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A TEXT BOOK
OF
HYDROSTATICS

FOR

B.A. AND B.SC. STUDENTS

BY

DR. M. RAY, D.Sc., F.N.I.

Principal, Agra College, Agra

AND

HAR SWARUP SHARMA, M.A.,

Associate Professor of Mathematics, Agra College, Agra

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Preface to the Third Edition

This edition contains a few more examples added here and there.

January, 1961

M. Ray.
H.S. Sharma.

Preface to the First Edition

This book on *Hydrostatics* is intended for the use of students preparing for the B.A. and B.Sc. Examinations of the Indian Universities and is so written as to cover the courses commonly prescribed for these Examinations of most of the Indian Universities. The Chapter on Centre of Pressure has been dealt in a different but simple and straight-forward way and the whole matter has been presented in a clear and lucid style and the propositions have been established in a perfectly logical manner.

Great care has been taken to explain the fundamental principles fully and rigorously. To illustrate these principles large number of examples have been worked out in as simple and straight-forward manner as possible. Numerous examples, mostly taken from University question papers, have been added after each important article so as to give the students an idea of what types of examples they can expect in the examinations. The book is complete as far as it goes and it is hoped that nothing of importance has been left out. Authors will consider their labour amply rewarded if this is of some value to those for whom it is intended.

Suggestions for improvements will be thankfully received.

Agra College,
Agra

M. Ray.
H.S. Sharma.

November, 1953.

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CHAPTER I

INTRODUCTORY

1. **Matter or substance** is broadly divided into two classes, (*i*) SOLIDS, (*ii*) FLUIDS.

A **Solid** *is a substance which retains its size and shape without lateral support when left to itself.*

A **Fluid** *substance is one which requires lateral support to maintain its shape or in other words it is a substance which flows or is capable of flowing.*

Fluids are again subdivided into two classes, *viz.*, **Liquids** and **Gases**.

Liquids require lateral support to maintain their shape. Their volume is the same whatever be the shape of the vessels containing them. Water or oil is a liquid, and while it may be poured from one vessel into another of a different shape, its volume does not alter but it moulds itself to the form of the new vessel.

Gases, on the other hand, are fluids which can easily be made to change their total volume, *i.e.*, which are, more or less easily, compressible.

Air and steam are the examples of gases. They can be made to expand indefinitely by increasing the volume to which they have access.

2. The distinction between **solid, liquid** and **gas** is sometimes formally stated as follows :—

A **solid** *has a definite size and a definite shape.*

A **liquid** *has a definite size but not a definite shape.*

A **gas** *has neither a definite size nor a definite shape.*

3. It has been seen experimentally that all gases can be converted into liquids by sufficient lowering of temperature and increase of pressure. When, however, the temperature of a gas is above a certain point, no increase of pressure, however great, will cause the gas to condense, whereas when the temperature is below this point, a sufficient increase of pressure produces condensation. This temperature which varies for different gases is called the *critical temperature.* At temperatures below the critical temperature, the gas is called a **vapour**, and when above, a **permanent gas**.

4. **Hydrostatics** *is the branch of Mathematics which deals with the conditions of equilibrium of masses of fluids or of solids in contact with fluids at rest.*

5. Fluids are again of two types :—

(*a*) PERFECT, (*b*) VISCOUS.

(*a*) **Perfect fluid** *is a substance such that its shape can be altered by any tangential force however small, if applied long enough, of which portions can be easily separated from the rest of the mass, and between different portions of which there is no* **tangential force**.

As in nature there are no perfectly smooth bodies, so no such fluids are to be found which are wholly devoid of exerting *tangential resistance*. If we set water revolving in a cup, we know that after some time it will come to rest. This is due to the frictional resistances between the water and the cup and between different elements of water. Had there been no tangential resistance the water would go on revolving and would not come to rest.

Thus, *a perfect fluid is an ideal substance which is incapable of exerting any tangential force.*

(*b*) *A fluid is said to be* **viscous** *when its particles exert friction on one another*, or *it offers resistance to the motion of a body in it.*

Fluids such as honey, tar, treacle are examples of this type. The viscosity of a fluid does not affect its equilibrium, but it does affect its motion.

Perfect liquids are incompressible fluids whose total volume, *i.e.*, the space they occupy, cannot be increased or diminished by the application of any force, however great, although any force, however small, would change their shape.

Liquids are also called **Inelastic Fluids**.

Gases are also known as **Elastic Fluids**.

6. **Fluid Pressure**. Suppose a hole be made in the side of a vessel containing fluid, and that this hole is covered by a plate which exactly fits the hole. The plate will only remain at rest when some external force is applied to it. The fluid must, therefore, exert a force on the plate. Suppose P be the force necessary to keep the area in position by counterbalancing the action of the fluid, then P is the measure of the **fluid pressure** or **fluid thrust** on the whole area of the plate.

7. **Uniform Pressure**. *If the fluid thrust on any portion of a plane area be proportional to the area of that portion, the pressure on the area is said to be* **uniform**.

If, however, the action of the fluid be different on equal portions of the area, the pressure is said to be *varying*.

8. **Mean Pressure**. *The mean pressure on a plane area is the uniform pressure on it which will give the same resultant thrust as the actual one.*

Hence, if A be the area, and P be the fluid thrust on it, the mean pressure is $\frac{P}{A}$.

9. Intensity of Pressure at a point. If the pressure is variable on a surface it is measured by the ratio of the pressure on a small area to the area itself, as that area becomes infinitesimally small in the limit.

Thus, if δP be the fluid pressure on a small area δA about a given point in the fluid, the mean pressure per unit area in the neighbourhood of that point is $\frac{\delta P}{\delta A}$.

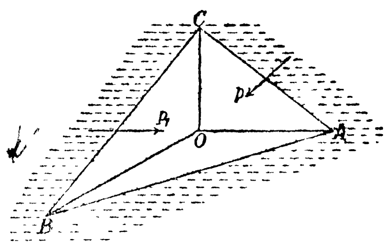
In the limit when δA tends to zero

$$\lim_{\delta A \to 0} \frac{\delta P}{\delta A} = \frac{dP}{dA} = p \text{ (say)}$$

then p is defined as the **Intensity of Pressure** or simply **Pressure** at that point.

Therefore, *the pressure at any point of an area is the limit of the mean pressure on an indefinitely small area enclosing the point.*

10. The pressure at any point of a fluid at rest is the same for all directions.



In the fluid draw a small tetrahedron OABC with three mutually perpendicular faces OBC, OCA, OAB.

Let p_1, p be the mean pressures on the surfaces OBC, ABC, so that the actual thrusts on these faces are $p_1 . \triangle$ OBC and $p . \triangle$ABC.

The forces acting on the fluid within this tetrahedron are :—

(1) The resultant of the external forces which will be proportional to the volume of the element, if gravity be the only external force acting on the fluid.

(2) The thrust on the face OBC which is $p_1 . \triangle$OBC acting along OA.

(3) The thrust on the face ABC which is $p . \triangle$ABC acting along the normal to the face ABC.

(4) The thrusts on the faces AOB and AOC which are acting along OC and OB respectively.

Since the $\triangle$BOC is the projection of the $\triangle$ABC on the plane BOC, we have

$$\triangle BOC = \triangle ABC \cos \theta$$

where θ is the angle between BOC and ABC or between OA and the normal to ABC.

Now the portion of the fluid enclosed within this tetrahedron is at rest because the whole mass of the fluid is at rest. Therefore resolving along OA,

$p_1 . \triangle OBC - p . \triangle ABC \cos \theta + k$ volume OABC $\cos \varphi = 0$ where φ is the angle between OA and the vertical line along which the weight of the fluid is acting.

or $p_1.\triangle OBC - p.\triangle OBC + k.\frac{1}{3}OA.\triangle\ OBC \cos\varphi = 0.$

or $p_1 - p + k.\frac{1}{3}OA \cos\varphi = 0.$

Now, let the elementary tetrahedron diminish indefinitely in size so that ultimately OA=0 and the tetrahedron dwindles into the point O. In that case, therefore, we have

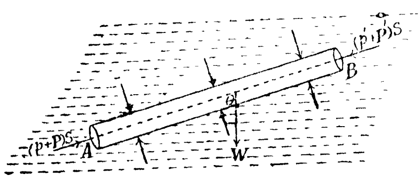
$$p_1 - p = 0 \qquad \text{or} \quad p_1 = p$$

i.e., the pressure at O along OA is equal to the pressure at O perpendicular to ABC.

In a similar way it can be shown that the latter is equal to the pressure along OB or OC, and by varying the magnitude of OA, OB and OC, it can also be shown that this holds whatever be the direction of the plane ABC.

Note. If there be any system of external forces instead of the weight, then instead of the vertical we shall have the direction of the resultant external force. Proceeding as before, the same result will be obtained.

11. Transmissibility of Liquid Pressure. *If a fluid at rest have any pressure applied to any point of its surface, the pressure is transmitted equally in all directions.**



Let A, B be any two points in the liquid.

Case I. *When the straight line* AB *lies entirely in the fluid.*

About the straight line AB as axis isolate a cylinder of small cross-section S with plane ends at right angles.

Let the axis of the cylinder be inclined at an angle θ to the vertical and the pressure intensity of the fluid on the plane ends A and B be p and p' respectively.

The fluid contained within the cylinder is kept in equilibrium by

(1) the pressures pS and p'S on the ends A and B respectively ;

(2) the pressures on the curved surface which are at every point perpendicular to AB ;

(3) its weight W acting downwards.

Now resolving these forces along AB, we have

$$pS - p'S - W\cos\theta = 0 \qquad \dots (1)$$

Apply an extra pressure of intensity P at the end A. In consequence, let the extra pressure of intensity at the other end B, be P'.

*This principle of perfect transmission of pressure of fluids appears to have been first discovered by STEVINUS ; but it was later rediscovered by **Pascal**, and, having been made generally known by his writings, is often called *Pascal's Principle*.

The system still being in equilibrium, resolving the forces along AB,

$$(p+P)S-(p'+P')S-W\cos\theta=0 \qquad \dots \quad (2)$$

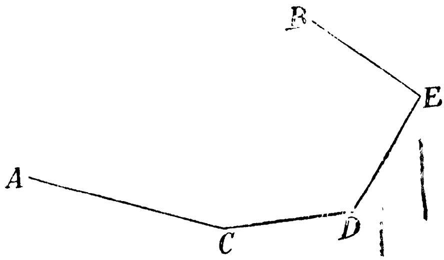
Subtracting (1) from (2)

$$P=P'$$

Hence a pressure of intensity P applied at the end A is transmitted with the same intensity P to the other end A.

Case II. *When the straight line AB is not entirely in the liquid.*

Join A, B by means of straight lines AC, CD, DE, EB each of which lies entirely within the liquid.

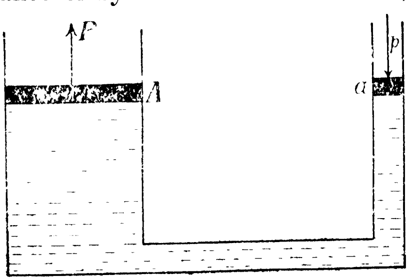


Then, by the proposition just proved, the additional pressure at A = the additional pressure at C = the additional pressure at D = ... = that at B.

Note. This may be illustrated by taking a rubber ball, pricking several holes in it with a needle and filling it with water. When the ball is squeezed, water jets of equal intensity spurt out in all directions normal to the surface of the ball at these points.

12. Hydraulic or Bramah's Press. The transmission of fluid pressure is the principle upon which *Hydraulics* or *Bramah's Presses* are constructed.

It consists of two vertical cylinders communicating with one another by a tube near their bases, and each fitted with a water-tight piston, the space below the pistons being filled with water.



If a downward force p be applied to the piston of area a, pressure $\frac{p}{a}$ will be transmitted throughout the liquid, and consequently if the second piston is of area A, the resulting upward thrust upon it is represented by

$$P=\frac{Ap}{a};$$

so that $$P:p=A:a$$

or *the thrust on the pistons are proportional to their areas.*

Hence by adjusting the ratio of the areas we arrive at the '**Hydrostatic Paradox**' *i.e., any force however small may, by its transmission through a liquid, be made to support any weight however large.*

CHAPTER II

DENSITY AND SPECIFIC GRAVITY

13. Density. *The density of a substance is defined as its mass per unit volume.*

The mass of a cubic foot of pure water at 4°*C* is found to weigh about $62\frac{1}{2}$ lbs. Hence the density of water is 62·5 in foot-pound system. The mass of a cubic centimetre of water at 4°*C* is 1 gm., hence density of water in C.G.S. system is 1.

Homogeneous. *A body is said to be homogeneous (or of uniform density) if equal volumes of it, however small, have equal masses.*

Otherwise, *the body is said to be* **heterogeneous** *(or of variable density).*

14. Weight in terms of density. *If* W *be the weight of a given substance in poundals, ρ its density in lbs. per cubic foot, V its volume in cubic ft., and g the acceleration due to gravity measured in foot-second units, then* $W=Vg\rho$.

If M be the mass of the substance, then we have by Dynamics,

$$W=Mg$$

Also from the definition of density

$$\begin{aligned} M &= \text{mass of V cubic feet of the substance} \\ &= V \times \text{mass of one cubic foot} \\ &= V\rho \end{aligned}$$

$$\therefore \quad W=Vg\rho$$

15. Specific Gravity or **Relative Density.** *The specific gravity of a substance is the ratio which its density bears to the density of some standard substance.*

Note 1. The standard substance usually taken is pure water at a temperature of 4°*C*.

Note 2. The specific gravity is always a *number*.

Note 3. The term specific gravity is generally shortened to sp. gr.

16. Weight in term of Specific Gravity. *If* W *be the weight of a volume* V *of a substance whose specific gravity is* **s**, *and* **w** *be the weight of a unit volume of the standard substance, then*

$$\mathbf{W=Vsw.}$$

Since $\quad s=\dfrac{\text{wt. of a unit volume of the substance}}{\text{wt. of a unit volume of the standard substance}}$

∴ wt. of unit volume of the substance $= sw$

∴ wt. of V units volume of the substance $= Vsw$

Hence $W = Vsw$.

17. Specific gravity of mixtures. *To find the specific gravity of a mixture of given volumes of different substances whose specific gravities are given.*

Let $V_1, V_2, V_3, \ldots$ be the volumes of the different substances and $s_1, s_2, s_3 \ldots$ their specific gravities, so that the weights of the different substances are

$$V_1 s_1 w,\ V_2 s_2 w,\ V_3 s_3 w \ldots\ldots$$

where w is the weight of a unit volume of the standard substance.

Let $\bar{s}$ be the specific gravity of the mixture, then the weight of the mixture is

$$(V_1 + V_2 + V_3 + \ldots)\,\bar{s}\,w\ ;$$

but this must be equal to the sum of the weights of the different substances, *viz.*,

$$V_1 s_1 w + V_2 s_2 w + V_3 s_3 w + \ldots$$

whence, equating and cancelling the factor w, we get

$$\bar{s} = \frac{\mathbf{V_1 s_1 + V_2 s_2 + V_3 s_3 + \ldots}}{\mathbf{V_1 + V_2 + V_3 + \ldots\ldots\ldots}}.$$

18. *To find the specific gravity of a mixture of given weights of different substances whose specific gravities are given.*

Let the given weights W_1, W_2, $W_3, \ldots$ have $s_1, s_2, s_3, \ldots$ as their respective specific gravities. Let s be the sp. gr. of the mixture and let w be the weight of a unit volume of the standard substance.

Then their respective volumes will be

$$\frac{W_1}{s_1 w},\ \frac{W_2}{s_2 w},\ \frac{W_3}{s_3 w}, \ldots\ldots\ldots$$

∴ The weight of the mixture is

$$\left(\frac{W_1}{s_1 w} + \frac{W_2}{s_2 w} + \frac{W_3}{s_3 w} + \ldots\right) s\,w \qquad \ldots \quad (1)$$

This must be equal to the sum of the weights of the substances of which the mixture is made, *viz.*,

$$W_1 + W_2 + W_3 + \ldots \qquad \ldots \quad (2)$$

Hence equating (1) and (2)

$$\mathbf{s} = \frac{\mathbf{W_1 + W_2 + W_3 \ldots\ldots}}{\mathbf{\frac{W_1}{s_1} + \frac{W_2}{s_2} + \frac{W_3}{s_3} \ldots\ldots}}$$

Note. It has been assumed in this article and the preceding that the volume of the mixture is the sum of the volumes of the component parts ; the results, therefore, will not hold for cases where there is an expansion or shrinkage in volume, as for instance when salt is dissolved in water.

19. Solved Examples.

Ex. 1. *When equal volumes of two substances are mixed the sp. gr. of the mixture is 6, when equal weights of the same substances are mixed, the sp. gr. of the mixture is 4. Find the specific gravities of the substances.* (*Utkal 1946*)

Let the specific gravities of the substances be s_1, s_2

$$\therefore \qquad 6=\frac{Vs_1+Vs_2}{V+V}=\frac{s_1+s_2}{2}$$

$$\text{and} \qquad 4=\frac{W+W}{\frac{W}{s_1}+\frac{W}{s_2}}=\frac{2}{\frac{1}{s_1}+\frac{1}{s_2}}$$

$$\therefore \qquad s_1+s_2=12 \qquad \dots (1)$$

$$\text{and} \qquad \frac{1}{s_1}+\frac{1}{s_2}=\tfrac{1}{2} \qquad \dots (2)$$

From (1) and (2) $s_1=6+2\sqrt{3}$ and $s_2=6-2\sqrt{3}$.

Ex. 2. *Show that the specific gravity of a mixture of n liquids is greater when equal volumes are taken than when equal weights are taken, assuming no change in volume as the result of mixing.* (*Utkal 1946*)

Let s be the sp. gr. of the mixture when the mixture of equal volumes is taken and s' be that when the mixture of equal weights is taken.

If $s_1, s_2, \dots s_n$ be the sp. gravities of the different substances, then we have

$$s=\frac{1}{n}(s_1+s_2+\dots\dots+s_n),$$

$$\text{and} \qquad \frac{1}{s'}=\frac{1}{n}\left(\frac{1}{s_1}+\frac{1}{s_2}+\dots+\frac{1}{s_n}\right)$$

Since the Arithmetic Mean of n unequal positive numbers is greater than their Geometric Mean, we have

$$s>(s_1s_2\dots s_n)^{\frac{1}{n}} \text{ and}$$

$$\frac{1}{s'}>\frac{1}{(s_1s_2\dots s_n)^{\frac{1}{n}}}$$

whence it follows that

$$s>s'.$$

Examples 1

1. 12 lbs. wt. of liquid of sp. gr. 1·1 is mixed with 20 lbs. of a liquid of sp. gr. ·9 ; what is the sp. gr. of the mixture ?

2. The specific gravity of cork being ·24, find what volume of water weighs as much as a cubic yard of cork.

3. A vessel full of water is fitted with a tight cork. How is it that a slight blow on the cork may be sufficient to break the vessel ? *(Agra 1954 ; U.P.S.C. 1940)*

(*Hint.*—Apply Principle of Transmission.)

4. The specific gravities of pure gold and copper are 19 3 and 8·62. Find the sp. gr. of standard gold which is an alloy of gold and copper in the proportion of 11 : 1. *(Patna 1940)*

5. What quantity of water must be mixed with a gallon of milk to reduce its sp. gr. from 1·03 to 1·02 ?

6. A mixture is formed of two fluids. The density ρ of the mixture, the ratio $m : 1$ of the volumes, and the ratio $n : 1$ of the densities are given. Prove that the densities of the fluids are

$$\frac{mn+n}{mn+1}\rho \text{ and } \frac{m+1}{mn+1}\rho.$$

7. Three equal vessels A, B and C are half full of liquids of densities ρ_1, ρ_2 and ρ_3 respectively. If now B be filled from A and then C from B, prove that the density of the liquid now contained in C is

$$\tfrac{1}{4}(\rho_1+\rho_2+2\rho_3)$$

the liquids being supposed to mix completely. *(Nagpur 1956)*

8. A mixture is formed of equal volumes of n liquids, the densities of which are in the ratio of the numbers 1, 2, 3, . n ; find the density of the mixture. Also find the density of the mixture when the volumes are in the ratio :—first, of the numbers 1, 2, 3, . n and second, of the numbers n, $n-1$,...3, 2, 1.

9. When equal volumes of two substances are mixed, the sp. gr. of the mixture is 4 ; when equal weights of the same substances are mixed, the sp. gr. of the mixture is 3. Find the sp. gr. of the substances. *(Calcutta 1945, 47, 49)*

10. Having given the sp. gr. ρ of a mixture formed of equal volumes of two fluids, and also the sp. gr. σ of a mixture formed by taking a volume of one fluid double that of the other, prove that the sp. gr. of the fluids are

$$3\sigma-2\rho \text{ and } 4\rho-3\sigma.$$

11. Two liquids of sp. gravities s, s' and of volumes v, v' having been mixed, the sp. gr. of the mixture is found to be σ. Prove that the volume of the mixture is

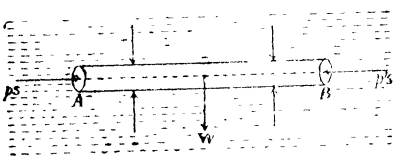
$$\frac{vs+v's'}{\sigma}.$$

CHAPTER III

THEOREMS ON FLUID-PRESSURES UNDER GRAVITY

20. Theorem. *The pressure of a heavy homogeneous fluid at all points in the same horizontal plane is the same.*

(*i*) Let A, B be any two points within the fluid in the same horizontal line, and *let the line AB be wholly within the fluid.* About AB as axis isolate a cylinder of the fluid of indefinitely small cross-section S.



Let p and p' be the intensities of the pressures at its two ends A and B.

The cylinder is in equilibrium under the action of the pressure of the surrounding fluid upon its bounding surfaces, and its weight acting downwards. The pressures on the curved surface will not contribute anything to the forces parallel to the length of the cylinder.

Hence for equilibrium resolving parallel to AB ; $pS - p'S = 0$

$$\therefore \quad p = p'.$$

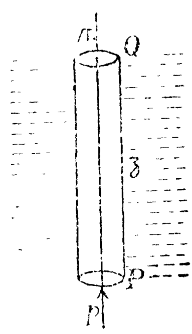
(*ii*) *If the st. line AB does not lie entirely in the fluid.* In the case join AB by a broken horizontal line ACDEB, the several parts AC, CD, DE, EB being straight and lying entirely in the fluid.

$\therefore$ The pressure at A = that at C = that at D = ... = that at B.

21. To find the pressure at a depth z below the surface.

(*a*) *If the liquid be homogeneous, the pressure at a depth z below its surface is given by $p = \omega z + \pi$, when π is the atmospheric pressure and ω the weight of a unit volume of the liquid.*

Take a point P within the liquid at a depth z below the surface and draw PQ perpendicular to the surface. With PQ as axis construct a thin cylinder of cross-section S. This cylinder of liquid is in equilibrium under the following forces :—



(1) thrust Sp on the face at P upwards,
(2) thrust $S\pi$ on the face at Q downwards,
(3) thrust on the curved surface horizontally
(4) the weight of the liquid inside the cylinder which is $Sz\omega$ downwards.

Hence for equilibrium resolving vertically we have $\quad S\pi + Sz\omega - Sp = 0$

$$\therefore \quad p = \pi + \omega z.$$

If there be no atmospheric pressure then $p = \omega z$, thus the *pressure at any point varies as the* *depth of the point.*

Also if p_1 be pressure at depth z_1 and p_2 that at depth z_2 then

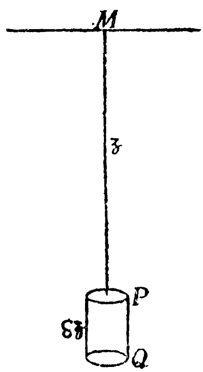
$$p_1 = \pi + \omega z_1 \quad \text{and} \quad p_2 = \pi + \omega z_2$$

$$\therefore \quad p_1 - p_2 = \omega(z_1 - z_2).$$

Hence *in homogeneous liquid at rest under gravity the difference between the pressures at two points varies as the vertical distance between them.*

(*b*) *When the liquid is heterogeneous.*

To find the pressure at a point P at a depth z below the surface, draw a perpendicular PM to the surface, and let MP be produced to Q so that $PQ = \delta z$. With PQ as axis draw a thin cylinder of cross-section S. Since this cylinder is very small, density of the liquid inside it may be taken to be constant and let the weight per unit volume of it be ω. It is to be noted that ω is not the same at every point of the liquid but varies from point to point.



If p be the intensity of pressure at the point P, then we can take $p + \delta p$ to be the intensity of pressure at Q.

Now this elementary cylinder of liquid is in equilibrium under the following forces :—

(1) thrust pS on the face at P downwards.
(2) thrust $(p + \delta p)S$ on the face at Q upwards.

(3) thrust on the curved surface horizontally,

(4) the weight of the liquid which is $S\delta z\omega$ downwards.

Hence for equilibrium resolving vertically we have

$$pS + S\omega\delta z - (p + \delta p)S = 0$$

$$\therefore \quad \delta p = \omega \delta z$$

or in the limit, $$\frac{dp}{dz} = \omega.$$

If however ω be constant, *i.e.*, liquid be homogeneous then we have by integration

$$p = \omega z + C.$$

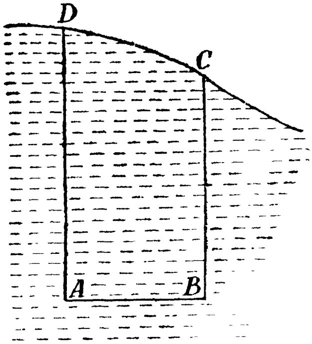
At the surface when $z = 0$, $p = \pi$ $\therefore$ $p = \pi + \omega z$.

This is part (*a*).

22. *The surface of a heavy liquid at rest is horizontal.*

Take two points, A and B, in the same horizontal plane in the liquid.

Then the pressure at A is equal to the pressure at B.



Hence from the previous theorem

$$w.AD+\Pi=W.BC+\Pi$$

where Π is the atmospheric pressure.

$$\therefore \quad AD=BC$$

Hence, since AB is horizontal, the line DC also must be horizontal.

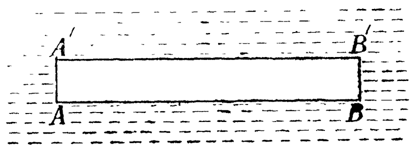
Since A and B are any two points in the same horizontal line, it follows that any line DC drawn in the surface of the liquid must be horizontal.

Hence the surface is a horizontal plane.

Note. The above theorem confirms the popular saying "*Water finds its own level.*"

23. Theorem. *The densities at two points in a fluid at rest under gravity and in the same horizontal plane are equal.*

Let A, B be two points in the fluid in the same horizontal line and A′, B′ two points at a short distance vertically above A, B and such that AA′=BB′.



Let ρ denote the mean density of the fluid between A and A′ and ρ' the mean density of the fluid between B and B′, so that

$$\text{pressure at A}-\text{pressure at A}'=g\rho AA'$$

Also $\quad \text{pressure at B}-\text{pressure at B}'=g\rho' BB'$

But $\quad \text{pressure at A}=\text{pressure at B}$

and $\quad \text{pressure at A}'=\text{pressure at B}'$.

$$\therefore \quad g\rho AA'=g\rho' BB'$$

or $$\rho=\rho'.$$

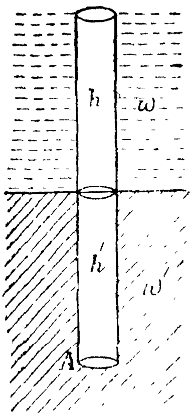
24. Theorem. *To find the pressure at any given depth in the lower of two given heavy homogeneous liquids which do not mix.*

Let the weights per unit volume of the upper and lower liquids be w and w' respectively. Let the depth of the upper liquid be h, and A be any point in the lower liquid at a depth h' below the common surface.

Isolate a vertical cylinder with its base at A, and its upper end in the free surface of the upper liquid.

Let S be the area of the cross-section of the cylinder, and p the intensity of pressure on the base.

The cylinder is in equilibrium under the pressures on its bounding surfaces and its own weight acting downwards.



The pressures on the vertical curved sides are horizontal and balance one another. For vertical equilibrium

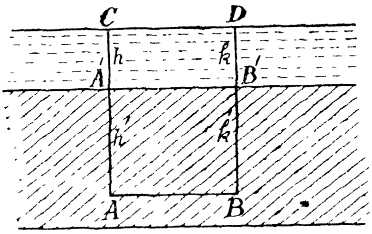
$$ps = whs + w'h's$$

or $$p = wh + w'h'.$$

25. Theorem. *The common surface of two heavy homogeneous liquids that do not mix is a horizontal plane.*

Let A, B be any two points in the lower liquid, the straight line being horizontal.

Draw AA′C and BB′D vertically to meet the common surface in A′ and B′ and the upper surface in C and D respectively.



Let CA′ $= h$, A′A $= h'$,

DB′ $= k$, B′B $= k'$.

Let w and w' be the weight per unit volume of the upper and the lower liquids respectively.

Since AB is horizontal,

Pressure at A = Pressure at B.

$\therefore \quad hw + h'w' = kw + k'w'$ ... (1)

Since both CD and AB are horizontal

$\therefore \quad$ CA = DB

or $\quad h + h' = k + k'$

or $\quad w'h + w'h' = w'k + w'k'$, ... (2)

Subtracting (1) from (2)

$$(w' - w)h = (w' - w)k$$

or $$h = k$$

Hence the common surface AB is horizontal.

26. Effective surface. In the formula $p = wz + \Pi$ when Π is not zero, let us imagine the atmosphere to be removed, and a stratum of the liquid of thickness $\frac{\Pi}{w}$ ($= z'$ say) to be placed above the original liquid. Then the pressure at a point at a depth h below the original surface is

$$w(h + z')$$

i.e., $wz + \Pi$ are the same as it was before the atmosphere was removed.

The upper surface of the supposed superimposed liquid is termed the **Effective surface** *or* **Surface of zero pressure**.

Hence, *the pressure at any point of a homogeneous liquid is proportional to the depth below the effective surface.*

27. Head of Liquid. The pressure to any point is often said to be due to such a depth of liquid, or to such a '**head**' of liquid, meaning that it is at that depth below the effective surface.

Thus we may say that a certain pressure is that due to a *head* of 10 feet of water. It means that it is the same as the pressure at a point in water 10 feet below the effective surface.

28. Characteristics of liquids.

From the theorems given above some of the characteristics of liquids can be summarized.

(1) The surface of a heavy liquid at rest is horizontal.
(2) The common surface of two heavy liquids that do not mix is horizontal.
(3) If there are two heavy liquids that do not mix, the density of the lower liquid is greater than that of the upper.
(4) Liquid always finds its own level.

Note. It must be remembered that some of the above Theorems are true only for small portions of liquids. For a large sheet of water like a big lake or sea, *the free surface is a curved surface*, as the attraction on each portion of the liquid is directed towards the centre of the Earth.

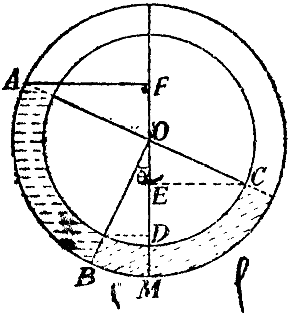
29. Solved Examples.

Ex. 1. *A small uniform tube is bent into the form of a circle whose plane is vertical: equal volumes of two fluids whose densities are* ρ, σ *fill half the tube. Shew that the radius passing through the common surface makes with the vertical an angle* θ *given by*

$$\tan\theta = \frac{\rho - \sigma}{\rho + \sigma}.$$

(*Agra 1955 ; Lucknow 1926 ; Bombay 1940 ; Rajputana 1949*)

Let AB be the portion of the tube occupied by the liquid of density σ and the portion BMC by the liquid of density ρ.



Let M be the lowest point of the tube. Let the radius through the common surface B make an angle θ with the vertical.

It is given that the volume of the liquid in AB = vol. in BC.

$\therefore$ arc AB = arc BC

But the total arc occupied is half of the circumference.

$$\therefore \quad \angle AOB=\frac{\pi}{2}, \ \angle BOC=\frac{\pi}{2}, \ \angle MOC=\frac{\pi}{2}-\theta.$$

Now the pressure at M must be the same on both the sides of the tube, and the pressures are equal to the vertical projections of the arcs containing the liquids multiplied by the product of the density and acceleration g.

$$\therefore \quad EM.g\rho=DM.g\rho+FD.g\sigma.$$

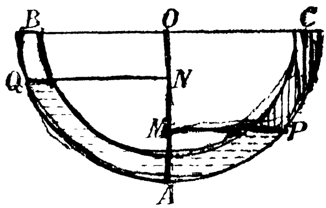
$$\therefore \quad a(1-\sin\theta)\rho=a(1-\cos\theta)\rho+a(\sin\theta+\cos\theta)\,\sigma.$$

Hence $$\tan\theta=\frac{\rho-\sigma}{\rho+\sigma}.$$

Ex. 2. *Three liquids, whose densities are in A.P., fill a semi-circular tube whose bounding diameter is horizontal. Prove that the depth of one of the common surfaces is double that of the other.*
(*M.T. 1861 ; Calcutta 1916 ; Patna 1927 ; Rangoon 1937 ; Utkal 1948*)

Let P and Q be the common surfaces of the liquids.

Let the densities of liquids be $\rho-\delta$, ρ, $\rho+\delta$. Let the column BQ contain the liquid of density $\rho-\delta$, QP contain that of $\rho+\delta$ and CP that of ρ.



The pressures on the two sides of A, the lowest point, must be the same,

$$\therefore \quad g\rho.OM+(\rho+\delta)g\,MA$$
$$=(\rho-\delta)gON+(\rho+\delta)gNA.$$

Let a be the radius of the tube.

$$\therefore \quad OA=a,\ MA=a-OM,\ NA=a-ON.$$

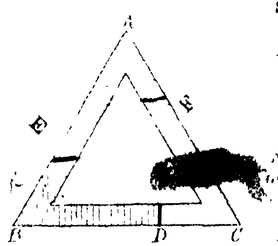
Hence $$\rho.OM+(\rho+\delta)(a-OM)=(\rho-\delta)ON+(\rho+\delta)(a-ON)$$

or $$\rho a+\delta a-\delta.OM=-2\delta.ON+a\rho+a\delta$$

Hence $$OM=2ON.$$

Ex. 3. *A closed tube in the form of an equilateral triangle contains equal volumes of three liquids which do not mix and is placed with its lowest side horizontal. Prove that if the densities of the liquids are in A.P., their surfaces of separation will be at points of trisection of the sides of the triangle.* (*Agra 1957*)

ABC is the equilateral triangular tube, D, E, F are the common surfaces.



Let CD or BE or AF $=x$, since the three volumes are equal.

Let the side of the triangle be a.

Let the portion EBD occupy the liquid of density $\rho+2\delta$, FCD that of $\rho+\delta$ and EAF that of ρ. The densities are in A.P.

Since B and C are in the same horizontal line

$\therefore$ The pressure at B = the pressure at C.

$$(\rho+2\delta)x \sin 60^\circ + \rho(a-x) \sin 60^\circ = \rho x \sin 60^\circ + (\rho+\delta)(a-x) \sin 60^\circ$$

$$\rho x + 2\delta x + \rho a - \rho x = \rho x + \rho a + \delta a - \rho x - \delta x$$

or $$3\delta x = \delta a$$

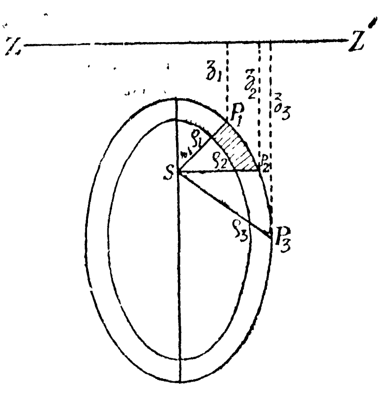
or $$x = \frac{a}{3}.$$

Hence at the common surface the side is trisected.

Ex. 4. *A fine tube bent in the form of ellipse is held with its plane vertical and is filled with n liquids whose densities are $\rho_1, \rho_2, \ldots, \rho_n$ taken in order round the elliptic tube. If $r_1, r_2, \ldots\ldots, r_n$ be the distances of the points of separation from either focus, prove that*

$$r_1(\rho_1-\rho_2)+r_2(\rho_2-\rho_3)+\ldots\ldots+r_n(\rho_n-\rho_1)=0.$$

Let S be the focus of the elliptic tube. $P_1, P_2, \ldots P_n$ are the points of the common surfaces.



Let the depths of $P_1, P_2 \ldots P_n$ from the directrix of the ellipse be $z_1, z_2, \ldots z_n$ respectively.

$SP_1 = r_1,\ SP_2 = r_2, \ldots, SP_n = r_n.$

Let $p_1, p_2, \ldots p_n$ be the pressures at the points of separation.

$\therefore\ p_2 - p_1 = g\rho_2(z_2 - z_1)$

(*i.e.*, vertical depth of the liquid)

$p_3 - p_2 = g\rho_3(z_3 - z_2)$

....

$p_n - p_{n-1} = g\rho_n(z_n - z_{n-1})$

$p_1 - p_n = g\rho_1(z_1 - z_n)$

Adding

$$0 = g[\rho_2(z_2 - z_1) + \rho_3(z_3 - z_2) + \ldots\ldots]$$

$$0 = \rho_2(r_2 - r_1) + \rho_3(r_3 - r_2) + \ldots\ldots + \rho_1(r_1 - r_n)$$

for the ratio of the distance of a point from the focus to that from its directrix is constant

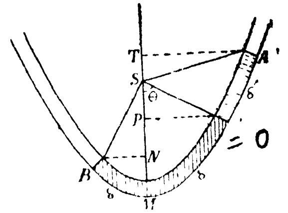
$$\therefore\ r_1(\rho_1-\rho_2)+r_2(\rho_2-\rho_3)+\ldots+r_n(\rho_n-\rho_1)=0.$$

Ex. 5. *A tube in the form of a parabola held with its vertex downwards and axis vertical, is filled with different liquids of densities δ and δ'. If the distances of the free surface of the liquids from the focus be r and r' respectively, show that the distance of their common surface from the focus is*

$$\frac{r\delta - r'\delta'}{\delta - \delta'}.$$

(*Vikram 1959 ; Jaipur 1951 ; Luc*[redacted]*gra 1930 ; Calcutta 1948 ; Nagpur 1957 ; Allahabad 1959 ; Saugar 1959*)

Let S be the focus. A and B be the free surfaces. O is the point of the common surface. Let α, β be the angles that SA and SB make with the axis of the parabola.



Let SO $(=r'')$ make an angle θ with SM where M is the lowest point of the parabola.

Let the equation of the parabola be

$$\frac{l}{R}=1+\cos\theta$$

Since A, O, B, are on the parabola

$$\therefore \quad \frac{l}{r}=1+\cos\beta$$

$$\frac{l}{r'}=1+\cos\alpha \qquad (2)$$

$$\frac{l}{r''}=1+\cos\theta \qquad (3)$$

The pressures on the two sides of M must be the same,

$$\therefore \quad \delta g\ \text{MN}=\delta g\text{MP}+\delta' g\text{PT}$$

or $\quad \delta(a-r\cos\beta)=\delta(a-r''\cos\theta)+\delta'(r''\cos\theta-r'\cos\alpha)$

or $\quad -\delta\, r\cos\beta=-\delta r''\cos\theta+\delta' r''\cos\theta-\delta' r'\cos\alpha$

From (1), (2) and (3)

or $$-\delta r\left(\frac{l}{r}-1\right)=-\delta r''\left(\frac{l}{r''}-1\right)+\delta'\left\{r''\left(\frac{l}{r''}-1\right)-r'\left(\frac{l}{r'}-1\right)\right\}$$

or $$-\delta l+\delta r=-\delta l+\delta r''+\delta' l-\delta' r''-\delta' l+\delta' r'$$

$$\therefore \quad r''=\frac{r\delta-r'\delta'}{\delta-\delta'}.$$

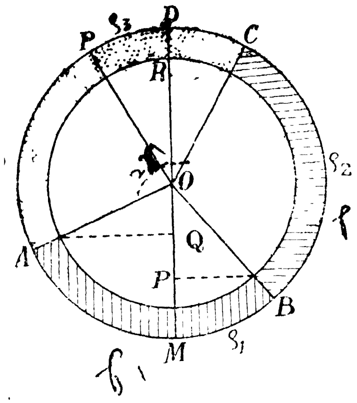
Ex. 6. *A circular tube centre O is filled with three fluids of densities ρ_1, ρ_2, ρ_3, (in descending order of magnitude) and placed in a vertical plane. If 2α, 2β, 2γ, be the angles subtended at the centres by the fluids and P be the point on the circumference midway between the ends of the lightest fluid, then angle θ which OP makes with the vertical is given by*

$$\frac{\rho_2-\rho}{\rho_1-\rho_3}=\frac{\sin\alpha}{\sin(\alpha+\theta)}\cdot\frac{\sin(\beta-\theta)}{\sin\beta}. \qquad \textit{(Agra 1946)}$$

Let AB[redacted]represent the three liquids of densities ρ_1, ρ_2, ρ_3, respecti[redacted]e centre.

$$\angle AOB=2\alpha,\ \angle BOC=2\beta,\ \angle AOC=2\gamma$$

OP is the bisector of the angle 2γ.



$\angle POD=\theta$, $\angle AOP=\gamma$,
$\angle MOA=\pi-\gamma+\theta$
$\angle BOM=2\alpha+\gamma+\theta-\pi$,
$\angle COD=\gamma-\theta$

Now the pressures at the lowest point M must be the same from both the sides.

$$\therefore \rho_1 g.MQ+\rho_3 g.QD.=\rho_1 gMP+\rho_2 gPR+\rho_3 gRD$$

$$\rho_1[a-a\cos\{\pi-(\gamma+\theta)\}]+\rho_3[a+a\cos(\pi-\gamma+\theta)=\rho_1[a-a\cos(2a+\gamma+\theta-\pi)]$$
$$+\rho_2[a\cos(\gamma-\theta)+a\cos(2a+\gamma+\theta-\pi)]+\rho_3\{a-a\cos(\gamma-\theta)\}$$

$$\text{or } \rho_1\cos(\gamma+\theta)-\rho_3\cos(\gamma+\theta)=\rho_1\cos(2\alpha+\gamma+\theta)+\rho_2\cos(\gamma-\theta)-\rho_2\cos(2\alpha+\gamma+\theta)-\rho_3\cos(\gamma-\theta)$$

$$\text{or } (\rho_1-\rho_3)\cos(\gamma+\theta)=(\rho_1-\rho_2)\cos(2\alpha+\gamma+\theta)+(\rho_2-\rho_3)\cos(\gamma-\theta)$$

Since $2\alpha+2\beta+2\gamma=2\pi$, $\alpha+\beta+\gamma=\pi$ or $\gamma=\pi-\alpha-\beta$,
hence eliminating γ, we get

$$(\rho_1-\rho_3)\cos(\pi-\alpha-\beta+\theta)=(\rho_1-\rho_2)\cos(\pi+\alpha+\theta-\beta)+(\rho_2-\rho_3)\cos(\pi-\alpha-\beta-\theta)$$

$$\text{or } (\rho_1-\rho_3)\cos(\alpha+\beta-\theta)=(\rho_1-\rho_2)\cos(\alpha+\theta-\beta)+(\rho_2-\rho_3)\cos(\alpha+\beta+\theta)$$

$$\rho_1\{\cos(\alpha+\beta-\theta)-\cos(\alpha+\theta-\beta)\}+\rho_3\{\cos(\alpha+\beta+\theta)-\cos(\alpha+\beta-\theta)\}=\rho_2\{\cos(\alpha+\beta+\theta)-\cos(\alpha+\theta-\beta)\}$$

$$\rho_1\sin(\beta-\theta)\sin\alpha+\rho_3\{\sin(\alpha+\theta)\sin\beta-\sin\alpha\sin(\beta-\theta)\}.=\rho_2\sin(\alpha+\theta)\sin\beta$$

$$\text{or } (\rho_1-\rho_3)\sin a\sin(\beta-\theta)=(\rho_2-\rho_3)\sin(\alpha+\theta)\sin\beta$$

$$\text{or } \frac{\rho_2-\rho_3}{\rho_1-\rho_3}=\frac{\sin\alpha}{\sin(a+\theta)}\cdot\frac{\sin(\beta-\theta)}{\sin\beta}.$$

Examples 2

1. If a cubic foot of water weighs 1000 ozs., what is the pressure per square inch at the depth of a mile below the surface of water ?

2. The pressure in a water-pipe at the base of a building is 34 lbs. wt. per. sq. inch and on the roof of it is 18 lbs wt. per sq inch ; find the height of the roof. (1 cu. ft. of water weighs 62·5 lbs.)

3. A water tank is 200 ft. above the level of the ground-floor of a house ; find the pressure of the water in a pipe at a height of 30 ft. above the ground-floor.

4. The pressure of the atmosphere is equivalent to that of 30 ft. of water. Assuming this to be true find the depth of a well the pressure at the bottom of which is 3 times that a depth of 3 feet. The well is full of water.
(*Banaras 1938 ; Rangoon 1947*)

5. The lower ends of two vertical tubes, whose cross-sections are 1 and ·1 sq. inch respectively, are connected by a tube. The tube contains mercury of sp. gr. 13·596. How much water must be poured into the larger tube to raise the level in the smaller tube by one inch? *(Utkal 1945)*

6. A glass tube of uniform bore is bent into the form of a U-tube, whose arms are vertical; it contains mercury of sp. gr. 13·6; water is poured into one arm to occupy a height of 408 cms. of it. Find the rise of the level of mercury in the other arm. *(Patna 1941)*

7. Two liquids that do not mix together meet in a bent tube. Prove that the heights of their surfaces above their common surface will be inversely proportional to their densities.

8. A vessel whose bottom is horizontal contains mercury whose depth is 30 inches and water floats on the mercury to a depth of 24 inches; find the pressure at a point on the bottom of the vessel, the sp. gr. of mercury being 13·6.

9. Two liquids A and B do not mix and have different densities. When A is poured in a vessel to a vertical height h, the pressure on the bottom is the same as when A stands to a height h_1 and B to a height h_2 above it. Show that the ratio of the density of A to that of B is

$$\frac{h_2}{h-h_1}.$$ *(Allahabad 1955 ; Banaras 1940)*

10. The pressures at two points A and B in a homogeneous liquid are p and p'. Prove that the pressure at the point R which divides AB in the ratio $m : n$ is

$$\frac{mp'+np}{m+n}.$$

11. If there be n liquids arranged in strata of equal thickness h, and the density of the uppermost being ρ, that of the next 2ρ, and so on, that of the lowest being $n\rho$. Prove that the pressure at the lowest point of the lowest stratum is

$$\tfrac{1}{2}n(n+1)g\rho h.$$ *(M. T.)*

12. If a triangle be immersed in a homogeneous liquid, prove that the sum of the pressures at the vertices is equal to three times the pressure at the centroid of the triangle.

13. If a parallelogram be immersed in any manner in a homogeneous liquid, prove that the sum of the pressures at the extremities of each diagonal is the same.

14. If a liquid be heterogeneous and of density kz at depth z, show that the pressure is

$$\Pi+\tfrac{1}{2}gkz^2$$

where Π is the atmospheric presssure. *(Lucknow 1950 Supp.)*

15. If ρ, ρ' be the densities of two fluids $(\rho<\rho')$ and the lengths of the arms of a U-tube in which they meet be m and n inches respectively; prove that in order that the tube may be completely filled, the height of the column of the lighter fluid above the horizontal plane in which they meet be

$$\frac{\rho'(m-n)}{\rho'-\rho} \text{ inches.}$$ *(M.T. 1859)*

16. A fine circular tube in the vertical plane contains a column of liquid, of density δ, which subtends a right angle at the centre, and a column of density δ' subtending angle α. If θ is the angle that the radius through the common surface makes with the vertical, prove that

$$\tan\theta=\frac{\delta-\delta'+\delta'\,\text{c[illegible]}\,\alpha}{\text{[illegible]}+\delta'\sin\alpha}.$$

(Jaipur 1953 ; Agra 1932, 48)

17. In the lower half of a uniform circular tube, one quadrant is occupied by a liquid of density 2ρ, and the other by two liquids of densities ρ and 3ρ. Prove that the volume of the lower of the two latter liquids is twice that of the other. (*Agra 1953 Supp.*; *Lucknow 1938*)

18. A circular tube of fine uniform bore is half filled with equal volumes of four liquids which do not mix and whose densities are as $1:4:8:7$ and is held with its plane vertical, show that the diameter joining the free surfaces makes an angle θ with the vertical such that

$$\tan\theta=2.$$ (*Jaipur 1952*)

19 Equal volumes of three fluids of different densities, which do not mix together, completely fill a circular tube which is kept in a vertical plane. Prove that, if the densities of the fluids are in A.P., the common surface of the lightest and heaviest fluids is at an extremity of a horizontal diameter of the circle.

20. A circular tube of uniform thin bore is half filled with equal volumes of three liquids of sp. gravities 3, 4 and 6, and is kept with its plane vertical. Prove that the diameter joining the two free surfaces makes an angle θ to the vertical such that

$$\tan\theta=\frac{19}{\sqrt{3}}.$$ (*Jaipur 1950*)

21. In a uniform circular tube two liquids are placed so as to subtend 90° each at the centre. If the diameter joining the two free surfaces be inclined at 60° to the vertical, prove that the densities of the two liquids are as

$$\sqrt{3}+1:\sqrt{3}-1.$$

22. A circular tube of uniform bore, whose plane is vertical, contains columns of two liquids, whose respective densities are ρ, σ, the respective columns subtending angles 2α, 2β at the centre of the circle. If θ be the angle which the portion of the tube intercepted between the lowest point and the common surface of the liquids subtends at the centre of the tube, prove that

$$\rho\sin\alpha\sin(\alpha\pm\theta)=\sigma\sin\beta\sin(\beta\mp\theta).$$ (*M.T. 1873*)

23. The two arms of a U-tube are close together. In one arm there is water and in the other mercury, so that their common surface is at the lowest point. One quarter of the water is taken out and is poured into the other arm over the mercury. Prove that in the new equilibrium position the difference of heights of the upper surfaces is one-half of what it was formerly.

24. A tube of the form of a complete circle fixed in a vertical plane, contains equal lengths of four liquids which do not mix and whose sp. gravities are as $1:2:4:3$ filling the tube in this order. Prove that the inclinations to the vertical of the two diameters joining the points of division are

$$\tan^{-1}\tfrac{1}{2} \quad \text{and} \quad \tan^{-1}2.$$

25. If three liquids which do not mix and whose densities are ρ_1, ρ_2, ρ_3, fill a circular tube in a vertical plane and if α, β, γ are the angles which the radii to the common surfaces make with the vertical diameter measured in the same direction, prove that

$$\rho_1(\cos\beta-\cos\gamma)+\rho_2(\cos\gamma-\cos\alpha)+\rho_3(\cos\alpha-\cos\beta)=0.$$

26. A closed tube in the form of an ellipse, with its major axis vertical, is filled with three liquids of densities ρ_1, ρ_2, ρ_3. P_1 is the point of separation of liquids of densities ρ_2, ρ_3; P_2 that of the liquids of densities ρ_3, ρ_1; and P_3 that of liquids ρ_1 and ρ_2. If the distances of P_1, P_2, P_3 from the same focus be r_1, r_2, r_3 respectively; prove that

$$r_1(\rho_2-\rho_3)+r_2(\rho_3-\rho_1)+r_3(\rho_1-\rho_2)=0.$$

27. A thin uniform cycloided tube contains equal weights of two fluids: if it be placed with its axis vertical, prove that the heights of the free surfaces of the fluids above the vertex of the tube are as

$$(3a+b)^2:(3b+a)^2,$$

where a and b are the lengths of the tube which the fluids occupy. (*Agra 1957*)

28. A fine glass tube in the form of an equilateral triangle is filled with equal volumes of three liquids which do not mix and whose densities are in A.P. The tube is held in a vertical plane, and the side that contains portions of the heaviest and lightest liquids make an angle θ with the vertical. Show that the surfaces of separation divide the sides in the ratio

$$\cos\left(\frac{\pi}{6}-\theta\right):\cos\left(\frac{\pi}{6}+\theta\right).$$

29. Three liquids, whose densities ρ_1, ρ_2, ρ_3 $(\rho_1>\rho_2>\rho_3)$ are in A.P., completely fill a circular tube in a vertical plane and occupy lengths of arcs subtending at the centre angles 2α, 2β, 2γ respectively. If the radius to the point midway between the ends of the lightest liquid makes with the vertical an angle θ, prove that

$$\cot\theta=\cot\alpha+2\cot\beta.$$ *(Bombay 1951)*

CHAPTER IV

PRESSURES ON PLANE SURFACES

30. We have seen that when a surface be in contact with a liquid, the pressure at any element of it is normal to the surface. If this surface be a plane area, the pressure at any point of it is perpendicular to the plane area. Pressures at all points on one side of this plane area, therefore constitute a system of parallel forces whose magnitudes are known. Hence all these parallel forces can be compounded into one single force acting at some definite point of the plane area.

This single force is called the **Resultant Fluid Pressure** or **Resultant Thrust**.

But since the thrusts are parallel, this resultant thrust will be the arithmetic sum of all the thrusts on the various elements of the surface. On this account the resultant fluid-pressure on a plane area is termed as the **whole pressure**.

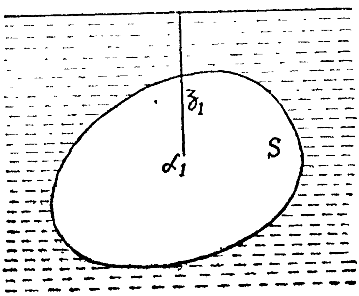
31. Definition. *The* **whole pressure** *of a fluid on a surface, is the sum of all the normal pressures exerted by the fluid on every element of the surface.*

Note 1. If the surface considered be a curved one the *whole pressure* is considered as the arithmetic sum of the thrusts at various elements of the area, though they are not in the same direction. In this case the whole pressure has no physical meaning whatever, and thus its calculation serves no useful purpose.

Note 2. The students must carefully note that in the case of a curved surface, the whole pressure is not the resultant thrust on the curved surface. The topic how to find out the resultant thrust will be considered later on.

32. Theorem. *The whole pressure of a heavy homogeneous liquid on a plane area is equal to the product of the area and the pressure at its centre of gravity.*

Let the surface S be divided into a great number of very small elements α_1, α_2, α_3...... and let z_1, z_2, z_3 be the depths of these elements from the free surface.



Since the elements are to be taken very small, the pressures on them will be respectively

$$w\alpha_1 z_1,\ w\alpha_2 z_2, \ldots\ldots$$

taking the pressure over each area to be uniform.

Hence the whole pressure

$$=w(\alpha_1 z_1+\alpha_2 z_2+\ldots\ldots)$$

But from statics we know that if $\bar{z}$ be the depth of the centre of gravity of the given plane area

$$\bar{z} = \frac{\alpha_1 z_1 + \alpha_2 z_2 + \ldots\ldots}{\alpha_1 + \alpha_2 + \ldots\ldots}$$

or $$\alpha_1 z_1 + \alpha_2 z_2 + \ldots = \bar{z}(\alpha_1 + \alpha_2 + \ldots) = \bar{z}\,S$$

Hence the whole pressure $= w\bar{z}S$

$=$ Pressure at the C. G. $\times$ area.

Note 1. The above theorem may also be stated as follows :—

(*i*) *If a plane surface be immersed in a liquid the whole pressure or thrust on it is equal to* $w\,S\,.\,\bar{z}$; *where* S *is the area of this plane surface and* $\bar{z}$ *is the depth of its centre of gravity below the surface of zero pressure and* w *the weight of the unit volume of the liquid.*

(*ii*) *The whole pressure of a liquid on a surface is equal to the weight of a column of liquid whose base is equal to the area of the given plane surface, and whose height is equal to the depth below the surface of the liquid of the centre of gravity of the given plane surface.*

Note 2. If the pressure of the air is not to be neglected, let Π be the pressure at any point of the surface of the liquid.

The pressure on the element

$$=(\Pi + zw_1)\alpha_1$$
$$=\Pi\alpha_1 + wz_1\alpha_1$$

$\therefore$ whole pressure

$$=\Pi(\alpha_2 + \alpha_1 + \ldots) + w(z_1\alpha_1 + z_2\alpha_2 + \ldots)$$
$$=\Pi S + wS\bar{z}$$
$$=(\Pi + w\,\bar{z})S.$$

If $\Pi = wh_1$, where h_1 is the height of the effective surface above the free surface, the whole pressure is given by

$$w(h_1 + \bar{z})S, \text{ or } w\,S\,z,$$

where z is the depth of the C.G. of the area below the effective surface.

Therefore *in this case also the above theorem holds provided if* z *is taken as the depth of the C.G. below the 'effective surface.'*

33. Layers of different Liquids. Let the fluid consist of layers of different liquids of densities $\rho_1, \rho_2, \cdots, \rho_n$ beginning from the top where pressure is zero. Let their depths be $h_1, h_2, \ldots, h_n$ respectively and let the *area be entirely in contact with the lowest liquid.*

Let z_1 be the depth of α_1, a very small element of the area below the lowest surface of separation.

