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CHAPTER I

UNITS AND DIMENSIONS

1. Units. The Physicist always seeks to reduce his physical concepts and conclusions to measurable quantities, in the spirit of Lord Kelvin's very shrewd and apt remark that '*when you can measure what you are speaking about and express it in numbers, you know something about it, but when you cannot measure it in numbers, your knowledge is meagre and unsatisfactory*',—a remark which is at once a challenge and an inspiration to men of science to sift and clarify their ideas and notions until they become quite precise and clear-cut.

Now, measurement inevitably involves comparison with a chosen *standard* or *unit* of a similar kind ; so that, the first essential step to be taken is the selection of a suitable standard or unit in accordance with the nature of the physical quantity to be measured, and the second, to determine its value in terms of the chosen unit. In other words, to form an exact idea of the magnitude of a physical quantity, it is necessary to express (*i*) *the standard or unit in which the quantity is measured and* (*ii*) *the number of times the quantity contains that unit.*

Thus, for example, when we speak of a distance as being equal to 5 *miles*, we mean that the standard or unit in which it is measured is the *mile*, and that the distance in question is *five times this unit.* If we choose the *yard* (which is 1/1760 of 1 *mile*) or the *foot* (which is $1/1760 \times 3$ of 1 *mile*) as our unit, the *same* distance will be equal to 8800 *yards* or 26,400 *feet* respectively,—*i.e.*, its numerical value will be 1760 times or 1760×3 times 5. Thus, *the larger the unit, the smaller the numerical value of the quantity ; and the smaller the unit, the larger its value. Or, the numerical value of a quantity is inversely proportional to the magnitude of the unit selected as the standard.* It follows, therefore, that *the product of the numerical value of the quantity and the magnitude of the unit in which it is expressed is a constant.* Thus,

$$5 \times 1 = 8800 \times \frac{1}{1760} = 26400 \times \frac{1}{1760 \times 3}.$$

Or, in general, if n_1 and n_2, be the numerical values of a given physical quantity, corresponding to the units x_1 and x_2 respectively, we have

$$n_1 x_1 = n_2 x_2.$$

2. Fundamental and Derived Units. For measuring different kinds of quantities, we must obviously have different kinds of units. If these be selected in any arbitrary manners they will be quite unrelated to each other, and their use will create difficulties and complications in actual practice. They are, therefore, all based on some *fundamental units*, so as to be interdependent and properly related to each other, the guiding principle in their choice being to see that

(*a*) they are *well-defined* and of a *suitable size*,

(*b*) they are *easily reproducible* at all places,

(*c*) they are *not subject to any secular changes* (*i.e.*, to changes with time),

(*d*) they *do not readily or appreciably vary with varying physical conditions, like temperature, pressure etc.*, and, if they do, their manner of variation is perfectly correctly known.

The *fundamental units* chosen, and internationally employed, are those of **mass**, **length** and **time** which *C. F. Gauss*, in 1832, termed as *absolute units**. The reason why these alone are chosen as the *'fundamental' units*, and not any others, would seem to be that they represent our elementary scientific notions and cannot be derived from one another ; nor can they be resolved into anything more basic or fundamental. All other units in Mechanics can be derived from them and are, therefore, called *'derived units'*. Thus, the units of area and volume are derived units, for they can both be derived from the unit of length,—the former being the area of a *square*, and the latter, the volume of a *cube*, each of unit length. Similarly, the unit of velocity is a derived unit and is the velocity of a body which covers unit distance, or length, in unit time, and so on.

3. Principal Systems of Units. There are three principal systems of units in vogue, *viz.*,

(*i*) the *Centimetre-Gramme-Second system* or the *C G.S. system*, (*ii*) the *Foot-Pound-Second system* or the *F. P. S. system* and (*iii*) the *Metre-Kilogramme-Second system* or the *M. K. S. system*.

(*i*) **The C. G. S. System.** In this system, the unit of length is the *centimetre*, that of mass, the *gramme* and that of time, the *second*.

The Centimetre *is one-hundredth part of a* **metre**, *which is the distance, at a temperature of 0°C, between two lines on a platinum-iridium bar, preserved at the International Bureau of Metric Weights and Measures at Sevres, near Paris.* Originally intended to be one-thousand millionth part of the longitude of the earth from the north pole to the equator, passing through Paris, it is found, however, to be slightly smaller.

The International Bureau of Weights and Measures has constructed a line standard *metre*, known as the *Prototype Metre*, copies or replicas of which have been supplied to various Governments.

The Gramme *is one-thousandth part of a lump of platinum-iridium, called a* **Kilogramme,** made by *Borda*, in accordance with a decree of the French Republic, and *also preserved at Sevres. It is equal to the mass of water, whose volume is one cubic centimetre, at 4°C, when it has its maximum density*, (*viz.*, 1 *gm./c.c.*)

The Second, or the *mean solar second*, as it is called, is $1/24 \times 60 \times 60th$, or $1/86400th$ *part of the mean solar day, which is the average value, for one year, of the solar day, or the time which elapses between two consecutive transits of the Sun across the meridian, at any place on the Earth's surface.*

*In connection with the measurements of the earth's magnetic field carried out by him at Gottingen.

Another unit of time, used in Astronomy, is the *mean siderial second*, which is 1/86400*th* part of the *siderial day*, or the true period of revolution of the Earth on its axis, *i.e.*, the interval which elapses between two consecutive passages of a fixed star across the meridian.

(*ii*) **The F. P. S. System.** Here, the unit of length is the *foot*, the unit of mass, the *pound* and the unit of time, the *second*.

The Foot *is one-third of the distance between two transverse lines, at a temperature of 62°F, on two gold plugs in a bronze bar*, kept at the Standards Office of the Board of Trade, London.*

The Pound (*avoirdupois*) *is the mass of a platinum-iridium cylinder, marked "P S., 1844, 1 lb." also kept at the Standards Office of the Board of Trade, London.*

And, the **Second**, or *the mean solar second*, is the same as defined above.

Other units, derived from those given above, are called the *Board of Trade units* or the **B. O. T. units.**

It may as well be mentioned here that we generally choose our units to suit the quantity to be measured. Thus, for example, for the measurement of very small lengths or distances, we have successively smaller units of length, *viz.*, *the* **micron** (μ) $= 10^{-3}$ *mm.*, *the* **millimicron** ($m\mu$) $= 10^{-6}$ *mm.* and the **Angstrom unit** (*A. U.* or, simply, *A*) $= 10^{-7}$ *mm.*; and, for the measurement of very large distances, like those of intersteller space, we have correspondingly larger units, like *the* **light year,** or the distance covered by light in vacuo, (with a velocity of $2\,99 \times 10^{10}$ *cm./sec.*) in one full year. Similar being the case with the units of mass and time.

(*iii*) **The M.K.S. System.** This is a comparatively new system, *very much akin to the C.G.S. system*, in which the units of length, mass and time are the *Metre*, the *Kilogramme* and the *Second* respectively.

The first system is the one invariably used in scientific work all over, the second is more or less confined in its use to only Great Britain and the third is now being increasingly adopted in electrical engineering etc., where it is found to be more convenient and useful.

4. Dimensions. Dimensional Formulae and Equations.

(*a*) **Dimensions.** The units of mass, length and time are denoted by the capical letters, $[M]$, $[L]$ and $[T]$†, which merely indicate their *nature* and *not their magnitude*. And, since the unit of area is the product of two unit lengths, we have the unit of area represented by $[L] \times [L]$ or $[L^2]$; and, similarly, the unit of volume, being the product of three unit lengths, is represented by $[L] \times [L] \times [L]$ or $[L^3]$. *We express this by saying that the unit of area is of two dimensions‡ in length, and the unit of volume, of three dimensions in length.*

*This bar has now also been replaced by a platinum-iridium one.

†The square brackets merely indicate '*dimension of*'. Once this is understood, they may as well be omitted, as we shall quite often do.

‡Which is the abbreviated form of '*exponent of dimension*', but is now commonly used and well understood.

Since neither the unit of area nor that of volume depends upon mass and time, their dimensions are said to be zero in both mass and time and we may, therefore, represent these units as $M^0 L^2 T^0$ and $M^0 L^3 T^0$, respectively.

"The dimensions of a derived unit may thus be defined as the powers to which the fundamental units of mass, length and time must be raised to represent it." Thus, if a derived unit depends upon the *n*th power of a fundamental unit, it is said to be of *n dimensions* in that fundamental unit. For example,

$$velocity = \frac{distance\ or\ length}{time} = \left[\frac{L}{T}\right] = LT^{-1},$$

and hence the dimensions of the unit of velocity are 1 in length and -1 *in time.* Since it is independent of mass, *its dimension in mass is zero*, and we may, therefore, represent it by M^0LT^{-1}.

Again, since $acceleration = \frac{velocity}{time}$, *the dimensions of the unit of acceleration are* $\frac{M^0\ L\ T^{-1}}{T} = M^0\ L\ T^{-2}$, and so on.

It will thus be seen that the dimensions of a physical quantity are obtained by simply defining it in terms of those physical quantities whose dimensions in mass, length and time are known,—the value of a derived unit depending upon the values of the fundamental units from which it is derived. Thus, if we take a *yard* as our unit of length, instead of a *foot*, the units of area and volume will respectively be 3^2 and 3^3 times as big as their units in the ordinary system. So that, *the dimensions of a physical quantity show how its nature and the value of its unit depend upon the fundamental units chosen.*

(*b*) **Dimensional Formulae and Equations.** *A dimensional formula is an expression, showing how and which of the fundamental units enter into the unit of a physical quantity.* Thus, all the expressions in the *Table* opposite, indicating the relation between the derived and fundamental units, are dimensional formulae. For example, the *dimensional formula for* work is ML^2T^{-2}. But when we put it in the form, $W = ML^2T^{-2}$, it is called a *dimensional equation* for work.

This idea of dimensional formulae for physical quantities, as we know it today, was first clearly given by **Fourier**, in the year 1822, although it originated initially with **Newton**, who refers to the *principle of similitude* in his famous and well celebrated *Principia*, (II, Proposition 32).

The student is no doubt aware that in Physics we come across two types of quantities, *viz.*, **variables** and **constants**, which may both be dimensional or non-dimensional (*i.e.*, dimensionless). Thus, we have

(*i*) *Dimensional Variables.*—These are quantities like acceleration, velocity, force and most of the others which the Physicist has to deal with, at every step. These are, so to speak, his 'current coin'.

(*ii*) *Dimensional Constants*—Quantities which have a constant value and yet have dimensions are called dimensional constants. As

examples of these may be cited G, the *Gravitational Constant, and c, the Velocity of Light* in vacuo, whose dimensions are $M^{-1}L^3T^{-2}$* and M^0LT^{-1} respectively.

(*iii*) *Non-dimensional Variables.*—These are quantities which are variables and yet have no dimensions; as, for example, *specific gravity, strain* or an *angle*, (see *Table* below).

Here, we also meet with groups of dimensional variables (with or without dimensisnal constants) such that *their dimension is zero in each of the fundamental quantities, i.e., in length, mass and time.* Thus, for example, the quantity $t\sqrt{g/l}$ has no dimensions; and so also the quantity $v\rho l/\eta$, called *Reynold's number*, can be shown to have zero dimensions in mass, length and time. Such quantities were given the name '*numerics*' by *James Thomson.*

(*iv*) *Non-Dimensional Constants.*—These are mere numbers like 3, 2, π etc.

Thus, *numerics, pure numbers* and quantities like *heat, electricity, temperature* and *dielectric constant* have no dimensions in MLT.

The following *Table* shows at a glance the dimensional formulae for some important physical quantities.

Physical quantity	Dimensional formula
1. Area = (length)^2	$M^0L^2T^0$, or simply $[L^2]$
2. Volume = (length)^3	$M^0L^3T^0$, or simply $[L^3]$
3. Velocity = length/time	L/T, or M^0LT^{-1}, or $[LT^{-1}]$
4. Acceleration = velocity/time	$\frac{L}{T\times T}=\frac{L}{T^2}$ or M^0LT^{-2} or $[LT^{-2}]$
5. Momentum = (mass × velocity)	$M\times L/T$, or $[MLT^{-1}]$
6. Force = (mass × acceleration) = rate of change of momentum	$M\times\frac{L}{T\times T}=\frac{M\times L}{T^2}$, or $[MLT^{-2}]$
7. Work* = (force × distance or length)	$\frac{M\times L\times L}{T^2}=[ML^2T^{-2}]$
8. Couple* = (force × length)	$M\times(L/T^2)\times L=[ML^2T^{-2}]$
9. Kinetic Energy* = ($\frac{1}{2}$ mass × velocity^2)	$M\times[L^2/T^2]=[ML^2/T^2]$ or $[ML^2T^{-2}]$
10. Potential Energy* = (mass × acceleration due to gravity × distance)	$M\times\frac{L}{T^2}\times L=[ML^2T^{-2}]$
11. Power (*or rate of doing work*) = work/time	ML^2T^{-2}/T, or $[ML^2T^{-3}]$
12. Density = mass/volume	M/L^3, or $[ML^{-3}T^0]$ or $[ML^{-3}]$
13. Specific gravity = a *mere ratio.*	*No dimensions*
14. Pressure = force/area	MLT^{-2}/L^2, or $[ML^{-1}T^{-2}]$
15. Stress = force/area	MLT^{-2}/L^2, or $[ML^{-1}T^{-2}]$

*See Solved Example 1 (*b*), page 13.

*It will be noted that the demensions of *couple, kinetic energy* and *potential energy* are the same as those for *work*, because they are *mutually convertible* and *energy is just work.* Same is the case witn *pressure* and *stress.*

Physical quantity	Dimensional formula
16. Strain $=\frac{\text{change of length or volume}}{\text{original length or volume}}$ $=\frac{\text{length}}{\text{length}}$ or $\frac{\text{volume}}{\text{volume}}$ =a *mere number*.	No dimensions *No dimensions.*
17. Coefficient of Elasticity = stress/strain	$[ML^{-1}T^{-2}]$
18. Coefficient of Viscosity $=\frac{\text{force}}{\text{area}} \div \text{velocity}$ distance	$\frac{MLT^{-2}}{L^2} \div \frac{LT^{-1}}{L}$ $= ML^{-1}T^{-1}$
19. Surface tension=force/length, or, = energy/area	MLT^{-2}/L, or ML^2T^{-2}/L^2 $= [ML^0T^{-2}]$ or $[MT^{-2}]$
20. Frequency = 1/time 21. Angle = length/length = a *number*.	$1/T = T^{-1}$, or $[M^0L^0T^{-1}]$ *No dimensions.*

5. Uses of Dimensional Equations. A careful examination of the dimensional equations of the various physical quantities involved in a relation, *i.e.*, *an analysis of their dimensions*, is of great help to us in more ways than one,—the process being known as **dismensional analysis**. Its three chief uses are the following :

(*a*) *conversion of one system of units into another,*

(*b*) *checking the results arrived at,*

and (*c*) *deriving a correct relationship between different physical quantities.*

Let us consider these in some detail.

(*a*) **Conversion of one system of units into another.** It is seen that a physical quantity is expressed in terms of an appropriate unit of the *same nature*, its value being equal to the product of a number and that particular unit. Further, as shown in §1, its value remains the *same* on all systems of units. This affords us an easy method of changing over from one system of units to another.

Thus, suppose there is a physical quantity of dimensions *a*, *b* and *c* in mass, length and time respectively, *i.e.*, whose dimensional formula is $M^aL^bT^c$. Then, if its numerical value be n_1 in one system in which the fundamental units are M_1, L_1 and T_1, it is clearly equal to $n_1[M_1^aL_1^bT_1^c]$.

Also, if its numerical value be n_2 in *another* system of fundamental units M_2, L_2 and T_2, it is equal to $n_2[M_2^aL_2^bT_2^c]$ in this system.

So that, $$n_2[M_2^aL_2^bT_2^c] = n_1[M_1^aL_1^bT_1^c],$$

whence, $$n_2 = n_1\frac{M_1^aL_1^bT_1^c}{M_2^aL_2^bT_2^c}.$$

Or, $$n_2 = n_1 {}^{*}\left[\frac{M_1}{M_2}\right]^a \times \left[\frac{L_1}{L_2}\right]^b \times \left[\frac{T_1}{T_2}\right]^c \quad \ldots (i)$$

So that, knowing the fundamental units in the two systems and the numerical value of the quantity in one of them, its numerical value in the other system can be easily determined. Care must, however, be taken to apply relation (*i*) above, *after expressing the given quantity in absolute units.* Let us consider an example or two.

(1) *To convert a poundal,* (the unit of force in the *F.P.S.* system), *into dynes,* (the unit of force in the *C.G.S.* system).

We know that *force* has dimensions MLT^{-2} and that 1 *lb.*=453·6 *gms.*, and 1 *ft.* = 12×2·54 *cms.*=30·48 *cms.* So that,

M units in *F.P.S.* system = 453 6 *M* units in *C.G.S.* system,

L units in *F.P.S.* system = 30·48 *L* units in *C.G.S.* system,

and *T* units in *F.P.S.* system= *T* units in *C.G.S.* system,

the fundamental unit of time being the same, *viz., the second,* in the two systems.

$$\therefore\ MLT^{-2}\ poundals = (453{\cdot}6M)(30{\cdot}48\ L)T^{-2}.$$

Or, $$1\ poundal = \frac{453{\cdot}6M \times 30{\cdot}48\ L \times T^{-2}}{M \times L \times T^{-2}}.$$

log 453·6 = 2·6567
log 30·48 = 1 4840
Antilog 4·1407
= 1·382×10⁴

$$= 453{\cdot}6 \times 30{\cdot}48$$

$$= 1{\cdot}382 \times 10^4 \text{ units in the } C.G.S. \text{ system.}$$

Thus, 1 *poundal* = $1{\cdot}382 \times 10^4$ *dynes.*

(2) *To convert one Horse Power, (F. P. S. system), into Watts (C.G.S. system).*—We know that

1 *H. P.* = 550 *ft. lbs./sec.* = 550×32·2 *ft. poundals/sec.*

and $g = 32{\cdot}2$ *ft./sec.*^{2}

Again, as shown in Ex. (1),

M units in *F.P.S.* system = 453·6 *M* units in *C.G.S.* system,

L units ,, ,, = 30·48 *L* ,, ,, ,,

and *T* units ,, ,, = *T* ,, ,, ,,

Since the dimensional formula for power is ML^2T^{-3}, we have

$$ML^2T^{-3}\ H.P. = 550 \times 32{\cdot}2(453{\cdot}6M) \times (30{\cdot}48L)^2 \times T^{-3}.$$

$$\therefore\quad 1\ H.P. = 550 \times 32{\cdot}2 \times \frac{453{\cdot}6M \times (30{\cdot}48L)^2 \times T^{-3}}{M \times L^2 \times T^{-3}}.$$

log 550 = 2·7404
log 32 2 = 1·5079
log 453·6 = 2·6567
2 log 30 48 = 2 9680
9·8730
7 log 10 = 7·0000
Antilog 2·8730
= 746·4

$$= 550 \times 32{\cdot}2 \times 453{\cdot}6 \times (30{\cdot}48)^2\ ergs/sec.$$

$$= \frac{550 \times 32{\cdot}2 \times 453{\cdot}6 \times (30{\cdot}48)^2}{10^7}.$$

$$= 746{\cdot}4\ watts.$$

[Because, as we know, 1 *joule/sec.* = 1 *watt.*]

Thus, 1 *Horse Power* = 746·4 *watts.*]

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*This ratio M_1/M_2, if M_1 be the unit of mass in the *C.G.S.* system M_2, that in the *F.P.S.* system, *i.e.*, the ratio '*gram to the pound*' is called **conversion factor.**

(*b*) **Checking the results arrived at.** This depends upon what is called the *principle of homogeneity of dimensions*, again due to *Fourier*, according to which *the dimensions of all the terms on the two sides of an equation must be the same.* This follows at once from the fact that it is not possible to compare two physical quantities of different natures, and that only quantities of the *same* nature can be added up together, their resultant being also of the *same* nature. *If, therefore, in a given relation the terms on either side have the same dimensions, the relation is a correct one*, but if they have not, there is a flaw somewhere, which must be diligently sought out.

Let us again take a couple of examples :

(1) *To check the accuracy of the relation,* $t = 2\pi\sqrt{l/g}$, *for a simple pendulum.*

Here, the term t on the left hand side has only *one dimension in time*, or the dimension of t is $[T]$, its dimensions in both mass and length being *zero*.

And, on the right hand side, 2π has no dimensions, being just a number ; l has *one dimension in length*, or its dimension is $[L]$, those in mass and time being *zero* ; and the dimensions of g, the acceleration due to gravity, are LT^{-2}, that in mass being *zero*. Hence the dimensions of the term, $2\pi\sqrt{l/g} = \sqrt{L/LT^{-2}}$ or $\sqrt{T^2} = [T]$, *i.e., it has only one dimension in time, the same as the term on the left hand side.* The relation $t = 2\pi\sqrt{l/g}$ is, therefore, a correct one.

(2) *To check the relation* $S = ut + \frac{1}{2}at^2$, *for the distance covered in t seconds by a body, having an initial velocity u and an acceleration a.*

Here, the dimension of the term S on the left hand side is *one in length*, or $[L]$, and taking the terms on the right hand side, we have

	(*i*) dimensions of u (*velocity*)	=	LT^{-1}
	(*ii*) dimensions of t (*time*)	=	T
	(*iii*) dimensions of $\frac{1}{2}$ (*a number*)	=	*Nil*
	(*iv*) dimensions of a (*acceleration*)	=	LT^{-2}
and	(*v*) dimensions of t^2 ($time^2$)	=	T^2.
	$\therefore$ dimensions of the term, $ut + \frac{1}{2}at^2$	=	$LT^{-1} \times T + LT^{-2} \times T^2$,
		=	$L + L$,

i.e., the dimension of each term on the right hand side is the same as that of the term on the left hand side ; hence the given relation is correct.

A similar dimensional homogeneity will be observed in the case of any other relation, representing a physical phenomenon. The method of dimensions has thus a very definite *mnemonical value** and enables the beginner to resolve his confusion between two alternative possibilities occurring to him regarding a particular half forgotten formula, as, for example, whether the time-period of a simple pendulum is given by $t = 2\pi\sqrt{g/l}$ or by $t = 2\pi\sqrt{l/g}$, or whether the formula $4\pi r^2$ gives the surface area or the volume of a sphere etc., etc.

* *i.e.*, value as an aid to memory.

(*c*) **Deriving a correct relationship between different physical quantities.** The principle of homogeneity of dimensions also enables us to deduce a relationship between different physical quantities, or, at any rate, a preliminary form of such a relationship For, knowing the factors on which a physical quantity may possibly depend*,—and this requires a little *physical insight* and a certain amount of '*horse sense*'—an expression for it can be obtained in terms of these factors, such that the dimensions of the terms on the two sides of the expression are the same, *the only acceptable form of the relationship being the one which remains true irrespective of the system of units employed.* A few examples will illustrate the point.

(1) *To deduce an expression for the time-period of a simple pendulum.*

The factors on which the time-period (t) *may* possibly depend are the following :

(*i*) *the mass of the bob* (m),
(*ii*) *the length of the pendulum* (l),
(*iii*) *acceleration due to gravity* (g) *and*
(*iv*) *the angle of swing of the pendulum* (θ).

Let t be proportional to m^a, l^b, g^c and θ^d. So that,

$t = K.m^a l^b g^c \theta^d$, where K is a *constant of proportionality.*

Taking dimensions of the terms on either side of the sign of equality, we have

$$[T] = [M^a][L^b][LT^{-2}]^c = M^a L^b L^c T^{-2c}. \quad \left[\begin{array}{l}K \text{ and } \theta \text{ having no} \\ \text{dimensions.}\end{array}\right.$$

Or, $$T = M^a L^{b+c} T^{-2c}.$$

Since the dimensions of the terms on the two sides must be the *same*, we have, equating the indices of M, L and T,

$$a = 0, \quad b+c = 0 \quad \text{and} \quad -2c = 1,$$

whence, $$c = -\tfrac{1}{2} \text{ and } \therefore\ b - \tfrac{1}{2} = 0, \text{ or } b = \tfrac{1}{2}.$$

Therefore, $$t = K.l^{\frac{1}{2}}.g^{-\frac{1}{2}}.$$

Or, $$t = K\sqrt{l/g}.$$

The value of K can be found out experimentally†, and comes to 2π ; so that, the required relation is $t = 2\pi\sqrt{l/g}$.

It will easily be noted, from the above, that

(*i*) *the time-period of the pendulum is independent of its mass,*—a fact we know to be true by actual experience ;

and (*ii*) *the expression* $t\sqrt{g/l}$, *has no dimensions,* as it is equal to the *dimensionless constant* K, and is thus a *numeric.*

An important deduction emerges from this latter point, *viz.*, that if two pendulums having different lengths, (l_1 and l_2), oscillating

* It is absolutely necessary to take into account all possible *major factors* on which our result may reasonably be expected to depend, though one or more of these factors may get eliminated later. The method, however, ceases to give any worthwhile result if the number of variables included is more than six.

†The value of K can be determined easily by substituting in the relation obtained, the observed value of t, for known values of l and g.

at two different places, where the values of the acceleration due to gravity are g_1 and g_2 respectively, take time τ_1 and τ_2 to describe *equal arcs*, they may have

$$\tau_1\sqrt{g_1/l_1} = \tau_2\sqrt{g_2/l_2},$$

i.e., the value of the *non-dimensional expression* or the *numeric* $\tau\sqrt{g/l}$ may be the *same* for both.

And, if this be so, it means that the two pendulums pass through *exactly the same phase* for the same value of $\tau\sqrt{g/l}$. This is a case of what is called **dynamical similarity**, and all moving systems of this type are said to be *dynamically* similar.

A very interesting and a classic example of this principle is the comparison of the speeds of fully grown animals with those of their young.

Very reasonably, *taking the density of the two animals to be the same and their muscular strengths directly proportional to the cross-section of their limbs*, we have the ratio between their *densities* equal to *one* and similarly that between their *strengths per unit area of cross-section of their limbs*, also equal to *one* ; so that, if subscripts 1 and 2 refer to the adult animal and to its young respectively, we have

$$\text{ratio of their densities, i.e., } \frac{M_1}{L_1^3} \Big/ \frac{M_2}{L_2^3} = 1$$

and also *ratio between their muscular strengths per unit area, i.e.,*

$$\left(\frac{M_1L_1}{T_1^2} \div L_1^2\right) \Big/ \left(\frac{M_2L_2}{T_1^2} \div L_2^2\right) = 1.$$

From these two relations then, we easily get

$$\frac{L_1}{T_1} = \frac{L_2}{T_2},$$

where L_1/T_1 is the *speed* of the full grown animal and L_2/T_2, that of its young.

The speeds of the two animals are thus the same,—a result which, at first sight, appears to be simply ridiculous. And yet it is an actual fact, *the shorter strides of the young being taken faster than the longer ones of the adult.*

(2) *To deduce a relationship for the velocity of sound in a material medium, the temperature of the medium remaining constant.*

The velocity V may depend upon (*i*) *the elasticity of the medium* E and (*ii*) *the density of the medium*, ρ ; so that,

$$V = K.E^a.\rho^b, \text{ where } K \text{ is a constant.}$$

Again, taking dimensions of the terms on both sides, we have

$$M^0LT^{-1} = \left[\frac{MLT^{-2}}{L^2}\right]^a\left[\frac{M}{L^3}\right]^b$$

$$= [ML^{-1}T^{-2}]^a[ML^{-3}]^b$$

$$= M^aL^{-a}T^{-2a}\,M^bL^{-3b}$$

$$= M^{a+b}L^{-a-3b}T^{-2a}.$$

[$\because$ *elasticity* = *stress*/*strain* = $\frac{\text{force}/\text{area}}{\text{a ratio}}$, and *density* = *mass*/*volume*.]

Since the dimensions on the two sides must be the *same*, it is clear that $a+b = 0$; $-a-3b = 1$, and $-2a = -1$,

whence, $a = \frac{1}{2}$ and $b = -a = -\frac{1}{2}$.

Hence, $V = K.E^{\frac{1}{2}}\,.\,\rho^{-\frac{1}{2}}$.

Or, $V = K\,.\,\sqrt{E/\rho}$.

The value of K is again determined by experiment, and it found to be 1 in this case ; so that, $V = \sqrt{E/\rho}$.

So far only simple cases have been considered. In other cases the method used above may not always be applicable. Let us consider one such typical example by way of illustration of the method adopted in such cases

(3) *To obtain a relation between the distance travelled by a body in time t, if its initial velocity be u and acceleration a.*

Let the distance covered by the body in time t be represented by $S = K.u^a.a^b.t^c$ Then, taking dimensions, we have

$$[L = \left[\frac{L}{T}\right]^a\left[\frac{L}{T^2}\right]^b\left[T\right]^c = [LT^{-1}]^a[LT^{-2}]^b\,T^c.$$

Or, $$[L] = L^aT^{-a} \times L^bT^{-2b} \times T^c = L^{a+b}T^{-a-2b+c}.$$

Since the dimensions on the two sides must be the *same*, we have

$$a+b = 1 \ldots(i) ; \qquad -a-2b+c = 0 ; \text{ or, } a+2b-c = 0 \quad \ldots(ii)$$

These two equations alone are *not enough* to give us the values of a, b and c. Hence we proceed as follows :

Suppose the body has no acceleration. Then,

$S = K'u^at^c$, where K' is another constant.

Taking dimensions, we have

$$L = L^aT^{-a}T^c = L^aT^{c-a},$$

whence, $a = 1$; and $c-a = 0$, or $c = a = 1$,

$$\therefore \qquad S = K' \,.\, ut. \qquad \ldots(A)$$

Now, suppose the body has no initial velocity. Then,

$S = K'' a^b t^c$, where K'' is yet another constant.

Again, taking dimensions, we have

$$L = L^b\,T^{-2b}\,T^c = L^b\,T^{c-2b}.$$

$$\therefore \qquad b = 1 ; \text{ and } \quad c-2b = 0, \text{ or } \quad c = 2b = 2.$$

Hence $$S = K''.at^2. \qquad \ldots(B)$$

If, therefore, a body has both, initial velocity as well as acceleration, its equation of motion contains both the expressions, (A) as well as (B) ; so that, we have

$$S = K'.ut+K''.at^2,$$

where the constants K' and K'' can be determined experimentally, and are found to be equal to 1 and $\frac{1}{2}$ respectively. Thus, the required relation comes to be $S = ut+\frac{1}{2}at^2$.

In addition to the three chief uses of dimensional analysis, discussed above, mention may also be made here of a couple of others. Thus,

(*iv*) *it is helpful in selecting experiments likely to give some useful information and avoiding others.* In this connection, *Lord Rayleigh's* remark is worth quoting. Says he, 'I have often been impressed by the scanty attention paid even by original workers in Physics to the great principle of similitude. It happens not infrequently that results in the form of 'laws' are put forward as novelties on the

basis of elaborate experiments which might have been predicted *a priori* after a few minutes' consideration.' How true, indeed !

(*v*) Then, again, it is a powerful aid to mathematical analysis, when the problem happens to be a complex one and when no experiments to solve it are possible. Even if the number of variables involved in the problem be a large one, dimensional analysis does help obtain at least a partial solution of it.

Limitations of Dimensional Analysis. It will be readily seen from the examples, given above, that the method of dimensional analysis is after all not quite so simple or straight in its application, except in obviously easy cases. Very helpful, as far as it goes, it has also its own limitations. Thus, for example :

(*i*) Its one obvious drawback is that *it gives little or no information about pure numerics* (like $t\sqrt{g/l}$ in Ex. 1) *and non-dimensional constants* (like K in Ex. 2), *involved in various physical relations*, and which, therefore, have to be determined by separate calculation or experiment.

(*ii*) Then, again, since at best *only three* equations can be obtained by equating the dimensions of $[M]$, $[L]$ and $[T]$; *the method is of no avail in deducing the exact form of a physical relation which happens to depend upon more than three quantities.* For, clearly, of a given number of quantities involved, the indices of only three can be expressed in terms of the rest, thus leaving us with a relation between the remaining number* of non-dimensional groups of terms ; so that, what we may ultimately succeed in obtaining is just an equation in terms of an undetermined function.

It will thus be clear that, while the method of dimensional analysis remains unrivalled and almost unique, in so far as conversion from one system of units into another and checking the correctness of physical relations are concerned, its use is not quite so safe or certain when it comes to establishing a definite or exact relationship between a given set of physical quantities and, particularly so, in the hands of beginners.

More often than not, the success of the method depends upon the proper choice of *dimensional constants* (like G or c), which have to be introduced as additional variables. And, it needs a trained, subtle and intuitive mind, with the solid background of a mature and a comprehensive knowledge of the subject, to decide, on the basis of analysis or experience or perhaps just on that of some sort of inspiration of the moment, what particular variables to select, and how, when and where to introduce them. A very apt illustration in support of these remarks is perhaps Raleigh's explanation, by the method of dimensions, as to *why the sky is blue.*

That the colour of the sky is due to the scattering of light by suspended drops of moisture and dust particles etc. (of molecular size) in the atmosphere is fairly well known. From this basic fact, Raleigh proceeds as follows :

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*viz., the given number of quantities *minus* three.

Let A_s be the *amplitude of the scattered wave.* Then, the *possible* factors on which it may depend are

(*i*) A_i the *amplitude of the incident wave of light,*

(*ii*) l, the *linear dimension of the scattering particle,*

(*iii*) r, the *distance from the particle,*

and (*iv*) λ, the *wave-length of light.*

So that, expressing A_s in terms of all these variables, we have

$$A_s = K \cdot A_i^a \, l^b \, r^c \, \lambda^d,$$

where K is a constant of proportionality.

Or, taking dimensions, we have

$$L = L^a \cdot L^b \cdot L^c \cdot L^d,$$

for, obviously, the dimensions of all these quantities are the same. *viz.*, L, and those of $K = 0$,

We, therefore, have

$$1 = a+b+c+d.$$

Now, we know that the amplitude of the scattered light is (*i*) directly proportional to that of the incident light and (*ii*) inversely proportional to its distance from the scattering particle. This at once gives us $a = 1$ and $c = -1$. And, therefore,

$$1 = 1+b-1+d, \text{ whence, } d = 1-b.$$

So that,
$$A_s = K \cdot A_i \, l^b \, r^{-1} \, \lambda^{1-b} = K \cdot \frac{A_i \, \lambda}{r} \left(\frac{l}{\lambda}\right)^b.$$

Now, as Rayleigh remarks, 'from what we know of the dynamics of the situation'* l varies directly as the volume of the scattering particle. And, therefore, $b = 3$.

Hence
$$A_s = K \cdot \frac{A_i \, l^3}{r \, \lambda^2}. \quad \text{Or,} \quad A_s \propto 1/\lambda^2.$$

And since *intensity* $\propto$ (*amplitude*)^2, we have

intensity of scattered light, $I_s \propto 1/\lambda^4$.

It thus follows, as a natural consequence, that the wave-length of blue light being roughly half that of red light, the intensity of scattered blue light is sixteen times that of scattered red light and that the sky, therefore, appears to us to be blue.

The student will appreciate how, in capable hands, the method of dimensional analysis can be made to yield results beyond the pale of elementary analysis.

SOLVED EXAMPLES

1. Deduce the dimensions of (*a*) the Coefficient of Viscosity, and (*b*) the Constant of Gravitation (G).

Obtain a formula for the time of swing of a simple pendulum from a knowledge of the dimensions of the physical quantities involved. (*Punjab*)

(*a*) We know that the coefficient of viscosity (η) of a liquid is given by the relation,

$$\eta = \pi P r^4 / 8vl,$$

where P is the *pressure difference* between the two ends of the capillary tube ; r, its *radius* ; l, its *length* and v, the *rate of flow of the liquid* through it, or the *volume of liquid flowing out per second*

---

viz., the ratio of the respective amplitudes of the incident and reflected light.

Therefore, taking dimensions of all these quantities, we have

dimensions of P = $ML^{-1}T^{-2}$ [see Table on page 5

,, ,, r^4 = L^4

,, ,, v = L^3T^{-1} [$\because$ *rate of flow* = volume/time.

,, ,, l = L

and ,, ,, π and 8 are *zero*. [Both being *numbers*.

Hence, dimensions of coefficient of viscosity η, are

$$\frac{ML^{-1}T^{-2}\times L^4}{LT^{-1}\times L} = ML^{-1}T^{-1}.$$

(*b*) We know that the value of G, the Garvitational Constant, is given by the relation, $G = C\theta d^2/M.m.l.$,

where C is *the restoring couple per unit twist of the wire;* θ, *the angle of twist of the wire* ; d, *the distance between the centres of the near large and small balls* ; M *and* m, *the masses of the large and small balls respectively* and l, *the length of the torsion rod, (Cavendish's experiment).*

Therefore, taking the dimensions of the quantities involved, we have

dimensions of C (*couple*) = ML^2T^{-2} [See Table on page 5.

,, ,, θ (*angle*) = 0

,, ,, d^2 = L^2

,, ,, M = M

,, ,, m = M

,, ,, l = L

Hence *dimensions of* G are $\frac{ML^2T^{-2}L^2}{MML}$ or $M^{-1}L^3T^{-2}$.

For answer to the second part of the question, see page 9, (Ex. 1).

2. Find the unit of length if one minute be the unit of time ; one stone, the unit of mass, and one pound-weight, the unit of force. (g = 32·2 ft per sec^2).

We know that 1 *lb. wt.* = *g poundals* = 32·2 *poundals*, and that the dimensions of force are MLT^{-2}.

Now, M units in the new system = $(14M)$ units in the ordinary system, [$\because$ 1 *stone* = 14 *lbs.*

and let T ,, ,, ,, ,, = $(60T)$ units in the ordinary system,

L ,, ,, ,, ,, = (xL) units in the ordinary system.

log 32·2 = 1·5079

2 log 60 = 3·5564

5·0643

log 14 = 1·1461

Antilog 3·9182

= 8283

Then, 32·2 MLT^{-2} = $14M\times xL\times[60T]^{-2}$ units in *this* system

= $14M.xL.60^{-2}T^{-2}$ units ,, ,,

Or, $32\cdot2=\frac{14x}{60^2}$. And $\therefore$ $x=\frac{32\cdot2\times60^2}{14}=8283$.

Or, the *unit of length in the new system would be* xL

$=8283\times L$, *i.e.*, equal to 8283 *ft.* [since $[L]=1$ *ft.*

3. If the acceleration due to gravity be represented by unity and one second be the unit of time, what must be the unit of length ?

In the ordinary system, in which the unit of length $[L]$ = 1 *cm.*, and that of time $[T]$ = 1 *sec* , we have

unit of acceleration, [dimensions LT^{-2}] = 1 *cm./sec.*^{2}, and acceleration due to gravity equal to 981 *cm./sec*^{2}.$=981\ LT^{-2}$.

If the unit of length, *in the new system*, be L_1, we have

L acceleration due to gravity, on this system, $= 1\times L_1\ sec^2. = L_1T^{-2}$, the unit of time being the *same, i.e.*, 1 *second*, in this system also.

$\therefore$ $L_1T^{-2} = 981\ LT^{-2}$. [§1.

Or, $L_1 = 981\ L$; that is L_1 = 981 *cms.*, since L = 1 *cm.*

Thus, the unit of length in the new system is equal to 981 *cms.*

4. Given that the unit of power is one million ergs per minute, the unit of force is 1000 dynes and the unit of time, 1/10 sec., what are the units of mass and length ?

Here (*a*), unit of power, [dimensions ML^2T^{-3}] = 1000,000 *ergs/mt.*

= 1000,000/60 *ergs per sec.*

(*b*) unit of force, [dimensions MLT^{-2}] = 1000 *dynes*,

and (*c*) unit of time [dimension T] = 1/10 *sec.*

$\therefore$ multiplying (a) by (c), we have

unit of power × unit of time = unit of work.

$$= [ML^2T^{-3}][T] = ML^2T^{-2}.$$

$$= \frac{1000,000}{60} \times \frac{1}{10} = \frac{10^4}{6}.$$

Dividing *this unit of work by the unit of force*, we have

$$\textit{unit of distance or length} = \frac{10^4}{6} \times \frac{1}{1000} = \frac{10}{6} = \frac{5}{3} \textit{ cms.}$$

Now, from (b) we have

$$\textit{unit of mass} = MLT^{-2} \div \frac{L}{T^2} = MLT^{-2} \times \frac{T^2}{L} \quad \left[\because \textit{mass} = \frac{\text{force}}{\text{acc.}} \right.$$

Or, $\textit{unit of mass} = \dfrac{1000 \times (1/10)^2}{5/3} = \dfrac{1000 \times 3}{100 \times 5} = 6 \textit{ gms.}$ $\left[\begin{array}{l} \because \textit{unit of force} = 1000 \textit{ dynes} \\ \text{and } \textit{unit of acc.} = \dfrac{5/3}{(1/10)^2} \end{array} \right.$

Therefore, the units of mass and length, in the given system, are 6 gms. and 5/3 cms., respectively.

5. If the fundamental units are the velocity of light in air, the acceleration of gravity at Greenwich, and the density of mercury at 0°C, find the units of mass, length and time. (*Velocity of light* $= 3 \times 10^{10}$ *cm./sec* ; *acceleration of gravity at Greenwich* $= 9{\cdot}81 \times 10^2$ *cm./sec.*^{2} ; *density of mercury* $= 13{\cdot}6$ *gm. per c.c.*).

Here, (a) *unit of velocity,* (dimensions LT^{-1}) $= 3 \times 10^{10}$ *cm./sec.*

(b) ,, ,, *acceleration,* (,, LT^{-2}) $= 9{\cdot}81 \times 10^2$ *cm./sec.*^{2}

(c) ,, ,, *density,* (,, ML^{-3}) $= 13{\cdot}6$ *gm./cm.*^{3}

log 3	$=0{\cdot}4771$
8 log 10	$=8\ 0000$
	$8\ 4771$
log 9·81	$=0\ 9917$
Antilog	$7{\cdot}4854$
	$=3{\cdot}058 \times 10^7$
log	$13{\cdot}6=1{\cdot}1335$
log	$(9{\cdot}174 \times 10^{17})$
	$=53\ 8878$
Antilog	$55{\cdot}0213$
	$1{\cdot}051 \times 10^{55}$

$\therefore$ dividing (a) by (b), we have

$$\textit{unit of time, } (T) = \frac{3 \times 10^{10}}{9{\cdot}81 \times 10^2} = \frac{3 \times 10^8}{9{\cdot}81}.$$

$$= 3{\cdot}058 \times 10^7 \textit{ seconds.}$$

Substituting this value in (a), we have

$$\textit{unit of length, } (L) = LT^{-1}.T.$$

$$= 3 \times 10^{10} \times 3{\cdot}058 \times 10^7 = 9{\cdot}174 \times 10^{17} \textit{ cms.}$$

And, from (c), we have

$$\textit{unit of mass, } (M) = ML^{-3} \times L^3 = 13{\cdot}6 \times (9{\cdot}174 \times 10^{17})^3.$$

$$= 1{\cdot}051 \times 10^{55} \textit{ gms.}$$

Thus, the required units of mass, length and time are $1{\cdot}051 \times 10^{55}$ *gms.*, $9{\cdot}174 \times 10^{17}$ *cms. and* $3{\cdot}058 \times 10^7$ *seconds, respectively.*

6. If the units of length and force be each increased four times, show that the unit of energy is increased sixteen times.

We have *unit of energy = unit of force × unit of distance.*

$$= [MLT^{-2}] \times L = ML^2T^{-2}.$$

If now, the units of force and distance be made *four times each*, they would be $4[MLT^{-2}]$ and $4L$ respectively, and, therefore, the *new unit of energy* would be $4MLT^{-2} \times 4L = 16ML^2T^{-2}$, which is *sixteen times* ML^2T^{-2}, the ordinary unit.

Thus, we see that by increasing the unit of force and length four times each, the unit of energy is increased sixteen times.

7. Show by the method of dimensions that the relation, $C = n\pi r^4/2l$ **for the couple per unit twist of a wire of length** l**, radius** r **and cefficient of rigidity** n**, is a correct one.**

Let us take the dimensions of the terms on the two sides of the sign of equality and see if they are the same. Thus,

dimensions of C (*couple*)	=	ML^2T^{-2}
,, ,, n (*rigidity*)	=	$ML^{-1}T^{-2}$ [*same as for elasticity*]
,, ,, r^4 (*radius*)^4	=	L^4
,, ,, l (*length*)	=	L

and, π and 2, being *numbers*, have no dimensions.

Therefore, the dimensions of the term $n\pi r^4/2l$ are

$$\frac{ML^{-1}T^{-2}\times L^4}{L}=ML^2T^{-2},$$

the *same as for C*, on the left hand side.

Hence, the relation $C = n\pi r^4/2l$ is a correct one.

8. Test by the method of dimensions the accuracy of the relation $t = 2\pi\sqrt{(k^2+l^2)/g}$, for the time-period of a compound pendulum.

If the relation be correct, the dimensions of the terms on either side of the sign of equality must be the same.

Let us put the relation as $t = 2\pi\sqrt{\frac{K^2}{lg}+\frac{l}{g}}$.

Now, the dimensions of $t = [T]$

,, ,, $K^2 = [L^2]$, K being the *radius of gyration.*

,, ,, $l = [L]$

,, ,, $g = [LT^{-2}]$.

Therefore, the dimensions of the term on the right hand side are

$$=\sqrt{\frac{L^2}{LLT^{-2}}+\frac{L}{LT^{-2}}}=\sqrt{T^2+T^2}=$$

Thus, we see that *the dimensions of the terms on either side are the same, viz.*, $[T]$. The relation is, therefore, a correct one.

9. Find the dimensions of velocity and acceleration. Assuming that when a body falls from rest under gravity the velocity v is given by Kg^ph^q, where h is the distance fallen through, g, the acceleration of gravity and K, p and q are constants. Show, by a consideration of the dimensions involved, that $v=K\sqrt{gh}$.
(London Higher School Certificate)

For answer to part one of the question, see *Table* on page 5.

We are given that $v = Kg^ph^q$. Taking dimensions, therefore, we have

dimensions of $v = \left[\frac{L}{T}\right] = LT^{-1}$.

,, ,, $g^p = \left[\frac{L}{T^2}\right]^p = [LT^{-2}]^p = L^pT^{-2p}$,

dimensions of $h^q = [L]^q = L^q$.

,, ,, $K = 0$, for it is a *constant* or *a mere number.*

Therefore, the dimensions of the term, Kg^ph^q,

$$=L^pT^{-2p}L^q=L^{p+q}T^{-2p}.$$

Since dimensions on both sides of the sign of equality must be the same, we have

$$LT^{-1} = L^{p+q}\,T^{-2p}.$$

Or, $p+q = 1$ and $-2p = -1$, whence, $p = \frac{1}{2}$ and $q = \frac{1}{2}$.

$\therefore$ $Kg^ph^q = Kg^{\frac{1}{2}}\,.\,h^{\frac{1}{2}} = K\sqrt{gh}$.

Or, $v = K\sqrt{gh}$.

10. The frequency of vibration (n) of a stretched string is a function of the tension (T), the length (l) and the mass per unit length (ρ). Prove that

$$n \propto \frac{1}{l}\sqrt{\frac{T}{\rho}}.$$

Let $n \propto T^a l^b \rho^c$. Then, taking dimensions, we have

dimensions of $n = 1/t = M^0L^0T^{-1}$, or $= T^{-1}$,

dimensions of T^a, $(force)^a = [MLT^{-2}]^a = M^aL^aT^{-2a}$,

,, ,, l^b, $(length)^b = L^b$,

,, ,, ρ^c, $\left[\frac{mass}{length}\right]^c = \left[\frac{M}{L}\right]^c = M^cL^{-c}$,

$\therefore$ dimensions of the term $T^aL^b\rho^c = M^aL^aT^{-2a}.L^b.M^cL^{-c}$,

Or, $T^{-1} = M^{a+c}.L^{a+b-c}.T^{-2a}$.

Since dimensions of n must be the same on both sides, we have $a+c=0$; $a+b-c=0$ and $-2a=-1$, or $2a=1$ *i.e.*, $a=\frac{1}{2}$, and hence $\frac{1}{2}+c=0$, or $c=-\frac{1}{2}$.

Also $\frac{1}{2}+b-(-\frac{1}{2})=0$, or $\frac{1}{2}+b+\frac{1}{2}=0$, *i.e.*, $1+b=0$, or $b=-1$.

Therefore, $T^aL^b\rho^c = T^{\frac{1}{2}}.l^{-1}.\rho^{-\frac{1}{2}} = \frac{1}{l}\sqrt{\frac{T}{\rho}}$.

And hence $n \propto \frac{1}{l}\sqrt{\frac{T}{\rho}}$.

11. The time of oscillation (t) of a small drop of liquid under surface tension depends only on the density (ρ), the radius (a), and the surface tension (T). Show that the period of oscillation is $K\rho^{\frac{1}{2}}.a^{\frac{3}{2}}.T^{-\frac{1}{2}}$, where K is a numeric.

Let $t = K\rho^a a^b T^c$. So that, taking dimensions, we have

dimensions of $t = T$,

,, ,, $\rho^a = \left[\frac{M}{L^3}\right]^a = M^aL^{-3a}$,

,, ,, $a^b = [L]^b = L^b$.

,, ,, $T^c = [MT^{-2}]^c = M^cT^{-2c}$, [See page 5.

and K has *no dimensions*, being a *numeric*.

$\therefore$ dimensions of the term $K\rho^a a^b T^c = M^aL^{-3a}L^bM_cT^{-2c}$.

Or. $= M^{a+c}L^{-3a+b}T^{-2c}$.

Since the dimensions of the terms on both sides must be the same, we have

$a+c=0$; $-3a+b=0$, and $-2c=1$.

Or, $c=-\frac{1}{2}$ and $\therefore$ $a=\frac{1}{2}$ and $b=\frac{3}{2}$.

$\therefore$ $K\rho^a a^b T^c = K\rho^{\frac{1}{2}}.a^{\frac{3}{2}}.T^{-\frac{1}{2}}$.

Hence *the time of oscillation of the drop*, $t = K\rho^{\frac{1}{2}}.a^{\frac{3}{2}}.T^{-\frac{1}{2}}$.

12. Explain the Principle of Homogeneity of dimensions in a physical equation.

Assuming that the mass M of the largest stone that can be moved by a flowing river depends on V, the velocity, ρ, the density of water and on g, show that M varies with the sixth power of the velocity, of flow in the river.

(Punjab)

Let M depend upon V^a, ρ^b and g^c.

So that, $M = KV^a\rho^b g^c$. [K being a constant.

Taking dimensions, we have

dimensions of $M = [M]$

,, ,, $V^a = \left[\frac{L}{T}\right]^a = L^aT^{-a}$

,, ,, $\rho^b = \left[\frac{M}{L^3}\right]^b = M^bL^{-3b}$

,, ,, $g^c = [LT^{-2}]c = L^cT^{-2c}$

and K has *no dimensions*, being a mere *number*.

$\therefore \quad M = L^a T^{-a} M^b L^{-3b} L^c T^{-2c}$. Or, $M = L^{a-3b+c} M^b T^{-a-2c}$.

Since the dimensions on both sides must be the same, we have

$$a-3b+c = 0,\ b = 1, \text{ and } -a-2c = 0, \text{ or } a+2c = 0,$$

So that, $a+c = 3$ and $a+2c = 0$. And $\therefore\ c = -3$ and $a = 6$,

Hence $M = KV^6 \rho . g^{-3}$.

Or, $M \propto V^6$.

i.e., the mass M varies with the sixth power of the velocity of flow.

EXERCISE 1

1. If 10000 *gms.* be the unit of mass, 60 *secs.*, the unit of time, and the acceleration due to gravity (981 *cms./sec².*), the unit of acceleration, what would be the unit of energy in *ergs* ? **Ans.** $3{\cdot}465 \times 10^{12}$ *ergs.*

(2) Convert by the method of dimensions, $4{\cdot}2 \times 10^7$ *ergs* into *foot-pounds* (1 *ft.* = 30·48 *cms.*, and 1 *lb.* =453·6 *gms.*). **Ans.** · 96 6 *ft. poundals.*

3. Deduce the dimensions of (*i*) *speeific gravity*, (*ii*) *surface density* and (*iii*) *angular velocity.*

Show that the kinetic energy of a body of mass *m*, moving with a velocity *v*, is given by kmv^2, where *k* is a constant.

Ans. (*i*) $M^0L^0T^0$; (*ii*) $ML^{-2}T^0$; (*iii*) $M^0L^0T^{-1}$.

4. Test, by the method of dimensions, the accuracy of the following relations :

(*i*) $v^2-u^2 = 2aS$, connecting *initial velocity u, final velocity v, acceleration a* and *distance S* covered by a body.

(*ii*) $S = ut + \frac{1}{2}at^2$, connecting *distance S with initial velocity u, time t* and *acceleration a* of the body,

(*iii*) $\rho = \frac{3g}{4\pi rG}$, where ρ is the *mean density of the earth, r,* its *radius g,* the *acceleration due to gravity* and *G,* the *gravitational constant.*

5. Assuming that the excess pressure (*p*) inside a soap bubble depends on (*i*) the surface tension (*T*) of the soap film and (*ii*) its radius (*r*), show, by your knowledge of dimensions, that it is directly proportional to the former and inversely proportional to the latter.

[**Hint** — *Simply show that* $p = k.T/r$, *whence it follows that p varies directly as T and inversely as r.*]

6. A drop of liquid is suspended in another liquid of the *same* density but with which it is *immiscible.* If the drop is distorted from the spherical shape and released, deduce, by dimensional methods, a formula for its period of oscillation (*t*), given that the latter depends on *surface tension T, density* ρ and *drop-radius r.*

Ans. $t = k\sqrt{\frac{\rho r^3}{T}}$, where *k* is a constant.

(7) Convert, by the method of dimensions, a pressure of 14 *pounds wt. per square inch into dynes per sq. cm.* **Ans.** $7{\cdot}912 \times 10^4$ *dynes/cm².*

8. Show that when bodies of geometrically similar form and of the same material, differing only in dimensions, vibrate in the same manner, the vibrations being due to the elasticity of the material, their periods are proportional to their dimensions.

[**Hint.**—*Proceeding in the usual manner, show that* $t = kl/\sqrt{\rho/E}$, *where l is the linear dimension of the body,* ρ, *its density and E, the elasticity of the material. Since* ρ *and E are the same for all bodies, t varies directly as l.*]

(9) Calculate, by the method of dimensions, the number of *foot-pounds* in *one calorie.* (Given that 1 *calorie*=$4{\cdot}2 \times 10^7$ *ergs* ; g=32 *ft./sec².* ; 1 *lb.*=453·6 *gms.*, and 1 *inch*=2·54 *cms.*). **Ans.** 3·115.

10. If in a system of units, the unit of length be 1 *mile* and that of time, 1 *hour,* what will be the value of '*g*' ? **Ans.** 14·88 *miles/sec².*

11. The time of oscillation *t* of a small drop of a liquid under surface tension depends upon the density *d,* radius *r* and surface tension *S.* Prove dimensionally that $t \propto \sqrt{\frac{dr^3}{S}}$. (*Punjab, 1947*)

12. Explain what you mean by the dimensions of a physical quantity; calculate the dimensions of Young's modulus.

Assuming that the period of vibration of a tuning fork depends upon the length of the prongs, and on the density and Young's modulus of the material, find, by the method of dimensions, a formula for the period of vibration.
(Calcutta, 1950)

Ans. $[ML^{-1}T^{-2}]$; $t \propto l\sqrt{d/Y}$,

(where t is the *period of vibration*; l, the *length*; d, the *density* and Y, the value of *Young's Modulus* for the material of the fork.)

*13. Using the method of dimensions, obtain an expression for

(*i*) the acceleration of a particle moving with a uniform speed v, in a circle of radius r;

(*ii*) the tension T in a uniform circular wire of radius r and mass m per unit length, rotating in its own plane with an angular velocity ω, about an axis passing through its centre and perpendicular to its plane;

(*iii*) the mass M of a planet round which a satellite completes its orbit of radius r, in a time-interval T.

Ans. (*i*) $K.v^2/r$; (*ii*) $K.mr^2\omega^2$, where K is a constant; (*iii*) $M \propto r^3/GT^2$.

*14. Obtain an expression for the height h to which a liquid, of density ρ and surface tension T will rise in a capillary tube, of radius r, given that $h \propto 1/r$.

Ans. $h = k.\dfrac{T}{r.\rho.g}$, ($k$ being a constant).

*15. Assuming that the viscosity η of a gas is proportional to the mean free path λ of its molecules, show that, if the temperature be kept constant, it is independent of the density ρ of the gas.

[**Hint.**—First obtain an expression for η, in terms of ρ, λ, c, (the root mean square velocity of the molecules) and D, the diameter of a molecule. Then, since $\eta \propto \lambda$, we shall have $\eta = k.\rho.c.\lambda$, (where k is a constant). Again, since ρ is inversely proportional to λ, *i.e.*, $\rho = k'/\lambda$, (where k' is another constant), we shall have $\eta = k.k'c$, showing that η is independent of ρ.]

*16. Show that if the linear dimensions of the whole of Cavendish's or Boys' method for the determination of G be changed, the sensitiveness of the apparatus remains the same.

*17 Show that the volume of a liquid, of coefficient of viscosity η, flowing per second through a tube of circular cross-section is given by $V = \pi pr^4/8\eta l$, where p is the excess pressure between the ends of the tube, r, its radius and l, its length.

*18. If the resistance of a liquid to the motion of a body through it with a velocity v, be proportional to v^2, show that it is quite independent of the viscosity of the liquid.

*19. A Nicholson's hydrometer of mass m, floating in a liquid of density ρ, is given a slight downward displacement and then released. Obtain an expression for the time-period T of its oscillation. (Assume the area of cross-section of its neck to be a.) Ans $T = 2\pi\sqrt{m/a\rho g}$.

*20. A U-tube of uniform cross-section contains mercury up to a height h in either limb. The mercury in one limb is depressed a little and then released. Obtain an expression for its time-period of oscillation. Ans $T = \pi\sqrt{2h/g}$.

[**Hint.**—Just put $T = k.d^a h^b g^c$ and show that

$$T = K.\left(\frac{d}{h}\right)^a \sqrt{\frac{h}{g}},$$

where T is the time-period of oscillation of mercury and d, its initial displacement, K being the usual constant of proportionality. For small values of d, $a = 0$ and experiment gives $K = \pi\sqrt{2}$. Substitute their values and obtain the result.]

Note.—The questions marked with an asterisk are of rather an advanced character and may be attempted when some confidence has been gained with others.

CHAPTER II

MOTION ALONG A CURVE—THE PROJECTILE

7. Rotation Suppose we have a rigid body, with a fixed axis, within or without it. Then, if a force be applied to it, it cannot move bodily, as a whole, relatively to the axis, *i.e.*, no *motion of translation* is possible ; but it simply moves round or *rotates* about the axis. such that every particle of it undergoes the *same* angular displacement. A body, so rotating about a fixed axis, is said to perform *rotatory* or *circular motion.*

The force, producing rotatory motion about the fixed axis, called the *axis of rotation*, is said to have a *moment about that axis*, which is measured by the *product of the force and the perpendicular distance between its line of action and the axis of rotation.* Obviously, therefore, if either of these be zero, *the moment, or the turning tendency of the force*, will be zero, for the product of the force and perpendicular distance between the axis and the line of action of the force is, then, zero.

If the rotation produced be *anti-clockwise*, the moment of the force is said to be *positive*, and if it be in the *clockwise* direction, the moment is said to be *negative*. And, since the moment of a force is a *vector quantity*, it follows that *if a number of forces act simultaneously on a body, the algebraic sum of their individual moments about the given axis of rotation will be equal to the moment of their resultant about it.*

8. Angular Velocity. Let a body rotate about a fixed axis through O, (Fig. 1). Then, the particles composing it, at any distance from O, such as at A, B, C, etc., complete one rotation in the *same* time *i.e.*, they describe the *same angle* in the *same time*, and, therefore, the angle described by them per unit time is the same. This angle described by a rotating body per unit time is called its *angular velocity* and is usually denoted by the Greek letter ω. Thus, if the rate of rotation of a body be *uniform*, *i.e.*, if its angular velocity be constant, and it describes an angle θ (*radians*) in time t (*seconds*), we have

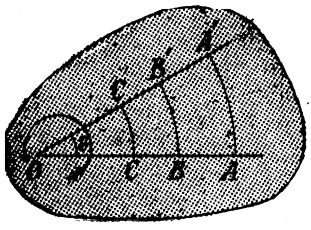


Fig. 1.

angular velocity of the body, $\omega = \theta/t$.

If the body makes n rotations in time t, the angle described by it is equal to $2\pi n$. Or, $\theta = 2\pi n$.

And, therefore, its *angular velocity* $\omega = 2\pi n/t$.

If, however, the velocity be not constant, it may, at a given instant, be expressed in the form $\omega = d\theta/dt$, where $d\theta$ is the small angle described by it in the small interval of time dt.

Now, although the angle θ described by all the particles of the body in a given time t is the *same*, the *linear distances* travelled by them are *different*. Thus, the particles at A, B, and C, (Fig. 1), cover the linear distances AA', BB' and CC' respectively, (which are *arcs of radii* OA, OB, OC), depending upon their respective distances from the axis of rotation through O.

If $OA = r_1$, $OB = r_2$ and $OC = r_3$,

clearly, arc $AA' = r_1\theta$, [$\because$ *arc = radius × angle subtended by it.*

,, $BB' = r_2\theta$,

and ,, $CC' = r_3\theta$.

$\therefore$ linear velocity of $A = r_1\theta/t$, that of $B = r_2\theta/t$, and that of $C = r_3\theta/t$.

Or, in general, *linear velocity v of a particle at a distance r from the axis of rotation is $r\theta/t$.*

Or, $v = r\omega$, [$\because \theta/t = \omega$.

i.e., linear velocity = distance from the axis of rotation × angular velocity.

9. Angular Acceleration. If the angular velocity of a rotating body be not constant, it is said to have an *angular acceleration*, which is defined as the *rate of change of angular velocity*. It is usually denoted by the symbol $d\omega/dt$. Thus, if the angular velocity of a particle about a given axis changes from ω to ω' in time t, its rate of change of angular velocity, or its angular acceleration is, clearly, $(\omega'-\omega)/t$, or $d\omega/dt$, if $d\omega$ be the change in angular velocity in time dt.

Now, if the distance of the particle from the axis of rotation be r, its linear velocity changes from $r\omega$ to $r\omega'$ in time t, and, therefore, rate of change of its linear velocity, or its *linear acceleration*, is given by

$$a = \frac{r\omega'-r\omega}{t} = \frac{r\omega(\omega'-\omega)}{t} = r.\frac{d\omega}{dt}.$$

Thus, *linear acceleration = distance from axis of rotation × angular acceleration.*

10. Couple. When two *equal*, *opposite*, *parallel* and *non-collinear* forces act on a body, (Fig. 2), bringing about rotation, (with no motion of translation), they are said to constitute a *couple*, the turning moment of the couple being measured by the *product of one of the forces and the perpendicular distance between them, or the arm of the couple*, as it is called.

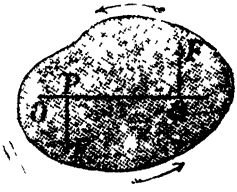


Fig. 2.

Thus, *moment of a couple, C*
= one of the forces × arm of the couple.

The moment of the couple (also sometimes referred to as the **torque**), *acting upon a body, is quite independent of the position of the axis of rotation.* For, if the two forces F and F, (Fig. 2), constituting a couple, act at points P and Q, and if the axis of rotation passes through P, there is no moment of the force acting at P about it and the moment of the force acting at Q is $F \times PQ$, and therefore, the

moment of the couple is $F \times PQ$. And, if the axis of rotation passes through any other point O, the moment of the couple about it is equal to the algebraic sum of the moments of the forces P and Q about it, *i.e.*, equal to $(F \times OQ) - (F \times OP) = F \times PQ$, as before.

The same will be true for any other position of the axis.

11. Work done by a Couple. Work is done by a couple in rotating the body on which it acts, the amount of work done being equal to the product of the couple and the angle of rotation of the body, as will be clear from the following :

Let the axis of rotation of a body, acted upon by a couple, pass through P, the point of application of one of the forces, constituting the couple, (Fig. 3).

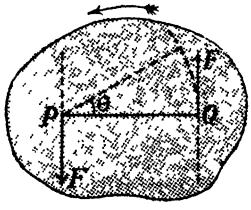


Fig. 3.

Now, if the body rotates through an angle $d\theta$, the point Q moves through a distance $PQ.d\theta$, where PQ is the perpendicular from P on to the line of action of the force F, acting at Q. Therefore, the work done by this force is equal to $F \times PQ.d\theta$. And, since the point P does not move, no work is done by the force at P. Thus, the work done by the two forces, *i.e.*, *by the couple*, in rotating the body through an angle $d\theta$, is equal to $F \times PQ.d\theta + 0 = F \times PQ.d\theta$.

Hence, work done by the couple in rotating the body through the whole angle θ is obtained by integrating this expression, for the limits $\theta = 0$ and $\theta = \theta$.

Or, work done by the couple in rotating the body through the whole angle θ is given by

$$W = \int_0^\theta F.PQ \, . \, d\theta = F.PQ \int_0^\theta d\theta.$$

Or,
$$W = F.PQ \left[\theta \right]_0^\theta = F.PQ.\theta.$$

Now, $F.PQ$ is the moment of the couple C, acting on the body.

$\therefore$ *work done by the couple in rotating the body through angle* θ, *i.e.*,

$$W = C.\theta = \textit{couple} \times \textit{angle of rotation.}$$

Now, in one complete rotation, the body describes an angle 2π; so that,

work done by the couple in one full rotation of the body $= 2\pi C$.

And $\therefore$ *work done by the couple in n full rotations of the body* $= 2\pi nC$.

12. Relation between Couple and Angular Acceleration. When the resultant couple acting on a body is *not* zero, it produces an angular acceleration in the body. Let us deduce the relation between the two.

In Fig. 3, the couple C, acting on the body, causes it to rotate about the axis of rotation through P.

Breaking up the couple and the body into small elements, let an element δC of the couple cause the rotation of an element of mass δm of the body situated at Q. Then, since *couple* = *force* $\times$ *distance*,

the *force* acting on the mass m at Q is $= \delta C/r$, where r is the arm of the couple. And, since a couple consists of two equal, opposite and parallel forces, it follows that an equal, opposite and parallel force is also acting at P.

Again, since *force=mass×acceleration*, the linear acceleration of the particle δm at $Q = \delta C/r.\delta m$. But, if angular acceleration of the particle be $d\omega/dt$, its linear acceleration is *also* equal to $r.d\omega/dt$, [see § 9].

$$\therefore \quad \frac{\delta C}{r.\delta m} = r.\frac{d\omega}{dt}. \quad \text{Or, } \delta C = \frac{d\omega}{dt}.r^2.\delta m.$$

$$\therefore \quad \Sigma\delta C = \Sigma(d\omega/dt).r^2\delta m.$$

$$\text{Or,} \quad C = (d\omega/dt)\Sigma.r^2.\delta m.$$

Now, $\Sigma r^2.\delta m = I$, the *moment of inertia of the body* about the axis of rotation, (see § 27).

$$\therefore \quad C = I.d\omega/dt.$$

Or, *Couple = moment of inertia × angular acceleration.*

13. The Hodograph. When a body describes a *curvilinear path*, so that its motion is accelerated and also changes in direction, its acceleration and its path may easily be determined by means of what is called the *hodograph* of its motion.

The hodograph may be defined as an auxiliary curve, obtained by joining the free ends of a moving vector representing the velocity of a moving particle along any path.

For instance, if a point P moves along a curve ABC, [Fig. 4 (*a*)] such that its velocities are v_1, v_2, and v_3...respectively at A, B and C etc., then, if we take any point O and draw straight lines, *i.e., vectors*, Oa. Ob and Oc, [Fig. 4 (*b*)], representing the velocities of P at A, B and C, in magnitude as well as in direction, the curve passing through a, b and c is the *hodograph* of the motion of P.

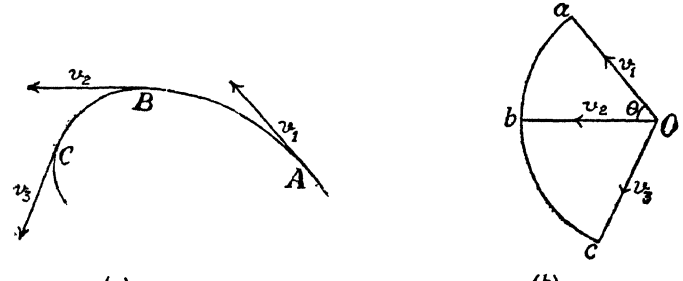


Fig. 4.

Now, different cases arise :

(*i*) If the point P be moving with a *uniform velocity along the same direction*, the points a, b, c, etc. will all lie in the same place and the hodograph will, therefore, be a *single point*.

(*ii*) If the point P be moving with a *variable velocity*, but *in the same direction*, the hodograph will be a *straight line*, passing through O. For example, in the case of a body falling freely under the action of gravity, the hodograph will be a *vertical line*, passing through O.

(*iii*) If P be projected with a horizontal velocity, the path described will be a *parabola*, (see §20), and both the direction and magnitude of the velocity will change. The horizontal velocity will throughout remain constant and equal to the initial horizontal velocity, because the acceleration due to gravity acts vertically downwards. The points, a, b, , etc. will, therefore, always be at the same horizontal distance from O, and the hodograph, in this case, will thus be a *vertical line, not passing through O.*

(*iv*) If the path of P be a *closed curve*, the hodograph will also be a *closed curve*. For example, if P moves in a circle with a uniform speed v, the hodograph will also be a circle of radius v, because all the lines, Oa, Ob, Oc, etc. will be of the same length v. If on, the other hand, it moves in a circle with a *variable speed*, the hodograph might be an *oval curve* about the point O.

14. Velocity in the Hodograph. An important property of the hodograph is that the acceleration of P at any point on the curve ABC is represented, in magnitude as well as in direction, by the velocity of the corresponding point on the hodograph, as can be seen from the following :

Let A and B be two points, close together, [Fig. 4 (*a*)], and let P move from A to B in time t_1, such that its velocity v_1 at A is changed to v_2 at B.

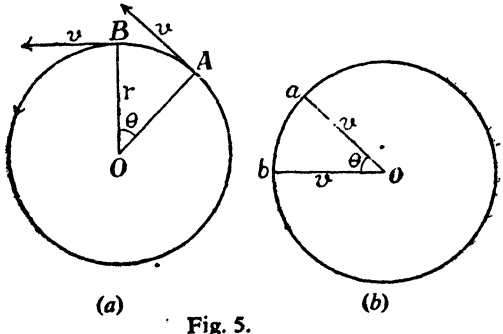
Further, let another point p describe the hodograph abc, [Fig. 4 (*b*)], while P describes the curve ABC.

Then, clearly, the point p moves from a to b in time t, and its velocity is, therefore, equal to ab/t.

But, since oa represents the velocity of P at A and ob, that at B, ab represents, in accordance with the law of triangle of velocities, the change in velocity of P in time t, and, therefore, *the rate of change of velocity*, or the *acceleration* of P, is represented by ab/t *i.e.*, by the *velocity of p* in the hodograph.

We thus see that, *at any instant, the acceleration of P is given by the velocity of p in the hodograph of its motion.*

15. Uniform Circular Motion. The above affords us a very simple method of determining the acceleration of a body, moving in a circle.



(*a*) (*b*)

Fig. 5.

Let P move in a circle, with centre O and radius r, with a uniform speed v, [Fig. 5 (*a*)]. Then, the hodograph is also a circle, of radius v, [Fig. 5 (*b*)].

Now, the velocity of P at any instant is *at right angles* to the radius of its circular path, passing through P. Therefore, *oa is perpendicular to OA* and *ob* is *perpendicular to OB* and $\angle AOB = \angle aob = \theta$ (in circular measure).

If P takes time t to describe the arc AB, its velocity $v = AB/t = r\theta/t$, whence, $\theta = vt/r$.

And, the velocity of the corresponding point p, in the hodograph, is $ab/t = v\theta/t$.

Since the velocity of p in the hodograph gives the acceleration of P in its actual path, we have *acceleration of P*

$$= \frac{v\theta}{t} = \frac{v}{t} \times \frac{vt}{r} = \frac{v^2}{r}.$$

And, since ab is small, it is, *in the limit, perpendicular to oa*, or *parallel to AO*.

Thus, the *acceleration* of P is v^2/r and is *directed along the radius or towards the centre of the circular path* in which it is moving.

Further, since $v = r.\omega$, (where ω is the *angular velocity* of P), we have

$$\textit{acceleration of } P, \text{ also } = r^2.\omega^2/r = r\omega^2.$$

Alternative Method. The acceleration of a body, executing a uniform circular motion may also be found out directly as follows :

Let a particle move with a uniform linear velocity v, in a circle of radius r, (Fig. 6), and let it cover the small distance from A to B in a small interval of time δt, describing an angle $\delta\theta$. Then, clearly, its angular velocity, $\omega = \delta\theta/\delta t$.

The direction of the *linear velocity* is at every point, *tangential to the circle at that point* and is, therefore, represented by the tangent AC at the point A, and by the tangent BD at the point B, where $AC = BD$.

Now at A, the entire linear velocity is along AC, there being no component of it along AO, which is at right angles to AC. And, revolving the velocity at B into two rectangular components, one along AO and the other, at right angles to it, we have the component along or parallel to AO, represented by $BE = v \sin \delta\theta$, and the component at right angles to AO, represented by $BF = v \cos \delta\theta$.

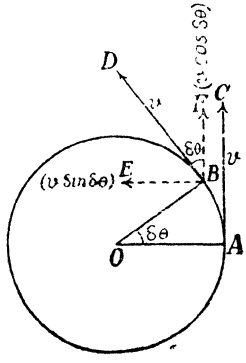


Fig. 6.

If $\delta\theta$ be very small, *sin* $\delta\theta = \delta\theta$ (in *radians*), and *cos* $\delta\theta = 1$.

So that, component BE, parallel to $AO = v.\delta\theta$,

and component BF, perpendicular to $AO = v$.

Thus, if B be very close to A, there is no change in the velocity of the particle along the perpendicular to AO, for it remains the same v, but an additional velocity $v\,\delta\theta$ is acquired by it along AO. And, since this velocity is acquired in time δt, the acceleration imparted to the particle is $v.\delta\theta/\delta t = v\omega$, where $\delta\theta/\delta t = \omega$, *the angular velocity* of the particle.

Or, *acceleration of the particle* $= v\omega = v.\frac{v}{r} = \frac{v^2}{r}$.

Now, because the magnitude of the velocity remains the *same* at every point on the circular path of the particle, it follows that the acceleration must be acting in a direction perpendicular to the direction of the velocity at that point, *i.e*, along the radius of the circle, or else it will also have a component along the tangent at the point, or along the direction of the velocity at that point, which will, therefore, no longer remain constant.

Since this acceleration acts along the radius of the circle, or towards the centre of the circle, it is called *radial* or *centripetal* — meaning *centre-seeking* — acceleration, (from *'peto' —I seek*).

Thus, *centripetal acceleration* $= \frac{v^2}{r}$, or, $= \frac{(r\omega)^2}{r} = r\omega^2$.

And, if n be the number of revolutions made by the particle per unit time, we have $\omega = 2\pi n$.

$\therefore$ *centripetal acceleration*, also $= r.(2\pi n)^2 = 4\pi^2 n^2 r$.

Even if the path be not exactly a circle, but any other curve, the value of the acceleration is v^2/r, where v is the linear velocity, and r, the radius of curvature of the path *at the point considered.*

16. Centripetal Force. According to Newton's first law of motion, a body must continue to move with a uniform velocity in a straight line, unless acted upon by a force. It follows, therefore, that when a body moves along a circle, some force is acting upon it, which continually deflects it from its straight or linear path ; and, since the body has an acceleration towards the centre, it is obvious that the force must also be acting in the direction of this acceleration, *i.e.*, along the radius, or towards the centre of its circular path. It is called the *centripetal force*, and its value is given by the product of the mass of the body and its centripetal acceleration. Thus, if m be the mass of the body, we have

centripetal force $= mv\omega = mv^2/r$, or, $= mr\omega^2 = 4\pi^2 n^2 mr$.

Numerous examples of centripetal force are met with in daily life. Thus, (*i*) in the case of a stone, whirled round at the end of a string whose other end is held in the hand, the centripetal force is supplied by the tension of the string ; (*ii*) in the case of a motor car or a railway train, negotiating a curve, it is supplied by the push due to the rails on the wheels of the train and (*iii*) in the case of (*a*) the planets revolving round the sun, or (*b*) the moon revolving round the earth, by the gravitational attraction between them.

If this force somehow vanishes at any point in its circular path, the body will fly off *tangentially* to it at that point, for it will no longer be compelled to move in the circular path.

17. Centrifugal Force. The equal and opposite *reaction* to the centripetal force is called the *centrifugal force*, because it tends to take the body away from the centre, (from *'fugo'—I flee*). Centripetal force and centrifugal force being just *action* and *reaction* in the sense

of Newton's third law of motion, the numerical values of the two are the same, *viz.*, $mv^2/r = mr\omega^2 = 4\pi^2n^2mr$.

Thus, in the case of a stone, whirled round at the end of a string, not only is the stone acted upon by a force, (the centripetal force), along the string towards the centre, but the stone also exerts an equal and opposite force, (the centrifugal force), on the hand, away from the centre, also along the string.

18. Practical Applications of Centripetal and Centrifugal Forces.

1. Road Curves. The centripetal force being directly proportional to the square of the linear velocity of the body and inversely proportional to the radius of its circular path, the radii of curvature of road curves must be large and the speed of the vehicles negotiating them slowed down, in order to keep the value of the centripetal force required within reasonable limits.

2. Rotating Machinery. The centrifugal force being proportional to n^2, where n is the number of rotations made by the body per second, the spokes of a wheel, joining its outer revolving parts to the axis of rotation, experience an outward force, *away from the centre*, and are, therefore, in a state of tension, and may give way if the value of n is very large. So is the case with the parts of other rotating machinery, connecting its outer revolving parts to its axis of rotation. In other words, there is a limit set to the value of n by the tension these connecting parts can withstand. This fact is always kept in view while designing highly rotating machinery, like armatures of motors and dynamos etc.

Let us, as a specific example, discuss the case of a belt or a string rotating at a high speed over a pulley etc.

Let the string rotate in a circle of *radius* r, (Fig. 7), and let its *angular velocity* be ω. Consider a small portion AB of the string, of length l and subtending an angle 2θ at the centre O of the circle. This portion is obviously subjected to a tension T, at either end, by the rest of the string as shown. Resolving these tensions T and T at A and B into two rectangular components along and at right angles to PO, (where PO passes through the mid-point of AB), we find that the components $T\cos\theta$ at right angles to PO are *equal and opposite* and thus neutralise each other, but the components $T\sin\theta$ along PO act in the *same* direction. So that, we have

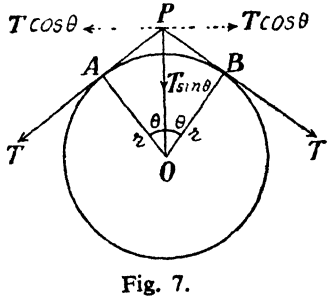


Fig. 7.

resultant tension on portion AB of the string
$= 2T\sin\theta$ in the direction PO.

And, the *centrifugal force acting on portion AB* of the string
$=$ *mass of* $AB \times r\omega^2$, in the direction OP.

If m be the *mass per unit length* of the string, clearly,
mass of $AB = m \times l$.

And $\therefore$ *centrifugal force acting on portion AB of the string*
$= m \times l \times r\omega^2$, in the direction OP,

For equilibrium, therefore, $2T\sin\theta = mlr\omega^2 = m.2r\theta.r\omega^2$. [$\because$ clearly, $l = AB = r.2\theta$.

If θ be small, we have $\sin\theta = \theta$. So that,
$2T\theta = m.2r\theta.r\omega^2$,

whence, $T = mr^2\omega^2$.

It will thus be seen that due to the centrifugal force, the tension in the string is very high. Indeed, if the rapidly rotating chain or belt be pushed off the pulley, it will run along like a rigid hoop.

The same is true about other rotating bodies which are always under a state of *elastic stress*. It is this stress which sets a limit to the speed up to which the flywheels can be rotated safely. Again, it is as a consequence of this stress that the tyres of racing cars get stretched and there is a danger of their being cast off the rims and flung out, at very high speeds.

3. Revolution of Planets and the Length of the Year. In the case of a planet revolving round the sun, it is the gravitational force of attraction between the two which supplies the centripetal force, necessary to keep it moving in its nearly circular orbit. Now, the gravitational force between two bodies is directly proportional to the product of their masses and inversely proportional to the square of the distance between them ; so that, if m and M be the masses of the planet and the sun respectively and r, the distance between them (or the radius of the planet's orbit round the sun), we have

$$\textit{gravitational pull} = \frac{m.M}{r^2}.G = \frac{k}{r^2}. \quad \left[\begin{array}{l}\text{Putting } m.MG = k,\\ \text{a constant.}\end{array}\right.$$

Or, $$4\pi^2n^2rm = \frac{k}{r^2}, \text{ whence, } n^2 = \frac{k}{4\pi^2r^3m}.$$

Or, $$n = \frac{1}{2\pi}\sqrt{\frac{k}{mr^3}},$$

i.e., $$t = \frac{1}{n} = 2\pi\sqrt{\frac{mr^3}{k}} = 2\pi\sqrt{\frac{mr^3}{mMG}} = 2\pi\sqrt{\frac{r^3}{MG}},$$

where t is the *time taken by one revolution of the planet round the sun, or the length of the year* for that planet.

Thus, t varies as $\sqrt{r^3}$, *i.e.*, the smaller the value of r, or the smaller the distance of the planet from the sun, the smaller the value of t, or the length of the year, for it. *A planet will, therefore, have a shorter year if nearer to the sun than when at a distance from it.*

4. Banking of Railway Lines and Roads. When a railway train goes round a level curve on a railway track, the necessary centripetal force is provided only by the force between the *flanges* or the rims of the wheels and the *rails*, the normal reaction of the ground or the track acting vertically upwards and supporting its weight. This results in a grinding action between the wheels and the rails, resulting in their wear and tear. Not only that, it may also prove dangerous in the sense that it may bring about a displacement of the rails and hence a derailment of the train.

To avoid these eventualities, *the level of the outside rail is raised a little above that of the inside one.* This is known as the **banking** of railway lines, and the angle that the track makes with the horizontal is called the **angle of banking.**

With the track thus *banked*, *i.e.*, with the outer rail thus raised above the level of the inner one, the reaction R acts perpendicularly to the track, as before, but is now inclined to the vertical at an angle

equal to the angle of banking and *its horizontal component* (and not the lateral thrust of the wheel flanges on the outside rail) *now supplies the necessary contripetal force to keep the train moving along the curve,* thereby eliminating all unnecessary wear and tear.

Thus, if θ be the angle of banking (Fig. 8), and R, the normal reaction acting perpendicularly to it, we have

$$\textit{vertical componet of } R = R \cos \theta,$$

and $$\textit{horizontal component of } R = R \sin \theta.$$

The former component balances the weight mg of the train and the latter supplies the required centripetal force mv^2/r, where v is the speed of the train and r the radius of the curve it negotiates.. So that,

$$R \sin \theta = mv^2/r$$

and

$$R \cos \theta = mg.$$

$$\therefore \quad \frac{R \sin \theta}{R \cos \theta} = \frac{mv^2/r}{mg}.$$

Or,

$$\tan \theta = \frac{v^2}{rg}.$$

Or,

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right).$$

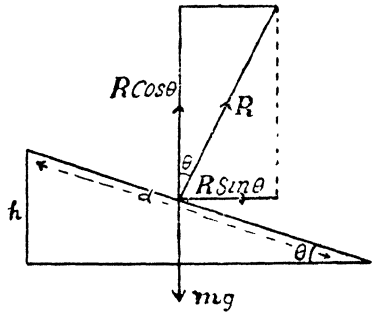


Fig. 8.

The angle of banking thus depends upon the speed (v) of the train and the radius (r) of the curve of the track. Obviously, therefore, a track can be *banked* correctly only for a particular speed of the train,—in practice, naturally for its *average* speed. At higher or lower speeds than this, there is again a lateral thrust due to the wheel flanges on the outer or the inner rail of the track respectively.

Clearly, the angle that the track makes with the horizontal is equal to θ, *i.e.*, equal to the angle of inclination of the train with the vertical, (Fig. 8).

Further, it will be readily seen that if the distance between the rails be d and the height of the outer rail above the inner one be h,

we also have

$$\sin \theta = \frac{h}{d}.$$

Or, *sine of the angle of banking*

$$= \frac{\textit{height of the outer rail over the inner one}}{\textit{distance between the rails}}.$$

Similarly, in the case of a car moving round a level corner, the centrifugal force is largely provided by the friction between the road and the tyres of the wheel. That is why, when the road is slippery and the frictional force not enough the car begins to slide or skid. Here, too, therefore, the roads are *'banked'*, the slope being generally steeper outwards,—more or less like a saucer—*the outer parts being meant to be used at higher speeds and the inner ones, at lower speeds.*

Again, an aeroplane, in order to turn, must also bank, the centripetal force here being supplied by the horizontal component of the lift L, (Fig. 9).

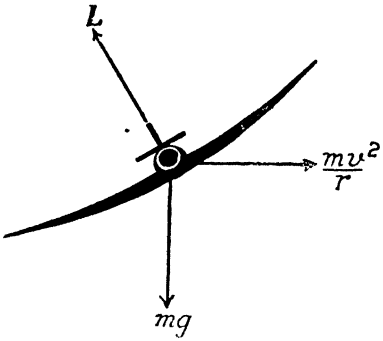


Fig. 9.

The same applies to a cyclist, when negotiating a curve or a corner, and he has to lean *inwards*, (*i.e.*, towards the centre of the curve), by an angle $\theta = tan^{-1}\, v^2/rg$; so that, the faster his speed and the sharper the curve, the more must he lean over. This will be clear from the following :

Let Fig. 10 represent a cyclist turning to the left in a circle of radius r, at a speed v. Then, the *normal reaction* R of the ground acts vertically upwards, with the *force of friction* F between the ground and the tyres and the *centrifugal force* mv^2/r in the directions shown,

where $R = mg$ and $F = mv^2/r$.

Then, for equilibrium, clearly, we have moment of mg about P *equal and opposite* to moment of mv^2/r about P.

Or, $$mg \times PQ = \frac{mv^2}{r} \times GQ.$$

Or, $$mg \times PG.sin\ \theta = \frac{mv^2}{r}.\ PG.cos\ \theta,$$

whence, $$\frac{sin\ \theta}{cos\ \theta} = tan\ \theta = \frac{v^2}{rg}.$$

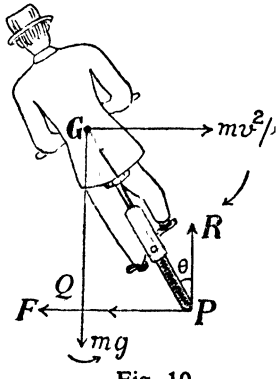


Fig. 10.

In other words, in order to keep himself in equilibrium, the cyclist must lean inwards from the vertical at an angle $\theta = tan^{-1}\,(v^2/rg)$.

If he were to remain vertical, his weight would act through P, having no moment about it, so that the moment of mv^2/r about P would remain unbalanced. In fact it will be readily seen that the system of forces acting on the cyclist form two pairs of couples, one due to F and mv^2/r and the other due to R and mg. So that, in the event of the latter couple vanishing (*i.e* , if the cyclist were vertical), the former alone will remain operative, resulting in the cyclist toppling over.

Further, since the maximum value of $F = \mu mg$, (where μ is the coefficient of friction between the ground and the tyres), the cyclist will skid when $mv^2/r > \mu mg$, or when $v^2 > \mu rg$.

Thus, *skidding will occur* (*i*) *if v is large, i.e., if the speed of the cyclist is large.* (*ii*) *if* μ *is small, i.e., if the road is slippery and* (*iii*) *if r is small, i.e., if the curve is sharp.*

Similar conditions apply in the case of a motor car or any other vehicle. For, here too, if we imagine it to be turning to the left, (Fig. 11), the various forces acting on it are the *normal reactions* R_1 *and* R_2, the *frictional forces* F_1 and F_2, its *weight* mg and the *centrifugal force* mv^2/r, as shown, the whole system being in equilibrium.

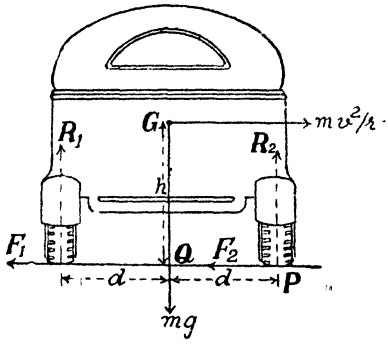


Fig. 11.

Obviously, in the event of the car being about to be upset, it will be moving on the wheels on one side only, so that the normal reaction on the wheels on the other side will be zero; say $R_1 = 0$. So that, it will overturn as soon as the moment of mv^2/r about P is greater than the opposing moment of mg about P,

i.e., as soon as $\frac{mv^2}{r} . GQ > mg . PQ$,

Or, when $\frac{mv^2}{r} . h > mg.d$,

where h is the height of the *c.g.*, G, of the car above the ground and $2d$, the distance between the two wheels.

For the car to be upset, therefore, we have $v^2 > \frac{d.rg}{h}$.

The car is, therefore, not likely to be upset if $2d$, the distance between the two wheels is large and if h, the height of the *c.g.* from the ground is small.

Again, the maximum value of the total frictional force

$$F_1 + F_2 = F = \mu.mg.$$

So that, as before, skidding will occur when $mv^2/r > \mu mg$.

Or, when $v^2 > \mu rg$.

To avoid skidding, therefore, while taking a turn at a fast speed, the corner must be cut so as to move along a comparatively flatter curve than that of the actual turning.

19. Other Effects and Applications of Centrifugal Force.

1. Rotation of the Earth—Its Effect. As we know already, the earth rotates or spins about its axis once during a day. It is this rotation of it which is responsible for its getting flattened at the poles and its bulging out at the equator, —a direct consequence of the centrifugal force $m\omega^2R$ acting on each particle of mass m of it, where ω is its angular velocity about the axis of rotation and R, the distance of the particle from this axis. The value of ω is obviously the *same* for each particle, but the distance R increases from *zero* for particles at the poles to a *maximum* for those at the equator. The centrifugal force pulling the earth outwards, as it were, is thus *zero at the poles*

and *the maximum at the equator* and it is this force which has made the earth (behaving like a plastic body) to bulge out at the equator and to flatten at the poles, thus bringing about an increase of about 13 miles in its equatorial, as compared with its polar radius.

This effect of the rotation of the earth had been first predicted by *Newton* and was duly verified by a French expedition to Lapland under the leadership of *Maupertius*, whose undue pomposity provoked *Voltaire* into making the caustic remark that he 'behaved as though he had flattened the poles himself.'

2. The Centrifuges. These are simple devices used to separate out substances of different densities suspended in a liquid, by rapidly rotating the liquid, when particles, whose density is greater than that of the liquid, are driven away from the axis of rotation, whereas those, with a density lower than that of the liquid, are drawn inwards towards it. Thus, for example, in the familiar cream-separator, when the vessel containing milk is rotated fast, the cream, being *lighter*, collects in a cylindrical layer round about the axis, whence it can be easily drawn off.

Since the centrifugal force ($m\omega^2 r$) increases with r, the pressure on the rotating liquid progressively increases as we move away from the axis, with the result that, on a heavier particle, the centrifugal force outwards is greater than the inward thrust of the liquid, whereas, on a lighter one, the reverse is the case. *The problem is more or less akin to the sinking or floating of a body in a liquid at rest*, depending upon the difference in the magnitude of the forces acting on the two sides of the body, *the inward thrust on it corresponding to the upthrust in the case of a stationary liquid.*

Since centrifuges have as high speeds of rotation as 40,000 revolutions (or more) per minute, the difference between the outward and inward forces acting on the heavier and lighter particles exceeds more than a thousand times the difference between their weights, so that quick and effective separation results. Sediments, precipitates and bacteria etc., may all be thus separated speedily.

Another familiar example is the **centrifugal drying machine**, which is just a cylindrical vessel, with perforations in its walls. When, with damp clothes placed inside it, it is rotated fast, the centrifugal force acting on them forces the water out through the perforations and the clothes thus get dried up quickly.

3. The Centrifugal Pump. Also known as the *Turbine Pump*, it consists of three essential parts, *viz.*, *(i)* an outer drum-shaped '*casing*', having an *inlet* near its axis and an *outlet* near its periphery, *(ii)* a paddle wheel (*i.e.*, a hollow wheel, fitted with vanes) called the '*impeller*', which can be rotated inside the casing, and *(iii)* the '*spindle*', through which energy is transmitted from the driving motor to the impeller.

If the casing be filled with water and the impeller rapidly rotated, it sets the water into similar rapid rotation, which, due to its centrifugal force, exerts high pressure against the outer wall of the casing, forcing the water out through the outlet at the periphery into

the rising discharge-tube connected to it. At the same time, there is decreased pressure near the axis, so that the atmospheric pressure forces fresh water in, from the reservoir, through the inlet. This is then again flung out through the outlet, in the manner explained, and the process goes on repeating itself over and over again.

The pump starts working only when the casing is full of water, but, once it has started working, it gives a *continuous* supply of water, unlike the ordinary piston pump, where we get only an *intermittent* supply.

Further, as there are no valves to operate, the pump can be used safely even if the water contains sledge or any other suspended matter, including sand or small-sized stones etc.,

20. The Projectile—Motion of a Projectile in a non-resisting medium. Before Galileo's time, it was supposed that a body thrown horizontally, travelled in a straight line until it had exhausted its force and then fell vertically down. It was he who first showed that it must take a *parabolic path**, realizing, as he did, the physical independence of its horizontal and vertical motions, so that each could be considered separately.

Such a body, subjected simultaneously to a uniform horizontal motion and a vertical uniform acceleration, is called a **projectile**, and the path it describes is called its **trajectory**. Let us study its motion in some detail.

Let a body be projected upwards with a velocity u, at an angle θ with the horizontal. Then, resolving u into two rectangular components, along the vertical† and along the horizontal, we have (*i*) *the vertical component* (along OY) $= u \sin \theta$ and (*ii*) *the horizontal component* (along OX) $= u \cos \theta$. The latter component, being perpendicular to the direction of gravity, is not accelerated, and hence

$$\frac{dx}{dt} = u \cos \theta.$$

Since at $t = 0$, $x = 0$, we have

$$x = ut \cos \theta. \quad \ldots \quad \ldots \quad \ldots (i)$$

And, because the vertical component is subjected to a downward acceleration due to gravity, we have

$$\frac{d^2y}{dt^2} = -g,$$

integrating which, we have $\frac{dy}{dt} = -gt + C_1$,

where C_1 is a constant of integration.

Now, at $t = 0$, $\frac{dy}{dt} = u \sin \theta$; so that $C_1 = u \sin \theta$.

$\therefore$ $\frac{dy}{dt} = u \sin \theta - gt.$

---

*He dropped objects from masts of moving ships, which fell vertically in relation to the ship but along parabolic paths in relation to the sea.

†*i.e.*, along the direction in which the force due to gravity acts.

Integrating this again, we have $y = ut \sin\theta - \frac{1}{2}gt^2 + C_2$, *where* C_2 *is another constant of integration.*

Since $y = 0$ at $t = 0$, we have $C_2 = 0$.

Or, $$y = ut \sin\theta - \tfrac{1}{2}gt^2. \quad \ldots \quad \ldots \quad \ldots(ii)$$

Now, from relation (*i*), we have $t = \dfrac{x}{u\cos\theta}$.

Substituting this value of t in relation (*ii*), we have

$$y = u.\frac{x}{u\cos\theta}.\sin\theta - \tfrac{1}{2}g.\left(\frac{x}{u\cos\theta}\right)^2.$$

Or, $$y = x\tan\theta - \tfrac{1}{2}g.\frac{x^2}{u^2\cos^2\theta}.$$

This is clearly an equation of the second degree in x and the first degree in y and thus represents a parabola, with its axis vertical. The trajectory of the particle is thus a *parabola.*

21. Horizontal Range of a Projectile. Clearly, the time taken by the body to reach the maximum height $= u \sin\theta/g$*, its vertical velocity being $u \sin\theta$.

And, *since time of ascent is equal to time of descent,* the total time taken by the body for the whole flight $= 2u \sin\theta/g$.

