt the above construction.]

27. X is the centre of a circle which touches a given circle, centre O, and also a given st. line PQ at a given point A. Prove that (i) X lies on the $\perp$ to PQ at A, (ii) X is equidistant from O and from one or other of two fixed points equidistant from A on the $\perp$ to PQ at A.

Hence find an alternative solution to the problem solved in No. 25.

28. Construct a right-angled isosceles $\triangle$XYZ, in which

$$XY = YZ = 4 \text{ cm.}$$

With centre X and radius 3 cm. describe a circle. Construct a circle touching the circle with centre X externally and also touching YZ at Z.

29. Construct a circle to touch two given circles, one of them at a fixed point A.

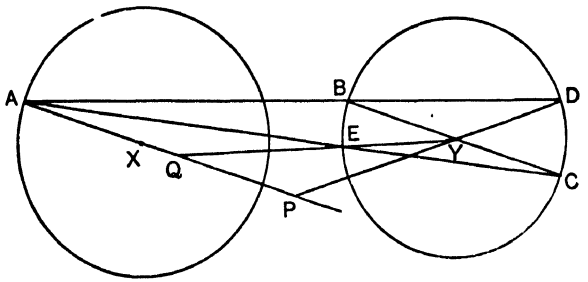


FIG. 197.

Join A to the centre X of the circle passing through A. The centre of the required circle must lie on this line. Through Y, the centre of the other circle, draw a straight line || AX cutting the circle, centre Y, at B, C.

Join AB and produce to meet the circle, centre Y, at D.

Join DY and produce to meet AX produced at P.

A circle, centre P, radius PA, is one of the required circles.

Join AC meeting the circle, centre Y, at E.

Join YE and produce to meet AX or AX produced at Q.

A circle, centre Q, radius QA, is the other circle satisfying the given conditions.

The pupil should supply the proof.

In the above figure one of the required circles encloses both the given circles, the second touches one of the given circles externally and the other internally. Other cases may arise if different positions are taken for the fixed point A.

CIRCLES CONNECTED WITH THE TRIANGLE

CONSTRUCTION 24

To construct the circumcircle of a triangle.

See the note on Theorem 34.

CONSTRUCTION 25

To construct the inscribed circle of a triangle.

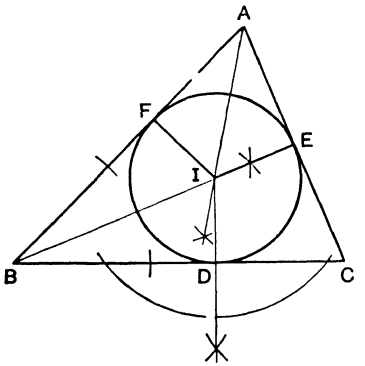


FIG. 198.

Let ABC be the given △.

It is required to construct the inscribed circle of the △ABC.

Construction. Bisect the ∠s BAC, ABC by straight lines meeting at I. (Construction 1.)

Draw ID ⊥ to BC. (Construction 4.)

With centre I and radius ID, draw a circle.

This is the required circle.

Proof. Let IE, IF be the ⊥s from I to AC, AB.

In the △s IBF, IBD,

$\angle IBF = \angle IBD$ (*construction*),

$\angle IFB = \angle IDB$ (*right angles, by construction*),

$IB = IB$ (*common*) ;

$\therefore$ the △s $\begin{matrix} IBF \\ IBD \end{matrix}$ are congruent (2 ∠ *s*, **corresponding** *side*) ;

$\therefore IF = ID.$

Similarly $IF = IE$;

$\therefore$ the circle drawn with centre I and radius IL passes through E and F.

Also, since ∠s IDB, IFB, IEA are right angles, BC, AB, CA are tangents to this circle, i.e. the circle is the required circle.

Note. It is not easy to draw a really accurate figure. To draw such a figure it is important that the angles should be very accurately bisected, and that the ⊥ from I to BC should be accurately drawn by a purely geometrical construction. In performing these operations it is advisable to draw arcs with radii as large as possible, so that their intersections may be accurately determined. These remarks apply also to the following construction.

CONSTRUCTION 26

To construct an escribed circle of a triangle.

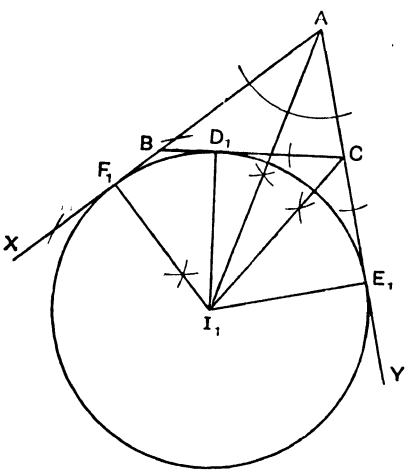


FIG. 199.

Let ABC be the given △ of which the sides AB, AC are produced to X, Y.

It is required to construct a circle touching BC, and AB, AC produced.

Construction. Bisect the ∠s BAC, BCY by straight lines meeting at I_1. (Construction 5.)

Draw I_1F_1 ⊥ to BX. (Construction 4.)

With centre I_1 and radius I_1F_1, draw a circle.

This is the required circle.

Proof. Let I_1D_1, I_1E_1 be the $\perp$s from I_1 to BC, CY.

In the $\triangle$s I_1AF_1, I_1AE_1,

$\angle I_1AF_1 = \angle I_1AE_1$ (*construction*),

$\angle I_1F_1A = \angle I_1E_1A$ (*right angles, by construction*),

$I_1A = I_1A$ (*common*);

$\therefore$ the $\triangle$s $\begin{matrix} I_1AF_1 \\ I_1AE_1 \end{matrix}$ are congruent ($2\ \angle s$, **corresponding** *side*);

$$\therefore\ I_1F_1 = I_1E_1.$$

Similarly $$I_1E_1 = I_1D_1;$$

$\therefore$ the circle drawn with centre I_1 and radius I_1F_1 passes through E_1 and D_1.

Also, since $\angle$s I_1F_1B, I_1D_1C, I_1E_1C are right angles, BX, BC, CY are tangents to this circle, i.e. the circle is the required circle.

EXERCISE 50

Nos. 1-40 are chiefly on the inscribed and escribed circles.

Nos. 41-56 are miscellaneous examples, chiefly on the properties of the orthocentre, pedal triangle, nine-point circle and Simson line.

In Nos. 1-12, 38, 39, D, E, F, I, D_1, E_1, F_1, I_1 have the same meaning as in Constructions 25 and 26. D_2, E_2, F_2, I_2 are respectively the points of contact with the sides and the centre of the escribed circle touching BA, BC produced. Similarly D_3, E_3, F_3, I_3. r, r_1, r_2, r_3 are the radii of the inscribed circle and of the escribed circles opposite A, B, C respectively; $2s$ is the perimeter of the $\triangle ABC$; $BC = a$, $CA = b$, $AB = c$.

1. Prove that $AE = AF = s - a$, $BD = BF = s - b$, $CD = CE = s - c$.

[Let $AE = AF = x$, $BD = BF = y$, $CD = CE = z$. Prove that $x + y + z = s$, and that $y + z = a$.

2. Prove that $AE_1 = AF_1 = s$, $BD_1 = BF_1 = s - c$, $CD_1 = CE_1 = s - b$.

3. Prove that (i) $\triangle = rs$, (ii) $\triangle = r_1(s - a)$.

[In the figure of Construction 25, $\triangle ABC = \triangle BIC + \triangle CIA + \triangle AIB$. In the figure of Construction 26, $\triangle ABC = \triangle AI_1B + AI_1C - \triangle BI_1C$.]

4. Prove that the middle point of DD_1 is the middle point of BC.

5. Prove that $EE_1 = a = E_2E_3$, and that $EE_3 = c = E_1E_2$.

6. Prove that the angles of the $\triangle DEF$ are $\frac{B}{2}+\frac{C}{2}, \frac{C}{2}+\frac{A}{2}, \frac{A}{2}+\frac{B}{2}$.

7. Prove that the angles of the $\triangle D_1E_1F_1$ are $90° + \frac{A}{2}, \frac{B}{2}, \frac{C}{2}$.

8. Prove that A, I, I_1 are collinear.

9. Prove that I_1, B, I_3 are collinear.

10. Find the angles of the $\triangle$s $I_1I_2I_3$ and II_2I_3.

11. Prove that, of the four points I, I_1, I_2, I_3, each is the orthocentre of the $\triangle$ whose vertices are the other three.

12. Prove that $E_1E_3 = c + a$, and $EE_2 = c \sim a$.

13. In a $\triangle$ whose sides are 6, 8, and 12 cm. long inscribe a circle, and calculate the lengths of the tangents from the angular points of the $\triangle$ to this circle.

14. The bisector of the interior $\angle B$ of a $\triangle ABC$ meets the bisector of the exterior $\angle C$ at a point E. The internal bisector of the $\angle A$ meets BE at D. Prove that the points A, D, C, E are concyclic.

15. I is the centre of the circle inscribed in the $\triangle ABC$, and twice the $\angle BIC$ is equal to three times the $\angle BAC$. Prove that $\angle BAC = 90°$.

16. Construct a $\triangle$ of which the sides are 6 cm., 7 cm., and 8 cm., and draw a circle which will intercept a chord of length 2·2 cm. on each side.

17. The inscribed circle of an isosceles $\triangle ABC$, in which the $\angle$s B and C are each double of the $\angle A$, touches the side BC at P and the sides AB, AC at R and Q respectively; show that the arc PQ is to the arc RQ as 3 : 4.

18. A cup in the form of a hollow right circular cone with its axis vertical is just full of water; its height is 8″ and its slant side is 10″. A ball is lowered slowly into the cup, of such a size that when it rests on the sides of the cup it is *just* covered with water. Find, by a geometrical construction, the radius of this ball. Measure it, and also find its length by calculation.

19. A, B, C are three points such that AB = 2·7″, BC = 2·3″, CA = 3·2″. With A, P, C as centres draw three circles each of which touches the other two externally.

20. If I is the centre of the inscribed circle of a $\triangle ABC$, and P is the point of contact with BC, prove that

$$IB^2 - IC^2 = BC(BP - PC) = BC(BA - AC).$$

21. Draw three semi-circles within a $\triangle$, each touching two sides of the $\triangle$ and having its diameter lying along the third side.

22. Show how to divide each side of a triangle into two parts whose difference is equal to the difference between the other two sides of the triangle.

23. A circle inscribed in a $\triangle ABC$ touches the sides BC, CA, AB at D, E, F respectively, and another circle touches BC, AC produced and AB produced at D_1E_1F respectively. Prove that the $\angle$s FDE, FD_1E_1 together are equal to two right angles.

24. The lengths of the sides AB, BC of a quad. circumscribing a circle are 5 and 6 cm. respectively and the angles at B and C are 80° and 66°. Construct the quad. and find the lengths of the remaining sides and the radius of the circle.

25. Construct a quad. PQRS in which the angles at P and S are right angles, RS = 4 cm., PS = 6 cm., PQ = 9 cm. Draw a circle such that the shortest distance from its circumference to each of the sides PQ, QR and PS is 2 cm.

26. If the inscribed circle of a $\triangle ABC$ touches BC at D, and the escribed circle which touches AB produced and AC produced touches BC at E, prove that $DE = AB \sim AC$.

27. The inscribed circle of a $\triangle ABC$ touches the side BC at X, and D is the mid-point of BC. Prove that $2DX = AB \sim AC$.

28. I is the centre of the inscribed circle of a $\triangle ABC$, and the bisector of the $\angle BAC$ meets BC at H and the circle circumscribing the $\triangle ABC$ at X. Prove that BX = IX and that BX touches the circle ABH.

29. The circumcircle and the inscribed circle of a $\triangle$ have the same centre. Prove that the $\triangle$ is equilateral.

30. If the inscribed circle of the $\triangle ABC$ touches the side BC at D, and the inscribed circle of the $\triangle ABD$ touches AD at E, prove that $AB - BD = AC - CD = AE - ED$. Also prove that the inscribed circles of the $\triangle$s ABD, ACD touch each other.

31. AP, AQ are two tangents to a circle whose centre is O. AO *produced* cuts the circle again at S. Prove that S is the centre of the circle which can be drawn to touch PQ, and AP, AQ both produced.

32. A circle is drawn touching the side BC of a $\triangle ABC$, and touching the sides AB, AC both produced. If I is the centre of this circle, and AI cuts the circumcircle of the $\triangle ABC$ at P, prove that

$$PB = PI = PC.$$

33. Show how to construct a $\triangle$ having given a side, the angle opposite to this side, and the radius of the inscribed circle. Is it always possible to construct a $\triangle$ given these elements ?

34. AB, AC are equal sides of a $\triangle ABC$, and D, E are points on AB, AC respectively such that $DE = BD + EC$. Prove that DE touches the circle to which AB and AC are tangents at B and C.

[Draw this circle and suppose that DE is not a tangent to it. Let DF be the second tangent from D to the circle, meeting AC at F. Then show that two sides of the $\triangle DEF$ are together equal to the third.]

35. Construct a $\triangle$, given the perimeter, altitude and vertical angle.

[Let BAC be the vertical $\angle$; along AB, AC mark off $AP = AQ =$ half the perimeter. Draw (i) the circle, touching AB, AC at P, Q, (ii) the circle, centre A, radius equal to the altitude. BC is the transverse common tangent of these circles.]

36. Construct a $\triangle$, given the centres of the three escribed circles.

37. Construct a $\triangle$, given the centres of the inscribed circle and of two of the escribed circles.

38. Construct a $\triangle$, given the position of B, D, C and the size of the $\angle BAC$.

[Either use No. 28 or consider the locus of I. Hence find the position of I.]

39. Construct a $\triangle$, given the position of B, D_1, C and the size of the $\angle BAC$.

[Either use No. 32 or consider the locus of I_1. Hence find the position of I_1.]

40. Construct a $\triangle$ given the vertical angle, perimeter, and radius of the inscribed circle.

41. O is the circumcentre of the $\triangle ABC$. Prove that

$$\angle OBC = 90° \sim \angle BAC.$$

42. PQRS is a quad. whose diagonals PR, QS intersect at X ; A, B, C, D are the circumcentres of the $\triangle$s PQX, QRX, RSX, SPX. Prove that ABCD is a parallelogram.

43. Prove that, if the circumcentre of the $\triangle XYZ$ lies on the bisector of $\angle XYZ$, the $\triangle$ is isosceles.

It has been previously proved that AD, BE, CF, the perpendiculars from A, B, C to the opposite sides of the $\triangle ABC$, meet at a point H, which is called the **orthocentre**. The $\triangle DEF$ is called the **pedal triangle** of the $\triangle ABC$. This notation will be used in Nos. 45-54.

44. The $\perp$s BE, CF of the $\triangle ABC$ meet at H; prove that

(i) AEHF is a cyclic quad.,

(ii) $\angle FAH = \angle FEH$,

(iii) BFEC is a cyclic quad.,

(iv) $\angle FEH = \angle FCB$,

(v) if AH is produced to meet BC at D, ACDF is a cyclic quad.,

(vi) AD is $\perp$ to BC.

Note. This provides an alternative proof of the fact that AD, BE, CF meet at a point.

45. Prove that the following quads. are cyclic: AFHE, BFHD, CEHD, AFDC, BFEC, AEDB.

46. Prove that AD bisects the $\angle FDE$ (i) if the $\triangle ABC$ is acute-angled, or (ii) if $\angle BAC$ is obtuse. Prove that AD bisects the $\angle$ between FD and ED produced (i) if $\angle ABC$ is obtuse, or (ii) if $\angle ACB$ is obtuse.

47. Prove that, if the $\triangle ABC$ is acute-angled, H is the in-centre of the pedal $\triangle$, but that, if it is obtuse-angled, H is one of the ex-centres of the pedal $\triangle$.

48. Prove that the angles of the pedal $\triangle$ of an acute-angled $\triangle$ are $180° - 2A$, $180° - 2B$, $180° - 2C$.

49. Prove that the angles of the pedal $\triangle$ of a $\triangle ABC$, in which $\angle BAC$ is obtuse, are $2A - 180°$, $2B$, $2C$.

50. Prove that the circumcircles of the $\triangle$s ABC, HBC are equal. Consider all possible cases.

51. If AH meets the circumcircle of the $\triangle ABC$ at P, prove that $HD = DP$. Consider all possible cases.

52. Of the four points H, A, B, C, each is the orthocentre of the $\triangle$ whose vertices are the other three.

53. Prove that OC is $\perp$ to DE, where O is the circumcentre of the $\triangle ABC$.

54. X, Y, Z are the mid-points of BC, CA, AB, and P, Q, R are the mid-points of AH, BH, CH; prove that (i) RY is $\parallel$ to BA, (ii) YZ is $\parallel$ to BC, (iii) $\angle RYZ = 90°$, (iv) $RXZ = 90°$, (v) R, Y, F, Z, X are concyclic, i.e. the circle XYZ passes through F, R, (vi) the circle XYZ passes through D, P, E, Q.

This circle is called the Nine-Point Circle of the Triangle.

55. P is any point on the circumcircle of the △ABC. PL, PM, PN are ⊥ to BC, CA, AB respectively. Prove that

(i) ∠PNL = 180° − ∠PBC,

(ii) ∠PNM = ∠PAM,

(iii) ∠PNL = 180° − ∠PNM,

(iv) **LMN is a straight line.**

The line LMN is called the **Simson line** of the △ABC for the point P.

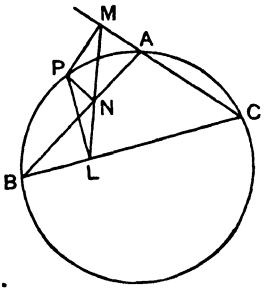


FIG. 200

56. P is any point ; PL, PM, PN are ⊥ to BC, CA, AB respectively. L, M, N lie on a st. line. Prove that P lies on the circumcircle of the △ABC.

*LOCI CONNECTED WITH THE CIRCLE

This section is difficult, and should be postponed until a late stage of the course. Much of it is beyond School Certificate standard.

Most locus questions connected with the circle depend upon the corollary to Theorem 39, but, as will be seen from the following examples, this result must be applied with great care. The treatment which follows does not profess to be exhaustive, but attention is drawn to most of the usual pitfalls for the beginner.

Ex. 1. Given the base BC and the vertical angle A of a triangle ; find the locus of the intersection of the bisectors of the ∠s ABC, ACB.

Let BC be the given base, and ∠BAC an angle equal to the given vertical angle. Let the bisectors of the ∠s ABC, ACB meet at X.

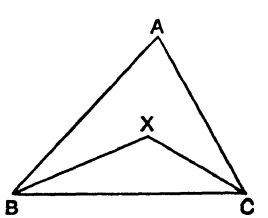


FIG. 201.

It is required to find the locus of X.

[Analysis. Consider first only △s on one side of the base BC.

Since $\angle A$ is given, the locus of A is a segment of a circle on BC as chord.

Consider a few special positions of A, in order to try to obtain a few fixed points on the locus. When A coincides with B, CA coincides with CB, $\therefore$ CX coincides with CB and meets at B any line passing through B, $\therefore$ X is at B. Similarly, when A coincides with C, X is at C, $\therefore$ B and C are fixed points on the locus of X.

Plot a number of positions of X. These appear to lie on the arc of a circle passing through B and C. This suggests that we should try to prove that $\angle BXC$ is constant.]

In the $\triangle BXC$,

$$\angle BXC = 180^\circ - (\angle XBC + \angle XCB) = 180^\circ - \tfrac{1}{2}(\angle ABC + \angle ACB)$$

$$= 180^\circ - \tfrac{1}{2}(180^\circ - \angle BAC) = 90^\circ + \tfrac{1}{2}\angle BAC,$$

$\therefore$ $\angle BXC$ is the same for all positions of A,

$\therefore$ the locus of X is a segment of a circle, with BC as chord, containing an angle equal to $90^\circ + \tfrac{1}{2}\angle A$.

Note 1. If we consider positions of A on the other side of BC we see that the complete locus of X is two equal segments, one on each side of BC, each standing on BC and containing an angle equal to $90^\circ + \tfrac{1}{2}\angle A$.

Note 2. It is not sufficient to prove that $\angle BXC$ is always the same. It should always be obtained in terms of other fixed angles. The reason for this will be clear after consideration of the following examples.

Note 3. The student should satisfy himself that, as A takes up all possible positions, X may be anywhere on the arc standing on BC. He should also satisfy himself that it is not possible to get a different value of the $\angle BXC$ by taking a different figure. The reason for this precaution will be clear after consideration of the following examples.

Ex. 2. Given the base BC and the vertical angle A of a triangle which is always on the same side of BC; find the locus of the intersection of the bisectors of the ∠s ABC, ACD, D being any point on BC produced.

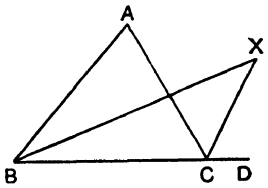


FIG. 202.

Let BC be the given base (produced to D), and ∠BAC an angle equal to the given vertical angle. Let the bisectors of the ∠s ABC, ACD meet at X.

It is required to find the locus of X.

In the △BXC,

$$\angle BXC = 180^\circ - \angle XBC - \angle XCB$$
$$= 180^\circ - \tfrac{1}{2}\angle ABC - \angle ACB - \tfrac{1}{2}(180^\circ - \angle ACB)$$
$$= 90^\circ - \tfrac{1}{2}\angle ABC - \tfrac{1}{2}\angle ACB = \tfrac{1}{2}\angle BAC,$$

which is the same for all positions of A.

It would, therefore, at first sight appear that the locus of X is a segment of a circle, with BC as chord, containing an angle equal to $\frac{1}{2}\angle A$. But if we consider the path of X, as A takes up all possible positions, we see that CX can never come to the left of the perpendicular at C to CD.

The complete locus in this case is that portion of the above segment which is cut off by the line through C ⊥ to CD.

Ex. 3. Given the base BC and the vertical angle A of a triangle which is always on the same side of BC, find the locus of the intersection of the perpendiculars from B, C to AC, AB respectively.

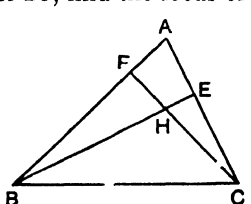


FIG. 203.

Let ABC be the △, with given base BC, and ∠BAC an angle equal to the given vertical angle. Let BE, CF, intersecting at H, be the ⊥s from B, C to AC, AB respectively.

[Analysis. Since ∠A is given, the locus of A is a segment of a circle on BC as chord. Consider a few special positions of A, in order to try to obtain a few fixed points on the locus.

When CA is $\perp$ CB, $\triangle ABC$ is right-angled at C and H coincides with C.

Similarly, when BA is $\perp$ BC, H coincides with B, $\therefore$ B, C are fixed points on the locus of H. Plot a number of positions of X. These appear to lie on the arc of a circle passing through B and C. This suggests that we should try to prove that $\angle BHC$ is constant.]

In the $\triangle BHC$,

$$\begin{aligned}\angle BHC &= 180^\circ - \angle HBC - \angle HCB \\ &= 180^\circ - (90^\circ - \angle ACB) - (90^\circ - \angle ABC) \\ &= \angle ACB + \angle ABC = 180^\circ - \angle A\ ;\end{aligned}$$

$\therefore$ it would at first sight appear that the locus of H is a segment of a circle on BC as chord, containing the constant angle $(180^\circ - A)$. But, if we consider the path of H, as A traces out the complete segment, which is its locus, we see that when the $\triangle ABC$ is an obtuse-angled $\triangle$, H falls below BC, and does not lie on what, at first sight, appeared to be its locus. This suggests that we should examine more carefully the figure when $\triangle ABC$ has an obtuse angle.

In this figure, since $\angle AFH = \angle AEH = 90^\circ$, the points A, F, E, H lie on a circle ;

$\angle EHF = \angle EAF$, being angles in the same segment ; i.e. $\angle BHC = \angle A$;

$\therefore$ When $\triangle ABC$ is obtuse-angled, H lies on a segment of a circle, on BC as chord, on the side of it remote from A, and containing the constant angle A.

(It should, however, be noticed that H cannot lie on that part of the segment which is cut off by the $\perp$s to BC at B and C.)

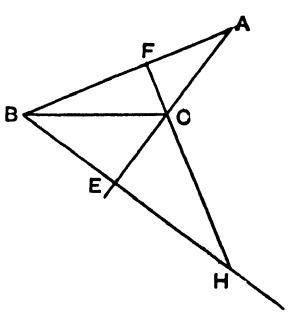


FIG. 204.

Also the two segments contain angles which are supplementary ;

$\therefore$ together they form a complete circle,

$\therefore$ The locus of H is part of a complete circle.

The reason for the warning contained in Note 2 will now be clear. Had we only proved that $\angle BHC$ was constant in each of the above cases we should not have been able to show that the segments combined to make one complete circle.

The student should consider the case in which the given angle A is obtuse.

At a suitable stage the teacher may point out that a more general statement of the solution is as follows: Let O be the centre of the fixed circle ABC, and let O′ be the image of O in BC. Then the locus of H is a segment of a circle, centre O′, equal in all respects to the segment of the circle, centre O, which is the locus of A.

Note. The student should verify, in each of the above examples, that any point on the specified arc is a possible position of the locus point.

EXERCISE 50A [Loci connected with the Circle]

1. Given the base BC and the vertical angle of a $\triangle ABC$; find (i) the locus of the intersection of the bisectors of the $\angle$s CBY, BCZ, Y, Z being any points on AB, AC produced; (ii) the locus of the intersection of the bisectors of the $\angle$s ACB, ABW, W being any point on CB produced.

2. Through two points P, Q on a st. line parallel lines PR, QS are drawn in the same sense. Find the locus of the bisectors of the $\angle$s RPQ, SQP.

3. Tangents are drawn from a point X to a set of circles which all have the same centre C. Find the locus of the points of contact of these tangents.

4. O is a point inside a given acute-angled $\triangle ABC$, and BO, CO, when produced, meet AC and AB at X and Y. If a circle can be described about the quad. AXOY, find the locus of O.

5. XY is a fixed chord of a given circle. X is joined to any point Z on the circumference and XZ is produced to P so that ZP=ZY. Find the locus of P. [Consider separately points Z on the major and minor arcs.]

6. Two unequal circles intersect at P and Q. Through Q any st. line is drawn cutting the first circle at X and the second circle at Y. A is a fixed point on the first circle, B is a fixed point on the second circle, and YA, YB (produced if necessary) meet at T.

Prove that the locus of T is a complete circle passing through A, B and P.

[Let α, θ be the acute angles subtended by the fixed arcs AP, PQ at X, and let β, ϕ be the acute angles subtended by the fixed arcs PB, PQ at Y. Calculate the $\angle$s ATB, APB in terms of α, β, θ, ϕ, and show that in all cases they are either equal or supplementary.]

7. A, B are fixed points on a fixed circle. P, Q are variable points on the circle such that the arc PQ is a fixed length. Find the locus of the intersection of AP, BQ.

[Consider all cases and notice that an additional point on the locus can always be obtained by interchanging P and Q.]

8. A semicircular protractor slides in its own plane with the ends of the base always on two fixed st. lines at right angles to each other. Prove that any mark on the rim (say the 20° mark) will move on a st. line through the intersection of the fixed lines.

9. Given the mid-points X, Y of two adjacent sides of a rectangle, prove that the loci of its vertices and of the intersection of its diagonals are all circles.

[Consider the distances of the vertices from the mid-point K of XY and from the reflection of K in X and Y.]

10. Two circles intersect at A and B. A st. line is drawn through A meeting the first circle at P and the second circle at X. The st. line through P, B meets the second circle again at Y. Find the locus of the intersection of AY and BX.

11. A chord PQ of a given circle subtends a constant angle at a fixed point A on the circumference. Find the locus of the mid-point of PQ and prove that PQ always touches a fixed circle.

12. Two variable circles touch each other at P, and also touch a fixed st. line at two fixed points A, B respectively. Find the locus of P.

13. ABCD is a square. P moves so that $\angle APB = \angle APD$. Prove that the locus of P consists of a semi-circle together with the whole of one diagonal of the square and parts of the other.

14. ABCD is a square, AB is produced to E and CB to F. P moves so that its distance from the nearest point of the segment AB of the line ABE is equal to its distance from the nearest point of the segment CB of the line CBF. Find the locus of P.

[Consider points in each of the four angles formed by ABE, CBF.]

PART IV

SQUARES AND RECTANGLES

This section of the book contains the substance of the Second Book of Euclid. Many of these results can be obtained more easily by Algebra than by Geometry. Most teachers prefer to use Algebra, but some prefer to retain purely geometrical methods on the ground that young pupils are unable to understand the proofs of the algebraical formulae used. Others have no choice, being tied by the requirements of examining bodies. It has therefore been thought desirable to insert geometrical proofs of all the important propositions contained in Euclid Book II, but the teacher may, at discretion, omit Theorems 50 to 55 and establish the results by Algebra.

AREAS OF RECTANGLES

DEFINITIONS

If AB, XY are two straight lines, any rectangle which has one side equal to AB and an adjacent side equal to XY is called **the rectangle contained by** AB **and** XY. All such rectangles are congruent and equal in area. The rectangle contained by AB and XY is often denoted by **the rect.** AB, XY or by the measure of its area, i.e. AB . XY. The rectangle PQRS is often called the rectangle PR or QS.

Similarly, a square drawn on the side AB is denoted by **the sq on** AB, or AB^2.

If a point X is taken on a straight line AB, or on AB produced either way, then X is said to divide AB into two **segments** AX, XB.

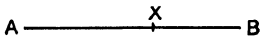


Fig. 1

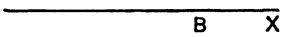


Fig. 2

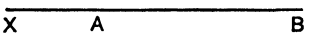


Fig. 3

FIG. 205.

Notice that in either case the segments are the distances of X from the extremities of the given line AB.

If X lies between A and B, as in Fig. 1, AB is said to be divided **internally** at X.

If X lies on AB produced, as in Fig. 2, AB is said to be divided **externally** at X.

If X lies on BA produced, as in Fig. 3, AB is said to be divided **externally** at X.

THE APPLICATION OF ALGEBRA TO GEOMETRY

The following examples illustrate the application of Algebra to Geometry.

Ex. 1. P, Q, R, S are four points in order on a straight line. Prove that

$$PR \cdot QS = PQ \cdot RS + PS \cdot QR.$$

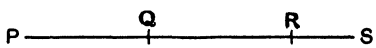


FIG. 206.

Let p, q, r stand for the number of units of lengths in PQ, QR, RS respectively.

Then PR, PS, QS contain $(p+q)$, $(p+q+r)$, $(q+r)$ units of length respectively,

$$\begin{aligned}
\therefore\ & PQ \cdot RS + PS \cdot QR - PR \cdot QS \\
&= pr + q(p+q+r) - (p+q)(q+r) \\
&= pr + qp + q^2 + qr - (pq + q^2 + pr + qr) \\
&= 0 ; \\
\therefore\ & PR \cdot QS = PQ \cdot RS + PS \cdot QR.
\end{aligned}$$

Ex. 2. PS is trisected at Q, R ; prove that

$$PS^2 = PQ^2 + 2QS^2.$$

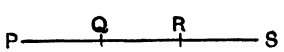


FIG. 206A.

Let the number of units of length in PS be $3p$.

Since PS is trisected at Q, R, PQ, QR, RS each contain p units of length. Also the number of units of length in QS is $2p$,

$$\begin{aligned}
\therefore\ & PQ^2 + 2QS^2 - PS^2 \\
&= p^2 + 2(2p)^2 - (3p)^2 \\
&= p^2 + 8p^2 - 9p^2 \\
&= 0 ; \\
\therefore\ & PS^2 = PQ^2 + 2QS^2.
\end{aligned}$$

EXERCISE 51

[The application of Algebra to Geometry]

These exercises may also be regarded as examples on Theorems 50-55 and done by purely geometrical methods.

1. A st. line XY is bisected at Z. W is any point on the line between X and Z. Prove that $2WZ = WY - XW$.

2. ABCD is a st. line; P, Q are the mid-points of AB, CD. Prove that $AD + BC = 2PQ$.

3. A st. line AB is bisected at C and produced to D, prove that $AD^2 = BD^2 + 4AC \cdot CD$.

4. A, B, C, D are four points in order on a st. line, such that $AB = CD$. Prove that $AC^2 = AB^2 + BC \cdot AD$.

5. A, B, C, D, X are five points in order on a st. line, such that $AB = BC = CD$. Prove that $AX^2 + 3CX^2 = 3BX^2 + DX^2$.

6. P, Q, R, X are four points taken in order on a st. line, such that $QR = 2PQ$. Prove that $2PX^2 + RX^2 = 3QX^2 + 2PQ^2 + QR^2$. Is the result true when X is any point not on the line PQR ?

7. P, X, Q, R are four points taken in order on a st. line, such that $QR = 2PQ$. Prove that $2PX^2 + RX^2 = 3QX^2 + 2PQ^2 + QR^2$. Is the result true when X is any point not on the line PQR ?

8. Four points P, Q, R, S are taken in order on a st. line, and on the line PS produced points Q′, R′ are taken, such that $QS = SQ'$ and $RS = SR'$. Prove that the rectangle $PQ \cdot PQ'$ is less than the rectangle $PR \cdot PR'$.

9. A st. line AB, of which C is the mid-point, is divided at D. Prove that the sum of the squares on AD and DB is least when D is at C.

10. Divide a st. line, three inches long, into two parts, the difference of the squares upon these parts being one square inch.

If further practice is needed, Theorems 50-55 may be proved algebraically.

THEOREM 50. [Euclid II. 1]

If there are two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.

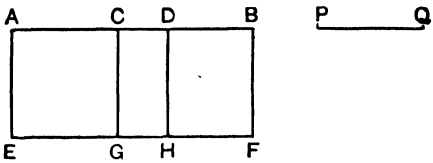


FIG. 207.

Let AB, PQ be the two straight lines, and let AB be divided into any number of parts, say three, at C, D.

It is required to prove that

rect. AB, PQ = rect. AC, PQ + rect. CD, PQ + rect. DB, PQ.

Construction. Draw AE $\perp$ to AB and equal to PQ.

Through C, D, B draw parallels to AE, meeting at G, H, F, respectively, the line drawn through E parallel to AB.

Proof. All the quadrilaterals in the figure are formed by parallels to two perpendicular lines,

$\therefore$ the figures AF, AG, CH, DF are rectangles,

and $PQ = AE = CG = DH = BF$,

$\therefore$ fig. AF = rect. AB, PQ ; fig. AG = rect. AC, PQ ;

fig. CH = rect. CD, PQ ; fig. DF = rect. DB, PQ.

But fig. AF = fig. AG + fig. CH + fig. DF,

$\therefore$ rect. AB, PQ = rect. AC, PQ + rect. CD, PQ + rect. DB, PQ.

Note. Let a, b, c, p stand for the number of units of length in AC, CD, DB, PQ respectively.

Then the above geometrical theorem is equivalent to the algebraic identity $(a+b+c)p = ap + bp + cp$.

Corollary 1. If a straight line is divided into two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line. [Euclid II. 2.]

This is a particular case of Theorem 50, when AE = AD and D coincides with B. **It is equivalent to the algebraic identity**

$$a(a+b)+b(a+b)=(a+b)^2.$$

Corollary 2. If a straight line is divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part, together with the rectangle contained by the two parts. [Euclid II. 3.]

This is a particular case of Theorem 50, when AE = AC and D coincides with B. **It is equivalent to the algebraic identity**

$$a(a+b)=a^2+ab.$$

THEOREM 51. [Euclid II. 4]

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the parts.