Then the pressure at the element α_1

$$=g\rho_1 h_1\alpha_1 + g\rho_2 h_2\alpha_1 + \ldots + g\rho_{n-1}h_{n-1}\alpha_1 + g\rho_n z_1\alpha_1$$

∴ Whole pressure on the area

$$=g\rho_1 h_1(\alpha_1+\alpha_2+\ldots)+g\rho_2 h_2(\alpha_1+\alpha_2+\ldots)+\ldots\ldots+g\rho_n(z_1\alpha_1+z_2\alpha_2+\ldots)$$
$$=g\rho_1 h_1 S+g\rho_2 h_2 S+\ldots+g\rho_{n-1}h_{n-1}S+g\rho_n\bar{z}\,S$$
$$=(g\rho_1 h_1+g\rho_2 h_2+\ldots\ldots+g\rho_{n-1}h_{n-1}+g\rho_n\bar{z})S$$

where $\bar{z}$ denotes the depth of the C.G. of the area below the lowest surface of separation.

Note. If the atmospheric pressure Π is taken into account, the above formula becomes

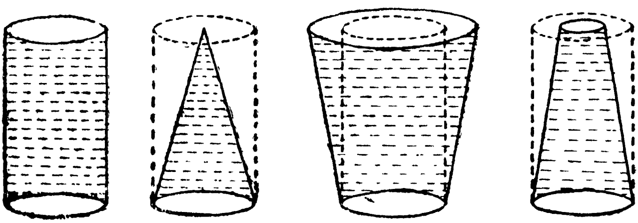
$$(\Pi+g\rho_1 h_1+\ldots\ldots+g\rho_{n-1}h_{n-1}+g\rho_n\bar{z})S$$

which can be easily proved.

34. A plane area in contace with more than one fluid.

If the plane area be in contact with more than one fluid, we divide it into several parts, each of which is in contact with one fluid only. The pressures on each part will be given by the formula given in the last Article and their sum is the *whole pressure* required.

35. Thrust on a horizontal area. Now it is evident that the thrust on the horizontal base of a vessel of the liquid which it contains does not depend on the shape of the vessel but only on the area of the base and the depth of the liquid.



Thus suppose we have four vessels of different shapes but having the same height and equal bases. They contain different amount of liquids. But the thrust on the base of each of the vessels is the same, for

the thrust $=w$. area. depth of the C.G. of the area, which are the same for all the vessels.

36. Solved Examples.

Ex. 1. *Find the whole pressure on a triangle, the depths of whose vertices are h_1, h_2 and h_3, the liquid being homogeneous.*

The depth of the C.G. of the triangle

$$=\tfrac{1}{3}(h_1+h_2+h_3)$$

∴ Whole pressure $=\frac{1}{3}\,w\,(h_1+h_2+h_3)$ S where S is the area of the triangle.

Ex. 2. *The width of a rectangular vertical dockgate is* 40 *ft. and on one side there is salt-water* (*sp. gr.* 1·026) *to a depth of* 24 *ft. On the other side there is fresh water. Find its depth if the thrusts on the two sides are equal.* (*Rangoon 1947*)

The area immersed under salt-water

$$=40\times 24 \text{ sq. ft.}$$

The depth of the C.G. of the immersed area

$$=12 \text{ ft.}$$

∴ Whole pressure on one side of the dock-gate

$$=40\times 24\times 12\times g\times 1{\cdot}026$$

If h be the depth of the fresh water on the other side, the whole pressure on the other side

$$=(40\times h)\ \frac{h}{2}\ g.$$

The thrust on the two sides being equal

$$\tfrac{1}{2}\times 40\ h^2 g=40\times 24\times 12\times g\times 1{\cdot}026$$

$$h^2=24\times 24\times 1{\cdot}026$$

$$h=24\times 1{\cdot}0129$$

$$=24{\cdot}3096 \text{ ft.}$$

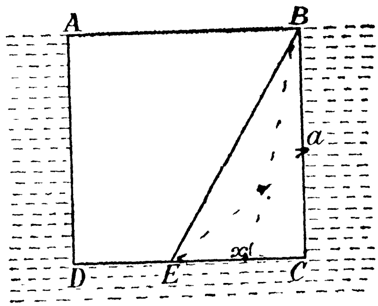
Ex. 3. *A square lamina* ABCD, *which is immersed in water, has the side* AB *in the surface. Draw a line* BE *to a point* E *in* CD *such that the pressures on the two portions into which it divides the lamina, may be equal.* (*Allahabad 1921 ; Agra 1931, 37, 51*)

Let the side of the square ABCD be a and let EC$=x$.

The pressure on BEC

$=$ pressure on ABED

$=\frac{1}{2}$ pressure on ABCD



Now pressure on BEC

$=w.\ \frac{2}{3}.a.\frac{1}{2}ax$

$=\frac{1}{3}a^2xw$

Pressure on ABCD

$=w.\frac{1}{2}a.a^2=\frac{1}{2}\ a^3w$

$\therefore\ \ \frac{1}{3}\ a^2xw=\frac{1}{2}.\ \frac{1}{2}.\ a^3w$

$\therefore\ \ x=\frac{3}{4}a$

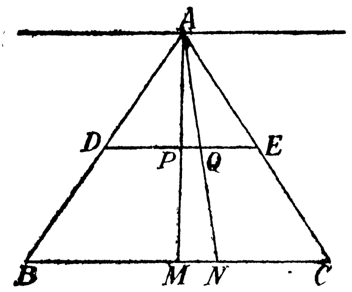
∴ EC : ED $=$ 3 : 1.

Ex. 4. *A triangle is immersed in a liquid with one vertex in the surface and the opposite side horizontal ; neglecting the atmospheric pressure, find the ratio in which the median through that vertex will be divided by a horizontal line which divides the triangle into two parts in which the total pressures are equal.* (*Allahabad 1922*)

Let DE divide the △ ABC into two portions such that

pressure on ADE$=$pressure on DECB$=\frac{1}{2}$ pressure on ABC

Let AM and AN be the perpendicular and the median of the side BC respectively. Let them cut DE in P and Q respectively.



Pressure on $\triangle$ ADE

$$=w.\tfrac{2}{3}AP.\ \tfrac{1}{2}AP.DE$$

Pressure on $\triangle$ABC

$$=w\ \tfrac{2}{3}\ AM.\ \tfrac{1}{2}\ AM.BC$$

$\therefore$ $$w.\tfrac{2}{3}AP.\tfrac{1}{2}AP.\ DE$$

$$=\tfrac{1}{2}w.\tfrac{2}{3}\ AM\ \tfrac{1}{2}\ AM.BC.$$

or $$AP^2.DE=\tfrac{1}{2}AM^2.BC$$

Now $$\frac{DE}{BC}=\frac{AP}{AM}=\frac{AQ}{AN}$$

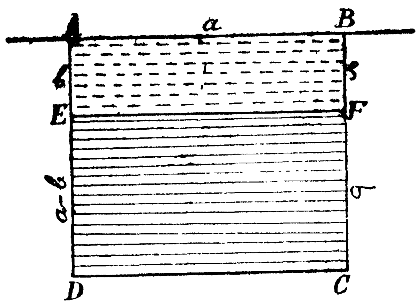
$\therefore$ $$\frac{AQ^2.AM^2}{AN^2}\cdot\frac{BC.AQ}{AN}=\tfrac{1}{2}AM^2.BC.$$

or $$AQ^3=\tfrac{1}{2}AN^3$$

$\therefore$ $$AQ=\frac{AN}{\sqrt[3]{2}}.$$

Ex. 5. *A square of side a is dipped vertically in two liquids of densities ρ and σ with the upper side in the free surface. The depth of the upper liquid of density ρ is b ($<a$). Find the thrust on the square.*

Let the square ABCD be immersed with AB in the free surface The portion ABFE is in the upper liquid.



Thrust on the area ABFE

$$=ab.\ \frac{b}{2}g\rho$$

Thrust on the area EFCD

$$=\{g\rho b+\tfrac{1}{2}g\sigma(a-b)\}(a-b)a$$

$\therefore$ Whole pressure on the square ABCD

$$=\tfrac{1}{2}ab^2g\rho+ab(a-b)g\rho+\tfrac{1}{2}g\sigma a\ (a-b)^2$$

$$=\tfrac{1}{2}ab\ g\rho\ (2a-b)+\tfrac{1}{2}g\sigma a\ (a-b)^2.$$

Aliter.—The thrusts of the liquids on the square will remain the same even if we consider that one liquid of density ρ is in contact with the whole area and another liquid of density $(\sigma-\rho)$ in contact with the lower part only of the rectangle.

The thrust due to the first liquid

$$=a^2.\frac{a}{2}\ \rho g=\tfrac{1}{2}\ a^3\ \rho g$$

The thrust due to the second liquid

$$=a\ (a-b)\ \frac{(a-b)}{2}\ g(\sigma-\rho)$$

$\therefore$ Total thrust on ABCD

$$=\tfrac{1}{2}a^3\rho g+\tfrac{1}{2}a(a-b)^2g(\sigma-\rho)$$
$$=\tfrac{1}{2}a(a-b)^2g\sigma+\tfrac{1}{2}ag\rho\{a^2-(a-b)^2\}$$
$$=\tfrac{1}{2}a(a-b)^2g\sigma+\tfrac{1}{2}ab\,g\rho(2a-b).$$

Ex. 6. *The lighter of the two liquids of density* ρ *rests on the heavier of density* σ*, to a depth b. If square of side a is immersed in a vertical position with one side in the surface of the upper liquid, if the thrusts on the two portions of the square in contact with the two liquids be equal, prove that*

$$b\rho(3b-2a)=\sigma(a-b)^2.$$

(Allahabad 1927 ; Sagar 1949 ; Jaipur 1953, 58)

From solved Ex. 5

thrust on ABFE=thrust on EFCD

$\therefore$ $\tfrac{1}{2}ab^2g\rho=\{g\rho b+\tfrac{1}{2}g\sigma(a-b)\}(a-b)a$

$\therefore$ $\tfrac{1}{2}ab\,g\rho\{b-2(a-b)\}=\tfrac{1}{2}g\sigma(a-b)^2a$

or $b\rho(3b-2a)=\sigma(a-b)^2.$

Examples 3

1. A cube of 30 cms. edge is suspended in water with its upper face horizontal and at a depth of 75 cms. below the surface. Find the thrust on each face of the cube.

2. A rectangular vessel, one face of which is of height 2 ft. and width 1 ft., is half filled with mercury (sp. gr. 13·5) and half with water. Find the thrust on this face.

3. One-third of a rectangular vessel is filled with mercury (sp. gr. 13·6) and the remainder with salt water (sp. gr. 1·04). If $3h$ be the depth of the vertical side and a be breadth, find the fluid thrust on a vertical side.

4. The lighter of two fluids, whose sp. gravities are as 2 : 3, rests on the heavier, to a depth of 4 inches. A square is immersed in a vertical position with one side in the upper-surface, determine the side of the square in order that the pressures on the portions in the two fluids may be equal. (*M.T. 1855*)

5. A rectangular vessel contains three liquids which do not mix and whose sp. gravities are 1·0, 1·2 and 1·6, the thickness of which are 4, 3 and 2 inches respectively. Compare the total thrusts of the liquids on the different parts of a side of the vessel. (*Andhra 1950*)

6. In the vertical side of a water tank there is a square plate whose upper edge is horizontal and at a depth of 8 ft. below the surface of the water. The depth of the plate is one foot ; find the resultant pressure on the plate, taking the weight of 1 cu. ft. of water to be 62·5 lbs. (*Calcutta 1910*)

7. A square plate, whose edge is 8 inches, is immersed in water, its upper edge being horizontal at a depth of 12 inches below the surface of the water. Find the thrust of the water on the surface of the plate when it is inclined at 45° to the horizon ; the mass of a cu. ft. of water being 64 lbs. (*Calcutta 1947*)

8. The resultant fluid pressure on a vertical circle of radius a is equal to twice the weight of a sphere of the same liquid of radius a. The circle is now lowered through a depth equal to $2a$. Prove that the fluid pressure in the new position is $\frac{7}{4}$ times the original one.

9. Find out the total thrust on one side of a [illegible]angular vertical dock-gate 45 feet wide, immersed in sea water to depth [illegible] feet, given that 1 c. ft. of sea-water weighs 1025 ozs.

If there is fresh water on the other side of the gate, find its depth so that the resultant thrusts on the two sides are equal. (*Madras 1934*)

10. A rectangular vessel is full of water. compare the fluid pressures on the lower and the upper halves of a vertical side of the vessel.

11. The side of a cistern are vertical. Its base is a horizontal regular hexagon each side of which is $\sqrt{3}$ ft. long. Find the depth if, when it is full of water, the thrust on each of its sides is the same as on its base (*Banaras 1927 ; Agra 1936, 50*)

12. A vessel containing some water rests on a horizontal table, if a person dips his hand in the water without touching, how is the pressure on the table affected ? (*Agra 1959 ; Lucknow 1940*)

13. An ellipse is placed with its major axis on the surface of water and its plane vertical ; a circle—the auxiliary circle—is described on the major axis as diameter. Prove that the thrust of water on the portion of the area enclosed between the auxiliary circle and the ellipse is

$$\tfrac{2}{3}wa(a^2-b^2)$$

where a and b are the major and minor axes of the ellipse respectively.

14. An ellipse is placed with its minor axis, on the surface of water and its plane vertical. A circle is described on the minor axis as diameter. Find the total pressure of water on the portion of the area enclosed between the ellipse and circle.

15. A square is placed in a liquid with one side in the surface. Show how to draw horizontal line in the square dividing it into two portions, the thrusts on which are the same. (*Lucknow 1959 ; Calcutta 1938*)

16. ABCD is a rectangle, AB being equal to 2BC. It is immersed in water with its plane vertical and AB in the surface. Show how to divide the area into two parts by a straight line from A such that the pressure on the two parts may be equal. (*Bombay 1947*)

17. ABCD is a rectangle immersed in a homogeneous liquid with AB in the free surface and AD vertical. Show how to draw a straight line through A, dividing the rectangular area into two portions equally pressed. (*Nagpur 1943*)

18. A parallelogram is immersed in a homogeneous liquid with one side in the surface ; show how to draw horizontal lines dividing it into n portions the thrusts on which are equal. (*Jaipur 1950*)

19. A triangle ABC has its plane vertical and the side AB is in the surface of the liquid in which the triangle is immersed ; divide it by the straight lines drawn from A into n triangles on each of which the pressure shall be the same.

20. An isosceles triangle is immersed vertically in a fluid with its vertex coincident with the surface of the fluid and its base horizontal ; determine how it must be divided by a line parallel to the base so that the pressures on the upper and lower parts may be respectively in the ratio $m : n$. (*Allahabad 1953*)

21. A closed hollow cone is just filled with liquid, and is placed with its vertex upwards and axis vertical ; divide its curved surface by a horizontal plane into two parts on which the whole pressures are equal.

22. A cone, full of water, is placed on its side on a horizontal table, show that the thrust on its base is $3 \sin \alpha$ times the weight of the contained fluid, where 2α is the vertical angle of the cone. (*Lucknow 1956 ; Patna 1940*)

23. A triangle ABC is immersed in a liquid, its plane being vertical and the side AB in the surface ; if O be the centre of the circumscribed circle of the $\triangle$ABC, prove that

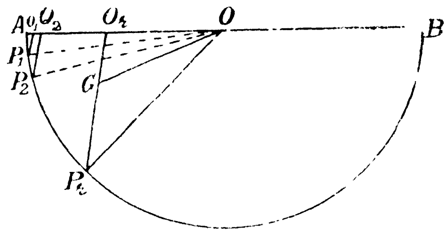
$$\frac{\text{pressure on } \triangle OCA}{\text{pressure on } \triangle OCB} = \frac{\sin 2B}{\sin 2A}.$$ (*Agra 1956*)

24. A triangle ABC is immersed vertically in a liquid with the vertex C in the surface, and the sides AC, BC equally inclined to the surface ; show that the vertical through C divides the triangle into two others, the fluid pressures upon which are as $b^3+3ab^2 : a^3+3a^2b$. (*Lucknow 1957*)

37. Harder Solved Examples.

Ex. 1. *A semi-circle is immersed vertically in a liquid with the diameter in the surface ; show how to divide it into n sectors, such that the thrust on each is the same.* (*M.T., Finance 1926 ; Utkal 1945*)

Let the semi-circle AP_rB of radius a be immersed. Let AOP_r consist of r sectors of the type AOP_1, P_1OP_2......such that the thrust on each of them is the same.



Let $\angle AOP_r = 2\alpha$. Let P_1O_1, P_2O_2,...P_rO_r...be the vertical heights of the required sectors from the free surface.

$$\therefore \frac{\text{Thrust on the area } AOP_r}{\text{Thrust on the semi-circle}} = \frac{r}{n}. \qquad \dots (1)$$

Area of sector $AOP_r = \alpha a^2$.

Depth of the C.G. of this area from the free surface

$$= OG \sin \alpha$$

$$= \frac{2}{3} \cdot \frac{a \sin \alpha}{\alpha} \cdot \sin \alpha = \frac{2}{3} \cdot \frac{a \sin^2 \alpha}{\alpha}$$

Thrust on the area $AOP_r = \alpha a^2 . \frac{2}{3} \frac{a \sin^2 \alpha}{\alpha} g\rho = \frac{2}{3} a^3 \sin^2 \alpha \, g\rho$.

Thrust on the semi-circle $= \frac{1}{2} \pi a^2 . \frac{4a}{3\pi} g\rho = \frac{2}{3} a^3 g\rho$

$\therefore$ From (1)

$$\frac{\frac{2}{3} a^3 \sin^2 \alpha g\rho}{\frac{2}{3} a^3 g\rho} = \frac{r}{n}$$

or $\sin^2 \alpha = \frac{r}{n}$

$\therefore \cos 2\alpha = 1 - 2 \sin^2 \alpha = 1 - \frac{2r}{n}$

$\therefore AO_r = a - a \cos 2\alpha = a - a\left(1 - \frac{2r}{n}\right) = \frac{2ra}{n}$

Giving $r = 1, 2, 3, \dots n$

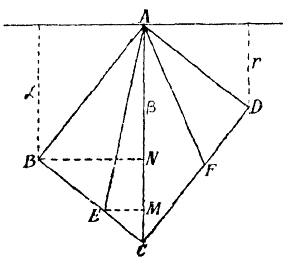
$$AO_1 = \frac{2a}{n}, \; AO_2 = \frac{4a}{n}, \; AO_3 = \frac{6a}{n} \; \dots\dots$$

$\therefore$ **The horizontal diameter is divided in n equal parts. The ordinates at these points will divide the arc of the semi-circle in the points of the required sectors.**

Ex. 2. *A parallelogram is immersed in a fluid with a diagonal vertical, one extremity of which is in the surface of the liquid. Through this point lines are drawn dividing the parallelogram into three equal parts. Compare the pressures on these three parts, and if P_2 be the pressure on the middle part and P_1, P_3 those on the other two prove that*

$$16P_2=11(P_1+P_3).$$

Let ABCD be the parallelogram with AC vertical and A in the surface. Let α, β, γ be the depths of B, C, D from the surface respectively. BN, EM are the perpendiculars on AC. Let E and F be such points so that pressure on the $\triangle$ABC= that on AECF=that on $\triangle$AFD. Let 3S be the area of the whole parallelogram.



$$\therefore \quad \triangle AEC = \triangle ABC - \triangle ABE = \tfrac{3}{2}S - S = \tfrac{1}{2}S = \tfrac{1}{3}\triangle ABC$$

$$\therefore \quad \tfrac{1}{2}AC.EM = \tfrac{1}{3}.\tfrac{1}{2}AC.BN$$

or $\quad EM=\tfrac{1}{3}BN.$

$\therefore$ In the $\triangle$CBN, $CM=\tfrac{1}{3}CN$ or $NM=\tfrac{2}{3}CN=\tfrac{2}{3}(\beta-\alpha)$

Depth of $E=\alpha+NM=\alpha+\tfrac{2}{3}(\beta-\alpha)=\dfrac{\alpha+2\beta}{3}$

$\therefore$ Depth of C.G. of the $\triangle$ABC from the free surface

$$=\tfrac{1}{3}\left(\alpha+\frac{\alpha+2\beta}{3}\right)=\frac{4\alpha+2\beta}{9}$$

$$\therefore \quad P_1=\text{Thrust on}\triangle ABE=g\rho S\left(\frac{4\alpha+2\beta}{9}\right)$$

Similarly $P_3=$Thrust on $\triangle AFD=\tfrac{1}{9}g\rho S(4\gamma+2\beta)$

$P_2=$Thrust on the parallelogram$-(P_1+P_3)$

$=g\rho 3S.\dfrac{\beta}{2}-P_1-P_3$ whence the thrusts can be compared.

Now $P_1+P_3=\tfrac{4}{9}g\rho S(\alpha+\beta+\gamma)=\tfrac{8}{9}gS\beta$, for $\alpha+\gamma=\beta$ for the parallelogram.

$$\therefore \quad P_2=\tfrac{3}{2}\times\tfrac{9}{8}(P_1+P_3)-P_1-P_3$$
$$=\tfrac{11}{16}(P_1+P_3)$$

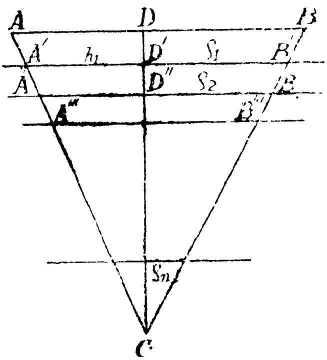
$$\therefore \quad 16P_2=11(P_1+P_3).$$

Ex. 3. *A vessel contains n different liquids resting in horizontal layers and of densities ρ_1, ρ_2,...ρ_n starting from the highest fluid. A triangle is held with its base in the upper surface of the highest liquid and with its vertex in the nth liquid. Prove that if $\triangle$ be the area of the triangle and h_1, h_2.. h_n be the depths of the vertex below the upper surface of the 1st, 2nd,...... nth liquids respectively, the thrust on the triangle is*

$$\frac{1}{3}\frac{g\triangle}{h_1^2}\{\rho_1(h_1^3-h_2^3)+\rho_2(h_2^3-h_3^3)+\ldots+\rho_n h_n^3\},$$

ABC is the triangle immersed in n liquids with AB in the upper surface of the highest liquid. Its vertex C in the nth liquid.

$CD=h_1$, $CD'=h_2$, $CD''=h_3$...



$\frac{1}{2}AB.h_1=\triangle$

$$AB=\frac{2\triangle}{h_1}$$

Also $\frac{A'B'}{AB}=\frac{h_2}{h_1}$

$$\therefore\ A'B'=\frac{2\triangle h_2}{h_1^2}$$

Similarly $A''B''=\frac{2\triangle h_3}{h_1^2}, \ldots\ldots$

Hence areas of different triangles are

$$\triangle,\ \frac{\triangle h_2^2}{h_1^2},\ \frac{\triangle h_3^2}{h_1^2},\ \frac{\triangle h_4^2}{h_1^2},\ldots\ldots,\ \frac{\triangle h_n^2}{h_1^2}$$

The thrust on the $\triangle ABC$ in this position will be the same as the sum of the thrusts on the $\triangle ABC$ in a liquid of density ρ_1, on $\triangle A'B'C'$ in a liquid of density $\rho_2-\rho_1$, on $\triangle A''B''C''$ in a liquid of density $\{\rho_3-\rho_1-(\rho_2-\rho_1)\}$ *i.e.*, $(\rho_3-\rho_2)$ and so on.

$\therefore$ Total thrust

$$=\triangle\cdot\frac{h_1}{3}\rho_1 g+\frac{\triangle h_2^2}{h_1^2}\cdot\frac{h_2}{3}\cdot(\rho_2-\rho_1)g+\frac{\triangle h_3^2}{h_1^2}\cdot\frac{h_3}{3}(\rho_3-\rho_2)g+\ldots$$

$$+\frac{\triangle h_n^2}{h_1^2}\ \frac{h_n}{3}(\rho_n-\rho_{n-1})g$$

$$=\tfrac{1}{3}\frac{g\triangle}{h_1^2}\{\rho_1 h_1^3+h_2^3(\rho_2-\rho_1)+h_3^3(\rho_3-\rho_2)+\cdots+h_n^3(\rho_n-\rho_{n-1})\}$$

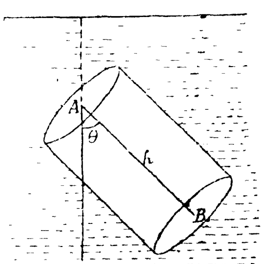
$$=\tfrac{1}{3}\frac{g\triangle}{h_1^2}\{\rho_1(h_1^3-h_2^3)+\rho_2(h_2^3-h_3^3)+\ldots\ \ldots+\rho_n h_n^3\}.$$

Ex. 4. *The inclinations of the axis of a submerged solid cylinder to the vertical in two different positions are complementary to each other. If P and P′ be the difference between the pressures on the two ends in the two cases, prove that the weight of the displaced fluid is equal to*

$$(P^2+P'^2)^{\frac{1}{2}}.$$

Let AB be a circular cylinder of radius a and length h.

Let the depth of A from the free surface be z. θ is the angle that the axis makes with the vertical.



$\therefore$ Thrust on the end $A=\pi a^2 g\rho z$

Thrust on the end $B=\pi a^2 g\rho(z+h\cos\theta)$

$\therefore$ P = Thrust on the end B − thrust on the end A

$$=\pi a^2 g\rho h\cos\theta.$$

In the second position the angle of inclination is $90°-\theta$

$\therefore$ $P'=\pi a^2 gh\rho\sin\theta$

$\therefore$ $(P^2+P'^2)^{\frac{1}{2}}=\pi a^2 g\rho h$

= weight of the displaced fluid.

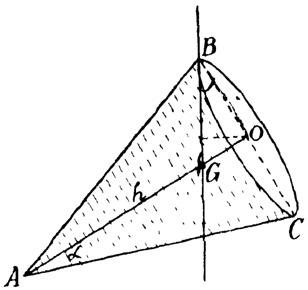
Ex. 5. *A hollow weightless cone of semi-vertical angle* α *and of height* h, *is filled with liquid and freely hung from a point on the rim of the base ; show that the thrust of the water on the base is*

$$\frac{\pi h^3 \tan^3 \alpha \sin \alpha \, g\rho}{\sqrt{1+15 \sin^2\alpha}}.$$

(*Nagpur 1954 ; Banaras 1938*)

Let ABC be the cone of height h and base of radius a.

$$\therefore \quad \tan \alpha = \frac{a}{h}.$$



The cone is suspended from the point B. Since there are only two forces *i.e.*, wt. of liquid contained in the cone and the tension in the string, the line BG, where G is the C.G. of the cone, must be vertical.

Let $\angle OBG = \theta$

$$\therefore \quad \tan \theta = \frac{OG}{BO} = \frac{h/4}{a} = \frac{h}{4a}$$

$$= \frac{1}{4a/h} = \frac{1}{4 \tan \alpha}$$

whence $\cos \theta = \dfrac{4 \sin \alpha}{\sqrt{1+15 \sin^2 \alpha}}$

$\therefore$ Thrust on the base

$$= \pi a^2 . a \cos \theta \, g\rho$$

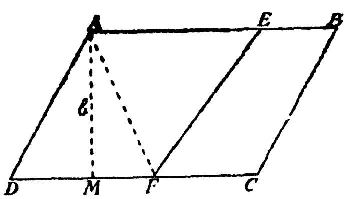
$$= \frac{4\pi h^3 \tan^3 \alpha \sin \alpha \, g\rho}{\sqrt{1+15 \sin^2 \alpha}}.$$

Ex. 6. *A parallelogram* ABCD *is immersed in homogeneous fluid of density* ρ, *open to the atmospheric pressure* Π, *with the side* AB *in the surface.* E *is a point in* AB *such that* AE $=\frac{2}{3}$ AB. *A straight line joins* E *to a point* F *in* CD. *Show that, if the thrusts on* AEFD *and* EBCF *are equal,* DF *is given by*

$$6 \text{ DF } (3\Pi + 2g\rho b) = \text{AB } (6\Pi + 5g\rho b),$$

where b *is the depth of* CD. (*M. T.*)

Let AM ($=b$) be the height of the parallelogram. Let z be the depth of C.G. of the area AEFD from AB. Join AF.



$$\therefore \quad \bar{z}. \tfrac{1}{2}(AE+DF)b$$

$$= \tfrac{1}{2}AE.\ b.\ \frac{b}{3} + \tfrac{1}{2}\ DF.\ b.\ \frac{2b}{3}$$

$$\therefore \bar{z}\ (AE+DF) = (AE+2DF).\frac{b}{3}$$

Thrust on AEFD $=\frac{1}{2}(\text{AE}+\text{DF})\, b.\, (\Pi+g\rho\,\bar{z})$

Thrust on ABCD $=$ AB. $b(\Pi+g\rho\,\frac{b}{2})$

$\therefore\quad \text{AB}\; b(\Pi+\frac{1}{2}\,g\rho b)=2.\;\; \frac{1}{2}(\text{AE}+\text{DF})\; b\; (\Pi+g\rho\,\bar{z})$

$$\text{or AB}(\Pi+\tfrac{1}{2}g\rho b)=(\tfrac{2}{3}\text{AB}+\text{DF})\left\{\Pi+g\rho\,\frac{b}{3}\left(\frac{\frac{2}{3}\text{AB}+2\text{DF}}{\frac{2}{3}\text{AB}+\text{DF}}\right)\right\}$$

$$=\Pi(\tfrac{2}{3}\text{AB}+\text{DF})+g\rho\,\frac{b}{3}\,(\tfrac{2}{3}\text{AB}+2\text{DF})$$

$$\text{or AB}\left(\Pi+g\rho\,\frac{b}{2}-\frac{2\Pi}{3}-\frac{2}{9}\,g\rho b\right)=\text{DF}(\Pi+\tfrac{2}{3}\,g\rho b)$$

$$\text{or}\quad \text{AB}\left(\frac{\Pi}{3}+\frac{5}{18}\,g\rho b\right)=\text{DF}(\Pi+\tfrac{2}{3}g\rho b)$$

$$\text{or}\quad \text{AB}\,(6\Pi+5g\rho b)=6\;\text{DF}\;(3\Pi+2g\rho b).$$

Examples 4

1. Find the thrust on a vertical quadrilateral which has one side of length a in the surface, and the opposite side of length b parallel to it at a depth h.

If the fluid consists of a top layer of density ρ and thickness $\frac{1}{2}h$, and the rest of density σ ; prove that the thrust on the quadrilateral is

$$\frac{1}{48}\,gh^2\{(7a+11b)\rho+(a+5b)\sigma\}.$$

2. A parallelogram ABCD is immersed vertically in a homogeneous liquid with the side AB of length $2a$ in the surface. Points F and E are taken upon AB and DC, such that EC$=x$ and AF$=y$. If EF divides ABCD into two parts of which the pressures are equal, show that

$$2x-y=a.$$

3. A hollow cone, whose axis is vertical and base downwards, is filled with equal volumes of two liquids whose densities are in the ratio $3:1$; show that the thrust on the base is $(3-\sqrt[3]{4})$ times as much as it is when the vessel is filled with the lighter liquid. *(Allahabad 1957 ; Lucknow 1946, 55)*

4. A hollow weightless hemisphere, filled with liquid is suspended freely from a point in the rim of its base ; show that the thrust on the plane base is to the weight of contained liquid as $12:\sqrt{73}$. *(Agra 1960)*

5. A rectangular area is immersed in a heavy liquid with two sides horizontal, and is divided by horizontal lines into strips on which the total thrusts are equal. Prove that if a, b, c are the breadths of the consecutive strips

$$a(a+b)(b-c)=c(b+c)(a-b).$$

(Banaras 1941 ; Delhi 1946 ; Utkal 1946 ; Calcutta 1948)

6. A cubical box with vertical sides is filled with equal volumes of n different liquids, which do not mix, the density of the uppermost being ρ, that of the next $2\rho, \ldots\ldots$ and that of the lowest $n\rho$. Show that the thrust on the base is $(n+1)$ times the thrust on that part of one of the sides which is in contact with the lowest liquid. *(Jaipur 1954)*

7. A cylindrical vessel on a horizontal circular base of radius a, is filled with a liquid of density w to a height h. If now a sphere of radius c and density greater than w is suspended by a thread so that it is completely immersed, prove that the increase of the whole pressure on the curved surface

is

$$\frac{8\pi}{3a} wc^3 \left(h+\frac{2c^3}{3a^2}\right).$$

8. A cylindrical tumbler, half filled with a liquid of density ρ, is filled up with a liquid of density ρ' which does not mix with the former one. Shew that the thrust on the base of the tumbler is to the whole pressure on its curved surface as

$$2r(\rho+\rho') \text{ to } h(\rho+3\rho')$$

where h is the height and r the radius of the base of the cylinder.

(*Allahabad 1943* ; *Agra 1953*)

9. A cylinder is filled with equal volumes of n different fluids which do not mix ; the density of the highest is ρ, that of the next is 2ρ, and so on, that of the lowest being $n\rho$. Shew that the whole pressure on the different portions of the curved surfaces of the cylinder are in the ratios

$$1^2 : 2^2 : 3^2 : \dots\dots\dots \;\dots\dots\dots : n^2.$$

10. A thin hollow cone, of negligible weight is filled with water, and is then suspended from a point in the rim and allowed gradually to take its position of equilibrium. Prove that if the vertical angle of the cone is $\cos^{-1}\frac{2}{3}$, the surface of water will divide the generating line through the point of suspension in the ratio 2 : 1.

(*Agra 1930*)

11. A square immersed vertically in a fluid with one side in the surface is divided by a straight line parallel to a diagonal so that the thrusts on the two parts are equal. If x be the depth of the point in which the dividing line cuts a vertical side a of the square, then prove that

$$2x^3-6a^2x+a^3=0.$$

12. A triangle, completely immersed in a liquid with the vertex C in its surface is divided into two parts by a straight line through A so that the thrusts on two parts are equal. If x be the depth of the point at which the dividing line cuts CB, and α, β be the depths of A and B, then prove that

$$2x^2+2\alpha x-\beta(\alpha+\beta)=0.$$

13. A triangle is immersed in a fluid with one of its sides in the surface ; find the position of a point within the triangle, such that, if it be joined to the angular points, the triangle will be divided into three others, the fluid pressures upon which are equal.

CHAPTER V

CENTRE OF PRESSURE

38. Centre of Pressure.—If a plane area be immersed in a liquid, the pressure at any point of it is normal to the plane surface and is proportional to the depth of the point from the free surface. The pressures at all the points on one side of it form a system of parallel forces which can be compounded into one single force acting at some definite point of the plane of the area. This single force is called the *Resultant Fluid-Pressure* or *Fluid Thrust* the magnitude of which, in the case of a plane area, is the sum of all the pressures acting at various elements of it. The definite point of the plane of the area where the Resultant Pressure acts is called the *Centre of Pressure*.

Definition. The **Centre of Pressure** *of a plane area in contact with a fluid is the point of the area at which the Resultant Thrust on one side of the area acts.*

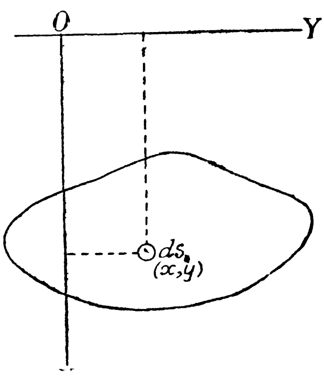
Note.—It has already been shown that the intensity of pressure in a heavy homogeneous liquid increases with the depth. If the area to be considered is horizontal, then the pressure at every point will be the same, and the resultant thrust will act through the centroid of the area. In all other cases when the pressures on all the points of the lower part of the area are greater than those on the upper one, the centre of Pressure will be below the centroid of the area. This leads to the establishment of the Proposition.

"*The depth of the centre of pressure of a plane area immersed in a liquid is greater than the depth of its C.G.*"

Henceforth Centre of Pressure will be shortened as C.P.

39. How to find the Centre of Pressure.

Let x, y be the coordinates of the small element dS of a plane area immersed in a liquid and let $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of Pressure, where the line of the intersection of the plane area and the free surface is taken as y-axis and a perpendicular line in the plane of the area as x-axis.



Let p be the intensity of pressure on the element dS. Therefore the pressure on the element is pdS and its moment about the axis of y is $xpdS$. Hence the sum of the moments of the pressures on all the elements of the type dS of the area is $\int xpdS$. The pressure on the whole area is $\int pdS$ acting at $\bar{x}$, $\bar{y}$ since all the forces are parallel, by taking moments about the axis of y, we obtain

$$\bar{x}\int p dS = \int x p dS$$

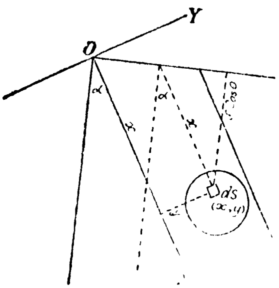
$$\therefore \qquad \bar{x} = \frac{\int x p dS}{\int p dS}.$$

Similarly, by taking moments about the axis of x

$$\bar{y} = \frac{\int y p dS}{\int p dS}.$$

40. *To prove that the position of the Centre of Pressure of a plane area is independent of the inclination of the area to the vertical.*

Let α be the inclination of the plane area to the vertical. Take the line of inter-section of the plane area with the surface of the liquid as y-axis and a perpendicular line OX in the plane of area as x-axis.



Let dS be a small element of the area at a point (x, y) and let p be the intensity of pressure at that element.

$\therefore$ The depth of the element dS from the free surface

$$= x \cos \alpha$$

$$\therefore \; p = g\rho\, x \cos \alpha$$

Hence
$$\bar{x} = \frac{\int x p dS}{\int p dS} = \frac{\int g\rho x^2 \cos \alpha \, dS}{\int g\rho x \cos \alpha \, dS} = \frac{\int x^2 dS}{\int x dS}.$$

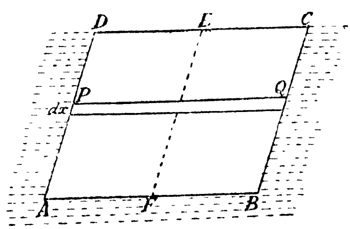
Similarly
$$\bar{y} = \frac{\int x y dS}{\int x dS}.$$

These values of $\bar{x}, \bar{y}$ do not involve α. Therefore they are the same whatever be the inclination of the plane to the vertical.

Note. Thus to find the Centre of Pressure of an area we may make the area vertical if not vertical, by rotating it about the line of intersection with the effective surface. And if P be the Centre of Pressure, then P will remain the Centre of Pressure in every case.

41. The C P. of a parallelogram with one side in the free surface.

Let the parallelogram ABCD be immersed in a liquid with DC in the free surface. If the parallelogram is not vertical, let us rotate it about DC so that it may be vertical ; and in doing so there is no loss of generality.



Join E, the middle point of DC, with F, the middle point of AB.

Take the line EF as x-axis and let α be the angle that EF makes with the horizontal

Now take a strip parallel to the free surface at a distance x from the origin E in the free surface along EF. Let dx be the breadth of the strip along EF. Let DC$=b$, DA$=a$.

If the area be divided into an infinite number of horizontal elementary strips like PQ, the thrusts on all such strips will be acting at their middle points which lie on EF. Hence the C.P. will lie on EF.

$\therefore \quad p=$intensity of pressure on the strip

$=g\rho x \sin\alpha$, where ρ is the density of the liquid

$dS=$area of the strip

$=bdx \sin\alpha$

$$\therefore \quad \bar{x}=\frac{\int xpdS}{\int pdS}=\frac{\int_0^a x.\, g\rho x \sin\alpha\, bdx \sin\alpha}{\int_0^a g\rho x \sin\alpha.\, bdx \sin\alpha}$$

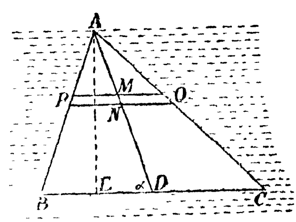
$$=\frac{\int_0^a x^2dx}{\int_0^a xdx}=\tfrac{2}{3}a.$$

Hence the C.P. is in EF at a distance $\frac{2}{3}$EF from E.

Note.—For a rectangle put $\alpha=\frac{\pi}{2}$, the result will be the same.

42. Centre of Pressure of a triangular area immersed in a liquid with its vertex in the surface and base horizontal.

Let AD, the median of the triangular area ABC immersed in a liquid, make an angle α with the horizon.



Take one strip PQ parallel to the base at a distance AM ($=x$) from A along AD. Let the intercept MN of AD, cut off by the strip, be dx.

Let BC$=a$ and AD$=r$

Now $\frac{PQ}{BC}=\frac{AM}{AD}$

$\therefore \quad PQ=\frac{ax}{r}$.

Also the depth of the strip from the free surface$=x \sin\alpha$ and $dS=$Area of the strip$=$PQ.MN $\sin\alpha$

$$=\frac{ax}{r}\, dx \sin\alpha.$$

$p=$the pressure at any point of the strip

$=x \sin\alpha g\rho$, where ρ is the density of the liquid.

If the area is divided into an infinite number of horizontal elementary strips like PQ, the thrusts on all such strips will be acting at their middle points which lie on the median AD.

Hence the C.P. of the triangle will be on AD.

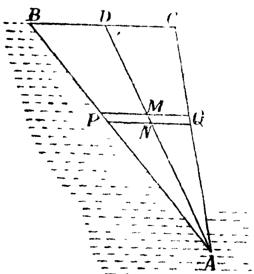
$$\therefore \quad x = \frac{\int xpdS}{\int pdS} = \frac{\int_o^r x.x \sin \alpha g\rho . \frac{ax}{r} dx \sin \alpha}{\int_o^r x \sin \alpha\, g\rho . \frac{ax}{r} dx \sin \alpha}$$

$$= \frac{\int_o^r x^3 dx}{\int_o^r x^2 dx} = \frac{\left[\frac{x^4}{4}\right]_o^r}{\left[\frac{x^3}{3}\right]_o^r} = \frac{3}{4} r$$

$= \frac{3}{4}$ AD, the C.P. of ABC is in AD at a distance $\frac{3}{4}$ AD from A.

43. The Centre of Pressure of a triangular area immersed in a liquid with one side in the surface.

Let DA, the median of the triangle ABC immersed in a liquid, make an angle α with the horizontal. As before it is evident that the C. P. will be on DA.



Take PQ any horizontal strip at a distance x from D along DA. Let the intercept MN of AD, cut off by the strip, be dx.

Let BC$=a$, DA$=r$

$$\frac{PQ}{BC} = \frac{AD - x}{AD} = \frac{r - x}{r}$$

$$\therefore \quad PQ = \frac{a(r-x)}{r}.$$

Also the depth of the strip from the free surface $= x \sin \alpha$

dS = area of the strip = PQ.MN $\sin \alpha$

$$= \frac{a(r-x)}{r} dx \sin \alpha$$

p = pressure at any point of the strip

$= g\rho x \sin \alpha$, where ρ is the density of the liquid.

$$\therefore \quad x = \frac{\int xpdS}{\int pdS} = \frac{\int_o^r xa \frac{(r-x)}{r} dx \sin \alpha\, g\rho\, x \sin \alpha}{\int_o^r \frac{a(r-x)}{r} dx \sin \alpha\, g\rho\, x \sin \alpha}$$

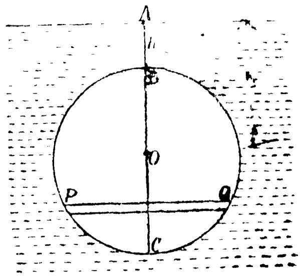
$$= \frac{\int_o^r (r-x)x^2 dx}{\int_o^r (r-x)x dx}$$

$$=\frac{\left[\frac{rx^3}{3}\right]_o^r-\left[\frac{x^4}{4}\right]_o^r}{\left[\frac{rx^2}{2}\right]_o^r-\left[\frac{x^3}{3}\right]_o^r}=\frac{r}{2}.$$

Hence the centre of pressure is the middle point of AD.

44. The centre of pressure of a vertical circular area of radius a wholly immersed with its centre at a depth h.

Let O, the centre of the circle, be the origin. Take a point Q on the circle whose co-ordinates are (x, y).



From symmetry it is clear that the C.P. will be on the vertical line ABOC.

Now take PQ, any horizontal strip of thickness dx, at a distance x from O.

Let the equation of the circle be

$$x^2+y^2=a^2.$$

$d\text{S}$=area of the strip

$=2y\,dx$

p=intensity of pressure on the strip PQ

$=g\rho(h+x)$, where ρ is the density of the liquid.

Therefore the depth of the C.P. of the circle from the centre

$$=\frac{\int xp\,d\text{S}}{\int p\,d\text{S}}=\frac{\int_{-a}^{a} xg\rho\, h+x)2y\,dx}{\int_{-a}^{a} g\rho(h+x)2y\,dx}$$

$$=\frac{\int_{-a}^{a} xy(h+x)dx}{\int_{-a}^{a} y(h+x)dx}$$

Putting $x=a\cos\theta$, $y=a\sin\theta$ and $dx=-a\sin\theta\, d\theta$

$$=\frac{a\int_o^{\pi}\cos\theta\sin^2\theta(h+a\cos\theta)d\theta}{\int_o^{\pi}\sin^2\theta\,(h+a\cos\theta)d\theta}$$

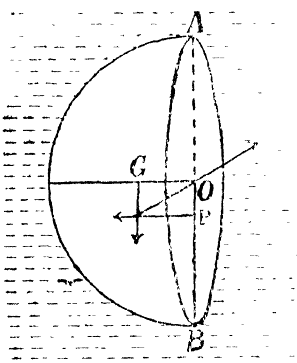
$$=\frac{ha\int_o^{\pi}\cos\theta\sin^2\theta\,d\theta+a^2\int_o^{\pi}\cos^2\theta\sin^2\theta\,d\theta}{h\int_o^{\pi}\sin^2\theta\,d\theta+a\int_o^{\pi}\sin^2\theta\cos\theta\,d\theta}$$

$$=\frac{a\int_0^{\pi}\cos^2\theta\ \sin^2\theta\ d\theta}{h\int_0^{\pi}\sin^2\theta\ d\theta}=\frac{a^2}{4h}.$$

Thus the depth of the C.P. from the free surface

$$=h+\frac{a^2}{4h}.$$

Aliter. Construct a hemisphere on the circle as base, and consider the equilibrium of the liquid contained in it.



The forces acting on this liquid are

(*i*) Its wt. $\frac{2}{3}\pi a^3 g\rho$, vertically downwards through G, the C.G. of it, where $OG=\frac{3}{8}a$.

(*ii*) The thrust on the circle $\pi a^2 hg\rho$ through the centre of pressure P which is clearly in AB.

(*iii*) The resultant thrust on the curved surface, which must pass through **O.**

Taking moments about O, we obtain

$$\tfrac{2}{3}\pi a^3 g\rho\times OG=\pi a^2 hg\rho\times OP$$

$$\therefore\ OP=\frac{2}{3}\times\frac{3}{8}\cdot\frac{a^2}{h}=\frac{a^2}{4h}.$$

Corollary. If the plane be not vertical, we shall obtain the same expression for OP, provided h denotes the distance of the centre from the effective surface measured in the plane of the circle.

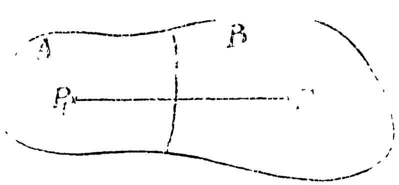
If, however, h is taken to denote the vertical depth of O then since its distance along the inclined plane of the circle will become h cosec θ, the distance OP will be given by

$$OP=\frac{a^2}{4h}\ \sin\theta$$

where θ is the inclination of the plane.

45. **Given the position of the C.P. and the pressure on each of the two portions into which a plane area is divided, to find the C.P. of the whole area.**

Let P_1, P_2 be the C.P. and p_1, p_2 be the pressures on the two portions A and B respectively. Now p_1 acts at P_1 and p_2 at P_2 and we are required to find where the total pressure (p_1+p_2) on the whole area, will act. This total pressure is clearly the resultant of p_1 and p_2 and therefore acts at P, (a point on the straight line joining P_1P_2) such that



$$\frac{P_1P}{PP_2}=\frac{p_2}{p_1}.$$

This gives position of P.

46. If the C.P. and the pressures on the whole area and on one of the parts be given, to find the C.P. of the other part.

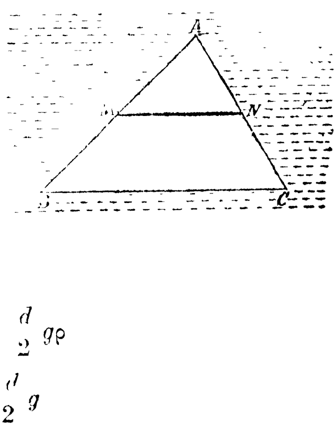
Let P, P_1 be the C.P., and p, p_1 be the pressures on the whole area and the part A respectively. Then the pressure p (on the whole area) will be resultant of the pressure p_1 (on A) and the pressure $p-p_1$ (on the other part B) ; let the last act at P_2. Then P_1PP_2 is a straight line and

$$\frac{PP_2}{PP_1}=\frac{p_1}{p-p_1}.$$

thus giving the C.P. required.

47. Solved Example.

Ex. 1. *ABC is a triangular lamina vertically immersed in a fluid with A in the surface and BC horizontal. M and N are the mid-points of AB and AC. Prove that the depth of the centre of pressure of the area MNCB is $\frac{45}{56}d$, where d is the depth of B.* (*Lucknow 1959*)



Depth of the horizontal side $MN=\frac{d}{2}$; $MN=\frac{1}{2}BC$.

If ρ be the density of the liquid, thrust on the triangle ABC

$$=\tfrac{1}{2}BC.d\,\frac{2d}{3}g\rho,$$

and it acts at a depth $\frac{3}{4}d$ below A.

Again the thrust on the $\triangle$AMN

$$=\frac{1}{2}MN.\frac{d}{2}\cdot\frac{2}{3}\,\frac{d}{2}g\rho$$

$$=\tfrac{1}{2}.\tfrac{1}{2}BC.\frac{d}{2}\cdot\frac{2}{3}\,\frac{d}{2}g$$

$$=\tfrac{1}{24}BCd^2g\rho$$

and it acts at a depth $\frac{3}{4}\cdot\frac{d}{2}$ below A.

Hence, the depth of the C.P. of the trapezium MNCB

$$=\frac{\frac{1}{3}BCd^2g\rho.\ \frac{3}{4}d-\frac{1}{24}BCd^2g\rho.\ \frac{3d}{8}}{\frac{1}{3}BCd^2g\rho-\frac{1}{24}BCd^2g\rho}$$

$$=\tfrac{45}{56}d.$$

Examples 5

1. Prove that the centre of pressure of triangle ABC immersed in a homogeneous liquid with the side BC in the surface, coincides with the centre of two equal forces each $\frac{1}{6} w \Delta \alpha$ acting at the middle point of AB and AC, where α is the vertical depth of A below BC, Δ the area of the triangle and w the specific wt. of the liquid. (*Allahabad 1931*)

2. Prove that the depth of the centre of pressure of a trapezium immersed in water with the side a in the surface and the parallel side b at a depth h below the surface is

$$\left(\frac{a+3b}{a+2b}\right)\frac{h}{2}.$$

(*Calcutta 1912 ; Lucknow 1923, 34, 40, 47 ; Agra 1934 ; Utkal 1948 ; Allahabad 1949, 51 ; Rajputana 1949*)

3. A triangle is wholly immersed in a liquid with its base on the surface. Shew that a horizontal straight line drawn through the centre of pressure of the triangle divides it into two parts, the pressures on which are equal. (*Banaras 1959 ; Calcutta 1937*)

4. Prove that the horizontal line through the centre of pressure of a rectangle, immersed in a liquid with one side in the surface, divides the rectangle in two parts, the fluid pressures on which are in the ratio 4 : 5. (*Jaipur 1956 ; Andhra 1953*)

5. P is the C.P. of the rectangle ABCD, the side AB being in the surface. Prove that the line through A and P divides the area into two portions the pressures on which are equal.

6. A quadrilateral ABCD, with sides AB, CD parallel (and DA, BC equal) is immersed in a homogeneous liquid with AB in the free surface. Show that the centre of pressure will be the intersection of the diagonals, if $AB=\sqrt{3}CD$. (*Bombay 1940*)

7. ABC is a triangular lamina immersed vertically in water with C in the surface and AB horizontal. Show how to divide the area by a horizontal line PQ into two portions, thrusts on which are equal.

If h be the length of the perpendicular from C on AB, prove that the height above AB of the centre of pressure of the area APQB in the above case is $\frac{1}{8}h(3\sqrt[3]{4}-4)$. (*Banaras 1933*)

8. A plane lamina consists of a circular disc (radius a and centre O) from which a circular portion (radius $a/2$ and centre P) has been cut. The lamina is completely immersed in a homogeneous fluid with its plane vertical and P vertically below O. If OP is equal to $a/2$ and the centre of pressure of the lamina is at O, prove that O is at a depth $(11/8a)$ below the surface of the fluid. (*Karnatak 1952 ; Agra 1946*)

9. A square lamina, of side a, has a portion of it in the form of the inscribed circle removed from it and the remaining figure is immersed vertically in water with one side of the square in the surface. Prove that the depth of the C.P. is

$$\frac{a}{24}\cdot\frac{64-15\pi}{4-\pi}.$$

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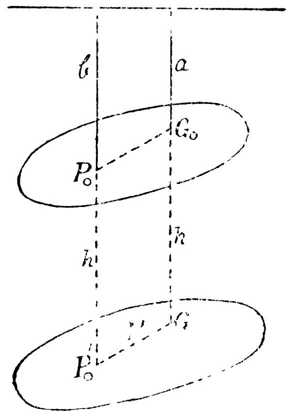
48. Effect of further immersion.

A plane area is immersed in a homogeneous liquid, and the depths of its centres of gravity and pressure are respectively a and b ; if the whole area be now lowered (without any rotation), to find the new position of the centre of pressure.

A plane lamina is immersed in a liquid with its centre of gravity, G_0, at a depth a and centre of pressure P_0 at a depth b from the free surface. If S is the area of the lamina, the thrust on it is $g\rho Sa$ acting at P_0.



It is then further dipped through h in its own plane. Therefore the depths of G and P_0' (the corresponding points of G_0 and P_0 in the new position) are now $a+h$ and $b+h$ respectively.

Due to this further immersion the pressure at every point increases by an amount $g\rho h$ and this extra pressure is uniform over the whole area. Therefore the resultant increased thrust due to this depth is $g\rho hS$ acting at G.

Thus in the second position there are two thrusts

(i) $g\rho Sa$ acting at P_0'

(ii) $g\rho Sh$ acting at G.

Hence by the rules for compounding parallel forces, it follows that the resultant of these two pressures acts at P which lies on P_0' G and divides it so that

$$\frac{GP}{PP_0'} = \frac{g\rho Sa}{g\rho Sh} = \frac{a}{h}.$$

If $\bar{z}$ be the depth of P from BA, we have by taking moments about BA,

$$\bar{z} = \frac{g\rho Sa\,(b+h) + g\rho Sh\,(a+h)}{g\rho Sa + g\rho Sh}$$

or

$$\bar{z} = \frac{h^2+2ah+ab}{a+h}.$$

Corollary 1. We have

$$\text{depth of P} - \text{depth of } P_0'$$
$$= \frac{h^2+2ah+ab}{a+h} - (b+h)$$
$$= -\frac{b-a}{h+a}.h$$

which is always negative for $b>a$.

Hence, *as a consequence of further immersion, in the area itself the centre of pressure is raised through the distance*

$$h\,\frac{b-a}{h+a}.$$

Corollary 2. The vertical distance between the centre of pressure and the centre of gravity in the second position

$$=\frac{h^2+2ah+ab}{a+h}-(a+h)$$

$$=\frac{ab-a^2}{a+h}$$

which varies inversely as $a+h$, i.e., as the depth of the C.G.

It follows that, *the greater the depth, the more nearly does the centre of pressure approach to the centre of gravity, and hence at an infinite depth the two centres coincide.*

Corollary 3. *If the centre of pressure of an area be known when the atmospheric pressure is neglected, its position may be found when this pressure is taken into account.*

For if Π is the atmospheric pressure its effect is equivalent to supposing that the given area has been further lowered through the distance $\frac{\Pi}{g\rho}$, because a height $\frac{\Pi}{g\rho}$ of that liquid would produce the same pressure Π as the atmosphere.

Corollary 4. *The above theorem is true if instead of the area being depressed through a distance h, a depth h of the liquid has been superimposed on the surface of the liquid.*

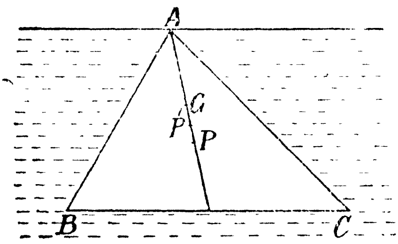
49. Solved Examples.

Ex. 1. *A triangle of height h is immersed in a liquid with the base horizontal and vertex in the surface. If the atmospheric pressure is equivalent to a head of H feet of the liquid, prove that the centre of pressure is raised to a height*

$$\frac{h\,\mathrm{H}}{4(2h+3\mathrm{H})}$$

in the plane of the triangle. (*Bombay 1935*)

Let the $\triangle$ABC be immersed with A in the free surface and BC horizontal. Let G be the C.G. and P the centre of pressure when the atmospheric pressure is neglected. Let S be the area of the triangle and ρ the density of the liquid. Since h is the height of the triangle, the depth of G is $\frac{2h}{3}$ and that of P is $\frac{3}{4}h$.