During this time, the *horizontal distance covered by the body,* with its uniform horizontal velocity $u\cos\theta$ is given by

$$u\cos\theta.\frac{2u\sin\theta}{g} = \frac{2u^2.\sin\theta.\cos\theta}{g} = \frac{u^2.\sin 2\theta}{g}. \quad [\because 2\sin\theta\cos\theta = \sin 2\theta.$$

This horizontal distance covered by a projectile is called its **horizontal range**, or, more usually, simply, its **range**. Denoting it by R, therefore, we have $R = u^2 \sin 2\theta/g$.

22. Maximum Height attained by a Projectile. We have the kinematic relation, $v^2 - u^2 = 2aS$, where the symbols have their usual meanings.

Here, $a = -g$ (the body being projected upwards), and, at the highest point, obviously, $v = 0$. So that, if the *maximum height attained by the projectile be* h, (*i.e.*, $S = h$), we have

$$0 - (u\sin\theta)^2 = 2.(-g).h,$$

whence, $$h = \frac{u^2.\sin^2\theta}{2g}.$$

[∵ the initial upward velocity here is $u \sin\theta$ and not u.

23. Angle of Projection for Maximum Range. It is obvious that for a given initial velocity (u) of the body, its horizontal range (R) will depend upon its angle of projection (θ).

Now, the horizontal range R, as we know, is given by

$$R = \frac{u^2.\sin 2\theta}{g},$$

Putting x for R, we have $x = \dfrac{u^2 \sin 2\theta}{g}$.

*We have the kinematic relation, $v = u + at$, where u is the initial velocity (here equal to $u \sin\theta$), $a = -g$ and v, the final velocity, equal to 0, (at the highest point). So that, $0 = u \sin\theta - gt$; or, $gt = u \sin\theta$. Or, $t = u \sin\theta/g$.

It is thus clear that the value of θ for maximum (horizontal) range of the projectile would be that for which $\sin 2\theta = 1$, *i.e.*, when $2\theta = 90°$, and, therefore, $\theta = 45°$.

Thus, for maximum range, the angle of projection should be 45°.

N.B. The following interesting result follows, however, from the relation

$$R = u^2 \sin 2\theta / g.$$

We know that the sine of an angle is the same as that of its supplement. And, therefore,

$$\sin 2\theta = \sin (180 - 2\theta),$$

from which it is clear that the *projectile will have the same range* (not the maximum), for the angles of projection θ and $(90-\theta)$,—the two paths taken being, however, different, and called the *high* (H) and the *low* (L) *trajectories* respectively, as shown in Fig 12, in view of the different maximum heights attained by the body.

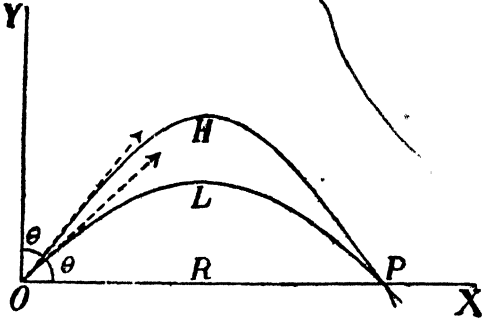


Fig. 12.

24. Range on an Inclined Plane. We have seen above, (§ 20), that the equation to the trajectory of a projectile is

$$y = x \tan \theta - \tfrac{1}{2} g \cdot \frac{x^2}{u^2 \cos^2 \theta}.$$

Let us consider a plane inclined to the horizontal at an angle α, (Fig. 13) ; so that, $y = x \tan \alpha$.

Now, to obtain the range on the inclined plane, we must determine the point where the trajectory of the projectile will meet the plane; and to do this, we must solve the above two equations. So that, substituting $y = x \tan \alpha$ in the equation for the trajectory, we have

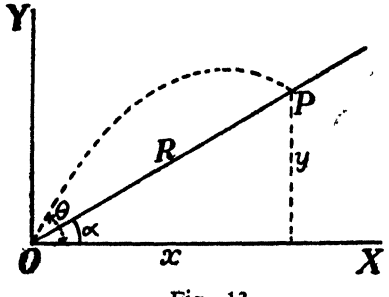


Fig. 13.

$$x \tan \alpha = x \tan \theta - \tfrac{1}{2} \cdot \frac{g x^2}{u^2 \cos^2 \theta}.$$

$$\text{Or, } \frac{gx}{2u^2 \cos^2 \theta} = \tan \theta - \tan \alpha.$$

$$\text{Or, } x = \frac{2(\tan \theta - \tan \alpha) . u^2 \cos^2 \theta}{g}.$$

And, therefore, substituting *this* value of x in $y = x \tan \alpha$,

we have $$y = \frac{2(\tan \theta - \tan \alpha) . u^2 \cos^2 \theta}{g} . \tan \alpha.$$

So that, the range R is clearly given by the relation,

$$R^2 = x^2 + y^2.$$

$$= \left[\frac{2(\tan\theta - \tan\alpha).u^2\cos^2\theta}{g}\right]^2 + \left[\frac{2(\tan\theta - \tan\alpha)\tan\alpha.u^2\cos^2\theta}{g}\right]^2$$

$$= \left[\frac{2(\tan\theta - \tan\alpha).u^2\cos^2\theta}{g}\right]^2 [1+\tan^2\alpha]$$

Now, $(1+\tan^2 a) = \sec^2\alpha = \dfrac{1}{\cos^2\alpha}$.

$\therefore$ $$R^2 = \left[\frac{2(\tan\theta - \tan\alpha)\,u^2\cos^2\theta}{g}\right]^2 \Big/ \cos^2\alpha.$$

Or, $$R = \frac{2(\tan\theta - \tan\alpha)\,u^2\cos^2\theta}{g\cos\alpha}.$$

25. Resultant Velocity of a Projectile at a given instant. The vertical and horizontal components of the velocity of a projectile, t *secs.* after its projection, are clearly dy/dt and dx/dt. So that, its *resultant* velocity v, at the instant, is given by the relation,

$$v = \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}.$$

Or, putting the values of dy/dt and dx/dt from above, (§ 20), we have

$$v = \sqrt{u\sin\theta - gt)^2 + (u\cos\theta)^2}.$$

$$= \sqrt{u^2\sin^2\theta + g^2t^2 - 2ugt.\sin\theta + u^2\cos^2\theta}.$$

$$= \sqrt{u^2(\sin^2\theta + \cos^2\theta) + g^2t^2 - 2ugt.\sin\theta}.$$

Or, $$v = \sqrt{u^2 - 2ugt.\sin\theta + g^2t^2}. \quad [\because \sin^2\theta + \cos^2\theta = 1.$$

And, the angle, β, say, that this resultant velocity makes with the horizontal, is obviously given by

$$\tan\beta = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{u\sin\theta - gt}{u\cos\theta} = \tan\theta - \frac{gt}{u\cos\theta}.$$

Or, $$\beta = \tan^{-1}\left[\tan\theta - \frac{gt}{u\cos\theta}\right].$$

N.B. For small values of t, $\tan\beta$, and, therefore, β is *positive*; but, for large values of t, it acquires a *negative* value. *And, obviously, when* $\beta = 0$, *the body is moving horizontally, i.e., at the very peak of the parabola.* In this case, since $\tan\beta = 0$, we have

$$\tan\theta - \frac{gt}{u\cos\theta} = 0.$$

Let this value of t be denoted by t'. Then, we have

$$\frac{gt'}{u\cos\theta} = \tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Or, $gt' = u\sin\theta$, whence, $t' = \dfrac{u\sin\theta}{g}$.

Clearly, the vertical distance covered by the body at an instant t is given by $$y = ut + \tfrac{1}{2}at^2.$$

$\therefore$ $y = h$, the *maximum height* attained by the body, when $t = t' = \dfrac{u\sin\theta}{g}$.

Substituting these values in the above relation, we have

$$h = u\sin\theta.t' - \tfrac{1}{2}gt'^2. \quad \left[\because \begin{array}{l}\text{initial upward velocity} \\ = u\sin\theta, \text{ and } a = -g.\end{array}\right.$$

$$= u \sin\theta . \frac{u \sin\theta}{g} - \tfrac{1}{2}g\left(\frac{u \sin\theta}{g}\right)^2.$$

$$= \frac{u^2 \sin^2\theta}{g} - \frac{u^2 \sin^2\theta}{2g}. \quad \text{Or,} \quad h = \frac{u^2 \sin^2\theta}{2g},$$

the same result, as obtained above in § 22 (page 34).

Again, the *horizontal range* may be easily obtained by equating the value of y to 0.

Let $t = t''$, when $y = 0$. Then, we have, from above,

$$0 = u \sin\theta . t'' - \tfrac{1}{2}gt''^2.$$

Or, $$u \sin\theta . t'' = \tfrac{1}{2}gt''^2. \quad \text{Or,} \quad u \sin\theta = \tfrac{1}{2}gt''.$$

Or, $$gt'' = 2u \sin\theta, \quad \text{whence,} \quad t'' = \frac{2u \sin\theta}{g}*.$$

Now, as we know, the *initial horizontal velocity* $= u \cos\theta$. And, therefore, the horizontal range R is given by the relation,

$$R = u \cos\theta . t'' = u \cos\theta . \frac{2u \sin\theta}{g}.$$

$$= \frac{u^2 . 2 \sin\theta \cos\theta}{g} = \frac{u^2 . \sin 2\theta}{g}, \qquad [\because\ 2 \sin\theta \cos\theta = \sin 2\theta.$$

the same result as obtained before in § 21, (page 34).

And, finally, if we substitute this value of t, (*i.e.*, $t'' = 2u \sin\theta/g$), in the expression for *tan* β above, we have

$$\tan\beta = \tan\theta - \frac{gt''}{u \cos\theta} = \tan\theta - \frac{g}{u \cos\theta} . \frac{2u \sin\theta}{g}.$$

Or, $$\tan\beta = \tan\theta - 2 \tan\theta = -\tan\theta,$$

showing that the projectile comes back to the horizontal surface at the *same* angle at which it was projected upwards. And, it is a further simple deduction that its *tangential velocity at this moment is the same as at the instant of projection.*

It must be emphasized again, however, that the above treatment applies to the motion of a projectile, only in a *non-resisting* medium. *i.e., in vacuo.* The presence of a medium, like air, offers a frictional resistance to its motion, which depends, to a great extent, upon the velocity of the body and is, for moderate velocities†, directly proportional to the square of the velocity, in accordance with the law of resistance given by *Newton*, in the year 1687. This alters the very character of the trajectory of a projectile, which no longer remains a parabola but becomes what is called a **ballistic curve****, *with its descending part much steeper than its ascending part and the height and range of the projectile considerably reduced, particularly at high initial velocities.*

At higher altitudes, well above the average level of the earth's surface, however, the air pressure, and hence also the air resistance, becomes much smaller and, therefore, if a projectile be shot up to such great maximum heights, it is quite possible to obtain a high range for it. This is perhaps the most probable explanation of the long range German canon, used with such conspicuous success in the historic Great War, and which could fire shots to a maximum height of about 54 kilometres and hence had a range of about 130 kilometres.

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*Another possible value of t'' is 0. We reject it, however, as it refers to the time when the body is just starting on its trajectory.

†For example, from a velocity of a few metres per second to about the velocity of sound, in the case of air.

***Ballistics* is the special name given to the science of the motion of

SOLVED EXAMPLES

1. A particle, moving in a circle of radius 105 cms., has its velocit increased in one minute by 120 rotations per minute. Calculate (*i*) its linea acceleration (*ii*) its angular acceleration.

Change in velocity of the particle in 1 *min.* = 120 rotations *per min.*
$= 120/60 = 2$ rotations/*sec.*

Now, in 1 rotation, distance covered $= 2\pi r$.

$\therefore$,, 2 rotations ,, ,, $= 2\times 2\pi r = 2\times 2\times \pi \times 105$ *cms./sec.*

$\therefore$ change in velocity of the particle *per minute* $= 420\times\pi$ *cms /sec.*

Or, change in velocity *per second* $= 420\times\pi/60 = 7\pi$ *cms./sec.*

Or, *rate of change of velocity* $= 7\times 22/7$ *cms./sec*^{2}. $= 22$ *cms./sec*^{2}.

i.e., *linear acceleration*, $a = 22$ *cms./sec*^{2}.

Now, linear acceleration, *i.e.*, $a = r.d\omega/dt$,

where r is the radius, and $d\omega/dt$, the angular acceleration.

So that, angular acceleration, $d\omega/dt = a/r$.

log 22 = 1·3424
log 105 = 2 0212
Antilog 1·3212 = 0·2095

Or, $d\omega/dt = \frac{22}{105} = \cdot 2095$ *radian/sec.*

Hence, the *linear* and *angular accelerations* of the particl are 22·0 *cms./sec*^{2}. and ·2095 *radian/sec*^{2}., respectively.

2. If the Earth be one-half its present distance from the Sun, how man days will there be in one year ?

Let the present distance of the Earth from the Sun (*i.e.*, the radius of its orbit round the Sun) be $= R$.

So that, half its present distance from the sun would be $= R/2$.

If a_1 and a_2 be the accelerations (*linear*) of the earth in the two cases respectively, we have

$$\frac{a_1}{a_2} = \frac{(R/2)^2}{R^2} = \frac{R^2}{4R^2} = \frac{1}{4}. \qquad [\because a \propto 1/R^2.$$

If v_1 and v_2 be the linear velocities of the Earth, corresponding to these two values of the acceleration, we have

$$a_1 = \frac{v_1^2}{R} \quad \text{and} \quad a_2 = \frac{v_2^2}{R/2} = \frac{2v_2^2}{R}.$$

So that, $\frac{v_1^2/R}{2v_2^2/R} = \frac{1}{4}$. Or, $\frac{v_1}{R}\times\frac{R}{2v_2^2} = \frac{1}{4}$.

Or, $\frac{v_1^2}{v_2^2} = \frac{1}{2}$. Or, $v_2^2 = 2v_1^2$.

$\therefore$ $v_2 = \sqrt{2v_1^2}$.

Now, $v_1 = \frac{2\pi R}{365}$ *per day.* $\left[i.e., = \frac{\text{circumference of Earth's orbit}}{\text{time taken}}\right.$.

And, therefore, $v_2 = \sqrt{2}v_1 = \sqrt{2}.\frac{2\pi R}{365}$ *per day.*

Now, circumference of the Earth's orbit, in the *second case* $= 2\pi.R/2$
$= \pi R$.

And, since one year is the time taken by the Earth to go once round the Sun, we have

log 365 = 2·5623 ..(*i*)
log 2 = 0·3010
½ log 2 = 0·1505
0·4515 ..(*ii*)
Subtracting (*ii*) from (*i*) we have 2·1108
Antilog 2·1108 = 129·0

one year, in the *second case* $= \pi R/v_2$.
$= \frac{\pi R\times 365}{\sqrt{2}\times 2\pi R}$.
$= \frac{365}{2\sqrt{2}} = 129$ *days.*

Hence, the number of days in one year, when the earth is half its present distance from the Sun, is 129.

3. Assuming that the Moon revolves uniformly in a circle round the Earth's centre, calculate the acceleration due to gravity at the Earth's surface from the following data :

Radius of the Earth $= 6{\cdot}4\times10^8$ **cms.**
Radius of the Moon's orbit $= 3{\cdot}84\times10^{10}$ **cms.**
Period of rotation of the Moon $= 27{\cdot}3$ **days.**

(The force of gravity on a particle is inversely proportional to the square of its distance from the Earth's centre).

(Oxford & Cambridge Higher School Certificate)

Here, velocity of the Moon, $v = \dfrac{\text{circumference of the orbit}}{\text{time taken}}$.

$$= \frac{2\pi\times\text{radius of its orbit}}{\text{time taken}}.$$

$$= \frac{2\pi\times3{\cdot}84\times10^{10}}{27{\cdot}3\times24\times60\times60}\ cms./sec.$$

$\therefore$ acceleration of the Moon towards the centre of the Earth $= v^2/r$, where r is the radius of the Moon's orbit

$$= \left[\frac{2\pi\times3{\cdot}84\times10^{10}}{27{\cdot}3\times24\times3600}\right]^2\times\frac{1}{3{\cdot}84\times10^{10}}.$$

$$= \frac{4\pi^2\times3{\cdot}84\times10^{10}}{(27{\cdot}3\times24\times3600)^2} = g_m\ cms./sec^2.,\ \textbf{say.}$$

If the acceleration due to gravity on the earth's surface be $g\ cms./sec^2.$, we have $g \propto \dfrac{1}{(\text{radius of Earth})^2}$ and $g_m \propto \dfrac{1}{(\text{radius of Moon's orbit})^2}$.

log 4	$= 0{\cdot}6021$
2 log π	$= 0{\cdot}9944$
3 log 3·84	$= 1{\cdot}7529$
3 log 10^{10}	$= 30{\cdot}0000$
	$33{\cdot}3494$ ...(*i*)
log 6·4	$= 0{\cdot}8062$
8 log 10	$= 8{\cdot}0000$
log 27·3	$= 1{\cdot}4362$
log 24	$= 1{\cdot}3802$
log 3600	$= 3{\cdot}5563$
	$15{\cdot}1789$
$2\times15{\cdot}1789$	$= 30{\cdot}3578$..(*ii*)
Subtracting (*ii*) from (*i*), we have	$2{\cdot}9916$
Antilog	$2{\cdot}9916$
	$= 980{\cdot}9$

$$\therefore \frac{g}{g_m} = \frac{(\text{radius of Moon's orbit})^2}{(\text{radius of Earth})^2}.$$

Or, $g = \left[\dfrac{(3{\cdot}84\times10^{10})^2}{(6{\cdot}4\times10^8)^2}\right]\times g_m$.

$$= \left[\frac{3{\cdot}84\times10^{10}}{6{\cdot}4\times10^8}\right]^2\times\left[\frac{4\pi^2\times3{\cdot}84\times10^{10}}{(27{\cdot}3\times24\times3600)^2}\right]$$

$$= \frac{4\pi^2\times(3{\cdot}84\times10^{10})^3}{(6{\cdot}4\times10^8\times27{\cdot}3\times24\times3600)^2}.$$

$$= 980{\cdot}9\ cms./sec^2.$$

Hence, acceleration due to gravity at the Earth's surface $= 980{\cdot}9\ cms./sec.$

4. The radius of curvature of a railway line at a place when the train is moving with a speed of 20 miles per hour is 800 yds., distance between the rails being 5 ft. Find the elevation of the outer rail above the inner rail so that there may be no side-pressure on the rails. *(Bombay)*

Here, we have $v = 20\ m.p.h. = \dfrac{20\times1760\times3}{60\times60} = \dfrac{88}{3}\ ft./sec.$

and, $r = 800\ yds. = 800\times3 = 2400\ ft.$

log 121	$= 2{\cdot}0828$
log 10800	$= 4{\cdot}0344$
Antilog	$= 2{\cdot}0484$
	$= {\cdot}0112$

So that, $\tan\theta = \dfrac{v^2}{rg} = \dfrac{88\times88}{3\times3\times2400\times32} = \dfrac{121}{10800} = {\cdot}0112.$

Or, $\theta = \tan^{-1}{\cdot}0112 = 38{\cdot}33'.$

Now, $d = 5\ ft.$ and $\sin 38{\cdot}33' = {\cdot}0112.$ [θ being small.

And, $\sin\theta = h/d.$ And $\therefore\ h = d\sin\theta.$

Or, $h = 5\times{\cdot}0112 = {\cdot}0560\ ft. = {\cdot}0560\times12 = {\cdot}6720\ inches.$

Therefore, the outer rail should be raised ·6720 *inches* above the inner one.

5. A stone of mass 10 lbs. is revolving in a vertical circle at the end of a string, 8 ft. long, the other end of which is fixed. When the stone is at the top of the circle, the velocity is 16 ft. per sec. Assuming g to be 32 ft./sec^2., find the stretching force in the string when the stone is (*i*) at the top, (*ii*) at the bottom (*iii*) at a level with the centre.

Here, mass of the stone, $m = 10$ *lbs.* ; radius of the circle, $r = 8$ *ft.*,
and velocity at the *top* of the circle $= 16$ *ft./sec.*

(*i*) Therefore, *force acting outwards, i.e., upwards along the string*, or along *OA*, (Fig. 14),

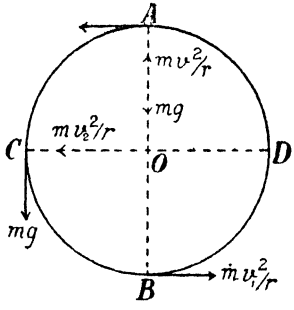


Fig. 14.

$$= \frac{mv^2}{r} = \frac{10\times16\times16}{8} = 320 \text{ poundals.}$$

And, downward force due to weight of the stone

$$= mg = 10\times32 = 320 \text{ poundals.}$$

$\therefore$ *resultant stretching force along the string* $= 320-320 = 0$.

(*ii*) By the time the stone reaches the *bottom* of the circle, it has acquired additional kinetic energy due to its having fallen a vertical distance 16 *ft*, (the diameter of the circle).

Therefore, *kinetic energy* of the stone at the bottom of the circle $= \frac{1}{2}mv^2 + mgh$.

$= [\frac{1}{2}\times10\times16\times16] + [10\times32\times16]$ [$\because$ $h = 16$ *ft.*

$= 1280+5120 = 6400$ *ft. poundals.*

This should be equal to $\frac{1}{2}mv_1^2$, where v_1 is the velocity at the bottom of the circle. So that,

$$\frac{1}{2}mv_1^2 = 6400. \quad \text{Or,} \quad mv_1^2 = 6400\times2 = 12800.$$

Or, $v_1^2 = 12800/10 = 1280.$ [$\because$ $m = 10$ *lbs*

Now, centrifugal force acting downwards along the string, *i.e.*, along *OB*

$$= \frac{mv_1^2}{r}, \text{ and, therefore, } = \frac{10\times1280}{8} = 1600 \text{ poundals.}$$

And the downward force due to the *weight* of the stone is equal to $10\times32 = 320$ *poundals* (as before).

$\therefore$ total stretching force in the string $= 1600+320 = 1920$ *poundals.*

$= 1920/32 = 60$ *pounds weight.*

(*iii*) Here, the additional *K.E.* acquired by the stone is due to the fall through a distance 8 *ft.*, the radius of the circle, and is

$$= 10\times32\times8 = 2560 \text{ ft. poundals.}$$

$\therefore$ total *K.E.* of the stone $= \frac{1}{2}mv^2+2560 = \frac{1}{2}\times10\times16\times16+2560.$

$= 1230+2560 = 3840$ *ft. poundals.*

Obviously, this must be equal to $\frac{1}{2}mv_2^2$, where v_2 is the velocity of the stone, in this position.

$\therefore$ $\frac{1}{2}mv_2^2 = 3840$, whence, $v_2^2 = \frac{3840\times2}{10} = 768.$

$\therefore$ centrifugal force acting outwards along the string, *i.e.*, along *OC*

$$= \frac{mv_2^2}{r} = \frac{10\times768}{8} = 10\times96 = 960 \text{ poundals.}$$

$$= 960/32 = 30 \text{ lbs. wt.}$$

And, since the weight of the stone is acting vertically downwards, it is acting at *right angles* to this centrifugal force and has no component along it. Therefore, the stretching force in the string, in this case, is equal to 960 *poundals* or 30 *lbs. wt.*

6. Four masses, each of 10 lbs., are fastened together with four strings, each 3 ft. long, so as to form a square. This square rotates in a horizontal plane on a smooth table at a speed of one revolution per second ; find in pounds-weight the tension in the string. (*Cambridge Higher School Certificate*)

Let m_1, m_2, m_3 and m_4 be the four masses of 10 *lbs.* each, forming a square, as shown in Fig. 15.

The radius of the circle in which they rotate will obviously be equal to $Om_1 = Om_2 = Om_3 = Om_4 = r$, where O is the point of intersection of the diagonals of the square.

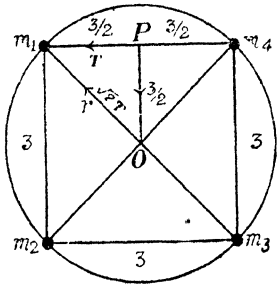


Fig. 15.

If P be the mid-point of $m_1 m_4$, we have

$$OP = Pm_1 = 3/2 \text{ ft.}$$

So that,
$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$$
$$= \sqrt{\frac{18}{4}} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} \text{ ft.}$$

Let T be the tension in each string, when the square is rotating on the smooth table.

Then, representing T, in magnitude and direction, by the straight line Pm_1, we have

force acting along $Om_1 = \sqrt{T^2 + T^2} = \sqrt{2T^2} = \sqrt{2}\,T$.

This, clearly, represents the *centrifugal* force acting along Om_1. Now, the *centrifugal* force acting along m_1O is also equal to $mr\omega^2$, *i.e.*,

$$= 10 \times \frac{3}{\sqrt{2}} \times (2\pi)^2 = \frac{30 \times 4\pi^2}{\sqrt{2}}.$$

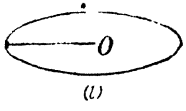
where m is the mass of $m_1 = 10$ *lbs.*, $r = 3/\sqrt{2}$ *ft.*, and $\omega = 2\pi$, because the square *makes one revolution per sec., i.e.*, describes an angle of 2π *radians per sec.*

So that,
$$\sqrt{2}T = \frac{30 \times 4\pi^2}{\sqrt{2}} = \frac{120\,\pi^2}{\sqrt{2}}.$$

log 15 = 1·1761	Or, $2T = 120 \times \pi^2$,
2 log π = 0·9944	whence, $T = 60\pi^2$ *poundals.*
$\bar{2}$·1705	
log 8 = 0·9031	$= \frac{60\pi^2}{32}$ *lbs. wt.*
Antilog $\bar{1}$·2674	
= 18·51	$= \frac{15\,\pi^2}{8} = 18{\cdot}51$ *lbs. wt.*

The tension in each string is, therefore, equal to 18 51 *lbs. wt.*

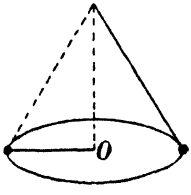
7. A certain string will break under a load of 50 k.gms. A mass of 1 k.gm. is attached to the end of a piece of the string, 10 metres long, and is rotated in a horizontal circle. Find the greatest number of revolutions per minute which the weight can make without breaking the string.



(*i*)

Here two cases arise, *viz.*,

(*i*) when the fixed end of the string, is itself the centre of the circle in which the load is rotating, *i.e.*, *the radius of the circle is the length of the string*, [Fig. 16 (*i*)], and



(*ii*)

Fig. 16.

(*ii*) when the string hangs vertically and a circular motion is given to the load at its end, the circle described being in the horizontal plane, [Fig. 16 (*ii*)].

First case.—The centrifugal force, acting on the string is given by

$$F = \frac{mv^2}{r}.$$

Now, the maximum value of F is 50 *k. gm. wt.* (given).
$= 50 \times 1000 \times 980$ *dynes.*

So that,
$$50 \times 1000 \times 980 = \frac{1000 \times v^2}{1000} = v^2.$$

Therefore,
$$v = \sqrt{50000 \times 980} = \sqrt{49000000}$$
$= 7000$ *cms./sec.*
$= 7000 \times 60$ *cms./minute.*
$= 420000$ *cms./minute.*

where m is the mass of the load
$= 1$ *k.gm.*
$= 1000$ *gms.*
$r = 10$ *metres.*
$= 1000$ *cms.*,
and v = ?

Now, the distance covered by the load in 1 rotation $= 2\pi r$.

$= 2\pi \times 1000 = 2 \times 3{\cdot}1416 \times 1000.$

$= 6283{\cdot}2$ *cms.* $= 6283$ *cms.*, say.

$\therefore$ number of revolutions made by the load *per minute* without breaking the string $= v/2\pi r = 420000/6283$

$= 66\,83.$

log 42000	= 5·6232
log 6283	= 3·7982
Antilog	1·8250
	= 66·83

Or, the number of *complete* rotations made by the load is 66 *per minute*.

Second case.—Here, two forces are acting on the load, *viz.*,

(*i*) mv^2/r, (*centrifugal force*) *horizontally*$=mr\omega^2$, since $v = r\omega$, where v and ω are the *linear* and *angular velocities* of the load respectively, and r, the *radius* of the horizontal circle in which it rotates.

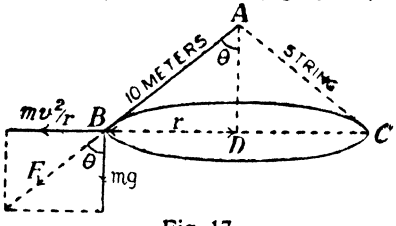


Fig. 17.

(*ii*) *mg*, (*weight of the load*) *vertically downwards.*

Obviously, for the load to be in steady motion, the resultant of these two must act along the string, as shown in Fig. 17.

Let θ be the angle, that the *resultant force* F (*the tension in the string*) makes with the vertical. Then,

$$\tan\theta = \frac{mr\omega^2}{mg} = \frac{r\omega^2}{g}. \quad \ldots \quad \ldots \quad \ldots (1)$$

Now, $r = AB \sin\theta = 1000 \sin\theta,$

$m = 1$ *k. gm.* $= 1000$ *gms.*

The maximum value of the tension of the string, *i.e.*,

$T = 50 \times 1000 \times 980$ *dynes.*

If n be the number of rotations made by the load *per minute*, the angular velocity $= 2\pi n$ *per minute.*

Or, $$\omega = \frac{2\pi n}{60} = \frac{\pi n}{30} \text{ per sec.}$$

Substituting the values of r and ω in relation (1) above, we have

$$\tan\theta = \frac{\pi^2 n^2 . 1000 \sin\theta}{900\, g}.$$

Or, $$\frac{\sin\theta}{\cos\theta} = \frac{\pi^2 n^2 . 1000 \sin\theta}{900\, g} = 10 . \frac{\pi^2 n^2 \sin\theta}{9\, g}.$$

Or, $$\cos\theta = \frac{9}{10} . \frac{g}{\pi^2 n^2}.$$

But, from the figure, we ha

$$\cos\theta = \frac{mg}{F} = \frac{1000\, g}{50 \times 1000 \times 980} = \frac{9}{50 \times 980}.$$

$\therefore$ $$\frac{9}{10} . \frac{g}{\pi^2 n^2} = \frac{g}{50 \times 980}.$$

Or, $$\frac{g}{10 . \pi^2 n^2} = \frac{1}{50 \times 980},$$

whence, $$n^2 = \frac{9 \times 50 \times 980}{10 \times \pi^2}.$$

Or, $$n = \frac{\sqrt{9 \times 5 \times 980}}{\pi} = 66{\cdot}83.$$

log 9	= 0·9542
log 5	= 0·6990
log 980	= 2 9912
	4·6444
	½ × 4·6444
	= 2·3222
log π	= 0·4972
Antilog	1·8250
	= 66·83

Hence, number of rotations = 66·83 *per minute* ; or, number of *complete* rotations made by the load *per minute* is 66.

8. A particle of mass m is attached to a fixed point by means of a string of length l and hangs freely. Show that if it is projected horizontally with a velocity greater than $\sqrt{5gl}$, it will completely describe a vertical circle.

Let OB, (Fig. 18), be the string, fixed at O and suspended freely, with a mass m at B. Let it be given a horizontal velocity u, when at rest at B. It will naturally move along an arc of radius l, the length of the string. Let B' be its position on the arc at a given instant, when its velocity is v.

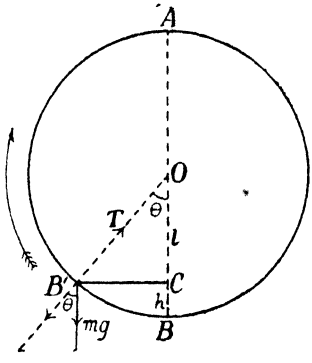


Fig. 18.

Then, clearly, $v^2 = u^2 - 2gh$,

where h is the vertical distance through which the mass has been raised up.

The *weight* mg of the mass is acting vertically downwards at B', and the *centripetal force* mv^2/l, along the string, in the direction $B'O$.

The component of mg, acting *along* the string in the *direction* OB', *i.e*, *opposite to that of the centripetal force*, is thus clearly $= mg.\cos\theta$.

$\therefore$ If T be the *tension* of the string

we have $$\frac{mv^2}{l} = T - mg\cos\theta.$$

Or, $$T = \frac{mv^2}{l} + mg\cos\theta. \quad \ldots \quad \ldots \quad \ldots(I)$$

Now, $$\cos\theta = \frac{OC}{OB} = \frac{OB-CB}{OB'} = \frac{l-h}{l}.$$

$\therefore$ substituting the values of v^2 and $\cos\theta$ in relation (I) above, we have

$$T = \frac{m(u^2-2gh)}{l} + mg\frac{(l-h)}{l} = \frac{m}{l}\left[\, u^2-2gh+g(l-h) \,\right]$$

$$= \frac{m}{l}\left[\, u^2-2gh+gl-gh \,\right]$$

Or, $$T = \frac{m}{l}\left[\, u^2-3gh+gl \,\right] \quad \ldots \quad \ldots(II)$$

At the highest point on the circle, *i.e.*, at A, when the mass completes half the vertical circle, we have $h = 2l$.

$\therefore$ at A, $$T = \frac{m}{l}\left[\, u^2-6gl+gl \,\right] = \frac{m}{l}\left(\, u^2-5gl \,\right).$$

This will clearly become equal to zero, if $u^2 = 5gl$.

Now, at A, $v^2 = u^2-4gl$. [$\because$ $h = 2l$, here.

Obviously, therefore, if the mass is to continue in motion along the circle, the tension T should not vanish, *i.e.*, should not become *zero*, which means that $u^2 > 5gl$. For this value of u^2, its velocity at A will also not vanish, and hence the mass will describe a complete circle of radius l.

Thus, *the condition necessary for the mass to complete a vertical circle is that* $u^2 > 5gl$, *or that* $u > \sqrt{5gl}$.

9. Assuming the law of Gravitation, and taking the orbit of the Earth round the Sun, and of the Moon round the Earth as circular, compare the masses of the Sun and the Earth, given that the Moon makes 13 revolutions per year and that the Sun is 390 times as distant as the Moon.

Let mass of the Sun be $= M_s$, mass of the Earth $= M_e$.

and mass of the Moon $= M_m$,

And, let distance of the Moon from the Earth be $= R$,

so that ... ... Sun ... ... $= 390\,R$.

Then, force of attraction between the Sun and the Earth $= \frac{M_s \times M_e}{(390R)^2} G$, where G is the *gravitational constant.*

$\therefore$ *centripetal acceleration of the Earth* $= \frac{force}{mass} = \frac{M_s . M_e}{(390R)^2 M_e} G.$

$$= \frac{M_s}{(390\ R)^2} G.$$

Similarly, force of attraction between the Earth and the Moon

$$= \frac{M_e\ M_m}{R} . G.$$

And $\therefore$ *centripetal acceleration of the Moon* $= \frac{M_e . M_m}{R^2 . M_m} . G = \frac{M_e}{R^2} . G.$

Let ω_e and ω_m be the angular velocities of the Earth and the Moon respectively. Then, clearly,

centripetal acceleration of the Earth is also $= 390\ R.\omega_e^2$.

and ,, ,, ,, ,, Moon ,, ,, $= R\omega_m^2$.

$\therefore$ $\frac{[M_s/(390\ R)^2]G}{(M_e/R^2)G} = \frac{390.R.\omega_e^2}{R\ \omega_m^2}$.

Or, $\frac{M_s}{(390)^2 . M_e} = \frac{390\omega_e^2}{\omega_m^2}$.

Now, the Earth goes round the Sun only *once in one year* ; and, therefore,

angular velocity of the Earth $= 2\pi$ *per year.*

And the Moon goes round the Earth 13 *times* in one year ; so that,

angular velocity of the *Moon* $= 13 \times 2\pi$ *per year.*

Therefore, $\frac{M_s}{(390)^2 . M_e} = \frac{390 \times (2\pi)^2}{(13 \times 2\pi)^2}$.

Or, $\frac{M_s}{(390)^2 . M_e} = \frac{390}{13^2}$. Or, $\frac{M_s}{M_e} = \frac{390^3}{13^2}$.

i.e., Mass of the sun : Mass of the Earth : : $(390)^3$: (13) .

10. Show that in the case of a liquid, rotating with a uniform angular velocity, (*i*) the pressure varies directly as the distance from the axis of rotation and (*ii*) the free surface of the liquid is a paraboloid of revolution.

(*i*) Imagine a *closed, vertical, cylindrical* vessel, *just* full of a liquid of density ρ, to be rotating about its axis with a *uniform angular velocity* ω.

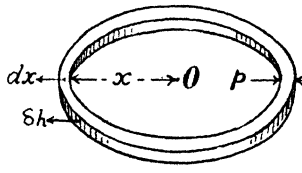


Fig. 19.

Now, consider a *ring* of the liquid, of *radius x, width* δx and *vertical height* δh, with its centre at *O on the axis of rotation,* (Fig. 19). Then, if the pressures in the liquid at distances x and $x+\delta x$ from the axis of rotation be p and $(p+\delta p)$ *outwards* and *inwards* respectively, we have

resultant inward thrust on the ring

$$= \delta p\ 2\pi x . \delta h,$$

and *centrifugal force outwards on the ring* $= m\omega^2 r$,

where m is the *mass* of the liquid in the ring.

Clearly, $m =$ *volume of liquid in the ring* $\times \rho = 2\pi x . \delta x . \delta h . \rho$.

So that, *centrifugal force outward on the ring* $= 2\pi x . \delta x . \delta h . \rho . \omega^2 . x$ [$\because$ $r = x$, here.

And, therefore, $\delta p . 2\pi x . \delta h = 2\pi x . \delta x\ \delta h . \rho . \omega^2 . x$.

whence, $\delta p = \rho \omega^2 . x . \delta x$.

Integrating this expression, we have

$$\int dp = \int \rho \omega^2 x . dx = \rho \omega^2 \int x . dx.$$

Or, $$p = \tfrac{1}{2}\rho\omega^2x^2 + C,$$

where C is a constant of integration.

This is then the expression for the pressure at a point distant x from the axis of rotation in a rotating liquid.

The value of C is obviously equal to that of p at $x = 0$, *i.e.*, equal to the pressure at O, which, as we know, is *zero* at the surface of the liquid, but increases with depth as in the case of a liquid at rest.

(*ii*) Consider a particle P in the liquid surface, of *mass m*, whose co-ordinates are (x, y) with respect to the axes OX and OY, (Fig. 20), the liquid being supposed to be rotating with a *uniform angular velocity* ω in the direction shown, about OY.

The forces acting on this particle P are, clearly, (*i*) *its weight mg, vertically downwards*, and (*ii*) *the centrifugal force* $m\omega^2x$, *outwards*. The resultant R of these two forces must act at right angles to the liquid surface, since there is no flow of the liquid taking place, and, obviously, it is counter-balanced by the thrust due to the rest of the liquid on P.

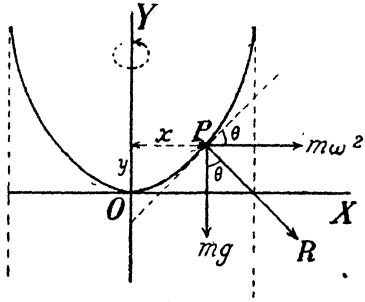


Fig. 20.

If, therefore, θ be the angle that the centrifugal force makes with the tangent at P, we have

$$\tan\theta = \frac{m\omega^2x}{mg} = \frac{\omega^2x}{g}.$$

Now, obviously, $\tan\theta = \text{slope at } P = dy/dx.$

$$\therefore \quad \frac{dy}{dx} = \frac{\omega^2x}{g}. \quad \text{Or,} \quad dy = \frac{\omega^2x}{g}.dx.$$

Integrating this, we have

$$\int dy = \int \frac{\omega^2x}{g}.dx = \frac{\omega^2}{g}\int x.dx.$$

Or, $$y = \frac{\omega^2x^2}{2g} + C,$$

where C is again a constant of integration.

Since $y = 0$, when $x = 0$, we have $C = 0$.

And, therefore, $$y = \frac{\omega^2x^2}{2g},$$ which is the equation to a *parabola*.

The free surface of a uniformly rotating liquid is thus clearly a paraboloid of revolution.

EXERCISE II

1. A particle of mass 1 *lb.* is whirled uniformly at the end of a string, 2 *ft.* long, and makes 3 revolutions in 1·2 *secs.* Find the tension in the string.
Ans. 15·43 *lb. wt.*

2. A half-pound weight is being whirled in a horizontal circle at the end of a string, 2 *feet* long, the other end of the string being fixed. If the breaking tension of the string is 112 *lb. wt.*, find the greatest speed which can be given to the weight. **Ans.** 119·7 *ft./sec.*

3. At what angle should a cyclist lean over, when negotiating a curve of 132 *ft.* radius at 15 *miles per hour*. **Ans.** 6° 32′.

4. A person skating on ice at the rate of 20 *ft. per second* describes a circle of 20 *ft.* radius. What is his inclination to the vertical ? **Ans.** 32·0°.

5. A train is going round a curve of 1000 *ft.* radius and the distance between the rails is 5 *feet.* By how much should the outer rail be raised above the inner one, so that a train running at 45 *miles per hour* may exert no lateral thrust on the outer rail ? **Ans.** 8·094".

6. A stone is suspended from the roof of a railway carriage by means of a string 5 *ft.* long. The angle through which the mass moves from the vertical is 10°, when the train moves along a curve of radius 600 *ft.* Calculate the speed of the train. **Ans.** 58·1 *ft./sec.*

7. Assuming that the Moon describes a circular orbit of radius $3{\cdot}84\times10^{8}$ *metres* in 27·3 days and the outer satellite of Mars describes a circular orbit of radius $= 2{\cdot}35\times10^{7}$ *metres* in 1·26 *days*, find the ratio of the mass of Mars to the mass of the Earth. (*Cambridge University*)
Ans. ·1076 : 1.

8. A curve on a railway line has a radius 1600 *ft.*, and the distance between the inner and outer rails is 5 *ft.* If the outer rail be 6" above the inner one, calculate the maximum speed of a train going along the curve, so that no side thrust is exerted on the outer rail. **Ans.** 48 89 *miles/hr.*

9. Calculate the increase in length of an elastic string of original length 10 *ft.*, at the end of which a stone of mass ·5 *lb.* is whirled at the rate of 4 revolutions *per second*, if a load of 25 *lbs.* increases the length of the string by 2%.
Ans. ·8576 *ft.*

10. A merry-go-round is revolving in a horizontal circle of radius 3 *ft.* at the rate of 7 revolutions in 11 *seconds.* A child of weight 20 *lbs.* rides a wooden horse suspended by a vertical string. Find the tension in the string and its inclination to the vertical. **Ans.** (*i*) 36 *lb. wt.*, nearly, (*ii*) $\tan^{-1}3/2$.

[**Hint** : *See solved example* 7—(*second case, page* 42.)]

11. A sea-plane of total mass 1000 *lb.* (including the pilot) rounds a pylon in a circular arc of radius half a mile at a speed of 300 *m.p.h.* Draw a diagram showing the forces acting on the sea-plane, and calculate the resultant force at right angles to its direction of motion exerted upon it by the air. Assuming that the pilot weighs 12 stones, calculate the force with which he is pressed against his seat during the "turn." (*Cambridge Higher School Certificate*)
Ans. 8×10^{4} *poundals* : 30 *stone-wt.*

12. Calculate the angle at which a curve of radius 352 *ft.* should be banked so as to avoid side-slip when a motor car is travelling round it at a speed of 30 *m.p.h.* **Ans.** 9° 45'.

13. A road over a bridge has the form of a vertical arc of radius 60 *ft.* What is the greatest speed in *m.p.h.* at which a car can cross the bridge without leaving the ground at the crest of the road ? **Ans.** 30 *m.p.h.*

14. A skater is moving on one foot in a circle of radius 20 *ft.* at 10 *m.p.h.* At what angle with the vertical will the line passing through his centre of gravity and the edge of his skate be inclined ? **Ans.** 18° 35'.

15. In a 'loop-the-loop' railway, the cars, after descending a steep incline, run round the inside of a vertical circular track, 20 *ft.* in diameter, making a complete turn over. Assuming there is no friction, find the minimum height above the top of the circular track from which the cars must start.
Ans. 5 *ft.*

16. A symmetrically loaded lorry weighs 5 *tons*, and the height of its centre of gravity is 5 *ft.* above the ground in a vertical plane midway between the wheels. The breadth of the wheel base may be taken to be 6 *ft.* 3 *in.* If there is no side-slip, what is the maximum speed at which the lorry can take a curve of mean radius 6 *yards* without beginning to overturn ? **Ans.** 13 *m.p.h.*

[**Hint** : *It will overturn only when the moment of the centrifugal force about the wheels on one side is greater than the moment of the weight about them,* (*see* § 13, *case* 4, *page* 31).]

17. An India rubber band has a mass of 4 *gm.* per metre when stretched on the circumference of a wheel of 10 *cm.* radius, the stretching force being

20,000 *dynes*. Find how many revolutions per second the wheel must make so that the band may not press upon the wheel. **Ans.** 11·3.

[**Hint** : *See* § 18, *case* 2, *page* 27.]

18. Discuss the possibility of a motor cyclist riding round the inside surface of a vertical cylinder. (*Cambridge Scholarship*)

19. Explain why a motor-cycle combination (side car on left) is liable to overturn when taking a left hand corner at speed. Assuming that the centre of mass of the combination is 2 *ft.* from the ground and 1 *ft.* to the left of the motor cycle, calculate the maximum speed of the combination in a circle of radius 50 *ft.* Assume that the road surface is horizontal and that there is no skidding. (*Oxford Scholarship*)
Ans. 19·3 *m.p.h.*

20. A closed cylindrical can of radius a and height h is first filled with a fluid of density ρ. It is then rotated with angular velocity ω about its own axis, held vertical. Prove that the total thrust on the top of the can will be $\pi\rho\omega^2a^4/4$.

CHAPTER III

MOMENT OF INERTIA—ENERGY OF ROTATION

26. Moment of Inertia and its Physical Significance—Radius of Gyration. We know that, according to Newton's *first law of motion*, a body must continue in its *state of rest* or of *uniform motion along a straight line*, unless acted upon by an external force. *This inertness or inability of a body to change by itself, its position of rest, or of uniform motion, is called* **inertia***, *and is a fundamental property of matter.* Thus, it is by virtue of its *inertia* that a body, at rest, resists or opposes being put into motion, and a body, in linear or translatory motion, opposes not only being brought to rest but also any change in the magnitude and direction of its motion And, we know, by experience, that the greater the mass of a body, the greater its inertia or opposition to the desired change ; for, the greater is the force required to be applied for the purpose. *The mass of a body is thus taken to be a measure of its 'inertia for translatory motion'*, as it is *this* that opposes the acceleration, (positive or negative), desired to be produced in it by the applied force.

Exactly in the same manner, in the case of *rotational motion* also, we find that a body, free to rotate about an axis, opposes any change desired to be produced in its state of rest or rotation, showing that it possesses '*inertia*' for this type of motion And, obviously, the greater the *couple or torque*, (see §28), required to be applied to a body to change its state of rotation, *i.e.*, to produce in it a desired angular acceleration, the greater its opposition to the desired change, or the greater its '*inertia for rotational motion*'. *It is this 'rotational inertia' of the body which is called its* **moment of inertia**** *about the axis of rotation*,—this name being given to it on the analogy of the moment of the couple, which it opposes.

It will thus be seen that the *moment of inertia* of a body, in the case of rotational motion, plays the same part as, or is the analogue of the *mass* of a body in the case of translatory motion ; and we may, therefore, for purposes of analogy, describe the moment of inertia of a body, in rotational motion as the '*effectiveness of its mass.*' Or, pushing the analogy a little further, we may define *mass* as the '*coefficient of inertia*† *for translatory motion*', and the *moment of inertia*, as the '*coefficient of rotational inertia*'.

Yet, with all this seeming similarity, there is all the difference between the two cases. For, in the case of translatory motion, the

*That is why the comparative slackness or sluggishness of the people of Eastern countries,—a consequence of climatic conditions—is dubbed by the Westerners as the '*Inertia of the East.*'

It is also sometimes referred to as the **'Spin inertia' of the body about its axis of rotation.

†The mass of a body being usually referred to as its **inertia coefficient.**

inertia of the body depends wholly upon its *mass* and is, therefore, measured in terms of it alone. In the case of rotational motion, on the other hand, the rotational inertia, or the moment of inertia, of the body, depends not only upon the mass (M) of the body but also upon the '*effective distance*' (K) of its particles from the axis of rotation, and is measured by the expression MK^2, (see next Article).

This '*effective distance*' (K) of the particles of a body from its axis of rotation is called its **radius of gyration** about that axis, and is equal to the *root mean square distance* of the particles from the axis, *i.e.*, equal to the square root of their *mean square distance* (not the square of their mean distance) from it. Or, to give it a clear cut definition, *the radius of gyration of a body about a given axis of rotation, may be defined as the distance from the axis, at which, if the whole mass of the body were to be concentrated, the moment of inertia of the body about the given axis of rotation would be the same as with its actual distribution of mass.*

Now, it is obvious that a change in the position or inclination of the axis of rotation of a body will bring about a corresponding change in the relative distances of its particles, and hence in their '*effective distance*', from the axis. *i.e.*, in the value of the *radius of gyration* of the body about the axis And, so will the transference of a portion of the matter (or mass) of the body from one part of it to another, or a change in the distribution of the mass about the axis, the total mass of the body remaining the same, in either case.

Thus, whereas the mass of a body remains the *same*, irrespective of the location or inclination of the axis of rotation, *the value of its radius of gyration about the axis depends upon*

(*i*) *the position and direction of the axis of rotation*, and

(*ii* *the distribution of the mass of the body about this axis*; so that, its value for the same body is different for different axes of rotation.

Further, it follows, as a converse of the above, that the radius of gyration of a body about a given axis of rotation gives an indication of the distribution of the mass of the body about it and hence, also, the effect of this distribution of mass on the moment of inertia of the body about that axis.

27. Expression for the Moment of Inertia. Suppose we have a body of mass M, (Fig. 21), and any axis YY'. Imagine the body to be composed of a large number of particles of masses m_1, m_2, m_3 etc., at distances r_1, r_2, r_3...etc from the axis YY'. Then, the moment of inertia of the particle m_1 about YY is $m_1r_1^2$, that of the particle m_2 is $m_2r_2^2$, and so on; and, therefore, the moment of inertia, I, of the whole body, about the axis YY', is equal to the sum of $m_1r_1^2$, $m_2r_2^2$, $m_3r_3^2$.....etc.

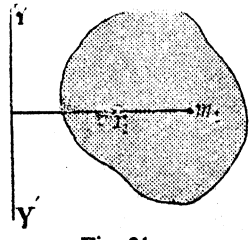


Fig. 21.

Thus, $I = m_1r_1^2+m_2r_2^2+m_3r_3^2+\ldots$

$= \Sigma mr^2$.

Or, $I = MK^2$,

[where M is the *mass* and MK^2 the summation ΣMr^2 for the whole body.

K being the *radius of gyration* of the body about the axis YY'.

28. Torque. If we wish to accelerate the rotation of a body, free to rotate about an axis, we have to apply to it a couple. *The moment of the couple, so applied, is called* **torque**, and we say that a torque is applied to the body.

Obviously, the *angular acceleration* of all the particles, *irrespective of their distances* from the axis of rotation, is the *same*, but because their distances are different from the axis, their *linear accelerations are different*, (the linear acceleration of a particle being the product of the angular acceleration and the distance of the particle from the axis of rotation).

If, therefore, $d\omega/dt$ be the angular acceleration of the body, or its particles, we have

linear acceleration of the particle distant r_1 from the axis $= r_1.d\omega/dt$,
" " " " " " r_2 " " " $= r_2.d\omega/dt$.
and so on.

Hence, if m be the mass of each particle of the body, the forces on the different particles are $mr_1.d\omega/dt$, $mr_2.d\omega/dt$, etc., and the *moments* o these forces about the axis of rotation will, therefore, be

$$(mr_1.d\omega/dt) \times r_1, \ (mr_2.d\omega/dt) \times r_2 \text{ and so on.}$$

Therefore, total moment for the *whole* body

$$= (mr_1.d\omega/dt) \times r_1 + (mr_2.d\omega/dt) \times r_2 + \ldots\ldots$$
$$= mr_1^2\, d\omega/dt + mr_2^2.d\omega/dt + \ldots\ldots$$
$$= \Sigma mr^2.d\omega/dt,$$
$$= (d\omega/dt).\Sigma mr^2. \qquad [d\omega/dt \text{ being constant.}$$

But $\Sigma mr^2 = I$, the moment of inertia of the body about the axis of rotation. And, therefore,

$$\textit{moment for the whole body} = I.d\omega/dt.$$

This must be equal to the *torque* applied to the body.

So that, $$\textit{torque} = I.d\omega/dt.$$

It will at once be clear that this relation corresponds to the familiar relation, $force = m \times a$, in the case of linear motion, where m is the *mass* and a, the *acceleration* of the body.

Thus, in the case of rotatory motion, torque, moment of inertia and angular acceleration are the analogues of force, mass and linear acceleration respectively in the case of linear or translatory motion.

Now, if $d\omega/dt = 1$, clearly, $torque = I$.

Or, *the moment of inertia of a body about an axis is equal to the torque, producing unit angular acceleration in it about that axis.*

Incidentally, the expression for torque, obtained above, furnishes us with a method of deducing an expression for the moment of inertia of a particle of mass m, about an axis, distant r from it.

For, if F be the force applied, we have

$$\textit{torque} = F \times r.$$
$$\textit{torque} \text{ is also } = I\, d\omega/dt.$$

And, therefore, $$I = \frac{torque}{d\omega/dt} = \frac{F \times r}{d\omega/dt} \quad \ldots (i)$$

Now, $$F = m \times a,$$

where a is the *linear acceleration* of the particle.

And, since $a = dv/dt$, (where v is the *linear velocity* of the particle), we have $F = m.dv/dt$.

Again, since $v = r\omega$, where ω is the *angular velocity* of the particle, we have

$$F = m.\frac{d(r\omega)}{dt} = m.\left(\frac{d\omega}{dt}.r + \frac{dr}{dt}.\omega\right).$$

Now, the component, $(dr/dt).\omega$, plays no part in the rotation of the body and may, therefore, be ignored ; so that, $F = mr.d\omega/dt$.

Substituting this value of F in relation (i) above, we have

$$I = \frac{mr.(d\omega/dt).r}{d\omega/dt} = mr^2.$$

Thus, the moment of inertia of a particle of mass m, about an axis distant r from it, is equal to mr^2.

29. General Theorems on Moment of Inertia. There are two general theorems of great importance on moment of inertia, which, in some cases, enable us to determine the moment of inertia of a body about an axis, if its moment of inertia about some other axis be known. We shall now proceed to discuss these.

(a) **The Principle or Theorem of Perpendicular Axes.**

(i) **For a Plane Laminar Body.** According to this theorem, the *moment of inertia of a plane lamina about an axis, perpendicular to the plane of the lamina, is equal to the sum of the moments of inertia of the lamina about two axes at right angles to each other, in its own plane, and intersecting each other at the point where the perpendicular axis passes through it.*

Thus, if I_x and I_y be the moments of inertia of a *plane lamina* about the perpendicular axes, OX and OY, which lie *in the plane of the lamina* and intersect each other at O, (Fig. 22), the moment of inertia about an axis passing through O and *perpendicular to the plane of the lamina*, is given by

$$I = I_x + I_y.$$

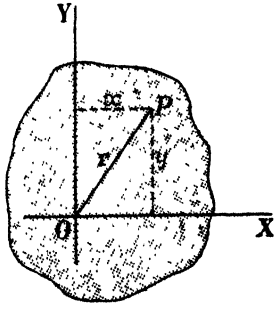


Fig. 22.

For, considering a particle of mass m at P, at distances x and y from OY and OX respectively, and at distance r from O, we have

$I = \Sigma mr^2$, $I_x = \Sigma my^2$ and $I_y = \Sigma mx^2$.

So that, $I_x + I_y = \Sigma my^2 + \Sigma mx^2$.

$= \Sigma m(y^2 + x^2)$.

$= \Sigma mr^2$. $[\because y^2 + x^2 = r^2$.

Or, $I_x + I_y = I$.

(ii) **For a Three-Dimensional Body*.** Suppose we have a *cubical* or a *three-dimensional* body, shown dotted in Fig. 23, with OX, OY and OZ as its three mutually perpendicular axes, representing its length, breadth and height respectively.

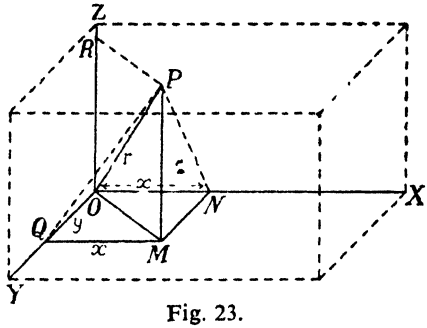


Fig. 23.

Consider a mass m of the body, at a point P, somewhere inside it. Drop a perpendicular PM from P on the xy plane to meet it in M. Join OM and OP, and from M draw MQ parallel to the x-axis and MN, parallel to the y-axis; also, from P draw PR, parallel to OM. Then, clearly, the co-ordinates of the point P are

$$x = ON = QM\,;\; y = OQ = NM \quad \text{and} \quad z = MP = OR.$$

Since the plane xy is perpendicular to the z-axis, any straight line drawn in this plane is also perpendicular to it, and, therefore, OM and PR are both perpendicular to the z-axis, ($\because$ PR is drawn parallel to OM).

Obviously, therefore, $\angle OMP$ is a right angle, because OM is parallel to PR and PM is parallel to OR. Hence, we have

$$OM^2+MP^2 = OP^2.$$

Or, $$OM^2+z^2 = r^2, \qquad \text{where, } OP = r. \qquad \ldots (i)$$

But $$OM^2 = QM^2+OQ^2.$$

Or, $$OM^2 = x^2+y^2.$$

[$\because$ OQM is a right angle, being the angle between the axes x and y.]

Therefore, substituting the value of OM^2 in relation (i) above, we have $$x^2+y^2+z^2 = r^2. \qquad \ldots (ii)$$

Join PN and PQ. Then, PN and PQ are the respective normals to the axes of x and y. For, $\angle PMN$ is a right angle, being the angle between the axes y and z, and, therefore,

$$PN^2 = MN^2+PM^2 = y^2+z^2.$$

So that, $$x^2+PN^2 = x^2+y^2+z^2 = r^2. \qquad [\text{From } (ii) \text{ above}$$

Or, $$ON^2+PN^2 = r^2, \qquad [\because\ x = ON.$$

from which it is clear that $\angle PNO$ is a right angle, and, therefore, PN is perpendicular to the x-axis.

Similarly, in the right-angled $\triangle PMQ$, we have

$$PQ^2 = MQ^2+PM^2 = x^2+z^2.$$

But $$x^2+y^2+z^2 = r^2.$$

$\therefore$ $$PQ^2+y^2 = r^2.$$

Or, $$PQ^2+OQ^2 = OP^2. \qquad [\because\ y = OQ. \text{ and } r = OP.$$

Or, $\angle PQO$ is a right angle, *i.e.*, PQ is perpendicular to the y-axis.

*Not strictly included in the B.Sc. (Pass or General) course.

Now, moment of inertia of mass m at P, about the z-axis
$$= m \times PR^2 = m.OM^2,$$
because $PR = OM$ is the perpendicular distance of the mass from the axis.

$\therefore$ moment of inertia of the whole body about the z-axis, *i.e.*,
$$I = \Sigma m.OM^2.$$
Or,
$$I_z = \Sigma m(x^2+y^2).$$
Similarly, the moment of inertia of the body about the y-axis, *i.e.*,
$$I_y = \Sigma m.PQ^2.$$
Or,
$$I_y = \Sigma m.(x^2+z^2).$$
[$\because$ PQ is the $\perp$ distance between the mass and the axis.

And, the moment of inertia of the body about the x-axis, *i.e.*,
$$I_x = \Sigma m.PN^2.$$
Or,
$$I_x = \Sigma m(y^2+z^2)$$
[$\because$ PN is the $\perp$ distance between the mass and the axis.

$\therefore$ adding up the moments of inertia of the body *about the three axes*, we have
$$I_x+I_y+I_z = \Sigma m(y^2+z^2)+\Sigma m(x^2+z^2)+\Sigma m(x^2+y^2).$$
$$= 2\Sigma m(x^2+y^2+z^2).$$
Or,
$$I_x+I_y+I_z = 2\Sigma mr^2.$$
[$\because$ $x^2+y^2+z^2=r^2$.

Hence the sum of the moments of inertia of a three-dimensional body about its three mutually perpendicular axes, is equal to twice the summation Σmr^2 about the origin.

The Principle or Theorem of Parallel Axes. This theorem (due to **Steiner**) is true both for a plane laminar body as well as a three-dimensional body and states that *the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis, through its centre of mass, plus the product of the mass of the body and the square of the distance between the two axes.*

(*i*) **Case of a Plane Laminar Body.** Let C be the *center of mass* of a body of mass M, (Fig. 24). and I_c, its moment of inertia about an axis through C, *perpendicular to the plane of the paper.*

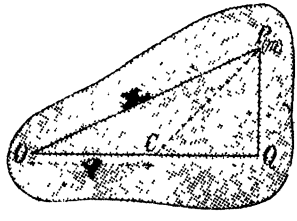


Fig. 24.

Now, let it be required to determine the moment of inertia I of the body about a *parallel* axis through O, distant r from C.

Consider any particle P of the body, of mass m, at a distance x from O.

Then, the moment of inertia of the body about O is given by
$$I = \Sigma mx^2.$$
[Since $OP^2 = x^2$.

From P drop a perpendicular PQ on to OC produced, and join PC. Then,
$$OP^2 = CP^2+OC^2+2OC.CQ.$$
[By simple geometry.

And $\therefore$ $m.OP^2 = m.CP^2+m.OC^2+2m.OC.CQ.$

$\therefore$ $\Sigma mx^2 = \Sigma m.CP^2+\Sigma mr^2+2r\Sigma m.CQ.$ [$\because$ $OP = x$ & $OC = r$.

Hence $I = I_c+Mr^2+2r\Sigma m.CQ.$ [$\because$ $\Sigma m.CP^2 = I_c$.

Now, since a body always balances about an axis passing through its centre of mass, it is obvious that the *algebraic sum* of the moments of the weights of its individual particles about the centre of mass must be *zero*. Hence, here, $\Sigma mg.CQ$, (the algebraic sum of such moments about C) and, therefore, the expression $\Sigma m.CQ$ is equal to 0, g being constant at a given place. Consequently,

$$2r.\Sigma m.CQ = 0.$$

So that, $$I = I_c + Mr^2.$$

(*ii*) **Case of a Three-Dimensional Body.** Let AB be the axis about which the moment of inertia of a body (shown dotted) is to be determined, (Fig. 25). Draw a parallel axis CD through the centre of mass G of the body, at a distance r from it.

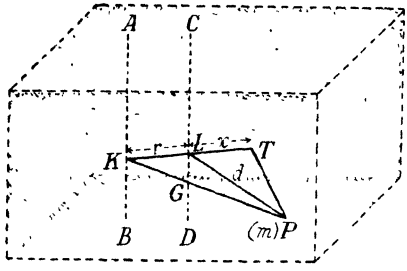


Fig. 25.

Imagine a particle of mass m at any point P, outside the plane of the axes AB and CD and let PK and PL be perpendiculars drawn from P on to AB and CD respectively and PT, the perpendicular dropped from P on to KL produced.

Put $PL = d$, $LK = r$, $LT = x$ and $\angle PLK = \theta$.

Then, if I be the moment of inertia of the body about the axis AB and I_c its moment of inertia about the axis CD (through G), we clearly have

$$I = \Sigma m.PK^2 \text{ and } I_c = \Sigma m.PL^2 = \Sigma m.d^2.$$

Now, from the geometry of the Figure, we have

$$PK^2 = PL^2 + LK^2 - 2PL.LK \cos PLK.$$
$$= d^2 + r^2 - 2d.r \cos \theta.$$

And, in the right-angled $\triangle PTL$, we have

$$\cos PLT = LT/PL,$$

where $\angle PLT = (180° - PLK) = (180° - \theta)$. So that,

$$\cos (180° - \theta) = x/d.$$

Or, $$-\cos \theta = x/d,$$

whence, $$d \cos \theta = -x.$$

Substituting this value of $d \cos \theta$ in the expression for PK above, we have

$$PK^2 = d^2 + r^2 + 2rx.$$

And, therefore, $$I = \Sigma m.PK^2 = \Sigma m(d^2 + r^2 + 2rx).$$

Or, $$I = \Sigma md^2 + \Sigma mr^2 + 2r\Sigma mx.$$
$$= I_c + Mr^2 + 2r\Sigma mx,$$

because $\Sigma mr^2 = Mr^2$, where M is the mass of the *whole* body and r, the distance between the two parallel axes and hence a constant. Clearly, $\Sigma mx = 0$, being the total moment about an axis through the centre of mass of the body.

We, therefore, have $$I = I_c + Mr^2,$$

the same result as obtained above in case (*i*) for a plane laminar body.

30. Calculation of the Moment of Inertia of a Body.—Its Units etc. In the case of a *continuous, homogeneous* body of a definite *geometrical shape*, its moment of inertia is calculated by first obtaining an expression for the moment of inertia of an *infinitesimal mass* of it about the given axis—*by multiplying this mass (m) by the square of its distance (r) from the axis,* (see page 51)—, *and then integrating this expression over the appropriate range, depending upon the shape of the body concerned making full use of the theorems of perpendicular and parallel axes, wherever necessary.*

In case, however, the body is not homogeneous or of a definite geometrical shape, the safest thing to do is to determine its moment of inertia by actual experiment, as explained later, in § 34 and in Chapter VIII.

Now, it will be seen that since the moment of inertia of a body about a given axis is equal to MK^2, where M is its *mass* and K, its *radius of gyration* about that axis, its demensions are 1 *in mass* and 2 *in length,* its dimensional formula being $[ML^2]$. If the mass of the body and its radius of gyration be measured in the *C.G.S.* units, *i.e.*, its mass in *grams* and radius of gyration in *centimetres*, the moment of inertia of the body is expressed in *gram-centimetre*^{2}, (*i.e.*, in *gm.-cm*^{2}.). And, if the two quantities be measured in the *F.P.S.* units, *i.e.*, the mass of the body in *pounds* and its radius of gyration in *feet*, the moment of inertia is expressed in *Pound-feet*^{2}, (*i.e.*, in *lb.-ft*^{2}).

And, finally, it must be carefully noted that since the moment of inertia of a body, *about a given axis*, remains unaffected by reversing its direction of rotation about *that* axis, it is just a *scalar quantity.** Thus, the total moment of inertia of a number of bodies, about a given axis, will be equal to the sum of their individual moments of inertia about that axis, in exactly the same manner as the total mass of a number of bodies will be equal to the sum of their individual masses.

Note. The argument is sometimes advanced that since the moment of inertia of a body changes with the direction of the axis of rotation, it is not a scalar quantity ; and, since it is independent of the sense or direction of rotation about that axis, it is not a vector quantity either, and that it is what is called a '*tensor*'.

The author begs to differ. For, the term, '*moment of inertia of a body*' has hardly any meaning unless clear mention is also made of the axis of rotation of that body. And, once the axis of rotation is fixed, the *moment of inertia* of the body, *about that particular axis*, becomes a *scalar quantity*, being independent of the sense of rotation about that axis. Indeed, it would be misleading to call it a *tensor* ; for, the fact is that the *moment of inertia* and the *products of inertia* (see below), at a point, together constitute the components of a *symmetric tensor of the second order*, which simply means that, knowing the system of moments and products of inertia at a point about any three mutually perpendicular axes we can, by means of certain simple transformations, obtain their values for any other set of three mutually perpendicular axes at that very point.

A general tensor, of the second order, in three-dimensional space, has, in general, nine components, say, C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33}. But, for a symmetric tensor, $C_{12} = C_{21}$, $C_{23} = C_{32}$ and $C_{31} = C_{13}$, so that it has only six distinct components, *viz.*, three moments of inertia and three products of inertia about the three perpendicular axes.

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Scalar quantities are those which *possess only magnitude,* but *no direction,—e.g., mass, time* etc. On the other hand, *vector quantities* are those which *possess both magnitude as well as direction,—e.g., acceleration, velocity, force,* etc.

us, if x, y, z be the co-ordinates of a particle of mass m, at P, in Fig. 23, we have

(*i*) *moments of inertia* about these three perpendicular axes respectively given by

$$I_x = \Sigma m(y^2+z)^2, \quad I_y = \Sigma m(z^2+x^2), \quad I_z = \Sigma m(x^2+y^2), \text{ and}$$

(*ii*) the *products of inertia* about these axes defined by

$$P_{yz} = \Sigma myz, \quad P_{zx} = \Sigma mzx, \quad P_{xy} = \Sigma mxy,$$

Then, $I_x, I_y, I_z, -P_{yz}, -P_{zx}$ and $-P_{xy}$ are the six components of the symmetric tensor at point P.

It will thus be seen that it is, at best, only a half-truth to say that the moment of inertia of a body about a given axis is a *tensor*.

31. Particular Cases of Moments of Inertia.

1. Moment of Inertia of a Thin Uniform Rod :

(*i*) *about an axis through its centre and perpendicular to its length.*—Let AB, (Fig. 26), be a thin *uniform* rod of length l and mass M, free to rotate about an axis CD through its centre O and perpendicular to its length. Then, its *mass per unit length is* M/l. Consider a small element of length dx of it, at a distance x from O. Its mass is clearly equal to $(M/l).dx$, and its moment of inertia about the axis through $O = (M/l).dx.x^2$.

The moment of inertia I of the *whole* rod about the axis is, therefore, obtained by integrating the above expression between the limits $x = -l/2$ and $x = +l/2$; or between $x=0$ and $x=l/2$ and multiplying the result by 2, to include both halves of the rod.

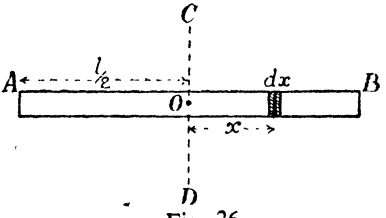


Fig 26.

$$\text{Thus, } I = 2\int_0^{l/2} \frac{M}{l}.x^2.dx.$$

$$= \frac{M}{l}\left[\frac{x^3}{3}\right]_0^{l/2}$$

$$= \frac{2M}{l}\left[\frac{(l/2)^3}{3} - 0\right]$$

$$\text{Or, } \mathbf{I = \frac{2M}{l}.\frac{l^3}{24} = \frac{Ml^2}{12}}.$$

(*ii*) *about an axis passing through one end of the rod and perpendicular to its length.*—The treatment is the same as above, except that, since the axis CD here passes through one end B of the rod, (Fig. 27), the expression for the moment of inertia of the element dx of the rod is now to be integrated between the limits, $x = 0$, at B and $x = l$, at A.

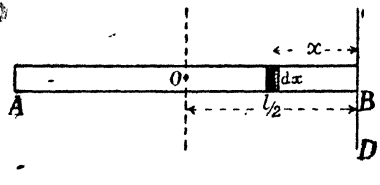


Fig. 27.

Thus, if I be the moment of inertia of the rod about CD, we have

$$I = \int_0^l \frac{M}{l}.x^2.dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$\mathbf{I = \frac{M}{l}.\frac{l^3}{3} = \frac{Ml^2}{3}}.$$

Or, we could have arrived at the same result by an application of the *principle of parallel axes*, according to which the moment of inertia of the rod about the axis through B is equal to the *sum* of its moment of inertia about a parallel axis through its centre of mass and the product of its mass and the square of the distance between the two axes.

Thus, $$I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}.$$

2. Moment of Inertia of a Rectangle.

(*i*) *about an axis through its centre and parallel to one of its sides.*—Let $ABCD$ be a rectangle, and let l and b be its length and breadth respectively, (Fig. 28). Let the axis of rotation YY' pass through its centre and be parallel to the side AD or BC.

If M be the mass of the rectangle, (supposed *uniform*), its *mass per unit length* will be M/l.

Consider a small *strip*, of width dx of the rectangle, parallel to the axis. The mass of the strip will obviously be $(m/l).dx$, and, therefore, the moment of inertia of the *strip*, about the axis YY' will be $(M/l).dx.x^2$.

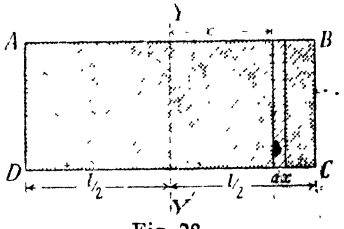


Fig. 28.

The whole rectangle may be supposed to be composed of *such like strips*, parallel to the axis, and therefore, the moment of inertia I of the *whole* rectangle about the axis YY' is obtained by integrating the expression $(M/l).dx.x^2$, for the limits $x=0$ and $x=l/2$ and multiplying the result by 2.

i.e., $$I = 2\int_0^{l/2} \frac{M}{l}.x^2.dx = \frac{2M}{l}\int_0^{l/2} x^2.dx = \frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{l/2}$$

Or, $$\mathbf{I = \frac{2M}{l}\cdot\frac{l^3}{24} = \frac{Ml^2}{12}}.$$

It will be seen at once that *if b be small*, the rectangle becomes a *rod*, of length l, whose moment of inertia, about the axis YY' passing through its centre and perpendicular to its length, would be $Ml^2/12$, [as obtained above, §31, case 1, (*i*)].

(*ii*) *about one side.*—We may proceed as above (in case we want an independent proof) except that the expression $(M/l).dx.x^2$ may here be integrated for $x = 0$ and $x = l$. Thus, the moment of inertia of the rectangle about the side AD or BC is given by

$$I = \int_0^l \frac{M}{l}.x^2.dx = \frac{M}{l}\int_0^l x^2.dx.$$

Or, $$\mathbf{I = \frac{M}{l}\left[\frac{x^3}{3}\right]_0 = \frac{M}{l}\cdot\frac{l^3}{3} = \frac{Ml^2}{3}}.$$

Alternatively, proceeding on the basis of the previous article, we may apply *the principle of parallel axes*, according to which the moment of inertia of the rectangle about side AD or BC is given by

$$I = M.I.\ about\ a \parallel axis\ through\ its\ centre + M\left(\frac{l}{2}\right)^2.$$

Or, $$\mathbf{I = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}}.$$

(iii) about an axis passing through its centre and perpendicular to its plane.—This may be obtained by an application of the *principle of perpendicular axes* to case (*i*) above, whence the moment of inertia of the rectangle about an axis through its centre O, perpendicular to its plane, is equal to its moment of inertia about an axis through O, parallel to its breadth b, *plus* its moment of inertia about a perpendicular axis through O, parallel to its length l,

i.e., $$I = \frac{Ml^2}{12} + \frac{Mb^2}{12} = \frac{M(l^2+b^2)}{12}.$$

The above relation is equally valid in the case of *thin* (*i.e.*, *laminar*) or *thick* rectangular plates or bars, no stipulation with regard to its thickness having been made in deducing it. And, after all, a thick rectangular plate or bar may be regarded as just a pile of thin (or laminar) plates or bars, placed one above the other.

The same argument will hold good in all other cases of a similar type, [see cases (*iv*) and (*v*) below].

(iv) about an axis passing through the mid-point of one side and perpendicular to its plane.—

(*a*) Suppose the axis of rotation passes through the mid-point of AD or BC, (Fig. 28). Then clearly, in accordance with the *principle of parallel axes*, we have

moment of inertia about *this* axis, *i.e.*,

$$I = \textit{moment of inertia about a parallel axis through its centre} + M \times (l/2)^2,$$

where $l/2$ is the distance between the two parallel axes.

i.e., $$I = \frac{M(l^2+b^2)}{12} + M.\left(\frac{l}{2}\right)^2$$

$$= \frac{M(l^2+b^2)}{12} + \frac{Ml^2}{4} = M.\left(\frac{l^2+b^2}{12} + \frac{l^2}{4}\right)$$

$$= M\frac{(l^2+b^2+3l^2)}{12} = \frac{M(4l^2+b^2)}{12}.$$

Or, $$\mathbf{I = M\left(\frac{l^2}{3} + \frac{b^2}{12}\right).}$$

(*b*) Similarly, if the axis of rotation passes through the mid-point of AB or DC, we have

$$I = M\frac{(l^2+b^2)}{12} + M\left(\frac{b}{2}\right)^2 \cdot \quad \left[\because \text{the distance between the two } \| \text{ axes is now } b/2.\right.$$

Or, $$\mathbf{I = M\left(\frac{l^2}{12} + \frac{b^2}{3}\right).}$$

3. Moment of inertia of a solid uniform bar of rectangular cross-section, about an axis, perpendicular to its length and passing through its middle point.*

Let *ABCDEFGH* (Fig. 29) be the rectangular uniform bar of length l, breadth b and thickness d, whose moment of inertia about the axis XX', passing through its centre and perpendicular to its length, is desired.

*This is really covered by case 2 above, but is given here for a clearer understanding of the student.

Imagine the whole bar to be made up of a large number of thin rectangular sheets, parallel to the face *CDEH* and perpendicular to the axis *XX'*, passing through the centre of mass of each sheet Consider one such sheet, (shown dotted), of mass *m*, of length and breadth *l* and *d* respectively, and centre of mass *O*, through which the axis *XX'* is passing perpendicular to its plane.

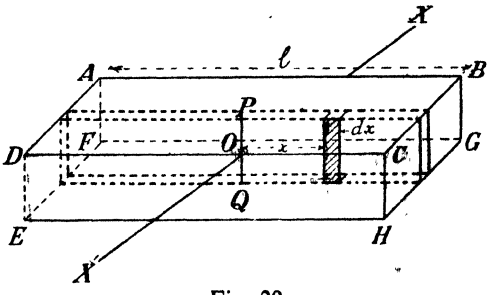


Fig. 29.

Then, the *M.I. of this* sheet about *XX'* = its *mass* × $(l^2+d^2)/12$, as can be seen from the following :

Let *PQ* be an axis through *O*; *in the plane of this sheet*, and *parallel to its breadth CH* or *DE*.

Take a thin strip of width *dx* of *this* strip, *parallel to*, and *at distance x from*, the axis *PQ* Then, mass of the strip = $(m/l).dx$ and, therefore, its moment of inertia about the axis *PQ* is

$$= (m/l)\, dx.x^2.$$

∴ moment of inertia *I* of the *whole* sheet about *PQ* is given by

$$2\int_0^{l/2} \frac{m}{l}.x^2.dx = \frac{2m}{l}\int_0^{l/2} x^2.dx$$

$$= \frac{2m}{l}\left[\frac{x^3}{3}\right]_0^{l/2} = \frac{2m}{l}\cdot\frac{l^3}{8\times 3}.$$

Or, $$\mathbf{I = \frac{Ml^2}{12}}.$$

Similarly, the moment of inertia of the sheet about an axis through *O*, *in its own plane and perpendicular to PQ, i.e., parallel to its length DC* or *EH* will be $\mathbf{Md^2/12}$.

Therefore, *by the principle of perpendicular axes*, the moment of inertia of the sheet about the axis *XX'* through *O* and perpendicular to its plane

$$= \frac{ml^2}{12} + \frac{md^2}{12} = m\left(\frac{l^2+d^2}{12}\right).$$

Hence, moment of inertia of the *whole* bar about the axis *XX'*

$$\textit{mass of the bar} \times \left(\frac{l^2+d^2}{12}\right).$$

Or, $$\mathbf{I = M\left(\frac{l^2+d^2}{12}\right)}$$ [*M* being the *mass* of the bar.

4. Moment of Inertia of a Thin Triangular Plate or Lamina about one side.—Let *ABC*, (Fig. 30), be a thin, triangular plate or lamina, of *surface density* or *mass per unit area*, ρ, whose moment of inertia is to be determined about the side *BC*.

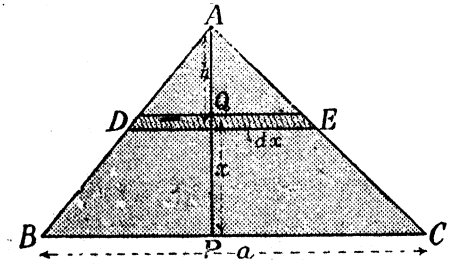


Fig. 30.

Then, if the *altitude* of the plate be $AP = H$, its *area* = ½ *base* × *altitude*

$= \frac{1}{2}.a.H$, [∵ $BC = a$, and, therefore, its *mass* $M = \frac{1}{2}.a.H.\rho$. ... (*i*

Now, let us imagine the triangular plate to be made up of a number of thin *strips*, parallel to BC, and placed side by side ; and, let us consider one such strip DE, of width dx, at a distance x from the base BC. Then, clearly, the *area of this* strip, (which may be considered to be almost rectangular, its width being infinitesimally small) $= DE.dx$. And, therefore,

$$\textit{mass of the strip} = DE.dx.\rho. \qquad .. \qquad .. \qquad ..(ii)$$

Now, in the similar triangles AQD and APB, we have

$$\frac{DQ}{BP} = \frac{AQ}{AP} = \frac{h}{H}, \qquad \text{where, } AQ = h,$$

whence, $$DQ = BP.\frac{h}{H}.$$

Similarly, from the similar triangles AQE and APC, we have

$$\frac{QE}{PC} = \frac{AQ}{AP} = \frac{h}{H}, \text{ whence, } QE = PC.\frac{h}{H}.$$

And, therefore, $$DQ+QE = BP.\frac{h}{H}+PC.\frac{h}{H}.$$

Or, $$DE = (BP+PC).\frac{h}{H} = BC.\frac{h}{H} = a.\frac{h}{H}.$$

$\therefore$ $$\textit{mass of the strip} = a.\frac{h}{H}.dx.\rho. \qquad \text{[From } (ii) \text{ above]}$$

Now, clearly, moment of inertia of strip DE about the side BC

$$= \textit{mass of the strip} \times x^2 = a.\frac{h}{H}.dx.\rho.x^2.$$

$$= a.\left(\frac{H-x}{H}\right).dx.\rho.x^2. \qquad [\because h = (H-x).$$

And, therefore, moment of inertia of the *whole* triangular plate about BC is equal to the integral of this expression, between the limits $x = 0$ and $x = H$. So that,

$$\textit{M.I. of the plate about } BC, \textit{ i.e., } I = \int_0^H a.\left(\frac{H-x}{H}\right).\rho.x^2.dx.$$

$$= \frac{a.\rho}{H}\int_0^H (H-x).x^2.dx = \frac{a.\rho}{H}\int_0^H (Hx^2-x^3).dx.$$

$$= \frac{a.\rho}{H}\left[\frac{H.x^3}{3}-\frac{x^4}{4}\right]_0^H = \frac{a.\rho}{H}\left(\frac{H^4}{3}-\frac{H^4}{4}\right)$$

$$= \frac{a.\rho}{H}\left(\frac{4H^4-3H^4}{12}\right) = \frac{a.\rho}{H}.\frac{H^4}{12} = \frac{a.\rho.H^3}{12}.$$

$$= \frac{1}{2}a.H.\rho.\frac{H^2}{6}.$$

But $\frac{1}{2}a.H\rho. = M$, *the mass of the plate.* [See (*i*) above.

$\therefore$ *M. I.* of the triangular plate about side BC, *i.e*, $\mathbf{I = \frac{M.H^2}{6}}$.

5. Moment of Inertia of an Elliptical Disc or Lamina.—(*i*) *about one of its axes.*—Let $XYX'Y'$ be a thin elliptical plate or lamina, of mass M, and surface density (*i.e.*, mass per unit area), ρ, and let its *major axis* XX' and *minor axis* YY' be equal to $2a$ and $2b$ respectively, (Fig. 31).

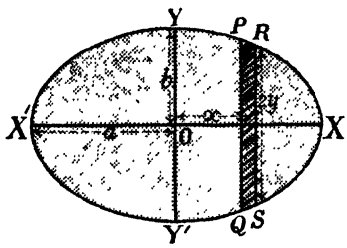


Fig. 31.

Consider a strip $PQSR$ of the plate of width dx, parallel to the minor axis YY' and at a distance x from it. Then, if $2y$ be the length of the strip, its area is clearly equal to $2y.dx$ and, therefore, its *mass* equal to $2y.dx.\rho$.

Obviously, then, $M.\ I.$ of the strip about the minor axis $YY' = 2y.dx.\rho.x^2$; and, therefore, $M.\ I.$ of the *whole* elliptical plate about the axis YY' is equal to twice the integral of the above expression, between the limits $x = 0$, and $x = a$. Or, denoting it by I_y, we have

$$I_y = 2\int_0^a 2y.\rho.x^2.dx = 4\rho\int_0^a y.x^2.dx. \quad \ldots \quad \ldots(I)$$

Now, with the centre of the ellipse as the origin, and with the co-ordinate axes coinciding with its major and minor axes respectively, we have $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, as the equation to the ellipse ; whence,

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}, \quad \text{or} \quad y^2 = b^2\left(1-\frac{x^2}{a^2}\right).$$

So that, $$y = b\sqrt{1-x^2/a^2}.$$

Substituting this value of y in relation (I) above, we have

$$I_y = 4\rho\int_0^a b\sqrt{(1-x^2/a^2)}.x^2dx.$$

$$= 4\rho b\int_0^a x^2.\sqrt{(1-x^2/a^2)}.dx. \quad \ldots \quad \ldots \quad \ldots(II)$$

Now, putting $x = a\ sin\ \theta$, we have $\frac{dx}{d\theta} = a\ cos\ \theta.$

Or, $$dx = a\ cos\ \theta.\ d\theta.$$

Substituting these values of x and dx in expression (II) above, we have

$$I_y = 4\rho.b\int_0^{\pi/2} a^2.sin^2\theta\sqrt{1-\frac{a^2.sin^2\theta}{a^2}}.a\ cos\ \theta.d\theta.$$

$$= 4\rho.b\int_0^{\pi/2} a^2.sin^2\theta\sqrt{1-sin^2\theta}.a\ cos\ \theta.d\theta$$

$$= 4\rho.b\int_0^{\pi/2} a^2.sin^2\theta.cos\theta.a\ cos\theta.d\theta.$$

$$[\because \sqrt{1-sin^2\theta}=cos\ \theta.$$

Or, $$I_y = 4\rho b.a^3.\int_0^{\pi/2} sin^2\theta.cos^2\theta\ d\theta.$$

$$= 4\rho b.a^3.\frac{1}{4}\int_0^{\pi/2}(2sin\theta.cos\theta)^2.d\theta$$

$$= \rho b.a^3.\int_0^{\pi/2} \sin^2 2\theta.d\theta.$$

$$= \rho b.a^3.\int_0^{\pi} \frac{1-\cos 4\theta}{2}.d\theta. \qquad \left[\because \cos 2\theta = 1-2\sin^2\theta.\right.$$

$$= \rho b.a^3.\frac{1}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \rho b.a^3.\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}\rho ba^3 = \pi.ab\rho.\frac{a^2}{4}.$$

Now, $\pi.a.b.\rho = M$, the *mass of the elliptical plate.*

$\therefore$ $$\mathbf{I_y = M.a^2/4.}$$

Similarly, the moment of inertia (I_x) of the elliptical plate about the major axis XX' is given by the expression,

$$\mathbf{I_x = Mb^2/4.}$$

(*ii*) *about an axis passing through the centre of the plate or lamina and perpendicular to its own plane* —The axis in this case will pass through O, (Fig. 31), and will be perpendicular to the plane of the paper, (or the plane of its two axes, XX' and YY'). Hence, if I be the moment of inertia of the elliptical plate about *this* axis, we have, by the *principle of perpendicular axes*,

$$I = I_x + I_y = I_y + I_x = Ma^2/4 + Mb^2/4.$$

Or, $$\mathbf{I = M.\left(\frac{a^2+b^2}{4}\right).}$$

6. Moment of Inertia of a Hoop or a Circular Ring.—

(*i*) *about an axis through its centre and perpendicular to its plane.*—Let the radius of the hoop or circular ring be R, and its mass, M, (Fig. 32).

Consider a particle of it, of mass m. Then, the moment of inertia of *this* particle about an axis through the centre O of the hoop, and perpendicular to its plane, will obviously be mR^2.

And, therefore, the moment of inertia I of the *entire* hoop about the axis will be ΣmR^2.

Or, $$\mathbf{I = MR^2.}$$ $\left[\because \Sigma m = M$ and R is the *same* for all particles.

(*ii*) *about its diameter.*—Let it be required to determine the moment of inertia of the hoop about the diameter AB, (Fig. 32). Obviously, the moment of inertia of the hoop will be the same about all the diameters. Thus, if I be the moment of inertia of the hoop about the diameter AB, it will also be its moment of inertia about the diameter CD, perpendicular to AB.

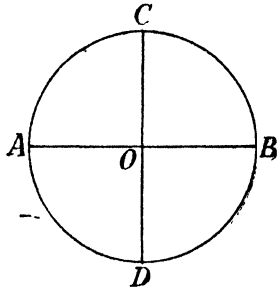


Fig. 32.

Then, by the *principle of perpendicular axes,* its moment of inertia about the axis through the centre O, and *perpendicular to its plane,* is equal to the *sum* of its moments of inertia about the perpendicular axes AB and CD, *in its own plane,* and intersecting at O.

And, therefore, $I+I = MR^2$. Or, $2I = MR^2$. [See case (*f*).

Or $\mathbf{I = MR^2/2.}$

7. Moment of inertia of a Circular Lamina or Disc.—

(*i*) *about an axis through its centre and perpendicular to its plane.*—Let M be the mass of the disc and R, its radius. Then, since the area of the disc is πR^2, its *mass per unit area* will be $M/\pi R^2$.

Consider a ring of the disc, distant x from O, *i.e.*, of radius x and of width dx, (Fig. 33). Its area is clearly equal to its circumference, multiplied by its width, or equal to $2\pi x \times dx$, and its mass is thus

$$= \frac{M}{\pi R^2}.\ 2\pi x.dx = \frac{2Mxdx}{R^2}.$$

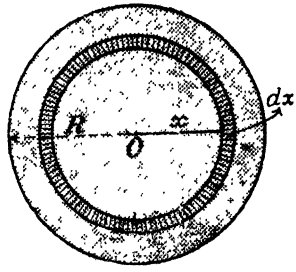


Fig. 33.

Hence, moment of inertia of *this* ring about an axis through O and perpendicular to its plane

$$= \frac{M.2\pi x.dx}{\pi R^2}.x^2 = \frac{2Mx^3dx}{R^2}.$$

Since the whole disc may be supposed to be made up of such like concentric rings of radii ranging from 0 to R, we can get the moment of inertia I of the disc by integrating the above expression for the moment of inertia of the ring, for the limits $x=0$, and $x=R$.

$$\therefore\ M.I. \text{ of the disc} = \int_0^R \frac{2M}{R^2}.x^3.dx. = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R = \frac{2M}{R^2}.\frac{R^4}{4}.$$

Or, $\mathbf{I = MR^2/2.}$

(*ii*) *about its diameter.*—Let AB and CD be two perpendicular diameters of a circular disc of radius R and mass M, (Fig. 34). Since the moment of inertia of the disc about one diameter is the same as about any other diameter, the moment of inertia about the diameter AB is equal to the moment of inertia about the diameter CD, perpendicular to AB. Let it be I.

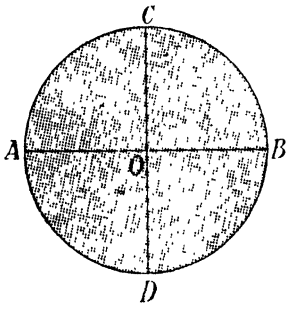


Fig. 34.

Now, we have, by the *principle of perpendicular axes*, $M.I.$ of the disc about AB + its $M.I.$ about CD

= its $M.I.$ about an axis through O and perpendicular to its plane.

Or, $I+I = \frac{MR^2}{2}$ or, $2I = \frac{MR^2}{2}$.

Or, $\mathbf{I = \frac{MR^2}{4}.}$

(*ii*) *about a tangent to the disc in its own plane.*—Let AB be the tangent to the circular disc of radius R and mass M, about which its

moment of inertia is to be determined, (Fig. 35). Let CD be a diameter of the disc, parallel to the tangent AB. The moment of inertia of the disc about this diameter is, clearly, equal to $MR^2/4$.

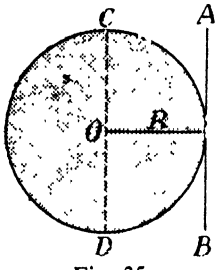


Fig. 35.

So that, by the *principle of parallel axes*, we have

$M.I.$ of the disc about AB = $M.I.$ of the disc about $CD + MR^2$.

Or, $$I = \frac{MR^2}{4} + MR^2 = \frac{5}{4}MR^2.$$

(*iv*) *about a tangent to the disc and perpendicular to its plane* —This tangent will obviously be parallel to the axis through the centre of the disc and perpendicular to its plane, the distance between the two being equal to the radius of the disc. Hence, *by the principle of parallel axes*, we have

M.I. about the tangent = *M.I. about the perpendicular axis* $+ MR^2$.

Or, $$I = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2.$$

8. Moment of Inertia of an Annular Ring or Disc.—

(*i*) *about an axis passing through its centre and perpendicular to its plane.*—An annular disc or ring is just an ordinary disc from which a smaller co-axial disc is removed, so that there is a concentric circular hole in it. Let R and r be the outer and inner radii of the disc, (Fig. 36), and M, its mass. Then, clearly,

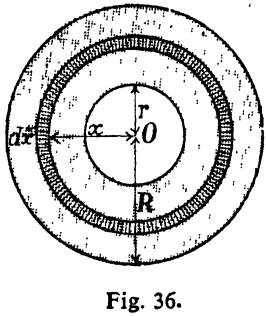


Fig. 36.

face-area of the annular disc = face-area of disc of radius R − face-area of disc of radius r,

$$= \pi R^2 - \pi r^2 = \pi(R^2 - r^2).$$

And $\therefore$ *mass per unit area of the disc*

$$= M/\pi(R^2 - r^2).$$

Imagine the disc to be made up of a number of thin circular rings, and consider one such ring of radius x and of width dx.

Then, *face-area of this ring* $= 2\pi x.dx$,

and its *mass* $$= \frac{M}{\pi(R^2 - r^2)}.2\pi x.dx = \frac{2Mx}{(R^2 - r^2)}.dx.$$

And, therefore, its moment of inertia about an axis through O and perpendicular to its plane $$= \frac{2Mx}{(R^2 - r^2)}.dx.x^2 = \frac{2Mx^3}{(R^2 - r^2)}.dx.$$

The moment of inertia of the *whole* annular disc may, therefore, be obtained by integrating the above expression for the limits $x = r$ and $x = R$. Or, moment of inertia of the disc about the axis through O and perpendicular to its plane

$$= \int_r^R \frac{2Mx^3}{(R^2 - r^2)}.dx = \frac{2M}{(R^2 - r^2)}\int_r^R x^3.dx.$$

$$= \frac{2M}{(R^2-r^2)}\left[\frac{x^4}{4}\right]_r^R = \frac{2M}{(R^2-r^2)}\left[\frac{(R^4-r^4)}{4}\right]$$

$$= \frac{2M}{(R^2-r^2)}\left[\frac{(R^2+r^2).(R^2-r^2)}{4}\right]$$

Or, $$\mathbf{I = M\left(\frac{R^2+r^2}{2}\right)}.$$

It follows at once from the above that if $r = 0$, *i.e.*, if there is no hole in the disc, or that it is just a *plane*, (and *not an annular*) disc, its moment of inertia is $MR^2/2$. [Case 7 (*i*), above.

Again, if $r = R$, *i.e.*, we have a hoop or a circular ring, of radius R, and its $M.I. = \frac{M(R^2+R^2)}{2} = \frac{M.2R^2}{2} = MR^2$. [Case 6 (*i*), above.

(*ii*) *about its diameter.*—Obviously, due to its *symmetrical* shape, the moment of inertia of the annular disc about one diameter will be the same as that about any other diameter. Let it be I. Then, the sum of its moments of inertia about two perpendicular diameters will, by the *principle of perpendicular axes*, be equal to its moment of inertia about an axis through O and perpendicular to its plane, *i.e.*, equal to $M(R^2+r^2)/2$.

Or, $$I+I = \frac{M(R^2+r^2)}{2}, \text{ i.e., } 2I = \frac{M(R^2+r^2)}{2},$$

whence, $$= \mathbf{\frac{M(R^2+r^2)}{4}}.$$

Now, if $r = 0$, *i.e.*, if the disc be a *plane* one, we have

$M.I.$ of the disc about a diameter $= MR^2/4$. [Case 7 (*ii*), above.