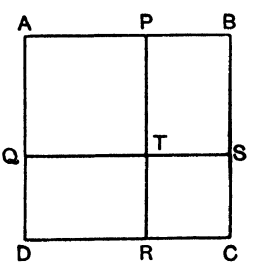


FIG. 208.

Let the straight line AB be divided into any two parts at P. It is required to prove that

sq. on AB = sq. on AP + sq. on PB + 2 rect. AP, PB.

Construction. On AB describe the square ABCD. Draw PR parallel to AD, meeting DC at R. Along AD cut off AQ equal to AP. Through Q draw QTS parallel to AB, meeting PR at T and BC at S.

Proof. All the quadrilaterals in the figure are formed by parallels to two perpendicular lines,

$\therefore$ the figures AC, AT, PS, QR, TC are rectangles.

But AQ = AP, $\therefore$ the fig. AT is the sq. on AP.

Also PB = AB − AP = AD − AQ = QD ; and $\therefore$ RC, TS, PB, QD, TR, SC are all equal, as are also DR, QT, AP, AQ, PT, BS ;

$\therefore$ the fig. TC = the sq. on PB ;

the fig. QR = the rect. AP, PB ;

the fig. PS = the rect. AP, PB.

But fig. AC = fig. AT + fig. TC + fig. QR + fig. PS,

$\therefore$ sq. on AB = sq. on AP + sq. on PB + 2 rect. AP, PB.

Note. Let a, b stand for the number of units of length in AP, PB respectively. Then the above theorem is equivalent to the algebraic identity $(a+b)^2 = a^2 + b^2 + 2ab$.

THEOREM 52. [Euclid II, 7]

If a straight line is divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole line and that part, together with the square on the other part.

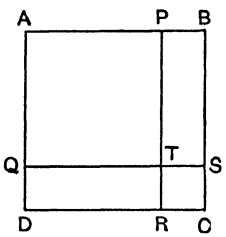


FIG. 209.

Let the straight line AB be divided into any two parts at P.

It is required to prove that

sq. on AB + sq. on PB = 2 rect. AB, PB + sq. on AP.

Construction. On AB describe the square ABCD. Draw PR parallel to AD, meeting DC at R. Along AD cut off AQ equal to AP. Through Q draw QTS parallel to AB, meeting PR at T and BC at S.

Proof. All the quadrilaterals in the figure are formed by parallels to two perpendicular lines,

∴ the figures AC, PC, TC, QC, AT are rectangles.

But AQ = AP ; ∴ the fig. AT is the sq. on AP.

Also PB = AB − AP = AD − AQ = QD ; and ∴ RC, TS, PB, QD, TR, SC are all equal ; as are also DC, QS, AB, AD, PR, BC ;

∴ the fig. TC = the sq. on PB ;

the fig. PC = the rect. AB, PB ;

the fig. QC = the rect. AB, PB.

But fig. AC − fig. PC + fig. TC − fig. QC

= fig. AR + fig. TC − fig. QC

= fig. ADCSTPA − fig. QC

= fig. AT ;

∴ sq. on AB − rect. AB, PB + sq. on PB − rect. AB, PB

= sq. on AP,

∴ sq. on AB + sq. on PB = 2 rect. AB, PB + sq. on AP.

Note. Let a, b stand for the number of units of length in AB, PB respectively. Then the above theorem is equivalent to the algebraic identity

$$a^2 + b^2 = 2ab + (a - b)^2 \quad \text{or} \quad (a - b)^2 = a^2 + b^2 - 2ab.$$

THEOREM 53

The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.

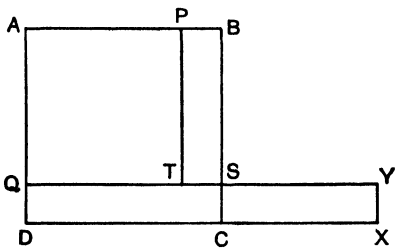


FIG. 210.

Let the given straight lines be AB, AP and let them be placed in the same straight line.

It is required to prove that

sq. on AB − sq. on AP = rect. (AB + AP), (AB − AP).

Construction. On AB describe the square ABCD. Along AD cut off AQ equal to AP. Through Q draw QTS parallel to AB, meeting at T a straight line through P parallel to AQ, and meeting BC at S. Produce QS to Y, making SY = AP. Produce DC to X, making CX = AP. Join XY.

Proof. The quadrilaterals AT, PS, QC are formed by parallels to two perpendicular lines, ∴ they are rectangles.

But AQ = AP ; ∴ the fig. AT is the sq. on AP.

Also PB = AB − AP = AD − AQ = QD ; and ∴ TS, PB, QD, SC are all equal ; as are also DC, QS, AB and BS, PT, AQ, AP.

Also SY = CX = AP, ∴ QY and DX are each equal to AB + AP ; but they are also parallel ;

∴ ▱DXY is a parallelogram and, since ∠QDX = 90°, it is a rectangle.

Also SCXY is a rectangle having its sides SC, SY respectively equal to PB, PT ; $\therefore$ rect. SCXY = rect. PBST ;

$\therefore$ sq. on AB − sq. on AP = fig. QDCBPTQ

= rect. QDCS + rect. PBST

= rect. QDCS + rect. SCXY

= rect. QDXY = rect. DX, QD

= rect. (AB + AP), (AB − AP).

Note. Let a, b stand for the number of units of length in AB, AP respectively. Then the above theorem is equivalent to the algebraic identity

$$a^2 - b^2 = (a + b)(a - b).$$

THEOREM 54. [Euclid II. 8]

If a straight line is divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and the first named part.

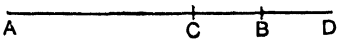


FIG 210A.

Let the straight line AB be divided into any two parts at the point C.

It is required to prove that

4 rect. AB, CB + sq. on AC = sq. on the st. line made up of AB and CB together.

Construction. Produce AB to D making BD = CB. Then AD is the st. line made up of AB and CB together.

Proof. By Theorem 51,

sq. on AD = sq. on AB + sq. on BD + 2 rect. AB, BD.

But BD = CB,

∴ sq. on AD = sq. on AB + sq. on CB + 2 rect. AB, CB. .....(1)

Also, by Theorem 52,

sq. on AB + sq. on CB = sq. on AC + 2 rect. AB, CB,

∴ substituting in (1),

sq. on AD = sq. on AC + 4 rect. AB, CB.

Note. Let a, b stand for the number of units of length in AB, CB respectively. Then the above theorem is equivalent to the algebraic identity

$$(a+b)^2-(a-b)^2=4ab.$$

The student should prove the theorem directly from the accompanying figure.

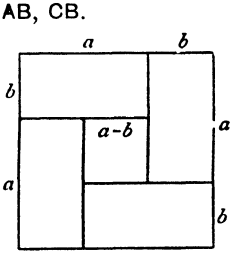


FIG. 211

*** THEOREM 54a. [Euclid II. 9 and 10]**

If a straight line is bisected and also divided, internally or externally, into two unequal parts, the sum of the squares on the unequal parts will be equal to twice the square on half the line and twice the square on the line between the points of section.

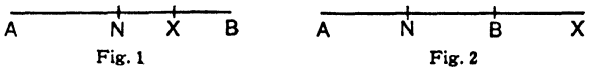


Fig. 1 Fig. 2

FIG. 211A

Let A, B be a straight line, bisected at N and divided at X, internally in Fig. 1, externally in Fig. 2.

It is required to prove that

sq. on AX + sq. on XB = 2 sq. on AN + 2 sq. on NX.

Proof. By Theorem 51,

sq. on AX = sq. on AN + sq. on NX + 2 rect. AN, NX. .....(1)

By Theorem 52,

sq. on BX = sq. on NB + sq. on NX − 2 rect. NB, NX

But AN = NB,

∴ sq. on BX = sq. on AN + sq. on NX − 2 rect. AN, NX; ....(2)

∴ adding (1) and (2),

sq. on AX + sq. on BX = 2 sq. on AN + 2 sq. on NX.

Note. Let a, b stand for the number of units of length in AN, NX respectively. Then the above theorem is equivalent to the algebraic identity

$$(a+b)^2+(a-b)^2=2a^2+2b^2.$$

EXTENSIONS OF PYTHAGORAS' THEOREM

Definition. If AB and CD are any two straight lines, and if AH, BK are the perpendiculars from A, B to CD (produced either way if necessary), then HK is called the projection of AB on CD.

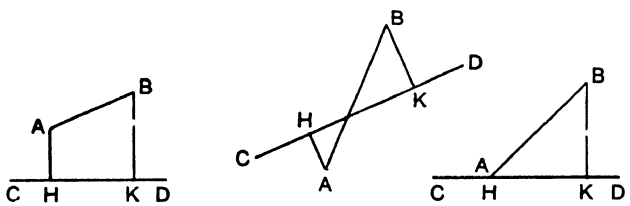


Fig. 1 Fig. 2 Fig. 3

FIG. 212.

In each of these figures HK is the projection of AB on CD. Note carefully Fig. 3. In this case A and H coincide.

THEOREM 55

In an obtuse-angled triangle, the square on the side opposite the obtuse angle is equal to the sum of the squares on the sides containing it, plus twice the rectangle contained by one of those sides and the projection on it of the other.

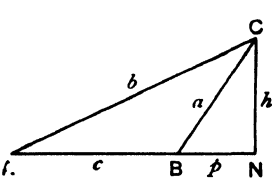


FIG. 213.

First Proof.

Let ABC be a triangle in which the angle B is an obtuse angle, and let CN be the perpendicular from C to AB produced.

It is required to prove that

$$CA^2 = BC^2 + AB^2 + 2AB \cdot BN.$$

Let $BC = a$ units, $CA = b$ units, $AB = c$ units, $BN = p$ units, $CN = h$ units.

Proof. Apply Pythagoras' Theorem to each of the right-angled $\triangle$s ACN, BCN,

$$\therefore\ b^2 = h^2 + (c+p)^2,$$

and

$$a^2 = h^2 + p^2;$$

$\therefore$ Subtracting, we get, $b^2 - a^2 = (c+p)^2 - p^2 = c^2 + 2cp$;

$$\therefore\ b^2 = a^2 + c^2 + 2cp,$$

i.e.

$$CA^2 = BC^2 + AB^2 + 2AB \cdot BN.$$

Note. In choosing the small letters always let the projection $= p$ units, the sides BC, CA, AB $= a$, b, c units respectively, and the height of the triangle $= h$ units. This rule applies also to Theorem 56.

Second Proof.

In the above figure,

$CA^2 = AN^2 + NC^2$ (*Pythagoras'*),

$= AB^2 + BN^2 + 2AB \cdot BN + NC^2$ (AN *is divided internally at* B),

$= BC^2 + AB^2 + 2AB \cdot BN$, since $BN^2 + NC^2 = BC^2$ (*Pythagoras'*).

THEOREM 56

In any triangle, the square on the side opposite an acute angle is equal to the sum of the squares on the sides containing it, minus twice the rectangle contained by one of those sides and the projection on it of the other.

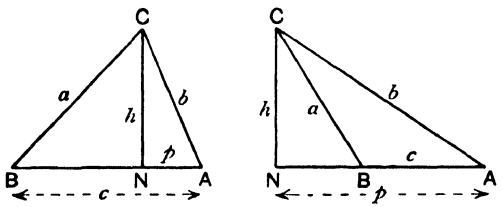


FIG. 214.

First Proof.

Let ABC be any triangle in which the angle A is an acute angle, and let CN be the perpendicular from C to AB or AB produced.

It is required to prove that

$$BC^2 = CA^2 + AB^2 - 2AB \,.\, AN.$$

Let $BC = a$ units, $CA = b$ units, $AB = c$ units, $AN = p$ units, $CN = h$ units.

Proof. Apply Pythagoras' Theorem to each of the right-angled $\triangle$s BCN, ACN,

$\therefore\ a^2 = h^2 + (c-p)^2$ in Fig. 1, or $a^2 = h^2 + (p-c)^2$ in Fig. 2,

i.e. in each case

$$a^2 = h^2 + c^2 - 2cp + p^2,$$

and

$$b^2 = h^2 + p^2;$$

$\therefore$ subtracting, we get,

$$a^2 - b^2 = c^2 - 2cp,$$

i.e.

$$a^2 = b^2 + c^2 - 2cp,$$

i.e.

$$BC^2 = CA^2 + AB^2 - 2AB \,.\, AN.$$

Second Proof.

In the above figures,

$BC^2 = BN^2 + NC^2$ (*Pythagoras'*),

$= AB^2 + AN^2 - 2AB \,.\, AN + NC^2$ (BN *is divided externally at* A),

$= CA^2 + AB^2 - 2AB \,.\, AN$, since $AN^2 + NC^2 = CA^2$ (*Pythagoras'*).

Corollary to Theorems 55 and 56. If A, B are fixed points and C is a variable point which moves so that $CA^2 - CB^2$ is constant, the locus of C is a straight line perpendicular to AB.

For in the above figures

$$CA^2 - CB^2 = AB^2 + 2AB \,.\, BN$$

so that BN is constant, i.e. the foot of the $\perp$ from C to AB is a fixed point. Conversely, if C lies on the $\perp$ to AB at this fixed point N, it is easily shown that $CA^2 - CB^2 = AN^2 - BN^2 =$ the given constant.

Note. If the lengths of the sides BC, CA, AB of a triangle are a units, b units, c units respectively, it follows from Theorems 29, 55, 56, that :

(1) If BC is opposite an acute angle, $a^2 < b^2 + c^2$.

(2) If BC is opposite a right angle, $a^2 = b^2 + c^2$.

(3) If BC is opposite an obtuse angle, $a^2 > b^2 + c^2$.

But if BC is the greatest side of a triangle and $a^2 < b^2 + c^2$, it is clear that $b^2 < c^2 + a^2$ and $c^2 < a^2 + b^2$, so that all the angles are acute ;

$\therefore$ If BC is the greatest side then

(1) If $a^2 < b^2 + c^2$, the triangle is acute-angled.

(2) If $a^2 = b^2 + c^2$, the triangle is right-angled.

(3) If $a^2 > b^2 + c^2$, the triangle is obtuse-angled.

These results may easily be adapted, if the sides of the triangle are not all unequal.

THEOREM 57

In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side, together with twice the square on the median which bisects the third side.

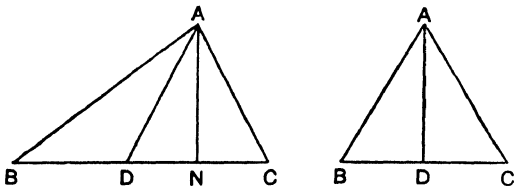


FIG. 215.

Let ABC be a triangle, and let AD be the median which bisects BC.

It is required to prove that $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

Proof. Of the angles ADB, ADC *either* one is acute and the other obtuse *or* both are right angles.

In the first case (Fig. 1), let $\angle ADB$ be the obtuse angle and let N be the foot of the perpendicular from A to BC.

Then, by applying the extensions of Pythagoras' Theorem to the $\triangle$s ABD, ACD, we get,

$$AB^2 = AD^2 + BD^2 + 2BD \cdot DN,$$

and $$AC^2 = AD^2 + DC^2 - 2DC \cdot DN.$$

But $$BD = DC \text{ (\textit{given})},$$

$$\therefore\ AC^2 = AD^2 + BD^2 - 2BD \cdot DN.$$

Hence, by addition,

$$AB^2 + AC^2 = 2AD^2 + 2BD^2.$$

In the second case (Fig. 2), by applying Pythagoras' Theorem to the right-angled $\triangle$s ABD, ADC, we get

$$AB^2 = AD^2 + BD^2,$$

and $$AC^2 = AD^2 + DC^2.$$

But $$BD = DC \text{ (\textit{given})},$$

$$\therefore\ AC^2 = AD^2 + BD^2.$$

Hence, by addition, $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

Corollary. If B, C are fixed points and A is a variable point which moves so that $AB^2 + AC^2$ is constant, the locus of A is a circle whose centre is the mid-point of BC.

For, in the above figures,

$$AB^2 + AC^2 = 2AD^2 + 2BD^2,$$

so that AD is a fixed length ;

$\therefore$ A lies on a circle, centre D having a fixed radius.

Conversely, if A is a point on this circle, it is easily shown that $AB^2 + AC^2$ is equal to the given constant.

Note. If $BC = 2a$ units and $AB^2 + AC^2 = 2k^2$ sq. units, the radius of the circle which is the locus of A is $\sqrt{k^2 - a^2}$ units.

EXERCISE 52 [Theorems 55-57]

1. In a $\triangle ABC$, AB, BC and CA are respectively equal to 5″, 4″ and 7″. Calculate the projection of BC upon AB.

2. The sides AB, BC, CA of a $\triangle ABC$ are 3·25″, 3·75″ and 3·5″ long. Calculate the length of the projection of the side AB upon the side BC.

3. In a $\triangle ABC$, BC = 4 cm., AB = 9 cm., and the projection of AC upon BC = 7 cm. Show that $\angle C$ must be acute, and find by calculation the length of AC and of the projection of AC upon AB.

4. The sides of a $\triangle$ are 5″, 7″, 9″ long. Prove that it is obtuse angled and calculate the lengths of the segments into which the longest side is divided by the $\perp$ from the opposite vertex.

5. Given four st. lines of lengths 3, 4, 5, 6 inches, find which of the triangles formed with three of them are obtuse-angled.

6. Without drawing, find what kind of $\triangle$ (if any) can be constructed with sides $\sqrt{3}$ in., $\sqrt{5}$ in., $\sqrt{13}$ in.

7. Determine by calculation whether the greatest angle is acute, right, or obtuse in each of the following $\triangle$s : (1) sides 15″, 12″, 7″ ; (2) sides 15″, 12″, 10″. In each case find by calculation the lengths of the portions into which the side of 15″ is divided by the foot of the $\perp$ drawn to it from the opposite vertex.

8. ABCD is a rectangle in which AB = 9′, BC = 2′. P is a point between A and B, distant 6″ from A. Find whether the $\angle$ CPD is acute, right or obtuse.

9. ABCD is a rectangle in which AB=16″, AD=7″. In AB a point P is taken so that AP=4″. Show that the angle subtended at P by DC is slightly less than 90°.

10. ABCD is a square, side 2″. The diagonal AC is produced to E, CE being also 2″. BE is joined, and EF drawn ⊥ to AB produced, meeting it at F. Obtain by calculation the lengths AE, BE, EF (to the nearest hundredth of an inch), and the ∠s AEB, ABE.

11. D is a point on the side BC of a △ABC, such that AB=4″, AD=BC=5″, AC=6″. Calculate the lengths of DB and DC to two places of decimals.

12. The sides of a △ are 7, 5 and 3 units long. Prove that the greatest angle is 120°.

13. In the △ABC, ∠C=60° and BC=4AC. Prove that

$$AB=\sqrt{13}\,AC.$$

14. OA, OB, OC are lines mutually at right angles and respectively 5, 4 and 2 in. long. Prove that the △ABC is acute-angled.

15. Each edge of a regular tetrahedron ABCD is 2 inches in length. AE is drawn ⊥ to BC, and AF ⊥ to DE. Calculate the lengths of AE, EF and AF.

16 The sides of a △ are 11″, 9″, 6″; without drawing the figure to scale, determine whether the centre of the circumcircle falls inside or outside the triangle.

17. RST is an isosceles △ in which RS=RT. If SQ is the ⊥ on RT, meeting it at Q, prove that $ST^2=2RT\,.\,QT$.

18. If the ∠A of a △ABC is 60°, prove that the sum of the squares on AB, AC is equal to the square on BC, together with the rectangle AB, AC.

19. BE, CF are altitudes of the △ABC; prove that

$$AE\,.\,AC=AF\,.\,AB.$$

20. N is the foot of the ⊥ from the vertex A to the base BC of a △ABC. Prove that $AB^2-AC^2=BC\,(BN-NC)$.

21. ABC is an equilateral △, D a point in BC produced. Prove that $AD^2=BC^2+CD^2+BC\,.\,CD$.

22. ABC is a △ in which AB=AC. D is any point on BC produced. Prove that $BD\,.\,CD=AD^2-AB^2$.

23. XYZ is a △ with XD ⊥ to YZ and E the mid-point of YZ. If $ED=\frac{1}{4}YZ$, prove that the difference of the squares on XY and XZ equals half the square on YZ.

24. The side BC of a square ABCD is produced to any point E, prove that $AE^2=2BC\,.\,BE+CE^2$.

25. Find the lengths of the medians of a Δ, if the sides are (i) $3''$, $5''$, $7''$; (ii) $4''$, $8''$, $10''$.

26. Two sides of a Δ are $6''$ and $8''$ and the median to the other side is $5''$. Calculate the length of the other side.

27. A parallelogram has its sides $6''$ and $7''$ and a diagonal $8''$. Calculate the length of the other diagonal.

28. The diagonals of a parallelogram are $11''$ and $4''$ and one side is $5''$. Calculate the length of the other side.

29. The base BC of a $\Delta ABC = 8''$ and X is its mid-point. What is the value of $AB^2 + AC^2 - 2AX^2$?

30. In the ΔABC, $AB = 7''$, $AC = 8''$, $\angle A = 60°$. Calculate the length of the median through A.

31. AB is a st. line 6 cm. long. P is a point which moves so that $PA^2 + PB^2 = 50$ sq. cm. Find the locus of P.

32. ABC is a Δ; D, E, F are the mid-points of BC, CA, AB respectively. AD, BE, CF intersect at G.

(i) If $AD = 4''$, $BE = 5''$, $CF = 3''$, calculate the lengths of BC, CA, AB.

(ii) If $BC = 10''$, $AD = 15''$, $BE = 12''$, calculate the lengths of CF, CA, AB.

(iii) If $AD = 12''$, $BE = 15''$, $CF = 9''$, prove that $\angle DGC = 90°$.

[Remember that G is a point of trisection of AD, BE, CF.]

33. PQR is a Δ in which X is the mid-point of QR; $PX = PR$. Prove that $PQ^2 = PX^2 + 2XR^2$.

34. Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

35. If X is a point in the same plane as a parallelogram PQRS, prove that $XP^2 - XQ^2 + XR^2 - XS^2$ is the same wherever X is, and that the expression is equal to half the difference of the squares on the diagonals of the parallelogram.

36. ABC is a Δ whose base BC is trisected at L and M, the order of the letters being B, L, M, C. Prove that

(i) $AB^2 + AC^2 = AL^2 + AM^2 + 4LM^2$,

(ii) $2AB^2 + AC^2 = 3AL^2 + 2BL^2 + CL^2$,

(iii) $AM^2 = \frac{1}{3}AB^2 + \frac{2}{3}AC^2 - \frac{2}{9}BC^2$.

EXERCISE 53 [Further Examples on Theorems 55-57]

1. PQR is an acute-angled $\triangle$, T the foot of the $\perp$ from P upon QR, X the mid-point of QR. Prove that

$$QP^2 \sim PR^2 = 2QR \cdot XT.$$

2. BCML is a square, and P is any point outside the square. Prove that $PL^2 - PB^2 = PM^2 - PC^2$. Deduce that if P is any point within a $\triangle ABC$ and squares BCML, CNRA, ASTB are described externally on its sides, then $PL^2 + PN^2 + PS^2 = PM^2 + PR^2 + PT^2$.

3. QR, PS are the parallel sides of any trapezium PQRS. Prove that $PR^2 + QS^2 = PQ^2 + RS^2 + 2QR \cdot PS$.

4. The $\perp$s AD, BE, CF drawn from the angular points of a $\triangle ABC$ to the opposite sides intersect at H and meet the sides at D, E, F respectively. Prove that $AH \cdot HD = BH \cdot HE = CH \cdot HF$.

5. The square on the side BC of a $\triangle ABC$ is less than the sum of the squares on CA and AB by four times the area of the $\triangle$. Prove that $\angle A = 45°$. What is the $\angle A$, if the square on the side BC is greater than the sum of the squares on CA and AB by four times the area of the $\triangle$?

6. ABC is an acute-angled $\triangle$; D is the foot of the $\perp$ from A to BC. Prove that the rectangle $BC \cdot BD$ is greater than twice the $\triangle ABC$, if $\angle B$ is less than $45°$.

7. In a $\triangle XYZ$, having an obtuse angle Z, YE is drawn $\perp$ to XZ produced and ZD $\perp$ to XY, meeting XZ, XY at E and D respectively. Prove that $XY \cdot XD = XZ^2 + XZ \cdot ZE$.

8. AB is any st. line divided internally at any point C. On AC, CB equilateral $\triangle$s APC, CQB are described on the same side of AB. Prove that $PQ^2 = AB^2 - 3AC \cdot BC$.

9. The sum of the squares on the opposite sides HK, LM of a quad. equals the sum of the squares on the other pair of opposite sides ; prove that the diagonals must be at right angles.

10. O is the centre of a circle of radius r ; AB is a chord of the circle whose distance from the centre is d, the $\angle AOB$ being obtuse. BC is a chord drawn through B $\perp$ to AO produced, cutting AO produced at D. Prove that $OD = \frac{r^2 - 2d^2}{r}$. If AB and BC are equal chords, prove that $r = 2d$.

11. If two medians of a triangle are equal, prove, by Theorem 57, that the $\triangle$ is isosceles.

12. Prove that, in any $\triangle$, four times the sum of the squares on the medians is equal to three times the sum of the squares on the sides.

13. ABCD is a rectangle. Find the locus of a point P which moves so that $PA^2+PR^2+PC^2+PD^2$ is constant.

[Apply Theorem 57 twice.]

14. ABCD is a quad.; L, M the mid-points of AC, BD. Prove that $AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4LM^2$.

15. If the sum of the squares on the sides of a quad. is equal to the sum of the squares on the diagonals, prove that the quad. is a parallelogram.

16. AOB is a st. line and $AO=2OB$. X is a point such that $AX^2+2BX^2=AB^2$. Prove that the locus of X is a circle, centre O.

17. Given two points B, C, on the same side of a given st. line XY, find a point P on XY such that PB^2+PC^2 is as small as possible.

18. PQR is a $\triangle$, the $\angle$R being obtuse. PN is drawn $\perp$ to QR produced. If $PQ^2=PR^2+3QR^2$, prove that R is the mid-point of QN.

19. XYZ is a $\triangle$ having $XZ=YZ$, and P is a point such that $PX^2+PY^2=2PZ^2$. Prove that P lies on a st. line parallel to XY.

20. P is any point in the hypotenuse BC of an isosceles right-angled $\triangle$ABC. Prove that $BP^2+PC^2=2AP^2$.

21. Construct a $\triangle$XYZ, given that its area is 7·5 sq. cm., $YZ=5$ cm., and $XY^2-XZ^2=18$ sq. cm.

22. Given two points B, C, within a circle, find the position of a point P on the circumference of the circle such that PB^2+PC^2 is (i) as great as possible, (ii) as small as possible.

23. X is a point between B and C on the base BC of a $\triangle$ABC, such that $m \cdot BX=n \cdot XC$. Prove that

$$m \cdot AB^2+n \cdot AC^2=m \cdot BX^2+n \cdot XC^2+(m+n) \cdot AX^2.$$

24. ABC is a $\triangle$; Y is a point on BC produced such that

$$m \cdot BY=n \cdot CY.$$

Prove that

$$m \cdot AB^2-n \cdot AC^2=m \cdot BY^2-n \cdot CY^2+(m-n)AY^2.$$

Area of a Triangle with given sides.

Ex. Find the area of a triangle whose sides are 21′, 20′, 13′.

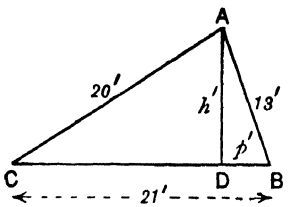


FIG. 216.

Let ABC represent the given triangle, and let AD be the perpendicular from A to BC.

Let AD = h feet, BD = p feet.

Since BC is the greatest side of the triangle, B and C are acute angles and AD will fall within the triangle;

$\therefore$ by the extension of Pythagoras' Theorem for an acute angle,

$$AC^2 = AB^2 + BC^2 - 2BC \cdot BD,$$

i.e. $$20^2 = 13^2 + 21^2 - 42 \cdot p;$$

$$\therefore 42p = 169 + 441 - 400 = 210;$$

$$\therefore p = 5;$$

$\therefore$ applying Pythagoras' Theorem to the $\triangle ADB$,

$$h^2 + 5^2 = 13^2,$$

i.e. $$h^2 = 144 \quad \text{and} \quad h = 12;$$

$\therefore$ The area of the $\triangle ABC = \frac{1}{2} \cdot 12 \cdot 21$ sq. feet

$= 126$ sq. feet.

Note. It is usual to draw the perpendicular to the greatest side of the triangle.

EXERCISE 54

Find by the above method the area of the triangles whose sides are as follows :

1. 21″, 17″, 10″.
2. 20 cm., 13 cm., 11 cm.
3. 13 m., 14 m., 15 m.
4. 56″, 25″, 39″.
5. $52a'$, $41a'$, $15a'$.
6. 4 yds., 13 yds., 15 yds.

7. In the above figure, if $BC=a$ units, $CA=b$ units, $AB=c$ units, $AD=h$ units, $BD=p$ units in length, prove

(1) $p=\dfrac{c^2+a^2-b^2}{2a}$;

(2) $h^2=c^2-\left[\dfrac{c^2+a^2-b^2}{2a}\right]^2=\left[c+\dfrac{c^2+a^2-b^2}{2a}\right]\left[c-\dfrac{c^2+a^2-b^2}{2a}\right]$;

(3) Area$=\sqrt{s(s-a)(s-b)(s-c)}$, where $2s=a+b+c$.

8. By using the above formula, find the areas of triangles whose sides are

(i) 2″, 3″, 4″ ; (ii) 8″, 9″, 11″.

9. A lawn is in the form of a trapezium ; the lengths of the parallel sides are 8 and 22 yards, the lengths of the other sides are 15 and 13 yards ; find the area.

10. Calculate the area of a trapezium whose parallel sides are 28″ and 11″, the other sides being 26″ and 25″.

11. OA, OB, OC are st. lines mutually at right angles and respectively 5, 4 and 3 in. long ; find the length of the ⊥ from A to BC and the area of the △ABC.

12. OA, OB, OC are edges of a rectangular solid, and are respectively 2, 3 and 4 in. long ; find the area of the △ABC.

13. OP, OQ, OR are three st. lines in space at right angles to one another. If $OP=a$ in., $OQ=b$ in., $OR=c$ in., calculate the area of △PQR and prove that it is always an acute-angled △.

14. ABCD is a tetrahedron having $AB=AC=AD=6''$, $BD=8''$, $CD=10''$, $BC=8''$. X is the mid-point of BD. Calculate the area of the △AXC.

THEOREM 58

If two chords of a circle intersect at a point within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

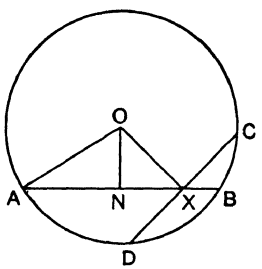


FIG. 217.

Let AB, CD be two chords of a circle, meeting at a point X, inside the circle. Let O be the centre of the circle.

It is required to prove that $AX \cdot XB = CX \cdot XD$.

Construction. Draw the perpendicular ON from O to AB. Join OA, OX.

Proof. Since ON is drawn from the centre perpendicular to the chord AB, it bisects AB, $\therefore AN = NB$;

$$\begin{aligned}\therefore AX \cdot XB &= (AN + NX)(NB - NX)\\ &= (AN + NX)(AN - NX) \quad (AN = NB, \textit{proved})\\ &= AN^2 - NX^2 \dagger\\ &= (AN^2 + NO^2) - (NX^2 + NO^2)\\ &= OA^2 - OX^2 \quad (\angle s\ ONA, ONX\ \textit{rt.}\ \angle s;\ \textit{Pythagoras'}).\end{aligned}$$

Similarly it may be proved that

$$CX \cdot XD = OC^2 - OX^2.$$

But $OA = OC$ (*radii of the same circle*),

$$\therefore AX \cdot XB = CX \cdot XD.$$

† **Note 1.** At this stage we have proved that, if AB is divided equally at N and unequally at X,

$$AX \cdot XB = AN^2 - NX^2,$$

or

$$AX \cdot XB + NX^2 = AN^2.$$

This is the equivalent of Euclid II. 5, and is sometimes enunciated as follows.

If a straight line is divided into two equal parts and also two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Note 2. We have proved above that, if X is any point inside a circle with centre O and radius r, the rectangle contained by the segments AX, XB of any chord AB drawn through X equals $r^2 - OX^2$.

Note 3. An easier proof of this theorem is given in Part V of this book. If this easier proof is adopted the student should learn how to obtain the results stated in Notes 1 and 2.

THEOREM 59

If two chords of a circle, when produced, cut at a point outside it, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other. Also, each rectangle is equal to the square on the tangent from the point of intersection.

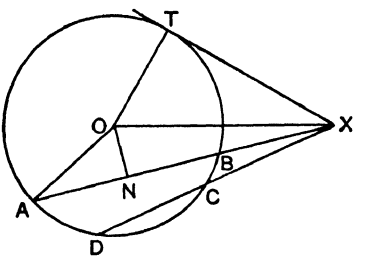


FIG. 218.

Let AB, DC be two chords of a circle, meeting, when produced, at a point X, outside the circle. Let XT be the tangent drawn to the circle from X. Let O be the centre of the circle.

It is required to prove that $XA \cdot XB = XC \cdot XD = XT^2$.

Construction. Draw the perpendicular ON from O to AB. Join OA, OX, OT.

Proof. Since ON is drawn from the centre perpendicular to the chord AB, it bisects AB, $\therefore$ BN = NA,

$$\therefore XA \cdot XB = (XN + NA)(XN - BN),$$
$$= (XN + NA)(XN - NA) \quad (BN = NA, \textit{proved})$$
$$= XN^2 - NA^2 \dagger$$
$$= (XN^2 + ON^2) - (NA^2 + ON^2)$$
$$= OX^2 - OA^2 \ (\angle s\ ONX, ONA\ \textit{rt.}\ \angle s\ ;\ \textit{Pythagoras'}).$$

Similarly it may be proved that $XC \cdot XD = OX^2 - OC^2$.

But OC = OA (*radii of the same circle*),

$$\therefore XA \cdot XB = XC \cdot XD.$$

Also, since XT is the tangent to the circle at T and OT is a radius drawn from the point of contact, $\angle OTX$ = a right angle,

$$\therefore XT^2 = OX^2 - OT^2 \text{ (}\textit{Pythagoras'}\text{)}.$$

But $OT = OA$ (*radii of the same circle*),

$$\therefore XA \cdot XB = XT^2,$$

$$\therefore XA \cdot XB = XC \cdot XD = XT^2.$$

† **Note 1.** At this stage we have proved that, if AB is bisected at N and produced to any point X,

$$XA \cdot XB = XN^2 - NA^2,$$

or

$$XA \cdot XB + NA^2 = XN^2.$$

This is the equivalent of Euclid II. 6, and is sometimes enunciated as follows.

If a straight line is bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the bisected line, is equal to the square on the straight line which is made up of the half and the part produced.

Note 2. We have proved above that, if X is any point outside a circle with centre O and radius r, the rectangle contained by the segments XA, XB of any chord AB drawn through X equals $OX^2 - r^2$.

Note 3. An easier proof of this theorem is given in Part V of this book. If this easier proof is adopted, the student should learn how to obtain the results stated in Notes 1 and 2.

THEOREM 58a

If two straight lines AB, CD intersect at X, so that the rectangle AX, XB is equal to the rectangle CX, XD, then the four points A, B, C, D lie on a circle.