When the atmospheric pressure is taken into account, the following are the thrusts acting on the triangle :—

(1) The whole pressure $g\rho\mathrm{S}\times\frac{2h}{3}$ acting at P whose depth from A is $\frac{3}{4}h$.

(2) The increased thrust $g\rho SH$ on account of the atmosphere acting at G whose depth from A is $\frac{2h}{3}$. This pressure is uniform on the whole triangle.

Now, if P′ be the new position of the centre of pressure, its depth from A

$$=\frac{g\rho S\frac{2h}{3}\times\frac{3}{4}h+g\rho SH\,.\,\frac{2h}{3}}{g\rho S\frac{2h}{3}+g\rho SH}$$

$$=\frac{3h^2+4hH}{2(2h+3H)}.$$

Hence the raised part

$$=\frac{3h}{4}-\frac{3h^2+4hH}{2(2h+3H)}$$

$$=\frac{1}{4}\frac{hH}{(2h+3H)}.$$

Examples 6

1. A triangle is wholly immersed in a liquid with its C.G. at a depth 2″ and C.P. at a depth 3″. If the triangle be lowered through a distance 2″, find the depth of the C.P. in the new position.

2. An equilateral triangle, each of whose sides is $6\sqrt{3}$ feet long, is immersed vertically in water with its side in the surface which is open to the air. If the water barometer stands at 34 feet, find the depth of the centre of pressure of the triangle.

3. A plane area of any shape is completely submerged in a fluid of uniform density. Its plane is vertical The depth of immersion is varied without rotating the area in its own plane. Show that the product of (i) the depth of the centre of gravity below the free surface and (ii) the depth of its centre of pressure below its centre of gravity is constant, the atmospheric pressure is to be neglected. (*I.C.S. 1928*)

4. A triangle has its base in the surface of a liquid and its vertex downwards; if the atmospheric pressure be equivalent to a height h of water, prove that the centre of pressure will be higher by a distance

$$\frac{1}{2}\frac{hh'}{h+h'}$$

than it is when the atmospheric pressure is neglected, where h' is the depth of the C.G. of the triangle from the surface of the water.

(*Lucknow 1932* ; *Allahabad 1938*)

5. A square lamina is just immersed vertically in water and is then lowered through a depth h ; if a is the length of the edge of the square, prove that the distance of the centre of pressure from the centre of the square is

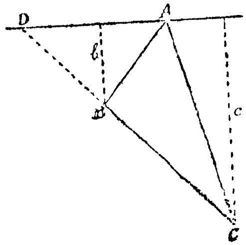
$$\frac{a^2}{6a+12h}.$$

(*Allahabad 1935* ; *Agra 1944*)

50. The depth of the C.P. of a triangle in terms of the depths of its three vertices.

Case I. *When one angular point A is in the free surface.*

Let ABC be the triangle immersed in a liquid with A in the free surface. Let the depths of B and C be b and c respectively.



Let CB be produced to meet the surface in D. Let ρ be the density of the liquid.

Thrust on the triangle ACD$=g\rho\frac{1}{2}$ AD $c.\frac{c}{3}$ acting at a point whose depth is $\frac{c}{2}$.

Thrust on the triangle ABD$=g\rho\frac{1}{2}$ AD. $b.\frac{b}{3}$ acting at a point whose depth is $\frac{b}{2}$.

$\therefore$ Thrust on the triangle ABC$=g\rho\frac{1}{6}$AD (c^2-b^2) acting at a point whose depth is z (say).

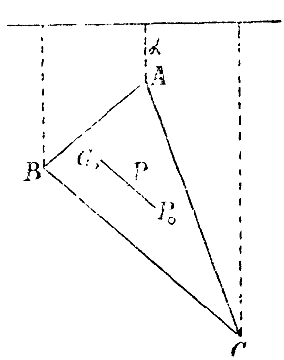
Taking moments about the line AD

$$z=\frac{g\rho\frac{1}{2}\text{AD}.\frac{c^3}{6}-g\rho\frac{1}{2}\text{AD}.\frac{b^3}{6}}{g\rho\frac{1}{6}\text{AD}(c^2-b^2)}$$

$$=\frac{1}{2}\frac{c^3-b^3}{c^2-b^2}$$

$$=\frac{1}{2}\frac{b^2+c^2+bc}{b+c}.$$

Case II. *When the depths of the vertices from the free surface are α, β, γ.*



Let the triangle of case I be now lowered through a distance α so that the depths of A, B and C are respectively α, $a+b$, $\alpha+c$.

Due to this additional depth α an additional thrust $g\rho\alpha$S will act at the centre of gravity G_0, where S is the area of the triangle.

Thus in this position there are two parallel thrusts.

(1) $\frac{1}{3}g\rho S(b+c)$ acting at P_0 whose depth is $\frac{1}{2}\frac{b^2+c^2+bc}{b+c}$ from A.

(2) $g\rho\,\alpha S$, acting at G_0, whose depth is $\frac{1}{3}(b+c)$ from A.

Therefore the depth of the final C.P. below the horizontal line through A is

$$=\frac{\frac{1}{3}g\rho S\,(b+c)\times\left(\frac{1}{2}\,\frac{b^2+c^2+bc}{b+c}\right)+g\rho S\alpha\times\left(\frac{b+c}{3}\right)}{\frac{1}{3}\,g\rho S(b+c)+g\rho S\alpha}$$

$$=\frac{1}{2}\,\frac{b^2+c^2+bc+2\alpha(b+c)}{b+c+3a}$$

$\therefore$ The depth of P below the free surface is

$$\alpha+\frac{1}{2}\,\frac{b^2+c^2+bc+2\alpha(b+c)}{b+c+3\alpha}$$

If the depths of B and C from the free surface be β and γ, then $b=\beta-\alpha$, $c=\gamma-\alpha$ and the above expression becomes

$$=\alpha+\frac{1}{2}\,\frac{(\beta-\alpha)^2+(\gamma-\alpha)^2+(\beta-\alpha)(\gamma-\alpha)+2\alpha(\beta+\gamma-2\alpha)}{\beta+\gamma-2\alpha+3a}$$

$$=\frac{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta}{2(\alpha+\beta+\gamma)}.$$

Cor. 1. The above expression can be written as

$$\frac{\frac{\alpha+\beta}{2}\cdot\frac{\alpha+\beta}{2}+\frac{\beta+\gamma}{2}\cdot\frac{\beta+\gamma}{2}+\frac{\gamma+\alpha}{2}\cdot\frac{\gamma+\alpha}{2}}{\frac{\alpha+\beta}{2}+\frac{\beta+\gamma}{2}+\frac{\gamma+\alpha}{2}}$$

which shows that *the C.P. coincides with the centre of parallel forces acting at the middle point of the sides and proportionate to the corresponding depths.*

Cor. 2. The above expression can also be written as

$$\frac{\alpha(2\alpha+\beta+\gamma)+\beta(\alpha+2\beta+\gamma)+\gamma(\alpha+\beta+2\gamma)}{(2\alpha+\beta+\gamma)+(\alpha+2\beta+\gamma)+(\alpha+\beta+2\gamma)}$$

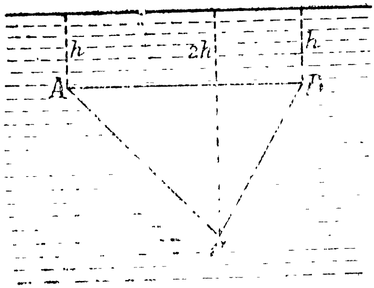
which shows that *the C.P. coincides with the centre of parallel forces acting at vertices A, B, C, and proportionate to* $2\alpha+\beta+\gamma$, $\alpha+2\beta+\gamma$, $\alpha+\beta+2\gamma$ *respectively.*

51. Solved Examples.

Ex. 1. *A plane triangular area is immersed in a liquid of uniform density with its plane vertical, one side horizontal and the opposite corner downwards. Its vertical altitude is h. and the horizontal side is at a depth h below the effective surface. Show that its centre of pressure is at a depth* $\frac{11}{8}$ *h below the surface.*

(*Travancore 1945, I.C.S. 1928*)

Let ABC be the triangle immersed in a liquid with AB horizontal.



According to the given conditions the depths of A, B, C are respectively (h, h, $2h$) from the free surface.

$\therefore$ Depth of C.P.

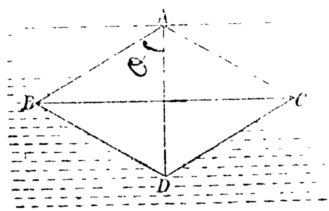
$$=\frac{h^2+h^2+4h^2+h^2+2h^2+2h^2}{2(h+h+2h)}$$

$$=\tfrac{11}{8}\,h.$$

Ex 2. *A rhombus is immersed in a liquid with a vertex in the surface and the diagonal through the vertex vertical. Prove that the centre of pressure divides the diagonal in the ratio 7 : 5.*

(*Lucknow, 1944 ; Nagpur 1930, 43, 55 ; Agra 1927, 36, 40, 52, 56 ; Jaipur 1955 ; Allahabad 1920, 27, 31, 54 : Gorakhpur 1959*)

Let ABCD be the rhombus immersed in the liquid with A in the surface and AD vertical.



Since the rhombus is symmetrical about the diagonal AD, the depth of the C.P. of the rhombus ABCD will be the same as that of the triangle ABD and will lie on AD.

Let one side of the rhombus be a and let AB make an angle θ with AD.

$\therefore$ Depths of A, B, D the three vertices of the $\triangle$ ABD are respectively a, $a\cos\theta$, $2a\cos\theta$

$$\therefore \quad \text{Depth of C.P.} = \frac{a^2\cos^2\theta+4a^2\cos^2\theta+2a^2\cos^2\theta}{2\,(a\cos\theta+2a\cos\theta)}$$

$$=\frac{7}{6}\,a\cos\theta$$

$\therefore$ Vertical distance between the C.P. and the point D,

$$=\text{AD}-\tfrac{7}{6}a\cos\theta$$

$$=2a\cos\theta-\tfrac{7}{6}a\cos\theta$$

$$=\tfrac{5}{6}a\cos\theta$$

Hence at the C.P. AD is divided in the ratio

$$\tfrac{7}{6}a\cos\theta : \tfrac{5}{6}a\cos a$$

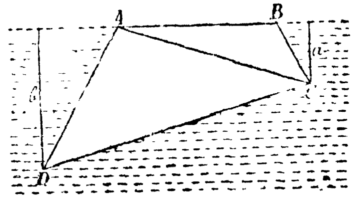
i.e., 7 : 5.

Ex. 3. *A quadrilateral is immersed in water with two angular points in the surface and the other two at depths a and b. If x and y are the depths below the surface of its centres of gravity and pressure respectively, show that*

$$6xy+ab=3x(a+b).$$

(Allahabad 1926 ; Nagpur 1942)

Let ABCD be the quadrilateral with AB in the surface. Let the depths of C and D be a and b respectively. Join AC. Let S_1, S_2 be the areas of the triangles ABC, ADC respectively.



The depths of their C.G's are $\frac{a}{3}$, and $\frac{a+b}{3}$ respectively.

Since x is the depth of the C.G. of the quadrilateral

$$(S_1+S_2)x=S_1.\frac{a}{3}+S_2.\frac{a+b}{3}$$

or $$\frac{3x-a}{b}=\frac{S_2}{S_1+S_2} \qquad \ldots (1)$$

The depths of the centres of pressures of the triangles ABC, ADC respectively are

$$\tfrac{1}{2}a,\ \tfrac{1}{2}.\frac{b^2+a^2+ab}{a+b}\ ;$$

and the thrust on them are

$$g\rho S_1\frac{a}{3},\ \tfrac{1}{3}g\rho S_2\,(a+b).$$

Since y is the depth of the C.P. of the quadrilateral

$$(S_1+S_2)\,xy=S_1\frac{a}{3}\cdot\frac{a}{2}+S_2.\frac{a+b}{3}.\frac{a^2+b^2+ab}{a+b}$$

or $$6\,(S_1+S_2)\,xy=a^2(S_1+S_2)+S_2b\,(a+b)$$

or $$6xy=a^2+\frac{S_2}{S_1+S_2}\,b\,(a+b)$$

$$=a^2+(3x-a)\,(a+b) \qquad \text{from (1)}$$

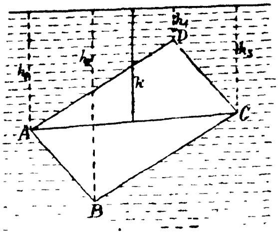
$$=3x(a+b)-ab.$$

Ex. 4. *A parallelogram has its corners at depths h_1, h_2, h_3, h_4 below the surface of a liquid, and its centre at a depth h ; show that the depth of its centre of pressure is*

$$\frac{h_1^2+h_2^2+h_3^2+h_4^2+8h^2}{12h}.$$

(Agra 1941 : Allahabad 1942 ; Utkal 1945 ; Lucknow 1951)

Let the parallelogram ABCD be immersed with its corners A, B, C, D at depths h_1, h_2, h_3, h_4 respectively from the surface. Join AC.



Now $h=\frac{h_1+h_3}{2}=\frac{h_2+h_4}{2}$

$$=\frac{h_1+h_2+h_3+h_4}{4} \quad \dots (1)$$

We know that the C.P. is the same as the centre of parallel forces acting at the middle points of the sides of these two triangles and proportional in magnitudes to the depths of these points.

Suppose forces proportional to $\left(\frac{h_1+h_2}{2}\right)$, $\left(\frac{h_2+h_3}{2}\right)$ and $\left(\frac{h_3+h_1}{2}\right)$ act as the middle points of the sides of the triangle ABC and forces proportional to $\left(\frac{h_1+h_4}{2}\right)$, $\left(\frac{h_3+h_4}{2}\right)$ and $\left(\frac{h_3+h_1}{2}\right)$ act at the middle points of the $\triangle$ ACD.

But $\frac{h_1+h_3}{2}=h$.

$\therefore$ Depth of C.P.

$$=\frac{\left(\frac{h_1+h_2}{2}\right)^2+\left(\frac{h_2+h_3}{2}\right)^2+\left(\frac{h_4+h_1}{2}\right)^2+\left(\frac{h_3+h_4}{2}\right)^2+2h^2}{h_1+h_2+h_3+h_4+2h}$$

$$=\frac{2[h_1^2+h_2^2+h_3^2+h_4^2+h_1(h_2+h_4)+h_3(h_2+h_4]+8h^2}{4.6h}$$

$$=\frac{(h_1^2+h_2^2+h_3^2+h_4^2+4h^2)+4h^2}{12h} \qquad \text{from (1)}$$

$$=\frac{h_1^2+h_2^2+h_3^2+h_4^2+8h^2}{12h}$$

Aliter. Let the area of each triangle be S.

$\therefore$ Thrust on the $\triangle$ ABC$=g\rho S\,\frac{h_1+h_2+h_3}{3}$ and acts at a point whose depth from the surface is

$$\frac{h_1^2+h_2^2+h_3^2+h_2h_3+h_3h_1+h_1h_2}{2(h_1+h_2+h_3)}.$$

Thrust on the $\triangle$ ACD$=g\rho S\frac{(h_1+h_3+h_4)}{3}$ and acts at a point whose depth from the free surface is $\frac{h_1^2+h_3^2+h_4^2+h_3h_4+h_4h_1+h_1h_3}{2(h_1+h_3+h_4)}$

$\therefore$ Depth of the C.P.

$$= \frac{g\rho S\left(\frac{h_1+h_2+h_3}{3}\right)\left\{\frac{h_1^2+h_2^2+h_3^2+h_2h_3+h_3h_1+h_1h_2}{2(h_1+h_2+h_3)}\right\}+g\rho S\frac{(h_1+h_3+h_4)}{3}\left\{\frac{h_1^2+h_3^2+h_4^2+h_3h_4+h_4h_1+h_1h_3}{2(h_1+h_3+h_4)}\right\}}{g\rho S.\frac{1}{3}(2h_1+2h_3+h_2+h_4)}$$

$$= \frac{h_1^2+h_2^2+h_3^2+h_2h_3+h_3h_1+h_1h_2+(h_1^2+h_3^2+h_4^2+h_3h_4+h_4h_1+h_1h_3}{12h}$$

Also $(h_1+h_2+h_3+h_4)^2=16h^2$ ... (1)

$(h_2+h_4)^2=4h^2$ ... (2)

$(h_1+h_3)^2=4h^2$ ... (3)

Subtracting the sum of (2) and (3) from (1)

$h_2h_3+h_3h_4+h_1h_2+h_1h_4=4h^2$ ... (4)

$\therefore$ The depth of the C.P.

$$= \frac{h_1^2+h_2^2+h_3^2+h_4^2+(h_1+h_3)^2+h_2h_3+h_3h_4+h_1h_2+h_1h_4}{12h}$$

$$= \frac{h_1^2+h_2^2+h_3^2+h_4^2+4h^2+4h^2}{12h} \quad \text{from (3) \& (4)}$$

$$= \frac{h_1^2+h_2^2+h_3^2+h_4^2+8h^2}{12h}.$$

Examples 7

1. A triangle of height 1″ is immersed vertically in a liquid with its base parallel to the surface and at a depth 3″. If the vertex be downwards, find the depth of the centre of pressure of the triangle.

2. A plane triangle (height 30 cm.) is immersed in a liquid, its base horizontal and vertex uppermost. Find the difference of level between the C.P. and C.G. of the triangle, given the vertex is 10 cm. below the surface. *(Madras 1926)*

3. Show that the centre of pressure of a triangular lamina, the depths of whose angular points are a, b, c, is at a depth

$$\frac{1}{6}\cdot\frac{a^2+b^2+c^2-bc-ca-ab}{a+b+c}$$

or

$$\frac{(b-c)^2+(c-a)^2+(a-b)^2}{12\,(a+b+c)}$$

below the centre of gravity of the lamina. *(Agra 1942, 60)*

4. Find the centre of pressure of an isosceles triangle immersed with its plane vertical and its base horizontal half as far below the surface as its vertex. *(Calcutta 1911)*

5. Find the centre of pressure of a square lamina immersed in a fluid, with one vertex in the surface and the diagonal vertical.

(Allahabad 1920 ; Patna 1926 ; Lucknow 1944 ; Agra 1931, 35, 50)

6. An equilateral triangle ABC of altitude h has one vertex fixed at a depth $2h$ below the surface of a liquid. From the position in which BC is horizontal above A, the triangle is turned till BC is again horizontal but below A,

the plane of the triangle always remaining vertical. Show that the shift of the centre of pressure relative to the triangle is $h/16$. *(Agra 1938)*

7. A parallelogram ABCD is immersed in a liquid with A in the surface, and BD horizontal. Prove that the centre of pressure P lies on AC such that AP : AC=7 : 12. *(Agra 1958 ; Patna 1935)*

8 A quadrilateral is immersed vertically having two sides of lengths $2a$, a parallel to the surface at depth h and $2h$ respectively. Show that the depth of the centre of pressure is $\frac{3}{2}h$. *(M.T.)*

9. A lamina in the shape of a quadrilateral ABCD has the side CD in the surface, and the sides AD, BC vertical and of lengths α and β respectively. Prove that the depth of the centre of pressure is

$$\frac{1}{2}\frac{\alpha^4-\beta^4}{\alpha^3-\beta^3} \quad \text{or} \quad \frac{1}{2}\frac{(\alpha+\beta)(\alpha^2+\beta^2)}{\alpha^2+\alpha\beta+\beta^2}$$

(Lucknow 1956 ; Banaras 1944 ; Agra 1947, 1951 ; Sagar 1959)

10. Show that the depth of centre of pressure of a rhombus totally mmersed with one diagonal vertical and its centre at a depth h, is

$$h+\frac{a^2}{24h},$$

where a is the length of the vertical diagonal.
(Agra 1959 ; Allahabad 1924, 34 ; Aligrah 1942 , Lucknow 1945, 57

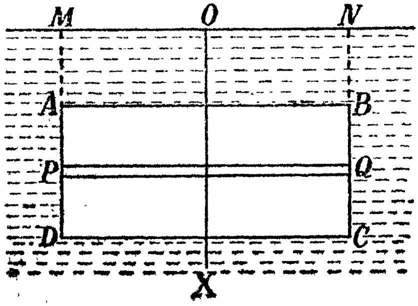
52. Solved Examples with Integration.

Ex. 1. *A rectangle immersed vertically in water with two sides horizontal and at depths h and a+h respectively below the effective surface ; prove that the distance of the C.P. from the upper side is*

$$\frac{a}{3}\cdot\frac{3h+2a}{2h+a}$$

Show also that the C.P. is below the C.G. of the area, and as the depth increases, approaches but never coincides with it.
(Allahabad 1923 ; Lucknow 1943)

Let ABCD be the rectangle immersed in water with two sides AB and CD horizontal and at depths h and $a+h$ respectively.



$\therefore$ MA$=h$. MD$=h+a$

Take OX, a vertical line dividing the rectangle into two equal parts as the axis of x with its point O in the free surface.

Divide the rectangular area into horizontal strips of thickness dx. Let PQ be one of them at a distance x from the free surface.

Let AB $=b$ and ρ the density of the water.

p=pressure at any point of PQ$=g\rho.\ x$

dS=area of the strip$=bdx$.

Hence the depth of C.P. below the free surface

$$\bar{x}=\frac{\int x\,pdS}{\int pdS}=\frac{\int_h^{a+h} x.\,g\,\rho\,x.\,b\,dx}{\int_h^{a+h} g\,\rho\,x.\,b\,dx}$$

$$=\frac{\int_h^{a+h} x^2\,dx}{\int_h^{a+h} x\,dx}=\frac{\frac{1}{3}[(a+h)^3-h^3]}{\frac{1}{2}[(a+h)^2-h^2]}$$

$$=\tfrac{2}{3}.\ \frac{a^2+3h^2+3ah}{a+2h}.$$

$\therefore$ The depth of the C.P. below the upperside

$$=\tfrac{2}{3}\ \frac{a^2+3h^2+3ah}{a+2h}-h$$

$$=\frac{a}{3}\cdot\frac{3h+2a}{a+2h}$$

And the depth below the C.G.

$$=\bar{x}-\left(h+\frac{a}{2}\right)$$

$$=\tfrac{2}{3}\ \frac{a^2+3ah+3h^2}{2h+a}-\frac{2h+a}{2}$$

$$=\tfrac{1}{6}\ \frac{4a^2-3a^2}{2h+a}$$

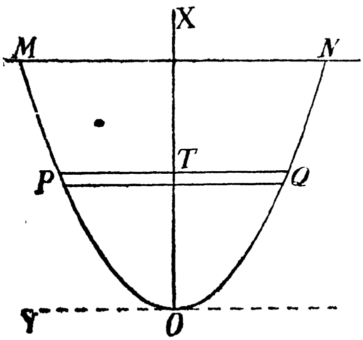
$$=\frac{a^2}{6(2h+a)}.$$

The above expression is positive hence the C.P. is below the C.G. of the area.

As h increases the above expression decreases but never actually vanishes until h is infinite.

Ex. 2. *A segment of a parabola cut off by a double ordinate in a distance h from the vertex is immersed with this ordinate in the surface of a liquid ; find the resultant thrust on it. Find also the position of the point where its line of action meets the lamina.* *(Jaipur 1955 ; Lucknow 1955 ; Allahabad 1925, 37 ; 52, 57 ; Agra 1949)*

Let the equation to the parabola be $y^2=4ax$ where a vertical line through O, the vertex, is x-axis and a perpendicular line OY through O as y-axis.



Let MN be in the free surface.

Take a horizontal strip PQ at a distance x from O and thickness dx.

Let ρ be the density of the liquid.

$$\therefore\ p=g\rho.\ XT=g\rho\ (h-x),$$

$$dS=2ydx$$

$\therefore$ The pressure on the strip

$$=2g\rho\ (h-x)y.dx$$

Hence the whole pressure

$$=2\int_0^h g\rho\ (h-x)\ y\ dx$$

$$=2g\rho\int_0^h (h-x)\ 2\sqrt{ax}\ dx$$

$$=4g\rho\sqrt{a}\left[\tfrac{2}{3}hx^{\frac{3}{2}}-\tfrac{2}{5}x^{\frac{5}{2}}\right]_0^h.$$

$$=4g\rho\sqrt{a}\ 2h^{\frac{5}{2}}\left(\tfrac{1}{3}-\tfrac{1}{5}\right)$$

$$=8g\rho\sqrt{ah}\,h^2\,\tfrac{2}{15}=\tfrac{16}{15}g\rho h^2\sqrt{ah}\ .$$

Taking moment about MN, we obtain if z be the depth of the CP.,

$$z\int_0^h 2g\rho y\ (h-x)dx=\int_0^h 2g\rho y(h-x)^2dx$$

$$\therefore\qquad z=\frac{\int_0^h y(h-x)^2\,dx}{\int_0^h y(h-x)dx}=\frac{\frac{64}{105}\sqrt{ah}g\rho h^2}{\frac{16}{15}g\rho h^2\sqrt{ah}}$$

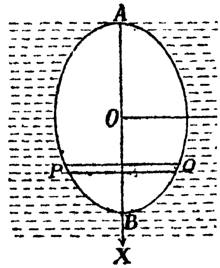
$$=\tfrac{4}{7}h.$$

Ex. 3. *An ellipse is just immersed in water with its major axis vertical. Show that if the centre of pressure coincides with the focus, the eccentricity of the ellipse must be* $\frac{1}{4}$.

(Nagpur 1956 ; Allahabad 1926, 43 ; Agra 1929, 43 ; Sagar 1949 ; Jaipur 1957)

Let the major axis and minor axis be respectively the axes of x and y. Then the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots (1)$$



By symmetry it is evident that the C.P. will lie on the line AOB.

Take an elementary strip PQ at a depth x from O, the centre of the ellipse and of thickness dx.

dS = area of the strip $=2ydx$

p = intensity of pressure at any point of the strip

$=g\rho\,(a+x)$

If $\bar{z}$ be the depth of the C.P. from O, we have

$$\bar{z}=\frac{\int_{-a}^{a} 2ydxg\rho(a+x)x}{\int_{-a}^{a} 2ydxg\rho(a+x)}$$

$$=\frac{\int_{-a}^{a} xy\,(a+x)dx}{\int_{-a}^{a} y\,(a+x)\,dx}$$

$$=\frac{\int_{-a}^{a} x\,(a+x)\sqrt{\left(1-\frac{x^2}{a^2}\right)}\,dx}{\int_{-a}^{a} (a+x)\sqrt{\left(1-\frac{x^2}{a^2}\right)}\,dx}$$

Now putting $x=a\sin\theta$, we get

$$\bar{z}=\frac{a\int_{-\pi/2}^{\pi/2}(1+\sin\theta)\cos^2\theta\sin\theta\,d\theta}{\int_{-\pi/2}^{\pi/2}(1+\sin\theta)\cos^2\theta\,d\theta}$$

$$=\frac{a\int_{-\pi/2}^{\pi/2}(\sin\theta\cos^2\theta+\sin^2\theta\cos^2\theta)\,d\theta}{\int_{-\pi/2}^{\pi/2}(\cos^2\theta+\sin\theta\cos^2\theta)\,d\theta}$$

$$=\frac{a\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta}{\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta}$$

$$=\frac{2a\int_{0}^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta}{2\int_{0}^{\pi/2}\cos^2\theta\,d\theta}$$

$$= \frac{a\,\dfrac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2})}{2\,\Gamma(3)}}{\dfrac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{2\,\Gamma(2)}} = \frac{a}{4}.$$

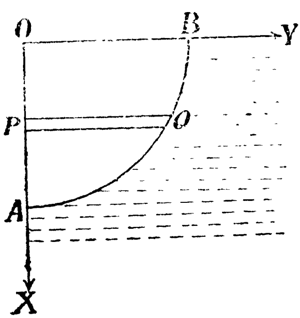
Now the C.P. will coincide with the focus, if

$$\tfrac{1}{4}a = ae$$

i.e., $e = \tfrac{1}{4}$.

Ex. 4. *A quadrant of a circle is just immersed vertically in a heavy homogeneous liquid with one edge in the surface. Find the centre of pressure.* (*Agra 1945 ; Allahabad 1948 : Jaipur 1949, 59*)

Let the edge OB of the quadrant be in the free surface. Let OA be the other vertical edge.



Let OA and OB be the axes of x and y respectively. If a be the radius ; the equation of the curve is

$$x^2 + y^2 = a^2 \qquad \dots (1)$$

Take one strip PQ parallel to OB at a depth x and of thickness dx.

$$\therefore \quad p = g\rho x$$
$$dS = y\,dx.$$

Denoting by $\bar{x}$ and $\bar{y}$ the x and y co-ordinates of the C.P.

$$\bar{x} = \frac{\int xp\,dS}{\int p\,dS} = \frac{\int_0^a g\rho x^2 y\,dx}{\int_0^a g\rho x y\,dx} = \frac{\int_0^a x^2 y\,dx}{\int_0^a xy\,dx}$$

$$= \frac{\int_0^a x^2\sqrt{a^2 - x^2}\,dx}{\int_0^a x\sqrt{a^2 - x^2}\,dx}$$

Putting $x = a \sin\theta$, $dx = a\cos\theta\,d\theta$.

$$= \frac{a\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta}{\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta\,d\theta} = \frac{\dfrac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2})}{2\,\Gamma(3)}}{\dfrac{\Gamma(1)\,\Gamma(\frac{3}{2})}{2\,\Gamma(\frac{5}{2})}} = \frac{3\pi a}{16}$$

Since the thrust on PQ will be acting at its middle point, taking moments about the axis of x we obtain

$$\bar{y}=\frac{\int \frac{1}{2}y\,p dS}{\int p dS}=\frac{\frac{1}{2}\int_0^a xy^2dx}{\int_0^a xydx}=\frac{\frac{1}{2}\int_0^a x(a^2-x^2)dx}{\int_0^a x\sqrt{a^2-x^2}.dx}$$

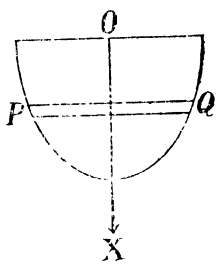
$$=\frac{1}{2}\frac{\left(\frac{a^4}{2}-\frac{a^4}{4}\right)}{\frac{a^3}{3}}=\frac{3}{8}a$$

Thus the C.P. is $(\frac{3}{16}\pi a,\ \frac{3}{8}a)$.

Ex. 5. *A semi-ellipse bounded by its minor axis is just immersed in a liquid, the density of which varies as the depth; if the minor axis be in the surface, find the eccentricity in order that the focus may be the centre of pressure.* (*Lucknow 1953 ; Allahabad 1933, 55 ; Agra 1938*)

Let the equation to the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



with reference to the major as the axis of x. It is clear that the C.P. lies on OX.

Take one strip PQ parallel to the minor axis at a depth x from O, the centre of the ellipse and of thickness dx.

dS = Area of the strip $=2ydx$.

Since the density at any point of PQ $=cx$, where c is any constant,

p = the intensity of pressure at any point of PQ.

$=gcx.x=gcx^2$

$$\therefore\ \text{Depth of C.P.}=\frac{\int xpdS}{\int pdS}=\frac{\int_0^a gcx^3\,(2ydx)}{\int_0^a gcx^2\,(2ydx)}$$

$$=\frac{\int_0^a x^3ydx}{\int_0^a x^2ydx}=\frac{\int_0^a x^3\sqrt{a^2-x^2}dx}{\int_0^a x^2\sqrt{a^2-x^2}\,dx}$$

Putting $x=a\sin\theta$, $dx=a\cos\theta\,d\theta$

$$\text{Depth of C.P.}=\frac{\int_0^{\pi/2} a^2\sin^3\theta.\ a^2\cos^2\theta\ d\theta}{\int_0^{\pi/2} a^2\sin^2\theta\ a^2\cos^2\theta\ d\theta}$$

$$=\frac{\dfrac{\Gamma(2)\,\Gamma(\frac{3}{2})}{2\,\Gamma(\frac{7}{2})}}{\dfrac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2})}{2\,\Gamma(3)}}a$$

$$=\frac{32a}{15\pi}.$$

In order that C.P. may coincide with the focus

$$ae=\frac{32a}{15\pi}$$

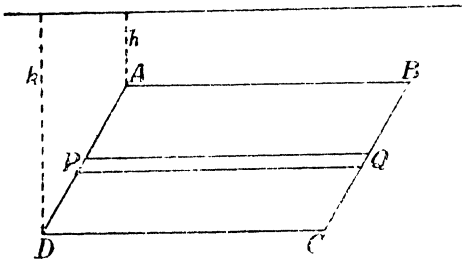
$$e=\frac{32}{15\pi}.$$

Ex. 6. *Prove that the depth of the C.P. of a parallelogram two of whose sides are horizontal and at depths h, k below the surface of a liquid whose density varies as the depth below the surface is*

$$\frac{3}{4}\cdot\frac{h^3+h^2k+hk^2+k^3}{h^2+hk+k^2}.$$

(Allahabad 1911 ; Calcutta ; 1914, 48 ; Agra 1955)

Let PQ be an elementary strip of area of small breadth dx, parallel to the horizontal side of the parallelogram, at a depth x from the free surface.



$\therefore$ The density at any point of this strip is $=cx$ where c is any constant.

($\because$ density varies as depth)

$p=$ intensity of pressure at any point of the strip $=gcx.x=gcx^2$

$dS=$ Area of the strip

$=PQ.dx=ABdx.$

$\therefore$ The depth of the C.P.

$$\bar{x}=\frac{\int xpdS}{\int pdS}=\frac{\int_h^k x.\ gcx^2.\ AB\ dx}{\int_h^k gcx^2.\ AB\ dx}$$

$$=\frac{\int_h^k x^3dx}{\int_h^k x^2dx}$$

$$=\frac{3}{4}\frac{(k^4-h^4)}{(k^3-h^3)}$$

$$=\frac{3}{4}\frac{h^3+h^2k+hk^2+k^3}{h^2+hk+k^2}.$$

Examples 8

1. A rectangle is immersed vertically in a heavy homogeneous liquid with two of its sides horizontal and at depths α and β below the surface. Show that the depth of the centre of pressure is

$$\frac{2}{3}\frac{\alpha^2+\alpha\beta+\beta^2}{\alpha+\beta}.$$

(Calcutta 1918 ; Patna 1942, 47 ; Utkal 1945, 47 ; Jaipur 1959)

2. A rectangular area of height h is immersed vertically in a liquid with one side in the surface ; show how to draw a horizontal line across the area so that the centres of pressure of the parts of the area above and below this line of division shall be equally distant from it. *(Nagpur 1937)*

3. A semi-circular lamina of radius a is immersed in a liquid with the diameter in the surface. Find the depth of the centre of pressure. *(Sagar 1959; Lucknow 1942, 45 ; Agra 1928, 48, 52, 54, 56 ; Allahabad 1939, 42, 59)*

4. A uniform elliptic lamina, whose axes are $2a$ and $2b$, is half immersed in water, the axis $2b$ being in the surface. Find the centre of pressure *(Allahabad ; 1953 Lucknow 1942, 48)*

5. An ellipse is completely immersed, with its minor axis horizontal and at a depth h ; find the positoin of the centre of pressure. *(Allahabad 1943)*

6. A semi-circular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of the centre of pressure is

$$\frac{1}{4}\cdot\frac{3\pi(a^2+4b^2)+32ab}{4a+3\pi b}. \qquad \textit{(Bombay 1948)}$$

7. Prove that the depth of the centre of pressure of the area bounded by two concentric semi-circles with their common bounding diameter in the surface is

$$\frac{3\pi}{16}\cdot\frac{a^4-b^4}{a^3-b^3},$$

a and b being their radii. *(Allahabad 1950 ; Jaipur 1956)*

8. A quadrant of a circle of radius a is immersed vertically with its bounding radius horizontal at a depth b.

Find the centre of pressure.

9. Find the C.P of a parabolic lamina immersed vertically in a liquid with its vertex at a depth k below the water surface, and the lamina is bounded by an ordinate at a distance h from the vertex which is below the ordinate.

10. A quadrant of a circle is just immersed vertically with one edge on the surface in a liquid whose density varies as the depth. Find out the centre of pressure.

11. Prove that the depth of the C.P. of a parabolic lamina from its vertex immersed vertically in a liquid with its vertex at a depth k below the water surface, and bounded by an ordinate at a distance h from the vertex is

$$\frac{3}{7}\frac{7k+5h}{5k+3h}h.$$

when the vertex is above the ordinate.

Consider also the case when the density of the liquid varies as the depth.

53. Liquids more than one.—Suppose we have a number of layers of different fluids which do not mix and we want to find the depth of the C.P. of an area immersed in all the liquids.

Let the densities of the n liquids be $\rho_1, \rho_2 \ldots, \rho_n$ in ascending order and let $A_1, A_2, \ldots A_n$ be the parts of the whole area in contact with these liquids respectively.

The thrust on the area will remain the same even if we consider that one liquid of density ρ_1 is in contact with the whole area $(A_1+A_2+\ldots\ldots+A_n)$, second liquid of density $\rho_2-\rho_1$ in contact with $(A_2+A_3+\ldots\ldots+A_n)$ area, third liquid of density $\rho_3-\rho_1-(\rho_2-\rho_1)$ *i.e.*, $(\rho_3-\rho_2)$ in contact with $(A_3+A_4+\ldots+A_n)$ area and so on and lastly the liquid of density $(\rho_n-\rho_{n-1})$ in contact with A_n area.

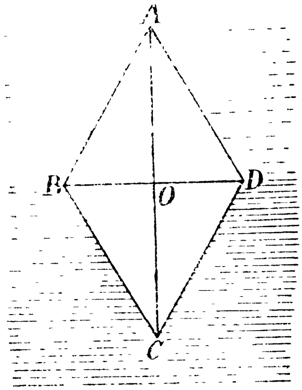
Then taking moments of these thrusts and the resultant thrust about some suitable line, the depth of the C.P. can be found.

The method is illustrated by means of examples in the next article.

54. Solved Examples.

Ex. 1. *Prove that the centre of pressure of a rhombus immersed in two liquids which do not mix, with a vertex in the upper surface and a diagonal in the common surface, divides the other diagonal in the ratio of 17 : 11 if the density of the lower liquid is twice that of the upper liquid.*

Let ABCD be the rhombus with A in the surface and BD in the common surface. Let ρ and 2ρ be the densities of the two liquids.



Let h be the depth of BD, therefore $2h$ is the depth of C. On account of the symmetry about the diagonal AC, the depth of the C.P. of the rhombus will be the same as that of the $\triangle$ABC. Let ABC be of area S.

Now for this triangle, thrust will be the same as that for a liquid of density ρ for the whole triangle ABC and that for a liquid of density $(2\rho-\rho)$ *i.e.*, ρ for the triangle BOC.

$\therefore$ Thrust on the triangle ABC

$=S.hg\rho$ acting at a point whose depth from the upper surface is $\frac{7}{6}h$.

Thrust on the $\triangle$BOC

$=\frac{S}{2}\cdot\frac{h}{3}g\rho$ acting at a point whose depth from the upper surface is $\left(h+\frac{h}{2}\right)$ *i.e.*, $\frac{3}{2}h$.

∴ Depth of the C.P. of the whole area

$$=\frac{Shg\rho\cdot\frac{7}{6}h+\frac{S}{2}\cdot\frac{h}{3}g\rho\cdot\frac{3}{2}h}{Shg\rho+\frac{S}{2}\cdot\frac{h}{3}g\rho}$$

$=\frac{17}{14}h.$

Depth of AC$=2h$.

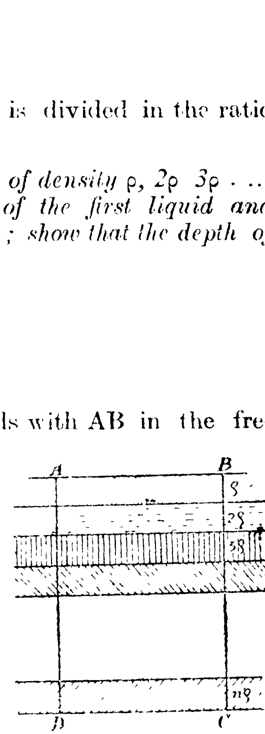
Hence at the C.P. the diagonal AC is divided in the ratio $\frac{17}{14}h$. $(2h-\frac{17}{14}h)$ *i.e.*, 17 : 11.

Ex. 2. *A rectangle is immersed in fluids of density* ρ, 2ρ 3ρ . ... $n\rho$; *the top of the rectangle being in the surface of the first liquid and the area immersed in each liquid being the same ; show that the depth of the centre of pressure of the rectangle is*

$$\frac{3n+1}{2n+1}\cdot\frac{h}{2}$$

where h is the depth of the lower side.

Rectangle ABCD is immersed in n liquids with AB in the free surface.



Thrust in this case is the same as that for a liquid of density ρ on the rectangle of height h+that for a liquid of density $(2\rho-\rho)$ on the rect. of height $h\left(1-\frac{1}{n}\right)$+ that for a liquid of density $(3\rho-2\rho)$ on the rect. of height.

$h\left(1-\frac{2}{n}\right)$+ ... +that for a liquid of density $n\rho-(n-1)\rho$ on the rect. of height

$$\frac{h}{n}.$$

The thrusts on these rectangles are

$$g\rho\,\text{AB}.\frac{h^2}{2},\ g\rho\,\text{AB}\frac{(n-1)^2}{n^2}\frac{h^2}{2},\ g\rho\,\text{AB}\frac{(n-2)^2}{n^2}\frac{h^2}{2},\ \ldots\ldots\ldots\ldots,$$

$g\rho\frac{\text{AB}h^2}{2n^2}$ respectively.

The distances of the centres of pressures of these rectangles from the lowest side DC are respectively

$$\frac{h}{3},\ \frac{(n-1)h}{3n},\ \frac{(n-2)h}{2n},\ \ldots\ldots\ldots\ \frac{h}{3n}.$$

If $\bar{z}$ be the height of the C.P. from DC, then taking moments about DC, we obtain

$$\bar{z} = \frac{g\rho AB\frac{h^2}{2}\cdot\frac{h}{3} + g\rho AB\frac{(n-1)^2}{n^2}\cdot\frac{h^2}{2}\cdot\frac{(n-1)h}{3n} + .. + g\rho AB\frac{h^2}{n^2}\frac{1}{2}\cdot\frac{h}{3n}}{g\rho AB\frac{h^2}{2} + g\rho AB\frac{(n-1)^2}{n^2}\cdot\frac{h^2}{2} + \ldots\ldots + g\rho AB\frac{h^2}{n^2}\frac{1}{2}\cdot}$$

$$= \frac{h}{3n}\left\{\frac{n^3+(n-1)^3+\ldots\ldots+1}{n^2+(n-1)^2+\ldots\ldots+1}\right\}$$

$$= \frac{h}{3n}\,\frac{\left\{\frac{1}{2}n(n+1)\right\}^2}{\frac{n(n+1)(2n+1)}{6}}$$

$$= \frac{h(n+1)}{2(2n+1)}.$$

$\therefore$ Depth of the C.P. from the free surface

$$= h - \frac{h(n+1)}{2(2n+1)}.$$

$$= \frac{3n+1}{2n+1}\,\frac{h}{2}.$$

Examples 9

1. A rectangle is immersed in two liquids of density ρ and 2ρ which do not mix; the top of the rectangle is in the surface of the first liquid and the area immersed in each is the same. Prove that the C.P. divides the rectangle in the ratio 7 : 3.

2. Prove that the C.P. of a rhombus immersed in two liquids which do not mix, with a vertex in the upper surface and a diagonal in the common surface, divides the other diagonal in the ratio of 5 : 3 if the density of the lower liquid is thrice that of the upper liquid.

3. A vessel contains three fluids of densities ρ, 2ρ and 3ρ. A triangle of area A is supported with one side in the surface of the fluid of density ρ and the opposite vertex in the fluid of density 3ρ. If $3h$, $2h$ and h are the depths of the vertex below the upper surface of the three fluids, prove that, neglecting the atmospheric pressure, the thrust on each face of the triangle is

$$\frac{4}{3}\,g\rho\,Ah.$$

Also prove that the depth of the C.P. of the triangle from the free surface is $59h/36$. (*M.T.*)

4. A fluid of depth $2a$ and uniform density ρ is superposed on a liquid of density 2ρ and depth greater than a. A circular lamina of radius a, is placed with its plane vertical and its centre in the surface common to the two liquids. Prove that the depth of the C P. below the centre of lamina, neglecting the atmospheric pressure, is

$$\frac{9\pi a}{16(3\pi+1)}.$$ (*M.T.*)

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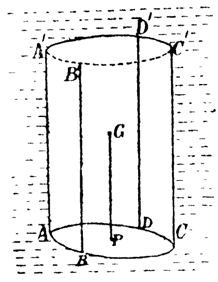
55. The centre of Pressure of a plane area lies vertically beneath the centre of Gravity of the superincumbent fluid.

Let the area ABCD be immersed not in a vertical position. From every point of the perimeter of the area draw vertical lines to meet the free surface in the curve A′B′C′D′. The volume of the fluid enclosed by these lines and the plane area is called *the superincumbent fluid*. Consider the equilibrium of the superincumbent fluid. The forces acting on it are :



(1) its wt. acting vertically downwards through its centre of gravity G ;

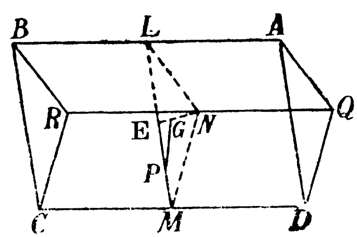
(2) normal fluid-thrust on ABCD acting at its centre of pressure P ;

(3) pressure on the curved surface, which is horizontal at every point.

The vertical components of these forces must themselves form a system in equilibrium *i.e.*, the wt. of the superincumbent liquid at G and the vertical component of the fluid thrust on the area acting at P must be equal and must act along the same line, so that GP is vertical.

56. **Application.** The above method can be illustrated by finding the C.P. of a rectangular plane area.

Let ABCD be rectangle inclined at a finite angle to the vertical plane through AB which is in the free surface.



Draw vertical lines through all the points on BC, CD, DA to meet the surface in BR, RQ, QA. Now the superincumbent liquid is in the shape of a triangular prism (BRC, AQD.)

If L, M, N be the middle points of AB, CD, RQ it is clear that the C.G of the superincumbent liquid coincides with the C.G. of the $\triangle$ LMN.

Hence bisecting LM in E, and taking EG $= \frac{1}{3}$ EN, we have G, the centre of gravity.

If we draw GP a vertical line to cut the rectangle in P, we have

$$EP : EM = EG : EN = 1 : 3.$$

$$\therefore \quad EP = \tfrac{1}{3}\, EM = \tfrac{1}{6}\, LM$$

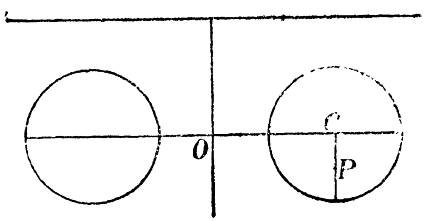
and $$LP = LE + EP = \tfrac{1}{2}\, LM + \tfrac{1}{6}\, LM = \tfrac{2}{3}\, LM.$$

Hence *the centre of pressure lies on the middle line of the rectangle at a distance down equal to two-thirds of it.*

57. **Harder Examples Solved.**

Ex. 1 *A system of coaxal circle is immersed in water with the line of centres at a given depth. Prove that the centres of pressure of those circular areas which are completely immersed, lie on a parabola.*

Taking the line of centres as x-axis and the radical line as y-axis, the equation to any such circle is



$$x^2+y^2-2gx+c=0,$$

where g is variable *i.e.*, different for different circles.

Its centre C is $(g, 0)$ and if r be its radius then

$$r^2=g^2-c$$

Now the C.P. P is below C such that $CP=\frac{r^2}{4h}$ where h is the depth of the line of centres.

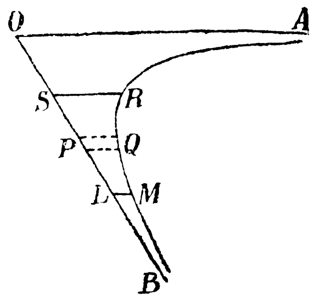
If (x, y) be the co-ordinates of P then

$$x=OC=g,\ y=CP=\frac{r^2}{4h}$$

$$\therefore\ y=\frac{g^2-c}{4h}=\frac{x^2-c}{4h}.$$ Hence the locus of P is parabola.

Ex. 2. *The asymptote of a hyperbola lies in the surface of a fluid; find the depth of the centre of pressure of the area included between the immersed asymptote, the curve, and two given horizontal lines in the place of the hyperbola.*

Asymptote OA is in the free surface. Let the depths of the given horizontal lines SR and LM be a and b respectively. We are required to find the C.P. of the area SRML. Divide the area into thin horizontal strips and let PQ be one of them at a depth x from the surface. Let its vertical thickness be dx.



If the angle between the asymptote is α, the area of this strip$=PQ.dx$

$\therefore$ The pressure on the strip

$=g\rho.PQ.dx.\ OP\sin\alpha.$

But PQ OP $\sin\alpha=c$, constant, by the property of the hyperbola.

$\therefore$ The pressure on the strip$=g\rho c dx$ and it acts at a depth x below the surface.

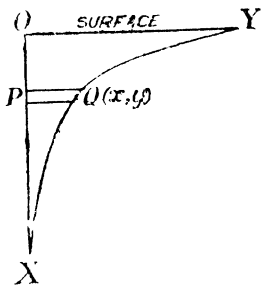
Hence depth of the C.P. of the area SRML

$$=\frac{\int_a^b g\rho cx dx}{\int_a^b g\rho c dx}=\frac{b^2-a^2}{2(b-a)}$$

$= \frac{1}{2}(a+b)$, *i.e.*, the arithmetic mean between the depths of the given lines.

Ex. 3. *Find the coordinates of the centre of pressure of the area between the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ and the axes, taking the axes to be rectangular and one of them in the surface.*

Take one strip PQ parallel to OY, the axis in the surface at a depth x from O and of thickness dx.



Let the coordinates of the point Q be (x, y)

$\therefore \quad dS = y\,dx$ and $p = g\rho x$

$$\therefore \quad \bar{x} = \frac{\int xp\,dS}{\int p\,dS} = \frac{\int_0^a x^2 y\,dx}{\int_0^a xy\,dx}$$

Putting $x = a\sin^4\theta$, $y = a\cos^4\theta$ and $dx = 4a\sin^3\theta\cos\theta\,d\theta$

$$\bar{x} = \frac{a\int_0^{\pi/2}\sin^{11}\theta\cos^5\theta\,d\theta}{\int_0^{\pi/2}\sin^7\theta\cos^5\theta\,d\theta}$$

$$= a\,\frac{\dfrac{\Gamma(6)\Gamma(3)}{2\Gamma(9)}}{\dfrac{\Gamma(4)\Gamma(3)}{2\Gamma(7)}} = \frac{5}{14}a.$$

Taking moments about the axis of x

$$\bar{y} = \frac{\frac{1}{2}\int yp\,dS}{\int p\,dS} = \frac{\frac{1}{2}\int_0^a xy^2\,dx}{\int_0^a xy\,dx}$$

$$= \frac{1}{2}a\,\frac{\int_0^{\pi/2}\cos^9\theta\sin^7\theta\,d\theta}{\int_0^{\pi/2}\sin^7\theta\cos^5\theta\,d\theta}, \text{ with the same substitution}$$

$$= \frac{1}{2}a\,\frac{\dfrac{\Gamma(5)\Gamma(4)}{2\Gamma(9)}}{\dfrac{\Gamma(4)\Gamma(3)}{2\Gamma(7)}} = \frac{3}{28}a.$$

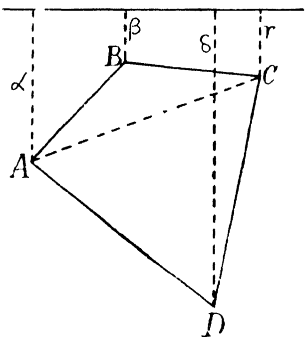
$\therefore$ The C.P. is $\left(\frac{5}{14}a, \frac{3}{28}a\right)$

Ex. 4. *If a quadrilateral area be entirely immersed in water, and α, β, γ, δ be depths of its four corners and h that of its centre of gravity,* *show that the depth of its centre of pressure is*

$$\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)-\frac{1}{6h}(\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta).$$

(*M.T. 1881 ; Patna 1927*)

Let ABCD be the quadrilateral. Join AC. Let S_1, S_2 be the areas of the triangles ABC, ADC respectively.



The depths of their centres of gravity are

$\tfrac{1}{3}(\alpha+\beta+\gamma)$ and $\tfrac{1}{3}(\alpha+\delta+\gamma)$ respectively. Since h is the depth of the C.G. of the quadrilateral.

$$h(S_1+S_2)=\tfrac{1}{3}(\alpha+\beta+\gamma)\,S_1+\tfrac{1}{3}(\alpha+\delta+\gamma)\,S_2.$$

$$\text{or}\quad \frac{S_1}{S_2}\Big\{3h-(\alpha+\beta+\gamma)\Big\}=(\alpha+\delta+\gamma)-h.$$

$$\text{or}\quad \frac{S_1}{S_2}=\frac{\alpha+\delta+\gamma-h}{3h-(\alpha+\beta+\gamma)} \qquad \dots (1)$$

The depths of the centres of pressure of the triangles ABC, ADC respectively, are

$$\frac{1}{2}\cdot\frac{\alpha^2+\beta^2+\gamma^2+\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha+\beta+\gamma},\quad \frac{1}{2}\cdot\frac{\alpha^2+\delta^2+\gamma^2+\alpha\delta+\alpha\gamma+\delta\gamma}{\alpha+\delta+\gamma}$$

and the thrusts on them are

$\tfrac{1}{3}\,S_1(\alpha+\beta+\gamma)g\rho$, $\tfrac{1}{3}S_2\,(\alpha+\delta+\gamma)g\rho$ respectively.

$\therefore$ z, the depth of the C.P. of the quadrilateral is given by

$$z\,h(S_1+S_2)=\tfrac{1}{6}S_1(\alpha^2+\beta^2+\gamma^2+\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$+\tfrac{1}{6}S_2(\alpha^2+\gamma^2+\delta^2+\alpha\gamma+\alpha\delta+\gamma\delta)\;;$$

$$\text{or}\quad z\,h\left(\frac{S_1}{S_2}+1\right)=\frac{1}{6}\,\frac{S_1}{S_2}\,(\alpha^2+\beta^2+\gamma^2+\alpha\beta+\alpha\gamma+\beta\gamma)\;;$$
$$+\tfrac{1}{6}(\alpha^2+\gamma^2+\delta^2+\alpha\gamma+\alpha\delta+\gamma\delta)\;.\;(2)$$

Putting the value of $\dfrac{S_1}{S_2}$ from (1) in (2)

$$z=\tfrac{1}{2}\,(\alpha+\beta+\gamma+\delta)-\frac{1}{6h}\,(\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta).$$

Examples 10

1. When the depth of the liquid is increased by an amount a, the depth of the centre of pressure is found to increase by y, and when instead, the depth of the liquid is increased by b, that of the centre of pressure is found to increase by z. Show that the depth of the centre of gravity of the area in the original state of the liquid is

$$\frac{ab(b-a+y-z)}{az-by}.$$

(*Utkal 1947, Poona 1932*)

2. Show that the ratio of the depths of the centre of pressure of an ellipse completely immersed in a liquid with its centre at a depth h below the free surface and with its minor axis horizontal in one case and vertical in

another case is

$$\frac{4h^2+a^2}{4h^2+b^2}.$$

3. Show that if a lamina totally immersed in a liquid is a quadrant of a circle of radius a, of which the centre is in the surface, the locus of the C.P. lies in a straight line

$$x+y=\tfrac{3}{16}a(\pi+2). \qquad \textit{(Poona 1949)}$$

4. A trapezium ABCD is immersed with the side AB in the surface of water and the sides AD $(=a)$, BC $(=b)$ are vertical. Prove that the vertical line through the C P. divides AB in the ratio

$$a^2+2ab+3b^2 : 3a^2+2ab+b^2. \qquad \textit{(Agra 1947)}$$

5. A parallelogram, whose plane is vertical and centre at a depth h below the surface, is totally immersed. Show that if a and b are the lengths of the projections of its sides on a vertical line, then the depth of its centre of pressure will be

$$h+\frac{a^2+b^2}{12h}. \qquad \textit{(M.T.)}$$

6. A square whose side is $2a$ is completely immersed in a homogeneous liquid is a vertical plane with its centre at a depth d Prove that the C P. is vertically below the centre of the square and at a distance $\frac{1}{3}a^2/d$ from it, whatever be the inclination of the side of the square to the vertical. *(M.T.)*

7. From a semi-circle whose diameter is in the surface of a liquid, circle is cut out, whose diameter is the vertical radius of the semi-circle. Prove that the depth of the C.P. of the remainder is

$$\frac{9\pi a}{8(16-3\pi)}. \qquad \textit{(Nagpur 1955)}$$

8. A rectangular hexagon of side a is immersed in water with one side in the surface. Show that the depth of its centre of pressure is to that of its centre of gravity as 23 : 18. *(U.P.S.C. 1939)*

9. A flat circular plate of radius a, lies on a plane inclined at 30° to the horizontal, and is subjected to water pressure on one face. The centre of pressure is at a distance $\frac{a}{16}$ from the geometric centre. Show that the geometric centre is at a depth $2a$ below the free surface of the water. *(M.T.)*

10. A cube, with edges of length $2a$, is immersed in a liquid and has one edge in the surface and two faces through that edge equally inclined to the horizontal. Find the centres of pressure of all the faces. *(M.T. Utkal 1946)*

11. APC is a triangle immersed in a liquid. The side AB is in the surface and is divided in D so that 6AD DB=AB². Lines DE, DF drawn parallel to AC and BC form the parallelogram DECF. Prove that the depths of the centre of pressure of DECF and ACB are in the ratio 11 : 9.