Or, if $r = R$, we have a *hoop* or *circular ring* of radius R, and its moment of inertia about its diameter

$$= \frac{M(R^2+R^2)}{4} = \frac{M.2R^2}{4} = \frac{MR^2}{2}.$$ [Case 6 (*ii*) above.

(*iii*) *about a tangent, in its own plane.*—The tangent being parallel to the diameter of the ring or disc, and at a distance R from it, we have, applying the *principle of parallel axes*,

$M.I.$ about the tangent $= M.I.$ about the diameter $+ MR^2$.

Or, $$\mathbf{I = M\frac{(R^2+r^2)}{4} + MR^2 = M\frac{(5R^2+r^2)}{4}}.$$

(*iv*) *about a tangent, perpendicular to its own plane.*—The tangent, in this case, is parallel to the axis through the centre of the ring or disc and perpendicular to its plane, the distance between the two being equal to R. Hence, by the *principle of parallel axes*, we have $M.I.$ about the tangent $= M.I.$ about the perpendicular axis $+ MR^2$.

Or $$\mathbf{I = M\frac{(R^2+r^2)}{2} + MR^2 = M\frac{(3R^2+r^2)}{2}}.$$

9. Moment of inertia of a Solid Cylinder.—

(*i*) *about its own axis, or its axis of cylindrical symmetry.*—A cylinder is just a thick circular disc, or *a number of thin circular*

discs, piled one upon the other, and, therefore, its moment of inertia about its axis is the same as that of a circular disc or lamina about an axis through its centre and perpendicular to its plane, *i.e.*, equal to $MR^2/2$, where M is its mass and R, its radius. [Case 7 (*i*) above.

(*ii*) *about an axis passing through its centre and perpendicular to its own axis of cylindrical symmetry.*—Let M be the mass of the cylinder, R its radius and l, its length, (Fig. 37). Then, obviously, if it be homogeneous, its *mass per unit length* will be M/l. Let YY' be the axis, passing through its centre O and perpendicular to its own axis XX', about which the moment of inertia is to be determined.

Imagine the cylinder to be made up of a number of thin discs and consider one such disc at a distance x from O, and of thickness dx.

Obviously, *the mass of the disc* is $(M/l).dx$ and its *radius*, equal to R; so that, its *moment of inertia about its diameter is equal to mass of the disc* $\times$ *(radius)*$^2/4$.

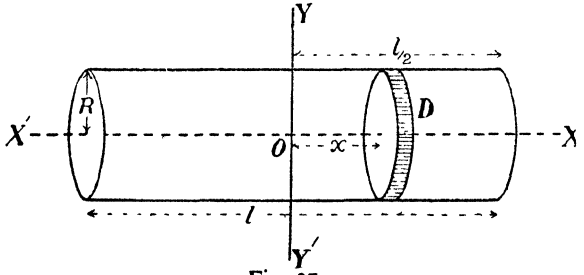


Fig. 37.

$$= \frac{M}{l} \cdot dx \cdot \frac{R^2}{4}.$$

And, its moment of inertia about the axis YY', by the *principle of parallel axes*, $= \frac{M}{l} \cdot dx \cdot \frac{R^2}{4} + \frac{M}{l} \cdot dx.x^2$.

Therefore, the moment of inertia of the *whole* cylinder about the axis YY' may be obtained by integrating this expression for the limits $x = 0$ and $x = l/2$ and multiplying the result by 2. Thus,

M.I. of the cylinder about the axis YY'

$$= 2\int_0^{l/2} \left(\frac{M}{l} \cdot \frac{R^2}{4}.dx + \frac{M}{l}.x^2.dx\right) = \frac{2M}{l}\int_0^{l/2} \left(\frac{R^2}{4}.dx + x^2.dx\right)$$

$$= \frac{2M}{l}\left[\frac{R^2x}{4} + \frac{x^3}{3}\right]_0^{l/2} = \frac{2M}{l}\left[\frac{R^2}{4} \cdot \frac{l}{2} + \frac{l^3}{8\times 3}\right]$$

Or, $$\mathbf{I = \frac{2M}{l}\left(\frac{R^2 l}{8} + \frac{l^3}{24}\right) = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right).}$$

(*iii*) *about a diameter of one of its faces.*—It is an easy deduction from the above; for, by the *principle of parallel axes*, we have

M.I. of the cylinder about the diameter of one face

$$= M\left(\frac{R^2}{4} + \frac{l^2}{12}\right) + M\left(\frac{l}{2}\right)^2.$$

$$= \frac{MR^2}{4} + \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{MR^2}{4} + \frac{Ml^2}{3}.$$

Or, $$\mathbf{I = M\left(\frac{R^2}{4} + \frac{l^2}{3}\right).}$$

10. Moment of Inertia of a Solid Cone.—

(*i*) *about its vertical axis.* Let mass of the cone be M, its vertical height, h and radius of its base, R, (Fig. 38).

Then, its *volume* $= \frac{1}{3}\pi R^2 h$.

And, if ρ be the *density* of its material, its *mass* $M = \frac{1}{3}\pi R^2 h\rho$, whence, $\rho = \frac{3M}{\pi R^2 h}$.

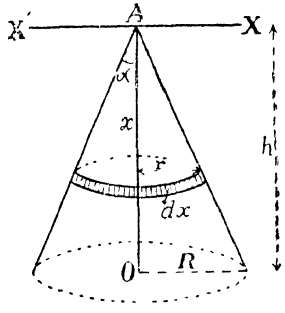


Fig. 38.

Imagine the cone to be made up of a number of discs, parallel to the base, and placed one above the other. Consider one such disc at a distance x from the vertex, and of thickness dx.

If r be the radius of *this* disc, we have

$$r = x \tan \alpha,$$

[where α is the *semi-vertical* angle of the cone.

And, *volume* of the disc $= \pi r^2.dx$.

$\therefore$ its *mass* $= \pi r^2.dx.\rho$.

Now, moment of inertia of the disc about the axis AO, passing through its centre and perpendicular to its plane, *i.e.*, about the vertical axis of the cone, is clearly equal to $\frac{\text{its mass} \times (\text{its radius})^2}{2}$.

$$= \pi r^2.dx.\rho.r^2/2 = \frac{\pi.\rho.r^4}{2}.dx = \frac{\pi.\rho.x^4.\tan^4\alpha}{2}.dx.$$

And, therefore, the moment of inertia of the *whole* cone about its vertical axis AO will be the integral of this expression, for the limits $x = 0$ and $x = h$.

i.e., *M.I.* of the cone about its vertical axis is given by

$$I = \int_0^h \frac{\pi\rho x^4.\tan^4\alpha}{2}.dx = \frac{\pi\rho.\tan^4\alpha}{2}\int_0^h x^4.dx.$$

$$= \frac{\pi\rho.\tan^4\alpha}{2}\left[\frac{x^5}{5}\right]_0^h = \frac{\pi\rho.\tan^4\alpha}{2}.\frac{h^5}{5}.$$

$$= \frac{\pi\rho.R^4}{2.h^4}.\frac{h^5}{5}. \qquad [\because \tan\alpha = R/h.$$

Or, substituting the value of ρ from above, we have

$$\mathbf{I} = \frac{\pi.3M.R^4}{\pi R^2.h.2h^4}.\frac{h^5}{5} = \frac{3MR^2}{10}.$$

(*ii*) *about an axis through its vertex and parallel to its base.*— Again, considering the disc at a distance x from the vertex of the cone, we have

M.I. of the *disc* about its diameter $= \pi r^2.dx.\rho.\frac{r^2}{4}$.

And, therefore, by the *principle of parallel axes,* its moment of inertia about a parallel axis XX', passing through the vertex of the cone

$$= \pi r^2.dx.\rho.\frac{r^2}{4} + \pi r^2.dx.\rho.x^2.$$

Therefore, the moment of inertia of the *whole* cone about the axis passing through the vertex and parallel to the base, *i.e.*, about XX', is obtained by integrating this expression for the limits, $x = 0$ and $x = h$. Thus,

M.I. of the cone about XX'

$$= \int_0^h \left[\frac{\pi(x \tan\alpha)^4.\rho}{4} . dx + \pi(x \tan\alpha)^2.x^2.\rho.dx.\right]$$

$$= \frac{\pi\rho \tan^4\alpha}{4}\int_0^h x^4.dx + \pi\rho \tan^2\alpha \int_0^h x^4.dx$$

$$= \frac{\pi\rho}{4} \cdot \frac{R^4}{h^4}\left[\frac{x^5}{5}\right]_0^h + \pi\rho. \frac{R^2}{h^2}\left[\frac{x^5}{5}\right]_0^h$$

$$= \frac{\pi\rho}{4} \cdot \frac{R^4}{h^4} \cdot \frac{h^5}{5} + \pi\rho. \frac{R^2}{h^2} \cdot \frac{h^5}{5}.$$

Or, substituting the value of ρ, we have

$$M.I. \text{ of the cone about } XX' = \frac{\pi.3M}{\pi R^2 h.4} \cdot \frac{R^4}{h^4} \cdot \frac{h^5}{5} + \frac{\pi.3M}{\pi.R^2 h} \cdot \frac{R^2}{h^2} \cdot \frac{h^5}{5}.$$

Or,
$$I = \frac{3MR^2}{20} + \frac{3Mh^2}{5}.$$

V. Moment of Inertia of a Hollow Cylinder.

(*i*) *about its own axis.*—A hollow cylinder may be considered to consist of a large number of annular discs or rings of the given internal and external radii, placed one above the other, the axis of the cylinder passing through their centre and being perpendicular to their planes, (Fig. 36).

The moment of inertia of the hollow cylinder about its own axis is, therefore, the same as that of an annular disc of the given external and internal radii about an axis through its centre and perpendicular to its plane, *i.e.*, equal to $M(R^2+r^2)/2$, where M is the mass of the cylinder, R and r, its external and internal radii respectively.

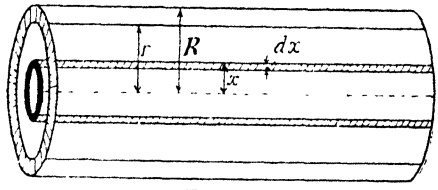


Fig. 39.

Alternative Proof.—Let M be the mass of the cylinder ; R and r, its external and internal radii and l, its length, (Fig. 39).

Then, *face-area* of the cylinder $= \pi(R^2-r^2)$.

And, *volume* of the cylinder $= \pi(R^2-r^2)l$;

so that, *its mass per unit volume* $= \frac{M}{\pi(R^2-r^2)l}$.

Now, imagine the cylinder to be made up of a large number of thin co-axial cylinders, and consider one such cylinder of radius x and thickness dx.

Then, its *face area* $= 2\pi x.dx$, its *volume* $= 2\pi x.dx.l$,

and its *mass* $= \frac{M}{\pi(R^2-r^2)l} \times 2\pi x.dx.l = \frac{2Mx\,dx}{(R^2-r^2)}$.

Since all its particles are equidistant from the axis, its moment of inertia about the axis $= \frac{2Mx.dx}{(R^2-r^2)}.x^2 = \frac{2M\,x^3.dx}{(R^2-r^2)}$.

And, therefore, the moment of inertia of the *whole* cylinder may be obtained by integrating the above expression for the limits, $x = r$ and $x = R$.

Or, *M.I.* of the cylinder about its axis, *i.e.*,

$$I = \int_r^R \frac{2M.x^3}{(R^2-r^2)}.dx = \frac{2M}{(R^2-r^2)}\int_r^R x^3.dx.$$

$$= \frac{2M}{(R^2-r^2)}\left[\frac{x^4}{4}\right]_r^R = \frac{2M}{(R^2-r^2)}\left[\frac{(R^4-r^4)}{4}\right]$$

$$= \frac{2M}{(R^2-r^2)}\left[\frac{(R^2+r^2)(R^2-r^2)}{4}\right].$$

Or, $$\mathbf{I = M\left[\frac{R^2+r^2}{2}\right]}.$$

(*ii*) *about an axis passing through its centre and perpendicular to its own axis.*—As before, let M be the mass of the cylinder, l, its length, and R and r, its external and internal radii respectively; and let YOY' be the axis through its centre O, and perpendicular to its own axis XX', (Fig. 40).

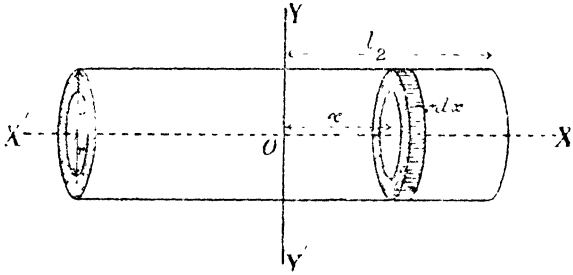


Fig 40.

Then, *face-area* of the cylinder $= \pi(R^2-r^2)$,

and its *volume* $= \pi(R^2-r^2)l$.

$\therefore$ its *mass per unit volume* $= M/\pi(R^2-r^2)l$.

Imagine the cylinder to be made up of a large number of annular discs of external and internal radii R and r, placed one by the side of the other, and consider one such disc at a distance x from the axis YY', and of thickness dx. Then, clearly,

surface area of the disc $= \pi(R^2-r^2)$,

its *volume* $= \pi(R^2-r^2).dx$, and $\therefore$ its *mass* $= M.dx/l$.

Now, moment of inertia of an annular disc of external and internal radii, R and r, about its diameter, is equal to *its mass* $\times (R^2+r^2)/4$. [Case 8 (*ii*), above.

$\therefore$ *M.I.* of the disc about its diameter $= \frac{M}{l}.dx \times \frac{(R^2+r^2)}{4}$.

And, therefore, its moment of inertia about the parallel axis YY' is, by the *principle of parallel axes*, given by

$$\frac{M}{l}.dx\times\frac{(R^2+r^2)}{4}+\frac{M}{l}.dx.x.$$

And, clearly, therefore, moment of inertia I, of the *whole* cylinder, about the axis YY', is twice the integral of this expression, for the limits, $x = 0$ and $x = l/2$.

$$i.e., \quad I = 2\int_0^{l/2}\left[\frac{M(R^2+r^2)}{4l}.dx+\frac{M}{l}.x^2.dx.\right]$$

$$= \frac{2M}{l}\int_0^{l/2}\left[\frac{(R^2+r^2)}{4}.dx+x^2.dx\right] = \frac{2M}{l}\left[\frac{(R^2+r^2)x}{4}+\frac{x^3}{3}\right]_0^{l/2}$$

$$\text{Or,} \quad \mathbf{I} = \frac{2M}{l}\left[\frac{(R^2+r^2)l}{4\times2}+\frac{l^3}{8\times3}\right] = \mathbf{M}\left[\frac{(R^2+r^2)}{4}+\frac{l^2}{12}\right].$$

It follows, therefore, that if $r = 0$, *i.e.*, if the cylinder be a *solid one*, we have

M.I. of the cylinder, (solid), about an axis through its centre and perpendicular to its own axis $= \mathbf{M(R^2/4 + l^2/12)}$. [Case 9, (*ii*) above.

12. Moment of Inertia of a Spherical Shell.

(*i*) *about its diameter.*—

First Method.—Let $ABCD$ be the section of a spherical shell through its centre O and let the mass of the shell be M, and its radius R, (Fig. 41).

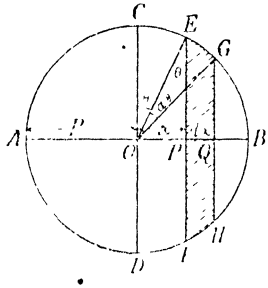


Fig. 41.

Then, area of the shell is equal to $4\pi R^2$, and

$\therefore$ *mass per unit area* of the shell $= M/4\pi R^2$.

Let it be required to determine its moment of inertia about the diameter AB.

Consider a *thin slice* of the shell, lying between two planes EF and GH, perpendicular to the diameter AB, and at distances x and $(x+dx)$ respectively from its centre O.

This *slice* is obviously a *ring* of radius PE, and width EG, (*not* PQ, which is equal to dx).

$\therefore$ area of *this* ring = its circumference × width.

$$= 2\pi.PE\times EG,$$

and, hence its *mass* = its *area* $\times M/4\pi R^2$.

$$= 2\pi.PE\times EG\times M/4\pi R^2. \qquad \ldots (i)$$

Join OE and OG, and let $\angle COE = \theta$ and $\angle EOG = d\theta$.

Then, $PE = OE.\cos OEP = R\cos\theta$

[$\because \angle OEP = \angle COE = \theta$.

Similarly, $OP = R\sin\theta$

Or, $x = R\sin\theta$, [$\because OE = R$ and $OP = x$.

Now, differentiating x with respect to θ, we have

$$dx/d\theta = R\cos\theta.$$

Or, $$dx = R\cos\theta.d\theta = PE.d\theta. \qquad [\because R\cos\theta = PE.$$

And, $$EG = OE.d\theta = R.d\theta. \qquad [\because \text{arc} = \text{radius} \times \text{angle subtended by the arc.}$$

$$\therefore \quad \text{mass of the ring} = 2\pi.PE.R.d\theta.M/4\pi R^2. \qquad [\text{from } (i).$$

$$= \frac{M.dx}{2R}. \qquad [\because PE.d\theta = dx.$$

Hence, moment of inertia of *the ring* about AB, (an axis passing through its centre and perpendicular to its plane), is equal to its mass $\times$ (its radius)^2, *i.e.*, $= \frac{M.dx}{2R}.PE^2$, where $PE^2 = (R^2-x^2)$.

$\therefore$ moment of inertia of *the ring* about $AB = \frac{M.dx}{2R}.(R^2-x^2)$.

And, therefore, the moment of inertia I, of the *whole* spherical shell about AB = twice the integral of $\frac{M.dx}{2R}.(R^2-x^2)$, between the limits, $x = 0$ and $x = R$.

Or, $$I = 2\int_0^R \frac{M}{2R}(R^2-x^2).dx = \frac{2M}{2R}\int_0^R (R^2-x^2).dx.$$

$$= \frac{M}{R}\left[R^2.x - \frac{x^3}{3}\right]_0^R = \frac{M}{R}\left[R^3 - \frac{R^3}{3}\right]$$

i.e., $$\mathbf{I} = \frac{M}{R}\cdot\frac{2}{3}R^3 = \frac{2}{3}\mathbf{MR^2}.$$

Second Method.—Let M be the mass of the shell and R, its radius.

Consider a particle, of mass m, anywhere on the shell. Then, since the thickness of the shell is negligible, the distance of the particle from the centre of the shell is the same as the radius of the shell, *i.e.*, R.

Obviously, therefore, the summation I_0, for all the particles of the shell, about its centre O, is given by the relation,

$$I_0 = \Sigma mR^2. \quad \text{Or,} \quad I_0 = MR^2 \qquad [\text{All particles being at distance } R \text{ from } O \text{ and } \Sigma m = M.$$

Now, if I be the moment of inertia of the shell about one diameter, it will be the same about any other diameter also, from the sheer symmetry of the shell. Hence, in accordance with the *principle of perpendicular axes for a three dimensional body*, [§29 (*a*), (*ii*), page 52] the sum of the moments of inertia of the shell about its three mutually perpendicular diameters must be equal to twice the summation I_0, for all its particles, about their point of intersection, *i.e.*, the centre of the shell O ; so that,

$$I+I+I = 2I_0, \quad \text{Or,} \quad 3I = 2MR^2, \qquad [\because I_0 = MR^2.$$

whence, $$\mathbf{I} = \frac{2}{3}\mathbf{MR^2}.$$

(*ii*) *about a tangent.*—Obviously, a tangent, drawn to the shell at any point, must be parallel to one of its diameters, and at a distance from it equal to R, the radius of the shell. Hence, applying the *principle of parallel axes*, we have

M.I. of the shell about a tangent

= its *M.I.* about a diameter $+ MR^2$.

Or, $$I = \tfrac{2}{3}MR^2 + MR^2 = \tfrac{5}{3}\mathbf{MR^2}.$$

13. Moment of Inertia of a Solid Sphere.

(*i*) *about its diameter.*—Let Fig. 42 represent a section of the sphere through its centre O.

Let *mass* of the sphere be M, and its radius, R.

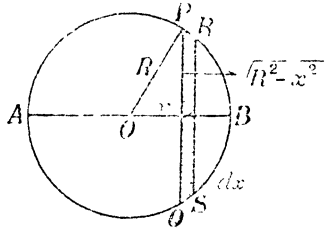


Fig. 42.

Then, clearly, its *volume* $= 4\pi R^3/3$.

And $\therefore$ its *mass per unit volume*

$= M/\frac{4}{3}\pi R^3 = 3M/4\pi R^3$.

Consider a thin circular *slice* of the sphere at a distance x from the centre O, and of thickness dx.

This slice is obviously a *disc* of radius $\sqrt{R^2-x^2}$, and of thickness dx.

$\therefore$ *surface area* of the slice $= \pi\sqrt{(R^2-x^2)^2} = \pi(R^2-x^2)$, and its *volume* $=$ area $\times$ thickness $= \pi(R^2-x^2).dx$.

And, $\therefore$ its *mass* $=$ its *volume* $\times$ *mass per unit volume* of the sphere

$$= \pi(R^2-x^2).dx \times \frac{3M}{4\pi R^3} = \frac{3M.(R^2-x^2)}{4R^3}.dx.$$

Now, the moment of inertia of *this* disc about AB, (an axis passing through its centre and perpendicular to its plane)

$=$ its *mass* $\times$ (*radius*)$^2/2$.

$\therefore$ moment of inertia of the disc AB

$$= \frac{3M.(R^2-x^2)}{4R^3}.dx.\frac{(R^2-x^2)}{2} = \frac{3M.(R^2-x^2)^2}{8R^3}.dx. \quad \ldots (1)$$

$\therefore$ moment of inertia I, of the *sphere* about the diameter AB is equal to twice the integral of expression (1) between the limits $x = 0$ and $x = R$.

$$i.e., I = 2\int_0^R \frac{3M(R^2-x^2)^2}{8R^3}.dx = \frac{2\times 3M}{8R^3}\int_0^R (R^2-x^2)^2.dx.$$

$$= \frac{3M}{4R^3}\int_0^R (R^4-2R^2.x^2+x^4).dx = \frac{3M}{4R^3}\left[R^4x-2R^2.\frac{x^3}{3}+\frac{x^5}{5}\right]_0^R$$

$$= \frac{3M}{4R^3}\left(R^5-\frac{2}{3}R^5+\frac{1}{5}R^5\right) = \frac{3M}{4R^3}\left[\frac{15R^5-10R^5+3R^5}{15}\right]$$

Or, $$\mathbf{I} = \frac{3M}{4R^3}\times\frac{8R^5}{15} = \frac{2}{5}\mathbf{MR^2}.$$

Now, the moment of inertia of the sphere about one diameter is the *same* as about another diameter, so that we have the moment of inertia of a solid sphere about any diameter given by $\mathbf{I} = \frac{2}{5}\mathbf{MR^2}$.

Alternative Method.—Let M be the mass of the sphere and ρ, the density of its material.

Imagining the whole sphere to be made up of a number of thin, concentric spherical shells, one inside the other, and considering one such shell of *radius* x and *thickness* dx, we have

surface area of the shell $= 4\pi x^2$,

and $\therefore$ *volume* of the shell $= 4\pi x^2.dx$ and its *mass* $= 4\pi x^2.dx.\rho$.

$\therefore$ moment of inertia of the *shell* about a diameter

$= \frac{2}{3} \times$ (its *mass*) $\times$ (its *radius*)$^2 = \frac{2}{3}.4\pi x^2.dx.\rho \times x^2 = \frac{8}{3}\pi\rho x^4\,dx$, [case 11 (*i*).

And, therefore, the moment of inertia I, of the *whole* sphere, about its diameter, is obtained by integrating the above expression between the limits. $x = 0$ and $x = R$.

i e.,
$$I = \int_0^R \frac{8}{3}\pi\rho x^4.dx = \frac{8}{3}\pi\rho \int_0^R x^4.dx.$$

$$= \frac{8}{3}\pi\rho \left[\frac{x^5}{5}\right]_0^R = \frac{8}{3}\pi.\rho.\frac{R^5}{5} = \frac{8}{15}\pi\rho.R^5$$

$$= \frac{4}{3}.\pi R^3.\rho \cdot \frac{2}{5}R^2.$$

t $4\pi R^3/3$ = the *volume* of the sphere ; and, therefore, $4\pi R^3\rho/3 = M$, its *mass*.

$\therefore$ *M.I.* of the sphere about its diameter, *i.e.*, $\mathbf{I = 2.MR^2/5}$.

(*ii*) *about a tangent*.—A tangent, drawn to the sphere at any point, will obviously be parallel to one of its diameters and at a distance from it equal to R, the radius of the sphere.

Therefore, in accordance with the *principle of parallel axes*, we have

M.I. of the sphere about a tangent

= its *M.I.* about a diameter $+ MR^2$.

Or, $\mathbf{I = 2\,MR^2/5 + MR^2 = 7MR^2/5}$.

14. Moment of Inertia of a Hollow Sphere or a Thick Shell.

(*i*) *about its diameter* —A hollow sphere is just a solid sphere from the inside of which a smaller concentric solid sphere has been removed. And so, the moment of inertia of the hollow sphere is equal to the moment of inertia of the bigger solid sphere *minus* the moment of inertia of the smaller solid sphere removed from it, (both about the *same* diameter). If R be the radius of the bigger sphere and r, that of the smaller sphere, *i.e.*, if R and r be the external and internal radii of the hollow sphere, and ρ, the density of its material, we have

volume of the bigger sphere $= \frac{4}{3}\pi R^3$, and $\therefore$ its mass $= \frac{4}{3}\pi R^3\rho$.

and, ,, ,, ,, smaller ,, $= \frac{4}{3}\pi r^3$ and ,, ,, ,, $= \frac{4}{3}\pi r^3\rho$.

$\therefore$ volume of hollow sphere $= \frac{4}{3}(R^3 - r^3)$ and its *mass* $= \frac{4}{3}\pi(R^3 - r^3)\rho$.

And $\therefore$ *M.I.* of the bigger sphere about its diameter

$$= \frac{2}{5}.(\frac{4}{3}\pi R^3\rho).R^2.$$

and *M.I.* of the smaller sphere *about the same diameter*

$$= \frac{2}{5}.(\frac{4}{3}\pi r^3.\rho).r^2.$$

$\therefore$ *M.I.* of the hollow sphere about that diameter

$$= \frac{2}{5}.(\frac{4}{3}\pi R^3.\rho).R^2 - \frac{2}{5}(\frac{4}{3}\pi r^3.\rho).r^2$$

$$= \frac{2}{5}.\frac{4}{3}\pi\rho.(R^5 - r^5) \qquad \ldots (1)$$

Now, mass of the hollow sphere, $M = \frac{4}{3}.\pi(R^3 - r^3).\rho$.

Or, $3M = 4\pi(R^3 - r^3).\rho$. And $\therefore$ $\rho = 3M/4\pi(R^3 - r^3)$.

Substituting this value of ρ in relation (1) above, we have moment of inertia I of the hollow sphere about its diameter

$$= \frac{2}{5} \cdot \frac{4}{3}\pi \cdot \frac{3M}{4\pi(R^3-r^3)} \cdot (R^5-r^5).$$

Or,
$$\mathbf{I = \frac{2}{5}.M.\frac{(R^5-r^5)}{(R^3-r^3)}}.$$

Alternative Method. As in the case of the solid sphere, so also here, we can imagine the sphere to be made up of a number of thin, concentric spherical shells, and considering one such spherical shell of radius x and thickness dx, we have, as before, mass of the shell $= 4\pi x^2.dx.\rho$. (ρ being the density of the material of the sphere.)

$\therefore$ *M.I.* of the shell about a diameter $= \frac{2}{3}.4\pi x^2.dx.\rho.x^2.$

$= \frac{8}{3}.\pi.\rho.x^4.dx.$

Hence, the moment of inertia of the *whole* sphere about its diameter is the integral of the above expression, between the limits, $x = r$ and $x = R$.

Or, *M.I.* of the sphere about a diameter *i.e.*, $I = \int_r^R \frac{8}{3}\pi\,\rho.x^4.dx.$

$$= \frac{8}{3}\pi\rho\int_r^R x^4.\,dx = \frac{8}{3}.\pi\,\rho\left[\frac{x^5}{5}\right]_r^R$$

$$= \frac{8}{3}\pi.\rho\left[\frac{R^5}{5}-\frac{r^5}{5}\right] = \frac{8}{15}.\pi.\rho\,[R^5-r^5]$$

$$= \frac{4}{3}\pi\,(R^3-r^3).\rho.\frac{2}{5}.\frac{(R^5-r^5)}{(R^3-r^3)}$$

But $\frac{4}{3}\pi(R^3-r^3).\rho = M$, the mass of the sphere. [See case (*i*) above.]

$\therefore$
$$\mathbf{I = \frac{2}{5}.M(R^5-r^5)/(R^3-r^3)}.$$

(*ii*) *about a tangent.* Again, as in the case of a solid sphere, the tangent to the sphere, at any point, will be parallel to one of its diameters, and at a distance equal to its *external radius* R from it. Hence, by the *principle of parallel axes*, we have

M.I. of the sphere about a tangent

$$= \text{its } M.I. \text{ about a diameter} + MR^2.$$

Or,
$$\mathbf{I = \left[\frac{2}{5}M(R^5-r^5)/(R^3-r^3)\right] + MR^2}.$$

15. Moment of Inertia of a Flywheel and Axle. A flywheel is just a *large heavy wheel*, with a long, cylindrical *axle*, passing through its centre. *Its centre of gravity lies on its axis of rotation*, so that, when properly mounted over *ball-bearings* (to minimise friction), it may continue to be at rest in any desired position.

Let M be the mass of the flywheel, and m, that of the axle; and let R and r be their respective radii.

Then, for our present purpose, we may regard the flywheel to be a disc, or a small cylinder, from which a smaller, *concentric* disc or cylinder, equal in radius to that of the axle, has been cut off. In other words, we may take it to be an *annular ring*, (or hollow cylinder) with an *outer radius equal to R*, and an *inner radius equal to r*, whose moment of inertia is to be determined about an axis passing through its centre and perpendicular to its plane.

The face area of this wheel or annular disc is clearly equal to the area of the whole disc of radius R *minus* the area of the disc of radius r.

i.e., *face area* of the wheel $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$.

And, if its *mass* be M, clearly,

mass per unit area of the wheel $= M/\pi(R^2 - r^2)$.

Now, consider a thin circular ring at a distance x from the centre, and of width dx.

Then, *face area of the ring* = *its circumference* × *its width* $= 2\pi x.dx$.

And, therefore, its mass $= 2\pi x.dx.M/\pi(R^2 - r^2)$.

Now, since the moment of inertia of a ring about an axis through its centre and perpendicular to its plane is equal to its *mass* × (*radius*)^2, we have

$$M.\ I.\ of\ the\ wheel\ about\ its\ axis = \int_r^R \frac{M}{\pi(R^2-r^2)}.2\pi x.dx.x^2.$$

$$= \frac{2\pi M}{\pi(R^2-r^2)}\int_r^R x^3.dx = \frac{2M}{(R^2-r^2)}\left[\frac{x^4}{4}\right]_r^R$$

$$= \frac{2M}{(R^2-r^2)}\left[\frac{R^4}{4} - \frac{r^4}{4}\right] = \frac{2M}{(R^2-r^2)}\left[\frac{R^4-r^4}{4}\right]$$

Or, $$M.I.\ of\ the\ wheel\ about\ its\ axis = \frac{M}{2}(R^2+r^2) = M.\frac{(R^2+r^2)}{2}.$$

The axle, again, is just a disc, (or solid cylinder), and its moment of inertia about its axis is, therefore, just the same as that of a disc or a cylinder about its axis, *i.e.*, = its mass × (radius)$^2/2$.

So that, $$M.I.\ of\ the\ axle = m.r^2/2.$$

Hence, $$M.I.\ of\ the\ wheel\ and\ axle = M.I.\ of\ the\ wheel + M.I.\ of\ the\ axle.$$

Or, $$\mathbf{I = [M(R^2+r^2)/2] + mr^2/2.}$$

32. Table of Moments of Inertia. The values of *moments of inertia* for the cases discussed above, together with some other important ones are given in the *Table* below for ready reference of the student, the mass of the body being taken to be M, in all cases.

BODY	AXIS (Position and Direction)	MOMENT OF INERTIA
1. Thin uniform rod, of length l.	(*i*) Through its centre and perpendicular to its length.	$Ml^2/12$
	(*ii*) Through one end and perpendicular to its length.	$Ml^2/3$
2. Thin and rectangular sheet or lamina, of sides l and b.	(*i*) Through its centre and parallel to one side	$Mb^2/12$ or $Ml^2/12$
	(*ii*) About one side.	$Mb^2/3$ or $Ml^2/3$
	(*iii*) Through its centre and perpendicular to its plane.	$M(l^2+b^2)/12$
	(*iv*) Through the mid-point of one side (l or b) and perpendicular to its plane.	$M(b^2/3+l^2/12)$ or $M(l^2/3+$

BODY	AXIS (Position and Direction)	MOMENT OF INERTIA
3. Thick uniform rectangular bar, of length l and thickness d.	Through its mid-point and perpendicular to its length.	$M\left(\frac{l^2+d^2}{12}\right)$
4. Thin triangular plate or lamina, of altitude H.	About one side.	$MH^2/6$
5. Elliptical disc or lamina, of major and minor axes $2a$ and $2b$.	(*i*) About one of the axes, (major or minor).	$Mb^2/4$ or $Ma^2/4$
	(*ii*) Through its centre and perpendicular to its plane.	$M(a^2+b^2)/4$
6. Hoop or circular ring, of radius R.	(*i*) Through its centre and perpendicular to its plane.	MR^2
	(*ii*) About a diameter.	$MR^2/2$
	(*iii*) About a tangent in its own plane.	$3MR^2/2$
	(*iv*) About a tangent, perpendicular to its plane.	$2MR^2$
7. Circular lamina or disc, of radius R.	(*i*) Through its centre and perpendicular to its plane.	$MR^2/2$
	(*ii*) About a diameter.	$MR^2/4$
	(*iii*) About a tangent, in its own plane.	$5MR^2/4$
	(*iv*) About a tangent perpendicular to its plane.	$3MR^2/2$
8. Annular ring or disc of outer and inner radii R and r.	(*i*) Through its centre and perpendicular to its plane.	$M(R^2+r^2)/2$
	(*ii*) About a diameter.	$M(R^2+r^2)4$
	(*iii*) About a tangent, in its own plane.	$M(5R^2+r^2)/4$
	(*iv*) About a tangent perpendicular to its plane.	$M(3R^2+r^2)/2$
9. Solid cylinder of length l and radius R.	(*i*) About its axis of cylindrical symmetry.	$MR^2/2$
	(*ii*) Through its centre and perpendicular to its axis of cylindrical symmetry.	$M\left(\frac{l^2}{12}+\frac{R^2}{4}\right)$
	(*iii*) About a diameter of one face.	$M\left(\frac{l^2}{3}+\frac{R^2}{4}\right)$
10. Solid cone, of altitude h and base radius R.	(*i*) About its vertical axis.	$3MR^2/10$
	(*ii*) Through its vertex and parallel to its base.	$\frac{3MR^2}{20}+\frac{3Mh^2}{5}$
11. Hollow cylinder, of length l and external and internal radii R and r.	(*i*) About its own axis, (*i.e.*, about its axis of cylindrical symmetry).	$M(R^2+r^2)/2$
	(*ii*) Through its centre and perpendicular to its own axis.	$M\left[\frac{R^2+r^2}{4}+\frac{l^2}{12}\right]$

	BODY	AXIS (Position and Direction)	MOMENT OF INERTIA
12.	Spherical shell, of radius R.	(*i*) About a diameter. (*ii*) About a tangent.	$2MR^2/3$ $5MR^2/3$
13.	Solid sphere, of radius R.	(*i*) About a dimeter. (*ii*) About a tangent.	$2MR^2/5$ $7MR^2/5$
14.	Thick shell or hollow sphere, of external and internal radii R and r.	(*i*) About a diameter. (*ii*) About a tangent	$\frac{2}{5}M\left(\frac{R^5-r^5}{R^3-r^3}\right)$ $\frac{2}{5}M\left(\frac{R^5-r^5}{R^3-r^3}\right)+MR^2$
15.	Flywheel, with radii of wheel and axle, R and r—(mass of axle, m).	About its own axis.	$\frac{M(R^2+r^2)}{2}+\frac{mr^2}{2}$
16.	Spheroid of revolution, of equatorial radius R.	About polar axis.	$2MR^2/5$
17.	Ellipsoid, of axes $2a$, $2b$ and $2c$.	About one axis, $(2a)$	$\frac{M(b^2+c^2)}{5}$
18.	Rectangular parallelopiped of edges l, b and d.	Through its centre and perpendicular to one face, (say, face $l-b$)	$\frac{M(l^2+b^2)}{12}$
19.	Rectangular prism, of dimensions $2l$, $2b$ and $2d$.	About axis $2l$.	$\frac{M(b^2+d^2)}{3}$
20.	Very thin hollow cylinder, of length l and mean radius R	Through its centre and perpendicular to its own axis	$M\left(\frac{l^2}{12}+\frac{R^2}{2}\right)$

33. Routh's Rule. This rule states that *the moment of inertia of a body about any one of the three perpendicular axes of symmetry passing through its centre of mass is given by*

(*i*) *the product of its mass and one-third of the sum of the squares of the other two semi-axes,* **in the case of a rectangular lamina or parallelopiped ;**

(*ii*) *the products of its mass and one-fourth of the sum of the squares of the other two semi-axes,* **in the case of a circular or an elliptical lamina ;**

(*iii*) *the product of its mass and one-fifth of the sum of the squares of the other two semi-axes,* **in the case of a sphere or a spheroid.**

Quite a few of the cases, dealt with in the proceeding pages, may be easily deduced by an application of this rule. Thus, for example,

(*i*) *moment of inertia of a uniform rectangular lamina* (of mass M, length l and breadth b), about an axis passing through its centre O and perpendicular to its plane

$$= M\left(\frac{(l^2/4)+(b^2/4)}{3}\right) = M\left(\frac{l^2+b^2}{12}\right);$$

for, here, the two *semi-axes* of the lamina are clearly, $l/2$ and $b/2$ respectively, (Fig. 43).

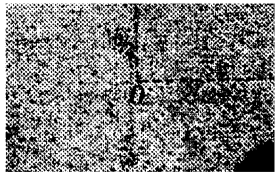


Fig. 43.

(*ii*) *Moment of inertia of a uniform circular lamina or disc*, (of mass M and radius R), about an axis, passing through its entre and perpendicular to its plane is equal to

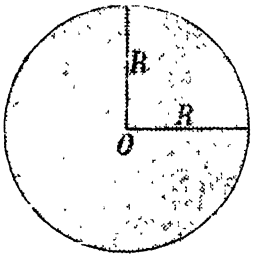


Fig. 44.

$$M.(R^2+R^2)/4 = MR^2/2,$$

because here the two semi-apes of the lamina or disc are obviously R and R, (Fig. 44).

And, again, *moment of inertia of a uniform elliptical lamina*, (of mass M, and with $2a$ and $2b$, as its major and minor axes respectively), about a perpendicular axis passing through its centre, is equal to

$$M(a^2+b^2)/4,$$

because (a) and (b) are the two semi-axes of the lamina, (Fig. 45).

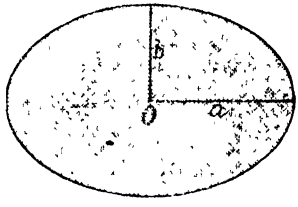


Fig. 45.

(*iii*) *moment of inertia of a solid sphere*, (of mass M and radius R) about its diameter is equal to

$$M.(R^2+R^2)/5 = 2MR^2/5,$$

because here the two semi-axes of the sphere are R and R.

34. Practical methods for the Determination of Moments of Inertia. The principle underlying the experimental determination of the moment of inertia I of a body, about a given axis, is to apply a known couple C to it and to measure the angular acceleration $d\omega/dt$ produced in it. Then, from the relation,

$$C = I.d\omega/dt, \qquad \text{we have} \qquad I = \frac{C}{d\omega/dt},$$

whence, I may be easily calculated.

(*i*) **Moment of Inertia of a Flywheel.**

First Method. The flywheel, whose moment of inertia is to be determined, is mounted on ball-bearings (to minimise friction), and its axle is arranged to be in the horizontal position at a convenient height from the ground, (Fig. 46).

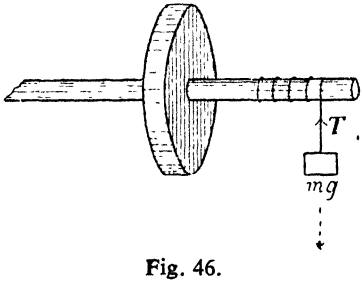


Fig. 46.

A small loop at the end of a small piece of fine cord is then slipped on to a tiny peg on the axle and *the entire length of the cord wound evenly round the latter*, with a suitable mass m suspended from its free end, and properly held in position.

As the mass is released and allowed to fall under the action of its own weight, the cord starts unwinding itself round the axle, thereby setting the wheel in rotation. The length of the cord is so adjusted that the moment the mass reaches the ground, the entire length of it gets *just* unwound from the axle and slips off the peg.

Obviously, the rotation of the wheel, (with the descent of

the mass), is due to a *couple* $T.r.$ where T is the tension in the cord and r, the radius of the axle*.

If, therefore, I be the moment of inertia of the flywheel about its axis of rotation and $d\omega/dt$, the angular acceleration produced in it, we have $I.d\omega/dt = T.r.$

The downward force due to the *weight* of the mass, when it has no acceleration, is mg ; but when it has a vertical acceleration a, the force due to it is equal to $m.a.$, and this must clearly be equal to $mg-T$.

Or, $m.a = mg-T$, whence, $T = m.(g-a)$,

$\therefore$ $I.d\omega/dt = m.(g-a)r.$

But $d\omega/dt = a/r$, And $\therefore$ $I.a/r = m(g-a)r.$ [$\because$ $a = r.d\omega/dt$]

Or, $$I = mr^2.\frac{(g-a)}{a} = mr^2\left(\frac{g}{a}-1\right) \quad ...(1)$$

The time-interval between the release of the mass and the slipping of the cord from the axle is carefully noted. Let it be t, and let the distance through which the mass falls down during this interval be S. Then, since the mass starts from rest, we have

$$S = \tfrac{1}{2}at^2, \quad \text{whence,} \quad a = 2S/t^2.$$

So that, substituting this value of a in relation (1) above, we have

$$I = mr^2.\left(\frac{g}{2S/t^2}-1\right) = mr^2.\left(\frac{gt^2}{2S}-1\right),$$

whence I, the moment of inertia of the flywheel about its axis of rotation, can be easily calculated.

Second Method.—Proceeding as above, *the loss of potential energy of the falling mass is equated against the gain in kinetic energy of the wheel, the K. E. of the mass itself and the work done against friction.* Thus, when the mass falls through distance S, the potential energy lost by it is equal to $mg.S$. And, if ω be the angular velocity of the wheel at the time, the *K.E.* gained by it is $\frac{1}{2}I\omega^2$, the *K.E.* acquired by the mass being $\frac{1}{2}mv^2$, where v is its velocity on descending through distance S.

$$\therefore\ mg.S = \tfrac{1}{2}I\omega^2+\tfrac{1}{2}mv^2+\textit{the work done against friction.} \quad ..(2)$$

To determine the work done against friction, we note the number of turns made by the wheel before coming to rest, *after the mass has been detached from the axle.* Then, obviously, the kinetic energy $\frac{1}{2}I\omega^2$, of the wheel, is used up in overcoming the frictional forces at the bearings. If the couple due to friction be C and the number of turns made by the wheel before coming to rest be n, work done by *this* couple is equal to $2\pi n \times C$, ($\because$ *work done* = *couple* $\times$ *angle*, and the angle, described by the wheel in *one* rotation is equal to 2π). So that, $\frac{1}{2}I\omega^2 = 2\pi nC$. Or, $C = I\omega^2/4\pi n$.

The couple due to friction being thus determined, we can easily calculate the work done against friction during the descent of the

*If the cord be appreciably thick, half of its thickness, added to the radius of the axle, gives the effective value of r.

mass through distance S. For, clearly, the number of turns made by the wheel during the fall of the mass through this distance is $S/2\pi r$; and, therefore, the total angle turned through by it is equal to $2\pi.S/2\pi r = S/r$.

Hence, work done against friction is equal to $C.S/r = SI\omega^2/4\pi nr$.

$\therefore$ our *energy equation* (2) now becomes

$$mg\,S = \tfrac{1}{2}I\omega^2+\tfrac{1}{2}mv^2+\frac{SI\omega^2}{4\pi nr} = \tfrac{1}{2}\,I\omega^2\left(1+\frac{S}{2\pi nr}\right)+\tfrac{1}{2}mv^2.$$

$$= \tfrac{1}{2}\,I.\frac{v^2}{r^2}\left(1+\frac{S}{2\pi nr}\right)+\tfrac{1}{2}mv^2. \qquad [\because \omega^2 = v^2/ar^2.$$

Or, $$I.\frac{v^2}{r^2}\left(1+\frac{S}{2\pi nr}\right)+mv^2 = 2mg.\,S.$$

Or, $$I.\frac{v^2}{r^2}\left(1+\frac{S}{2\pi nr}\right) = 2mg.S-mv^2. \qquad ..(3)$$

Now, if t be the time taken by the mass to fall through the distance S, its *average velocity* $= S/t$; and since average velocity $=$ (*initial velocity* $+$ *final velocity*)$/2$, we have

$$\textit{final velocity},\ v = 2S/t. \qquad \text{Or,} \qquad v^2 = 4S^2/t^2.$$

[$\because$ the initial velocity is *zero*, the mass starting from rest.]

Substituting this value of v^2 in expression (3) above, we have

$$\frac{I}{r^2}\cdot\frac{4S^2}{t^2}\left(1+\frac{S}{2\pi nr}\right) = 2mg.S-m.\frac{4S^2}{t^2}\cdot$$

$$= 2mS\left(g-\frac{2S}{t^2}\right) = 2mS\left(\frac{gt^2-2S}{t^2}\right),$$

whence, $$I = 2mS\left(\frac{gt^2-2S}{t^2}\right)\cdot\frac{r^2t^2}{4S^2(1+S/2\pi nr)}.$$

Or, $$I = \frac{mr^2(gt^2-2S)}{2S(1+S/2\pi nr)}. \qquad ..(4)$$

Alternative Calculation.—Let the number of rotations made by the wheel, before the cord and the mass slip off from the axle, (*i.e.*, after the mass has fallen through a distance S), be N.* Then, *taking the frictional force to be uniform*, and the work done against it *per rotation* of the wheel to be w, we have

work done against friction during N rotations of the wheel $= N.w$.

Thus, our energy equation (2) becomes

$$mg.S = \tfrac{1}{2}\,I\omega^2+\tfrac{1}{2}mv^2+N.w. \qquad ..(5)$$

Now, after the detachment of the mass from the axle, the wheel comes to rest after n rotations, and, therefore, work done against friction during these n rotations of the wheel $= n.w$ and this must obviously be equal to $\frac{1}{2}\,I\omega^2$, the *K.E* of the wheel *at the instant that the mass gets detached from it*. Thus,

$$n\,w = \tfrac{1}{2}\,I\omega^2, \qquad \text{whence,} \qquad w = \tfrac{1}{2}\,I\omega^2/n.$$

Substituting this value of w in equation (5) above, we have

$$mg.S = \tfrac{1}{2}\,I\omega^2+\tfrac{1}{2}\,mv^2+\tfrac{1}{2}NI\omega^2/n = \tfrac{1}{2}mv^2+\tfrac{1}{2}\,I\omega^2(1+N/n).$$

Or, $$2mg.S = mv^2+I\omega^2(1+N/n).$$

*This is obviously equal to the number of turns of the cord on the axle at the very start.

Or, $$2mg.S - mv^2 = I\omega^2(1+N/n),$$

whence, $$I = \frac{2mg.S - mv^2}{\omega^2(1+N/n)} = \frac{2mg.S - mr^2\omega}{\omega^2(1+N/n)}. \quad [\text{Since } v = r\omega.$$

Or, by dividing both the numerator and the denominator of this expression by ω^2, we have

$$I = \frac{(2mg\ S/\omega^2) - \ ^2}{1+N/n} = \frac{m[(2g.S/\omega^2) - r^2]}{1+N/n}. \quad ..(6)$$

Now, the angular velocity of the wheel at the instant that the mass gets detached from it is ω, and becomes *zero* , when the wheel comes to rest, after time t', say. Hence, if the fractional force *uniformly* retards the rotation of the wheel, its *average* angular velocity, during this interval of time t', may be taken to be equal to $(\omega+0)/2$, *i.e.*, equal to $\omega/2$. And, since the wheel makes n rotations before coming to rest, it describes an angle equal to $2\pi n$ in time t',

$\therefore$ $\omega/2 = 2\pi n/t'$, whence, $\omega = 4\pi n/t'$.

So that, substituting this value of ω in relation (6) above, we have

$$I = \frac{m[(2g\ S \times t'/16\pi^2n^2) - r^2]}{1+N/n} = \frac{m[(g\ S \times t'^2/8\pi^2n^2) - r^2]}{(n+N)/n},$$

$$= m \times \left(\frac{n}{n+N}\right)\left(\frac{g.S.t'^2}{8\pi^2n^2} - r^2\right)$$

Or, $$I = \frac{m}{n+N}\left(\frac{g\ S.t'^2}{8\pi^2 n} - nr^2\right), \quad ..(7)$$

whence I, the moment of inertia of the flywheel, about its axis of rotation, can be easily calculated.

Accurate value of ω.—In the above treatment, the angular velocity ω of the wheel has been obtained on the supposition that the frictional force remains constant during the time t' that the value of ω falls to zero, after the detachment of the mass from the axle. Obviously, this is by no means a valid assumption, because, as we know, the frictional force decreases with increase of velocity ; so that, the value of I, the moment of inertia of the wheel, deduced on the basis of the above calculations, cannot possibly be quite accurate.

If we aim at accuracy, therefore, we must adopt a *sensitive* method for determining the value of ω, and the one method, which at once suggests itself, is to make use of a tuning fork, as explained below :

A tuning fork, of a known frequency n, is arranged horizontally, (Fig. 47), with a slightly bent metallic style, attached to one of its prongs, such that, when desired, it can be made to lightly press against, or taken off, a strip of smoked paper, wrapped round the rim of the wheel.

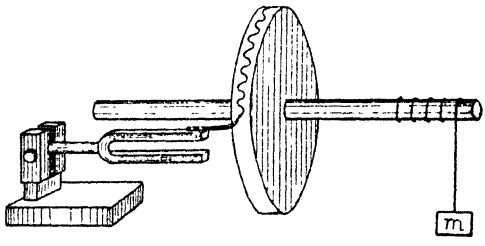


Fig. 47.

Now, with the style kept off the paper-strip, the mass m is allowed to fall down, thus setting the wheel in rotation, and just a second or so before the mass is due to get detached from the axle, the tuning fork is set into vibration (by smartly drawing a bow across it), and the style pressed lightly on to the strip, taking care to take it off soon after the detachment of the mass. A long wavy curve is thus traced out by the style on the smoked strip. The *mean* wavelength λ of this wave is then determined by dividing the tota distance occupied by the wavy curve by the total number of waves constituting it.

Since one wave is traced out by the style during one vibration of the prong or the fork, we have linear distance covered by the wheel during *one* vibration of the fork $= \lambda$. So that, distance covered by the wheel during n vibrations of the fork $= n\lambda$.

Again, since n vibrations are made by the fork in *one second*, it follows that *distance covered by the wheel in* 1 *second, i.e., the linear velocity* $v = n\lambda$.

But $v = R\omega$, where R is the radius of the wheel and ω, its angular velocity ; so that, we have $R\omega = n\lambda$; whence, $\omega = n\lambda/R$.

Thus, knowing n, λ and R, we can easily calculate the value of ω for the wheel.

This value of ω, substituted in relation (6) above, then gives a much more accurate value of I, the moment of inertia of the flywheel about its axis of rotation.

Note.—The student may, as an interesting exercise, show that expression (4) above can also be reduced to the same form as expression (7). This may be easily done by remembering (*i*) that when the wheel makes *one* full turn, the mass descends through a distance $2\pi r$, the circumference of the axle, and, therefore, when the mass descends through a distance S, the number of rotations made by the wheel is equal to $S/2\pi r$; so that, $S/2\pi r = N$; and further (*ii*) that $t = 2S/v = 2S/r\omega$, where $\omega = 4\pi n/t'$, (see page 81).

(*ii*) **Moment of inertia of a disc about an axis passing through its centre and perpendicular to its plane.**—

(*a*) *Disc suspended by two parallel threads.*—The disc, with a metal axle, is supported on two cords, wound uniformly on the axle on either side, (Fig. 48). On releasing the disc, it begins to fall down until the whole cord is unwound from the axle, say through a distance S.

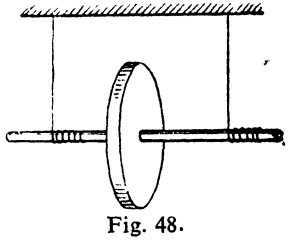


Fig. 48.

Then clearly, $P.E.$ lost by the disc $= mg.S$, where m is the mass of the disc and the axle. This energy will obviously be gained by the disc in the form of kinetic energy of rotation and translation.

If ω be the angular velocity acquired by it after falling through this distance S, its $K.E.$ of rotation will clearly be $\frac{1}{2}I\omega^2$, where I is its moment of inertia about an axis passing through its centre and parallel to the axle, (*i.e.*, perpendicular to its plane) ; and its kinetic energy of translation will be $\frac{1}{2}mv^2$.

$\therefore$ $mg.S = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$, ($v$ being its final *linear velocity*),

$\therefore$ $= (\frac{1}{2}I.v^2/r^2) + \frac{1}{2}mv^2$.

[$\because$ $\omega^2 = v^2/r^2$, where r = radius of the disc.

Or, $\frac{1}{2}I\,v^2/r^2 = mg.S - \frac{1}{2}mv^2$,

whence, $I = (mgS - \frac{1}{2}mv^2).2r^2/v^2$.

Now, *average velocity* $= S/t$, where t is the time taken by the disc in falling through distance S ; and, therefore, velocity v of the disc $= 2S/t$, and $\therefore$ $v^2 = 4S^2/t^2$. So that,

$$I = \left(mg.S - \tfrac{1}{2}m.\frac{4S^2}{t^2}\right) \cdot \frac{2r^2.t^2}{4S^2} = \left(\frac{mg.S.t^2 - 2mS^2}{t^2}\right) \cdot \frac{r^2t^2}{2S^2}.$$

$$= \frac{mgSt^2}{t^2} \times \frac{r^2.t^2}{2S^2} - \frac{2mS^2}{t^2} \times \frac{r^2.t^2}{2S^2} = \frac{mgr^2.t^2}{2S} - mr^2.$$

Or, $$I = mr^2\left(\frac{gt^2}{2S} - 1\right).$$

(*b*) *Disc mounted on axle, rolling on inclined rails.*—Here, the disc, of *mass M* and *moment of inertia I*, is allowed to roll down along inclined rails, as shown in Fig. 49. Let it acquire a *linear velocity* v and an *angular velocity* ω, when it descends a *vertical distance h*, as it rolls down a distance *S along* the rails.

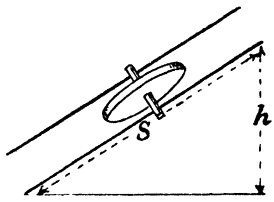


Fig. 49.

Then, clearly, *loss of P.E. of the disc*
= *K.E. of translation gained by disc*
+ *K E. of rotation gained by disc.*

Or, $Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$.

So that, $Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I.\frac{v^2}{r^2}$.

[where r = radius of the axle
And $\therefore\ \omega = v^2/r^2$.

Or, $I.\frac{v^2}{2r^2} = M(gh - \frac{1}{2}v^2)$, whence, $I = \frac{2Mr^2}{v^2}.(gh - \frac{1}{2}v^2)$

Or, $I = \frac{Mr^2}{v^2}(2gh - v^2)$.

Or, substituting the value of $v = 2S/t$, (see page 82), where t is the time taken by the disc to cover the distance S, we have

$$I = \frac{Mr^2.t^2}{4S^2}\cdot\left(2gh - \frac{4S^2}{t^2}\right) = \frac{Mr^2t^2.gh}{2S^2} - Mr^2,$$

whence the value of I, the moment of inertia of the disc can be easily calculated.

Note :—For other methods for the determination of moment of inertia, see under Torsional Pendulum, (Chapter VIII)

35. Angular Moment and Angular Impulse.—In the case of linear motion, the *momentum* of a body, as we know, is the product of its mass and velocity. On the same analogy, we have, in the case of rotational motion, *the product of the moment of inertia and the angular velocity as the angular momentum of a rotating body.*

Thus, *angular momentum* $= I.\omega$,

where I is the moment of inertia and ω, *the angular velocity of the body about the axis of rotation.*

For, suppose we have a body, rotating about an axis with a velocity ω. Then, all its particles will have the *same* angular velocity ω, but their linear velocities will depend upon their respective distances from the axis of rotation, being equal to the product of the angular velocity and the distance from the axis. Thus, the linear velocity of a particle, distant r_1 from the axis, will be $r_1\omega$; of that distant r_2 from the axis will be $r_2\omega$ and so on.

And, therefore, if m be the mass of each particle, we have, *linear momentum* of the particle, distant r_1 from the axis, equal to $m.r_1\omega$ and, therefore, the *moment of its momentum* about the axis would be $m.r_1.\omega \times r = m.r_1^2.\omega$. Similarly, the *moment of momentum* of the particle, distant r_2 from the axis, would be $m.r_2^2.\omega$ and so on.

Therefore, the *moment of momentum* of the *whole* rotating body about the axis $= mr_1^2\omega + mr_2^2\omega + \ldots\ldots$

$= \Sigma mr^2\omega = I.\omega$, $\quad [\because \Sigma mr^2 = I.$

where I is the *moment of inertia* of the body about the axis of rotation.

Thus, the *angular momentum* of a rotating body about its axis of rotation is the sum of the moments of the linear momenta of its particles about that axis. For this reason, it is also referred to as its *moment of momentum* about the axis.

Now, we have $I.d\omega/dt = C.$ Or, $I.d\omega = C.dt$,

where C is the *torque* or the *couple* acting on the body.

Integrating this with respect to t, between the limits 0 and t, we have

angular momentum, $I\,\omega = \int_0^t C.dt$,

an expression which is true, however C may vary with time.

If C be constant, we have $I.\omega = \int_0^t C.dt = C.t$,

which gives the *angular momentum* acquired by the body *in time t*.

If t be very small and C quite large, the expression $\int_0^t C.dt$ stands for the *angular impulse* given to the body, which again becomes equal to $C.t$, if C be constant.

36. Law of Conservation of Angular Momentum. Just as we have the law of conservation of momentum for linear motion, we have, for rotational motion also, the *law of conservation of angular momentum*, which states that *the angular momentum of a rotating body about an axis remains constant, if no external torque be applied to it.*

For, suppose the angular velocity of a body is changed by $d\omega$, by a torque C, applied to it for a very small interval of time dt. Then, we have $C.dt = I.d\omega$,

where I is the moment of inertia of the body about the given axis.

Hence, $C = I.d\omega/dt$, assuming I to remain constant.

If, however, I also changes, we have $C = d(I\omega)/dt$,

i.e., the torque is equal to the rate of change of angular momentum.

Obviously, therefore, if $C = 0$, *i.e.*, if there be no external torque applied to the body, $d\omega/dt$ or $d(I\omega)/dt$ is also equal to *zero*, or the rate of change of angular momentum remains constant.

It is obvious from tha above that in the case when I is not constant, and no external torque is applied to the body, the angular velocity must change in the inverse ratio to I, in order to keep its angular momentum constant.

This may be clearly seen by whirling round a stone tied to one end of a string, whose other end is held in the hand. On stopping the application of any force to it, *i.e.*, on removing the external torque,

the string begins to wind itself on the hand, with continuously increasing velocity, because as the distance of the stone from the hand decreases, its moment of inertia about its axis of rotation also *decreases*, resulting in a proportionate *increase* in its angular velocity.

Another good illustration is provided by an acrobat executing a somersault. For, as we know, he instinctively curls himself up in air, thereby decreasing his moment of inertia and consequently increasing his speed of rotation. But, before his feet touch the ground, he slows it down by straightening himself up and increasing his moment of inertia.

37. Laws of Rotation.—Corresponding to Newton's three laws in the case of linear motion, we have also three laws of rotational motion, *viz.*,

1. Unless an external torque be applied to it, the rate of rotation of a rigid body, about a fixed axis in it, remains unaltered.

An obvious example of this is the *constant rotation of the Earth about its axis.* The force of attraction due to the Sun is certainly there, but it acts at the centre of the earth and hence produces no effect on its rotation.

2. The rate of change of rotation of a body, about a fixed axis in it, is directly proportional to the external torque applied and takes place in the direction of the torque.

3. If a torque be applied by one body upon another, an equal and opposite torque is applied by the latter upon the former, about the same axis of rotation. In other words, a change in the angular momentum of one body brings about an equal and opposite change in the angular momentum of the other body.

It is useful to remember that *the moment of inertia* (I), *in rotational motion, corresponds to mass* (m), *and the angular velocity* (ω) *to linear velocity* (v), *in the case of linear motion.*

The following *Table* gives the linear and rotational analogues at a glance :

Linear Motion	Rotational Motion
1. Mass covered ..... m.	Moment of Inertia ..... I.
2. Distance ..... S.	Angle described ..... θ.
3. Velocity ..... v or dS/dt.	Angular velocity ..... ω, or $d\theta/dt$.
4. Acceleration ..... a or d^2S/dt^2.	Angular acceleration ..... $d\omega/dt$. or $d^2\theta/dt^2$.
5. Force, $F = m.a$.	Torque C, or (moment of the couple). $= I.d\omega/dt$.
6. Linear $K.E. = \frac{1}{2}mv^2$.	Angular $K.E. = \frac{1}{2}I\omega^2$.
7. Linear momentum $= mv$.	Angular momentum or moment of momentum $= I\omega$.
8. Work done by force, *i.e.*, $W = F.S$	Work done by couple, *i.e.*, $W = C.\theta$.
9. $v = u + at$.	$\omega_2 = \omega_1 + (d\omega/dt).t$.
10. $S = ut + \frac{1}{2}at^2$	$\theta = \omega_1 t + \frac{1}{2}(d\omega/dt)t^2$.
11. $v^2 - u^2 = 2aS$.	$\omega_2^2 - \omega_1^2 = 2(d\omega/dt)\theta$, where ω_1 and ω_2 are the initial and final angular velocities and $d\omega/dt$, the angular acceleration.

38. Kinetic Energy of Rotation.—

(*a*) *Kinetic energy of a body about an axis through its centre of mass.*—Suppose we have a body of mass M rotating about an axis AB, passing through its centre of mass O, (Fig. 50). It, obviously, possesses kinetic energy due to its motion ; this energy of the body is called its *energy of rotation*, because it is due to its motion of rotation.

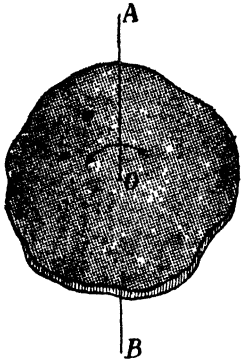


Fig. 50.

Imagine the body to be divided up into a large number of small particles, of masses m_1, m_2, m_3, etc., at distances r_1, r_2, r_3 .... etc., respectively from the axis AB. Then, we have

linear velocity of $m_1 = r_1\omega = v_1$;

of $m_2 = r_2\omega = v_2$; of $m_3 = r_3\omega = v_3$ and so on,

$\therefore$ *kinetic energy* of mass $m_1 = \frac{1}{2}m_1v_1^2$; of mass $m_2 = \frac{1}{2}m_2v_2^2$; of mass $m_3 = \frac{1}{2}m_3v_3^2$ and so on.

Or, total *K.E.* of the body $= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + \ldots\ldots$

$= \frac{1}{2}\omega^2[m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots]$

$= \frac{1}{2}\omega^2\Sigma mr^2 = \frac{1}{2}\omega^2MK^2$. $[\because \Sigma mr^2 = MK^2$.

Or, **K.E. of the body** $= \frac{1}{2}\mathbf{M}\mathbf{K}^2\omega^2 = \frac{1}{2}\mathbf{I}\omega^2$, $[\because MK^2 = I$.

where I is the *moment of inertia* of the body about axis AB.

Now, if $\omega = 1$, then, obviously, *K.E.* of the body $= \frac{1}{2}I$.

Or, $I = 2$ *K.E.*

Thus, the moment of inertia of a body, rotating with unit angular velocity, is equal to twice its kinetic energy of rotation.

(*b*) *K.E. of body which is not only rotating but whose centre of mass has also a linear velocity v.*—A body which is rotating as well as moving forwards with a velocity v, has both types of kinetic energy, *viz.*, (*i*) *energy of rotation*, because of its motion of rotation about a perpendicular axis through its centre of mass, and (*ii*) *energy of translation*, because of its linear motion. And, clearly, therefore, we have *K.E. of rotation* of the body $= \frac{1}{2}I\omega^2$,

and its *K.E. of translation* $= \frac{1}{2}Mv^2$.

$\therefore$ *total K.E. of the body* = *K.E. of rotation* + *K.E. of translation*,

$= \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 = \frac{1}{2}MK^2\omega^2 + \frac{1}{2}Mv^2$.

$= \frac{1}{2}M(K^2\omega^2 + v^2) = \frac{1}{2}MK^2(v^2/r^2) + \frac{1}{2}Mv^2$

because $\omega^2 = v^2/r^2$ where r is the radius of the body.

Or, *total kinetic energy of the body* $= \frac{1}{2}Mv^2[(K^2/r^2) + 1]$.

39. Acceleration of a body rolling down an inclined plane.— Let a body of mass M roll freely down an inclined plane, of inclination α to the horizontal, (Fig. 51). The plane is supposed to be rough enough, so that there may be *no slipping*, and hence no work done by friction.

Then, if v be the velocity acquired by the body after traversing a distance S along the plane, we have

vertical distance through which it has descended $= S.\sin\alpha$.

And, therefore,

$P.E.$ *lost by the body* $= Mg.S.\sin\alpha$.

This must, obviously, be equal to the $K.E.$ gained by the body.

Now, $K.E.$ *of rotation of the body* $= \frac{1}{2}I\omega^2$,

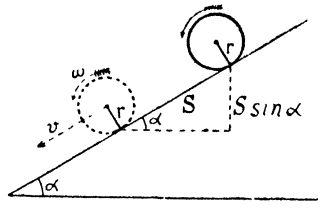


Fig. 51.

where ω is its *angular velocity* about a perpendicular axis through its centre of mass.

And, its $K.E.$ *of translation* $= \frac{1}{2}Mv^2$,

because its centre of mass has a *linear velocity* v.

$\therefore$ *total K.E. gained by the body* $= \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2$.

$= \frac{1}{2}Mv^2[(K^2/r^2)+1]$. [See §38.

Since *gain in K.E of the body is equal to the loss in its P.E.*, we have

$$\frac{1}{2}Mv^2[(K^2/r^2)+1] = Mg.S\sin\alpha.$$

Or, $$Mv^2[(K^2/r^2)+1] = 2Mg.\sin\alpha\, S.$$

Or, $$v^2(K^2+r^2)/r^2 = 2g.\sin\alpha.S,$$

whence, $$v^2 = 2(r^2/K^2+r^2).g\sin\alpha.S.$$

Comparing this with the *kinematic relation*, $v^2 = 2aS$, for a body starting from rest, we have

acceleration of the body down the plane, i e.,

$$a = (r^2/K^2+r^2).g\sin\alpha.$$

Or, the acceleration is proportional to $r^2/(K^2+r^2)$ for a given angle of inclination α.

This show that

(*i*) *the greater the value of K, as compared with r, the smaller the acceleration of the body coming down the plane* and, therefore, the greater the time it takes in rolling down along it and *vice versa.*

(*ii*) *the acceleration and, therefore, the time of descent is independent of the mass of the body.*

Thus, a solid sphere, for which $K^2 = 2r^2/5$, will roll down faster than a disc, for which $K^2 = r^2/2$, and, similarly, a disc will roll down faster than a hoop, for which K^2 is equal to r^2.

Since K^2 for a hollow sphere about the diameter is greater than that for a solid sphere of the *same mass and radius*, they can be distinguished from each other by allowing them to roll down the plane, Obviously, the solid sphere will roll down faster than the hollow one. The same test may be applied in the case of a hollow and a solid cylinder etc.

Some particular cases : —

(*i*) Case of a Spherical Shell. Let α be the angle of inclination of the plane, down which the spherical shell is rolling and let the velocity be v when it has moved a distance S along the plane, (Fig. 51),

Clearly, the vertical distance covered by the shell $= S \sin \alpha$.

$\therefore$ loss in *P.E.* of the shell $= Mg.S \sin \alpha$.

This loss in *P.E.* must be equal to the gain in *K.E.* of the shell, for no work is done by friction, as there is no slipping.

Now, *K.E* of the shell $= \frac{1}{2} I\omega^2 + \frac{1}{2}Mv^2 = \frac{1}{2}MK^2\omega^2 + \frac{1}{2}Mv^2$.

$= \frac{1}{2}M.\frac{2}{3}r^2\omega^2 + \frac{1}{2}Mv^2$. [$\because$ $K^2 = \frac{2}{3}r^2$ for a shell.

$= \frac{1}{3}Mr^2\omega^2 + \frac{1}{2}Mv^2 = \frac{1}{3}Mv^2 + \frac{1}{2}Mv^2$.

$= \frac{2Mv^2 + 3Mv^2}{6} = \frac{5Mv^2}{6}$.

Since *gain in K.E. of shell = loss in its P.E.*,

we have $5Mv^2/6 = Mg.S.\sin \alpha$. Or, $5v^2 = 6\, g\, S.\sin \alpha$.

Or, $v^2 = \frac{6}{5}.g.S.\ \sin \alpha = 2(\frac{3}{5}\ g \sin \alpha)S$

Comparing it with the relation, $v^2 = 2a\ S$, (when $u = 0$), we find that the value of acceleration a of the shell, down the plane $= \frac{3}{5}g\ \sin\ a$.

(*ii*) **Case of a Solid Shere.** We know that the acceleration (a) of a body down an inclined plane $= (r^2/K^2 + r^2)\ g\ \sin \alpha$, where K is the *radius of gyration* of the body, and α, the angle of inclination of the plane.

$\therefore$ $a = (r^2/\frac{2}{5}r^2 + r^2)\ g\ \sin \alpha$. [$\because$ $K^2 = \frac{2}{5}r^2$, in this case.

$= (r^2/\frac{7}{5}r^2)g\ \sin \alpha = \frac{5}{7}.g\ \sin \alpha$.

Thus, the acceleration of a solid sphere down the inclined plane is equal to $\frac{5}{7}\ g\ \sin \alpha$.

40. Graphical Representation of Plane Vectors.—We are already familiar with the two types of physical quantities, *viz.*, (*i*) *scalar* and (*ii*) *vector*,—the former possessing *only magnitude*, but no direction ; and the latter, possessing *both magnitude and direction*, (see foot note on page 55). These latter can, as we know, be represented by a straight line, drawn to a chosen scale, whose length and direction respectively represent the magnitude and direction of the quantity.

Any other quantity, either derived from a vector, or obtained by combining a vector with a scalar quantity, is also vectorial in nature. Thus, for example, the acceleration of a body, depending upon the velocity of the body, (a vector quantity), is also a vector quantity.

The vector quantities referred to above are, strictly speaking, *linear vectors*, and must be clearly distinguished from what are called *plane vectors*,—a term applied, in rotational dynamics, to such quantities as angular velocity, angular momentum and torque etc , which are all directional in the sense that they are confined to one plane.

Such a *plane* or *two dimensional* vector is also represented by a straight line, drawn *normal to its plane of rotation, or parallel to its axis of rotation*, its clockwise or anticlockwise rotation being indicated, according to an agreed and established convention, by the straight line being directed towards, or away from, the observer respectively.

Further, corresponding to the parallelogram law for the composition of linear vectors, we have, here, a modified form of it to determine the resultant of two plane vectors, *viz.*, that

"if there be two plane vectors acting simultaneonsly on a body in two different planes, such that they can be represented in magnitude and

direction by the two adjacent sides of a parallelogram, drawn perpendicular to those planes, their resultant is represented completely by the diagonal of the parallelogram, passing through their point of intersection,—this diagonal representing a plane vector in a third plane, perpendicular to itself."*

Thus, if *OA* and *OB*, (Fig. 52), represent two couples, *in two different planes*, acting simultaneously on a body, where *OA* and *OB* are drawn *perpendicular to those planes*, their resultant is given completely by the diagonal *OC* of the parallelogram, which represents a couple *in a third plane*, perpendicular to itself.

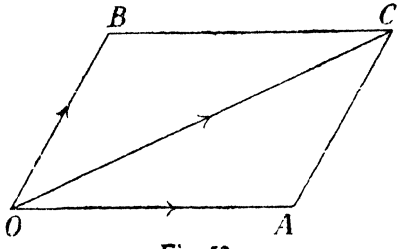


Fig. 52.

And, obviously, what is true about the composition of couples is equally true for the composition of any other plane or two-dimensional vector quantities.

41. Precession.—Just as in the case of linear motion, we may have a constant acceleration acting on a body, without changing its constant speed, (*e g.*, the *centripetal* acceleration acting on a body, moving with a uniform speed in its circular orbit), so also, in rotational motion, *we may have a constant angular acceleration acting on a body, having a constant angular speed. This is rendered possible by the plane of rotation changing direction at a given rate*, without, in any way, affecting the rate of rotation of the body about its axis of rotation, or *axis of spin*, as it is also sometimes referred to. *This change in the plane of rotation is called* **'precession'**, and is caused by a couple or torque, called the *precessional torque*, acting in a plane, perpendicular to the immediate or instantaneous plane of rotation (or spin) of the body. In other words, *the axis of the torque is, at any given instant, perpendicular to the rotation-axis of the body*, as will be clear from the following :

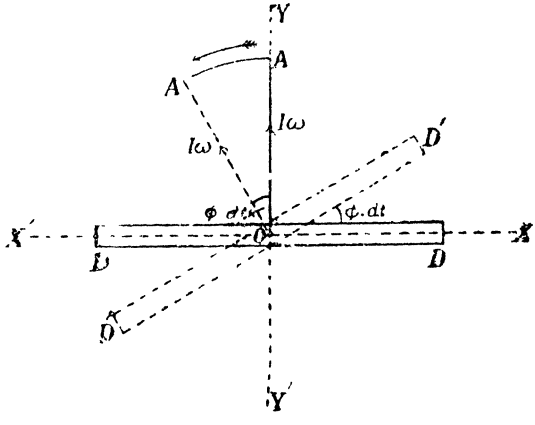


Fig. 53.

Let *DD*, (Fig. 53), be the edge of a disc, with its plane revolving about its geometric axis, with an angular velocity ω. Then, if its moment of inertia about this axis be *I*, its angular momentum will clearly be $I\omega$. Let this be represented by the straight line *OA*, drawn perpendicular to the plane of rotation of the disc.

*In accordance with the convention, stated above.

Now, let the axle of the disc also rotate, *i.e.*, let there be a *precessional motion*, about an axis, perpendicular to the plane of the paper at a (precessional) rate ϕ ; so that, after a small interval of time dt, the disc takes up the position $D'D'$, making an angle $\phi.dt$ with its original position Its angular momentum, again equal to $I\omega$, is now represented by the straight line OA'.

The change in the angular momentum of the disc is thus represented *vectorially* by $AA' = I\omega\,\phi.dt$. [$\because$ arc = radius × angle.

This change has, clearly, been brought about in time dt, and therefore,

rate of change of momentum of the disc $= I\omega.\phi.dt/dt = I\omega.\phi$.

And, since the rate of change of momentum of a rotating body is equal to the torque applied to it, we have

$$T_1 = I\omega\,\phi,$$

where T_1 is the torque applied to the disc.

So that, the *rate of precession*, $\phi = T_1/I\omega$.

Now, since the change in the angular momentum of the disc is along AA', it is clearly parallel to its plane of rotation, or perpendicular to its axis of rotation, and AA' is thus the axis of the torque applied. In other words, the axis of the torque lies along OX.

Thus, we see that *if the axis of rotation of a body be along OY and the axis of the applied torque along OX, the body* '**precesses**' *about the third mutually perpendicular axis OZ.* This will be readily understood from Fig. 54, which shows the disc *in perspective.*

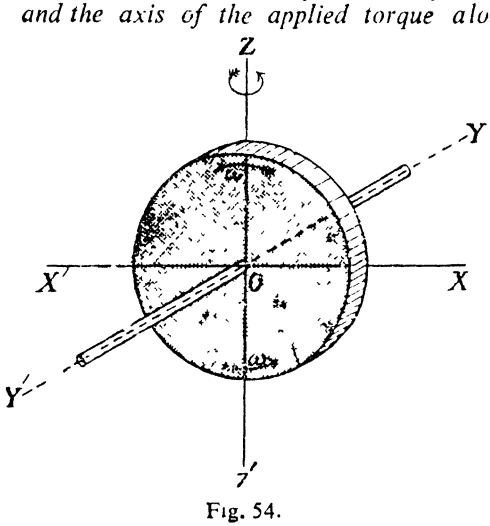


Fig. 54.

Here, *OY is the axis of rotation* and, therefore, *XOZ* is the plane of rotation ; *OX is the axis of the torque or couple applied, and, therefore, YOZ is the plane of the torque and, since the axle of the disc turns towards OX, i.e., about the axis OZ, the plane of precession is XOY.* In other words, the axis of rotation (OY) turns in *this* plane, which is, clearly, perpendicular to the first two planes,—its direction of rotation (towards OX, here), depending upon the direction of rotation of the disc and that of the torque or the couple applied.

42. The Gyrostat. A gyrostat is just a disc or a flywheel, having a large moment of inertia, rotating at a high speed about an axle, passing through its centre of mass, and mounted, as shown in Fig. 54, so that the wheel and the axle are both free to turn, as a whole, about any axis, perpendicular to the axle itself.

As explained above in §41, if a torque or couple be applied to the wheel, with its axis perpendicular to the axis of rotation of the wheel, the wheel '**precesses**' about the third mutually perpendicular axis, at a precessional rate, given by $\phi = T_1/I\omega$, where T_1 is the torque or couple applied, I, the moment of inertia of the wheel about it rotation-axis and ω, its angular velocity about this axis.

Clearly, therefore, for a given torque (T_1) applied to the wheel, the *precessional rate is inversely proportional* (*i*) *to the moment of inertia of the wheel about its axle, and* (*ii*) *to the angular velocity of the wheel* ; so that, the larger the moment of inertia of the wheel about its axle, and the higher its angular velocity, the smaller the rate of precession of the axle, and *vise versa*.

The following simple experiment will beautifully illustrate the above results :

Take a fairly large and heavy disc, (Fig. 55), free to rotate about its axis YY' passing through its centre, and fitted inside two sockets at the ends of the horizontal diameter of a bigger ring, suspended by means of a string vertically above its centre of gravity.

(*i*) Now, if with the disc quite stationary, a weight Mg be suspended at Y', the torque due to it will tilt the ring, the end Y' moving down and the end Y moving up, *i e*, the ring will turn about OX.

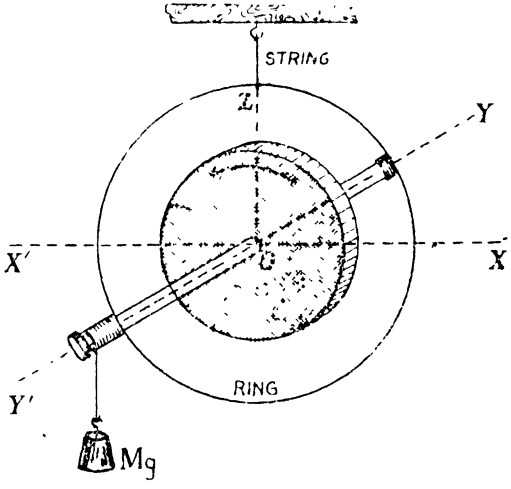


Fig 55.

But, if instead of suspending the weight, the ring be simply pushed horizontally at Y', from in front or behind, it will turn about OZ.

(*ii*) Let the disc be now set into rotation about its axle, in the direction shown, with the weight Mg kept properly supported, so as to exert no downward pull at Y'. It will be found that the ring remains quite steady and a twist, given to the string either way, hardly produces any tendency in it to rotate about OZ, as it certainly would, if the disc were stationary.

(*iii*) With the disc in motion let the weight Mg be released, so as to exert a downward pull at Y', thus producing a torque about OX. It will be found that the ring at once rotates about OZ, with the end Y' slightly tilted downwards. On pushing the ring horizontally at Y', as before, the axle, instead of turning more rapidly about OZ, as might be expected, simply gets tilted a little, raising the weight Mg slightly upwards, *clearly showing thereby that the horizontal*

rotation of the axle YY' opposes the torque due to Mg, which, therefore, descends comparatively slowly now. So long, however, as the downward descent of the weight continues, just so long does the rate of rotation of the axle about *OZ* also continue to increase, thereby increasingly opposing the torque due to the weight, until a stage is reached where the two exactly balance each other. After this, the downward descent of the weight naturally ceases, and the ring continues to turn about *OZ* at a constant rate, with the axle *YY'* slightly tilted.

(*iv*) It will be found that the greater the moment of inertia of the disc about its rotation-axis or the axle, and the higher its angular velocity about it, the smaller the rate of rotation of the axle about *OZ*, *i.e.*, the smaller the precessional motion about it.

(*v*) Since the torque or a couple is needed to produce this precessional motion of the rotation-axis of the disc, it is clear that a rotating body offers resistance to a precessional motion of its axis. This resistance to a precessional motion is called *gyrostatic resistance*, and is *equal and opposite to the precessional torque.*

43. Gyroscope. In a majority of cases, a body, subject to precessional motion, is supported at a point, away from the vertical line through its centre of gravity.

A *gravitational torque or couple* thus acts upon the body, which, *in its stationary condition*, simply tends to rotate it into a position of a lower potential energy, *i.e.*, simply tends to lower its centre of gravity. But, *if the body be rotating about some axis*, this gravitational torque supplies the necessary precessional torque equal in value to its own, provided there is no other couple acting on the body. The rate of precession ϕ, maintained by this gravitational torque T_2, is given by the relation,

$$\phi = T_2/I\omega,$$

where I and ω stand, as usual, for the moment of inertia of the body and its angular velocity about its axis of rotation.

Such a body is called a *gyroscope*, its motion being appropriately termed '*gyroscopic*'.

Thus, consider a heavy disc D, revolving with a high angular velocity ω about its physical axis POQ, itself resting on a vertical pivot at P, (Fig. 56).

Then clearly, its weight Mg, acting vertically downwards at its *c g.*, O, exerts a *gravitational torque* T_2 on it, given by $T_2 = Mg.OP = Mg.l$. [Putting $OP = l$]. So that, if ϕ be the *rate of precession of the disc* maintained by it, we have

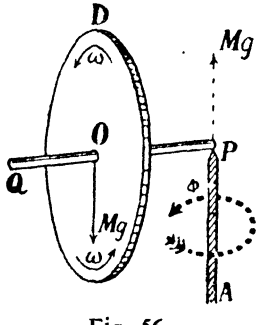


Fig. 56.

$$\phi = \frac{T_2}{I.\omega} = \frac{Mg.l}{MK^2.\omega} = \frac{gl}{K^2.\omega},$$

putting $I = MK^2$, where K is the *radius of gyration* of the disc about the axis POQ.

Hence, if t be the *time-period* of its precessional motion, *i.e.*, if it takes time t to complete its one full cycle of precessional motion, we have

$$t = \frac{2\pi}{\phi} = \frac{2\pi}{gl/K^2.\omega} = 2\pi \frac{K^2.\omega}{gl}.$$

This precession, once started, can be maintained, *at this very rate*, by the gravitational torque *alone*. A higher rate of precession than this will make axis *POQ* rise and a lower rate will make it fall. This rise and fall of the axis of rotation, or its oscillation up and down about its position of dynamic equilibrium, accompanied by a correspondingly changing precessional rate, is termed **nutation.**

Further, there is a *centrifugal* force acting on the disc along *POQ* and an equal *centripetal* force in the opposite direction *QOP*, their net effect, if they act along the same line, being to increase the frictional resistance at the pivot *P*. If, however, their lines of action be different, we have yet another couple T_3, formed by them, aptly known as the *centrifugal torque*.

In order to prevent the disc, or a precessing body, in general, from moving outwards from the centre of precession, it is necessary that the centrifugal torque on it must be balanced by an equal and opposite centripetal torque, this balancing effect being supplied by part of the gravitational torque, the remaining part of it producing precession. Thus, if T_3 be the centripetal torque and T_1 and T_2, the gyrostatic and gravitational torques, we have

$$\mathbf{T_2 - T_3 = T_1}, \quad \ldots \quad \ldots \quad \ldots (1)$$

where the different torques are given their proper sings, (*i.e.*, anti-clockwise, positive and clockwise, negative), all acting in the same direction in the case shown.

A general rule to determine the sense of the torque, producing precession in a given direction, is given by **Lanchester's rule**, which may be stated as follows :

If the gyrostat be viewed from a point in its own plane, with the line of sight perpendicular to the axis of the given precession, it is seen to describe an ellipse, the sense of whose path gives the direction of the precessional torque, with the line of sight as its axis.

44. The Gyrostatic Pendulum. A gyrostatic pendulum is a small and heavy disc or gyrostat (*D*), revolving with uniform angular velocity (ω) about a light rigid rod, (*SD*) as axis and *precessing* about the vertical (*SO*) at a uniform rate (ϕ) as shown in Fig. 57.

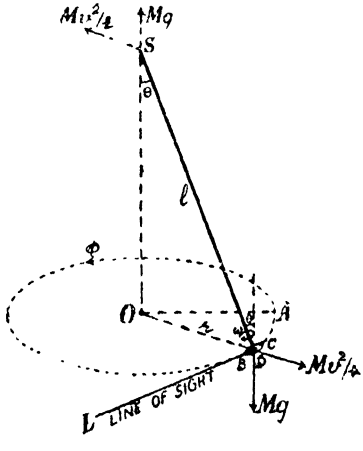


Fig. 57.

Obviously, there are the three following torques acting on the pendulum.

(*i*) **A gyrostatic torque, T_1, due to the *gyrostat* or disc *D* possessing two simultaneous rotatory motions.**

Since the plane of rotation of *D* is always perpendicular to the rod *SD*, ***its rate of precession*** ϕ **is the same as the angular velocity of *SD*, *i.e.*, equal to** v/l. For, in time dt, *SD* **traces out an arc** $v.dt$, **where** v **is its *linear velocity* in the horizontal circle**

of radius r, which is described by it in its precessional motion about SO ; or, the angle described by it in time dt is equal to $v.dt/l$, and hence the angle described by it in *unit* time is clearly equal to $(v.dt/l) \div dt = v/l$. Thus, $\phi = v/l$.

But, we know that $\phi = \frac{T_1}{I\omega}$. So that, $\frac{T_1}{I\omega} = \frac{v}{l}$, [See page 90.

whence, $$T_1 = \frac{I.\omega.v}{l} = \frac{MK^2.\omega.v}{l},$$

where K is the *radius of gyration* of the gyrostat about the axis SD.

To determine the direction of this torque, let us apply *Lanchester's rule, i.e.*, let us look at a point B on the edge of the disc along LB, where LB is perpendicular to both OD and SD, when B clearly appears to move in the anticlockwise direction, indicating that the direction of the torque T_1 is *anticlockwise.*

If t be the *periodic time* of precession of the disc or the gyrostat, we have $v.t = 2\pi r$, or $v = 2\pi r/t$. So that, substituting this value of v in the relation for T_1 above, we have

$$T_1 = MK^2\omega.2\pi r/t.l.$$

And, since $r/l = \sin\theta$, we have

$$T_1 = +MK^2.\omega.\sin\theta.(2\pi/t),$$

the +ve sign indicating that its direction is *anticlockwise.*

(*ii*) **A gravitational torque T_2**, due to the weight Mg of the gyrostat acting vertically downwards at D, (where its whole mass is supposed to be concentrated).

Clearly, the moment of this gravitational torque $= Mg \times BO = Mg.SD \sin\theta = Mg.l \sin\theta$, where BO is the perpendicular distance between Mg and an equal and opposite reaction at S ; l, the length of the rod SD and θ, the angle that it makes with the vertical.

So that, $$T_2 = -Mg.\ l \sin\theta,$$

the −ve sign indicating that its direction is *clockwise.*

(*iii*) **A centripetal torque T_3**, due to the centrifugal force Mv^2/r, acting on the gyrostat, outwards along OD.

And, the moment of this torque T_3 is obviously equal to

$$\frac{Mv^2}{r}.SO,$$

where SO is the perpendicular distance between Mv^2/r and an equal and opposite reaction at S. Or, since $SO = l\cos\theta$, we have

$$T_3 = -\frac{Mv^2}{r}.\ l\cos\theta,$$

the −ve sign again indicating that the direction of the torque is *clockwise.*

Or, substituting the value of $v = 2\pi r/t$ [see (*i*) above], we have

$$T_3 = -\frac{M}{r}\left(\frac{2\pi r}{t}\right)^2.\ l\cos\theta = -\frac{M}{r}.r^2\left(\frac{2\pi}{t}\right)^2.\ l\cos\theta = -Mr.l\cos\theta\left(\frac{2\pi}{t}\right)^2.$$

Now, $$\frac{r}{l} = \sin\theta\ ;\ \text{or}\ r = l\sin\theta.$$

Hence $$T_3 = -Ml^2.\sin\theta.\cos\theta\left(\frac{2\pi}{t}\right)^2.$$

Now, from relation (1), (page 93), we have

$$-T_2-(-T_3)=+T_1,$$

Or, $$-T_2+T_3=T_1.$$

So that, substituting their values, we have

$$-Mgl\sin\theta+Ml^2.\sin\theta.\cos\theta\left(\frac{2\pi}{t}\right)^2=MK^2\omega.\sin\theta\left(\frac{2\pi}{t}\right)$$

Or, $$-gl+l^2\cos\theta\left(\frac{2\pi}{t}\right)^2=K^2\omega\left(\frac{2\pi}{t}\right).$$

[Dividing by $M\sin\theta$ throughout.]

Now, putting $(2\pi/t)=p$, we have

$$-gl+p^2l^2\cos\theta=pK^2\omega.$$

Or, $$p^2l^2\cos\theta-pK^2\omega-gl=0,$$

which is a *quadratic equation in p.*

Therefore, $$p=\frac{+K^2\omega\pm\sqrt{K^4\omega^2-4gl^3.\cos\theta}}{2l^2.\cos\theta},$$

which, obviously, gives two values of p.

To decide between the two values, we put $\omega=0$, so that there is no rotation of the disc about SD, and the whole arrangement reduces to a *conical pendulum*, with

$$p=\frac{\pm\sqrt{4gl^3.\cos\theta}}{2l^2.\cos\theta}=\pm\sqrt{\frac{4gl^3.\cos\theta}{4l^4.\cos^2\theta}}=\pm\sqrt{\frac{g}{l\cos\theta}}.$$

i.e., $$\frac{2\pi}{t}=\pm\sqrt{\frac{g}{l\cos\theta}}.$$

But, since $2\pi/t$ must necessarily be *positive* for a conical pendulum, the negative value becomes inadmissible and, we, therefore, have

$$\frac{2\pi}{t}=\sqrt{\frac{g}{l\cos\theta}}.$$

It follows, therefore, that, in the expression for p above, only the positive value must be taken. So that,

$$p=\frac{2\pi}{t}=\frac{+K^2\omega+\sqrt{K^4\omega^2+4gl^3.\cos\theta}}{2l^2.\cos\theta}.$$

whence, $$t=\frac{4\pi l^2\cos\theta}{K^2\omega+\sqrt{K^4\omega^2+4gl^3.\cos\theta}}.$$

45. Case of a Rolling Disc or Hoop.—A simple and a familiar example of gyroscopic motion is that of a thin circular *disc* or a *hoop*, set rolling over a plane horizontal surface. If its velocity be large enough, it continues to roll along a straight path in a stable vertical position. But, as its velocity decreases, due to friction between it and the horizontal surface, its plane inclines progressively to one side and its path becomes curved towards the '*side of lean*',—the curvature of the path constantly increasing with the decrease in its velocity

so that it follows a spiral path, until, finally, it falls *flat* on the surface. Let us study this motion of the disc in some detail.

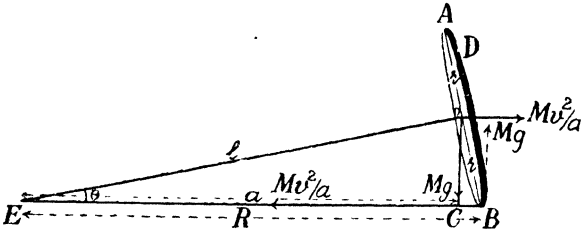


Fig. 58.

Let D be the circular disc, (Fig. 58), of mass M and radius r, rolling along a horizontal surface with v, as the linear velocity of its centre O, and with its plane AB inclined at an angle θ to the vertical. Then, the three torques acting on it are :

(*i*) *Gyrostatic torque* $\mathbf{T}_1$, due to its simultaneous rotation about its point of contact and about E, such that $T_1 = I.\omega.\phi$, where I is its *moment of inertia about the axis* OE through its centre O and perpendicular to its plane ; ω, its *angular velocity about the same axis* and ϕ, its *rate of precession.*

Now, $I = MK^2$, where K is its *radius of gyration* about OE,

$$\omega = v/r \quad \text{and} \quad \phi = v/l. \qquad [\text{See § 44 } (i).$$

So that, $$\mathbf{T}_1 = -MK^2 \cdot \frac{v}{r} \cdot \frac{v}{l} = -MK^2 \cdot \frac{v^2}{r^2} \cdot \frac{r}{l} = -MK^2 \frac{v^2}{r^2} \cdot \tan\theta,$$

where $r/l = \tan\theta$, and the $-ve$ sign of $\mathbf{T}_1$ shows that the torque is *clockwise.*

(*ii*) *Gravitational torque* $\mathbf{T}_2$, due to its weight Mg, acting vertically downwards at O, such that

$$\mathbf{T}_2 = -Mg.CB = -Mg.r \sin\theta,$$

also acting in the *clockwise* direction.

[$\because$ in the right-angled $\triangle OCB$ $\sin\theta = \frac{CB}{OB} = \frac{CB}{r}$.]

(*iii*) *Centrifugal torque* $\mathbf{T}_3$, due to its rotation about E, such that

$$\mathbf{T}_3 = \frac{-Mv^2}{a} \cdot OC = \frac{-Mv^2}{l\cos\theta} \cdot r\cos\theta,$$

where $EC = a$.

[$\because$ in the rt.-angled $\triangle OCE$ $\cos\theta = a/l$. and in the rt.-angled $\triangle OCB$ $\cos\theta = OC/r$.]

Or, $$\mathbf{T}_3 = -Mv^2 \cdot \frac{r}{l} = -Mv^2 \cdot \tan\theta,$$

the $-ve$ sign again indicating the *clockwise* direction of the torque.

Substituting these values of $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ in relation (1), (page 93) we have, for equilibrium,

$$-Mg.r \sin\theta - (-Mv^2.\tan\theta) = -MK^2 \cdot \frac{v^2}{r^2} \cdot \tan\theta.$$

Or, $$-Mg.r \sin\theta + Mv^2.\tan\theta = -MK^2 \cdot \frac{v^2}{r^2} \cdot \tan\theta.$$

Or, $$Mv^2.\tan\theta + MK^2 \cdot \frac{v^2}{r^2} \cdot \tan\theta = -Mg.r \sin\theta.$$

Or, $$v^2 \tan\theta + K^2 \cdot \frac{v^2}{r^2} \cdot \tan\theta = gr \cdot \sin\theta$$

Or, $v^2 \tan\theta \left(1+\frac{K^2}{r^2}\right) = gr.\sin\theta$. Or, $v^2\left(1+\frac{K^2}{r^2}\right) = gr.\frac{\sin\theta}{\tan\theta}$.

Or, $v^2\left(1+\frac{K^2}{r^2}\right) = gr.\cos\theta$. Or, $\cos\theta = \frac{v^2}{gr^2}\left(1+\frac{K^2}{r^2}\right)$,

whence the angle of *'lean'* of the disc for a given velocity v of it is clearly given by

$$\theta = \cos^{-1}\left[\frac{v^2}{gr}\left(1+\frac{K^2}{r^2}\right)\right] \qquad ..(A)$$

and its *velocity* by $v^2 = \frac{gr\cos\theta}{1+K^2/r^2}$ for equilibrium in the *leaning* position.