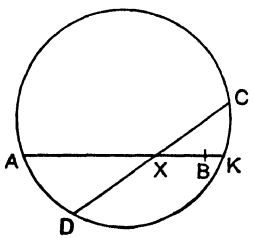


FIG. 219.

Let the straight lines AB, CD meet at X, and let

$$AX \cdot XB = CX \cdot XD.$$

It is required to prove that A, B, C, D lie on a circle.

Proof. If possible, let the circle through the non-collinear points A, C, D not pass through B, but let it cut AX produced at K.

Then $AX \cdot XK = CX \cdot XD$ (*rectangle property of circle*).

But $AX \cdot XB = CX \cdot XD$ (*given*),

$$\therefore AX \cdot XK = AX \cdot XB,$$

$$\therefore XK = XB,$$

i.e. the whole is equal to the part, which is absurd ;

$\therefore$ our supposition that the circle ACD does not pass through B is wrong ;

$\therefore$ A, B, C, D lie on a circle.

THEOREM 59a

(1) If two straight lines AB, CD are both produced to meet at X, and, if the rectangle XA, XB is equal to the rectangle XC, XD, then the four points A, B, C, D lie on a circle.

(2) If X is a point on AB produced and XT is any straight line through X, such that the rectangle XA, XB is equal to the square on XT, then the line XT is a tangent at T to the circle which passes through A, B, T.

(1)

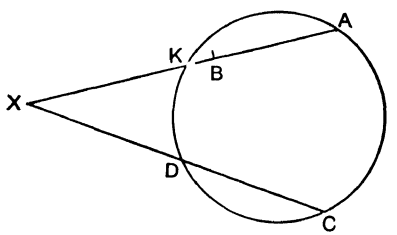


FIG. 220

Let the straight lines AB, CD, when both produced, meet at X, and let XA . XB = XC . XD.

It is required to prove that A, B, C, D lie on a circle.

Proof. If possible, let the circle through the non-collinear points A, C, D not pass through B, but let it cut XA (or XA produced) at K.

Then $XA \cdot XK = XC \cdot XD$ (*rectangle property of circle*).

But $XA \cdot XB = XC \cdot XD$ (*given*),

$\therefore XA \cdot XK = XA \cdot XB,$

$\therefore XK = XB,$

i.e. the part is equal to the whole, which is absurd ;

$\therefore$ our supposition that the circle ACD does not pass through B is wrong ;

$\therefore$ A, B, C, D lie on a circle.

(2)

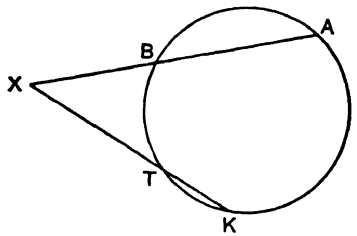


FIG. 221.

Let X be a point on AB produced, and let XT be a straight line through X such that $XA \cdot XB = XT^2$.

It is required to prove that XT is the tangent at T to the circle which passes through A, B, T.

Proof. If possible, let the circle through the non-collinear points A, B, T not touch XT at T, but let it cut XT or XT produced at K.

Then $XA \cdot XB = XT \cdot XK$ (*rectangle property of circle*).

But $XA \cdot XB = XT^2$ (*given*),

$\therefore XT \cdot XK = XT^2$,

$\therefore XK = XT$, i.e. the whole is equal to the part, which is absurd ;

$\therefore$ our supposition that the circle ABT does not touch XT at T is wrong ;

$\therefore$ XT is the tangent at T to the circle ABT.

EXERCISE 55 [Theorems 58-59a]

1. In the figure of Theorem 58 :
   (i) if AX = 4″, XB = 3″, CX = 2″, find XD ;
   (ii) if AB = 22″, XB = 6″, CX = 8″, find CD ;
   (iii) if AB = 30″, XB = 6″, CX = XD, find CD ;
   (iv) if OA = 8 cm., OX = 5 cm., find AX . XB ;
   (v) if OA = 9 cm., AX = 5 cm., OX = 4 cm., find XB.

2. In the figure of Theorem 59 :

(i) if AB=7″, BX=3″, DX=7·5″, find CD ;
(ii) if AB=18 cm., BX=6 cm., find TX ;
(iii) if TX=10″, CX=5″, find DX ;
(iv) if OX=14, XA . XB=96, find OA ;
(v) if XB=5·6″, XA=12·6″, find XT.

3. The tangent to a circle of radius 1·5″ from an external point T is 2″ long. Calculate the distance of T from the nearest point of the circumference.

4. OAB, OC are the two arms of an angle. If OA=2″, OB=4·5″ OC=5″, show that the circle which passes through A, B, C touches OC at C.

5. A vertical wheel touches the ground at T, and is propped by a brick 4″ high, placed a foot from T. Find the height of the wheel.

6. In a circle are two chords AKC, BKD at right angles. AK=20″, KC=30″, AB=25″. Calculate KB, KD and the radius of the circle.

7. A, B are two points on a circle 6 cm. apart. The chord AB is produced to C making BC=2 cm. Find the length of the tangent drawn from C to the circle.

8. ACB is an arc of a circle, C being the mid-point of the arc. The length of the chord AB is 3·2″ and that of the chord AC is 2″. Find the length of the radius of the circle.

9. A chord AB of a circle of unknown radius is 5 cm. long. AB is produced to C so that BC is 3 cm. From C another st. line CDE is drawn cutting the circle at D and E. CD is 4 cm. long. What is the length of the chord DE ? If the centre of the circle is 7 cm. from C, what is the radius of the circle ?

10. O is the centre of a circle of radius 6·5 cm., and P is a point 5·5 cm. distant from O. A chord AB is drawn through P so that BP=3AP. Find the length of the chord.

11. AB is a chord of length 9·6″ in a circle of radius 5″. The diameter bisecting AB meets the circle at X and Y. Calculate the lengths of AX and AY.

12. Two chords ACB, XCY of a circle intersect at C within a circle. AC=12″, CB=3″, XY=20″. Calculate the length of XC and of CY, if XC>CY.

13. OA, OBC are two lines at right angles, and a circle touches OA at A and passes through B and C OA=5 cm., OP=3 cm Calculate the length of OC and the radius of the circle.

14. P is a point inside a circle. The shortest chord which can be drawn through P is 6 cm. in length, and the shortest distance from P to the circumference is 1 cm. Calculate the length of the diameter of the circle.

15. From a point 8 cm. distant from the centre of a circle of 5 cm. radius tangents are drawn. Calculate the length of the chord of contact.

16. Taking the earth's radius to be 3,960 miles, find the distance of the visible horizon to an airman who is 6,000 feet up.

(**The distance of the visible horizon** from a point above the surface of the earth is the length of the tangent from that point to the earth's surface. **If the distance of the point from the surface of the earth is d miles, d is usually so small that d^2 may be neglected.**)

17. How high above sea-level must a person be who can just see a boat on the distant horizon, which is 6 miles away ?

18. Mount Everest is approximately 29,000 feet high. What is the distance of the visible horizon ?

19. Mount Kenya is approximately 17,000 feet high. Can a station 160 miles away be seen from the top ?

20. Prove that, if the radius of the earth be taken as 3,960 miles, the distance in miles of the visible horizon from a height of h feet is $\sqrt{\frac{3h}{2}}$.

21. The radius of a circular arch is 50 feet, and its height is 36 feet ; find the span of the arch. If the height is reduced by 16 feet, the radius remaining the same, by how much will the span be reduced ?

22. Two circles cut one another at A and B. Show that the tangents drawn to these two circles from a point P on AB produced are equal.

23. Two non-intersecting circles being given, a third circle is drawn cutting the first at A, B, and the second at C, D ; and AB, CD produced meet at P. Show that the tangents from P to all three circles are equal.

24. Two circles intersect at X and Y. Prove that XY, when produced, bisects the common tangents of the circles.

25. Water is flowing in a channel of circular section whose radius is r. The width of the stream is $2k$ and the depth at the middle point is h. Prove that $h^2 - 2rh + k^2 = 0$, and find the depth

of the stream, if the radius of the channel is 10′ and the width of the stream 16′. Show by a rough sketch the meaning of the two solutions.

26. AB, CD are chords of a circle ; E, F are points outside the circle on AB, CD, produced respectively, such that

$$AE \cdot BE = CF \cdot DF.$$

Prove that E and F are equidistant from the centre of the circle.

Prove the corresponding result, if E, F are points, inside the circle, on AB, CD respectively, such that $AE \cdot EB = CF \cdot FD$.

27. Find the locus of the point of contact of tangents drawn from a point A on a st. line ABC to the circles which can be drawn through the points B and C.

28. The ⊥s AD, BE, CF drawn from the angular points of a △ABC to the opposite sides intersect at O and meet the sides at D, E, F respectively. Prove that $AO \cdot OD = BO \cdot OE = CO \cdot OF$.

29. LM is a chord of a circle and it is bisected at K. DKJ is another chord. On DJ as diameter a semi-circle is drawn and KS, ⊥ to DJ, meets this semi-circle at S. Prove that KS = KL.

30. Two circles A and B intersect one another and C is a third circle. Show how to find the point or points on C from which equal tangents can be drawn to A and B. Can such a point always be found ?

31. P is a point outside a circle of which LM is a fixed diameter. PM meets the circle again at N. If P moves in such a way that the rectangle $PM \cdot PN$ is constant, prove that the locus of P is a circle and find its centre.

32. P, Q, R are points on the sides BC, CA, AB of a △ABC, such that $BP \cdot PC = CQ \cdot QA = AR \cdot RB$. O is the centre of the circle passing through A, B, C. Prove that O is also the centre of the circle passing through P, Q, R.

33. The common chord of two circles is produced to a point P ; from P a st. line is drawn cutting one of the circles at Q and R, and PT is drawn touching the other circle at T. Prove that PT touches the circle QTR.

34. AXB is a diameter of a circle, PXQ a chord ⊥ to AB. Prove that $AX \cdot XB = XP^2$. Hence also, if $AB = b$, $AX = x$, show how to find the greatest value of $x(b-x)$ when x varies.

35. ABC is a △ and AN, the ⊥ from A to BC, falls within the △. If $BN \cdot NC = AN^2$, prove that $\angle BAC = 90°$.

36. The tangent at A to the circle ABC meets the diameter BC produced at D, and AL is drawn $\perp$ to BC. Prove that

$$DB \cdot DC - LB \cdot LC = DL^2.$$

37. PQ is a chord of a circle with centre C, and O is a point on PQ. A is a point on the circumference of the circle such that $AO^2 = PO \cdot OQ$. Prove that AO is $\perp$ to OC. If $PO = 1{\cdot}6''$, $OQ = 2{\cdot}5''$ and $\angle POA = 45°$, calculate the lengths of CO and the radius of the circle.

38. AB is a diameter of a circle, O the point of trisection of AB nearer to A. Show that the shortest chord of the circle through O is equal to the diagonal of the square described on OB.

39. One vertex P of an equilateral $\triangle PQR$ lies on the circumference of a circle, and the other two vertices are inside the circle. PQ, PR produced meet this circle again at A and B respectively, and QR produced both ways meets the circle at C and D. Prove that the difference between PA and PB equals the difference between QC and RD.

40. Through X, a point of intersection of two circles, two st. lines PXR, QXS are drawn, each passing through a centre and terminated by the circumferences; prove that $PX \cdot XR = QX \cdot XS$.

41. Show how to draw a rectangle equal to a given rectangle ABCD and having a side equal to a given st. line EF.

42. Construct a $\triangle ABC$ having $AB = 3{\cdot}5''$, $AC = 2{\cdot}5''$, $\angle BAC = 54°$. Construct a rectangle with base $2''$, and area equal to that of the $\triangle$.

43. Draw an equilateral $\triangle ABC$ of side $3''$, and a median AD. Construct, without calculation, a rectangle having one side equal to AD, and its area equal to the square on BC.

44. A chord AB is drawn through a point P within a circle; describe how to draw through P a chord QPR, such that

$$QP^2 = AP \cdot PB.$$

45. Show how to draw a rectangle equal to a given square and having the difference of its sides equal to a given st. line.

EXERCISE 56 [Further Examples, Theorems 58-59a]

1. A piece of paper is bounded by two equal st. lines PA, PB, each a inches long, and a circular arc ACB to which PA, PB are tangents. The shortest distance of P from the arc is b inches and $b < a$. Show that the radius of the arc is $\frac{a^2 - b^2}{2b}$ inches.

2. The centre of the inscribed circle of a $\triangle ABC$ is I and E is the centre of the escribed circle touching BC, not produced. If AE cuts BC at D, prove that $BD \cdot DC = ID \cdot DE$.

3. M is the mid-point of a st. line AB. Any st. line through A cuts the circle on AM as diameter at P and the st. line through B which is $\perp$ to AB at Q. Prove that the rectangle AP . AQ is equal to eight times the square on the radius of the circle.

4. ABC is an isosceles $\triangle$ having AB = AC and each of these sides greater than BC. If a point K is taken in AB so that $BC^2 = BK \cdot BA$, prove that the $\triangle BCK$ is isosceles. Construct the figure accurately, taking $\angle A = 32°$ and $AB = 4''$. Measure $\angle AKC$.

5. The $\triangle ABC$ has AB = AC, and a circle is drawn to touch AC at C and BA produced at Q. If this circle cuts BC produced at P, prove that $BC \cdot BP = 4AB^2$, and deduce that $BC \cdot CP = 4AD^2$, where D is the foot of the $\perp$ from A to BC.

6. The diagonals of a cyclic quad. ABCD intersect at E; the circumscribing circle of the $\triangle AEB$ cuts DA, produced if necessary, at F; through E a line is drawn $\parallel$ to DC, meeting DA at G. Prove that $GE^2 = GA \cdot GF$.

7. B is a fixed point on a diameter AC of a given circle, PBQ is any chord through B. AP, AQ, produced when necessary, meet the $\perp$ through B to AC at R and S. Prove that a circle can be drawn to pass through P, Q, R and S; prove also that the rect. BR . BS is constant.

8. From a point A the two tangents are drawn to a circle to touch it at B and C. A st. line APSQ cuts the circle at P, Q, and cuts BC at S. If O is the mid-point of BC, prove

(i) that $PS \cdot SQ = OB^2 - OS^2$, (ii) that $AB^2 = AS^2 + PS \cdot SQ$.

9. Two circles intersect at B and R. Through B a st. line is drawn, meeting the first circle at A and the second circle at C. Through R is drawn a st. line, meeting the first circle at Q, the second circle at P and the line BAC at X. If PX = XQ, prove that CX = XA.

10. AB is a chord of a circle whose centre is O. The tangent at A and a chord AC are equally inclined to AB. If BO (produced if necessary) and AC meet at D, prove that $\angle BDA = 90^\circ$, and that $2BD \cdot BO = AB^2$.

11. B, E are the opposite ends of a diameter of a circle. A, D are points on the circle on opposite sides of BE, and EA, DB, when produced, meet at C. Prove that $DE^2 = BD \cdot DC \sim AE \cdot EC$.

12. Two circles whose centres are O and O′ intersect at A and B. Prove that, if the tangent at A to the former circle passes through O′, the tangent at A to the latter circle must pass through O. (Such circles are said to cut at right angles or to cut orthogonally.) Prove that, if OO′ cuts the circle, centre O′, at P and Q, then *any* circle through P and Q cuts the circle whose centre is O at right angles.

13. A, B, C, D are four points in order on a st. line. Find a point O in the line so that $AO \cdot OC = BO \cdot OD$.

14. C is a point on a st. line AB, situated between A and B. Find a point X on AB (between A and B) such that $XC^2 = AX \cdot XB$. Show that there are two solutions.

15. Given a $\triangle ABC$ in which $AB > AC$, construct (i) a point X on BC produced such that $XA^2 = XB \cdot XC$, and (ii) a point Z on BC itself such that $ZA^2 = BZ \cdot ZC$, the $\triangle$ being such that the latter construction is possible.

16. The inscribed circle of a square ABCD touches the side AB at E. From the middle point of AE a tangent is drawn to the circle and is produced to meet CD produced at F. Show that $DF = AE$.

17. ABC is a $\triangle$ inscribed in a circle. D is the foot of the $\perp$ from A to BC and the diameter of the circle through A meets BC at G. Prove that $AD^2 : AG^2 = BD \cdot DC : BG \cdot GC$.

[This example should not be attempted until Part V has been done.]

CONSTRUCTION 27

To construct a square equal in area to a given rectangle.

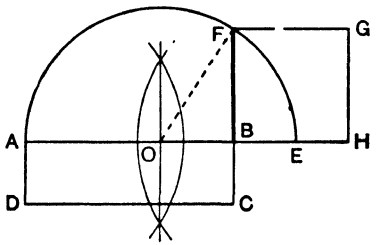


FIG. 222.

Let ABCD be the given rectangle.

It is required to construct a square equal in area to ABCD.

Construction. Produce AB to E, making BE equal to BC.

Bisect AE at O. (*Construction* 2.)

With centre O and radius OA, describe a circle.

Produce CB to cut the circle at F.

On FB describe a square.

This is the required square.

Proof. Let r be the radius of the circle. Join OF.

Then the rect. $ABCD = AB \cdot BC = AB \cdot BE$

$= (AO + OB)(OE - OB)$

$= (r + OB)(r - OB)$

$= r^2 - OB^2 = OF^2 - OB^2 = FB^2$ (*Pythagoras'*)

= the sq. on FB.

Corollary. To construct a square equal in area to any rectilineal figure.

Construct a triangle equal in area to the given figure. (*Const.* 13.)

Construct a rectangle equal in area to this triangle. (*Const* 14.)

Construct, as above, a square equal in area to this rectangle.

EXERCISE 57 [Construction 27, etc.]

1. Describe an equilateral $\triangle$ of sides 2″, and construct a rectangle and hence a square equal to it in area.

2. Construct a $\triangle$ with sides 1·9″, 2·4″, 1·6″. Then construct a square equal in area to the $\triangle$.

3. ABCD is a quad. AB=6 cm., BC=4 cm., $\angle A=63°$, $\angle B=72°$, $\angle C=147°$. Construct the quad.; then construct a square equal to it in area.

4. Draw an equilateral $\triangle ABC$ of side 3 in. From AB, AC mark off AD, AE respectively, each equal to 1·2 in. Construct a square equal in area to the quad. BDEC.

5. Construct a square equal to a quad. ABCD in which AB = 5 cm., AC=7·5 cm., CD=9 cm. $\angle ABC=100°$, $\angle BAD=130°$.

6. **Divide a given straight line internally so that the rectangle contained by its parts shall be equal to a given square.** When is this impossible?

[Let AB be the given st. line and r the side of the given square. Draw a semi-circle on AB as diameter. Draw a st. line $\parallel$ to AB, distant r from it. Let this line meet the semi-circle at C, D. The projections of C, D on AB are the required points of division.]

7. Divide a st. line 2″ long into two parts which will contain a rectangle of area equal to a quarter of a square inch.

8. Draw a st. line AB of length 7·5 cm.; in it find by construction a point P such that the area of the rectangle AP . PB may be equal to the area of a square whose side is 3·2 cm.

9. Divide a st. line 5″ long into two parts, so that the rectangle they contain may be equal in area to 3 square inches.

10. **Divide a given straight line externally so that the rectangle contained by its parts shall be equal to a given square.** Is this always possible?

[Let AB be the given st. line and r the side of the given square. Draw BE $\perp$ to AB and equal to r. Bisect AB at C. With centre C and radius CE draw a semi-circle to cut AB produced at X and BA produced at Y. X, Y are the required points of division.]

11. AB is 2″ long. Find geometrically a point P in AB produced such that the rectangle PA . PB may be equal to the square on AB.

12. AB is 3″ long. Divide AB externally so that the rectangle contained by its parts shall be equal to a square of area 5 square inches.

13. The sum of two adjacent sides of a rectangle is 5″, and its area is equal to that of a square of side 1·5″. Construct the rectangle.

14. The perimeter of a rectangle is 12″ and its area is equal to to that of a square of side 1·7″. Construct the rectangle.

15. Construct a square whose area is the difference of the areas of two given polygons.

16. Divide a given st. line, internally and externally, into two parts, so that the rectangle contained by the whole line and one part shall be equal to the square on the other part. A straight line so divided is said to be divided in Medial Section.

Let AB be the given st. line.

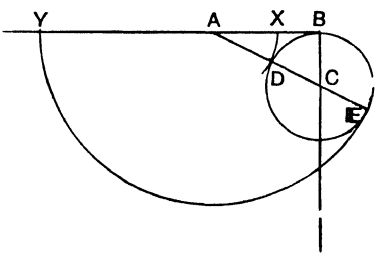


FIG. 223.

It is required to find points X, Y such that $AB \cdot XB = AX^2$ and $BA \cdot BY = AY^2$.

Draw BC ⊥ to AB and equal to half AB. Join AC and produce it. With centre C and radius CB draw a circle cutting AC at D and AC produced at E. With centre A and radius AD draw a circle cutting AB at X. With centre A and radius AE draw a circle cutting BA produced at Y. The X divides AB internally and Y divides AB externally in medial section. To prove this, let $AB = 2a$. All the lengths in the figure may then be calculated by Pythagoras', etc., and it can be verified that X, Y satisfy the required condition. The pupil should also obtain the position of X and Y by solving the algebraic equation $2a(2a - x) = x^2$, where $AX = x$.

17. Divide a st. line 8 cm. long internally in medial section. Measure the greater segment, and find its length algebraically.

18. Divide a st. line 2″ long externally in medial section. Measure the greater segment, and find its length algebraically.

19. A st. line AB is divided at D so that $AB \cdot BD = AD^2$. With centres B and D circles are described, each with radius equal to AD, cutting at C. Prove that $\angle BCD = 36°$. **Hence construct an isosceles △ having each of the angles at the base double of the vertical angle. If AB = 2″, calculate the length of BC.**

20. Construct with ruler and compasses only angles of 72° and 9°.

21. Explain how to graduate the face of a clock in hours and minutes by a purely geometrical construction.

22. Construct an isosceles $\triangle$ having each angle at the base one-third of the vertical angle.

23. In an isosceles $\triangle ABC$, AB is 1″ long, and the angles at A and B are each twice the $\angle$ at C. A circle is described round ABC, and a tangent to this circle at A meets CB produced at D. Find by calculation BD, $\angle ADB$, AD.

24. ABC is an isosceles $\triangle$ having the $\angle A = 36°$. Prove that, if E and F are the mid-points of the sides AC and AB, then EF is the side of a regular pentagon inscribed in the circle described round the $\triangle DEF$, where D is the mid-point of BC.

25. Z is a point on a st. line XY, such that $XZ^2 = XY \cdot ZY$. Prove that $XY^2 + ZY^2 = 3XZ^2$.

26. Show how to inscribe in a given circle (i) a regular pentagon, (ii) a regular decagon, (iii) a regular 15-sided figure, in each case by a purely geometrical construction.

27. Show how to describe about a given circle (i) a regular pentagon, (ii) a regular decagon, (iii) a regular 15-sided figure.

28. Prove the following construction for solving graphically the quadratic equation $x^2 - 2px + q = 0$.

Draw a line $OX = p$ units. Draw $XY \perp$ to OX and equal to 1 unit. Join OY and produce YX. Draw $OZ \perp OY$ to meet YX produced at Z. From Z towards Y mark off $ZA = q$ units. Draw a circle on YA as diameter meeting OX at B and C. Then OB and OC are the roots of the equation.

[Prove (i) $OX^2 = YX \cdot XZ$, (ii) $AX = p^2 - q$, (iii) $BX = \sqrt{p^2 - q}$, (iv) $OB = p - \sqrt{p^2 - q}$ and $OA = p + \sqrt{p^2 - q}$.]

Note. The student who is acquainted with the Theory of Quadratics will see that Examples 6 and 10 give a simple construction for solving quadratic equations graphically.

Consider the solution of the equation $x^2 - 11x + 5 = 0$; we have to find two numbers whose sum is 11 and whose product is 5, i.e. we have to divide a st. line of length 11 units internally, so that the rectangle contained by its parts shall be equal to 5 square units. Again, to solve the equation $x^2 - 6x - 13 = 0$, we have to find two numbers whose difference is 6 and whose product is 13, i.e. we have to divide a line of length 6 units externally, so that the rectangle contained by its parts shall be equal to 13 square units.

* THE CONSTRUCTION OF CIRCLES TO SATISFY GIVEN CONDITIONS

1. To construct a circle to pass through two given points A and B and to touch a given straight line CD which is not parallel to AB.

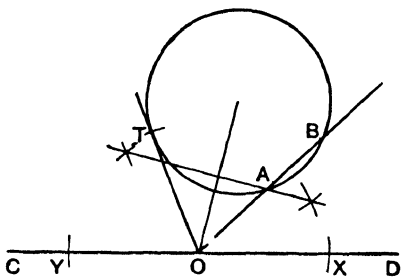


FIG 224.

Let BA produced meet CD at O.

Draw any circle passing through A and B.

Draw the tangent OT from O to this circle, touching it at T.

With centre O and radius OT, describe a circle cutting CD at X and Y.

Draw the circles ABX, ABY. These are the circles required

For $OX^2 = OY^2 = OT^2 = OA \cdot OB$,

$\therefore$ OX and OY are tangents to the circles ABX, ABY respectively.

2. To construct a circle to pass through two given points A and B and to touch a given straight line CD which is parallel to AB.

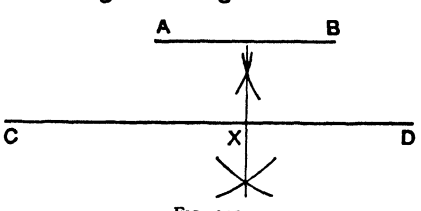


FIG. 225.

Let the perpendicular bisector of AB meet CD at X. Draw the circle ABX. This is the required circle. The pupil should supply the proof.

3. To construct a circle to pass through two fixed points A and B and to touch a fixed circle C.

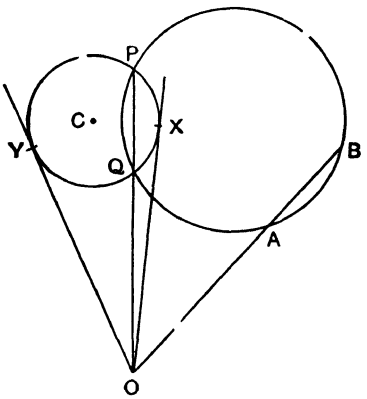


FIG. 226.

Through AB draw any circle to cut the given circle at P, Q.

Join BA, PQ and produce to meet at O. From O draw the tangents OX, OY to the given circle, touching it at X, Y. Draw the circles ABX, ABY. These are the required circles.

For $OX^2 = OY^2 = OP \cdot OQ$ from the given circle,

$= OA \cdot OB$ from the other circle ;

$\therefore$ OX and OY are tangents to the circles ABX, ABY respectively, i.e. the given circle touches ABX at X and ABY at Y.

4. To construct a circle to pass through a given point P, and to touch two given intersecting straight lines AB, AC.

Let the bisector of the arm of the angle between AB, AC in which P lies be AD. Let Q be the image of P in AD.

Then, by symmetry, the required circle passes through Q,

$\therefore$ we have only to draw a circle to pass through P, Q and touch AB. This is No. 1 above. There are two solutions.

If P lies on AD, Q coincides with P and the above construction fails. In this case the required circles touch AB, AC and the straight line through $P \perp$ to AD.

Note. It has previously been shown how to draw a circle to pass through a given point P, and to touch two parallel straight lines between which P lies.

5. To construct a circle to touch two given straight lines and a given circle.

Let AB, CD be the given st. lines, X the centre of the given circle of radius r. Draw two st. lines $\parallel$ to AB, CD, and at a distance r from them. Draw, as above, a circle to pass through X and to touch these st. lines. The required circle is a circle concentric with this The student should supply the proof, and show that in the most general case there are eight circles satisfying the given conditions.

PART V

SIMILAR FIGURES

RATIO AND SIMILARITY

If the areas of two triangles are 8 sq. in. and 12 sq. in., we say that the first is $\frac{8}{12}$ or $\frac{2}{3}$ of the second, and that $\frac{2}{3}$ is the ratio of the areas of the two triangles; this is sometimes written 2 : 3. More generally, if two quantities contain respectively x units and y units **of the same kind,** we may say that their ratio is $\frac{x}{y}$ or $x : y$. The ratio of the two quantities is thus a comparison of their magnitudes. The quantities must of course be of the same kind; we cannot compare £5 with 3 miles, but we can compare £5 with 5 pence, since they may both be expressed in pence.

If we take four quantities a, b, c, d such that the ratio of a to b equals the ratio of c to d, i.e. $\frac{a}{b}=\frac{c}{d}$, the quantities are said to be **in proportion,** and d is said to be the **fourth proportional** to a, b, c.

If we take three quantities a, b, c such that a, b, b, c are in proportion, i.e. $\frac{a}{b}=\frac{b}{c}$, the three quantities are said to be in **continued proportion.** Also c is said to be the **third proportional** to a and b, and b is said to be the **mean proportional** between a and c.

THEOREM 60

Triangles of equal altitude have areas which are proportional to their bases.

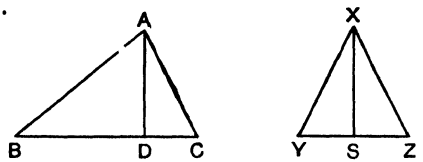


FIG. 227.

Let ABC, XYZ be two triangles whose altitudes AD, XS are equal.

It is required to prove that $\frac{\triangle ABC}{\triangle XYZ} = \frac{BC}{YZ}$.

Proof. Since the area of a triangle is measured by one-half the product of the measures of its base and altitude,

$$\therefore \triangle ABC = \tfrac{1}{2}BC \cdot AD,$$

and

$$\triangle XYZ = \tfrac{1}{2}YZ \cdot XS;$$

$$\therefore \frac{\triangle ABC}{\triangle XYZ} = \frac{\tfrac{1}{2}BC \cdot AD}{\tfrac{1}{2}YZ \cdot XS} = \frac{BC}{YZ}, \text{ since } AD = XS \text{ (\textit{given})}.$$

Note 1. This proof is only valid if the bases and altitudes of the triangles are commensurable, because it depends upon Theorem 27, which has only been proved if the base and altitude of the triangle are commensurable.

A strict proof, applicable to incommensurable magnitudes, is given on p. 380. This is outside the syllabuses of all School Certificate Examinations, and candidates are only expected to learn proofs which are valid for commensurable magnitudes.

Note 2. An important special case of this theorem occurs when two triangles have a common vertex and a common altitude, e.g. in the figure $\frac{\triangle BCA}{\triangle CDA} = \frac{BC}{CD}$. It is this case which is used in the proof of Theorem 61.

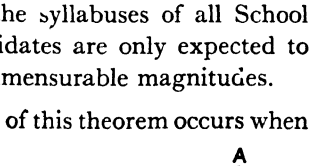


FIG. 228.

If AB is a straight line divided internally at P (Fig. 1), its segments are AP and PB, and it is said to be **divided internally** in the ratio $\frac{AP}{PB}$, or AP : PB.

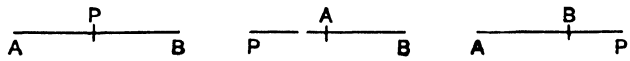


Fig. 1 Fig. 2 Fig. 3

FIG. 228A.

If AB is a straight line divided externally at P (Figs. 2 and 3), its segments are AP and PB, and it is said to be **divided externally** in the ratio $\frac{AP}{PB}$, or AP : PB.

It should be particularly noticed that in **all** cases AB is the whole line and does **not** appear in the ratio AP : PB.

The teacher may at this stage, if it is considered advisable, discuss the convention of signs and the interpretation of positive and negative ratios.

[The segments AP, PB of a straight line have the same sign, if a person travelling along the line from A to P goes in the same direction as a person travelling along the line from P to B. They have opposite signs, if such persons travel in opposite directions. It will thus be seen that in Fig. 1 the ratio AP : PB is positive and that in Figs. 2 and 3 it is negative.]

The pupil is probably already familiar with the use of ratio in Arithmetic and Algebra, but the following examples, most of which may be discussed orally, should be mastered before proceeding to the theorems.

EXERCISE 58

1. What is the value of the following ratios :

(i) 4 in. to 3 ft. ; (ii) 5d. : £10 ;

(iii) 4 min. 30 sec. to $2\frac{1}{2}$ hr. ;

(iv) 3 cm. 25 mm. to 4 m. (v) 63° to 4 rt. ∠s ?

2. If $a : b = c : d$, prove that

(i) $ad = bc$; (ii) $a : c = b : d$;

(iii) $b : a = d : c$; (iv) $\frac{a+b}{b} = \frac{c+d}{d}$;

(v) $\frac{a-b}{b} = \frac{c-d}{d}$; (vi) $\frac{a}{a+b} = \frac{c}{c+d}$;

(vii) $\frac{a-b}{a+b} = \frac{c-d}{c+d}$; (viii) $\frac{ka+lc}{kb+ld} = \frac{ma-nc}{mb-nd}$.

Prove the converses of these statements.

3. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that $\frac{a}{b} = \frac{ka+lc+me}{kb+ld+mf}$.

4. Are the following quantities in proportion :

(i) 7″, 1′, 1′ 9″, 3′ ;

(ii) 8 oranges, 6 pence, 12 pence, 9 oranges ?

5. What is wrong with the following statement :

£12 : 36″ = £5 : 15″ ?

6. Find the fourth proportional to

(i) 6, 9, 16 ; (ii) pq, qr, rs.

7. Find the third proportional to

(i) 8, 32 ; (ii) 9, 4 ; (iii) ab^2, a^2b.

8. Find a mean proportional between

(i) 8, 32 ; (ii) 9, 4 ; (iii) ab^2, a^2b.

9. A st. line AB 10 cm. long, is divided internally at P in the ratio 3 : 2 ; find AP.

10. How many points are there which divide AB internally in the ratio $p : q$?

11. A st. line AB, 6 cm. long, is divided externally at P in the ratio 1 : 3 ; find AP.

12. A st. line AB, 6 cm. long, is divided externally at P in the ratio 3 : 1 ; find AP.

13. How many points are there which divide AB externally in the ratio $p : q$?

14. A st. line, a inches long, is divided internally at P and externally at Q in the ratio $l : m$. Calculate the length of PQ.

THEOREM 61

1. If a straight line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.

2. If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

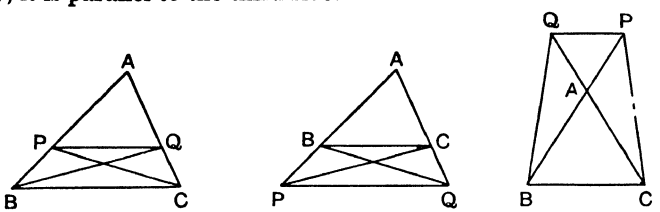


FIG. 229.

Let ABC be a triangle and let a straight line PQ cut the sides AB, AC (or these sides produced) at P, Q respectively.

1. If PQ is parallel to BC, then it is required to prove that

$$\frac{AP}{PB} = \frac{AQ}{QC}.$$

Construction. Join PC, QB.

Proof. Since triangles of equal altitudes have areas which are proportional to their bases,

$$\therefore \frac{AP}{PB} = \frac{\triangle APQ}{\triangle PBQ} = \frac{\triangle APC}{\triangle PBC},$$

and

$$\frac{AQ}{QC} = \frac{\triangle AQP}{\triangle QCP} = \frac{\triangle AQB}{\triangle QCB}.\dagger$$

† **Note.** It may seem unnecessary to include both the fractions equal to $\frac{AP}{PB}$ and $\frac{AQ}{QC}$, but experience has shown that beginners often get confused if confronted with a less familiar form of the theorem. It is important that they should realise that not only are $\frac{AP}{PB}$ and $\frac{AQ}{QC}$ equal, but also $\frac{AP}{AB}$ and $\frac{AQ}{AC}$, and $\frac{AB}{BP}$ and $\frac{AC}{CQ}$.