12. An area bounded by the curve $ay^2=x^3$, the x-axis and the ordinate $x=a$, is immersed in water with the axis in the surface. Find the co-ordinates of the centre of pressure. *(Agra 1948, 51, Jaipur 1959)*

13. An area bounded by the curve $ay^2=x^3$, an abscissa of length h and the ordinate at its extremity, is placed in water with the ordinate in the surface. Prove that the depth of the C.P. is $\frac{4}{9}h$.

14 A circular disc of radius a is completely immersed with its plane vertical in a homogeneous fluid. If h is the depth of the centre below the free surface of the liquid, prove that the distance between the centres of pressure of the two semi-circles into which the disc is divided by its horizonal diameter is

$$6\pi a.\frac{4h^2-a^2}{9\pi^2h^2-16a^2}. \qquad \textit{(M.T.)}$$

15. If the C.P. of a triangle ABC, completely immersed, coincide with its circumcentre, show that the depths of the angular points are as

$$1-2\cot B\cot C : 1-2\cot C\cot A : 1-2\cot A\cot B.$$

16. Given that the centre of pressure of a circular disc of radius r with one point in the surface, is at a distance p from the centre, show that for a disc of radius R wholly immersed with its centre at a distance h from the surface, the distance between the centre of the circle and the centre of pressure is

$$\frac{pR^2}{hr}. \qquad (M.T.)$$

17. A plane quadrilateral ABCD is entirely immersed in water with the side AB in the surface. If the depths of C and D below the surface are γ and δ respectively, and that of the C.G. is h, prove that the depth of the centre of pressure is

$$\frac{\gamma+\delta}{2}-\frac{1}{6}\frac{\gamma\delta}{h}.$$

18. A triangular lamina ABC right-angled at C, is just immersed in a fluid with the vertex C in the surface and the side CA inclined at 30° with the surface. If the centre of pressure is vertically below C, prove that the angle B is $\frac{1}{2}\tan^{-1}(2\sqrt{3})$. (*Nagpur 1940*)

19. A circular sector centre O and radius a, symmetrical about the radius OP, is completely immersed with P in the surface and OP vertical. Determine the depth of its centre of pressure.

20 The angular points of a triangle immersed in a liquid whose density varies as the depth, are at distances α, β, γ respectively below the surface. Show that the centre of pressure is at the depth

$$\frac{3}{5}\frac{(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2)+\alpha\beta\gamma}{\alpha^2+\beta^2+\gamma^2+\alpha\beta+\beta\gamma+\alpha\gamma}. \qquad (Patna\ 1926)$$

CHAPTER VI

THRUSTS ON CURVED SURFACES

58. Thrust on a Curved Surface. So far we have confined ourselves to plane surfaces in contact with fluids. But in this chapter we shall be considering the curved surfaces in contact with fluids at rest under gravity.

Suppose a curved surface is in contact with a fluid, This surface can be divided into very small elements the fluid thrust on each of which is normal to that element In this manner these fluid thrusts form a system of forces which are neither in the same direction, nor generally in the same plane. Therefore such a system can be reduced in general to *a single force together with a single couple*, and not to a *single force*. But this totality of pressure can be found in the form of three (or two) forces acting along lines which may not necessarily be coplanar in the following manner.

59. Resultant Vertical Thrust. Let the fluid thrust on each element be firstly resolved into vertical and horizontal components. Since all the vertical components are parallel they now form a system of parallel forces which can be compounded into a single resultant vertical force, say Z. This resultant is called the **Resultant Vertical Thrust** of the fluid on the surface.

60. Resultant Horizontal Thrust. Next consider the horizontal components. They are not parallel to each other since horizontal lines can be drawn in infinite directions. Hence we resolve them further along Ox and Oy, two conveniently chosen horizontal directions at right angles to one another. The components parallel to Ox will form a system of parallel forces which can be compounded into a single resultant (say X) acting parallel to Ox. Similarly the resolved components parallel to Oy can be compounded into a single resultant (say Y) along Oy. These resultants X and Y which can be thus determined in magnitude and direction are called the **Resultant Horizontal Thrusts** parallel to Ox and Oy respectively.

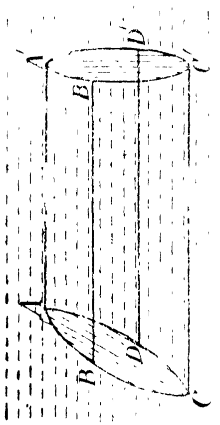
Three resultants X, Y and Z will together give us the resultant of the fluid thrust on the given surface. These thrusts will in general act along non-coplanar lines.

Note 1. If in a particular case these three forces are concurrent, they can be compounded into a single resultant force, which may be called **Resultant Fluid Thrust** on the surface.

Note 2. If only two forces X and Y intersect, these two can be compounded and thus the horizontal fluid thrust will be obtained.

61. How to find Resultant Vertical Thrust.

Let a curved surface ABCD be immersed in a heavy homogeneous liquid. Through each point of the bounding edge of the surface ABCD draw vertical straight lines so as to isolate a vertical cylinder*(ABCD, A′B′C′D′) of the liquid, resting on the curved area ABCD and cutting the surface of the liquid in A′B′C′D′. This cylinder of liquid may be called the '**superincumbent liquid**'.



For vertical equilibrium the forces acting on this cylinder are

(*i*) The weight of the cylinder of the liquid acting downwards through its centre of gravity.

(*ii*) The vertical component of the reaction of the surface ABCD upon it acting upwards.

But the reaction of the surface is equal and opposite to the thrust of the fluid upon it. Therefore the vertical component of the thrust of the fluid upon the surface is equal to the weight of the superincumbent liquid.

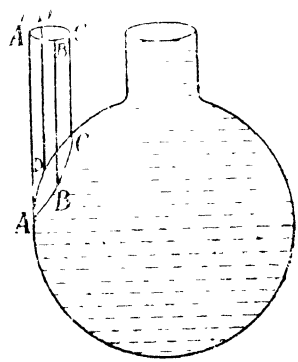
Hence

The **Resultant Vertical Thrust** *on any surface immersed in a liquid is equal to the weight of the superincumbent liquid and acts through the centre of gravity of this superincumbent liquid.*

62. Special Cases.

(*a*) When the liquid presses upwards.

Suppose a vessel (*surahi*) as shown in the adjoining figure is filled with a liquid and it is required to find a vertical thrust on the portion ABCD This thrust is pressing upwards. As in the last article through every point of the bounding edge of the surface ABCD draw vertical straight lines to meet the plane of zero pressure in curve A′B′C′D′.

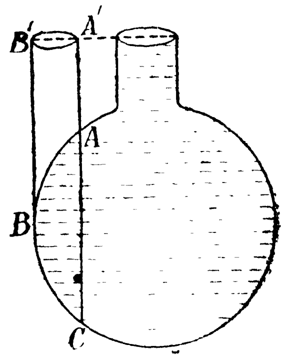


Since pressure at each point of the surface ABCD depends only on the depth of the point below the surface of the liquid, the vertical component of the upward thrust of the fluid on ABCD is equal to the weight of the fluid which would fill the cylinder (ABCD, A′B′C′D′). Hence in such cases

The **Resultant Vertical Thrust** *on the given portion of surface is equal to the weight of the liquid that could lie upon it upto the level of the surface of the liquid, and acts vertically upwards through the C.G. of the liquid.*

(*b*) **When the liquid presses partly upwards and partly downwards.**

Suppose we are to consider the pressure on the area ABC where liquid presses partly upwards on the area AB and presses partly downwards on the area BC.



The resultant vertical thrust on the area AB is upwards and is equal to the weight of the liquid that would fill the column ABB'A' vertically above it.

The resultant vertical thrust on the area BC is downwards and is equal to the weight of the liquid that would fill the column BCA'B' vertically above it.

∴ The vertical thrust on the surface ABC
= Difference of the above two weights
= wt. of the liquid CBA
= wt. of the contained liquid acting downwards.

Note.—The above is a particular case of the next Article.

63. Liquid contained in a vessel.

If liquid be contained in a vessel, then the resultant fluid thrust on the vessel is equal to the weight of the liquid in the vessel and acts vertically downwards through the C.G. of this liquid.

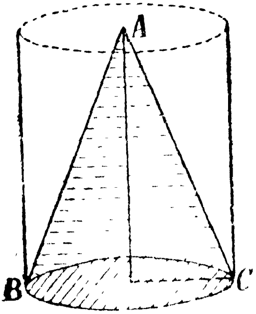
Consider the equilibrium of the liquid contained in the vessel. There are two forces acting on it, *viz.*, its weight and the reaction of the vessel on the liquid. Hence these two must be equal and opposite ; but the reaction of the vessel on the liquid is equal and opposite to the thrust of the liquid on the vessel. Therefore the fluid-thrust on the vessel is equal to the weight of contained liquid, whence, the theorem.

64. Solved Examples.

Ex. 1. *A conical wine-glass is filled with water and placed in an inverted position upon a table ; show that the resultant vertical thrust of the water on the glass is two-thirds that on the table.*

(M.T. 1858, Rangoon 1947 ; Agra 1950)

Let ABC be the conical glass of height h and radius of the base r.



Thrust on the base $=\pi r^2 hg\rho$

Vertical thrust on the curved surface

$=$wt. of the superincumbent liquid

$=$wt. of the cylindrical liquid of height h and the same base $-$ wt. of the liquid contained in the glass.

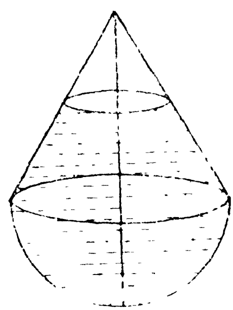
$=\pi r^2 hg\rho-\frac{1}{3}\pi r^2 hg\rho$

$=\frac{2}{3}\pi r^2 hg\rho$

$=\frac{2}{3}$ thrust on the base.

Ex. 2. *A vessel in the shape of a hollow hemisphere surmounted by a cone is held with the axis vertical and vertex uppermost. If it be filled with a liquid so as to submerge half the axis of the cone in the liquid, and height of the cone be double the radius of its base, show that the resultant downward thrust of the liquid on the vessel is $\frac{15}{8}$ times the weight of the liquid that the hemisphere can hold.*

Let the radius of the hemisphere be r.



Height of the cone $=2r$. Hence the radius of the free surface can easily be found to be $\frac{r}{2}$.

Weight of the liquid that the hemisphere can hold

$=\frac{2}{3}\pi r^3 g\rho$ where ρ is the density of the liquid.

The resultant vertical downward thrust on the vessel.

$=$wt. of the liquid contained in the vessel.

$=\frac{1}{3}\pi r\{r^2+\frac{1}{4}r^2+\frac{1}{2}r^2\}g\rho+\frac{2}{3}\pi r^3 g\rho$

$=\frac{5}{4}\pi r^3 g\rho$

$=\frac{15}{8}\cdot\frac{2}{3}\pi r^3 g\rho$

$=\frac{15}{8}$. the wt. of the liquid that the hemisphere can hold.

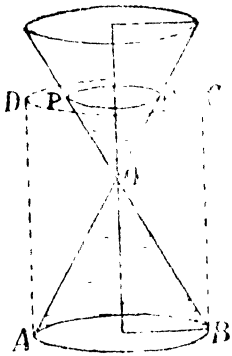
Ex. 3. *A double-funnel is formed by joining two equal hollow cones at their vertices and stands on a horizontal plane with the common axis vertical, liquid is poured into the cone until its surface bisects the axis of the upper cone. If the liquid be on the point of escaping between the lower cones and the table, prove that the weight of either cone is to that of the liquid it can hold as 27 : 16.* (*M.T. Agra 1949*)

Let each cone be of height h and radius r. The height of the quid is $\frac{3}{2}h$,

$\therefore$ Vertical downward thrust on the upper cone $=\frac{1}{24}\pi r^2hg\rho$ where ρ is the density of the liquid.

Vertical upward thrust on the cone
$=$wt. of the superincumbent liquid
$=$wt. of the cylindrical liquid ABCD$-$wt. of the liquid in the cone OAB
$=\frac{3}{2}\pi r^2hg\rho-\frac{1}{3}\pi r^2hg\rho=\frac{7}{6}\pi r^2hg\rho$

$\therefore$ Resultant vertical upward thrust on the vessel $=(\frac{7}{6}-\frac{1}{24})\ \pi r^2hg\rho=\frac{27}{24}\ \pi r^2gh\rho$.



Since the liquid is on the point of escaping between the lower cone and the table, the vertical upward thrust must be equal to the weight of the vessel.

Let W be the wt. of either cone

$\therefore\quad \frac{27}{24}\ \pi r^2hg\rho=2\text{W}$

or $\qquad \text{W}=\frac{27}{48}\ \pi r^2hg\rho$.

The weight of the liquid that a cone can hold $=\frac{1}{3}\pi r^2hg\rho$

$\therefore$ W : wt. of the liquid a cone can hold $=\frac{27}{48}\ \pi r^2hg\rho\ .\ \frac{1}{3}\pi r^2hg\rho$
$\qquad 27\ .\ 16.$

Examples 11

1. A conical vessel filled with water, stands with its plane base on a horizontal table. Prove that the thrust of the liquid on the base of the cone is three times the weight of the contained water. How do you explain the result ?
(Agra 1933, 58 , Calcutta 1948)

2. A quantity of water which just fills a cone, whose height is h and the radius of whose base is r, is poured into a cylinder, the radius of whose base is also r. Compare the thrust on the two bases, the axis of each vessel being vertical.

3. A hemispherical bowl is filled with water and inverted and placed with its plane base in contact with a horizontal table. Show that the resultant thrust on its surface is one-third of the thrust on the table.
(Agra 1944, 49 , Patna 1948)

4. A hollow cone filled with water and closed, is held with its axis horizontal, find the resultant vertical pressure on the upper half of its curved surface.

Find it on the lower half as well. *(Calcutta 1915 , Aligarh 1942)*

5. A right circular cylinder is just immersed in water with its axis horizontal. Compare the vertical thrusts on the two parts of the curved surface into which it is divided by the horizontal plane through the axis.

6. A solid right cone, height h and radius a (of the base) is completely immersed in a liquid with its axis horizontal and at depth b. Compare the vertical fluid thrust on the two halves into which the curved surface is divided by a horizontal plane through the axis. *(Utkal 1947)*

7. A bucket in the form of a frustum of a cone, the radii of its top and bottom being a and b $(b<a)$ inches respectively, is of height h and is full of water. Find the resultant vertical thrust on its curved surface.

8. A hollow sphere of radius a is just filled with water ; find the resultant vertical thrusts on the two portions into which the surface is divided by a horizontal plane at depth c below the centre. (*Allahabad 1946*)

9. A hollow closed vessel in the shape of a cylinder surmounted by a cone is filled with liquid. If the axis of the cone be three times as long as that of the cylinder, prove that the resultant thrust on the surface of the cone will be the same in the two positions in which the vessel can be placed with its axis vertical.

10. A conical glass whose weight is $\frac{5}{8}$th of the weight of water which would just fill it, is placed vertex upwards on a smooth table and water is gradually poured in through a hole made in the top. Show that the cup will be on the point of rising from the table when the water reaches half the height of the cup. (*Sagar 1949 ; Agra 1956*)

11. A hollow cone is placed with its vertex upwards on a horizontal table and a liquid is poured in through a small hole in the vertex ; if the cone begins to rise when the weight of the liquid poured in is equal to its own weight, prove that its weight is to the weight of the liquid required to fill the cone as

$9-3\sqrt{3} : 4.$ (*Nagpur 1956 ; Patna 1941*)

12. A vessel in the shape of the greater segment of a sphere of radius a cut off by a plane at a distance c from the centre rests with its base on a table. If the segment is filled with water, show that the resultant vertical thrust on the curved surface will be an upward force if $c<\frac{1}{2}a$.

13. The shape of the interior of a vessel is a double cone, the ends being open and the two portions connected by a minute aperture at a common vertex ; it is placed with one circular rim fitting close upon a horizontal plane and is filled with water ; find the resultant vertical thrust on the vessel and show that it will be zero if the length of the axis of the upper portion be twice that of the lower.

14. A double funnel, formed of two equal cones with a common axis communicating at their common vertex, is placed on a horizontal plane with the axis vertical and is filled with water ; prove that the resultant vertical thrust on the curved surface of the lower cone is $2\frac{1}{2}$ times the wt. of the water.

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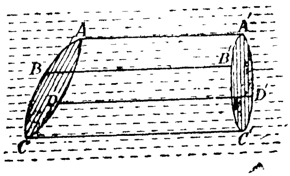
65. How to find Resultant Horizontal Thrust.

Suppose ABCD is a surface on which the horizontal thrust is to be found in a given horizontal direction.



Through each point of the boundary of the given surface, draw horizontal lines AA′, BB′, CC′, DD′ in the given direction, and let these lines meet a vertical plane perpendicular to the given direction in a closed curve A′B′C′D′. Then A′B′C′D′ will be the projection of the surface on this plane.

Now consider the equilibrium of the mass of fluid enclosed between the given surface ABCD, its projection A′B′C′D′ and the cylindrical surface generated by lines drawn in the given horizontal direction. For horizontal equilibrium of this cylinder, the only forces are (*i*) the horizontal component of the thrust on the surface ABCD in this direction, and

(*ii*) the fluid-thrust or the whole pressure on the plane area A′B′C′D′ acting through its centre of pressure.

Therefore these two fluid-thrusts must be equal in magnitude and act along the same line on opposite directions.

Hence, *the* **Resultant Horizontal Thrust** *on a given surface in contact with a fluid in any given direction is equal, in magnitude and line of action, to the whole pressure on the projection of the surface upon a vertical plane perpendicular to the given direction.*

Note 1. The direction of the resultant horizontal thrust is from the liquid to the side of the surface under consideration.

Note 2. In the above theorem each of the lines AA′ cuts the surface in one point only. If the case be otherwise, the given surface should be divided into two or more parts in each of which the condition is satisfied. Then the above theorem may be applied to obtain the horizontal component in the given direction on each part separately and then all these components can be compounded into a single one.

Note 3. If a continuous surface can be divided into two parts only, the projections of which on the plane perpendicular to the given direction are identical, there will be no horizontal thrust on the surface in that direction (for the horizontal thrusts on two parts will balance).

For an example we may consider the curved surface of a right circular solid cone immersed in a liquid with its axis vertical. There is no horizontal fluid thrust on this solid cone. The only fluid thrust is wholly vertical.

66. Resultant Thrust. As has been stated in the beginning of this chapter, the resultant horizontal thrust is obtained by compounding, where ever possible, the horizontal components in two convenient directions at right angles (say X and Y).

The Resultant Vertical Thrust Z is also determined in both magnitude and line of action.

If these three thrusts can he compounded into one single resultant (as is generally the case with symmetrical bodies) the resultant thrust on the curved surface can be found.

67. Solids immersed in fluids

A solid can be immersed either entirely or in part in a liquid. Whatever be the case, let us consider the position of equilibrium of the solid in the liquid and suppose for a moment that the solid is removed and its place filled by the same liquid which may be called

the displaced liquid.

This liquid occupying the space of the solid is subject to the same resultant pressure due to the surrounding liquid as the original solid.

Now the liquid, which is supposed to occupy the position of the solid, is in equilibrium under the action of its own weight acting vertically downwards through its centre of gravity, and the thrusts upon its bounding surfaces due to the surrounding liquid.

Hence the resultant thrust, to which the solid is subject, is equal to the weight of the liquid displaced and acts vertically upwards through the centre of gravity of the liquid displaced.

The results of the above explanations were formulated by the Greek scientist Archimedes and may be included in the following principle known as the **Principle of Archimedes.**

"**When a solid is wholly or partially immersed** *in fluid at rest, the resultant thrust of the fluid on the solid is equal and opposite to the weight of the fluid displaced by the solid and acts vertically upwards through the centre of gravity of the fluid displaced.*"

Or in simple words, *every body immersed in a liquid loses a portion of its weight equal to the weight of the liquid displaced.*

Note. It is not necessary for the truth of this theorem that the fluid should be homogeneous. The density of the fluid may vary from one horizontal layer to another. In such cases the displaced fluid must be taken to be of the same density at any point as that of the surrounding fluid at the same horizontal level.

68. Definitions.

Force of Buoyancy. *The resultant fluid thrust on a body wholly or partially immersed in a fluid is called the* **Force of Buoyancy.**

In other words *the force of buoyancy is equal to the wt. of the liquid displaced and acts upwards.*

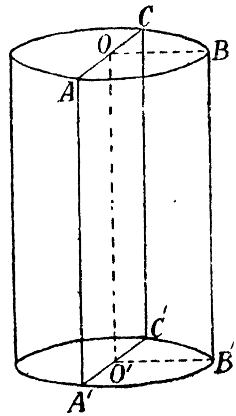
Ceutre of Buoyancy. *The Centre of Gravity of the displaced fluid is called the* **Centre of Buoyancy.**

69. Solved Examples.

Ex. I. *A hollow cylinder closed by a plane base is filled with liquid and held with its axis vertical. Find the resultant horizontal thrust on half the cylinder cut off by a vertical plane through the axis.*

(A′B′C′. ABC) is half the cylinder cut off by a vertical plane ACC′A′ through its axis OO′.

Let OBB′O′ be another vertical plane perpendicular to the plane ACC′A′.



Choosing AC and its perpendicular OB, as the two horizontal directions along which components of horizontal thrust may be taken we note that the horizontal thrusts in the direction AC must be zero. For, if the surfaces AB, B'A' and BC, C'B' be projected on the vertical plane OB, B'O', which is a plane of symmetry for the semi-cylindrical surface, the projections will be identical ; consequently the horizontal thrusts for the surfaces ABB'A' and BCC'B', being equal and opposite, will cancel one another. Therefore the horizontal thrust will be parallel to OB only.

Horizontal thrust = whole pressure on the projection of the semi-cylindrical surface (ABC, A'B'C') on the vertical plane ACC'A'.

= whole pressure on the rectangle ACC'A'

$=2ah.\dfrac{h}{2}\,g\rho.$

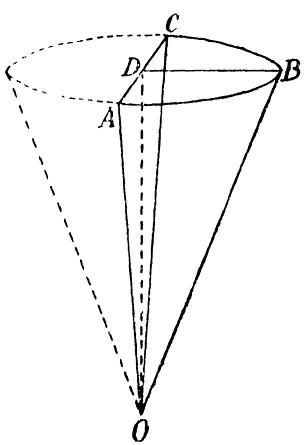
$=ah^2g\rho$, where h is the height and a the radius of the base of the cylinder.

Ex. 2. *A solid right circular cone is divided into two parts by a plane through its axis and one of these portions is just immersed, vertex downwards and axis vertical in water. Find the resultant thrust on its curved surface and show that it is inclined at an angle* $\tan^{-1}(\frac{1}{2}\pi\tan\alpha)$ *to the horizontal, where* α *is the semi-vertical angle of the cone.*

(*Agra 1929, 49 ; Allahabad 1933 ; Banaras 1944, 59*)

Let OAC be the vertical plane which cuts the cone into halves.

Let OABC be one of them immersed in the liquid. Let a be radius of the base of the cone.



Let AC and BD which are perpendicular to each other be the two convenient horizontal directions along which the components of horizontal thrust may be taken. Here we see that the horizontal thrust in the direction AC must be zero. For if the surfaces OAB and OBC be projected on the vertical plane ODB, which is a plane of symmetry for the semi-conical surface, the projections will be identical ; consequently the horizontal thrusts for the surfaces OAB and OBC being equal and opposite will balance one another.

Therefore the horizontal thrust will be parallel to BD only. Let X and Z denote the resultant horizontal and vertical thrust respectively.

$\therefore$ X = whole pressure on the projection of the semi-conical surface OABC on the vertical plane OAC

$$= \text{whole pressure on the triangle OAC}$$
$$= g\rho a . a \cot\alpha . \tfrac{1}{3} a \cot\alpha$$
$$= \tfrac{1}{3} g\rho a^3 \cot^2\alpha.$$
$$Z = \text{wt. of the supereincumbent liquid}$$
$$= \text{wt. of the vol. OABC of liquid}$$
$$= \tfrac{1}{2} . \tfrac{1}{3}\pi a^3 \cot\alpha\, g\rho = \tfrac{1}{6}\pi a^3 \cot\alpha g\rho.$$

Now since resultant horizontal thrust is normal to the plane AOC through some point on OD, its line of action will be in the plane ODB.

Since the resultant vertical thrust acts downwards through the centre of gravity of the semi-cone which is in the symmetrical plane ODB, its line of action is also in the plane ODB as it is also vertical.

Therefore these two thrusts are coplanar and since they are not parallel, they must be concurrent.

$\therefore$ Resultant Thrust $= (X^2 + Z^2)^{\frac{1}{2}}$

$$= \tfrac{1}{6} g\rho a^3 \cot\alpha \sqrt{4\cot^2\alpha + \pi^2}$$

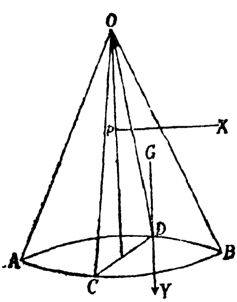
If θ be the inclination of this resultant thrust to the horizontal, we have

$$\tan\theta = \frac{Z}{X} = \frac{\frac{1}{6}\pi a^3 \cot\alpha\, g\rho}{\frac{1}{3} a^3 \cot^2\alpha g\rho} = \frac{\pi}{2}\tan\alpha$$

$$\theta = \tan^{-1}\left(\frac{\pi}{2}\tan\alpha\right).$$

Ex. 3. *A thin hollow vessel in the shape of a right cone with a circular base is cut in two by a plane through the axis, and the two parts are hinged together at the vertex and the edges greased so as to be water-tight. The vessel is then hung up by the hinge and filled with water through a small aperture near the hinge. Show that the water will not flow out if the vertical angle of the cone exceeds 120°.*

OCBD is half of the cone cut off by the plane OCD. Let a be the radius of the base and α the semi-vertical angle of the cone.



Let X and Y be the horizontal and vertical thrusts respectively.

$\therefore$ X = whole pressure on the $\triangle$OCD, by symmetry there is no component along the perpendicular to X.

$$X = a^2 \cot\alpha . \tfrac{2}{3} a \cot\alpha\, g\rho$$
$$= \tfrac{2}{3} a^3 \cot^2\alpha\, g\rho \text{ acting at P where}$$
$$OP = \tfrac{3}{4} a \cot\alpha.$$

Y = vertical thrust on the curved surface.

$= \frac{1}{6}\pi a^3 \cot\alpha g\rho$ acting at G where the distance of G from the axis of the cone is $\frac{a}{\pi}$.

From Statics we know that the water will not flow out if Moment of Y about the hinge > moment of X about the hinge

$$i.e., \tfrac{1}{6}\pi a^3 \cot\alpha\, g\rho.\ \frac{a}{\pi} > \tfrac{2}{3} a^3 \cot^2\alpha\, g\rho.\ \tfrac{3}{4}a\cot\alpha$$

or $\tan^2\alpha > 3$

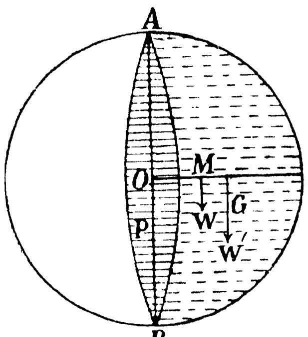
or $\tan\alpha > \sqrt{3}$

or $\alpha > 60°$

or $2\alpha > 120°$.

Ex. 4. *Two closely fitting hemispheres made of a material of uniform thickness are hinged together at a point on their rims, and are suspended from the hinge. The rims are greased so that the whole forms a water-tight vessel. It is being filled with a liquid through a small hole near the hinge. Prove that the contact will not give way if the weight of the shell exceeds three times that of the water it contains.* (*M.T.*)

Let A be the hinge and O the centre of the hemispheres whose radius is a.



The forces acting on one of the hemispheres are

(*i*) Reaction at the hinge A.

(*ii*) Its weight W′ acting vertically downwards at G where $OG = \frac{1}{2}a$.

(*iii*) The vertical fluid thrust equal to the weight of the hemispherical volume of the liquid, say W acting downwards through M where $OM = \frac{3}{8}a$.

(*iv*) The horizontal fluid thrust equal to the whole pressure on the vertical circle acting through P, its centre of pressure where $OP = \frac{1}{4}a$.

This thrust is $= \pi a^3 g\rho = \frac{3}{2}W$.

When we take moments about A, we notice that the force (*i*) can be neglected, that the forces (*ii*) and (*iii*) tend to keep the hemispheres together and that the force (*iv*) tries to separate them.

Hence the contact will not give way if the sum of the moments of (*ii*) and (*iii*) are greater or equal to that of (*iv*)

or $W'.\frac{a}{2} + W.\frac{3}{8}a \geqslant \frac{3}{2}W\left(a + \frac{a}{4}\right)$

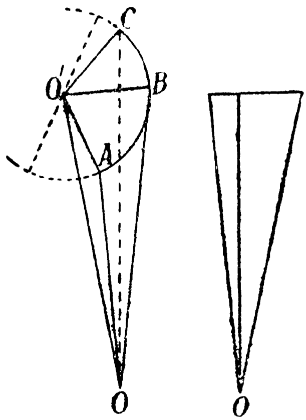
or $W' \geqslant 3W$

or wt. of the shell $\geqslant$ 3 (wt. of the liquid contained).

Ex. 5. *A hollow right circular cone, filled with water is held with its axis vertical and vertex downwards. Find the resultant pressure on the portion of its surface contained between two vertical planes through its axis, and show that if the inclination of these planes to each other be 2θ and the vertical angle of the cone 2α, the direction of the resultant pressure makes with the vertical an angle equal to*

$$\tan^{-1}\left(\frac{\sin\theta}{\theta\tan\alpha}\right).$$ (*Allahabad 1919*)

Let h be the height of the cone and a the radius of its base. Let OABCO′ be the portion of the cone between the two vertical planes OO′A and OO′C.



$$\therefore \quad \angle O'OA = \alpha = \tan^{-1}\frac{a}{h}.$$

$$\angle AO'B = \tfrac{1}{2}\, AO'C = \theta$$

Area of the sector $O'ABC = \theta a^2$

Let V and H be the resultant vertical and resultant horizontal thrusts respectively.

V = weight of the superincumbent liquid

$= g\rho$ volume OABCO′

$= g\rho\theta a^2 . \tfrac{1}{3} h$

H = whole pressure on the horizontal projection of the curved surface on a plane perpendicular to O′B *i.e.*, on the plane OAC

$= g\rho\tfrac{1}{3} h .\, ha \sin\theta$, for $AC = 2a\sin\theta$

(The horizontal component of the thrust along a line perpendicular to O′B will be zero by symmetry.)

$\therefore$ The required angle

$$= \tan^{-1}\frac{H}{V} = \tan^{-1}\frac{g\rho\frac{1}{3}h^2 a\sin\theta}{g\rho\theta\, a^2\frac{1}{3}h}$$

$$= \tan^{-1}\frac{h\sin\theta}{\alpha\theta}$$

$$= \tan^{-1}\left(\frac{\sin\theta}{\theta\tan\alpha}\right).$$

Examples 12

1. An unsymmetrical cask with plane horizontal ends of equal area is just filled with water. Prove that the thrust on its curved surface is the same in magnitude whichever end is uppermost. *(Delhi 1927)*

2. A right circular cone is just immersed in a liquid with its axis horizontal. Find the resultant horizontal thrust on the curved surface of the lower half of the cone cut off by a horizontal plane through the axis.

3. A right circular cone filled with liquid is held with its axis vertical. Prove that the horizontal thrust on half the curved surface cut by a plane through the axis, when the vertex is upwards, is twice that when the vertex is downwards.

4. A right circular cone is just immersed in a liquid with its axis horizontal. Find the resultant horizontal thrust on the curved surface of half the cone cut off by a vertical plane through the axis.

5. A solid right circular cone is placed with its axis horizontal and at a depth h below the surface of water ; find the horizontal thrust on half of the cone imagined to be cut off by a vertical plane through the axis.

6. A hollow cylinder closed by a plane base is filled with liquid and held with its axis vertical ; find the magnitude and the line of action of the

resultant thrust on half the sylinder cut off by a vertical plane through the axis. (*Utkal 1947*)

7. A solid circular cylinder is divided in two equal parts by a plane through the axis. If it is held just immersed in a liquid with the axis horizontal and the plane section vertical, what will be the resultant horizontal thrust on the curved surface ?

8. If a hollow right circular cylinder is filled with liquid and held with its axis horizontal, find the magnitude and the line of action of the resultant thrust on half the curved surface cut off by a vertical plane through the axis. (*Jaipur 1957* ; *Agra 1943, 55*)

9. A solid hemisphere, of radius a, is placed with its centre at a depth h below the surface of water, and has its plane base in a vertical plane ; what is the horizontal thrust on its curved surface ? Find also the resultant thrust on it.

10. A hemispherical bowl with its lowest point downwards and the plane base horizontal is filled with water. The water is poured into a cylindrical tumbler, the radius of whose base is equal to that of the hemisphere. Prove that the horizontal thrusts on half of the curved surfaces in which they may be divided by vertical planes through their axis, are in the ratio of 3 : 2.

11. A hemispherical bowl is filled with water ; find the horizontal fluid pressure on one half of the surface divided by a vertical diametral plane, and show that it is $\frac{1}{\pi}$ of the magnitude of the resultant fluid thrust on the whole surface (*Calcutta 1918* ; *Lucknow 1941*)

12. A right hollow cylinder with a circular base is filled with liquid and closed at the top. Prove that the ratio of the height of the cylinder to the radius of the base is 2 : 1 if the resultant horizontal thrust on half the curved surface cut off by a vertical plane through the axis, when the axis is horizontal, is equal to that when the axis is vertical.

13. A horizontal trough is semi-circular in section and is filled with water whose weight is w ; if the trough be imagined to be divided into halves along the middle, show that the water will tend to push them asunder horizontally with a force $\frac{w}{\pi}$.

Show also that the resultant thrust of the water on either half of the trough makes with the vertical angle θ so that

$$\cot\theta=\frac{\pi}{2}.$$

14. A closed cylindrical vessel with hemispherical ends is filled with water, and placed with its axis horizontal. Find the resultant thrust on each of the ends and determine its line of action. (*Allahabad 1920* ; *Patna 1940* ; *Agra 1956*)

15. A hollow right circular cone filled with liquid is held with its axis vertical and vertex downwads. Prove that the magnitude of the resultant fluid thrust in half the surface of the cone cut off by a vertical plane through the axis is $\frac{1}{6}\,wrh\,\sqrt{(\pi^2r^2+4h^2)}$ and that it is inclined at an angle

$$\tan^{-}\left(\frac{\pi r}{2h}\right) \text{ to the horizontal.}$$

16. A right circular cylinder is just immersed in water with its axis horizontal. Compare the horizontal fluid thrusts on the four parts of the curved surface made by the horizontal and vertical planes through the axis.

17. A spherical shell formed of two halves in contact along a vertical plane is filled with water ; show that the resultant pressure on either half of the shell is $\frac{\sqrt{13}}{4}$ of the total weight of the liquid.

18. The end of a horizontal pipe is closed by a sphere of the same radius a as the internal section of the pipe. The sphere is hinged at its highest point. If the pipe is just full of liquid of density ρ, prove that the moment about the hinge of the liquid pressure on the sphere is $g\rho\pi a^4$. (*M.T.*)

19. A right cone whose base is an ellipse of semi-axes a, b is divided into two equal parts by a plane through its axis and the major axis of the base. One part is then removed and just immersed in a liquid with its vertex upwards and the axis of the cone vertical. Find the horizontal thrust on one half of its curved surface made by plane perpendicular to the vertical face of the solid.

20. A solid circular cone of uniform material and height h and of vertical angle 2α, is made of uniform material and floats in water with its axis vertical and vertex downwards and a length h' of the axis is immersed. The cone is bisected by a vertical plane through the axis and the two parts are hinged together at the vertex. Show that the two parts will remain in contact if $h' > h \sin^2 \alpha$. (*Allahabad 1938 : Agra 1934, 52 Supp., 54 ; Banaras 1943 ; Jaipur 1954, 56 ; Gorakhpur 1959 ; Lucknow 1944*)

21. The stem of a funnel has a radius a and its mouth has a radius b : the height of the stem is h and that of the slanting part is l. The mouth is placed in a horizontal plane (being greased so as to remain water-tight) and the funnel is filled with water to the top of the stem ; prove that the upward pressure on the funnel is equal to

$$\pi w(b-a)\{h(a+b)+\tfrac{1}{3}l(a+2b)\}$$

where w is the weight of unit volume of water.

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70. Thrust on a curved surface bounded by a plane curve.

If the solid consists partly of a curved surface and partly of known plane areas, the resultant fluid pressure may be easily obtained as follows :—

Let the components of the resultant thrust on the curved surface be X and Y in the horizontal and vertical directions respectively.

Let the components of the thrust on the plane area be X′, Y′ in the same directions respectively.

Now, the resultant thrust on all the surfaces of the body is a vertical force $g\rho V$, acting through the centre of gravity of the solid, and there is no horizontal component to the resultant thrust on the solid.

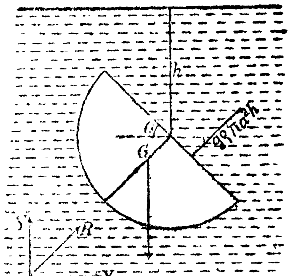
Hence
$$Y-Y'=g\rho V$$
$$X-X'=0$$

$$\therefore \quad Y=Y'+g\rho V$$
$$X=X'.$$

71. Solved Examples.

Ex. 1. *A hemispherical surface of radius a is immersed in liquid of density ρ with its centre of depth h and its base inclined at an angle θ to the horizontal. Find the resultant thrust on the curved surface and its direction.* (*M.T., I.C.S. 1935 ; Delhi 1947 ; Rajputana 1949*)

Let X, Y be the horizontal and vertical components of the resultant thrust on the curved surface.



By Archimedes' Principle the resultant thrust on the whole surface of the body is equal to the weight of the liquid displaced, *viz.* $\frac{2}{3}g\rho\pi a^3$.

But since the surface of a hemisphere is partly curved and partly plane, this thrust is the resultant of

(1) X, (2) Y,

(3) The whole pressure P on the circular plane base acting normal to it. where $P = g\rho\pi a^2 h$.

Resolving horizontally and vertically X, Y and P which together make up the total vertical thrust $\frac{2}{3}g\rho\pi a^3$, we get

$$X - P\sin\theta = 0$$
$$Y - P\cos\theta = \tfrac{2}{3}g\rho\pi a^3$$

whence $\quad X = g\rho\pi a^2 h\sin\theta \quad \dots (1)$

$\quad Y = \frac{2}{3}g\rho\pi a^3 + g\rho\pi a^2 h\cos\theta \quad \dots (2)$

∴ Resultant thrust on the curved surface is

$$\left(X^2+Y^2\right)^{\frac{1}{2}} = g\rho\pi a^2\left(h^2+\frac{4}{9}a^2+\frac{4}{3}ah\cos\theta\right)^{\frac{1}{2}}.$$

Let ϕ be the angle which this resultant thrust makes with the horizontal.

$$\therefore \quad \tan\phi = \frac{Y}{X} = \frac{2a+3h\cos\theta}{3h\sin\theta}$$

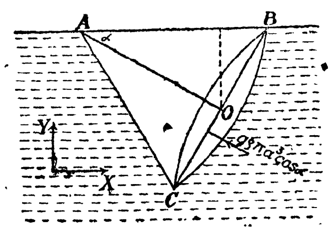
If the curved surface of the hemisphere be taken to be uppermost the last term in the results of the resultant thrust and the angle will have a minus sign.

Ex. 2. *A solid cone is just immersed with a generating line in the surface. if θ be the inclination to the vertical of the resultant thrust on the curved surface, prove that*

$$(1-3\sin^2\alpha)\tan\theta = 3\sin\alpha\cos\alpha,$$

2α being the vertical angle of the cone. *(Raj. 1960 ; Agra 1952)*

Let AB, a generating line of the cone ABC, be in the free surface. Let a be the radius of the base of the cone.



Let X and Y be the horizontal and vertical components of the resultant thrust on the curved surface.

By Archimedes' Principle the resultant thrust on the whole surface of the cone is equal to the weight of the liquid displaced, *viz.* $\frac{1}{3}g\rho\pi a^3\cot\alpha$.

Since the surface of the cone is partly curved and partly plane, this thrust is the resultant of

(*i*) X, (*ii*) Y,

(*iii*) The whole pressure P on the circular end acting **normal to** it

where $P = g\rho\pi\, a^3 \cos\alpha$.

Resolving horizontally and vertically X, Y and P which together make up the total vertical thrust $\frac{1}{3}\, g\rho\pi a^3 \cot\alpha$, we get

$$X - P\cos\alpha = 0$$

$$Y + P\sin\alpha = \tfrac{1}{3} g\rho\pi a^3 \cot\alpha$$

whence $$X = g\rho\pi a^3 \cos^2\alpha$$

$$Y = \tfrac{1}{3} g\rho\pi a^3 \cot\alpha - g\rho\pi a^3 \cos\alpha \sin\alpha$$

since θ is the inclination to the vertical of the resultant thrust

$$\tan\theta = \frac{X}{Y} = \frac{\cos^2\alpha}{\frac{1}{3}\cot\alpha - \cos\alpha\sin\alpha}$$

$$= \frac{3\cos\alpha\sin\alpha}{1 - 3\sin^2\alpha}$$

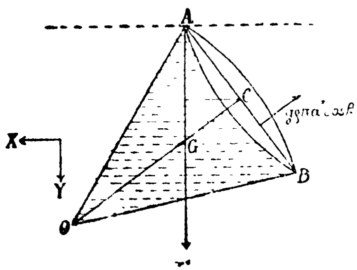
$$\therefore\quad (1 - 3\sin^2\alpha)\tan\theta = 3\sin\alpha\cos\alpha$$

Ex. 3. *A hollow cone without weight, closed and filled with some* ***liquid****, is suspended from a point in the rim of its base ; if* θ *be the* ***angle*** *which the direction of the resultant pressure makes with the vertical* ***then*** *show that*

$$\cot\theta = \frac{28\cot\alpha + \cot^3\alpha}{48},$$

α *being the semi-vertical angle of the cone.*

Let a be the radius and A the point of suspension of the cone **OAB** ; then AG will be vertical, where G is the C.G. of the enclosed conical mass of water, so that



$$\text{C.G.} = \tfrac{1}{4}\text{OC} = \tfrac{1}{4}\, a \cot\alpha.$$

Let X, Y be the horizontal and vertical components of the resultant thrust on the curved surface.

If the diameter AB makes an angle β with AG, then

$$\tan\beta = \frac{GC}{AC} = \frac{\frac{1}{4}\, a\cot\alpha}{a} = \tfrac{1}{4}\cot\alpha$$

whence $\sin\beta = \dfrac{\cot\alpha}{\sqrt{16 + \cot^2\alpha}}$, and $\cos\beta = \dfrac{4}{\sqrt{16 + \cos^2\alpha}}$.

Here the resultant thrust on the vessel is equal to the wt. of the liquid contained, *viz.*, $\frac{1}{3}\pi a^3 \cot\alpha g\rho$ where ρ is the density of the liquid and it acts vertically downwards through G.

P, the whole pressure on the circular end

$$=g\rho\pi a^2.\ a\cos\beta$$

Resolving horizontally and vertically X, Y, and P which together make up the total vertical thrust $\frac{1}{3}g\rho\pi a^3\cot\alpha$, we get

$$X-P\cos\beta=0$$

$$Y-P\sin\beta=\tfrac{1}{3}\pi a^3\cot\alpha\, g\rho$$

whence $X=g\rho\pi a^3\cot^2\beta$ ... (1)

$Y=\frac{1}{3}g\rho\pi a^3\cot\alpha+g\rho\pi a^3\cos\beta\sin\beta$ ... (2)

or $$X=\frac{16\,g\rho\,\pi a^3}{16+\cot^2\alpha}$$

$$Y=\tfrac{1}{3}g\rho\pi a^3\cot\alpha+\frac{4g\rho\pi a^3\cot\alpha}{16+\cot^2\alpha}$$

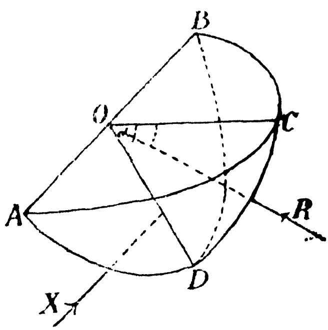
$$\therefore\quad \cot\theta=\frac{Y}{X}=\frac{\frac{1}{3}\cot\alpha+\dfrac{4\cot\alpha}{16+\cot^2\alpha}}{\dfrac{16}{16+\cot^2\alpha}}$$

$$=\frac{28\cot\alpha+\cot^2\alpha}{48}.$$

Ex. 4. *A solid is formed by the revolution of a semi-circle of radius a about its bounding diameter through an angle α and the solid is immersed with one plane face in the surface of a liquid; prove that the magnitude of the resultant thrust on the curved surface of the solid is*

$$\tfrac{2}{3}a^3g\rho\{(\alpha-\sin\alpha\cos\alpha)^2+\sin^4\alpha\}^{\frac{1}{2}}.$$

Let the plane face ABC be in the surface of the liquid. Let R be the resultant thrust on the curved surface of the solid. The lower plane end is ADB inclined at an angle α with the horizontal.



The C.G. of the semi-circle ADB is at a distance $\frac{4a}{3\pi}$ from the centre.

Therefore thrust X on the plane area ADB is equal to

$$g\rho.\tfrac{1}{2}\pi a^2.\ \frac{4a}{3\pi}\ \sin\alpha=\tfrac{2}{3}g\rho a^3\sin\alpha$$

Also $$\frac{\text{volume of this solid}}{\text{volume of the whole sphere}}=\frac{\alpha}{2\pi}$$

or volume of the solid ABCD$=\dfrac{3\alpha\, a^3}{3}$

∴ Wt. of the liquid displaced $=\frac{2}{3}\alpha\, a^3\, g\rho$ which is the resultant of X and R. Therefore

$$R\cos\theta = X\cos(90-\alpha)$$
$$= X\sin\alpha = \tfrac{2}{3}g\rho a^3\sin^2\alpha \quad \ldots (1)$$

and
$$R\sin\theta = \tfrac{2}{3}\alpha\, a^3\, g\rho - \tfrac{2}{3}\, g\rho\, a^3\sin\alpha\cos\alpha \quad \ldots (2)$$

where θ is the angle R makes with the horizontal.

$$\therefore \quad R = \tfrac{2}{3}g\rho\, a^3\left\{(\alpha-\sin\alpha\cos\alpha)^2+\sin^4\alpha\right\}^{\frac{1}{2}}.$$

Examples 13

1. Find the direction and magnitude of the resultant thrust on the curved surface of a hemisphere of radius 3″, placed with its base vertical and centre at a depth of 6″ below the free surface of a liquid of which one cubic inch weighs w grammes. *(Agra 1938)*

2. A hemisphere of radius a is immersed in a liquid of density σ. The plane of the base is vertical and its centre at a depth $a\sqrt{5}$ below the surface. Show that the resultant force on the curved surface is $\frac{7}{3}\pi\sigma g a^3$ and that its direction makes with the horizontal an angle θ, where

$$\tan\theta = \frac{2}{\sqrt{45}}.$$ *(Alld., 1957 ; Lucknow 1950 ; Jaipur 1952)*

3. A solid hemisphere is immersed in a liquid with the highest point of its plane base in the surface, and the base is inclined at a given angle α to the horizon ; show that the resultant thrust on the curved surface will be equal to twice the weight of the liquid displaced if $\tan\alpha = 2$. *(Jaipur 1959 ; Agra 1958)*

4. A cylindrical vessel full of water is held with its axis inclined at an angle of 45° to the vertical. Find the magnitudes of the pressures on the ends and show that the resultant pressure on the curved surface is equal to the difference between the pressures on the ends. *(Lucknow 1948 ; Patna 1931)*

5 A cylinder closed at both ends is entirely immersed in water with its axis inclined at a given angle θ to the horizontal ; if its height be h and the radius of its base be a, prove that the resultant thrust on its curved surface is $\cos\theta$ times the weight of the liquid displaced inclined at an angle θ to the vertical.

6. A closed cylinder whose base diameter is equal to its length is full of water, and hangs freely from a point in its upper rim, if the weight of the cylinder is neglected, prove that the vertical and horizontal components of the resultant thrust on the curved surface are each half the weight of the water. *(M.T.)*

7. A cone floats with its axis horizontal in a liquid of density double its own ; find the pressure on its base and prove that if θ be the inclination to the vertical of the resultant thrust on the curved surface, and α the semi-vertical angle of the cone, then

$$\tan\theta = \frac{4}{\pi}\tan\alpha.$$

(Saugar 1959 ; Agra 1942, 48, 53 Supp. ; Lucknow 1938, 43, 50, 54)

8. A right cone of semi-vertical angle 30° has its uppermost generator in the surface of a liquid. Show that the resultant thrust on its curved surface passes through the centre of its base. *(Delhi H.M. 1927)*

9. A solid right circular cone of vertical angle 2α is just immersed in water so that one generator is in the surface of the liquid ; prove that the resultant pressure on the curved surface of the cone is to the weight of the fluid displaced by the cone, as

$$\sqrt{1+3\sin^2\alpha} : 1,$$

and that it is inclined to the axis of the cone at an angle $\cot^{-1}(2\tan\alpha)$. *(Allahabad 1951, 57 ; Lucknow 1952 ; Nagpur 1957)*

10. A right cone of vertical angle 2α is filled with water, it is then closed and laid with a generating line in contact with a table. Prove that the resultant vertical and horizontal thrusts upon the curved surface are respectively

$$W(1+3\sin^2\alpha) \text{ and } 3W\sin\alpha\cos\alpha$$

where W is the weight of the contained liquid.

11. A cone whose vertical angle is 2α, has its lowest generator horizontal and is filled with liquid; prove that the resultant pressure on the curved surface is

$$W\sqrt{1+15\sin^2\alpha}$$

where W is the weight of the liquid.

(Allahabad 1956 ; Lucknow 1953 ; Jaipur 1951, 55 ; Agra 1937)

12. A solid cone is just immersed in water with a generating line in the surface; prove that the inclination to the vertical of the resultant thrust on the curved surface is

$$\tan^{-1}\frac{3\tan\alpha}{1-2\tan^2\alpha}$$

where 2α is the vertical angle of the cone. *(Nagpur 1955 ; Agra 1936, 45)*

13. A solid cone, whose vertical angle is 2α, is immersed in a liquid with its vertex in the surface and axis vertical. Prove that if P be the whole pressure on the curved surface and base, and P′ the resultant pressure, then

$$\frac{P}{P'}=\frac{2+3\sin\alpha}{\sin\alpha}.$$

(Bombay 1936)

14. A spherical shell is filled with liquid, find the magnitude and line of action of the resultant pressure on each of the hemispheres divided by a diametral plane making an angle θ to the horizon. *(Agra 1960)*

15. A hollow weightless hemisphere with a plane base is filled with water and hung up by means of a string, one end of which is attached to a point of the rim of its base. Prove that the whole pressure on the curved surface and on the base are in the ratio of

$$19 : 8.$$

Also prove that the resultant thrust on its curved surface makes an angle $\tan^{-1}\frac{10}{9}\frac{9}{6}$ with the horizontal. *(Rajputana 1949 ; Utkal 1945)*

16. A solid sphere of density ρ is placed at the bottom of a vessel which is horizontal, and a liquid of density σ $(<\rho)$ is poured in so as just to cover up the sphere. The sphere is then cut along the vertical diametral plane. Prove that the two parts will not separate if

$$\rho<4\sigma.$$ *(Agra 1947 ; Calcutta 1948 ; Alld. 1959)*

17. A portion of a sphere cut off by two planes through its centre inclined at an angle $\frac{\pi}{4}$ is just immersed in a liquid with one face in the surface. Find the resultant thrust on the curved surface and show that it makes an angle θ with the horizontal so that

$$\tan\theta=\frac{\pi}{2}-1.$$ *(I.C.S. 1937 ; Jaipur 1953)*

18. A right cone of semi-vertical angle α is just immersed with a slant side of length l in the free surface. Show that the resultant thrust on the curved surface will cut this slant side at a distance

$$\frac{3l}{4(1-3\sin^2\alpha)}$$

from the vertex, and find the magnitude of this thrust. *(Bombay 1940)*

19. A solid octant of a sphere is immersed with a plane face in the surface, prove that the resultant pressure on the curved surface is $\left(1+\frac{8}{\pi^2}\right)^{\frac{1}{2}}$ times the weight of water displaced by the octant.

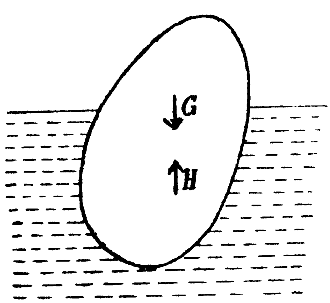
(Jaipur 1953 ; Sagar 1951)

CHAPTER VII

FLOATING BODIES

72. Conditions of equilibrium of a body freely floating in a liquid.

Consider the equilibrium of a body floating freely wholly or partially immersed in a liquid which is at rest under gravity.



The only forces acting on the body are

(1) its weight acting vertically downwards through G, the C.G. of the body,

(2) the force of buoyancy which is equal to the weight of the liquid displaced and acts vertically upwards through H, its centre of buoyancy.

The necessary and sufficient conditions of equilibrium are that these two forces should be equal and opposite and act along the same straight line.

Hence the required conditions are :

(1) *The weight of the body must be equal to the weight of the liquid displaced by it.*

(2) *The centre of gravity and the centre of buoyancy must be in the same vertical line.*

73. Volume immersed. *When a solid of volume V and mean density* ρ *is floating in a liquid of density* σ, *the volume immersed is* $\frac{V\rho}{\sigma}$

Let V′ be the volume immersed.

Then the weight of the liquid displaced $= g\,\sigma V'$.

The weight of the solid $= g\,\rho\,V$

$$\therefore \qquad g\,\sigma\,V' = g\rho V$$

$$\text{or} \qquad V' = \frac{V\rho}{\sigma}.$$

Note 1. (*i*) Since V′ cannot be greater, it follows from the above formula that σ cannot be less than ρ or a solid cannot float in a liquid of less density than its own. If the density of the liquid be less than that of the solid, the latter will sink, as for instance, a piece of lead dropped in water.

(*ii*) If the density of the solid be less than that of the liquid, the body will rise partly out of the liquid until the weight of the liquid displaced is equal to its own weight as for instance, a piece of cork.

Note 2. In the above formula it should be quite clear what is meant by the *mean density* of the solid. For instance, we know that though the density of iron is much greater than that of water, yet a hollow iron ball may be made to float in water. The reason is that in finding out the mean density of the ball, we have to take into account the hollow space inside. We know that for a given weight $W=V\rho g$, ρ will depend upon the volume and will differ from the actual density of the iron. In this way the mean density of the iron ball may be made less than that of water.

74. Solved Examples.

Ex. 1. *A cylinder floats vertically with 8 feet of its length above the fluid ; find the whole length of the cylinder, the sp. gr. of the fluid being three times that of the cylinder.*

Let the sp. gr. of the cylinder be ρ. Therefore the sp. gr. of the fluid is 3ρ. Let h be the whole length and S be the cross-section of the cylinder.

Since wt. of the body = wt. of the liquid displaced,

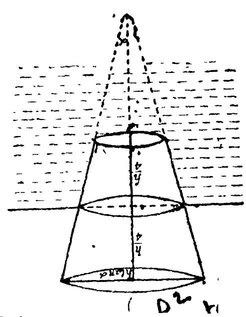
therefore $\quad Shg\rho=S\ (h-8)\ g\ 3\rho.$

or $\quad h=3h-24$

or $\quad h=12.$

Ex. 2. *A frustum of a right circular cone, cut off by a plane bisecting the axis perpendicularly, floats with its smaller end in water and its axis just half immersed. Prove that the sp. gr. of the cone is* $\frac{19}{56}$.

Let α be the semi-vertical angle and h be the height of complete cone. Height of the frustum is $\dfrac{h}{2}$ and the height of the part of the frustum in contact with water is $\dfrac{h}{4}$.



Radius of the lower end is $\dfrac{h}{2}\tan\alpha$ and that of the upper one is $h \tan \alpha$.