Now, for the *critical velocity* v_c, *i.e.*, the minimum velocity at which the disc can move along a straight path, *with its plane vertical*, clearly, $\theta = 0°$, so that $\cos\theta = 1$.

And, $\therefore$ in this case, $v_c^2 = \frac{gr}{1+K^2/r}$ Or, $v_c = \sqrt{\frac{gr}{1+K^2/r^2}}$.

For a value of v less than v_c, the upright position would obviously be unstable ; for, on the slightest displacement, it will be tilted over by the force of gravity until θ attains the value given above, (by expression A), corresponding to the leaning position.

Now, for a *(uniform) disc*, $K^2 = r^2/2$. Hence,

for a *disc*, $$v_c = \sqrt{\frac{gr}{1+r^2/2r^2}} = \sqrt{\frac{gr}{3/2}} = \sqrt{\frac{2}{3}gr}.$$

And, for a *hoop*, $$v_c = \sqrt{\frac{gr}{1+r^2/r^2}} = \sqrt{\frac{gr}{2}}. \qquad \left[\because \text{for a } \textit{hoop}\ K^2 = r^2.\right.$$

Let us now calculate the radius of curvature R of the *path of the disc on the horizontal surface.* It is clearly equal to $EB = R$. And, in the right-angled triangle EOB, we have

$$\sin\theta = OB/EB = r/R \text{ ; so that, } r = R\sin\theta.$$

$$\therefore \quad R = \frac{r}{\sin\theta} = \frac{r}{\sqrt{1-\cos^2\theta}}. \quad [\because \sin\theta = \sqrt{1-\cos^2\theta}.$$

Or, substituting the value of $\cos\theta$ deduced above, we have

$$R = \frac{r}{\sqrt{1-\frac{v^4}{g^2r^2}\cdot\left(1+\frac{K^2}{r^2}\right)^2}},$$

which, with the substitution of the appropriate value of K, gives the radius of curvature of the path of the disc or the hoop along the horizontal surface.

46. Gyrostatic and Gyroscopic Applications. The tendency of a rapidly rotating disc or wheel (and, in fact, any rigid body), to preserve its axis of rotation endows a gyrostat with a stability of direction, which is made use of in a

number of ways for the steadying of motions. Among the more important and familiar applications of this may be mentioned the following :

(*i*) **The Gyrostatic or the Gyro-Compass.** It is a special type of compass used in aeroplanes and ships, and, more particularly, in submarines. In essentials, it consists of a disc or a flywheel, of a large moment of inertia, (*i.e.*, a gyrostat), suspended in frictionless *gimbals* inside a supporting frame, which is kept rotating at a high speed by means of an electric motor about a horizontal axis, lying in the geographic meridian (*i.e.*, in the vertical plane passing through the geographic north and south of the earth) Its directional stability and the conservation of its angular momentum make its axis always lie in the direction of the metridian, *i.e.*, along the geographic north and south. And, since the arrangement is such that the disc or the flywheel has three degrees of freedom, irrespective of any porsition of the supporting frame, a movement of the latter produces no deflecting toque or couple on it, and *this particular direction* of its axis continues to be maintained in space all the time, despite any changes in the direction of the ship or the submarine, or any tossings or pitchings of it. It is, therefore, preferred to the ordinary magnetic compass and is more dependable than the latter, in view of the additional advantage of its remaining altogether unaffected by any type of magnetic diturbances.

The Pendulum Gyro-Compass. The above arrangement, with a small mass, suitably suspended below the rotating disc or flywheel, constitutes what is called the *pendulum Gyro-compass,—the small mass supplying the necessary restoring torque* to bring its axis back to its original direction, should it get displaced due to some disturbance. In the absence of this simple but ingenious device, the instrument would lack its restorative action, due to the inherent stability of a gyrostat in any position.

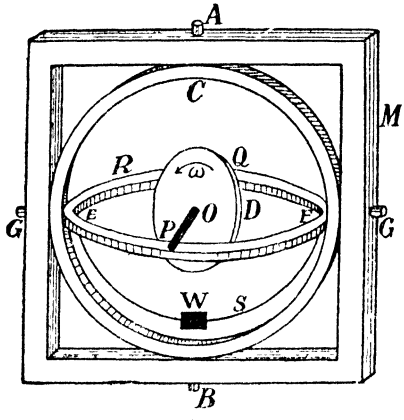


Fig. 59.

The essential features of the construction of the *Pendulum Gyro-compass* will be clear from Fig 59, where the rotating disc or gyrostat *D* has its axle *PQ* mounted in a horizontal ring *R*, free to rotate about the axis *EF* inside a vertical ring *C* which, in its turn, rotates freely about the axis *AB* within a frame work *M*, carried on horizontal gimbals, (of which *GG* forms one pair), to ensure the fullest freedom of movement.

The horizontal ring *R* has a stirrup *S*, fixed rigidly to it, which is loaded with a weight *W*, *immediately below O*, the centre of the disc or the gyrostat.

It can be shown that this arrangement would be stable, at any given place, *only along true north and south*, *i.e.*, when the end *P* points truly north,—any accidental displacement of it calling into play a directive force, restoring it back to its original direction.

(*ii*) **Rifling of barrels of Guns and Rifles.** This is another well-known application of the directional stability of a rapidly revolving body. For, it is found that if a shot or a bullet be given a rapid '*spin*', about an axis along its direction of motion, its uniformity of flight is greatly improved by making it less responsive to small deflective forces during its passage through air. This is achieved by '*rifling*' the barrel, *i.e.*, by cutting spiral grooves inside it so that the shot or bullet is first forced to move along these, before it emerges out into the air, thus acquiring the necessary '*spin*' to ensure an almost uniform linear motion.

(*iii*) **Riding of Bicycle and Rolling of Hoops or Discs.** These are both cases of what is called '*statical instability* ; for, neither of the two, at rest, can possibly remain in equilibrium in the position in which it does, when it is in motion. Here, again, it is the gyroscopic action that does the trick, by appropriately deflecting their axes of rotation and thereby changing their planes of rotation, to counterbalance the disturbing effect due to gravity.

Thus, when a person rides a bicycle, without holding its handle, he has simply to tilt to one side in order to turn to *that* side ; for, by so doing, he produces a couple about the horizontal direction of motion of the front wheel of his bicycle, which, here, acts as a rotating gyrostat. This couple, then, turns the axle of the wheel about the vertical, and hence its plane of rotation, into the desired direction.

The same is true about a hoop or a disc, projected, with its plane vertical, to roll over a horizontal surface, which we have discussed fully in §45, above. As explained there, so long as its linear or translational velocity remains above a certain *critical value*. it continues to advance along a straight path, but as soon as its velocity falls below this critical value, its plane gets inclined to the vertical, or it begins to '*lean*' from the vertical and its path gets curved towards its 'side of lean'. And, then, as its velocity goes on progressively decreasing, due to friction, the curvature of its path goes on increasing correspondingly, so that it follows a more or less spiral path until, finally, it falls flat on the surface.

(*iv*) **Precession of the Equinoxes.** The earth, as we know, is not an exact sphere, but bulges out slightly at the equator, (or has the shape of a '*flattened ellipsoid of revolution*') Further, the *Sun* and the *Moon* do not usually lie in its equatorial plane but rather in the plane of the ecliptic, which is inclined at an angle of 23 5° to the former, with the result that the gravitational attraction due to the Sun and the Moon, on this equatorial bulge gives rise to a torque, bringing about the precession of the axis of the earth, which, acting as a gigantic top*, describes a *cone*†, *relative to the fixed stars, e.g., the pole star, similar in manner to the cone described by the axis of a precessing top, due to its weight*,—the phenomenon being spoken of as the '*precession of the equinoxes*'. This couple on the earth due to the attractive force of the Sun and the Moon is, however, very small, so that it takes 25,800 years for the earth's axis to describe the complete cone, at which rate of rotation, the star *Vega* will be the pole star in about 12,000 years hence.

It is interesting to observe that atoms too have the mechanical properties of tops, and, at least in one special case, their gyrostatic moment has been demonstrated experimentally by *Einstein* and *De Haas*.

(*v*) **Other Recent Applications.** The modern aircraft appliances, like the *automatic pilot*, *the artificial horizon* and the *turn and bank indicators etc.*, all depend for their action on *gyrostatic* principles.

The function of all these instruments is to record the effects of a change of orientation between a relatively fixed plane, provided by a fast rotating gyrostat, serving as the reference or the *datum plane*, and some other movable plane in the machine, and this they do with a degree of precision which makes their indication far more safe to rely upon than mere human judgement, howsoever trained or mature.

SOLVED EXAMPLES

1. A flywheel of mass 500 k. gms. and 2 metres diameter, makes 500 revolutions per minute. Assuming the mass to be concentrated at the rim, calculate the angular velocity, the energy and the moment of inertia of the flywheel.

(*i*) No. of revolutions made by the flywheel = 500 *per minute*.

Angle described in one rotation = 2π *radians*.

$\therefore$ angle described by the wheel per minute = $2\pi.500$,,

And ,, ,, ,, ,, ,, ,, *second* = $2\pi.500/60$.

$= 50\pi/3$ *radians*.

Or, the *angular velocity* (ω) *of the flywheel* = $50\pi/3$ *radians/sec*.

(*ii*) Moment of inertia $I = MK^2$.

Here, mass $M = 500 \times 1000$ *gms*.

And K, the radius of gyration = 1 *metre* or 100 *cms*.

---

*A '*top*', in Physics, is the name given to a rotating body, either completely free to move, or fixed at the most at just *one* point with absolute freedom of rotation, and it must not, therefore, be confused with the toy that goes by that name.

†In the *clockwise* direction, as seen from the north.

because the mass is concentrated at a distance, *equal to its radius* from the axis of rotation, which passes through its centre.

$\therefore$ *moment of inertia of the flywheel* $= 500\times1000\times(100)^2$.

$= 500\times1000\times10000 = 5\times10^9$ *gm-cm*^{2}.

(iii) The energy possessed by the flywheel is due to its rotation, *i.e., it possesses only rotational energy*, which is equal to $\frac{1}{2}I\omega^2$.

log 5	= 0·6990
log 10^9	= 9·0000
log 1250	= 3·0969
2 log π	= 0·9944
	13·7903
log 9	= 0·9542
Antilog	12·8361
	$= 6857\times10^9$
	$= 68\cdot57\times10^{11}$

Or, *energy of the flywheel* $= \frac{1}{2}\times5\times10^9\times\left(\frac{50\pi}{3}\right)^2$.

$= \frac{1}{2}\times5\times10^9\times\frac{2500\pi^2}{9}$.

$= \frac{5\times10^9\times1250\pi^2}{9}$.

$= 68\cdot57\times10^{11}$ *ergs.*

2. A flywheel weighs 10 tons, and the whole of the weight may be considered as concentrated at a distance 3 ft. from the axis. What is the amount of energy stored in the flywheel when rotating at a speed of 100 revolutions per minute ? *(Punjab, 1934)*

log 5	= 0·6990
log 2240	= 3·3502
log 100	= 2·0000
2 log π	= 0·9944
	7·0436
log 32	= 1·5051
Antilog	5·5385
	= 345500
	$= 34\cdot55\times10^4$.

Here, $M = 10$ *tons* $= 10\times2240$ *lbs.*

$\omega = 100\times2\pi$ *radians/min.*

$= 100\times2\pi/60 = 10\pi/3$ *radians/sec.*

and $K = 3$ *ft.* *(Given)*

Since *K.E.* of rotation of a body $= \frac{1}{2}\times I\omega^2 = \frac{1}{2}MK^2\omega^2$, we have

K.E. of the flywheel

$= \frac{1}{2}\times10\times2240\times3^2\times(10\pi/3)^2$.

$= 5\times2240\times9\times100\times\pi^2/9$ *ft.poundals.*

$= \frac{5\times2240\times9\times100\times\pi^2}{9\times32}$ *ft.pounds.*

Or, kinetic energy stored up in the flywheel $= 34\cdot55\times10^4$ *ft.-lbs.*

3. A flywheel of mass 100 k. gms. and radius of gyration 20 cms. is mounted on a light horizontal axle of radius 2 cms., and is free to rotate on bearings whose friction may be neglected. A light string wound, on the axle carries at its free end a mass of 5 k. gms. The system is released from rest with the 5 k. gms. mass hanging freely. Prove that the acceleration of this mass is g/2001 cm./sec^2.

If the string slips off the axle after the weight has descended 2 metres, prove that a couple of moment 31·8 k. gms. wt.-cm. (approximately) must be applied in order to bring the flywheel to rest in 5 revolutions *(Cambridge H. S. Certificate)*

(i) The *mass* of the flywheel $(M) = 100$ *k. gms.* $= 100\times1000$ *gms.*

and its *radius of gyration* $(K) = 20$ *cms.*

So that, its *moment of inertia* $(I) = MK^2 = 100\times1000\times20^2$.

$= 100\times1000\times400 = 4\times10^7$ *gm.-cm*^{2}.

Let angular acceleration of the flywheel be $= d\omega/dt$.

Then, linear acceleration of the mass of 5 *k. gm.* $= r.d\omega/dt$, where r is the radius of the axle.

Or, acceleration of the mass, *i.e.*, $a = r.d\omega/dt$,

whence, $d\omega/dt = a/r = a/2$.

$\therefore$ rotational couple acting on the flywheel $= I.d\omega/dt$.

$= 4\times10^7\times a/2$ *gm.wt.-cm.* $= 2\times10^7\times a$ *gm. wt.-cm.*

This must, obviously, be equal to the couple applied to the wheel by the tension in the string.

If T be the tension in the string, the couple due to it $= T.r$.

If the mass of 5 *k. gms.* had no acceleration, the tension in the string would be equal to its *weight* $= 5\times1000\times g$ *dynes.*

But, since it has an acceleration a, we have

$ma = mg - T$. Or $T = m(g-a)$.

$\therefore$ *couple applied to the wheel* $= T.r = m(g-a)r.$

$= 5 \times 1000(g-a) \times 2$ *dynes-cm.*

Now, $I.d\omega/dt = T\,r.$

$\therefore$ $2\times10^7 a = 5\times1000(g-a)\times2 = 10000g - 10000a.$

Or, $2\times10^7 a + 10000a = 10000g.$ Or, $a(2\times10^7+10000) = g\times10^4,$

whence, $a = \dfrac{g\times10^4}{(2\times10^7+10000)} = \dfrac{g\times10^4}{2001\times10^4}.$ Or $a = g/2001$ *cm./sec*^{2}.

Thus, the acceleration of the mass $= g/2001$ *cm./sec*^{2}.

(*ii*) When the weight has descended 2 *metres*, it has lost some *P.E.* This must be equal to the gain in *K.E.* of the wheel and the weight ; so that,

K.E. of the wheel + *K.E. of the weight* = *P.E. lost by the weight.*

Now *P.E.* lost by the weight = work done by it in falling through 2 *metres* distance = *mgh*, (where h = 2 *metres* = 200 *cms*) $= 5\times1000\times981\times200$ *ergs.*

$\therefore$ *K.E.* of the wheel and the weight $= 1000000\times981 = 981\times10^6$ *ergs.*

This must, therefore, be the work that the couple applied to the wheel must do in order to stop it. If C be the couple required for the purpose, we have

work done by the couple $= C\theta$, (where θ is the angle of rotation).

Since the wheel comes to rest after 5 revolutions, it describes an angle $\theta = 2\pi\times5$ *radians.*

$\therefore$ *work done by the couple* $= 2\pi\times5\times C.$ And, $\therefore$ $2\pi\times5\times C = 981\times10^6.$

Or, $C = \dfrac{981\times10^6}{2\pi\times5} = \dfrac{981}{\quad}\times10^5$ *dynes-cm.* $= \dfrac{981\times10^5}{981\times\pi} = \dfrac{10^5}{\pi}$ *gm.wt.-cm.*

2 log 10 =	2·0000
log π =	0·4972
Antilog	1·5028
	= 31·83

$= \dfrac{10^5}{1000\times\pi} = \dfrac{10^2}{\pi}$ *k.gm.wt.-cm.*

$= 31{\cdot}83$ *k.gm.wt.-cm.*

Hence, a couple of moment 31·83 *k.gm.wt-cm.* will bring the flywheel to rest in 5 revolutions.

4. A flywheel of weight 200 lbs. which may be regarded as a uniform disc of radius 1 ft. is set rotating about its axis with an angular velocity of 5 revolutions per second. At the end of 40 secs., this velocity, owing to the action of a constant frictional couple, has dropped to 4 revolutions per second. What constant couple must now be applied so that in further 20 secs., the angular velocity will be 8 revolutions per second.

Find the total angle turned through during the minute.

(*Cambridge Higher Secondary School Certificate*)

Here mass of the flywheel, M = 2000 *lbs.* and its radius r = 1 *ft.*

Since it is a uniform circular disc, its moment of inertia about its axis, *i.e.*,

$$I = \tfrac{1}{2}\,Mr^2 = \tfrac{1}{2}.200.l^2 = 100\ lb.\ ft^2.$$

Its angular velocity, to start with = 5 *revolutions/sec.* $= 2\pi\times5$ *radians/sec.*

and ,, ,, ,, after 40 *sec.* = 4 *revolutions/sec.* $= 2\pi\times4$ *radians/sec.*

$\therefore$ change in angular velocity in 40 ,, $= 2\pi\times5 - 2\pi\times4 = 2\pi$ *radians/sec*^{2}.

$\therefore$ rate of change of angular velocity $= 2\pi/40 = \pi/20$ *radians/sec*^{2}.

Or, angular retardation, *i.e.*, $d\omega/dt = \pi/20$ *radians/sec*^{2}.

Now, couple $= I.d\omega/dt.$

$\therefore$ frictional couple acting on the wheel $= 100\times\pi/20 = 5\pi$ *poundal-ft.*

Again, the velocity of the wheel is now desired to be raised from 4 to 8 revolutions *per sec.*, in 20 *seconds.*

$\therefore$ initial angular velocity $= 2\pi\times4$ *radians/sec.*

And, final angular velocity in 20 *secs.* $= 2\pi \times 8$ *radians/sec.*

$\therefore$ change in velocity in 20 *secs.* $= 2\pi \times 8 - 2\pi \times 4 = 8\pi$ *radians/sec.*

And, $\therefore$ rate of change of velocity $= 8\pi/20 = 2\pi/5$ *radians/sec*^{2}.

Or, angular acceleration, $d\omega/dt = 2\pi/5$ *radians/sec.*

And, the couple required to produce this acceleration

$= 100 \times 2\pi/5 = 40\pi$ *poundal-ft.* [$\because$ *couple* $= I.d\omega/dt$.

$\therefore$ *total couple required to be applied*

log 45 = 1·6532	= *this couple of* 40π + a *couple of* 5π (*to overcome frictional couple*).
log π = 0·4972	
Antilog $\bar{2}$·1504 = 141·4	Or, *total couple required* $= 40\pi + 5\pi = 45\pi = 141{\cdot}4$ *poundal-ft.*

Now, let the angle described by the wheel in the first 40 *sec.* be θ_1

Then, from the relation $\theta_1 = \omega t + \frac{1}{2} \cdot \frac{d\omega}{dt} \cdot t^2$, we have [See table on page 85.

$$\theta_1 = (2\pi \times 5) \times 40 - \tfrac{1}{2} \cdot \frac{\pi}{20} \cdot (40)^2 = 400\pi - 40\pi = 360\pi \text{ radians.}$$

And, if θ_2 be the angle described by the flywheel in the next 20 *secs.*, we have, as above,

$$\theta_2 = (2\pi \times 4) \times 20 + \frac{1}{2} \cdot \frac{2\pi}{5} \cdot (20)^2 = 160\pi + 80\pi = 240\pi \text{ radians.}$$

$\therefore$ the total angle turned through by the flywheel in one full minute

$$= \theta_1 + \theta_2 = 360\pi + 240\pi = 600 \text{ radians.}$$

Now, since a rotation through 2π radians means one revolution, a rotation through 600π radians means $600\pi/2\pi = 300$ revolutions.

Thus, the flywheel makes 300 revolutions during the minute.

5. A pulley of radius 2 ft. has hanging from it, a rope with masses of 60 lbs. and 52 lbs. attached to its two ends, the masses being kept at rest initially by holding one of them. If the moment of inertia of the pulley be 320 lb -ft^2., what will be the velocity of the masses, when they have moved a distance of 6 ft. from their position of rest ? It may be assumed that there is no slip between the rope and the pulley and that friction at the axle of the pulley is negligible.

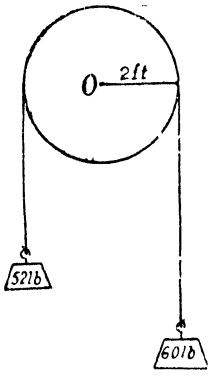


Fig. 60.

Here, obviously, the *motive force, i.e.*, the force which makes the masses and the pulley move, is the *weight of the excess mass of* (60−52) or 8 *lbs. wt.* at one end of the rope $= 8 \times 32 = 256$ *poundals.* (Fig. 60).

When the masses have moved through a distance of 6 *ft.*, *the loss of potential energy* suffered by this excess mass is clearly $= 256 \times 6 = 1536$ *ft. poundals.*

This loss of *P.E.* of the excess mass is equal to the gain in the *K.E.* of the system consisting of the two masses and the pulley.

Let v be the velocity of the masses at this instant.

Then, *K.E.* of the two masses

$= \frac{1}{2}(60+52)v^2 = \frac{1}{2}.112.v^2 = 56v^2$ *ft. poundals.*

And, *K.E.* of the pulley $= \frac{1}{2}I\omega^2 = \frac{1}{2} \times 320 \times v^2/r^2 = \frac{1}{2} \times 320 \times v^2/4$

$= 40v^2$ *ft. poundals.* [$\because \omega = v/r$ and $r = 2$ *ft.*

$\therefore$ total gain in *K.E.* of the system $= 56v^2 + 40v^2 = 96v^2$ *ft. poundals.*

Since *gain in K.E. of the system = loss in P E. of the excess mass.*

we have $96v^2 = 1536$. Or, $v^2 = 1536/96 = 16$.

whence, $v = \sqrt{16} = 4$ *ft./sec.*

Thus, the velocity of the masses when they have moved through a distance of 6 *ft.* will be 4 *ft./sec.*

6. A flywheel of mass 65·4 K. gms. is made in the form of a circular disc of radius of 18 cms. ; it is driven by a belt whose tensions at the points where it runs on and off the rim of the wheel are 2 K. gms. and 5 K. gms. weight respectively. If the wheel is rotating at a certain instant at 60 revolutions per minute, find how long will it be before the speed has reached 210 revolutions per minute. While the flywheel is rotating at this latter speed, the belt is slipped off and a brake applied. Find the constant braking couple required to stop the wheel in 7 revolutions.

(Cambridge Higher School Certificate)

(*i*) Here, obviously,

tension T_1, where the belt runs *on* the rim $= 2K.$ *gms. wt.*

and ,, T_2, ,, ,, ,, ,, *off* ,, ,, $= 5$,, ,, ,,

$\therefore$ the resultant tension in the belt $= T_2 - T_1 = (5-2)$ *K. gms. wt.*

$= 3$ *K. gms. wt.* $= 3 \times 1000 \times 981$ *dynes.*

$\therefore$ moment of the couple due to this tension $= 3 \times 1000 \times 981 \times 18$ *dynes-cm.*

And, if $d\omega/dt$ be the angular acceleration of the wheel, the couple acting on the wheel $= I.d\omega/dt$.

Now, $I = M.r^2/2 = (65{\cdot}4 \times 1000 \times 18^2)/2 = 65400 \times 18 \times 9$ *gm cm*^{2}.

$\therefore$ moment of the couple $= 65400 \times 18 \times 9 \times d\omega/dt$ *dynes-cm.*

This must be equal to the moment of the couple due to tension in the belt. Hence,

$$65400 \times 18 \times 9 \times d\omega/dt = 3 \times 1000 \times 981 \times 18.$$

Or, $$d\omega/dt = \frac{3 \times 1000 \times 981 \times 18}{65400 \times 18 \times 9} = 5 \text{ radians/sec}^2.$$

Now, we ave the relation, $\omega_2 = \omega_1 + (d\omega/dt)t$ ..(*i*) [See page 85.

where ω_2 is the final angular velocity ; ω_1, the initial angular velocity ; $d\omega/dt$, the angular acceleration and t, the time.

Here, $\omega_2 = 210$ *rev./mn.* $= 210 \times 2\pi/60 = 7\pi$ *radians/sec.*,

$\omega_1 = 60$,, $= 60 \times 2\pi/60 = 2\pi$,,

and $d\omega/dt = 5$ *radians/sec*^{2}.

$\therefore$ from relation (*i*) above, we have $7\pi = 2\pi + 5t$. Or, $5\pi = 5t$.

whence, $t = \pi = 3{\cdot}142$ *secs.*

So that, the flywheel will obtain a speed of 210 revolutions per minute after 3·142 *seconds.*

(*ii*) Let the angular retardation produced in the wheel by the *braking couple* be $d\omega/dt$, the angle turned through by it before coming to rest being equal to $7 \times 2\pi = 14\pi$ *radians.*

Then, applying the relation $\omega_2^2 - \omega_1^2 = 2(d\omega/dt)\theta$, [See page 85.

we have $0^2 - (7\pi)^2 = 2(d\omega/dt) \times 14\pi$. Or, $28\pi.d\omega/dt = -49\pi^2$. [$\because \theta = 14\pi$.

Or, $$d\omega/dt = -49\pi^2/28\pi = -7\pi/4 \text{ radians/sec}^2.$$

Or, the angular *retardation* required $= 7\pi/4$ *radians/sec*^{2}.

Now, since couple $= I.d\omega/dt$, we have

braking couple required, $C = \frac{1}{2} \times 65400 \times 18 \times 18 \times 7\pi/4$ *dynes-cm.*

log 327 = 2 5145	Or, $C = \frac{65400 \times 18 \times 18 \times 7\pi}{2 \times 4 \times 981}$ *gms. wt.-cm.*
log 63 = 1 7993	$= \frac{65400 \times 18 \times 18 \times 7\pi}{2 \times 4 \times 981 \times 1000}$ *K. gms. wt.-cm.*
log π = 0·4972	$= \frac{327 \times 63\pi}{1090} = 59\ 37$ *K. gms. wt.-cm.*
4·8110	
log 10 0 = 4	
Antilog of 1 7736 = 59 37	

Hence, the required *braking couple* $= 59\ 37$ *K. gms. wt.-cm.*

7. A flywheel, which can revolve on a horizontal axis weighs 900 lbs. and its radius is r ft. A rope is coiled round its rim and a weight of 90 lbs. hung from its free end, turns the wheel by its descent. Find the speed at which the weight is moving after descending 20 ft. from rest.

Let the acceleration of the weight be *a ft. per sec²*., and let *M* be the mass of the weight suspended, (Fig. 61).

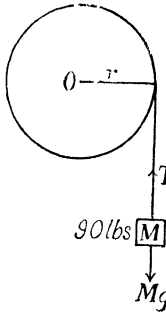


Fig. 61.

Then, if *T* be the tension in the string, we have

$M.a=(Mg-T)$. Or, $T = M(g-a) = 90(32-a)$ *poundals*

Now, moment of the couple acting on the wheel due to tension *T* in the rope $= T.r$.

$\therefore \quad T.r = 90(32-a).\ r$ *poundal-ft.*

Also, rotational couple on the flywheel $= I.d\omega/dt$.

Here, $I = \frac{1}{2}Mr^2 = \frac{1}{2}.900.r^2.$ $= 450r^2.$ { Considering the flywheel to be a uniform circular disc of *mass M* and *radius r*.

And $d\omega/dt = a/r.$

$\therefore$ *rotational couple on the wheel* $=450r^2\ (a/r) = 450\ a.r$ *poundal-ft*

We have *couple due to tension T = rotational couple.*

$\therefore \quad 90(32-a)\ r = 450\ a.r.$ Or, $90(32-a) = 450\ a.$

Or, $2880-90\ a = 450\ a.$ Or, $540\ a = 2880.$

whence, $a = 2880/540 = 16/3$ *ft /sec²*.

If *v* be the velocity of the weight after it has descended 20 *ft*., we have

log	640 = 2·8062	$v^2-u^2 = 2.aS.$	
log	3 = 0 4771	Or, $v^2-0 = 2\times(16/3)\times 20.$	[$\because\ S = 20$ *ft.*
	$\frac{1}{2}\times 2\cdot3291$	Or, $v^2 = 640/3,$	
Antilog	1·1645	whence, $v = \sqrt{640/3} = 14\cdot61$ *ft./sec.*	
	= 14·61	Therefore, the speed of the weight would be 14·61 *ft./sec.*	

8. A sphere of mass 50 gms., diameter 2 cms., rolls without slipping with a velocity of 5 cms. per sec. Calculate its total kinetic energy in ergs.

Here, mass of the sphere, $M = 50$ *gms*., and radius of the sphere, $r = 1$ *cm.*

Now, moment of inertia of the sphere (solid) is given by

$$I = \tfrac{2}{5}.Mr^2 = \tfrac{2}{5}\times 50\times 1 = 20\ gm.\text{-}cm^2.$$

As the sphere rolls, it rotates about its own diameter as axis as well as its centre of mass moves with a velocity of 5 *cms /sec.* It has, therefore, both kinetic energy of rotation as well as kinetic energy of translation ; and, therefore, its total energy is the sum of both.

Now, *K.E. of rotation* $= \frac{1}{2}I\omega^2 = \frac{1}{2}I.v^2/r^2 = \frac{1}{2}\times 20\times 5^2/1^2 = 250$ *ergs.* [$\because\ \omega = v/r$.

And *K E of translation* $= \frac{1}{2}Mv^2 = \frac{1}{2}\times 50\times 5^2 = 25\times 25 = 625$ *ergs.*

$\therefore$ *total kinetic energy of the sphere* $= 250 + 625 = 875$ *ergs.*

9. A flywheel of mass 10 K. gms. and radius 20 cms. is mounted on an axle of mass 8 K. gms. and radius 5 cms. A rope is wound round the axle and carries a weight of 10 K. gms. The flywheel and the axle are set into rotation by releasing the weight. Calculate (*i*) the angular velocity and the kinetic energy of the wheel and axle and (*ii*) the velocity and kinetic energy of the weight, when the weight has descended 20 cms. from its original position.

The flywheel, here, (Fig. 62), is just a hollow circular disc or cylinder, (as it has been cut in the centre for the axle to pass through) ; its moment of inertia about its axis, therefore, is equal to $M(R^2+r^2)/2$, where *M* is its mass and *R* and *r* its outer and inner radii, (*r* being the radius of the axle). (See page 78).

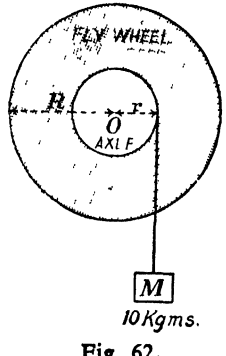


Fig. 62.

Now, *M* for the flywheel, is equal to 80 *K. gms.* or 80×10^3 *gms*., and *R* and *r*, equal to 20 *cms*. and 5 *cms*. respectively ; so that, the *moment of inertia of the wheel*

$= 80\times 10^3\times(20^2+5^2)/2 = 80\times 10^3\times 425/2.$

$= 17000\times 10^3 = 17\times 10^6\ gm.\text{-}cm^3.$

And, the axle is just a disc or cylinder whose moment of inertia about its axis is equal to $\frac{1}{2}Mr^2$, where *M* is its mass and *r*, its radius. [See pp. 63 & 66.

So that, *moment of inertia of the axle*

$= \frac{1}{2}\times 8\times 10^3\times 5^2 = 10^3\times 10^2 = 10^5\ gm.\text{-}cm^2.$

$\therefore$ total moment of inertia of the wheel and axle, *i.e.*,

$I = 17\times 10^6 + 10^5.$ Or, $= 171\times 10^5\ gm.\text{-}cm^2.$

When the weight descends through a distance h, it loses potential energy $= mgh$, and this loss in *P.E.* of the weight is, obviously, equal to the gain in *K.E.* of the wheel and axle and the weight itself.

Now, since $m = 10\times10^3$ *gm.*, $g = 981$ *cm./sec²*. and $h = 20$ *cms.*, we have loss in *P.E.* of the weight $= 10\times10^3\times981\times10 = 1962\times10^5$ *ergs.*

If ω be the angular velocity of the wheel and axle, when the weight has descended through 20 *cms.*, the velocity v (*linear*) of the weight will be $r\omega$, where r is the radius of the axle ; *i.e.*, $v = 5\omega$. [$\because$ $r = 5$ *cms.*

$\therefore$ *K.E.* of the wheel and axle $= \frac{1}{2} I.\omega^2 = \frac{1}{2}\times171\times10^5\times\omega^2 = 855\times10^4\times\omega^2$ *ergs.*

and, *K.E.* of the weight $= \frac{1}{2}mv^2 = \frac{1}{2}\times10\times10^3\times(5\omega)^2 = 125\times10^2\times\omega^2$ *ergs.*

$\therefore$ total gain in *K.E.* of the wheel and the axle and the weight.

$= 855\times10^4\times\omega^2+125\times10^3\times\omega^2 = 8550\times10^3\times\omega^2+125\times10^3\times\omega^2$.

$= (8550+125)\times10^2\times\omega^2 = 8675\times10^3.\omega^2$ *ergs.*

Since total gain in *K.E.* of the wheel and axle and the weight is equal to the loss in *P.E.* of the weight, we have

log 1962 = 3·2927		$8675\times10^3.\omega^2 = 1962\times10^5$.
log 8675 = 3 9383	Or,	$\omega^2 = \frac{1962\times10^5}{8675\times10^3} = \frac{1962\times10^2}{8675}$,
$\frac{1}{2}\times\bar{1}\cdot3544$		
Antilog $\bar{1}\cdot6772$	whence,	$\omega = 10\times\sqrt{1962/8675}$.
$= \cdot4755$		$= 10\times\cdot4755$.
		$= 4\ 755$ *radians/sec.*

Or, *angular velocity* of the wheel and axle, *i.e.*, $\omega = 4\cdot755$ *radians/sec.*

and *linear velocity* of the weight, *i.e.*, $v = r\omega = 5\omega = 5\times4\ 755 = 23\cdot775$ *cms./sec.*

Now, *K.E.* of the wheel and axle is given by

log 855	=	2·9320
4 log 10	=	4 0000
2 log 4·755	=	$\bar{1}\cdot3544$
Antilog		8 2864
	=	$19\cdot34\times10^7$
log 5	=	0 6990
3 log 10	=	3·0000
log 25	=	1·3979
2 log 4·755	=	1·3544
Antilog		6·4513
	=	$28\cdot27\times10^5$

$\frac{1}{2} I.\omega^2 = \frac{1}{2}\times171\times10^5\times\omega^2$.

$= 855\times10^4\times(4\cdot755)^2 = 19\cdot34\times10^7$ *ergs.*

And, *K.E.* of the weight is given by

$\frac{1}{2} mv^2 = \frac{1}{2} m(5\omega)^2 = \frac{1}{2}\times10^1.(5\times4\ 755)^2$

$= 5\times10^3\times25\times(4\ 755)^2$.

$= 28\cdot27\times10^5$ *ergs.*

Thus, (*i*) the angular velocity and kinetic energy of the wheel and axle are 4 755 *radians/sec.* and 19 34$\times10^7$ *ergs* and (*ii*) the velocity and kinetic energy of the weight are 23·775 *cms./sec.* and $28\cdot27\times10^5$ *ergs* respectively.

10. If the pulley in an Atwood's machine be of moment of inertia 1500 c.g.s. units and radius 5 cms., what should be the acceleration of the system in which the weights at the two ends of the string passing over the pulley be 200 and 250 gms. respectively ? (Given that g = 981 cm./sec².)

Let a be the acceleration of the system and v, the velocity of the weights, when they have moved a distance S *cms.* from the starting position, (Fig. 63).

Then, clearly, $v^2-u^2 = 2aS$. Or, $v^2 = 2aS$. [$\because$ $u=0$.

And $\therefore$ $a = v^2/2S$.

Now, *loss in P.E of the heavier weight M*, falling through distance $S = Mg.S = 250\times981\times S$ *ergs*,

and, *gain in P.E. of the lighter weight m* $= mg.S = 200\times981\times S$ *ergs.*

$\therefore$ net *loss in P.E. of the system* $= (250\times981\times S)-(200\times981\times S) = 50\times981\times S$ *ergs.*

This must be equal to the gain in *K.E.* of the pulley as well as the weights themselves.

Clearly, gain in *K.E.* of the pulley $= \frac{1}{2} I\omega^2 = \frac{1}{2} I.v^2/r^2$.

$\because$ $v = r\omega$, where ω is the angular velocity and r, the radius of the pulley.

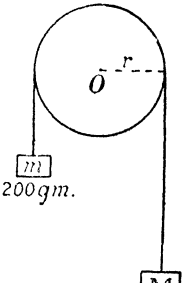


Fig. 63.

Or, *gain in K.E. of the pulley* $= \frac{1}{2} I.v^2/25 = \frac{1}{2} \times 1500 \times v^2/25 = 30v^2$ *ergs.*

And, *gain in K.E. of the two weights* $= \frac{1}{2}Mv^2 + \frac{1}{2}mv^2 = \frac{1}{2}(M+m)\ v^2$.

$= \frac{1}{2} \times 450 \times v^2 = 225\ v^2$ *ergs.*

$\therefore$ total gain in *K.E.* of the pulley and the weights,

$= 30\ v^2 + 225v^2 = 255v^2$ *ergs.*

Since this gain in *K.E.* = loss in *P.E.* of the system and the weights,

we have $255v^2 = 50 \times 981 \times S$. Or, $v^2 = \dfrac{50 \times 981 \times S}{255}$.

log 327 = 2·5145	Now, the acceleration of the system, *i.e.*, $a = v^2/2S$.
log 5 = 0·6990	
3·2135	$\therefore \quad a = \dfrac{50 \times 981 \times S}{255 \times 2S} = \dfrac{327 \times 5}{17}$.
log 17 = 1·2304	
Antilog 1·9831	$= 96{\cdot}18$ *cms./sec*^{2}.
= 96·18	

Or, the acceleration of the system of weights is 96·18 *cms /sec*^{2}.

11. A narrow uniform metal bar, 1 metre long, weighing 3 K. gms., rotates once per second. What is the K.E., if the axis of rotation passes through (*i*) its centre of gravity, (*ii*) one extreme end ?

Here, $M = 3$ *K.gms.* $= 3 \times 1000 = 3000$ *gms.*, and $l = 1$ *metre* $= 100$ *cms.*

$\therefore$ *M.I.* of the bar about an axis through its *c.g.*

$= Ml^2/12 = 3000 \times (100)^2/12$ *gm. cm*^{2}.

And, its *M.I.* about an axis through one end

$= Ml^2/3 = 3000 \times (100)^2/3$ *gm. cm*^{2}.

Angular velocity of the bar $\omega = 2\pi$ *radians/sec.* [$\because$ it is 1 *rotation/sec.*

Now, *K.E.* of a rotating body $= \frac{1}{2} I\,\omega^2$.

$\therefore$ *K.E.* of the bar in case (*i*) $= \frac{1}{2} \times [3000 \times (100)^2/12] \times (2\pi)^2$.

$= \frac{1}{2} \times [3000 \times 10000/12] \times 4\pi^2 = 500 \times 10000 \times \pi^2 = 5 \times 10^6 \times \pi^2$ *ergs.*

And, its *K.E.* in case (*ii*) $= \frac{1}{2}(3000 \times 10000/3) \times 4\pi^2$.

$= 2000 \times 10000 \times \pi^2 = 20 \times 10^6 \times \pi^2$ *ergs.*

12. Find the moment of inertia of a homogeneous circular cylinder of length $2l$, radius of cross-section r, about (*i*) the axis of the cylindrical symmetry ; (*ii*) a generating line ; (*iii*) a diameter of cross-section at a distance $x\ l$, and $2l$ (or 0) from one base.

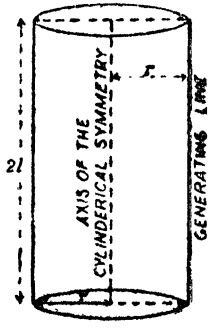


Fig. 64.

(*i*) The moment of inertia of the cylinder about the axis of cylindrical symmetry is the same as that of a disc about an axis passing through its centre and perpendicular to its plane, (for a cylinder is nothing but a thick disc), and is equal to $MR^2/2$, where M is the *mass* of the disc or cylinder and R, its *radius*.

$\therefore$ if M be the mass of the cylinder, and r, its radius of cross-section, (Fig. 64), we have *moment of inertia of the cylinder about its axis of cylindrical symmetry* equal to $Mr^2/2$.

(*ii*) The generating line is parallel to the axis of symmetry, passing through the *c.g.* of the cylinder, and is at a distance r from it. Therefore, by the *principle of parallel axes*, moment of inertia of the cylinder *about the generating line* is equal to its moment of inertia about the axis of symmetry *plus* mass of the cylinder $\times$ (distance from the axis)^2, *i.e.*, $= \frac{1}{2}Mr^2 + Mr^2 = 3Mr^2/2$.

(*iii*) (*a*) The moment of inertia of the cylinder about an axis, passing through its centre and perpendicular to its length

$= M[(length^2/12) + (radius^2/4)]$

$= M[(4l^2/12) + (r^2/4)]$ [$\because$ *length* $= 2l$.

$= M[l^2/3 + r^2/4]$

(*b*) Since an axis at a distance x from one base is at a distance $(l-x)$ from the axis through the centre, we have, by the *principle of parallel axes*,

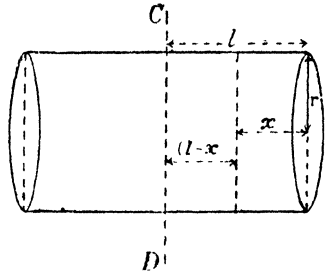


Fig. 65.

M.I. about this axis = *M. I.* about a parallel axis through the centre $+ M(l-x)^2$.

$= M[(l^2/3) + (r^2/4) + (l-x)^2]$

$= M[(4l^2/3) + (r^2/4) + (x^2-2lx)]$.

(*c*) Similarly, by *the principle of parallel axes*,

M.I. about the diameter of cross-section

$= M[(l^2/3) + (r^2/4)] + Ml^2 = M[(4l^2/3) + (r^2/4)]$.

13. Find the moment of inertia of a sphere about a diameter.

You are given two spheres of the same mass and size and appearance, but one of them is hollow at the centre and the other is solid throughout. How will you find which is hollow and which is solid ? (*Delhi*)

For answer to first part, see § 31, (case 13), pages 72.

The moment of inertia of a solid sphere about its diameter is, as we know, $2MR^2/5$, where M is its *mass* and R, its *radius*, and that of a hollow sphere $2/5[M(R^5-r^5)/(R^3-r^3)]$, where R and r are its *outer* and *inner radii* respectively, R and M being the *same* in the two cases.

$\therefore$ the radius of gyration for the *solid sphere* $= \sqrt{2R^2/5}$, and the radius of gyration for the *hollow sphere* is

$= \sqrt{2/5[R^5-r^5)/(R^3-r^3)]}$, their masses being the same.

Since acceleration of a body rolling down an inclined plane is given by

$$a = (R^2/R^2+K^2)\, g \sin \alpha,$$

[α being the inclination of the plane, (page,..)

it is clear that the greater the value of K^2 as compared with R^2, the less the acceleration of the body.

Now, K^2, for the *solid sphere* $= 2R^2/5$,

and K^2, for the *hollow sphere* $= \frac{2}{5}\left[\frac{R^5-r^5}{R^3-r^3}\right]$

$$= \frac{2}{5}\cdot\frac{R^5[1-r^5/R^5]}{R^3[1-r^3/R^3]} = \frac{2}{5}\cdot R^2\left[\frac{1-r^5/R^5}{1-r^3/R^3}\right]$$

Obviously, $r^5/R^5 < r^3/R^3$. And $\therefore$ $(1-r^5/R^5) > (1-r^3/R^3)$.

So that, the fraction, $\dfrac{1-r^5/R^5}{1-r^3/R^3} > 1$.

And $\therefore$ $K^2 = \frac{2}{5}\cdot R^2 \times$ *a quantity greater than* 1.

Or, K^2 *for a hollow sphere is greater than* $2R^2/5$.

Thus, the fraction (R^2/R^2+K^2) is less for a hollow sphere than for a solid sphere and, therefore, the acceleration of the hollow sphere is *less* than that of a solid sphere. In other words, the solid sphere will come down the inclined plane faster than the hollow sphere, and the two can thus be easily distinguished from each other.

EXERCISE III

1. Define Moment of Inertia and Radius of Gyration. Explain their physical significance. State the laws of (*i*) parallel and (*ii*) perpendicular axes and prove any one of them. (*Bombay, 1945*)

2. Calculate the moment of inertia of a thin circular disc of mass M and radius r (*i*) about its diameter (*ii*) about a parallel axis to the diameter and tangential to the disc. **Ans.** $Mr^2/4$; $5Mr^2/4$,

3. Define Moment of Inertia of a body about an axis. Show that the moment of inertia of a body about an axis through the centre of gravtty is less than that about any other parallel axis.

A uniform circular disc of radius r is free to oscillate in a vertical plane about an axis perpendicular to it and distant x from its centre. Calculate the periodic time. (*Madras, 1950*)

Ans. $t = 2\pi\sqrt{(r^2+x^2)/2gx}$.

4. A flywheel of mass $2\frac{1}{2}$ *tons* and diameter 8 *ft.* makes 250 revolutions *per minute.* Find (*i*) its *angular velocity* (*ii*) its *energy* (*iii*) its *moment of inertia.* Assume its mass to be concentrated at the rim.

Ans. (*i*) $25\pi/3$ *radians/sec.* (*ii*) $3\,07\times10^7$ *ft. poundals.* (*iii*) 89600 *lb.-ft*^{2}.

5. Show that the acceleration of a disc rolling down an inclined plane of angle θ is $2g\ sin\ \theta/3$, while that of a ball is $5g\ sin\ \theta/7$.

6. A uniform rod 4 *ft.* long and weighing 9 *lbs.*, revolves 60 times a minute about one end. Calculate its *kinetic energy.* **Ans.** 29 61 *ft. lbs.*

7. A hoop of mass 5 *k. gms.* and radius 50 *cms.* rolls along the ground at the rate of 10 *metres per second.* Calculate its kinetic energy in *ergs.*

Ans. 5×10^9 *ergs.*

8. Explain clearly what you understand by 'Moment of Inertia' and 'Angular momentum'. State the principle of conservation of angular momentum, illustrating your answer by an example.

Find the moment of inertia of a circular lamina about a tangent in its plane. (*Patna, 1949*)

Ans. $5Mr^2/4$.

9. A solid spherical ball rolls on a table. What fraction of its total kinetic energy is rotational ? **Ans.** 2/7th.

10. Show that the *K.E.* of a uniform cylinder or disc of mass M, rolling so that its centre has a velocity v is $\frac{3}{4}\ Mv^2$. In the case of a sphere, show that the *K.E.* would be $7Mv^2/10$.

11. A thin hollow cylinder, open at both ends and of mass M (*a*) *slides* with a velocity v *without rotating*, (*b*) *rolls without slipping*, with the *same* speed Compare the kinetic energies it possesses in the two cases. **Ans.** 1 : 2.

12. Define (*i*) radius of gyration, (*ii*) moment of inertia. Find the moment of inertia of a circular disc about the axis perpendicular to its plane.

A circular disc of mass m and radius r is set rolling on a table. If ω is its angular velocity, show that its total energy E is given by

$$E = \tfrac{3}{4}.\ mr^2.\omega^2.$$ (*Punjab, 1950*)

13. Derive an expression for the kinetic energy of a body rotating about an axis.

A flywheel is in the form of a uniform circular disc ; its radius is 2 *ft.*, and mass 2 *lbs.* Find the work which must be done on the flywheel to increase its speed of rotation from 10 to 20 revolutions per second. (*Madras B.A., 1947*).

Ans. $\frac{1}{2}I\omega^2$; $2\cdot366\times10^4$ *ergs.*

14. Five masses, each of 2 *k. gms.*, are placed on a horizontal circular disc, (of negligible mass) which can be rotated about a vertical axis passing through its centre. If all the masses be equidistant from the axis and at a distance 10 *cms.* from it, what is the moment of inertia of the whole system ?

Hint.: *M.I.* of each mass about the axis = Mr^2, and $\therefore$ total *M.I.* of the system = sum of the *M.I.* of the masses. **Ans.** 10^6 *gm. cm*^{2}.

15. Define 'Moment of Inertia' and 'Radius of Gyration.' State the law of parallel axes, and prove it.

A wheel of radius 6 *cms.* is mounted so as to rotate about an horizontal axis through its centre. A string of negligible mass, wrapped round its circumference carries a mass of 200 *gms.* attached to its free end. When let fall, the mass descends through 100 *cms.* in the first 8 seconds. Calculate the angular acceleration of the wheel and its moment of inertia. (*Bombay, 1947*)

Ans. Angular acceleration $= \frac{1}{8}$ *radians/sec*^{2}. ; *M.I.* $= 8{\cdot}748 \times 10^4$ *gm.-cm*^{2}.

16. What is meant by moment of inertia of a body? Show with necessary theory how the moment of inertia of a flywheel may be determined. (*Allahabad, 1948*)

17. The free end of a string wrapped round the axle of a flywheel, of moment of inertia $27{\cdot}61 \times 10^5$ *gm.-cm*^{2}., carries a weight of 5 *k.gms.*, which is allowed to fall. What is the number of revolutions *per second* made by the wheel, when the weight has fallen through 1 *metre*? The kinetic energy of the weight may be neglected. Ans. 3.

18. If in question 17, the wheel be mounted on an axle of half its moment of inertia (*i.e.*, $13{\cdot}80 \times 10^5$ *gm. cm*^{2}.) and radius 5 *cms.*, and the *K.E.* of the weight be taken into account, what will be the number of revolutions per second made by the wheel? Ans. 2·413.

19. Masses of 95 *gm.* and 105 *gm.*, hanging freely are connected by a light string which passes over a pulley of mass 20 *gm.* when released, the system moves with an acceleration of 46·7 *cm. per sec*^{2}. Calculate a value of *g* if the mass of the pulley is (*a*) neglected, (*b*) taken into account. Regard the pulley as a simple disc of moment of inertia $\frac{1}{2} Mr^2$, and assume that no kinetic energy is lost in friction. (*Northern Universities Higher School Certificate*)

Ans. (*a*) 934 *cm.sec*$^{-2}$., (*b*) 980·7 *cm.sec*$^{-2}$.

20. (*a*) Four spheres, each of diameter 2*a* and mass *m*, are placed with their centres on the four corners of a square of side *b*. Calculate the moment of inertia of the system about one side of the square. (*Punjab, 1951*)

(*b*) A flat thin uniform disc of radius *a* has a hole of radius *b* in it at a distance *c* from the centre of the disc, $[c < (a-b)]$. If the disc were free to rotate about a smooth circular rod of radius *b* passing through the hole, calculate its moment of inertia about the axis of rotation. (*Punjab, 1953*)

Ans. (*a*) $\frac{2}{5}\, m(4a^2+5b^2)$; (*b*) $M\,\frac{(a^2+b^2)}{2} + Mc^2$,

where *M* is the mass of the disc.

21. Describe the experiment to determine the moment of inertia of a flywheel. Derive the formula used in the experiment, without neglecting the friction at the bearings of the flywheel. (*Allahabd, 1948* ; *Gujrat, 1951*)

22. A flywheel, which can turn about a horizontal axis, is set in motion by a 500 gm. weight hanging from a thin string that passes round the angle. After the wheel has made 5 revolutions, the string is detached from the axle and the weight drops off. The wheel then makes 7 revolutions before being brought to rest by friction. The radius of the axle is 2·0 *cm.*, and at the instant when the weight drops off the angular velocity of the wheel is 10 radians per sec. Assuming that the work done against friction in each revolution is always the same, calculate the moment of inertia of the flywheel about its axis of rotation. (*Oxford and Cambridge Higher School Certificate*)

Ans. $3{\cdot}59 \times 10^5$ *gm.cm*^{2}.

23. A pair of rails is supported in a horizontal position and the axle of a wheel rests on the rails. A thread is wrapped round the axle and a weight hung on the end of the thread. As the weight falls the wheel moves along the rails. How would you determine the moment of inertia of the wheel with this arrangement?

24. A circular disc, starting from rest, rolls (without slipping) down an inclined plane of 1 in 8, and covers a distance of 5·32 *ft.* in 2 *secs.* Calculate the value of '*g*'. Ans. 31·92 *ft /sec*^{2}.

25. Two gear wheels, of equal thickness, of the same material and having radii in the ratio 2 : 1, are mounted on parallel frictionless spindles, but are separated so as not to mesh with one another. The larger wheel is set spinning

at a speed of 10 *rev. per sec.*, and the wheels are then brought into mesh. What is the resulting speed of each wheel ? (*Cambridge Schorlaship Examination*)

Ans. 8·9 and 17·8 *rev.* sec^{-1}.

26. What do you understand by the term "*precession*' ? Show that if the axis of the torque applied to a body be perpendicular to its axis of rotation, the body precesses about an axis perpendicular to either of the first two axes.

27. What is (*i*) a *gyrostat* and (*ii*) a *gyroscope* ? Describe suitable experiments to illustrate their action.

What is meant by the term *nutation* ?

28. Explain the theory underlying a *gyrostatic pendulum* and obtain an expression for its time-period.

29. Discuss in detail the case of a thin disc or hoop set rolling over a plane horizontal surface and obtain expressions for (*i*) its *critical velocity*, and (*ii*) the *radius of curvature of its path on the surface*.

30. Write short notes on the following :

(*i*) *Gyro-compass*. (*ii*) *Pendulum Gyro-compass*, (*iii*) *Rifling of barrels of guns* and (*iv*) *Precession of the Equinoxes*.

CHAPTER IV

SIMPLE HARMONIC MOTION

47. Definitions. A simple harmonic motion is a particular case of periodic motion, *i.e.*, a motion which repeats itself over and over again after regularly recurring intervals, called its *time-period*, and is so called because of its association with musical instruments. Common in nature, it is in fact the most fundamental type of periodic motion, as all other periodic motions, (harmonic as well as non-harmonic), can be obtained by a suitable combination of two or more simple harmonic motions.

If the acceleration of a body be proportional to its displacement from its position of equilibrium, or any other fixed point in its path and be always directed towards it, the body is said to execute a **simple harmonic motion,** (*written, for short, as S.H.M.*).

Now, a simple harmonic motion may be (*i*) *linear*, or (*ii*) *angular*, according as the body moves along a linear path, under the action of a constraining force constantly acting upon it, or rotates about an axis, under the action of a constant torque or couple.

The time-period of a body, executing a *S. H. M.*, is quite independent of the extent of its motion to either side of its mean position, (*i.e.*, of its amplitude), and the motion is, therefore, said to be **isochronous.**

Mathematically, *a linear S.H.M. may be regarded as the projection of a uniform circular motion, or of a rotating vector, on the diameter of the circle, or any other fixed line in the plane of the circle,—this circle being referred to as the* **circle of reference**, *and may, in many a case, be purely imaginary.*

Thus, if a particle P (Fig. 66), moves with a *uniform speed* v along a circle of radius a, and another particle M, along the diameter YOY', such that when P is at X, M is at O, and, as P starts along the circle in the *anti-clockwise* direction, M starts along OY, so that when P reaches Y, M also reaches Y. As P continues to travel further along YX', M starts back towards O.

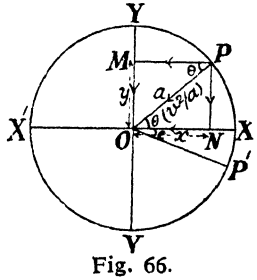


Fig. 66.

And, when P reaches X', M reaches O. As P now traverses the lower half of the circle along $X'Y'$, M proceeds downwards along OY', so that both P and M reach Y' together ; and, finally, when P travels further on along $Y'X$, M starts back along $Y'O$, reaching its mean position O when P reaches X.

Thus, the particle M moves along the diameter YOY' from O to Y, from Y to Y' and back to O (*i.e.*, completes one vibration), in the *same time* in which P moves *once* round the circle, such that, *at*

any given instant, the line joining the positions of P and M is perpendicular to the diameter YOY'; or, the position of M on the diameter YOY' corresponding to the position of P on the circle of reference, at any instant, is given by the foot of the perpendicular from P on to YOY'.

This particle M is said to be performing a linear S.H.M. along YOY', and is obviously, the *projection* of the particle *P*, moving uniformly along the *circle of reference XYX'Y'*, or is the projection of the rotating vector *OP* on the diameter *YOY'* of the circle.

If the motion of *M* be due, *not to P*, but to any other force *along its path*, the circle of reference will, as indicated above, be purely an *imaginary* one. Further, the path of *M* need not necessarily be straight and may as well be curved.

Since a force acting on a body is proportional to the acceleration it produces in it, it is obvious that the force acting on a body executing a *S. H. M.* must correspond to the changes in its acceleration. In other words, it must also be proportional to the displacement of the body from its mean position and must always be directed towards it.

Some familiar examples of simple harmonic motion.—

(*a*) **Linear.—**

(*i*) *The up and down oscillations of the piston of a cylinder, containing a gas, when suddenly pressed down and released,* (see solved example 1).

(*ii*) *The oscillations of mercury or water contained in a U-tube, when the column in one limb is depressed and released,* (see solved example 7).

(*iii*) *The vertical oscillations of an elastic string (or a spiral spring) suspended from a rigid support and loaded at its lower end,* (see solved example 10).

(*b*) **Angular.—**

(*i*) *The oscillations of a pendulum, provided the amplitude be small,* (see Chapter VI).

(*ii*) *The oscillations of a magnet suspended in a magnetic field,* (see solved example 8).

(*iii*) *Torsional oscillations, in general,* (see Chapter VIII).

48. Characteristics of a Linear S.H.M.

1. Amplitude. The maximum distance covered by the body on either side of its mean or equilibrium position is called its *amplitude.* It is, obviously, *equal to the radius of the circle of reference.*

Thus, the amplitude of the particle *M*, in the case above, is $OY = OY' = a$, the radius of the circle *XYX'Y'*.

2. Displacement. The distance of a body from its mean position, at any given instant, *measured along its path,* gives its *displacement* at that instant.

Thus, the displacement of *M*, in the position shown, (Fig. 66), is equal to *OM*, or *y*, such that

$OM = OP \sin \theta$. Or, $y = a \sin \theta$. (where $\angle XOP = \angle MPO = \theta$.

If ω be the angular velocity of P and t, the time taken by it in traversing the distance OP along the circle, *i.e.*, in describing the angle θ, we have $\theta = \omega t$; so that, $y = a \sin \omega t$.

This relation, giving the value of the displacement of a body, executing a *S.H.M.*, in terms of its amplitude and the angular velocity of the rotating vector, (or of the particle in the circle of reference) is referred to as its **equation of motion**. Thus, the equation of motion of M along YOY' is represented by $y = a \sin \omega t$.

If we consider the motion of a particle N along the diameter XOX', such that both P and N are together at X and as P goes round the circle in the anticlockwise direction, N starts along XOX', so that when P reaches Y, N reaches O, and when P reaches X', N also reaches X', and when P goes along the lower half of the circle, N starts back along $X'O$, reaching O when P reaches Y', and finally both arrive together at X, then the motion of N along XOX' is also a *S. H. M.* And its displacement $ON = x$ is clearly given by

$$ON = OP \cos \theta.$$

Or, $x = a \cos \theta$. [$\because ON = x$ and $OP = a$.

$\therefore$ its *equation of motion* is $x = a \cos \omega t$. [$\because \theta = \omega t$.

The position of the particle M executing a *S.H.M.* along the diameter YOY' may at any time, be found with the help of its **displacement curve**, *which is a graph, showing the relation between the time that elapses since the particle was at its mean position O, and its displacement from O during this time.*

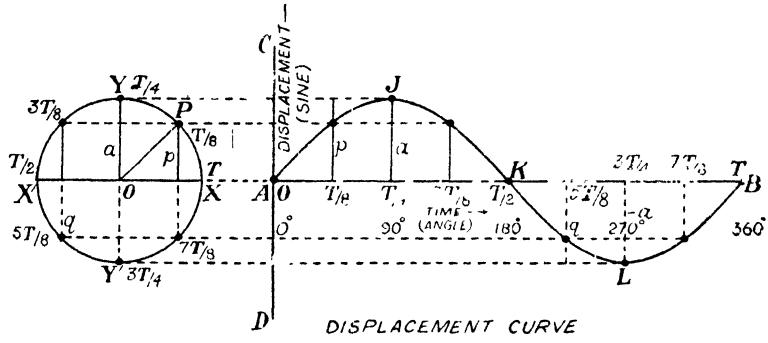


Fig. 67.

Let time be represented along the horizontal axis AB and displacement along the vertical axis DC, (Fig. 67).

Let the circle $XYX'Y'$ be divided up into an equal number of parts, say 8, representing equal intervals of time $T/8$, where T is the time taken by the particle P to go *once* round the circle. Let these intervals of time be also marked along the axis AB, taking A as the origin or the starting point. Then, the perpendiculars drawn from the points on the circle on to XOX' give the displacements of M along YOY', corresponding to the intervals of time represented by them, as shown in tabular form below :

Time	0	$T/4$	$T/2$	$3T/4$	T
Displacement	0 (*min.*)	a (*max.*)	0 (*min.*)	$-a$ (*max.*)	0 (*min.*)

Ordinates equal to these perpendiculars are then erected at the corresponding points on AB. Thus, the ordinate p represents the displacement of M after a time $T/8$ of its starting from O; the ordinate a, after time $T/4$; the ordinate q after time $5T/8$, and so on. The extremities of all these ordinates are then joined and a smooth curve $AJKLB$ is obtained, which is a *harmonic* or a *sine curve*, because it is of the same form as would be obtained for the relation between angles from 0° to 360°, and their sines, the maximum value, *viz.*, 1, being at 90°, and -1 at 270° and the least, *i.e.*, zero at 0°, 180° and 360°. The displacement curve shows at a glance how the displacement of the particle M changes along the diameter YOY' and its value can be readily obtained from the curve at any given instant.

3. Velocity. The velocity of the particle M is clearly given by the component of the velocity of P, along the diameter YOY', (Fig. 68).

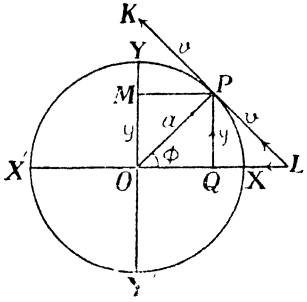


Fig. 68.

Now, the velocity of P is v in a direction *tangential to the circle at P.*

Representing it in magnitude and direction by the straight line PK, we may resolve it into two rectangular components PN and PM along and perpendicular to the diameter YOY', as shown. The component PN represents the velocity of M along YOY'.

Now clearly, component $PN = v \cos\phi = v \cos\omega t$. [$\because \phi = \omega t$.

Since $\cos\phi = OQ/OP = \sqrt{OP^2 - QP^2}/OP = \sqrt{a^2 - y^2}/a$, [$\because QP = OM = y$.

we have $\text{component } PN = v \cdot \sqrt{a^2 - y^2}/a = a\omega \sqrt{a^2 - y^2}/a$. [$\because v = a\omega$.

Or, $\text{velocity of } M = \omega\sqrt{a^2 - y^2}$.

Alternatively, the velocity of M may be obtained by differentiating its displacement y with respect to t (because *velocity is rate of change of displacement*). Thus, since $y = a \sin\omega t$, we have

$dy/dt = a\omega.\cos\omega t$. Or, $dy/dt = a\omega.\sqrt{a^2 - y^2}/a = \omega\sqrt{a^2 - y^2}$.

Or, $\text{velocity of } M = \omega\sqrt{a^2 - y^2}$.

Thus, the velocity of M would be different at different points along its path, depending upon its displacement, or distance from its mean position O,—*being a maximum when y is a minimum*, and a *minimum, when y is a maximum.*

So that (*i*) when $y = 0$, *i.e.*, when M is at O, (or, its displacement is *zero*), its velocity $= \omega\sqrt{a^2} = \omega.a = v$, *the same as that of P*;

and (*ii*) when $y = a$, *i.e.*, when M is at Y or Y', (*i.e.*, its displacement is *maximum*), its velocity $= \omega\sqrt{a^2 - a^2} = \omega.0 = 0$.

Or, the velocity of the particle varies inversely as its displacement.

It will thus be readily seen that the velocity of M varies from a maximum (v) at O to a minimum (*zero*) at Y, then increases from *zero*

to v at O, decreasing to *zero* again at Y', and again becoming v when it comes back to O. In other words, at time 0, it is maximum; at time $T/4$, a minimum; at $T/2$, a maximum; at $3T/4$, again a minimum, and finally at time T, again a maximum, as shown in the table below, where the velocities of M are shown at different times.

Time	0	$T/4$	$T/$	$3T/4$	T
Velocity	v (*max.*)	0 (*min.*)	$-v$ (*max.*)	0 (*min.*)	v (*max.*)

If, therefore, a graph be plotted between time and velocity of the particle M, we get the **velocity curve** of the particle, shown in Fig. 69, where time is shown along the horizontal axis and velocity along the vertical axis. The curve obtained is a *cosine curve*, for it is of the same form as the curve plotted between angles from 0° to 360° and their cosines.

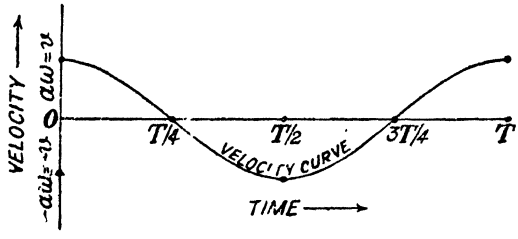


Fig. 69.

It should be noted that the maximum value of the velocity of the particle is $a\omega$ or v, and occurs when it passes its mean position to either side, or when its displacement is zero; and its velocity is zero, when it has attained its maximum displacement on either side.

4. Acceleration. As in the case of velocity, so also here, the acceleration of M is the resolved part of the acceleration of P along YOY', (Fig. 67). Now, the acceleration of P is v^2/a or $a^2\omega^2/a$, or $a\omega^2$, and is directed towards O. Resolving it into two rectangular components, along PM and MO, we have the component along MO equal to $a\omega^2.\sin\theta$ or $= a\omega^2.y/a = \omega^2 y$. And, as is clear from the figure, it is directed towards O, the mean or equilibrium position of M.

Thus, *acceleration of* $M = -\omega^2 y$,

the negative sign being put to indicate that it is directed toward O, in a direction opposite to that of y, its displacement.

Alternatively, we may obtain the acceleration of M by differentiating its velocity with respect to time, for *acceleration is the rate of change of velocity.*

Thus, since $dy/dt = a\omega.\cos\omega t$, [see page 114.

we have $d^2y/dt^2 = -a\omega^2.\sin\omega t = -\omega^2.a\sin\omega t = -\omega^2 y$.

Or, *acceleration of* $M = -\omega^2 y$.

Or, we may put it as *acceleration of* $M = -\mu y$*,

where $\omega^2 = \mu$, *a constant of proportionality.*

Or, *acceleration of* $M \propto y$.

*Similarly, in the case of angular *S.H.M*, we shall have *angular acceleration of the particle* $= \mu.\theta$, where θ is its angular displacement.
So that, here, *angular acceleration* $\propto \theta$.

Thus, we see that the acceleration of the particle M is proportional to y, its displacement from O, and is always directed towards it.

Obviously, *this acceleration has its maximum value, $a\omega^2$, the same as that of P, at the extreme positions Y and Y', where $y = a$, and its minimum value, i.e., zero, at O, where $y = 0$.*

It should be noted that the constant of proportionality μ is equal to ω^2, or the *square of the angular velocity* of the particle P in the circle of reference.

Further, if $y = 1$, acceleration of $M = \mu$. Thus, μ may be defined as the *acceleration per unit displacement of M.*

Tabulating acceleration of M against time, we have

Time	0	$T/4$	$T/2$	$3T/4$	T
Acceleration	0 (*min.*)	$-a\omega^2$ (*max*)	0 (*min.*)	$a\omega^2$ (*max.*)	0 (*min.*)

If, therefore, a graph be plotted between time and acceleration, we get the **acceleration curve** of the particle, as shown in Fig. 70, which is of a type similar to the *displacement curve*, (Fig. 67), but is reciprocal in form, *for acceleration is directed in the opposite sense to displacement.*

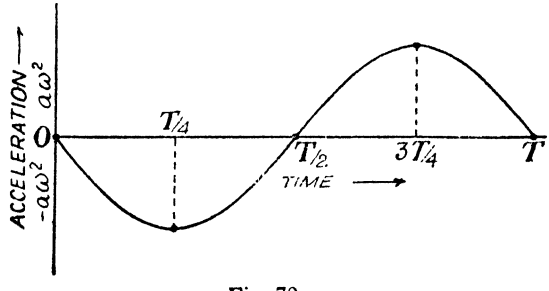


Fig. 70.

5. Time-Period and Frequency. The time taken by the particle M, in completing one vibration, (or one cycle), *i.e.*, in going from O to Y, Y to Y', and finally back to O, is called its *time-period, period of vibration* or *periodic time*, usually denoted by the letter T. Obviously, it is the *same* as the time-period of the particle P, *i.e.*, equal to the time taken by P in making *one* full round of the circle, (from X back to X), or in describing an angle 2π. Therefore,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega^2}} = \frac{2\pi}{\sqrt{\mu}}, \quad \left[\because \omega = \mu, \text{ the acceleration of } M \text{ per unit displacement.} \right.$$

whence, $$T = 2\pi\sqrt{\frac{1}{\mu}}.$$

Or, $$\textit{time-period of } M = 2\pi\sqrt{\frac{1}{\textit{acceleration per unit displacement}}}.$$

The number of vibrations made by the body per second is called the *frequency* of vibration of the body, and is denoted by the

letter n. Clearly, therefore,

$$\text{frequency, } n = 1/T = \sqrt{\mu}\,/2\pi.$$

$$= \frac{\sqrt{\textit{acceleration per unit displacement}}}{2\pi}.$$

6. Phase. The term, '*phase*' applied to a vibrating particle, has a meaning similar to the one associated with it when we talk of the '*phases*' of the moon. Just as the phase of the moon—*i.e.*, whether it is a *crescent*, (or new moon), *half moon* or *full moon*—tells us about its position etc., so also the phase of a particle, executing a *S.H.M.*, enables us to form an idea about its state of vibration.

Thus, the phase of a vibrating particle, at any given instant, may be defined as its state or condition as regards its position and direction of motion at that instant. It tells us in what stage of vibration the particle is.

It is indicated either (*i*) in terms of the angle θ, described by the rotating vector, measured as a fraction of the whole angle 2π that it describes in one full rotation, or (*ii*) in terms of time t that has elapsed since the particle last passed its mean position, *in the positive direction*, measured as a fraction of its time-period T.

Thus, taking O as the starting position of the particle M (Fig. 66), if its phase be zero, it indicates that the particle is at O, tending to move towards Y. And, if the phase be $\pi/2$, or $T/4$, it indicates that it is at Y, the position of the maximum positive displacement ; for, the radius of the circle of reference, or the rotating vector, has, up to this instant, described one-fourth of the total angle 2π, *i.e.*, an angle $2\pi/4$ or $\pi/2$; or that one-fourth of the time-period, *i.e.*, $T/4$, has elapsed since the particle last passed its mean position O in the positive or upward direction, towards Y.

Hence, when we talk of a **'phase difference'** between two simple harmonic motions, we mean to indicate how much the two are out of step with each other, or by how much angle, (measured as a fraction of 2π), or by how much time (measured as a fraction of T), one is ahead of the other.

Now, because the phase of a vibrating particle merely indicates its *actual stage of vibration*, it is clear that two vibrating particles, if they happen to be in *identical stages* of their respective vibrations, at any given instant, will be said to be in the *same phase*, at that particular instant, irrespective of their amplitudes and velocities being the same or different.

Thus, for example, they will be in the same phase, if they both simultaneously attain their maximum displacements, positive or negative ; or, when the two pass through their respective mean positions at the same time and in the same direction. Similarly, if one of the particles attains its maximum *positive* displacement simultaneously with the other particle attaining its maximum *negative* displacement, or when the two cross each other simultaneously in *opposite* directions at their mean positions, they are said to be in ***opposite phases.***

7. Epoch or Initial Phase. We have deduced the relation $y = a \sin \omega t$ for the displacement of the particle M, executing a *S.H.M.*, [§ 48, (2)] on the assumption that the starting position of the rotating vector is OX, or that the starting point of the particle in the circle of reference is X, *i.e.*, we start counting time when P crosses the axis of x at X.

Sometimes, however, the starting position of the rotating vector, or the position of the particle P in the circle of reference, is fixed, *not* in some standard position, as on the axis of x or y, but anywhere, *in an arbitrary manner*, such as at P', (Fig. 66), *i.e.*, the time is counted from the instant when P is at P', such that the angle $XOP' = e$.

Then, clearly, $\angle POP' = \omega t = \theta + e$. Or, $\theta = (\omega t - e)$.

So that, $y = a \sin \theta = a \sin (\omega t - e)$, where ωt is the *phase angle* of P. This angle e is called the **'epoch'** or the **'initial phase'** of the particle. It may also be measured, in terms of the time taken by the particle P in describing this angle, *i.e.*, by the time $(T/2\pi).e$.

It should not be confused with '*phase*' ; for, whereas the '*phase*' of the particle *continuously changes with time, its epoch or initial phase remains the same all through.*

49. Equation of Simple Harmonic Motion. Let y be the *displacement* from its mean position of a particle, executing a *S.H.M.* Then, if v be its *velocity* at that instant, we have

$$v = dy/dt.$$

So that, *acceleration of the particle* $= \dfrac{d^2y}{dt^2} = \dfrac{dv}{dt} = \dfrac{dv}{dy} \times \dfrac{dy}{dt} = v.\dfrac{dv}{dy}$.

Now, acceleration is also given by $-\mu.y$, where μ is the constant of proportionality and is equal to ω^2, (ω being the angular velocity of the rotating vector, or the particle in the circle of reference).

Thus, $d^2y/dt^2 = -\omega^2.y.$
Or, $v.dv/dy = -\omega^2.y,$
whence, $v.dv = -\omega^2.y.dy.$

{ the negative sign indicating that acceleration and displacement (y) are *oppositely directed.*

Integrating this expression, we have

$$\int v.dv = \int -\omega^2.y\, dy = -\omega^2 \int y.dy.$$

Or, $$\tfrac{1}{2}v^2 = -\tfrac{1}{2}\omega^2 y^2 + C, \qquad \ldots(i)$$

where C is the *constant of integration*, and has to be determined from the condition of the particle at the instant considered.

Obviously, *the velocity of the particle is zero, when it has its maximum displacement* a, or $-a$, *i.e.*, $v = 0$, when $y = a$, or $-a$.

So that, we have, from relation (i), above,

$$0 = -\tfrac{1}{2}\omega^2 a^2 + C. \qquad \text{Or,} \qquad C = \tfrac{1}{2}\omega^2 a^2.$$

Hence, $\tfrac{1}{2}v^2 = -\tfrac{1}{2}\omega^2 y^2 + \tfrac{1}{2}\omega^2 a^2 = \tfrac{1}{2}\omega^2(a^2 - y^2)$. Or, $v^2 = \omega^2(a^2 - y^2)$.

And $\therefore$ $$v = \omega\sqrt{a^2 - y^2}. \qquad \ldots (ii)$$ [See § 48 (3).

Thus, the velocity of the particle can be determined for any value of its displacement y. Clearly, *the value of* v *is the maximum, i.e.,* $= \omega a$, *when* $y = 0$, *i.e., when the displacement is zero, or the particle is in its mean or equilibrium position ;* and, *it is a minimum, i.e.,* 0, *when* $y = a$, or, $-a$, *i.e.,* when the *particle has its maximum displacement, positive or negative.*

Now, since $v = dy/dt$, we have, from (*ii*) above,

$$dy/dt = \omega\sqrt{a^2-y^2}.$$

Or, $$\frac{dy}{\sqrt{a^2-y^2}} = \omega.dt.$$

Integrating this, we have $\int \frac{1}{\sqrt{a^2-y^2}}\,dy = \int \omega.dt.$

Put $y = a \sin\theta$; then, $dy = a, \cos\theta.d\theta$, and $\sqrt{a^2-y^2} = a\cos\theta$, So that, $\int \frac{dy}{\sqrt{a^2-y^2}} = \int d\theta = \theta = \sin^{-1}\frac{y}{a}$.

Or, $$\sin^{-1} y/a = \omega t + C',$$

where C' is another *constant of integration.*

Or, $$y = a \sin(\omega t + C'). \qquad \ldots(iii)$$

(*a*) Now, *if we start counting time when* $y = 0$, *i.e., when the particle is in its mean position, moving in the positive direction, i.e.,* $y = 0$, *when* $t = 0$, we have, from relation (*iii*) above

$$0 = a \sin(0 + C').$$

Or, $$a \sin C' = 0, \text{ whence, } C' = 0.$$

$\therefore$ substituting this value of C' in (*iii*) above, we have $y = a \sin \omega t$.

(*b*) *If, on the other hand, we start counting time when the particle has its maximum displacement, i.e.,* $t = 0$, *when* $y = a$, we have

$$a = a \sin(0 + C'), \text{ or, } \sin C' = a/a = 1 \text{ or } C' = \pi/2 ;$$

so that, in this case, $y = a \sin(\omega t + \pi/2)$. Or, $y = a \cos \omega t$.

(*c*) Again, *if we start counting time from an instant* t' *before the particle has passed through its mean or equilibrium position,* we have $t = t'$, when $y = 0$. Therefore, from (*iii*) above,

$$0 = a \sin(\omega t' + C'), \text{ or, } \omega t' + C' = 0.$$

Or, $$C' = -\omega t' = -e,$$

where e is the **epoch** of the particle in the circle of reference.

Substituting this value of C' in (*iii*), we have $y = a \sin(\omega t - e)$. ...(*iv*)

(*d*) And, *if we start counting time from an instant* t' *after the particle has passed through its mean position,* we have

$$y = 0, \text{ when } t = -t', \text{ or, } 0 = a \sin(-\omega t' + C'),$$

whence, $$-\omega t + C' = 0, \text{ or, } C' = \omega t' = e.$$

And, therefore, $$y = a \sin(\omega t + e).$$

A mere glance at the above relations for y indicates that these simple harmonic vibrations of the particle are a case of *periodic motion.*

(*e*) Now, *if the time* t *be increased by* $2\pi/\omega$, we have, from relation (*iv*) above,

$$y = a \sin[\omega(t + 2\pi/\omega) - e] = a \sin(\omega t + 2\pi - e),$$

whence, $y = a \sin(\omega t - e)$, *i.e., the same as be ore,* in *(iv)*,

showing that the position and direction of motion of the particle is the same as $2\pi/\omega$ *seconds* earlier, *i.e., the particle repeats its movements after every* $2\pi/\omega$ *seconds.* In other words, *the time-period of the particle is* $2\pi/\omega$. Further, since this value of the time-period is quite independent of a and e, it is clear that the vibrations or oscillations of the particle are also *isochronous.*

The above results will also be true for *angular S.H.M.*, if we consider *angular displacement, acceleration, velocity,* etc., in place of the linear ones.

Important Note.—We have seen how the acceleration of a particle executing a *S.H.M.* is given by

$$\frac{d^2y}{dt^2} = -\omega^2.y.$$

The general solution of this equation is of the form

$$y = a \sin \omega + b \cos \omega t.$$

Thus, if the displacement of a vibrating particle be given by a relation of the form $y = a \sin \omega t + b \cos \omega t$, it is executing a *Simple Harmonic Motion.*

Clearly, as t takes up the values $0, 2\pi/\omega, 4\pi/\omega, 2n\pi/\omega$, etc., ωt assumes the values $0, 2\pi, 4\pi, 2n\pi$, etc., with y assuming the same value over and over again.

In other words, the time-period of the motion $= 2\pi/\omega$.

Further, this equation can easily be reduced to the simple *sine form* as follows :

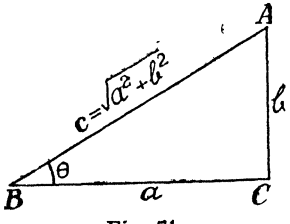


Fig. 71.

Let $\theta = \tan^{-1}\frac{b}{a}$, (Fig. 71).

Then, clearly, $a = c \cos\theta = \sqrt{(a^2+b^2)}\cos\theta$

and $b = c \sin\theta = \sqrt{(a^2+b^2)}.\sin\theta$.

Now, $y = a \sin \omega t + b \cos \omega t$.

So that, substituting the values of a and b, obtained above, we have

$$y = \sqrt{a^2+b^2}\sin\omega t \cos\theta + \sqrt{(a^2+b^2)}.\cos\omega t \sin\theta.$$

Or, $$y = \sqrt{a^2+b^2}\,(\sin\omega t \cos\theta + \cos\omega t \sin\theta).$$

Or, $$y = \sqrt{a^2+b^2}\sin(\omega t+\theta),$$

which is the usual form of the displacement of a body executing *S.H.M.*

Obviously, the displacement will have the maximum value $\sqrt{a^2+b^2}$, when $(\omega t+\theta) = 90°$ and, therefore, $\sin(\omega t+\theta) = 1$.

And, since *the maximum displacement of the particle is equal to*

its amplitude, we have

amplitude of the vibrating particle, here $= \sqrt{a^2+b^2}$.

Again, the velocity of the particle is given by

$$v = \frac{dy}{dt} = \sqrt{a^2+b^2}.\ \omega \cos(\omega t+\theta)\ ;$$

so that, the *maximum value of* $v = \sqrt{a^2+b^2}.\ \nu$.

And, finally, *acceleration of the particle* $= \frac{d^2y}{dt^2}$.

$$= -\omega^2\sqrt{a^2+b^2}.\sin(\omega t+\theta).$$

Hence, *maximum value of acceleration of the particle*

$$= \omega^2\sqrt{a^2+b^2}.$$

50. Composition of Two Simple Harmonic Motions. Just as a particle may be subjected to two forces or two velocities simultaneously, so also we may have a particle under the action of two simple harmonic motions *at the same time*. Its final motion will then be the resultant of the two simultaneous simple harmonic motions impressed upon it. It does not, of course, mean that it will execute both the motions simultaneously, any more than a particle, having two velocities impressed upon it, will move in both the directions at the same time. All it means is that its resulting motion would be one as though it were simultaneously executing the two motions together.

It should be clearly understood, however, that the simultaneous execution of two rectilinear simple harmonic motions by a particle is no guarantee that the resultant motion of the particle will necessarily be rectilinear or harmonic. Indeed, if their time-periods be incommensurable, it may not even answer to the definition of a vibration.

We shall now take up first the simpler case of the composition of a *S.H.M.* along one direction with a linear motion in a perpendicular direction and then pass on to the composition of simple harmonic motions along the same straight line and at right angles to each other, both graphically and analytically.

1. Graphical Method.

(*i*) **Composition of a S.H.M. with a Uniform Linear Motion perpendicular to it.** The resultant motion will, in this case, be a *sine curve*. This may be easily seen by attaching a small spike or style to the prong of a tuning fork (at right angles to its length) and then drawing it* uniformly over a smoked plate of glass, with the style just touching the plate, in a direction at right angles to that of the vibrations of the fork. It will be found that a series of sine curves are traced out on the plate, with the direction of motion of the fork as the horizontal or the *time-axis* and the direction of vibration of the prong as the vertical or the *displacement-axis*.

Composition of two linear simple harmonic motions in the same direction. If two simple harmonic motions take place in the same direction, their resultant is also a simple harmonic motion, defined by the resultant of the vectors which define the two motions, this *resultant vector* being obtained by the ordinary law of vector addition. This will be clear from the following :

*Or, holding the fork in position and moving the plate.

Let two simple harmonic motions, having the *same time-period* but *different amplitudes and phases*, be represented by the projections of the vectors

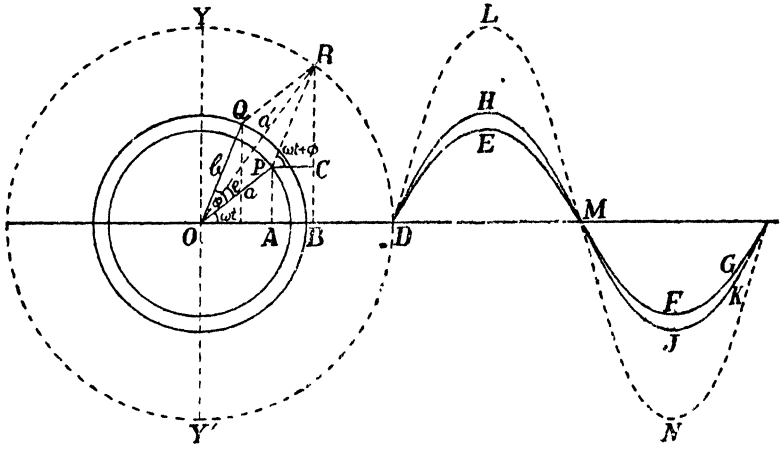


Fig. 72.

OP and *OQ* respectively, on the axis of *y*, (Fig. 72), and let their equations of motion be

$$y = a \sin \omega t \quad \text{and} \quad y = b \sin (\omega t + \phi),$$

where *a* and *b* are their amplitudes, and ωt and $(\omega t + \phi)$ equal to angles *AOP* and *AOQ* respectively.

Then, if *DEFG* be the *sine-curve* for the first motion, and *DHJK* for the second, *we obtain the sine-curve DLMN* for *the resultant motion by adding up the ordinates of the two curves at all points*, because the displacements of the two are in the same direction and can be added up algebraically.

Now, the curve *DLMN* is the same as would be obtained for the rotation of the resultant vector *OR*, whose projection on the axis of *y*, therefore, gives the resultant of the two motions.

For,
$$RB = CB + RC = PA + RC.$$
$$= OP.\sin AOP + PR.\sin CPR.$$
$$= a \sin \omega t + b \sin (\omega t + \phi),$$

because $OP = a$, $PR = OQ = b$, $\angle AOP = \omega t$ and $\angle CPR = \angle AOQ = \omega t + \phi$.

Thus, the resultant motion is also a S H.M. and takes place along the same line and, (since the rotating vectors *OP*, *OQ* and *OR* have the same velocities), *it has the same time-period as the two component motions.*

The amplitude *a'* of the resultant motion is, clearly, equal to *OR*.

Now,
$$OR^2 = OP^2 + OQ^2 + 2OP.OQ.\cos POQ.$$
[By the ‖ *gram law of vectors,*

Or,
$$a'^2 = a^2 + b^2 + 2ab.\cos \phi,$$

whence,
$$a' = \sqrt{a^2 + b^2 + 2ab.\cos \phi}.$$

Now, if $\phi = 0$, *i.e.*, if there be no phase difference between the two motions, $\cos \phi = 1$, and, therefore,

$$a' = \sqrt{a^2 + b^2 + 2ab} = (a + b),$$

i.e., equal to the algebraic sum of the amplitudes of the two component motions.

The *phase angle* of the resultant motion is, obviously, given by the angle *ROB*, such that

$$\tan ROB = \frac{RB}{OB} = \frac{CB + RC}{OA + AB} = \frac{PA + RC}{OA + PC}.$$

Or, $$\tan ROB = \frac{a \sin \omega t + b \sin (\omega t + \phi)}{a \cos \omega t + b \cos (\omega t + \phi)}.$$

Now, $$\angle ROB = \angle POA + \angle POR = \omega t + e,$$

where $\angle POR = e$, the *phase angle* by which the resultant motion is ahead of the first motion.

$$\therefore \quad \tan ROB = \tan (\omega t + e) = \frac{a \sin \omega t + b \sin (\omega t + \phi)}{a \cos \omega t + b \cos (\omega t + \phi)}.$$

$$= \frac{a \sin \omega t + b \sin \omega t \cos \phi + b \cos \omega t \sin \phi}{a \cos \omega t + b \cos \omega t \cos \phi - b \sin \omega t \sin \phi}.$$

Now, *just at the start*, $t = 0$, and $\therefore \omega t = 0$; so that,

$$\sin \omega t = 0, \text{ and } \cos \omega t = 1.$$

Hence $$\tan e = \frac{b \sin \phi}{a + b \cos \phi}.$$ Or, $$e = \tan^{-1} \frac{b \sin \phi}{a + b \cos \phi}.$$

Thus, the resultant motion is ahead of the first motion by a phase angle e,

where $$e = \tan^{-1} = \frac{b \sin \phi}{a + b \cos \phi}.$$

(*ii*) **Resolution of a S. H. M. into two components in the same direction.** The converse of the above is also true, *viz.* that *a simple harmonic motion may be resolved into two , by resolving its rotating vector into two vectors, in accordance with the law of resolution of vectors, each vector defining a component simple harmonic motion.*

(*iii*) **Composition of two linear simple harmonic motions at right angles to each other.** The resultant of two, *S.H.M*'s, impressed simultaneously on a particle, along directions at right angles to each other, is a curve lying in the plane containing the two motions and its character depends upon the amplitudes, time-periods (or frequencies) and the phase difference of tee two component motions. Let us consider the different cases that arise.

(*a*) **When the time-periods (or frequencies) of the two motions and their phases are the same, but their amplitudes are different.** Let the two motions be defined by the rotation of the vectors in circles (*i*) and (*ii*) respectively, (Fig. 73), *i.e.*, let (*i*) and (*ii*) be the *circles of reference* of the two motions, with radii equal to their amplitudes respectively, say *a* and *b*. Divide the two circles into a number of equal parts in the ratio of the frequencies of the two motions, in this case, 1 : 1, as shown, (each circle being divided into eight equal parts, for the sake of convenience), *the starting point of the rotating vector being marked zero.* Then, draw straight lines passing through points bearing the *same numerals* in the two circles, and parallel to the axes *OX* and *OY* respectively, along which the motions take place in the two cases.

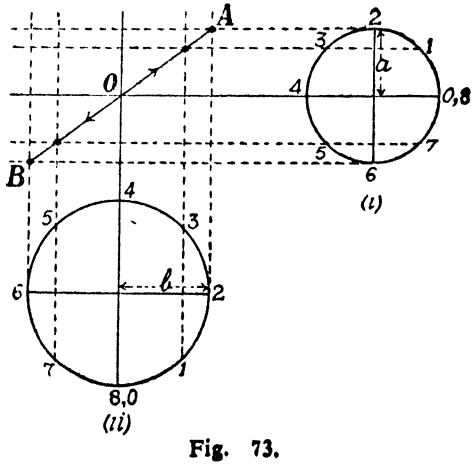


Fig. 73.

Mark the points where these lines intersect and join all these points of intersection. It will be found that, in this case, the straight line *AB* is obtained as the path along which the resultant motion takes place, the arrow heads indicating the direction of motion about *O*. And, as will be readily seen, this straight line is the diagonal of the rectangle with sides $2a$ and $2b$; and the amplitude of the resultant vibration of the particle, *i.e.*, *OA* or *OB* is, therefore, clearly equal to

$$\sqrt{a^2+b^2}.$$

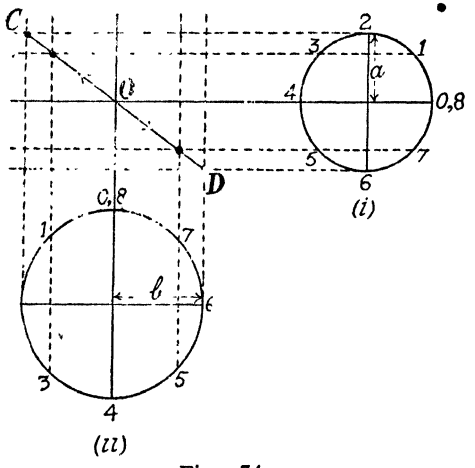


Fig. 74.

(*b*) **When the time-periods or frequencies are the same, plitudes are different and phase difference is** π. Here, the starting position ,(Fig. 74), of the vector in circle (*ii*) is at the top of the circle, as shown, instead of at the bottom, (as in the first case),—the second motion being ahead of the first by a distance equal to *half its path*,—the other numerals being shifted accordingly. Again, drawing straight lines through the same numerals and parallel to the corresponding axes *OX* and *OY*, along which the two motions take place, and joining their points of intersection, we get the straight line *CD*, *inclined in the opposite direction*, showing that the resultant motion is again a *straight line motion*, about *O*, but *inclined the other way*, (*i.e.*, the other diagonal of the rectangle of sides $2a$ and $2b$), the direction of motion of the particle being as indicated by the arrow-heads.

(*c*) **When the time-periods are the same, amplitudes different, and the phase difference is** $\pi/4$. We again proceed exactly as above, with the only difference that, here, we shift the zero, or the starting position of the radius vector, by *one-eighth* of its path in the case of the second (lower) circle of reference, (Fig. 75), the second motion being ahead of the first by $\pi/4$,

Joining smoothly the points of intersection of the straight lines through the same numerals, parallel to the two axes respectively, we get an *oblique ellipse* as the resultant path of motion of the particle, the direction of motion along it being indicated by the arrow-head.

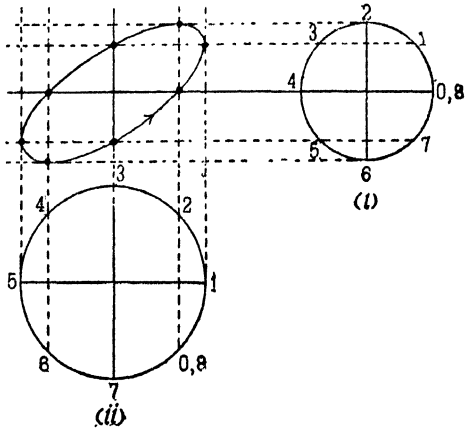


Fig. 75.

(*d*) **When the time-periods are the same, amplitudes different, and the phase difference is $\pi/2$.** In this case, the starting point of the radius vector in the second circle of reference is taken a *quarter of its path* ahead of its original position, the phas difference being $\pi/2$.

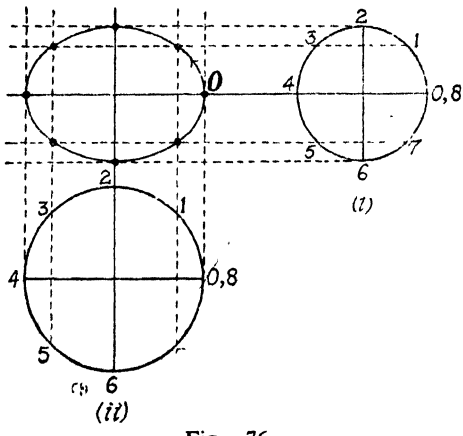


Fig. 76.

Then, proceeding as before, we get an *ellipse* as the path of the resultant motion, *with its axes coincident with the directions of the component motions*, the starting point being *O* and the direction *anticlockwise*, as shown in Fig. 76

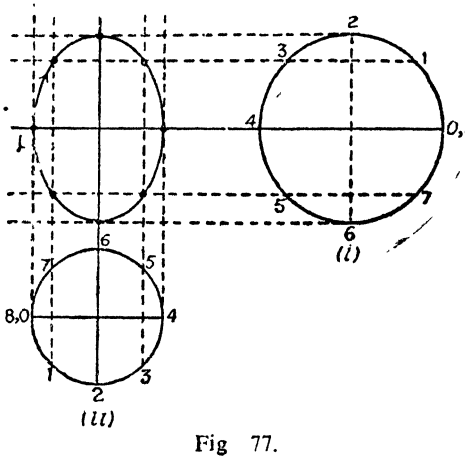


Fig 77.

(*e*) **When the time-periods or frequencies are the same, amplitudes different, and the phase difference is $3\pi/2$,** Here, (Fig. 77), the starting point of the radius vector in the second circle is taken *three-fourths of its path* ahead of the original position., and we get, as in the last case, an *ellipse* as the resultant path the direction of motion being *clockwise*, as shown, and the starting point being *O*.

(*f*) **When the time-periods or the frequencies are the same, amplitudes equal, and the phase difference is $3\pi/2$.** In this case, (Fig. 78), we take both the circles of reference *of the same radii* and proceed as in case (*c*), when a *circle* is obtained as the path of the resultant motion, the direction of travel along it being *anticlockwise* and the starting point being *O*.