The theorem is used in this form in proving Theorems 63 and 62 respectively, and the direct proof of the equality of these ratios should

But it is given that PQ is parallel to BC,

$\therefore \triangle PBQ = \triangle QCP$, being on the same base PQ and between the same parallels PQ, BC,

$$\therefore \frac{\triangle APQ}{\triangle PBQ} = \frac{\triangle AQP}{\triangle QCP},$$

$$\therefore \frac{AP}{PB} = \frac{AQ}{QC}.$$

2. If $\frac{AP}{PB} = \frac{AQ}{QC}$, it is required to prove that PQ is parallel to BC.

Construction. Join PC, QB.

Proof. Since triangles of equal altitudes have areas which are proportional to their bases,

$$\therefore \frac{AP}{PB} = \frac{\triangle APQ}{\triangle PBQ} = \frac{\triangle APC}{\triangle PBC},$$

and

$$\frac{AQ}{QC} = \frac{\triangle AQP}{\triangle QCP} = \frac{\triangle AQB}{\triangle QCB}.\dagger$$

But it is given that $\frac{AP}{PB} = \frac{AQ}{QC}$,

$$\therefore \frac{\triangle APQ}{\triangle PBQ} = \frac{\triangle AQP}{\triangle QCP},$$

$$\therefore \triangle PBQ = \triangle QCP.$$

But these $\triangle$s are on the same base PQ and on the same side of it, $\therefore$ they are between the same parallels,

$\therefore$ PQ is parallel to BC.

---

be carefully considered. It often happens that, when asked to do this, pupils begin correctly by writing down

$$\frac{AB}{BP} = \frac{\triangle ABQ}{\triangle BPQ} \text{ and } \frac{AC}{CQ} = \frac{\triangle ACP}{\triangle CQP},$$

but then fail to continue, because the fractions obtained have neither numerator nor denominator the same.

If the pupil is taught to write down both the fractions equal to $\frac{AB}{BP}$ and $\frac{AC}{CQ}$, he will find that in all cases there will be **at least one pair** of fractions with either the same numerator or the same denominator. If this pair is selected the remainder of the work is easy.

† See note on previous page.

EXERCISE 59 [Theorems 60 and 61]

1. P is a point on the side AB of a $\triangle ABC$ such that $5AP = 2AB$. Find (i) $\triangle APC : \triangle ABC$, (ii) $\triangle APC : \triangle CPB$.

2. P is a point on the side AB of a $\triangle ABC$ such that $4AP = PB$ and Q is a point on the side CA such that $3CQ = 2QA$. Find

(i) $\triangle APQ : \triangle PQC$, (ii) $\triangle APQ : \triangle ABC$.

3. ABC is a $\triangle$, and PQ is drawn $\parallel$ to BC, cutting AB at P and AC at Q.

(i) If $AB = 7{\cdot}2''$, $AC = 4{\cdot}8''$, and $AP = 4{\cdot}2''$, find AQ.

(ii) If $AB = 6$ cm., $AC = 4{\cdot}5$ cm., and $AQ = 2{\cdot}7$ cm., find BP.

(iii) If $AP = 2{\cdot}8''$, $PB = 0{\cdot}8''$, $AQ = 4{\cdot}2''$, find QC.

(iv) If $AP = 5{\cdot}4''$, $AB = 6''$, $QC = 1{\cdot}5''$, find AC.

4. ABC is a $\triangle$ and PQ is drawn $\parallel$ to BC, cutting BA produced at P and CA produced at Q.

(i) If $AB = 8''$, $AC = 5''$, $AQ = 3''$, find AP.

(ii) If $AB : PB = 8 : 11$ and $QC = 3{\cdot}3''$, find QA.

5. PQR is a $\triangle$ in which $PQ = 3''$ and $QR = 2{\cdot}7''$. X is a point on PQ such that $PX = 1''$; XY, drawn $\parallel$ to QR, meets PR at Y; YZ, drawn $\parallel$ to PQ, meets QR at Z. Calculate the length of QZ.

6. Prove that $\triangle$s on equal bases are to one another as their altitudes.

7. AB, CD, EF are three $\parallel$ st. lines which are cut by two transversals ACE and BDF. Prove that $AC : CE = BD : DF$.

8. A point P is taken on the side AB of a $\triangle ABC$ such that $AP : PB = m : n$. A line PQ, drawn $\parallel$ to BC, meets AC at Q, and a line QR, drawn $\parallel$ to AB, meets BC at R. Find the ratio $BR : RC$.

9. X is a point in the side PQ of a $\triangle PQR$; XY is drawn $\parallel$ to QR and meets PR at Y. RZ is drawn $\parallel$ to YQ and meets PQ produced at Z. Prove that $PX : XQ = PQ : QZ$.

10. PQRS is a trapezium, the parallel sides being PQ, SR. Prove that any line $\parallel$ to PQ divides PS, QR proportionally. [Join PR.]

11. D is the mid-point of the side BC of a $\triangle ABC$; AD is bisected at E; and BE produced cuts AC at X. Prove that $BE : EX = 3 : 1$. [Draw EF $\parallel$ AC, meeting BC at F.]

12. ABC is a $\triangle$. X is a point on BC such that $BX : XC = 3 : 5$; Y is a point on BA such that $BY : YA = 1 : 2$. AX, CY meet at K. Prove that $CK : CY = 5 \cdot 7$. [Draw YZ $\parallel$ to AX, meeting BC at Z.]

13. ABC is a △. Y is a point on CA such that CY : YA = 1 : 10 ; Z is a point on BA such that BZ : ZA = 3 : 2. BY, CZ meet at X. Prove that CZ : XZ = 7 : 6. [Draw ZK || BY, meeting CA at K.]

14. ABC is a △. X is a point on BC such that BX : BC = 2 : 5 ; Y is a point on CA such that CY : CA = 1 : 3. AX, BY meet at Z. Prove that Z is the mid-point of BY.

15. ABC is a △. Z is a point on AB such that AZ : ZB = 3 : 4 ; Y is a point on AC such that AY : YC = 3 : 2. ZY produced meets BC produced at X. Prove that C is the mid-point of BX.

16. ABC is a △. X is a point on BC such that BX : BC = 1 : 3 · Y is a point on AC such that AY : YC = 7 : 4. YX produced meets AB produced at Z. Prove that AZ : BZ = 7 : 2.

17. E is any point in the side AD of the quad. ABCD ; AE is divided at P and ED at S so that AP : PE = DS : SE ; PQ is drawn || to AB meeting EB at Q ; and SR || to DC meeting EC at R ; show that QR is || to BC.

18. Two △s PQR, SQR stand on the same side of the common base QR. From any point X in QR lines are drawn || to QP, QS meeting PR, SR at Y and Z. Prove that YZ is || to PS.

19. Two △s PQR, SQR stand on opposite sides of the common base QR. From any point X in QR lines are drawn || to QP, QS meeting PR, SR at Y and Z. Prove that YZ is || to PS.

20. D is the mid-point of the side BC of a △ABC ; and from any point P in BC lines PX, PY are drawn || to AC, AB cutting AD at X, Y. Prove that D is the mid-point of XY.

THEOREM 62

1. The bisector (internal or external) of the vertical angle of a triangle divides the base (internally or externally) in the ratio of the sides containing the vertical angle.

2. If a straight line through the vertex of a triangle divides the base (internally or externally) in the ratio of the sides containing the vertical angle, it bisects the vertical angle (internally or externally).

First Proof.

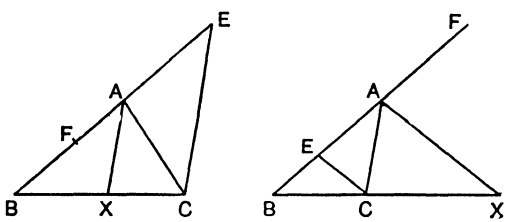


FIG. 230.

1. Let AX bisect the angle A of the $\triangle ABC$, either internally, as in Fig. 1, or externally, as in Fig. 2, and meet BC or BC produced at X.

It is required to prove that $\frac{BX}{XC} = \frac{BA}{AC}$.

Construction. In Fig. 1, let F be any point in BA, and in Fig. 2, let F be any point in BA produced. Draw through C a straight line parallel to XA to meet BA produced, or BA, at E.

Proof. In the $\triangle BCE$, XA is parallel to the base CE,

$\therefore$ it divides the other two sides proportionally,

$$\therefore \frac{BX}{XC} = \frac{BA}{AE}.$$

Also, since XA is parallel to CE,

$$\therefore \angle FAX = \angle AEC$$

and

$$\angle CAX = \angle ACE.$$

But $$\angle FAX = \angle CAX \text{ (\textit{given})},$$

$$\therefore \angle AEC = \angle ACE ;$$

$$\therefore \text{ the } \triangle AEC \text{ is isosceles and } AC = AE,$$

$$\therefore \frac{BX}{XC} = \frac{BA}{AC}.$$

2. Let X be a point in BC, or in BC produced, such that

$$\frac{BX}{XC} = \frac{BA}{AC}.$$

It is required to prove that AX bisects the angle BAC, internally or externally.

Construction. In Fig. 1, let F be any point in BA, and in Fig. 2, let F be any point in BA produced. Draw through C a straight line parallel to XA to meet BA produced, or BA, at E.

Proof. In the $\triangle BCE$, XA is parallel to the base CE,

$\therefore$ it divides the other two sides proportionally,

$$\therefore \frac{BX}{XC} = \frac{BA}{AE}.$$

But $$\frac{BX}{XC} = \frac{BA}{AC} \text{ (\textit{given})},$$

$$\therefore \frac{BA}{AE} = \frac{BA}{AC},$$

$$\therefore AE = AC ;$$

$$\therefore \angle AEC = \angle ACE \text{ (\textit{base angles of an isosceles } $\triangle$)}.$$

Also, since XA is parallel to CE,

$$\therefore \angle FAX = \angle AEC \text{ and } \angle CAX = \angle ACE.$$

But $$\angle AEC = \angle ACE \text{ (\textit{proved})},$$

$$\therefore \angle FAX = \angle CAX ;$$

$\therefore$ AX bisects the angle BAC, internally or externally.

Second Proof.

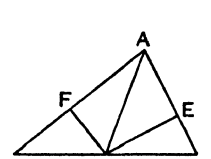


Fig. 1

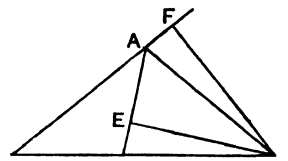


Fig. 2

FIG. 231.

(1) Let AX bisect the angle A of the $\triangle ABC$, either internally, as in Fig. 1, or externally, as in Fig. 2, and meet BC or BC produced at X.

It is required to prove that $\frac{BX}{XC} = \frac{BA}{AC}$.

Construction. Draw XF, XE perpendicular to AB, AC (produced either way if necessary), meeting them at F, E respectively.

Proof. The $\triangle$s ABX, AXC have the same altitude (the perpendicular from A to BC),

$$\therefore \frac{BX}{XC} = \frac{\triangle ABX}{\triangle AXC}.$$

Also AX is the bisector (internal or external) of the $\angle BAC$,

$\therefore$ it is part of the locus of points equidistant from AB, AC,

$$\therefore XF = XE,$$

$\therefore$ The $\triangle$s ABX, AXC have equal altitudes XF, XE to bases BA, AC,

$$\therefore \frac{BA}{AC} = \frac{\triangle ABX}{\triangle AXC},$$

$$\therefore \frac{BX}{XC} = \frac{BA}{AC}.$$

(2) Let X be a point in BC, or in BC produced, such that

$$\frac{BX}{XC} = \frac{BA}{AC}.$$

It is required to prove that AX bisects the angle BAC, internally or externally.

Construction. Draw XF, XE perpendicular to AB, AC (produced either way, if necessary) meeting them at F, E respectively.

Proof. The △s ABX, AXC have the same altitude (the perpendicular from A to BC),

$$\therefore \frac{BX}{XC} = \frac{\triangle ABX}{\triangle AXC}.$$

But
$$\frac{BX}{XC} = \frac{BA}{AC} \text{ (given)},$$

$$\therefore \frac{BA}{AC} = \frac{\triangle ABX}{\triangle AXC} = \frac{\frac{1}{2}BA \cdot XF}{\frac{1}{2}AC \cdot XE},$$

$$\therefore XF = XE,$$

∴ X lies on the locus of points equidistant from AB, AC, i.e. on one of the bisectors of the angle BAC ;

∴ AX bisects the angle BAC, internally or externally.

EXERCISE 60 [Theorem 62]

1. The sides of a △ are 5″, 6″, 7″ ; find the lengths of the segments into which the smallest side is divided by the internal bisector of the opposite angle.

2. The sides of a △ are 4″, 6″, 7″ ; find the lengths of the segments into which the longest side is divided by the external bisector of the opposite angle.

3. The internal and external bisectors of ∠QPR meet QR and QR produced at X and Y ; QX = 10 cm., XR = 6 cm. ; find RY.

4. The sides BC, CA, AB of a △ are 4″, 3″, 5″ long respectively. The internal and external bisectors of ∠BAC meet BC and BC produced at P, Q respectively. Calculate the lengths PC, CQ. Find also PA^2, AQ^2 and show that $PA^2 + AQ^2 = PQ^2$.

5. ABC is a △ right-angled at A, I is the centre of the inscribed circle and AI meets BC at D. Prove that if AB : AC = 3 : 4 then DI · IA = 5 : 7.

6. A, X, B, Y are four points in order on a st. line ; AX = 1″, AB = 1½″, AY = 3″, P is a point 2″ from A and 1″ from B. Prove that ∠XPY = 90°.

7. ABC is a △ in which BC = CA = 2AB. The internal bisector of ∠BAC meets BC at D, and DE drawn ‖ to CA meets AB at E. Find the ratio of DE to AB.

8. ABC is a $\triangle$, D the mid-point of the side BC ; AD is joined, and the $\angle$s ADC, ADB are bisected by st. lines which meet CA, AB at E, F respectively ; prove that FE is $\parallel$ to BC.

9. In a $\triangle$ABC the st. line AD, bisecting the $\angle$BAC, meets the side BC at D. DE, drawn $\parallel$ to BA, meets AC at E ; and CF, drawn $\parallel$ to EB, meets AB produced at F. Prove that BF=AC.

10. ABCD is a quad. having AB=AD. The $\angle$s BAC, CAD are bisected by lines meeting BC at E and CD at F. Prove that EF is $\parallel$ to BD.

11. ABCD is a quad. such that AB . CD=BC . AD, and the bisector of the $\angle$B meets AC at E ; show that DE bisects the $\angle$D.

12. Apply Theorem 62 and its converse to show that, in any triangle, the internal bisectors of the three angles are concurrent.

13. PQRS is a parallelogram whose diagonals meet at O ; the bisector of the $\angle$POQ meets PQ at X and the bisector of the $\angle$POS meets PS at Y. Prove that XY is $\parallel$ to QS.

14. X is a point inside the $\triangle$PQR and the bisectors of the $\angle$s QXR, RXP, PXQ meet BC, CA, AB at D, E, F respectively. Prove that QD . RE . PF=DR . EP . FQ.

EXERCISE 61 [Further Examples on Theorems 60-62]

1. ABC is a $\triangle$; X, Y are points on AB, AC such that 2AY=XB, and AY=YC. If BY and CX intersect at O, prove that 2CO=3OX, and BO=4OY.

2. From a point X on the side QR of a $\triangle$PQR a line is drawn $\parallel$ to RP to meet PQ at Y. Prove that the area of $\triangle$XPQ is a mean proportional between the areas of the $\triangle$s PQR and XQY.

3. A point O is taken within a quad. ABCD, and the lines OX, OY drawn through O $\parallel$ to CB, CD, meeting AB, AD at X, Y respectively. Prove that, if XY is $\parallel$ to BD, then O lies on AC.

4. O is a point within a $\triangle$ABC, and the bisectors of the $\angle$s AOB, AOC cut AB, AC at P, Q. Prove that, if PQ is $\parallel$ to BC, the locus of O is a st. line which bisects BC at right angles.

5. ABCD is a quad. in which the rect. AB, CD=the rect. AC, BD. Prove that the bisectors of the $\angle$s ABD, ACD meet on AD.

6. From a point O outside a circle st. lines OLM, OPQ are drawn to cut the circle at L, M and P, Q. If OL=2OP, prove that the bisector of the $\angle$MOQ passes through a point of trisection of each of the lines PL and QM.

7. The base BC of a $\triangle ABC$ is bisected at D. DE, the bisector of the $\angle ADB$, meets AB at E and EF is drawn $\parallel$ to BC meeting AC at F. Prove that DF is the bisector of $\angle ADC$.

8. CD is the diameter of a circle which bisects a chord AB, and E is any point on AB. CE produced meets the circle at P. Prove (*a*) that AP : PB = AE : EB ; (*b*) that, if DE produced meets the circle at Q, then PA : AQ = PB : BQ.

9. PQR is a $\triangle$ and O is any point inside it. X is any point on OP, and through X lines are drawn $\parallel$ to PQ, PR to meet OQ, OR at Y, Z respectively. Prove that YZ is $\parallel$ to QR.

10. PQR is a $\triangle$ and O is any point inside it. PO produced meets QR at X, QO produced meets PR at Y, RO produced meets PQ at Z. Prove that $\triangle OPQ : \triangle OPR = QX : XR$, and deduce that

$$QX \cdot RY \cdot PZ = XR \cdot YP \cdot ZQ.$$

This is known as Ceva's Theorem.

11. With the figure of Ex. 10, prove that

$$\triangle PQR : \triangle OQR = PX : OX.$$

12. PQRS is a trapezium, PQ being $\parallel$ to SR. X is the mid-point of PS, Y is the mid-point of QR. Prove (i) that $\triangle XSR = \frac{1}{2}\triangle PSR$; (ii) that $\triangle YSR = \frac{1}{2}\triangle QSR$. Deduce that XY is $\parallel$ to SR.

13. ABC is a $\triangle$; P, Q, R points on BC, CA, AB such that AP, BQ, CR meet at a point O. If BR : RA = 2 : 3, and AQ : QC = 4 : 5, prove that BP : PC = 8 : 15. [Use Ceva's Theorem.]

14. Use Ceva's Theorem to prove that the medians of a triangle meet at a point.

15. The inscribed circle of a $\triangle PQR$ touches the sides QR, RP, PQ at X, Y, Z respectively. Prove that PX, QY, RZ meet at a point.

16. A, B, C are three points in order on a st. line, BC being greater than AB. Circles are drawn with centre B, radius BA and with centre C, radius CB. BP, CQ are parallel radii of these circles. QP produced cuts BA produced at T. Prove that TB is the mean proportional between TC and TA.

17. In a $\triangle PQR$ a transversal is drawn cutting the sides QR, RP, PQ (produced, if necessary) at X, Y, Z respectively. If it makes equal angles with PQ and PR, prove that QX : RX = QZ : RY. [Draw RS $\parallel$ YZ to meet PQ produced at T.]

18. PQRS is a parallelogram. A st. line is drawn through S to meet QP produced at Z, QR produced at Y, and PR produced at X. Prove that XS is a mean proportional between XY and XZ.

19. PQRS is a parallelogram. A st. line is drawn through P to meet QS at X, SR at Y, and QR produced at Z. Prove that PX is a mean proportional between XY and XZ.

20. ABC is a △ right-angled at A. On the sides AB, AC squares ABNM AKLC are described external to the △ABC. CN, BL cut AB, AC at P, Q respectively. Prove that AP=AQ, and hence show that each of them is a mean proportional between BP, QC.

SIMILAR FIGURES

DEFINITIONS

1. Two rectilineal figures are said to be **equiangular** to one another when the angles of the first **taken in order** are equal respectively to those of the second taken in order.

2. Rectilineal figures which are equiangular, and in which the ratio of the side adjacent to two angles in one figure to the side adjacent to the equal angles in the other figure is the same for all the pairs of sides are said to be **similar.**

Thus the figures ABCDE, PQRST are similar, if the angles A, B, C, D, E are equal to the angles P, Q, R, S, T respectively, and if also

$$AB : PQ = BC : QR = CD : RS = DE : ST = EA : TP.$$

It must be carefully noticed that the order of the letters is important, and that when two figures ABCDE ..., PQRST ... are said to be similar, it is always understood that A and P, B and Q, C and R, ... are equal angles.

The sides of equiangular figures which are adjacent to equal angles are called **corresponding** sides.

Similar figures are said to have the same **shape.**

Note. It should be particularly noticed that both the conditions stated in the definition are necessary; one without the other is not sufficient, e.g. (i) a rectangle and a square are equiangular, but their corresponding sides are not proportional and they are not similar; (ii) a square and a rhombus have their corresponding sides proportional, but they are not equiangular and they are not similar; (iii) two rectangles are equiangular, but they are not, in general, similar; to decide whether they

are similar it is necessary to know the ratio of corresponding sides.

But although both the conditions stated in the definition are necessary in order that two rectilineal figures may be similar, **in the case of two triangles one condition is a consequence of the other.** The four separate cases (corresponding to the four cases of congruence—Theorems 4, 10, 13, 14) are considered in the next group of theorems.

It will be shown that two triangles are similar :

(1) if they are equiangular ;

(2) if the three sides of one triangle are proportional to the three sides of the other ;

(3) if one angle of one triangle is equal to an angle of the other and if the sides about the equal angles are proportional ;

(4) if one angle of one triangle is equal to an angle of the other and if the sides about one other angle in each are proportional, so that the sides opposite the equal angles correspond, and the remaining angles are both acute, both right or both obtuse.

The first case is the most important, and the third case is also of frequent application. The second case is not often used, and the fourth case is very rarely of use.

It is recommended that a first course on similar triangles should be given early in the school course, and that the treatment should be informal and mainly oral. Most of the examples should be numerical. All children are familiar with similar figures, and during the first course it is sufficient for them to realise that drawings to scale and photographs are examples of similar figures, and that, if two figures are similar, one is an enlargement of the other. Most numerical examples can be solved mentally by consideration of the ratio in which the sides of one figure have been enlarged or diminished in order to obtain the other.

THEOREM 63

If two triangles are equiangular, their corresponding sides are proportional.

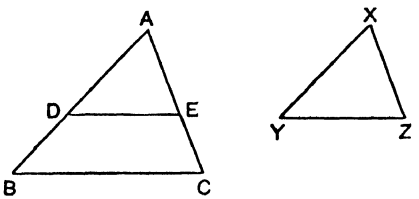


FIG. 232

Let ABC, XYZ be two triangles, in which

$$\angle A = \angle X,\ \angle B = \angle Y,\ \angle C = \angle Z.$$

It is required to prove that

$$\frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ}.$$

Construction. From AB, AC cut off AD, AE equal to XY, XZ respectively.

Join DE.

Proof. Then in the $\triangle$s ADE, XYZ,

$$AD = XY \text{ (\textit{construction})},$$

$$AE = XZ \text{ (\textit{construction})},$$

$$\angle DAE = \angle YXZ \text{ (\textit{given})};$$

$\therefore$ the $\triangle$s $\begin{matrix} ADE \\ XYZ \end{matrix}$ are congruent (2 *sides*, **included** $\angle$);

$$\therefore \angle ADE = \angle XYZ.$$

But $$\angle ABC = \angle XYZ \text{ (\textit{given})},$$

$$\therefore \angle ADE = \angle ABC.$$

But these are corresponding angles, $\therefore$ DE is parallel to BC, $\therefore$ it divides the sides AB, AC proportionally,

$$\therefore \frac{AB}{AD} = \frac{AC}{AE}.$$

But $AD = XY$ and $AE = XZ$ (*construction*),

$$\therefore \frac{AB}{XY} = \frac{AC}{XZ}.$$

Similarly, by cutting off lengths from BA, BC, equal to YX, YZ respectively, it can be proved that

$$\frac{AB}{XY} = \frac{BC}{YZ};$$

$$\therefore \frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ}.$$

Corollary. Equiangular triangles are similar.

Note 1. In proving two triangles equiangular, it is sufficient to prove that two angles of the one are equal to two angles of the other, for, since the three angles of a triangle are together equal to two right angles, the third angles must also be equal.

Note 2. The proofs of Theorems 58 and 59 (see pp. 357-359) furnish worked examples on Theorem 63.

THEOREM 64

If the three sides of one triangle are proportional to the three sides of the other, the triangles are equiangular.

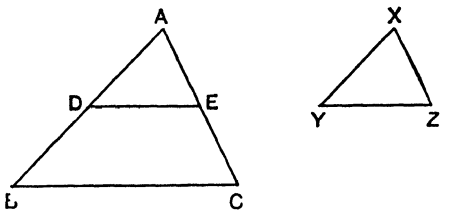


FIG. 233.

First Proof.

Let ABC, XYZ be two triangles, in which

$$\frac{AB}{XY}=\frac{AC}{XZ}=\frac{BC}{YZ}.$$

It is required to prove that $\angle A=\angle X$, $\angle B=\angle Y$, $\angle C=\angle Z$.

Construction. From AB, AC cut off AD, AE equal to XY, XZ respectively. Join DE.

Proof.

$$\frac{AB}{AD}=\frac{AB}{XY}\ (\textit{construction}),$$

$$=\frac{AC}{XZ}\ (\textit{given}),$$

$$=\frac{AC}{AE}\ (\textit{construction});$$

$\therefore$ DE divides the sides AB, AC of the $\triangle ABC$ proportionally,

$\therefore$ DE is parallel to BC,

$\therefore$ $\angle ABC=\angle ADE$ and $\angle ACB=\angle AED$,

$\therefore$ The triangles $\frac{ABC}{ADE}$ are equiangular,

$\therefore$ their corresponding sides are proportional;

$$\therefore \frac{AB}{AD}=\frac{AC}{AE}=\frac{BC}{DE}.$$

Then $$\frac{BC}{DE} = \frac{AC}{AE} \text{ (\textit{proved})},$$

$$= \frac{AC}{XZ} \text{ (\textit{construction})},$$

$$= \frac{BC}{YZ} \text{ (\textit{given})},$$

$$\therefore DE = YZ;$$

$\therefore$ in the $\triangle$s ADE, XYZ,

$$AD = XY \text{ (\textit{construction})},$$

$$AE = XZ \text{ (\textit{construction})},$$

$$DE = YZ \text{ (\textit{proved})};$$

$$\therefore \text{ the } \triangle\text{s } \frac{ADE}{XYZ} \text{ are congruent (3 \textit{sides})},$$

$$\therefore \angle DAE = \angle YXZ, \ \angle ADE = \angle XYZ \text{ and } \angle AED = \angle XZY.$$

But $\angle ABC = \angle ADE$ and $\angle ACB = \angle AED$ (*proved*),

$$\therefore \angle BAC = \angle YXZ, \ \angle ABC = \angle XYZ, \ \angle ACB = \angle XZY.$$

Corollary. Triangles which have their sides proportional are similar.

Note. This proof has the advantage that the figure and construction are the same as those of Theorems 63 and 65. On the other hand some will prefer to retain the traditional proof which follows, because the figure is analogous to that of the corresponding congruence theorem.

Second Proof.

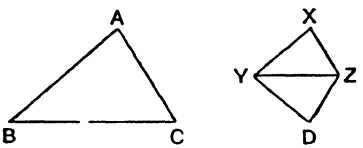


FIG. 234.

Let ABC, XYZ be two triangles in which

$$\frac{AB}{XY}=\frac{AC}{XZ}=\frac{BC}{YZ}.$$

It is required to prove that $\angle A=\angle X$, $\angle B=\angle Y$, $\angle C=\angle Z$.

Construction. On the side of YZ opposite to X, make $\angle ZYD$ equal to $\angle B$ and $\angle YZD$ equal to $\angle C$.

Proof. In the $\triangle$s ABC, DYZ, $\angle B=\angle ZYD$ and $\angle C=\angle YZD$;

$\therefore$ the third angle $A=\angle YDZ$,

$\therefore$ the $\triangle$s $\begin{matrix}ABC\\DYZ\end{matrix}$ are equiangular,

$\therefore$ their corresponding sides are proportional,

$$\therefore \frac{AB}{DY}=\frac{BC}{YZ}=\frac{AC}{DZ}.$$

But
$$\frac{AB}{XY}=\frac{AC}{XZ}=\frac{BC}{YZ}\ (\textit{given}),$$

$$\therefore \frac{AB}{DY}=\frac{AB}{XY} \quad \text{and} \quad \frac{AC}{DZ}=\frac{AC}{XZ},$$

$$\therefore DY=XY \quad \text{and} \quad DZ=XZ;$$

$\therefore$ in the $\triangle$s DYZ, XYZ,

$DY=XY$ (*proved*),

$DZ=XZ$ (*proved*),

$YZ=YZ$ (*common*);

$\therefore$ the $\triangle$s $\begin{matrix}DYZ\\XYZ\end{matrix}$ are congruent (3 *sides*),

$\therefore \angle YDZ=\angle YXZ$, $\angle ZYD=\angle ZYX$ and $\angle YZD=\angle YZX$.

But $\angle A=\angle YDZ$ (*proved*),

and $\angle B=\angle ZYD$, $\angle C=\angle YZD$ (*construction*),

$\therefore \angle A=\angle YXZ$, $\angle B=\angle ZYX$ and $\angle C=\angle YZX$.

THEOREM 65

If two triangles have an angle of the one equal to an angle of the other, and the sides about these equal angles proportional, the triangles are similar.

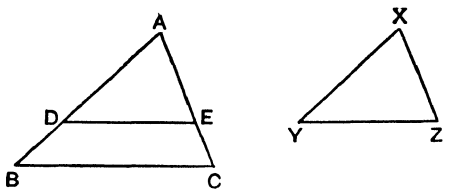


FIG. 235.

Let ABC, XYZ be two triangles, in which $\angle A = \angle X$ and $\frac{AB}{XY} = \frac{AC}{XZ}$.

It is required to prove that the $\triangle$s are similar.

Construction. From AB, AC cut off AD, AE equal to XY, XZ respectively.

Join DE.

Proof. In the $\triangle$s ADE, XYZ,

$AD = XY$ (*construction*),

$AE = XZ$ (*construction*),

$\angle DAE = \angle YXZ$ (*given*);

$\therefore$ the $\triangle$s $\frac{ADE}{XYZ}$ are congruent (2 *sides*, **included** $\angle$);

$\therefore \angle ADE = \angle XYZ$ and $\angle AED = \angle XZY$.

Also $\frac{AB}{XY} = \frac{AC}{XZ}$ (*given*),

$\therefore \frac{AB}{AD} = \frac{AC}{AE}$ (*construction*),

$\therefore$ DE divides the sides AB, AC of the triangle ABC proportionally,

$\therefore$ DE is parallel to BC,

$\therefore \angle ADE = \angle ABC$ and $\angle AED = \angle ACB$.

But $\angle ADE = \angle XYZ$ and $\angle AED = \angle XZY$ (*proved*),

$\therefore \angle ABC = \angle XYZ$ and $\angle ACB = \angle XZY$,

$\therefore$ the triangles $\frac{ABC}{XYZ}$ are equiangular,

$\therefore$ they are similar.

* THEOREM 65a

If two triangles have an angle of one equal to an angle of the other, and the sides about another angle in one proportional to the corresponding sides of the other, then the third angles are either equal or supplementary, and in the former case the triangles are similar.

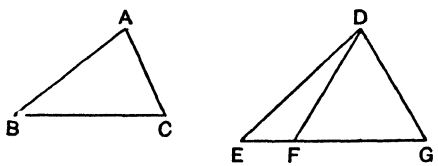


FIG. 236.

Let DEF, ABC be two triangles in which $\angle E = \angle B$ and $\frac{DE}{AB} = \frac{DF}{AC}$.

It is required to prove that $\angle DFE$ is either equal or supplementary to $\angle ACB$, and that, in the former case the $\triangle$s DEF, ABC are similar.

Proof. The $\angle EDF$ is either equal to the $\angle BAC$ or it is not.

If $\angle EDF = \angle BAC$, then the third $\angle$s DFE, ACB must be equal,

$\therefore$ the $\triangle$s $\frac{DEF}{ABC}$ are equiangular, and therefore similar.

If $\angle EDF$ is not equal to $\angle BAC$, let a straight line be drawn through D, on the same side of ED as F, making $\angle EDG = \angle BAC$ and meeting EF (produced, if necessary) at G.

Then in the $\triangle$s EDG, BAC,

$\angle DEG = \angle ABC$ (*given*),

$\angle EDG = \angle BAC$ (*construction*),

$\therefore$ the third $\angle EGD$ = the third $\angle BCA$;

$\therefore$ the $\triangle$s $\frac{EDG}{BAC}$ are equiangular,

$\therefore$ their corresponding sides are proportional,

$$\therefore \frac{DE}{AB} = \frac{DG}{AC}\left(= \frac{EG}{BC}\right).$$

But $$\frac{DE}{AB} = \frac{DF}{AC} \text{ (given)},$$

$$\therefore \frac{DF}{AC} = \frac{DG}{AC},$$

$$\therefore DF = DG,$$

$$\therefore \angle DGF = \angle DFG \text{ (\textit{base} } \angle s \text{ \textit{of an isosceles} } \triangle).$$

Then $\angle DFE$ = the supplement of $\angle DFG$ (*adjacent* $\angle s$; E, F, G *a st. line*)

= the supplement of $\angle DGF$ (*proved*),

= the supplement of $\angle ACB$ (*proved*).

The $\angle$s DFE, ACB are, therefore, either equal or supplementary, and, in the former case, the $\triangle$s DEF, ABC are similar.

EXERCISE 62 [Theorems 63-65A]

Nos. 1-33 are numerical.

1. In a $\triangle ABC$, XY is drawn $\parallel$ to BC, cutting AB, AC at X and Y;
(i) if AB = 4·2 cm., BC = 7·2 cm., AX = 2·8 cm. ; find XY.
(ii) if AX = 4·5, AY = 3·5″, XY = 5·5″, BC = 7·7″ ; find AB, YC.
(iii) if AX = 3″, AB = 5″, AC = 4″, BC = 7″ ; find XY, YC.

2. The sides of a triangular field are 450 yards, 300 yards and 250 yards. In a plan of the field the smallest side measures 1·5″ ; find the lengths of the other sides.

3. A man 6′ high casts a shadow 10′ 3″ long : how high is a pole which casts a shadow 41′ long ?

4. What will be the sides of a $\triangle$ similar to the $\triangle$ whose sides are 2″, 4″, 5″, if its longest side is 30 yards ?

5. A hill ascends 450 feet in one mile. How far up the hill must a man walk to reach a height of 180 feet ?

6. A ladder has 20 rungs and the top rung is 15 feet above the ground. Find the height of the 9th rung.

7. A ladder has 25 rungs and the 4th rung is 1′ 8″ above the ground. Find the height of the top rung.

8. ABC is a $\triangle$ and X a point in AB such that AX : XB = 2 : 3. The line XS, drawn $\parallel$ to BC, meets AC at S. If BC = 15″, calculate the length of XS.

9. AB is a chord of a circle of radius 5″ ; AC is a diameter of the circle. If the distance of AB from the centre is 3″, find the length of BC.

10. ABC is a △ in which AB=2″, AC=1·5″, BC=3″. D is a point on the side of BC remote from A such that ∠CBD=∠ACB, ∠BCD =∠BAC. Find BD and CD.