Also the radius of the circular section which the water level cuts the frustum is $\frac{3}{4}h \tan\alpha$.

wt. of the cone

$$=\frac{\pi h}{6}\left(h^2\tan^2\alpha+\tfrac{1}{4}h^2\tan^2\alpha+\tfrac{1}{2}h^2\tan^2\alpha\right)g\rho,$$

where ρ is the density of the cone.

$$=\tfrac{7}{24}\pi h^3\tan^2\alpha\, g\,\rho$$

wt. of the water displaced

$$=\frac{\pi h}{12}\left(\tfrac{1}{4}h^2 \tan^2\alpha+\tfrac{9}{16} h^2 \tan^2\alpha+\tfrac{3}{8} h^2 \tan^2 \alpha\right)g$$

$$=\tfrac{19}{192} \pi h^3 \tan^2 \alpha g$$

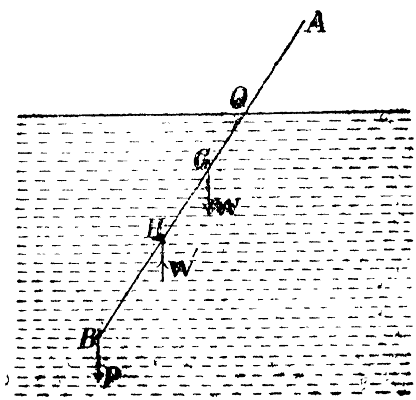
$$\therefore \quad \tfrac{7}{24} \pi h^3 \tan^2\alpha\, g\rho=\tfrac{19}{192} \pi h^3 \tan^2 \alpha\, g.$$

or $$\rho=\tfrac{19}{56}.$$

Ex. 3. *A thin rod, of weight W, is loaded at one end with a weight P of insignificant volume. If the rod floats in an inclined position with $\frac{1}{n}$th of its length out of the water, prove that*

$$(n-1)\,P=W. \qquad \textit{(Patna 1948)}$$

Let AB be the rod of length $2a$. Let G be the C.G. of the rod and H, the centre of buoyancy. Let W′ be the force of buoyancy.



Portion BO is in the water.

$BG=a$,

$$BH=\tfrac{1}{2}BO$$

$$=\tfrac{1}{2}\cdot 2a\left(1-\frac{1}{n}\right)$$

$$=a\left(1-\frac{1}{n}\right)$$

$$HG=a-a\left(1-\frac{1}{n}\right)$$

$$=\frac{a}{n}.$$

Let θ be the angle the rod makes with the horizontal.

The only forces acting on the rod are P and W acting vertically downwards and W′ acting upwards.

Taking moments about H,

$$W.\,HG\cos\theta=P.\,BH\cos\theta$$

or $$W=P.\frac{BH}{HG}=P.\frac{a\left(1-\frac{1}{n}\right)}{\frac{a}{n}}$$

$$=(n-1)P.$$

Ex. 4. *A ship sailing out of the sea into a river sinks through a distance b feet ; on unloading a cargo of weight P, the ship rises through c feet. Show that the weight of the ship after unloading is*

$$\left\{\frac{b\sigma}{c(\sigma-\rho)}-1\right\}P,$$

where σ and ρ are sp. gr. of sea water and river water respectively. The cross-section of the ship near the level of water may be assumed to be uniform. (*Lucknow 1931, 40*)

Let S be the cross-section of the ship and W be the wt. of the ship before unloading. Let h be the height to which the ship is already sinking in the sea.

Since the wt. of the ship is equal to the wt. of the liquid displaced,

$$\therefore \quad W = Shg\sigma \qquad \ldots (1)$$

When it goes to the river from the sea it sinks b feet

$$\therefore \quad W = S(h+b)g\rho \qquad \ldots (2)$$

When a weight P is unloaded, it rises through c feet

$$\therefore \quad W - P = S(h+b-c)g\rho \qquad \ldots (3)$$

Dividing (1) by (2)

$$1 = \frac{h\sigma}{(h+b)\rho} \quad \text{whence } h = \frac{b\rho}{\sigma-\rho} \qquad \ldots (4)$$

Subtracting (3) from (2)

$$P = Scg\rho \quad \text{whence} \quad Sg = \frac{P}{c\rho} \qquad \ldots (5)$$

From (1), (4) and (5)

$$W = \frac{P}{c\rho} \cdot \frac{b\rho}{\sigma-\rho}\,\sigma = \frac{Pb\sigma}{c(\sigma-\rho)}$$

Hence the required weight $= W - P = \left\{ \dfrac{b\sigma}{c(\sigma-\rho)} - 1 \right\} P.$

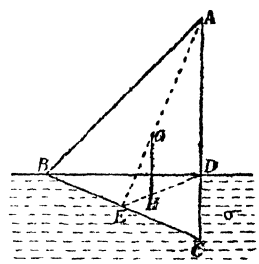
Ex. 5. *A triangular lamina ABC of density ρ floats in a liquid of density σ with its plane vertical, the angle B being in the surface of the liquid, and the angle A not immersed. Show that*

$$\frac{\rho}{\sigma} = \frac{\sin A \cos C}{\sin B} = \frac{a^2+b^2-c^2}{2b^2}$$

a, b, c being the lengths of the sides of the triangle.

(*Lucknow 1927, 33 ; Patna 1926 ; Agra 1952 ; Alld. 1954*)

The portion BCD of the $\triangle$ABC is in contact with the liquid. BD, being the level of the liquid, is horizontal. Let G and H be the centres of gravity and buoyancy respectively. E is the mid-point of BC.



The conditions of equilibrium are

(1) The line GH must be vertical.

(2) The weight of the lamina must be equal to the weight of displaced liquid.

Since EG $= \frac{1}{3}$EA, EH $= \frac{1}{3}$ED, GH is parallel to AD.

But GH is vertical from (1), hence AC also must be vertical.

Now applying the second condition

$\triangle ABC\, g\, \rho = \triangle BDC\, g\sigma$

$$\therefore \quad \frac{\rho}{\sigma} = \frac{\triangle BDC}{\triangle ABC} = \frac{\frac{1}{2}BD \cdot DC}{\frac{1}{2}BD \cdot AC} = \frac{DC}{AC}.$$

But $\dfrac{AC}{\sin B} = \dfrac{BC}{\sin A}$

or $\quad AC = BC \cdot \dfrac{\sin B}{\sin A}$

$$\text{Hence} \quad \frac{\rho}{\sigma} = \frac{DC \sin A}{BC \sin B} = \frac{\cos C \sin A}{\sin B}.$$

Examples 14

1. What fraction of a solid (sp. gr. ·7) must be outside the water in which it floats ?

2. A man of weight 160 lbs. floats in water with 4 cu. inches of his body above the surface. What is his volume ?

3. A piece of iron weighing 275 grammes floats in mercury of density 13·6 with $\frac{5}{9}$ of its volume immersed. Find out the volume and density of the iron.

4. (*a*) How much water will overflow from the edges of a cup just full of water when a cork 2 cu. inches in volume is gently placed in it so as to float ? The sp. gr. of the cork is 0·24.

(*b*) How much water will overflow from a tumbler full of water when a piece of ice 1 cubic inch in volume is gently put in it so as to float ? (Specific gravity of ice may be taken to 0·9). (*Jaipur 1959*)

5. A spherical shell of copper (sp. gr. 8) floats in water with half its surface immersed. Find its internal radius if the outer one be 10″.

6. A man whose weight is equal to 160 lbs. and whose sp. gr. is 1·1, can just float in water with his head above the surface by the aid of a piece of cork which is wholly immersed. Having given that the volume of his head is $\frac{1}{16}$-th of his whole volume and that the sp. gr. of the cork is 0·24, find the volume of the cork, the weight of a cu. ft. of water being 62·5 lbs. (*Allahabad 1931*)

7. A cylindrical lead pencil floats in water with $\frac{7}{8}$th of its volume immersed. If the lead is a cylinder whose radius is one-fourth that of the pencil and the sp. gr. of the wood is ·78, find the sp. gr. of the lead.

8. A cubical box of one foot external dimension is made of a material of thickness one inch and floats in water immersed to a depth of $3\frac{3}{4}$ inches. Not taking air into consideration, determine the weight of the box and the amount of water that must be poured in so that the water inside and outside may stand at the same level. (*Agra 1938*)

9. A hollow conical vessel floats in water with its vertex downwards and a certain depth of its axis immersed ; when water is poured into it upto the level originally immersed. It sinks till its mouth is on a level with the surface of the water. What portion of the axis was originally immersed ? (*Agra 1936 ; M.T., Banaras 1926 ; 48*)

10. A solid homogeneous cone of sp. gr. σ and height h floats in a given liquid of sp. gr. ρ. Find the position of equilibrium when

(*i*) the vertex is down and the base up,

(*ii*) the base is down and the vertex up.

11. A ship of mass 1000 tons, goes from fresh water to salt water. If the area of the section of the ship at the water line be 15,000 sq. ft. and its side vertical where they cut the water, find how much the ship will rise, taking the sp. gr. of sea water to be 1·026.

12. A steamer in going from salt water into fresh water was observed to sink 2 inches, but after burning 50 tons of coal to rise one inch. Supposing the densities of salt and fresh water to be as 65 : 64, find the displacement of the steamer in tons. (*Lucknow 1934* ; *Agra 1951*)

13. A ship sailing from the sea into a river sinks a inches, and on discharging x tons of her cargo rises b inches. If sea water be one-fortieth heavier than river water, prove that the weight of the ship is $41\left(\frac{a}{b}\right) x$ tons. Assume the sides of the ship to be vertical at water level.
(*Lucknow 1949* ; *Banaras Eng. 1943* ; *Banaras 1944* ; *Jaipur 1959*)

14. A thin cylindrical rod, weighed at one end, floats in water with half it length immersed and inclined at any angle to the horizon. Prove that the weight which is added is equal to the weight of the rod.

15. A solid displaces $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ of its volume respectively when it floats in three different fluids, find the volume it displaces when it floats in a mixture formed of

(*i*) equal volumes of the fluids,

(*ii*) equal weights of the fluids.

(*Agra 1935* ; *Rangoon 1938* ; *Utkal 1948*)

16. A rod, of small section and of density ρ, has a small portion of metal of weight $\frac{1}{n}$th that of the rod attached to one extremity ; prove that the rod will float at any inclination in a liquid of density σ if

$$(n+1)^2\rho = n^2\sigma$$

(*Lucknow 1936, 42* ; *Banaras 1948* ; *Agra 1929, 35* ; *Jaipur 1956*)

17. Shew that a homogeneous body in the shape of right circular cone can float in a liquid of twice its own density with its axis horizontal.
(*Allahabad 1927*)

18. A uniform prism, whose cross-section is an isosceles triangle of vertical angle 2α, floats freely in a liquid with its base just immersed, one edge being in the surface ; show that the ratio of its density to that of the liquid is

$$2 \sin^2\alpha : 1. \qquad (M.T.)$$

19. A prism, whose section is a triangle ABC is made of uniform material, and floats freely in water with the vertex C in the surface ; prove that its sp. gr. is either

$$\frac{\sin A \cos B}{\sin C} \text{ or } \frac{\sin B \cos A}{\sin C}.$$

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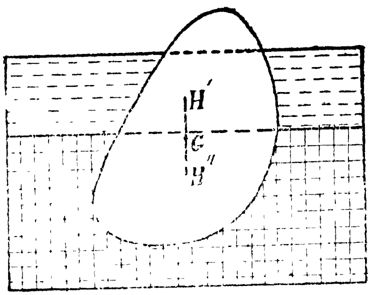
75. Bodies floating in more than one liquid. *A body floats partly immersed in one liquid and partly in another, to find the conditions of equilibrium.*

Let W be the wt. of the body and W′, W″ be the wts. of the two displaced liquids. W′ and W″ are also the forces of buoyancy



acting vertically upwards.

Now there are only three forces acting on the body, all of them vertical.

$\therefore$ For equilibrium

$$W = W' + W''$$

Hence the required conditions are

(1) *The wt. of the body must be equal to the sum of the wts. of the displaced portions of the two liquids.*

(2) *The C.G of the body must lie on the line joining the centres of gravity of the two displaced liquids.*

Note 1. The above theorem includes the case of a body floating partly immersed in liquid and partly in air.

Note 2. The above theorem is also true if there are more than two liquids.

76. Solved Examples.

Ex. 1. *A mass composed partly of gold (sp. gr. 19·25) and partly of silver (sp. gr. 10·5) floats with $\frac{15}{16}$th of its volume immersed in mercury (sp. gr. 13·6) and the remainder in water. Compare the weights of gold and silver in the mass.*

Let V_1 and V_2 be the volumes of the gold and silver respectively.

$\therefore$ wt. of the mass $= V_1 . g\ 19{\cdot}25 + V_2 g\ 10{\cdot}5$

wt. of the mercury displaced $= \frac{15}{16}(V_1 + V_2)g\ (13{\cdot}6)$

wt. of the water displaced $= \frac{1}{16}(V_1 + V_2)g$

$\therefore\ V_1 g\ (19{\cdot}25) + V_2 g\ (10{\cdot}5) = \frac{15}{16}(V_1 + V_2)g\ (13{\cdot}6) + \frac{1}{16}(V_1 + V_2)g$

or $\quad V_1\ (19{\cdot}25) + V_2 (10{\cdot}5) = \frac{205}{16}(V_1 + V_2)$

or $\quad 103 V_1 = 37 V_2$

or $\quad \dfrac{V_1}{V_2} = \dfrac{37}{103}$

$$\therefore \quad \frac{\text{wt. of the gold}}{\text{wt. of silver}} = \frac{V_1 g (19{\cdot}25)}{V_2 g (10{\cdot}5)} = \frac{37 \times 19{\cdot}25}{103 \times 10{\cdot}5} = \frac{407}{618}.$$

Ex. 2. *A cylinder of wood (sp. gr. $\frac{3}{4}$) of length h, floats with its axis vertical in water and oil (sp. gr. $\frac{1}{2}$), the length of the solid in contact with oil being a $(< \frac{1}{2}h)$. Find how much of the wood is above the liquid.*

Also find to what additional depth must oil be added so as just to cover the cylinder. (*Jaipur 1952 ; I.C.S. 1940*)

Let x be the lengths of the cylinder above the liquids ; since a length a is in contact with oil, the remainder $h - a - x$ is in water. Let

h be the height and r radius of the base of the cylinder.

Since the wt. of the solid=sum of the wts. of the liquids displaced.

$\therefore \quad \pi r^2 hg.\frac{3}{4} = \pi r^2.ag\frac{1}{2} + \pi r^2 (h-a-x)\, g.\, 1$

or $\quad \frac{3}{4}h = \frac{1}{2}a + h - a - x$

whence $\quad x = \frac{1}{4}h - \frac{1}{2}a.$

Since a is given to be less than $\frac{1}{2}h$, the value of x is positive. If, however a were greater than $\frac{1}{2}h$, the value of x would have been negative, showing thereby that the hypothesis as well as the above equations were wrong. In that case no part of the cylinder would be above the surface of the oil, its length l in contact with oil being given by

$$\frac{3}{4}h = \frac{1}{2}h + (h - l)$$

whence $l = \frac{1}{2}h$.

Second Part. Now let y be the additional depth of oil which is necessary just to cover up the cylinder.

$\therefore$ The new length of the solid in contact with oil$=a+y$.

Its length in water$=h-a-y$.

Hence $\quad \pi r^2 hg\frac{3}{4} = \pi r^2 (a+y)\, g.\frac{1}{2} + \pi r^2 (h-a-y)g$

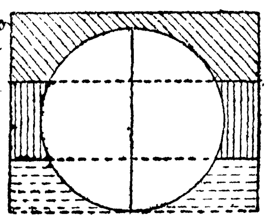
or $\quad \frac{3}{4}h = \frac{1}{2}(a+y) + h - a - y$

or $\quad y = \frac{1}{2}h - a.$

Ex. 3. *A solid sphere floats just immersed in heterogeneous liquid composed of three liquids which do not mix and whose densities are as 1 : 2 : 3. If the thickness of the two upper liquids be each one-third of the diameter of the sphere, show that the density of the liquid in the middle is equal to the density of the sphere.* (*Allahabad 1924, 38*)

Let the density of the sphere be σ and those of the liquids be ρ, 2ρ, 3ρ respectively.

Let a be the radius of the sphere. The volume of the uppermost and lowest liquids displaced are equal and each



$$= \frac{1}{3}\pi \frac{2a}{3}\left\{3a^2 - \left(a^2 + \frac{a^2}{3} + \frac{a^2}{9}\right)\right\}$$

$$= \frac{2}{9}\pi a^3 \times \frac{14}{9} = \frac{28}{81}\pi a^3$$

The volume of the middle liquid displaced is

$$= \frac{4}{3}\pi a^3 - \frac{56}{81}\pi a^3 = \frac{52}{81}\pi a^3$$

Hence the wts. of the three displaced liquids are respectively

$$\frac{28}{81}\pi a^3 g\rho,\ \frac{52}{81}\pi a^3.2\rho g,\ \frac{28}{81}\pi a^3.3\rho g.$$

By the conditions of equilibrium of floating, the wt. of the sphere must be equal to the sums of the wts. of the liquids displaced.

$$\therefore \quad \frac{4}{3}\pi a^3 g\sigma = \frac{28}{81}\pi a^3 \rho g + \frac{52}{81}\pi a^3.2\rho g + \frac{28}{81}\pi a^3.3\rho g$$

$$= \frac{8}{3}\pi a^3 g\rho$$

$$\therefore \quad \sigma = 2\rho.$$

Examples 15

1. An alloy of gold (sp. gr. 19) and silver (sp. gr. 10·5) floats with $\frac{47}{48}$ of its volume in mercury (sp. gr. 13·6) and the remainder in a liquid of sp. gr. 0·8. Find the proportion of gold and silver in the alloy. (*Utkal 1946*)

2. A piece of wood floats with $\frac{9}{10}$th of its volume in water and the remaining in air. Find the additional volume which will be immersed when the vessel is placed in vacuum, the sp. gr. of air being 0·0013.

3. If a body be floating partially immersed in a liquid and the air in contact with it be suddenly removed, will the body rise or sink ? Give reasons. (*Agra 1959*)

4. A cylinder floats completely with its axis vertical immersed in two liquids ; the densities of the upper and lower liquids being respectively ρ and 2ρ, and the density of the cylinder $\frac{5}{4}\rho$. Find what portion of the cylinder is in the lower liquid.

5. A cylinder of sp. gr. 2ρ floats with its axis vertical between two liquids of sp. gr. ρ and 3ρ respectively, the height of the cylinder being equal to the depth of the upper liquid. Prove that the pressures on its ends are as 1 : 5. (*Rangoon 1938, 48 ; Bombay 1947*)

6. A right circular cylinder of sp. gr. σ floats in water with its axis vertical, one-third being above the water. If ρ be the sp. gr. of air, prove that

$$3\sigma=2+\rho.$$ (*Banaras 1959*)

7. A cylinder of sp. gr. σ floats with its axis vertical partly in one fluid of sp. gr. σ_1 and partly in another of sp. gr. σ_2. Show that the common surface divides the axis in the ratio of

$$\sigma-\sigma_2 : \sigma_1-\sigma.$$ (*Lucknow 1959*)

8. A solid sphere of radius a is just immersed in three liquids whose densities are 1 : 3 : 5 ; the two surfaces of separation of the liquids are at depths $\frac{2}{3}a$ and $\frac{4}{3}a$ from the top. Prove that the density of the sphere to that of the topmost liquid are in the ratio of 71 : 27.

9. Two liquids which do not mix are placed in the same vessel ; the density of the lower liquid is ρ and that of the upper is $m\rho$. A cylinder floats in them with its axis vertical and is completely submerged ; its density being $n\rho$. Prove that the condition that it may be half in the upper and half in the lower liquid is

$$n=\tfrac{1}{2}(m+1).$$

10. A vertical cylinder of density ρ, floats in two liquids, the density of the upper liquid being ρ_1 and that of the lower ρ_2. If the length h of the cylinder be n times the depth of the upper liquid, prove that the depth of immersion of the upper face of the cylinder is

$$\frac{h}{n}-h\,\frac{\rho_2-\rho}{\rho_2-\rho_1}$$

provided the necessary relations between the densities exist for floating.

11. A right circular cone, of density ρ, floats just immersed with its vertex downwards in a vessel containing two liquids, of densities σ_1 and σ_2 respectively. Show that the plane of separation of the two liquids cuts off from the axis of the cone a fraction

$$\left(\frac{\rho-\sigma_2}{\sigma_1-\sigma_2}\right)^{\frac{1}{3}}$$

of its length.

(*Lucknow 1944, 45, 48 ; Allahabad 1925, 39, 50 ; Agra 1949, 52 Supp.*)

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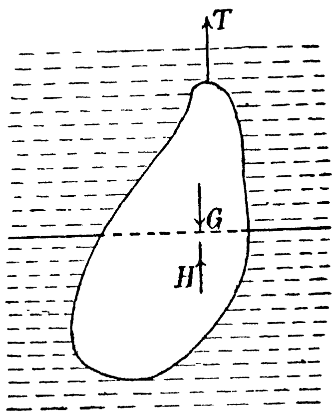
77. Tension in the string supporting a body.—*A body is entirely immersed in a liquid, being supported by a string, to find the tension of the string.*

Let T denote the tension of the string, W the wt. of the body and W′ the wt. of the liquid displaced.



In this case the vertical forces (and they are the only) acting on the body are :

(1) T, the tension of the string acting upwards.

(2) W′, the force of buoyancy acting upwards.

(3) W, the wt. of the body acting downwards.

Hence, for equilibrium, we have

$$T+W'=W$$

or $$T=W-W'$$

i.e., *Tension of the string*

=wt. of the body−wt. of the displaced liquid.

78. **Weighing in a liquid.**—Suppose a body is weighed while immersed in a liquid by means of a balance. The tension of the string supporting the body will be equal to the true weight of the body minus the weight of the displaced liquid.

The tension is termed as the **apparent weight** of the body.

∴ True weight of a body=apparent weight+wt. of the displaced liquid.

79. **Weighing in air.** To obtain a perfectly accurate weight the body should be weighed *in vacuo.* Otherwise a slight discrepancy, as when weighed in a liquid, will arise from the fact that the quantities of air displaced, by the body and by the "*weights*" that are used are not the same.

However, to be more accurate, if the densities of the body and the "*weights*" are given the true weight can be determined from the apparent weight as in the next Article.

80. *A body, whose density is* ρ, *is weighed in air by means of weights, whose density is* ρ'. *If* σ *be the density of air, to find the true weight corresponding to an apparent weight* W_0.

Let W be the true weight of the body corresponding to its apparent weight W_0 which is the sum of the *weights* used.

Then. the balance being assumed to be pure, the tensions of the two supporting strings of the scale-pans are equal, *i.e.*,

wt. of the body−wt. of the air it displaces

=wt. of the 'weights'−wt. of the air the 'weights' displace.

i.e., $W-\frac{W}{\rho g}\cdot\sigma g=W_0-\frac{W_0}{\rho' g}\cdot\sigma g$, for the volume of the body of wt.

W and density ρ is $\frac{W}{g\rho}$ and the wt. of the air it displaces is $\frac{W}{g\rho}.g\sigma$ and similarly the wt. of the air displaced by the 'weights' is $\frac{W}{\rho' g}.g\sigma$.

$$\therefore\quad W = W_0 \frac{1-\frac{\sigma}{\rho'}}{1-\frac{\sigma}{\rho}}$$

$$= W_0\left(1-\frac{\sigma}{\rho'}\right)\left(1-\frac{\sigma}{\rho}\right)^{-1}$$

$$= W_0\left(1-\frac{\sigma}{\rho'}\right)\left(1+\frac{\sigma}{\rho}+\frac{\sigma^2}{\rho^2}+\ldots\ldots\right)$$

Since, in general, σ is very small compared with ρ and ρ', neglecting the square and higher powers of $\frac{\sigma}{\rho}$, a sufficiently approximate value of the true weight is given by

$$W = W_0\left(1-\frac{\sigma}{\rho'}+\frac{\sigma}{\rho}\right).$$

81. Solved Examples.

Ex. 1. *A body immersed in a liquid is balanced by a weight P to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by a weight 2P. Prove that the densities of the body and liquid are as 3 : 2.*

Let ρ and σ be the densities of the body and the liquid respectively. Let V be the volume of the body. If T be the tension in the string in the first case, then

$$P = T = \text{wt. of the body} - \text{wt. of the liquid displaced}$$
$$= Vg\rho - Vg\sigma \qquad \ldots (1)$$

If T′ be the tension in the second case, then

$$2P = T' = Vg\rho - \tfrac{1}{2}Vg\sigma \qquad \ldots (2)$$

Dividing (1) by (2)

$$\tfrac{1}{2} = \frac{\rho-\sigma}{\rho-\frac{1}{2}\sigma}$$

or $\quad \rho - \frac{1}{2}\sigma = 2\rho - 2\sigma$

or $\quad \frac{3}{2}\sigma = \rho$

or $\quad \frac{\rho}{\sigma} = \frac{3}{2}.$

Ex. 2. *A body consists of an alloy of two metals of sp. gr. s_1 and s_2 respectively ; its weight in vacuo is w and in water is W. Show* *that the proportion of the two metals by volume is*

$$s_2 W-(s_2-1) w : (s_1-1) w-s_1 W.$$

Let V_1 and V_2 be the volumes of the two metals of the body.

$$\therefore \quad w=V_1 s_1 g+V_2 s_2 g$$

$$=(V_1 s_1+V_2 s_2)g \quad \ldots (1)$$

W=wt. of one metal−wt. of the water displaced by it +wt. of the other metal−wt. of the water displaced by the other metal.

$$=V_1 s_1 g-V_1 g+V_2 s_2 g-V_2 g$$

$$=[V_1(s_1-1)+V_2(s_2-1)]g \quad \ldots (2)$$

Dividing (2) by (1)

$$\frac{W}{w}=\frac{V_1(s_1-1)+V_2(s_2-1)}{V_1 s_1+V_2 s_2}$$

or $V_1(W s_1-w s_1+w)=V_2(w s_2-w-W s_2)$

or
$$\frac{V_1}{V_2}=\frac{W s_2-w(s_2-1)}{w(s_1-1)-W s_1}$$

Ex. 3. *Two solids are each weighed in succession in three homogeneous liquids of different densities, if the weights of the one are* w_1, w_2, w_3 *and those of the other are* W_1, W_2 *and* W_3 *prove that*

$$w_1(W_2-W_3)+w_2(W_3-W_1)+w_3(W_1-W_2)=0.$$

(*Banaras 1935 ; Jaipur 1954 ; Agra 1954*).

Let v, V be the volumes of the two solids and let σ_1, σ_2 be their respective densities. Let ρ_1, ρ_2, ρ_3 be the densities of the three liquids.

Since the apparent weight=wt. of the body−wt. of the liquid displaced

$$\therefore \quad vg\sigma_1-vg\rho_1=w_1 \quad \ldots (1)$$

$$vg\sigma_1-vg\rho_2=w_2 \quad \ldots (2)$$

$$vg\sigma_1-vg\rho_3=w_3 \quad \ldots (3)$$

and

$$Vg\sigma_2-Vg\rho_1=W_1 \quad \ldots (4)$$

$$Vg\sigma_2-Vg\rho_2=W_2 \quad \ldots (5)$$

$$Vg\sigma_2-Vg\rho_3=W_3 \quad \ldots (6)$$

$\therefore$ From (4), (5) and (6)

$$W_2-W_3=Vg(\rho_3-\rho_2)$$

$$W_3-W_1=Vg(\rho_1-\rho_3)$$

$$W_1-W_2=Vg(\rho_2-\rho_1)$$

Substituting these values

$$w_1(W_2-W_3)+w_2(W_3-W_1)+w_3(W_1-W_2)$$

$$=vg(\sigma_1-\rho_1)Vg(\rho_3-\rho_2)+vg(\sigma_1-\rho_2)Vg(\rho_1-\rho_3)+vg(\sigma_1-\rho_3)Vg(\rho_2-\rho_1)$$

$$=vVg^2\{(\sigma_1-\rho_1)(\rho_3-\rho_2)+(\sigma_1-\rho_2)(\rho_1-\rho_3)+(\sigma_1-\rho_3)(\rho_2-\rho_1)\}$$

$=vVg^2.0$, on opening the brackets.

$=0.$

Examples 16

1. A body, whose wt. is 100 lbs. and whose sp. gr. is 10, is suspended by a string in a liquid of sp. gr. 7. What is the tension of the string ?

2. A solid (sp. gr. 21) is held suspended by string so that $\frac{19}{24}$th of its volume is immersed in a liquid of sp. gr. 13 and the remainder in water. Prove that the tension of the string is half the weight of the solid.

3. A mixture of gold (sp. gr. 19·25) and silver (sp. gr. 10·5) lost $\frac{1}{14}$ th of its weight when weighed in water. Prove that the ratio of the volume of the two metals is 2 : 3.

4. A piece of lead and a piece of wood balance one another when weighed in air ; which will really weigh the most and why ?

5. A body is found to sink in each of liquids whose sp. gr. are ·7 and 1·3 and its apparent weight in the two liquids is found to be 14 and 8 grammes respectively. Find the sp. gr. of the body and show that it will float half immersed when placed in a liquid of sp. gr. 4·2. *(Banaras 1927)*

6. An iron (sp. gr. 7·2) spheroid shell is found to lose half its weight when weighed in water. If the external radius is 12″, find out the thickness of the metal.

7. A vessel containing water, the whole weighing 3 lbs. is suspended from a spring balance. A mass of iron weighing 1·3 lbs. is suspended by a thin string from a second spring balance, which is moved so that the iron is immersed in the water, without touching the sides or bottom of the vessel. State what variations and their amounts, you would expect to observe in the readings on the spring balances, and explain why they occur. The sp. gr. of the iron may be taken as 7·7. *(Allahabad 1923)*

8. A glass, partly filled with a liquid, is placed on one pan of a common balance and is counterpoised by suitable weight-pieces placed on the other. A solid which is suspended by means of a string is now brought and immersed in the liquid without touching the sides of the glass, being still in suspension from the string. Neglecting the effect of the atmosphere, explain why some more weights are now needed on the second pan to balance the glass of water.

9. A graduated glass vessel in the form of a cylinder with vertical walls contains water to a height of 20 cm. from the bottom. A body weighing 20 gms. is placed in it and floats, and the water level rises to 24 cm. The body is then completely submerged with some force, the water level rises to 25 cm. Find the mean sp. gr. of the body, the force required to submerge it and volume of water in the vessel.

10. A body of sp. gr. σ when weighed against a weight of sp. gr. ρ in water—the whole balance being immersed—appears to have a weight W. Show that its true weight is

$$\frac{\sigma}{\sigma-1} \cdot \frac{\rho-1}{\rho} \text{ W.}$$ *(Jaipur 1955 ; I.A.S. 1948 ; Lucknow 1952)*

11. If W and w be the weights of a body in vacuo and water respectively, prove that the weight in the air of sp. gr. σ will be

$$W-\sigma(W-w).$$ *(Jaipur 1957 ; Patna 1941 ; Lucknow 1951)*

12. A given mass weighed in air on a spring balance indicates a weight W, it is then compressed to $\frac{1}{n}$th of its former volume and appears to weigh W′ ; prove that its weight in vacuo is

$$\frac{nW'-W}{n-1}.$$ *(M.T.)*

13. A solid of weight W is weighed in air by a spring balance and its weight is found to be w. Prove that

$$\frac{W}{w}=\frac{\rho}{\rho-\sigma}$$

where σ and ρ are the sp. gravities of ths air and the solid respectively.

14. If a body of mass m_1 and density ρ_1 is balanced by a mass m_2 density ρ_2 when placed on the pans of a common balance, show that

$$m_1=m_2\,\frac{\rho_1}{\rho_2}\left(\frac{\rho_2-\sigma}{\rho_1-\sigma}\right)$$

where σ is the density of the air. *(Banaras 1940)*

15. If the sp. gr of air be s and W W′ be the weights of a body in air and water respectively, prove that its weight in vacuo is

$$W+\frac{s}{1-s}\,(W-W').$$

(Lucknow 1953)

16. A body floats in a fluid of sp. gr. ρ with as much of its volume out of the fluid as would be immersed in a second fluid of sp. gr. σ, if it floated in that fluid. Prove that the sp. gr. of the body is

$$\frac{\rho\sigma}{\rho+\sigma}.$$

17. If w_1, w_2, w_3 be the apparent weights of a given body in fluids whose sp. gravities are s_1, s_2, and s_3 respectively, then shew that

$$w_1(s_2-s_3)+w_2(s_3-s_1)+w_3(s_1-s_2)=0.$$

(Jaipur 1957 ; Agra 1933, 50, 58 ; Utkal 1945 ; Allahabad 1948)

18. Prove that, if volumes v and V of two different substances balance in vacuum and volumes v', V′ balance when weighed in a liquid, the densities of the substances and the liquid are as

$$\frac{v'-V'}{v}:\frac{v'-V'}{V}:\left(\frac{v'}{v}-\frac{V'}{V}\right).$$

19. A substance whose density is ρ, is weighed by means of weights, the density of which is ρ' ; if σ be the density of air, find what is the true weight corresponding to any apparent weight.

If the density of air increases from σ to σ', prove that the apparent weight of the body is less than its former weight by a fraction

$$\frac{(\rho'-\rho)\,(\sigma'-\sigma)}{(\rho-\sigma)\,(\rho'-\sigma')}$$

of the latter, ρ' being greater than ρ.

(Allahabad 1932 ; Lucknow 1951)

20. A body floating in water has volumes V_1, V_2, V_3 above the surface when the densities of the surrounding air are respectively ρ_1, ρ_2, ρ_3. Prove that

$$\frac{\rho_2-\rho_3}{V_1}+\frac{\rho_3-\rho_1}{V_2}+\frac{\rho_1-\rho_2}{V_3}=0.$$

(Allahabad 1951 ; Jaipur 1951 ; Lucknow 1952 ; Utkal 1947 ; Agra 1937, 44, 50 ; Saugar 1959)

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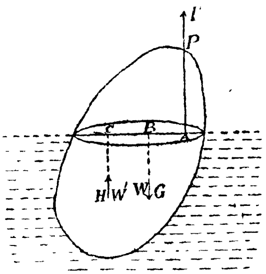
82. To find the conditions of equilibrium of a body floating with a string attached to a point of it.

Let P be the point at which the string is attached to the body and let T be its tension acting vertically upwards.



Let W be the weight of the body acting vertically downwards through its C.G., G.

Let W′ be the force of buoyancy *i.e.*, the weight of the liquid displaced acting vertically upwards through H, its centre of buoyancy.

Let the vertical lines through P, G, H meet the surface of the liquid in the points A, B and C respectively.

Since the vertical forces acting on the body are only T, W and W′ and the system is in equilibrium, the points A, B and C are in the same straight line and by Statics the sum of the moments about A must be zero.

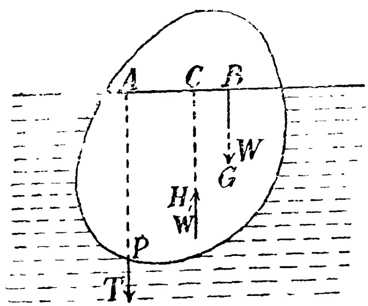
Hence *the conditions for equilibrium are*

$$W = T + W' \quad \dots (1)$$

and

$$W.AB = W'.AC. \quad \dots (2)$$

83. Tension acting downwards. If a body be wholly immersed in a liquid whose sp. gr. is greater than that of the body, the force of buoyancy will be greater than the weight of the body. In this case the body will ascend unless it is prevented from doing so by applying some downward force. Therefore T will be acting downwards.



Hence the corresponding conditions of equilibrium (as found in the last Art.) are

$$T = W' - W$$

$$W'.AC = W.AB.$$

Note 1. It should be carefully noted that in this case the line of action of W′ will be between those of T and W.

Note 2. The ascent of a **balloon** depends on the principle of the previous article. A balloon consists of a very large envelope made of silk or some other strong and light substance. It is filled with a gas or hydrogen. Since the weight of the displaced air is greater than the weight of the balloon, the balloon ascends until air around is not of sufficient density to support its weight.

The force with which the balloon ascends is called **ascensional force**. It is equal to

wt. of the air displaced — wt. of the balloon.

84. Bodies floating about a fixed point. If a body be free to turn about a fixed point O, there will be some reaction R at that point. Since the only forces acting are W, W′ and R and W, W′ are both vertical, R will also be vertical. Hence the conditions of equilibrium will be the same as obtained previously.

85. Bodies floating about two fixed points.

When two points in the body are fixed, it means that a line is fixed in the body, and the necessary and sufficient condition of equilibrium is that the moments of the weights of the body and the force of buoyancy about this line must be equal and opposite.

86. Solved Examples.

Ex. 1. *The mass of a balloon and the gas which it contains is 3,500 lbs. If the balloon displaces 48,000 cu. ft. of air and the mass of a cu. ft. air be 1·25 ozs.; find the acceleration with which the balloon commences to ascend.*

The wt. of the air displaced by the balloon

$=48000\times 1{\cdot}25$ oz. wt.

$=3750$ lb. wt.

Hence ascensional force

$=$wt. of the displaced air$-$wt. of balloon

$=(3750-3500)$ lb. wt.

$=250g$ poundals.

∴ Initial acceleration of balloon

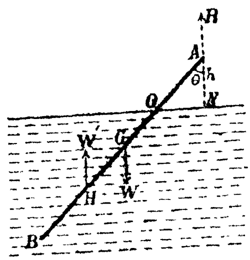
$$=\frac{\text{Force}}{\text{Mass}}=\frac{250g}{3500}=\frac{g}{14}.$$

Ex. 2. *A uniform rod of length 2a, can turn freely about one end which is fixed at a height h (<2a) above the surface of the liquid; if the densities of the rod and liquid be ρ and σ show that the rod can rest either in a vertical position or inclined at an angle θ to the vertical such that*

$$\cos\theta=\frac{h}{2a}\sqrt{\frac{\sigma}{\sigma-\rho}}.$$

(*Jaipur 1955; Nagpur 1942; Lucknow 1932, 37, 46, 47; Agra 1928, 31, 43, 49; Allahabad 1935*)

Let the rod AB of cross-section S be movable about the end A. Portion BO is in the liquid. Let W and W′ be the wt. of the rod and the wt. of the displaced liquid respectively. AN$=h$; G and H are respectively the centres of gravity of the rod and the displaced liquid.



$BO=AB-AO=(2a-h\sec\theta)$

$AH=a+\frac{1}{2}h\sec\theta$

∴ $W=2aSg\rho$, $W'=(2a-h\sec\theta)Sg\sigma$.

The rod is in equilibrium under the forces R (reaction), W and W′, all vertical.

Taking moments of the forces about the point A,

$$2aSg\rho \, AG \sin\theta = (2a - h \sec\theta)\, Sg\sigma \, AH \sin\theta \qquad \dots (1)$$

or $$2a\rho . AG = (2a - h \sec\theta)\, \sigma . AH$$

or $$2a\rho . a = (2a - h \sec\theta)\, \sigma\, (a + \tfrac{1}{2}h \sec\theta)$$

or $$2a^2\rho = \sigma(2a^2 - \tfrac{1}{2}h^2 \sec^2\theta)$$

or $$\tfrac{1}{2}h^2 \sec^2\theta\sigma = 2a^2(\sigma - \rho)$$

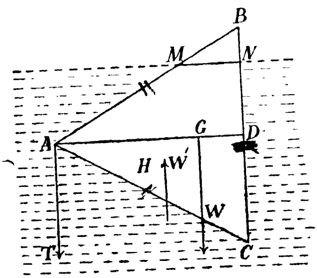
or $$\sec^2\theta = \frac{4a^2}{h^2} \cdot \frac{\sigma - \rho}{\sigma}$$

or $$\cos\theta = \frac{h}{2a}\sqrt{\frac{\sigma}{\sigma - \rho}} .$$

The other position will be vertical when taking $\sin\theta$ common in (1) ; $\sin\theta = 0$ *i.e.*, $\theta = 0$.

Ex. 3. *A triangular lamina* ABC, *of which the sides* AB, AC *are equal, floats in water with* BC *vertical, and three quarters of its length immersed, being kept in equilibrium in this position by means of a string fastened to* A *in the bottom of the vessel. Find the sp. gr. of the lamina, and show that the tension of the string is* $\frac{1}{27}$*th of the weight of the lamina.* (*Alld. 1959*)

Let T be the tension, W be the wt. of the lamina and W′ be the force of buoyancy acting vertically as shown in the figure. G is the C.G. and H the centre of buoyancy.



Let AD be the perpendicular on BC since $BN = \frac{1}{4} BC = \frac{1}{2} BD$ and MN and AD are parallel being horizontal, therefore $MN = \frac{1}{2} AD$.

Area of the $\triangle BMN$

$$= \tfrac{1}{2} MN.BN = \tfrac{1}{2}.\tfrac{1}{2} AD . \tfrac{1}{2} BD$$
$$= \tfrac{1}{8} AD.BD$$
$$= \tfrac{1}{8} \triangle ABC.$$

∴ The volume of the lamina $AMNC = \frac{7}{8}$ that of the lamina ABC.

If W be the wt. of the lamina ABC and ρ be its density

Volume of the lamina AMNC

$$= \frac{7}{8} \cdot \frac{W}{\rho g}$$

$$\therefore \quad W' = \frac{7}{8} \cdot \frac{W}{\rho g} . g = \frac{7}{8} \frac{W}{\rho}$$

For equilibrium

$$T + W = W' = \frac{7}{8} \frac{W}{\rho}$$

$$\therefore \quad T = \left(\frac{7}{8\rho} - 1\right) W \qquad \dots (1)$$

We know that the distance of the C.G. of the $\triangle$ABC from A is $\frac{2}{3}$ AD and that of $\triangle$BMN is $\frac{1}{2}$ AD$+\frac{2}{3}$ MN

$$=\frac{1}{2}AD+\frac{2}{3}\cdot\frac{AD}{2}=\frac{5}{6}\,AD$$

Let the distance of H *i.e.*, C.G. of AMNC$=x$ from A.

$$\therefore\quad \triangle ABC.\ \tfrac{2}{3}\,AD=\tfrac{7}{8}\,\triangle ABC x+\tfrac{1}{8}\,\triangle ABC\ \tfrac{5}{6}\,AD$$

or $$\tfrac{2}{3}\,AD=\tfrac{7}{8}\,x+\tfrac{5}{48}\,AD$$

or $$x=\tfrac{9}{14}\,AD$$

Now taking moments about A

$$W.\tfrac{2}{3}\,AD=W'.\tfrac{9}{14}\,AD$$

or $$W.\frac{2}{3}AD=\frac{7}{8}\ \frac{W}{\rho}\cdot\frac{9}{14}AD$$

or $$\rho=\tfrac{27}{32}$$

$$T=(\tfrac{7}{8}\cdot\tfrac{32}{27}-1)\,W$$
$$=\tfrac{1}{27}\,W.$$

Examples 17

1. A piece of wood of weight $1\frac{1}{2}$ lbs. and sp. gr. $\frac{3}{4}$, is tied by a string to the bottom of a vessel of water so as to be totally immersed. Prove that the tension of the rope is $\frac{1}{2}$ lb. wt.

2. A cylinder of wt. 15 lbs. and length 3 ft. floats in water with its axis vertical and half immersed in water. Prove that a force of 5 lbs. wt. will be required to depress it $\frac{1}{2}$ ft. more.

3. A uniform rod capable of turning about one end, which is out of the water, rests inclined to the vertical with one-third of its length in some water. Prove that its sp. gr. is $\frac{5}{9}$. (*Calcutta 1938* ; *Lucknow 1950 Supp.*)

4. A thin rod, of sp. gr. $\frac{3}{4}$ and of length 4 ft. floats partly immersed in water, being supported by a string fastened to one end of the rod ; how much of the rod is immersed ?

If the upper end of the rod be 1 ft. above the surface of water, prove that the rod is inclined at an angle 30° to the horizontal. (*Allahabad 1922*)

5. A uniform rod, of length $2a$, floats partly immersed in a liquid, being supported by a string to one of its ends. If the density of the liquid be $\frac{4}{3}$ times that of the rod, prove that the rod will rest with half its length out of the liquid.

Find also the tension of the string. (*Jaipur 1950* ; *Agra 1932*)

6. A uniform rod is suspended by two vertical strings attached to its extremities and half of it is immersed in water ; if its sp. gr. be 2·5, prove that the tensions of the strings will be as 9 : 7.

7. A solid hemisphere floats in a liquid, completely immersed with a point of the rim joined to a fixed point by means of a string. Prove that the inclination of the base to the vertical is

$$\tan^{-1}\tfrac{3}{8}.$$

Also prove that the tension of the string is

$$\tfrac{2}{3}\,\pi(\rho-\sigma)a^3 g$$

where ρ and σ are the densities of the solid and the liquid respectively.

(*Agra 1955*)

8. A uniform rod is bent in the form of the three sides AB, BC, CD of a square. A is attached to a hinge fixed on the surface of water and the frame rests in a vertical plane with AB, BC and half of CD immersed in water. Prove that the sp. gr. of the rod is $\frac{31}{40}$.

9. A semi-circular cylinder floats in water with its axis fixed in the surface of water. If this cylinder be movable about the fixed axis and if its density be half of that of water, show that it will be in equilibrium in any position. (*Agra 1939*)

10. A rod of length $2l$ and sp. gr. ρ is hinged at one end at a height h above the surface of liquid of sp. gr. $n\rho(n>1)$. The rod remains in equilibrium in an inclined position, partly immersed in the liquid. Prove that the inclination of the rod to the vertical is

$$\cos^{-1}\frac{h}{2l}\sqrt{\frac{n}{n-1}}$$

and the length immersed is

$$2l\left(1-\sqrt{1-\frac{1}{n}}\right).$$

(*Nagpur 1941, 55 ; Agra 1953*)

11. A uniform rod of length $2a$ and density ρ is movable in a vertical plane about one end which is fixed in a liquid of density ρ' at a depth $2h$ below the surface. A liquid of smaller density σ is added on the top of the first liquid ; if in the oblique position of equilibrium the rod is just covered by the liquid, prove that its inclination to the vertical is

$$\cos^{-1}\left[\frac{h}{a}\left(\frac{\rho'-\sigma}{\rho-\sigma}\right)^{\frac{1}{2}}\right].$$

(*Agra 1946 ; Nagpur 1942 ; Jaipur 1959*)

12. An equilateral triangular lamina suspended freely from A, rests with the side AB vertical and the side AC bisected by the surface of a heavy fluid ; prove that the density of the lamina is to that of the fluid as 15 : 16.

13. An equilateral triangular lamina ABC of sp. gr. $n^2(<1)$ is movable about a fixed hinge at A and is in the equilibrium when the corner C is immersed in water and AB horizontal and above the water. If the triangle be turned about A in its vertical plane and kept in the position with the side BC immersed and horizontal, prove that the action of the hinge in this position is

$$\frac{2}{n}(1-n)\,W$$

where W is the wt. of the lamina.

14. A uniform rod, loaded at one end so as to float upright, floats in a liquid of density ρ_1 with a length a unimmersed, in a liquid of density ρ_2 with a length b unimmersed. Show that the length that is unimmersed when the rod floats in a liquid of density ρ is

$$\frac{a\rho_1(\rho-\rho_2)-b\rho_2(\rho-\rho_1)}{\rho(\rho_1-\rho_2)}.$$

15. A rod of density σ and length l is freely movable about one end fixed at a depth h below the surface of a liquid of density ρ ; show that the rod may rest in a position inclined to the vertical provided that

$$1<\frac{\rho}{\sigma}<\frac{l^2}{h^2}.$$

16. An equilateral triangle ABC of weight W and sp. gr, σ is movable about a hinge at A and is in equilibrium when the angle C is immersed in water

and the side AB is horizontal and above water. It is then turned about A in its own vertical until the whole of the side BC is in the water and horizontal ; prove that the pressure on the hinge in this position is

$$2\left(\frac{1-\sqrt{\sigma}}{\sqrt{\sigma}}\right)W. \qquad \textit{(Nagpur 1956; Agra 1945)}$$

17. A semi-circular lamina has one of the ends of its diameter smoothly hinged to a fixed point above the surface of a liquid, and floats with its plane vertical and its diameter half-immersed. If the inclination of the diameter to the horizon is $\frac{1}{4}\pi$, prove that the ratio of density of the liquid to that of the lamina is

$$4(3\pi-4) : (9\pi-8). \qquad \textit{(M.T.)}$$

87. Harder Solved Examples

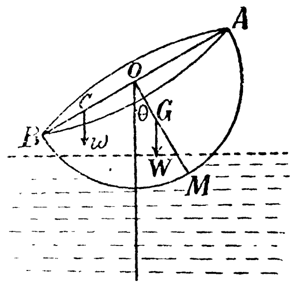
Ex. 1. *A solid hemisphere of radius a and weight W is floating in liquid, and at a point on the base at a distance c from the centre rests a weight w ; show that the tangent of the inclination of the axis of the hemisphere to the vertical for the corresponding position of equilibrium, assuming the base of the hemisphere entirely out of the fluid, is*

$$\frac{8}{3} \cdot \frac{c}{a} \frac{w}{W} .$$

If w is attached to the rim, then prove that $\tan\theta = \frac{8}{3}\frac{w}{W}$.

(Allahabad 1956)

OM is the axis of the hemisphere making an angle θ with the vertical ; G is the C.G. so that $OG=\frac{3}{8}a$. C is the point on the base at a distance c from the centre, w is acting on C.



The pressure at every point being along the normal passes through the centre. Therefore the force of buoyancy also passes through the centre and is vertical because w and W both are vertical.

Taking moments about O

$$W.\ OG \sin\theta = w.\ c\cos\theta$$

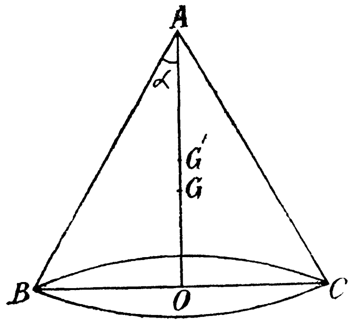
$$\tan\theta = \frac{w}{W} \cdot \frac{c}{OG} = \frac{8}{3} \cdot \frac{c}{a}\frac{w}{W} .$$

For the second part put $c=a$.

Ex. 2. *A thin hollow cone, with a base, floats completely immersed in water wherever it is placed. Show that the vertical angle is*

$$2\ \sin^{-1}(\tfrac{1}{3}). \qquad \textit{(Agra 1928, 48 ; Allahabad 1937)}$$

The condition that the cone may float in equilibrium in any position is that the C.G. of the whole surface of the cone must coincide with the C.G. of solid cone *i.e.*, the centre of buoyancy in this case.



Let G′ be the C.G. of the curved surface so that $AG'=\frac{1}{3}h$ where h is the height of the cone. O is the centre of the base.

α is the semi-vertical angle.

Now the curved surface

$$=\pi AB.BO$$

$$=\pi h^2 \sec\alpha \tan\alpha$$

Area of the base

$$=\pi BO^2=\pi h^2 \tan^2\alpha$$

To find the C.G. of the whole surface of the cone, let us take moments about the point O. Let G be the C.G. of the whole surface.

$$OG.(\pi h^2 \sec\alpha \tan\alpha+\pi h^2 \tan^2\alpha)=\pi h^2 \sec\alpha \tan\alpha.\ OG'$$

$$OG=\frac{\sec\alpha.\ OG'}{\sec\alpha+\tan\alpha}$$

$$=\frac{h}{3}\cdot\frac{1}{1+\sin\alpha}$$

Since the C.G. of the surface coincides with the C.G. of the conical volume of the fluid displaced which is at a distance $\frac{h}{4}$ from O, therefore, we have

$$\frac{h}{3}\cdot\frac{1}{1+\sin\alpha}=\frac{h}{4} \text{ or } \sin\alpha=\frac{1}{3}$$

Hence the vertical angle is $2\sin^{-1}(\frac{1}{3})$.

Ex. 3. *A thin uniform wooden rod* AB *is in equilibrium in an inclined position with one end* AC *immersed in a bowl of water, one point* D *supported on the edge of the bowl. Shew that the sp. gr.* σ *of the rod is*

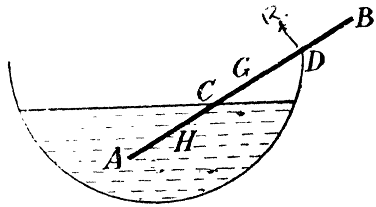
$$\frac{AC}{BC}\cdot\frac{2\ AD-AC}{2\ AD-AB}.$$

Also prove that the greatest fraction of the length of the rod which can remain immersed is

$$1-\sqrt{(1-\sigma)}.$$

(*M.T., Agra 1947*)

Let G and H be the centres of gravity and buoyancy respectively. Let S be the cross-section of the rod.



Weight of the rod$=S.AB.\,g\sigma$
and the force of buoyancy$=S.AC.g$.
Now $GD=AD-AG=AD-\frac{1}{2}AB$
and $HD=AD-\frac{1}{2}AC$.

Taking moments about the point D to cancel the reaction at D), we have

$S.AB.\,g\sigma\,(AD-\frac{1}{2}AB)\cos\theta$
$=S.AC.\,g\,(AD-\frac{1}{2}AC)\cos\theta$

where θ is the angle the rod makes with the horizontal.

$$\therefore \quad \sigma\,AB\,(2\,AD-AB)=AC\,(2\,AD-AC)$$

or $$\sigma=\frac{AC}{AB}\cdot\frac{2\,AD-AC}{2\,AD-AB}.$$

Second Part. Since the weight of the part BD has tendency to lift the rod out of water, so the length immersed will be greatest when BD is least *i.e.*, when B is at D.

If $AC=x.\,AB$ where x is a proper fraction, the result of the first part becomes

$$\sigma=x\,(2-x)$$

or $$1-\sigma=(1-x)^2$$

or $$x=1-\sqrt{(1-\sigma)}.$$

Ex. 4. *A uniform isosceles triangular lamina (sp. gr. σ) floats in water with its plane vertical, its vertical angle (which is equal to 2α) immersed and the base wholly above the water. Prove that in the position of equilibrium in which the base is not horizontal the sum of the lengths of the immersed portions of the two sides is*

$$2a\cos^2\alpha$$

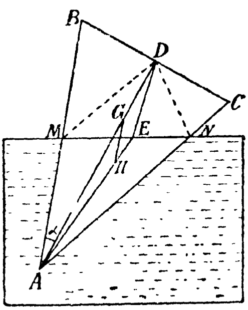
where a is one of the equal sides.

Also prove that $\sigma<\cos^4\alpha$ *as well as* $\cos 2\alpha$.

(*Delhi 1947*)

Let MN be the level of water, therefore horizontal.

$AB=AC=a$, BC is not horizontal. D and E are the mid-points of BC and MN, $\angle BAD=\alpha$.



Let G and H be the centres of gravity and buoyancy. Therefore for the equilibrium GH must be vertical.

In the △ ADE, GH ∥ DE

∴ DE is also vertical. Also E is the mid-point of MN

∴ DM=DN

From △ DMA

$$DM^2=AM^2+AD^2-2AM.AD\cos\alpha.$$

Similarly from the $\triangle$DNA

$$DN^2 = AD^2 + AN^2 - 2\,AD.AN \cos\alpha$$

$\therefore$ Equating the values of DM^2 and DN^2

$$AM^2 - AN^2 = 2AD(AM - AN)\cos\alpha$$

$\therefore$ Either AM$=$AN in which case BC is horizontal but the base is not so by hypothesis.

or $\quad AM + AN = 2AD\cos\alpha$

$$= 2a\cos^2\alpha \qquad \dots (1)$$

Second Part. Since the wt. of the lamina$=$wt. of the water displaced

$$\therefore \quad \sigma g\,\tfrac{1}{2}\,AB.AC\sin 2\alpha = g.\,\tfrac{1}{2}\,AM.AN\sin 2\alpha$$

or $$\sigma = \frac{AM.AN}{a^2} \qquad \dots (2)$$

or $$\sigma < \frac{(AM+AN)^2}{4a^2}, \text{ from the inequality } 4ab < (a+b)^2$$

or $$< \cos^4\alpha \text{ from (1).}$$

Third Part. We know that greater σ is, the greater will be the portion in water, *viz.*, the $\triangle$AMN. But by hypothesis, the base BC must always be wholly above water. Therefore σ will be greatest when MN passes through the lower point C. Hence

$$AN = AC = a$$

From (1) $AM = 2a\cos^2\alpha - AN = 2a\cos^2\alpha - a = a\cos 2\alpha$

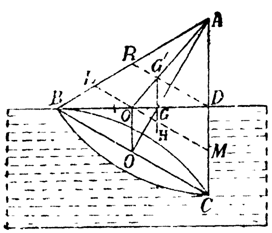
$$\therefore \quad \text{from (2)} \quad \sigma = \frac{AN.AM}{a^2} = \frac{a.\,a\cos 2\alpha}{a^2} = \cos 2\alpha$$

But this is the greatest value of σ

Hence $\qquad \sigma \leqslant \cos 2\alpha.$

Ex. 5. *The vertical angle of a solid right circular cone is 60°; prove that it can float in a liquid with its vertex above the surface and one point of its base in the surface, if the densities of the cone and the liquid are in the ratio of $2\sqrt{2}-1 : 2\sqrt{2}$.* (*Allahabad 1920*)

Let the cone of radius r float with BD as the surface of floatation. G is the C.G. of the complete cone so that $OG = \frac{1}{4}OA$.



Similarly G′ is the C.G. of the portion of the cone above the liquid on AO′ such that $O'G' = \frac{1}{4}AO'$.

Hence the C.G. of the portion submerged *i.e.*, the centre of buoyancy will be on the line G′G (produced) at H. O and O′ are the middle points of BC and BD respectively.