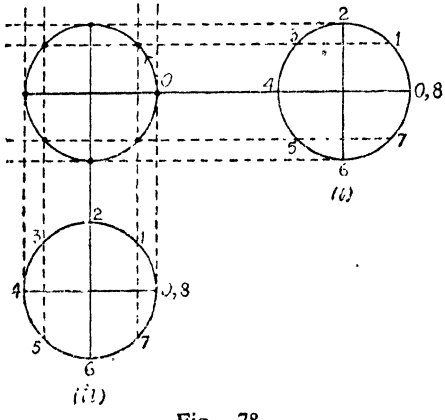


Fig. 78.

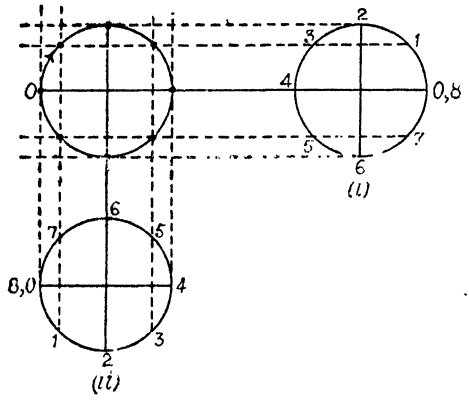


Fig. 79.

(*g*) **When the time-periods or frequencies are the same, amplitudes equal and the phase difference is $3\pi/2$** —Here, again, we take the radii of the two circles to be the same, and proceed as in case (*d*), when we obtain a *circle*, as the path of resultant motion the direction of travel along it being *clockwise*, in this case, as shown, (Fig. 79), and the starting point being *O*.

(*h*) **When the time-periods or frequencies are in the ratio of 2 : 1, amplitudes are different, and the phase difference is zero.** In this case, (Fig. 80), we divide the two circles into equal parts, *in the ratio* 2 : 1, (*e.g.*, the first one into 8 parts, and the second one into four parts). Then, proceeding as before, we get the path of the resultant motion of *the form of the figure* 8, as shown, the direction of motion along it being indicated by the arrow-heads, and the starting point being *O*.

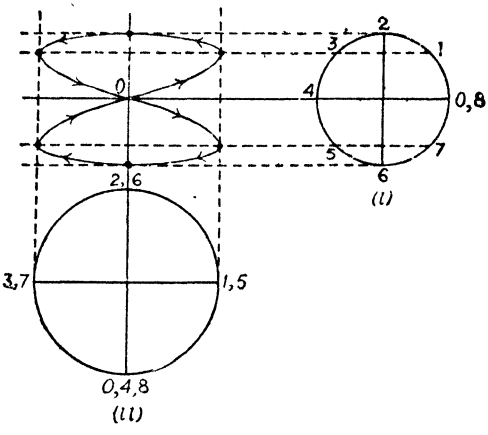


Fig. 80.

(*j*) **When the time-periods or frequencies are in the ratio 2:1, amplitudes are different and there is an initial phase difference equal to a quarter of the smaller time-preiod.** As in the case above, we divide the two circles here also into equal parts, in the ratio 2 : 1, but shift the *zero* of the second circle, *one-fourth part* ahead. Then, proceeding as before, we obtain, in this case, a *parabola* as the resultant path of motion of the particle, (Fig. 81), the direction of motion being as indicated by the arrow-heads.

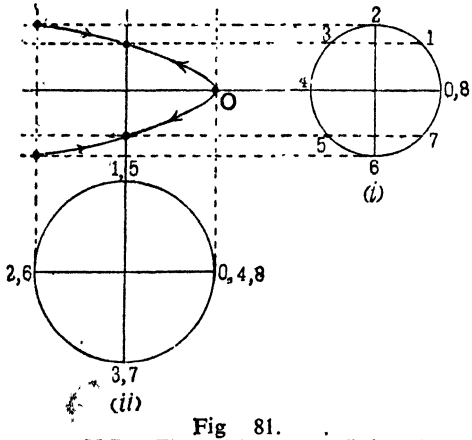


Fig 81.

N.B —The epithet '*initial*' has been deliberately used here with '*phase difference*' to emphasize that the time-periods being different, *the phase does not remain constant*, even though we start with the same phase originally. Inevitably, therefore, a phase difference comes in between the two motions.

Precisely in the same manner, we can obtain the path of the resultant motion of a particle, subjected simultaneously to two simple harmonic motions, perpendicular to each other, whatever their frequency ratio or the phase difference between them.

The student may, as an exercise, try to determine the resultant path of a particle, subjected to two simple harmonic motions, at right angles to each

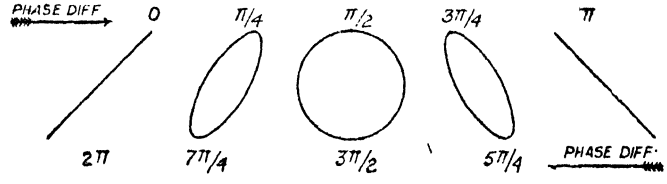


Fig. 82.

other, with the time-periods and amplitudes equal, but with phase difference changing from 0 to 2π, when he will find that, as the phase difference changes from 0 to π, the resultant path changes from a *straight line, inclined one way*, through an *oblique ellipse* inclined the same way, a *circle*, and, again, an *ellipse, inclined the other way*, to finally, *a straight line, inclined at right angles to the first one*, as shown in Fig. 82. And, as the phase difference changes from π to 2π, the same figures are repeated in the reverse order, as shown.

The superposition of such rectangular vibrations is of particular importance in the subject of sound, since it serves as a test for the equality of the periods of two vibrating bodies like tuning forks etc. The method was first adopted by *Lissajous* and hence the various curves thus obtained, *viz.*, those in Figs. 73 to 81 and others, are usually referred to as *Lissajous' figures*.

II. Analytical method.

(1) Composition of two linear simple harmonic motions along the same line. Let two simple harmonic motions, having the *same time-period*, be represented by the equations.

$$y_1 = a \sin \omega t \quad \text{and} \quad y_2 = b \sin (\omega t + \phi),$$

where ϕ is the phase angle by which the second motion is ahead of the first.

The phase difference will throughout remain constant, because the time-periods of the two motions are the same.

Now, since the two displacements are along the same line, the resultant displacement y will, at any given instant, be equal to the algebraic sum of the displacements of the two component vibrations.

Thus, $$y = y_1 + y_2.$$

Or, $$y = a \sin \omega t + b \sin (\omega t + \phi).$$
$$= a \sin \omega t + b \sin \omega t \cos \phi + b \cos \omega t \sin \phi.$$

Or, $$y = \sin \omega t \ (a + b \cos \phi) + \cos \omega t . b \sin \phi.$$

Putting $(a + b \cos \phi) = a' \cos e$ and $b \sin \phi = a' \sin e$, we have

$$y = a' \sin \omega t \cos e + a' \cos \omega t \sin e.$$

Or, $$y = a' \sin (\omega t + e),$$

i.e., the resultant motion is also a S.H.M., along the same line, and has the same time-period, its amplitude being a', and its phase angle e, by which it is ahead of the first motion.

The values of a' and e may be deduced as follows :

We have $$a' \sin e = b \sin \phi \quad \text{and} \quad a' \cos e = (a + b \cos \phi).$$

So that, squaring and adding the two, we have

$$a'^2 \sin^2 e + a'^2 \cos^2 e = b^2 \sin^2 \phi + a^2 + b^2 \cos^2 \phi + 2ab \cos \phi.$$

Or, $$a'^2 (\sin^2 e + \cos^2 e) = a^2 + b^2 (\sin^2 \phi + \cos^2 \phi) + 2ab \cos \phi.$$

Or, $$a'^2 = a^2 + b^2 + 2ab \cos \phi. \quad [\because \sin^2 e + \cos^2 e = 1 ; \text{ also, } \sin^2 \phi + \cos^2 \phi = 1.$$

Or, $$a' = \sqrt{a^2 + b^2 + 2ab \cos \phi}.$$

And $$\tan e = \frac{\sin e}{\cos e} = \frac{a' \sin e}{a' \cos e} = \frac{b \sin \phi}{a + b \cos \phi}.$$

Or, $$e = \tan^{-1} \frac{b \sin \phi}{a + b \cos \phi}.$$ [See page 123]

Now, (*i*) if $\phi = 0$, *i.e., if the two motions be in the same phase,* we have $e = 0$; so that, $y = a' \sin \omega t$.

Or, *the resultant motion will also be in phase with the two component motions,* with its *amplitude* given by

$$a' = \sqrt{a^2 + b^2 + 2ab} = \sqrt{(a+b)^2} = (a+b), \quad [\because \cos 0 = 1.$$

i.e., equal to the sum of the amplitudes of the two component motions.

[See page 122.

And (*ii*) if $\phi = \pi$, *i e., if the two motions be in opposite phases,* we have again $e = 0$. So that, again, $y = a' \sin \omega t$.

Or, the resultant motion will be in phase with the first motion, with its *amplitude* now given by

$$a' = \sqrt{a^2 + b^2 - 2ab} = \sqrt{(a-b)^2} = (a-b), \quad [\because \cos \pi = -1.$$

i e., equal to the difference between the amplitudes of the two motions.

Further, in this case, *if* $a = b$, *i.e., if the amplitudes of the two component motions be also equal,* we shall have $a' = a - b = 0$.

Or, *the amplitude of the resultant motion will be zero.* In other words, *there will be no resultant motion at all.*

Note.—In the above treatment, we have, for the sake of simplicity, taken one motion a phase angle ϕ ahead of the other.

The same result may be obtained, however, if we take the phase angles of the two motions to be ϕ_1 and ϕ_2 respectively. For, in this case, we have

$$y_1 = a \sin (\omega t + \phi_1) \text{ and } y_2 = b \sin (\omega t + \phi_2).$$

So that, $$y = y_1 + y_2 = a \sin (\omega t + \phi_1) + b \sin (\omega t + \phi_2).$$

$$= a \sin \omega t \cos \phi_1 + a \cos \omega t \sin \phi_1 + b \sin \omega t \cos \phi_2 + b \cos \omega t \sin \phi_2.$$

$$= \sin \omega t (a \cos \phi_1 + b \cos \phi_2) + \cos \omega t (a \sin \phi_1 + b \sin \phi_2).$$

Now, putting $(a \cos \phi_1 + b \cos \phi_2) = a' \cos e$ and $(a \sin \phi_1 + b \sin \phi_2) = a' \sin e$, we have $y = a' \sin \omega t \cos e + a' \cos \omega t \sin e$.

Or, $y = a' \sin (\omega t + e)$, as before ;

i e., the resultant motion, is also *simple harmonic,* with the same time-period as of the two component motions.

And, $$\tan e' = \frac{a' \sin e}{a' \cos e} = \frac{a \sin \phi_1 + b \sin \phi_2}{a \cos \phi_1 + b \cos \phi_2}.$$

Again, the values of the amplitude a' of the resulting motion may be obtained in exactly the same manner, as before.

Thus, $$a'^2 \sin^2 e + a'^2 \cos^2 e = (a \sin \phi_1 + b \sin \phi_2)^2 + (a \cos \phi_1 + b \cos \phi_2)^2.$$

Or, $$a'^2 (\sin^2 e + \cos^2 e) = a^2 \sin^2 \phi_1 + b^2 \sin^2 \phi_2 + 2ab \sin \phi_1 \sin \phi_2. + a^2 \cos^2 \phi_1 + b^2 \cos^2 \phi_2 + 2ab \cos \phi_1 \cos \phi_2.$$

Or, $$a'^2 = a^2 (\sin^2 \phi_1 + \cos^2 \phi_1) + b^2 (\sin^2 \phi_2 + \cos^2 \phi_2) + 2ab (\sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2),$$

Or, $a'^2 = a^2+b^2+2ab\cos(\phi_1-\phi_2)$, whence, $a' = \sqrt{a^2+b^2+2ab\cos(\phi_1-\phi_2)}$, which, when $(\phi_1-\phi_2) = \phi$, gives the same result, as above, (page 80).

Proceeding again, as before, if, (*i*) $\phi_1 = \phi_2$ or $(\phi_1-\phi_2) = 0$, *i.e., if the two motions be in the same phase*, we have

$$a' = (a+b);$$

(*ii*) if $\phi_1-\phi_2 = \pi$, *i.e., if the two motions be in opposite phases*, we have

$$a' = (a-b);$$

and, further, if $a = b$, *i.e.*, if the amplitudes of the two motions be also equal, we have $a' = 0$; *i.e.*, we obtain the same results as above.

(2) **Composition of two linear simple harmonic motions at right angles to each other.** Let the two simple harmonic motions be along the axes of co-ordinates XOX' and YOY', and let a and b be their amplitudes respectively and ϕ, the *phase difference* between them. Then, if their displacements at any instant t be x and y,

we have $x = a\sin\omega t$, ...(*i*) and $y = b\sin(\omega t+\phi)$, ...(*ii*)

$\therefore$ $\sin\omega t = x/a$; [from (*i*), above.

and, since $\sin^2\omega t+\cos^2\omega t = 1$, we have $\cos^2\omega t = 1-\sin^2\omega t$.

Or, $\cos^2\omega t = 1-x^2/a^2$, so that, $\cos\omega t = \left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}$.

Now, $y = b(\sin\omega t\cos\phi+\cos\omega t\sin\phi)$, [from (*ii*), above

Or, $y/b = \sin\omega t\cos\phi+\cos\omega t\sin\phi$.

Or, $\frac{y}{b} = \frac{x}{a}\cos\phi+\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}\sin\phi$.

Or, $\left(\frac{y}{b}-\frac{x}{a}\cos\phi\right) = \left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}\sin\phi$.

So that, squaring both sides, we have

$$\left(\frac{y}{b}-\frac{x}{a}\cos\phi\right)^2 = \left(1-\frac{x^2}{a^2}\right)\sin^2\phi.$$

Or, $\frac{y^2}{b^2}+\frac{x^2}{a^2}\cos^2\phi-\frac{2xy}{ab}\cos\phi = \sin^2\phi-\frac{x^2}{a^2}\sin^2\phi.$

Or, $\frac{y^2}{b^2}+\frac{x^2}{a^2}\cos^2\phi+\frac{x^2}{a^2}\sin^2\phi-\frac{2xy}{ab}\cos\phi = \sin^2\phi.$

Or, $\frac{y^2}{b^2}+\frac{x^2}{a^2}(\cos^2\phi+\sin^2\phi)-\frac{2xy}{ab}\cos\phi = \sin^2\phi.$

Or, $$\frac{y^2}{b^2}+\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\phi = \sin^2\phi. \quad \text{...(iii)}$$

This is the equation to an *ellipse*, inclined to the axes of the co-ordinates, and may be inscribed in a rectangle of sides $2a$ and $2b$.

Now, a number of special cases arise :

(*a*) **When $\phi = 0$, i.e., when there is no phase difference between the two motions.** In this case, $\sin\phi = 0$, and $\cos\phi = 1$.

So that, substituting these values in relation (*iii*) above, we have

$$\frac{y^2}{b^2}+\frac{x^2}{a^2}-\frac{2xy}{ab} = 0.$$

Or, $$\left(\frac{y}{b}-\frac{x}{a}\right)^2 = 0. \quad \text{Or,} \quad \frac{y}{b}-\frac{x}{a} = 0.$$

Or, $$\frac{y}{b} = \frac{x}{a}. \quad \text{Or,} \quad \frac{y}{x} = \frac{b}{a}.$$

This is the equation to a *straight line, passing through the origin, such that it meets the axis XX′ at an angle* $tan^{-1}\, b/a$, [see case I, *(iii)*, *(a)*].

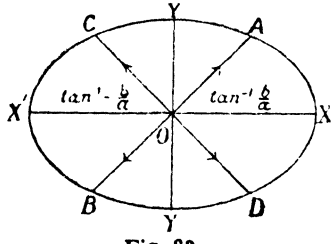


Fig. 83.

The resultant motion, is, therefore, along the straight line *AB*, (Fig. 83), *i.e., the particle describes a S.H.M. along this line, with the same time-period as that of the two component motions* and an amplitude equal to $\sqrt{a^2+b^2}$. If the amplitudes of the two motions be equal, the straight line *AB* is *inclined at an angle of* 45° to the axes of x and y.

(b) **When $\phi = \pi$, i.e., when the phase difference between the two motions is π.** Here, $sin\ \phi = 0$, and $cos\ \phi = -1$. So that, from *(iii)* above, we have

$$\frac{y}{b} + \frac{x}{a} = 0. \quad \text{Or,} \quad \frac{y}{x} = -\frac{b}{a}.$$

This too is an equation to a *straight line, passing through the origin but inclined to the x-axis at an angle* $tan^{-1}-b/a$; so that, *the resultant motion is again a S.H.M., with the same time-period but along the straight line CD*, (Fig 83), inclined the other way, [see case I, *(iii)*, *(b)*], the amplitude being again $\sqrt{a^2+b^2}$.

(c) **When $\phi = \pi/2$, i.e., when the phase difference between the two motions is $\pi/2$.**

Here, $$sin\ \phi = 1 \quad \text{and} \quad cos\ \phi = 0.$$

Substituting these values of $sin\ \phi$ and $cos\ \phi$ in relation *(iii)*, above, we have

$$\frac{y^2}{b^2}+\frac{x^2}{a^2} = 1.$$

This is the equation to an ellipse, whose major and minor axes coincide with the directions of the two given motions, and whose semi-axes are equal to b and a respectively. The resultant path is, therefore, an ellipse, (Fig. 84), which it describes once in the time-period of each component *S.H.M.*, [see case I, *(iii)*, *(d)*, above].

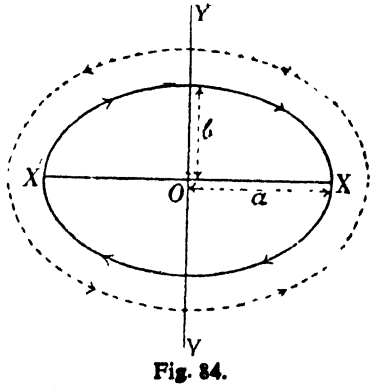


Fig. 84.

The direction of motion of the particle along the ellipse may be determined as follows :

Since $\phi = \pi/2$, and $x = a\ sin\ \omega t$, we have $y = b\ sin\ (\omega t+\phi) = b\ sin\ (\omega t+\pi/2)$, whence, $y = b\ cos\ \omega t$.

$\therefore$ differentiating y with respect to t, we have

$$\textit{velocity of the particle} = dy/dt = -\omega.b \sin \omega t.$$

Now, x and, therefore, $\sin \omega t$, is positive in the right half of the figure. And, therefore, the negative sign of dy/dt means that the velocity of y is *negative, i.e., it is directed downwards in the right half of the figure* In other words, *the direction of the particle along the ellipse is clockwise.*

If, on the other hand, $\phi = -\pi/2$, we have, again, $\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$.

i.e., the resultant motion is again an *ellipse.*

But, since $y = b \sin(\omega t - \pi/2) = -b \cos \omega t$, we have $dy/dt = \omega.b \sin \omega t$, *i.e., the velocity of y is now positive in the right half of the figure*, and is, therefore, *directed upwards.* In other words, *the direction of motion of the particle along the ellipse is now anticlockwise.*

(*d*) **When $\phi = \pi/2$, and $b = a$, i.e , when the phase difference is $\pi/2$ and the amplitudes are equal.** In this case, obviously, $\sin \phi = \sin \pi/2 = 1$, and $\cos \phi = \cos \pi/2 = 0$. $\therefore$ substituting these values in relation (*iii*) above, we have

$$\frac{y^2}{a^2} + \frac{x^2}{a^2} = 1. \quad \text{Or,} \quad \frac{y^2 + x^2}{a^2} = 1,$$

whence, $y^2 + x^2 = a^2$.

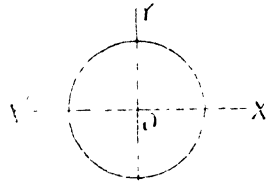


Fig. 85.

This is the equation to a *circle, whose radius is equal to the amplitude of either simple harmonic motion* ; so that, in this case, *the particle describes a circle*, (Fig. 85), *once in the same time as that taken by any one of the two component motions*, [see case I, (*iii*), (*g*), above]—the direction of motion along the circle being determined as explained above, in the case of the ellipse.

A uniform circular motion may thus be regarded as a combination of two equal or similar simple harmonic motions, at right angles to each other, and differing in phase by $\pi/2$.

(*e*) **When $\phi = \pi/4$, i.e., when the phase difference between the two motions is $\pi/4$.**

Here, $\cos \phi = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$; and so also, $\sin \phi = \frac{1}{\sqrt{2}}$.

Hence, from relation (*iii*) above, we have

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} - \frac{2xy}{ab} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}. \quad \text{Or,} \quad \frac{y^2}{b^2} + \frac{x^2}{a^2} - \frac{\sqrt{2}xy}{ab} = \frac{1}{2},$$

which is the equation to an *oblique ellipse.* (See case I (*iii*), (*c*) above.) So that, the resultant motion, in this case, is an *oblique* ellipse.

Thus, we see that the two perpendicular linear simple harmonic motions compound into a straight line motion, when they differ in phase by 0 or π, and into an ellipse or a circle, when the phase difference is $\pi/2$.

For any other phase difference, the motion is still an *ellipse*, with its major and minor axes no longer coinciding with the directions of the two component motions. but being inclined to them.

(*f*) **When the amplitudes are different and the time-periods or frequencies are nearly equal.** In the cases, dealt with above, where the time-periods of the two component vibrations are *identical*, the elliptical paths of the resultant vibrations remain fixed. But if the two time-periods differ slightly from each other, there comes about a gradual but progressive change in the relative phase (ϕ) of the two vibrations and the elliptical path consequently undergoes a corresponding cycle of changes, whose frequency is equal to the difference between the frequencies of the two component vibrations.

Thus, (*i*) *when* $\phi = 0$, *the ellipse coincides with one diagonal of the rectangle of sides* $2a$ *and* $2b$, *within which the ellipse lies*; for, here,

$$\frac{x}{a} - \frac{y}{b} = 0.$$

(*ii*) *When* ϕ *increases from* 0 *to* $\pi/2$, *the ellipse opens out to the form* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *passing through intermediate oblique positions.*,

And, if $a = b$, *the ellipse is reduced to the form of a circle.*

(*iii*) *When* ϕ *increases from* $\pi/2$ *to* π, *the ellipse closes up again and finally coincides with the other diagonal of the rectangle*; for now

$$\frac{x}{a} + \frac{y}{b} = 0.$$

And, the same changes take place, in the reverse order, when ϕ *increases from* π *to* 2π, *the ellipse ultimately coinciding with the same first diagonal as in case* (*i*),—all these changes being shown in Fig. 82, above.

(*g*) **When the frequencies are in the ratio 2 : 1, or the periods are in the ratio, 1 : 2, and the amplitudes are different.** In this case, the angular velocity of the particle in the circle of reference of one will be double of that of the particle in the circle of reference of the other.

$\therefore$ if the two motions be represented by

$$x = a \sin \omega t \quad \text{and} \quad y = b \sin (2\omega t + \phi),$$

where ϕ is the *phase angle* by which the second motion is ahead of the first, we have

$$x/a = \sin \omega t, \text{ and } \therefore \cos \omega t = \sqrt{1 - \sin^2 \omega t};$$

and $$y/b = \sin (2\omega t + \phi) = \sin 2\omega t \cos \phi + \cos 2\omega t \sin \phi.$$

Or, $$y/b = 2 \sin \omega t \cos \omega t \cos \phi + (1 - 2 \sin^2 \omega t) \sin \phi.$$

$[\because \sin 2\omega t = 2 \sin \omega t . \cos \omega t \text{ and } \cos 2\omega t = (1 - 2 \sin^2 \omega t)]$.

$\therefore$ substituting the values of $\sin \omega t$ and $\cos \omega t$ from above, we have

$$\frac{y}{b} = 2 \frac{x}{a} \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \cos \phi + \left(1 - 2\frac{x^2}{a^2}\right) \sin \phi.$$

$$= \frac{2x}{a} \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \cos \phi + \sin \phi - \frac{2x^2}{a^2} \sin \phi.$$

Or, $$\frac{y}{b}-\sin\phi+\frac{2x^2}{a^2}\sin\phi=\frac{2x}{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}\cdot\cos\phi.$$

Or, $$\left(\frac{y}{b}-\sin\phi+\frac{2x^2}{a^2}\sin\phi\right)^2=\frac{4x^2}{a^2}\left(1-\frac{x^2}{a^2}\right).\cos^2\phi.$$

Or, $$\left\{\left(\frac{y}{b}-\sin\phi\right)+\frac{2x^2}{a^2}\sin\phi\right\}^2=\frac{4x^2}{a^2}\cos^2\phi-\frac{4x^4}{a^4}\cos^2\phi.$$

Or, $$\left(\frac{y}{b}-\sin\phi\right)^2+\frac{4x^4}{a^4}\sin^2\phi+2\left(\frac{y}{b}-\sin\phi\right).\frac{2x^2}{a^2}\sin\phi.$$
$$=\frac{4x^2}{a^2}\cos^2\phi-\frac{4x^4}{a^4}\cos^2\phi.$$

Or, $$\left(\frac{y}{b}-\sin\phi\right)^2+\frac{4x^4}{a^4}\sin^2\phi+\frac{4x^4}{a^4}\cos^2\phi+\frac{4x^2}{a^2}\left(\frac{y}{b}-\sin\phi\right)\sin\phi$$
$$-\frac{4x^2}{a^2}\cos^2\phi=0.$$

Or, $$\left(\frac{y}{b}-\sin\phi\right)^2+\frac{4x^4}{a^4}(\sin^2\phi+\cos^2\phi)+\frac{4x^2y}{a^2b}.\sin\phi-\frac{4x^2}{a^2}.\sin^2\phi$$
$$-\frac{4x^2}{a^2}\cos^2\phi=0.$$

Or, $$\left(\frac{y}{b}-\sin\phi\right)^2+\frac{4x^4}{a^4}+\frac{4x^2}{a^2}\left[\frac{y}{b}\sin\phi-(\sin^2\phi+\cos^2\phi)\right]=0.$$

Or, $$\left(\frac{y}{b}-\sin\phi\right)^2+\frac{4x^4}{a^4}+\frac{4x^2}{a^2}\left(\frac{y}{b}\sin\phi-1\right)=0.$$

Or, $$\left(\frac{y}{b}-\sin\phi\right)^2+\frac{4x^2}{a^2}\left(\frac{x^2}{a^2}+\frac{y}{b}\sin\phi-1\right)=0. \quad ..(A)$$

This is the general equation for a curve having two loops, for any values of phase difference and amplitude.

Let us now take some particular cases :

(*i*) **If the phase difference, i.e., ϕ, be eqnal to o or π.**

Here, $\sin\phi=0$, and, therefore, $\frac{y^2}{b^2}+\frac{4x^2}{a^2}\left(\frac{x^2}{a^2}-1\right)=0$,

which is the equation to the figure of 8. [See case I (*iii*), (*h*), above.

(*ii*) **If the phase angle $\phi=\pi/2$, i.e., $\sin\phi=1$,**

we have $$\left(\frac{y}{b}-1\right)^2+\frac{4x^2}{a^2}\left(\frac{x^2}{a^2}+\frac{y}{b}-1\right)=0.$$

Or, $$\left(\frac{y}{b}-1\right)^2+\frac{4x^2}{a^2}\left(\frac{y}{b}-1\right)+\frac{4x^4}{a^4}=0.$$

Or, $$\left(\frac{y}{b}-1\right)^2+2\frac{2x^2}{a^2}\left(\frac{y}{b}-1\right)+\left(\frac{2x^2}{a^2}\right)^2=0.$$

Or, $$\left(\frac{2x^2}{a^2}\right)^2+2\frac{2x^2}{a^2}\left(\frac{y}{b}-1\right)+\left(\frac{y}{b}-1\right)^2=0.$$

Or, $$\left(\frac{2x^2}{a^2}+\frac{y}{b}-1\right)^2=0.$$

This *represents two coincident parabolas*, each having the equation

$$\left(\frac{y}{b}-1\right)+\frac{2x^2}{a^2}=0.$$

Or, $$\frac{2x^2}{a^2}=-\left(\frac{y}{b}-1\right)=-\left(\frac{y-b}{b}\right)=-\frac{1}{b}\left(y-b\right).$$

Or, $$2x^2=-\frac{a^2}{b}\left(y-b\right). \quad \text{Or,} \quad x^2=-\frac{a^2}{2b}\left(y-b\right).$$

If, however, the frequencies differ slightly from the ratio 2 : 1, (*i.e.*, the time-periods differ from 1 : 2), the variation in the resultant path of the particle may be obtained by substituting the consequent changes in the value of ϕ in the general relation above. The changes occurring when ϕ changes from 0 to π and then from π to 2π are shown in Fig. 82, above.

Note. Alternatively, the student may, without deducing the general equation (*iii*), [§50, (2), page 129], obtain the resultant motions in simple cases as follows :

Taking displacements of the two *S.H.M's*, at right angles to each other, as $x = a \sin \omega t$, and $y = b \sin(\omega t + \phi)$,

where ϕ is the phase angle by which the second motion is ahead of the first, we have the following cases :

(*i*) **When $\phi = 0$.** We have $x = a \sin \omega t$ and $y = b \sin \omega t$.

So that, $x/a = \sin \omega t = y/b$, whence, $y/x = b/a$, [See §50, II, (2), (*a*).

which is the equation to a *straight line*, passing through the origin ; and inclined to the x-axis at an angle, $\tan^{-1} b/a$ (straight line AB, in Fig. 83); *i.e., the resultant motion, here, is along the straight line AB.*

(*ii*) **When $\phi = \pi$.** We have $x = a \sin \omega t$ and $y = -b \sin \omega t$.

So that, $x/a = \sin \omega t = -y/b$, whence, $y/x = -b/a$, [See § 50, II, (2), (*b*).

which is, again, the equation to a *straight line*, inclined to the x-axis at an angle, $\tan^{-1} -b/a$, (*i.e.*, straight line CD, in Fig. 83), at right angles to that in case (*i*). *The resultant motion is thus along a straight line, at right angles to that in the first case.*

(*iii*) **When $\phi = \pi/2$.** We have $x = a \sin \omega t$ and $y = b \cos \omega t$.

So that, $\sin \omega t = x/a$ and $\cos \omega t = y/b$.

And $\therefore$ $x^2/a^2 + y^2/b^2 = (\sin^2 \omega t + \cos^2 \omega t) = 1$,

Or, $(y^2/b^2) + (x^2/a^2) = 1$, [See § 50, II, (2), (*c*)

which is the equation to an *ellipse*, with its major and minor axes coinciding with the directions of the two given perpendicular motions, and whose *semi-axes* are equal to b and a (Fig. 84). *The resultant motion is thus an ellipse here, described once in the time-period of each component motion.*

(*iv*) **When $\phi = \pi/2$ and $b = a$, i.e., the amplitudes are also equal.**

Here, $x = a \sin \omega t$ and $y = b \cos \omega t = a \cos \omega t$.

So that, $\sin \omega t = \frac{x}{a}$ and $\cos \omega t = \frac{y}{a}$.

And, $\frac{y^2}{a^2}+\frac{x^2}{a^2} = \sin^2 \omega t + \cos^2 \omega t = 1$. Or, $y^2+x^2 = a^2$, [See §50, II, (2), (*d*).

which is the equation to a *circle*, with a radius equal to the amplitude of either of the two motions. *The resultant motion, in this case, therefore, is a circle, described once in the time-period of each component motion.*

51. Composition of two equal circular motions in opposite directions. Let two particles P_1 and P_2 move with equal velocities along the same circle $XYX'Y'$ of radius $2a$, in opposite directions, as shown, (Fig. 86), such that when P_1 passes the point X, P_2 passes X'.

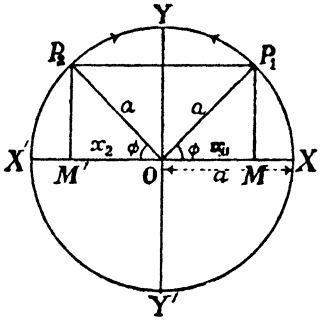


Fig 86.

Let the positions of P_1 and P_2 be as shown, at any given instant t, after starting from X and X' respectively ; so that, $XOP_1 = X'OP_2 = \phi = \omega t$, where ω is the angular velocity of P_1 and P_2.

Now, *we know that a circular motion is equivalent to two equal linear simple harmonic motions with a phase difference* $\pi/2$ *and along perpendicular directions to each other*, (see page 131).

The circular motions of both P_1 and P_2 may, therefore, be resolved along perpendicular directions XX' and YY'. Then, the displacements x_1 and x_2, of P_1 and P_2, will, at the given instant, be equal in magnitude but *opposite in direction* along XOX' and will, therefore, cancel each other out, but their displacements y_1 and y_2 along YOY' will be equal and *in the same direction*, so that the resultant displacement is given by $y = y_1 + y_2$, along YOY'.

Since $y_1 = y_2 = a \sin \phi = a \sin \omega t$, we have

$$y = y_1 + y_2 = a \sin \omega t + a \sin \omega t = 2a \sin \omega t.$$

Or, *the resultant motion is a linear simple harmonic motion along the diameter* YOY', *at the extremities of which the particles* P_1 *and* P_2 *cross each other as they describe their circular motions.*

And clearly, the amplitude of the resultant motion is $2a$, and its time-period the same as that of the two constituent circular motions.

52. Energy of a Particle in Simple Harmonic Motion. The acceleration of a particle, executing a *S.H.M*, is, as we know, directed *towards its equilibrium position, or in a direction opposite to that in which y, the displacement of the particle, increases. Hence, work is done during its displacement, or the particle has potential energy.* Also, *the particle possesses velocity and, therefore, has kinetic energy.* Thus, it has both potential as well as kinetic energy, or *its energy is partly potential and partly kinetic.* And, if there be no dissipative force at work, *i.e.*, if the energy is not dissipated away in any way, the sum total of the two remains constant, although as the displacement increases and the velocity decreases, the potential energy increases and the kinetic energy decreases.

Now, when the particle has its maximum displacement, positive or negative, its velocity is zero and, therefore, its kinetic energy is then *zero* ; so that, in this position, *the whole of its energy is present in the form of potential energy.* And, *when the particle is in the equilibrium position,* its displacement is *zero* and its velocity, maximum ; so that, *the whole of its energy is now present in the form of kinetic energy.*

If m be the mass of the particle ; a, its amplitude and $2\pi/\omega$, its time-period (T), *i.e*, if ω be the angular velocity of the rotating vector, or that of the particle in the circle of reference, we have

velocity of the particle in its equilibrium position $= a\omega$, *a maximum.*

And .·. its *kinetic energy* $= \frac{1}{2}m.(a\omega)^2 = \frac{1}{2}ma^2\omega^2$,

and its *potential energy* $= 0$, [·.· its displacement is zero.

Or, its *total energy* $= \frac{1}{2}ma^2\omega^2 + 0 = \frac{1}{2}ma^2\omega^2$.

In other words, the whole of its energy, here, is present in the *kinetic form*. Similarly, when the particle has its maximum displacement, the whole of its energy is in the *potential form*, which, therefore, is also equal to $\frac{1}{2}\ ma^2\omega^2$, in *this* position of the particle.

For any other position of the particle, its displacement is given by $y = a \sin \omega t$.

And .·. its velocity is given by $dy/dt = a\omega.\cos \omega t$.

Hence, its *kinetic energy* $= \frac{1}{2}m.(a\omega \cos \omega t)^2 = \frac{1}{2}m.a^2\omega^2.\cos^2 \omega t$.

And, since its *total energy* $= \frac{1}{2}ma^2\omega^2$, we have

its *potential energy* $= \frac{1}{2}ma^2\omega^2 - \frac{1}{2}ma^2\omega^2.\cos^2\omega t = \frac{1}{2}ma^2\omega^2.\sin^2\omega t$.

Alternatively, we may proceed as follows :

We have, acceleration of the particle, executing a *S. H. M.*, given by

$$d^2y/dt^2 = -\omega^2 y.$$

·. if m be the mass of the particle, the force F required to maintain this displacement y is equal to $m\omega^2 y$

And, therefore, work done by the force for a small displacement dy is equal to $F.dy = m\omega^2 y\ dy$.

Now, this work is also a measure of the potential energy of the particle at this displacement.

·. *P. E.* of the particle for a displacement $dy = F.dy = m\omega^2 y.dy$.

Hence, total work done for displacement y and, therefore, total *P. E.* of the particle for a displacement y is given by

$$\int_0^y m\omega^2.y.dy = m\omega^2 \int_0^y y.dy.$$

Or, *P. E.* of the particle for a displacement $y = \frac{1}{2}m\omega^2.y^2$,

i.e., *Potential Energy* $\propto y^2$.

Thus the *P. E.* of a particle, executing a *S.H.M.* is, at any given instant, directly proportional to the square of its displacement from its mean or equilibrium position, at that instant.

Now, *velocity of the particle at displacement* $y = dy/dt$.

$$= \frac{d}{dt}(a \sin \omega t) = a\omega \cos \omega t.$$

And, .·. *K. E. of the particle* $= \frac{1}{2}m(a\omega \cos \omega t)^2 = \frac{1}{2}ma^2\omega^2.\cos^2\omega t$.

$$= \frac{1}{2}ma^2\omega^2(1 - \sin^2\omega t) = \frac{1}{2}ma^2\omega^2(1 - y^2/a^2).$$

$$= \frac{1}{2}ma^2\omega^2(a^2 - y^2)/a^2 = \frac{1}{2}m\omega^2.(a^2 - y^2).$$

Hence *tolal energy* of the particle $=$ *its potential energy* $+$ *its kinetic energy*.

$$= \frac{1}{2}m\omega^2y^2 + \frac{1}{2}m\omega^2(a^2 - y^2) = \frac{1}{2}m\omega^2a^2.$$

And, since $\omega = 2\pi/T = 2\pi n$, where T is the *time-period* of the particle and n, the *frequency* of its vibration, we may also say that

$$\textit{total energy of the particle} = \frac{1}{2}m\left(\frac{2\pi}{T}\right)^2 a^2 = \frac{2\pi^2ma^2}{T^2} = 2\pi^2n^2ma^2,$$

Now, since in any *conservative system* the sum total of the kinetic and potential energies of the system must be a *constant*, it is clear that the former can only increase at the expense of the latter, and, therefore, attains its maximum value, when the latter is reduced to its minimum value or *zero*, and *vice versa*. Thus, the maximum value of any one of the two forms of energy measures the total energy of the system, (see page 136).

53. Average Kinetic and Potential Energies of a Particle in S.H.M. We have seen above that, at any given instant, the *P.E.* of a particle is equal to $\frac{1}{2}ma^2\omega^2 \sin^2\omega t$ and its *K.E.* equal to $\frac{1}{2}ma^2\omega^2 \cos^2\omega t$. Now, the mean or average value of both $\sin^2\omega t$ and $\cos^2\omega t$ for a whole cycle, (from 0 to 2π), *i.e.*, for a whole time-period is equal to $\frac{1}{2}$*, and, therefore, the mean or average *K.E.* of the particle over the whole of its period of vibration is equal to its mean or average *P.E.* over the whole period, each being equal to $\frac{1}{2} \times \frac{1}{2}\, ma^2\omega^2 = \frac{1}{4}ma^2\omega^2$. Thus,

$$\textit{average K.E of the particle} = \textit{its average P.E.} = \tfrac{1}{4}ma^2\omega^2.$$

And, $\therefore$ *total energy* of the particle $= 2 \times \frac{1}{4}ma^2\omega^2 = \frac{1}{2}ma^2\omega^2$.

We may express this by saying that *the energy of a particle, executing a S.H M., is, on the average, half kinetic and half potential in form*,—the whole being present in the kinetic form at its mean or equilibrium position, and in the potential form at its extreme position, on either side.

The results obtained above for linear *S.H.M.* are equally valid for angular *S.H.M.* Only, the linear displacement x or y of the particle or the body, and its mass m, are replaced by their rotational analogues, *viz.*, the angular displacement θ and its moment of inertia I about its axis of rotation, respectively.

SOLVED EXAMPLES

1. A quantity of gas is enclosed in a cylinder, fitted with a smmooth heavy piston. The axis of the cylinder is vertical. The piston is thrust downwards to compress the gas, and then let go. Is the ensuing motion of the piston as S.H.M. ? If so, what is its time-period ?

Let original *volume* of the gas be $= V$ and *its pressure* $= P$.

Let a be the *area of cross-section* of the piston, (and cylinder). Then, if

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*This is so, because the mean value of $\sin^2\omega t$ for a whole cycle from 0 to 2π is given by

$$\frac{\int_0^{2\pi} \sin^2\omega t.dt}{\int_0^{2\pi} d\omega t.} = \int_0^{2\pi} \frac{1-\cos 2\omega t}{2}.d\omega t/2\pi$$

$$= \frac{\left[\omega t/2 - \sin 2\omega t/4\right]_0^{2\pi}}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}$$

Similar is the case with $\cos^2\omega t$ for a whole time-period.

the piston be displaced through a distance x, (Fig. 87), the *change in volume* produced is given by $x.a$, the corresponding *change in pressure* being p. By *Boyle's law*, therefore,

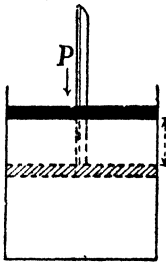


Fig. 87.

$$PV = (P+p)\times(V-x.a) = PV - Px.a + pV - p.x.a.$$

Or, $$0 = -P.x.a + pV - p.x.a.$$

Neglecting $p\ x.a$ as the product of very small quantities, compared with the other terms in the expression, we have

$$0 = -P.x.a + pV. \quad \text{Or,} \quad pV = P.x.a, \text{ whence, } p = \frac{P.a}{V}.x.$$

Now, the *restoring force on the piston*, which is equal to the disturbing force, is obviously equal to change in pressure *into* area of cross-section of the piston $= p.a$.

$$\therefore \quad p.a. = \frac{P.a}{V}.x.a = \frac{Pa^2}{V}.x.$$

Since, *acceleration* $=$ *force/mass*, the *acceleration of the piston* $= p.a/m$.

$$= \frac{Pa^2}{V}\cdot\frac{x}{m} = \frac{P.a^2}{Vm}.x. \qquad \left[\begin{array}{l}\text{substituting the value of}\\ p.a., \text{ from above.}\end{array}\right.$$

Or, *acceleration of the piston* $= \mu.x$,

where $Pa^2/Vm = \mu$, *a constant of proportionality, which is equal to the acceleration of the piston per unit displacement*, (*i.e.*, when $x = 1$).

Or, *acceleration of the piston is proportional to* x, *its displacement.*

Hence, the motion of the piston is a *S. H. M.*

And $\therefore$ its *time-period*, $T = 2\pi\sqrt{\dfrac{1}{\text{acceleration per unit displacement}}}$.

$$= 2\pi\sqrt{\frac{1}{\mu}} = 2\mu\sqrt{\frac{1}{P.a^2/V.m}} = 2\pi\sqrt{\frac{V.m}{P.a^2}}.$$

Or, $$T = \frac{2\pi}{a}\sqrt{\frac{V.m}{P}}.$$

2. A body describing a simple harmonic motion executes 100 complete vibrations per minute, and its speed at its mean position is 15 ft. per second. What is the length of its path ? What is its velocity when is its half way between its mean position and an extremity of its path ?

Here, *time-period* T of the body $= 1/100\ mt. = 60/100 = \cdot 6\ secs.$

and *velocity of body at its mean position* $= 15\ ft./sec.$

Since velocity of a body executing a *S.H.M.* $= a\omega$, *at its mean position*, where a is its *amplitude and* ω, its *angular velocity, we have*

$$a\omega = 15 \qquad \text{Or,} \quad a.2\pi/T = 15, \qquad [\ \because \omega = 2\pi/T.$$

whence, $$a = \frac{15\times T}{2\pi} = \frac{15\times\cdot 6}{2\pi} = \frac{9}{2\pi}. \qquad \text{Or,} \quad a = 1\cdot 432\ ft.$$

Now, *length of path of the body* $=$ *twice its amplitude*, because it goes the same distance on either side of its mean position.

Hence, *length of path* $= 2\times 1\cdot 432 = 2\cdot 864\ ft.$

Again, velocity of a body at a displacement y is given by

$$v = \omega\sqrt{a^2 - y^2}.$$

Here, displacement of the body $=$ its amplitude/2.

Or, $$y = a/2 = 1\cdot 432 = \cdot 716\ ft.$$

So that, $$v = \frac{2\pi}{T}\sqrt{(1\cdot 432)^2 - (\cdot 716)^2} = \frac{2\pi}{T}\sqrt{(1\cdot 432 + \cdot 716)(1\cdot 432 - \cdot 716)}.$$

$$= \frac{2\pi}{\cdot 6}\sqrt{2\cdot 148\times\cdot 716} = 12\cdot 99\ ft./sec.$$

Thus, the *length of path* of the body is 2 864 *ft.*, and its *velocity* when it is half way between its mean position and an extremity of its path, is 12·99 *ft./sec.*

3. If the earth were a homogeneous sphere, and a straight hole were bored in it through its centre, show that a body dropped into the hole will execute a S.H.M., and calculate the time-period of its vibration. [Radius of the earth = 4000 miles, and value of g on its surface = 32 ft. per sec. per sec.]

We know that the force with which a body is attracted by the earth towards its centre is equal to the weight of the body, (mg), and also equal to $G.m.M/R^2$, where m is the mass of the body ; M, the mass of the earth ; R, the radius of the earth ; g, the acceleration due to gravity on the surface of the earth and G, the gravitational constant.

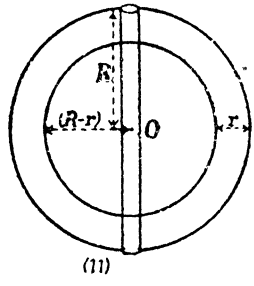
$\therefore$ $mg = G.m.M/R^2$. Or, $g = G.M./R^2$.

Since the earth is supposed to be a sphere, its *volume* $= 4\pi R^3/3$, and, therefore, if Δ be its *density*, we have

its *mass*, $M = 4\pi R^3.\Delta/3$. So that, $g = \frac{4}{3} \cdot \frac{\pi R^3.\Delta}{R^2}. G = 4.\pi R\Delta.G/3.$ ..(*i*)

If the value of acceleration due to gravity at a distance r below the surface of the earth, (Fig. 88), be g', we have, as above,

$$g' = \frac{4}{3}. \pi . \frac{(R-r)^3}{(R-r)^2}.\Delta.G = 4\pi.(R-r).G\Delta/3. \quad ..(ii)$$



(11)

Fig. 88.

Dividing (*ii*) by (*i*), we have

$$g'/g = \tfrac{4}{3}.\pi.(R-r)\Delta.G/\tfrac{4}{3}.\pi.R.\Delta.G.$$

Or, $g'/g = (R-r)/R.$

Or, $g' = g(R-r)/R = (R-r).g/R.$

Thus, the acceleration of a body at a distance $(R-r)$ from the centre of the earth is equal to $(R-r).g/R$; and since g/R is constant, *this acceleration is proportional to* $(R-r)$, *the displacement of the body* from the centre O of the earth. The body, therefore, *executes a S.H.M.*, and its time-period is given by

$$t = 2\pi \sqrt{\frac{1}{g/R}} = 2\pi\sqrt{\frac{R}{g}}. \quad .. \quad .. \quad ..(iii)$$

Now, $R = 4000$ *miles* $= 4000 \times 1760 \times 3$ *ft.*

and $g = 32$ *ft./sec*^{2}.

$\therefore$ substituting the values of R and g in relation (*iii*) above, we have

$$t = 2\pi\sqrt{\frac{4000 \times 1760 \times 3}{32}} = 2\pi \sqrt{\frac{12 \times 176 \times 10^4}{32}}.$$

$$= 5105 \text{ secs.} = \frac{5105}{60} = 85{\cdot}07 \text{ minutes}$$

Thus, the *time-period* of vibration of the body would be 85·07 *minutes.*

4. If a body executes a simple harmonic vibration in time T_1, under one constraining force, and in time T_2, under another, what will be its time-period under both forces together ?

Let mass of the body be $= m$,

and let its acceleration under the *first force* F_1 be $= a_1$,

" " " " *second* " F_2 " $= a_2$

and " " " " *both the forces* F_1+F_2 be $= a$.

Then, clearly, $a_1 = F_1/m$; $a_2 = F_2/m$; $a = (F_1+F_2)/m$ $\left[\because \text{acc.} = \frac{\text{force}}{\text{mass}} \right.$.

Also, the acceleration of a body executing a *S.H.M.* is proportional to its displacement x from the equilibrium position, *i.e.*, acceleration $a \propto x$.

$\therefore$ $a_1 = \mu_1 x$; $a_2 = \mu_2 x$; $a = \mu x$,

where μ_1, μ_2 and μ are the constants of proportionality in the three cases respectively.

Again, because the time-period of a body executing a *S.H.M.* is given by $T = 2\pi\sqrt{1/\mu}$, where μ is a constant of proportionality, we have

$$\sqrt{\frac{1}{\mu}} = \frac{T}{2\pi}, \text{ or } \frac{1}{\mu} = \left(\frac{T}{2\pi}\right)^2. \text{ Or, } \mu = \left(\frac{2\pi}{T}\right)^2.$$

So that, $a_1 = \frac{F_1}{m} = \mu_1.x = \left(\frac{2\pi}{T_1}\right)^2.x$; $a_2 = \frac{F_2}{m} = \mu_2.x = \left(\frac{2\pi}{T_2}\right)^2.x$

and $a = \frac{F_1+F_2}{m} = \mu.x = \left(\frac{2\pi}{T}\right)^2.x,$

where T is the time-period of the body under both the forces acting together.

Since acceleration a, under both the forces acting together, will obviously be equal to the sum of the accelerations under the individual forces, we have $a = a_1 + a_2$,

And, therefore, $\left(\frac{2\pi}{T}\right)^2 . x = \left(\frac{2\pi}{T_1}\right)^2 . x + \left(\frac{2\pi}{T_2}\right)^2 . x.$

Or, dividing by $(2\pi)^2 . x$ throughout, we get

$$\left(\frac{1}{T}\right)^2 = \left(\frac{1}{T_1}\right)^2 + \left(\frac{1}{T_2}\right)^2.$$

Or, $\frac{1}{T^2} = \frac{1}{T_1^2} + \frac{1}{T_2^2} = \frac{T_2^2 + T_1^2}{T_1^2 \, T_2^2}$. Or, $T^2 = \frac{T_1^2 \, . \, T_2^2}{T_1^2 + T_2^2}$.

Or, $T = \sqrt{\frac{T_1^2 \, . \, T_2^2}{T_1^2 + T_2^2}} = T_1 . T_2 \sqrt{\frac{1}{T_1^2 + T_2^2}}$.

Thus, the time-period of the body under both the forces together will be $T_1 T_2 \sqrt{1/(T_1^2 + T_2^2)}$.

5. A test tube of weight 6 gms and of external diameter 2 cms. is floated vertically in water by placing 10 gms of mercury at the bottom of the tube. The tube is depressed by a small amount and then released Find the time of oscillation. (*Oxford and Cambridge Higher School Certificate*).

Here, mass of the tube and mercury $= 10 + 6 = 16$ *gms.*

and external radius of the tube $= 2/2 = 1$ *cm.*

$\therefore$ area of cross-section of the tube $= \pi r^2 = \pi . 1^2 = \pi$ *sq. cms.*

Let the tube be depressed through a distance x *cms.*

Then, *volume of the water displaced* $= \pi \times x = \pi x$ *c.cs.*

and *weight* ,, ,, ,, ,, $= \pi x . 1 . = \pi . x$ *gms.wt.* [Taking density of water = 1 *gm./c.c.*

Therefore, *upward thrust* experienced by the tube is equal to the weight of the water displaced, *i.e.*, $\pi . x\, g$ *dynes.*

Hence *acceleration* of the tube $= \frac{\pi x g}{16} = \frac{\pi g}{16} x.$ $\left[\because \text{acc.} = \frac{\text{force}}{\text{mass}}\right.$.

Since $\pi g/16$ is a constant, say, μ, we have

acceleration of the tube $= \mu x$; *i.e., acceleration of the tube is proportional to its displacement* ; and, therefore, it executes a *S.H.M.*

Hence, its *time-period* is given by $t = 2\pi \sqrt{\frac{1}{\mu}}$.

$$= 2\pi \sqrt{\frac{1}{\pi g/16}} = 2\pi \sqrt{\frac{16}{\pi g}} = 2 \sqrt{\frac{\pi^2 16}{\pi g}}.$$

$$= 2 \sqrt{\frac{\pi . 16}{g}} = 2 \sqrt{\frac{\pi . 16}{981}} = .4527 \text{ sec.}$$

Or, *the time of oscillation* of the tube $= .4527$ *sec.*

6. A particle executing a S H.M. has a maximum displacement of 4 cms. and its acceleration at a distance of 1 cm from its mean position is 3 cm /sec^2. What will its velocity be when it is at a distance of 2 cms from its mean position ?

Here, amplitude of the particle $= 4$ *cms.*

and its acceleration, when its displacement is 1 *cm.*, is equal to 3 *cms./sec*^{2}.

Now, *acceleration* $= x . \omega^2$. Or, $3 = 1 \omega^2$,

where x is the *displacement* and ω, the *angular velocity* of the particle.

Or, $\omega^2 = 3$, whence, $\omega = \sqrt{3}$ *radians/sec.*

Now, the velocity of a particle executing a *S.H.M.* is given by

$$v = \omega . \sqrt{a^2 - x^2},$$

where a is the amplitude and x, the displacement of the particle.

$\therefore$ when the displacement of the particle, *i.e.*, $x = 2$ *cms.*, we have

$$v = \sqrt{3} . \sqrt{4^2 - 2^2} = \sqrt{3} . \sqrt{16 - 4} = \sqrt{3} . \sqrt{12}.$$

$$= \sqrt{36} = 6 \text{ cms./sec.}$$

$\therefore$ the velocity of the particle at a distance of 2 *cms.* from its mean position will be 6 *cms./sec.*

7. A vertical U-tube of uniform cross-section contains water up to a height of 30 cms. Show that if the water on one side is depressed and then released, its motion up and down the two sides of the tube is simple harmonic, and calculate its time-period. *(Delhi, 1947)*

Let AA', (Fig. 89), represent the level of the water in the two limbs of the U-tube, to start with, and let the column on the left be depressed up to B, through a distance x *cms.* Then, the column on the right will naturally go up, say to the level C, such that the difference of levels in the two limbs is now, $B'C$, where B' is at the same level as B.

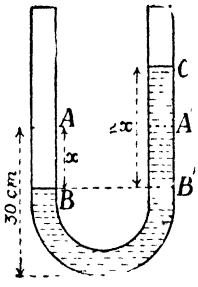


Fig. 89.

The weight of this column of length $2x$ will now act on the mass of the water in the tube, as a result of which it will oscillate up and down.

Now, obviously, the weight of this column of water

= its *mass* $\times g$ = its *volume* $\times$ its *density* $\times g$.

$= a \times 2x \times 1 \times g$ *dynes.*

Or, *force* acting on the mass of water in the tube

$= 2.x.a.g.$ *dynes.*

And, *mass of water* in the tube (in both its limbs)

$= 2 \times 30 \times a \times 1 = 60a$ *gms.*

$\therefore$ *acceleration* produced in the mass of water in the tube

$$= \frac{2\,x\,a.g}{60a} = \frac{x.g}{30} = \frac{g}{30}.x = \mu.x, \qquad \text{where } g/30 = \mu, \text{ a constant.}$$

Or, *acceleration is proportional to x, the displacement of the water column.*

Hence, the motion of the water column is *simple harmonic*, and its *time-period* is given by

$$t = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{1}{g/30}},$$

$$= 2\pi\sqrt{\frac{30^*}{g}}, \text{ or, } t = 2\pi\sqrt{\frac{30}{981}}. \quad \text{Or, } t = 1{\cdot}098 \text{ secs.}$$

The water in the U-tube will thus oscillate with a time-period equal to $2\pi\sqrt{30/g}$ or 1·098 *secs.*

8. Show that the time-period for the swing of a magnet in the earth's field is given by $t = \sqrt{I/MH}$, where M is the magnetic moment of the magnet, I, its moment of inertia about the axis of suspension and H, the earth's field.

Let a magnet NS, of pole strength m, be suspended so as to make an angle α with the earth's field H, (Fig. 90).

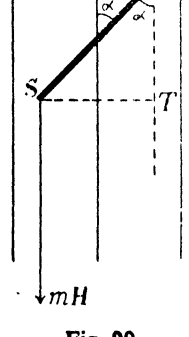


Fig. 90.

Then, clearly, the forces acting on its two poles are mH and mH, as shown. These two forces being *equal*, *opposite* and *parallel*, constitute a *couple*, whose moment is equal to the product of one of the forces and the perpendicular distance between them.

So that, *couple* C, acting on the magnet $= mH \times ST$.

$= mH \times NS \sin \alpha$, $\quad [\because ST = NS \sin \alpha.$

Or, $\quad C = MH \sin \alpha,$

$\therefore \quad m \times NS = M$, the *magnetic moment* of the magnet.

If α be small, we have $\sin \alpha = \alpha$ (*in radian measure*).

So that, $\quad C = MH.\alpha.$

Since the magnet is in equilibrium, this must be balanced by the restoring couple set up in the suspension *i.e.*, by $I.d\omega/dt$, where $d\omega/dt$ is the *angular acceleration of the magnet* and I, its *moment of inertia* about the axis of suspension So that, $\quad I.d\omega/dt = MH.\alpha,$

Or, $d\omega/dt = (MH/I).\alpha = \mu.\alpha$, where $\mu = MH/I$, a constant,

Or, $\quad d\omega/dt \propto \alpha,$

*It should be noted that the time-period is the same as that of a simple pendulum *of the same length as the height of the water column, i.e.,* of length equal to 30 *cms.*

i.e., the *acceleration of the magnet is proportional to its displacement*. The motion of the magnet is, therefore, simple harmonic and its time-period is given by

$$t = 2\pi\sqrt{\frac{I}{\mu}} = \sqrt{\frac{I}{mH/I}} = 2\pi\sqrt{I/MH}.$$

9. A particle is moving in a straight line with simple harmonic motion. Its velocity has the values 5 ft./sec. and 4 ft./sec., when its distances from the centre-point of its motion are 2 ft. and 3 ft. respectively. Find the length of its path, the frequency of its oscillation, and the phase of its motion, when it is at a distance of 2 ft. from the centre. (*London Higher School Certificate*)

Length of path. We know that $v = \omega\sqrt{a^2-x^2}$.

So that, in the *first case*, when $v=5$ *ft./sec*, and $x=2$ *ft.*, we have $5=\omega\sqrt{a^2-2^2}$.

Or, $$25 = \omega^2.(a^2-4). \qquad ...(i)$$

And, in the *second case*, when $v = 4$ *ft./sec.*, and $x=3$ *ft.*, we have $4=\omega\sqrt{a^2-3^2}$.

Or, $$16 = \omega^2.(a^2-9). \qquad ..(ii)$$

$\therefore$ dividing (i) by (ii), we have $\frac{25}{16} = \frac{a^2-4}{a^2-9}$. Or, $25(a^2-9)=16(a^2-4)$.

Or, $25a^2-225 = 16a^2-64$. Or, $25a^2-16a^2 = 225-64$.

Or, $9a^2 = 161$, Or, $a^2 = \frac{161}{9}$.

And $\therefore$ $$a = \sqrt{\frac{161}{9}} = 4{\cdot}23 \text{ ft.}$$

Or, the amplitude of the particle is 4·23 *ft.*

Since the length of the path traversed by the particle is twice the amplitude, (as it travels equal distances on either side), we have

$$\textit{length of path of the particle} = 2.a = 2\times4{\cdot}23 = 8{\cdot}46 \text{ ft.}$$

Frequency. The frequency of the particle, $n = 1/t$, where t is its time-period ; and since $t = 2\pi/\omega$, we have

$$n = \frac{1}{2\pi/\omega} = \frac{\omega}{2\pi}. \qquad ...(iii)$$

The value of ω may be obtained by substituting the value of a in either (i) or (ii). Thus, we have from (i),

$$\omega^2 = \frac{35}{a^2-4}. \quad \text{Or,} \quad \omega = \frac{5}{\sqrt{a^2-4}} = \frac{5}{\sqrt{(4{\cdot}23)^2-(2)^2}}.$$

$$= \frac{5}{\sqrt{(4{\cdot}23+2)(4{\cdot}23-2}} = \frac{5}{\sqrt{6{\cdot}23\times2{\cdot}23}}.$$

And, therefore, substituting the value of ω in (iii) above, we have

$$n = \frac{5}{2\pi\sqrt{6{\cdot}23\times2{\cdot}23}} = 0{\cdot}2135.$$

Thus, the *frequency* of the particle is ·2135.

Phase angle. We have the relation, $x = a\sin\theta$. [where x is the *displacement* of the particle, a, its *amplitude* and $\theta = \omega t$, the *phase angle*.]

Here, $x = 2$ *ft.* and $a = 4{\cdot}23$ *ft.*

$\therefore$ $2 = 4{\cdot}23\sin\theta$. Or $\sin\theta = \frac{2}{4{\cdot}23} = {\cdot}4729$.

$\theta = \sin^{-1}{\cdot}4729 = 28°\ 13'$.

Hence, the phase of the motion of the particle, when its distance is 2 *ft.* from the centre, is 28° 13′.

10. A light elastic string is suspended vertically from a point and carries a heavy mass at its lower free end, which stretches it through distance l cms. Show that the vertical oscillations of the system are simple harmonic in nature, and of a time-period equal to that of a simple pendulum of length l cms.

Let original length of the string AB, (Fig. 91), be $= L$ *cms*, and let the mass attached to its lower end be $= mg$ *dynes*.

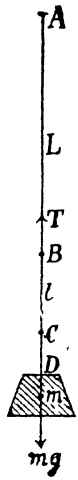


Fig. 91.

Then, the downward force acting on it $= mg$ *dynes*.

And, if T be the tension of the string (upwards), we have

$T = mg$, because the string is in equilibrium.

Now, *Young's modulus* for the string, *i.e.*, $Y = \frac{stress}{strain}$.

Or, $stress = Y \times strain$.

Obviously, $stress = T/a$, and $strain = l/L$,

where a is the *area of cross-section* of the string.

So that, $\frac{T}{a} = Y.\frac{l}{L}$. Or, $T = Y.\frac{l.a}{L}$.

Since $mg = T$, we have $mg = \frac{Y.a}{L}.l$,

Or, $\frac{mg}{l} = \frac{Y.a}{l}$. ...(*i*)

If the string be pulled down a little through a distance x, the tension in the string acting upwards will, clearly, be

$$= \frac{Y.a}{L}.(l+x) = \frac{mg}{l}.(l+x). \qquad \text{[See (}i\text{) above.}$$

And, since the downward force acting on the string $= mg$, the resultant upward force acting on the string will now be

$$\frac{mg\,(l+x)}{l} - mg.$$

Or, *resultant upward force* $= \frac{mg\,(l+x) - mgl}{l} = \frac{mg.x}{l}$.

Now, *acceleration* $= \frac{force}{mass} = \frac{mg\,x}{l.m} = \frac{g\,x}{l} = \mu.x$, where $g/l = \mu$, a constant.

Or, *acceleration* $\propto$ *displacement*.

$\therefore$ the oscillations are *simple harmonic* in nature ; and since the *time-period* of a body executing a *S. H. M.* is given by $t = 2\pi\sqrt{1/\mu}$, we have

$$t = 2\pi\sqrt{\frac{1}{g/l}} = 2\pi\sqrt{\frac{l}{g}},$$

i.e., the same as that of a simple pendulum of length l cms.

11. A particle moving in a straight line with simple harmonic motion, of period $2\pi/\omega$, about a fixed point O, has a velocity $\sqrt{3}\,b\omega$, when at a distance b from O. Show that its amplitude is $2b$ and that it will cover the rest of its distance in time $\pi/3\omega$.

We know that the velocity of a particle executing a *S.H.M.*, at a displacement y, is given by $\omega\sqrt{a^2-y^2}$, where a is its amplitude.

Here, displacement of the particle is b, and its velocity at this displacement is $\sqrt{3}.b\omega$.

$\therefore$ $\sqrt{3}.b\omega = \omega\sqrt{a^2-b^2}$. Or, $\sqrt{3}\,b = \sqrt{a^2-b^2}$, [$\because y = b$.

Or, $3b^2 = a^2-b^2$, whence, $a^2 = 4b^2$.

Or, $a = 2b$, *i.e.*, the *amplitude* of the particle is $2b$.

Now, we have $y = a \sin \omega t$.

Here, $y = b$, and $a = 2b$. And, therefore, $b = 2b \sin \omega t$.

Or, $\sin \omega t = b/2b = \frac{1}{2}$, Or, $\omega t = \sin^{-1}\frac{1}{2} = \pi/6$.

Or, $t = \pi/6\omega$.

Hence the time taken by the particle in covering the distance b from O is equal to $\pi/6\omega$.

Now, since it takes time $2\pi/\omega$ to complete one vibration, it will take $\frac{1}{4}$ of $2\pi/\omega$ or, time $\pi/2\omega$ to complete $\frac{1}{4}th$ of its vibration, *i.e.*, its amplitude, $2b$. And, time *already taken by it* is $\pi/6\omega$.

$\therefore$ the time it will take to cover the *rest of its distance* is clearly equal to

$$\frac{\pi}{2\omega} - \frac{\pi}{6\omega} = \frac{3\pi - \pi}{6\omega} = \frac{2\pi}{6\omega} = \frac{\pi}{3\omega} \text{ secs.}$$

EXERCISE IV

1. Deduce the equation for the simple harmonic motion of a particle.

Two simple harmonic motions, having the same period but differing in phase and amplitude, are acting in the same direction on a particle Show that the resultant motion is simple harmonic, and deduce the expression for the resulting amplitude and phase. (*Calcutta 1940*)

2 Find the resultant of two mutually perpendicular *S. H.* motions which agree in period but differ in phase. Consider the important cases for phase difference varying from 0 to 2π (*Punjab, 1953*)

3. A particle executes a *S. H M.* of period 10 *seconds* and amplitude 5 *ft.* Calculate its maximum acceleration and velocity.

Ans. 1·974 *ft /sec*^{2}. ; 3·142 *ft./sec.*

4. The path of a body executing a *S. H. M.* along a straight line is 4 *cms.* long : and its velocity, when passing through the centre of its path, is 16 *cms./sec.* Calculate its time-period. Ans. ·7854 *sec.*

5. The maximum velocity of a particle undergoing a *S. H M.* is 8 *ft./sec.* and its acceleration at 4 *ft.* from the mean position is 16 *ft./sec*^{2}. What is (*i*) its *amplitude* and (*ii*) its *period of vibration.* Ans. (*i*) 4 *ft.* ; (*ii*) 3·142 *secs.*

6. Explain the characteristics of a simple harmonic motion and show how to find the velocity at any phase of the motion.

A particle executes simple harmonic motion of period 16 *secs.* Two *seconds* after it passes the centre of oscillation, its velocity is found to be 4 *ft. per second.* Find the amplitude. (*Madras. 1949*)

Ans. 14·41 *ft.*

7. Define simple harmonic motion and show that if the displacement of a moving point at any time is given by an equation of the form

$$x = a \cos \omega t + b \sin \omega t,$$

the motion is simple harmonic. If $a = 3$, $b = 4$, and $\omega = 2$, determine the period, amplitude, maximum velocity and maximum acceleration of the motion. (*Madras, 1949*)

Ans. (*i*) 3·142, (*ii*) 5, (*ii*) 10, (*iv*) 20.

8. Find the velocity and acceleration of a point executing simple harmonic motion.

A point describes simple harmonic motion in a line 4 *cms.* long. Its velocity, when passing through the centre of the line, is 12 *cm. per second.* Find the period. (*Calcutta, 1945*)

Ans. 1·047 *secs.*

9. Define a *S.H.* motion, explaining the meanings of the terms, period, amplitude and phase.

A particle is subjected simultaneously to two *S.H.* vibrations of the same period but of different amplitudes and phases, in perpendicular directions. Find an expression for the resultant motion and show that the path traced by the particle is an ellipse.

For what conditions may the path be a circle and a straight line ? (*Calcutta, 1950*)

10. A mass of 15 *lbs.* is suspended from a fixed point by a light spring. In the equilibrium position, the spring is extended by 8 *inches.* The mass is then pulled down by 4 *inches* and released from rest. Show that it executes a *S.H.M.* and calculate its time-period. Also calculate the energy of its mass.

Ans. ·9069 *sec.* ; 40 *ft.-poundals.*

11. Show that a compound pendulum would swing most rapidly when the distance of its *c.g.*, from the axis of oscillation equals its radius of gyration.

12. A thin and square metal plate, of a side $2l$, is suspended from one corner so as to swing in a vertical plane. Calculate the length of the *equivalent simple pendulum.* **Ans.** $4\sqrt{2l/3}$.

13. Calculate the time-period of a circular disc of radius r, oscillating about an axis through a point, distant $r/2$ from its centre and perpendicular to its plane. **Ans.** $2\pi\sqrt{3r/2g}$.

14. Find the velocity, acceleration and the periodic time of a point executing Simple Harmonic Motion.

A particle is moving with simple harmonic motion in a straight line. When the distance of the particle from the equilibrium position has the values x_1 and x_2, the corresponding values of the velocity are u_1 and u_2. Show that the period is

$$2\pi[(x_2^2-x_1^2)/(u_1^2-u_2^2)]^{\frac{1}{2}}.$$

Find also (*i*) the maximum velocity and (*ii*) the amplitude.

(*Madras, 1949*)

15. A particle moves with uniform speed in a circle. Show that the motion may be resolved into two simple harmonic motions at right angles to each other. How do they differ in phase and amplitude? Show how the potential and kinetic energies of a particle executing simple harmonic motion vary. (*Calcutta*)

16. Show that the total energy of a particle executing simple harmonic motion is proportional to (*a*) the square of its amplitude, (*b*) the square of its frequency.

Show how, on an average, its energy is half *kinetic* and half *potential* in form.

17. In the *HCl* molecule the force required to alter the distance between the atoms from its equilibrium value is $5{\cdot}4\times10^5$ dynes per cm. What is the fundamental frequency of the vibration of the molecule, assuming the vibration to be simple harmonic, and the mass of the *Cl* atom to be infinite compared to that of the *H* atom which is $1{\cdot}66\times10^{-24}$ gm.?

(*Cambridge Scholarship Certificate*)
Ans. $9{\cdot}1\times10^{13}$.

18. Find graphically the resultant of two simple harmonic motions at right angles to one another (*a*) when the amplitudes and periods are equal and one vibration differs in phase by $\pi/2$ from the other, (*b*) when the amplitudes are equal, the period of one is twice that of the other and the slower vibration is $\pi/2$ ahead of the other.

19. The total energy of a particle executing a *S.H.M.* of period 2π *sec.* is 10,240 *ergs* $\pi/4$ *sec.* after the particle passes the midpoint of the swing its displacement is $8\sqrt{2}$ *cm.* Calculate the amplitude of the motion and the mass of the particle. (*Oxford and Cambridge Higher School Certificate*)
Ans. 16 *cms.*; 80 *gms.*

20. Show that the motion of the piston of a steam engine is approximately simple harmonic if the connecting rod is long compared with the crank.

CHAPTER V

MEASUREMENT OF MASS—THE BALANCE

54. Mass and Weight. The *mass* of a body is the *quantity of matter* contained in it and is an *inherent*, *invariable* and *fundamental* property of it, quite independent of the presence or absence of any other neighbouring bodies or of the place where the body happens to be situated. Thus the mass of a given body will be the same at the equator, at the poles of the earth, or, for that matter, anywhere else in the whole of the universe.

The *weight* of a body, on the other hand, is the *force* with which it is attracted by the earth towards its centre, and is equal to the *product of its mass and the acceleration due to gravity.*

Thus, if m be the mass of a body, and g, the acceleration due to gravity, its weight is given by $w = m.g$.

Since the value of g changes from place to place, being inversely proportional to the square of the distance from the centre of the earth*, the weight of the *same* body differs from one place to another, being about half a per cent greater at the poles than at the equator, twenty-eight times its weight on earth, on the sun and about one-sixth its weight on earth, on the moon.

It will thus be seen that the weight of a body in a *variable* property of it, depending not only upon its own mass but also on its distance from the centre of the earth, *i.e., on its position, relative to the earth.*

Then, again, since the *mass* of a body endows it with the *property of inertia* or of reluctance to change of both rest and motion, we may also define it as the *degree of resistance of matter to changes of motion.*. As against this, the weight of a body, being a force, directed towards the centre of the earth, *tends to accelerate its own motion* in that direction. Thus, *whereas the one resists, the other tends to produce, motion.*

Nevertheless, at a place, since g is constant†, at any rate, within a small space, the weights of two bodies are directly proportional to their masses. For, if w and w' be their *weights* and m and m', their *masses*, we have

$$w = mg \text{ and } w' = m'g.$$

So that

$$w/w' = mg/m'g = m/m'.$$

If follows, therefore, that the common physical balance may be used to compare masses. For, although, strictly speaking, it really compares weights—indicating a measure of their equality or want of equality,— but since the value of g, for the body as well as the standard weights, placed in its two pans respectively, is the *same*, the forces exerted at the two ends of the beam, in its equilibrium

*See Chapter VI.

†This was first shown by *Galileo* in 1591, by his famous experiment of dropping simultaneously two *unequal* masses from the top of the *Leaning Tower of Pisa*, when they reached the ground together. The same fact was confirmed by *Newton* and later by *Bessel*, by using pendulums with hollow bobs, filled with materials of different densities and, observing no variations in the value of g beyond those within experimental error. And finally, it has been shown conclusively by *Eotvos* by his experiments with an ingenious modification of the Torsion Balance.

position, are evidently equal, thus indirectly establishing the equality of the two masses, irrespective of the value of g.

If it be desired, however, to determine the weight of a body, we make use of a *spring balance*, the stretch of the spiral spring of which, if not unduly large, is proportional to the force applied to it by the weight of the body suspended from it, and this, as can be readily seen, will be different for the *same* body at different places, depending on the value of g.

55 The Common Balance. It is, in essentials, *an equi arm lever of the first order* and depends, for its action, on the principle of moments.

The essential feature of its construction is a symmetrical *rigid* beam usually in the form of a *triangular lattice girder*, as shown in Fig 93, (to ensure lightness with strength), pivoted centrally, so as to be free to rotate in the vertical plane about the horizontal axis provided by a knife-edge of steel or agate, resting on an agate plane carried by a stout vertical pillar. A long and light pointer, fixed at right angles to it moves over a small ivory scale below, whose central division marks its normal position, when the beam is in equilibrium or at rest. A screw, worked upwards and downwards, at the top of the pointer, enables the *c.g.* of the beam (together with the pointer) to be raised or lowered, as desired*.

Two other knife edges, similar to, and equidistant from, the central one, are carried by the beam itself on either side, with two identical scale pans, of equal mass, suspended from the agate planes resting on them.

The whole instrument is enclosed in a glass case, with side-windows and a sliding front, to safeguard against disturbance due to air draughts or temperature variations,—all weighings being carried out with the glass case properly closed on all sides.

The body† to be weighed is placed in the left-hand pan and standard weights from the weight box, in the right hand pan‡, starting with the seemingly heavier ones, until the pointer swings evently on either side of the central mark on the ivory scale If the *arms* (*i.e.*, the two halves of the beam, on either side of the central knife edge) be of the *same length* and the scale pans be of the *same weight*, the beam will come to rest in the horizontal position, but if the weights of the scale pans differ even slightly, it will be tilted towards the side of the heavier weight, with the pointer moving correspondingly over the scale below.

The use of the Rider. Since the weight boxes are not provided with weights smaller than 1 *milligram*, the final adjustment for the equilibrium of the beam is made with the help of what is called a '*rider*', which is just a piece of wire, weighing 1 *centigram*, and bent into the form shown, (Fig. 92), and can be moved over the right half of the beam by a lever-device, manipulated from outside the case, —this arm of the balance being graduated into 100 equal divisions from the central to the end knife-edge. With the rider at the 100th division, the effect is equivalent to placing a centigram weight in the right-hand pan ; so that, when it is, say, at the *nth* division, the effect is equivalent to adding a weight of $n/100$ *centigram* or $10n/100$ or $n/10$ *milligram* to the pan.

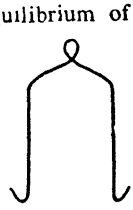


Fig. 92.

56. Essentials or Requisites of a Good Balance. There are three essentials of a good physical balance, *viz.*, (*i*) *Truth*, (*ii*) *Sensitiveness* (or *Sensitivity*), and (*iii*) *Stability* or *Quickness*.

*If the beam were to be pivoted exactly at its *c.g.*, it would be in neutral equilibrium, and will remain at rest at any angle with the horizontal. Its *c.g.* is, therefore, arranged to be below the central knife-edge, because as it tilts one way or the other, the *c g.* rises upward, and the beam is thus in stable equilibrium.

†Too heavy bodies, likely to break or bend the beam, should be avoided.

‡It is purely a matter of convenience, with no principle involved in it.

1. Truth. *A balance is said to be true, when, with its scale pans unloaded, or equally loaded, the beam remains horizontal.*

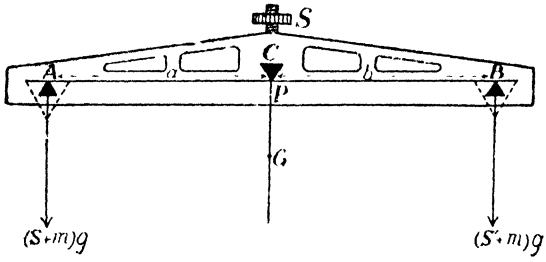


Fig. 93.

Let a and b be the lengths of the two arms of the balance, (Fig. 93), and S and S', the weights of its two scale pans.

Then, with the scale pans unloaded, the beam will remain horizontal, when the moments on either side of the central knife-edge C balance each other, *i.e.*, when

$$S \times a = S' \times b. \qquad ..(i)$$

Now, let the scale pans be loaded with *equal* masses m and m.

Then, for equilibrium, we have $(S+m).a=(S'+m).b.$ ..(*ii*)

Subtracting relation (*i*) from (*ii*), we have $m.a=m.b.$

whence, $a=b.$ ..(*iii*)

Substituting this in relation (*i*) we have

$$S=S'. \qquad (iv)$$

Thus, a *true* balance must have (*i*) *arms of equal lengths* and (*ii*) *pans of equal weights.*

2. Sensitiveness. *A balance is said to be sensitive when, for a small difference of loads in the two scale pans, the beam (and, therefore, the pointer) swings through an appreciable angle,* it being assumed that the balance is true.

Thus, the ratio between the deflection of the beam or the pointer and the difference of load, (usually 1 *m.gm.*) causing it, measures the sensitiveness or the sensitivity of the balance. So that, the greater the displacement of the pointer for a given difference of load, —or, conversely, the smaller the difference of load required to produce a given displacement of the pointer,—the greater the sensitiveness of the balance. Usually, a balance is regarded to be quite sensitive, when a difference of 1 *m.gm.* in load causes the pointer to be displaced through 1 division on the scale.

(*i*) **Case of a Balance with the three knife-edges in one plane.** Let Fig. 94 represent a vertical section of the balance through the centre of the beam, passing through the three knife-edges at A, B and C, *all in one plane.*

Let a be the length of each arm, d, the depth of the *c.g.* (G) of the beam from the central knife-edge C and M, its mass. Further, let S and $(S+m)$ be the masses of the two scale pans

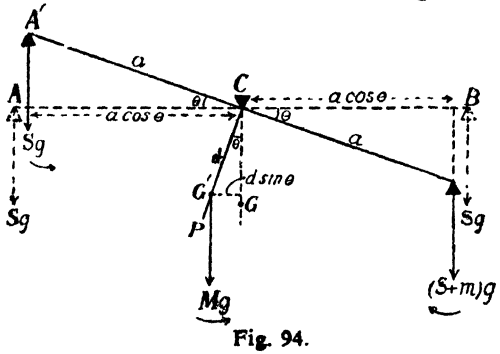


Fig. 94.

together with their loads, the difference of load m between them being small.

Then, if θ be the deflection of the beam from its initial horizontal position, so that it takes up the position $A'B'$, with its *c.g.* shifted to G', we have, taking moments about C,

$$(S + m)g.a \cos \theta = Sg.a \cos \theta + Mg.d \sin \theta.$$

Or, $$(S + m).a \cos \theta = S.a \cos \theta + M.d \sin \theta.$$

Or, $$m.a \cos \theta = M.d \sin \theta.$$

Or, $$\frac{\sin \theta}{\cos \theta} = \frac{m.a}{Md}. \quad \text{Or,} \quad \tan \theta = \frac{m.a}{Md}. \qquad ..(i)$$

And, if θ be small, $\tan \theta = \theta$; so that, in that case,

$$\theta = \frac{m.a}{M.d}. \quad .. \quad ..(ii) \text{ Or, } \frac{\theta}{m} = \frac{a}{M.d}, \qquad ..(ii)$$

where θ/m measures the *sensitiveness* of the balance,

It is thus clear that *to increase the sensitiveness* of the balance.

(*i*) a must be large, *i e., the arms (or the beam) must be long,*

(*ii*) M must be small, *i.e., the beam must be light,* and

(*iii*) d must be small, *i.e., the c.g. of the beam must be close to the central knife-edge.*

Now, a cannot be increased beyond reasonable limits. For, as *Birge* correctly pointed out, the bending of a beam being proportional directly to the cube of its length and inversely to the cube of its thickness or depth, its thickness will also have to be increased in the same proportion with its length if its original stiffness is to be maintained, and this will inevitably increase its mass in a much greater proportion, thereby seriously impairing sensitiveness.

Nor can the beam be made light beyond a limit, or else it will break or bend permanently ; so that, the only workable alternative is to decrease d. This may be done with the help of the vertical movement of the screw, provided at the top of the pointer, (Fig. 93), though, carried to an excess, this too has its own drawbacks, *viz.*, (*i*) *loss of stability,* and (*ii*) *a longer period of swing of the beam.* We have, therefore, to content ourselves with a judicious compromise between all these factors.

Further, if l be the length of the pointer, its displacement s on the scale will obviously be $l\theta$. And, therefore, if θ be small, so that $\tan \theta = \theta$, we have from relation (*ii*) above,

$$s = l \cdot \frac{m.a}{M.d}.$$

To determine the displacement of the pointer, it is by no means necessary to wait until it actually comes to rest ; it can be easily estimated from its swings to the right and the left.

Thus, suppose we have a scale with its zero at one end and the successive turning points of the pointer occur on it at the $10th$, the $2nd$ and the $8th$ divisions, (Fig. 95). Then, clearly, if S be its resting-point, *i.e.*, the division where it will eventually come to rest, it is clear that the successive displacements of the pointer from this division are $s_1 = (10 - S)$, $s_2 = (S - 2)$ and $s_3 = (8 - S)$.

Now, although, theoretically, as we shall soon see, (page 152), the oscillations of the pointer must be simple harmonic, it really seldom happens that the oscillations of any vibrating system remain truly so. The oscillations always die down and their amplitude goes on progressively decreasing due to air-resistance and other causes, but the ratio between the successive swings to the left and the right, —very aptly called **'decrement'**,—is found to remain constant.

Thus, with s_1, s_2 and s_3 as the successive swings of the pointer, we have

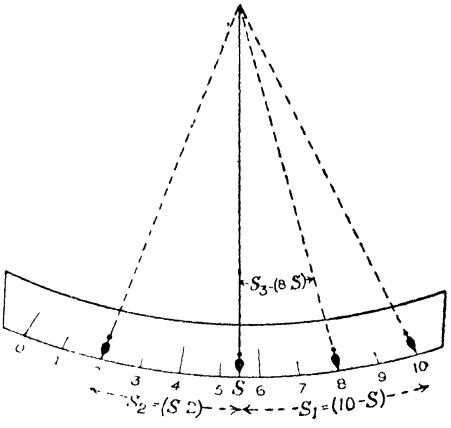


Fig. 95.

$\frac{s_1}{s_2} = \frac{s_2}{s_3}$, and so on. So that,

$$\frac{10-S}{S-2} = \frac{S-2}{8-S}$$

Or, $(S-2)^2 = (10-S)(8-S)$

Or, $S^2-4S+4 = S^2-18S+80.$

Or, $14S = 76,$

whence, $S = 76/14.$

$= 5{\cdot}43.$

Thus, the pointer will ultimately come to rest at the $5{\cdot}43rd$ division on the scale.

(*ii*) **Case of a Balance with the end knife-edges in a different plane from the central one** The balance, discussed above, with all the three knife-edges lying in the same plane, is really only an ideal one,—this condition being hardly ever attainable in ordinary balances. For, the beam does yield, however so little, to the forces acting at its two ends, so that the end knife-edges do get depressed a little below the central one, and no longer remain in the same plane with it.

Let us see how the sensitiveness is affected when the three knife-edges are *not* co-planar.

Let h be the height of the end knife-edges A and B, above the central one, C, (Fig. 96), and let the beam be deflected through an angle θ, as before, for an extra mass m in the right-hand pan. Then, for equilibrium, we have here,

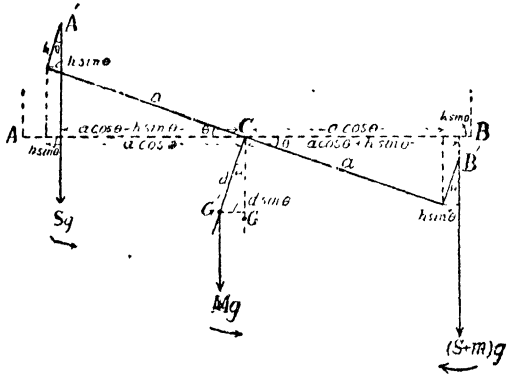


Fig. 96.

$$(S+m).g.(a\cos\theta+h\sin\theta) = S.g.(a\cos\theta-h\sin\theta) + Mg.d\sin\theta.$$

Or, $$(S+m)(a\cos\theta+h\sin\theta) = S(a\cos\theta-h\sin\theta)+M.d\sin\theta.$$

Or, $S.a \cos\theta + S.h \sin\theta + m\,a\cos\theta + m.h\sin\theta = S.a\cos\theta - S.h\sin\theta + M.d\sin\theta.$

Or, $2\,S\,h\sin\theta + m.a\cos\theta + m.h\sin\theta = M.d\sin\theta.$

Now, if θ be small, so that $\sin\theta = \theta$, and $\cos\theta = 1$, we have

$$2S.h.\theta + ma + mh\theta = M.d.\theta.$$

Neglecting $mh\theta$ as the product of very small quantities, we have

$2S\,h.\theta + ma = M\,d.\theta$. Or, $ma = M.d.\theta - 2S\,h.\theta = \theta(Md - 2Sh)$,

whence, **sensitiveness**, $\dfrac{\theta}{m} = \dfrac{a}{Md - 2Sh}$ ... (*iv*)

Now, three possible cases arise :

(*i*) **When h = o, i.e., when the three knife-edges are co-planar.** Here, $\theta/m = a/Md$, [See relation (*iii*) above], and *the sensitiveness is quite independent of the total load* (2*S*).

(*ii*) **When h is positive, i.e., when the end knife-edges are higher than the central knife-edge.** In this case, obviously, *the sensitiveness increases with the total load.* But, as will be readily seen from Fig. 96, the effective length of the arm, in the tilted position of the beam, becomes greater on the side of the heavier, and smaller on the side of the lighter, pan than its true length ; so that, for a given value of the excess load *m*, and for a given deflection θ, the difference of moments due to the pans on either side of *C* is greater for a heavier than for a lighter load, with the result that *the greater the load in the pans, the longer the beam takes to attain equilibrium.*

(*iii*) **When h is negative, i e., when the end knife-edges are lower than the central one.** In this case, clearly, *the sensitiveness decreases with increase in the total load.*

N.B. We have seen above how, in the ideal balance, with its three knife-edges in the same straight line, the end knife-edges get depressed with the beam a little below the central one when the pans are loaded. This results in a decrease in the sensitiveness with increasing load. If, therefore, the end knife-edges could be arranged at the correct height above the central knife-edge, the decrease in sensitiveness due to flexure could be just offset by its increase due to the latter, and the balance thus made *equally sensitive for all loads.* The method has actually been used with the success in building balances whose sensitiveness is quite independent of the total load placed in their pans.

3. Stability or Quickness *A balance is said to be stable (or quick), if, with the pans unloaded or equally loaded, the beam be disturbed, its time of swing is small and it comes back to rest quickly,* thus making for convenience in weighing.

Now, as we have seen above, with the three knife-edges in the same horizontal plane, the condition for equilibrium is that

$$(S+m)g.a\cos\theta = S.g\,a\cos\theta + Mg\,d\sin\theta.$$ [See page 149.

Therefore, if the two pans be *equally loaded, i.e.*, if $m = 0$, or there be no extra load in the right hand pan, the only *restoring moment* about *C*, tending to bring the beam back to its original position, is $Mg.d\sin\theta$, or $Mgd\theta$, if θ be small. This, obviously, tends to accelerate the motion or swing of the beam ; so that, if α be the *angular acceleration* it produces in the beam and *I*, the moment of inertia of the moving system about the central knife-edge *C*, we have

$$\alpha = \frac{Mg.d\theta}{I}.$$

And, clearly, I = *moment of inertia of the beam about C* + *moment of inertia of the two scale pans about C.*

Or, $$I = Mk^2 + 2S.a^2,$$

where k is the *radius of gyration* of the beam about C.

Thus, $$\alpha = \frac{Mg.d}{Mk^2 + 2Sa^2}.\theta.$$ Or, $\alpha = \mu\theta$, where μ is a constant.

Putting $\frac{Mg.d}{Mk^2+2Sa^2} = \mu$, a constant for a given balance.

Or, $$\alpha \propto \theta,$$

i.e., the angular acceleration of the beam is proportional to its angular displacement. The swing of the beam is thus a *simple harmonic motion*, and its time period t is, therefore, given by the relation,

$$t = 2\pi\sqrt{I/\mu}.$$

Or, $$t = 2\pi\sqrt{\frac{1}{\frac{Mg.d}{Mk^2+2Sa^2}}} = 2\pi\sqrt{\frac{Mk^2+2Sa^2}{Mg.d}}.$$

Or, $$t = 2\pi\sqrt{\frac{k^2}{gd} + \frac{2Sa^2}{Mg.d}}.$$

In order, therefore, that t be small, *i.e.*, the balance be *stable*, k, S *and a should be small* and M *and* d *should be large.* We thus see that *a balance would be stable when*

(*i*) *its arms are short*;

(*ii*) *its beam is heavy, with its c.g. far below the central knife-edge*;

(*iii*) *the radius of gyration of the beam about the central knife-edge is small*;

and that the *stability diminishes with increasing load.* It will be seen at once that almost all these conditions are opposed to those for sensitiveness. So that, sensitiveness and stability of a balance are, to a great extent, *mutually exclusive*, and we have, therefore, to strike a working balance between the two.

57. Faults in a Balance—Determination of True Weights.

1. Arms unequal in lengths and pans unequal in weights. The commonest fault in a balance is that it may appear to be true, *i.e.*, the beam may swing evenly on either side of the central knife-edge, with the moments on either side balancing each other, and yet the arms may have unequal lengths and the pans, unequal weights.

Thus, if S_1 and S_2 be the weights of the two scale pans and a and b, the lengths of the two arms respectively, we have

$$S_1.a = S_2.b. \quad \ldots (i)$$

The true weight w of a body may be determined with such a balance by the method of *double weighing*, *i.e.*, by weighing the body first in one pan and then in the other.

Let the counterpoising weight in the right hand pan be w_1, when the body is placed in the left hand pan. Then, clearly,

$$(w+S_1).a = (w_1+S_2).b \quad \ldots (ii)$$

And, $\therefore$ subtracting relation (*i*) from (*ii*) we have

$$w.a = w_1.b. \qquad \ldots(iii)$$

Now, let the body be placed in the right hand pan and let the counterpoising weight required in the left hand pan be w_2. So that,

$$(w+S_2).b = (w_2+S_1)a. \qquad \ldots (iv)$$

Again, subtracting relation (*i*) from (*iv*), we have

$$w.b = w_2.a. \qquad \ldots (v)$$

Multiplying relations (*iii*) and (*v*), we, therefore, have

$$w^2.ab = w_1.w_2.ab, \qquad \text{whence,} \qquad w^2 = w_1.w_2.$$

Or, $$w = \sqrt{w_1.w_2},$$

i.e., the true weight of the body is the geometric mean of its apparent weights in the two pans.

The same will be true if the pans be equal in weight and the arms slightly different in lengths.

Note. If we multiply relation (*iii*) and (*v*) above, *crosswise*, we have

$$w.w_2.a^2 = w.w_1.b^2. \qquad \text{Or,} \quad w_2.a^2 = w_1.b^2.$$

Or, $$\frac{a^2}{b^2} = \frac{w_1}{w_2}. \quad \text{Or,} \quad \frac{a}{b} = \sqrt{\frac{w_1}{w_2}}.$$

And, since from relation (*i*) above, $a/b = S_2/S_1$, we have

$$\frac{a}{b} = \frac{S_2}{S_1} = \sqrt{\frac{w_1}{w_2}}.$$

Thus, we can determine the ratio between the lengths of the two arms or that between the weights of the two scale pans.

2. Scale pans unequal in weights. Another common fault in a balance is that whereas the arms may be equal in length, the scale pans may not be truly equal in weights, so that the beam does not remain perfectly horizontal.

To determine the true weight of a body with such a balance, we again resort to *double weighing*, *i.e.*, to weighing the body first in one pan and then, in the other.

Let its apparent weights in the two scale pans respectively be w_1 and w_2. Then, if the length of each arm be a and the weights of the two scale pans, S_1 and S_2, we have

in the first case, $(S_1+w).a = (S_2+w_1).a$, or $S_1+w = S_2+w_1 \ldots(i)$

And, *in the second case*, $(S_2+w).a=(S_1+w_2).a$, or $S_2+w=S_1+w_2 \ldots(ii)$

Adding relations (*i*) and (*ii*), we have

$$S_1+S_2+2w = S_1+S_2+w_1+w_2. \quad \text{Or,} \quad 2w = w_1+w_2,$$

whence, $$w = \frac{w_1+w_2}{2},$$

i.e., the true weight of the body is now the arithmetic mean of its apparent weights in the two scale pans.

3. Inaccuracy of the Brass Weights. A possible source of error may also be the inaccuracy of the brass or '*standard*' weights, supplied in the weight box, due to their getting worn out by use or getting slightly rusted by discuse. The probability of error due to such causes is presumably the least in the case of the larger weights and the greatest in the case of the smaller ones. So that, assuming the largest among them to be accurate, others, of smaller denominations, are counterpoised against it ; these smaller ones are then counterpoised against others smaller than them, the process being continued up to the very smallest ones, and, in this manner, the errors in the smaller weights are easily

detected. Thus, for example, a weight of 100 *gms.* is first counterpoised against 50 + 20 + 20 + 10 ; then, the weight of 50 *gms.* against 20 + 10 + 10 + 5 + 2 + 2 + 1, and again the weight of 20 *gms.* against 10 + 5 + 2 + 2 + 1 and so on. To make sure, the weights assumed to be correct must also be tested against a weight known accurately in terms of the *International Standard.*

4. Blunting of the Knife-Edges. Due to constant use, the knife edges get blunted or rounded off, in course of time ; so that, with the tilt of the beam, the point of contact with the plane of support may shift slightly. This tantamounts to a slight change in the lengths of the arms and must also be corrected for.

58. Correction for Buoyancy. Ordinarily, we make all our weighings in air. But air, in common with all other gases and liquids, exerts an upward thrust on a body immersed in it, in strict obedience to the *Principle of Archimedes.* So that, the body to be weighed, as well as the brass weights, against which it is weighed, are subject to this upthrust or buoyancy due to the air displaced by them, which is equal to the weight of the displaced air, in either case.

If the body weighed happens to have the *same* density as that of the material of which the standard weights are made their volumes too would obviously be the *same*, when they are counterpoised against each other, and the volume of air displaced by both, and hence the buoyancy or upthrust due to it, would just be counter-balanced and the standard weights used would straightaway give the true weight of the body in vacuum. This is, however, rarely the case. But, it makes it clear why, in realby true and accurate balances, we insist upon the arms and the pans being identical in length, volume and mass.

More often than not, the density of the body is quite different from that of the material of the standard weights, and, therefore, even when they counterpoise each other, their volumes, and hence also the weights of the air displaced by them are altogether different. Let us, therefore, deduce the necessary correction in this commonly occurring case.

Let the *true* mass of the body weighed be M and its density ρ. And, let the mass of the weights required to counterpoise it be M', and the density of their material ρ'. And, finally, let the density of air be δ.