11. A point O on the side AB of a △ABC is such that AO : OB =3 : 4, and P, Q are points on AC, CB such that OP, OQ are ‖ respectively to BC, AC and OP=1·5″, OQ=1·6″. Find AC and BC.

12. ABC is a △ in which AB=4 cm., AC=6 cm., BC=8 cm. The bisector of the internal vertical angle A of the △ meets BC at N and NK is drawn ‖ to CA to meet BA at K. Find BN, CN, NK.

13 A path 2′ wide runs all round a rectangular lawn, 15 yd. long, 12 yd. wide. Is the rectangle formed by the outer edge of the path the same shape as the lawn ?

14. Are all isosceles △s similar ?

15. AOB, COD are two chords of a circle cutting at any point O inside the circle :

(i) if AO=4 cm., BO=6 cm., CO=5 cm., find DO ;

(ii) if AD=6 cm., AO=7·5 cm., DO=9 cm., OB=3 cm., find CO, BC ;

(iii) if AB=18″, CO=8″, DO=9″, find AO ;

(iv) if AC=BD, and AO=3·5″, BO=2·3″, find CD.

16. ABO, CDO are two chords of a circle which, when produced, meet at O ;

(i) if AO=6 cm., BO=4 cm., CO=5 cm., find DO ;

(ii) if AD=6 cm., AO=7·5 cm., DO=9 cm., BO=3 cm., find CO, BC.

17. A chord BA of a circle when produced cuts the tangent at a point T at X ;

(i) if XA=4″, AB=5″, AT=6″, find XT, BT ;

(ii) if XA=4″, XT=5″, AT=2″, find BX, BT.

18. What will be the height of the image of a man whose height is 5′ 9″ in a photograph taken by a pinhole camera, if he is 12′ in front of the pinhole and the plate is 6″ behind it ?

19. A light is 8 feet above the floor. A pencil 6·6 cm. long is held horizontally and its shadow on the floor is 9·9 cm. long. Find the height of the pencil above the floor.

20. A light is 7′ 6″ above the floor. A pencil 7·2 cm. long is held horizontally 3′ 6″ above the floor. Find the length of its shadow.

21. A cone stands on a base of 6″ diameter and is 12″ high. A section parallel to the base is made 3″ from it. Find the radius of the section.

22. A cone stands on a base of 8″ diameter. A section parallel to the base and 4·5″ from it has its radius 2·5″. Find the height of the cone.

23. AB is ‖ to DC and AC, DB meet at O. If AB=6′, DC=4″, DB=7·5″, find DO.

24. ABC is a △ and P is a point on AB such that AP : PB=3 : 5. A point R on BC is such that BR : RC=1 : 3. Lines PQ, drawn ‖ to BC, and RS, drawn ‖ to BA, cut AC at Q, S respectively ; (i) if AC=16″, calculate SQ, (ii) if PQ cuts RS at V, calculate the ratio RV : VS.

25. The parallel sides AB, DC of a trapezium ABCD are respectively 6″ and 4″ long ; the lengths of the sides AD and BC are 8″ and 7″ respectively. The sides AD and BC are produced to meet at E. Find ED and EC.

26. Draw a st. line OAB, making OA=1″, AB=3″ ; draw any circle through A, B ; and draw the tangent OP, touching it at P. Prove that PB=2PA.

27. The flame of a candle is 2′ 8″ above the ground and 3′ 9″ from a vertical wall. A vertical pole of height 5′ 10″ stands between the candle and the wall. The top of the shadow of the pole on the wall is 10′ 7″ from the ground. Find the distance of the pole from the wall.

28. ABCD is a rectangle, in which AB=8″, BC=6″. From O, the mid-point of the diagonal AC, OE is drawn ⊥ to AC to meet AB at E. Find AE and the area of the △AOE.

29. ABC is a △, ∠B being an acute angle. A point P is taken on AC such that AP : PC=1 : 2. PR, drawn ‖ to AB, meets BC at R, and CQ, drawn ‖ to RP, meets BP produced at Q. RQ meets AC at S. Calculate the ratio RP : CQ and prove that CS=SA.

30. ABC is a △ in which AB=24″, BC=14″, CA=18″. The bisector of ∠BAC meets BC at D. From BA, a part BP is cut off equal to BD, and from CA, a part CQ is cut off equal to CD. Calculate the lengths of AP, AQ, PQ.

31. The internal bisector of the ∠BAC of a △ABC meets BC at R, and RS, drawn ‖ to BA, meets AC at S. If 2AC=AB, find the ratio (i) of AS to SC, (ii) of AB to SR.

32. Two △s DHE, EGF lie on the same side of the straight line DEF. If DH, HE, ED are 2″, 4″, 3″ respectively and EG, GF, FE are 4″, 8″, 6″ respectively, prove that HE is ‖ to GF and calculate where GH produced meets FD produced.

33. ABC is a △ in which AB=2″, AC=1·5″, BC=3″. D is a point on the side of BC remote from A such that ∠DBC=∠CBA and BD=4·5″. Find DC.

34. PQRS is a trapezium, PQ and SR being the parallel sides. PR, QS meet at X. If PQ=2SR, prove that X is a point of trisection of both diagonals.

35. XY is a diameter and XZ is a chord of a circle. P is any point on XY and PQ is drawn ⊥ to XY to meet PZ at Q. Prove that XQ : XP=XY : XZ.

36. TP, TQ are tangents to a circle whose centre is O, OT cuts PQ at K. Prove that OK . OT=OQ².

37. Two circles TPQ, TXY touch internally at T; through T st. lines TPX, TQY are drawn to cut the circles. Prove that

$$TQ \cdot XY = TY \cdot PQ.$$

38. In a △ABC the bisector of the vertical angle A meets the base at D and the circumference of the circumcircle at E; if EC is joined, prove that AB . AC=AE . AD.

39. Two circles intersect at P and Q; at P tangents are drawn, one to each circle, to meet the circumferences at X and Y; if XQ, QY, PQ are joined, prove that QX : QP=QP : QY.

40. The ⊥s AD, BE, CF, drawn from the angular points of a △ABC to the opposite sides, intersect at O and meet the sides at D, E, F respectively. Prove that AO . OD=BO . OE=CO . OF.

41. Two △s ABC, DBC on the same side of the base BC are equal in area; the sides AC, BD cut at E. The line through E ‖ to BC cuts AB at X and DC at Y. Prove that XY is bisected at E.

42. Prove that, if D is the foot of the ⊥ from A to BC, the rectangle AB . AC is equal to the rectangle contained by AD and the diameter of the circle circumscribing the △ABC.

43. PQR is an equilateral △ inscribed in a circle and *any* point S is taken in the arc QR remote from P. QS and RS are produced to meet PR and PQ, produced respectively at H and K. Prove that the △s QRK and RHQ are similar and that KQ . RH=QR².

44. ABC is a △; D is the mid-point of AB and E any point on BC nearer to C than to B. DE and AC produced meet at X; prove that XC : XA=CE : EB. [From C draw a line ‖ to AB.]

45. Two △s ABC, AQR (with a common vertex) are directly similar, and BQ, CR are joined; prove that △ABQ is similar to △ACR.

46. ANB is a st. line; NP is ⊥ to AB and NP²=AN . NB. Prove that ∠APB=90°.

47. In the quads. ABCD, PQRS, the ∠s A, B, C, D are equal respectively to the ∠s P, Q, R, S, and AB, BC are proportional to PQ, QR. Prove that the quads. are similar.

48. AOB, COD are two st. lines. Prove that, if AO=5OB and CO=5OD, then △AOD=△BOC.

49. The base BC of an isosceles △ABC is produced both ways to D, E. Prove that, if AE²=DB . CE, the △s EAC, ADB are equiangular.

50. In a △PQR, PQ=PR=2QR. In the side PQ a point S is taken such that 4QS=QP. Prove that the △QRS is isosceles.

51. Two st. lines POQ, XOY intersect at O; prove that, if OP · OQ=OX : OY, then △POY=△QOX.

52. Two △s ABC, DEF are equiangular to one another; I, J are the centres of the inscribed circles of the triangles and IX, JY are the radii of the inscribed circles. Prove that IX : JY=BC : EF.

53. PQRST is a regular pentagon; QT, PS intersect at X. Prove that PS : PT=PT : TX.

54. ABC is a △, P is any point in AB. Through A on the side AC away from B draw AE ⊥ to AC and equal to it. Through P draw PM ‖ to AC to meet BC at M and draw PK ‖ to AE to meet BE at K. Prove that PK=PM.

55. ABCD is a quad., BE is drawn within the quad. so that ∠CBE=∠ABD, and CE is drawn to meet BE at E so that ∠BCE=∠BDA. Prove that the △s ABE, DBC are similar.

56. A △ABC has AB=AC and BC=AX, AB being divided at X in medial section, i.e. so that AX²=AB . BX. Prove that the △s ABC, CXB are similar.

57. The diagonals AC, BD of a quad. ABCD meet at O, and lines OX, OY are drawn ‖ to CB, CD meeting AB, AD at X, Y. Prove that XY is ‖ to BD, and that XY : BD=AO : AC.

58. A quad. ABCD is inscribed in a circle and the diagonals AC, BD meet at O. Prove that AB . AD : CB . CD=AO : OC.

59. X, Y are points in the sides AB, AC of a △ABC such that XY is ‖ to BC. BY, CX meet at O, and a line through B ‖ to CA meets CX produced at Z. Prove that OX . OZ=OC².

60. ABCD is a quad. in which AB is ‖ to DC; and AC, BD meet at O. Prove that the st. line joining the mid-point of AB to O, if produced, bisects CD.

61. BC, QR are unequal segments of the same st. line; ABC, PQR, △s equiangular to one another on the same side of the line. PA produced meets the line at O. Prove that OB : OC=OQ : OR.

62. ABCD is a square; a variable st. line through A meets BC at X and a point Y is taken on AX such that $AY \cdot AX = AB^2$. Prove that $\angle AYB = 90°$.

63. In a trapezium ABCD, AB, DC are the || sides of length 10″ and 7″ respectively, their ⊥ distance apart being 9″. The sides BC, AD are equal. PQ, drawn || to AB and distant x inches from it, cuts BC, AD at P and Q. Express the length PQ in terms of x. Find the length of the side of the square which could be drawn with two vertices on AB, and the other two on BC, AD respectively.

64. At the mid-point D of the hypotenuse AB of a right-angled △ABC a ⊥ to AB is drawn, meeting AC at E and BC produced at F. Prove (i) $DE \cdot DF = DA^2$, (ii) $BC \cdot BF = 2DE \cdot DF$.

65. OX, OY are two fixed st. lines. (i) P is a variable point on a fixed st. line passing through O and PM, PN are the ⊥s from P to OX, OY, meeting them at M, N respectively. Prove that PM : PN is the same for all positions of P. (ii) P moves so that the ratio of the ⊥s from P to OX, OY is constant. Find the locus of P.

66. A and B are the centres of two circles, and C is the point which divides AB in the ratio of the radii. A st. line through C meets one circle at P and R and the other circle at Q and S, the points being in the order RPCQS. Prove that PQ is divided at C in the ratio of the radii and that the tangents to the circles at P and Q are ||. [Use Theorem 65A.]

EXERCISE 63 [Further Examples on Theorems 63-65A]

1. If two circles intersect at P and the tangents at P form chords PQ, PR, prove that PQ and PR are proportional to the diameters of the circles.

2. ABC is a △ and P is a point in AB. PR, drawn || to AC, meets BC at R; AR and CP intersect at S; and SQ, drawn || to CA, meets AB at Q. Prove AQ : QP = AB : PB.

3. AB, CD are two || st. lines drawn in the same direction. BC, AD meet at O. Through O a st. line is drawn || to AB meeting AC at E and BD at F. Prove that EO = OF.

4. In a △ABC, in which AB = AC, CD is cut off from CA so that CD is a third proportional to AC and BC. Prove that BD = BC.

5. From points A, B st. lines AL and BM are drawn in the same direction ⊥ to AB so that $AL = a$, $BM = b$. If the distance of the point of intersection of AM and BL from AB is x, prove that $\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$. What does the relation become, if BM is drawn in the opposite direction to AL, in the case where BM is less than AL?

6. The side AB of a $\triangle$ is fixed and the length of BC is given. The bisector of $\angle ABC$ meets AC at D, and DO is drawn $\parallel$ to CB to meet AB at O. Prove (*a*) that O is a fixed point; (*b*) that the locus of D is a circle.

7. ABC is a $\triangle$, FAE is a st. line through A $\parallel$ to BC and bisected at A; EB meets AC at H, FC meets AB at G. Prove that GH is $\parallel$ to BC.

8. PQR is a st. line; PL, QM, RN are $\perp$s to it, all on the same side and such that PL=QM. If LR, PN meet at X and MR, QN meet at Y, prove that XY is $\parallel$ to PQR.

9. The $\triangle$ DEM, with sides DE, EM, DM, 2″, 3″, and 4″ respectively, is inscribed in a circle. At E the tangent EN is drawn away from D, 6″ long. Prove that the $\triangle$s EDM, MEN are equiangular and that DM is a tangent to the circle circumscribing $\triangle$ ENM. Also calculate the length of MN.

10. ABC is any $\triangle$; on the line drawn through A $\parallel$ to BC two points Y, Z are taken such that $\angle AYB=\angle AZC=\angle BAC$. Prove that $YA : AZ=AB^2 : AC^2$.

11. Prove that, if the rectangles contained by opposite sides of a quad. are equal, and also one angle is equal to the opposite angle, the diagonal joining the other two angles divides the quad. into two congruent $\triangle$s.

12. ABCD is a quad. such that the diagonal AC bisects the $\angle A$. E is a point in AC such that $\angle ADE=\angle ACB$. Prove that the $\triangle$s DAC, EAB are similar.

13. ABC is a $\triangle$; through A, the line XAY is drawn $\parallel$ to BC and AX=AY. XB and AC, produced if necessary, meet at P, and YC and AB at Q. Prove that QP is $\parallel$ to XY.

14. P is any point on the circumference of a circle described with centre O and radius OC; also A is a point on OC, and B a point on OC produced, such that $OA \cdot OB=OC^2$. Prove that (i) OP touches the circle APB at P, (ii) PC bisects the $\angle APB$, (iii) the ratio PA : PB is constant for all positions of P.

15. The direct common tangent to two circles, whose centres are C and C′, meets CC′ produced at O. CP, C′P′ are $\parallel$ radii of the two circles on the same side of CC′. Prove that O, P, P′ are in a st. line.

16. PQRSTUV is a regular heptagon; QV cuts PR, PS at X, Y respectively. Prove that $PX \cdot PR=PY \cdot PS$.

17. AB, AC are the equal sides of an isosceles $\triangle$ABC, D is the mid-point of BC. Points X, Y are taken on AB, AC produced such that $\angle XDY=\angle DCY$. Prove that the $\triangle$s XBD, XDY and DCY are similar.

18. A point P is taken on the side BC of a rectangle ABCD, and PQ is drawn $\perp$ to AP to meet CD at Q. If AB$=a$, BC$=b$, AP$=c$, PQ$=x$, prove that $(ax-bc)^2=c^2(c^2-a^2)$.

19. ABCD is a quad. ; AB and DC meet at E and AD and BC meet at F. If 2BC=CF and 2DC=CE, show that AD . BE=AB . DF.

20. ABC and A'B'C' are two $\parallel$ st. lines. If AA', BB', CC' are concurrent, prove that AB : A'B'=BC : B'C' and if

$$AB : A'B'=BC : B'C'$$

prove that AA', BB', CC' are either concurrent or parallel.

21. From any point X on the common chord of two circles produced, two st. lines are drawn to cut one of the circles at P, Q and the other at R, S. Prove that $\angle XRQ=\angle XPS$ and that

$$\angle XQS=\angle XRP.$$

22. P, Q, R, S are four consecutive corners of a regular polygon ; PR, QS intersect at X ; prove that QR is a mean proportional between PR and RX.

[Remember that a regular polygon can be inscribed in a circle.]

23. CD, EF are each of them common tangents to two circles whose centres are A and B. C and D are on the same side of AB ; E and F are on opposite sides of AB. CD cuts EF produced at P. Prove that $\angle APB=90°$, and that the rect. PC . PD=the rect. contained by the radii of the circles.

24. The $\angle A$ of a $\triangle ABC$ is obtuse. Points P and Q are taken between B and C on the side BC such that BP . BC$=BA^2$ and CQ . CB$=CA^2$. Prove that AP=AQ.

25. B, C are the points of contact of the tangents from A to a circle. From any point P on the circumference, on the same side of BC as A, $\perp$s PL, PM, PN are drawn to BC, CA, AB. Prove that $PL^2=PM . PN$.

26. If a st. line cuts the sides BC, CA, AB of a $\triangle ABC$, or the sides produced, at X, Y, Z respectively, then

$$BX . CY . AZ=CX . AY . BZ.$$

This is known as Menelaus' Theorem. [Draw $\perp$s AD, BE, CF from A, B, C to XYZ meeting it at D, E, F respectively. Prove that AZ : BZ=AD : BE, etc.]

27. Points Y, Z are taken in the sides CA, AB of a $\angle ABC$, such that AY$=\frac{2}{5}$AC, AZ$=\frac{3}{7}$AB. If YZ, CB are produced to meet at X, prove that BX$=\frac{7}{4}$CB.

28. ABC is a $\triangle$. X is the mid-point of BC, Z is a point on CA such that CZ$=2$ZA. AX, BZ meet at Y. Prove that (i) AY=YX, (ii) BZ$=4$YZ.

29. ABC is a Δ. Z is a point on AB such that AZ : ZB = 3 : 4 . ZC is joined and a point Y is taken on it such that ZY : YC = 5 : 6. AY is joined and produced to cut BC at X. Find (i) the ratio BX : XC, (ii) the ratio AY : YX.

30. ABC and PQR are two similar Δs. D is the mid-point of BC and S is the mid-point of QR. Prove that AB : PQ = AD : PS.

31. Draw any ΔPQR and find a point X within it such that the $\perp$s from X to QR, RP, PQ are in the ratio 1 : 2 : 2. [See Ex. 62, No. 65.]

32. In a circle a diameter AB and a chord AC are drawn ; in AC a point P is taken such that if PQ be drawn $\perp$ to AB, then PQ bisects the $\angle$CPQ. Prove that AQ . QB = AC . PQ.

33. PQRS is a cyclic quad. and QP : PR = RS : SQ. Prove that QP = RS and PR = SQ.

34. AXB, AYC are two Δs such that AX = XB, AY = YC,

$$\angle AXB = \angle AYC,$$

and $\angle$BAY is an acute angle. Prove that Δs AXY, ABC are similar.

35. If A, B, C are three points in order on a st. line, and AP, BQ, CR are drawn $\perp$ to this line on the same side of it, of such lengths that CP, AR intersect on BQ, prove that, if PB, RB are joined, the Δs PAB, RCB are similar.

36. ABC is a Δ right-angled at A, and the altitude AD is produced to E so that DE is a third proportional to AD, DC. Prove that the Δs BDE, ADC are equal in area.

37. D is any point in the base BC of a ΔABC. Prove that the ratio of the radii of the circles circumscribing the Δs ADB, ADC is the same, wherever the point D be taken in the base BC, being always equal to the ratio of AB : AC.

38. A st. line CD is drawn bisecting the right angle of a right-angled ΔABC and cutting the hypotenuse at D ; from D a $\perp$ DE is drawn to the side BC. Prove that $\frac{1}{AC}+\frac{1}{BC}=\frac{1}{DE}$. Show also that, if CF is drawn $\perp$ to AB, AD . FB = DB . CF.

39. A square QRST is described on the base QR of a ΔPQR, and on the side opposite to P. If PS, PT cut QR at X, Y, prove that if a square is described on XY, on the same side of QR as P, its other vertices will lie, one on PQ and the other on PR.

40. DC, BA are two chords of a circle which, when produced, meet at P. PQ is drawn $\parallel$ to AD to meet BC produced at Q. Prove that $PQ^2 = QB \cdot QC$.

THEOREM 66

If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.

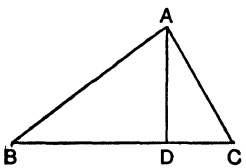


FIG. 237.

Let ABC be a triangle, in which $\angle A$ is a right angle, and let AD be perpendicular to BC, meeting it at D.

It is required to prove that the triangles DBA, DAC are similar to the whole triangle ABC and to one another.

Proof. In the $\triangle$s DBA, ABC,

$\angle BDA = \angle BAC$ (*rt.* $\angle s$, *given*),

$\angle DBA = \angle ABC$ (*the same angle*),

$\therefore$ the third angles DAB, ACB are equal,

$\therefore$ the $\triangle$s $\begin{matrix}DBA\\ABC\end{matrix}$ are equiangular and, therefore, similar.

In the same way it can be proved that the $\triangle$s $\begin{matrix}DAC\\ABC\end{matrix}$ are similar,

$\therefore$ the $\triangle$s $\begin{matrix}DBA\\DAC\end{matrix}$ are similar.

Corollary 1. The square on the perpendicular is equal to the rectangle contained by the segments of the base, i.e. $AD^2 = BD \cdot DC$.

For the $\triangle$s $\begin{matrix}DBA\\DAC\end{matrix}$ are similar,

$\therefore$ their corresponding sides are proportional,

$$\therefore \frac{DA}{DC} = \frac{DB}{DA}\left(= \frac{BA}{AC}\right),$$

$$\therefore DA^2 = DB \cdot DC.$$

Corollary 2. The square on either of the sides containing the right angle is equal to the rectangle contained by the hypotenuse and the segment of the hypotenuse adjacent to that side,

i.e. $BA^2 = BC \cdot BD$ **and** $CA^2 = CB \cdot CD$.

For the $\triangle$s $\begin{matrix} ABC \\ DBA \end{matrix}$ are similar,

$\therefore$ their corresponding sides are proportional,

$$\therefore \frac{BA}{BD} = \frac{BC}{BA}\left(= \frac{AC}{DA}\right),$$

$$\therefore BA^2 = BC \cdot BD.$$

Similarly the $\triangle$s $\begin{matrix} ABC \\ DAC \end{matrix}$ are similar and $CA^2 = CB \cdot CD$.

Corollary 3. The square on the hypotenuse is equal to the sum of the squares on the other two sides, i.e. $BC^2 = BA^2 + CA^2$.

For $BA^2 = BC \cdot BD$ and $CA^2 = CB \cdot CD$,

$$\therefore BA^2 + CA^2 = BC \cdot BD + CB \cdot CD$$
$$= BC(BD + DC) = BC \cdot BC = BC^2.$$

Note. This is an independent proof of Pythagoras' Theorem.

EXERCISE 64 [Theorem 66]

1. In the figure of Theorem 66,
   (i) if $AB = 6''$, $AC = 8''$, find BD, DC, DA;
   (ii) if $BD = 9''$, $DC = 4''$, find AD, AB, AC;
   (iii) if $AD = 8''$, $AB = 17''$, find AC, BC.

2. The line drawn from the intersection of the diagonals of a rhombus $\perp$ to a side divides the side into segments of lengths $4''$ and $1''$. Find the area of the rhombus.

3. ABC is a $\triangle$, right-angled at A. D is the foot of the $\perp$ from A to BC; E, F are the feet of the $\perp$s from D upon AC, AB respectively. Prove that (i) $AD \cdot AE = DE \cdot DB$, (ii) $AC \cdot AE = AB \cdot AF$.

4. PQ is a diameter of a circle, XY a chord $\perp$ to it. PR, any chord through P, cuts XY at S. Prove that $PS \cdot PR = PY^2$.

5. PQ is a st. line divided internally at R. Show how to draw through R a st. line so that the sq. of the ⊥ upon it from P may be equal to the rectangle QP . QR. When is this impossible ?

6. XOY is a diameter of a circle, centre O. At X, Y tangents to the circle are drawn. These are cut at Q, R by the tangent at any other point P on the circle. Prove that (i) $\angle QOR = 90°$, (ii) $QP \cdot PR = OP^2$.

7. From any variable point X on the circumference of a given circle a ⊥ XK is drawn to the tangent at a fixed point L, meeting it at K. Prove that $XL^2 \div XK$ is constant.

THEOREM 67

The areas of similar triangles are proportional to the squares on corresponding sides.

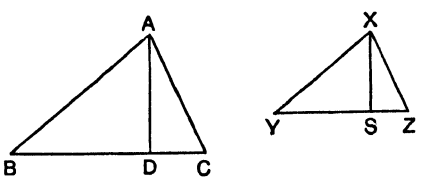


FIG. 238.

Let ABC, XYZ be two similar triangles.

It is required to prove that $\frac{\triangle ABC}{\triangle XYZ} = \frac{BC^2}{YZ^2}$.

Construction. Draw the perpendiculars from A, X to BC, YZ meeting them at D, S respectively.

Proof. .Since the area of a triangle is measured by one-half the product of the measures of its base and altitude,

$$\therefore \quad \triangle ABC = \tfrac{1}{2}BC \cdot AD,$$

and

$$\triangle XYZ = \tfrac{1}{2}YZ \cdot XS;$$

$$\therefore \quad \frac{\triangle ABC}{\triangle XYZ} = \frac{\tfrac{1}{2}BC \cdot AD}{\tfrac{1}{2}YZ \cdot XS} = \frac{BC \cdot AD}{YZ \cdot XS}.$$

But, in the Δs ABD, XYS,

$$\angle ABD = \angle XYS \left(\textit{since the } \Delta s \frac{ABC}{XYZ} \textit{ are given similar}\right),$$

$$\angle ADB = \angle XSY \ (\textit{rt. } \angle s, \textit{ construction}),$$

$$\therefore \text{ the third } \angle BAD = \text{the third } \angle YXS;$$

$$\therefore \text{ the } \Delta s \frac{ABD}{XYS} \text{ are equiangular,}$$

$$\therefore \text{ their corresponding sides are proportional,}$$

$$\therefore \frac{AD}{XS} = \frac{AB}{XY} \left(= \frac{BD}{YS}\right).$$

But $\frac{AB}{XY} = \frac{BC}{YZ}$, since the Δs $\frac{ABC}{XYZ}$ are given similar,

$$\therefore \frac{AD}{XS} = \frac{BC}{YZ};$$

$$\therefore \frac{\Delta ABC}{\Delta XYZ} = \frac{BC}{YZ} \cdot \frac{BC}{YZ} = \frac{BC^2}{YZ^2}.$$

Note 1. The following result, which is sometimes useful, may be proved in a similar manner :

If two triangles have one angle of one equal to one angle of the other, their areas are proportional to the rectangles contained by the sides about the equal angles.

The proof is identical, except for the omission of one step.

Note 2. The following facts are sometimes useful :

(i) **The ratio of the volumes of similar solids is equal to the ratio of the cubes of corresponding lengths.**

(ii) **The ratio of the surfaces of similar solids is equal to the ratio of the squares of corresponding lengths.**

EXERCISE 65 [Theorem 67]

1. Two similar Δs have corresponding sides 1·8″ and 2·7″ respectively. The area of the first is 3·2 sq. in. Find the area of the second.

2. The areas of two similar Δs are 80 and 125 sq. cm. If one side of the first is 3·2 cm., find the corresponding side of the second.

3. The sides of a $\triangle$ are 2·3″, 3·7″, 4·1″. Find the sides of a similar $\triangle$ whose area is sixteen times as great.

4. The area of the $\triangle ABC$ is 102·4 sq. in., and PQ drawn $\parallel$ to BC cuts AB at P and AC at Q. If AP : PB = 5 : 3, find the area of the $\triangle APQ$.

5. ABC and PQR are two similar $\triangle$s whose areas are respectively 45 sq. in. and 151·25 sq. in. If PQ = 8·8″, find AB.

6. In two $\triangle$s ABC, PQR, $\angle A = \angle P$. If AB, AC are 3·6″ and 2·1″ respectively and PQ, QR are 1·4″ and 2·7″, prove that

$$\triangle ABC = 2\triangle PQR.$$

7. ABC is a $\triangle$ having $\angle A = 90°$, and D is the foot of the $\perp$ from A on BC. If BD = 4DC, find the ratio AB : AC.

8. The sides BC, CA, AB of a $\triangle ABC$ are divided at X, Y, Z so that BX : XC = CY : YA = AZ : ZB = 2 : 1 ; calculate the ratio of the area of the $\triangle AYZ$ to the area of the $\triangle ABC$; and prove that $3\triangle XYZ = \triangle ABC$.

9. Two $\triangle$s are equiangular. The area of one is 8 sq. in. and one side is 2″ long. What is the length of the corresponding side of the other $\triangle$ if its area is 50 sq. in. ?

10. Through each of two opposite corners of a rectangle $\perp$s are drawn to the diagonal which joins the other two corners. Prove that, if the three parts, into which the diagonal is thus divided, are equal, the squares on the sides of the rectangle are in the ratio 2 : 1.

11. The side AB of a $\triangle ABC$ is trisected at P and Q ; the lines drawn through P, Q $\parallel$ to BC cut AC at R and S respectively. Prove that the area of PQRS is one third that of $\triangle ABC$.

12. AOB, COD are intersecting chords of a circle ; if AO = 9″, OB = 4″, OD = 8″, find the ratio $\triangle AOD : \triangle BOC$.

13. ABC is a $\triangle$ in which AB : AC = 5 : 4, BC = 2·5″ and D is a point in AB such that $\angle ACD = \angle ABC$. Find (i) CD, (ii) the ratio $\triangle ABC : \triangle ACD$.

14. Each side of a $\triangle ABC$ is divided into five equal parts. The first and fourth points of division on BC are L, M ; the first and fourth on CA are P, Q ; the first and fourth on AB are R, S. Find the ratio of the area of the hexagon LMPQRS to the area of the $\triangle ABC$.

15. E is the point in the side AC of the $\triangle ABC$, such that

$$AE \cdot EC = 1 : 4.$$

EF is drawn $\parallel$ to CB, meeting AB at F ; and ED is drawn $\parallel$ to AB, meeting BC at D. Show that the area of the parallelogram BDEF is $\frac{8}{25}$ of the area of the $\triangle ABC$.

16. The bisectors of the ∠s A and B of a △ABC meet the opposite sides at D and E respectively; if BC=9″, CA=8″, AB=7″, prove that △CDE : △ABC=3 : 10.

17. A light is 8′ above the ground. A picture measuring 9″ by 6″ is held 5′ above the ground. What is the area of its shadow?

18. A cylindrical tin 6″ high holds ½ lb. of cocoa; how much cocoa will a similar tin 4″ high hold?

19. It cost £1 5s. 0d. to whitewash a room 10 feet long. What will it cost to whitewash a room similar in all respects and 12 feet long?

20. It cost 16 shillings to paint the outside of a chest 8 feet long. What will it cost to paint the outside of a similar chest 6 feet long?

21. A bottle 8″ high is full of milk and the price of the milk is 6d. What should be the price of the milk which will fill a similar bottle 12″ high?

22. The weight of a cricket ball, 4″ in diameter, is 5 oz. What would be the weight of a similar ball 3½″ in diameter?

23. A trunk 10″ high has a volume of 375 cu. in. What is the height of a similar trunk whose volume is 648 cu. in.?

24. Two circles have their centres A and B 13″ apart, their radii being respectively 11″ and 6″. A common tangent touches the larger circle at C and the smaller at D, and meets the st. line AB produced at E. Find the areas of ABCD and DBE.

25. ABC is a △ right-angled at B. D is the foot of the ⊥ from B to AC, and 3AD=5DC. Prove that $3AB^2=5BC^2$.

26. An equilateral △ is described on one side of a square and another equilateral △ is described on a diagonal of the same square. Find the ratio of the areas of the two △s.

27. The base YZ of a △XYZ is divided internally at P in the ratio 3 : 4, PX is divided internally at S in the ratio 3 : 2, and SQ, SR are drawn ‖ to XY, XZ respectively to meet YZ at Q, R. Find the ratio △SQR : △XYZ. What part of the data is irrelevant?

28. Two unequal circles intersect at O and Z. Straight lines AOP, BOS, COT through O cut one circle at A, B, C and the other at P, S, T. Prove that $\triangle ABC : \triangle PST = AB^2 : PS^2$.

29. If X be the mid-point of the side AC of a △ABC, and Z be taken between X and C, so that AZ is a mean proportional between AX and AC, prove that a line through Z ‖ to BC bisects the area of the △.

30. Prove that the areas of similar triangles have the same ratio as the squares of (i) corresponding medians; (ii) the radii of their circum-circles.

EXERCISE 66 [Further Examples on Theorems 66-67]

1. From a point B on the circumference of a semi-circle ABC a $\perp$ BD is drawn to the diameter AC ; the $\angle$ADB is bisected by DE, meeting the chord AB at E and the $\angle$BDC by DF, meeting the chord BC at F. Prove that AE : EB = BF : FC = ED : DF.

2. PQ is a chord of a circle, centre O. The tangents at P, Q to the circle meet at T and TO is joined, cutting the circle at A and B and the chord PQ at N. Prove that $\frac{2}{TN} = \frac{1}{TA} + \frac{1}{TB}$.

3. A chord AB of a circle with centre O is bisected at C, and XY is any other chord drawn through C ; the tangents to the circle at A, B meet at P. Prove that (i) $OX^2 = OC \cdot OP$,

(ii) $\angle APX = \angle BPY$.

4. ABC is a $\triangle$ in which $\angle B > \angle C$; BD is drawn, meeting AC at D, so that $\angle ABD = \angle C$. Prove that $AD : AC = BD^2 : BC^2$.

5. O is the centre of a circle, OPQ any st. line through O ; P, Q two points on it, on the same side of O, such that

$$OP \cdot OQ = OR^2,$$

where R is any point on the circle. Prove that

$$\triangle OPR : \triangle ORQ = PR^2 : RQ^2.$$

6. In a $\triangle$ABC, right-angled at A, AN is drawn $\perp$ to BC, meeting it at N. The internal bisector of $\angle$BAC meets BC at D. Prove that $BD^2 : DC^2 = BN : NC$.

7. P is a point on a circle of which AB is a diameter. AP produced meets the tangent at B in T. Prove that, if C is the foot of the $\perp$ from P on AB, $AB^2 : BT^2 = AC : CB$.

8. The tangents to a circle at P and Q meet at T and C is the centre of the circle. TC meets the circle at A and B, and meets PQ at X. Prove that AX : XB = TA : TB.

9. AB is the chord of a segment of a circle containing an angle of 45°. From any point P, on the arc of the segment, st. lines PA, PB are drawn, cutting again the circle described on AB as diameter at L and M. Prove that $\triangle PAB = 2\triangle PLM$.

10. ABC is a $\triangle$ in which AB > AC. On AB and AC produced are cut off lengths AT, AS, each equal to the mean proportional between AB and AC. Prove that the $\triangle$s ATS, ABC are equal in area and that, if ST cuts BC at N, the ratios in which ST, BC are divided at N are equal.

11. Regular hexagons are inscribed in and circumscribed about the same circle. Find the ratio of their areas.

12. AB is a diameter of a circle, and C is a point on the circumference ; F is the foot of the $\perp$ from B to the tangent at C. Prove that the $\triangle$s BFC, BCA are similar. In AB a point D is taken such that AD=AC and DE, drawn $\parallel$ to BC, meets AC at E. Prove that the $\triangle$BFC is equal in area to the quad. BDEC.

THEOREM 58. [ALTERNATIVE PROOF]

If two chords of a circle intersect at a point within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

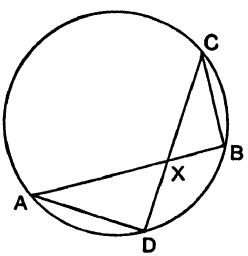


FIG. 239.