$\therefore$ In the $\triangle$BCD, OO′ will be parallel to DC. Hence G′GH will also be parallel to DC.

For equilibrium to exist GH must be a vertical line. Hence the line AC is also vertical.

Vol. of the complete cone$=\frac{1}{3}\pi r^2.\sqrt{3}r$ (vertical angle is 60°)

$$=\frac{1}{\sqrt{3}}\pi r^3.$$

The section of the cone by the plane of floatation is an ellipse ; let us obtain the volume of the fluid displaced.

Let BD be the major axis of the ellipse of the surface of floatation ; O′ its centre and a, b its semi-axes.

Through O′ and D draw LM and DR parallel to BC.

Then BD$=2a$ and AO$=\sqrt{3}\,r$

Also it is proved in the books of geometry that

$$b^2=\text{LO}'.\text{O}'\text{M} \text{ (Euclid, Book III, Pr. 35)}$$

$$=\frac{\text{DR}}{2}\cdot\frac{\text{BC}}{2}$$

$$=\frac{r}{2}\cdot\frac{2r}{2}=\frac{r^2}{2}$$

$$\therefore \quad b=\frac{r}{\sqrt{2}}.$$

Height of the cone above the surface is AD$=r$.

$\therefore$ Volume of the cone above the surface

$$=\pi ab.\tfrac{1}{3}\text{AD}$$

$$=\pi\frac{\sqrt{3}}{2}r.\frac{1}{\sqrt{2}}r.\tfrac{1}{3}r$$

$$=\pi r^3\frac{1}{2\sqrt{6}}.$$

Hence the volume submerged$=\pi r^3\left(\frac{1}{\sqrt{3}}-\frac{1}{2\sqrt{6}}\right)$.

Let σ and ρ be the densities of the solid and the liquid respectively.

For equilibrium the wt. of the body must be equal to the wt. of the liquid displaced.

$$\therefore \quad \pi r^3\frac{1}{\sqrt{3}}g\sigma=\pi r^3\left(\frac{1}{\sqrt{3}}-\frac{1}{2\sqrt{6}}\right)g\rho$$

or

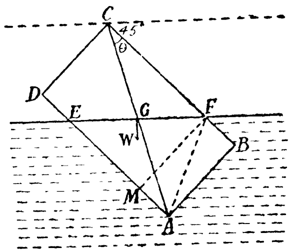
$$\frac{\sigma}{\rho}=1-\frac{1}{2\sqrt{2}}=\frac{2\sqrt{2}-1}{2\sqrt{2}},$$

Ex. 6. *A rectangle movable about an angular point floats with its area immersed in a liquid. If the angular point lies outside the liquid, and if the rectangle floats with its sides equally inclined to the* *vertical, shew that the ratio of the density of the rectangle to that of the liquid is*

$$3b+a : 4b$$

where a and b are the sides of the rectangle.

Let ABCD be the rectangle movable about C. EF is the horizontal line in which the surface of the liquid cuts the rectangle.



BC$=b$, CD$=a$. Let BFE$=x$, $\angle$ACB$=\theta$.

Draw FM perpendicular to AD. Since EM, MF are equally inclined to the horizon, EM$=$MF$=a$.

Since half of the rectangle is in the liquid, the surface EF of the liquid must pass through G, the C.G. of the rect.

$\therefore \frac{1}{2}ab=$rect. AMFB$+\triangle$EFM
$=x.a+\frac{1}{2}a^2$

$\therefore \quad x=\frac{1}{2}(b-a)$, and AE$=\frac{1}{2}(a+b)$

Let ρ and σ be the densities of the rectangle and the liquid respectively.

The wt. W of the rectangle$=abg\rho$ and acts vertically downward through G.

The wt. W′ of the liquid ABFM$=ax.g\sigma=\frac{1}{2}a(b-a)g\sigma$ and acts vertically upwards through the mid-point of AF.

The wt. W″ of the liquid EFM$=\frac{1}{2}$ EM.MF$g\sigma=\frac{1}{2}a^2g\sigma$.

Also W″ acting at the C.G. of the $\triangle$EFM is equivalent to $\frac{1}{3}$W″ at each of the angular points E, F, M.

Taking moments about C

$$\tfrac{1}{2}W.AC\cos(\theta+45°)=\frac{W''}{3}\{(b-x)\cos 45°+AC\cos(\theta+45°)-x\cos 45$$
$$-a\cos 45°+(b-a-x)\cos 45°\}+\frac{W'}{2}\{AC\cos(\theta+45°)$$
$$+(b-x)\cos 45°\}$$

or $$\frac{W}{2\sqrt{2}}(b-a)=\frac{W''}{3\sqrt{2}}(b-x+b-a-x-a+b-a-x)$$
$$+\frac{W'}{2\sqrt{2}}(2b-a-x)$$

or $$\frac{W}{2}(b-a)=W''(b-a-x)+\frac{W'}{2\sqrt{2}}(2b-a-x)$$

or $$\frac{W}{2}(b-a)=\frac{W''}{2}(b-a)+\frac{W'}{4}(3b-a)$$

or $$\frac{ab}{2}(b-a)\rho g=\tfrac{1}{4}a^2\sigma(b-a)g+\tfrac{1}{8}a(b-a)(3b-a)\sigma g$$

or $\rho b = \frac{1}{2}a\sigma + \frac{1}{4}(3b-a)\sigma$

$\therefore$ $\rho : \sigma = 3b+a : 4b.$

Examples 18

1. A hollow conical vessel floats in water with its axis vertical and $\frac{1}{n}$ of its axis immersed. Find to what depth must the vessel be filled with water so that it may just sink till its mouth is on the level with the surface of the water outside.

2. A sphere of radius r made of wood of sp. gr. σ floats in water with its lowest point at a depth $\frac{r}{h}$ below the surface of water. Find an expression for the fraction of the volume of the sphere which is immersed and show that h is the root of the equation $4\sigma x^3 - 3x + 1 = 0$. (*I.C.S. 1926*)

3. A thin uniform rod of weight W has a particle of weight w attached to one end. It is floating, in an inclined position, in water with this end immersed. Prove that the length of the rod above water is $\frac{w}{w+W}$ times its whole length and that the sp. gr. of the rod is

$$\frac{W^2}{(w+W)^2}.$$

(*Delhi 1959 ; Agra 1955 ; Lucknow 1950, 57*)

4. A thin uniform rod AB, of length $2a$ and weight W, has a particle of weight W′ attached to a point D near the end A, AD being of length b. The rod floats, in an inclined position, freely in water with a length AC $(=2x)$ immersed. Prove that

$$x = \frac{aW + bW'}{W+W'}$$

and that the sp. gr. σ of the rod is given by

$$a\sigma(W+W')^2 = W(aW+bW').$$

5. A cone of given weight and volume, floats in a given fluid with vertex downwards ; show that the surface of the cone in contact with the fluid is least, when the vertical angle of the cone is

$$2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right).$$

(*Raj. 1960*)

6. ABC, an isosceles triangle right angled at A, composed of two heavy roads AB, AC hinged together at A and a light string BC, floats with the angle A immersed in water. Show that the tension of the string is $\frac{a-b}{2a}$ W, where $2a$ is the length of a rod, $2b$ the length immersed and W the weight of each rod. (*Bombay 1940 ; Agra 1948*)

7. A hollow closed cone of semi vertical angle $\sin^{-1}\frac{1}{3}$ of metal whose specific gr. is σ is made of such uniform thickness that it will float in all positions wholly submerged in liquid of sp. gr. ρ. Show that the thickness must be

$$\tfrac{1}{4}h\left\{1 - \left(1 - \frac{\rho}{\sigma}\right)^{\frac{1}{3}}\right\}$$

where h is the external height of the cone. (*Nagpur 1943*)

8. A sphere of density σ floats just immersed in three liquids. The densities of the liquid in descending order are ρ, 4ρ, 9ρ and the thickness of the two upper liquid layers are each one-third of the sphere. Prove that

$$27\sigma = 122\rho.$$

9. A cylindrical piece of cork, of height h, is floating with its axis vertical in a basin of water. If the basin be placed under the receiver of an air-pump and the air be pumped out, prove that the cork will sink through a distance $\frac{\sigma}{1-\sigma}(1-\rho)h$, where σ and ρ are respectively the sp. gravities of air and cork.

10 A rod floats upright partially immersed in a homogeneous liquid. Prove that a small increase of atmospheric density will produce a small rise of the rod proportional to the square of the length of the unimmersed portion. (*Allahabad 1943*)

11. A triangular lamina ABC of sp. gr. σ floats with its plane vertical in water, A being outside the surface and BC not horizontal. The angle A is a right angle and AC=AB. If θ be the inclination of AB to the vertical, prove that

$$\sin 2\theta = \frac{2-2\sigma}{2\sigma \sim 1}$$

given $\sigma > \frac{3}{4}$. (*Gorakhpur 1959 ; Nagpur 1935*)

12. Two uniform straight rods of length $2a$, $2b$ and sp. gr ρ, σ respectively, are joined together to form one straight rod. If it floats freely in water in an inclined position, the rod of length $2a$ and part of the other being immersed, prove that

$$a^2\rho(\rho-1)+2ab\sigma(\rho-1)+b^2\sigma(\sigma-1)=0.$$

13. Two equal and similar rods AB, BC, fixed at an angle α at B rest in a fluid of twice the sp. gr. with the angle B out of the fluid, and the bisector of the angle ABC makes an angle θ with the horizon. Prove that

$$\cos 2\theta + \sec \alpha = 2. \qquad \textit{(Bombay 1937)}$$

14. ABC is a right-angled triangular lamina and floats with its plane vertical, and the right angle immersed in water ; prove that if its sp. gr. be to that of the water as 2 : 5, and CB : CA=5 : 4, CB is cut by the surface of the water at a distance from C equal to CA.

15 A rectangle, movable about an angular point which is fixed below the surface of a liquid, floats with its sides equally inclined to the vertical and with half its area immersed in the liquid. If the lengths of the sides be a and b and one of the sides of length b be entirely immersed in the liquid, show that the ratio of the density of the body to that of the liquid is

$$a-b : 4a$$

16. A uniform log whose cross-section is a square floats horizontally with one edge in the surface and one edge above the surface of a homogeneous liquid. Show that the ratio of their sp. gr. is as 3 is to 4. (*M.T., Rangoon 1947*)

17. A prism of square section floats in water with its long edges horizontal and the centre line of one of its faces hinged to an axis fixed in the surface of the water. Show that, if the sp. gr. of the prism is $\frac{39}{64}$, the opposite face of the prism will be $\frac{3}{4}$ immersed. (*M.T.*)

18. Shew that a right circular cone of density ρ and semi-angle α can float vertex downwards in a liquid of density σ with one generator vertical and the base just clear of the liquid if

$$\rho = \sigma(\cos 2\alpha)^{\frac{3}{2}}. \qquad \textit{(I A S., 1959)}$$

19. Two buckets containing water, the mass of each bucket with the contained water being M, balance each other over a smooth pulley. Two pieces of wood masses m and m' and specific gravity σ, σ' are then tied to the bottoms of the buckets so as to be wholly immersed prove that the tension of the string attached to the masses is

$$\frac{2m(M+m')g}{2M+m+m'}\left(\frac{1}{\sigma}-1\right). \qquad \textit{(Jaipur 1953)}$$

CHAPTER VIII

PROPERTIES OF GASES

X **88. Gases and Liquids.** As already mentioned gases have several properties in common with liquids, and both are commonly termed fluids. Gases like liquids, transmit pressure and possess elasticity of volume but not shape. It possesses no free surface as a liquid. On the other hand, gases being much lighter than liquids, are very compressible and have got the property of indefinite expansion. A certain volume of a gas like that of liquid possesses mass and so it has got some weight.

89. Air has weight. On weighing a flask full of air, and then exhausting the air by means of an exhaust-pump, and weighing again, *Otto Von Guericke* found that the empty flask weighed somewhat less than when full. This experiment proved that *air has weight.*

90. Air exerts pressure. That air or gases exert pressure can be shown from several experiments.

(1) If a rubber balloon be filled with a gas or air, the envelope expands owing to the pressure exerted by the gas or air within it on its surface.

(2) If a glass tumbler be pushed with its mouth downwards into water, it will be seen that the level of the water inside the tumbler is lower than that outside ; the pressure of the air inside the glass forces the water down.

(3) If the two hollow hemispheres are fitted together very accurately so that they are air-tight and if the air is drawn out, a great force will be required to separate the hemispheres. The reason is that the air from outside presses the hemispheres from all sides and keep them very tightly pressed against each other. This experiment is called as that of Magdeburgh.

91. Specific Gravities of some principal Gases. It can be proved experimentally that the density of a given quantity of gas does not remain constant. It changes with the change of temperature and pressure. In the case of solids and liquids, temperature has some slight effect on their densities, but in gases the effect is much considerable. In order to compare the densities and sp. gravities of gases it is, therefore, necessary that they should be reduced to the same temperature and pressure. For this purpose the standard temperature is usually taken to be 0°C and the standard pressure to be the pressure due to a column of 76 cms. of mercury. The following are the sp. gravities of some principal gases at the above standard when water is taken as the standard substance.

Air	...	...	0·001293
Oxygen	...	...	0·001430
Hydrogen	...	...	0·000089
Nitrogen	...	...	0 001256
Carbon dioxide	...	...	0·001977
Steam	...	...	0·000802

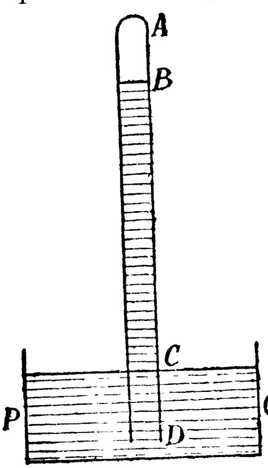
Hydrogen is the lightest gas known. For this reason it is taken as the standard substance for measuring the sp. gr. of gases. Thus the values of the sp. gravities of the above gases will be

Air	...	14·5	Oxygen	...	16
Hydrogen	...	1	Nitrogen	...	14
Carbon dioxide	...	22	Steam	...	9

92. Measurement of the Atmospheric Pressure.

Torricelli's Experiment.—After having reached at a definite conclusion that there is atmospheric pressure, the scientists had been busy in finding some methods to measure it. Evengelista Torricelli, a pupil of Galileo, was the first who in 1643 performed a simple experiment to determine the measure of the atmospheric pressure.

A glass tube AD about 3 feet in length, closed at one end and open at the other, is filled with mercury. The tube is then temporarily closed and inverted and re-opened with its lower end D below the surface of mercury contained in a basin PQ. The mercury column then sinks down to some extent and comes to rest at a point B. The height of the point B above the mercury surface C in the basin is found to be nearly 29 or 30 inches.



Now the pressure inside the tube at level C is the same as the pressure outside the tube at the surface of mercury. The pressure inside is due to a depth BC of mercury, *i.e.*, $g\rho$.BC, where ρ is the density of mercury. The pressure outside the tube at the surface of mercury is the pressure due to the atmosphere. Hence the atmospheric pressure is $g\rho$. BC.

The space AB *within the tube above the mercury surface contains no air and is usually referred to as* **Torricellian Vacuum.**

Note 1. Really, however, the above space AB contains some mercury vapour and is not a perfect vacuum. But as the pressure due to mercury vapour at the ordinary temperature is extremely small, its presence may practically be neglected.

Note 2. Other liquids than mercury might be used, but mercury is the most convenient because of its high sp. gr. If water were used, the tube would need to be at least 35 feet in length.

93. **Pascal's Experiment.** Pascal suggested that since the atmospheric pressure decreases as we go up, the height of the mercury column would also decrease accordingly. In 1648, Perrier went up to the top of Puy-de-Dome, a mountain in France about 1000 metres in height and found that the mercury column actually came down through 8 cms. It was also found that with different liquids, columns stood at different heights.

94. **Height of the Homogeneous Atmosphere.** The atmosphere is not homogeneous. Its density gradually decreases as one proceeds up from the surface of the earth. If the atmosphere be replaced by a column of homogeneous air of the same density as at the earth's surface and of height such as would give the same pressure at the earth's surface as the actual atmosphere does, then this height is called *the height of the homogeneous atmosphere.*

Method to find this height :—

The wt. of a cubic foot of air $= \cdot 0013 \times$ wt. of a cu. ft. of water $= \cdot 0013 \times 62\frac{1}{2}$ lbs. wt. nearly.

If x be the height of the homogeneous atmosphere $x \times$ sp. gr. of air $=$ height of mercury barometer $\times$ sp. gr. of mercury

$$\therefore \quad x = \frac{\text{sp. gr. of mercury}}{\text{sp. gr. of air}} \times \text{ht. of the mercury barometer}$$

$$= \frac{13 \cdot 596}{\cdot 0013} \times \frac{30}{12} \text{ ft.}$$

$$= 26146 \text{ ft. nearly}$$

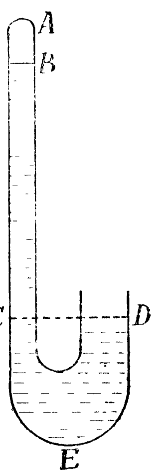
$$= 5 \text{ miles nearly.}$$

95. **The Barometer.** *The* **barometer** *is an instrument which is used for measuring the atmospheric pressure.*

Cistern Barometer. The Torricellian tube standing vertically on a cistern of mercury constitutes the simplest type of a barometer which is usually known as the **Cistern barometer.** The height of the mercury column measured from the mercury surface in the cistern gives the atmospheric pressure at the time. When the atmospheric pressure changes, the height of the mercury column in the tube and consequently the level of mercury in the cistern also changes.

Fortin's Barometer. Since the barometric height is always measured above the mercury-level in the cistern, to avoid the necessity of measuring from different levels means are devised by which the mercury-level in the cistern can be brought to a definite position : such as a flexible bottom to the cistern, whose capacity can be increased or decreased by turning a screw attached to its base. *Fortin's Barometer* is one of such types.

Siphon Barometer. This type of barometer is more convenient and portable than a simple cistern one. It consists of a U-tube, one arm of which AE is long and narrow and the other arm DE short and wide. The longer arm, which is closed at the top, is filled with mercury as in the Cistern Barometer, while the shorter arm, which is open, serves as a cistern. Above the mercury in the long arm there is a vacuum AB.

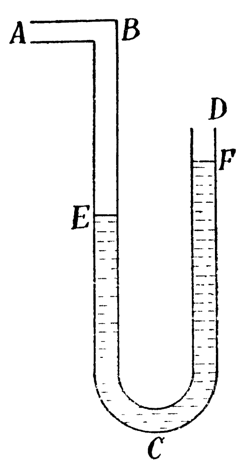


When the surfaces of the mercury in the two arms are at B and D respectively, the pressure of the air is measured by the wt. of a column of mercury whose height is equal to the vertical distance between B and D *i.e.*, BC (as in the case of a Cistern Barometer).

Therefore the pressure of the atmosphere is $g\rho$.BC, where ρ is the density of mercury.

Aneroid Barometer. This barometer derives its name from the fact that no liquid is used in its construction. Its consists of a cylindrical metal box exhausted of air, the top of which is made of thin corrugated metal, so elastic that it really yields to alternations in the pressure of the atmosphere.

When the pressure increases, the top is pressed inwards ; when it decreases the elasticity of the lid, aided by a spring tends to move it in the opposite direction. Then motions are transmitted by delicate multiplying levers to an index which moves on a scale. It is graduated empirically by comparing its indications, under different pressures, with those of a standard mercurial barometer. It is portable and so sensitive as to be able to indicate even small heights. Arrangements for allowing for temperature and the elasticity of the disc are also attached to the instrument. They are used with great advantage in the aeroplanes for measuring heights but a sudden variation in height impairs the sensitiveness of the disc which should be corrected from time to time, by comparing its readings with those of a standard mercurial barometer.



Manometer. Manometer is sometimes used to determine the pressure of some enclosed volume of a gas. It is just like a U-tube with a horizontal connection AB. The lower part ECF is filled with mercury. The horizontal arm AB is connected with the vessel containing the gas, and the end D is open. The difference of mercury-levels in the two arms is then measured. Let it be *d*.

Then pressure of the gas

$$= \text{pressure at the level E}$$

=the pressure at the level F$\pm$wt. of column, *d*, of the liquid according as E is below or above F ; the pressure at F is that due to atmosphere whose value can be found by means of a barometer.

96. Graduation of a Barometer. In graduating a barometer we must keep in view the fact that if the mercury level rises in one arm, its level in the cistern (or in the shorter arm in case of Siphon Barometer) must fall and the height of the barometric column is the difference of those two levels.

Cistern Barometer. Let a and A be the uniform sectional areas of the tube and the cistern respectively.

Let x be the rise in the level of the mercury in the tube. The level in the cistern will, therefore, fall by $\frac{ax}{A-a}$, since the area of the surface of mercury in it is $A-a$.

Hence an apparent increase of x in the height of the barometric column would correspond to a real increase of

$$x+\frac{vx}{A-a} \quad i.e., \quad \frac{A}{A-a}x.$$

So an apparent increase $\frac{A}{A-a}x$ would correspond to a real increase of x.

Siphon Barometer. In Siphon Barometer if A be the cross-section of the shorter arm, a rise of x in the level of the mercury in the longer arm would cause a fall of $\frac{a}{A}x$ in the shorter arm.

Hence an apparent increase of x in the height of the barometric column would correspond to a real increase of

$$x+\frac{a}{A}x=\frac{a+A}{A}x.$$

So an apparent increase of $\frac{A}{a+A}x$ would correspond to a real increase of x.

For this reason, in the graduation of a barometer the distances between successive markings in the longer tube are generally kept shorter and they are marked in the ratio

$$A : a+A.$$

In a particular case if a and A be $\frac{1}{8}$ and 1 sq. inch respectively, the height between the two levels has increased by $(1+\frac{1}{8})$ *i.e.*, $\frac{9}{8}$ of an inch corresponding to an apparent rise of 1 inch in the longer arm. So the distance given as one inch by the graduations on the longer arm must really be kept only $\frac{8}{9}$ inch long.

97. Correction to the Barometer reading.

There are a number of corrections which are applied to an observation with the mercury barometer.

(1) *Correction for capacity.*

If the graduations on a barometer be marked at their true value, then to get the true height we must multiply the apparent height by $\frac{a+A}{A}$ as shown in the last article.

(2) *Correction of temperature.*

Since the mercury and the measuring rod both expand with a rise of temperature, the following account is taken of the temperature at which observations are made.

If h be the observed height of the mercury at temperature $t°$ centigrade, the true height h_0 is given by

$$h_0 = h - 0{\cdot}00016 \times h \times t.$$

(3) *Correction for height above the sea level.*

If the observation is taken over the sea-level, the height is less than it would have been if it had been taken at the same time and place but at the sea-level.

For an apparent height h, this correction is

$$h\left(e^{\frac{gz}{k}} - 1\right)$$

where z is the altitude above sea-level, and k is the constant ratio of pressure and density.

(4) *Corrections for Capillarity.*

On account of capillarity, the top of the mercury in the tube is not flat but is convex. Therefore this is also taken into consideration for the correct reading.

(4) *Correction for Vapour Pressure.*

The mercury in the tube gives off a certain amount of vapour in the *Torricellian Vacuum.* Its pressure depresses the mercury column.

Note 1. In a mercury barometer the effect caused by the last two errors given above is very small.

Note 2. For the detailed description of these corrections students are advised to consult standard books on Physics.

98. **Expansion of Gases.** Gases like solids and liquids change in volume due to changes in temperature, although at a greater rate. Again in considering the changes of volume of solids or liquids, the effect of pressure is not taken into account, but in case of gases even a small change in pressure produces an appreciable change in volume, although the temperature is kept constant.

99. **Relation between Pressure and Volume.**

Boyle's Law. *If the temperature of a given mass of gas is* *constant, the pressure varies inversely as its volume, i.e.,*

$$p \propto \frac{1}{v}.$$

where p denotes the pressure and v the volume.

Note 1. The above formula also gives **pv=k**, where k is constant. This equation is called the **Isothermal Equation** of state.

Note 2. The above law is also called as **Marriotte's Law.**

Note 3. Strictly speaking Boyle's Law is not perfectly true. For gases like air, oxygen, hydrogen, and nitrogen which are very hard to liquefy, the law is very nearly true. But for easily liquefiable gases, such as carbon dioxide, water-vapour etc. the volume is found to decrease more rapidly than the pressure increases and, therefore, they do not behave quite in accordance with this law.

Note 4. *A gas which accurately obeys Boyle's Law is called a* **Perfect Gas.**

✓ **100. Relation between Pressure and Density.** Let m denote the given mass of a gas, and p, v and ρ denote respectively its pressure, volume and density. Let p', v', ρ' denote the corresponding quantities when the pressure is changed from p to p', temperature being the same in the two cases.

Boyle's law then gives

$$pv=p'v' \quad \dots (1)$$

But since the given mass of the gas remains unaltered, we get

$$\rho v=m=\rho' v' \quad \dots (2)$$

Therefore from (1) and (2), by division

$$\frac{p}{\rho}=\frac{p'}{\rho'} \quad \dots (3)$$

Hence $\frac{p}{\rho}$ is always the same for a given mass. Let its value be denoted by k, so that

$$\mathbf{p=k\rho} \quad \dots (4)$$

i.e., if the temperature remains unaltered, the pressure of gas varies as its density.

Note. The equation $p=k\rho$ may be taken as another mode of stating Boyle's Law.

101. Solved Examples.

Ex. 1. *The diameter of the tube of a mercurial barometer is $\frac{1}{6}$ in. and that of cistern is $1\frac{1}{2}$ ins. When the surface of the mercury rises 1 in., find the real alteration in the height of the barometer.*

Area of the cross-section of the tube

$$=\pi \cdot \tfrac{1}{144} \text{ sq. in.}$$

Area of the cross-section of the cistern

$$=\pi \cdot \tfrac{9}{16} \text{ sq. ins.}$$

For an apparent increase of 1 in. in the height of the barometric column, the real increase is

$$=1+\frac{\frac{1}{144}}{\frac{9}{16}} \cdot 1=1+\frac{1}{144}\times\frac{16}{9}=1\frac{1}{81} \text{ ins.}$$

Ex. 2. *When the water barometer is standing at 33 ft. a bubble at a depth of 10 ft. from the surface of water has a volume of 3 cub. ins. At what depth will its volume be 2 cub. ins.?*

Let h be the required depth.

The pressure at a depth of 10 ft. from the surface of water is due to a column of water $=(33+10)$ ft.

Applying Boyle's Law, therefore, we get

$$(h+33)\cdot 2=(33+10)\cdot 3$$

or $$2h=63$$

$\therefore$ $$h=31\tfrac{1}{2} \text{ ft.}$$

Ex. 3. *At a depth of 10 ft. in a pond the volume of an air bubble is 0·0001 of a cubic inch; find approximately what it will be when it reaches the surface, if the height of the barometer and the specific gravity of mercury is 13·5.* (*Lucknow 1934*)

The height of the water barometer

$$=\tfrac{30}{12}\times 13{\cdot}5=\tfrac{135}{4} \text{ ft.}$$

The pressure at a depth of 10 ft. below the surface of water is due to a column of water

$$=(10+\tfrac{135}{4}) \text{ ft.}$$
$$=\tfrac{175}{4} \text{ ft.}$$

If V cu. in. be the volume of the bubble at the water surface, then applying Boyle's Law we get

$$\frac{V}{12\times12\times12}\cdot\frac{135}{4}=\frac{\cdot0001}{12\times12\times12}\times\frac{175}{4}$$

$$V=\frac{\cdot0001\times175\times}{4\times135}$$

$$=\cdot00013 \text{ nearly.}$$

Ex. 4. *A hollow closed conical vessel of height h, floats partially immersed in water with vertex downwards and axis vertical. A hole is then made very near the vertex and water allowed to come into the vessel so that no air escapes from within. If the vertex was originally at a depth b and* H *is the height of the water barometer, prove that the new depth c of the vertex is given by*

$$c^3-b^3=\left\{c-\frac{H(c^3-b^3)}{h^3-(c^3-b^3)}\right\}^3$$

Let W be the wt. of the cone and α the semi-vertical angle. Therefore originally

$$W = \text{wt. of the water displaced}$$
$$= \tfrac{1}{3}\pi b^3 \tan^2 \alpha . g \qquad \dots (1)$$

Let x be the depth of water inside the vessel in the second position. Therefore wt. of the water inside $= \frac{1}{3}\pi x^3 \tan^2 \alpha . g$.

$\therefore$ In this case

wt. of the cone + wt. of the water inside
= wt. of the water displaced.

Hence from (1)

$$\tfrac{1}{3}\pi b^3 \tan^2 \alpha\, g + \tfrac{1}{3}\pi x^3 \tan^2 \alpha . g = \tfrac{1}{3}\pi c^3 \tan^2 \alpha . g$$

or $$x^3 = c^3 - b^3 \qquad \dots (2)$$

Again, the air which occupied the whole volume of the cone at the atmospheric pressure H, now occupies a volume $\frac{1}{3}\pi(h^3 - x^3)\tan^2 \alpha$ under a pressure $H + c - x$ which is the pressure at the level of water inside the vessel.

$\therefore$ By Boyle's Law

$$\tfrac{1}{3}\pi h^3 \tan^2 \alpha . H = \tfrac{1}{3}\pi (h^3 - x^3) \tan^2 \alpha . (H + c - x)$$

or $$h^3 H = (h^3 - x^3)(H + c - x)$$

or $$x = c - \frac{x^3 H}{h^3 - x^3}$$

$\therefore$ $$x^3 = \left(c - \frac{x^3 H}{h^3 - x^3}\right)^3.$$

Hence $$c^3 - b^3 = \left\{ c - \frac{H(c^3 - b^3)}{h^3 - (c^3 - b^3)} \right\}^3 \text{ from (2).}$$

Examples 19

1. The diameter of the tube of a mercurial barometer is 1 cm. and that of the cistern is 4·5 cms. If the surface of the mercury in the tube rise through 2·5 cms., find the real alteration in the height of the barometer.

2. The volume of an air bubble increases ten-fold in rising from the bottom of a lake to its surface. If the barometric height is 30 inches, what is the depth of the lake ? (sp. gr. of mercury = 13·6). *(Dacca 1941)*

3. A good barometer reads 75 cms. On admitting 1 c.c. of air, the reading is 70 cms. Find the volume of the space above the mercury at the end. *(Punjab 1931)*

4. Find the height of the homogeneous atmosphere corresponding to a barometric height of 760 mm. of mercury, taking the sp. gr. of air 0·0013 and that of mercury 13·596. *(Nagpur 1933)*

5. The sp. gr. of mercury is 13·6 and the barometer stands at 30 ins. A bubble of gas, the volume of which is 1 cub. in. when it is at the bottom of a lake 170 ft. deep, rises to the surface. What will be its volume when it reaches the surface ?

6. Bubbles of air rise from a depth of 20 ft. in water. Find the ratio in which the radius of bubble is increased when it reaches the surface, given that the height of the mercury barometer is 30 inches and the sp. gr. of mercury is 13·6.

7. A balloon half filled with coal-gas just floats in the air when the mercury barometer stands at 30 ins. What will happen when the barometer sinks to 28 ins. ? (*Utkal 1945*)

8. A conical glass, height h, is immersed mouth downwards in water, the height of the water barometer being H. If the water inside the glass is to rise half way within it, prove that it must be depressed through a distance

$$7\mathrm{H}-\frac{h}{2}.$$

(*M.T. 1859 ; Calcutta 1916*)

9. A hollow cylinder, of height h and open at the top, is inverted and partially immersed so that a length k of it is under water. Prove that the air inside it occupies a length x given by the equation

$$x^2+x(\mathrm{H}+k-h)=\mathrm{H}.h,$$

where H is the height of the water barometer. (*Lucknow 1954*)

10. A hollow cylinder of height b, open at the top is inverted, and partly immersed in water with its length a in air. Find the difference between the water levels outside and inside the cylinder. (*Banaras 1942*)

11. A thin closed cylindrical vessel, of height a, contains air at atmospheric pressure and floats in water with axis vertical and length b immersed. If a small hole is made in the bottom of the vessel, shew that water leaks in until there is a depth

$$\frac{ab}{h+b}$$

inside, h being the height of the water barometer.

(*Allahabad 1950 ; Lucknow 1956, 57*)

12. If a body floats in a liquid with volume v_1, v_2 and v_3 above the surface when the barometric heights are h_1, h_2 and h_3, prove that

$$h_1\, v_1\, (v_2-v_3)+h_2\, v_2\, (v_3-v_1)+h_3\, v_3\, (v_1-v_2)=0.$$

(*Lucknow 1959*)

13. A heavy sphere of weight W and radius r is placed in a vertical cylinder, filled with atmospheric air of density σ. Prove that the density of the air in the cylinder when the sphere is a position of permanent rest is

$$\left(1+\frac{\mathrm{W}}{\pi r^3\Pi}\sigma\right)$$

where Π is the atmospheric pressure. (*M.T.*)

14. A U-tube with equal arms of fine uniform bore is placed so that the arms are vertical and the end of one of the arms is closed. The tube is partially filled with mercury, so that there is a length $\frac{h}{2}$ of vacuum above the surface of the mercury in the closed arm, the barometric height being h. Prove that, if it is inclined at an angle of 60° to the vertical, the mercury reaches the top of the closed end ; also that if the end that is open when the tube is vertical is closed before inclining it, the length of vacuum becomes $\frac{7h}{40}$ roughly.

15. A closed air-tight cylinder, of height $2a$, is half-full of water and half-full of air at atmospheric pressure, which is equal to that of a column, of height h, of the water. Water is introduced without letting the air escape so as to fill an additional height k of the cylinder, and the pressure of the base is thereby doubled. Prove that

$$k=a+h-\sqrt{ah+h^2}.$$

16. A thin conical surface of weight W just sinks to the surface of a fluid, when immersed with its open end downwards ; but when immersed

with its vertex downwards, a weight equal to mW must be placed within it to make it sink to the same depth as before. If x be the length of the axis, h the height of the barometric column of the fluid, show that

$$x=hm(1+m)^{\frac{1}{3}}.$$

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102. Charles' Law. *If the pressure of a given mass of gas be constant, the volume increases by a definite fraction* α *of the volume at 0°C, for every degree centrigrade by which the temperature is raised.*

Hence, if V_0 be the volume at 0°C and V be the volume at t°C, then

$$V-V_0=V_0.\alpha t,$$

or

$$V=V_0\ (1+\alpha t).$$

Note 1. The above law is sometimes called as **Gay-Lussac's** or **Dalton's Law.**

Note 2. For air and most of the gases, the value of α, which is called the *coefficient of expansion*, is 0·003665 or $\frac{1}{273}$ approximately.

Note 3. Like Boyle's law, this law is also based on the results of experiments, description of which can be found in any standard book on Physics.

Note 4. The law does not apply to vapours.

103. Relation between Densities at different temperatures.

Let ρ and ρ_0 be the respective densities at the temperatures t°C and 0°C ; and let V and V_0 be the corresponding volumes. Since mass remains unaltered we have

$$\rho V=\rho_0 V_0$$

or

$$\frac{\rho_0}{\rho}=\frac{V}{V_0}=(1+\alpha t), \text{ from Charles' Law}$$

$$\therefore \quad \rho_0=\rho\ (1+\alpha t).$$

104. Relation between Pressure, Density and Temperature.

Let p and ρ_0 be the pressure and density of a given mass of gas at 0°C temperature, therefore

$$p=k\rho_0 \qquad \ldots \quad (1)$$

where k is a constant which depends upon the nature of the gas.

Let ρ be the density when the temperature of the gas is raised to t°C, while the pressure p remains the same.

$$\therefore \quad \rho_0=\rho(1+\alpha t) \qquad \ldots \quad (2)$$

from the last article.

From (1) and (2), therefore, we have

$$p=k\rho\ (1+\alpha t)$$

105. Absolute Temperature. If we imagine that the temperature of a gas is lowered till its pressure becomes equal to zero and that in this process it neither liquefies nor suffers any alteration in vòlume, we shall arrive at what is known as the absolute zero of the temperature.

Therefore $1+at=0$

or $$t=-\frac{1}{\alpha}=-273°\text{C}.$$

This temperature, $-273°$C, is called the **absolute zero,** and the temperatures measured from this zero point are called **absolute temperature.**

The absolute temperature is generally denoted by **T**, so that

$$T=\frac{1}{\alpha}+t=273+t.$$

106. Relation between Pressure, Volume, and Absolute Temperature.

From above we have

$$\begin{aligned} p &= k\rho(1+\alpha t) \\ &= k\rho\alpha\left(\frac{1}{\alpha}+t\right) \\ &= k\rho\alpha T \end{aligned}$$

Therefore, if V be the volume of a certain quantity of gas, we have

$$\frac{p.V}{T}=k\alpha.(V.\rho)=k.\alpha\times\text{mass of the gas}$$

$=$a constant, mass of the gas being constant.

or $pV=R.T$, where R is a constant depending on the nature of the gas.

i.e., the product of the volume of any given mass of gas and its pressure varies as its absolute temperature.

107. Mixture of Gases *:*

It has been experimentally seen that

'If two gases, occupying different vessels, be at the same temperature and pressure, they will, when one vessel is allowed to communicate with the other, form a mixture whose pressure is the same as before, provided no chemical action takes place.'

Pressure of a mixture of equal volumes.

The pressure of the mixture of two gases occupying a given volume would be the sum of the pressures that each gas would have when occupying alone the same volume at the same temperature.

Let p_1, p_2 be the pressures of the two gases when the volume (of each) is v. Let the pressure of the second gas be changed to p_1 before mixing, so that its volume changes from v to v'. Then, by Boyle's Law,

$$p_1 v' = p_2 v. \qquad \ldots \quad (1)$$

When the gases are mixed, the volume of the mixture is $v+v'$ and by the experimental law stated in the beginning, its pressure is p_1. Now if the volume of the mixture is changed to v, then the required pressure p is given by

$$\begin{aligned} \mathrm{P}v &= p_1(v+v') \\ &= p_1 v + p_2 v \qquad \text{from (1)} \\ \mathrm{P} &= p_1 + p_2 \end{aligned}$$

Note 1. This result is known as Dalton's Law for the pressure of a mixture of gases.

Note 2. It is applicable to the mixture of any number of gases.

108. Pressure of a mixture of gases of different volumes.

Two volumes v_1 and v_2 of different gases at different pressures, p_1 and p_2, are mixed together to form a mixture, of volume V ; to find the pressure of the mixture, the temperature being constant.

Let us suppose that before mixing, the volume of each gas is changed to V ; then their pressures are, by Boyle's Law,

$$\frac{p_1 v_1}{\mathrm{V}} \quad \text{and} \quad \frac{p_2 v_2}{\mathrm{V}}$$

respectively.

Now if they are mixed ; the pressure P of the mixture when its volume is V is, by the last Art., given by

$$\mathrm{P} = \frac{p_1 v_1 + p_2 v_2}{\mathrm{V}}.$$

Aliter. Let the pressure of the second gas be changed from p_2 to p_1 ; so by Boyle's Law its volume changes from v_2 to $\frac{v_2 p_2}{p_1}$.

Thus there are now two gases, of volumes v_1 and $\frac{v_2 p_2}{p_1}$, each of pressure p_1.

When the gases are mixed, the volume of the mixture is $v_1 + \frac{v_2 p_2}{p_1}$ of pressure p_1.

Let P be the pressure when the volume of the mixture is changed. Therefore, by Boyle's Law,

$$P.\ V = p_1 \times \left(v_1 + \frac{p_2 v_2}{p_1}\right)$$
$$= p_1 v_1 + p_2 v_2$$

or
$$P = \frac{p_1 v_1 + p_2 v_2}{V}.$$

Note. This result also holds true for a mixture of more than two gases.

109. Solved Examples.

Ex. 1. *If the volume of a certain quantity of air at a temperature of 10°C, be 300 cu. cm., what will be its volume when its temperature is 20°C ?* (*Calcutta 1938*)

If m be the mass of the air and α be the coefficient of expansion of air, we have by Charles' Law

$$p = k\rho(1+\alpha t)$$

or
$$pv = p\rho v\ (1+\alpha t)$$
$$= km\ (1+\alpha t)$$

∴ From given conditions

$$p.300 = km\ (1+\alpha.10) \qquad \dots (1)$$

and
$$p.V = km\ (1+\alpha.20) \qquad \dots (2)$$

where V is the required volume.

Dividing (2) by (1)

$$\frac{V}{300} = \frac{1+20\alpha}{1+10\alpha}$$
$$= \frac{273+20}{273+10}$$

∴
$$V = \frac{293}{283} \times 300$$
$$= 310\tfrac{170}{283} \text{ c.cs.}$$

Ex. 2. *In the morning a graduated bottle when placed mouth downwards on the surface of water was found to have 200 c.cs. of air, atmospheric temperature and pressure being 10°C and 74 cm. respectively. How many gm. of weight must be placed on the bottle so that the volume of air inside it may be halved, temperature and atmospheric pressure being 20°C and 75 cm. respectively ? Density of mercury is 13·6 gm. per c.c.*

In the second case volume becomes 100 c.cs. at 20°C at total pressure say P_2, then

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

or
$$\frac{74 \times 200}{(273+10)} = \frac{P_2 \times 100}{(273+20)}$$

or
$$P_2 = \frac{74 \times 200 \times 293}{283 \times 100}$$
$$= 153{\cdot}2 \text{ cms. of mercury column.}$$

Out of this total pressure there is already atmospheric pressure of 75 cm. of mercury column.

Therefore, the additional pressure must be

$$=153\cdot2-75=78\cdot2 \text{ cm. mercury column,}$$

$\therefore$ Mass of this mercury column

$$=78\cdot2\times13\cdot6 \text{ gms.}$$
$$=1063\cdot52 \text{ gms.}$$

Ex. 3. *Masses m, m' of two gases in which the ratio of the pressure to the density are respectively k and k' are mixed at the same temperature. Prove that the ratio of the pressure to the density in the compound is*

$$\frac{mk+m'k'}{m+m'}.$$

(*M.T., Calcutta 1929*)

Let ρ, ρ' and p, p' be the densities and the pressures of the masses m, m' respectively.

We have $p=k\rho$, and $p'=k\rho'$

Let v, v' be the corresponding volumes.

$$\therefore \quad v=\frac{m}{\rho},\ v'=\frac{m'}{\rho'}.$$

If P be the pressure of the compound and V be the volume, then

$$P=\frac{pv+p'v'}{V}$$

$$=\frac{\rho k.\frac{m}{\rho}+\rho' k.\frac{m'}{\rho'}}{\frac{m+m'}{\sigma}},$$ if σ be the density of the mixture.

$$\therefore \quad \frac{P}{\sigma}=\frac{mk+m'k'}{m+m'}$$

= ratio of the pressure to the density in the compound.

Ex. 4. *If p_1, ρ_1, t_1; p_2, ρ_2, t_2; p_3, ρ_3, t_3 be the corresponding values of the pressure, density and temperature of the same gas, show that*

$$t_1\left(\frac{p_2}{\rho_2}-\frac{p_3}{\rho_3}\right)+t_2\left(\frac{p_3}{\rho_3}-\frac{p_1}{\rho_1}\right)+t_3\left(\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right)=0$$

(*M.T.*)

From the formula

$$p=k\rho\,(1+\alpha t)$$

we have $\quad p_1=k\rho_1\,(1+\alpha t_1)$

i.e., $$\frac{p_1}{\rho_1}=k+\alpha k t_1 \quad \ldots (1)$$

Similarly $$\frac{p_2}{\rho_2}=k+\alpha k t_2 \qquad \ldots (2)$$

and $$\frac{p_3}{\rho_3}=k+\alpha k t_3 \qquad \ldots (3)$$

Subtracting (3) from (2)

$$\frac{p_2}{\rho_2}-\frac{p_3}{\rho_3}=\alpha k(t_2-t_3) \qquad \ldots (4)$$

Similarly $$\frac{p_3}{\rho_3}-\frac{p_1}{\rho_1}=\alpha k(t_3-t_1) \qquad \ldots (5)$$

and $$\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}=\alpha k(t_1-t_2) \qquad \ldots (6)$$

Multiplying (4) by t_1, (5) by t_2, (6) by t_3 and adding we get

$$t_1\left(\frac{p_2}{\rho_2}-\frac{p_3}{\rho_3}\right)+t_2\left(\frac{p_3}{\rho_3}-\frac{p_1}{\rho_1}\right)+t_3\left(\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right)$$
$$=\alpha k(t_1t_2-t_1t_3+t_2t_3-t_2t_1+t_3t_1-t_3t_2)$$
$$=0.$$

Ex. 5. *If volumes $v_1, v_2, \ldots\ldots, v_n$ of gases at pressures $p_1, p_2, \ldots\ldots, p_n$ and absolute temperatures $t_1, t_2, \ldots\ldots, t_n$ are mixed without chemical change and V, P, T denote the volume, pressure and absolute temperature of the mixture, then prove that*

$$P=\left(\frac{v_1p_1}{t_1}+\frac{v_2p_2}{t_2}+\ldots\ldots+\frac{v_np_n}{t_n}\right)\frac{T}{V}.$$

Let us suppose that the volumes and absolute temperatures of all different gases are changed to V and T ; then their corresponding pressures are

$$\frac{T}{V}\cdot\frac{v_1p_1}{t_1},\ \frac{T}{V}\cdot\frac{v_2p_2}{t_2},\ldots\ldots,\ \frac{T}{V}\cdot\frac{v_np_n}{t_n}.$$

Now when the temperature remains T, let the different gases be mixed in a vessel of volume V. In this new case the pressure P is the sum of the pressures of all gases, so that

$$P=\frac{T}{V}\left(\frac{v_1p_1}{t_1}+\frac{v_2p_2}{t_2}+\ldots\ldots+\frac{v_np_n}{t_n}\right).$$

Examples 20

1. At what temperature will the volume of a given mass of gas be exactly double of what it is at 17°C ? (The coefficient of expansion $=\frac{1}{273}$).

2. At 0°C the density of gas A is to that of gas B as 28 is to 44. Prove that the density of B at 156°C is equal to that of A at 0°C., the pressures being the same. The coefficient of expansion is taken as $\frac{1}{273}$.

3. A gas was found to have a volume of 100 c.c. at 18°C, and 72 cm. pressure, and a volume of 200 c.c. at 90°C. and 45 cm. pressure. Assuming that the gas is perfect, find out at what temperature it would have a volume of 400 c.c. at 100 cm. pressure.

4. A mass of air at a temperature of 39°C. and a pressure of 32 ins. of mercury occupies a volume of 15 cub. ins. What volume will it occupy at a temperature of 78°C, under a pressure of 54 inches of mercury ?

5. At sea-level the barometer stands at 750 mm. and the temperature is 7°C., while on the top of a mountain it stands at 400 mm. and the temperature is 13°C. Compare the weights of a cubic meter of air at the two places. *(Calcutta 1937)*

6. The radius of a sphere containing air is doubled and the temperature raised from 0°C to 455°C. Show that the pressure of the air is reduced to one-third of its original value, the coeff. of expansion of air per 1°C being $\frac{1}{273}$. *(U.P.C.S. 1938)*

7. The sp. gr. of the gas inside a balloon is 0·1, the air at 76 cm. pressure at 0°C. being taken as the standard substance. If the atmospheric pressure changes from 76 cm. to 75 cm. while the temperature remains at 0°C., prove that the lifting power of the balloon is reduced in the ratio 337 : 342, the volume of the balloon remaining unaltered and the weight of its envelope neglected.

8. The same quantities of atmospheric air are contained in two hollow spheres, their internal radii being r, r' and the temperatures t, t' respectively. Prove that the whole pressures on the surfaces are in the ratio

$$r'(1+\alpha t) : r(1+\alpha t')$$

with the usual notation of α.

9. A volume v of a gas at temperature t and pressure p is mixed with a volume v' of another gas at temperature t' and pressure p'. If the volume of the mixture is V and the temperature T, prove that the pressure is,

$$\frac{T}{V}\left(\frac{pv}{t}+\frac{p'v'}{t'}\right)$$ *(Patna 1935)*

10. For a gas of which at temperature t°C. the mass of V cubic feet is W lbs., the height of the barometer being h inches. Prove that in the formula $p=k\rho\,(1+\alpha t)$, the value of k in ft. lb. sec. units is

$$\frac{Vh\sigma}{3W}\cdot\frac{273000}{273+t}$$

where σ is the sp. gr. of mercury.

11. A volume of air of any magnitude, free from the action of force, and of variable temperature, is at rest. If the temperature at a series of points be in A.P., prove that the densities at these points are in H.P. *(M.T.)*

12. Two vessels contain air having the same pressure Π but different temperatures t, t' : the temperature of each being increased by the same quantity, find which has its pressure most increased.

If the vessels be of the same size and the air in one be forced in to the other, prove that the pressure of the mixture at zero temperature is

$$\Pi\left(\frac{1}{1+t\alpha}+\frac{1}{1+t'\alpha}\right)$$

with the usual notation of α.

13. Prove that if volumes v_1 and v_2 of atmospheric air are forced into vessels of volumes V_1 and V_2 and a communication is opened between them, a mass of air of volume

$$\frac{(V_1 v_2 \sim V_2 v_1)}{V_1+V_2}$$

at atmospheric pressure will pass from one vessel to the other.

14. A cylinder contains two gases which are separated from each other by a movable piston. The gases are both at 0°C. and the volume of one gas is double that of the other. If the temperature of the first be raised t°, prove that

the piston will move through a space

$$\frac{2lat}{9+6\alpha t},$$

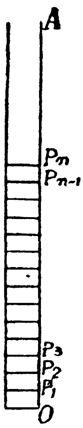
where l is the length of the cylinder, and α is the coeff. of expansion per 1°C.

15. A piston accurately fits a cylinder and moves freely in it; initially it is placed in the middle of the cylinder and the ends are closed. The cylinder being placed vertically the distance of the piston from the top is $\sqrt{2}$ times its original distance. The temperature in the two parts being raised to the absolute temperature t_1 and t_2, the piston goes back to the middle of the cylinder. Show that the original absolute temperature of the cylinder is

$$t_1 \sim t_2.$$

✓110. **Theorem.** *If the atmosphere be at rest and if its temperature and force of gravity be constant, then, if points be taken whose heights above the earth are in Arithmetic Progression, the common difference being small, the densities at these points are in Geometrical Progression.*

Let a column of air, of small horizontal section, be taken with a vertical axis $OP_1P_2 \ldots P_n$. Let P_1, P_2, $P_3 \ldots\ldots$ be a series of points on OA in A.P. so that



$$OP_1 = P_1P_2 = \ldots\ldots P_{n-1}P_n$$
$$= \beta, \text{ where } \beta \text{ is very small.}$$

If n be taken indefinitely great *i.e.*, if β is taken very small, we may suppose the density of each layer OP_1, P_1P_2, $P_2P_3 \ldots\ldots$ constant throughout. Let the densities of the layers OP_1, P_1P_2, $P_2P_3 \ldots\ldots P_{n-1}P_n$ be ρ_1, ρ_2, $\rho_3 \ldots\ldots$, ρ_n respectively and, therefore, their corresponding pressures are $k\rho_1$, $k\rho_2$, $k\rho_3 \ldots\ldots$, $k\rho_n$ which may be taken as the pressures at the points O, P_1, $P_2 \ldots\ldots$, P_n.

The difference of the pressures on the faces at O and P_1 supports the element OP_1. Hence

$$k\rho_1 - k\rho_2 = g\rho_1\beta$$

Similarly, for the columns P_1P_2, P_2P_3, ...we have

$$k\rho_2 - k\rho_3 = g\rho_2\beta,$$
$$k\rho_3 - k\rho_4 = g\rho_3\beta,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$k\rho_{n-1} - k\rho_n = g\rho_{n-1}\beta,$$

Hence
$$\rho_2 = \rho_1\left(1 - \frac{g\beta}{k}\right)$$
$$\rho_3 = \rho_2\left(1 - \frac{g\beta}{k}\right) = \rho_1\left(1 - \frac{g\beta}{k}\right)^2$$
$$\rho_4 = \rho_3\left(1 - \frac{g\beta}{k}\right) = \rho_1\left(1 - \frac{g\beta}{k}\right)^3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\rho_n = \rho_{n-1}\left(1 - \frac{g\beta}{k}\right) = \rho_1\left(1 - \frac{g\beta}{k}\right)^{n-1}$$

Hence *as the altitudes increase in A.P., the densities and, therefore, the corresponding pressures diminish in G.P.*

111. Density of air at given altitude.

If ρ be the density just above P_n, we have as in the last Art.

$$\rho=\rho_n\left(1-\frac{g\beta}{k}\right)=\rho_1\left(1-\frac{g\beta}{k}\right)^n$$

If we put $n\beta=h$ so that ρ is the density at a height h above the ground, we have

$$\rho=\rho_1\left(1-\frac{gh}{nk}\right)^n$$

Now, if $\frac{nk}{gh}=z$, so that $n=\frac{ghz}{k}$

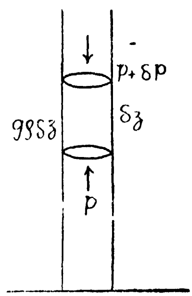
then $$\rho=\rho_1\left(1-\frac{1}{z}\right)^{\frac{ghz}{k}}=\rho_1\left|\left(1-\frac{1}{z}\right)^{-z}\right|^{\frac{-gh}{k}}$$

Now when $n\to\infty$, h being kept constant, we get in the limit

$$\rho=\rho_1.e^{-gh/k}$$

This formula gives the density at a height in terms of that at the surface of the earth.

Aliter. *Calculus method.*



Let p be the pressure at a height z, $p+\delta p$ that at a height $z+\delta z$, and ρ the density at height z so that $p=k\rho$ ... (1)

Then considering the equilibrium of the element δz of the thin column, we have

$$p=p+\delta p+g.\rho.\delta z$$

Hence the limit

$$\frac{dp}{dz}=-g\rho \qquad \text{... (2)}$$

Hence differentiating (1) with respect to z and combining the result with (2)

we get

$$\frac{d\rho}{dz}=-\frac{g}{k}\rho$$

$$\therefore \quad \frac{d\rho}{\rho}=-\frac{g}{k}dz$$

$$\therefore \quad \log\rho=-\frac{g}{k}z+c$$

But when $z=0$, $\rho=\rho_1$

$\therefore$ $c=\log\rho_1$

Hence $$\log\frac{\rho}{\rho_1}=-\frac{g}{k}z$$

i.e., $$\rho=\rho_1 e^{-gz/k}$$

Cor. *If p and p_1 be the pressures corresponding to densities ρ and ρ_1 respectively, then we have*

$$p=p_1 \, e^{-gz/k}.$$

112. Determination of Heights by Barometer.

Let h, h_1 be the heights of the barometer at the upper and lower stations respectively. Let z be the difference of their altitudes.

Let p, p_1 be the pressures, ρ, ρ_1 the densities at the two stations.

Then if the temperature be constant throughout

$$\frac{h}{h_1}=\frac{p}{p_1}=\frac{\rho}{\rho_1}=e^{-gz/k}$$

$$\therefore \qquad \frac{-g}{k}z=\log\frac{h}{h_1}$$

$$\text{or} \qquad \mathbf{z}=\frac{\mathbf{k}}{\mathbf{g}}\log\frac{\mathbf{h}_1}{\mathbf{h}}.$$

This formula gives the height of one station above another by means of a barometric observation at each.

113. The height of the homogeneous atmosphere is the same at all places at the same temperature.

Let H be the height of the homogeneous atmosphere, ρ the density of the air and p the pressure at that point.

Then $\qquad g\rho\,\mathrm{H}=p$

$$\therefore \qquad \mathrm{H}=\frac{p}{g\rho}=\frac{k\rho}{g\rho}=\frac{k}{g}=\text{constant}.$$

114. Solved Examples.

Ex. 1. *The readings of a faulty barometer containing same air are 29·4 and 29·9 inches, the corresponding readings of a correct instrument being 29·8 and 30·4 inches respectively. Prove that the length of the tube occupied by the air is 2·9 inches, when the reading of the barometer is 29 inches and find the corresponding correct reading.*

(M.T. 1879 ; Allahabad 1919 ; Banaras 1927)

Let the total length of the tube be x.

$\therefore$ In the reading no. 1 the length occupied by the air

$$=x-29{\cdot}4$$

The actual pressure $=29{\cdot}8$

Hence the pressure due to the air column

$$=29{\cdot}8-29{\cdot}4={\cdot}4''$$

3. A barometer stands at 30 inches, the vacuum above the mercury being perfect, and the area of the cross-section of the tube is 0 2 sq. inch. If 0 2 cubic inch of ordinary air be introduced into the vacuum, the mercury is seen to fall through 3 inches. Find the length of the original vacuum.

(*Lucknow 1942*)

4. A barometer tube rises to a height of 33 inches above the external mercury surface, and contains as much air as would, at atmospheric pressure, occupy 1 inch of the tube. The true barometric height is known to be 29 6 inches ; find the height of the tube actually occupied by the air.

5. The readings of a true barometer and of a barometer which contains a small quantity of air in the upper portion of the tube are respectively 30 and 28 inches. When both barometers are placed under the receiver of an air-pump from which the air is partially exhausted, the readings are observed to be 15 and 14·6 inches respectively. Prove that the length of the tube of the faulty barometer measured from the surface of the mercury in the basin is 31·35 inches.