Then, clearly, *volume* of the body and hence the volume of air displaced by it $= M/\rho'$.

And, so the *weight* of this displaced air and, therefore, the upthrust due to it $= M.\delta.g/\rho$.

So that, their *apparent* (or *observed*) *weight of the body*

$$= \left(Mg - \frac{M}{\rho} .\delta.g. \right).$$

Similarly, the volume of the standard weights and, therefore, of the air displaced by them $= M'/\rho'$;

and the upward thrust due to this displaced air $= M'.\delta.g/\rho'$.

So that, *their apparent weight* $= \left(M'g - \frac{M'}{\rho'} .\delta.g \right)$.

Since the body and the standard weights counterpoise each other in air, their apparent weights must be equal. And, therefore,

$$\left(Mg - \frac{M}{\rho} .\delta.g. \right) = \left(M'g - \frac{M'}{\rho'} .\delta.g \right).$$

Or,
$$M\left(1 - \frac{\delta}{\rho} \right) = M' \left(1 - \frac{\delta}{\rho'} \right).$$

whence,
$$M = M' \left(\frac{1 - \delta/\rho'}{1 - \delta/\rho} \right) = M' \left(1 - \frac{\delta}{\rho'} \right)\left(1 + \frac{\delta}{\rho} \right).$$

$$= M' \left[1 + \left(\frac{\delta}{\rho} - \frac{\delta}{\rho'} \right) \right]. \quad \left\{ \begin{array}{l} \text{Neglecting the product of} \\ \delta/\rho \text{ and } \delta/\rho', \text{ compared with} \\ \text{the other terms.} \end{array} \right.$$

Or,
$$M = M' \left[1 + \delta \left(\frac{1}{\rho} - \frac{1}{\rho'} \right) \right].$$

From the above it follows at once that

$$M > = < M' \text{ according as } \rho < = > \rho',$$

i.e., the true weight of a body (*i e., its weight in vacuum*) is greater than, equal to or less than, its observed or apparent weight in air, according as its density is less than, equal to or greater than that of the material of the standard weights used.

SOLVED EXAMPLES

1. The arms of a balance are unequal in length but, without the scale pans, the beam and the scale-pan holders are correctly balanced. The scale pans A and B are of weights $2w_1$ and $2w_2$ respectively. A body placed in pan A has an apparent weight W_1 and placed in pan B has an apparent weight W_2. Show that the true weight of the body is

$$\sqrt{[W_1 W_2 + 2(w_1 W_1 + w_2 W_2) + (w_1 + w_2)^2]} - (w_1 + w_2).$$

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Let the true weight of the body be W and the lengths of the left-hand and right hand arms be a and b respectively.

Then, since equilibrium is attained with the body in the left-hand pan and a weight W_1 in the right-hand pan, the moments on either side of the central knife-edge must be equal ; so that, neglecting moments due to the pan holders, which already balance each other, we have

$$(2w_1 + W).a = (2w_2 + W_1).b. \quad \ldots \quad \ldots \quad \ldots \ (i)$$

Again, equilibrium is attained with the body in the right-hand pan and a weight W_2 in the left-hand pan ; so that,

$$(2w_2 + W).b = (2w_1 + W_2).a. \quad \ldots \quad \ldots \ (ii)$$

Multiply the corresponding sides of relation (*i*) and (*ii*), we have

$$(2w_1 + W).(2w_2 + W).ab = (2w_2 + W_1).(2w_1 + W_2).ab.$$

Or, $$(2w_1 + W)(2w_2 + W) = (2w_2 + W_1).(2w_1 + W_2).$$

Or, $$4w_1w_2 + 2w_1W + 2w_2W + W^2 = 4w_1w_2 + 2w_2W_2 + 2w_1W_1 + W_1W_2.$$

Or, $$W^2 + 2W(w_1 + w_2) = W_1W_2 + 2(w_1W_1 + w_2W_2).$$

Adding $(w_1 + w_2)^2$ to both sides, we have

$$W^2 + 2W(w_1 + w_2) + (w_1 + w_2)^2 = W_1W_2 + 2(w_1W_1 + w_2W_2) + (w_1 + w_2)^2.$$

The left hand expression is clearly the complete square of $W + (w_1 + w_2)$; so that,

$$[W + (w_1 + w_2)]^2 = W_1W_2 + 2(w_1W_1 + w_2W_2) + (w_1 + w_2)^2.$$

Or, $$W + (w_1 + w_2) = \sqrt{[W_1W_2 + 2(w_1W_1 + w_2W_2) + (w_1 + w_2)^2]}.$$

And, $\therefore$ $$W = \sqrt{[W_1W_2 + 2(w_1W_1 + w_2W_2) + (w_1 + w_2)^2]} - (w_1 + w_2).$$

Or, the true weight of the body $= \sqrt{[W_1W_2 + 2(w_1W_1 + w_2W_2) + (w_1 + w_2)^2]} - (w_1 + w_2).$

2. The arms of a balance are similar and of equal length, a. The scale pans are similar and of equal weight, P. When the beam of the balance is horizontal, the central knife-edge is a distance x vertically above the middle of the line joining the knife-edges of the scale pans, and the c.g. of the balance is a distance y vertically below the same point Assuming that the weight of the moving system of the balance is W, derive an expression for the angle of deflection of the beam when weights w_1 and w_2 are placed on the scale pans. $[w_1 > w_2]$.

(Joint Matriculation Board High School Certificate)

Let AB, (Fig. 97) be the position of the beam, when the pans are yet unloaded, C, that of the central knife-edge, P and P, of the scale pans and p, that of the pointer, with G, as the *c.g.* of the beam.

Let the heavier weight w_1 be now placed in the right-hand pan and the lighter weight w_2, in the left-hand pan and let the beam, and, therefore, also the pointer, deflect through an angle θ, into the positions $A'B'$ and p', with the *c.g.* of the beam at G', where $OG' = y$ (O being the mid-point of the line joining the knife-edges of the two scale pans). Let the central knife-edge be now at C', where $OC' = x$ (*given*).

Then, the different forces,—all acting vertically downwards,—on the beam are

(*i*) $(P+w_1)$ at B', (*ii*) $(P+w_2)$ at A' and (*iii*) W, the weight of the moving system, at G'.

Since the beam is in equilibrium in this position, the moments about the knife-edge C', on either side, must be equal.

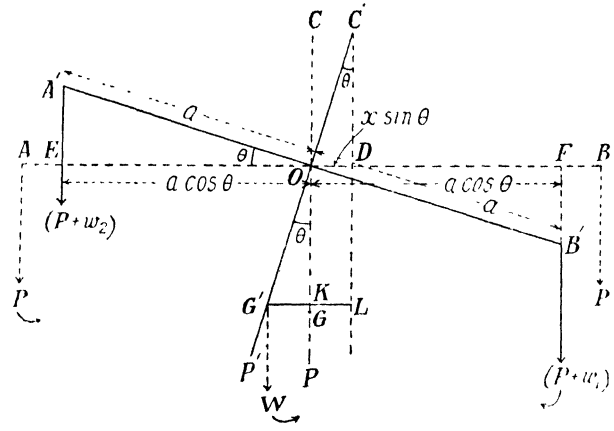


Fig. 97.

Or, $$(P+w_2).DE+W.G'L=(P+w_1).DF.$$

But $$DE=OE+OD=OA'\cos\theta+OC'\sin\theta=a\cos\theta+x\sin\theta,$$

$$G'L=G'K+KL=G'K+OD=OG'\sin\theta+x\sin\theta,$$

$$=y\sin\theta+x\sin\theta=(x+y)\sin\theta,$$

and $$DF=OF-OD=OB'\cos\theta-OC'\sin\theta=(a\cos\theta-x\sin\theta).$$

So that, $(P+w_2).(a\cos\theta+x\sin\theta)+W.(x+y).\sin\theta=(P+w_1).(a\cos\theta-x\sin\theta).$

Or, $$P.a\cos\theta+P.x\sin\theta+w_2.a\cos\theta+w_2.x\sin\theta+W.(x+y).\sin\theta.$$
$$=P.a\cos\theta-P.x\sin\theta+w_1.a\cos\theta-w_1.x\sin\theta.$$

Or, $$2P.x\sin\theta+w_1.x\sin\theta+w_2.x\sin\theta+W.(x+y).\sin\theta$$
$$=w_1.a\cos\theta-w_2.a\cos\theta.$$

Or. $$\sin\theta[2Px+x(w_1+w_2)+W(x+y)]=a\cos\theta.(w_1-w_2).$$

Or, $$\frac{\sin\theta}{\cos\theta}=\tan\theta=\frac{a(w_1-w_2)}{2Px+x(w_1+w_2)+W(x+y)},$$

whence, $$\theta=\tan^{-1}\frac{a(w_1-w_2)}{2Px+x(w_1+w_2)+W(x+y)}.$$

This, then, is the angle of deflection of the beam, when weights w_1 and w_2 are placed on the two scale pans.

3. With a balance of which the arms were 10 cm. long, it was found that 0·010 gm. extra-load on one pan deflected the beam of mass 20 gms. through 1° and that this deflection was independent of the loads placed on the pans. What can you deduce from these measurements ? (*Oxford Local Higher School Certificate*)

Since the deflection of the beam for the given extra load of ·01 *gm.* is quite independent of the loads placed on the pans, it is clear that *the three edges are co-planar.*

And, since, in the question, every other factor is given except the depth of the *c.g.* of the beam below the central knife-edge or the centre of the beam, we are obviously expected to determine its value. Let it be *h.*

Since the beam is in equilibrium at angle $\theta=1°$ from the initial horizontal position, it is evident that the moment about the central knife-edge due to its

We know that sensitivity of a balance is given by the relation,

$$\frac{\theta}{m} = \frac{a}{Md - 2S.h}, \text{ where } 2S \text{ is the total load.} \quad \text{[Page 151.}$$

Now, *in the first case*, $\theta = 3{\cdot}0$, $m = 1$ *mg.* $= {\cdot}001$ *gm.* and $2S = 0$.

And, $\therefore \quad \frac{3}{{\cdot}001} = \frac{a}{Md}$. ... ...(*i*)

In the second case, $\theta = 2{\cdot}70$, $m = 1$ *mg.* $= {\cdot}001$ *gm.* and $2S = 100$ *gms.*

So that, $\frac{2{\cdot}7}{{\cdot}001} = \frac{a}{Md - 100h}$. ... ...(*ii*)

$\therefore$ dividing relation (*i*) by (*ii*), we have

$$\frac{3}{2{\cdot}7} = \frac{Md - 100h}{Md}. \quad \text{Or, } 3\,Md = 2{\cdot}7\,Md - 270h.$$

Or, $270h = -{\cdot}3\,Md$, whence, $h = -{\cdot}3\,Md/270$.

The negative value of h thus clearly indicates that *the end knife-edges are below the central knife-edge.*

Now, let the sensitivity of the balance be x divisions per *mg.* for a load of 200 *gms.* Then, we have

$$\frac{x}{{\cdot}001} = \frac{a}{Md - 200h} = \frac{a}{Md - 200\left(\frac{-{\cdot}3\,Md}{270}\right)}$$ [Substituting the value of h from above.

Or, $\frac{x}{{\cdot}001} = \frac{a}{Md + \frac{60\,Md}{270}} = \frac{270a}{330\,Md}$. ... ...(*iii*)

$\therefore$ dividing relation (*iii*) by (*i*), we have

$$\frac{x}{{\cdot}001} \times \frac{{\cdot}001}{3} = \frac{270a}{330\,Md} \times \frac{Md}{a}. \quad \text{Or, } \frac{x}{3} = \frac{270}{330},$$

whence, $x = \frac{270 \times 3}{330} = \frac{27}{11} = 2{\cdot}455.$

Thus, the sensitivity of the balance for a load of 200 *gms.* is 2·455 divisions per *mg.*

6. A piece of metal weighs 300 gms in air If the densities of the metal and the brass weights used by 19 gms. c.c , and 8 gms./c.c., respectively, and that of air ·00123 gm./c.c., calculate the true mass of the piece in vacuum and the correction due to buoyancy.

We know that the true mass of a body in vacuum is given by the relation,

$$M = M'\left[1 + \delta\left(\frac{1}{\rho} - \frac{1}{\rho'}\right)\right], \quad \text{[See page 154.}$$

where M is the true mass of the body, ρ, its density, M', its apparent mass, ρ', the density of the weights used and δ, the density of the air.

Here, $M' = 300$ *gms.*, $\rho = 19$ *gms /c.c.*, $\rho' = 8$ *gms./c.c.* and $\delta = {\cdot}00123$ *gms./c.c.* Substituting these values in the relation for M above, we have

$$M = 300\left[1 + {\cdot}00123\left(\frac{1}{19} - \frac{1}{8}\right)\right],$$

$$= 300\left(1 - {\cdot}00123 \times \frac{11}{152}\right) = 300\left(1 - \frac{{\cdot}01353}{152}\right)$$

$$= 300(1 - {\cdot}00008902) = 300 - {\cdot}026706 = 299{\cdot}973294\ gms.$$

Thus, the true mass of the body in vacuum will be 299 973294 *gms.*, the buoyancy correction being obviously ·026706 *gm.*

EXERCISE V

1. What are the essentials of a good balance and how are they secured in actual practice ?

Show that the sensitiveness of a balance is independent of the load in the two scale pans, if the three knife-edges be co-planar. How does the position of the centre of gravity of the beam affect the working of the balance ?

2. The arms of a balance are each 7 *cms.* long, the length of the pointer is 12 *cms.* and the mass of the beam is 50 *gms.* If the knife-edges are in a plane and the centre of gravity of the beam is 0·02 *cm.* below the centre knife-edge, how much will the end of the pointer be deflected when the difference in load in the pans is 1 milligram ? (*Cambridge Local Higher School Certificate*)

Ans 0·84 *cm.*

3. A certain balance has a beam weighing 200 *gms.*, with knif-edges carrying the pans 15 *cms* from the central knife-edge. What is the depth of the centre of gravity of the beam below this knife-edge, if a weight of 1 *mg.* placed on one of the pans displaces the end of the pointer through a distance of 0·5 *cm.*, the pointer being 15 *cms.* long ? (*Br Inter*)

Ans. 0·0225 *mm.*

4. Two balances, made of the same material, are alike in all respects except that the linear dimensions of one are *n* times those of the other. Compare the angular deflections of the beams for a given difference in load

(*Cambridge Local Higher School Certificate*)

Ans. $1/n^3 : 1$.

5. A body is weighed first in the left and then in the right hand pan of a balance, the respective weights being 9·842 *gms.* and 9·833 *gms.* Find the true weight of the body and the ratio of the lengths of arms of the balance.

Ans. *True weight* 9 837 *gms ; Ratio of arms* 1·0005 : 1.

6. Discuss the points to be taken into consideration in the design of an accurate, sensitive and convenient balance. If the arms are of unequal lengths, show how the error on this account can be avoided. How would you except the sensitiveness to vary with the load ? (*Bombay, 1933*)

7. What are the requisites of a balance ? Obtain the general expression used for determining the conditions for these requisites and show that the conditions for two of these are mutually contradictory. (*Punjab, 1933*)

8. Sketch the essential parts of a balance in which the two end knife-edges are *h cms.* below the central knife-edge and discuss the conditions of its sensitiveness.

9. Obtain an expression for the true mass of a body in vacuum, when its apparent mass in air is M' *gm* , its density ρ, the density of the standard weights used ρ', and the density of air δ. Would the same treatment be applicable, if the body be weighed in a liquid ?

10. Given the apparent weights of a body in two different liquids, of known densities, similar standard weights being used in both the cases, how will you proceed to calculate the density of the body ?

11. A glass stopper (density 2·5 *gms /c c.*) is first weighed in water of density ·995 *gm./c.c* and then in oil. Its apparent weights in the two liquids are found to be 18·6 *gms.* and 20·4 *gms.* respectively, the brass weights used being similar in either case. Calculate the density of oil. Ans. ·8489 *gm /c.c.*

CHAPTER VI

ACCELERATION DUE TO GRAVITY

59. Acceleration due to Gravity. Galileo was the first person to have performed in 1590, the then bold and spectacular experiment of dropping a cannon and a musket ball from the *Leaning Tower of Pisa*, which, contrary to the teachings of *Aristotle*, reached the ground simultaneously. He thus clearly showed that, *at any given place*, all bodies, big or small, when dropped so as to fall freely, do so at the *same uniform rate*, neglecting, of course, the resistance to their motion due to air. That is to say, *all bodies, irrespective of their mass or nature, falling freely in vacuum, will have the same acceleration at a given place.* This acceleration is called the *acceleration due to gravity*, as it is due to the gravitational attraction of the body by the earth, towards its centre, (see Chapter VII). It is denoted by the letter g, and is *numerically equal to the force with which a unit mass is attracted by the earth towards its centre, i.e., equal to the weight of unit mass**.

The value of g differs from place to place, being the *greatest at the poles* and the *least at the equator*. Its value, for all practical purposes, is however, taken to be 981 *cms./sec*^{2}., in the *C.G.S.* system, and 32 *ft./sec*^{2}., in the *F.P.S.* system. Due to this comparatively large value of g, bodies fall much too quickly to the surface of the earth, when dropped freely, and hence it becomes difficult to measure it directly with any great accuracy. It is, therefore, determined indirectly with the help of a simple or a compound pendulum, or by other methods. We shall now proceed to consider some of these in proper detail.

60. The Simple Pendulum. A *simple* (or, a *mathematical*) *pendulum* is just a heavy particle, (*ideally, only a point-mass*), suspended from one end of an *inextensible, weightless* string, whose other end is fixed to a rigid support,—the point where the string is fixed to the support being known as *the point of suspension* of the pendulum. In practice, we usually take a small and a heavy metallic spherical bob, tied to a *fine* silk thread.

The motion of the pendulum is simple harmonic and isochronous, f i.e., *its time-period is quite independent for its amplitude of small swings, and is given by the relation,* $t = 2\pi\sqrt{l/g}$, where l is the *length* of the pendulum, (or the *distance between its point of suspension and*

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*This is so, because the force with which a body of mass m is attracted by the earth, towards its centre, is equal to its weight mg; and, therefore, if, $m = 1$, *i.e.*, if the body be of *unit* mass, *this force of attraction* on it, or its *weight*, is equal to g.

For the discovery of this property of the pendulum, we are again indebted to *Galileo*, who noticed a swinging lamp in the cathedral at Pisa and timed its oscillations against his own pulse beats. The time taken for each swing was found to be the same and, as far as Galileo could judge, quite independent of the size of the swing.

the centre of gravity of the bob), and, *g, the acceleration due to gravity at the place.* This will be clear from the following :

Let *S* be the point of suspension and *O*, the mean or equilibrium position of the bob, (Fig. 99). Then, if the bob be given a *small* angular displacement θ, *in the vertical plane*, (or the plane of the pendulum itself), so as to occupy the position *A*, it is clear that it will be under the action of two forces, *viz.*, (*i*) its weight *mg, acting vertically downwards*, (*m* being the *mass* of the bob), and (*ii*) *the tension T of the string acting, along the string, towards its point of suspension S.*

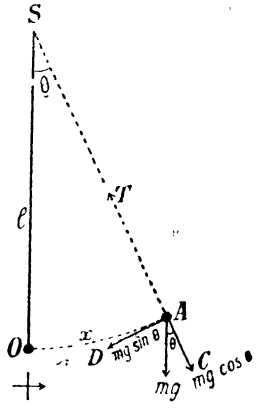


Fig. 99.

Resolving the weight (*mg*) of the bob, into two rectangular components, we have

(*a*) component *mg cos* θ, acting along the string, as shown, and

(*b*) component *mg sin* θ, at right angles to it, *i.e.*, along the tangent to the arc *OA*, in the direction *AO*.

Obviously, the former component (*mg cos* θ) is just balanced by the tension (*T*) of the string, there being no motion along it either way; so that, the only force left acting on the bob is *mg sin* θ, towards its mean or equilibrium position *O*.

Now, *acceleration = force/mass.*

And, therefore, *acceleration of the bob = mg sin* $\theta/m = g \sin\theta$ in the direction *AO*, or towards the mean or equilibrium position *O* of the bob. And, if θ be small, we have $\sin\theta = \theta$,

Hence, *acceleration of the bob* $= g\theta$, directed towards *O*.

Again $\theta = OA/SO = x/l$, [$\because$ *angle = arc/radius.*

where, $OA = x$, the *displacement of the bob*, and *l*, the *length of the pendulum.*

$$\therefore \quad \text{acceleration of the bob} = g.\frac{x}{l} = \frac{g}{l}.x.$$

Or, putting $g/l = \mu$, a constant, for a given pendulum at a given place, we have

$$\text{acceleration of the bob} = \mu x.$$

Or, *acceleration of the bob* $\propto x$, *and is directed towards O*,

i.e, the acceleration of the bob is proportional to its displacement from its mean or equilibrium position, and is always directed towards that position. The bob thus executes a simple harmonic motion, and its *time-period t* is given by the relation,

$$t = 2\pi\sqrt{1/\mu} = 2\pi\sqrt{\frac{1}{g/l}} = 2\pi\sqrt{\frac{l}{g}}.$$

Alternative Methods.

Method (*i*). Let the bob be given a small angular displacement α in the vertical plane, so as to occupy the position *A*, (Fig. 100). In other words, let α be the *angular amplitude* of the bob.

Then, clearly, the *c.g.* of the bob has been raised up through a vertical distance OC, when AC is the perpendicular dropped from A on to SO.

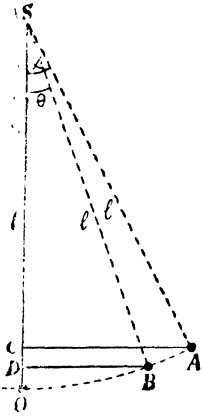


Fig. 100.

$\therefore$ *potential energy of the bob at* $A = mg.OC$.

As the bob is released at A, it starts moving towards O, thus acquiring *kinetic energy*, (due to its motion), at the expense of its potential energy.

Consider the bob to be at B, on its way towards O, and let its angular displacement here be θ. Then, clearly,

potential energy of the bob at $B = mg.OD$,

where BD is the perpendicular dropped from B on to SO.

$\therefore$ *loss in P.E. of the bob in moving from position* A *to position* $B = mg.OC - mg.OD = mg\ (OC - OD)$.

$= mg\ [(SO - SC) - (SO - SD)]$.

Now, $SO = SA = SB = l$, the *length of the pendulum*;

and $\therefore$ $SC = SA \cos \alpha = l \cos \alpha$ and $SD = SB \cos \theta = l \cos \theta$.

So that, *loss in P. E. of the bob* $= mg[(l - l \cos \alpha) - (l - l \cos \theta)]$

$= mgl\ (\cos \theta - \cos \alpha)$.

This must, therefore, be equal to the gain in *K.E.* of the bob at B, or equal to the *K.E.* of the bob at B, (since its *K E* at A was equal to zero). If I be the moment of inertia of the bob about the axis of suspension through S, and ω, its angular velocity at B, we have

$$K\,E.\ of\ the\ bob\ at\ B = \tfrac{1}{2} I\omega^2.$$

And $\therefore$ $$\tfrac{1}{2} I\omega^2 = mgl\ (\cos \theta - \cos \alpha). \quad .. \quad .. \quad (i)$$

Differentiating this expression with respect to time (t), we have

$$I\omega . \frac{d\omega}{dt} = -mgl \sin \theta . \frac{d\theta}{dt},$$

the angular amplitude (α) of the bob being a constant quantity.

Now, $d\theta/dt = \omega$, the angular velocity of the bob at B.

And $\therefore$ $d\omega/dt$ is its angular acceleration, here.

Thus, $I\omega . d\omega/dt = -mgl \sin \theta . \omega$.

Or, $I . d\omega/dt = -mgl \sin \theta$,

whence, $d\omega/dt = -mgl \sin \theta / I$.

And since θ is small, $\sin \theta = \theta (radians)$, very nearly.

Hence, $$acceleration\ of\ the\ bob = -\frac{mgl}{I} . \theta, \quad .. \quad .. \quad (ii)$$

the —ve sign merely indicating that it is directed towards O, opposite to that in which the angular displacement (θ) *increases.*

Or, neglecting the $-ve$ sign, and putting $mg.l/I = \mu$, a constant for a given pendulum at a given place, we have

$$acceleration\ of\ the\ bob = \mu . \theta,$$

and, therefore, proportional to θ, its angular displacement from its mean or equilibrium position.

The bob thus executes a *S.H.M.*, and its time-period is given by

$$t = 2\pi\sqrt{1/\mu} = 2\pi\sqrt{\frac{I}{mgl/I}} = 2\pi\sqrt{\frac{I}{mg.l}}.$$

Now, if r be the radius of the bob, we have, by the principle of parallel axes, $I = 2mr^2/5 + ml^2$.

And, therefore, $$t = 2\pi\sqrt{\frac{2mr^2/5 + ml^2}{mgl}}.$$

Since the bob is a small one, its radius r is negligible compared with l, the length of the pendulum; so that, we have

$$t = 2\pi\sqrt{ml^2/mgl}. \quad \text{Or,} \quad t = 2\pi\sqrt{l/g}$$

Method (ii). See Borda's pendulum, (next article).

Calculation for g. Squaring the expression for the time-period of the pendulum, we have

$$t^2 = 4\pi^2 l/g, \quad \text{whence,} \quad g = 4\pi^2 l/t^2.$$

Thus, knowing l, the *length* of the pendulum, and t, its *time-period*, we can easily calculate out the value of g at the given place.*

Drawbacks of Simple Pendulum. Though simple in theory, and easy to perform, this method of determining the value of g is not quite an accurate one, due to its numerous drawbacks, the more important ones of which are the following :

1. *A simple pendulum is just an ideal conception, not realizable in actual practice, for we can neither have a point-mass, nor a weightless string; so that, the string too has a moment of inertia about the suspension-axis.*

2. *The resistance and the buoyancy of air appreciably affect the motion of the bob.*

3. *The relation for the time-period (t), obtained above, is true only for oscillations having an infinitely small amplitude.*

4. *The motion of the bob is not, strictly speaking, a motion of translation, for it also has a rotatory motion about the axis of suspension passing through the point of suspension.*

5. *The bob has also a relative motion with respect to the suspension-thread at the extremities of its amplitude on either side.*

61. Borda's Pendulum. In this pendulum, the bob is a sphere of *large radius* and, assuming that it is rigidly fixed to the string and oscillates only about the axis of suspension, (there being no relative motion between the bob and the string), its time-period is given by $$t = 2\pi\sqrt{\frac{l + \frac{2}{5}r^2/l}{g}},$$

where l is the length of the pendulum, and r, the radius of the bob. This relation for t may be deduced as follows :

*The earliest determinations of the value of g were all made by means of simple pendulums. Thus, **Picard**, in 1669, used a pendulum in which a copper *bob* of diameter 1 inch was suspended by an *aloe fibre* (which remains unaffected by moisture) and, in the year 1792, **Borda** and **Cassini** used one, with a platinum ball of diameter 1·5 inches, suspended by an iron wire about 12·75 *ft.* long.

Let *SO* be a Borda's pendulum, (Fig. 101), suspended from the point of suspension *S* and let *m* be the mass of the bob, of radius *r*.

Imagine the bob to oscillate in the plane of the paper, and let it be displaced from its original position *A* to the position *B*, through an angle θ, at any given instant.

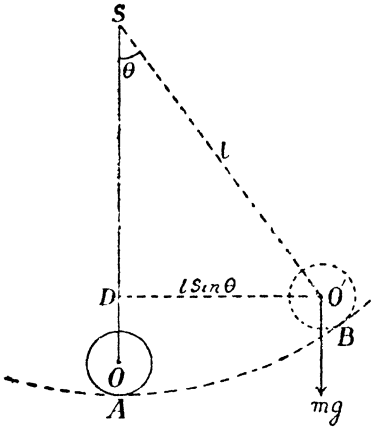


Fig. 101.

The only restoring force on the bob in this position is its weight *mg*, acting vertically downwards, which has, obviously, a *moment about the axis of suspension* through *S* and perpendicular to the plane of the paper,

$$= mg \times O'D = mg.l \sin\theta,$$

where l is the length of the pendulum, and $\therefore$ $O'D = l \sin\theta$.

This, then, is the *restoring couple*, acting on the bob in the position *B*, tending to bring it back to its original position.

If $d\omega/dt$ be the *angular acceleration* produced in the bob, and *I*, its *moment of inertia* about the axis of suspension through *S*, the *couple* is also equal to $I.d\omega/dt$.

$\therefore$ $I.d\omega/dt = -mg\,l \sin\theta = -mg.l\theta$. [$\because$ if θ be small, $\sin\theta = \theta$. $\because$ the couple due to *mg* is clockwise and $\therefore$ negative.]

Or, $d\omega/dt = -\theta.mg\,l/I = -\mu\theta$,

where $mgl/I = \mu$, a constant.

Or, $d\omega/dt \propto \theta$,

i.e., the angular acceleration of the bob is proportional to θ, *its angular displacement; so that, its motion is simple harmonic*; and, therefore, the *time-period* of the pendulum is given by

$$t = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{1}{mgl/I}}.$$

Or,
$$t = 2\pi\sqrt{\frac{I}{mgl}}.$$

Now, I = the *M. I.* of the bob about an axis through its *c.g.* and parallel to the axis of suspension $+ ml^2$. [Principle of ∥ axes.

$$= M.\ I.\ of\ the\ bob\ about\ its\ diameter + ml^2.$$

Or, $I = \frac{2}{5}mr^2 + ml^2$, ($\because$ *M. I.* of a sphere of mass *m*, about its diameter is equal to $\frac{2}{5}\,mr^2$).

$\therefore$
$$t = 2\pi\sqrt{\frac{ml^2 + \frac{2}{5}mr^2}{mgl}} = 2\pi\sqrt{\frac{l^2 + \frac{2}{5}r^2}{gl}}.$$

Or,
$$t = 2\pi\sqrt{\frac{l + \frac{2}{5}r^2/l}{g}}.$$

Thus, as will readily be seen, *the time-period of this pendulum is the same as that of a simple pendulum of length* $(l+\frac{2}{5}r^2/l)$, which, for this reason, is called the *length of an equivalent simple pendulum*, or the *reduced length* of the pendulum.

Further, if r be equal to zero, *i.e.*, if the bob be just a point, or the pendulum be a *simple pendulum*, we have, substituting $r = 0$ in the relation for t, above,

$$t = 2\pi\sqrt{\frac{l+0}{g}}. \quad \text{Or, } t = 2\pi\sqrt{\frac{l}{g}},$$

which is the expression for the time-period of a simple pendulum, given above, (§ 60).

This pendulum too cannot give an accurate value of g, as, in the first place, *the string has also a moment of inertia about the axis of suspension* and secondly, *there is relative motion between the bob and the string*, the bob oscillating about it at each extreme end.

62. Compound Pendulum. *A compound, rigid* (or, a *physical*) *pendulum is just a rigid body, capable of oscillating freely about a horizontal axis passing through it.* Its vibrations are also simple harmonic and its time-period is given by the relation,

$$t = 2\pi\sqrt{\frac{I}{mg.\,l}},$$

where I is its *moment of inertia about the axis of suspension* ; m, its *mass* and l, its *length* (or, the distance between its axis of suspension and its centre of gravity).

This may be seen from the following :

Let S be the *point of suspension* of the body, (or the pendulum), through which passes a horizontal axis, perpendicular to the plane of the paper, about which the body oscillates. Its *c.g.*, G, will obviously lie vertically below S, in its normal position of rest, (Fig. 102).

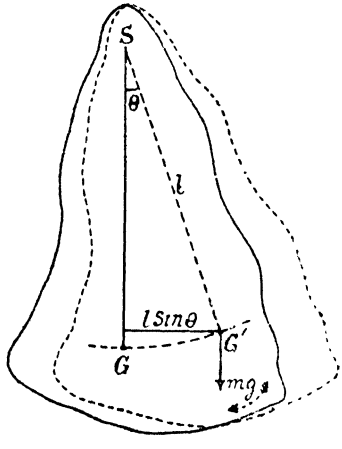


Fig. 102.

Let the body be displaced through an angle θ, into the dotted position shown ; so that, its *c.g.* is now at G'.

Then, the *couple* acting on the body due to its weight mg will, obviously, be $mg.l \sin\theta$, tending to bring it back into its original position, (SG' being equal to l, the length of the pendulum).

If the angular acceleration produced in the body by this couple be $d\omega/dt$, the couple will also be equal to $I.d\omega/dt$, where I is the moment of inertia of the body about an axis through the point of suspension S, and perpendicular to the plane of the paper.

So that, $I.\frac{d\omega}{dt} = -mg.l \sin\theta = -mg.l\theta$...(A) [$\because$ if θ be small, $\sin\theta = \theta$.

Or, $\frac{d\omega}{dt} = -\frac{mg.l}{I}.\theta = -\mu\theta$. [where $mg.l/I = \mu$, a constant.

Thus, the angular acceleration, $(d\omega/dt)$, of the body is proportional to its angular displacement, (θ). The body, therefore, executes a *S.H.M.*, and its time-period is given by

$$t = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{1}{mgl/I}} = 2\pi\sqrt{\frac{I}{mgl}}. \quad \ldots (B)$$

If I_g be the *M. I.* of the body *about an axis through G., its c.g.*, and parallel to the axis of suspension through *S*, we have, by *the principle of parallel axes*,

$$I = I_g + ml^2.$$

And, if k be the *radius of gyration* of the body about the axis through G, obviously $I_g = mk^2$: so that,

$$I = mk^2 + ml^2.$$

$\therefore$ substituting the value of I in relation (B) above, we have

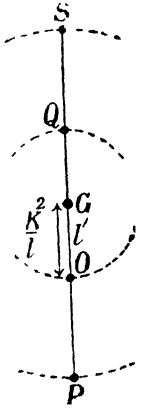
$$t = 2\pi\sqrt{\frac{mk^2+ml^2}{mgl}} = 2\pi\sqrt{\frac{k^2+l^2}{lg}}. \quad \ldots \ldots \quad (C)$$

Or, $$t = 2\pi\sqrt{\frac{k^2/l + l}{g}}.$$

i.e., the period of viberation is the same as that of a simple pendulum of length $\frac{k^2}{l} + l$, or $\frac{k^2+l^2}{l}$, which is, therefore, **the length of an equivalent simple pendulum**, or the **reduced length** of the pendulum. **It** is sometimes denoted by the letter L.

Since k^2 is always greater than zero, *this length of an equivalent simple pendulum is always greater than l.*

Centre of Oscillation. A point O, on the other side of G, at a distance k^2/l from G, is called the *centre of oscillation*, and a horizontal axis through it, parallel to the axis of suspension, is known as the *axis of oscillation* of the pendulum.



Thus, $GO = k^2/l$, (Fig. 103). Putting it equal to l', we have $SO = \frac{k^2+l^2}{l} = l + \frac{k^2}{l} = l + l'$.

And, $\therefore$ $$t = 2\pi\sqrt{\frac{l+l'}{g}} = 2\pi\sqrt{\frac{L}{g}}.$$

63. Interchangeability of the Centres of Suspension and Oscillation. If the pendulum is inverted and suspended about *the axis of oscillation* through O, its time-period of vibration will obviously be given by

$$t = 2\pi\sqrt{\frac{k^2+l'^2}{l'g}}.$$

And, since $k^2/l = l'$, we have $k^2 = l.l'$;
so that, the expression for the time-period t becomes

$$t = 2\pi\sqrt{\frac{ll'+l'^2}{l'g}} = 2\pi\sqrt{\frac{l+l'}{g}},$$

which is the same as about the axis of suspension through S.

Or, *the centres of suspension and oscillation are interchangeable, i.e.,* are reciprocal to each other,—a property of the pendulum, first discovered by Huyghens.

Thus, we get the same values for the time-period and the length of the *equivalent simple pendulum* whether the pendulum be suspended at S or at O, *i.e., at a distance l from the c.g., (G), or at a distance* k^2/l *from it.*

If, therefore, we draw two circles or arcs, with G as centre, and radii equal to l and k^2/l respectively, they will cut SG produced at S and Q above, and at O and P below, G.

Then, clearly, $SG = GP = l$ and $GQ = GO = k^2/l$.

And, therefore, $QP = GP + GQ = l + k^2/l = l + l' = SO$.

Thus *we have four points, S, Q, O and P, collinear with G, the c.g. of the pendulum, about which the time-period is the same.*

If, therefore, we can determine, by experiment, these four points, we can easily find out the length L or $(l+l')$ of the *equivalent simple pendulum*, and hence the value of g at the given place, with the help of the relation, $t = 2\pi\sqrt{L/g}$, where t is the time-period of the pendulum.

64. Centre of Percussion. Fig. 104 shows a section of a rigid body, of mass m, by a vertical plane passing through its *c.g.*, G with S, as its point of suspension, the axis of suspension through which is perpendicular to the plane of the paper.

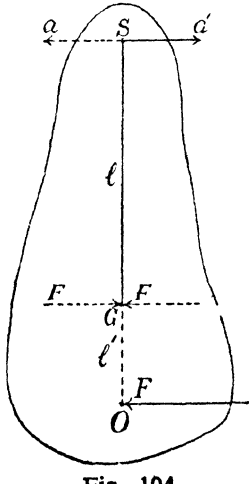


Fig. 104.

Let a force F be applied at O, in the direction shown, so as to be perpendicular to both the line SGO and the axis of suspension through S. Then, this force is equivalent to (*i*) *an equal, and a like parallel force F at G, and* (*ii*) *a clockwise couple, formed by the force F at O and an equal and opposite force F at G,* the moment of which is clearly equal to $F \times l'$ where l' = distance GO.

Now, this force F at G tends to produce linear acceleration a, say, in all the particles of the body, including the centre of suspension S ; so that, $a = F/m$,

in the direction of the force, *i.e.*, from right to left.

This, then, is the acceleration produced at S by the force F at G. The couple, on the other hand, tends to produce an angular acceleration a, say, in the body, about a parallel axis through G. If I be the moment of inertia of the body about this axis, we have

$I.a = F \times l'$, whence, $a = F \times l'/I = F \times l'/mk^2$,

where $I = mk^2$, (k being the *radius of gyration* of the body about this axis).

Now, linear acceleration = angular acceleration × distance from the axis.

Hence, linear acceleration produced by this couple is given by $a' = l \times a = F \times l' \times l/mk^2$, in the direction left to right, *i.e.*, opposite to that of a.

In order, therefore, that the force F applied at O may produce no effect at S, the linear acceleration a due to force F at G must be equal and opposite to the linear acceleration a' due to this couple, *i.e.*, $a = a'$. In other words, $F/m = F \times l' \times l/mk^2$,

whence, $l' \times l/k^2 = 1$. Or, $l' = k^2/l$.

This, therefore, is the distance of the point O from the *c.g.* of the body, and the point O is thus the centre of oscillation, (see page 166), and is here called the *centre of percussion*, with respect to S.

It is thus clear that *if a body be struck at the centre of percussion, or the centre of oscillation, in a direction perpendicular to its axis of suspension, it does not move bodily, as a whole, at its point of suspension, but simply turns about the axis passing through it.*

This explains why when a ball strikes against a bat such that the point where it strikes the latter is the centre of oscillation, or the centre of percussion, corresponding to the point where it is held in the hand as the point of suspension, no *sting* or *shock* of any kind is felt. Similarly, a good hammer should be so constructed that its centre of percussion lies in a line with the driving force.

65. Other points, collinear with the c.g., about which the time-period is the same. Squaring the expression, $t = 2\pi\sqrt{l^2+k^2/lg}$, for the *time-period* of a compound pendulum, where l is its *length* and k, its *radius of gyration* about the *c.g.*, we have

$$t^2 = \frac{4\pi^2.(l^2+k^2)}{lg}. \quad \text{Or,} \quad 4\pi^2(l^2+k^2) = l.g.t^2.$$

Or, $$4\pi^2l^2+4\pi^2k^2 = l.g.t^2.$$

Dividing both sides by $4\pi^2$, we have $l^2+k^2 = \frac{gt^2}{4\pi^2}.l$.

Or, $l^2 - \frac{gt^2}{4\pi^2}.l + k^2 = 0$, which is clearly a *quadratic equation in* l.

So that, $$l = \frac{\frac{gt^2}{4\pi^2} \pm \sqrt{\frac{g^2t^4}{16\pi^4} - 4k^2}}{2}.$$

Thus, l has two values, *viz.*,

$$l = \frac{gt^2}{8\pi^2} + \sqrt{\frac{g^2t^4}{64\pi^4} - k^2} = (a+b), \text{ say,}$$

$$l = \frac{gt^2}{8\pi^2} - \sqrt{\frac{g^2t^4}{64\pi^4} - k^2} = (a-b).$$

putting $\frac{gt^2}{8\pi^2} = a$, and $\sqrt{\frac{g^2t^4}{64\pi^4} - k^2} = b$

Therefore, there are two values of l at distances $(a+b)$ and $(a-b)$ from the *c.g.*, for which the time-period is the same.

even less than one-thousandth of that of the bar, and yet maintaining its strength and rigidity. The values of g and k may then be determined in the usual manner.

68. Kater's Reversible Pendulum. Devised and first used by *Captain Kater*, in the year 1817, to make the celebrated determination of the value of g in London, it is a compound pendulum, consisting of a brass or steel bar with a fixed *heavy bob* B and fitted with two *adjustable* and mutually facing knife-edges F_1 and F_2* near its two ends, *so that the pendulum may be suspended from either.* (Fig. 111). Two weights, W_1 and W_2, can be made to slide along the length of the bar and clamped in the position desired, the smaller weight W_2 having a *micrometer screw arrangement* for finer adjustment of its position. The position of the *c.g.* of the pendulum can be altered by changing the relative positions of the two weights, their positions being, however, so chosen that *the c.g. always lies in-between the two knife-edges* †

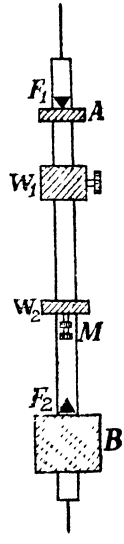


Fig. 111.

The pendulum is first suspended from the knife-edge F_1, and its time-period determined. It is then suspended from the knife-edge F_2, and its time-period determined again. If there be a divergence in the two values of the time-period, the heavier weight W_1 is moved up or down and a proper position found for it so that the time-period is very nearly the same, whether the pendulum be suspended from F_1 or F_2. The smaller weight W_2 is then adjusted by means of the micrometer screw M, until the time-periods, in the two cases, are as nearly equal as possible (say, differing only by ·01 *sec.* or less, *i.e.*, until the number of oscillations made by the pendulum in 24 hours, in the two cases, differs by just a fraction of one full oscillation). *When this is so, we have, obviously, one knife-edge at the centre of oscillation of the other.* The distance between the two knife-edges is measured carefully**. This gives the length L of the *equivalent simple pendulum*‡ and the value of g is then calculated from the relation $g = 4\pi^2 L/t^2$, (see page 171), where t is the mean of the time-periods about the two knife-edges, which *Kater* determined by the method of coincidences, (see § 69).

The values of L and t may, however, be determined more easily and accurately as follows :

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*It is gratifying to observe that in Kater's own pendulum, these were made in India, from a special variety of steel, called '*wootz*'.

†In fact, the heavier weight W_1 is there to ensure that this is so.

**This is done by means of a travelling microscope. The pendulum is laid horizontally on a table with its knife-edges lying alongside, and on a level with, a standard steel scale, and the positions of the knife-edges read on the scale with the help of the microscope. Or, a better method is to use a vertical *comparator* (an instrument carrying two microscopes fitted on two massive stone slabs), when the cross wires of the microscopes are first focused on the two knife-edges of the suspended pendulum, and then on a standard steel scale, fixed vertically in place of the pendulum. The distance between the positions of the two cross-wires then gives the distance between the two knife-edges.

‡This deduction from the reciprocal nature of the points of suspension and oscillation was first pointed out by *Bohnenberger*, in the year 1811.

First a near equility in the time-periods of the pendulum is obtained about either knife-edge, by adjusting W_1 and W_2, as explained above, the time-period being slightly greater, say about F_1 than about F_2, *i.e.*, when the bob B is down than when it is up. The weights are now kept fixed in their positions and one of the knife-edges, say F_1, moved up or down a bit to further narrow down the discrepancy in the two time-periods until a position is attained when a little more displacement of F_1 makes the time period greater about F_2 (with the bob up) than about F_1, (with the bob down), *i.e.*, a reversal in the relative magnitudes of the time-periods about the two knife-edges takes place. The time-periods of the pendulum for two slightly different distances between F_1 and F_2 are noted, *just before this happens*, and similarly for two slightly different distances between them, *after this happens*. These four distances between F_1 and F_2 are measured accurately and the time periods corresponding to them plotted on an exaggerated scale, as shown in Fig. 112, where *a* and *b* represent the distances and time-periods for oscillations about F_1 in the *first two cases* and *c* and *d* in the *second two cases*. The coordinates of the point of intersection *O* of *ab* and *cd* then give the true length *L* of the *equivalent simple pendulum* and the true time-period *t* corresponding to it.

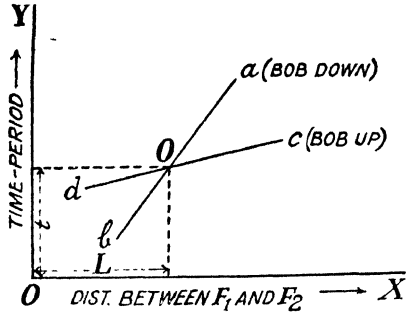


Fig. 112.

69. Kater's Method of Coincidences. Kater determined the time-period of his pendulum by what is known as the *method of coincidences*, in which the oscillations of the experimental pendulum, (Kater's, or any other), are compared with those of a *standard second's pendulum*, (*i.e.*, *a pendulum of time-period two seconds*), which may be a simple pendulum or a clock pendulum. This gives better results than those obtained by simply timing the oscillations against

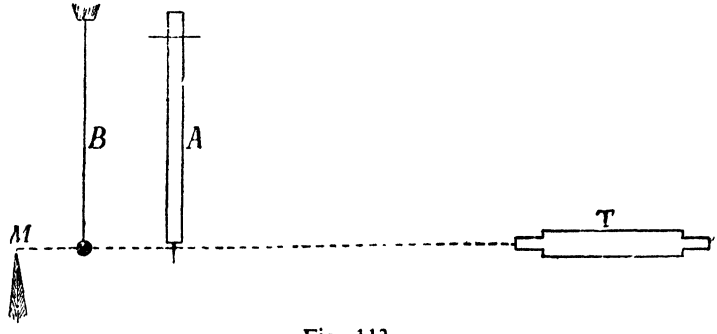


Fig. 113.

a stop watch or clock, the accuracy of which can hardly be expected to go beyond ·5 *sec.**

*This means that the time taken for several thousand of swings will have

The experimental pendulum *A* (Fig 113) here is suspended in front of the second's pendulum *B*, with its knife-edge resting on a rigid support, *care being taken to see that the lower ends of the two pendulums lie in the same level and exactly coincide with each other when viewed from in front, in their mean or equilibrium position.*

A suitable marking device* *M* is arranged behind the pendulum *B*, such that when the two pendulums are in their mean position, this mark *M* is *just* covered by their lower ends and is thus not visible to the observer viewing them through a telescope *T* some distance away.

The whole idea is to enable the observer, watching the oscillations, to judge as accurately as possible as to when exactly do the two pendulums come into coincidence, *i.e., as to when exactly do they simultaneously pass a particular reference point, in the same direction.* And this is perhaps best done by using a cross-wire in the eye-piece of the telescope itself, (in which case the marking device *M* becomes quite unnecessary).

The two pendulums are set oscillating, and, if they start together, and have identical time-periods, they continue to oscillate '*in step*', *i.e.*, they appear to the observer as *just* '*one*' pendulum,—pendulum *B* being *just* hidden behind pendulum *A*, all the time. But if their time-periods differ, ever so slightly, they soon get '*out of step*', and their oscillations are watched carefully until they both simultaneously pass the reference point fixed upon, (say, their lowest positions), in the same direction. When this happens, a '*coincidence*' is said to occur, and the mark *M* behind the pendulums is *just not visible* to the observer. After this, the pendulums again get out of step and *the next coincidence occurs when one of them gains or loses a whole swing or oscillation over the other.* The oscillations made by both between two successive coincidences are carefully counted.

Let n and $(n+1)$ be the oscillations made by the *seconds* and the *experimental* pendulums respectively.

Then, if t' and t be their respective time-periods, we have

$$t(n+1) = t'.n,$$

whence, $$t = t'\left(\frac{n}{n+1}\right) = t'\left(\frac{1}{1+1/n}\right) = t'\left(1+\frac{1}{n}\right)^{-1}$$

$$= t'\left(1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \ldots\ldots\right)$$

$$= t'\left(1 - \frac{1}{n}\right) \textit{ very nearly,}$$

neglecting the terms involving the second and the higher powers of n; for, with t and t' nearly the same, n is sufficiently large, (about 500 or more), and these terms become negligibly small.

Now, $t' = 2$ *secs.*, so that, $t = 2\left(1 - \frac{1}{n}\right)$, [pendulum *B* being a *second's pendulum*]

whence t, the time-period of the experimental pendulum can be easily

*A white paper pointer or just a black (iron) stand, with a white chalk-mark on it, would do

calculated out. This value of t is then used in the expression for g, above.

It will be easily seen that with all the care taken, it is not really possible to determine with absolute certainty the particular oscillations at which the *exact* coincidence occurs. Luckily, however, even a difference of a couple of oscillations, this way or that, hardly matters, *if n be fairly large*, as it usually is. For, supposing there is an error of *two* whole oscillations made in judging the point of coincidence. Then, clearly,

$$t=t'\left(1-\frac{1}{n\pm 2}\right)=t'\left[1-\frac{1}{n(1\pm 2/n)}\right]$$

Or,
$$t=t'\left[1-\frac{1}{n}\left(1\pm\frac{2}{n}\right)^{-1}\right]=t'\left(1-\frac{1}{n}\pm\frac{2}{n^2}\right).$$

The error introduced is thus $\pm 2/n^2$, which, if $n=500$, works out to be $+\cdot 0008\%$ and is, therefore, not of much consequence.

Note. In Kater's own determination, made at Portland place, the pendulum was set in front of a standard clock and the swings of the two pendulums observed by means of a telescope from a distance of 9 feet.

The standard clock was checked every 24 hours by stellar measurements, so that the rate of swing of the pendulum was, in effect, compared with the rate of rotation of the earth itself.

Successive coincidences occurred every 530 *secs.*, during which time the reversible pendulum completed 528 swings or half oscillations. The error thus worked out to 1 part in 1,00,000 and the length of seconds pendulum at sea level, in the latitude of London, came to 39·13829, inches.

70. Computed Time—Bessel's Contribution. *Kater*, in his reversible pendulum, made the time-period about the two axes *exactly* equal, which, as we have seen, is an extremely tedious process. But **Bessel** showed that it was by no means necessary to make them *exactly equal* and that it was enough to make them only *nearly equal.*

Thus, suppose the time-periods about the two axes are t_1 and t_2, respectively, (both being very nearly equal), and that l and l' are their respective distances from the *c.g.* of the pendulum.

Then, we have

$$t_1=2\pi\sqrt{\frac{k^2+l^2}{lg}} \quad \ldots (i)$$

[where k is the radius of gyration of the pendulum *about its c.g.*

and
$$t_2=2\pi\sqrt{\frac{k^2+l'^2}{l'g}} \quad \ldots (ii)$$

Squaring and re-arranging (i) and (ii), we have

$$t_1^2lg=4\pi^2(k^2+l^2), \quad \ldots (iii) \text{ and } t_2^2l'g=4\pi^2(k^2+l'^2). \quad \ldots (iv)$$

So that, subtracting (iv) from (iii), we have

$$g.(t_1^2.l-t_2^2.l')=4\pi^2(k^2+l^2)-4\pi^2(k^2+l'^2).$$
$$=4\pi^2(l^2-l'^2)=4\pi^2(l+l')(l-l').$$

$$\therefore \quad \frac{4\pi^2}{g}.(l+l')=\frac{t_1^2.l-t_2^2\,l'}{(l-l')}=\frac{2t_1^2.l-2t_2^2.l'}{2(l-l')}.$$
$$=\frac{(t_1^2+t_2^2)(l-l')+(t_1^2-t_2^2)(l+l')}{2(l-l')}.$$

Or, $$\frac{4\pi^2}{g}.(l+l') = \frac{t_1^2+t_2^2}{2} + \frac{t_1^2-t_2^2}{2} \cdot \frac{l+l'}{l-l'}.$$

This quantity, $\frac{t_1^2+t_2^2}{2} + \frac{t_1^2-t_2^2}{2} \cdot \frac{l+l'}{l-l'} = \tau^2$, [*cf* $\frac{4\pi^2 l}{g} = t^2$ for a simple pendulum]

where τ is called the *computed time* of the pendulum.

This distance $(l+l')$, between the two knife-edges, can be determined accurately, but $(l-l')$, the difference between the distances of the two axes from the *c.g* cannot be determined to any high degree of accuracy, because of the difficulty of locating the position of the *c.g.* of the pendulum correctly. Since, however, $(t^2-t_2^2)$ is much too small a quantity compared with $(t_1^2+t_2^2)$, this can only introduce an inappreciable error which does not matter.

71. Errors in the Compound Pendulum and their Remedies. Besides the difficulty of adjusting the time-periods to be *exactly the same* about either knife-edge, and of correctly measuring the distance between them, there are a number of other sources of error in a Kater's pendulum (or the compound pendulum, in general) for which proper corrections must be applied to obtain an accurate result, the chief among them being the following :

(*i*) **The Finite Amplitude of the Pendulum.** The expression for the time-period has been deduced on the assumption that the amplitude of swing of the pendulum is *vanishingly small* ; for, then alone, will its motion be truly simple harmonic in nature. In actual practice, however, it has always a finite value. This reduces its acceleration and thus increases its time-period The observed time period may be corrected for this error by multiplying it with $(1-\alpha_1.\alpha_2/16)$, where α_1 and α_2 are *half-swings* (in radians) at the beginning and at the end of the experiment, respectively, as can be seen from the following :

Proceeding in a manner similar to that discussed in connection with a simple pendulum (Alternative method (*i*), page 162), we have

$$\tfrac{1}{2}I\omega^2 = mgl\,(\cos\theta - \cos\alpha), \qquad \text{[Relation (}i\text{) page 162.}$$

Or, $$I\omega^2 = 2mgl\,(\cos\theta - \cos\alpha),$$

where α is the *angular amplitude* of the pendulum and θ, its angular displacement at time t.

Now, $$I = mk^2 + ml^2 = m\,(k^2+l^2), \quad \text{and} \quad \omega = -\frac{d\theta}{dt}.$$

So that, $$m(k^2+l^2)\left(-\frac{d\theta}{dt}\right)^2 = 2mgl(\cos\theta - \cos\alpha).$$

Or, $$(k^2+l^2)\left(\frac{d\theta}{dt}\right)^2 = 2gl(\cos\theta - \cos\alpha).$$

Or $$\left(\frac{d\theta}{dt}\right) = \sqrt{\frac{2gl}{k^2+l^2}(\cos\theta - \cos\alpha)}.$$

Or, $$dt = \frac{d\theta}{\sqrt{\frac{2gl}{k^2+l^2}(\cos\theta - \cos\alpha)}} = \sqrt{\frac{k^2+l^2}{2gl}} \cdot \frac{d\theta}{(\cos\theta - \cos\alpha)^{\frac{1}{2}}}.$$

Integrating this expression for the limits 0 to $t/4$ and 0 to α, where t is the observed time-period of the pendulum, we have

$$\int_0^{t/4} dt = \sqrt{\frac{k^2+l^2}{2gl}} \int_0^{\alpha} \frac{d\theta}{(\cos\theta - \cos\alpha)^{\frac{1}{2}}}.$$

$$= \sqrt{\frac{k^2+l^2}{2gl}} \int_0^{\alpha} \frac{d\theta}{\left(1-2\sin^2\frac{\theta}{2}-1+2\sin^2\frac{\alpha}{2}\right)^{\frac{1}{2}}}.$$

$$= \sqrt{\frac{k^2+l^2}{gl}} \cdot \frac{1}{\sqrt{2}} \int_0^{\alpha} \frac{d\theta}{\left[2\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}\right]}.$$

Or $$\frac{t}{4} = \sqrt{\frac{k^2+l^2}{gl}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \int_0^{\alpha} \frac{d\theta}{\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}.$$

Or $$\frac{2t}{4} = \sqrt{\frac{k^2+l^2}{lg}} \int_0^{\alpha} \frac{d\theta}{\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}.$$

Putting $\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\phi$, we have $\frac{1}{2}\,d\theta\cos\frac{\theta}{2} = d\phi\sin\frac{\alpha}{2}\cos\phi$.

Or, $$d\theta = 2d\phi\sin\frac{\alpha}{2}\cos\phi\cos\frac{\theta}{2}.$$

Substituting the value of $d\theta$ in the above expression, we have

$$\frac{t}{2} = \sqrt{\frac{k^2+l^2}{lg}} \int_0^{\pi/2} \frac{2\sin\frac{\alpha}{2}\cos\phi.d\phi}{\cos\frac{\theta}{2}\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}.$$

Now, $$\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}} = \left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\sin^2\phi\right)^{\frac{1}{2}}$$

$$= \left[\sin^2\frac{\alpha}{2}\left(1-\sin^2\phi\right)\right]^{\frac{1}{2}} = \left(\sin^2\frac{\alpha}{2}\cos^2\phi\right)^{\frac{1}{2}} = \sin\frac{\alpha}{2}.\cos\phi.$$

So that, $$\frac{t}{2} = \sqrt{\frac{k^2+l^2}{lg}} \int_0^{\pi/2} \frac{2\sin\frac{\alpha}{2}\cos\phi.d\phi}{\cos\frac{\theta}{2}\sin\frac{\alpha}{2}\cos\phi}.$$

$$= \sqrt{\frac{k^2+l^2}{lg}} \int_0^{\pi/2} \frac{2d\phi}{\cos\frac{\theta}{2}}.$$

Or, $$t = 4\sqrt{\frac{k^2+l^2}{lg}} \int_0^{\pi/2} \frac{d\phi}{\cos\frac{\theta}{2}}.$$

Since $\cos\frac{\theta}{2} = \sqrt{\cos^2\frac{\theta}{2}} = \sqrt{1-\sin^2\frac{\theta}{2}} = \left(1-\frac{\sin^2\alpha}{2}\sin^2\phi\right)^{\frac{1}{2}}$,

we have $$t = 4\sqrt{\frac{k^2+l^2}{lg}} \int_0^{\pi/2} \frac{d\phi}{\left(1-\sin^2\frac{\alpha}{2}\sin^2\phi\right)^{\frac{1}{2}}}.$$

$$= 4\sqrt{\frac{k^2+l^2}{lg}} \int_0^{\pi/2} \left(1-\sin^2\frac{\alpha}{2}\sin^2\phi\right)^{-\frac{1}{2}}.d\phi.$$

$$= 4\sqrt{\frac{k^2+l^2}{lg}} \int_0^{\pi/2} \left(1+\tfrac{1}{2}\sin^2\frac{\alpha}{2}\sin^2\phi + \frac{1.3}{2.4}\sin^4\frac{\alpha}{2}\sin^2\phi + \ldots\right)d\phi.$$

Or, $$t = 4\sqrt{\frac{k^2+l^2}{lg}} \cdot \frac{\pi}{2}\left[1+(\tfrac{1}{2})^2 \sin^2\frac{\alpha}{2} + \left(\frac{1\,3}{2.4}\right)^2 \sin^4\frac{\alpha}{2}+\ldots\right]$$

Or, $$t = 2\pi\sqrt{\frac{k^2+l^2}{lg}}\left[1+\tfrac{1}{4}\sin^2\frac{\alpha}{2}+\ldots\right].$$

Now, $2\pi\sqrt{\frac{k^2+l^2}{lg}}$ would be the time-period of the pendulum, for an infinitely small amplitude. Denoting it by t_0, we have

$$t = t_0\left(1+\tfrac{1}{4}\sin^2\frac{\alpha}{2}\right).$$

If α be small, $\sin\frac{\alpha}{2} = \frac{\alpha}{2}$, and we have

$$t = t_0\left(1+\frac{1}{4}\cdot\frac{\alpha^2}{4}\right) = t_0\left(1+\frac{\alpha^2}{16}\right).$$

Since the amplitude (or half swing) does not remain constant but decreases from α_1 in the beginning to α_2 at the end, both being small, we replace α^2 by $\alpha_1\alpha_2$. So that

$$t = t_0\left(1+\frac{\alpha_1\alpha_2}{16}\right), \quad \text{whence,} \quad t_0 = t\Big/\left(1+\frac{\alpha_1\alpha_2}{16}\right).$$

Or, $$t_0 = t\left(1-\frac{\alpha_1\alpha_2}{16}\right).$$

(*ii*) **Air-effects.** There are *three* distinct ways in which the presence of air affects the time-period of a pendulum :

(*a*) *Buoyancy of the air.* This tends to reduce the restoring couple acting on the pendulum, due to slight decrease in its weight, similar to the one produced in a body immersed in a liquid. For, if m be the mass of the pendulum and m', that of the air displaced by it*, the restoring couple is reduced from the value $mg.l \sin\theta$ to $(ml-m'h).g \sin\theta$, where h† is the distance between the *c.g.* of the displaced air and the axis of rotation of the pendulum, and can be obtained sufficiently accurately from its physical dimensions.

The equation of motion of the pendulum thus becomes

$$m(k^2+l^2)\cdot\frac{d^2\theta}{dt^2} = -g \sin\theta(ml-m'h).$$

Or, $$\frac{d^2\theta}{dt^2} = -\frac{(ml-m'h)g}{m(k^2+l^2)}\cdot\theta. \qquad \left[\text{Putting } \sin\theta=\theta.\right.$$

This is obviously the equation of a simple harmonic motion, of time-period t, given by

$$t = 2\pi\sqrt{\frac{m(k^2+l^2)}{(ml-m'hg)}},$$

which is clearly greater than $t = 2\pi\sqrt{(k^2+l^2)/lg}$, the expression for the time-period in vacuo.

The time-period of the pendulum is thus slightly increased due to buoyancy of the air.

This was the only correction taken account of by *Newton*, followed by *Kater*, and it was left to *Bessel* to show that other corrections due to air-effects were also called for.

---

*This can easily be obtained from the volume of the pendulum and the density of the air at the time.

†The value of h may not be the same as that of l, unless the pendulum has a uniform density.

(b) Some air being dragged along with the pendulum, during its motion, **(Du Buat's Correction).** The pendulum during its 'to and fro' motion, "*carries air with it*" and this increases it effective mass, and hence its moment of inertia, making the observed time-period greater than the true one, as will be clear from the following :

That the pendulum does 'carry' some air with it as it oscillates, can be shown by a simple experiment, *viz*, by attaching a feather to its bob, in a direction at right angles to its direction of motion. It will be found that the feather tilts in a direction opposite to that of the motion of the bob, showing that the air surrounding it is at rest. *If, however, the feather be sufficiently close to the bob, it is not found to tilt at all, clearly indicating that the air in immediate contact with the bob moves along with it, or that it 'carries air with it'.*

Let the mass of this air 'carried' by the pendulum be m'' and let the distance of its centre of mass from the point of suspension of the pendulum be d. Then, clearly, the *effective* moment of inertia of the pendulum, *i.e*, the moment of inertia of the pendulum and the adherent mass of air with it, is equal to $m(k^2+l^2)+m''d^2$. And, therefore, the equation of motion of the pendulum now becomes

$$\left[m(k^2+l^2)+m''d^2\right]\frac{d^2\theta}{dt^2} = -(ml-m'h)g\theta.$$

Or,
$$\frac{d^2\theta}{dt^2} = -\frac{(ml-m'h)g}{m'k^2+l^2)+m''d^2}\cdot\theta.$$

The time-period of the pendulum in thus given by

$$t = 2\pi\sqrt{\frac{m(k^2+l^2)+m''d^2}{(ml-m'h)g}}.$$

It follows at once from the above that

$$\frac{gt^2}{4\pi^2} = \left[\ m(k^2+l^2)+m''d^2\ \right]\frac{1}{ml}\left[\ 1-\frac{m'h}{ml}\ \right]^{-1}$$

$$= \frac{1}{ml}\left[m(k^2+l^2)+m''d^2\right]\left[\ 1+\frac{m'h}{ml}\ \right] \quad \left[\begin{array}{l}\text{Neglecting}\\ \text{higher terms.}\end{array}\right.$$

Or,
$$\frac{gt^2}{4\pi^2} = \frac{k^2+l^2}{l}+\frac{m''d^2}{ml}+\frac{k^2+l^2}{l}\cdot\frac{m'h}{ml}, \quad \left[\begin{array}{l}\text{Neglecting second}\\ \text{order terms.}\end{array}\right.$$

assuming the time-period here to be already corrected for the finite arc of its swing.

Now, if l_1 and l_2 be the distances of the two points of suspension from the centre of gravity, on either side, such that the time-periods in the two cases are *nearly the same*, then, if h_1, h_2 and d_1, d_2 be the respective distances from the point of suspension of the centres of buoyancy and the centres of mass of the air adhering to the bob, in the two positions, we have

$$\frac{gt_1^2}{4\pi^2}l_1 = l_1^2+k^2+\frac{m''d_1^2}{m}+\frac{k^2+l_1^2}{ml_1}.m'h_1. \qquad \ldots.(i)$$

and
$$\frac{gt_2^2}{4\pi^2}l_2 = l_2^2+k^2+\frac{m''d_2^2}{m}+\frac{k^2+l_2^2}{ml_2}.m'h_2. \qquad \ldots.(ii)$$

So that, subtracting relation (*ii*) from (*i*), we have

$$\frac{g}{4\pi^2}(t_1^2l_1-t_2^2l_2) = (l_1^2-l_2^2)+\frac{m''}{m}(d_1^2-d_2^2)+\frac{(k^2+l_1^2)}{l_1}\cdot\frac{m'h_1}{m}-\frac{(k_2+l_2^2)}{l_2}\cdot\frac{m'h_2}{m}.$$

Since t_1 is very nearly equal to t_2, we have $k^2=l_1l_2$.

And, therefore,

$$\frac{g}{4\pi^2}\frac{(t_1^2l_1-t_2^2l_2)}{l_1-l_2} = (l_1+l_2)+\frac{m''}{m}\left(\frac{d_1^2-d_2^2}{l_1-l_2}\right)+\left(\frac{l_1+l_2}{l_1-l_2}\right)\frac{m'}{m}(h_1-h_2).$$

Here, $\dfrac{(t_1^2l_1-t_2^2l_2)}{l_1-l_2}$ is the square of the *computed time*, (see § 70). Denoting it by τ^2, we have

$$\frac{g\tau^2}{4\pi^2} = (l_1+l_2)+\frac{m''}{m}\left(\frac{l_1^2-d_2^2}{l_1-l_2}\right)+\left(\frac{l_1+l_2}{l_1-l_2}\right)\frac{m'}{m}(h_1-h_2). \qquad \ldots.(iii)$$

Thus, the obvious method to eliminate this correction is to make $d_1=d_2$ and $h_1=h_2$, (*i.e.*, to make the pendulum symmetrical in shape), which will reduce the two expressions on the right-hand side of relation (*iii*) to zero. This is precisely what has been done in Repsold's reversible pendulum, (see § 72, page 188).

Both effects (*a*) and (*b*) due to air can, however, be made almost negligible by arranging to swing the pendulum in reduced pressure,—a procedure now being increasingly adopted—for the residual effect in low pressures is found to be a linear function of the pressure. The required correction can thus be directly obtained by plotting a graph between pressure and time-period and obtaining the value of the latter by extrapolating the graph to *zero* pressure.

(*c*) *Viscosity of the air.* The viscous drag due to air produces a damping effect on the pendulum and tends to reduce its amplitude, thereby increasing its time-period.

For, taking the viscosity-drag for small velocities to be proportional to velocity, the equation of motion would be of the type

$$\frac{d^2\theta}{dt^2}+r.\frac{d\theta}{dt}+\mu\theta=0.$$

Let the solution of this equation be $\theta=Ae^{\omega t}$. Then, clearly,

$$A\omega^2+Ar\omega+\mu A=0. \quad \text{Or,} \quad \omega^2+\omega r+\mu=0,$$

which is a quadratic equation in ω.

So that, $$\omega=\frac{-r\pm\sqrt{r^2-4\mu}}{2}=\frac{-r}{2}\pm j\sqrt{\mu-\frac{r^2}{4}}. \qquad \left[\begin{matrix}\text{where } j=\sqrt{-1}\\ \text{and } \mu>r.\end{matrix}\right.$$

Hence the general solution is

$$\theta=Ae^{\left[-\frac{r}{2}+j\sqrt{\mu-r^2/4}\right]t}+Be^{\left[-\frac{r}{2}-j\sqrt{\mu-r^2/4}\right]t}.$$

$$=e^{-\frac{r}{2}t}\left[(A+B)\cos\sqrt{\left(\mu-\frac{r^2}{4}\right)}.t+j(A-B)\sin\sqrt{\left(\mu-\frac{r^2}{4}\right)}t\right]$$

$$=A_0e^{-\frac{r}{2}t}\cos\left(\sqrt{\mu-\frac{r^2}{4}}t+\phi\right),$$

which is a simple harmonic motion of decaying amplitude, of a time-period

$$t=2\pi/\sqrt{\mu-r^2/4}.$$

Thus, $$t=\frac{2\pi}{\sqrt{\mu}}\left[1-\frac{r^2}{4\mu}\right]^{-\frac{1}{2}}=\frac{2\pi}{\sqrt{\mu}}\left(1+\frac{r^2}{8\mu}\right).$$

Now, $2\pi/\sqrt{\mu}=t_0$, the time-period of the pendulum in the absence of any viscous drag.

So that, $$t=t_0\left[1+\frac{r^2}{8\mu}\right].$$

And, if we make use of the approximate relation $t'=2\pi/\sqrt{\mu}$ for the time-period of the pendulum, we have $\mu=4\pi^2/t'^2$. So that, substituting this value of μ in the expression for t above, we have

$$t=t_0\left[1+\frac{r^2}{8\times4\pi^2/t'^2}\right]=t_0\left[1+\frac{r^2t'^2}{32\pi^2}\right].$$

So that, $$t_0=t\left[1-\frac{r^2t'^2}{32\pi^2}\right], \text{ very nearly.}$$

This correction due to viscosity is however much too small,—being of the order of 10^{-9} and is, therefore, usually neglected.

(*iii*) **Non-rigidity of the Support.** Due to the yielding of the support, the time-period tends to be greater than the correct value, as explained below :

It might, at first sight, appear improbable that the support should yield by the mere swinging of a pendulum suspended from it. This is, however, not so, for the simple reason that *no support is perfectly rigid*. In fact, any ordinary support, we consider to be rigid, would yield under a weight of 100 *k.gms.* or so. True, a pendulum is seldom as heavy as that, but in view of the fact that we can measure lengths and time-periods to an accuracy of one in several thousand, it is only in the fitness of things that we must take into account even this slight yielding of the support, if we really aim at a high degree of precision in our work.

Again, it is also true that we can adopt ways and means of eliminating this error altogether (as well as that due to the presence of air) in so far as the pendulum is concerned, as explained in §72, below, we should, nevertheless, acquaint ourselves with the method of deducing a proper correction for it should it become necessary in other similar cases, where its outright elimination is not feasible or possible.

Now, then, let us pass on to a brief consideration of it.

We know that a vibrating body tends to set into vibration any other body in contact with it the degree of response of the latter depending upon how nearly its natural time-period agrees with that of the vibrating body,—the closer this agreement between the two, the greater the response and *vice versa*.

In the case of the pendulum, therefore, the support carrying it also yields a little to its vibrations and is forced to oscillate co-periodically with it.

This oscillation of the support may be resolved into two rectangular components, (*i*) *along the vertical* and (*ii*) *along the horizontal*, the latter having a more pronounced effect on the time-period of the pendulum than the former.

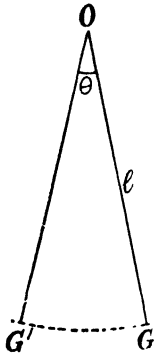


Fig. 114.

Thus, if OG be the mean or equilibrium position of the pendulum, (Fig 114), of length l, with the axis of suspension passing through O, then, as it swings through an angle θ into the position OG', there are two forces acting upon it, (*i*) along the arc of its swing, to which its motion is due and (*ii*) the other at right angles to it, *i.e.*, along $G'O$, (the centripetal force). So that,

acceleration of the pendulum along the arc

$$= \frac{d^2s}{dt^2} = \frac{d}{dt} \cdot \frac{ds}{dt} = l \cdot \frac{d^2\theta}{dt^2}, \qquad \left[\text{where } s = l\theta. \right.$$

and *acceleration along the length of the pendulum*

$$= \frac{v^2}{l} = \frac{1}{l} \left(\frac{ds}{dt} \right)^2 = l \left(\frac{d\theta}{dt} \right)^2. \qquad \left\{ \begin{array}{l} \therefore v = ds/dt \\ = l\left(\frac{d\theta}{dt} \right). \end{array} \right.$$

These accelerations, obviously, act at the support O, with the *component in the horizontal plane*

$$= l \frac{d^2\theta}{dt^2} . \cos\theta + l\left(\frac{d\theta}{dt} \right)^2 \sin\theta,$$

and the *component in the vertical plane*

$$= l \frac{d^2\theta}{dt^2} . \sin\theta - l\left(\frac{d\theta}{dt} \right)^2 \cos\theta.$$

So that, if θ be small we have

horizontal component of the acceleration

$$= l . \frac{d^2\theta}{dt^2} + l\left(\frac{d\theta}{dt} \right)^2 . \theta \qquad \ldots\ldots (i) \qquad \left[\begin{array}{l} \because \cos\theta = 1 \\ \text{and } \sin\theta = \theta. \end{array} \right.$$

and *vertical component of the acceleration*

$$= l . \frac{d^2\theta}{dt^2} . \theta - l\left(\frac{d\theta}{dt} \right)^2 \qquad \ldots\ldots \qquad (ii)$$

Now, from the equation of the pendulum, $I.\frac{d^2\theta}{dt^2} = mg.l.\theta$, we have

$$\frac{d^2\theta}{dt^2} = \frac{mgl}{I}\cdot\theta = \frac{mgl}{m(k^2+l^2)}\cdot\theta = \left(\frac{gl}{k^2+l^2}\right).\theta.$$

And, from the energy equation of the pendulum, *viz.*,

$$mgl\ (cos\ \theta - cos\ \alpha) = \tfrac{1}{2}m\ (k^2+l^2)\ \left(\frac{d\theta}{dt}\right)^2,$$

we have

$$\left(\frac{d}{dt}\right)^2 = \frac{2gl\ (cos\ \theta - cos\ \alpha)}{k^2+l^2},$$

whence, expanding *cos* θ and *cos* α and retaining only the first two terms, in view of the small values of θ and α, we have

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{gl\ (\alpha^2-\theta^2)}{k^2+l^2}.\quad\left[\because\ cos\ \theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}\cdot\right.$$

Substituting these values of $\frac{d^2\theta}{dt^2}$ and $\left(\frac{d\theta}{dt}\right)^2$ in relations (*i*) and (*ii*) above, we therefore have

$$horizontal\ acceleration = l.\left(\frac{gl}{k^2+l^2}\right).\theta + l\theta.\frac{gl(\alpha^2-\theta^2)}{k^2+l^2},$$

and

$$vertical\ acceleration = l.\left(\frac{gl}{k^2+l^2}\right).\theta^2 + l.\frac{gl(\alpha^2-\theta^2)}{k^2+l^2},$$

Or, neglecting the second term as being extremely small, we have

$$horizontal\ acceleration = \left(\frac{gl^2}{k^2+l^2}\right).\theta$$

and

$$vertical\ acceleration = \left(\frac{gl^2}{k^2+l^2}\right).\theta^2$$

And, therefore, *horizontal force on the support* $= \left(\frac{mgl^2}{k^2+l^2}\right).\theta$

and

$$vertical\ force\ on\ the\ support = \left(\frac{mgl^2}{k^2+l^2}\right).\theta^2$$

i.e., *horizontal force on the support* $\propto \theta$

and *vertical* ,, ,, ,, ,, $\propto \theta^2$.

Since, θ is small, θ^2 is comparatively very much smaller. In other words, the horizontal force on the support is very much greater than the vertical force on it and the latter may, therefore, be easily ignored.

Now, as the pendulum moves from G to G', the point of suspension moves from O to O', say, so that its displacement is equivalent to shifting its axis of suspension to Q, (Fig 115).

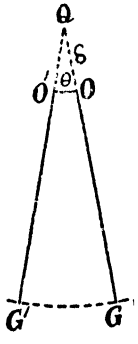


Fig. 115

Then, if β be the displacement of the support *per unit force, in the horizontal direction*, when displaced through an angle θ, *its displacement due to a horizontal force* $(mgl^2/k^2+l^2)\theta$ wil clearly be equal to $(mgl^2\beta/k^2+l^2)\theta$.

Since Q is now the *effective* axis of suspension, $\theta = \angle GQG'$ and hence the displacement of the point of suspension

$= OO' = OQ.\theta. = \delta.\theta.$ [Putting $OQ = \delta$.

We, therefore, have $\delta.\theta = \frac{mgl^2\beta}{k^2+l^2}.\theta$, whence, $\delta = \frac{mgl^2\beta}{k^2+l^2}$.

It follows, therefore, that the *effective length* of the pendulum due to this yielding of the support is

$$(l+\delta) = l + \frac{mgl^2\beta}{k^2+l^2}.$$

And, therefore, the time-period of the pendulum is now given by

$$= 2\pi\sqrt{1\Big/\frac{g(l+\delta)}{k^2+(l+\delta)^2}} = 2\pi\sqrt{\frac{k^2+(l+\delta)^2}{g(l+\delta)}}. \qquad \left[\text{Writing } (l+\delta) \text{ for } l.\right.$$

From this it follows at once that

$$\frac{gt^2}{4\pi^2} = \frac{k^2+(l+\delta)^2}{(l+\delta)} = (l+\delta)+k^2\cdot\frac{1}{l}\left(1+\frac{\delta}{l}\right)^{-1}$$

$$= (l+\delta)+\frac{k^2}{l}\left(1-\frac{\delta}{l}\right).$$

So that, if l_1 and l_2 be the lengths of the pendulum for which the time-periods t_1 and t_2 are very nearly the same, we have

$$\frac{gt_1^2}{4\pi^2} = (l_1+\delta_1)+\frac{k^2}{l_1}\left(1-\frac{\delta_1}{l_1}\right)$$

and

$$\frac{gt_2^2}{4\pi^2} = (l_2+\delta_2)+\frac{k^2}{l_2}\left(1-\frac{\delta_2}{l_2}\right)$$

$\left[\right.$ where δ_1 and δ_2 are the corresponding additions to the two lengths of the pendulum.

whence, $\frac{g}{4\pi^2}\left(\frac{t_1^2l_1-t_2^2l_2}{l_1-l_2}\right)=(l_1+l_2)+\frac{\delta_1l_1-\delta_2l_2}{l_1-l_2}+k^2\left(\frac{\delta_2}{l_2}-\frac{\delta_1}{l_1}\right)\left(\frac{1}{l_1-l_2}\right)$.

Since t_1 is very nearly equal to t_2, we have $k^2 = l_1l_2$. And, therefore,

$$\frac{g}{4\pi^2}\left(\frac{t_1^2l_1-t_2^2l_2}{l_1-l_2}\right) = (l_1+l_2)+\frac{\delta_1l_1-\delta_1l_2}{l_1-l_2}+\frac{\delta_2l_1-\delta_1l_2}{l_1-l_2}.$$

$$= (l_1+l_2)+\frac{(\delta_1+\delta_2)(l_1-l_2)}{l_1-l_2} = (l_1+l_2)+(\delta_1+\delta_2).$$

Now, $\delta_1+\delta_2 = \frac{mgl_1^2\beta}{k^2+l_1^2}+\frac{mgl_2^2\beta}{k^2+l_2^2}$.

$$= mg\beta\left(\frac{l_1^2}{l_1l_2+l_1^2}+\frac{l_2^2}{l_1l_2+l_2^2}\right) = mg\beta\left(\frac{l_1}{l_1+l_2}+\frac{l_2}{l_1+l_2}\right). \qquad [\because k = l_1l_2$$

Or, $\delta_1+\delta_2 = mg\beta$. $\qquad \left[\because \frac{l_1}{l_1+l_2}+\frac{l_2}{l_1+l_2} = \frac{(l_1+l_2)}{l_1+l_2} = 1.\right.$

So that, putting $(t_1^2l_1-t_2^2l_2)/(l_1-l_2) = \tau^2$, (where τ is the *computed time*), we have

$$\frac{g\tau^2}{4\pi^2} = l_1+l_2+mg\beta = \frac{g\tau^2}{4\pi^2} = L+mg\beta, \quad \left[\text{For, } l_1+l_2 = L, \text{ the length of the } \textit{equivalent simple pendulum}.\right.$$

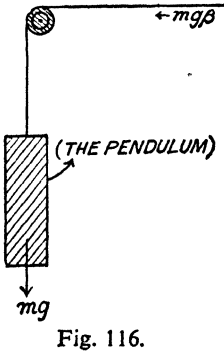


Fig. 116.

Clearly, here, mg is the weight of the pendulum; so that, *the correction factor* $mg\beta$ *is the displacement of the support due to a horizontal force equal to the weight of the pendulum* and can be determined directly by suspending the pendulum from a string passing over a pulley and attached horizontally to the support at O, as shown, (Fig. 116), when the displacement $OO' = mg\beta$ can be read accurately by means of a microscope.

Vening Meinesz suggested a method by which this correction could be considerably reduced, *viz.*, that of using two pendulums, swinging from the same support but in opposite phases with each other. This involves however, the difficulty of having to adjust their time-periods to very near equality. The correction is thus best eliminated as explained in § 72, (page 187).

(*iv*) The knife-edges not being perfectly sharp, (*but more or less rounded*). Due to this also the effective length of the pendulum is altered.

For, the axis of suspension is *not a mathematical line* we have so far tacitly assumed it to be In actual practice, it has a definite shape,—generally symmetrical—with a finite radius of curvature.

Let us, as a first approximation, assume the edge to be the part of a cylinder,—a *cylinderical cone*, as shown in Fig. 117. Then, if O be the *centre of curvature* of the edge, a line perpendicular to the plane of the paper and passing through O represents the axis of the cylindrical edge.

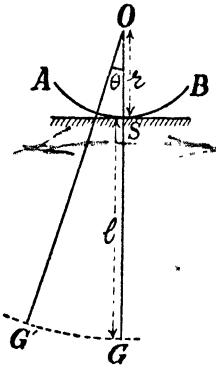


Fig. 117.

As the pendulum (of length l) swings, the edge also moves along with it about *this* axis through O, so that the axis of suspension is, in effect, shifted from S to O, *i.e.*, through the distance $SO = r$, the radius of curvature of the edge, and the effective length of the pendulum thus becomes $(l+r)$.

Since, however, the instantaneous axis of rotation of the pendulum still passes through the bottom of the knife-edge (S), the moment of inertia (I) is still to be taken about this axis In other words, we still have $I = m.(k^2+l^2)$, where m is the *mass* of the pendulum.

The equation of motion of the pendulum thus becomes

$$m(k^2+l^2)\frac{d^2\theta}{dt^2} = -mg(l+r)\sin\theta = -mg(l+r)\theta. \qquad [\theta \text{ being small.}$$

And, therefore, its time-period is given by $t = 2\pi\sqrt{\frac{k^2+l^2}{g(l+r)}}$,

whence, $\frac{gt^2}{4\pi^2} = \frac{k^2+l^2}{l+r} = \frac{k^2+l^2}{l\left(1+\frac{r}{l}\right)}$. Or, $\frac{gt^2l}{4\pi^2} = \frac{k^2+l^2}{1+\frac{r}{l}} = (k^2+l^2)\left(1-\frac{r}{l}\right)$.

If we find two lengths of the pendulum, say l_1 and l_2 on the two sides of its *c.g.*, such that the time-periods t_1 and t_2 for them are nearly the same, (with, of course l_1 not equal to l_2), then, if r_1 and r_2 be the radii of the two knife-edges respectively, we have

$$\frac{gl_1t_1^2}{4\pi^2} = (k^2+l_1^2)\left(1-\frac{r_1}{l_1}\right) \text{ and } \frac{gl_2t_2^2}{4\pi^2} = (k^2+l_2^2)\left(1-\frac{r_2}{l_2}\right).$$

So that, subtracting the second expression from the first, we have

$$\frac{g}{4\pi^2}(l_1t_1^2-l_2t_2^2) = (l_1^2-l_2^2)-(l_1^2+k^2)\frac{r_1}{l_1}+(l_2^2+k^2)\frac{r_2}{l_2}.$$

Since t_1 is very nearly equal to t_2, we have $k^2 = l_1l_2$. And, therefore,

$$\frac{g}{4\pi^2}(l_1t_1^2-l_2t_2^2) = (l_1^2-l_2^2)-(l_1^2+l_1l_2)\frac{r_1}{l_1}+(l_2^2+l_1l_2)\frac{r_2}{l_2}.$$

Or,
$$\frac{g}{4\pi^2}(l_1t_1^2-l_2t_2^2) = (l_1^2-l_2^2)-(l_1+l_2)r_1+(l_2+l_1).r_2.$$
$$= (l_1-l_2)(l_1+l_2)-(l_1+l_2)(r_1-r_2).$$
$$= (l_1-l_2)\left[(l_1+l_2)-\left(\frac{l_1+l_2}{l_1-l_2}\right)(r_1-r_2)\right].$$

Or,
$$\frac{g}{4\pi^2}\cdot\frac{l_1t_1^2-l_2t_2^2}{l_1-l_2} = (l_1+l_2)-\left(\frac{l_1+l_2}{l_1-l_2}\right)(r_1-r_2).$$

Again, putting $\frac{l_1t_1^2-l_2t_2^2}{l_1-l_2} = \tau^2$, where τ is the *computed time-period*, we have

$$\frac{g\tau^2}{4\pi^2} = (l_1+l_2)-\left(\frac{l_1+l_2}{l_1-l_2}\right)(r_1-r_2).$$

Here, clearly, the correction term $\left(\frac{l_1+l_2}{l_1-l_2}\right)(r_1-r_2)$ becomes *zero*, only if $r_1=r_2$, *i.e.*, only if the two knife-edges have the *same* radius of curvature.

Since it is difficult to make the two knife-edges of exactly the same radius of curvature, the suggestion at once comes to the mind that the same knife-edge may be used at both the two points of suspension. But this may affect the position of the *c.g.*, which might be different for the two positions of the knife-edge. And, then, it would disturb the symmetry of the pendulum, necessitating the troublesome air-corrections. This difficulty may be tided over by arranging two knife-edges of the same shape and mass, and by *using only one of them for suspension*, *i.e.*, by interchanging them when we change the side of the pendulum. Here too, however, an error may creep in if we do not succeed in replacing one knife-edge with the other exactly in its true or original position. This difficulty too may be got over, however, by performing the experiment *four-times*,—first taking the two observations for l_1 and l_2 with one position of the knife-edges and then two similar observations with the knife-edges interchanged. Thus, if τ_1 and τ_2 be the respective computed time-periods in the two cases, we have

$$\frac{g\tau_1^2}{4\pi^2} = (l_1+l_2) - \left(\frac{l_1+l_2}{l_1-l_2}\right)(r_1-r_2) \quad \text{and} \quad \frac{g\tau_2^2}{4\pi^2} = (l_1+l_2) - \left(\frac{l_1+l_2}{l_1-l_2}\right)(r_2-r_1).$$

So that, adding the two, we have

$$\frac{g}{4\pi^2}(\tau_1^2+\tau_2^2) = 2(l_1+l_2), \qquad \left\{ \begin{array}{l} \because \text{the sum of the correction terms (involving } r_1 \\ \text{and } r_2) = 0. \end{array} \right.$$

whence, $$\frac{g}{4\pi^2}\frac{\tau_1^2+\tau_2^2}{2} = (l_1+l_2).$$

But, even this correction does not help much. For, as the pendulum swings to and fro about its mean position, the edges invariably get chipped off, resulting in the loss of weight of the pendulum.

The one and only way of eliminating this correction, now being increasingly used, is to replace the two knife-edges in the pendulum by just plane bearings, *i.e.*, by flat plates, and to provide a fixed knife edge on the support,—the latter being carefully ground to a sharp edge and the former being accurately plane or flat and always placed in the same position on the knife-edge.

N.B. In a bid for an extremely high degree of accuracy, the effect of the elasticity of the pendulum was also investigated at *Potsdam* under the supervision of **Helmert**, *viz.*, its periodic extension under the varying longitudinal strain and its flexure under the changing bending moment to which it is subjected as it swings—the latter being the more important of the two and resulting in a reduction in the effective length of the pendulum.

Thanks to the work of *Clark*, *Heyl* and *Cook*, an accuracy of one in a million is now more or less easily attainable

(*v*) **Change of Temperature during the Experiment.** This results in a corresponding change in the length and hence the time-period of the pendulum.

A correction for it can, however, be readily applied, if we know the coefficient of expansion of the material of the pendulum. Or, the error may be eliminated altogether by using what are called invariable pendulums, (see §76).

(*vi*) **Other Errors.** We have considered above the errors and corrections in so far as they relate to the pendulum itself. To obtain an accurate value of *g* at a place, however, certain other corrections must also be applied, *viz.*, the corrections (*a*) for *rotation of the earth*, (*b*) for *latitude*, (*c*) for *altitude*, (*d*) for *elevated masses* and (*e*) for

the terrain or the *topography of the place*, all of which are discussed in the succeeding chapter.

Nevertheless, as a method for determining the value of g, a compound pendulum, (*e.g.*, the Kater's pendulum), is distinctly superior to a simple pendulum. For,

(*i*) whereas a simple pendulum is just an *ideal* conception, not realizable in actual practice, *the length of an equivalent simple pendulum, and hence the value of g, can be easily and accurately determined with its help* ;

(*ii*) *it vibrates as a whole, there being no lag between the bob and the string*, as in the case of a simple or a Borda's dendulum ;

(*iii*) *the length to be measured here is clearly defined, viz., the distance between the two knife-edges, and can thus be easily and accurately measured* whereas, in the case of a simple pendulum, the point of suspension and the *c.g.* of the bob are both more or less *indefinite* points, and hence its true length can hardly be expected to be determined correctly ;

(*iv*) *due to its large mass, the compound pendulum keeps on oscillating for a fairly long time, thus enabling its time-period to be determined with accuracy.* In a simple pendulum, on the other hand, the oscillations die down much too soon due to the comparatively small mass of the bob, and it becomes difficult to determine its time-period to an equivalent degree of accuracy.

The one obvious disadvantage in the case of a compound pendulum, however, is that during its vibrations to and fro, about its mean position, some air is dragged along with it, as mentioned above, thus increasing its *effective* mass and hence its moment of inertia. But it has been clearly shown by Bessel that if it be of a form, *symmetrical about the centre of its geometrical shape (which is not the same thing as its centre of gravity)*, this error is automatically eliminated. This explains the symmetrical shapes of various types of compound pendulums we use, though, theoretically, a rigid body of any shape whatever would do.

72. Other Improvements due to Bessel. Not only has Bessel done away with the trouble and the tedium of having to make the time-periods about the two axes identical, but he has also succeeded in removing quite a few other important errors. Thus, for example :

(*i*) The error due to some air being dragged along with the pendulum is removed by the *symmetrical physical form* of the instrument, as suggested and shown by him.

(*ii*) The error due to the knife-edges not being perfectly sharp, for which a correction, proportional to their radii of curvature, would be necessary, (unless they be of the same radii of curvature), [see §71, (*iv*), page 184],—has also been eliminated by him. For, he has shown that this error would automatically vanish if the two knife-edge, at the two ends of the pendulum, could be made interchangeable. Then, if t_1 and t_2 be the *computed times*, before and after the inter-

change of knife-edges, the true time-period t is given by the relation

$$t=\sqrt{(t_1^2+t_2^2)/2}.$$

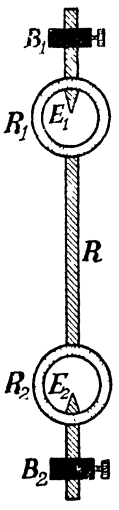


Fig 118.

Bessel, unfortunately, died before he could put his theory into actual practice but, later, **Repsold** did actually construct in the year 1860, a *reversible pendulum* of this type and used it with success.

Repsold's Pendulum is more or less a Kater-type pendulum but is *symmetrical in geometrical form about its mid-point*, (Fig 118). Here, we have a rod R fixed on to two rings R_1 and R_2 at its two ends, which, in their turn, have two short rods screwed into them, terminating in knife-edges E_1 and E_2 inside the rings and carrying two bobs B_1 and B_2,—one *solid* and the other *hollow**.

The time-period of the pendulum can be made nearly equal about either knife-edge by moving the bobs up and down and screwing them into the desired position.

With the symmetrical form of the pendulum, the error due to air-effects is automatically eliminated, as explained in § 71, (*ii*) above.

And, (*iii*) finally, the error, due to the yielding of the support has been eliminated by **Defforges**, by using two reversible *Repsold type* pendulums, of the *same mass* but *different lengths*, the *same ratio of l to l'* and having a *common pair of knife-edges*, (to be used with either of them). He has shown that if L_1 and L_2 be the *reduced lengths* of the two reversible pendulums (*i.e.*, the lengths between the knife-edges) and τ_1 and τ_2, their computed times, then $\sqrt{(\tau_1^2-\tau_2^2)}$ *gives the* correct time-period of a pendulum of length (L_1-L_2), as can be seen from the following :

We have $$\frac{g\tau_1^2}{4\pi^2}=L_1+mg.\beta-\left(\frac{l_1+l_2}{l_1-l_2}\right)(r_1-r_2)$$

and $$\frac{g\tau_2^2}{4\pi^2}=L_2+mg.\beta-\left(\frac{l_1'+l_2'}{l_1'-l_2'}\right)(r_1-r_2),$$

[See pages 184 and 186.

where l_1 and l_2 are the two lengths on the two sides of the *c g.* in the case of one pendulum and l_1' and l_2' in that of the other.

So that, subtracting the second expression from the first, we have

$$\frac{g}{4\pi^2}(\tau_1^2-\tau_2^2)=L_1-L_2+\left[\frac{l_1+l_2}{l_1-l_2}-\frac{l_1'+l_2'}{l_1'-l_2'}\right](r_2-r_1).$$

Clearly, the correcting term, (*i.e.*, the second term) in this expression can be made zero if $l_1 : l_2 :: l_1' : l_2'$, and this is easily done by adjusting the positions of the bobs of the two pendulums. With this adjustment made, we have

$$\frac{g}{4\pi^2}\left(\tau_1^2-\tau_2^2\right)=L_1-L_2.$$

*This is to ensure that the lengths l_1 and l_2 of the pendulum on the two sides of the *c.g.* are not very nearly equal, or else the correcting terms for the error due to air effects, (page 180), will not be small. This is the reason why in a Kater's pendulum one bob is made smaller than the other.

Or, if we use only one knife-edge, (fixed on the support, as explained in § 71 (*iii*), page 181), we have

$$\frac{g\tau_1^2}{4\pi^2} = L_1 + mg.\beta \quad \text{and} \quad \frac{g\tau_2^2}{4\pi^2} = L_2 + mg\,\beta,$$ [See page 184.

where L_1 and L_2 are the reduced lengths and τ_1 and τ_2, the computed time-periods of the two pendulums respectively.

Thus, subtracting the second expression from the first, we have, straightaway,

$$\frac{g}{4\pi^2}\left(\ \tau_1^2 - \tau_2^2\ \right) = L_1 - L_2.$$

This removes at one stroke the errors due to yielding of the support and curvature of the knife-edges, as also those due to air-effects, and we have

$$g = \frac{4\pi^2(L_1 - L_2)}{\tau_1^2 - \tau_2^2}.$$

This is about the most accurate method of determining the value of g at a given place.

73. Conical Pendulum. A simple conical pendulum is just *a simple pendulum*, (*i.e.*, a small heavy bob attached to a light, inextensible string), which is given such a motion that *the bob describes a horizontal circle and the string traces out a cone.* The length of the pendulum is the distance between the point of suspension and the *c.g.* of the bob.

Let m be the mass of the bob B; v, its velocity and r, the radius of the circle it describes, (Fig. 119). Then, its centripetal acceleration towards O, the centre of the circle, is equal to v^2/r, and the centripetal force on it is, therefore, mv^2/r in that direction.