Let AB, CD be chords of a circle intersecting at a point X inside the circle.

It is required to prove that AX . XB = CX . XD.

Construction. Join AD, BC.

Proof. In the $\triangle$s AXD, CXB,

$\angle AXD = \angle CXB$ (*vertically opposite* $\angle$*s*),

$\angle XAD = \angle XCB$ ($\angle$*s in the same segment*),

$\therefore$ the third $\angle XDA$ = the third $\angle XBC$;

$\therefore$ the $\triangle$s $\frac{AXD}{CXB}$ are equiangular,

$\therefore$ their corresponding sides are proportional,

$\therefore \frac{AX}{CX} = \frac{XD}{XB} \left(= \frac{AD}{CB} \right)$;

$\therefore$ AX . XB = CX . XD.

THEOREM 59. [ALTERNATIVE PROOF]

If two chords of a circle, when produced, cut at a point outside it, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other. Also each rectangle is equal to the square on the tangent from the point of intersection.

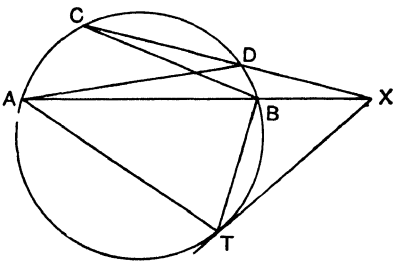


FIG. 240

Let AB, CD be two chords of a circle meeting, when produced, at a point X, outside the circle.

Let XT be the tangent drawn to the circle from X.

It is required to prove that

(1) $XA \cdot XB = XC \cdot XD$,

(2) $XA \cdot XB = XT^2$.

Construction. Join AD, BC, AT, BT.

Proof. (1) In the $\triangle$s XBC, XDA,

$\angle BXC = \angle DXA$ (*the same* $\angle$),

$\angle BCX = \angle DAX$ ($\angle$*s in the same segment*),

$\therefore$ the third $\angle XBC$ = the third $\angle XDA$;

$\therefore$ the $\triangle$s $\begin{matrix} XBC \\ XDA \end{matrix}$ are equiangular,

$\therefore$ their corresponding sides are proportional,

$\therefore \frac{XB}{XD} = \frac{XC}{XA} \left(= \frac{BC}{DA} \right)$;

$\therefore XA \cdot XB = XC \cdot XD$.

(2) In the $\triangle$s XBT, XTA,

$\angle BXT = \angle TXA$ (*the same* $\angle$),

$\angle BTX = \angle TAX$ (*in the alternate segment*),

$\therefore$ the third $\angle XBT$ = the third $\angle XTA$;

$\therefore$ the $\triangle$s $\frac{XBT}{XTA}$ are equiangular,

$\therefore$ their corresponding sides are proportional,

$$\therefore \frac{XB}{XT} = \frac{XT}{XA}\left(= \frac{BT}{TA}\right);$$

$$\therefore XA \,.\, XB = XT^2.$$

Examples on these theorems will be found on pp. 302-308.

FURTHER CONSTRUCTIONS AND LOCI

CONSTRUCTION 28

To construct a fourth proportional to three given straight lines.

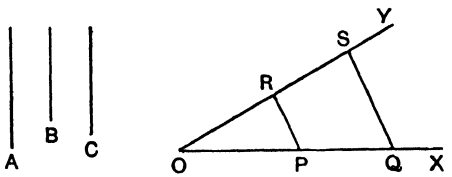


FIG. 241.

Let A, B, C be the three given straight lines.

It is required to construct a fourth proportional to A, B, C.

Construction. Draw two straight lines OX, OY, containing any angle. From OX cut off OP equal to A, and PQ equal to B. From OY cut off OR equal to C. Join PR. Through Q draw a line parallel to PR, meeting OY at S.

Then RS is the fourth proportional to A, B, C.

Proof. In the $\triangle OSQ$, PR is by construction parallel to QS,

$\therefore$ it divides the sides OQ, OS proportionally,

$\therefore$ OP : PQ = OR : RS, i.e. A : B = C : RS (*construction*),

$\therefore$ RS is the fourth proportional to A, B, C.

Note. An alternative construction is to make OP = A, OQ = B, OR = C. Then OS is the fourth proportional to A, B, C. This has the advantage of requiring less space and, if drawing to scale is necessary, a larger scale can be taken and a more accurate result obtained. It is usually found, however, that beginners find the method given in the text easier to understand.

CONSTRUCTION 29

To construct a third proportional to three given straight lines.

Let A, B be the two lines to which the third proportional is required.

Then we have to construct a fourth proportional to A, B, B.

Proceed as in Construction 28.

CONSTRUCTION 30

To divide a given straight line internally and externally in a given ratio.

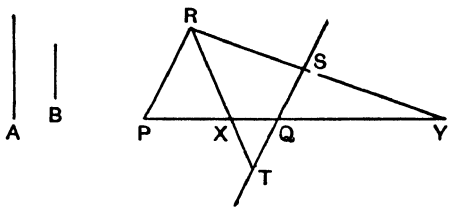


FIG. 242.

Let A, B be two lengths which are in the given ratio, and PQ the given straight line.

It is required to divide PQ internally and externally in the ratio of A to B.

Construction. Through P draw any st. line PR of length A. Through Q draw a line parallel to PR and on it mark off QS equal to B. Produce SQ to T making QT equal to B. Join RS and produce it to meet PQ produced at Y. Join RT, meeting PQ at X. Then X divides PQ internally and Y divides PQ externally in the ratio of A to B.

Proof. Since QS is, by construction, parallel to PR, and PQY, RSY cut them,

$$\therefore \angle RPY = \angle SQY \text{ and } \angle PRY = \angle QSY,$$

$$\therefore \text{ the } \triangle s \begin{matrix} PRY \\ QSY \end{matrix} \text{ are equiangular,}$$

$$\therefore \text{ their corresponding sides are proportional,}$$

$$\therefore PY : QY = PR : QS = A : B \text{ (\textit{construction}),}$$

i.e. Y divides PQ externally in the ratio of A to B.

Similarly, the $\triangle s \begin{matrix} PRX \\ QTX \end{matrix}$ are equiangular,

$$\therefore \text{ their corresponding sides are proportional,}$$

$$\therefore PX : QX = PR : QT = A : B \text{ (\textit{construction}),}$$

i.e. X divides PQ internally in the ratio of A to B.

Note. PR may make any angle with PQ, but if squared paper is available it is convenient to make the $\angle RPQ$ a right angle.

CONSTRUCTION 31

To construct the mean proportional between two given straight lines.

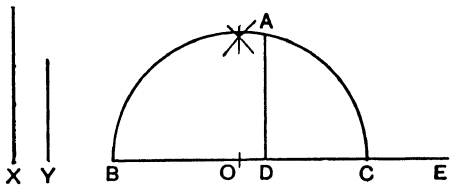


FIG. 43.

Let X, Y be the given straight lines.

It is required to construct a mean proportional between X and Y.

Construction. Draw any straight line BE, and from it cut off BD = X and DC = Y. On BC as diameter describe a semi-circle.

At D draw a st. line ⊥ to BC to meet the semi-circle at A.

Then AD is the mean proportional between X and Y.

This is Theorem 66, Cor. 1.

Note 1. An alternative construction is to make BC = X, BD = Y. Draw the semi-circle on BC as diameter, and let the ⊥ to BC at D meet the semi-circle at A. Then BA is the mean proportional between BD and BC, i.e. between X and Y (Theorem 66, Cor. 2). This has the advantage of requiring less space and, if drawing to scale is necessary, a larger scale can be taken and a more accurate result obtained. It is usually found, however, that beginners find the method given in the text easier to understand.

Note 2. By using this construction it is possible to construct a line representing the square root of any number, e.g. to construct a line representing $\sqrt{33}$; we notice that $33 = 5{\cdot}5 \times 6$, and

it is only necessary to construct a mean proportional between 5·5 and 6. It is usual to choose factors as nearly equal as is convenient in order to avoid taking up too much space, e.g. it is better to find the mean proportional between 5·5 and 6 (or 5 and 6·6) than between 1 and 33.

EXERCISE 67 [Constructions 28-31]

1. Construct a fourth proportional to (i) 4·2″, 3″, 2·8″, (ii) 6 cm., 3·6 cm., 3 cm., (iii) 13·5, 6, 9.

2. Construct a third proportional to (i) 5 cm. and 4 cm., (ii) 6·3 and 4·2.

3. Construct graphically lengths equal to

(i) $\frac{1\cdot4\times1\cdot8}{1\cdot2}$, (ii) $\frac{1\cdot7}{1\cdot1}$, (iii) $\frac{0\cdot9^2}{1\cdot5}$, (iv) $\frac{1}{2\cdot7}$.

4. Divide a st. line 2·1 cm. long internally and externally in the ratio 3 : 4. Measure the segments.

5. Divide a st. line 4·8″ long internally and externally in the ratio 3 : 1. Measure the segments.

6. Construct a mean proportional between (i) 3·6″ and 2·5″, (ii) 5·5″ and 2″.

7. Find graphically the approximate values of

(i) $\sqrt{13}$, (ii) $\sqrt{23}$, (iii) $\sqrt{58}$.

8. Draw a st. line 5·5″ long and divide it into three parts which are proportional to 5, 4, 2.

9. Divide a given st. line internally into two parts so that the square on one part is five times the square on the other part.

10. AB is a st. line 7″ long. Construct a △ABC such that AC=8″, BC=6″. Construct the internal and external bisectors of the ∠ACB, and let them meet AB and AB produced at X, Y respectively. Prove that X, Y divide AB internally and externally in the ratio 4 : 3. **Deduce an alternative construction for dividing a given st. line in the ratio *a* to *b*.**

11. Construct a △ given the perimeter and the ratio of the three sides.

* CONSTRUCTION 32

On a given straight line PQ, draw a polygon, similar to a given polygon ABCDE, so that PQ and AB may be corresponding sides.

First Method.

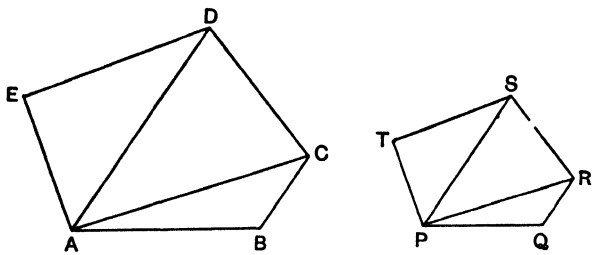


FIG. 244.

Construction. Join AC, AD. Make the ∠s QPR, PQR equal to the ∠s BAC, ABC respectively, thus obtaining the △PQR. (Construction 5.)

Make the ∠s RPS, PRS equal to the ∠s CAD, ACD respectively, thus obtaining the △PRS.

Make the ∠s SPT, PST equal to the ∠s DAE, ADE respectively thus obtaining the △PST. Then PQRST is the required polygon.

Proof. By construction, the polygons ABCDE, PQRST are equiangular.

Also, the △s $\frac{ABC}{PQR}$ are equiangular,

∴ their corresponding sides are proportional,

∴ AB : PQ = BC : QR = AC : PR. .....(1)

But the △s $\frac{ACD}{PRS}$ are equiangular,

∴ their corresponding sides are proportional,

∴ AC : PR = CD : RS = AD : PS. .....(2)

∴ From (1) and (2), BC : QR = CD : RS.

Also, the Δs $\begin{matrix}ADE\\PST\end{matrix}$ are equiangular,

$\therefore$ AD : PS = DE : ST = EA : TP. .....(3)

$\therefore$ From (2) and (3), CD : RS = DE : ST = EA : TP;

$\therefore$ the sides of the polygons ABCDE, PQRST about any pair of equal angles are proportional, and, since the polygons are also equiangular, they are similar.

Note. If it is merely required to construct a polygon similar to a given polygon ABCDE, so that the side corresponding to AB may be of a given length, it is easier to proceed as follows. From AB cut off AP equal to the given length. Through P draw PQ $\parallel$ to BC, cutting AC at Q. Through Q draw QR $\parallel$ to CD, cutting AD at R. Through R draw RS $\parallel$ to DE, cutting AE at S. Then APQRS is the required polygon.

Second Method.

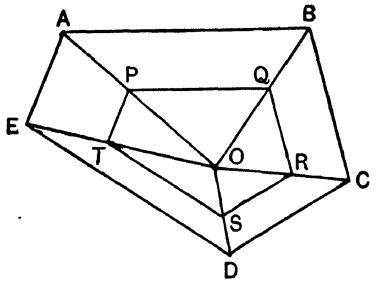


FIG. 245.

Construction. Draw a line PQ || to AB and equal to the given length.

Let AP, BQ meet at O. Join OC, OD, OE.

Draw QR || to BC meeting OC at R, RS || to CD meeting OD at S, ST || to DE meeting OE at T. Join TP. Then PQRST is the required polygon.

Outline of Proof. Since PQ, QR, RS, ST are respectively || to AB, BC, CD, DE,

$$\therefore \text{OP} : \text{OA} = \text{OQ} : \text{OB} = \text{OR} : \text{OC} = \text{OS} : \text{OD} = \text{OT} : \text{OE},$$

$$\therefore \text{PT is } \| \text{ to AE},$$

$$\therefore \text{the polygons are equiangular.}$$

Also the $\triangle$s OPQ, OAB ; OQR, OBC ; ORS, OCD ; OST, ODE ; OTP, OEA ; are similar,

$$\therefore \text{their corresponding sides are proportional,}$$

$$\therefore \text{PQ} : \text{AB} = \text{OQ} : \text{OB} = \text{QR} : \text{BC, etc.,}$$

$\therefore$ the sides of the polygons ABCDE, PQRST about any pair of equal angles are proportional ;

$$\therefore \text{the polygons are similar.}$$

* CONSTRUCTION 33

To construct a figure similar to a given rectilineal figure, and equal to a given fraction of it in area.

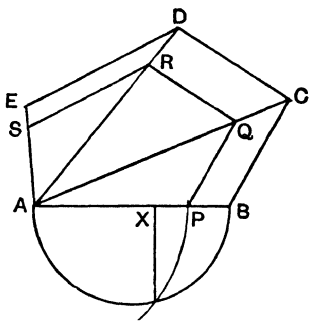


FIG. 246.

Let ABCDE be the given figure, to which a similar figure is to be constructed, having its area a given fraction (say five-eighths) of that of ABCDE.

Construction. Make AX five-eighths of AB. (Construction 12.)

Construct AP the mean proportional between AB and AX. (Construction 31, Note 1.)

On AP construct APQRS similar to ABCDE. (Construction 32.)

Then APQRS is the required figure.

Outline of Proof. By construction $AP^2 = AX \cdot AB$ and $AX = \frac{5}{8}AB$.

By construction the $\triangle$s APQ, ABC are similar,

$$\therefore \frac{\triangle APQ}{\triangle ABC} = \frac{AP^2}{AB^2} = \frac{AX \cdot AB}{AB^2} = \frac{AX}{AB} = \frac{5}{8},$$

$$\therefore \triangle APQ = \tfrac{5}{8}\triangle ABC.$$

Similarly, it may be proved that

$$\triangle AQR = \tfrac{5}{8}\triangle ACD \quad \text{and} \quad \triangle ARS = \tfrac{5}{8}\triangle ADE,$$

$\therefore$ by addition,

$$\triangle APQ + \triangle AQR + \triangle ARS = \tfrac{5}{8}(\triangle ABC + \triangle ACD + \triangle ADE),$$

i.e.

$$APQRS = \tfrac{5}{8}ABCDE.$$

EXERCISE 68 [Constructions 32, 33]

1. Draw any quad. ABCD and then draw a similar quad. PQRS such that (i) $PQ=\frac{3}{4}AB$, (ii) $PQ=\frac{4}{3}AB$.

2. Draw any pentagon ABCDE and then draw a similar pentagon PQRST such that (i) $PQ=\frac{3}{5}AB$, (ii) $PQ=\frac{6}{5}AB$.

3. **Construct a $\triangle ABC$ such that $AB : AC = 4 : 3$, $\angle BAC = 50°$, $BC = 10$ cm.**

[Draw a $\triangle BDE$ such that $DB = 4$ cm., $DE = 3$ cm., $\angle BDE = 50°$. On BE (produced, if necessary) take a point C such that $BC = 10$ cm. Through C draw CA $\parallel$ to ED to meet BD (produced, if necessary) at A. ABC is the required $\triangle$.]

4. Construct a $\triangle ABC$ in which $BC = 2''$, $\angle BAC = 60°$ and

$$AB : AC = 3 : 2$$

5. Construct a rectangle equal in area to a given square and having its sides in the ratio 3 : 5.

[Let ABCD be the given square. Take points P, Q on BC and DC produced such that $PC : CQ = 3 : 5$. Draw a semi-circle on PQ as diameter cutting CB or CB produced at R. Join RP, RQ. Draw BX, BY $\parallel$ to RP, RQ cutting DC (produced either way, if necessary) at X, Y respectively. Then CX, CY are the sides of the required rectangle.]

6. In a $\triangle ABC$, whose base BC is $1{\cdot}9''$ and whose base angles B, C are 68° and 66°, inscribe an equilateral $\triangle$ such that the side which meets AB and AC is inclined to AB at an angle of 35°.

[At any point R in BA, draw a st. line RP making an angle of 35° with AB and meeting BC (produced, if necessary) at P. On RP describe an equilateral $\triangle RPQ$, B and Q being on opposite sides of RP. Join BQ and let it (produced, if necessary) cut AC at Y. Through Y draw YX, YZ $\parallel$ to QP, QR, meeting BC, BA at X, Z respectively. XYZ is the required $\triangle$.]

7. In a given $\triangle ABC$ inscribe a $\triangle$ similar to a given $\triangle DEF$.

8. Construct a square EFGH, with two of its vertices E, F on the side BC of a given $\triangle ABC$ and with G on CA and H on AB. [Such a square is said to be inscribed in the triangle.]

[On BC and on the opposite side of it to A, construct a square BCSR. Join AR, AS meeting BC at E, F. EF is the side of the required square.

OR, Let Q be the foot of the $\perp$ from A to BC. Draw AP $\perp$ to AQ and equal to it, P and B being on opposite sides of AQ. Join B, P, cutting AC at G. Draw GH $\parallel$ to CB, meeting AB at H. Then GH is a side of the required square.]

9. Given a $\triangle ABC$, construct a line $\parallel$ to BC to cut off a $\triangle$ of three-sevenths of the area of the $\triangle$ ABC.

10. Given a $\triangle ABC$, construct a line $\parallel$ to BC to cut off a trapezium of four-fifths of the area of the $\triangle ABC$.

11. Given a $\triangle ABC$, construct a st. line $\perp$ to BC to cut off a $\triangle$ of one-third of the area of the $\triangle ABC$.

[Suppose that PQ is the required line, P being on AB, Q on BC. Let Y be a point on BC such that $3BY = BC$. Then $\triangle ABY = \triangle BPQ$. Draw YZ $\perp$ to BC to meet AB at Z. Let X be the foot of the $\perp$ from A to BC. Then $\triangle ABY : \triangle ABX = BY : BX = BZ : BA$. Also $\triangle BPQ : \triangle ABX = BP^2 : BA^2$. But $\triangle ABY = \triangle BPQ$;

$$\therefore\ BZ : BA = BP^2 : BA^2;\quad \therefore\ BP^2 = BZ \cdot BA,$$

i.e. BP is a mean proportional between BZ and BA, i.e. the position of the point P can be found by construction.]

12. Construct a rectilineal figure equal in area to a given rectilineal figure A and similar to a given rectilineal figure B.

[Construct squares P, Q respectively equal in area to A, B (Construction 27, Cor.) Let the sides of these squares be p, q respectively, and let b be one of the sides of B.

Find a fourth proportional x to q, p, b (Construction 28), so that $q : p = b : x$.

On x draw a figure X similar to the figure B, so that x and b are corresponding sides. Then X is the figure required; for

$$\frac{X}{B} = \frac{x^2}{b^2} = \frac{p^2}{q^2} = \frac{P}{Q} = \frac{A}{B},$$

$$\therefore\ X = A.]$$

13. Construct an equilateral $\triangle$ equal in area to a given $\triangle ABC$.

14. Construct geometrically an isosceles $\triangle$ whose sides are in the ratio 2 : 2 : 1 equal in area to a square whose side is 2″.

15. Construct a regular hexagon of area 25 sq. cm.

16. Construct a $\triangle$ similar to a $\triangle$ with sides 4·2″, 4·7″, 5·3″ and having an area of 6·25 sq. in.

SOME IMPORTANT LOCI

* THEOREM G

The Circle of Apollonius

To find the locus of a point which moves so that its distances from two fixed points are in a constant ratio.

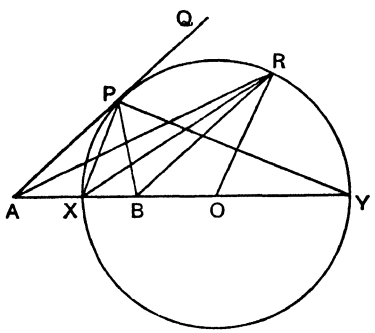


FIG. 247.

Let A, B be the two fixed points.

It is required to find the locus of a point P which moves so that PA : PB is constant.

1. Let X, Y be the points which divide AB internally and externally in the given ratio. Then X, Y are two **fixed** points on the required locus. Let P be **any** other point such that PA : PB = the given ratio. Join AP, PB, PX, PY and produce AP to Q.

Then $PA : PB = AX : XB$,

$\therefore$ PX is the bisector of the $\angle APB$.

Also $PA : PB = AY : YB$,

$\therefore$ PY is the bisector of the $\angle BPQ$,

i.e. PX and PY are the bisectors of adjacent supplementary angles;

$\therefore \angle XPY = 90°$.

But X, Y are fixed points,

$\therefore$ P lies on the circle on XY as diameter, i.e. on a fixed circle.

2. Let R be any other point on this circle, and let O be the centre of the circle. Join RA, RX, RB, RO.

Let $OA = a$ units, $OB = b$ units, $OX = OY = OR = r$ units.

By construction, $AX : XB = AY : YB$,

$$\therefore \frac{a-r}{r-b} = \frac{a+r}{r+b}$$

or

$$\frac{a-r}{a+r} = \frac{r-b}{r+b},$$

$$\therefore \frac{a}{r} = \frac{r}{b} \text{ (see Converse of Ex. 58, 2 (vii))},$$

$$\text{i.e. } \frac{OA}{OR} = \frac{OR}{OB},$$

and $\angle AOR$ is the $\angle ROB$,

$$\therefore \text{ the } \triangle s \frac{AOR}{ROB} \text{ are similar,}$$

$$\therefore \angle OAR = \angle ORB.$$

But $OX = OR$; $\therefore \angle OXR = \angle ORX$,

$$\therefore \angle OXR - \angle OAR = \angle ORX - \angle ORB = \angle BRX.$$

But $\angle OXR - \angle OAR = \angle XRA$, since $\angle OXR$ is the exterior angle of the $\triangle AXR$,

$$\therefore \angle XRA = \angle BRX, \text{ i.e. } RX \text{ bisects } \angle ARB,$$

$$\therefore AR : RB = AX : XB,$$

i.e. R is a point on the required locus.

We have therefore proved that the locus of a point which moves so that its distances from two fixed points are in a constant ratio is a fixed circle. This is called the circle of Apollonius.

* THEOREM H

O is a fixed point and P moves along a fixed st. line. On OP (produced, if necessary) a point Q is taken so that OQ : OP is a constant ratio, k. It is required to find the locus of Q.

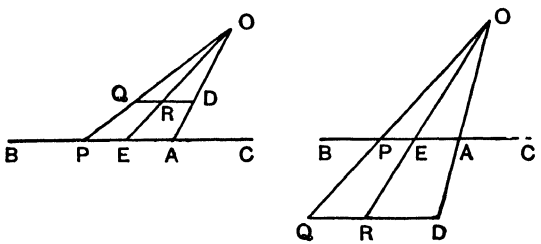


FIG. 248.

1. Let BC be the fixed st. line along which P moves. Let A be a **fixed** point on BC. On OA (produced, if necessary), let a point D be taken so that OD : OA is equal to k. Then D is a **fixed** point on the locus. Let P be **any** other point on BC, and let Q be a point on OP (produced, if necessary) such that

$$OQ : OP = k.$$

Then Q is **any** point on the locus. Join QD.

Then $OQ : OP = OD : OA$ (each $= k$),

i.e. in the $\triangle OAP$ the sides OA, OP are divided proportionally at D and Q, $\therefore$ DQ is parallel to AP ;

$\therefore$ Q lies on a line parallel to the fixed line AP, through the fixed point D,

i.e. Q lies on a fixed straight line.

2. Let R be **any** point on the line through D, parallel to BC. Join OR and let it (produced, if necessary) meet BC at E.

Then in the $\triangle OAE$, DR is parallel to AE,

$\therefore$ it divides the sides OA, OE proportionally,

$\therefore$ $OR : OE = OD : OA = k$,

$\therefore$ R is a point on the locus.

We have therefore proved that the locus of Q is a fixed straight line parallel to the fixed straight line along which P moves.

* THEOREM I

O is a fixed point and P moves on a fixed circle. On OP (produced, if necessary) a point Q is taken so that OQ : OP is a constant ratio, k. It is required to find the locus of Q.

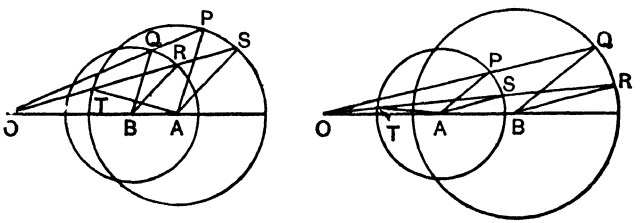


FIG. 249.

1. Let the centre of the fixed circle be A. On OA (produced, if necessary) let a point B be taken such that $OB : OA = k$. Then B is a **fixed** point. Let P be **any** point on the circle and let Q be a point on OP (produced, if necessary) such that $OQ : OP = k$. Then Q is **any** point on the locus. Join QB, PA.

Then $OQ : OP = OB : OA$ (each $= k$),

i.e. in the $\triangle OAP$ the sides OA, OP are divided proportionally at B and Q,

$\therefore$ BQ is parallel to AP ;

$\therefore \angle OBQ = \angle OAP$ and $\angle OQB = \angle OPA$;

$\therefore$ the $\triangle$s $\frac{OBQ}{OAP}$ are equiangular,

$\therefore$ their corresponding sides are proportional,

$$\therefore \frac{BQ}{AP} = \frac{OB}{OA} = k, \text{ i.e. } BQ = k \,.\, AP.$$

But AP is the radius of a fixed circle,

$\therefore$ the length of PQ is constant,

i.e. the distance of Q from a fixed point B is constant,

i.e. Q lies on a fixed circle.

2. Let R be **any** point on this circle. Let OR (produced, if necessary) meet the first circle at S and T. Join AS, AT, BR.

Then $OB : OA = BR : AS$ (or AT) (each $= k$),

$\therefore$ In the $\triangle$s OBR, OAS (or OAT),

$OB : OA = BR : AS$ (or AT) (*proved*),

and $\angle BOR = \angle AOS$ (or $\angle AOT$) (*same angle*),

$\therefore$ $\angle$ ORB and $\angle OSA$ (or $\angle OTA$) are either equal or supplementary (Theorem 65a); i.e. one of the $\angle$s OSA, OTA is equal to $\angle ORB$ and the other is supplementary to it.

Let $\angle OSA$ be the angle which is equal to $\angle ORB$.

Then the $\triangle$s $\begin{matrix} OBR \\ OAS \end{matrix}$ are similar,

$\therefore$ $OR : OS = OB : OA = k$,

i.e. R is a point on the locus.

We have therefore proved that the locus of Q is a fixed circle.

Note. If O is a fixed point and P moves on any fixed curve, and if on OP (produced, if necessary) a point Q is taken such that $OQ : OP =$ a constant ratio, k, the locus of Q is said to be **homothetic** to the locus of P. O is called the **homothetic centre** and k the **homothetic ratio.** We have proved above that the locus homothetic to a straight line is a straight line and to a circle is a circle. These results are very important. These results can easily be extended to three dimensions; the locus homothetic to a plane is a plane and to a sphere is a sphere

* THEOREM J

The Radical Axis

To find the locus of a point P which moves so that the tangents drawn from it to two given circles are equal.

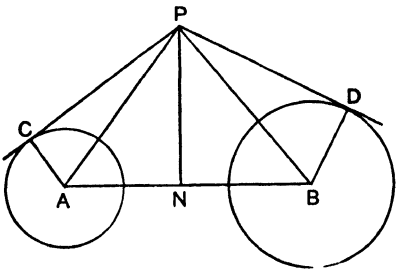


FIG. 250.

Let A, B be the centres of the given circles (radii a, b) and let PC, PD be the equal tangents drawn from a point P which satisfies the given condition. Join AC, AP, BD, BP, and draw PN $\perp$ to AB, meeting it at N.

Since CP is a tangent and CA a radius, $\angle ACP = 90°$. Similarly

$$\angle BDP = 90° ;$$

$$\therefore\ PA^2 - a^2 = PC^2, \text{ and } PB^2 - b^2 = PD^2 \ (\textit{Pythagoras'}).$$

But $PC = PD$ (*given*), $\therefore\ PC^2 = PD^2$, $\therefore\ PA^2 - a^2 = PB^2 - b^2$,

$$\therefore\ PA^2 - PB^2 = a^2 - b^2 = \text{constant} ;$$

$\therefore$ the locus of P is a straight line $\perp$ to AB (Th. 56, Cor.).

Note. The locus of a point from which the tangents to two circles are equal is called the **Radical Axis** of the circles.

We have shown above that the Radical Axis is a st. line $\perp$ to the line of the centres.

EXERCISE 69 [Theorems G-J]

1. AB is a st. line 6″ long ; P is a variable point which moves so that PA : PB = 2 : 1. Find the radius of the circle on which P moves. How far is its centre from A ?

2. AB is a st. line 10 cm. long and PA : PB = 2 : 3. Find the radius of the circle on which P moves.

3. O, A, B are three points in a st. line, such that AB = 3OA. P is a point which moves so that PB = 2PA. Prove that the locus of P is a circle with centre O.

4. In a circle of radius 1·4″ inscribe a △ having its base 2·5″ long and the ratio of its remaining sides 2 : 3.

5. Inscribe in a circle of diameter 6 cm. a rectangle whose unequal sides are in the ratio 5 : 3.

6. Construct a △ of base 10 cm. and vertical angle 50°, so that the ratio of the other sides may be 4 : 3.

[The method given in No. 3 of the previous exercise is the best method.]

7. Construct a △, given the base, the altitude and the ratio of the other sides.

8. Construct a △, given the base, the ratio of the other sides, and the length of the median which bisects the base

9. Given the base and the circumscribing circle of a △ and the ratio of the other two sides, state at what points the bisector of the angle opposite the base cuts (i) the base, (ii) the circumcircle. If the given side is 3″ long, the radius of the circle 2·5″ and the ratio of the sides is 2 : 1, construct the △ and measure the lengths of the unknown sides.

[N.B. This gives a third solution of the problem, "To construct a △ABC, given BC, ∠A, and the ratio of AB to AC."]

10. Draw a △ABC, such that BC = 10 cm., ∠B = 50°, ∠C = 70°. Draw through X, the mid-point of BC, a st. line cutting CA at Y and AB produced at Z, such that YX : XZ = 3 : 7.

11. AOB is a right angle, P a point 2″ from OA and 3″ from OB. Draw through P a st. line cutting OA at X and OB at Y, so that XP may be to PY as 5 : 3.

12. DEF is a △ and X is any point outside the △. Draw a line XZY meeting DE, DF (produced, if necessary) at Y, Z respectively, such that 5XY = 6XZ.

13. Given two intersecting circles and a point O, through O draw a st. line OPQ to meet one circle at P and the other at Q, so that

$$OQ : OP = 2 : 3.$$

[Consider the case (i) when O is outside both circles, (ii) when

O is outside one circle and inside the other, (iii) when O is inside the two circles.]

14. Given two circles and a point O, through O draw a st. line OPQ to meet one circle at P and the other at Q, so that PQ=3OP.

15. ABC is a △. The sides BC, CA have fixed lengths, while the length of the side AB varies. If the base BC is also fixed in position, find the locus of the intersection with AB (i) of the internal bisector of ∠ACB, (ii) of the external bisector of ∠ACB.

16. ABC is a △. Draw a st. line AXY through A, meeting the circumscribing circle at X and meeting BC produced at Y, so that AY may be four times AX.

17. ABC is a △, having the base BC fixed in position and the ∠BAC=62°. Find the locus of the centroid of the △ABC.

18. O is a fixed point outside a given circle; P is a variable point on the given circle. OP meets the circle again at S. By considering the connection between R and S, prove that, if R is taken on OP so that OR . OP is constant, the locus of R is a circle.

19. Prove the second parts of Theorems G, H, I by "reductio ad absurdum."

20. OPQ is a △ whose angles are given. O is a fixed point and P moves on a fixed st. line; find the locus of Q.

[Take a point R on OP or OP produced such that OR=OQ. Consider the connection between the locus of R and the locus of Q.]

21. AB is a fixed st. line; X, Y divide AB internally and externally in a given ratio. P is any point on the circle on XY as diameter. BL, BM are drawn ∥ to YP, XP, meeting AP or AP produced at L, M respectively. Prove that PB=PL=PM, and deduce a proof of the second part of Theorem G.

22. Prove that, if two circles intersect, their radical axis is the common chord produced.

23. What is the radical axis of two circles which touch one another?

24. Show that the radical axis of two circles bisects any one of their common tangents.

25. Prove that the radical axes of three circles taken in pairs are concurrent.

26. If any circle is drawn to cut one given circle at E, F and a second given circle at G, H, and if EF, GH meet at P, prove that P is a point on the radical axis of the two given circles. Hence deduce a construction for finding the radical axis of two given non-intersecting circles.

* NOTE ON INCOMMENSURABLE MAGNITUDES

1. If a greater magnitude contains a smaller magnitude a definite number of times exactly, the greater magnitude is called a **multiple** of the smaller; the smaller is called a **sub-multiple,** or a **measure**, of the greater.

2. Two magnitudes of the same kind are said to be **commensurable** when they have a common measure, and to be **incommensurable** when they have no common measure. If A and B are commensurable magnitudes and if C is their common measure, we have $A = pC$, $B = qC$, where p and q are integers. Thus the ratio of A to B can be expressed as an exact fraction $\frac{p}{q}$.

The definition of ratio given at the beginning of Part V is applicable only to commensurable magnitudes. It has also been previously pointed out that Theorems 25 and 60 (and therefore all theorems which depend upon them) are only valid if the ratios of the magnitudes dealt with are commensurable.

Magnitudes which are incommensurable are of frequent occurrence in Geometry, in fact **incommensurability is the rule and commensurability the exception.**

The best-known example is that a side and a diagonal of a square are incommensurable. In practice the ratio of two incommensurables is represented by some suitable approximate value.

Consider the diagonal of a square of side 1 unit. The length of the diagonal is $\sqrt{2}$ units. It can be shown that there is no fraction of the form $\frac{p}{q}$ (where p and q are integers) which equals $\sqrt{2}$. Numbers which can be put in the form $\frac{p}{q}$ are called **rational numbers**; numbers, such as $\sqrt{2}$, which cannot be so expressed are called **irrational numbers**. But although $\sqrt{2}$ cannot be exactly represented by a fraction of the form $\frac{p}{q}$, we can, by applying the arithmetical rule, show that it equals 1·41421 ....