6. A barometer stands at 39 inches, and the space occupied by the Torricellian vacuum is then 2 inches ; if now a bubble of air which would at atmospheric pressure occupy half an inch of the tube be introduced into it, show that the surface of the mercury in the tube will be lowered 3 inches. Show also that the height of a correct barometer, when the incorrect one stands at 21 inches, is

$$x+\frac{15}{32-x} \text{ inches.}$$

7. The height of the Torricellian vacuum in a barometer is a inches, and the instrument indicates a pressure b inches, of mercury when the true reading is c inches. The faulty readings are due to an imperfect vacuum. Prove that the true reading corresponding to an apparent reading of d inches is

$$d+\frac{a(c-b)}{a+b-d}.$$

(*Nagpur 1955 ; Lucknow 1928, 47, 51 ; Allahabad 1941, 54*)

8. The barometer falls from 30 to 29 inches as a balloon rises 900 feet from the surface, prove that $\frac{k}{g}$ is approximately 26,500 ft. (*M.T., Delhi 1927*)

9. Prove that the height of the *homogeneous atmosphere* is equal to $\frac{k}{g}$, where k is given by the formula $p=k\rho$ and g is the acceleration due to gravity, being assumed to be constant.

10. Prove that $p=k\rho(1+\alpha t)$ for a gas. If the law connecting the pressure and density of air were $p=k\rho^2$ prove that, neglecting changes of temperature and gravity, the height of the atmosphere would be twice the height of the homogeneous atmosphere. (*Patna 1926*)

11. If a change from 30 inches to 27 inches in the barometric height corresponds to a rise in the altitude of 2290 ft. Prove that the altitude which corresponds to the barometric height of 24 inches is 4850 ft. Given that $\log 2=0·3010$, $\log 3=0·4771$.

(*Lucknow 1952 ; Bombay 1935 ; Allahabad 1932, 40*)

12. The density of mercury is 13·6 and that of air at 760 mm. pressure 0·001293. Prove that when the reading of the mercury barometer is 750 mm., its reading at an altitude 500 metres is about 704·7 mm., the variation of the temperature being neglected.

$\log 2=0·301$, $\log 3=0·4771$, $\log_e 10=2·3026$, $\log 70·47=18·48$.

(*Bombay 1937*)

13. The vertical distance between two points, whose pressures are p_0 and p, is h. Prove that the pressure at an intermediate point which is at a vertical distance x above the lower point of pressure p_0 is

$$p_0\left(\frac{p}{p_0}\right)^{x/h},$$

assuming the temperature to be uniform.

14. A cylindrical well of depth h and section A is maintained at constant temperature. If ρ_0 and ρ_1 are the densities of the air at the top and bottom, show that the total amount of air contained is

$$\frac{Ah(\rho_1-\rho_0)}{\log \rho_1-\log \rho_0}.$$

(*Allahabad 1953, 56* ; *Bombay 1936*)

15. A heavy gas at constant temperature is confined in a vertical cylinder of height h. If ρ_0 be the density at the base and k be the ratio of the pressure to the density of the gas, prove that the mean density is

$$\frac{k\rho_0}{gh}\left(1-e^{-gh/k}\right)$$

(*Lucknow 1955* ; *Delhi 1947*)

16. The fall in temperature in the atmosphere is proportional to the increase in height above the earth's surface ; h, h' are the readings of mercury barometers at two stations, the former of which is at a height z above the other. Prove that

$$z=\lambda\left\{1-\left(\frac{h}{h'}\right)^m\right\},$$

where λ and m are some constants.

17. A thin heavy cylinder, hollow and open at its lower end, is found when depressed from the atmosphere successively into three liquids, to remain at rest when its higher end is at respective depth h, h', h'' below the surfaces. If s, s', s'' be the sp. grs. of the fluids. Prove that

$$s(s'-s'')h+s'(s''-s)h'+s''(s-s')h''=0,$$

neglecting the wt. of the air inside the cylinder. (*M.T. 1864*)

18. If the pressure of the air at any height varied as the mth power of its density, show that the height of the atmosphere would be

$$\frac{m}{m-1}\,H$$

where H is the height of the *homogeneous atmosphere*.

19. A hollow gas-tight balloon, containing helium, weighs W lbs. When its lowest point touches the ground, it requires a force of w lbs. to prevent it from rising. Show that it can float in equilibrium at a height

$$H\log_e\left(1+\frac{w}{W}\right)$$

where H is constant. (*Allahabad 1924*)

20. The tube of a barometer rises to 34 inches above the mercury in the trough, and the mercury column is 30 inches high. Find what changes are produced in the height of the column by the following operations performed successively :—

(*i*) As much air is allowed to rise through the mercury as would, at atmospheric pressure, occupy 2 inches of the tube ;

(*ii*) **A rod of iron whose volume equals that of 5 inches of the tube is allowed to float to the top of mercury column.**

(sp. gr. of mercury 13·5, and that of iron to be 7·5).

(*M.T., I.C.S. 1937*)

21. A balloon carrying a self-registering barometer records pressures equivalent to h and h_1 inches of mercury when it ascends to heights equal to fractions α and α_1 of the earth's radius a respectively. Taking into account the variation of gravity with height and assuming that the temperature of the air is constant at all height, prove that

$$\frac{\alpha g_0}{k} \cdot \frac{\alpha_1-\alpha}{(1+\alpha)(1+\alpha_1)} = \log\frac{h}{h_1} + 2\log\frac{1+\alpha_1}{1+\alpha}$$

where $k=\frac{r_0}{\rho_0}$ and r_0, ρ_0, g_0 are the values of the pressure, density and gravity at the earth's surface. (*I.A.S. 1959*)

22. A hollow sphere containing hydrogen requires a force mg to prevent it from rising when the lowest point touches the ground ; the total mass of sphere and hydrogen is M. Show that the sphere can float in equilibrium with its lowest point at a height h above the ground, where

$$h=\frac{r_0}{g\rho_0}\log\frac{M+m}{M}$$

ρ_0, r_0 being the density and pressure at the surface of the earth.

(*Patna 1954 ; M.T., Indian Finance 1927*)

CHAPTER IX

STABILITY OF FLOATING BODIES

115. Stable, Unstable and Neutral Equilibrium. Suppose a floating body be slightly displaced from its position of equilibrium and then left to move under the influence of the forces acting on it. If the forces tend to bring the body back into its original position, the equilibrium is said to be **stable**, if they tend to send the body still further from its original position the equilibrium is said to be **unstable** ; and if the body remains at rest in the displaced position, the equilibrium is **neutral**.

116. Displacements. Since the body may be stable, for one displacement and unstable for another, we must define the stable, unstable and neutral equilibriums for the different types of displacements.

(*a*) **Vertical displacement**. Suppose the body is to undergo a small vertical displacement say downwards from the equilibrium position without any angular motion. In this case the excess fluid displaced will exert an additional upward pressure which will be equal to the weight of the excess fluid displaced and naturally the acting forces will try to restore the body to its original position, or, in other words the body will remain in stable equilibrium and a slight displacement will make it swing about this original position.

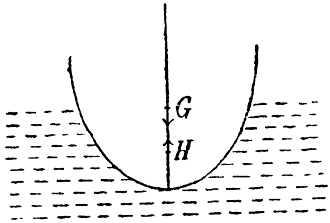
Similarly, if the body be raised up, it would displace less liquid and its weight being now greater than the fluid pressure would urge the body downwards *i.e.*, towards the original position. Hence the equilibrium is *stable.*

(*b*) **Horizontal displacement**. Suppose the body is to undergo a small horizontal displacement without any angular motion. In this case, since there is no resultant horizontal force on the body, it would have no tendency to move horizontally from any position in which it is placed. For such displacements, therefore, the equilibrium is *neutral.*

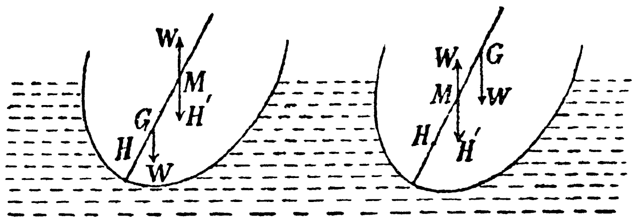
(*c*) **Rotational or Angular displacement.** Lastly remains for consideration the case of small rotational or angular displacement in which the mass of the displaced liquid remains unaltered while the shape changes. This case we shall consider at length in what follows.

117. Meta-centre. Let W be the weight of the floating body, G its centre of gravity and H its centre of buoyancy in the position of equilibrium ; then GH is vertical in that position.

Let the body be slightly displaced so that H′ is the new centre of buoyancy since the mass of liquid displaced remains unaltered the force of buoyancy remains equal to W but in the displaced position of the body it acts upward through H′. If the displacement takes



place in a plane of symmetry of the body, the vertical through H′ will intersect the line GH, which was vertical before the displacement, in the point M. Then the point M is called the **Meta-centre** and GM is called the **meta-centric height** of the floating body.



The forces acting on the body in the displaced position are (*i*) its weight, W, acting vertically downwards through G and (*ii*) the force of buoyancy, W, acting vertically upwards in the line H′M. These forces constitute a couple. This couple is generally known as the **Restoring couple**.

(1) **If the meta-centre M is above G**. In this case, as in the left-hand figure, the restoring couple tends to bring GH to the vertical *i.e.*, tends to bring the body back to its original position. *The equilibrium is, therefore, stable.*

(2) **If the meta-centre is below G**. In this case, as in the right-hand figure, the restoring couple causes the body to recede further from its original position. *The equilibrium is, therefore, unstable.*

(3) **If M coincides with G**. In this case GH′ is vertical and so the condition of equilibrium is satisfied. The body remains at rest in the displaced position. *The equilibrium is, therefore, unstable.*

Hence we may conclude from above that *a floating body is in* **stable** *or* **unstable** *equilibrium according as the meta-centre is above or below the centre of gravity of the body.*

If the meta-centre coincides with the centre of gravity the equilibrium is **neutral.**

In other words, **the equilibrium is stable, unstable or neutral according as HM>, <, or=HG.**

118. DEFINITIONS.

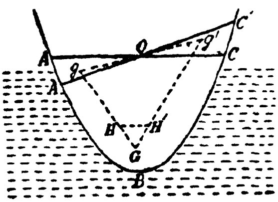
Surface of Buoyancy. *If a body floating in a homogeneous liquid be supposed to take in turn every possible position for which the volume displaced remains constant, the locus of the centre of buoyancy is termed the* **Surface of Buoyancy.**

Plane of Floatation. *If a body be floating partially immersed in a liquid, the section of the body made by the plane of the surface of the liquid is called the corresponding* **Plane of Floatation.**

Surface of Floatation. *The surface enveloped by the planes of floatation is called the* **Surface of Floatation.**

119. Theorem. *The tangent-plane at any point of surface of buoyancy is parallel to the corresponding plane of floatation.*

Let H be the centre of buoyancy corresponding to the plane of floatation AOC. Let H′ be the centre of buoyancy corresponding to a consecutive plane of floatation A′OC′.



Since the volume ABC=the volume A′BC′, the volume of the wedge AOA′ =that of COC′. ... (1)

Let g, g', G be the centres of gravity of AOA′, COC′, A′BC respectively. Join Gg, Gg'.

Now H, the C.G. of ABC must lie in g H, and divide it so that

$$\frac{gH}{HG}=\frac{\text{vol. A}'\text{BC}}{\text{vol. AOA}'}. \quad \ldots (2)$$

Similarly, H′ is in Gg', so that

$$\frac{GH'}{Hg'}=\frac{\text{vol. COC}'}{\text{vol. A}'\text{BC}} \quad \ldots (3)$$

∴ From (1), (2) and (3)

$$\frac{gH}{HG}=\frac{H'g'}{H'G}$$

Hence HH′ is parallel to gg' *i.e.*, to AC, ultimately when the displacement is very small.

In a similar way we can show that the line joining any point on the surface of buoyancy near H with H is parallel to the plane AB.

Hence the tangent plane at H to the surface of buoyancy is parallel to the corresponding plane of floatation, AB.

120. Theorem. *The positions of equilibrium of a floating solid are obtained by drawing normals from the centre of gravity of the solid to the surface of buoyancy.*

When the solid is in equilibrium the vertical through the corresponding centre of buoyancy passes through the centre of gravity. But this vertical line is perpendicular to the corresponding plane of floatation which is horizontal, and, therefore, by the last theorem is normal to the surface of buoyancy. Hence the normal to the surface of buoyancy at every centre of buoyancy corresponding to a position of equilibrium passes through the centre of gravity of the solid.

Note 1. Since we know that for certain rotations the lines through adjacent centres of buoyancy perpendicular to the corresponding planes of floatation meet, the point of intersection being the metacentre, a *meta-centre,* therefore, is a *point where two consecutive normals to the surface of buoyancy meet.* It is in fact **a centre of curvature of the surface of buoyancy**.

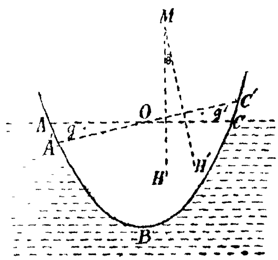
Note 2. If the displacements are confined to rotations about axes perpendicular to the plane of symmetry in the solids like a cylinder, the centres of buoyancy will lie on **a curve** in the plane of symmetry. This curve may be called the **Curve of Buoyancy.**

Note 3. From the last Theorem *the locus of the meta-centre for these displacements will be the evolute of the curve of Buoyancy.*

121. Determination of the Meta-centre.

Let the plane of the paper be the plane of symmetry.

Let AOC be a plane of floatation, cutting off the volume V. Let H be the corresponding centre of buoyancy. Let H′ be the new centre of buoyancy when A′OC′ is the new plane of floatation inclined at a small angle θ with AOC.



Since vol. ABC = vol. A′BC′,

$\therefore$ vol. AOA′ = vol. COC′.

Draw HM $\perp$ AOC and H′M $\perp$ A′OC′, then the limiting position of M is the metacentre. Let g, g' be the centres of gravity of AOA′ and COC′.

vol. A′BC′ = vol. ABC + vol. COC′ − vol. AOA′, and H′ is the centre of gravity of A′BC′.

$\therefore$ Taking the moments about the line H′M

$$V.HM \sin \theta - \text{vol. } AOA' \, gg' = 0$$

$$\therefore \qquad HM = \lim_{\theta \to 0} \frac{\text{vol. } AOA' . gg'}{V \sin \theta}$$

Analytical Expression. Let the line AOC in the plane of the paper be as the axis of y, the perpendicular to the plane of the paper through O that of x. Let $dxdy$ be the area of an elementary portion of the area AOC at a point (x, y).

Now the height of the small column on this portion, cut off by A′OC′ is $\pm\, y \tan \theta$.

$\therefore$ Volume of that element $= \pm y dx dy \tan\theta$

Hence the above formula becomes

V.AM $\sin\theta - \tan\theta . \int\int y^2 dx dy = 0$, where the integration is extended over the area AOB.

$$\therefore \qquad \textbf{HM} = \frac{\int\int y^2 dx dy}{V}, \text{ as } \theta \to 0$$

$$= \frac{\textbf{AK}^2}{V},$$

where AK^2 is the moment of inertia of the area AOC $(=A)$ about the line through O perpendicular to the plane of the paper ; (or in other words K is the radius of gyration of the area A of the section in the plane of floatation, and V is the volume immersed).

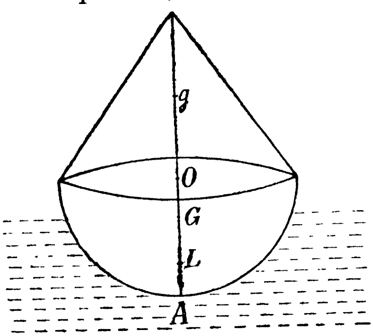
Particular Cases. If a solid which is in contact with a liquid be spherical, it is clear that the centre of the sphere is the *Meta-centre* because the pressure at each point of the spherical surface is normal to the surface and so passes through the centre.

If the immersed surface be the curved surface of a circular cylinder whose axis is horizontal, the *Meta-centre* will be on the axis of the cylinder, for the thrust on each point of the surface meets the axis.

122. Solved Examples.

Ex. 1. *A solid body consists of a right cone joined to hemisphere on the same base and floats with the spherical portion partly immersed ; prove that the greatest of the cone consistent with stability is $\sqrt{3}$ times the radius of the base.* (*Nagpur 1957 ; Banaras 1943*)

Let a be the radius of the hemisphere whose centre is O and lowest point A, and let h be the height of the cone.



If ρ be the density of the material, the wt. of the hemisphere is $\frac{2}{3}\pi a^3 g\rho$ acting through L where $OL = \frac{3a}{8}$.

Wt. of the cone is $\frac{1}{3}\pi a^2 h g\rho$ acting through g where $Og = \frac{h}{4}$.

$\therefore$ The height of the centre of gravity G of the whole body above the lowest point A is

$$AG = \frac{\frac{2}{3}\pi a^3 g\rho \left(a - \frac{3a}{8}\right) + \frac{1}{3}\pi a^2 h g\rho \left(a + \frac{h}{4}\right)}{\frac{2}{3}\pi a^3 g\rho + \frac{1}{3}\pi a^2 h g\rho}$$

$$=\frac{\frac{5a^2}{4}+ha+\frac{1}{4}h^2}{2a+h}.$$

In this case the *meta-centre* is at the centre O of the spherical part. Hence for stability, we must have

$$AG \leqslant AO$$

or $$\tfrac{5}{4}a^2+ha+\frac{h^2}{4}\leqslant a\,(2a+h)$$

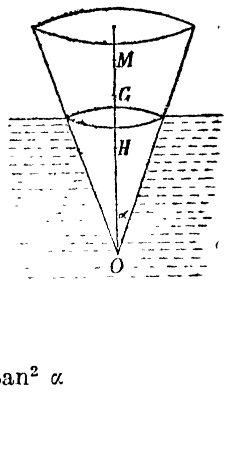
or $$h^2 \leqslant 3a^2$$

or $$h \leqslant \sqrt{3}a$$

$\therefore$ The greatest height of the cone is $\sqrt{3}$ times the radius of the base.

Ex. 2. *A solid cone, of semi-vertical angle α sp. gr. σ floats in equilibrium in the liquid of sp. gr. ρ, with its axis vertical and vertex downwards. Determine the condition for which the equilibrium is stable* (*Nagpur 1941 ; Aligarh 1942*).

Let h be the height of the cone and h' the length of axis immersed.



In this case A = area of the circle of radius $h' \tan\alpha$

$\therefore$ $$Ak^2=\pi h'^2 \tan^2\alpha.\tfrac{1}{4}h'^2\tan^2\alpha$$
$$=\tfrac{1}{4}\pi\, h'^4 \tan^4\alpha$$

and $$V=\tfrac{1}{3}\pi\, h'^3 \tan^2\alpha$$

$\therefore$ $$HM=\frac{Ak^2}{V}=\frac{\frac{1}{4}\pi\, h'^4\tan^4\alpha}{\frac{1}{3}\pi\, h'^3\tan^2\alpha}$$
$$=\tfrac{3}{4}\,h'\tan^2\alpha.$$

Now $$OH=\tfrac{3}{4}\,h'.$$

$\therefore$ $$OM=OH+HM=\tfrac{3}{4}\,h'+\tfrac{3}{4}\,h'\tan^2\alpha$$
$$=\tfrac{3}{4}\,h'\sec^2\alpha.$$

Hence for stability, we must have

$$OM>OG, \text{ where G is the C.G. of the cone.}$$

or $$\tfrac{3}{4}\,h'\sec^2\alpha > \tfrac{3}{4}\,h.$$

But since the cone floats, $\frac{1}{3}\,\rho h'^3\tan^2\alpha=\frac{1}{3}\,\sigma h^3\tan^2\alpha$

or $$\frac{h'}{h}=\left(\frac{\sigma}{\rho}\right)$$

$\therefore$ The equilibrium is stable if

$$\frac{h'}{h} > \cos^2\alpha \text{ or } \frac{\sigma}{\rho} > \cos^6\alpha.$$

Examples 22

1. A wooden ball is floating in water; show that its equilibrium will become unstable if any weight, however small, be placed upon its highest point.

2. A solid cylinder floats in a liquid with its axis vertical. If the ratio of the sp. gr. of the cylinder to that of the fluid be σ, prove that the equilibrium is stable if the ratio of the radius of the base to the height be greater than

$$\sqrt{2\sigma(1-\sigma)}.$$

3. A uniform circular cylinder, whose radius is two-thirds of its height, floats in water with its axis vertical. Provo that the equilibrium cannot be stable if the sp. gr. of the cylinder lies between $\frac{1}{3}$ and $\frac{2}{3}$.

4. Show that a uniform circular cylinder, of sp. gr. $\frac{1}{3}$, cannot be in stable equilibrium, when floating upright in water, if its length exceeds three-quarter of its diameter.

5. A right circular cylinder of radius a and length h floats with its axis vertical in a liquid whose density is $\frac{4}{3}$ that of the cylinder. Prove that the height of the metacentre above the base is

$$\frac{3}{8}h+\frac{1}{3}\frac{a^2}{h}.$$

6. Show that in the case of a right circular cylinder of radius a and height h, floating with its axis vertical in any liquid the equilibrium will be stable whatever be the sp. gr, if $\sqrt{2}\,a>h$. (*Nagpur 1942, 43*)

7. A solid and homogeneous body consists of a cylinder joined to a hemisphere on the same base and floats with the hemispherical portion partly immersed in water. Prove that greatest height of the cylinder consistent with stability is $\frac{1}{\sqrt{2}}$ times the radius of the base of the cylinder. (*Nagpur 1939*)

8. A right prism whose base is an equilateral triangle, floats in water with the lateral edges horizontal and one of them below the surface. Show that the equilibrium is stable for all displacements into which the lateral edges remain horizontal, if it be given that the sp. gr. of the prism $>\frac{9}{16}$.

(*Banaras 1940*)

9. A hollow buoy is made of a hemisphere and a cone joined at their bases, the thickness of the metal being the same throughout. Show that it can float in stable equilibrium with the cone uppermost if the semi-vertical angle of the cone be 45° but not if it be 30°.

10. A uniform right circular cone of semi-vertical angle 30° is floating with its axis vertical and its vertex downwards in a liquid whose density is $\frac{3}{8}$ its own. Show that the equilibrium is stable. (*Nagpur 1932*)

11. Show that when a uniform hemisphere of density ρ and radius a floats with its plane base immersed in homogeneous liquid of density σ, the equilibrium is stable and the meta-centric height is

$$\frac{3}{8}a\frac{(\sigma-\rho)}{\rho}.$$ (*M.T.*)

12. A homogeneous circular cylinder of length h, radius a, and sp. gr. floats in water. Prove that the position with the axis vertical is stable if

$$\frac{a^2}{h^2}>2\rho(1-\rho).$$

Also prove that the position with the axis horizontal is stable if $h>b$, where b is the breadth of the rectangular water section.

If $\rho=\frac{1}{2}$, and $8a^2=3h^2$, also show that the position of stable equilibrium is one in which one end of the cylinder is just immersed, and the other is just out of the water.

13. A thin uniform lamina whose shape is that of an isosceles triangle, floats in water with its plane vertical and the base horizontal and above the surface. Show that the equilibrium is stable, if

$$\sigma > \cos^4 \alpha,$$

where σ is the sp. gr. of the lamina and 2α the vertical angle.

14. A right circular cylinder floats in a liquid with its axis (of length h) horizontal and at a height c above the surface. Show that the equilibrium will be stable if

$$h^2 > 4(a^2 - c^2)$$

where a is the radius of the base of the cylinder.

15. A solid in the shape of two equal cones placed with their vertices coincident and axes in the same straight line, floats in a liquid of double its density, the common axis being horizontal. Prove that the equilibrium is stable if the semi-vertical angle of each cone is less than 60°.

(*Indian Finance 1927*)

16. A square lamina of density ρ floats in water, of density σ, with its plane vertical and one angular point below the surface. If $9\sigma > 32\rho$, prove that there are three positions of equilibrium in two of which neither diagonal is vertical.

17. A hemispherical bowl rests on the top of a sphere of double its radius, and water is slowly poured in the bowl; prove that the equilibrium will be stable until a weight of water, equal to half the weight of the bowl, has been poured in.

18. A thin uniform lamina whose shape is that of an isosceles triangle floats in water with its plane vertical and the base horizontal and below the surface. If 2α be the vertical angle of the triangle and ρ its sp. gr. prove that the equilibrium is stable if $\rho < 1 - \cos^4 \alpha$.

19. Prove that if the ratio of the length to the diameter of a uniform solid circular cylinder of sp. gr. $\frac{1}{2}$ lies between $\frac{1}{2}\sqrt{2}$ and 1, the cylinder cannot float in a stable equilibrium in water with its axis either horizontal or vertical.

20. A thin uniform rod of weight w and sp. gr. s floats in a vertical position in stable equilibrium, when a particle of weight W is attached to its lower end, prove that

$$W > w\left(\frac{1}{\sqrt{s}} - 1\right).$$

21. If a segment of a sphere of density σ floats in a liquid of density, ρ, prove that the position in which the plane force is immersed and horizontal is a position of stable equilibrium and that the metacentric height is $\frac{\rho - \sigma}{\sigma}$ times the distance of the centre of gravity of the segment from the centre of the sphere.

22. A solid paraboloid of sp. gr. σ floats in a liquid of sp. gr. ρ with its axis vertical and vertex downwards. Prove that the metacentric height is constant, and the equilibrium will be stable if

$$3a > \left(1 - \sqrt{\frac{\sigma}{\rho}}\right)h$$

where $4a$ is the latus rectum of the generating parabola and h is the height of the solid.

23. Prove that a circular cylinder of radius a and length a/n cannot float upright in water in stable equilibrium if its sp. gr. lies between

$$\frac{1}{2}\left[1 - \sqrt{1 - 2n^2}\right] \text{ and } \frac{1}{2}\left[1 + \sqrt{1 - 2n^2}\right].$$ (*Nagpur 1955*)

24. Two equal uniform rods, whose density is ρ are joined together at an angle 2α. If they be immersed with angle downwards in a fluid of density σ find the position of equilibrium and prove that the curve of buoyancy is a parabola. Also show that the rods cannot float with the line joining their ends inclined to the horizon unless

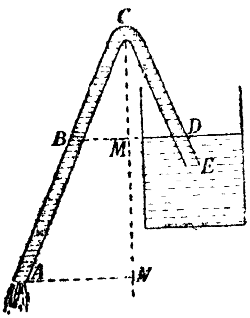
$$\frac{\sigma - \rho}{\sigma + \rho} > \sin^2 \alpha.$$ (*I.A.S. 1959*)

CHAPTER X

HYDROSTATIC MACHINES

123. There is a large number of machines which depend for their working on the properties of fluids, which have been discussed in the previous chapters. Here we shall describe some of them.

The Siphon. *The* **siphon** *is an instrument by means of which a vessel, full of liquid, can be emptied without moving the vessel.*



It consists of a bent tube ABCDE usually of glass, open at both ends with one arm ABC longer than the other CDE. To use the siphon, it is first filled with the same kind of liquid as is contained in the vessel, and both the ends E and A are temporarily closed. The tube is then inverted and placed with the end E under the level of the liquid in the vessel while the longer arm ABC projects outside with its end A below the level of the liquid in the vessel. Now as soon as the ends are opened, the liquid begins to flow in a continuous stream so long as the end E is below the surface of the liquid.

Principle of Action. Let CMN be a vertical line through the highest point and let it meet the level of the surface of the liquid in M and a horizontal line through A in N.

Let B be the point in which the horizontal plane through D meets the longer arm. Let the atmospheric pressure be π and ρ the density of the liquid.

Now just before the commencement of the motion, we have

The pressure at B = the pressure at D

$$=\pi$$

The pressure at A = the pressure at $B+g\rho.MN$

$$=\pi+g\rho.MN.$$

$\therefore$ The pressure at A is greater than the atmospheric pressure *i.e.*, π.

Hence the liquid at A will flow out and the liquid in the arm CA will follow.

It is clear that the siphon will not work if the height of C above D exceeds the barometric height for the liquid concerned.

LIQUID-PUMPS

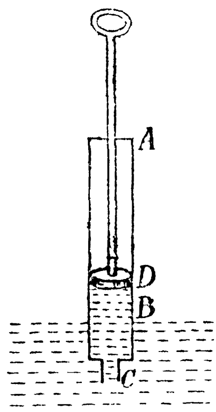
124. **Principle of Pumps.** The principle of all pumps is that of suction. A partial vacuum is created and the atmospheric pressure on the surface outside forces the liquid in to fill up the partial vacuum. A familiar instance of this is when liquid is sucked into the mouth through a straw.

Valves are used in all pumps. They are placed at the openings or in the connecting pipes. They open or close with change of pressure and allow the passage of air or water *in one direction only* but prevent any fluid to pass through in the opposite direction. Valves differ in forms according to the purpose for which they are meant. They are generally made of metal, leather or oiled silk. The difference of pressure at the two sides of a valve causes it to open or close.

Theoretically a valve should open with the smallest excess of pressure on one side and permit no leakage the way, but practically in all valves a definite excess of pressure is required, and a certain amount of leakage occurs.

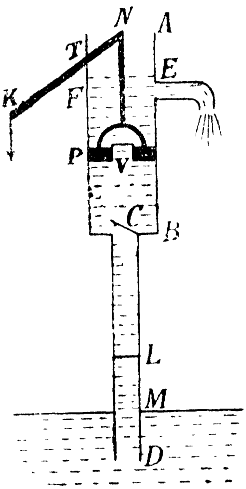
135. **The Syringe.** An ordinary *syringe* is an example of the pump in its simplest form.

It consists of a hollow cylinder AB, ending in a nozzle C. An air-tight piston D works within this cylinder. With the piston at its lowest position, the nozzle is dipped under a liquid. When the piston is drawn up, a partial vacuum is formed within the cylinder below the piston. In consequence of this the atmospheric pressure forces the liquid up into the cylinder to fill the vacuum. When the piston is again pushed down, the pressure on the liquid becomes greater than the outsider pressure and so the liquid is forced out.



126. **The Common or Suction Pump.** It consists of cylinder AB called the *barrel* within which an air-tight piston P works smoothly. The barrel is connected at its lower side with another longer cylinder BD which communicates with the liquid to be raised. There are two *valves*, both opening upwards, one of which, V, is at an opening in the piston, while the other C is at the junction of the two cylinders. Near the top of the barrel there is a *spout* E through which the liquid flows top. In the case of a hand pump, the piston is worked by means of a handle KN which is attached to the piston-rod working as a lever with T as fulcrum and K the end at which the force is applied.

(*a*) **Action of the Pump.** Suppose the piston to be at its lowett position B and that the water has not risen inside the lower cylinder. Let the piston P be moved upwards. As it moves a partial vacuum is created between it and B. The air in BD being at a greater pressure forces the valve C upwards, and some air passes from the lower cylinder to the upper. The pressure of the air inside BD and PB becomes less than the atmospheric pressure and so the liquid rises in the lower cylinder. This rising is continued till the piston reaches the topmost position. When the piston is pushed down the air in PB, is compressed and in consequence the increased pressure closes the valve C and opens V in the piston so that the air escapes through the piston valve. If some liquid had been raised within the barrel during the upstroke, this water finds its way above the piston through the valve V.



During the next *up stroke*, the valve V remains closed due to atmospheric pressure and any liquid above it, while due to the vacuum produced between PB, the valve C opens and the liquid is forced in the barrel. The liquid above the piston flows out through the spout E. It may often be necessary to work the piston *a few times* to drive off air from inside the pump and hence to get the *first* supply of the liquid. In all this action the height of ME must be less than that of the water barometer.

(b) Tension of the Piston rod.

The tension of the piston rod is equal to the weight of a column of water, whose sectional area is equal to that of piston, and whose height is equal to the distance between the levels of the water within and without the pump.

Let A be the area of the piston, h the height of the water barometer, and w the wt. of a unit volume of water. Let T denote the tension of the piston rod.

The tension of the piston rod must be such that it would balance the difference of pressures on the upper and lower surfaces of the piston.

Case I. *Let the water level be at L, a point between B and M.*

The pressure of the air above L

$=$the pressure of the water at L

$=$the pressure at M$-w$.ML

$=w\ (h-\text{ML})$

Therefore the pressure on the lower surface of the piston

$=\text{A}\times w\ (h-\text{ML})$,

and that on the upper one

$=\text{A}\times wh.$

Hence, we have

$$T + A \times w\,(h - ML) = Awh$$

$$\therefore \qquad T = A \times w.ML.$$

Case II. *Let the water have risen to a point F.*

The pressure at a point on the upper surface of the piston

$$= w.PF + w.h = w(h + PF)$$

The pressure at a point on the lower one

$$= wh - w.MP = w\,(h - MP)$$

Hence, we have

$$T + A.w\,(h - MP) = A.w(h + PF)$$

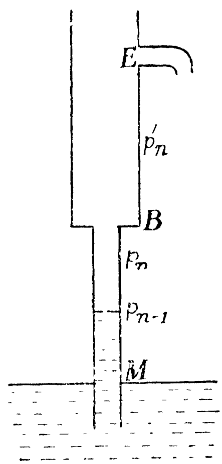
$$\therefore \qquad T = A\,w\,MF.$$

Hence the theorem is established.

(c) To find the water level raised during the nth stroke.

Let P_{n-1}, P_n be the water surfaces at the beginning and end of the nth stroke.

Let $MP_{n-1} = x_{n-1}$, $MP_n = x_n$ where M is the water-level.



Let $MB = c$, $BE = d$ and let h be the height of the water barometer.

Let A be the cross-section of the barrel BE, and a that of the pipe MB.

Case I. *When P_n is below B.*

At the beginning of the nth stroke when the piston is at B, the volume of air in the tube MP was $(c - x_{n-1})a$, and its pressure was

$$w(h - x_{n-1}).$$

At the end of the nth stroke when the piston was at L, this air occupied the volume $= (C - x_n)\,a + dA$ and its pressure $= w(h - x_n)$.

Hence from Boyle's Law, we have, after cancelling the common factor w

$$\{(c - x_n)\,a + dA\}\,(h - x_n) = \{(c - x_{n-1})\,a\}\,(h - x_{n-1}) \qquad \ldots (1)$$

From this equation x_n would be obtained if x_{n-1} is known and hence the value of $x_n - x_{n-1}$ *i.e.*, the *rise* in the water-level in the nth stroke.

Case II. *When P_n is above B.*

At the end of the nth stroke, in this case, the volume occupied by the air is $(c + d - x_n)A$ and its pressure is $(h - x_n)$.

Hence we shall have, as in formula (1)

$$(c + d - x_n)A.(h - x_n) = \{(c - x_{n-1})a\}(h - x_{n-1}) \qquad \ldots (2)$$

which gives the value of x_n in terms of x_{n-1}.

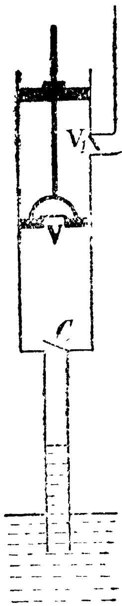
Cor. Giving n in succession the values 1, 2, 3......in (1) or in (2) and noting that $x_0 = 0$ we have the heights to which the surface of the water has risen at the end of the 1st, 2nd, 3rd,......strokes.

Therefore from (1), $\{(c - x_1)\,a + dA\}\,(h - x_1) = cah$

$\{(c - x_2)\,a + dA\}\,(h - x_2) = (c - x_1)a(h - x_1)$

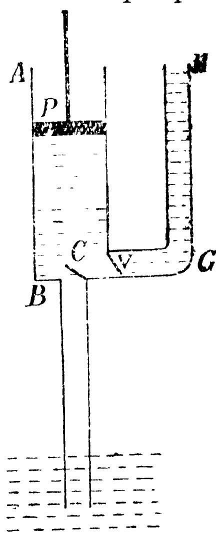
$\{(c - x_3)\,a + dA\}\,(h - x_3) = (c - x_2)a(h - x_2)$, etc.

127. The Lifting Pump. This pump is used to raise water to any desired height. It is only a modification of the common pump having a third valve V_1 opening outwards at the junction of the spout and the barrel. Thus the water instead of being discharged at the spout, rises up through the pipe connected with the spout turned upwards.



As the piston rises, third valve V_1 opens and the water enters the spout and passes into the pipe above. When the piston descends this valve closes and does not allow the water of upper pipe to come back in the barrel. This process is repeated and more and more water is forced up the tube through the valve V_1. In this way water can be lifted to any height, provided the different parts of the pump are strong enough to bear the strain caused by the weight of water above.

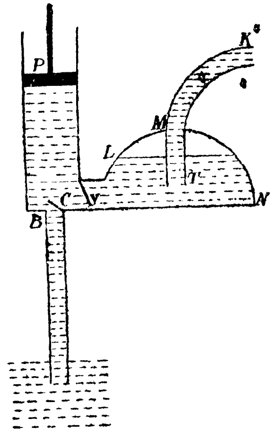
128. The Forcing Pump. It is another modified form of the common pump with a strong barrel AB. In this pump there is no valve on the piston, which has a solid base fitting the cylinder tightly. The spout is fixed at the bottom B and consists of a long pipe as in the lifting pump, having a valve V opening outward at the junction of the barrel and the pipe. When the water has entered the barrel PB as in other pumps, the downward stroke of the piston drives the air or water between itself and B through the valve V into the pipe GH, the valve C remaining closed during the action. During the *upward stroke*, C opens and V closed and water rises in BA.



Thus the process is repeated and the water passes in the pipe GH.

The Forcing Pump with an Air-chamber. In order that the forcing pump may throw up a continuous stream of water, a pipe from V is introduced to a chamber LMN partially filled with air. From this chamber a tube TMK, whose end T is well below the air in the chamber, leads up to the height required.

Action. When the piston moves downwards the water is forced into TMK. So there is flow of water during this half-stroke and the air in the chamber is compressed.



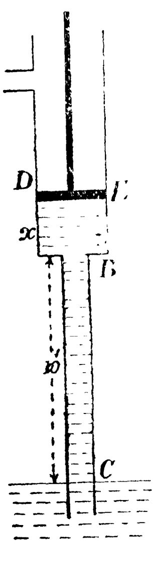
When the piston moves upwards, the valve V is closed and the compressed air of the chamber tries to expand and, therefore, forces some water from the chamber, to go up in the pipe TMK. Thus in the second half-stroke also the flow of water continues.

129. The Fire Engine. The *fire-engine* is essentially a forcing up with air-chamber described as above. It is formed of two force-pumps communicating with the same air chamber.

Both the pumps draw from a common reservoir ; when the piston in one moves down, that in the other moves up and *vice versa*. Then the air-chamber maintains the continuity of flow as in the forcing pump with an air chamber.

130. Solved Examples.

Ex. 1. *The length of the lower pipe of a common pump above the surface of the water is 10 feet, and the area of the section of the upper pipe is 4 times that of the lower. Taking 33 feet as the height of the barometer, prove that, if at the end of the first stroke the water just rise into the upper pipe, the length of the stroke must be very nearly 3 feet 7 inches.* (*Allahabad 1921*)



Initially the piston is at B, and encloses a column of air 10′ height in tube BC at atmospheric pressure.

Let the piston now rise through a height x feet, such that all the air is sucked up in the bigger barrel and the water rises in the lower column filling it up to the top B.

Let p be the pressure due to the air enclosed in the bigger barrel.

$\therefore \quad p+10=33$

$=$ pressure on the surface of the reservoir.

$\therefore \quad p=23.$

Initially the volume of air was 10A, where A is the area of the cross-section of the lower pipe, and its atmospheric pressure is 33′. After the first stroke, the volume is $x.4.$A and the pressure 23′ of water.

The temperature is assumed to be constant throughout.

$\therefore$ $10A.\ 33 = 4A.\ x.\ 23$

or $x = \frac{10.33}{4.23}$

$= 3$ ft. 7 in. nearly.

Ex. 2. *Prove that in the common pump the water will just rise into the upper cylinder at the end of the second stroke, if*

$$h^2\left(1-\frac{a}{nb}\right)\left(2-\frac{a}{nb}\right)-h\left(4a+nb-\frac{3a^2}{nb}\right)+a(2a+nb)=0$$

where a, b are the lengths of the lower and upper cylinders, n is the ratio of the sectional area of the latter to that of the former and h is the height of the water barometer. (*M.T. 1889 ; Calcutta 1912 ; Patna 1927*)

Let x_1, x_2 be the heights to which water is raised during the first and the second strokes. Here x_2 is given to be a.

Hence applying the formula of the Art. in which the water level raised during the nth stroke was found, we have

$$[(a-x_1)+nb](h-x_1)=ah \qquad \ldots (1)$$

and $$nb(h-a)=(a-x_1)(h-x_1) \qquad \ldots (2)$$

From (1) and (2), we get

$$nb(h-a)+nb(h-x_1)=ah_1$$

or $$h-x_1=\frac{ah}{nb}-(h-a)\ ;$$

$\therefore$ $$a-x_1=\frac{ah}{nb}-2(h-a).$$

Substituting these values in (2) we get

$$nb(h-a)=\left[\frac{ah}{nb}-(h-a)\right]\left[\frac{ah}{nb}-2(h-a)\right]$$

or $$h^2\left(1-\frac{a}{nb}\right)\left(2-\frac{a}{nb}\right)-h\left(4a+nb-\frac{3a^2}{nb}\right)+a(2a+n)=0.$$

Examples 23

1. A tall cylinder 90 cms. in length is full of mercury. What is the greatest depth of it which can be emptied out by means of a siphon, the barometric height being 756 mms. of mercury. (*Punjab 1924*)

2. A siphon is used to empty a cylindrical vessel filled with mercury. The shorter limb of the siphon reaches to the bottom of the vessel which is 45 inches deep, but it is found that the mercury ceases to run before the vessel is empty. Explain this (*Calcutta 1919 ; Dacca 1930*)

3. A cylindrical vessel, whose height is that of the water barometer, is three-quarters full of water and is fitted with an air-tight lid. If a siphon, whose highest point is in the surface of the lid and the end of whose longer arm is on a level with the bottom of the vessel, be inserted through an air-tight hole in the lid, prove that one-third of the water may be removed by the action of the siphon.

4. The lengths of the arms of a siphon measured vertically are h, k $(h > k)$. The siphon is filled with liquid of density σ greater than the density ρ of the liquid in the vessel. Prove that the siphon will begin to work with the end of the longer arm immersed in the liquid in the vessel provided the depth of that end below the surface of the liquid in the vessel exceeds $(h-k)\dfrac{\sigma}{\rho}$.

5. The spout of a common pump is 7 ft. above the level of water in a well. The diameter of the piston is 4 inches. What is the force on the piston rod when the pump is working? Wt. of 1 cu. ft. of water is 62·5 lbs.

(*Allahabad 1931*)

6. A lift pump is used to pump oil of sp. gr. 0·8 from a lower into an upper tank. What is the maximum of possible height of the pump above the lower tank when the pressure of the atmosphere is 76 cms. of mercury.

(*Patna 1929 ; Calcutta 1928*)

7. The length of the lower pipe of a common pump above the surface of water is 20 ft., the cross-section of the barrel of the pump is 36 times that of the pipe, and the length of the stroke is 1 ft. Find how far the water will rise at the end of the 1st stroke, taking the height of the water-barometer to be 30 ft. (*Patna 1934*)

8. The lower valve of a common pump is 27 ft. above the surface of water in a well, and the area of the cross-section of the upper cylinder is twices that of the lower. If the length of the upper cylinder be 38 ft., find how much water rises at the end of first and second strokes, the height of the water barometer being 34 ft.

9. The lower valve of a common pump is a piece of brass 8 ozs. in weight, resting over a hole 2 sq. inches in area. Find the greatest height at which the pump can be installed, if the height of the water baron eter is 34 ft. and the wt. of the 1 cubic foot of water is 1000 ozs. (*Lucknow 1939*)

10. In a common pump the length of the barrel is 18″ and that of the lower pipe 21 ft. above the surface of water ; if the section of the pipe is $\frac{3}{14}$ of that of the barrel, find the height through which water would rise at the end of the first stroke, taking the height of the water barometer to be 32 ft.

(*Patna 1931*)

11. A pump consists of a pipe 2 inches in diameter dipping into a well 12 ft. below, together with a barrel 6 inches in diameter. When the pipe is full of air at atmospheric pressure, a closely fitting piston is raised 16 inche from the bottom of the barrel. How far will the water then rise in the pipe? The height of the water barometer is 33 ft.

12. If the barrel of a common pump be 2 ft. long, and its lower end be 26 ft. above the surface, and if the area of the section of the barrel be 6 times that of the pump, find in how many strokes the water will reach the barrel, the height of the water-barometer being 32 ft.

13. In a common pump, if the piston does not go home up to the lower valve of the barrel, prove that no water can in general be lifted into the upper cylinder unless the range of the piston is greater than

$$\frac{ab}{h}$$

where a, b are the lengths of the lower and upper cylinders and h is the height of the water barometer.

14. Prove that if h, h' be the heights at which the water stands in the lower cylinder of a common pump before and after a stroke, then

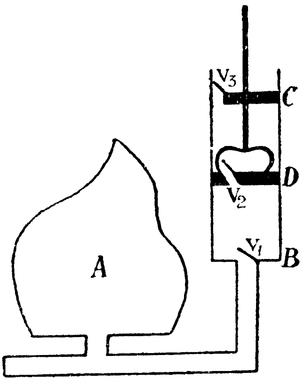
$$(h'-h)(h'+h-\mathrm{H}-a)+nb(\mathrm{H}-h')=0,$$

where a, b are the lengths of the lower and upper cylinders, n is the ratio of the sectional area of the latter to that of the former, and H is the height of the water-barometer.

AIR-PUMPS

131. Air-Pumps. The air-pumps are used to pump the air out of a vessel in which a vacuum is desired.

132. Smeaton's Air-Pump. The most common form of vacuum pump is known as *Smeaton's Pump.* In this type of pump there is a cylinder CB with valves V_3, V_1 at C and B opening upwards. A piston D works in CB and is lifted at the bottom with a valve opening upwards. Valves are all air-tight.



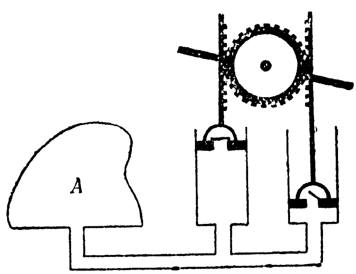
From the bottom of the cylinder CB a pipe leads into the vessel A which is to be exhausted of air. Suppose the piston D resting at B is raised ; a vacuum will be created between D and B and air from A rushes through the pipe into the cylinder after opening the valve at B. When the piston is at C and is pressed down again the air in D escapes through the valve in the piston since the valve at B is closed and from the piston this first instalment of air escapes into the atmosphere. This is the working of the first stroke and as these strokes are repeated more and more air from A escapes into the atmosphere though at a diminishing pressure.

The process is continued till the pressure in the receiver is too weak to raise the valves.

Sometimes another valve, opening upwards is introduced at C. Its advantage is that during the downward stroke of the piston the pressure of the air above it becomes much less than the atmospheric pressure and hence the piston valve is more easily varied than would otherwise be the case.

133. Hawkskee's or Double-barrelled Air-Pump.

To make the action of the pump continuous, a double-barrelled or Hawkskee's Air-Pump is used. This machine consists of two cylinders each similar to the single cylinder in Smeaton's Pump and each furnished with a piston. The pistons are worked by means of a toothed-wheel in such a manner that as one piston goes up the other goes down. The barrels are connected to a common tube leading to the receiver A. Thus the air is taken out continuously and consequently the rate of rarefaction becomes twice as rapid as a pump with a single barrel.



To Calculate the Density of the Air left in the Receiver after the nth stroke.

Let V be the volume of the receiver including that of the pipe, and v that of the barrel between extreme positions of the piston. Let ρ be the density of air *initially* in the receiver ; and let ρ_1, ρ_2, $\rho_3 \ldots \rho_n$ be the density of the air in the receiver, after 1, 2,......n complete strokes.

At the end of each up-stroke, a volume V of air expands to occupy a volume $(V+v)$ and thus is reduced in density to ρ_1.

$$\therefore \quad (V+v)\rho_1 = V\rho \quad \text{or} \quad \rho_1 = \frac{V}{V+v}\rho.$$

At the end of the second up-stroke, the air of density ρ_1 in the receiver expands to a volume $(V+v)$ and thereby becomes reduced in density to ρ_2. Hence

$$\rho_2 = \frac{V}{V+v}\rho_1 = \left(\frac{V}{V+v}\right)^2 \rho.$$

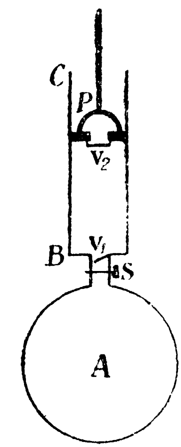
Proceeding in this way the density of air left in the receiver after the nth stroke is given by

$$\rho_n = \left(\frac{V}{V+v}\right)^n \rho.$$

Note. Since ρ_n can never be zero, a perfect vacuum can never be obtained.

134. The Air-Condenser or Condensing Air-Pump. The object of this type of pump is exactly opposite to that of the Air-Pump, *viz.*, to increase the pressure of the air in a given vessel or receiver. This is the common bicycle-pump or inflator.

It consists of a cylinder AB into which works a piston P. The cylinder is open at the top. At the end B of the cylinder is a valve V_1 which opens outwards. In the piston there is also another valve V_2 opening downwards. The pump is connected to the receiver A into which the air is to be pumped, by a tube below the valve V_1, which is fitted with a stop-cock S.



When the piston is pressed down, the air between B and the piston is compressed; the valve V_1 opens and the air is forced into the receiver A.

When the piston reaches B, its action is reversed and in doing so a partial vacuum is created between B and the piston. In consequence of this the valve V_1 closes and V_2 opens, and the barrel is filled with air from outside. When the piston is pushed down again, the air is driven into the receiver. Thus at every stroke the air full of the barrel is forced into the receiver.

Density of Air in the Receiver after the nth Stroke.

Let ρ be the density of the air at atmospheric pressure, ρ_n the density of the air in the receiver after the nth stroke. Let V be the volume of the receiver with that of the connecting tube and v that of the barrel.

After n strokes the total volume of air at atmospheric pressure in the receiver is $V+nv$.

$\therefore$ From Boyle's Law

$$\rho_n V=(V+nv)$$

or

$$\rho_n=\left(1+\frac{nv}{V}\right)\rho.$$

135. Solved Examples.

Ex. 1. *After 4 strokes the density of the air in the receiver of an air pump is found to bear to its original density the ratio of 256 to 625. What is the ratio of the volume of the barrel to that of the receiver?*

(Calcutta 1923)

Let the volume of the receiver and that of the barrel be V and v respectively.

Then
$$\frac{256}{625}=\left(\frac{V}{V+v}\right)^4$$

or
$$\frac{V}{V+v}=\frac{4}{5} \text{ whence } \frac{v}{V}=n.$$

Ex. 2. *The mass of air in the receiver of an air-pump is m at beginning, and it becomes m' after n complete strokes of the piston. If V and v denote the volumes of the receiver and the barrel, prove that*

$$n=\frac{\log m-\log m'}{\log(V+v)-\log V}.$$

(Allahabad 1938)

Let ρ and ρ_n be the densities of the air in the receiver in the beginning and after n strokes.

Since the volume of the receiver will remain the same, we have

$$\frac{m}{V}=\rho \text{ and } \frac{m'}{V}=\rho_n \qquad \dots (1)$$

After the nth stroke the density ρ_n is given by

$$\rho_n=\left(\frac{V}{V+v}\right)^n\rho \qquad \dots (2)$$

$\therefore$ Replacing ρ and ρ_n from (1) in (2)

$$\frac{m'}{V}=\left(\frac{V}{V+v}\right)^n\frac{m}{V}$$

or

$$\frac{m'}{m}=\left(\frac{V}{V+v}\right)^n$$

or $$\log \frac{m'}{m} = n \log \frac{V}{V+v}$$

or $$\log m' - \log m = n\{\log V - \log (V+v)\}$$

or $$n = \frac{\log m - \log m'}{\log (V+v) - \log V}.$$

Examples 24

1. If the cylinder of an air-pump is one-third the size of the receiver, what fractional part of the original air will be left after 5 strokes? (*Patna 1926, 29*)

2. The volumes of the barrel and the receiver are 25 and 75 cu. inches respectively. Find the pressure of the air after 3 strokes. (*Bombay 1935*)

3. Describe briefly the action of air-pump in its simplest form, and explain how the degree of rarefaction produced by a given number of strokes can be calculated. (*Punjab 1931*)

4. If the volume of the receiver in a Smeaton's air-pump is 5 times that of the barrel, find the pressure in the receiver after 3 strokes of the piston, the barometric height being 30 inches. (*Calcutta 1912*)

5. If the pressure in a pump were reduced to $\frac{1}{3}$ of the atmospheric pressure in 4 strokes, to what it would be reduced in 6 strokes? (*Punjab 1923*)

6. In one pump the barrel has $\frac{1}{12}$th of the volume of the receiver and in another it has $\frac{1}{6}$th. How many strokes of the latter are required to produce the same degree of exhaustion as six of the former? (*Patna 1935*)

7. If the receiver of an air-pump be 6 times as large as the barrel, find the number of strokes required before the density of the air is less than $\frac{1}{4}$th of the original density. (*Patna 1941*)

8. The barrel and the receiver of a condensing pump have capacities of 75 c.cs. and 1000 c.cs. respectively. How many strokes will be required to raise the pressure of the air in the receiver from one to four atmospheres. (*Calcutta 1925*)

9. If the volume of B of the cylinder of a condenser only C is traversed by the piston, prove that the pressure in the receiver cannot be made to exceed $\frac{B}{B-C}$ atmosphere.

10. In a condenser the piston can only work through a length, a, the distance of the piston in its lowest position from the lower valve being b. Show that the density of the air in the receiver cannot exceed $\frac{a+b}{b}$ times the atmospheric density.

11. Air is pumped from a vessel of volume V and forced into a vessel of volume C by means of a Smeaton pump of volume B. Show that if the whole of the air have originally a density ρ, and if the piston be originally at the end of the cylinder nearest to the vessel A, then after the piston has made $2n-1$ strokes, reaching the end C for the nth time, the density of the air in the two vessels will be

$$\left(\frac{A}{A+B}\right)^n \rho \text{ and } \left[A+B+C-\frac{A^n}{(A+B)^{n-1}}\right]\frac{\rho}{C}.$$

12. If h be the range of the piston in a Smeaton's air pump, a, its distance from the top of the barrel in its highest position, b its distance from the bottom in its lowest position, and ρ the density of the atmosphere, prove that

the limiting density of the air in the receiver will be

$$\frac{ab}{(h+a)(h+b)}\rho.$$

(*M.T. 1861; Calcutta 1912, 14, 15*)

13. A body weighing n lbs. in air, weighs $(n+1)$ lbs. in the receiver of an air-pump after n strokes of the piston. If the capacity of the receiver be n times that of the barrel, prove that the wt. of the body in vacuum will be

$$\frac{(n+1)^{n+1}-n^{n+1}}{(n+1)^n-n^n}.$$

(*Lucknow 1939*)

14. In Smeaton's air-pump, find the position of the piston in its $(r+1)$th ascent when the highest valve begins to open ; and show that in that position the tension of the piston rod : thrust of the atmosphere on the piston is as

$$1-\frac{A}{(A+B)^n} : 1-\frac{A}{(A+B)^n}\cdot\frac{B}{(A+B)}.$$

(*M.T. 1856*)

15 Prove that if the piston of Hawkskee's air-pump cannot traverse the whole length of the cylinder, the density in the receiver after n strokes will be

$$1-\left\{1-\left(\frac{A}{(A+B)}\right)^n\right\}\frac{C}{B}$$

of the density of the atmosphere, supposing A to denote the volume of the receiver, B that of the cylinder, and C that of the part traversed by the piston.

(*Patna 1941 ; M.T. 1883*)

16. The volume of the barrel of a condenser is V and volume v of it is below the piston when the latter is pushed down as far as it will go If the valves open when the difference of pressure between the two sides is p, and Π the atmospheric pressure, prove that the greatest pressure that can be produced in the air in the receiver is

$$(\Pi-p)\frac{V}{v}-p.$$

(*M.T. 1872*)

17. A condenser, whose clearance may be neglected, and its receiver is connected in an air-tight room. The air in the room and in the receiver is initially of density ρ. Prove that after n strokes the density of the air in the receiver becomes

$$\rho\left[1+\frac{V+u}{v}-\frac{V+u}{v}\left(\frac{V}{V+u}\right)^n\right],$$

where u is the volume of the barrel of the condenser, v that of the receiver and $V+u+v$ that of the room.

(*M.T.*)

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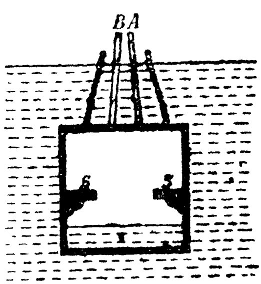
136. The Diving Bell. It is used to enable persons to go down a great depth under water to lay the foundations of piers or do some such work there. It consists of a large bell-shaped or cylindrical vessel made of metal closed at the top and open at the bottom. To its upper extremity chains are fastened and inside the bell there are seats S for the divers.



The bell is lowered into the deep water after the divers have taken their seats within it by means of the chain. As it descends it carries with it the air enclosed within, but the volume of air diminishes and water rises within it. To supply this diminution of the volume of the contained air and also to drive out polluted air, a fresh stream of air is constantly being pumped in through a tube A which opens above the surface of water and polluted air keeps flowing out through another tube B.