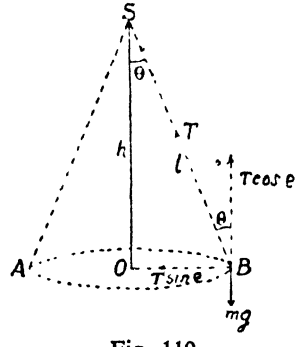


Fig. 119.

Let SO be equal to h.

Clearly, the forces acting on the bob are (*i*) *its weight*, mg, vertically downwards, and (*ii*) *the tension of the string* T, in the direction BS.

The weight mg is balanced by the vertical component $T\cos\theta$, of the tension T of the string, and its horizontal component $T\sin\theta$ provides the centripetal force mv^2/r towards O, where θ is the *semi-vertical angle* of the cone

Thus, $T\sin\theta = mv^2/r$ and $T\cos\theta = mg.$

Or, $\dfrac{T\sin\theta}{T\cos\theta} = \tan\theta = \dfrac{mv^2/r}{mg} = \dfrac{v^2}{rg}.$

Or, $v^2/rg = \tan\theta.$ Or, $v^2 = r.g.\tan\theta.$

Since $v^2 = r^2.\omega^2$, where ω is the angular velocity of the bob, we have $r^2.\omega^2 = r.g.\tan\theta = r.g.r/h = r^2g/h,$ [$\because \tan\theta = r/h.$

whence, $\omega^2 = g/h,$ and $\therefore\ \omega = \sqrt{g/h},$ .. .. .. (*i*)

Now, the time-period of the pendulum is given by

$$t = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/h}} = 2\pi\sqrt{\frac{h}{g}}.$$

$$t = 2\pi\sqrt{\frac{l\cos\theta}{g}}, \qquad ..(ii) \qquad [\because h = l\cos\theta.$$

where l is the length of the pendulum.

It will thus be noted that the time-period is the same as that, of a simple pendulum of length h, the **axial height** *of the cone.*

If θ be very small, $\cos\theta$ is nearly equal to 1 ; so that $h = l$, *i.e.*, the time period is almost independent of θ. *In other words, the time-period remains the same whether the bob moves along a circular or a linear path.*

74. Steam Engine Governor. It will be seen from relation (*i*) above, (§ 73), that the angular velocity (ω) of the bob of a conical pendulum varies inversely as the square root of the depth (h) of its c.g. from the point of suspension ; or, conversely, that *the depth of the c.g. of the bob, below its point of suspension, varies inversely as the square of its angular velocity.*

This is made use of in the construction of what is called the '*governor*' is a steam engine, which is just a device to maintain the speed of the engine constant by regulating, or '*governing*', the supply of steam from the boiler to the steam-chest.

In essentials, it is just a combination of two similar conical pendulums, mounted on either side of the vertical shaft (with a common point of suspension), rotated by the engine, and consists of two rods OP and OQ hinged together at their upper end O to the shaft OS, and carrying two spherical metallic bobs P and Q at their lower ends, (Fig. 120). Two other smaller rods connect OP and OQ to a metallic collar C, which slides freely along the shaft, thus operating a lever which controls the *throttle valve*, or the steam valve, opening it partially or fully, according as the collar moves up or down the shaft.

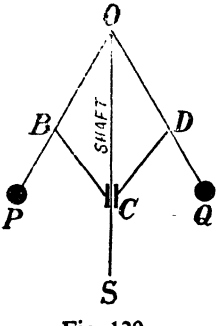


Fig. 120.

Now, when due to a greater supply of steam to the cylinder, the shaft, and, therefore, the bobs rotate faster, *i.e.*, ω increases, h proportionately decreases, or the bobs rise up, thus partially closing the steam valve, thereby partially cutting off the supply of steam to the cylinder. This automatically results in a falling off of the speed of rotation (ω) of the shaft or the bobs, and when this happens, h increases, *i.e.*, the collar slides down with the bobs, thus opening the steam valve more fully, allowing more steam into the cylinder, which then, increases the rate of rotation of the shaft. So that, by proper adjustment, the rate of supply to the steam chest or cylinder, and, consequently, the rate of rotation of the shaft, can be maintained at any constant value desired.

The sensitiveness of the device, however, decreases with the increasing speed of the engine. For, we have the relation

$$\omega^2 = g/h, \qquad \ldots(iii)$$

differentiating which, we have $2\omega.d\omega = -\frac{g}{h^2}\,dh.$ $\ldots(iv)$

And, therefore, dividing relation (*iv*) by (*iii*), we have

$$\frac{2\omega.d\omega}{\omega^2} = -\frac{g.dh}{h^2} \times \frac{h}{g}. \quad \text{Or,} \quad 2d\omega/\omega = -dh/h,$$

whence, $$dh = -\frac{2h}{\omega}.d\omega.$$

Or, substituting the value of h, from relation (*iii*) above, we have

$$dh = -\frac{2g}{\omega^3}.d\omega.$$

Thus, it is clear that $dh \propto 1/\omega^3$, *i.e.*, dh decreases as ω increases. In other words, the change in the position of the *c.g.* decreases with increasing angular velocity of the bobs or the shaft, thus slowing down the 'up and down motion' of the collar along the shaft or decreasing the sensitiveness of the device.

75. Other methods for the determination of 'g'. The following are a few other methods that may be used to determine the value of g at a place. Although they do not compare favourably with the pendulum methods in point of accuracy or ease of performance, they are, nevertheless, valuable laboratory exercises, affording good illustrations of the various principles employed for the purpose. Here, then, are these different methods :

(1) The Inclined Plane. We have seen before, in § 39, (page 88), how the acceleration a of a body, rolling down an inclined plane, (without slipping), is given by the expression,

$$a = [r^2/(k^2+r^2)]g \sin \alpha,$$

where r is the *radius* of the body ; k, its *radius of gyration* about its axis of rotation ; α, the *angle of inclination of the plane*, and g, *the acceleration due to gravity* at the place.

It follows, therefore, that $g = \frac{a(k^2+r^2)}{r^2 \sin \alpha}$.

So that, knowing r, (*by means of a vernier calliper*) k, (*from the geometrical shape of the body*) $\sin \alpha$, *from the height and length of the plane*) and a, (*by direct experiment, as explained below*), we can easily calculate out the value of g at the place.

The value of a can be easily and accurately obtained by noting the distances covered by the body, down along the plane, in successive equal intervals of time*, and plotting the distance-time curve for it. The equation of the curve being $S = \frac{1}{2}at^2$, (u being zero, because the body starts from rest), it will, obviously, be *parabolic* in

*This may be easily done if the angle of inclination (α) of the plane be small ; for, then the acceleration of the body will also be small and the time taken by it in rolling down the plane will be fairly accurately measured by means of a stop watch.

form, with its axis coinciding with the distance-axis. So that, substituting the co-ordinates of any suitable point on the curve in the relation $S = \frac{1}{2}at^2$, the value of a can be easily determined.

(2) The Dynamical Spherometer. Let S be the centre of curvature of a spherical surface, arranged horizontally *with its concavity upwards*, and, R, its radius, (Fig. 121) ; and let a steel ball, of mass m and radius r, be allowed to roll to and fro on it, *without slipping*. Then clearly, the ball oscillates on the inside of the surface *as though it were a compound pendulum, with its centre of suspension at S and its centre of oscillation at O, the centre of the ball ; so that, SO = (R−r).*

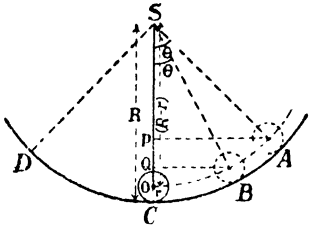


Fig. 121.

Let the angular amplitude of the ball be θ', (*i.e.*, the angle that it makes at S, when in its extreme positions). Let it be in the position B, at any given instant, such that the angle it now makes at S is θ.

Then, clearly, *work done on the ball by the force of gravity in* bringing it down from A to B is equal to *weight of the ball × the vertical distance through which the ball has fallen down.*

This must, therefore, be clearly equal to the loss in its potential energy, *i e.*, equal to $mg \times PQ$ where mg is the weight of the ball.

Or, $$\text{loss in P.E.} = mg(SQ-SP).$$

Now, $SQ = (R-r).\cos\theta$ and $SP = (R-r).\cos\theta'$.

And $\therefore$ $$SQ-SP = (R-r).(\cos\theta-\cos\theta').$$

So that, work done by the force of gravity in moving the ball from A to B is equal to $mg.(R-r)\ (\cos\theta-\cos\theta')$ and is equal to the loss in the potential energy of the ball.

This must, clearly, be equal to the *gain in the kinetic energy* of the ball *i.e.*, equal to $\frac{1}{2}I.\omega'^2$, where I is the moment of inertia of the ball and ω', its angular velocity about the line of contact.

So that, $$mg\,(R-r).(\cos\theta-\cos\theta') = \tfrac{1}{2}I\,\omega'^2.$$

If ω be the angular velocity of the ball about a horizontal axis through S, we have $(R-r).\omega = r\omega'$. Or, $\omega' = \left(\frac{R-r}{r}\right).\omega$.

$$\therefore\ mg.(R-r).(\cos\theta-\cos\theta') = \frac{1}{2}I.\left[\left(\frac{R-r}{r}\right)\omega\right]^2,$$

whence, *differentiating with respect to time*, we have

$$mg.(R-r).\sin\theta.\frac{d\theta}{dt} = 2\times\frac{1}{2}\cdot I\left(\frac{R-r}{r}\right)^2\omega.\frac{d\omega}{dt}. \quad \left[\because\ \theta' \text{ is a constant.}\right.$$

And, since $d\theta/dt = \omega$, we have

$$mg.(R-r).\sin\theta.\omega = I.\frac{(R-r)^2}{r^2}.\omega.\frac{d\omega}{dt}.$$

Or, $$mg.(R-r).\sin\theta = I.\frac{(R-r)^2}{r^2}.\frac{d\omega}{dt}.$$

Or, $$\frac{d\omega}{dt} = \frac{mg.r^2(R-r)\sin\theta}{I(R-r)^2}.$$

$$= \frac{mg.r^2.\sin\theta}{I(R-r)} = \frac{mg.r^2}{I.(R-r)}.\theta. \qquad \left[\because \sin\theta=\theta, \text{ if } \theta \text{ be small.}\right.$$

Now, $$I = m\left(\frac{2}{5}r^2 + r^2\right) = \frac{7mr^2}{5}.$$

$\therefore$ $$\frac{d\omega}{dt} = \frac{5mg.r^2}{7mr^2(R-r)}.\theta = \frac{5g}{7(R-r)}.\theta.$$

Here, clearly, $d\omega/dt$ is the angular acceleration of the ball; so that, *angular acceleration of the ball* $= \frac{5g}{7(R-r)}.\theta$,

i.e., the angular acceleration of the ball is proportional to θ.

[$\because$ $5g/7(R-r)$ is a constant].

The ball thus executes a *simple harmonic motion*, and its time-period is, therefore, given by

$$t = 2\pi\sqrt{\frac{1}{5g/7(R-r)}}. \quad \text{Or, } t = 2\pi\sqrt{\frac{7(R-r)}{5g}}.$$

Squaring this expression, we have $t^2 = \frac{4\pi^2.7(R-r)}{5g} = \frac{28\pi^2(R-r)}{5g}$,

whence, $$g = \frac{28\pi^2(R-r)}{5t^2}.$$

Thus, knowing the radii of the concave surface and the ball (with the help of a spherometer and a vernier calliper, respectively), and noting the time-period of oscillation of the ball, we can easily calculate the value of g at the given place.

N.B. Re-arranging the expression for t^2, obtained above, we have $5gt^2 = 28\pi^2R - 28\pi^2r$. Or, $28\pi^2R = 5gt^2 + 28\pi^2r$.

Or, $$R = \frac{5gt^2+28\pi^2r}{28\pi^2} = \frac{5gt^2}{28\pi^2} + r.$$

So that knowing r, the radius of the ball, t, its period of oscillation and the value of g, we can easily calculate the *radius of curvature* (R) of the given spherical surface.

(3) The Atwood's Machine. In the ribbon-type of machine, [Fig. 122 (*a*)], a *strip* or *ribbon* R is passed round the *flat* rim of a light and *frictionless pulley*, (running on ball-bearings), with two equal masses M and M at its two ends, one of which is initially kept resting on a platform P.

An identical ribbon is attached to the lower ends of the two masses, as shown, so that when the system is set into motion, an

additional length of the first ribbon, passing on to the right side of the pulley, is exactly balanced by an *equal* length of the second, passing on to its left side, *thus ensuring that no extra mass is transferred from one side of the pulley to the other.*

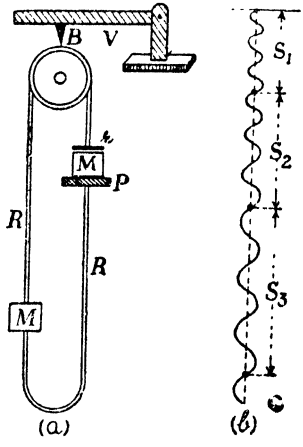


Fig. 122.

A steel strip or *vibrator V*, of a known time-period T, is clamped horizontally at one end, and carries a light *style*, or an *inked brush B*, which *just* touches the paper ribbon going round the pulley.

A small *rider r*, of mass m, is placed on the mass M, resting on the platform, to make it slightly heavier than the other.

Then, with the masses not yet in motion, the brush is moved across the paper ribbon to mark a horizontal line on it, *indicating the starting point.*

The platform P is now suddenly made to fall, (by means of *'trigger releases'*) and the vibrator simultaneously set vibrating. Naturally, the mass, loaded with the rider, moves *down* and the other *up* with a *common acceleration*, say a. And, as the ribbon runs past the brush, a wavy curve, due to the transverse vibrations of V, gets traced on it, and goes on gradually lengthening out, [Fig. 122 (*b*)], on account of the accelerated motion of the masses, and hence that of the ribbon.

Since *one wave* is traced out on the ribbon *during one vibration of V*, the distances occupied by successive waves represent the distances covered by the masses during successive time periods of it.

Thus, if S_1, S_2, S_3 etc., be the distances covered by the masses in the first, second and third etc., time-periods of V, we have

$$S_1 = \tfrac{1}{2}aT^2 \quad [\because u = 0, \text{ the masses starting from rest.}$$

and $S_2 = aT.T + \tfrac{1}{2}aT^2 = aT^2 + \tfrac{1}{2}aT^2$, [$\because$ here $u = aT$, the velocity after time T.

i.e., $S_2 = 3aT^2/2.$

And, similarly, $S_3 = 2aT.T. + \tfrac{1}{2}aT^2$ [$\because$ now $u = 2aT$, the velocity after time $2T$.

$$= 2aT^2 + \tfrac{1}{2}aT^2 = 5aT^2/2.$$

So that, $S_2 - S_1 = S_3 - S_2 = aT^2 = x$, say. Or, $a = x/T^2$.

Thus, T being known, the value of a can be easily calculated out.

Now, if v be the velocity acquired by the masses, when they have covered a distance, h we have

gain in K.E. of the pulley and the masses (together with the rider)
= loss in P.E. of the masses and the rider,

i.e., $\tfrac{1}{2}I\omega^2 + \tfrac{1}{2}(2M+m)v^2 = (M+m)gh - Mgh = mgh,$

where I is the *moment of inertia of the pulley* about its axis of rotation and ω, its *angular velocity* at the time.

Now, if R be the *radius* of the pulley, $v = R\omega$, or $\omega = v/R$.

Hence, $\frac{1}{2}I.v^2/R^2 + \frac{1}{2}(2M+m).v^2 = mgh.$

Or, $Iv/R^2 + (2M+m)v^2 = 2mgh.$

Or, $v^2(I/R^2 + 2M+m) = 2mgh,$

whence, $$g = \frac{v^2(I/R^2 + 2M + m)}{2mh}.$$

But $v^2 = 2ah.$ [$\because u = 0$ and $S = h$.

$$\therefore \quad g = \frac{2ah(I/R^2 + 2M + m)}{2mh} = \frac{a(I/R^2 + 2M + m)}{m}, \qquad ...(i)$$

whence, the value of g can be easily calculated.

It is, however, desirable to eliminate I from this expression, by repeating the experiment with the *same masses* but a *different rider*, of mass m'. If a' be now the acceleration of the masses, (determined as before), we have

$$g = \frac{a'(I/R^2 + 2M + m')}{m'}. \qquad .. (ii)$$

Re-arranging relations (i) and (ii), we have

$$mg/a = (I/R^2 + 2M + m), \qquad .. (iii)$$

and $$m'g/a' = (I/R^2 + 2M + m'). \qquad .. (iv)$$

So that, subtracting relation (iv) from relation (iii), we have

$mg/a - m'g/a' = (m-m')$. Or, $g(m/a - m/a') = (m-m')$,

whence, $$g = (m-m')/\left(\frac{m}{a} - \frac{m'}{a'}\right)^* \qquad .. (v)$$

Thus knowing m, m', a and a', the value of g can be easily obtained.

A possible source of error, here, is the *frictional force* encountered by the pulley as it rotates about its axle, which, obviously, tends to lower its angular velocity. This may be easily remedied by placing another *auxiliary* rider on the loaded mass, such that, with the main rider (r) removed from it, if an initial velocity be given to it, to set the system in motion, it continues to move with the *same uniform velocity*, *i.e.*, (its motion, is neither accelerated nor retarded). Obviously, then, the *weight* of this auxiliary rider exactly counteracts the retarding force due to friction. If, therefore, kept on the mass throughout the experiment, it completely eliminates the error due to friction, and, clearly neither its weight nor the frictional force need enter into our calculations.

(4) **The Dropping Plate.** A plate of glass, P, smoked by holding it over burning camphor, is suspended with its plane vertical, by means of a thread, as shown in Fig 123 (a), and a tuning fork F, of a known frequency N is mounted close to it, so that a light aluminium stylo, (or better still, a *hog's bristle*, such as may be obtained from a discarded hair brush), attached to one of its prongs, *just* touches the surface of the plate.

---

*Or, we could use *different masses* M and M' but the *same rider*, in which case we shall have $g = 2(M-M')/m(1/a - 1/a')$, where a' is the acceleration of the masses in the second case.

The fork is set vibrating by lightly drawing a bow across it, or by simply pinching it strongly, and the thread, supporting the plate, burnt or cut *simultaneously*. Thus released, the plate starts falling, with an acceleration, equal to the value of g at the place, and the style traces out a wavy line on it, of the form shown in Figs. 123 (*b*) and (*c*),—the waves being smaller and closer together at first, but gradually lengthening out and getting further apart, due to the accelerated motion of the plate, though the time taken to trace each wave remains the *same*, *viz.*, equal to $1/N$, the *time-period* of the fork.

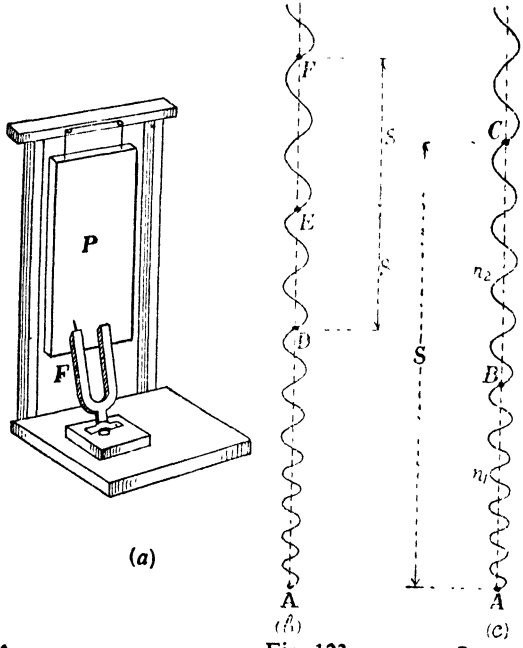


Fig. 123.

Three points D, E and F, [Fig. 123 (*b*)], are then marked on this wavy line, such that DE and EF contain the *same number of waves*, say, n, each Let distances DE and EF be S_1 and S_2 respectively, as measured by means of a travelling microscope, both being covered by the plate in the *same interval of time* $t = n/N$, taken by the fork to complete n vibrations.

Then, clearly, $S_1 = ut + \frac{1}{2}gt^2$. Or, $2S_1 = 2ut + gt^2$, ... (*i*)

where u is the velocity of the plate at D.

And, $(S_1+S_2) = 2ut + \frac{1}{2}g.(2t)^2$.

because, here, distance $= (S_1+S_2)$, and time $= (2t)$.

Or, $(S_1+S_2) = 2ut + 2gt^2$. ... (*ii*)

So that, subtracting equation (*i*) from (*ii*), we have

$$(S_1+S_2)-2S_1 = 2ut+2gt^2-(2ut+gt^2).$$

Or, $(S_2-S_1) = gt^2$,

whence, $$g = \frac{(S_2-S_1)}{t^2}.$$ ...(*iii*)

Or, substituting the value n/N for t, in relation (*iii*), we have

$$g = \frac{S_2-S_1}{(n/N)^2} = \frac{N^2(S_2-S_1)}{n^2}.$$ ... (*iv*)

Thus, knowing N, n and (S_2-S_1), we can easily calculate out the value of g at the place.

N.B. It will be readily seen that the mass of the style (or the *'hog's bristle'*), attached to the prong of the fork, together with the friction it encounters at the plate, will slightly lower its frequency, so that it will actually be somewhat less than N. For greater accuracy, therefore, the frequency of the fork (with the style attached to it) must be determined by the method of *'beats'*, by sounding it

with another fork of an accurately known frequency. The frequency, thus determined, should then replace N in relation (*iv*) for g, above.

Alternative Calculation. The following is a comparatively more accurate method of calculating the value of g, because, here, the possible error in correctly counting n is eliminated.

Three points A, B and C are marked on a portion of the wavy line, [Fig. 123 (*c*)], where the waves are clearly visible and can be distinctly counted. Let there be n_1 waves (and, therefore, n_1 vibrations) made by the fork in-between A and B, and n_2 waves (or n_2 vibrations) in between B and C, and let the total distance AC be S.

Then, clearly, (n_1+n_2) waves or vibrations are made by the fork in time $(n_1+n_2)/N$. So that,

$$S = \frac{1}{2}g\left(\frac{n_1+n_2}{N}\right)^2. \text{ Or, } \sqrt{S} = \sqrt{\frac{g}{2}} \cdot \frac{n_1+n_2}{N}. \quad \left[\because u = 0,\ a = g,\ \text{and}\ t = (n_1+n_2)/N.\right.$$

Plotting $\sqrt{S}$ against n_1, therefore, we obtain a straight line, (Fig. 124), of slope $\sqrt{\frac{g}{2}}\Big/N$, from which the value of g can at once be calculated, *without knowing n.*

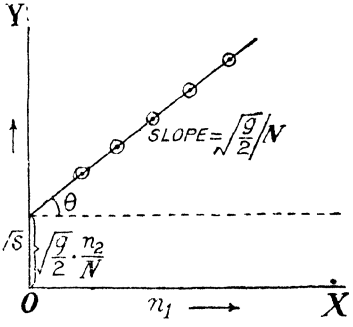


Fig. 124.

Incidentally, the dropping plate method also shows that a freely falling body is subjected to a constant acceleration due to gravity,—a fact, not easy to demonstrate otherwise.

(5) Vertical Oscillations of a Flat Spiral Spring. A spiral spring is just a uniform wire or ribbon, designed to have, in its normal, unstrained condition, the form of a regular *helix*, such as may be obtained by winding the wire closely and uniformly round a cylinder, of a diameter much greater than its own.

If the plane of each coil of the spiral is perpendicular to the axis of the cylinder, it is called a *flat spiral*, but if it be inclined at a small angle to this axis, it is spoken of as an *inclined spiral*. We shall concern ourselves here only with the flat spiral of a wire of *circular cross-section*.

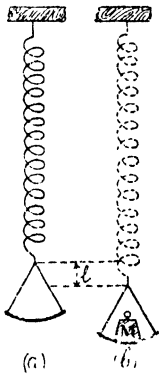


Fig. 125

If a small force be applied to a flat spiral, *along its axis, (i.e., along the straight line passing through the centre of each coil of it)*, and perpendicular to its plane, it increases in length a little, but still preserves its helical form, as will be clear from Figs. 125 (*a*) and (*b*).

Let us consider a flat spiral, of length L and radius R, (where R is much greater than its pitch), suspended from a rigid support, with its upper and lower ends (A and B), bent as shown in Fig. 126 (*a*), so as to lie along its axis.

Then, if a mass m be suspended from its lower free end, a force equal to mg (the weight of the mass) acts vertically downwards along its axis, producing a *statical* extension l in its length, (*i.e.*, an extension, with the mass m at rest).

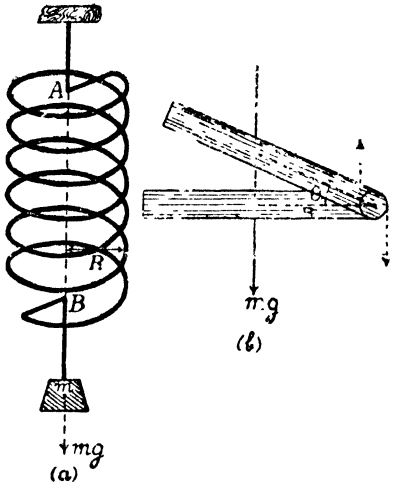


Fig. 126.

The effect of this force mg, acting along the axis of the spiral is to produce a *turning moment*, equal to $mg\,R$, at every section of it, [Fig. 126 (*b*)]. And, this, in its turn, subjects the wire to a uniform twist θ, say, per unit length of it.

Now, the *twisting* or *torsional couple** *per unit length of the wire* is equal to $n\pi r^4\theta/2$, where n is the *coefficient of rigidity* of the material of the wire; r, its *radius* and θ, its *angle of twist*.

This, therefore is the *torsional resistance*, opposing the turning moment $mg.R$ due to the weight mg. For equilibrium, therefore, we have

$$mg.R = n\pi r^4\theta/2. \qquad \ldots (i)$$

Now, a twist θ per unit length of the wire corresponds to a displacement or extension $R.\theta$ *per unit length* of it, [Fig 126 (*b*)]. Clearly, therefore, the extension produced in the whole length of the wire is equal to $L.R.\theta$.

Or, $l = L.R.\theta$, whence, $\theta = l/LR$. [$\because$ l is the total extension produced.

Substituting this value of θ in relation (*i*) above, we have

$$mg.R = \frac{\pi nr^4}{2} \cdot \frac{l}{LR} = \frac{\pi nr^4 l}{2LR}, \text{ whence, } mg = \frac{\pi nr^4\, l}{2LR^2}.$$

The expression $\pi nr^4.l/2LR^2$ thus represents the force of *elastic reaction* for an increase l in the length of the spiral, and, therefore, the *elastic reaction per unit increase* in the length of the spiral $= \pi nr^4/2LR^2$, (because $l-1$). Denoting this by K, we have

$$mg = K\,l, \quad \text{whence,} \quad K = mg/l.$$

If the suspended mass (m) be now displaced or pulled vertically downwards through a distance x and then released, so as to produce vertical oscillations in the spiral, the restoring force F, acting on the mass may *for small oscillations*, be taken to be directly proportional to its displacement. So that, $F = K.x$.

And, if d^2x/dt^2 be the acceleration of the mass at the given

*See chapter VII, where it is shown that the twisting couple on a cylinder (or wire) is equal to $n\pi r^4\theta/2l$, where θ is the angle of twist and l, its length.

instant, when its displacement is x, the *inertial reaction* of the mass is clearly equal to $m.d^2x/dt^2$.

Hence, by Newton's third law of motion, we have

$$m.d^2x/dt^2 = -K\,x, \qquad \text{whence,} \qquad d\,x/dt^2 = -x.K/m.$$

Now, since K and m are both constant quantities for the given spiral, we have K/m = a constant, μ, say.

So that, $d^2x/dt^2 = -\mu.x$. Or, $d^2x/dt^2 \propto x$;

i.e., the acceleration of the mass is directly proportional to its displacement x; and it, therefore, executes a simple harmonic motion, its time-period being given by the expression

$$t = 2\pi\sqrt{1/\mu} = 2\pi\sqrt{\frac{1}{K/m}}, \text{ Or, } = 2\pi\sqrt{m/K} = 2\pi\sqrt{\frac{m}{mg/l}} \qquad ...(ii)$$

Or, $$t = 2\pi\sqrt{l/g};$$

i.e., the time-period is the same as that of a simple pendulum of length l, the extension produced in the spiral.

Squaring and re-arranging this expression for t, we have $g = 4\pi^2.l/t^2$, whence the value of g, at the given place, can be easily calculated out.

In the above treatment, we have not taken into consideration the mass of the spring, assuming it to be negligible, compared with the suspended mass m. For greater accuracy, however, it must also be taken into account. So that, if the *effective mass* of the spring be m_s, the total mass acting downwards along the axis of the spring becomes $m+m_s$ and the expression (*ii*) above, for the time-period of the spiral, becomes $t = 2\pi\sqrt{(m+m_s)/K}$.

It is, however, best to eliminate m_s altogether. This is done by performing the experiment with two different suspended masses, m_1 and m_2.

Then, if t_1 and t_2 be the respective time-periods of oscillation in the two cases, we have

$$t_1 = 2\pi\sqrt{(m_1+m_s)/K} \quad \text{and} \quad t_2 = 2\pi\sqrt{(m_2+m_s)/K}.$$

So that, squaring and subtracting the second from the first, we have

$$(t_1^2-t_2^2) = \frac{4\pi^2}{K}(m_1-m_2) = 4\pi^2.\frac{(m_1-m_2)}{K} \qquad ...(iii)$$

Now, if l_1 and l_2 be the *statical* extensions corresponding to the two masses, we have

$$m_1.g = Kl_1 \text{ and } m_2.g = Kl_2,$$

whence, $m_1.g-m_2.g = Kl_1-Kl_2$. Or, $(m_1-m_2).g = K(l_1-l_2)$.

Or, $$\frac{(m_1-m_2)}{K} = \frac{(l_1-l_2)}{g}.$$

Substituting this value of $(m_1-m_2)/K$ in expression (*iii*) above, we have $(t_1^2-t_2^2) = 4\pi^2.\frac{(l_1-l_2)}{g}$, whence, $g = 4\pi^2.\left(\frac{l_1-l_2}{t_1^2-t_2^2}\right)$.

Thus, observing t_1 and t_2 directly, and measuring l_1 and l_2, by noting the positions of a light pointer, attached to the spiral, on a

vertical centimetre scale fixed alongside it, the value of g can be easily calculated.

(6) **The Bifilar Suspension.** If a heavy and uniform bar or cylinder, (or, in fact, any rigid body), be suspended horizontally by means of two *equal, vertical, flexible* and *inelastic* threads, *equidistant from its centre of gravity*, the arrangement constitutes what is called a *bifilar suspension*. On being displaced a little *in its own plane, i.e.*, in the horizontal plane), and then released, the bar or cylinder executes a simple harmonic motion about the vertical axis through its centre of gravity.

Now, two cases, arise, (*i*) *when the two suspension threads are parallel,* and (*ii*) *when they are not.* Let us consider both.

(*i*) **Bifilar Suspension, with Parallel Threads.** Let AB [Fig. 127, (*a*)], represent the original or equilibrium position of a cylinder, of mass m, and with its *c.g.* at O, where its weight mg acts vertically downwards. Let the two suspension threads PA and QB be parallel to each other, and distance $2d$ apart ; and let the length of each be l.

Now, if the cylinder be displaced a little into the position $A'B'$, through a small angle θ, about the vertical axis through O, the suspension threads take up the position PA' and QB' at an angle ϕ with their original positions, where ϕ is small.

Let T be the tension in in *each* thread, acting upwards along it. Then, resolving it into its two rectangular components, we have

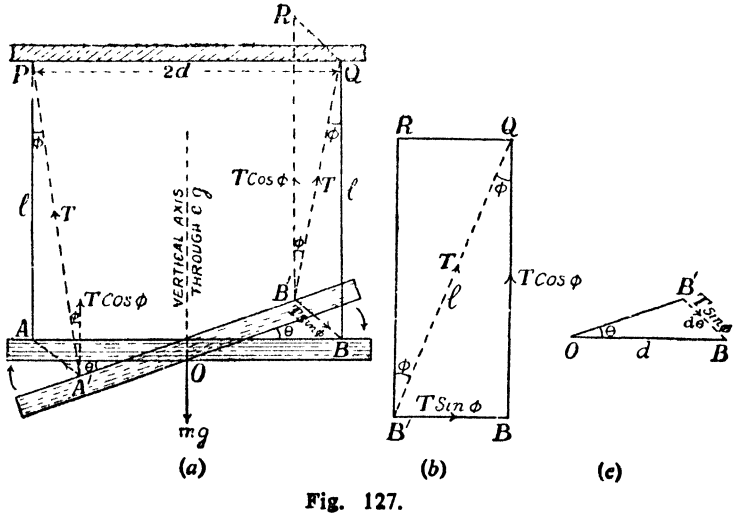


Fig. 127.

(*i*) the *component* $T \cos \phi$, *acting vertically upwards* ;

and (*ii*) the *component* $T \sin \phi$, *acting horizontally along* $B'B$ *and* $A'A$, [*Fig. 127* (*b*)].

Obviously, the vertical components support the weight of the cylinder. Hence,

$$2T \cos \phi = mg. \qquad \text{Or,} \qquad T \cos \phi = mg/2.$$

And, since ϕ is small, $\cos\phi = 1$, very nearly.

So that, $T = mg/2$.

The components, $T \sin\phi$, (acting at A' and B'), on the other hand being *equal, oppostie* and *parallel*, constitute a *couple*, tending to bring the cylinder back into its original position. And, since $A'A$ and $B'B$ are practically at right angles to $A'B'$, we have

moment of this restoring couple $= T.\sin\phi.2d = T.\phi.2d.$ [$\because$ ϕ being small, $\sin\phi = \phi$.

Now, $\theta = BB'/OB = BB'/d$; so that, $BB' = \theta.d$. [Fig. 127 (*c*).

And, therefore, $\phi = BB'/l = \theta.d/l$. [Fig 127 (*b*).

Hence, *restoring couple* $= T.\frac{\theta.d}{l}.2d = \frac{mg}{2}.\frac{\theta\,2d^2}{l} = \frac{mg.d^2}{l}.\theta$. But the restoring couple is also $= I.d^2\theta/dt^2$,

where I is the *moment of inertia* of the cylinder about the vertical axis through O, (its *c.g.*), and $d^2\theta/dt^2$, its *angular acceleration.*

$\therefore \quad I.\frac{d^2\theta}{dt^2} = \frac{-mg.d^2}{l}.\theta$, whence, $\frac{d^2\theta}{dt^2} = \frac{-mg\,d^2}{I\,l}.\theta$.

Now, $mg.d^2/Il$ is a constant quantity, in a given case, and, therefore, putting it equal to μ, we have,

$$d^2\theta/dt^2 = \mu.\theta. \qquad \text{Or,} \qquad d^2\theta/dt^2 \propto \theta,$$

i.e., the angular acceleration of the cylinder is proportional to its angular displacement, and is clearly directed towards its mean position. The cylinder, therefore, executes a simple harmonic motion and *its time-period is given by*

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{1}{mgd^2/Il}}.$$

Or, $$T = 2\pi\sqrt{\frac{I.l}{mgd^2}} = 2\pi.\frac{1}{d}\sqrt{\frac{I.l}{mg}}. \qquad \ldots(i)$$

Or, if we put $I = mk^2$, where k is the *radius of gyration* of the cylinder about the vertical axis through O, we have

$$T = 2\pi\frac{1}{d}\sqrt{\frac{mk^2.l}{mg}} = 2\pi.\frac{1}{d}\sqrt{\frac{k^2.l}{g}},$$

whence, $$T = 2\pi.\frac{k}{d}\sqrt{\frac{l}{g}}. \qquad \ldots(ii)$$

Now, squaring relation, (*i*) or (*ii*), and re-arranging, we have, from relation (*i*), $g = 4\pi^2\,I.l/m.d^2.T^2$ ... ...(*iii*)

and, from relation (*ii*), $g = 4\pi^2.k^2.l/d^2.T^2$. ... ... (*iv*)

And, thus, the value of g at the given place may be easily calculated out.

(*ii*) **Bifilar Suspension with Non-Parallel Threads.** Let the rod or cylinder AB, [Fig. 128 (*a*)], be suspended symmetrically by two *equal* but *non-parallel* threads*, each of length l, and let the distance between the threads at the top and at the bottom be $2d_1$, and $2d_2$ respectively, where $(d_2 - d_1) = x$.

*The threads, in this position of the rod, are not shown, to avoid complicating the Figure

If the cylinder be displaced through an angle θ, *in its own plane*, into the position $A'B'$, the suspension threads take up positions PA' and QB'. Then, as before, tension T acts upwards along each thread, and may be resolved into two rectangular components, *viz.*,

(*i*) $T \cos \phi$, *acting vertically upwards, as shown, and* (*ii*) $T \sin \phi$, *acting horizontally, along* $A'K$ *and* $B'L$ [Fig. 128 (*a*)], *where* ϕ is the angle that *each thread makes with the vertical, or the perpendiculars* PK *and* QL, *from* P *and* Q *on to* AB.

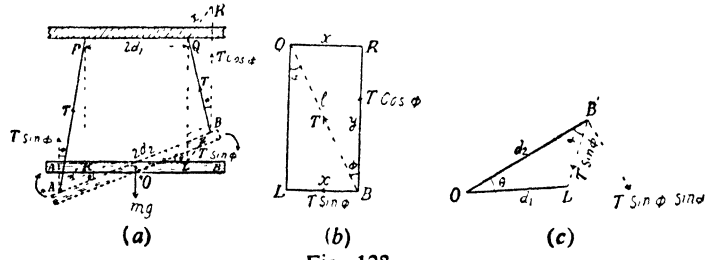


Fig. 128.

The vertical components $T \cos \phi$ support the weight mg of the cylinder, acting vertically downwards at its *c.g.*, O ; and, therefore,

$$2T \cos \phi = mg. \qquad \ldots(v)$$

Now, clearly, $LB' = \sqrt{d_2^2 + d_1^2 - 2d_2 . d_1 \cos \theta}$. [Fig 128 (*c*)

$= \sqrt{d_2^2 + d_1^2 - 2d_2 . d_1}$, approximately. [θ being small, and $\therefore \cos \theta = 1$, nearly.

$= \sqrt{(d_2 - d_1)^2}$.

Or, $LB' = (d_2 - d_1) = x$, approximately.

And, $\cos \phi = \dfrac{B'R}{B'Q} = \dfrac{B'R}{l} = \dfrac{\sqrt{B'Q^2 - RQ^2}}{l} = \dfrac{\sqrt{l^2 - x^2}}{l}$. [Fig 128 (*b*).

Now, $T = \dfrac{mg}{2 \cos \phi} = \dfrac{mg}{2\sqrt{l^2 - x^2}/l} = \dfrac{mg.l}{2\sqrt{l^2 - x^2}/l}$ ...(*vi*) [From relation (*v*) above.

And, resolving forces $T \sin \phi$, *acting at* A' and B' along $A'K$ and $B'L$ respectively, into their rectangular components along and at right angles to $A'B'$, we have

the components *at right angles to* $A'B' = T \sin \phi . \sin \alpha$, [Fig. 128 (*c*).

Since these two components act in opposite directions at A' and B', they constitute a *couple*, tending to rotate the cylinder back into its original position AB ; and, clearly,

moment of this restoring couple $= T \sin \phi . \sin \alpha . A'B'$.

Or, *restoring torque on the cylinder* $= T \sin \phi . \sin \alpha . 2d_2$.

$$= 2d_2 . \frac{mg.l}{2\sqrt{l^2 - x^2}} . \frac{x}{l} . \sin \alpha. \qquad \left[\because \sin \phi = \frac{x}{l}, \text{ from Fig. 128 (b).} \right.$$

$$= \frac{d_2 . mg\, l}{\sqrt{l^2 - x^2}} . \frac{x}{l} \sin \alpha.$$

Now, the sides in a triangle, being proportional to the sines of the angles opposite to them, we have, from Fig. 128 (*c*),

$$d_1 / \sin \alpha = LB' / \sin \theta,$$

So that, $d_1/\sin\alpha = x/\theta$. And $\therefore \sin\alpha = \frac{d_1}{x}.\theta$. [$\because LB' = x$, and θ is small.

Hence, *restoring torque on the cylinder*

$$= \frac{d_2.mg\,l}{\sqrt{l^2-x^2}}.\frac{x}{l}.\frac{d_1}{x}.\theta = \frac{d_1 d_2.mg}{\sqrt{l^2-x^2}}.\theta.$$

But, *restoring torque is also* $= I.\frac{d^2\theta}{dt^2}$,

where I is the *moment of inertia of the cylinder* about the vertical axis through its *c.g.*, and $d^2\theta/dt^2$, its *angular acceleration.*

$$\therefore \quad I.\frac{d^2\theta}{dt^2} = -\frac{d_1.d_2.mg}{\sqrt{l^2-x^2}}.\theta. \text{ Or, } \frac{d^2\theta}{dt^2} = -\frac{d_1 d_2.mg}{I.\sqrt{l^2-x^2}}.\theta. = -\mu\theta,$$

where, $\frac{d_1.d_2.mg}{I.\sqrt{l^2-x^2}} =$ a constant μ.

Thus, $$\frac{d^2\theta}{dt^2} \propto \theta.$$

Or, *the angular acceleration of the cylinder is proportional to its angular displacement.* It, therefore, executes *a simple harmonic motion* and its *time-period* is given by

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{1}{d_1.d_2\,mg/I.\sqrt{l^2-x^2}}}.$$

Or, $$T = 2\pi\sqrt{\frac{I\sqrt{l^2-x^2}}{d_1 d_2.mg}}. \qquad \ldots(vii)$$

And, if we put $I = mk^2$, where k is the *radius of gyration* of the cylinder about the vertical axis through its *c.g.* we have

$$T = 2\pi\sqrt{\frac{mk^2\sqrt{l^2-x^2}}{d_1.d_2\,mg}} = 2\pi\frac{k}{\sqrt{d_1.d_2}}.\sqrt{\frac{\sqrt{l^2-x^2}}{g}}.$$

Now, if y be the *vertical* distance between the two ends of each suspension thread, we have $y = B'R = LQ = \sqrt{l^2-x^2}$. [Fig. 128 (*b*).

So, that, $$T = 2\pi.\frac{k}{\sqrt{d_1 d_2}}\sqrt{\frac{y}{g}}. \qquad \ldots \quad (viii)$$

Again, squaring and re-arranging relations (*vii*) and (*viii*), we have

from relation (*vii*) $$g = \frac{4\pi^2.I.\sqrt{l^2-x^2}}{d_1 d_2.m.T^2}. \quad \ldots \quad \ldots \quad \ldots(ix)$$

and, from relation (*viii*), $$g = \frac{4\pi^2 k^2.y}{d_1.d_2.l^2}. \quad \ldots \quad \ldots \quad \ldots(x)$$

The value of g, at the place, can thus be calculated from either of these relations

It will be readily seen that if $d_1 = d_2 = d$, and $y = l$, so that $x = 0$, *i.e.*, *when the two threads are parallel* and *vertical* in the original equilibrium position of the cylinder, we have

(*a*) relation (*vii*) reduced to $T = 2\pi.\frac{1}{d}\sqrt{\frac{I.l}{mg}}$, the same as relation (*i*), for parallel threads, and

(*b*) relation (*viii*) reduced to $T = 2\pi.\frac{k}{d}\sqrt{\frac{l}{g}}$, the same as relation (*ii*) for parallel threads, (see page 201).

Note. It will be clear from the above that the bifilar suspension may also be used to determine the value of I for the suspended cylinder etc. For, relation (*iii*) above, when re-arranged, gives $I = \frac{m\,d^2\,T^2\,g}{4\,\pi^2.l}$,

and relation (*ix*), when re-arranged, gives $I = \frac{m\,d_1.d_2T^2.g}{4\pi^2\,\sqrt{l^2-x^2}}$.

In fact, this method is more suitable for determining the moment of inertia of a body than for determining the value of g.

76. Variation of the value of '*g*'. The value of g at a given place is affected by a number of factors, *viz.*,

(*i*) *latitude* of the place ; (*ii altitude* and (*iii*) *depth*. We shall now proceed to study the effect due to each of these factors a little in detail.

(*i*) **Effect due to Latitude.** The effect of *latitude* on the value of g may be considered under two headings, *viz.*, (*a*) the *effect of the rotation of the earth*, and (*b*) *the effect due to the bulge at the equator.*

Let us consider each separately.

(*a*) **Effect of Rotation.** We know that the earth is rotating about its axis from west to east. If it were at rest, and were a homogeneous sphere, the acceleration due to gravity would be the same for a body at all points on its surface and would be directed towards its centre. Due to its rotation, however, part of the force of gravity on the body is used up in overcoming the centripetal force acting on it, and thus the resultant acceleration on it is different, both in magnitude and direction, at different places,— *i.e.*, the apparent value of '*g*' is different in different latitudes, as will be clear from the following :

Let *NWSE*, (Fig. 129), be a section of the earth, (supposed to be a perfect sphere), through its polar diameter *NS*, and let its radius be r. Then, if ω be the angular velocity of the earth about the axis of rotation *NS*, all points on its surface rotate about this axis with the angular velocity ω. The linear velocity of each point will depend, however, on its distance from the axis. Thus, the linear velocity of a particle at the points *E*, *N*, *W* and *S* will be $r.\omega$, and that at a point *P*, distant $PM = x$ from the axis, will be $x.\omega$, where x is the radius of the circle that the point *P* describes as it rotates with the earth.

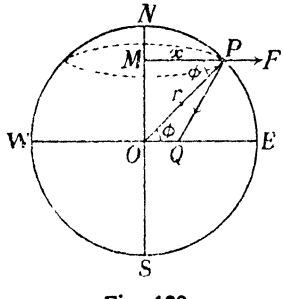


Fig. 129.

Let ϕ be the latitude in which the point *P* is situated. Then, since the radius *PM*, of the circle described by *P*, is $r\ cos\ \phi$, the linear velocity of $P = r\ cos\ \phi.\omega$; so that, the centrifugal force acting on *P*, *away from the centre* (*M*) *of the circle it describes, and acting*

along MP, is clearly given by $m.r.\cos\phi.\omega^2$. Let it be represented in magnitude as well as direction by the straight line *PF*.

The force of gravity *mg*, which would act on the body if the earth were at rest, (*g* being the acceleration due to gravity, with the earth at rest), would obviously act towards the centre of the earth *O*. Let it be represented in magnitude as well as direction by the straight line *PO*.

Thus, there are two forces acting simultaneously at the point *P*, *viz.*, (*a*) the centrifugal force $m.r.\cos\phi.\omega^2$, *along PF*, and (*b*) the force *mg* due to gravity *along PO*. Completing the triangle of forces *POQ*, where *PO* represents the gravitational force *mg*, and *OQ*, the centrifugal force $m.r.\cos\phi.\omega^2$, we have the resultant force at *P* represented by the third side *PQ* of the triangle, both in magnitude and direction, where

$$PQ = \sqrt{PO^2+OQ^2-2PO\ OQ\ \cos POQ}. \quad \text{[See Appendix 1·7 (2),}$$
$$= \sqrt{m^2g^2+m^2.r^2.\cos^2\phi.\omega^4-2mg.mr.\cos\phi.\omega^2.\cos\phi}.$$
$$= \sqrt{m^2g^2+m^2.r^2\cos^2\phi\ \omega^4-2m^2.g\ r\ \cos^2\phi.\omega^2}.$$

Now, the value of $r.\omega^2$ comes to be about 3·39 *cms./sec*^{2}., or about 1/288 of the value of *g*; for $r = 6\,378\times10^8$ *cms.*, and $\omega = 2\pi/86164$, (where 86164 is the number of *mean solar seconds* in one day).

Thus, the expression $m^2.r^2.\cos^2\phi.\omega^4$ is negligible, compared with the other terms involving *g*, and, therefore,

$$PQ = \sqrt{m\ g^2-2m^2.g.r\ \cos^2\phi.\omega^2} = \sqrt{m^2(g^2-2g.r.\cos^2\phi.\omega^2)}.$$
$$= \sqrt{m^2g^2\left(1-\frac{2r\ \cos^2\phi.\omega^2}{g}\right)} = mg\left(1-\frac{2r\ \cos^2\phi\omega^2}{g}\right)^{\frac{1}{2}}.$$

Or, $$PQ = mg\left(1-\frac{1}{2}\times\frac{2r\ \cos^2\phi.\omega^2}{g}+\textit{some other negligible terms}\right).$$

Or, $$\textit{the resultant force on } P = mg\left(1-\frac{r.\omega^2.\cos^2\phi}{g}\right).$$

∴ if acceleration of the point *P*, in latitude ϕ, be g_ϕ, we have

$$g_\phi = g\left(1-\frac{r.\omega^2\ \cos^2\phi}{g}\right) \quad \ldots\text{I} \quad [\because \textit{acc.} = \textit{force/mass}.$$

This value is obviously smaller than *g*, and is *directed towards Q*, and *not towards O*, the centre of the earth; the angle *OPQ*, or *the change in direction of the gravitational force* is, however, very small.

For points on the equator, since $\phi = 0^0$, and, therefore, $\cos\phi = 1$, the value of the centrifugal acceleration $= r.\omega^2$, *i.e., a maximum.* And, *for points on the poles*, because $\phi = 90°$, and, therefore, $\cos\phi = 0$, the value of the centrifugal acceleration is *zero, i e., a minimum.*

It follows, therefore, that *the apparent acceleration of a body is the least at the equator, and the greatest at the poles, with intermediate values in-between.*

Substituting the value of $r.\omega^2/g = 3{\cdot}39'978 = 1/288$ in the expression for g above, (the value of g being 978·03 *cms./sec*^{2}. at the equator), we have

$$g_\phi = g\left(1 - \frac{\cos^2\phi}{288}\right).$$

This is a result, not quite in agreement with the experimental value.

The discrepancy may, however, be ascribed (*i*) to *the ellipticity of the earth*, its radius increasing as we proceed from the poles towards the equator, so that points in the higher altitudes are nearer to its centre than those near the equator ; (*ii*) to *the non-homogeneity of its composition*, the density of its different layers being different, *with the inner layers comparatively much denser than* (about more than twice as dense as) *the outer ones.*

(*b*) **Effect of Bulge at the Equator.** It was **Richer**, whose experiments in 1672 first showed a variation in the value of '*g*' at two different places.

Determining the length of a seconds pendulum at Cayenne (in French Guiana) and at *Paris*, he found its length at Paris to be just over one-tenth of an inch greater than at Cayenne, clearly showing the value of *g* to be greater at *Paris.*

Newton soon explained this variation on the assumption that the earth behaved as though it were a *'uniformly gravitating fluid globe'* ; so that, by virtue of its very rotation, it was bound to have a spheroidal shape, with a bulge or a protuberance at the equator, and comparative flattening of the poles under the influence of the centrifugal force acting on it, the value of which varies from *zero* at the poles to a maximum at the equator. In fact, even if the earth were perfectly rigid, it should have assumed this shape before it actually cooled down. As a consequence, the equatorial radius is about 13 miles greater than its polar radius. Hence, all bodies in the equatorial regions are farther from the centre than those in the polar regions, and the force of attraction due to gravity on the latter is, therefore, greater than that on the former.

It can be shown that the true value of g at a place in latitude λ is given by the relation,

$$g = (980{\cdot}61 - {\cdot}025\cos 2\lambda)\ cms./sec^2.$$

These changes in the value of g due to latitude are of great help in determining the figure or the shape of the earth.

(*ii*) **Effect of Altitude.** The correction for altitude we really owe to *Laplace* and *Stokes*,—particularly to the latter.

Let g be the value of acceleration due to gravity on the surface of the earth and g'. its value at a height h above the surface. Then, if the earth be considered to be a sphere of *homogeneous composition*, the acceleration due to gravity at any point above its surface will vary inversely as the square of the distance of that point from its centre ; so that

$$\frac{g}{g'} = \frac{(r+h)^2}{r^2} = \frac{r^2+2rh+h^2}{r^2} = 1 + \frac{2h}{r} + \frac{h^2}{r^2}.$$

[where r is the radius of the earth,

If h be small, compared with r, the quantity h^2/r^2 will be negligibly small, and we shall, therefore, have

$$g/g' = 1+2h/r. \quad \text{Or, } g'/g = 1/(1+2h/r) = 1-2h/r \text{ approx.}$$

Or, $$g' = g\,(1-2h/r),$$

i.e., the greater the value of h, the smaller the value of g'.

Or, *the value of g decreases with altitude.*

The general expression for the acceleration due to gravity at altitude h and in latitude λ. thus becomes

$$g' = (1-2h/r)(980{\cdot}61-{\cdot}025\ cos\ 2\lambda)\ cms./sec^2.$$

(*iii*) **Effect of Elevated Masses** The correction term $(1-2h/r)$ for altitude would only be valid when there is nothing but space between the surface of the earth and the point h above, *e.g.*, for an observer in an aeroplane at height h But if we consider the point to lie on the top of a mountain, of height h, a complication comes in due to the effect of the attraction by the mountain.

Bouguer suggested the correction $\left(1-\frac{2h}{r}+\frac{3.h.\rho}{2.r.\Delta}\right)$, known as *Bouguer's Rule*, where Δ is the mean density of the earth and ρ, that of the mountain.

It is now found, however, that Bouguer somewhat over-estimated the effect of the mountain and his correction*, therefore, gives the upper limit, as it were, of its effect, the lower limit being that in which its attraction is neglected altogether. The *Board of Trade* have, therefore, adopted the following relation for the combined effect of latitude and altitude :

$$g' = (980{\cdot}61-{\cdot}025\ cos\ 2\lambda)(1-5h/4r)\ cms./sec^2.$$

(*iv*) **Effect of Depth.** Again, *imagining the earth to be a homogeneous sphere*, let us take a body of mass m. *inside the earth*, at a depth h below the surface, so that its distance from the centre of the earth is $(r-h)$, where r is the radius of the earth. Imagine a sphere of this radius $(r-h)$ to be drawn concentric with the earth, (Fig. 130).

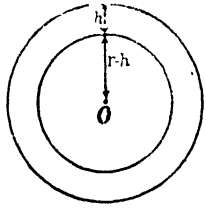


Fig. 130.

Then, clearly, the body lies on the surface of this inner sphere, and inside the outer hollow spherical shell, of thickness h. Let g and g' be the accelerations due to gravity at the surface of the earth and at a depth h below it, respectively. And since the force of attraction on a body inside a hollow shell is *zero*. the only force of attraction on the body is that due to the inner solid sphere, of radius $(r-h)$, and is directed towards its centre, its magnitude being clearly given by

$$mg' = \frac{mass\ of\ the\ sphere \times mass\ of\ the\ body}{(r-h)^2} \times G,$$

where G is the *gravitational constant.*

*This correction by Bouguer was prompted by the same idea which inspired his Mountain experiment for the determination of the Gravitational Constant G, (See page 231), *viz.*, that the attraction on a mass due to the mountain could simply be added up to that due to the earth, (taken to be a sphere).

Now, *mass of the sphere* = *its volume* × *its density*,

$$= \tfrac{4}{3}.\pi.(r-h)^3 \times \triangle,$$

where $\triangle$ is the *density* of the earth, *supposed uniform.*

$\therefore$ force of attraction on the body at a depth h inside the surface of the earth is equal to

$$mg' = \frac{4}{3}.\frac{\pi.(r-h)^3 \times \triangle \times m}{(r-h)^2} \times G = \tfrac{4}{3}.\pi.(r-h).\triangle m.G,$$

whence, $$g' = \tfrac{4}{3}.\pi.\triangle.G.(r-h). \qquad ..II$$

And, if the body were kept on the surface of the earth, the force of attraction towards the centre of the earth would be given by

$$mg = \frac{4}{3}.\frac{\pi.r^3.\triangle.m}{r^2}.G, \quad \text{whence,} \quad g = \frac{4}{3}.\pi.\triangle.G.r. \qquad ...III$$

Dividing relation II by relation III, we have

$$\frac{g'}{g} = \frac{\frac{4}{3}\pi.\triangle\, G\,(r-h)}{\frac{4}{3}\pi.\triangle.G\,r} = \frac{r-h}{r} = \left(1-\frac{h}{r}\right)$$

Or, $$g' = g\left(1-\frac{h}{r}\right), \qquad ...IV$$

i.e., the value of g decreases with depth from the surface of the earth. And it follows at once, from relation IV above, that at the centre of the earth, where $h = r$, the value of g will be *zero ; i.e., the acceleration due to gravity and, therefore, the weight of a body at the centre of the earth will be zero.*

(*v*) **Effect of Terrain—(Topographical Correction).** This correction consists in reducing the result at any given station to that we would obtain if the land or the terrain in which it is situated were just a horizontal plane, instead of its actual form.

Obviously, some parts of this terrain would be above and others below the horizontal plane, so that the former would exert an upward attraction, thus decreasing the value of g and the latter, a downward attractive force, thereby increasing the value of g.

It so turns out, however, that this correction is always a positive one.

77. **Determination of the value of *g* at Sea.** Until comparatively recently, the value of g at sea was determined indirectly, because it was not considered possible to use a pendulum on board a ship. The method, suggested by **Hecker** and **Duffield**, and usually adopted, was to determine the atmospheric pressure in two different ways, one of which involved g and the other did not, so that, by equating the two, the value of g could be easily calculated out.

Thus, for example, the atmospheric pressure P could be obtained (*i*) from a barometer *which involved g*, because $P = H\rho.g.$, where H is the height of the mercury column and ρ, its density, and (*ii*) from the boiling point of water, *which did not involve g*, because it could be calculated from the *Tables*, giving the relation between temperature and the saturated vapour pressure of water vapour ; or directly from an *aneroid barometer*, (again, without involving g). Then, equating $H.\rho.g$, against P, as obtained from method (*ii*), we have

$$g = \frac{P}{H.\rho}.$$

The results obtained by this method give us an error of about $\pm$ ·01 *cm./sec*^{2}., which is considerably greater than that given by pendulum methods on land, the chief source of error being the oscillations or 'bumpings' of the mercury columns in the barometer, caused by the movement of the vessel,—the ship or the submarine.

Vening Meinesz has shown, however, that pendulums can be used for the purpose with far greater accuracy, particularly in a submerged submarine. His argument is as follows :

A pendulum is subject to four types of disturbances on board a ship, *viz.*, (*i*) *the point of suspension having a horizontal acceleration,* (*ii*) *vertical acceleration of the support,* (*iii*) *angular movement of the support or 'rocking' of the plane of oscillation, and* (*iv*) *slipping or sliding of the knife-edges on their agate planes.*

Of these, the first disturbance is the most marked, but it can be completely eliminated by simultaneously oscillating two identical *half-second pendulums**, suspended from the *same support*, oscillating in the *same vertical plane*, but with *different phases*, and noting their angular displacements θ_1 and θ_2. Then, it can be easily shown that $(\theta_1-\theta_2)$ gives the angular displacement of a pendulum, altogether unaffected by this disturbance.

The vertical acceleration of the support can, however, not be eliminated, without eliminating g itself, but the disturbance due to this can be greatly minimised by taking the mean of a large number of observations. For, the value of g seems to be affected only by the mean value of the vertical acceleration during the whole period of observation. And, since the vertical motion is alternately up and down the zero position, the mean value of this acceleration becomes almost inappreciable.

The error due to '*rocking*' can be easily corrected for, by noting different values of the rocking angle and computing the necessary correction, which is usually quite small.

And, with all these errors eliminated, or minimised, to an extent that the total angular deviation due to them does not exceed 1°, the fourth error, *viz.*, the slipping of the knife-edges gets automatically eliminated.

What is done, in actual practice is that *three* half-second pendulums are suspended from the same support and set oscillating, and continuous photographic records of (*i*) the difference $(\theta_1-\theta_2)$ between the angular displacements of the first and the second, and (*ii*) the difference $(\theta_2-\theta_3)$, between those of the second and the third pendulums obtained on a sensitized paper, by means of a suitable optical arrangement. The value of g is then calculated from each of these two sets of observations and their mean taken.

The whole system is suspended in *gimbals*, to avoid external disturbances due to small angular movements of the ship or the submarine ; and, further, to avoid any possible errors due to any slight change in temperature, the whole apparatus is kept properly thermally insulated and any small correction, still necessary, applied. And, finally, to make sure that no magnetic disturbances affect the result, the pendulums are made, not of *invar-steel*†, (which would be so helpful in minimising any temperature corrections), but of brass.

The probable error in the value of g thus obtained is claimed to lie within $\pm$ ·0018 *cm./sec*^{2}.,—obviously, a marked improvement over *Duffield's* earlier indirect method.

A recent and comparatively much more accurate method consists in measuring the change in the frequency of transverse vibrations of a wire under tension, due to a weight suspended from it For, whereas, any variations in the value of g produce next to no effect on the density of the wire (density being the ratio of mass to volume), they do naturally affect the pull of the earth on the suspended weight and hence the tension in the wire, resulting in corresponding changes in its frequency of vibration These can be easily detected to just a fraction of a vibration in a frequency of several thousands, by comparison with the vibrations of a quartz-crystal oscillator by the methods of beats. This explains the high accuracy of the method, which is obviously equally applicable to the measurement of the value of g on land, particularly at places where it is difficult or impracticable to use the usual method, as for example, at the bottoms of boreholes etc.

78. Local and Temporal Changes in the value of g. The value of g at a point is also affected, to some slight extent, by local causes,

i.e., pendulums, whose full time-period is *one* second.

†It is an alloy of nickel and steel, whose co-efficient of thermal expansion is *exceptionally low*, and which is, therefore, used in the construction of what are called *inveriable pendulums*, *i.e.*, pendulums whose lengths remain practically unaffected by temperature variations,—the name '*invar*' for the alloy being suggested by the word '*invariable*'.

like *small geological deposits near about, the topography of the region* or even by *masses like buildings etc., in the neighbourhood.* It is also affected by time, because deformations in the earth's surface take place periodically, thus bringing about a change in the equipotential surface, and hence in the direction of the force of gravity, which is always perpendicular to this surface. These changes are, however, much too small to be measured by ordinary pendulum methods whose accuracy is limited to within 10^{-4} *cms./sec².*, or 10^{-1} *milligals,* (where 1 *cm./sec².* $=1$ *gal* $=$ 1000 *milligals**). More sensitive methods have, therefore, to be used for the purpose. A detailed study of these is beyond the scope of this volume, and we shall, therefore, deal with them only briefly here.

(*i*) **Local Changes.** Small changes in the value of g due to local causes are measured with the help of (*i*) what are called *invariable pendulums* and (*ii*) *gravity-meters or balances.* The former are suitable only for the measurement of place-to-place variations in the value of g in regions, free of all marked local abnormalities, and the latter, for changes due to abnormal conditions like irregularities in the density of surface constituents and such other causes. For the most accurate determination of small variations in the value of g, however, a still more sensitive instrument *viz.*, the *Eotvos gravity balance* must be used.

(1) **The Invariable Pendulums.** These pendulums are so called, because of their being standardised to such an extent that their time periods (t) vary solely due to variations in the value of g and to no other factor.

They are usually rigid pendulums of *invar-steel,* suspended from a massive tripod in a partially evacuated chamber, with a *specified air pressure* inside it, to make all air-corrections constant. And the variations due to temperature already small on account of the use of invar-steel, (with its negligible coefficient of expansion), are further corrected for by a direct determination of the change in time-period with temperature.

The time-period of such a pendulum is first determined at a chosen *base station, i.e.*, at a place where the value of g is known and then at the field station, *i.e.*, at the place where it is to be determined. Then, clearly, the *gravity ratio,* or the ratio between the values of g at the two stations, will be given by the inverse ratio of the squares of its time-periods there, since

$$t \propto 1/\sqrt{g}. \quad \text{Or,} \quad g \propto 1/t^2.$$

The only error possible, after all this standardisation, is that in noting the time-periods of the pendulum at the two stations, or in the '*timing operation*', as it may be called, and the utmost accuracy is attempted to be secured here by arranging to have precise time-signals broadcast at frequent intervals. In the ultimate analysis, however, the results obtained will be restricted to the same order of accuracy to which the time-period of the pendulum and the other constants involved have been determined at the base station.

The use of the time-signals at the field station may be obviated by the technique used by **Bullard** in his determination of the value of g in East Africa (in 1933), *viz.*, that of using *two* pendulums—one at the base station, (in his case, Cambridge) and the other at the field station and recording an agreed *Morse signal,* alongside the oscillations of the pendulum, at each station, on a photographic film, repeating the same an hour or so later. The time-periods of the pendulums can then be compared with the *equal* time-intervals given by the Morse signals, and a high degree of accuracy thus attained in their measurement.

(2) **Gravity Meters.** Next in sensitivity come the *gravity meters,* various forms of which are now in commercial use as *prospecting instruments* and otherwise. We shall consider here only a few of them.

*The *milligal* is a new unit, now increasingly being used to express small changes in the value of g.

The Threlfall and Pollock Gravity Meter. Used first by *Threlfall and Pollock*, in 1899, it is perhaps the earliest *gravity meter* and consists of a fine *quartz thread AB*, (Fig. 131), stretched horizontally, with the end *A* fixed or *'anchored'* and the end *B* attached to an axle which can be rotated, in line with the thread (the latter being thus twisted) by means of a pointer (or a vernier) which moves over a circular scale *S*. A small metal rod *R* is fused athwart the thread near about its midpoint and is so *weighted* (by a bob or weight *w*) that its *c.g.* lies on one side of the thread.

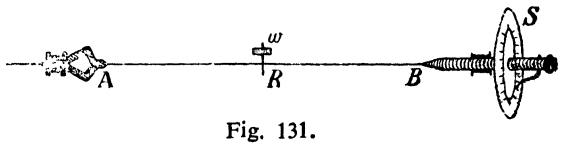


Fig. 131.

The end *B* is twisted by means of the pointer, *until the rod becomes horizontal*—about three full turns of the thread being necessary for the purpose—in which position it is *just* stable under the balancing forces due to the tension of the thread and the gravitational pull on itself, *i.e.*, when the torsional couple due to thread just balances that due to the pull of the earth. The position of the ponter is now read off on the scale,—the slightest further movement of it making the rod lose its precarious balance and turn right over. This is safeguarded against by a suitable stop or arrester, but the very fact of this tending to occur enables its position of approaching instability to be readily determined.* Thus, with a change in the value of *g*, the rod will no longer remain horizontal and the end *B* of the thread will have to be twisted to restore it to that position. The angular twist thus given to the thread can be read on the scale from the position of the pointer, and is a measure of the variation in the value of *g*, the pressure being kept constant and proper correction for temperature effects (*i.e.*, for expansion and change in the rigidity of the thread etc.) being made.

The instrument is made direct-reading by first noting the positions of the pointer at two stations, where the value *g* is accurately known, its variation with temperature being determined at one of them. So that, if now the instrument be carried from place to place, the various positions of the pointer indicate the values of *g* on the scale straightaway.

With proper precautions taken, this simple appliance can yield results of a fairly high degree of accuracy.

(*ii*) **The Boliden Gravity Meter.** A later form of gravity-meter, shown in Fig. 132 is due to *Boliden* (1938), in which two pieces of spring *S*, *S*, support a mass *M* which ends in two flat plates *D* and *E* above and below, each forming one plate of the parallel plate condensers *AD* and *BE*, whose other plates *A* and *B* are properly insulated from the framework of the instrument by means of insulating slabs *F* and *G*. The condenser *AD* above forms part of an oscillatory (or *LC*) circuit, whose frequency (*N*) is compared with a standard oscillator.

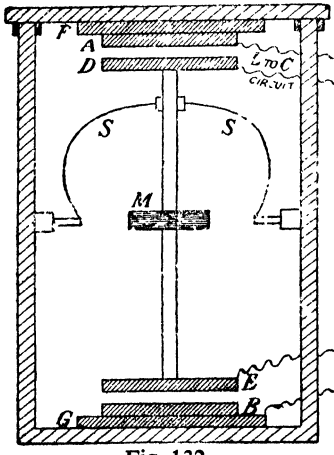


Fig. 132.

A change δg in the value of *g* brings about a change in the flexure of the springs and hence a proportionate change δx in the air gap (*d*) between the condenser plates *A* and *D*. This, in its turn, results in a change δC in the capacity (*C*) of the condenser, such that $C\delta/C = \delta x/d$. And, finally, the change in the capacity of the condenser is then responsible for a corresponding change δN in the frequency (*N*) of the oscillatory circuit.† So that,

$$\delta g \propto \delta x \propto \delta N,$$

whence δg can be easily calculated out.

*For, with the approach of the position of instability, the net couple acting on the thread varies only slowly with the change in its inclination and hence the time-period of the torsional vibrations of the thread about its equilibrium position goes on increasing.

†Because the frequency of an oscillatory circuit depends upon the *capacitance* (*C*) and the *inductance* (*L*) included in it.

The instrument is calibrated by applying known potential differences to the plates *B* and *E* of the lower condenser, calculating the force of attraction between them—(and hence on *D*)—and the attendant frequency change of the oscillatory circuit and plotting a graph between the latter two.

The sensitivity of the instrument is rather low, being only about 1×10^{-9} *cms./sec*^{2}., or just 1 *milligal*.

(*iii*) **The Gulf Gravity Meter.** This is a more recent (1941) and sensitive type of gravity-meter and depends upon the same principle as a spring balance, *viz.*, that the weight of a body is proportional to the acceleration due to gravity, so that a mass suspended from a spring will exert a different pull on it for different values of *g*, *the stretch of the spring thus indicating the variations in g*.

The method fails in the case of the ordinary spring balance purely for want of requisite sensitiveness. In the case of the present instrument, however, this sensitiveness is well assured, as much by the choice of a suitable type of spring as by the accuracy of the means of observation.

We use here a *flat, metallic ribbon-spiral spring*, fastened to a torsion head, at the top, and carrying a load at its free end below, including a mirror *m*, (Fig. 133), which untwists the spring by about 8 full revolutions.



Fig. 133.

Any change in the value of *g* will bring about a proportionate change in the *weight* of the suspended mass and the consequent pull on the spring, resulting in a corresponding rotation of the mirror, which can be measured by the deviation of a beam of light from an illuminated slit, reflected from it. The angle of deviation is magnified by making the beam travel four times between *m* and a fixed reflector and the image of the slit finally observed by means of a microscope, fitted with a micrometer eye-piece. The slight changes in the value of *g* corresponding to these deviations can thus be easily determined.

The sensitivity of the instrument is found to be about 5×10^{-8} *cms./sec*^{2}., or 5×10^{-2} *milligals*.

(*iv*) **Eotvos Balance.** None of the above appliances possess the necessary sensitivity to be able to measure the small change in the value of *g* due to neighbouring buildings or small geological deposits etc. Instruments far more responsive to small variations in the value of *g* must be employed for these delicate measurements and the gravity balance, devised by **Baron Eotvos,** admirably answers this requirement. It is not only used for a comparative or an absolute determination of *g*, but also for the measurement of other important quantities connected with the earth's gravitational field and for purposes of gravitational survey, the accuracy claimed for the instrument being 10^{-9} *cms./sec*^{2}., or 10^{-6} *milligals*.

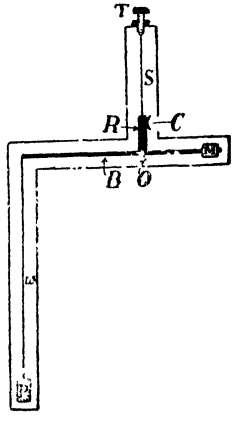


Fig. 134.

In essentials, the **Eotvos Balance** consists of a *rectangular torsion beam B*, (Fig. 134), of aluminium about 40 *cms.* in length, and between 3×10^{-4} to 4×10^{-4} *mms.* in diameter, suspended from a torsion head *T*, by means of a fine suspension *S*, about 60 *cms.* long, of the alloy *platinum-iridium*, through an aluminium rod *R*, fixed on to the beam at its *c.g*, *O*. The rod carries a small concave mirror *C*, to enable the deflections of the beam to be read by the

lamp and the scale method, with the help of a telescope. A small cylindrical weight P, of platinum, gold or silver, of mass about 30 *gms.* is suspended from one end of the beam, by means of a fine wire (w) of platinum, and a counterpoise weight M, of mass about 25 *gms*, is slid on to, or suspended from, the other end of the beam, as shown.

If the instrument be taken to a place, where the value of g varies from point to point, its suspended system experiences a *couple*, producing a twist in the wire and deflecting the beam from the position, (not known), that it would occupy if the value of g were constant.

Let the beam, *in its equilibrium position*, make an angle θ with the x-axis, (lying along the north-south direction),—*i.e.*, let θ be the '*azimuth angle*', as it is called,—and let S_1 be the reading on the scale in this position. Then, if S_0 be the scale reading in the (unknown) position, in which there would be no gravitational torque on the beam, it can be shown that

$$S_1-S_0 = A\left[\frac{\partial^2 U}{\partial y^2}-\frac{\partial^2 U}{\partial x^2}\right] sin\ 2\theta+2A\left[\frac{\partial^2 U}{\partial x\,\partial y}\right].cos\ 2\theta$$
$$-C\left[\frac{\partial^2 U}{\partial z\,\partial x}\right]sin\ \theta+C\left[\frac{\partial^2 U}{\partial y.\partial z}\right].cos\ \theta, \qquad ...(1)$$

where A and C are the *constants of the instrument*, U, *the gravitational potential*, and $\partial U/\partial x$, $\partial U/\partial y$ and $\partial U/\partial z$, the values of the gravitational attraction along the *North*, the *East*, and the *vertical* directions respectively, (this last one being the value of g). The origin of the three axes, *along these three directions*, is taken to be the mid-point O of the beam.

Putting $A\left[\frac{\partial^2 U}{\partial y^2}-\frac{\partial^2 U}{\partial x^2}\right]=A'$, $\quad 2A\left[\frac{\partial^2 U}{\partial x.\partial y}\right]=B'$,

$$-C\left[\frac{\partial^2 U}{\partial z.\partial x}\right]=C' \text{ and } C\left[\frac{\partial^2 U}{\partial y.\partial z}\right]=D',$$

we have, from relation (1) above,

$$S_1-S_0 = A'\ sin\ 2\theta+B'\ cos\ 2\theta+C'\ sin\theta+D'\ cos\ \theta. \qquad ..(2)$$

Now, taking $\theta = 0^0$, 60°, 120°, 180,° 240° and 300°, in equation (2) above, and taking the corresponding values of S_1 to be S_1, S_2, S_3, S_4, S_5, S_6, we have

$$S_1+S_3+S_5 = S_2+S_4+S_6 = 3S_0,$$
$$2\sqrt{3}\,A' = (2S_2-S_3)-(S_1-S_4),$$
$$2B' = (S_1+S_4-2S_0),$$
$$2\sqrt{3}C' = 2(S_2-S_5)-(S_1-S_4)$$

and
$$2D' = S_3-S_4.$$

Thus, all the constants of relation (1) being known, the rate of change of g northwards, (given by $-C'/C$), as also that in the eastern direction, (given by D'/C), can be easily determined.

On account of its high sensitivity, the balance is used for geophysical prospecting (see § 80). And. *Shaw* and *Lancaster Jones* have successfully mapped out with its help the local gravitational field in a laboratory.

(*ii*) **Temporal Changes—The Horizontal Pendulum.** As we have seen above, a deformation of the earth's surface and its gravitational equipotential surface results in a change in the direction of the force of gravity and hence in that of g. Since a plumb line always sets itself normally to the gravitational equipoential surface of the earth, it is clear that measuring a change in the direction of g at a point, is tantamount to measuring a change in the direction of the plumb line at that point. These changes, however, are much too small—seldom exceeding 1″,—to permit of their accurate measurement by means of a plumb line. The most commonly used device to measure these is what is called the **horizontal pendulum,** devised by **Hengler,** in the year 1832.

This *horizontal pendulum* essentially consists of a rod AB, (Fig. 135), carrying a knob or bob at B, with G as its $c.g.$ It is supported in an inclined position, by means of two pieces of a light string, AP and CQ, attached to a *rigid* support at P and C respectively, such that the straight line CP, joining the two, meets AB in O, and makes an angle ϕ with the direction of the force of gravity. *The pendulum thus takes up a position in a plane parallel to the force of gravity.* On giving the bob a slight lateral displacement (towards or away from the observer), it begins to oscillate slowly, with a small amplitude, along an arc with O as its centre and OG as radius. Its period of vibration is deduced as follows :

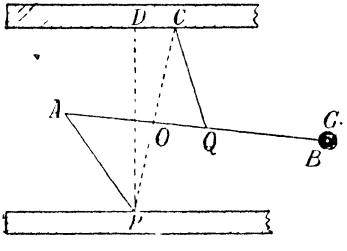


Fig. 135.

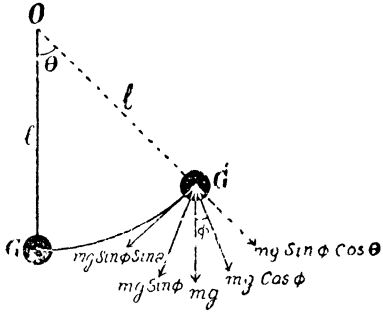


Fig. 136.

Let the bob oscillate along an arc GG', (Fig. 136), which lies in a plane, making an angle ϕ with the normal to the plane of the force of gravity.

If the bob be displaced through an angle θ *in this inclined plane*, into the position shown, its weight mg acts at *its c.g.*, G' ($\because$ G is now at G') *in the direction of the force of gravity.* Resolving it into its two rectangular components, (*i*) *in the inclined plane of its rotation*, and (*ii*) *perpendicular to it*, we have the former component $= mg \sin \phi$, and the latter $= mg \cos \phi$.

Further, resolving the component $mg \sin \phi$ into two rectangular components, along and perpendicular to OG', we have

the component along $OG' = mg \sin \phi \cos \theta$.

and *the component perpendicular to* $OG' = mg \sin \phi \sin \theta$.

This latter component ($mg \sin \phi \sin \theta$) has, clearly, a restoring

moment about the inclined axis $= mg \sin\phi \sin\theta.l$, (where $OG' = OG = l$), tending to bring the pendulum back to its original position.

If θ be small, $\sin\theta = \theta$, very nearly. And, therefore,

$$\textit{the restoring moment} = mg \sin\phi.\theta.l.$$

And, if the *angular acceleration* of the pendulum be $d\omega/dt$, and its *moment of inertia about O* be I, we have

$$\textit{restoring moment (or torque)} \text{ also} = I.d\omega/dt.$$

So that, neglecting frictional and viscous forces, we have, for equilibrium,

$$I.d\omega/dt = mg\ l.\sin\ \phi.\theta, \text{ whence, } d\omega/dt = \frac{mg\ l.\sin\phi}{I}.\theta.$$

Or, putting $mg.l.\sin\phi/I = \mu$, a constant, we have

$$d\omega/dt = \mu.\vartheta. \qquad \text{Or,} \qquad d\omega/dt \propto \theta,$$

i.e., the *angular acceleration of the pendulum is proportional to its angular displacement.*

It is, therefore, a case of simple harmonic motion, and its time-period T is given by the relation,

$$T = 2\pi\sqrt{1/\mu}.$$

Or, $$T = 2\pi\sqrt{1\Big/\frac{mg.l\sin\phi}{I}} = 2\pi\sqrt{\frac{I}{mg.l.\sin\phi}}.$$

But $I = mk^2$, where k is the *radius of gyration* of the pendulum about O.

Hence $$T = 2\pi\sqrt{\frac{mk^2}{mg\ l\sin\phi}} = 2\pi\sqrt{\frac{k^2}{g.l.\sin\phi}}.$$

In practice, to make T large, ϕ is made as small as possible.

Now, if $\phi = 90°$, $\sin\phi = 1$, and, therefore, the time-period T', in this case, is given by the relation $T' = 2\pi\sqrt{k^2/g.l}$.

Hence $T/T' = \sqrt{1/\sin\phi} = (1/\sin\phi)^{\frac{1}{2}}$. And $\therefore\ T^2/T'^2 = 1/\sin\phi$, whence, $\sin\phi = T'^2/T^2$.

Thus, knowing T' and T, we can easily calculate $\sin\phi$, and hence ϕ, *which represents the change* in the direction of the force of gravity, and, therefore, that of g, in the equilibrium plane of the pendulum.

79 Gravity Survey. The purpose of a gravity survey is two-fold, *viz.*, (*i*) the main one being to determine the value of the force of gravity and its direction at various points of the sea-level surface of the earth, or the '*geoid*', as it is called, and (*ii*) a secondary one being to deduce from it the possible distribution of matter in the earth, and thus to form an idea about its structure and internal condition.

Now, in any gravity survey, it is found necessary to introduce two new quantities, connected with the earth's gravitational field. These are (1) the *gravity gradient*, denoted by the letter G, and (2) the *horizontal directive tendency* (written as *H. D. T.*, for brevity) denoted by the letter R. Let us try to understand their meaning.

(1) **The Gravity Gradient (G).** It is a convenient abbreviation for "*maximum gradient of g in a horizontal direction, near a point*", where g is the *vertical gravitational intensity* at the point, *i.e.*, $g = \partial U/\partial z$ *The gravity gradient may, therefore, be denoted by* $\partial g/\partial s$, *the rate of variation of g per unit distance, in the direction of the maximum rate of change in its value,* and it is thus obviously a *vector quantity.*

Now, if the gravity gradient G, $(=\partial g/\partial s)$ makes an angle ϕ with the axis of x (or the north-south direction), and if its components along the axes of x and y be $\partial g/\partial x$ and $\partial g/\partial y$ respectively, then, clearly,

$$\partial g/\partial x = G\cos\phi \quad \text{and} \quad \partial g/\partial y = G\sin\phi.$$

And, since $g = \partial U/\partial z$, we have $\partial g/\partial x = \partial^2 U/\partial x\partial z = U_{xz}$,

and $\partial g/\partial y = \partial^2 U/\partial y\partial z = U_{yz}$.

So that, $U_{xz} = G\cos\phi$ and $U_{yz} = G\sin\phi$,

$\therefore$ squaring and adding the two, we have

$G^2\cos^2\phi + G^2\sin^2\phi = U^2_{xz} + U^2_{yz}$. Or, $G^2(\cos^2\phi + \sin^2\phi) = U^2_{xz} + U^2_{yz}$.

Or, $G^2 = U^2_{xz} + U^2_{yz}$, whence, $G = (U^2_{xz} + U^2_{yz})^{\frac{1}{2}}$.

Now, U_{xz} and U_{yz} can be easily determined by means of an *Eotvos Balance*, (see page 212), and thus the value of G can be calculated out from the above expression.

(2) **The Horizontal Directive Tendency (R).** It is what is called a '*curvature vector*', *i.e.*, a directed quantity, though not a true vector. Its value at a point is given by the relation,

$$R = g(1/r_1 - 1/r_2),$$

where r_1 and r_2 stand for the maximum and the minimum radii of curvature of the level surface, or the gravitational equipotential surface at the point. Its direction, according to an agreed convention, is taken to be the direction *in which the level surface has the least downward curvature and, therefore, the maximum radius of curvature.*

If the direction of *H.D.T.* makes an angle θ with the axis of x, or the north-south direction, it can be shown that

$$R\sin 2\theta = 2U_{xy},$$

and

$$R\cos 2\theta = U^2_{xx} - U^2_{yy}.$$

where, $U_{xy} = \partial^2 U/\partial x\partial y$, $U_{xx} = \partial^2 U/\partial x^2$ and $U_{yy} = \partial^2 U/\partial y^2$.

The dimensions of both G and R are $[T]^{-2}$ and they are generally expressed in what are called *Eotvos units*,—where one Eotvos unit is equal to $10^{-9}/sec^2$.

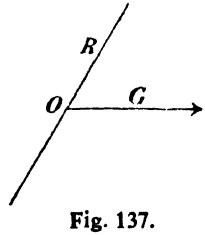


Fig. 137.

In survey maps, the *gravity gradient* at a point is represented, in magnitude and direction, by an arrowhead drawn from the point, whereas the *horizontal directive tendency* is just represented in magnitude and direction, by a straight line, passing through that point, without any arrowhead or feathered tail, as shown in Fig. 137, where O is the point in question.

Further, points, where the value of g is the same, are joined by curves, which are called **isogams**,—*G being always directed along the normals to these.*

80. Geophysical Prospecting. We have seen above how, due to the presence of local geological deposits, (*i.e.*, minerals etc.), inside the earth, small variations are produced in the value of g. Similar changes are produced, by their presence in the normal values of the other quantities, magnetic, electrical, seismic* etc., associated with it; so that, by measuring these variations, with the help of specially designed instruments, we can detect their presence. This is technically called '*geophysical prospecting.*' We are concerned here only with the gravitational methods adopted for the purpose, the principle

*See article on, '*Earthquakes*' in the next chapter.

underlying which is to measure the *gravity gradient G*, and the *horizontal directive tendency R*, at various points in the region under survey, with the help of a sensitive instrument, like an *Eotvos gravity balance*, as explained above in § 78, (*iv*) the instrument, when so used, being called a '**gradiometer**'.

Thus, if θ be the angle that R makes with the axis of x, or the north-south direction at a point, we have

$R \sin 2\theta = 2U_{xy}$, and $R \cos 2\theta = U^2_{xx} - U^2_{yy}$. [§ 79, (2), above

And, therefore, $\tan 2\theta = 2U_{xy}/U^2_{xx} - U^2_{yy}$,

the two solutions of which, θ_1 and θ_2, differ by $2/\pi$ and give the directions of the two principal axes of curvature of the equipotential surface, (*i.e.*, the level surface), at the point. The values of G and R at various points are then plotted, the direction of θ_2 being such that *secant* 2θ and $(U^2_{xx} - U^2_{yy})$ are of opposite signs.

A graphical representation of the variations of g over the region, under examination, is thus obtained, and closed curves or *isogams*, [see §79, (2), above], are then clearly marked out on it, (which, as we know, are at every place perpendicular to G), so that we have an **isogam chart** of the region in question.

Interestingly enough, the physical form of these *isogams* almost faithfully represents the physical form of the subterranean deposits. Thus, for example, a *uniformly monoclinic* type of region would give *isogams*, which are all *parallel and equally spaced*, whereas if the subterranean deposits form a dome-like structure, the isogams obtained also resemble the outline or the contour of a surface dome, as it were.

This method can, however, succeed only in the hands of those well-trained in the use of the delicate instruments employed and in the proper interpretation of the results obtained from them.

SOLVED EXAMPLES

1. A metal disc oscillates in its own plane about an axis passing through a point on its edge. What is the length of the equivalent simple pendulum ?

Let the disc of radius r oscillate about an axis through the point P on its edge, (Fig. 138).

Then, clearly, the time-period of the disc is given by the relation, $t = 2\pi \sqrt{I/Mg\,l}$,

where I is its *M.I.* about the axis through P ; M, its *mass* and l, its *length*.

Or, $t = 2\pi \sqrt{I/Mg.r}$, for $l = r$, the distance between the point of suspension (P) and the *c.g.* (O) of the disc.

Now, $I = I_q + Mr^2$.

where I_q is the *M I.* of the disc about a parallel axis through O, *i.e.*, $I_q = Mr^2/2$.

So that, $I = (Mr^2/2) + Mr^2 = 3Mr^2/2$.

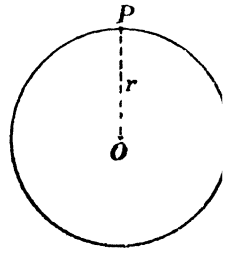


Fig. 138.

And, therefore, $t = 2\pi \sqrt{\frac{3Mr^2/2}{Mg.r}} = 2\pi \sqrt{\frac{3r/2}{g}}$,

i.e., the same as that of a simple pendulum of length $l = 3r/2$.

Or, *the length of the equivalent simple pendulum is* 3/2 *times the radius of the disc.*

2. Find the period of small oscillations of a rigid body, free to turn about a fixed horizontal axis, and also find a formula for the length of the equivalent simple pendulum.

Three particles of the same mass m are fixed to a uniform circular hoop of mass M and radius a at the corners of an equilateral triangle. The hoop is free to move in a vertical plane about the point on the circumference opposite to one of the masses m. Prove that the equivalent simple pendulum is equal in length to the diameter of the circle. (*London Higher School Certificate ; Patna, 1948*)

For first part ; see §62 (page 165).

Let the three equal masses, m, m and m, be fixed to the hoop, of *radius* a, as shown, (Fig. 139), so as to lie at the corners of an equilateral triangle. Since they are all *equidistant from the centre, the c.g.* of the triangle is at O, the centre of the circle. The whole arrangement is thus equivalent to a hoop of mass $(M+3m)$, with its centre of gravity at its centre O.

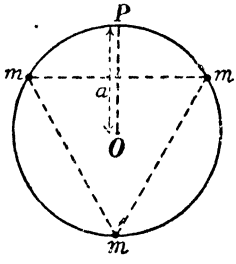


Fig. 139.

Clearly, then, the moment of inertia of this loaded hoop about O, (*i.e.*, about an axis through O and perpendicular to its plane) $= (M+3m)a^2$. And, therefore, its moment of inertia about a parallel axis through the point of suspension P is (*by the principle of parallel axes*), given by

$$I = (M+3m)a^2+(M+3m)a^2 = 2(M+3m)a^2.$$

Now, the *time-period* of the hoop about P is given by

$$t = 2\pi\sqrt{\frac{I}{(M+3m)ga}} = 2\pi\sqrt{\frac{2(M+3m)a^2}{(M+3m)ga}}. \quad [\because I = a.$$

Or,

$$t = 2\pi\sqrt{\frac{2a}{g}},$$

i.e., the same as that of a simple pendulum of length $2a$, the diameter of the hoop.

Or, *the length of the equivalent simple pendulum is equal to the diameter of the circular hoop.*

3. How much faster than its present rate should the earth revolve about its axis in order that the weight of a body on the equator may be zero, and how long would it take to make one revolution then ? What would happen if (*i*) the rotation became faster still, (*ii*) the rotation were stopped altogether ? (g = 978 cms./sec^2.)

We have seen (page 203), that the value of g is different in different latitudes, due to the rotation of the earth, and that, assuming the earth to be a perfect sphere,

$$g_\phi = g\left(1 - \frac{r.cos^2\phi.\omega^2}{g}\right). \quad [\text{where } g_\phi \text{ is the value of '}g\text{' in latitude } \phi, \text{ (see page 205)}$$

Now, at the equator, $\phi = 0$, and $\therefore$ $cos^2\phi = 1$; so that,

$$g_0 = g\left(1 - \frac{r.\omega^2}{g}\right).$$

where, g_0 is the value of 'g' at the equator.

With the actual value of g, the value of $r.\omega^2$ comes out to be 3·39 and, therefore, we have

$$\frac{r\omega^2}{g} = \frac{3{\cdot}39}{978} = \frac{1}{288}.$$

Thus, in order that the weight of a body may be *zero*, the value of g_ϕ should be *zero*, *i.e.*, $r.\omega^2/g$ should be equal to 1, or the value of $r\omega^2$ should be 288 times greater than its present value, r being a constant. It follows, therefore, that ω should be $\sqrt{288}$ times, *i.e.*, 16·97 times greater than its present value. When this is so, *the outward centrifugal force on the body will, obviously, be just balanced by the inward force due to gravity.*

If, therefore, the earth rotates, 16·97 *times, or* 17 *times, faster than at present, the weight of the body at the equator will be zero.*

Now, the earth makes one complete revolution in 24 hours, *i.e.*, discribes an angle of 2π in 24 hours. But, in the case considered, *viz.*, when the weight of the body at the equator is *zero*, it rotates 17 times faster, and will, therefore, describe an angle $17\times 2\pi$ in 24 hours, or an angle 2π in 24/17 hours or 1·412 hours.

$\therefore$ *the earth will then make one rotation in* 1·42 hours.

If the rotation became *faster still, i.e.*, faster than 16·97 or 17 times its normal rate, obviously, all objects kept loose on the equator will start leaving the surface of the earth ; for, the increased centrifugal acceleration on them will be greater than that due to gravity, and, therefore, *a resultant force will be acting on them outwards, away from the centre of the earth.*

If, on the other hand, the rotation of the earth about its axis were stopped altogether, we shall have $\omega = 0$; so that, substituting this value of ω in the relation,

$g_0 = g\left(1-\frac{r.\omega^2}{g}\right)$, for the value of '$g$' at the equator, we have

$$g_0 = g\,(1-0) = g.$$

i.e., the value of g increases by $(g-g_0) = r.\omega^2/g$ times g.

Or, $= (3{\cdot}39/978)$ times $g = 1/288$ times g, or $= g/288$. [For $r.\omega^2 = 3{\cdot}39$.

Thus, *if the motion of the earth were stopped altogether, the value of g would increase by* 1/288 *of its normal value.*

4. Assuming that the whole variation of the weight of a body with its position on the earth's surface is due to the rotation of the earth, find the difference in the weight of a gram as measured at the equator and at the poles. (Radius of the earth $= 6{\cdot}378\times10^8$ cms.)

We have the relation,

$g_\phi = g\left(1-\frac{r\,\omega^2.\cos^2\phi}{g}\right)$, for the value of '$g$' in latitude ϕ.

Since $r.\omega^2 = 6{\cdot}378\times10^8\times\left(\frac{2\pi}{86164}\right)^2 = 3{\cdot}39$, and $\therefore\ r.\omega^2/g = \frac{3{\cdot}39}{978} = 1/288$,

we have $g_\phi = g(1-\cos^2\phi/288)$.

Now, at the equator, $\phi = 0$, so that, $\cos\phi$ and $\therefore\ \cos^2\phi = 1$.

Hence $g_\phi = g(1-1/288)$,

where g_ϕ is the value, of 'g' at the equator.

And, at the poles, $\phi = 90°$, so that, $\cos\phi$ and $\therefore\ \cos^2\phi = 0$.

Hence $g_\phi = g(1-0) = g$,

where g_ϕ is the value of 'g' at the poles.

Since the weight of body is mg, where m is its mass, we have

weight of 1 *gm. at the equator* $= w = 1\times g_0$.

$= 1\times g(1-1/288) = g(1-1/288)$, ... *(i)*

and *its weight at the poles* $= w' = 1\times g_\phi = 1\times g = g$. ... ... *(ii)*

Hence, the *difference in the weights of this mass at the poles and at the equator* $= w'-w$.

$= g-g(1-1/288) = g-g+g/288 = g/288 = 978/288 = 3{\cdot}395$ *dynes.*

Or, the difference in *weights* of a *gram* at the poles and at the equator is 3·395 *dynes.*

5. The mass of a railway train is 100 tons. What will be its weight when (*a*) stationary, (*b*) travelling due east, (*c*) travelling due west, along the equator at 60 miles per hour ? Radius of the earth is 4000 miles. (*Punjab*)

(*a*) **When the train is stationary.** When the train is at rest, its *apparent weight* is 100 *tons wt.* $= 100\times2240 = 224\times10^3$ *lbs. wt.*, (because 1 *ton* $= 2240$ *lbs.*).

(*b*) **When the train is moving East.** When the train travels due east, its angular velocity about the axis of rotation of the earth *increases*, because the earth itself is rotating about the axis from west to east. The centrifugal force on the train, therefore, increases, (being proportional to $r.\omega^2$), and hence the apparent force of gravity on it and, therefore, the apparent acceleration towards the centre of earth, *i.e.*, *apparent acceleration due to gravity, decreases.* And, since *weight* $=$ *mass* $\times$ *acceleration due to gravity*, the apparent weight of the train decreases. Let us see by how much.

The radius of the earth, $r = 4000$ *miles* $= 4000\times1760\times3$ *ft.*, and, therefore, linear velocity (v) of a point on the earth $= 2\pi r/24\times60\times60$ *ft./sec.*, since the earth makes one complete rotation in 24 *hours.*

A point on the equator will also thus describe a distance $2\pi r$ in 24 *hours* ; its velocity is, therefore, given by

$$v = \frac{2\pi \times 4000 \times 1760 \times 3}{24 \times 60 \times 60} = 1536\ ft./sec.$$

$\therefore$ *centrifugal acceleration* of the point, when at rest relative to the earth, is given by

$$\frac{v^2}{r} = \frac{1536 \times 1536}{4000 \times 1760 \times 3} = 0{\cdot}1116\ ft./sec^2.$$

Since velocity of the train $= 60\ m./hr. = \frac{60 \times 1760 \times 3}{60 \times 60} = 88\ ft./sec.$,

the resultant velocity of the train, say, v'

$$= 1536 + 88 = 1624\ ft./sec.$$

And $\therefore$ centrifugal acceleration on the train moving at 60 $m/hr. = v'^2/r.$

$$= \frac{1624 \times 1624}{4000 \times 1760 \times 3} = 0{\cdot}1248\ ft./sec^2.$$

$\therefore$ *increase in centrifugal acceleration, or decrease in acceleration towards the centre of the earth, i.e., in the acceleration due to gravity*

$$= 0{\cdot}1248 - 0\ 1116 = 0{\cdot}0132\ ft./sec^2.$$

$\therefore$ decrease in the *weight* of the train = mass $\times$ decrease in acceleration due to gravity.