Thus $1\frac{207}{500}$, $1\frac{2071}{5000}$, $1\frac{41421}{100000}$ are all good approximations to $\sqrt{2}$, and if it is known what degree of accuracy is required, $\sqrt{2}$ (and therefore the ratio of the diagonal to the side of the square) can be represented by a fraction, **to a sufficient degree of accuracy.**

Again, the area of the rectangle contained by a side and a diagonal of a square, of side 1 unit, is the area of a rectangle contained by sides of length 1 unit and $\sqrt{2}$ units. This is easily seen to be greater than a rectangle contained by sides of length 1 unit and 1·41421 units and less than the rectangle contained by sides of length 1 unit and 1·41422 units, so that the area of the rectangle lies between 1·41421 and 1·41422 sq. units. It is therefore 1·4142 sq. units, correct to 4 places of decimals. It is clear that the method may be extended to obtain an approximation to any desired degree of accuracy. The formal proof that the area of the rectangle is $1 \times \sqrt{2}$ sq. units involves the theory of irrational numbers and is beyond the scope of this book. Theorem 25 has not been proved, if the sides of the rectangle are incommensurable magnitudes. The first proofs of Theorems 26-28 are, however, valid, but all results concerning the measurement of area depend upon Theorem 25 and have only been established when the lengths involved are commensurable. But, although the properties of incommensurable magnitudes and irrational numbers are of the greatest importance in Mathematics, they are of no consequence when measurements are made. For all measurement is approximate only, and we can always obtain our results to as great a degree of accuracy as is justified by our data.

The proof of Theorem 60, which is valid for incommensurables, involves the following exact definition of the equality of ratios:

" The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth: if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth: or, if the multiple

of the first be equal to that of the second, the multiple of the third is also equal to the multiple of the fourth: or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth."

This is Euclid's definition. It may be more simply stated thus:

The ratio of A to B is equal to the ratio of P to Q if

$$\left.\begin{matrix} mA > = \text{ or } < nB \\ mP > = \text{ or } < nQ \end{matrix}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

according as

for all integral values of m and n.

It is clear that no fraction $\frac{c}{d}$, where c and d are integers, can lie between $\frac{A}{B}$ and $\frac{P}{Q}$, for if $\frac{P}{Q} < \frac{c}{d} < \frac{A}{B}$, we should have

$$dP < cQ, \text{ but } dA > cB,$$

which is contrary to (1).

By using this definition a general proof of Theorem 60 may now be given.

If two triangles have the same altitude, they have areas which are proportional to their bases.

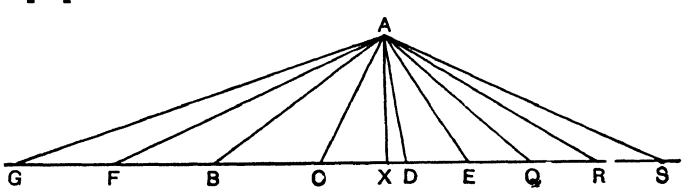


FIG. 251.

Let ACB, ADE be △s which have the same altitude, i.e. the ⊥ AX from A on their common base BCDE.

Produce CB and mark off any number of parts BF, FG, ... each equal to CB.

Produce DE and mark off any number of parts EQ, QR, RS ... each equal to DE

Join AF, AG ... , AQ, AR, AS, ...

Then since CB = BF = FG = ... , △ACB = △ABF = △AFG = ...

Similarly, △ADE = △AFQ = △AQR = △ARS = ...

Consider any pair of △s such as △ACG and △ADS.

$\triangle ACG$ and the line CG are equimultiples of the $\triangle ACB$ and the line CB.

Also $\triangle ADS$ and the line DS are equimultiples of the $\triangle ADE$ and the line DE, and

$$\triangle ACG \gtrless \text{ or } < \triangle ADS,$$

according as $$CG \gtrless \text{ or } < DS.$$

But, by definition, this is the criterion that

$$\triangle ACB : \triangle ADE = CB : DE;$$

$\therefore$ this is true.

The proof may easily be extended to the case of $\triangle$s with equal altitudes. Theorems 61-66 are therefore valid in all cases, but the proof of Theorem 67, which depends upon the measure of the area of a $\triangle$ is not. The following is an outline of a proof valid in all cases.

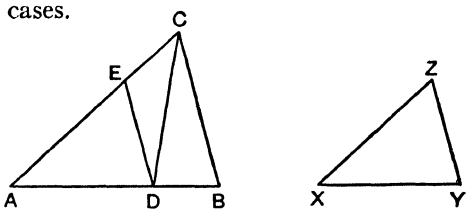


FIG. 252.

Let ABC, XYZ be two similar $\triangle$s.

To prove that $\triangle ABC : \triangle XYZ = AB^2 : XY^2$.

On AB mark off $AD = XY$; on AC mark off $AE = XZ$.

Join DE, DC.

Then $\triangle$s $\begin{matrix} ADE \\ XYZ \end{matrix}$ are congruent and $\therefore$ similar;

$$\therefore \angle ADE = \angle XYZ = \angle ABC,$$

$$\therefore DE, BC \text{ are } \|.$$

$$\therefore \frac{\triangle ABC}{\triangle XYZ} = \frac{\triangle ABC}{\triangle ADE} = \frac{\triangle ABC}{\triangle ACD} \cdot \frac{\triangle ACD}{\triangle ADE}$$

$$= \frac{AB}{AD} \cdot \frac{AC}{AE} \text{ (Th. 60)} = \frac{AB}{AD} \cdot \frac{AB}{AD}$$

(Th. 61, since DE, BC are $\|$)

$$= \frac{AB^2}{AD^2} = \frac{AB^2}{XY^2}.$$

MISCELLANEOUS EXAMPLES

These examples have not been arranged in order of difficulty. Nos. 1-20 are on Part I, Nos. 21-30 on Part II, Nos. 31-60 on Part III, Nos. 61-70 on Part IV, Nos. 71-80 on Part V. All these are of moderate difficulty only. The remainder are mixed, and some of them are of considerable difficulty.

1. Prove that the st. lines which bisect the four angles of a parallelogram form a rectangle.

2. ABCD is a square ; P, Q are the mid-points of AB and AD respectively. Prove that the $\triangle$s ABQ, BCP are congruent and that BQ is $\perp$ to CP.

3. ABCDE is a regular polygon. AC, BE meet at T. Prove that ETCD is a rhombus.

4. ABC is an acute-angled $\triangle$. D is the image of B in the line AC, E is the image of C in the line AD. Prove that the $\triangle$s ABC, ADE are congruent.

5. The base of a $\triangle$ is 3″ long, the sum of the lengths of the other two sides is 4·5″, and one of the base angles is 35°. Construct the $\triangle$.

6. Find two points E, F in the sides AB, AC of a $\triangle$ABC so that EF is $\parallel$ to BC and equal to the sum of BE and CF.

7. If the external angles of the $\triangle$ABC be bisected, prove that the bisecting lines form a $\triangle$ such that A, B, C are the feet of the $\perp$s from its vertices to the opposite sides.

8. On a given base describe a quad., such that each of its three sides shall be half the base.

9. Show that the lines joining the mid-points of the sides of a regular polygon form another regular polygon.

10. Find the locus of a point which moves so that its distance from a given st. line is half its distance from a given point on the line.

11. ABCD is a parallelogram ; PAB, QBC, RCD, SDA are equilateral $\triangle$s described outside the parallelogram on bases AB, BC, CD, DA. Prove that PQRS is a parallelogram.

12. A quad. ABCD has AB=3 cm., BC=8 cm., CD=4 cm. State two extreme lengths between which the side AD must lie. If the side AD is $\parallel$ to BC and at $\perp$ distance 2 cm. from it, construct all the possible quads. and describe the various cases which may arise for other values of the $\perp$ distance.

13. P is any point, X'OX, Y'OY, any two st. lines. Show how to draw a st. line APB meeting X'OX at A, Y'OY at B, such that AB=PB.

14. ABC is a △ obtuse-angled at C. AC, BC are bisected at D and E. Through D, E ⊥s DF, EG are drawn to AC, BC meeting AB at F, G. Show that the perimeter of the △CFG is equal to the side AB, and that the ∠s GFC, CGF are twice the angles at A and B respectively. Deduce a construction for a △ when the perimeter and two angles are given.

15. O is a point within a △ABC, such that OA=OB=OC. Through A lines are drawn ‖ to OB, OC; through B lines are drawn ‖ to OC, OA; through C lines are drawn ‖ to OA, OB. Prove that (i) these lines form an equilateral hexagon, (ii) the opposite angles of the hexagon are equal in pairs, (iii) each angle of the hexagon is double of an angle of the △ABC.

16. ABCD is a parallelogram; P, Q, R, S are the mid-points of the sides AB, BC, CD, DA respectively. Prove that the four lines AQ, BR, CS, DP are the four sides of a parallelogram. Show also that the diagonals of this parallelogram intersect in the same point as do the diagonals of ABCD.

17. The sides BA, CA of a △ABC are produced to E and F respectively, so that AE=½AB, AF=½AC. Through E and F are drawn lines EG, FG ‖ respectively to CA, BA. Prove that GA passes through the mid-point of BC.

18. Construct a quad. ABCD, having the sides AB, BC, CD equal to 2, 3, 4 in. respectively, and the ∠s A, D equal to 70°, 80° respectively. Measure the remaining ∠s B, C and the side AD.

19. ABC is an equilateral △; P is any point inside it. On PC an equilateral △PCQ is drawn (so that PQ cuts AC and not BC). Prove that the △s ACQ, BCP are congruent. Hence, by first drawing the △APQ, construct an equilateral △ABC, being given the distances PA, PB, PC of a point P from the vertices to be 1″, 1·6″, 2″.

20. ABCDE is a pentagon in which AB=BC=CD. If each of the ∠s at A, B, C and D is 120°, prove that AE=ED=2BC.

21. From a point P, inside a parallelogram ABCD, parallels are drawn to the sides, dividing the whole into four parallelograms. If the point P lies on the diagonal BD, prove that the parallelograms PA, PC have equal areas. Conversely, if the areas PA, PC are given equal, prove that P must lie on BD.

22. ABCDEF is a regular hexagon inscribed in a circle, centre O. Prove that (i) ABOF is a parallelogram; (ii) △FBD is an equilateral △ of half the area of the hexagon.

23. Two rectangles of equal area ABCD, AEFG are placed so that AB falls on AE, AD falls on AG. If EF, CD meet at H and FG, BC meet at K, prove that HK passes through A.

24. Construct a quad. ABCD with AB=1″, AC=2″, ∠ABC=90°, and the △ADC equilateral. Prove that the ∠BCD is 90°, and reduce the area ABCD to a right-angled △ on BC. Hence prove that the area of ABCD equals $\frac{1}{2}$(AB+CD) . BC.

25. ABC is any △. AB is produced to D, so that AB=BD and D is joined to the mid-point E of AC. If DE meet BC at H, show that the △s BHD, EHC are equal in area.

26. AB, CD are two ‖ lines, and a point E is taken upon AB produced. Prove that, if the sum of the △s CED, BED is equal to the △AEC, AC=BD.

27. Prove that, if one diagonal of a quad. divides it into two parts of equal area, the quad. need not be a parallelogram, but it must be one, if in addition one pair of opposite sides are parallel.

28. Construct a regular octagon, four of whose sides lie along the four sides of a square of side 2″.

29. Show how, without using a protractor, to circumscribe a regular octagon to a given square, so that the corners of the square shall be the mid-points of alternate sides of the octagon.

30. Given a quad. ABCD with AB ‖ to DC, and with E the mid-point of AD, prove that the area of the △EBC is half the area of the quad.

31. If the inscribed and circumscribed circles of a given △ have a common centre, prove that the △ must be equilateral.

32. A parallelogram ABCD has the diagonal AC equal to the side AD; the circumcircle of the △ABC cuts DA, produced, if necessary, at F. Prove that BC . DF=AB^2.

33. From any point within a regular hexagon ⊥s are drawn to the sides. Prove that the sum of the ⊥s is six times the radius of the circle inscribed in the hexagon.

34. AB, AC are two chords of a circle; BD is drawn ⊥ to AC, meeting it at D and the circle at L. CE is drawn ⊥ to AB, meeting it at E and the circle at M. Prove that DE is ‖ to LM.

35. Prove that a common tangent to two circles, measured between its points of contact, cannot be greater than the line joining their centres.

36. A circle touches the sides BC, CA, AB of a △ at the points D, E, F respectively. Prove that 2BD=BC−CA+AB.

37. Show how to inscribe a regular 12-sided polygon in a given circle, without using a protractor. If O is the centre of the circle, and AB, BC, CD are three successive sides of the polygon, and the tangents at A, D meet at T, prove that (i) TAOD is a square, (ii) BOD is an equilateral $\triangle$, (iii) TB is equal to a side of the polygon.

38. POQ is the tangent to a circle at O; chords OA, OB are drawn through O; OL, OM are chords bisecting the angles POA QOB. Prove that the $\perp$s from L to OA and from M to OB meet at the centre of the circle.

39. Draw a circle of diameter AB and centre O, and produce AB to a point P, take a point R in OB such that $OP \cdot OR = OA^2$. Prove that Q, the mid-point of RP, is such that PQ = the tangent from Q to the circle.

40. Prove that the shortest st. line that can be drawn to a given circle, from a point O outside it, is the one that passes through the centre of the circle when produced. Prove that the mid-points of all chords which pass through O when produced lie on another circle.

41. A circle touches a second circle externally, and the tangents to it from the centre of the second circle contain an angle of 60°. Prove that the circles have equal radii.

42. Describe a $\triangle$DBC right-angled at B, equal in area to a given $\triangle$ABC. Produce CB to N making BN equal to BD. On CN as diameter describe a circle intersecting BD at E. Prove that the square on BE is equal to twice the $\triangle$ABC in area.

43. P is a point inside a circle centre O. Show that, of all chords through P, the one $\perp$ to OP is the shortest. The radius of the circle being 2 in., and OP = 1·5 in., give a ruler-and-compass construction for a chord through P of length 3·2 in. Measure the distance from P to the middle point of this chord, and check by calculation.

44. ABC is a $\triangle$ inscribed in a circle. The internal and external bisectors of the angle A meet the circumference again at D, E respectively. Prove that DE is a diameter, and that it is $\perp$ to BC.

45. Draw a $\triangle$ABC having AC = 9 cm., AB = 7 cm., $\angle BAC = 35°$. Construct the circle ABC. Find, by construction, a point D on the circle ABC, such that BD is bisected by AC. How many solutions are there?

46. If, from a given point O, a st. line be drawn intersecting a given circle in P and Q, prove that the rectangle OP . OQ is the same for all such lines. If O be such that OP . OQ is equal to the corresponding rectangle obtained by drawing a st. line through it to intersect another given circle, find the locus of O, (i) when the two circles intersect one another, (ii) when they do not intersect one another.

47. Draw seven equal circles, arranged as a central one surrounded by the six others in contact ; and prove that the interstices are equal to the difference of the area of the hexagon formed by the centres of the outside circles and of the area of three circles.

48. PQ is a chord of a circle, PA and QB are chords making equal ∠s APQ, BQP with PQ on the same side of it. Prove that PA = QB.

49. A, B are two fixed points ; P is a point moving on the semicircle described on AB as diameter ; Q, R are the mid-points of AP, BP ; find the locus of the point of intersection of AR and BQ.

50. Two circles touch at O. Through O is drawn OHK cutting the circles at H and K ; P is the mid-point of HK. Find the locus of P as OHK rotates around O.

51. The bisector of the ∠A of a △ABC meets BC at D. At D the ∠ CDE is made equal to the ∠BAC, DE cutting CA at E. Prove that DE = BD.

52. Two parallel st. lines cut by a transversal in the points A, B, and two circles are drawn, each touching the three st. lines. Prove that the distance between the centres of the circles is equal to AB.

53. Prove that the altitude of an equilateral △ inscribed in a circle is three-quarters of the diameter of the circle.

54. Prove that, if an equilateral △ is inscribed in a circle, the radii drawn ⊥ to the sides of the △ are bisected by those sides.

55. AOB is a quadrant of a circle bounded by the arc AB and by radii OA, OB. On OA as hypotenuse a right-angled isosceles △OPA is drawn. Prove that the circle, centre O, radius OP bisects the area of the quadrant. Show also how to bisect the area of the quadrant by drawing a semicircle.

56. Prove that each of the equal sides of a right-angled isosceles △ is equal in length to the sum of the radii of the inscribed and circumscribed circles of the △.

57. ABCD is a square, L and M the mid-points of BC and CD respectively. AM, DL meet at O. Prove that ABLO is a cyclic quad. Prove also that, if BO is joined, BO = AB.

58. ABCD is a quad. inscribed in a circle ; AC, BD meet at P ; AB, DC meet at a point Q on AB produced. Prove that the bisectors of the ∠s PBQ, PCQ meet on the circle ABCD.

59. Two circles touch externally at a point O. A chord AB of the first touches the second at C ; AO produced meets the second circle again at E. Prove that the ∠s BOC, COE are equal. What happens in the special case in which CE is a diameter of the second circle ?

60. ABCD is a parallelogram. Any circle passing through A and D cuts DC, AB at E, F respectively. Prove that EFBC is a cyclic quad., and that EF is equal to one of the sides of the parallelogram.

61. If two equal chords of a circle, AB, CD, intersect at O, show that AO is equal either to OD or to OC.

62. If P, Q are two points on the line AB, and if the rectangles AP, PB and AQ, QB are equal, prove that AP = QB.

63. Find the locus of a point from which the tangents drawn to two given circles are equal.

64. P, Q are any points on a circle, centre O, and A, B are two points on the tangents at P and Q, such that the circles, centre A, radius AP, and centre B, radius BQ, intersect. Prove that, if K is one of the points of intersection, OK is $\perp$ to AB.

65. O is any point on a circle and a st. line $\parallel$ to the tangent at O meets the circle at X, Y. If through O a st. line be drawn, meeting XY at P and the circle again at Q, prove that $OP \cdot OQ = OX^2$.

66. PQR is an isosceles $\triangle$ whose vertex is R, and the base PQ is produced to any point O. Prove that, if OR cuts the circle circumscribing the $\triangle$ at S, $OS \cdot OR = OR^2 - QR^2$.

67. Find the locus of a point X which moves in the plane of a $\triangle ABC$ so that $2XA^2 = XB^2 + XC^2$.

68. If XAB be any st. line drawn through a fixed point X and cutting a given circle, whose centre is O, at A and B, prove that the circle passing through O, A and B cuts OX at a fixed point and that, if C be this point, the angles ACX, BCO are equal.

69. Two equal circles are drawn with centres A and B. Through any point P on the $\perp$ bisector of AB st. lines are drawn cutting one circle at X, Y and the other at Q, R. Prove that Q, R, X, Y are concyclic.

70. Two circles are drawn having the same centre O. From a point A on the outer circle tangents are drawn to touch the inner circle and are produced to meet the outer at B, C. Prove that, if BC is a tangent to the inner circle, the radius of the outer is twice the radius of the inner.

71. Prove that the feet of the $\perp$s from the vertices of a parallelogram to the diagonals are the vertices of a similar parallelogram.

72. ABC is a $\triangle$ right-angled at A. Semi-circles are described on AB and AC outside the $\triangle$, and on BC so as to pass through A. Show that the two parts of the smaller semi-circles which are outside the larger semi-circle are together equal in area to the $\triangle ABC$.

73. P is a point outside a circle of which LM is a diameter. PM meets the circle again at N. If P moves in such a way that the rectangle PN . PM is constant, prove that the locus of P is a circle and find its centre. Find also the locus of the mid-point of PL.

74. P is a point on the circle circumscribed to the $\triangle ABC$, and PL, PM, PN are drawn $\perp$ to BC, CA, AB respectively. Prove that the $\triangle PLM$ is similar to the $\triangle PBA$ and that

$$PL \cdot PA = PM \cdot PB = PN \cdot PC.$$

75. Prove that the centroids of the four $\triangle$s into which the diagonals divide any quad. are the vertices of a parallelogram.

76. ABC is a $\triangle$, D any point in BC; DF and DE are drawn $\parallel$ to CA and BA to meet AB, AC at F, E respectively. Prove that the area of the $\triangle DEF$ is a mean proportional between the areas of the $\triangle$s DBF and DCE.

77. PQRS is a cyclic quad., and PS, QR produced meet at X. Through X a st. line is drawn meeting SR and PQ at Y and Z respectively, so that $\angle XYS = \angle XZQ$. Prove that PZ : ZQ = RY : YS.

78. The tangents at A and B, the extremities of a diameter AB of a circle, centre C, are cut by another tangent at points R and S respectively. Prove that the $\triangle$s CAR, SBC are equiangular, and that $RA \cdot SB = CA^2$.

79. AB is a st. line 2″ long. Find by a ruler-and-compass construction a point C at which AB subtends an angle of 45°, and such that the bisector of this angle divides AB in the ratio 1 : 2. How many such points are there?

80. AOB is the diameter of a circle with centre O; chords AC, BD are drawn, on the same side of the diameter, each equal to half the radius, and AC is produced to E so that AE = AO. The line EB cuts OD at F; prove that OF = BF.

81. ABC is an isosceles $\triangle$ with AB = AC. P is any point in the base BC. D, E are the feet of the $\perp$s from P to AB, AC respectively. Prove that PD + PE is the same for any position of P between B and C.

82. Two quads. ABCD, EFGH have the $\angle$s at A, B and C equal to the $\angle$s at E, F, G respectively. Prove that they are certainly equal in all respects, if AB = EF and BC = FG, and that they are generally equal in all respects, if AB = EF and CD = GH. What possible exception is there in this last case?

83. E is a point on the side DC of a parallelogram ABCD. DC is produced to F so that CF = DE, and AE is produced to G so that EG = AE. Prove that EGFR is a parallelogram, and that its area equals that of ABCD.

84. Each of three equal circles touches the other two externally. Prove that the difference of the perimeters of the two circles that can be drawn to touch all the three given circles is twice the perimeter of one of the given circles. Prove also that the difference of the areas of the two circles so drawn is $\frac{8\sqrt{3}}{3}\pi r^2$, where r is the radius of the given circles.

85. Through a point O are drawn three equal circles to intersect again at A, B, C. The centres of the circles OBC, OCA, OAB are respectively X, Y, Z. Prove that (1) OA is bisected by YZ ; (2) the sides of the $\triangle$ABC are equal and $\parallel$ to those of the $\triangle$XYZ.

86. C and D are points on a circle on opposite sides of a diameter AB. AC, DB are produced to meet at E ; AD, CB are produced to meet at F. Prove that EF is $\perp$ to AB produced.

87. A, B are two points fixed on a moving arm OAB, movable about a point O. C is a fixed point. Find the locus of the centre of the circle ABC as OAB moves round.

88. ABC is an isosceles $\triangle$ right-angled at C. A circle is described with centre C and radius CA ; Q is any point on the arc AB. Through Q is drawn PQR $\parallel$ to AB and meeting CA and CB produced at P and R. Prove that $PQ^2+QR^2=AB^2$.

89. PQRS is a trapezium, PQ being $\parallel$ to SR. PR, QS meet at X ; SP and RQ produced meet at Y. Join XY and prove that (i) $\triangle XPS=\triangle XQR$, (ii) $\triangle XSY=\triangle XRY$, (iii) XY bisects PQ and SR.

90. Construct a quad. ABCD from the following data. AB=2″, AD=3″, $\angle$BAD=90°, the diagonal AC=3·75″ and the area of the quad.=6 sq. in.

91. ABCD is a quad. circumscribed to a circle. If $\angle$ABC=$\angle$ADC, prove that AC is a diameter of the circle ; and if BD is also a diameter, prove that ABCD is a rhombus.

92. ABCD is a quad. and E, F, G, H are the mid-points of the sides AB, BC, CD, DA respectively. Prove that the area of EFGH is one-half of the area of the quad.

93. Two unequal circles intersect at A, B ; a st. line through B cuts the circles again at P, Q. Find the direction in which the line must be drawn so that the area of the $\triangle$APQ shall be as great as possible.

94. Construct a quad. ABCD from the following data : the $\angle$s A and C are right angles, AC=6 cm., BD=7 cm., and AC and BD intersect at an angle of 60°.

95. If the sides of a convex quad. are AB=2·2″, BC=1·6″, CD=1·9″, DA=2·7″, find the least and the greatest possible lengths of the diagonals AC, BD.

96. Points P, Q, R, S are taken upon two st. lines, APQ, ARS, so that AP=AR and AQ=AS. If T is the point of intersection of PS and RQ, prove that AT bisects the ∠QAS.

97. OPQRS is a pentagon in which OP=OS, PQ=RS, and the diagonals OQ, OR are equal. Prove that the diagonal PS is ‖ to QR.

98. Two points A and B are 10 cm. apart, and their distances from a st. line LM are 7 and 13 cm. respectively. Find a point P in LM so that AP, PB make equal angles with LM.

99. Find a point P in the base BC of a given △ABC so that CP is equal to the ⊥ from P to AB.

100. AMNB is a horizontal line 60″ long and AM=NB=6″. P, Q are points above AB, such that ∠AMP=∠BNQ=90°, and

$$MP=NQ=12''.$$

AP, BQ are joined by two circular arcs centred on AB, and PQ is joined by a third circular arc having a common tangent with the first arc at P and with the second arc at Q. Calculate the radii of these arcs.

101. ABC is a △ right-angled at A. From a point D in BC ⊥s DE, DF are drawn to AB, AC meeting them at E, F respectively. Prove that BD . DC=AE . EB+AF . FC.

102. The △ABC has the sides AB, AC equal, and D, E are the mid-points of these sides. If the circle through D, E, A touches BC, prove that the △ is right-angled.

103. Regular hexagons are inscribed in and circumscribed to a given circle. Prove that the area of one is three-quarters of the area of the other.

104. In a △PQR, X, Y are the feet of the ⊥s from P, Q to the opposite sides. If PX and QY intersect at Z, prove that the st. line joining the mid-points of PQ and RZ bisects XY at right angles.

105. Two unequal circles intersect at A, B, and O is any point on AB produced. A circle with centre O cuts one of the circles at P and Q, and the other at L and M. OP, OM, produced, if necessary, cut the circles PQA, LMA at S and T respectively. Prove that PS=TM, and that the centres of the original circles lie on the bisectors of the ∠s POQ, LOM.

106. AB, BC are two sides of a regular ten-sided figure inscribed in a circle whose centre is O. AB, OC meet, when produced, at P. Prove (1) that BP=the radius of the circle; (2) that CP=a side of the figure; (3) that, if BQ be drawn ‖ to CO to cut the circle again at Q, then BQ=the radius of the circle+a side of the figure.

107. In a $\triangle ABC$ the $\angle A = 60°$. Prove that $CB + \frac{1}{2}BA > CA$.

108. The $\perp$s drawn from B, C to the sides AC, AB of the $\triangle ABC$ meet at P. Prove that $AC^2 + BP^2 = AB^2 + CP^2$.

109. Two circles whose centres are A and B intersect at C and D. Show how to draw through C a st. line meeting the circles again at P and Q, such that $PC = CQ$.

110. Ox, Oy are $\perp$ st. lines. A, a point on Ox, is joined to B, a point on Oy. A square ABDC is described on AB on the opposite side of AB to O. If $OA = a$, $OB = b$, find the distances from Ox and Oy of the points C, D and the centre of the square. Hence, or otherwise, prove that, if the square is moved so that its vertices A and B always remain on Ox and Oy, the centre of the square describes a st. line.

111. If AP, AQ are tangents to a circle of which PS is a diameter, prove that AS bisects the $\perp$ from Q to SP.

112. The $\angle BCA$ of the $\triangle ABC$ is twice the $\angle CAB$, and its bisector meets AB at N. The $\perp$ from N to AC meets CB produced at F. Prove that the $\triangle CFA$ is isosceles and that N is the centre of its inscribed circle.

113. AOB, COD are two $\perp$ chords of a circle; prove that the $\triangle$s AOC, BOD cannot have equal areas unless these chords are equal. Is this result true if the chords are not perpendicular?

114. Prove that if p, q, r are the altitudes of a $\triangle$, then

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{r}.$$

Prove that a $\triangle$ with altitudes of 3″, 4″ and 1·5″ is not possible, but that one with altitudes 3″, 4″, 2″ is possible. Construct this latter $\triangle$.

115. Calculate the lengths of the diagonals of a cyclic quad. whose successive sides are 1, 2, 3, 4 inches.

116. ABCD is a rhombus and P is a point on the diagonal AC. Two circles are described touching one another externally at P. One of these circles also touches AB and AD, while the other touches CB and CD. Prove that the sum of the radii of these circles is the same for all positions of P between A and C. If each side of the rhombus is 13″ and the diagonal is 24″, find the sum of the radii of the two circles.

117. Prove that, if the angles of a polygon inscribed in a circle are all equal, the sides must be all equal when the number of sides is an odd number. What can be inferred when it is an even number?

118. A wire 8″ long is to be cut into two pieces which can be bent respectively into a square and the circle inscribed in the square. Find to the nearest tenth of an inch the lengths of the two pieces (take $\pi = 3{\cdot}142$).

119. ABC is any △, and D and E are points on BC such that ∠BAD = ∠CAE. The circle circumscribing the △ADE cuts AB at P and AC at Q; prove that PQ is ‖ to BC and that

$$BD \cdot BE : CD \cdot CE = AB^2 : AC^2.$$

120. A wire is in the form of a circle of area A square inches. If the wire were bent into an equilateral △, find the area of this △ in terms of A and π.

121. Y, Z are the feet of the ⊥s drawn from A to the bisectors BY, CZ of the ∠s B, C of a △ABC. Prove that YZ is ‖ to BC.

122. Show that, of all △s on the same base and of the same altitude, an isosceles triangle has the least perimeter.

123. ABC is a △, right-angled at A, and the bisector of the angle A cuts BC at D. If DF is drawn ⊥ to BC to meet AB at E and AC at F, prove that DE = DC, DF = DB, and BE = CF.

124. On each side of a parallelogram as hypotenuse an isosceles right-angled triangle is drawn outwards, and the outer vertices of these triangles are joined. Prove that the resulting quad. is a square.

125. In a given △ABC a length BP is set off along BA and an equal length CQ is set off along CA. The parallelogram PCQR, with CP, CQ as adjacent sides, is completed. What is the locus of R for varying lengths of BP?

126. ABCD is a cyclic quad. and X is a point on the diagonal BD, such that ∠BAC = ∠XAD. Prove that (i) AD . BC = AC . DX; (ii) AB . CD = AC . BX; (iii) AD . BC + AB . CD = AC . BD. **(This is called Ptolemy's Theorem.)**

127. D is a point on the minor arc BC of the circumcircle of an equilateral △ABC. D is joined to A, B, C. Prove that DB + DC = DA.

128. A cylindrical vessel contains water 2 feet deep. A cylindrical stick is thrust vertically into the water till it rests on the bottom of the vessel; the water rises 1 inch without covering the stick. Find the ratio of the internal diameter of the vessel to the diameter of the stick

129. Through a fixed point O a st. line PQ is drawn, cutting at P and Q a given circle whose centre is C, and segments are drawn on OP and OQ touching the given circle at P and Q. Prove that the locus of the intersection of the segments is a circle on OC as diameter.

130. TA and TC are st. lines which touch a circle at A and C, and TBD is a st. line which cuts the circle at B and D. Prove that the bisectors of the $\angle$s A and C of the quad. ABCD intersect on BD.

131. If two given circles intersect at A, and AC, AD are diameters, and E is the mid-point of CD, prove that a circle with centre E and radius EA will cut the given circles again at points P, Q such that AP, AQ are the tangents at A to those circles.

132. A, P, Q, B are points on a circle of which AB is a diameter. AM and BN are drawn $\perp$ to PQ produced both ways, meeting it at M and N. Prove that PM=QN.

133. A hollow cone, the height of which is 12″ and the radius of the base 9″, is fixed with its vertex downwards and axis vertical, and is filled with water. A sphere is then lowered into the water until it rests on the inner surface of the cone, its centre being 5″ above the vertex. If the sphere is withdrawn, find, to the nearest tenth of an inch, the distance through which the surface of the water will fall. Take $\sqrt[3]{3}=1\cdot44$.

134. Two circles touch externally at O; through O are drawn st. lines POQ, LOM, meeting one circle at P and L, the other at Q and M. Prove that the $\triangle$s POM, QOL are equal in area.

135. A cyclic quad. PQRS is such that PQ . RS=QR . PS. SP is produced to X so that PX=SP. Prove that $\angle$PQX=$\angle$SQR. If QR is produced to Y, so that QY is equal to RQ, and if PY meets QX at Z, prove that PZ=ZQ.

136. In the $\triangle$ABC, AB > AC. The bisector of the $\angle$BAC meets BC at D. Prove that BD > DC. Prove also that, if X is the mid-point of BC, AX > AD.

137. ABCD is a quad. such that the $\angle$s BAC, DAC are equal; the sides CB, CD are also equal, but the sides AB, AD are unequal. Prove that a circle can be described about the quad.

138. The points B, E, A, C lie on a semi-circle of which BC is a diameter. AD is drawn $\perp$ to BC meeting it at D, and BE=AD. If DA is produced to F so that DF=BA, prove that CF=CE.

139. XYZ is a $\triangle$, X being an obtuse angle. R, S are points on YZ, in the order Y, R, S, Z, such that RX=RY and SX=SZ. Prove that the circumcentre of XYZ coincides with one of the ex-centres of XRS.

140. Two circles intersect at a point O. One common tangent touches them at A and B, the other at C and D respectively. Prove that the circles OAB and OCD touch one another.

141. ABC is a $\triangle$ in which $\angle$B=2$\angle$C. The bisector of $\angle$B meets AC at E; and EF, drawn $\parallel$ to CB, meets AB at F. Prove that AE . BE=AF . BC.

142. Two parallelograms have a common diagonal; show that the other angular points are those of a parallelogram.

143. O is a fixed point within a given $\triangle ABC$ and P a variable point on BC. The circle OPC cuts AC at Q, and the circle OPB cuts AB at R. Prove that AQOR is a cyclic quad. and that the $\triangle$s PQR are similar to one another for all positions. When is the length PQ least?

144. AC is a chord of a circle. From a point B on the minor arc AC lines BD, BE are drawn $\parallel$ respectively to the tangents at A and C, cutting AC at D and E. Prove that the $\triangle$s ABD, BCE are similar and that $AD \cdot CE = BD^2$.

145. ABCD is a quad. inscribed in a circle, AB being $\parallel$ to DC. AD and BC meet at X, and the tangents at A and C intersect at Y. Prove that XY is $\parallel$ to AB or CD.

146. Show that the perimeter of a polygon is always greater than that of a polygon of which the successive corners lie in the sides of the first polygon taken in order. Show also that the perimeter of a polygon is greater than the perimeter of any polygon lying wholly within it, provided that the latter is not re-entrant.

147. If the internal $\angle$s A and D of a parallelogram ABCD are bisected by lines AP, DQ, meeting the diagonals of the parallelogram at P and Q, prove that the $\triangle$s ABP and DCQ are equal in area.

148. The bisector of the $\angle ABC$ of a $\triangle ABC$ cuts the opposite side at D and the circumcircle at E. Prove that (i) $AB \cdot BC = AD \cdot DC + BD^2$, (ii) $CE^2 = AD \cdot DC + DE^2$.

149. From a point O on the diagonal AC of a parallelogram ABCD parallels are drawn to the sides, meeting the other sides at P, Q and R, S (P on AD, Q on BC, R on AB, S on CD). Prove that PQ, RS are divided similarly at O, and that SP and QR produced meet on CA produced.