The diving bell is so heavy that it can sink down to a very great depth by its own weight and as soon as the divers have finished their operations below, they indicate the intention by giving a jerk to the chains.

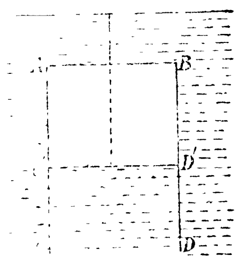
137. Problems. *A diving bell is lowered into water of given density. If no air be supplied from above, find*

*(i) the compression of the air at a given depth **a**,*

(ii) the tension of the chain at this depth, and

(iii) the amount of air at atmospheric pressure that must be forced in so that at this depth the water may not rise within the bell.

(*i*) Let b be the height of the bell ABCD. Let a be the depth of the upper side AB from the free surface. Let AC′ the length of the bell occupied by the air be x and let h be the height of the water barometer in atmospheric pressure.



Now $CC' = b - x$. Also, let A be the cross-section of the bell. Let w be the wt. of unit volume of water and

Π the atmospheric pressure

$$\therefore \quad \Pi = wh \quad \ldots \quad (1)$$

The pressure of compressed air enclosed in ABD′C′

$= $ the pressure at the water level C′D′

$= \Pi + (a+x)w$

$= (h + a + x)w.$

The original volume ABCD of air was at atmospheric pressure Π. Hence by Boyle's Law

$$x\text{A}.(h+a+x)w = b\text{A}.wh$$

or $$\mathbf{x^2 + (a+h)x - bh = o}$$

which is a quadratic equation in x having one positive and one negative root. If the positive root be x_1, the required rise of water in the bell will be $b - x_1$.

(*ii*) If T be the tension of the chains and W be the wt. of bell, we have

$$\mathbf{T} = \text{wt. of the bell} - \text{wt. of the water displaced}$$
$$= \mathbf{W - w.A.x}.$$

Note. To be more accurate, the wt. of the air contained in it may be added. But it is very small in comparison with the wt. of the diving bell.

(*iii*) Let V be the volume of the diving bell, and V′ the volume of atmospheric air that must be forced in to keep the water level at CD.

In this case the pressure of the air within the bell

$$= \text{pressure of the water at the level CD}$$
$$= w(b+a)+\pi$$
$$= w(a+b+h).$$

Since a volume $(V+V')$ of air at pressure $\pi(=wh)$ has been compressed to volume V at pressure $w(a+b+h)$, it follows from Boyle's Law that

$$(V+V')\,hw = V(a+b+h)\,w$$

or

$$V' = V\frac{a+b}{h}.$$

138. Solved Examples.

Ex. 1. *A cylindrical diving bell, whose height is 6 ft., is let down till its top is at a depth of 80 ft. Find the pressure of the contained air, the height of the water barometer being $33\frac{1}{3}$ ft.*

(*Patna 1941 ; Agra 1928*)

If a be the depth of the top of the bell below the water, b its height, h the height of the water barometer and x the length of the bell occupied by the compressed air, then from the first result of the last Article we have

$$x^2+(a+h)x-hb=0.$$

Putting the values of a, b and h in the above, we get

$$x^2+(80+33\tfrac{1}{3})x-33\tfrac{1}{3}\times 6=0$$

or

$$x=\frac{-170+10\sqrt{307}}{3}$$

Hence the pressure inside is equivalent to $x+80+33\frac{1}{3}$ ft. of water

or

$$=\frac{x+80+33\frac{1}{3}}{33\frac{1}{3}}\text{ atmospheres}$$
$$=\frac{17+\sqrt{307}}{10}\text{ atmospheres}$$
$$=3\cdot45\text{ atmospheres.}$$

Ex. 2. *A cylindrical diving bell, whose cross-section is of area A, is suspended in water with its flat top at a distance a below the surface, the air inside the bell then occupying a length b of the bell. A man of volume kA and sp. gr. σ who has been sitting on a platform inside the bell, falls into the enclosed water and floats. Find the consequent change in the level of the water inside the bell and the total amount of water inside the bell after the fall of the man.*

Find also the change in the tension of the supporting chain.

(*I.C.S.*)

The volume of the air inside the bell initially

$$=Ab-kA=A(b-k).$$

Let $b-x$ be the length of the bell occupied by the air finally.

then the volume of the air inside the bell

$=A(b-x)-$(portion of the man above water in floating)
$=A(b-x)-(Ak-A\sigma k$, since $A\sigma k$ is the volume of the water displaced by the man
$=A(b-x-k+\sigma k)$

Let p and p' be the pressures of the air initially and finally, we have by Boyle's Law

$$A(b-k)p=A(b-x-k+\sigma k)p'$$

But as the pressure inside is due to the level of water inside the bell below the effective surface,

$p=w(b+a+h)$ and
$p'=w(b-x+a+h)$, where h is the ht. of the water barometer

Hence $(b-k)(b+a+h)=(b-x-h+\sigma k)(b-x+a+h)$

$$\therefore \quad x^2-x(2b+a+h-k+\sigma k)+k\sigma(b+a+h)=0. \qquad \ldots (1)$$

which is a quadratic equation for x. Its second term is clearly negative, and its third term is positive. Hence both the roots are positive. Thus x being positive, the level of the water rises in the bell.

Let c be the total height of the bell the amount of water in it initially $=A(c-b)$.

The amount finally

$=A\{c-(b-x)\}-$amount of water displaced by the man
$=A(c-b+x)-A\sigma k$

Thus the amount of water initially $-$ amount finally

$$=A\sigma k-Ax$$
$$=-A(x-\sigma k) \qquad \ldots (2)$$

Hence from equation (1)

$$(x-\sigma k)\,(x-2b-a-h+k)=\sigma k(b-k).$$

The second factor on the left-hand side is clearly negative, and the right-hand side is positive

$\therefore \quad x-\sigma k$ is negative.

Here the right-hand side of (2) is positive *i.e.*, initially the amount of the water inside the bell is greater than the final amount.

Tension of the chain initially

$=$wt. of the bell$+$wt. of the man$-$wt. of water displaced
$=W+Ak\sigma w-Abw$

Tension of the chain finally

$=W-A(b-x)w$

$\therefore$ Initial Tension $-$ Final Tension

$=Ah\sigma-Axw$
$=Aw(\sigma k-x)$
$=$ positive as in (2).

Hence in the second case the tension of the chain is decreased.

Ex. 3. *In a diving bell a soda water bottle is opened, which in the external air would liberate a volume V of gas ; show that the tension of the rope is diminished by*

$$\frac{whV}{\sqrt{(a+h)^2+4bh}},$$

where squares of $\frac{Vh}{A}$ *are neglected, w being the wt. of a unit volume of the water, A the cross-section, a the depth and b the height of the bell, and h the height of the water barometer in atmospheric air.*

(*Banaras 1935, 44*)

With the usual notation, we have

$$x^2+x(a+h)-hb=0 \qquad \ldots (1)$$

When the bottle has been opened, let x become $x+y$. The original volume of the gas and air inside the bell would at pressure h occupy a length $b+\frac{V}{A}$ of the bell.

Hence, by Boyle's Law,

$$(x+y)(x+y+a+h)=\left(b+\frac{V}{A}\right)h. \qquad \ldots (2)$$

Subtracting (1) from (2), we have

$$y(y+2x+a+h)=\frac{Vh}{A}. \qquad \ldots (3)$$

Since squares of $\frac{Vh}{A}$ are to be neglected, it follows from (3) that the squares of y are also to be neglected.

Hence (3) becomes

$$y(2x+a+h)=\frac{Vh}{A}. \qquad \ldots (4)$$

If T and T′ be the original and final tensions respectively, we have

$$T=W-Axw \text{ and } T'=W-A(x+y)w$$

$$\therefore \quad T-T'=W-Axw-W+A(x+y)x$$

$$=A.y.w=\frac{Vwh}{2x+a+h}, \text{ from (4)}$$

$$=\frac{Vwh}{\sqrt{(a+h)^2+4hb}}, \text{ from (1)}.$$

Examples 25

1. A bottle whose volume is 500 c.cs. is sunk mouth downwards below the surface of a tank containing water. How far must it be sunk for 100 c cs. of water to run up into the bottle ? The height of the barometer at the surface of the tank is 760 mm. and the sp. gr. of mercury is 13·6. (*Patna 1928*)

2. A diving-bell is lowered in a lake until one half of it is filled with water. Prove that if d be the depth of the top of the bell below the surface, the height of the bell is $2(h-d)$ where h is the height of water barometer.

3. A cylindrical diving bell weighs 2 tons and has an internal capacity of 200 cu. ft., while the volume of the material composing it is 20 cu. fit. The bell is made to sink by attached weights. At what depth may the weights be removed and the bell just not ascend, the height of the water barometer being 33 feet ?

4. A cylindrical diving bell, whose height is 9 ft. is lowered till the level of the water in the bell is 17 ft. below the surface. The height of the water barometer being 34 ft. ; find the depth of the bottom of the bell. If the area of the section of the bell be 25 sq. ft , find how much air at atmospheric pressure must be pumped into the bell to drive out all the water.

5. The height of the water barometer being 33 feet 9 inches and the sp. gr. of mercury 13 5, find at what height a common barometer will stand in a cylindrical diving bell which is lowered till the water fills one-tenth of the bell. How far will the surface of the water in the bell be below the external surface of the water ?

6. A cylindrical diving bell is lowered in water and it is observed that the depth of the top when the water fills $\frac{1}{3}$ of the inside is $3\frac{1}{3}$ times the depth when the water fills $\frac{1}{4}$ of the inside ; prove that the height of the cylinder is $\frac{1}{3}$ of the height of the water barometer.

7. A cylindrical bell 4 ft. long whose volume is 20 cu. ft. is lowered into water until its top is 14 ft below the surface of the water, and air is forced into it until it is $\frac{3}{4}$th full. What volume would the entire quantity of air occupy under atmospheric pressure, the water barometer standing at 33 ft. ? (*Calcutta 1912*)

8. If a diving bell in the shape of an inverted cone, of height a, be lowered till the vertex is at depth d, prove that the height x of the part of the bell occupied by the air is given by

$$x^4+x^3(h+d)=a^3h,$$

where h is the height of the water barometer. (*Allahabad 1932*)

9 A diving bell is made of a substance whose sp. gr. is 4 and its interior will contain a quantity of water whose weight is twice that of the bell. If the bell be lowered in water till the tension of the rope is half the weight of the bell, prove that the density of the air within it will be eight times that of atmosphere. (*M.T. 1870, Nagpur 1939, Calcutta 1924*)

10. If h, h' be the heights of mercury in a barometer placed within a cylindrical diving bell of length b, at the beginning and the end of a descent ; H the height of the mercury barometer at the surface and ρ the sp. gr. of mercury, prove that the depth of the bell is

$$(h'-h)\left(\rho+\frac{bH}{hh'}\right).$$

(*Allahabad 1933*)

11. A circular cone, hollow, but of great weight is lowered into the sea by means of a rope attached to the vertex. If h be the height of the cone, c the depth of the vertex below the free surface, k the height of the water barometer and T, T′ the absolute temperature of the air at the surface and of the water, prove that the depth of the water surface below the vertex of the cone is given by

$$T\{x^4+x^3(k+c)\}-kh^3T'=0$$

(*Bombay 1935*)

12. A cylindrical diving bell of height $\frac{h}{4}$ is sunk into water till its lower end is at a depth nth below the surface ; if the water fill $\frac{1}{5}$th of the bell, prove that the bell contains air whose volume at atmospheric pressure would be

$$\frac{4}{5}\left(n+\frac{19}{20}\right)V,$$

where V is the volume of the bell and h is the height of the water barometer.

13. A cylindrical diving bell of height a, is lowered till its top is at a depth h below the surface of the water. If the bell be now half full of water

and air be pumped in till all the water is expelled, prove that the bell must be lowered a further distance $4H-2h$, before the bell is again half full of water, H, being the height of the water barometer. *(Nagpur 1938)*

14. A hemispherical diving bell, of radius r, is lowered in water with its base horizontal till the water rises up to the middle point of the vertical radius of the bell. Show that the depth of the base of the bell below the surface is

$$\frac{11}{5}h+\frac{r}{2},$$

h being the height of the water barometer. *(Nagpur 1942)*

15. A diving-bell having the form of a cylinder surmounted by a hemisphere, the height of the cylinder being equal to the radius of the common base, is lowered in the sea until the water occupies the whole cylinder. The height of the water barometer is 32 ft. and the sp. gr. of the sea-water is 1 025. Find the depth of the water level inside the bell below the surface of the sea.

16. A diving bell is in the form of a cylinder of length a surmounted by a cone of height h. If no air is pumped in when it is immersed, find how far it must be lowered for all the air to be forced into the conical part. Show that the volume of air at atmospheric density which must now be pumped in that the bell may be filled is

$$\left(\frac{a}{H}+\frac{3a}{h}\right)V,$$

where H is the height of the water barometer, and V is the volume of the bell. *(Nagpur 1955 ; Indian Police 1926)*

17. The weight in water of the material of a cylindrical diving bell is W, and the tension of the chain is P. A length a of the bell is occupied by air. Show that, if a weight w of water is drawn up in a bucket into the air space, and the bell is raised through a distance $\frac{wa}{W-P}$, the tension of the chain will be the same as before. *(M. T.)*

18. A diving bell, whose height is b feet, contains a mercury barometer, whose height is h inches when the bell is above the surface of the water, and h' inches when it is below ; to what height is the top of the bell submerged when its shape is

(*i*) conical, and

(*ii*) cylindrical.

19. A cylindrical diving-bell of height 10 ft. and internal radius 3 ft. is immersed in water so that the depth of the top is 100 ft. Show that if the temperature of the air in the bell be now lowered from 20°C to 15°C and if 30 ft. be the height of the water-barometer at the time, the tension of the chain is increased by about 67 lbs. (1 cu. ft. of water weighs $62\frac{1}{2}$ lbs.) *(Patna 1926 ; Banaras 1926 ; M. T.)*

20. A diving bell is immersed in water so that its top is at a depth a below the surface, the height of the air within the bell being then x and the height of the water barometer being h. If a bucket of water, of small weight W, be now drawn up into the bell, prove that the tension of the chain is increased by $\frac{W.x}{h+a+2x}$ approximately.

21. If a diving bell descends from the surface with uniform velocity V, show that the water will ascend a height b in the bell in time

$$\frac{b}{V}\left(1+\frac{H}{l-b}\right)$$

where l is the length of the bell and H the height of the water barometer. *(Nagpur 1957)*

22. A diving bell is lowered into water at a uniform rate, and air is supplied to it by a force pump, so as to keep the bell full without allowing any

air to escape. How must the quantity *i.e.*, mass per second, vary as the bell descends? (*Agra 1930*)

23. A cylindrical diving bell is lowered to a given depth in water by means of a chain and is completely immersed. If it be lowered to the same depth in a lighter liquid, will the tension of the chain be greater or smaller? (*Bombay 1935*)

24. If a cylindrical diving bell, of height a and of such internal volume that it would contain a weight W of water, be lowered so that the depth of its highest point is d, prove that, when the temperature is raised from t°C to t_1°C, the tension of the supporting chain is diminished by

$$\frac{\alpha(t_1-t)}{1+\alpha t}\cdot\frac{Wh}{\sqrt{(h+d)^2+4ah}} \text{ nearly,}$$

h being the height of water barometer and $\alpha=\frac{1}{273}$. (*Patna 1927*)

25. A cylindrical diving bell, of height b, is immersed in water with its highest point at a depth a below the surface; if the barometer rises so that the increase of the pressure on its top is P, show that the alteration in the tension of the chain is approximately

$$\frac{P}{2}\left[1-\frac{a+h}{\sqrt{(a+h)^2+4bh}}\right]$$

where h is the height of the water barometer.

26. If a cylindrical diving bell of height a whose chamber could contain a weight W of water, be lowered so that the depth of the highest point is d, prove that when the temperature is raised to t°C, the tension of the supporting chain is diminished by

$$\frac{htW}{T\sqrt{\{(h+d)^2+4ah\}}} \text{ nearly,}$$

h being the height of the water barometer and T the absolute temperature. (*M. T.*)

27. A cylindrical diving bell fully immersed, is in equilibrium without a chain. Show that if the exterior atmospheric pressure increases slightly, the ratio of the distance moved through by the bell, to that moved through by the surface of water in the bell when held fixed, is $Hh+x^2 : x^2$ approximately where H is the height of the water barometer, h the height of the bell, and x the height of that part of it which is filled with air.

CHAPTER XI

DETERMINATION OF SPECIFIC GRAVITY

139. Different Instruments. In chapter II the specific gravity of a substance was defined as the ratio of the weight of any volume of that substance to the weight of an equal volume of some standard substance, usually water. The weight of a body can be measured with very great accuracy, but it is often impracticable to measure its volume directly. Therefore in such cases some indirect methods are adopted.

The sp. gr. of solids or liquids are usually determined with the help of the following instruments :

(1) **The Specific Gravity Bottle**,

(2) **The Hydrostatic Balance**,

(3) **Hydrometers**, and

(4) **The U-tube**.

140. The Specific Gravity Bottle. The Specific Gravity Bottle is a glass which can hold a definite volume of liquid ; for this purpose it is fitted with a glass-stopper through which a perforation runs so that, when the bottle is quite full of liquid and the stopper is pushed in, the excess of the liquid passes out through this hole and the stopper is always able to occupy the same position when pushed home.

141. (*a*) *To determine the specific gravity of a given liquid by means of the Specific Gravity Bottle.*

Let the bottle be weighed when empty, next, when it is filled with water, and again when filled, with the given liquid. Let these weights be W_1, W_2 and W_3 respectively.

Then the wt. of the water filling the bottle $= W_2 - W_1$,

The wt. of the liquid filling the bottle $= W_3 - W_1$

Hence, the sp. gr. of the given liquid

$$= \frac{W_3 - W_1}{W_2 - W_1}.$$

(*b*) *To determine the sp. gr. of a given solid.*

Let the solid be broken into small pieces so that they can be put inside the bottle.

Let W_1 = wt. of the solid

W_2 = wt. of the bottle when filled with water

W_3=wt. of the bottle when it contains the solid and is filled up with water.

∴ W_1+W_2=total wt. of the solid and of the bottle when filled with water.

Also W_3=total wt. of the solid and of the bottle when filled with water—wt. of the water displaced by the solid.

Hence, by subtraction,

$W_1+W_2-W_3$=wt. of the water displaced by the solid.

Therefore W_1 and $W_1+W_2-W_3$ are the wts. of equal volumes of the solid and water.

Hence, the sp. gr. of the solid

$$=\frac{W_1}{W_1+W_2+W_3}.$$

Note. If the body be, like sugar, soluble in water, it must be placed in a liquid in which it is insoluble and the sp. gr. of which is known. If the solid be lighter than water another liquid should be taken into which the solid sinks. By determining the sp. gr. of this new liquid (*a*), the sp. gr. of the solid can be easily calculated. That liquid should also not be used in which a given solid produces any chemical changes.

142. The Hydrostatic Balance. This is an ordinary balance, one scale-pan of which is suspended by shorter chains than the other. The first scale-pan has a hook attached to it so that a solid can be suspended from it and immersed in a liquid.

143. (*a*) *To determine the sp. gr. of a solid by means of the Hydrostatic Balance.*

(*i*) *When the solid is heavier than the water.*

Let

W_1=wt. of the solid

W_2=apparent wt. of the solid in water

∴ wt. of the water displaced$=W_1-W_2$, which has the same volume as the solid

Hence the sp. gr. of the solid$=\frac{W_1}{W_1-W_2}$.

(*ii*) *When the solid is lighter than the water.*

In this case the body must be attached to another body, called a *sinker*, which when attached to the solid will cause both to sink in water.

Let

W_1=wt. of the solid

W_2=wt. of the sinker in water

W_3=wt. of the sinker and body together when placed in water.

$\therefore$ W_3=wt. of the sinker in water+wt. of the solid−wt. of the water displaced by the solid.

$=W_2+W_1-$wt. of the water displaced by the solid

$\therefore$ wt. of the water displaced by the solid
$=W_1+W_2-W_3$

Hence, the sp. gr. of the solid$=\dfrac{W_1}{W_1+W_2-W_3}$.

(b) *To determine the sp. gr. of a liquid.*

Let W_1 be the wt. of a solid which will sink both in the given liquid and water without causing any chemical changes.

Let

W_2=wt. of the solid in water

W_3=wt. of the solid in the given liquid

$\therefore$ W_1-W_2=wt. of the water displaced by the solid

W_1-W_3=wt. of the given liquid displaced by the solid

Hence, the sp. gr. of the given liquid

$$=\frac{W_1-W_3}{W_1-W_2}.$$

142. Jolly's Balance. This is practically a hydrostatic balance which consists of a long spiral carrying two scale-pans, one above the other, and is so arranged that the lower scale-pan is in water. The body whose sp. gr. is required is first placed in the upper pan and the point to which the spring is extended is noted by means of the scale. The body is removed and the '*weights*' are placed to produce the same extension as before. Thus the wt. of the body is determined. The body is next placed in the lower pan and '*weights*' are placed in the upper to produce the same extension as before. This gives the wt. of the water displaced by the body. Now since the wt. of the body and the wt. of the water displaced have been found, the sp. gr. of the body can be determined.

145. Solved Examples.

Ex. 1. *A Specific Gravity Bottle weighs* 212 *grains when it is filled with water* ; 50 *grains of metal are put into it* ; *the overflow of the water is removed and bottle now weighs* 254 *grains. Find the sp. gr. of the metal.*

The wt. of the bottle when filled with water only
=212 grains.

Wt. of the solid=50 grains

Wt. of the bottle when it contains the solid and is filled with water
=254 grains

$\therefore$ wt. of the water displaced by the solid
=(52+212−254) grains
=8

$$\therefore \text{ specific gravity} = \frac{\text{wt. of the solid}}{\text{wt. of the water displaced by the solid}}$$

$$= \tfrac{50}{8} = 6\tfrac{1}{4}.$$

Ex. 2. *Some copper weighs 72 grammes and is coated with 18 grammes of wax whose sp. gr. is ·9. If the whole weighs 62 grammes in water, find the sp. gr. of the copper.*

Wt. of the water displaced by the copper and wax

$$= 72 + 18 - 62 = 28 \text{ grammes}.$$

Wt. of the water displaced by the wax

$$= \frac{1}{\cdot 9} \times 18 \text{ grammes} = 20 \text{ grammes}.$$

$\therefore$ wt. of the water displaced by the copper

$$= 28 - 20 = 8 \text{ grammes}.$$

$\therefore$ sp. gr. of the copper $= \frac{72}{8} = 9$.

Examples 26

1. A body weighs 7·55 gms. in air, 5·17 gms. in water and 6·35 gms. in another liquid. Find the density of the body and the sp. gr. of the liquid. *(Calcutta 1932)*

2. A specific gravity bottle weighs 12·64 gms. when empty and 61·54 gms. when some pieces of iron are put into it. On being filled up with water only, it is found to weigh 37·6 gms. Find the sp. gr. of the iron.

3. A sp. gr. bottle weighs 63 gms. when it is filled with water; some pieces of metal (sp. gr 8·4) are put into it when some water overflows. Then the bottle is found to weigh 100 gms. Find the weight of the water that has overflowed.

4. A piece of metal which weighs 15 ozs. in water is attached to a piece of wood which weighs 20 ozs. in vacuum, and the two together weigh 10 ozs. in water. Find the sp. gr. of the wood. *(Calcutta 1913)*

5. Two bodies are in equilibrium when suspended in water from the arms of a balance. The mass of one body is 28 gms. and its density is 5·6 gm./c.c. If the mass of the other body is 36 gms. what is its density? *(Dacca 1942)*

6. The effect of the air being neglected, the sp. gr. of a solid body is found by a specific gravity bottle to be σ; if ρ be the sp. gr. of the air, show that the real specific gravity of the body is

$$\sigma - \rho(\sigma - 1).$$

7. A specific gravity bottle weighs 14·72 grams when empty; 39·74 grams when filled with water, and 44·15 grams when filled with a solution of common salt. Find the specific gravity of the solution. *(Calcutta 1934)*

8. A solid, which would float in water, weighs 4 lbs., and when the solid is attached to a heavy piece of metal the whole weighs 6 lbs. in water: the weight of the metal in water being 8 lbs., find the sp. gr. of the solid.

9. A piece of sugar weighing 40 grammes is coated with 5·76 grammes of wax whose sp. gr. is ·96. If the whole weighs 14·76 grammes in water, find the sp. gr. of the sugar.

10. The apparent wt. of a *sinker* when weighed in water is 5 times the wt. in vacuo of a portion of a certain substance; the apparent wt. of the sinker and substance together is 4 times the same weight. Prove that the specific gravity of the substance is $\frac{1}{2}$. *(M.T. 1855)*

11. The sp. gr. of a body found by the Hydrostatic Balance, the observation being in air and its effect neglected, is σ. Prove that this result is too great by

$$\rho(\sigma-1),$$

where ρ is the sp. gr. of air.

12. A specific gravity bottle of oil weighs 42·5 grammes. A lump of metal weight 11·2 grammes is placed in the bottle and the bottle filled up with oil, the whole now weighing 52·1 grammes. The bottle is now emptied and 20 grammes of mercury (sp. gr. 13·5) poured into it: on filling up with oil, the combined weight is found to be 61·9. Find the specific gravities of the oil and the metal.

13. The sp. gr. of a solid heavier than water is found to be ρ by the Hydrostatic Balance when the effect of air is neglected. Prove that if this effect be taken into consideration, the true sp. gr. is

$$\rho(1-\sigma)+\sigma$$

where σ is the sp. gr. of the atmospheric air.

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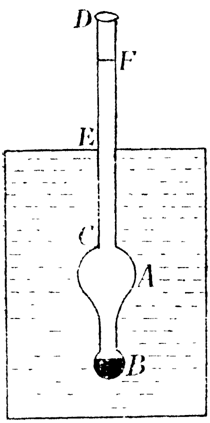
146. **Hydrometers.** Hydrometer is an instrument which is used to determine the sp. gr. of the liquids or of solids. There are various forms of the hydrometer: but here we shall confine ourselves to only two varieties *i.e.*,

(*i*) *Common Hydrometer*

(*ii*) *Nicholson's Hydrometer.*

147. **Common Hydrometer.** The specific gravities of liquids only can be determined by the Common Hydrometer. It consists of a straight glass stem CD ending in a bulb A, below which there is another smaller bulb B. The bulb B is loaded with mercury, so that the hydrometer will float with the stem vertical.



It is clear that the hydrometer always displaces liquids whose weights are equal to that of the instrument; for this reason it is sometimes called the *constant weight hydrometer.*

148. *To determine the sp. gr. of a given liquid.*

Let V be the volume of the hydrometer. Let the portion DE of the stem remain above the surface when the instrument floats in equilibrium immersed in the given liquid and let DF remain above when it floats in water. Let α be the cross-section of the stem, which is constant throughout.

$\therefore$ The volumes of the given liquid and the water displaced by the hydrometer are

$$V-\alpha.DE \text{ and } V-\alpha.DF$$

respectively.

In each case the wt. of the displaced liquid is equal to the wt. of the hydrometer.

Hence, if s be the sp. gr. of the given liquid, we have

$$s.(V-\alpha.DE)=V-\alpha.DF$$

$$\therefore \quad s=\frac{V-\alpha.DF}{V-\alpha.DE}.$$

149. To graduate a Common Hydrometer.

Let the stem be produced, if necessary, to a point O such that the volume of the length DO is equal to V, the total volume of the hydrometer.

Hence $\quad V=\alpha.DO.$

$\therefore$ The sp. gr. s of the liquid is given by the last Art.

$$s=\frac{V-\alpha.DF}{V-\alpha.DE}$$

$$=\frac{\alpha.DO-\alpha.DF}{\alpha.DO-\alpha.DE}=\frac{OF}{OE}$$

$$\therefore \quad OE=\frac{OF}{s}.$$

As O would be a definite point on the prolongation of the stem, we may take O as our origin ; and since F denotes the point up to which the hydrometer sinks in water, OF may be taken to be of definite length.

$$\therefore \quad OE \propto \frac{1}{s}.$$

$$s_1 \propto \frac{1}{OE_1},\ s_2 \propto \frac{1}{OE_2},\ s_3 \propto \frac{1}{OE_3},\ \text{etc.}$$

Hence the hydrometer may be graduated thus :—

Let F be the point at which the instrument would float in water, let O be the point on the stem or stem produced, such that the volume of the length OF of the stem is equal to that of water displaced by the instrument when floating in water, then the graduation E_1, E_2, E_3......corresponding to sp. gravities s_1, s_2, s_3......are given by

$$OE_1=\frac{OF}{s_1},\ OE_2=\frac{OF}{s_2},\ OE_3=\frac{OF}{s_3},\ \text{respectively.}$$

Hence it follows from above *that if the sp. gr. s vary in A.P., the corresponding distances OE would be in H.P., whilst if the distances OE are in A. P. the corresponding sp. gravities are in H.P.*

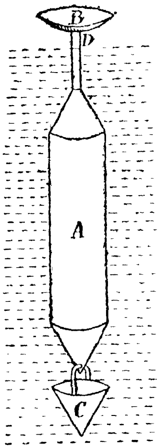
Thus there are two types of the common hydrometer :—

(1) **Twaddell's Hydrometer.**—This type of hydrometer is used in England. In it the specific gravities ascend in A.P. and the corresponding values of OE descent in H.P., so that the marks of graduation become closer together the lower they are on the stem.

(2) **Beaume's Hydrometer.**—This type of hydrometer is used in Europe. In it the values of OE are in A.P., so that the distances

between the marks of graduation are the same ; the corresponding values of s are now in H.P.

150. Nicholson's Hydrometer. This hydrometer is used for determining the specific gravity of liquids as well as of solids. It may also be used for weighing small bodies and for comparing the specific gravities of two liquids. It consists of a hollow cylinder A, generally made of brass, which supports by a thin stem a small cup B on which weights can be placed. Another cup C below A is connected with it by a wire and loaded sufficiently so that the instrument may float with its stem vertical.



On the stem there is a well-defined mark D and in using the hydrometer weight-pieces are placed on the pan till the instrument sinks in the liquid so that the point D is in the surface. Thus the volume of the liquid displaced by the instrument is always the same. For this reason the hydrometer is sometimes called the **Constant Volume Hydrometer**.

151. *To determine the sp. gr. of a given liquid.*

Let

W = wt. of the hydrometer

W_1 = wt. required to be put on the pan B to sink the instrument in the given liquid up to the mark D.

W_2 = wt. required to sink the hydrometer in water up to the mark D.

So $W+W_1$ = wt. of the liquid displaced by the hydrometer.

$W+W_2$ = wt. of water displaced by the hydrometer.

Hence $W+W_1$ and $W+W_2$ are the wts. of equal volumes of the given liquid and water.

Hence, the sp. gr. of the liquid

$$=\frac{W+W_1}{W+W_2}.$$

152. *To determine the sp. gr. of a solid.*

Let

W_1 = wt. required to be put on the pan B to sink the instrument to D in water.

W_2 = wt. required to sink the instrument to D, when the solid is also placed upon the pan,

W_3 = wt. required to sink the instrument to D, when the solid is placed in the cup C underneath the water.

The wt. of the solid together with W_2 has, therefore, the same effect as the wt. of the solid in water together with W_3.

$\therefore$ wt. of the solid in water$+W_3$=wt. of the solid$+W_2$

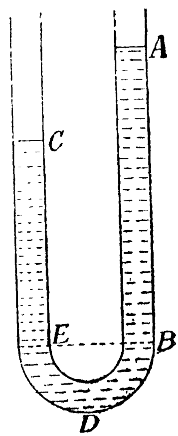
$\therefore$ W_3-W_2=wt. of the solid$-$wt. of the solid in water
=wt. of the water displaced by the solid.

Also W_1-W_2=wt. of the solid.

Hence, the sp. gr. of the solid

$$=\frac{W_1-W_2}{W_3-W_2}.$$

153. The U-Tube Method.—By this method the sp. grs. of two liquids which do not mix can be compared.



Let ABCD be a U-shaped tube of glass. Let the two liquids be poured into the two arms, one in each, with their common surface at B.

Let E be the point of the other arm in the same horizontal level as B.

$\therefore$ The pressure at B = the pressure at E

$\therefore$ $gs_1AB+\pi=gs_2CE+\pi$

where π is the atmospheric pressure

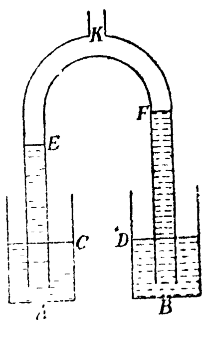
$$\therefore \quad \frac{s_1}{s_2}=\frac{CE}{AB}$$

i e., the specific gravities are inversely proportional to the heights of the free surfaces from their common surface.

Note. *If the liquids mix with each other.*—In each case some mercury is first poured into the U-tube. Let BDE be the portion occupied by mercury, BE being horizontal. Next liquids are poured in the two arms so that the levels of the mercury are the same as before. Measure CE and AB. The remaining calculation is the same as before.

154. The Inverted U-tube or Hare's Hydrometer.

This instrument is used for comparing the sp. grs. of two liquids. It consists of an inverted U-tube connected by an open tube K as shown in the figure.



The two vertical arms of the tube are dipped in two vessels A and B containing two liquids whose sp. grs. are to be compared. The tube K is connected by an air-pump which draws out a certain quantity of air. In doing so let Π, the atmospheric pressure be reduced to Π' and let the liquids rise in the two arms up to E and F.

Let s_1, s_2 be the sp. grs. of the liquids in A and B.

$\therefore$ $\Pi=\Pi'+gs_1.CE$,

and $\Pi=\Pi'+gs_2.DF$

$$\therefore \qquad s_1.\mathrm{CE} = s_2.\mathrm{DF}$$

$$\text{or} \qquad \frac{s_1}{s_2} = \frac{\mathrm{DF}}{\mathrm{CE}}.$$

155. Solved Examples.

Ex. 1. *A common hydrometer sinks to the points A, B, C in liquids whose densities are ρ_1, ρ_2, ρ_3, respectively. If $AB=a$, $BC=b$, and $AC=a+b$, prove that*

$$\frac{b}{\rho_1} + \frac{a}{\rho_3} = \frac{a+b}{\rho_3}. \qquad \textit{(Bombay 1937)}$$

We know that

$$s = \frac{\mathrm{V} - \alpha.\mathrm{DF}}{\mathrm{V} - \alpha.\mathrm{DE}} \quad \text{(see Common Hydrometer)}$$

$$\therefore \qquad \rho_1 = \frac{\mathrm{V} - \alpha.\mathrm{DF}}{\mathrm{V} - \alpha.\mathrm{DA}} \qquad \dots (1)$$

$$\rho_2 = \frac{\mathrm{V} - \alpha.\mathrm{DF}}{\mathrm{V} - \alpha.\mathrm{DB}} = \frac{\mathrm{V} - \alpha.\mathrm{DF}}{\mathrm{T} - \alpha(\mathrm{PA} + a)} \qquad \dots (2)$$

$$\rho_3 = \frac{\mathrm{V} - \alpha.\mathrm{DF}}{\mathrm{V} - \alpha.\mathrm{DC}} = \frac{\mathrm{V} - \alpha.\mathrm{DF}}{\mathrm{V} - \alpha(\mathrm{FA} - a - b)} \qquad \dots (3)$$

Cross-multiplying and subtracting the result of (2) from (1)

$$\mathrm{V}\,(\rho_1 - \rho_2) = \alpha.\ \mathrm{DF}\,(\rho_1 - \rho_2) + \rho_2 a \alpha$$

$$\text{or} \qquad \mathrm{V} = \alpha.\mathrm{DF} + \frac{\rho_2 a \alpha}{\rho_1 - \rho_2} \qquad \dots (4)$$

Similarly from (1) and (3)

$$\mathrm{V} = \alpha.\mathrm{DF} + \frac{\rho_3 (a+b)\alpha}{\rho_1 - \rho_3} \qquad \dots (5)$$

Subtracting (5) from (4)

$$\frac{a\rho_2}{\rho_1 - \rho_2} = \frac{\rho_3(a+b)}{\rho_1 - \rho_3}$$

$$\text{or} \qquad a\rho_1\rho_2 + b\rho_2\rho_3 = (a+b)\rho_1\rho_3$$

$$\text{or} \qquad \frac{a}{\rho_3} + \frac{b}{\rho_1} = \frac{a+b}{\rho_2}$$

Ex. 2. *One limb of a U-tube of uniform cross-section 1 sq. cm. is open to the atmosphere and the other is closed. The tube is partly full of mercury and the space above the mercury in the closed limb is occupied by air. The mercury stands at a higher level in the closed limb than in the other. In what follows the temperature may be assumed constant.*

When the atmospheric pressure is equal to that due to $h_1 + x$ cms. of mercury the difference in the level of the mercury in the two limbs is h_1 cms.; and when the atmospheric pressure is $h_2 + y$ cms. of mercury the difference in the level is h_2 cms. Show that if h_1 is greater than h_2, x is greater than y.

Show further that when the difference of level is h cms. the atmospheric pressure will be

$$h+\frac{xy(h_1-h_2)}{x(h_1-h)+y(h-h_2)}\ cms.$$

of mercury. (*I.C.S. 1926*)

Let v_1 be the length of the tube occupied by air in the first reading, v_2 in the second and v in the third.

Evidently the pressure due to the air column will be x, y, z respectively in the three readings, where $x+h$ is the atmospheric pressure in the third reading.

$$\therefore \qquad v_1x=x_2\,y=vz$$

Now as the pressure decreases, some mercury is forced down in the tube and the same amount increases in the other tube.

$$\therefore \qquad v_1+\tfrac{1}{2}(h_1-h_2)=v_2$$

Similarly for the third reading

$$v_2+\tfrac{1}{2}(h_2-h)=v$$

and $$v_1-\frac{x}{y}v_1=-\tfrac{1}{2}(h_2-h_1)$$

and $$\frac{x}{y}v_1-\frac{x}{z}v_1=\tfrac{1}{2}(h-h_2)$$

or $$\frac{y-x}{zx-xy}\cdot\frac{y^2}{y}=\frac{h_2-h_1}{h-h_2}$$

or $$\frac{z}{z-y}=\frac{x(h_2-h_1)}{(y-x)(h-h_2)}$$

or $$\frac{z}{-y}=\frac{x(h_2-h_1)}{-x(h_2-h_1)+(y-x)(h-h_2)}$$

$$\therefore \quad z=\frac{xy(h_1-h_2)}{x(h_1-h_2+h_2-h)+y(h-h_2)}=\frac{xy(h_1-h_2)}{x(h_1-h)+y(h-h_2)}$$

$\therefore$ The atmospheric pressure

$$=h+\frac{xy(h_1-h_2)}{x(h_1-h)+y(h-h_2)}.$$

Ex. 3. *The specific gravity of a body found by Nicholson's hydrometer is σ when the effect of the air is neglected; prove that the real specific gravity is* $\sigma-\rho(\sigma-1)$, *where* ρ *is the sp. gr. of the air. Also if W be the apparent weight of the body as found from the experiment, find its real weight.*

Let s_1 and s_2 be the real sp. grs. of the body and the '*weights's* respectively. Let w_1 be the real wt. of the body and w_2 the wt. of the hydrometer.

Let W_1=wt. required to put on the pan to sink the instrument to the definite point in water

W_2 = wt. required to sink the hydro. to that point, when the solid is also placed upon the pan

W_3 = wt. required to sink the hydro. to that point when the solid is placed in the cup underneath the water.

Hence $W = W_1 - W_2$, $\sigma = \dfrac{W_1 - W_2}{W_3 - W_2}$.

$\therefore \quad w_2 + W_1 - \dfrac{W_1\rho}{s_2}$ = wt. of the water displaced,

$w_2 + W_2\left(1 - \dfrac{\rho_1}{s_2}\right) + w_1\left(1 - \dfrac{\rho_1}{s_1}\right)$ = wt. of the water displaced;

and $w_2 + W_3\left(1 - \rho_1\right) + w_1\left(1 - \dfrac{\rho}{s_1}\right)$ = wt. of the water displaced

$\therefore$ by subtraction,

$$w_1\left(1 - \frac{\rho}{s_1}\right) = (W_1 - W_2)\left(1 - \frac{\rho}{s_2}\right) \quad \dots (1)$$

and
$$w_1\left(\frac{1-\rho}{s_1}\right) = (W_3 - W_2)\left(1 - \frac{\rho}{s_2}\right) \quad \dots (2)$$

$\therefore$ By division

$$\frac{s_1 - \rho}{1 - \rho} = \frac{W_1 - W_2}{W_3 - W_2} = \sigma$$

$$\therefore \quad s_1 = \rho + \sigma(1 - \rho) = \sigma - \rho(\sigma - 1)$$

From (1)

$$w_1 = W\frac{1 - \frac{\rho}{s_1}}{1 - \frac{\rho}{s_1}} = \frac{W\left(1 - \frac{\rho}{s_2}\right)}{s_1 - \rho}s_1$$

$$= \frac{W\left(1 - \frac{\rho}{s_2}\right)}{\sigma(1 - \rho)}\left[\sigma - \rho(\sigma - 1)\right]$$

$$= W\left(1 - \frac{\rho}{s_2}\right)\left[1 + \frac{\rho}{\sigma(1 - \rho)}\right].$$

Examples 27

1. The whole volume of a common hydrometer is 6 cubic inches and the stem, which is a square, is $\frac{1}{8}$ inch in breadth; it floats in one liquid with 2 inches of its stem above the surface and in another with 4 inches above the surface. Prove that their sp. grs. are in the ratio 190 : 191

2. When a common hydrometer floats in water $\frac{9}{10}$th of its volume is immersed, and when it floats in milk $\frac{90}{103}$ of its volume is immersed. Prove that the sp. gr. of the milk is 1 03.

3. The whole volume of a common hydrometer is 15 cms. and its stem is 3 mm. in diameter. The hydrometer floats in a liquid with 3 cm. of the stem above the surface, and in another liquid with 6 cm. above the surface. Compare the densities of the liquid. *(Madras 1936)*

4 A common hydrometer floats in water with half the stem immersed. A lighter liquid which does not mix with water is poured into the containing vessel until one-fourth of the stem remains above the liquid. If the depth of the upper liquid be $\frac{1}{n}$ th of the stem, n being <4, find the sp. gr. of the liquid.

5. A common hydrometer has a small portion of its bulb rubbed off from frequent use. In consequence, when placed in the water, it appears to indicate the sp. gr. of water is 1·002; find what fraction of its weight has been lost.

6. A common hydrometer being graduated upwards, its readings for two different fluids are x', and x'' and for a mixture of equal volumes of these readings is x; show that the volume of a unit of length of the stem is to the volume of the whole instrument below zero point as

$$x'+x''-2x \; : \; xx'+xx''-2x'x''.$$

(M.T. 1872 ; Lucknow 1926)

7. A Nicholson's hydrometer sinks to a certain mark in liquid of sp. gr. 0·6; but it takes 120 grammes to sink it to the same mark in water. What is the weight of the hydrometer ? *(Calcutta 1948)*

8. A Nicholson's hydrometer weighing 150·6 gms. requires 20·4 gms. to be placed on the upper pan to sink it up to mark, when made to float in water. What weight must be added to sink it up to mark when floated in a liquid of sp. gr. 1·2 ?

9. A hydrometer marks graduations a, b, c, in liquids whose densities are ρ_1, ρ_2, ρ_3 respectively. Prove that

$$\frac{b-c}{\rho_1}+\frac{c-a}{\rho_2}+\frac{a-b}{\rho_3}=0.$$

(Patna 1926, 33)

10. If w, w_1, w_2 are the weights required to sink a Nicholson's hydrometer in liquids of sp. grs. s, s_1, s_2, respectively, prove that s can be determined in terms of the other quantities without knowing the weight of the instrument and its value is

$$\frac{w-w_1}{w_2-w_1}s_2+\frac{w-w_2}{w_1-w_2}s_1.$$

11. A Nicholson's hydrometer is used to determine the weight and specific gravity of a solid, and W and s are the results when the effect of the air is neglected. Prove that the actual weight is

$$W[1+k/s(1-k)]\,(1-k/d)$$

where k and d are specific gravities of air and of the material of the known weights employed. *(Patna 1927)*

12. It is found that a solid whose true weight is w, placed on the upper pan, sinks a Nicholson's hydrometer to the mark ; as also another solid whose true weight is W, when placed in the lower cup. If the two solids have the same sp. grs. show that its value is

$$\frac{W-w\sigma}{W-w}$$

where σ is the sp. gr. of the air.

13. If a common hydrometer be accurately graduated for use in a vacuum, show that the error due to using it in air of sp. gr. σ will be an apparent increase of σ' of sp. gr., where σ' is to σ in the ratio of the volume of the hydrometer unimmersed to that immersed.

14 A common hydrometer whose volume is V and the cross section of whose stem is $\frac{V}{c}$, has lengths a and b of its stem uncovered when floating in

one or the other of two liquids. In a mixture of weights of the two liquids in the proportion $m:1$, x is uncovered, and in a mixture of volumes in the proportion $m:1$, y is uncovered. Prove that

$$(x-y)(c-y)=(a-y)(b-y). \qquad \textit{(Allahabad 1927)}$$

15. If the reading of a common hydrometer when placed in fluid at the same temperature as itself be x, and if, when it is placed in the same fluid at a higher temperature than itself its reading be at first x_1, but afterwards the reading rises to x_2, the ratio of the expansions of the fluid and of the hydrometer for the same change of temperature is approximately

$$x-x_1 : x_2-x_1 \qquad (M.\ T.)$$

16. A common hydrometer is used to determine the sp. gr. of a liquid which is at a temperature higher than that of water. When the hydrometer is transferred from water to the liquid the sp. gr. appears at first to be σ but afterwards to be σ_1. Show that the true sp. gr. at the temperature of the water is

$$\sigma+\frac{\alpha'}{\alpha}(\sigma_1-\sigma),$$

where α and α' are the co-efficients of expansions of the hydrometer and the fluid respectively. *(Banaras 1937)*

17. A common hydrometer has a piece of its bulb chipped off. When placed in fluids of density α and β, it indicates densities α', β' respectively. Find the proportion of the weight which has been chipped off, and show that if in any fluid the apparent density is x, the true density is

$$\frac{x\alpha\beta(\beta'-\alpha')}{\alpha'\beta'(\beta-\alpha)-x(\alpha'\beta-\alpha\beta')}. \qquad \textit{(Calcutta 1927, M.T.)}$$

18. Supposing a common hydrometer to be immersed in a liquid less dense than water as far as the point to which it would sink in water prove that, if let go, it will sink through a distance

$$\frac{2w}{k}\cdot\frac{1-s}{s},$$

w being the wt. of the hydrometer, k the section of its stem and s the sp. gr. of the liquid. The hydrometer is supposed to be never entirely immersed. *(M. T.)*

19. A U-tube of cross-section α with equal vertical legs contains a liquid of density ρ. In the liquid on one side there is floating freely a solid body of volume ab and density $\sigma(<\rho)$. The length of the other leg unoccupied by liquid is c. Another liquid of density $\mu(<\sigma)$ is then poured into the leg in which the solid is floating till that leg is full. Show that the length which is still unoccupied of the other leg is

$$b\frac{\mu(\rho-\sigma)}{\rho(2\rho-\mu)}+2c\frac{\rho-\mu}{2\rho-\mu}.$$

ANSWERS

Examples 1

1. ·9658. **2.** 6·48 cu. ft. **4.** 18·41. **5.** 0·5 gall. (nearly)

8. $\frac{1}{2}(n+1)$, $\frac{1}{3}(2n+1)$, $\frac{1}{3}(n+2)$ (Ratio). **9.** 6, 2.

Examples 2

1. $2291\frac{2}{3}$ lbs. wt. **2.** 36·864 ft.

3. $73\frac{113}{144}$ lbs. wt. per. sq. inch. **4.** 69 ft.

5. 14·9556 cu. in. **6.** 30 cms.

8. $1\frac{5}{48}$ lbs. wt. per sq. inch.

Examples 3

1. 67500 grams. wt. on the upper face, 94500 grams wt. on the lower face, 81000 grams wt. on each vertical face.

2. $515\frac{5}{8}$ lbs. wt. **3.** 10·96 gah^2. **4** $\frac{4}{3}(1+\sqrt{10})$.

5. 40 : 87 : 92. **6.** 531·25 lbs. wt. **7.** 35·148 lbs. wt. nearly.

9. 1, 297, 265·625 lbs., 30·372 ft. nearly. **10.** 3 : 1.

11. 9 ft. **12.** Increased. **14.** $\frac{2}{3}b(a^2-b^2)w$.

15. At a depth of $\frac{1}{\sqrt{2}}$ times its side.

16. Join A to a point P in DC so that DP : PC = 3 : 1.

17. As the answer of Q. 16.

18. Draw lines at distance $\sqrt{\frac{1}{n}}$, $\sqrt{\frac{2}{n}}$, $\sqrt{\frac{3}{n}}$ ...times the height of the par^m from the top.

19. Distances from B of the points of division are in the ratio $1 : \sqrt{2} : \sqrt{3} :$............

20. ~~The plane cuts of $\sqrt[3]{\frac{1}{2}}$ of the~~ axis. ✗

Examples 4

1. $\frac{1}{6}\,g\rho h^2(a+2b)$.

13. The point lies in the line from the vertex bisecting the base and at a depth $\frac{1}{\sqrt{3}}$ times the depth of the vertex.

Examples 6

1. $4\frac{1}{2}$. **2.** $3\frac{9}{74}$.

Examples 7

1. $\frac{67}{20}$. **2.** $\frac{5}{2}$. **4.** $\frac{11}{8}$ times the depth of the base.

5. Divides the diagonal in the ratio 7 : 5.

Examples 8

2. $(-1+\sqrt{5})\dfrac{h}{2}$ is the depth of the line.

3. $\frac{3}{16}\pi a$. **4.** $\frac{3}{16}\pi a$.

5. $\dfrac{a^2}{4h}$ below the centre along the major axis, a being the semi-major axis.

8. $\left(\dfrac{a}{2}\cdot\dfrac{3a+8b}{4a+3\pi b},\ \dfrac{1}{4}\ \dfrac{3\pi(a^2+4b^2)+32ab}{4a+3\pi b}\right)$.

9. $\dfrac{1}{7}\cdot\dfrac{35k^2+15h^2-42hk}{5k-3h}$. **10.** $\dfrac{32a}{15\pi}, \dfrac{16a}{15\pi}$.

11. $\dfrac{h}{3}\left(\dfrac{63k^2+35h^2+90kh}{35k^2+15h^2+42kh}\right)$ from the vertex.

Examples 10

10. Vertical face at depth $\dfrac{7a}{3\sqrt{2}}$, upper slant faces $\frac{1}{3}a$ from centre of face, lower faces $\frac{1}{9}a$ from centre.

12. $\dfrac{4a}{5}, \dfrac{16a}{33}$.

19. $\dfrac{a}{4}\cdot\dfrac{15\alpha-16\sin\alpha+3\sin\alpha\cos\alpha}{3\alpha-2\sin\alpha}$, where 2α is the angle of the sector.

Examples 11

2. 3 : 1. **4.** $ha^2\left(1+\dfrac{\pi}{6}\right)w,\ ha^2\left(1+\dfrac{\pi}{6}\right)w$, h and a being height and radius of the cone respectively.

5. $4-\pi : 4+\pi$. **6.** $6b-\pi a : 6b+\pi a$.

7. $\frac{1}{3}\pi h\ (a^2+ab-2b^2)w$.

8. $\frac{1}{3}\pi\ (a-c)\ (5a^2+5ac+2c^2)\ w,\ \frac{1}{3}\ \pi w\{a^3\sim(3ac^2+2c^3)\}$.

13. $\frac{1}{3}\ \pi r^2 g\rho(3h_1+2h\sim h_1^3/h^2)$, where h and h_1 are the heights of the lower and upper cones, and r the radius of the base of the lower cone.

Examples 12

2. $\frac{1}{2}\pi a^2 w\left(a+\dfrac{4a}{3\pi}\right)$, a being the radius of the base.

4. $a^2 w\left(h^2+\pi^2\dfrac{a^2}{4}\right)^{\frac{1}{2}}$, a and h are the radius of the base and height of the cone respectively.

5. $ahh'w$, where h' is the height and a the radius of the base of the cone.

6. $whr\sqrt{h^2+\frac{1}{4}\pi^2r^2}$. $\tan\theta = \frac{\pi r}{2h}$. 7. $2r^2hw$.

8. $\frac{1}{2}a^2hw\sqrt{\pi^2+16}$ through the centre at an angle $\tan^{-1}\frac{\pi}{4}$ to the horizontal.

9. πa^2hw, $\pi a^2w\sqrt{h^2+\frac{4a^2}{9}}$ at $\tan^{-1}\frac{2a}{3h}$ to the horizontal.

11. $\frac{2}{3}a^3w$.

14. $\frac{1}{3}\sqrt{13}\,\pi a^2w$ at an angle $\tan^{-1}(\frac{2}{3})$ to the horizontal passing through the centre of the hemisphere whose radius is a.

15. Thrust on a lower part is 3 times than on an upper.

18. $\frac{1}{3}wh^2\sqrt{a^2+b^2}$ where h is the height of the cone.

Examples 13

1. $\tan^{-1}(\frac{1}{3})$ with the horizontal, 178·88 w grams.

4. $\pi a^2w/\sqrt{2}$, $\pi a^2(h+a)\,w/\sqrt{2}$, a is base-radius.

14. $\frac{1}{3}\pi a^3w\sqrt{(13-12\cos\theta)}$, $\frac{1}{3}\pi a^3w\sqrt{(13+12\cos\theta)}$.

15 $\frac{1}{6}g\rho a^3(\pi^2-4\pi+8)^{\frac{1}{2}}$.

16. $\frac{1}{3}g\rho\pi l^3\sin^2\alpha\cos\alpha\sqrt{1+3\sin^2\alpha}$.

Examples 14

1. $\frac{3}{10}$. 2. $2\frac{6073}{10800}$ cu. ft. 3. 36·4 cu. cm. nearly, 7·5.

4. 0·48 cu. in. 5. 2·3.

6. $\frac{104}{209}$ c. ft. 7. 2·3.

8. 19·5 lbs., 900 cu. ins.

9. $\sqrt[3]{\frac{1}{2}}$. 10. $\left(\frac{\sigma}{\rho}\right)^{\frac{1}{3}}h$, $\left[1-\left(1-\frac{\sigma}{\rho}\right)^{\frac{1}{3}}\right]h$

11. ·726...inch. 12. 6,500 tons.

15. $\frac{3}{a+b+c}$, $\frac{ab+bc+ca}{3abc}$.

Examples 15

1. 1 : 2. 2. ·00013. 3. It will sink. 4. $\frac{1}{4}$.

Examples 16

1. 30 lbs. 4. Wood. 5. 2·1. 6. 1·23″.

7. $3\frac{13}{17}$ lbs., $\left(1\cdot3-\frac{13}{77}\right)$ lbs. 9. ·8, wt. of 5 gms., 100 c.cs.

Examples 17

4. 2 ft. 5. $\frac{1}{3}$ wt. of the rod.

Examples 18

1. $\frac{h(n^3-1)^{\frac{1}{3}}}{n}$.

Examples 19

1. 2·623..... cms. 2. 306 ft. 3. .15 c.cs.

4. 7948·43 metres. 5. 6 cu. ins. 6. $3 : 17^{\frac{1}{3}}$.

7. It would float.

10. The positive root of $x^2+(a+h)x-(b-a)h=0$, h being the height of water barometer,

Examples 20

1. 307°C. 3. 1330°C. 4. 10 cu. inches. 5. 429·224.

6. The one whose temperature is the lower.

Examples 21

1. 0·0686 of the original V. 3. 7 in. 4. 7·4 in.

Examples 23

1. 756 mm. 5. 38·2 lbs. wt. 6. 12·92 metres.

7. 16 ft. 8. 22·58 ft., 33·20 ft. nearly.

9. 33 ft. 10. 4 ft. 11. 8·1 ft. 12 6.

Examples 24

1. $\frac{243}{1024}$. 2. ·422 atmos. 4. 17·36 ins.

5. $\frac{\sqrt{3}}{9}$ of the initial pressure. 6. 4. 7. 9. 8. 40.

Examples 25

1. 258·4 cms. 2. $94\frac{228}{323}$ ft.

4. 20 ft , $132\frac{6}{17}$ cu. ft. 5. $33\frac{1}{3}$ inches., 3 ft. 9 inches.

6. $22\frac{8}{11}$ cu ft. 15. 46·83 ft. nearly.

18. (*i*) $\frac{\sigma}{12}(h'-h)-b\left(\frac{h}{h'}\right)^{\frac{1}{3}}$

(*ii*) $\frac{\sigma}{12}(h'-h)-\frac{bh}{h'}$, where σ is sp. gr. of mercury.

22. It remains constant.

Examples 26

1. 3·17 gms./c.c., 0·5. 2. 7·54. 3. 5 gms.

4. 0·8. 5. 2·77. 7. 1·17.

8. $\frac{2}{3}$. 9. 1·6. 12. ·405, 2·835.

Examples 27

3. 36 : 43. 4. $1-\frac{n}{4}$.

5. $\frac{1}{501}\left[\frac{\sigma}{\sigma-1}\right]$, σ being the sp. gr. of the substance of the bulb.

7. 180 gms. 8. 51·6 gms.