150. Prove that, if equilateral $\triangle$s are drawn on the exterior of the sides of any $\triangle$, their centres will always form an equilateral $\triangle$.

STANDARD LOCI

Law satisfied by the moving point	Locus in two dimensions	Locus in three dimensions	Reference
1. A, B are fixed points; PA=PB.	The ⊥ bisector of AB.	The plane bisecting AB at rt. ∠s.	Theorem 31.
2. ACB, COD are fixed intersecting st. lines; P is equidistant from them.	The bisectors of the ∠s between the fixed st. lines.	The planes ⊥ to the plane containing the fixed st. lines and passing through the bisectors of the ∠s between them.	Theorem 32.
3. AB, CD are fixed ‖ st. lines; P is equidistant from them.	A st. line ‖ to AB, CD and mid-way between them.	A plane ⊥ to the plane containing the fixed st. lines and passing through the st. line ‖ to AB, CD and mid-way between them.	Theorem 2[illegible].
4. AB is a fixed st. line; P moves at a fixed distance (=k) from it.	Two st. lines ‖ to AB, one on either side of it, at a fixed distance (=k) from it.	A rt. circular cylinder with AB as axis, and the fixed distance (=k) as radius of the circular section ⊥ to the axis.	Theorem 21.

STANDARD LOCI—*Continued*

Law satisfied by the moving point	Locus in two dimensions	Locus in three dimensions	Reference
5. AB is a fixed st. line ; P moves so that its distance (measured ‖ to a given st. line) from AB is constant.	Two st. lines ‖ to AB, one on either side of it, at a fixed distance from it.	——	Theorem 21.
6. A, B are fixed points ; P moves so that the area of the △ABP is constant.	Two st. lines ‖ to AB, one on either side of it, at a fixed distance from it.	A rt. circular cylinder with AB as axis.	Theorem 27.
7. O is a fixed point. P moves so that its distance from O is constant ($=r$).	A circle, centre O, radius r.	A sphere, centre O, radius r.	Definition of a circle or sphere.
8. P is the mid-point of parallel chords of a fixed circle.	A diameter of the circle, ⊥ to the chords.	——	Theorem 33.
9. P is the mid-point of equal chords of a fixed circle.	A concentric circle, touching the chords.	——	Theorems 35, 46.
10. A, B are fixed points ; ∠APB=constant.	Equal segments of two circles one on each side of AB, each standing on AB and containing the given angle.	The surface obtained by rotating the plane locus about AB as axis.	Theorem 39.

11. A, B are fixed points; $\angle APB = 90°$.	A circle on AB as diameter.	A sphere on AB as diameter.	Theorem 41.
12. P is the centre of a circle touching the fixed st. line AB at a fixed point X.	A st. line through X $\perp$ to AB.	A plane through X $\perp$ to AB.	Theorem 46.
13. P is the centre of a circle touching the fixed st. line AB, and having a given radius.	Two st. lines $\parallel$ to AB, one on each side of it, and at a distance from it equal to the given radius.	A right circular cylinder, with axis AB, and having the given radius as radius of the circular section $\perp$ to AB.	Theorem 46.
14. P is the centre of a circle touching two fixed $\parallel$ st. lines.	A st. line $\parallel$ to the given st. lines and mid-way between them.	— —	Theorem 46.
15. P is the centre of a circle touching at X a fixed circle, centre O.	The st. line OX, produced indefinitely either way.	——	Theorem 48.
16. P is the centre of a circle of given radius a, touching a fixed circle, centre O and radius b.	Two circles, centre O, one with radius $a+b$, the other with radius $a \sim b$.	——	Theorem 48.
17. A, B are fixed points; P moves so that $PA^2 - PB^2 =$ constant.	A fixed st. line $\perp$ to AB.	A fixed plane $\perp$ to AB.	Theorems 55, 56.
18. A, B are fixed points; P moves so that $PA^2 + PB^2 =$ constant.	A circle whose centre is the mid-point of AB.	A sphere whose centre is the mid-point of AB.	Theorem 57.

STANDARD LOCI—*Continued*

Law satisfied by the moving point	Locus in two dimensions	Locus in three dimensions	Reference
19. A, B are fixed points; P moves so that PA : PB=constant.	The circle of Apollonius.	The sphere of Apollonius.	Theorem G.
20. O is a fixed point; P moves along a fixed st. line. Q moves so that OQ : OP=constant.	The locus of Q is a st. line ‖ to the st. line along which P moves.	——	Theorem H.
21. O is a fixed point; P moves along a fixed plane. Q moves so that OQ : OP=constant.	——	The locus of Q is a plane ‖ to the plane along which P moves.	
22. As in No. 20, but P moves along a fixed circle.	The locus of Q is a fixed circle.	——	Theorem I.
23. As in No. 21, but P moves on a fixed sphere.	——	The locus of Q is a fixed sphere.	
24. P moves so that the tangents from P to two fixed circles are equal.	The Radical Axis of the two circles.	——	Theorem J.

INDEX

The numbers refer to the pages

(*See also Table of Contents*)

Acute angle, 4
Acute-angled triangle, 10
Adjacent angles, 3
Algebraical identities, 275-283
Alternate angles, 22
Alternate segment, 233
Altitude (angular), 55
  of △ or par^{m}, 119
Ambiguous case, 80
Angle (and arms of), 3
  between line and circle, 220
  between two circles, 220
  of depression, 55
  of elevation, 55
  in a segment, 186
  in alternate segment, 233
  in a semi-circle, 192
  subtended at a point, 186
Angular point of △, 8
Apollonius' Circle, 370
  Theorem, 288
Arc of a circle, 5
Area, 8, 117, 271
  of circle, 211
  of par^{m}, 121, 127
  of rectangle, 118
  of rectilineal figure, 134
  of square, 117
  of trapezium, 135
  of triangle, 126, 128, 294
Axiom, 1

Base of triangle, 8
Bearing of an object, 54
Between same parallels, 120
Bisection, 6

Cardinal Points of the Compass, 54
Centre of a circle, 5
Centroid, 112
Ceva's Theorem, 331
Chord of a circle, 5
Circle, 5
  Area of, 211
Circumcentre, 186
Circumcircle (and radius), 186
Circumference of a circle, 5, 211
Circumscribed circle, 186
  figures, 220
Collinear points, 110
Commensurable magnitudes, 378
Common tangents, 226
Complement, 12
Complementary angles, 12
Complements of a par^{m}, 139
Concentric circles, 6
Conclusion, 8
Concurrency theorems, 109-113
Concurrent lines, 110
Concyclic points, 186
Congruent triangles, 11
Conjugate arcs, 186
Construction, 7, 8
Contact, 220
Continued proportion, 317
Converse theorems, 8
Convex polygon, 9
Co-planar, 22
Corollary, 3
Corresponding angles, 22
  sides and angles of a △, 11
Cyclic figure, 186
  quadrilateral, 195

Decagon, 9
Degree, 4
Depression, angle of, 55
Described about, figures, 186, 220
Diagonal, 82
Diagonal Scale, 100
Diagrams to scale, 55
Diameter of circle, 5
Distance between two points, 7

Elevation (angle of), 55
Equality, test of, 7
Equal in all respects, 11
Equiangular triangles, 334
Equilateral figures, 9
  triangles, 9
Equivalent (or equal), 120
Escribed circles, 220
Ex-centre, 220
Exterior angle, 22
  of triangle, 27
External bisector, 27
  contact, 220
  division, 272, 319, 361

Fourth proportional, 317, 360

General enunciation, 7

Height of Δ or par^{m}, 119
Heptagon, 9
Hexagon, 9, 252
Homothetic centre, 374
  figures, 374
  ratio, 374
Horizontal, 54
Hypotenuse, 10
Hypothesis, 8
Hypothetical constructions, 6

Image of a point in a line, 77, 175
In-centre, 220
Incommensurable magnitudes, 378
Inequalities, 70-74, 114-115
Inscribed circle, 220
  figure, 186
  regular polygon, 253
Intercept theorems, 93-96
Interior angles, 22
  opposite angles, 27
Internal contact, 220
  division, 272, 319, 361

Irrational numbers, 378
Isosceles Δ, 9
  base and vertex of, 9

Join, 3

Kite, 83

Line, 2
Locus, 157

Major arc, 186
  segment, 186
Mean proportional, 317 362
Medial section, 311
Median, 10
Menelaus' Theorem, 348
Mid-point, 6
Minor arc, 186
  segment, 186
Minute, 4

Nine-point circle, 263
Non-convex, 31

Obtuse angle, 4
Obtuse-angled triangle, 10
Octagon, 9
Order, in, 9
Orthocentre, 113, 263
Orthogonal circles, 157, 308

Parallel lines, 22
Parallelogram, 82
Particular enunciation, 7
Parts of a triangle, 10
Pedal (or Simson) line, 264
Pedal triangle, 263
Pentagon, 9
Perimeter, 8
Perpendicular, 4
  bisector, 61
Plane, 3
  figure, 5, 8
Playfair's Axiom, 22
Point, 2
Points of compass, 54
Point of contact, 218
Polygon, 9
  Inscribed regular, 253
Postulate, 2
Produce, 7, 9

Projection, 284
Proof, 8
  by Exhaustion, 72
Proportion, 317
Proposition, 7
Ptolemy's Theorem, 392
Purely geometrical constructions, 58

Quadrilateral, 8, 82

Radical Axis, 375
Radius of circle, 5
Ratio, 317
Rational numbers, 378
Rectangle, 82
Rectilineal figure, 8
Reductio ad absurdum, 24
Re-entrant, 31
Reflex angle, 4
Regular polygon, 9
Rhombus, 82
Right angle, 4
Right-angled triangle, 10
Ruler and compass constructions, 58

Scale diagrams, 55
Scalene triangle, 10
Secant of a circle, 218
Second, 4
Sector of a circle, 6
Segment of a circle, 5
  line, 7, 272
Semicircle, 5
Sense, 7
Side of a triangle, 8
Similar figures, 332
  segments, 186
  triangles, 333 seq.
Simson (or pedal) line, 264
Square, 82
Straight line, 2
Subtended, angle, 186
Superposition, 7
Supplement, 12
Supplementary angles, 12
Surface, 2
Symmetrical, 38
Symmetry, 38, 175
  axis of, 38
  centre of, 175

Tangent to a circle, 218
Theorem, 7
Third proportional, 317, 360
Touch, 220
Transversal, 22
Trapezium, 83
Triangle, 8

Vertex of angle, 3
  triangle, 8
Vertical, 54
Vertically opposite angles, 4

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ANSWERS

ANSWERS TO NUMERICAL EXERCISES

THE answers given here have been found by calculation and are accurate as far as they go. This degree of accuracy cannot be expected in Geometrical Drawing, but the answers given furnish a standard by which the pupil may test the correctness of his drawing and measurement. In Calculations the pupil's answers should be exactly the same as those given here.

PART I

EXERCISE 1 (P. 14)

1. $1\frac{1}{4}$ rt. $\angle$, 137°, 62°, 86° 25′, 143° 17′, 18° 51′, $\frac{3}{5}$ rt. $\angle$.

2. $\frac{5}{7}$ rt. $\angle$, 63°, 57° 34′, $48\frac{1}{3}$°, 10° 27′, 40° 22′ 48″.

3. 117°, 63°. **4.** 59°. **5.** $\angle BOD = 126°$, $\angle AOD = \angle BOC = 54°$.

6. $\angle BOC = 36°\ 17'$, $\angle AOC = \angle BOD$ 143° 43′. **7.** $90° + x°$, $90° - x°$.

8. $\angle BOD = x°$, $\angle AOD = \angle BOC = 180° - x°$.

9. 30°, 120°, 150°, $30x°$ or $360° - 30x°$.

10. (i) 90°, 240°, $6x°$; (ii) $7\frac{1}{2}$°, 20°, $\frac{x°}{2}$.

11. 110°, 150°. **13.** $82\frac{1}{2}$°, 185°. 4.30 p.m. **14.** 177°, $302\frac{3}{8}$°.

EXERCISE 3 (Pp 26, 27)

1. (i) $c = f = g = 53°$, $a = d = e = h = 127°$;
(ii) $a = d = h = 114°$, $b = c = f = g = 66°$.

2. $d = g = l = r = t = w = y = 117°$, $a = c = h = m = q = s = x = z = 63°$,
$e = k = j = 93°$, $f = n = v = i = 87°$.

EXERCISE 4 (Pp. 29, 30)

1. 54°. **2.** 55°. **3.** 72°.

4. 40°, 60°, 80°. **5.** 52°, 60°. **6.** 30°.

7. 50°; (i) 80° (or 100°), 30° (or 150°), (ii) 85° (or 95°), 15° (or 165°).

8. 100°, 80°. **9.** $50\frac{1}{2}$°. **11.** 72°.

13. A = 75°, B = 70°, C = 35° **14.** 100°, 60°, 20°.

15. P = 81°, Q = 27°, R = 72°. **16.** 42°, 54°, 84°.

EXERCISE 5 (Pp 33, 34)

1. 98°. **2.** 138°, 69°. **3.** 60°, 120°.

4. 45°, 36°, 30°, 18°, 72°, $\frac{360°}{n}$.

135°, 144°, 150°, 162°, 108°, $180° - \frac{360°}{n}$.

5. 9. **6.** 18. **7.** 30°, 48° **8.** Yes, No, No, Yes

9. Yes, 10; No; Yes, 72; No **10.** 18°, 54°, 126°, 162°

11. $\frac{2n-2}{n}$. **12.** 70°, 82°, 110°, 98°. **13.** 42°, 84°, 126°, 168°, 120°

14. One is 150°, the others are each 125°.

15. 170°, 140°, 160°, 160°. **16.** $112\frac{1}{2}°$.

17. 118°, 108°, 62°, 72°. **18.** A = C = E = 160°, B = D = F = 80°.

19. 60°, 35°, 25°, 25°, 35°.

20. $(80+x-y)°$, $(60+y-z)°$, $(40+z-x)°$, $x=70°$, $y=z=90°$

21. 108° 72°, 108°, 144°, 108°. **22.** $146\frac{1}{4}°$. **23.** 5.

24. C = 155°, D = 70°, F = F = 135°.

EXERCISE 7 (Pp. 40, 41)

1. (i) 44°, 68°, 68°; (ii) 34°, 73°, 73°;
(iii) 128° 34′, 25° 43′, 25° 43′; (iv) 152° 54′, 13° 33′, 13° 33′;
(v) 105°, 37° 30′, 37° 30′, (vi) 98° 9′, 40° $55\frac{1}{2}$′, 40° $55\frac{1}{2}$′

2. (i) 54°, 54°, 72°, (ii) 80°, 80°, 20°. **5.** 108°, 72°.

6. 18°, 18°. **7.** 36° each **8.** 90°.

9. ∠ABD = ∠BDE = ∠EDC = 35°, ∠BED = 82°, ∠DCE = 47°, ∠DBE = 63°, ∠ADB = 110°, ∠ABC = 98°

13. 56°, 100°, 38°, 62°. **14.** 21°. **15.** 36°, 168°. **16.** 162°, 81°

17. 128° 34′; 77° 9′, 51° 26′, 51° 26′, 12° 51′, 25° 43′, 141° 26′.

18. All are either 36° or 72° or 108° **19.** $54\frac{1}{2}°$ **21.** 38°

EXERCISE 9 (P 49)

11. (i) 30°, 75°, 75°, (ii) 150°, 15°, 15°.

EXERCISE 11 (Pp. 51, 52)

1. 105°, 135°, 125°, 100°, 75°.

5. ∠OCA = A°, ∠OBC = ∠OCB = 90° − A°.

EXERCISE 12 (Pp. 56, 57, 58)

2. 31·8 yds. **3.** 33·3 ft. **4.** 7·39 cm.

5. (i) 21 ft., (ii) 36·4 ft. **6.** 5·58 miles, S. 32° 33′ E.

7. 25·45 miles, N. 25° 37′ W. **8.** 14·2 miles.

9. 9·6 miles. **10.** N. 31° 45′ W., 11·1 miles. **11.** 471 yds.
12. 2·44 miles, 63° W. of S. **13.** Just over 12 m p.h.
14. 18·2 miles, S. 37° 40′ W. **15.** (*a*) 660 ft., (*b*) 560 ft.
16. 185 ft. **17.** 36° 52′. **18.** 4·47 miles.
19. 6020 ft. **20.** 330 ft. **21.** 64·3 ft., 31·7 ft.
22. 17·1 yds. **23.** 2·99 miles, N. 52° 24′ E.
24. 3·34 miles, S. 66° 41′ W **25.** 149 ft.

EXERCISE 13 (P. 69)

10. 9·01 miles N 11° 11′ E and N. 56° 11′ W.
12. 11·6 miles S 15° E.

EXERCISE 17 (P. 89)

7. 60°, 60°, 120°, 120°; 1′ **15.** 110°, 130°, 60°, 170°

EXERCISE 19 (P. 97)

1. $1\frac{1}{2}''$, $\frac{1}{2}''$. **2.** $4\frac{1}{2}''$, 3″

EXERCISE 22 (P. 105)

48. 14·5 cm. or 9·73 cm.; 8·05 cm., 11·6 cm.

PART II

EXERCISE 25 (Pp. 130, 131)

1. 10 sq. in. **2.** 14·7 sq. cm. **3.** 7·5 sq. in **4.** 1·62 sq. cm.
5. 5·4 sq. in. **6.** 14·4 sq. cm. **7.** 1·6 sq. in. **8.** 2·88 sq. in
9. 4·9 sq. in. **10.** 14·4 sq. cm. **11.** 3 cm. **12.** 3″.
13. 4·16 sq. in. **14.** 14·3 sq. cm. **15.** 6·68 sq. cm. **16.** 1·42 sq. in
17. 78·4 sq. cm **18.** 2·61 sq. in. **20.** 3 sq in., 4·5 sq. in.
21. 44 sq. in **22.** $\frac{1}{4}$, $\frac{1}{4}$. **24.** 72600. **25.** $\frac{1}{8}$.
26. 25″. **29.** $5\frac{1}{4}$ sq. in., 14 sq. in. **30.** $\frac{2}{3}$.

EXERCISE 26 (P. 132)

3. $\frac{1}{8}$, $\frac{1}{2}$, $\frac{3}{8}$.

EXERCISE 27 (Pp. 135, 136)

1. 5·79 sq. in. **2.** 7·69 sq. in. **3.** 50·4 sq. cm.
5. 7·16 sq. in. **6.** 6·89 sq. cm

EXERCISE 30 (Pp. 149, 150, 151, 152)

1. 13″, 6·5″, 17 cm., 8·4″, 7.08 miles.
2. 1·5″, 2·7″, 15·6 cm., 140·7 yds. **3.** 4·036″. **4.** 9·9 cm.
5. $4+\sqrt{7}=6{\cdot}646''$. **6.** 23·41 ft. **7.** 20 ft.
8. 33 miles. **9.** 3·25 cm **10.** 1·6 cm.
11. 1·5″. **12.** (ii), (iii), (v). **14.** 30 sq. cm
16. 1764 sq. ft. **19.** 24 ft. from the foot of the wall, $3\frac{1}{18}$ ft.
20. 8·39 miles. **21.** 34 miles.
22. 5″, 1″, $\sqrt{24}$ = 4·899″, 14·697 sq in **23.** 4·36 miles, 5·60 miles
24. 26·9 ft., 0·29 ft. **25.** 44·87 ft, **27.** 1″, $\sqrt{34}-5=0{\cdot}831''$.
28. $\sqrt{52}=7{\cdot}21''$, $\sqrt{142{\cdot}16}=11{\cdot}9''$.
29. $\sqrt{45}=6{\cdot}708$ ft., $\sqrt{380{\cdot}8}=19{\cdot}51$ cm. **30.** $\sqrt{57}=7{\cdot}55''$, 0·92 cm.
31. 30″. **32.** 4·5″. **33.** $\sqrt{61}$ ft $=7'\ 10''$.
34. VB = 17 cm. VA = VC = $\sqrt{322}=17{\cdot}94$ cm. **35.** $\sqrt{17}=4{\cdot}123''$.
36. $\sqrt{13{\cdot}5}=3{\cdot}674''$. **37.** 4 cm. **38.** $3\sqrt{22}=14{\cdot}07$ cm
39. 8·13″. **40.** $3\sqrt{2}=4{\cdot}24''$. **41.** $\sqrt{6}''$, $\sqrt{6}''$, 2″; $\sqrt{5}=2{\cdot}236$ sq. in.

EXERCISE 33 (Pp. 156, 157)

1. 15·6 ft. **3.** 2 $\sqrt{2}=0{\cdot}586$. **4.** 3 cm., 7 cm.
9. 2·9″. **12.** 1 : 3 or 2 : 3. **13.** 4·8″.

EXERCISE 36 (Pp. 169, 170, 171)

4. 210 yds., 434 yds., 169 yds., 349 yds.
17. P 1·87″, Q 1·5″ and 2·5″. **18.** AP = 2·96″, AQ = 2·81″.
19. PB = 6″, PC = 5″. **25.** 7·8 cm. or 15·5 cm.

PART III

EXERCISE 37 (Pp. 177, 178, 179)

1. 0·7 cm. **2.** 3·4 cm. **3.** 12″. **4.** $3\sqrt{3}=5{\cdot}196''$.
5. 14 cm. **6.** 5″. **7.** $2\sqrt{35}=11{\cdot}83''$. **8.** 2·5″.
9. 2·5″. **17.** 150 ft. **21.** 5″, 13″. **22.** 2″. **23.** 6″.

EXEPCISE 39 (P. 184)

5. 4·5″, 2·4″.

ANSWERS

EXERCISE 40 (Pp. 188, 189)

1. 40°, 32°. **2.** 27°, 116°. **3.** 82°, 36°, 36°, 41°.
4. 43°, 22°, 100°. **5.** 66°, 95°. **6.** 45°. **7.** 98°.
8. 95°, 105°, 85°, 75°, 80°, 100°. **9.** 130°, 25°. **10.** 25°.
11. 230°, 200°, 160°, 130° **12.** $\angle C = 115°$, $\angle B = 70°$.
13. 106°, 66°, 74°, 114° **14.** 55°. **16.** 66°.
25. $\frac{B}{2}+\frac{C}{2}$, $\frac{C}{2}+\frac{A}{2}$, $\frac{A}{2}+\frac{B}{2}$.

EXERCISE 41 (Pp. 197, 198, 199)

1. (i) $\angle BCX$ 72°, $\angle ADC$ $\angle CBX$ 77°, $\angle ABC$ 103°, $\angle DCB$ 108°
(ii) $\angle BCX$ $x°$, $\angle ADC$ $= \angle CBX = 149° - x°$, $\angle ABC$ $31° + x°$,
$\angle DRC = 180° - x°$
2. 4″. **3.** 30° E of N. **4.** 61°, 96° **5.** 112°, 68°
6. 50°, 80°, 130°, 100°. **7.** 75°, 87°, $75° - x°$, $87° - x°$, 36°.
8. 70°, 30°, 110°. **9.** 108°, 115°, 110°, 102°, 105°. **10.** 124°.
11. 90°, 140°, 80°, 90°. **12.** 55°, 85°, 125°, 95°. **13.** 120°.
15. 34°. **28.** 9·6 cm.

EXERCISE 42 (P. 202)

1. 31°. **2.** 124°, 127°. **3.** 98°, 128°.

EXERCISE 43 (P. 212)

1. 87°, 108°. **2.** 72° **3.** 15° or 105° **4.** 10°.
5. 60°. **6.** 90°, $\frac{7}{18}$ **7.** 75°, 30°, 30°. **8.** 90°.
9. 15°, 30°, 60°. **10.** $\frac{1}{18}$

EXERCISE 44 (P. 214)

9. Each $= 180° - A°$.

EXERCISE 45 (Pp. 228, 229, 230, 231, 232)

1. 20″. **2.** 25 cm. **3.** 24″. **4.** 58°, 55°, 67°.
5. 4″, 5″. **6.** 1·7″, 1·6″. **29.** 2·4″. **30.** 0·7″.
31. (i) 1·6″. **32.** $\sqrt{30 \cdot 16} = 5 \cdot 49$ cm., 4·4 cm. **43.** 2·5 cm.
54. 3·09 cm.

EXERCISE 46 (P. 236)

1. 76°. **2.** 68°, 44°, 68°. **3.** 76°, 56° 48°
4. 119°. **5.** 78°. **6.** 46°, 80°. **7.** 1″ **9.** 12°

EXERCISE 47 (P 239)

17. $\frac{a}{3}$.

EXERCISE 48 (P 247)

39. 1″ **47.** $81\frac{1}{2}°$, 115°, $98\frac{1}{2}°$, 65°.

48 $x < \sqrt{5} + 1$, no solution, $x = \sqrt{5} + 1$, one solution;
$\sqrt{5} + 1 < x < \sqrt{5} - 1$, two solutions,
$x = \sqrt{5} - 1$, three solutions; $x < \sqrt{5} - 1$, four solutions

EXERCISE 49 (P. 254)

8. PA = 5·1 m., PB = 8·7 m., PC = 10·9 m **16.** 3·4772 sq in

EXERCISE 50 (Pp 260, 261)

10. $\frac{B}{2} + \frac{C}{2}$, $\frac{C}{2} + \frac{A}{2}$, $\frac{A}{2} + \frac{B}{2}$, $90° + \frac{A}{2}$, $\frac{C}{2}$, $\frac{B}{2}$. **13.** 7, 5, 1 cm. **18.** 3″.

24. 4·49 cm., 3·49 cm., 2·20 cm.

PART IV

EXERCISE 52 (Pp 289, 290 291)

1. 0·8′ **2.** 1·65″ **3.** 11 cm, $10\frac{1}{3}$ cm **4.** $5\frac{5}{6}$″, $3\frac{1}{6}$″.

5. (i) 3″, 4″, 6″, (ii) 3″, 5″, 6″ **6.** Obtuse angled

7. (i) Obtuse $10\frac{2}{3}$″, $4\frac{1}{3}$″ (ii) Acute. $8\frac{29}{30}$″, $6\frac{1}{30}$″ **8.** Obtuse

10. $2 + 2\sqrt{2} = 4{\cdot}83$″, $\sqrt{8} + 4\sqrt{2} = 3{\cdot}70$″, $2 + \sqrt{2} = 3{\cdot}41$″, $22\frac{1}{2}°$, $112\frac{1}{2}°$.

11. DB = 3·54″, DC = 1·46″.

15. $\sqrt{3} = 1{\cdot}732$″, $\frac{1}{\sqrt{3}} = 0{\cdot}577$″, $\frac{2\sqrt{2}}{\sqrt{3}} = 1{\cdot}63$″

25. (i) $\frac{1}{2}\sqrt{91} = 4{\cdot}77$″, $\frac{1}{2}\sqrt{19} = 2{\cdot}18$″, $\frac{1}{2}\sqrt{139} = 5{\cdot}9$″, (ii) $\sqrt{78} = 8{\cdot}83$″, $\sqrt{42} = 6{\cdot}48$″, $\sqrt{15} = 3{\cdot}87$″.

26. 10″. **27.** $\sqrt{106} = 10{\cdot}3$″ **28.** $\sqrt{43{\cdot}5} = 6{\cdot}60$″.

29. 32 sq. in. **30.** 6·5″.

32. (i) $\frac{4\sqrt{13}}{3} = 4{\cdot}81$″, $\frac{10}{3} = 3{\cdot}33$″, $\frac{2\sqrt{73}}{3} = 5{\cdot}70$″, (ii) 9″, $4\sqrt{13} = 14{\cdot}4$″, $2\sqrt{73} = 17{\cdot}1$″

EXERCISE 53 (P. 292)

5. 135°

EXERCISE 54 (P. 295)

1. 84 sq. in. **2.** 66 sq cm. **3.** 84 sq cm. **4.** 420 sq. in.
5. $234a^2$ sq in **6.** 24 sq yds.
8. (i) $\frac{1}{4}\sqrt{135} = 2{\cdot}9$ sq in, (ii) $\sqrt{1260} = 35{\cdot}5$ sq in. **9.** 180 sq yds.
10. 468 sq in **11.** $\sqrt{30{\cdot}7^5 - 5{\cdot}546''}$, 13·87 sq. in
12. 7·81 sq in. **13.** $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$. **14.** $\frac{1}{2}\sqrt{695} = 13{\cdot}2$ sq. in.

EXERCISE 55 (Pp. 302, 303, 304, 305)

1. (i) 6″, (ii) 20″, (iii) 24″, (iv) 39 sq. cm, (v) 13 cm
2. (i) 3·5″, (ii) 12 cm, (iii) 20″, (iv) 10, (v) 8·4″ **3.** 1″ **5.** 40′.
6. 15″, 48″, 32·5″. **7.** 4 cm **8.** $1\frac{2}{3}$″. **9.** 2 cm, 5 cm
10. 8 cm. **11.** 6″, 8″. **12.** 18″, 2″.
13. $8\frac{1}{3}$ cm., $5\frac{5}{8}$ cm. **14.** 10 cm. **15.** $\sqrt[5]{4}\sqrt{39} = 7{\cdot}81$ cm.
16. $\sqrt{9000} = 95$ miles nearly. **17.** 24′ approx.
18. 208·6 miles **19.** No. **21.** 96′, 16′.
25. 4, 16′. **37.** $\frac{9\sqrt{2}}{20} = 0{\cdot}636''$, $\frac{\sqrt{1762}}{20}$ 2·10″.

EXERCISE 56 (P. 307)

4. 106°.

EXERCISE 57 (Pp. 311, 312)

17. $4\sqrt{5} - 4 = 4{\cdot}944$ cm. **18.** $\sqrt{5} + 3 \cdot 5{\cdot}236''$.
19. $\sqrt{5} - \cdot = 1{\cdot}236''$. **23.** 1″, 36°, $\frac{\sqrt{5}+1}{2} = 1{\cdot}62''$.

PART V

EXERCISE 58 (Pp 320, 321)

1. (i) 1 : 6 (ii) 1 : 480, (iii) 3 · 100, (iv) 13 1600, (v) 7 : 40.
6. (i) 24, (ii) $\frac{r^2s}{p}$. **7.** (i) 128, (ii) $\frac{1}{16}$, (iii) a^3.
8. (i) ± 16, (ii) ± 6, (iii) $\pm a^{\frac{3}{2}}b^{\frac{3}{2}}$. **9.** 6 cm. **10.** One.
11. 3 cm. **12.** 9 cm. **13.** One **14.** $2alm - (l^2 \sim m^2)$.

EXERCISE 59 (P. 324)

1. 2 : 5 ; 2 : 3. **2.** 3 . 2 ; 3 : 25.
3. (i) 2·8″ ; (ii) 2·4 cm. ; (iii) 1·2″, (iv) 15″.
4. (i) 4·8″ ; (ii) 0·9″. **5.** 0·9″. **8.** $m : n$.

EXERCISE 60 (P. 329)

1. $2\frac{4}{13}''$, $2\frac{9}{13}''$. **2.** 21″, 14″. **3.** 24 cm.
4. $1\frac{1}{2}''$, 6″, $11\frac{1}{4}$ sq. in., 45 sq. in. **7.** 2 : 3.

EXERCISE 62 (Pp. 341, 342, 343, 344, 346)

1. (i) 4·8 cm., (ii) 6·3″, 1·4″, (iii) 4·2″, 1·6″. **2.** 2·7″, 1·8″.
3. 24 ft. **4.** 12 yds., 24 yds., 30 yds. **5.** 704 yds.
6. 6′ 9″. **7.** 10′ 5″. **8.** 6″. **9.** 6″.
10. 6″, 4″. **11.** 2·8″, 3·5″ **12.** 3·2 cm., 4·8 cm., 2·4 cm.
15. (i) 4·8 cm., (ii) 2·5 cm., 2 cm., (iii) 6″ or 12″, (iv) 5·8″
16. (i) 4·8 cm., (ii) 2·5 cm., 2 cm. **17.** (i) 6″, 9″, (ii) $6\frac{1}{4}''$, $2\frac{1}{2}''$
18. $2\frac{7}{8}''$. **19.** 2′ 8″ **20.** 13·5 cm. **21.** $2\frac{1}{4}''$.
22. 12″. **23.** 3″. **24.** (i) 2″, (ii) 5 : 1
25. 16″, 14″. **27.** 2′ 3″. **28.** $6\frac{1}{4}''$, $9\frac{3}{8}$ sq. in. **29.** 1 : 3.
30. 16″, 12″, $9\frac{1}{3}$. **31.** 2 : 1 ; 3 : 1. **32.** 3″ from D **33.** $2\frac{1}{4}''$.
63. $10 - \frac{x}{3}$; 7·5″.

EXERCISE 63 (Pp. 346, 347, 349)

5. $\frac{1}{x} = \frac{1}{b} - \frac{1}{a}$. **9.** $4\frac{1}{2}''$. **29.** 35 · 18 ; 53 · 24.

EXERCISE 64 (P. 351)

1. (i) 3·6″, 6·4″, 4·8″, (ii) 6″, $3\sqrt{13} = 10{\cdot}8''$, $2\sqrt{13} = 7{\cdot}2''$, (iii) $8\frac{8}{8}''$, $19\frac{4}{13}''$
2. 20 sq. cm.

EXERCISE 65 (Pp. 353, 354, 355)

1. 7·2 sq. in. **2.** 4 cm. **3.** 9·2″, 14·8″, 16·4″.
4. 40 sq. in. **5.** 4·8″. **7.** 2 : 1. **8.** 2 · 9.
9. 5″. **12.** 4 : 1. **13.** 2″ ; 25 : 16 **14.** 22 · 25.
17. $2\frac{8}{9}$ sq. ft. **18.** $\frac{4}{27}$ lb. **19.** £1 16s. 0d. **20.** 9s.
21. 1s. $8\frac{1}{4}$d. **22.** $3\frac{17}{512}$ oz **23.** 12″.
24. 192 sq. in., 43·2 sq. in. **26.** 1 : 2. **27.** 9 : 25.

EXERCISE 66 (P. 356)

11. 3 : 4.

EXERCISE 67 (P. 363)

1. (i) 2″, (ii) 1·8″, (iii) 4. **2.** (i) 3·2 c n., (ii) 2·8.
3. (i) 2·1, (ii) 1·$\frac{5}{8}$, (iii) 0·54, (iv) 0·37.
4. 0·9 cm., 1·2 cm., 6·3 cm., 8·4 cm.
5. 3·6″, 1·2″, 7·[illegible], 2·4″ **6.** (i) 3″, $\sqrt{11} = 3{\cdot}317$″.
7. (i) 3·606, (ii) 4·796, (iii) 7·[illegible]6.

EXERCISE 69 (P. 376)

1. 4″, 8″. **2.** 12 cm. **9.** $\sqrt{5} = 2{\cdot}236$″, $2\sqrt{5} = 4{\cdot}472$″.

MISCELLANEOUS EXAMPLES

(Pp. 382, 383, 385, 389, 390, 391, 392, 393)

12. 1 cm., 15 cm. **18.** B = 153°, C = 57°, AD = 3·56″ **43.** 0·9″
95. AC, 1·7″ and 3·8″, BD, 1·3 and 3·5″. **100.** 15″, 15″, 40″
110. C is $(a+b, a)$, D is $(b, a+b)$. Centre of square is $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$, i.e. $x = y$
115. 3·92″ or 2·80″. **116.** $6\frac{8}{13}$″ **118.** $\frac{8}{4+\pi} = 1{\cdot}1$″.
120. $\frac{\pi A\sqrt{3}}{9}$. **128.** 5 . 1. **133.** 0·5″.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
AT THE UNIVERSITY PRESS, GLASGOW.