$= 100 \times 2240 \times 0{\cdot}0132$ *poundals* $= 100 \times 2240 \times 0{\cdot}0132/32$ *lbs. wt.*

$= 100 \times 2240 \times 0{\cdot}0132/32 \times 2240 = 100 \times 0{\cdot}0132/32$ *ton wt.* $= 0{\cdot}0412$ *ton wt.*

$\therefore$ *apparent weight of the train* $= 100 - 0{\cdot}0412 = 99{\cdot}9588$ *tons wt.*

(*c*) **When the train is moving West.** In this case, since the train is moving from east to west, *opposite to the direction of rotation of the earth*, its angular velocity about the axis of rotation of the earth *decreases* and, therefore, the centrifugal acceleration on it also decreases, with the result that the acceleration towards the centre of the earth, *i e , the acceleration due to gravity increases.* The apparent weight of the train on the equator, therefore, increases. Let us calculate this apparent increase.

As before, velocity of a point on the equator, *i.e.*,

$$v = \frac{2\pi r}{24 \times 60 \times 60} = \frac{2\pi \times 4000 \times 1760 \times 3}{24 \times 60 \times 60} = 1536\ ft./sec.$$ [See above.

$\therefore$ centrifugal acceleration of the point, when at rest relative to the earth

$$= \frac{v^2}{r} = \frac{1536 \times 1536}{4000 \times 1760 \times 3} = 0{\cdot}1116\ ft./sec^2.$$

Hence, the resultant velocity of the train, say, $v'' = 1536 - 88 = 1448\ ft./sec.$

And $\therefore$ the centrifugal acceleration on the train moving at 60 *m./hr.* is clearly given by

$$\frac{v''^2}{r} = \frac{1448 \times 1448}{4000 \times 1760 \times 3} = 0{\cdot}0993\ ft./sec^2.$$

$\therefore$ *decrease in cenrtifugal acceleration or increase in acceleration towards the centre of the earth, i.e., increase in acceleration due to gravity*

$$= 0{\cdot}1116 - 0{\cdot}0993 = 0{\cdot}0123\ ft./sec^2.$$

And $\therefore$ increase in apparent weight of the train

$= 100 \times 2240 \times 0{\cdot}0123$ *poundals* $= 100 \times 2240 \times 0{\cdot}0123/32$ *lbs. wt.*

$= 100 \times 2240 \times 0{\cdot}0123/32 \times 2240 = 100 \times {\cdot}0123/32$ *ton wt.*

$= 0{\cdot}03844$ *ton wt.*

Thus, the apparent weight of the train

$= 100 + 0{\cdot}03844 = 100{\cdot}03884 = 100{\cdot}04$ *tons weight.*

EXERCISE VI

1. What is a simple pendulum ? Is it obtainable in actual practice ? Deduce an expression for its time-period and show how the value of g may be determined with its help. What are the drawbacks of this pendulum ?

2. Deduce the formula for the time of vibration of a compound pendulum and show that this is a minimum when the length of the compound pendulum equals its radius of gyration about a horizontal axis through the centre of gravity of the compound pendulum. (*Punjab, 1951*)

3. Distinguish between a simple and a compound pendulum.

For a given compound pendulum, show that the centres of oscillation and suspension are interchangeable.

How is the value of '*g*' determined with the help of a compound pendulum ? (*Agra, 1948*)

4. Give the theory of Kater's pendulum and find an expression for the acceleration due to gravity in terms of two nearly equal periods of oscillation about the two parallel-knife-edges.

Indicate the sources of error in an experimental determination of *g*.
(*Bombay, 1940-41* ; *Punjab, 1948*)

5. A Borda's pendulum has a bob of radius 12 *cms*., which is suspended by a fine wire, 94 *cms*. long. Calculate the length of the equivalent simple pendulum. **Ans.** 100·144 *cms.*

6. If a pendulum beats seconds at a place where $g = 32{\cdot}2$ *ft./sec*^{2}., how much would it gain or lose per day at a place where $g = 32{\cdot}18$ *ft./sec*^{2}.
Ans. Gains 3 *min.* 36 *secs.*

7. Explain the Dropping Plate method for the determination of the value of *g*. If there be an error of 1% in measuring the distance covered by the plate as also in measuring the frequency of the fork, how would it affect the result ? **Ans.** *The result will be wrong by* 3%.

8. The length between the knife-edges of a Kater's pendulum is 89·28 *cms*., while the times of oscillation about the two edges are 1·920 *secs*. and 1·933 *secs*., respectively. The *c.g.* of the pendulum is about 54·4 *cms*. from one edge What is the value of *g* ? **Ans.** 979 *cms./sec*^{2}.

9. An Atwood's machine has a pulley of radius *a* and moment of inertia *I* ; the masses attached to the ends of the string are each *M* and the rider is of mass *m*.

Prove that the acceleration *f* of the masses is given by

$$\frac{g}{f} = 1 + \frac{2M}{m} + \frac{I}{ma^2},$$

assuming that the string does not slip on the pulley, and neglecting axle friction.
(*Madras, 1949*)

10. A uniform rod of length 100 *cms* can rotate about a horizontal axis through one end. Find the angular velocity which will enable the rod just to make a complete rotation. (*Madras, 1947*)
Ans. 3·83 *radians/sec.*

11. A solid cylinder, of radius 4 *cms*. and mass 250 *gms*., rolls down an inclined plane, with a slope of 1 in 10. Find the acceleration and the total energy of the cylinder after 5 *secs*. (*Bombay, 1944*)
Ans. 65·4 *cms./sec*^{2}. ; 4·799 *Joules.*

12. A cylinder, of mass 100 *lbs*. and diameter 12 *inches*, rolls from rest down a smooth inclined plane of 1 in 8 and 20 *feet* long. Calculate the total kinetic energy and its energy due to rotation, when it reaches the bottom.
(*Madras, 1949*)
Ans. (*i*) $8{\cdot}0 \times 10^2$ *ft. poundals.*
(*ii*) $2{\cdot}6 \times 10^2$ *ft. poundals.*

13. Define 'centre of suspension' and 'centre of oscillation'. Show that in a compound pendulum they are interchangeable.

What is the distance between the centre of suspension and the centre of oscillation on a uniform cylindrical metal bar used as seconds pendulum ?
(Diameter of the bar=1 *cm*., to density, 8 *gms./c.c.* and g=978 *cms./sec*^{2}.)
(*Allahabad, 1949*)
Ans. 99·19 *cms.*

14. Obtain an expression for the time-period of a compound pendulum, and show that

(*i*) there are four points, collinear with its *c.g.*, about which its time period is the same.

(*ii*) its time-period remains unaffected by the fixing of a small additional mass to it at its centre of suspension.

15. Obtain an expression for the period of vibration of a compound pendulum and show that the centre of suspension and the centre of oscillation are interchangeable.

A thin uniform bar of length 120 *cms.* is made to oscillate about an axis through its end. Find the period of oscillation and other points about which it can oscillate with the same period. (*Punjab, 1953*)

Ans. 1·795 *secs.* ; at 40 *cms* , 80 *cms.* and 120 *cms.* from the top.

16. Derive an expression for the period of oscillation of a circular disc, supported on a horizontal rail passing through a narrow hole, which is bored through the disc half-way between the centre and the periphery. (*Bombay, 1946*)

Ans. $T = 2\sqrt{3r/2g}$.

17. A uniform rectangular sheet of metal is supported by frictionless hinges, attached to one edge which is horizontal. Determine the period of oscillation of the sheet if l denotes the length of the side of the rectangle which hangs downwards. (*Patna, 1951*)

Ans. $T = 2\pi\sqrt{13l/12g}$.

18. A metal bar is suspended in turn from two parallel axes on the same side of its *c.g.*, and its time-periods are found to be 1·42 *secs* in each case. If the distance of the two axes be 10·8 *cms.* and 39·2 *cms.* respectively from the *c.g.*, calculate the value of g and the radius of gyration of the bar about a parallel axis through the *c.g.*

Ans. $g = 979{\cdot}2$ *cms./sec*^{2}., and $K = 20{\cdot}58$ *cms.*

19. A narrow uniform bar of mass 1000 *gms.* oscillates about an axis, 40 *cms.* from the centre, with a period of 7·48 *secs.*, and about a parallel axis, 10 *cms.* from the centre, with a period of 1·67 *secs.* Find the value of g, the moment of inertia of the bar about its *c.g.* and the length of the bar.

Ans. 990 *cms./sec*^{2}., ; $6{\cdot}02 \times 10^5$ *gm.-cm*^{2}. ; 85 *cms.*

20. What is meant by a simple equivalent pendulum ? If the periods of a Kater's pendulum in the erect and inverted positions are equal, prove that the distance between the knife-edges is equal to the length of the simple equivalent pendulum.

A uniform circular rod, with a radius of 2 *cms.* oscillates when suspended from a point on its axis at a distance of 4 *cms.* from one end. If the length of the rod is one *metre*, find the point or points from which, if suspended, the periodic time would remain unaltered. (*Bombay, 1942*)

Ans. At 31·87 *cms.* and 68·13 *cms.*, also at 96·0 *cms. from the same end.*

21. Define a conical pendulum, and show that, for a small amplitude, its period equals that of a "*plane*" pendulum of the same length. Do simple pendulums exist ? What are the nearest approximations to them ? Why are they discarded in favour of compound pendulums and what are the main applications of pendulums ? (*Bombay, 1947*)

22. Describe a conical pendulum and derive an expression for its frequency. Explain how it is used to regulate the speed of steam engines. Show that the sensitiveness of the pendulum used as a governor increases with diminishing speed. (*Bombay, 1937*)

Ans. $n = 1/2\pi\sqrt{h/g}$,

(where h is *axial height* of the cone described by it, and equal to $l \cos \theta$, where l is the *length* of the pendulum and θ, its angular displacement ; see §74).

23. What is a steam engine governor ? Explain clearly the principle underlying its action, and discuss its limitations.

24. If the earth were to cease rotating about its axis, what will be the change in the value of g at a place of latitude 45°, assuming the earth to be a sphere of radius $6{\cdot}38 \times 10^8$ *cms.* ? (*Madras, 1947*)

Ans. 1·6895 *cms./sec*^{2}.

25. Find an expression for the period of swing of a compound pendulum.

A disc of metal, of radius R, with its plane vertical, can be made to swing about a horizontal axis passing through any one of a series of holes, bored along a diameter. Show that the minimum period of oscillation is given by

$$T = 2\pi\sqrt{1{\cdot}414\,R/g}. \qquad \textit{(Saugar, 1948)}$$

26. Give the theory of the compound pendulum and show that the centres of suspension and oscillation are reversible.

In a reversible pendulum, the periods about the two knife-edges are t and $(t+T)$, where T is a small quantity. The knife-edges are distant l and l' from the centre of gravity of the pendulum. Prove that

$$l+l' = \frac{gt}{4\pi^2}\left(t+\frac{2l'}{l'-l}T\right). \qquad \textit{(Madras, 1949)}$$

27. A heavy uniform rod, 30 *cms.* long, oscillates in a vertical plane, about a horizontal axis passing through one end. When a concentrated mass is fixed on to it at a distance x from its point of suspension, its time-period remains unaffected. Calculate the value of x. Ans 20 *cms.*

28. Explain how the length of the simple pendulum which has the same period as a given compound pendulum may be found experimentally.

A uniform cube is free to turn about one edge which is horizontal. Find in terms of a seconds pendulum, the length of the edge, so that it may execute a complete oscillation in 2 *secs.* *(Central Welsh Board Higher School Certificate)*

Ans. $3\sqrt{2}l$.

29. A body of mass 200 *gms.* oscillates about a horizontal axis at a distance of 20 *cms.* from its centre of gravity. If the length of the equivalent simple pendulum be 35 *cms.*, find its moment of inertia about the axis of suspension. *(Patna, 1954)*

Ans. 14×10^6 *gms.-cm*^{2}.

30. A pendulum, whose period slightly exceeds 2 *secs*, is compared with a standard seconds pendulum by the method of coincidences. Successive coincidences occurred at times 0 *min.*, 2 *mins* 58 *secs.*, 5 *mins.* 48 *secs.*, 8 *mins* 48 *secs.* Find the exact period of the pendulum. Ans. 2·0224 *secs.*

31. A thin rod is suspended by means of two threads parallel to each other and tied to its two ends. Compare the time-period of the rod when it oscillates thus in its own plane with that when it oscillates as a compound pendulum about a horizontal axis, passing through one of its ends.

Ans. 1 : 1 414.

32. Give the theory of the compound pendulum and show that the centres of suspension and oscillation are interchangeable.

A uniform thin rod AB, of mass 100 *gms.* and length 120 *cms.*, can swing in a vertical plane about A, as a pendulum. A particle of mass 200 *gms.* is attached to the rod at a distance x from A. Find x such that the period of vibration is a minimum. *(Madras, 1951)*

Ans 2 748 *cms.*

33. How does g (acceleration due to gravity) vary with latitude and height? Obtain a general relation, assuming the earth to be a homogeneous sphere. Does the relation agree with observed values? Give reasons. *(Punjab, Sept, 1955)*

34. Give the theory of Kater's pendulum and mention the errors to which pendulum experiments are liable. How is the value of g compared at different places? *(Punjab, Sept., 1956)*

35. What are *gravity meters* and *balances*? Describe the construction and working of one you consider to be the best.

36. Write short explanatory notes on :

(*i*) Gravity survey, and (*ii*) Geophysical prospecting.

CHAPTER VII

GRAVITATION

81. Historical. The celestial bodies have been an object of interest to scientists all through the ages, and the first astronomical observations, of which we have any definite knowledge, were perhaps made by the Chinese, as far back as 2,000 *B.C*, though the Babylonian astronomers are credited with having mapped out the constellations even earlier, near about 2700 *B.C.* The first authoritative treatise on the subject, however, was due to **Ptolemy,** working in Alexanderia, about 100 *A.D.* who formulated his theory on the basis of the catalogue showing the nightly positions of planets and some 1000 stars, prepared earlier by the Greek astronomer *Hipparchus.* Ptolemy's book, the *Almagest*, enjoyed the authority of the Bible and reigned supreme for 1400 years. According to him, the whole of the heavens, carrying the stars, revolved round the earth, supposed stationary. The forward and retrograde motion of the planets* among the stars was explained by postulating that the planets revolved in circles, with their centres revolving in larger circles round the earth,—the former circles being termed *epicycles* and the latter ones, *deferents.* And, it stands to his credit that, with a suitable choice of radii and velocities, he could explain quite accurately the observed facts of the day.

The Ptolemaic theory was first challenged in 1543, by the famous Polish monk, **Nicolaus Copernicus**, in his book, *'Concerning the Revolutions of the Heavenly bodies'*†, his geometrical solution being much neater than that of Ptolemy, involving only *thirty four* epicycles as against the *eighty* of the latter. In it he propounded his **heliocentric theory**,‡ according to which the planets moved in perfect circles round the Sun, which was supposed to be fixed. The theory was, however, received with reserve and scepticism, being objected to on the ground that (*i*) the rotation of the earth should result in bodies being hurled from its surface, and (*ii*) with greater justification that, no parallax (or relative motion) could be noticed between stars as was always observed between objects at different distances from a moving ship. This parallax has since been shown to *actually* exist, and was first measured by **Bessel,** in 1838. It is, however,

*From the Greek word, meaning '*wanderer*',—because a planet moves forwards and backwards or 'wanders' about among the stars.

†He hesitated and deferred publishing his book until he was dying, dedicating it to the Pope, who, not taking it seriously, expressed himself pleased with it. And *Martin Luther* was positively contemptuous towards it. 'Did not Joshua, (in the Bible) command the Sun to stand still and not the earth ?', he asked.

‡This was really a revival of the theory, first propounded by the Greek Astronomer *Aristarchus*, that the earth was not the centre of the universe but revolved round the Sun, as also did the other planets.

extremely small, on account of the enormous distances of the stars from the earth.

As is so well known, **Galileo** was compelled to recant his belief in it, a century later, and legend has it that blind and helpless, he was thrown back into prison for murmuring *'And still it moves'*, (*E pur si muove*), until he died nine years later, and that **Giordano Bruno** was actually mercilessly burnt at the stake for refusing to do so.

Then, appeared on the scene, twenty-five years later, in **1569**, **Tycho Brahe**, an imperious nobleman and a brilliant astronomer*, who rejected the *Copernician theory* and made careful observations of the motions of heavenly bodies, on every clear night for thirty long years, particularly of the motion of the planet *Mars*, from his observatory in Denmark, with his celebrated wooden quadrant, (about 10 *ft.* in radius), carrying a brass scale. In view of the fact that the telescope was yet to come, some forty years later, we cannot but marvel at the unprecedented accuracy of his observations. No wonder, they were used by navigators for centuries together, much in the manner of the *Nautical Almanack* today. With all his great mathematical and experimental skill and his 'infinite capacity for taking pains', however, Tycho Brahe could not somehow piece his results together into a proper theory. But later, **Kepler**†, his assistant at the Royal Observatory at Prague,—an impecunious but a gifted mathematician—into whose hands passed all his data on the subject, carried on the work and, accepting the Copernician theory‡, which his chief had rejected, worked on the latter's figures and finally succeeded, after twenty-two years of ceaseless work, in evolving the famous three laws, known after him,—the first two in the year 1609 and the third, ten years later, in 1619.§

82. Kepler's Laws. The following are the three laws, formulated by *Kepler*.

1. *The path of a planet is an elliptical orbit, with the Sun at one of its foci.*

2. *The radius vector, drawn from the Sun to a planet sweeps out*

*He was reputed to be 'an unsurpassed practical astronomer' and made his own instruments for his well-equipped laboratory at *Uraniborg*, built for him by Frederick II, King of Denmark. He had, however, a violent temper and lost part of his nose in a duel, while still young, going about for the rest of his life with this lost part replaced by an artificial one of an alloy of silver. On the death of Frederick, he had to flee and seek asylum at Prague, under the patronage of *Rudolph II*, Emperor of Bohemia. It was here that Kepler joined him as his assistant.

†He actually succeeded *Tycho Brahe*, who died after a little over one year of his migration to Prague, under the impressive designation of 'Imperial Mathematician', at a high salary which was, however, seldom paid. He had, therefore, to supplement his income by practising astrology, 'the foolish and disreputable daughter of astronomy, without which the wise old mother would starve'. He was also the founder of Geometrical Optics.

‡And, for this he had to migrate to a Protestant country to save himself from persecution.

§He was so filled with ecstasy at his success in enunciating his third law that he declared 'I will indulge in my sacred fury ; I will triumph over mankind by the honest confession that I have stolen the golden vases of the Egyptians to build up a tabernacle for my God.'

equal areas in equal time, i.e., its areal velocity, (or the area swept ou by it per unit time), is a constant.

3. *The square of a planet's year, (i.e., its time-period, or its time of revolution round the Sun), is proportional to the cube of the major axis of its orbit.*

Unfortunately, Kepler was not aware of the property of inertia and so could not proceed any further. For him, it was necessary to suppose a power acting continuously on a body, in order to make it move. Most of the fellows of the Royal Society*, which included among others, men like *Robert Boyle*, *Edmund Hailey* and *Samuel Papys*, were convinced, by the year 1685, that a planet could move in an elliptical orbit, only if it were attracted by the Sun with a force, varying inversely as the square of its distance from the Sun, but they could not prove it mathematically. **Newton**, who was also a member, was at this time *Lucesian Professor of Mathematics* at Cambridge and seldom attended the meetings of the society, mostly held at London and Oxford. *Edmund Hailey*, therefore, went all the way to Cambridge to ask him if he could furnish the required proof and was simply astonished to learn that he had already done so years earlier, but had somehow lost his papers.

Realising that no other member of the Royal Society could hope to provide the required proof and also that Newton had really already achieved something much more than this, Hailey pleaded with him to reproduce his papers in book form and, though not a rich man himself, offered to bear all the cost of publication of the same, which ultimately resulted in the appearance of the celebrated *Principia*, in the year 1687.

Newton knew that both rest and uniform motion along a straight line were equally natural and, after a careful study of Kepler's laws, he showed (*i*) that it follows from his second law that only a *central force* acts on the planet and is directed towards the Sun, it alone being responsible for keeping the planet in its orbital path,† (*ii*) that it can be deduced from his first and third laws that this force between the planet and the Sun is inversely proportional to the square of the distance between them†, and (*iii*) that it is an easy further deduction from the above that this force of attraction between the two is also directly proportional to the product of their masses.

He further proceeded to verify these deductions from Kepler's laws by comparing the value of acceleration of the Moon towards the Earth, calculated on their basis, with its value obtained experimentally, the two values showing a close agreement with each other, as will be seen from the following :

If g_m be the acceleration of the Moon towards the Earth, v, its

*The Royal Society, the fellowship of which today is considered to be a very high honour, really grew out of informal group meetings of men interested in natural philosophy, about the year 1645, and received its Royal Charter, in 1662 from Charles II, who, according to Samuel Papys, 'mightily laughed at them for spending time only in weighing air'.

†See *'Note on Newton's deductions from Kepler's laws'* on page 228.

linear velocity in its orbit about the Earth and R, the distance between the centres of the Earth and the Moon, we have

$$g_m = v^2/R = (\omega R)^2/R, = \omega^2 R,$$

where ω is the *angular velocity* of the Moon.

Since $\omega = 2\pi/T$, where T is the time taken by the Moon in going *once* round the Sun, we have

$$g_m = \left(\frac{2\pi}{T}\right)^2 \times R.$$

Now, $T = 27{\cdot}3$ *days* $= 27{\cdot}3 \times 24 \times 60 \times 60$ *secs.*, and

$R = 60$ *times the radius of the Earth*

$= 60 \times 4000$ *miles.*

$= 60 \times 4000 \times 1760 \times 3$ *ft.* [$\because$ radius of the Earth is equal to 4000 miles.

Hence $$g_m = \frac{4\pi^2 \times 60 \times 4000 \times 1760 \times 3}{(27{\cdot}3 \times 24 \times 3600)^2} = {\cdot}00899 \text{ ft./sec}^2.$$

Again, if the acceleration due to gravity be g on the surface of the Earth, its value at the distance of the Moon from it would, in accordance with deduction (*ii*) above, be equal to $g/60^2$,

i.e., $$g_m = g/60^2.$$

So that, taking the value of g to be $32{\cdot}2$ *ft./sec*^{2}., on the surface of the Earth, we have

$$g_m = 32{\cdot}2/60^2 = {\cdot}00084 \text{ ft./sec}^2.,$$

which is practically the same as the one deduced above, thus fully vindicating the deductions made by Newton, and convincing him of the existence of a *universal* and *mutual* force of attraction between any two masses.

Not only this, but Newton also put to test his assumption that in so far as the attraction at external points is concerned, both the Earth and the Moon behave *as though their masses were concentrated at their respective centres.* He actually showed that the force of attraction, exerted at an external point, by a uniform sphere, or by a sphere consisting of a number of concentric uniform shells, one inside the other, is the same as that exerted by an *equal point-mass*, occupying the same position as its centre. In other words, the sphere behaves as though the whole of its mass were concentrated at its centre.

Thus fortified with a clear and complete confirmation of his deductions and assumptions, Newton announced to the world, in the year 1687, his celebrated *Law of Gravitation* in his monumental work, the *Principia**, which the entire scientific world later hailed, in the words of *Langrange*, as 'the greatest production of the human mind'.

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*Having lost his papers, as mentioned already, Newton had to recreate the whole, step by step, all over again and accomplished the almost superhuman feat in only 18 months. He used geometrical methods, partly due to his admiration for ancient geometers and partly to avoid being baited by 'little smatterers in mathematics'. Perhaps he had *Robert Hooke* in mind, who had claimed priority in the discovery of the Inverse Square Law ; the Royal Society had, however, sided with Newton.

83. Note on Newton's deductions from Kepler's laws.

1. Let A be the position of a planet, (Fig 140), at a given instant t in its elliptical path round the sun S, situated at one of its foci. Then, if the planet moves on to B in a small interval of time dt, the area swept out by the radius vector SA, in this interval of time, is equal to the area of the triangle SAB.

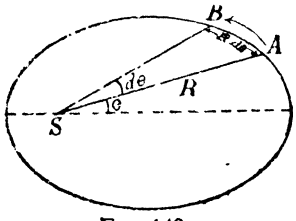


Fig. 140.

i.e., equal to $\frac{1}{2}\, SA.AB = \frac{1}{2}\, R.R\, d\theta.$

$= \frac{1}{2}\, R^2.d\theta,$

because $SA = R$ and $AB = R\, d\theta$.

$\therefore$ *areal velocity of the planet* $= \frac{1}{2}R^2.d\theta/dt.$

But, this, according to Kepler's second law, must be a constant. Putting it equal to $h/2$, therefore, we have $R^2.d\theta/dt = h$.

Now, the fact that the planet moves in a curved path and thus continually changes its direction, means, in accordance with Newton's first law of motion, that it must be under the action of a force, and must consequently be possessing an acceleration in the direction of the force Resolving this acceleration into its two rectangular components, along and at right angles to the radius vector, we have

(*i*) component a_1, *along the radius vector, i.e.*, the *radial acceleration of the planet*, given by

$$a_1 = \frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2,$$

[a result obtained from simple dynamical consideration.

and (*ii*) component a_2, *at right angles to the radius vector, i.e.*, the *transverse acceleration of the planet*, given by

$$a_2 = \frac{1}{R} \cdot \frac{d}{dt}\left(R^2 \cdot \frac{d\theta}{dt}\right).$$

But, we have seen above that $R^2.\frac{d\theta}{dt}$ is a constant (h), and hence its differential coefficient must be equal to zero. It follows, therefore, that $a_2 = 0$. In other words, the planet has no transverse acceleration ; so that, the only acceleration it has is the *radial one*, and, therefore, *the only force acting on it is towards the Sun.*

2. Now, since $R^2.d\theta/dt = h$, it is clear that $d\theta/dt = h/R^2$. Or, putting $1/R = u$, (or $R = 1/u$), we have $d\theta/dt = hu^2$.

It follows, therefore, that

$$\frac{dR}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2} \cdot \frac{du}{dt} = -\frac{1}{u^2} \cdot \frac{du}{d\theta} \cdot \frac{d\theta}{dt} = -h \cdot \frac{du}{d\theta},$$

because $\frac{1}{u^2} \cdot \frac{d\theta}{dt} = R^2 \cdot \frac{d\theta}{dt} = h.$

Differentiating this again with respect to t, we have

$$\frac{d^2R}{dt^2} = -h\,\frac{d^2u}{d\theta^2} \cdot \frac{d\theta}{dt} = -h^2u^2 \cdot \frac{d^2u}{d\theta^2}, \qquad \left[\because \frac{d\theta}{dt} = \frac{h}{R^2} = hu^2.\right.$$

Substituting the values of $d\theta/dt$ and d^2R/dt^2 in the expression for a_1 above, we have

$$a_1 = -h^2u^2 \cdot \frac{d^2u}{d\theta^2} - R(hu^2)^2 = -h^2u^2\left(R.u^2 + \frac{d^2u}{d\theta^2}\right).$$

Or, $$a_1 = -h^2u^2\left(\frac{1}{u}.u^2 + \frac{d^2u}{d\theta^2}\right) = -h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) \qquad \ldots(i)$$

Now, let the equation of the elliptical orbit of the planet be

$$l/R = 1 + e\cos\theta. \quad \text{Or,} \quad lu = 1 + e\cos\theta, \qquad \ldots(ii)$$

where l is its *latus rectum* and e, its *eccentricity.*

Differentiating this expression *twice*, with respect to θ, we have

$$l\,\frac{d^2u}{d\theta^2} = -e\cos\theta. \qquad \ldots(iii)$$

And, adding relations (*ii*) and (*iii*), we have

$$lu + l.\frac{d^2u}{d\theta^2} = 1 + e\cos\theta - e\cos\theta = 1.$$

Or, $$l\left(u + \frac{d^2u}{d\theta^2}\right) = 1, \quad \text{whence,} \quad \left(u + \frac{d^2u}{d\theta^2}\right) = \frac{1}{l}.$$

Substituting this value of $\left(u + \frac{d^2u}{d\theta^2}\right)$ in relation (*i*), above, we have

$$a_1 = -h^2u^2/l = -\frac{h^2}{l} \cdot \frac{1}{R^2}. \qquad \left[\text{Putting } \frac{1}{R^2} \text{ for } u^2.\right.$$

Or, denoting the constant h^2/l by K, we have $a_1 = -K/R^2$.

Or, $$a_1 \propto -1/R^2. \qquad \ldots (iv)$$

i.e., the acceleration, and hence *the force acting on the planet is inversely proportional to the square of its distance from the Sun*, (the $-ve$ sign merely indicating that the force in question is one of attraction).

Now, the *time-period* (T) *of the planet* (*i.e.*, the time taken by it to complete its one full revolution round the Sun) is given by

$$T = \frac{\text{area of the ellipse}}{\text{areal velocity of the radius vector}} = \frac{\pi.ab}{\frac{1}{2}.R^2.\frac{d\theta}{dt}},$$

where a and b are the *semi-major* and *semi-minor axes* of the elliptical orbit of the planet.

Or, $$T = \frac{\pi ab}{h/2}. \quad \text{And} \quad \therefore \quad T^2 = 4\pi^2.a^2b^2/h^2. \qquad \left[\because \tfrac{1}{2}R^2.\frac{d\theta}{dt} = \frac{h}{2}.\right.$$

Now, clearly, $b^2/a = l$, the *latus rectum* of the ellipse, and, therefore, $b^2 = al$; so that,

$$T^2 = \frac{4\pi^2 l}{h^2} \cdot a^3 = \frac{4\pi^2}{K} \cdot a^3. \qquad \left[\because \frac{h^2}{l} = K.\right.$$

But, since, in accordance with Kepler's third law, $T^2 \propto a^3$ for every planet, it follows that $4\pi^2/K$ is a constant, or that K is a constant for every planet. In other words, *K is quite independent of the nature of a planet.*

3. Finally, if m and M be the respective masses of the planet and the Sun, and F and F', the force of attraction, exerted by the Sun on the planet, and the reaction of the planet on the Sun respectively, we have, from relation (*iv*) above,

$$F = km/R^2 \quad \text{and} \quad F' = KM/R^2,$$

where k and K are constants.

And, since by Newton's third law of motion, action and reaction are equal and opposite, we have $F = F'$; so that,

$$k.m = K.M. \quad \text{Or,} \quad k/M = K/m = \text{a constant, say, } G.$$

So that, $$k = M.G.$$

Substituting this value of k in the expression for F above, we have

$$F = \frac{m\,M}{R^2}.G,$$

showing that *the force of attraction between the planet and the Sun is directly proportional to the product of their masses.*

84. Newton's Law of Gravitation.—This law states that *every particle of matter in the universe attracts every other particle with a force which is directly proportional to the product of their masses, and inversely proportional to the square of the distance between them.*

Thus, if m and m' be the masses of two particles, distance r apart, and F, the force of attraction between them, we have

$$F \propto m\,m'/r^2. \quad \text{Or,} \quad F = G.m.m'/r^2,$$

where G is a *universal constant*, called the **Gravitational Constant.** Obviously, if $m = m' = 1$ *gm.*, and $r = 1$ *cm.*, then, $F = G$.

Thus, *the* **Gravitational Constant** *is equal to the force of attraction between two unit masses of matter, unit distance apart.* Its dimensions are $M^{-1}L^3T^{-2}$, and its latest, accurate value, (as determined by *Heyl* in 1930), is taken to be $6{\cdot}669 \times 10^{-8}$ *C.G.S.* units. The gravitational constant is also sometimes referred to as the **astronomical unit of force.**

The law is *universal* in the sense that it holds good, right from huge interplanetary distances to the smallest terrestrial ones. The minimum distance up to which it is valid is probably not yet known with absolute certainty, but it seems to break down at molecular distances, which are as small as 10^{-7} *cm.* We shall discuss latter, in this chapter, some of the overwhelming evidence in favour of this law, as well as the small deviations from it and the proper explanation for them, on the basis of the new ideas put forth by **Einstein.**

85. Determination of the Gravitational Constant. The methods for the determination of the *gravitational constant*, (and, therefore, also those for the determination of the density and the mass of the earth), may be divided into two categories, *viz* ,

(*i*) *Mountains and Mine Methods.*—which involve the measurement of the force of attraction exerted by a large natural mass, like a mountain or the earth's *crust**, on a plumb line suspended on one side of it, which is then compared with the force of attraction on it due to the earth, as a whole.

(*ii*) *Laboratory Methods,*—which involve the more delicate measurement of attraction between small masses.

We shall deal here only briefly with the former,—more for their historical interest than otherwise ; for, the results obtained were not,—indeed, they could not be,—very accurate. The latter, *i.e.*, the laboratory methods, we shall however study in proper detail.

(*i*) Mountain Methods.—

1. **Bouguer** was the first to have attempted a determination of the value of *G*. While engaged in geographical measurements in the *Andes* (*Peru*), in the year 1740, he suspected a deflection of his *plumb line* due to large mountain-masses. He decided to verify this, and selected a mountain, (*humborazo*, 20,000 *ft.* high, (in the *Andes*) for the purpose. Shorn of experimental details, his method was the following :

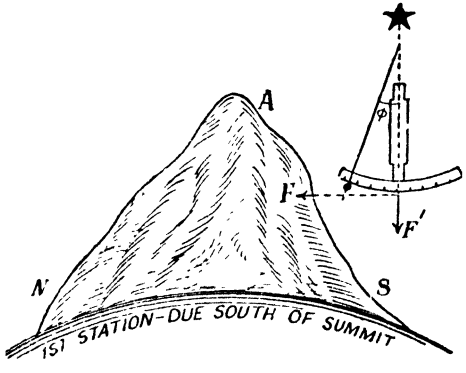


Fig. 141 (*i*).

He chose two stations *A* and *B* [Fig. 141, (*i*) and (*ii*)], the former *due south* of the summit of the mountain and *close to it*, and the latter, *in the same latitude*, and at about the same altitude some distance to its west, *away from its influence.* At station *B*, he observed a star passing the meridian directly overhead, so that the plumb line

*The word is probably a relic of the times when the earth was supposed to be a globe of water, bounded by a solid shell or crust. It is now used, however, to signify the rigid surface layer of the earth, which is heterogeneous and, more or less, in a state of permanent stress and strain.

hung *exactly* vertically parallel to the telescope. But at station *A* he observed that it was attracted by the huge mountain-mass, (due to its nearness). He measured this *deflection of the plumb line* at *A* and thus compared the horizontal pull of the mountain with the vertical pull of the earth. For, if F and F' be the forces of gravitational attraction acting on the plumb line due to the mountain and the earth respectively, and ϕ, its deflection from the vertical, [Fig. 141, (*i*)], we have

$$\tan \phi = F/F'.$$

Now, clearly, $F' = mg$, where m is the mass of the plumb line and g, the acceleration due to gravity. So that,

$$F = mg.\tan \phi.$$

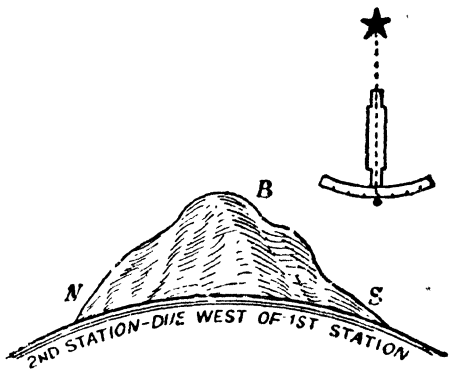


Fig. 141 (*ii*).

And, if V be the volume and ρ, the density of the mountain, and r, the distance of its *c g.* from the plumb line, we have

$$F = \frac{m.V.\rho}{r^2}.G. \qquad \left[\because \text{mass of the mountain} = V.\rho. \right.$$

Hence $m.V.\rho.G/r^2 = mg \tan \phi$. Or, $G = g.r^2 \tan \phi/V.\rho$.

Thus, a knowledge of the volume, density and shape of the mountain, and hence its *centre of gravity*, was needed to determine the value of *G*. *Bouguer*, therefore, proceeded to do so, but did not quite succeed ; for, due to the most adverse conditions of snow and storm under which he had to work, he could not properly survey the mountain, and the results he obtained were very much wide of the mark. Thus, for example, he found that his plumb line was drawn aside by about 8″, and his calculation showed that if the mountain were as dense as the earth itself, the deflection of the plumb line would have been twelve times as great, indicating that the earth was about twelve times as dense as the mountain. And this, as we know, is very much beyond the truth. Nevertheless, he had the satisfaction of showing that the attraction due to the mountain-masses did actually exist and that the method was, therefore, possible. Not only that, but he also deserves the fullest credit for proving conclusively that the earth was not just a globe of water or a hollow shell, as was fairly widely supposed at the time.

2. **Maskelyne**, later in the year 1774* repeated, at the request of the Royal Society, Bouguer's experiment on the mountain *Schiehallion*, in Perthshire (Scotland), 3547 feet high, an elaborate survey of which was first made to determine as accurately as possible, its volume and density (and hence its mass) and centre of gravity.

Two stations were then chosen *at equal distances* from the *c g.* of the mountain, on the north-south line (Fig. 142), and the *same* star was observed, (as in Bouguer's experiment), by means of a special type of telescope, called the *Zenith Sector*†, first at the South Station and, a month later, at the North Station. At the former Station, the star which, in the absence of the mountain, would be directly overhead, appeared to shift slightly to the north, because the plumb line was pulled by the mountain towards the north, (and the *zenith*

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*He was the *Astronomer Royal* at the time.

†The instrument could rotate about a horizontal (East and West) axis at its object-glass end, pointing upwards, and was provided with a plumb line, suspended from this axis, over a scale, graduated in degrees, so that the angular distance of the telescope from the vertical could be directly read on it.

thus shifted to the south). At the other Station, on the other hand, the exact opposite was the case, (the plumb line being pulled towards the south, the *zenith* thus shifting to the north) and the star, therefore, appearing to shift *equally* to the south. Thus, the total shift of the star was double of the deflection of the plumb line at either station due to its attraction by the mountain. This was carefully measured and was found to be 55″. Out of this, a shift of 43″ was calculated to be due to the curvature of the earth's surface so that the *net shift* or *deflection of the plumb line, due to the gravitational pull of the mountain*, was (55″−43″) = 12″. In other words, *the plumb line, at each of the two stations, was deflected by 6″ due to the mountain-mass* The value of G was then calculated, as explained above (in Bouguer's experiment), and was found to be $7{\cdot}4\times10^{-9}$ *C.G.S units.*

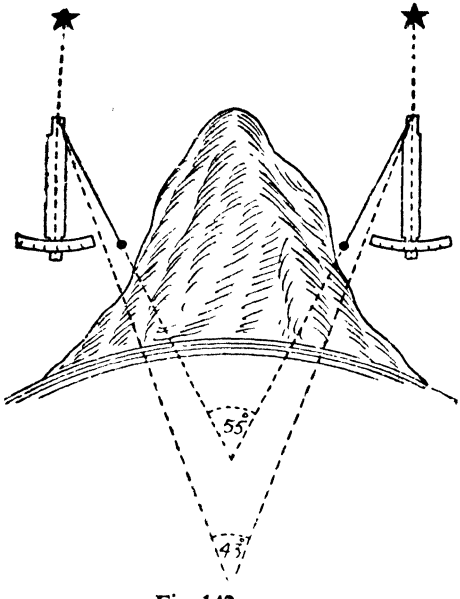


Fig. 142.

Further, it was estimated that if the mountain had the same density as that of the earth, the deflection of the plumb line, due to its attraction, would have been 9/5 times the observed deflection, showing that the earth was 9/5 times denser than it. And, since the density of the mountain, determined from pieces of rocks composing it, was found to be 2 5 *gms /c c.*, the density of the earth came to be $9\times2{\cdot}5/5$ or 4 5 *gms./c c.* This was corrected and increased to 5·0 *gms./c c.* after a careful re-survey of the mountain, some thirty years later,—a result not very much wide of the mark.

Since, however, it is almost impossible to determine correctly the mass and position of the *c.g.* of a huge natural mass like a mountain, the value of G, obtained by the above methods, is far from reliable, and is at best only an approximation.

(*ii*) **Mine Methods.** In these, the time-periods of a pendulum, (say, *a seconds pendulum*), are compared on the surface of the earth and at the bottom of a mine. It is obviously greater in the latter case, the value of g being less there than on the surface of the earth, [see page 206]. The change in its time-period enables a comparison to be made between the values of acceleration due to gravity, and hence between the density of the layers immediately above the pendulum-bob and that of the rest of the earth, which, in its turn, leads to a determination of M, the mass of the earth and thus to a calculation of the value of G, as will be clear from the following :

We know that the weight of a body at a place is the force with which it is attracted by the earth towards its centre, and is numerically equal to the product of its mass (m) and the acceleeration due to gravity at the place.

Thus, if g be the acceleration due to gravity on the surface of the earth and g', that at the bottom of a mine of depth h, we have its weights at the two places given respectively by

$$mg = \frac{mM}{R^2}.G \qquad \text{and} \qquad mg' = \frac{m(M-m')}{(R-h)^2}.G,$$

where M is the mass of the earth, R its radius, and m', the mass of the outer shell of the earth, of thickness h.

So that,
$$g = \frac{M}{R^2}.G \qquad \text{and} \qquad g' = \frac{(M-m')}{(R-h)^2}.G,$$

hence,
$$\frac{g'}{g} = \frac{M-m'}{M}\cdot\left(\frac{R}{R-h}\right)^2 \cdot \cdot \qquad \ldots(i)$$

And, clearly, if ρ be the density of the outer shell of the earth, its mass m' is obviously given by the relation,

$$m' = \tfrac{4}{3}\pi[R^3-(R-h)^3].\rho. \qquad ...(ii)$$

Now, the average value of ρ was obtained by determining the densities of the samples of rock at different levels, down to the bottom of the mine, and thus the mass (m') of the outer shell evaluated. Substituting this value of m' in relation (*i*) above, the mass (M) of the earth was easily determined. And, then, putting the value of M in the expression for g above, the value of G could be calculated out straightaway.

Airy was the first person to have made a successful attempt of this nature in the Harton coal pit in *Sunderland*, in the year 1854, two earlier attempts made by him, in a Cornish copper-mine, as early as 1826 and 1828, having come to naught, due to unfortunate accidents in the mine. *Airy's* value of G came to $5{\cdot}7\times10^{-8}$ *C.G.S. units*, and that of the density of the earth's surface, to 6·5 *gms./c.c.*

Like the earlier Mountain experiments, these experiments by *Airy* too gave far from satisfactory results, due mainly to the difficulty of determining accurately the density of the outer shell of the earth. His methods, however, with improved modifications, now find a wide and useful scope in the branch of *Geophysics*, [§ 80, (page 216)].

(*iii*) **Laboratory Methods.** In these methods, the attraction between the masses is inevitably feebler due to their smallness. But this is more than compensated for by the high degree of accuracy with which the masses and their sizes can be determined. The first successful attempt at an accurate method of this type, for the determination of the Gravitational Constant (G) was made by *Cavendish*, in the year 1798, in which he made use of the *Torsion Balance*.

It will be of some interest to recall that *Cavendish* was probably also associated with *Maskelyne* in his Mountain experiment, performed some twenty five years earlier. He, however, took his cue from *Rev. John Michell*, who had devised an apparatus almost similar to Cavendish's own, but was not destined to use it, due to his sudden death. His apparatus fell into the hands of *Prof. Wollaston*, who passed it on to *Cavendish*.

(*a*) **Cavendish's Method.** The apparatus used by **Cavendish**, and installed in an outhouse in his garden on Clapham Common, was as shown in Fig. 143. A long *cross bar PQ*, about 6 *ft.* (or about 180 *cms.*) long, was suspended from the ceiling of a room and was free to turn about a vertical axis by means of an arrangement, manipulated from outside. It carried two large and *equal lead spheres C* and *D*, about 8 to 10 *inches* (or 20 to 25 *cms.*) in diameter and weighing about 350 *lbs.* each, at the ends of two metal rods attached to its two ends.

Immediately below the mid-point of the cross-bar was a *torsion head M* which could also be worked from outside, from which was suspended a *deal-rod RS*. (slightly bigger than *PQ*), by means of a *fine torsion wire W*, of silvered copper. Two wires (*w. w*), fastened the ends of the deal-rod to a vertical rod *r* in the middle, which was attached to the suspension wire. This increased the strength of the rod, *without increasing its moment of inertia*. Two *small lead balls*, 2 *inches* (or 5 *cms*) in diameter and weighing about a pound and a half (or 680 *gms.*) each, were suspended from the two ends of the deal-rod *RS* such that *the centres of the four balls lay in the same horizontal plane*, roughly in a horizontal circle of about 3 *ft.* (or 90

cms.) radius. The arrangement was such that, when the line joining the centres of the large lead spheres was at right angles to the torsion rod, there was no twist or torsion in the suspension wire *W*. Each end of the torsion rod carried a vernier, (of five divisions), which moved over a fine ivory scale, fixed to vertical stands, and with each division equal to ·05″.

To guard against any changes of temperature, and consequent air-draughts, which would otherwise mask the gravitational effect, the room was closed and observations were taken with the help of telescopes *T* and *T*, fixed into the walls of the room, as shown. And, further, to avoid the effect of any outside electric charges, the whole apparatus was enclosed in a gilded glass case, supported on four levelling screws.

The method of procedure was the following :

The rod *PQ* was rotated until the line joining the centres of the large spheres was at right angles to the torsion rod, *i.e.*, *in the position in which there was no twist in the suspension wire*, and the reading on the verniers, attached to the torsion rod at either end, taken.

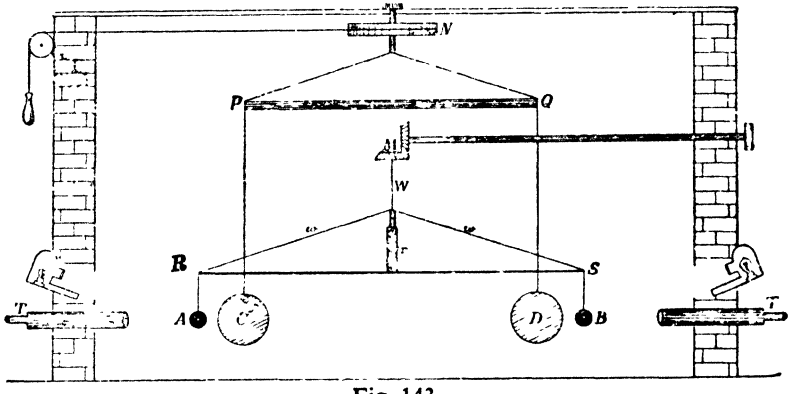


Fig. 143.

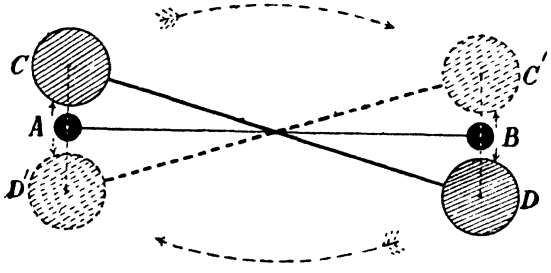


Fig. 144.

The large spheres were then rotated until they lay on *opposite sides* of the torsion rod and near to the small balls at either end, *i.e.*, in the positions *C* and *D*, as shown in Fig. 144, such that the lines joining the centres of each pair of near balls were *equal in length* and *perpendicular* to the torsion rod. Obviously, then, the forces exerted by the big spheres on the corresponding near small balls were *equal* and *opposite*, thus constituting a *couple*, tending to rotate the torsion rod. This was resisted by the *torsional couple* set up in the suspension wire, and *equilibrium was attained when the deflecting couple, due to the forces between the two*

pairs of balls, was just balanced by the restoring torsional couple, set up in the suspension wire. The position of the verniers was again noted on the scales by the method of oscillation, as in the case of an ordinary physical balance In Cavendish's own experiment, this distance between each pair of balls was 8″ (or 20 *cms.*), and the small balls were displace l through ·7661″ (or 1·915 *cms.*). The rod *PQ* was then rotated again about its vertical axis, until the large spheres now occupied the positions *C′* and *D′* respectively, and the *same* adjustment was made as before, *viz.*, that the lines joining the centres of the two near balls were of the *same equal lengths* as before and *perpendicular* to the torsion rod. The positions of the verniers were read on the scales, as before, and their mean taken as the deflection of the torsion rod.

The value of *G* was then calculated as follows :

Let *M* be the mass of each large sphere, *m*, that of each small ball and *d* the distance between each pair of near balls. Then, the force of attraction between each pair of balls is, clearly, equal to $G.M.m/d^2$; and, therefore, if *l* be the length of the torsion rod, the deflecting couple formed by this pair of equal, opposite and parallel forces, is equal to $\frac{M.m}{d^2}.G.l.$ [$\because l \cos\theta \approx l.$]

And, if θ be the twist (in *radians*) in the suspension wire, and *C*, the restoring couple per unit (*radian*) twist set up in it, the restoring couple due to torsion is equal to $C.\theta$. Since the torsion rod is in equilibrium under these two couples, we have

$$\frac{M.m}{d^2}.G.l = C.\theta. \quad \text{whence,} \quad G = \frac{C.d^2}{M.m.l}.\theta.$$

In order to determine the value of *C*, the torsion rod alone was set into torsional vibrations about the suspension wire, and its time-period was measured. Cavendish found it to be 28 *minutes* in his apparatus.

Then, if *I* be the *moment of inertia of the torsion rod* (together with the small balls, about the wire as axis, and *t*, the time-period of the rod, we have

$$t = 2\pi\sqrt{\frac{I}{C}}, \quad \text{whence,} \quad C = \frac{4\pi^2 I}{t^2}.$$

Substituting this value of *C* in the relation for *G* above, we have

$$G = \frac{4\pi^2 I.d^2}{M.m.l.t^2}.\theta.$$

Or, *if we ignore the mass of the torsion rod*, we can put

$$I = 2m(l/2)^2 = ml^2/2.$$

So that, substituting this value of *I* in the above expression, we have

$$G = \frac{2\pi^2.ld^2}{Mt^2}.\theta.$$

Corrections and Sources of Error in the Experiment. Corrections were applied for the following :

(1) *force of attraction between each large sphere and the distant small ball ;*

(2) *force of attraction between the two large spheres and the torsion rod ;* and

(3) *forces exerted by the rods carrying the large balls.*

The following are the *sources of error* in the experiment :

(1) The gravitational forces between a pair of balls being small, the torsion rod had to be made long to increase the deflecting couple. This also minimised the force of attraction between a large sphere and the distant small ball, but required a large chamber, and thus *convection currents could not be avoided.* In addition to this, the torsion rod was also disturbed by heavy traffic on a nearby road.

(2) For a given deflecting couple, *the deflection of the torsion rod was small,* because the suspension wire required a large *torque* per unit deflection.

(3) The torsion wire, being not perfectly elastic, did not return to its normal position when the applied forces were removed, and thus *the torque was not strictly proportional to the angle of deflection.*

(4) *The distant large spheres decreased the angle of deflection while the rods carrying them increased it.*

(5) *The method of measuring the angle of deflection was not sensitive enough.*

(6) The time-period of the torsion rod, with the attached small balls was much too big, *i.e.*, its swings were much too sluggish and impaired, rather than improved, the accuracy of measurement.

The value of G obtained by Cavendish, as the mean of twenty nine observations, was $6{\cdot}754 \times 10^{-8}$ *C.G.S.* units *

Many other attempts were made since Cavendish's time, by *Jolly* and *Poynting*, among others, to obtain the volue of G more accurately but the method adopted by *Boys*, with his newly invented quartz fibre, used as suspension for the torsion rod, was by far the best of these. We shall, therefore, discuss that first.

(*b*) **Boys' method. Sir Charles Vernon Boys** removed, almost a century later, in the year 1895, all the defects of Cavendish's experiment by

(*i*) reducing greatly the size of the chamber, thus considerably *minimising convection currents* and making it easier to control its temperature ;

(*ii*) arranging the pairs of balls at different levels, thus *making the attraction between the distant large and small balls almost negligible* ;

(*iii*) using, for the suspension wire, a *quartz-fibre*, which required a comparatively very small torque per unit deflection ; and

(*iv*) which being *almost perfectly elastic,* besides being fine and strong†. *the angular deflection produced was appreciably large and also proportional to the torque* ; and

(*v*) *measuring the angular deflection by the telescope and scale method,* which greatly enhanced the accuracy of measurement.

---

*Amusingly enough, Cavendish made a slip in calculating this mean and it was later pointed out by Baily.

†A quartz-fibre is found to be much stronger than a steel wire of the *same* dimensions. Boys was thus able to use fibres having a diameter as small as ·0125 *mm.*

Thus, Boys greatly reduced the size of the apparatus and yet increased its sensitiveness. This may, at first sight, appear to be a contradiction in terms ; for, it is commonly believed that the larger a piece of apparatus, the greater the degree of accuracy obtained from it. Boys clearly showed, however, that this was not so, that the sensitivity of Cavendish's apparatus was quite independent of its dimensions and that there was no point, therefore, in attempting a larger version of it. He argued as follows :

$$\textit{The sensitivisy of the apparatus} = \frac{\textit{Deflecting couple}}{\textit{Restoring couple}}.$$

Clearly, deflecting couple $\propto \frac{M.m.l}{d^2}$, *i.e.*, $\propto \frac{(\textit{mass})^2 \times \textit{length of torsion rod}}{\textit{distance between near balls}}$,

Now, $\textit{mass} \propto \textit{volume} \propto 4\pi\,(\textit{radius})^3/3 \propto r^3$.

So that, $\textit{deflecting couple} \propto \frac{(L^3)^2 \times L}{L^2}$, *i.e.*, $\propto \frac{L^6 \times L}{L^2} \propto L^5$.

And, $\textit{restoring couple} \propto \frac{I}{t^2} \propto \frac{\textit{moment of inertia}}{(\textit{time of swing})^2}$.

Now, there is a practical limit to the time of swing which should not exceed 5 *minutes*, whatever the size of the apparatus, or else the swings become very sluggish, thus impairing the accuracy of measurement. This being so, we have

$$\textit{restoring couple} \propto \frac{MK^2}{1} \propto \frac{L^3 \times L^2}{1}. \qquad \left[\begin{array}{l} t \text{ being the same} \\ \text{for any apparatus.} \end{array}\right.$$

Or, $\textit{restoring couple} \propto L^5$. And $\therefore$ $\textit{sensitivity} \propto \frac{L^5}{L^5}$.

In other words, the sensitivity is independent of the size (L) of the apparatus.

Thus, if, for example, we double the dimensions of Cavendish's apparatus, *keeping the time of swing* (t) *the same*, we find that M is increased by $(2)^3$, because it is proportional to $(\textit{radius})^3$, d is increased by $(2)^3$ and l by 2.

The net result is that the value of $\theta = Mt^2.G/2\pi^2d^2l$ remains the same, *i.e.*, no advantage is derived by doubling the size of the apparatus, *i.e.*, by increasing the dimensions of all its parts in the same ratio.

What Boys, therefore did was to reduce the dimensions of the different parts of the apparatus *in different ratios*, as will be clear from the following :

A small mirror strip S, (Fig. 145), about 2·5 *cms.* long which acts as the torsion bar, is suspended by means of a *fine quartz-fibre*, from a torsion head T inside a glass tube, about 4 *cms.* in diameter. From the two ends of the strip are suspended, by *similar* quartz-fibres, two small *gold** balls A and B, about 0·5 *cm.* in diameter and 2·65 *gms.*, in mass each, *one ball being about* 15 *cms. above the level of the other.* In an outer coaxial tube, which can be rotated about the common axis, two large *lead balls* C and D each about 11·0 *cms.* in diameter, (and about 74 *kgm.* in mass), are

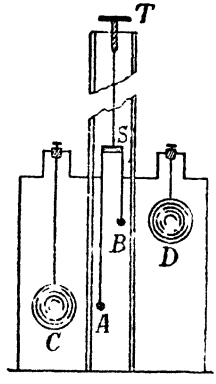


Fig. 145.

*On account of the higher density of gold (19·3 *gms /c.c.*) compared with that of lead (11·3 *gms /c.c.*), the spheres of gold for the same mass are smaller than spheres of lead and thus enable the distance d between the centres of the large and small balls to be reduced.

suspended such that *the centre of C is in level with that of A, and that of D, in level with that of B*, (to ensure greater precision in the measurement of the distance between each pair), *the distances between the centres of the pairs A and C, and B and D, being exactly equal.* The deflection is measured by the telescope and scale method, a *half-millimetre scale* being placed at a distance of about seven metres from the strip.

The experiment is performed by rotating the outer cylindrical tube until the large (lead) balls lie on the *opposite sides* of the two gold balls (*but not in a line with the mirror strip**), so as to exert the maximum moment on the suspended system, *i.e.*, when the angle of deflection is the largest.† The tube is then rotated, so that the lead balls now lie on the *other sides* of the gold balls in a *similar* position, again exerting the maximum moment, or producing the greatest deflection. The mean of the two is then taken. Let it be θ.

The calculation for the value of G are then made as follows :

Let A, B, C and D, (Fig. 146), be the four balls, when they are *in equilibrium*, in the position of *maximum deflection* θ. To visualise the balls in these positions, we must remember that to start with, the centres of all the four balls lie *in the same vertical plane*, there being no twist in the suspension fibre and hence no couple acting on the suspended system If we now rotate the larger balls C and D through a certain angle, the plane containing their centres will also rotate through the *same* angle ; with the result that a gravitational couple now comes into play on the suspended system, tending to rotate it into a position of equilibrium in which, once again, the centres of the small balls come to lie in the same vertical plane with the centres of the large balls.—this being equally true when the suspended system suffers its maximum deflection.

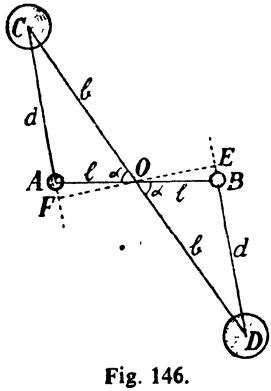


Fig. 146.

Here, the small balls A and B are shown in their initial positions, corresponding to $\theta=0°$ and the large balls in their final positions when they have been rotated into a position so as to exert the maximum couple on the suspended system, tending to make the latter suffer its maximum deflection θ. Obviously, equilibrium will be attained again only when the centres of the small balls come to lie in the vertical plane of the centres of the large balls. To bring the small balls back to their original positions, (shown in the Figure), therefore, the torsion head will have to be rotated in a direction opposite to that in which they have been deflected by this couple. In other words, the deflecting gravitational couple exerted by the large

*For, in this position, the gravitational forces due to large balls on the small balls near to them will act in opposite directions along *the same straight line* and will thus neutralise each other.

†This position of the lead balls is chosen because when the couple on the suspended system is the maximum, the rate of variation of the couple is small and the relative positions of the balls need not be known with any great accuracy.

balls on the suspended system, in the position shown, is just balanced by the restoring torsional couple set up in the quartz suspension.

Now, let O be the mid-point of the mirror strip, and let l be its *half-length* (*i.e.*, let $OA=OB=l$). Let $OC=OD=b$, $AC=BD=a$ and let $\angle AOC=\angle BOD=\alpha$. Let OE be the perpendicular drawn from O on to DB produced.

Then, clearly, in the triangle OBD, we have

$$BD = \sqrt{OD^2+OB^2-2OD.OB\cos\alpha}. \qquad \text{[See Appendix 1, 7, (ii)]}$$

Or, $$d = \sqrt{b^2+l^2-2bl\cos\alpha} = (b^2+l^2-2bl\cos\alpha)^{\frac{1}{2}}.$$

Also, $$\frac{\sin\alpha}{\sin BDO} = \frac{BD}{OB} = \frac{d}{l}\ ; \quad \left[\because \text{in a triangle, the sides are proportional to the sines of the angles opposite to them. [See Appendix I, 7, (i).]}\right.$$

and, therefore, $$\sin BDO = \frac{l\sin\alpha}{d}. \qquad .. \qquad .. \qquad .. \qquad (i)$$

Now, in the right-angled triangle OED, we have

$$OE = OD.\sin EDO = b.\sin BDO.$$

Or, $$OE = \frac{b.l\sin\alpha}{d} \qquad \text{[From (i) above.}$$

Obviously, the attraction between the two balls of each of the pairs, A, C and B, D is equal to $M\,m.G/BD^2 = M.m.G/d^2$, where M and m are the masses of *each* large ball and small ball respectively.

These two forces, being *equal*, *opposite* and *parallel*, constitute a *couple*, tending to rotate the mirror strip; and, quite clearly, the moment of the couple $= G.\frac{M.m}{d^2}.EF = G.\frac{M\,m}{d^2}\,2.OE$,

where $EF = 2.OE$ is the *perpendicular distance between the lines of action of the two forces.*

Or, the *deflecting couple* $= G.\frac{M.m}{d^2}.\,2.\,\frac{b.l\sin\alpha}{d}.$ [Substituting the value of OE, from above.

$$= G.\frac{2M.m.b.l\sin\alpha}{d^3} = G.\frac{2M.m\,b.l\sin\alpha}{(b^2+l^2-2bl\cos\alpha)^{\frac{3}{2}}}.$$

Now, this deflection of the mirror strip is resisted by the torsion or twist, set up in the suspension-fibre, and the mirror comes to rest, when the deflecting couple due to the attraction between the two pairs of the balls is just balanced by the restoring torsional couple set up in the suspension fibre. If C be the *torsional couple per unit deflection*, set up in the suspension-fibre*, the restoring torsional couple is equal to $C.\theta$, and, therefore,

$$\frac{2G.M\,m\,b\,l\sin\alpha}{(b^2+l^2-2bl\cos\alpha)^{\frac{3}{2}}} = C.\theta.$$

Or, $$G = \frac{C.\theta.(b^2+l^2-2bl\cos\alpha)^{\frac{3}{2}}}{2M.m\,b.l\sin\alpha}, \qquad .. \qquad .. \qquad .. \qquad (I)$$

whence the value of G may be easily calculated out.

* This is determined, as in Cavendish's experiment, by oscillating the moving system, *in the absence of the lead balls*, and noting its time-period.

† The value of α is clearly the angle through which the torsion head T must be rotated to bring the small balls back into their original-positions and can be easily read on a circular scale attached to it. Since a quartz-fibre is nearly perfectly elastic, the value of α may be taken to be the same as that of θ.

The value of G obtained by *Boys* was equal to $6{\cdot}6576 \times 10^{-8}$ *C.G.S.* units.

Alternative Calculation. The value of G may alternatively be calculated out as follows, using the same symbols as above.

Gravitational pull between each pair of near lar·e and small balls $= \frac{M.m}{d^2}.G$,

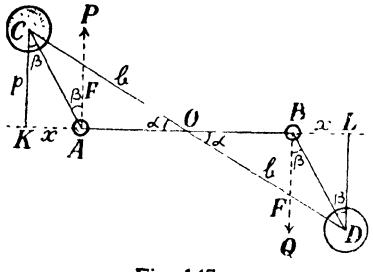


Fig. 147.

in the directions A to C and B to D respectively, (Fig. 147). Resolving these into their two rectangular components each, along and perpendicular to AB, we have *the latter components of each* equal to

$$F = G.\frac{M.m}{d^2}\cos\beta = G.\frac{M.m}{d^2}.\frac{p}{d},$$

$$= G.\frac{M.mp}{d^3},$$

where $CK = DL = p$ are the perpendiculars drawn from C and D respectively on AB produced.

Now, $AC^2 = BD^2 = d^2 = p^2+x^2$, where $p = b \sin\alpha$ and $x = (OK-OA)$
$= (b\cos\alpha - l)$.

So that, $d^2 = p^2+x^2 = (b\sin\alpha)^2+(b\cos\alpha-l)^2$.
$= b^2\sin^2\alpha+b^2\cos^2\alpha+l^2-2bl\cos\alpha$.

Or, $d^2 = b^2+l^2-2bl\cos\alpha$, $\quad[\because \sin^2\alpha+\cos^2\alpha = 1.$

And, therefore, $d^3 = (d^2)^{\frac{3}{2}} = (b^2+l^2-2bl\cos\alpha)^{\frac{3}{2}}$.

Thus, $F = G.\dfrac{M.m.b\sin\alpha}{(b^2+l^2-2bl\cos\alpha)^{\frac{3}{2}}}$.

Hence, the *deflecting couple on AB due to these forces F and F,*

$$= F.AB = F.2l = G\frac{M.m.b\sin\alpha.2l}{(b^2+l^2-2bl\cos\alpha)^{\frac{3}{2}}}.$$

Now, if C be the torsional couple set up in the suspension fibre per unit twist of it, (*i e.*, for unit deflection of the suspended system), the total restoring torsional couple for a twist θ in it $= C.\theta$.

Since, for equilibrium, the two couples must balance each other, we have

$$G.\frac{M.m.b\sin\alpha.2l}{(b^2+l^2-2bl\cos\alpha)^{\frac{3}{2}}} = C\theta,$$

Or, $$G = \frac{C\theta(b^2+l^2-2bl\cos\alpha)^{\frac{3}{2}}}{M.m.b.2l\sin\alpha},$$

the same as expression I in the case above, whence the value of G can be easily obtained.

N B. In case the centres of the near large and small balls do not lie in the same horizontal plane, but a vertical distance h apart, as shown in Fig. 148, then, if d' be the actual distance between them, we have

gravitational couple on the suspended system $= \dfrac{M.m.G.p}{d'^3}$.

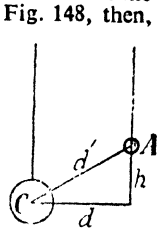


Fig. 148.

But, clearly, $d'^2 = (d^2+h^2)$, where d is the horizontal (or the perpendicular) distance between the balls, or rather between their centres. So that,

gravitational couple on the suspended system $= M.m.G.p/(d^2+h^2)^{\frac{3}{2}}$.

And, therefore, proceeding as above, we have

$$G = \frac{C\theta(b^2+l^2-2bl\cos\alpha+h^2)^{\frac{3}{2}}}{M.m.b.2l\sin\alpha}, \qquad \ldots(II)$$

an expression which, when $h = 0$; when the centres of the near large and small balls lie in the same horizontal plane, is reduced to relation (I), above.

(*iii*) **Balance Methods.** These methods do not compare in accuracy with *Cavendish's*, *Boys'* or *Heyl's* methods, but are given here only for their historical importance.

(*a*) **Jolly's Method.** As early as 1881, **Von Jolly** had suggested that a common balance could be used to measure directly the gravitational force of attraction exerted on a mass, placed in one pan, by a large lead mass, placed immediately below it.

He actually performed this experiment in Munich, where he had a common balance fixed at the top of a tower, 21 *metres* high, and suspended two long wires from its two scale pans, carrying two other pans at their lower ends.

Two equal masses were then placed in the two upper scale pans and balanced against each other. One of the masses was then moved down into the lower scale pan, on the *same* side, so that, being now comparatively (about 20 *metres*) nearer to the centre of the earth than the other mass, its weight increased a little,—this increase (due to the earth's attraction) being equal to the extra weights needed in the other scale pan to balance the beam.

A large lead sphere (of *known* mass) was then placed immediately below the lower pan carrying the mass, so that due to the additional attraction of it by the lead sphere, its weight again increased a little. This increase was also determined, as before, by putting some more weights in the other scale pan. The attraction of the mass by the lead sphere could thus be compared with its attraction by the earth. And, since the distance between the centres of gravity of the lead sphere and the earth was known, the masses of the two could also be compared. Then, the mass of the lead sphere being known, the mass of earth could be easily calculated out. And, once the mass of the earth was obtained, the value of G could be deduced as in § 85 (*ii*), page 232.

(*b*) **Poynting's Method.** The balance method has perhaps been used to the best advantage by **Prof. J.H. Poynting**, whose arrangement was much more elaborate and susceptible of a much higher degree of accuracy. He performed his experiment in the year 1891, in the basement of the University of Birmingham.

The apparatus used by him, (shown diagrammatically in Fig. 149), consisted of a strong and sensitive bullion type of balance, with a gun-metal beam, provided with steel knife-edges and planes. The whole apparatus was fully enclosed in the room and all necessary manipulations were made from the outside.

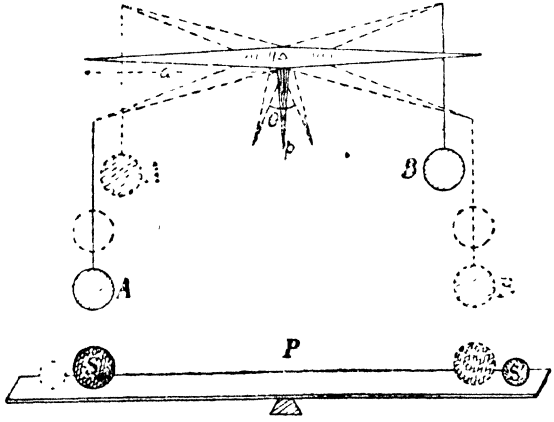


Fig. 149.

Two equal spherical balls, A and B, of an alloy of lead and antimony, weighing about 50 *lbs*, each, were suspended from the two ends of the beam. A large sphere S, of the *same* alloy and weighing about 350 *lbs*. was arranged below, on a turn-table, which could be turned about its vertical pivot P, so that the sphere could be brought to lie immediately under the ball A or B, as desired. To guard against the tilting of the turn-table due to the weight of the sphere S, a smaller sphere S', *of half the mass of* S, was placed on the other side of the pivot, at double the distance of S from it, so that, in accordance with the principle of moments, the turn-table was kept in equilibrium.

To start with, the sphere S was brought to lie under the ball A, so that A was attracted downwards with a force equal to $G.M.m/r^2$, where M and m are the respective masses of S and A, and r, the distance between their centres, (which was about one foot).

The turn-table was next rotated about its pivot until the sphere S came from under A to under B, and the balancing sphere S' moved on to the other side, (into the dotted positions shown), so that S now exerted a pull on B instead of on A, resulting in the beam being tilted in the opposite direction to that in the first case, the angle of tilt being now obviously twice* that due to S on A or B. Let this deflection or tilt of the beam be θ.

Then, if a be the length of each arm of the balance, (*i.e.*, if $2a$ be the length of the beam), we have

change in torque or couple due to the shifting of S from the first position (under A) to the second position (under B)

$$= \frac{G.M.m}{r^2}.2a. \qquad [\because \cos \alpha = 1.$$

And, if C be the torque or couple required per unit deflection of the beam, the torque for deflection θ of the beam is also equal to $C\theta$.

Hence $$G\frac{M\,m}{r^2}.2a = C.\theta \quad \text{So that, } G = \frac{r^2\,C.\theta}{M.m.2a}. \qquad \ldots (i)$$

Thus, knowing M, m, a, r, C and θ, the value of G could be easily calculated.

To determine the value of C, a *centigram rider* was moved along the arm of the balance and the deflection α of the beam, for a shift l *cms.* of it along the arm, was noted. Then, clearly,

$$\cdot 01\ g.l = C\alpha, \text{ whence, } C = \frac{\cdot 01\ g.l}{\alpha}.$$

Substituting this value of C in relation (i) above, we have

$$G = \frac{r^2\theta}{M.m\,2a}.\frac{\cdot 01\ g.l}{\alpha} = \frac{\cdot 01\ g\,r^2.l\ \theta}{2M.m\,a\cdot\alpha},$$

whence, G can be easily evaluated.

The effect of S and S' on the beam was eliminated by repeating the experiment with A and B, a foot higher up in the dotted positions shown, and proper corrections were also applied for the cross effect of S and S' on A and B.

Both the angles, θ, and α, being very small, (θ being only about *one second*) were measured by Kelvin's double suspension mirror method, as illustrated in Fig. 150, where M is a small mirror, suspended by means of a bifilar suspension (w and w') from two horizontal brackets B and B', in level with each other, with a small adjustable gap between them,—the former being a movable one, attached to the pointer (p) of the balance, and the latter, a fixed one. Thus, when the beam turned, the wire w also turned with it, making the mirror M turn about the stationary wire w'. This, with the gap BB' suitably adjusted, magnified the deflection of the beam about 150 times. The scale, graduated in *half-millimetres*, was arranged about 5 *metres* in front of the mirror, and its image in the latter viewed from the room above by means of a vertical telescope, fixed up in the ceiling.

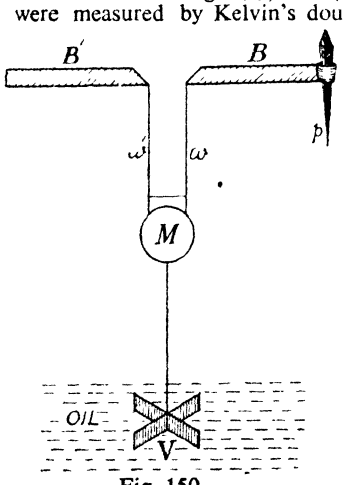


Fig. 150.

The effect of the air draughts or currents was eliminated by using what are called damping vanes V, suspended from the mirror and kept immersed in oil.

Poynting obtained the value of G to be $6{\cdot}6984\times10^{-8}$ *C.G.S.* units.

*This is so, because the beam first comes to its original position from its first tilt and then gets tilted *equally* in the opposite direction, due to the attraction of S on B.

(*c*) **Heyl's Method.** The value of *G* obtained by **P.R. Heyl**, in the year 1930, is taken to be the most accurate one so far. His method is a modification of *Braun's Torsion Balance* experiment, which, in its turn, was a revised version of *Boys'* earlier experiment, referred to above, (page 236).

Heyl performed his experiment in a constant temperature enclosure*, with the pressure inside reduced to about 2 *mms.* of mercury column, in order to minimise convection currents. The attracting large masses, used by him, were massive steel cylinders, each of mass about 66 3 *k.gms.* suspended from a system, free to rotate about a vertical axis midway between the two. The smaller masses, each weighing 2·44 *gms.*, were balls of gold, platinum and optical glass, in three different sets of experiments respectively, and were suspended from the two ends of a light aluminium torsion rod *R*, 28·6 *cms.* long, (Fig. 151), supported by a tungsten thread, *T.W.*, (1 *metre* long and 0·25 *mm.* in diameter), and two inclined copper wires, (*w*, *w*), so that *almost the whole of the moment of inertia remains in the balls themselves.* He chose as suspension a tungsten thread in preference to a quartz-fibre, because the latter is sometimes found to break quite unexpectedly and for no apparent reason.

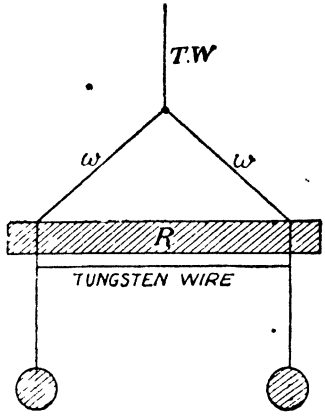


Fig. 151.

This suspension system (of the torsion rod and the two small masses) was made to oscillate in the gravitational field of the two large masses, which were arranged once with the centres of all the four masses lying *along the same horizontal line*, and then, with the horizontal line joining their centres, *along the right bisector of the torsion rod*, the two positions being referred to as the '*near*' and the '*distant*' positions respectively, [Figs. 152 (*a*) and (*b*)]—the gravitational attraction accelerating the oscillations in the first case and retarding them, in the second. Or, as a variation of this, the time-period of the suspension system was first determined, *with no other masses in its neighbourhood* and then with the large masses brought in the near position, shown in Fig. 152 (*a*).

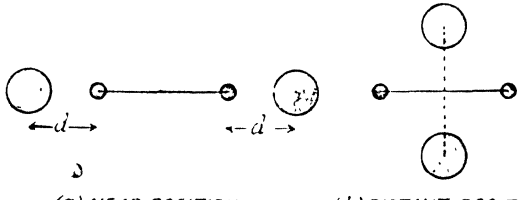


Fig. 152.

To set the system oscillating, bottles of mercury were brought near the smaller masses for a short while and then removed, when, for

*It was, in fact, the constant-temperature room of the American Bureau of Standards, 35 *ft.* below ground level.

an angular displacement of 4°, the system continued oscillating for about 20 hours.

The usual telescope, lamp and scale method, was employed to observe the oscillations, and the passages of the lines on the image of a scale across a vertical cross wire of the telescope were recorded automatically by a pen on a chronograph, another pen marking down on it the '*second*' signals from a standard clock.

From the time-periods of the suspension system, in the two cases, the value of G was then calculated out as indicated in brief outline below :

Let T_1 be the time-period of the suspension system, when the large masses are not yet brought in its neighbourhood. Then, if G be the torsional couple per unit twist of the suspension wire and I, the moment of inertia of the system, we have $T_1 = 2\pi\sqrt{I/C}$.

The large masses are then brought into the near position, shown in Fig. 152 (*a*), such that the distance between the centres of the neighbouring large and small balls is the same on either side, say, equal to d. This will obviously result in a gravitational pull F by each large mass over the corresponding small one, towards itself, given by $F = \frac{M\,m}{d^2}\,G$,

where M and m are the values of each large and small mass respectively.

Considering the gravitational pull between the neighbouring large and small masses to remain unaltered by any small displacements of the small spheres from their initial or equilibrium positions, the gravitational pull of each large mass over the small one, when the small masses A and B are deflected a little through an angle θ, into the positions A' and B', (shown in Fig. 153) will also be equal to

$$F = \frac{M.m}{d^2}.G,$$

towards itself, along the line joining the centres of the two masses. And, clearly, resolving this gravitational pull on both sides into two rectangular components, along and perpendicular to $A'B'$, we have the components perpendicular to $A'B'$ (the line joining the centres of the two small masses) equal to $F\cos\alpha$, represented by $A'C$ and $B'E$ respectively, where $\angle OA'C = \angle O'B'E = \alpha$.

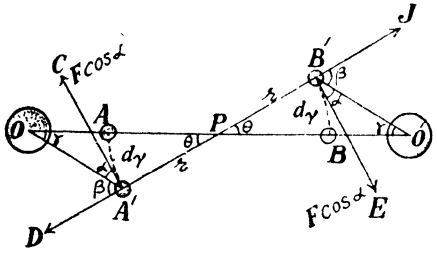


Fig. 153.

So that, we have $F\cos\alpha = \frac{M.m\,G}{d^2}\cos\alpha = \frac{M\,m.G}{d^2}\cdot\sin\beta$,

where $\beta = \angle OA'D = \angle O'B'J = (90-\alpha)$.

Now, since $\angle A'OP = \angle B'O'P = \gamma$, we have $\beta = (\theta+\gamma)$.

And, therefore, $F\cos\alpha = \frac{M.m.G}{d^2}.\sin(\theta+\gamma)$.

Or, $F\cos\alpha = \frac{M.m.G}{d^2}.(\theta+\gamma)$, ..(I)

If θ and γ be small, as they *are* in actual practice.

Now, clearly, $A'A = B'B = d\gamma = r\theta$, whence, $\gamma = r\theta/d$.

Substituting this value of γ in relation I above, we have

$$F\cos\alpha = \frac{M.m.G.}{d^2}.\left(\theta + \frac{r\theta}{d}\right) = \frac{M.m.G}{d^2}\left(1+\frac{r}{d}\right)\theta.$$

These two forces acting at A' and B' obviously form a couple, tending to bring the small balls back into their original positions A and B ; and, clearly,

moment of this couple $= \frac{M.m.G}{d^2}\cdot\left(1+\frac{r}{d}\right).\theta\times 2r.$ [where $A'B' = 2r$.

Since C is the torsional couple per unit twist of the suspension wire, the torsional couple set up in the suspension wire, also tending to bring the small balls back into their original positions A and B is equal to $C\theta$.

Thus, the *total couple acting on the suspended system of the small balls*

$$= C\theta + \frac{M.m.G}{d^2}\left(1+\frac{r}{d}\right).\theta.2r.$$

$$= \left[C + \frac{2M\,m\,G}{d^3}\left(\frac{rd+r^2}{d}\right)\right].\theta.$$

If, therefore, T_2 be the time-period of the suspension system *now*, we have

$$T_2 = 2\pi\sqrt{I\ C + \frac{2M\,m\,G}{d^3}\left(\frac{rd+r^2}{d}\right)}.$$

And, thus, $\frac{T_1^2}{T_2^2} = \left[C + \frac{2MmG(rd+r^2)}{d^3}\right]\cdot C = 1 + \frac{2MmG(rd+r^2)}{Cd^3}$.

Or, $$\frac{2MmG(rd+r^2)}{Cd^3} = \frac{T_1^2}{T_2^2} - 1 = \frac{T_1^2-T_2^2}{T_2^2}.$$

Or, $$G = \frac{T_1^2-T_2^2}{T_2^2}\cdot\frac{Cd^3}{2Mm(rd+r^2)}.$$

whence the value of G can be easily calculated.

The mean of Heyl's results (for the different small masses, mentioned above) gave the value of G to be $6{\cdot}670\times10^{-8}$ *C.G.S.* units.

Birge estimated the probable error in this result to be ·005. So that, the best value of G obtained so far is $6\ 67\pm0\ 005)\times10^{8}$ *dynes* cm^2. gm^{-2}.

86. Density of the Earth. We know that the weight of (or, the force acting on) a body of mass m, on the surface of the earth, is equal to mg, where g is the acceleration due to gravity at the place.

Also, if M be the mass of the earth and R, its radius, the force acting on the mass is, by Newton's Law of Gravitation, equal to $M\,m.G/R^2$.

So that, $$mg = \frac{M.m}{R^2}.G. \quad \text{Or,} \quad g = \frac{M}{R^2}.G.$$

Taking the Earth to be a *homogeneous sphere*, its volume $V = \frac{4}{3}\pi R^3$.

Therefore, if $\triangle$ be its density, its mass $M = 4.\pi R^3.\triangle/3$.

Hence $$g = \frac{4\pi R^3\triangle}{3R^2}.G. \quad \text{Or,} \quad g = \frac{4}{3}.\pi R.\triangle.G$$

Or, $$\triangle = \frac{3g}{4\pi.R.G.},$$

whence, the value of $\triangle$ may be easily obtained.

This gives the value of $\triangle$ to be 5 5270 *gms./c.c.*, taking the value of G to be $6{\cdot}6576\times10^{-8}$ *C.G.S.* units, (the value obtained by Boys). And, with Poynting's and Heyl's values of G, the values of $\triangle$ come respectively to 5 4934 *gms./c.c.* and $(5{\cdot}515\pm0{\cdot}004)$ *gms./c.c.*

The most probable value of $\triangle$ is, however, taken to be 5·5247 *gms./c.c.* ; and, since the density of the upper layers of the earth is found to be only 2·7 *gms./c.c.*, it follows that the density of its inner layers must be very much greater than 5·52 *gms./c.c.*

It is interesting to observe how Newton intuitively made a lucky guess at the probable density of the earth, placing it so aptly between

5 and 6,—a truly inspired '*Newtonian*' guess, which stands so amply justified to-day ! As to the reasoning that led him to it, we can do no better than to listen reverently to as he puts it himself in his celebrated *Principia* :

'But that our globe of earth is of greater density than it would be if the whole consisted of water only, I thus make out. If the whole consisted of water only, whatever was of less density than water, because of its less specific gravity, would emerge and float above. And upon this account, if a globe of terrestrial matter, covered on all sides with water, was less dense than water, it would emerge somewhere ; and the subsiding water falling back would be gathered to the opposite side. And such is the condition of our earth, which, in great measure, is covered with seas. The earth, if it were not for its greater density, would emerge from the seas, and, according to its degree of levity would be raised more or less above their surface, the water and the seas flowing backwards to the opposite side. By the same argument, the spots of the sun which float upon the lurid matter thereof, are lighter than that matter. And however the planets have been formed, while they were yet fluid masses all the heavier matter subsided to the centre. Since, therefore, the common matter of our earth on the surface thereof is about twice as heavy as water, and a little lower, in mines, is found to be three or four or even five times more heavy, it is probable that the quantity of the whole matter of the earth may be five or six times greater than if it consisted all of water, especially since I have before showed that the earth is about four times more dense than Jupiter.'

The attempts made by different workers to determine the values of G and $\triangle$ are tabulated below in chronological order, for the convenience of the student.

Year	Name of Experimenter	Types of Experiment	Value of G	Value of $\triangle$
1775	Maskelyne	Mountain method	$7{\cdot}4\times10^{-8}$ *dynes* $cm^2.gm.^{-2}$	5·0 *gms* /*c.c.*
1898	Cavendish	Torsion Balance	$6{\cdot}754\times10^{-8}$,,	5·448 ,,
1854	Airy	Mine method	$5{\cdot}7\times10^{-8}$,,	6·5 ,,
1881	Von Jolly	Sensitive Common Balance	$6{\cdot}465\times10^{-8}$,,	5·692 ,,
1891	Poynting	,, ,,	$6{\cdot}6984\times10^{-8}$,,	5 493 ,,
1895	Boys	Torsion Balance	$6{\cdot}6576\times10^{-8}$,,	5 5270 ,,
1896	Eotvos	,, ,,	$6{\cdot}66\times10^{-8}$,,	5·53 ,,
1901	Burgess	,, ,,	$6{\cdot}64\times10^{-8}$,,	5·55 ,,
1930	Heyl	Oscillation method	$6{\cdot}670\times10^{-8}$,,	5·517 ,,

87. Qualities of Gravitation. We shall now proceed to see whether the gravitational attraction between two bodies is in any way affected by the nature of the intervening medium between them, by the nature of the masses themselves or by physical conditions, like temperature etc.

1. **Permeability.** From the similarity of the formula ($F = G\,m.m'/r^2$) for gravitational attraction between two bodies, with those for magnetic and electrostatic attraction, it might appear that, like the constants μ and K there, the value of G might also depend upon the nature of the intervening medium between them.

That *this is not so* has been clearly shown by **Austin** and **Thwig**, who performed a direct experiment with a modified form of *Boys'* apparatus, in which they placed slabs of different materials in between the two attracting masses and could detect no change whatever in the value of G, within the limits of their experiment.

This stands further confirmed by (*i*) the fact that whereas in the experiments for the determination of G, discussed above, *air* is the intervening medium in the case of planets, the intervening medium is just *free space*, and yet the astronomical predictions, deduced on the basis of the same law, come out so sur

prisingly true, showing clearly that the value of G cannot possibly be very different in the two cases ; and (*ii*) the very close agreement between the values of G, obtained by different pendulum experiments, with their bases of different materials, so that different materials lie between the pendulum and the earth. We, therefore, conclude that *little or no effect is produced in the gravitational attraction between the masses by the nature of the medium interposed in-between them.*

2. **Selectivity.** The law simply states that the force of attraction between two masses depends *only upon their magnitude*, having nothing to do with their nature, or their chemical combination, etc. This is amply borne out by the large volume of experimental evidence in its favour. For, it has been shown by *Eotvos* and others, by their experiments with *Boys'* apparatus, using a large variety of materials as the attracting masses that the values of G obtained in the different cases all agree admirably, *even in the case of radio active substances*, thus showing clearly that *gravitational attraction is by no means a selective phenomenon.*

3. **Directivity.** We know that in the case of *anisotropic** substances, their physical properties, like refractive index, conductivity for heat and electricity etc. depend upon their orientation, *i.e.*, upon the direction of their crystallographic axes. **Dr. Mackenzie** and a host of other workers, therefore, tried to investigate as to whether it had any effect upon the gravitational attraction also, and they all obtained *negative* results. An additional obvious proof of the independence of the value of G of the orientation of crystals is the fact that their *weight* (which is just another name for the force of gravitational attraction between them and the earth) is exactly the same whatever the orientation, *showing that the phenomenon of gravitational attraction is far from directive.*

Poynting and **Gray** confirmed this fact in an ingenious experiment, in which two quartz crystals or spheres were suspended close to each other,—one enclosed in a case, whereas the other, outside it, being free to rotate. If there were even a trace of a directive influence in gravitational attraction, the rotation of one crystal would set up forced vibrations in the other ; so that, if their time-periods agreed, the enclosed crystal or sphere would be set into appreciably large resonant or sympathetic oscillations with the outer one. Nothing of the kind, however, was found to occur.

4. **Temperature.** **Poynting** and **Phillips,** together with a whole lot of other workers, tried to investigate the effect of temperature on the value of G, and, once again, the results obtained were absolutely negative. Only **Shaw**, experimenting with a *Boys-Cavendish* type of Torsion Balance, observed that the value of G, increased slightly with the temperature of the attracting bodies†, the value of the coefficient of increase (α) being negligibly small,—being only about $1{\cdot}6\times10^{-4}$ between 0 °C and 250°C.

All the above mass of evidence thus goes to suggest that *gravitational attraction is purely a function of the masses of the attracting bodies and of the distance between them, being quite independent of all other factors.*

No wonder, then, that Newton's theory of gravitation held such an unquestioned sway over the minds of scientists all over the world. And, for the non-scientific people in general, it had equally spectacular predictions which, when found true, could not but impress them deeply as to its unerring truth.

Thus, for example, **Adams**, in 1845, predicted, from his calculations, based on the disturbance of the orbit of *Uranus*, the presence of a hitherto unknown planet. Unfortunately, **Airy**, the then Astronomer Royal, did not as much as care to look for this planet, perhaps from sheer scepticism ; and **Challis**, the Director of the Cambridge Observatory, though he actually saw a new star, looking like a, disc did not care to verify whether it was the one predicted by

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**Anisotropic substances* are those whose properties are different in different directions,—*e.g.*, crystals, in general.

†According to **Shaw**, $G = G_0(1+\alpha t)$, where G and G_0 stand for the values of the Gravitational Constant at t°C and 0°C respectively.

Adams. A year later, **Leverrier,** a French mathematician, made similar calculations to those of Adams and communicated his results to the Berlin Observatory and, lo and behold, the planet we call *Neptune*, was there for all to see at the very spot predicted !

And, once again, in 1930, the disturbance of the orbit of Neptune itself led to the discovery, by American astronomers, of the planet *Pluto.*

These two profound discoveries put Newton's theory beyond the pale of any doubt or scepticism and it came to be looked upon as *infallible.* Indeed, it continued to enjoy its 'infallible' status until the arrival on the scene of that genius of modern times, **Albert Einstein,** who showed it to be no better than a close approximation to the actual law of gravitation propounded by him, as we shall see in the next article.

88. Law of Gravitation and the theory of Relativity. Although, as we have seen above, the Newtonian Law of Gravitation is found to be valid over a wide range and is supported by a large mass of experimental evidence, there are certain small divergences, not quite in conformity with it. But such, indeed, has been the general faith in the infallibility of the law that any divergences from it were ascribed to some hitherto undiscovered disturbing influences rather than to any possible discrepancy or flaw in the law itself. It was only after **Einstein** put forward his "*Theory of Relativity*" that it came to be realised that Newton's Law was only an approximation—although an extremely close one—to the true or the fundamental Law of Gravitation.

A detailed discussion of this theory is beyond our present scope, and we shall, therefore, consider here only one or two salient points of it, to show how Newton's formula lays itself open to criticism in the light of these :

(i) One consequence of the theory, fully confirmed experimentally, is what may be called the '*inertia of energy*', *viz.*, that wherever a change in the energy of a body is brought about, a corresponding change takes place in its mass. In other words, *energy and mass are mutually convertible, one into the other,* the relation between the two being the following :

Change in mass (in grams) = Change of energy (in ergs)$/c^2$,

where c is the velocity (in *cms./sec.*) of light in vacuo, (equal to 3×10^{10} *cms./sec.*). Further, according to this theory, we have

$$m = m_0/\sqrt{1-u^2/c^2},$$

where m is the mass of a body, moving with a velocity u, (called its *moving mass,* and m_0, its mass when at rest, (called its *rest mass*).

Thus, the mass of a body is different when in motion from that when at rest, *i.e.*, it changes with its velocity, and *Newton* has not specified which one of these is to be used in his formula for gravitational attraction,—a noticeable omission.

(ii) **Another consequence of the theory is that the numerical value of the distance between two points varies according to the system of space-time co-ordinates chosen, so that the distance bet-**

ween them changes with the circumstances of the observer, making the measurement.

The effect of these two discrepancies is, of course, only slight, but it is there, nevertheless. *Einstein* took both these factors into account and was thus led to the formulation of his famous Theory, which explains satisfactorily the deviations from the Newtonian law, among which may be mentioned the following :

(*a*) *The precessional motion of the perihelion of the orbit of the planet Mercury, with a period of* 3×10^6 *years.*

On the Newtonian theory of gravitation, it could be explained only as being due to the influence of another planet, for which the name *Vulcan* was chosen but which has never been located.

(*b*) *The deflection of a ray of light in a gravitational field.*—A ray of light, due to its great velocity, behaves like a material body, possessing mass and momentum, etc,, [see (*i*) above], and it must, therefore, suffer deflection in a gravitational field. Thus, a ray of light from a star would get deflected near the edge of the Sun, due to its high gravitational field, resulting in an apparent shift in the position of the star. Calculated on the basis of both these laws, the value of this shift, by *Einstein's theory*, comes out to be twice that by Newton's law, and actual experimental observation* fully supports the former result.

(*c*) *The shift in the spectral lines in the solar spectrum.*—Due to the Sun's gravitational field, the spectral lines in the solar spectrum must have different positions, and, therefore, different wave-lengths, from those they would have, when emitted by some terrestrial source. This has been fully verified in the year 1924,—the shift being only very slight, about one-hundredth of an *Angstrom Unit*, (or $= \frac{1}{100}th$ of 10^{-8} *cm., i.e.,* $= 10^{-10}$ *cm.*).

This has been further confirmed by *Dr. Adams*, who measured a shift of as much as half an *Angstrom Unit* in the spectrum of the dwarf companions of *Sirius*, due to the greater gravitational field at the surface of these stars than at that of the Sun.

It is thus clear that whereas the correct Law of Gravitation is that due to *Einstein*, the Newtonian Law is a sufficiently close approximation to it for our ordinary experimental purposes, except in a few rare cases, here and there.

89. Gravitational Field—Intensity of the Field. The area round about a body, whithin which its gravitational force of attraction

*This was made on the *Island of Principe* (on the African coast) and at Sobral, in Brazil, on May 29th, 1919, during a solar eclipse, (the stars being not visible otherwise). Two well-equipped expeditions at these two places obtained photographs of the portions of the sky, near the Sun, just before the eclipse and again after the eclipse, when the Sun had shifted away from its earlier position. The stars were found to have been displaced, with respect to the Sun, the respective values of this displacement at the edge of the Sun obtained by the two expeditions being 1·61″ and 1·98″. Their mean, (1·795″), agreed admirably with the value predicted by *Einstein.*

A similar observation was made again at another solar eclipse, three years later, in 1922,—this time at Wallal in West Australia,—when the shifts of as many as eighty stars were observed. The mean shift was found to be 1·74″ which was only about ·01″ short of the calculated value.

is perceptible, (no other body being near about it), is called its *gravitational field.*

The *intensity* or strength of a gravitational field at a point is defined as the *force experienced by a unit mass, placed at that point in the field.* It may also be defined as the *rate of change of gravitational potential* or *the potential gradient,* at the point, (see § 91). Thus, if I be the intensity of a gravitational field at a point, we have

$$I = -\frac{dV}{dx},$$

where dV is a small change of potential for a small distance dx.

N B. The strength or intensity of a field at a point is often spoken of merely as the *field* at that point.

90. Gravitational Potential—Potential Energy. Consider a body A, with its gravitational field around it. It will naturally attract any other body B, placed at any point in its field, in accordance with the *Law of Gravitation,* and this force of attraction will *decrease* with the *increase* in the distance of B from A, so that at an infinite distance from A it will be *zero.* But, as B is moved away from A, work has to be done against this force of mutual attraction and, therefore, the potential energy of B increases, its value depending upon the masses of A and B and their distance apart. *The work done in moving a unit mass from infinity to any point in the gravitational field of body A is called the gravitational potential of that point due to the body A,* and is an important gravitational property of that point. It is usually denoted by the letter V. Obviously, *it will also be the potential energy of the unit mass at that point, with its sign reversed,* (for whereas the *potential decreases, the energy increases,* with the increases in the distance from A). If we, therefore, replace the unit mass by the body B, the potential energy of B will, clearly, be equal to the product of its mass and the gravitational potential (with its sign reversed) at that point. Thus, *the potential energy of a body at a point in a gravitational field is equal to the product of the mass of the body and the gravitational potential (with the sign reversed) at that point.*

91. Gravitational Potential at a Point distant r from a Body of mass m. Let a mass m be situated at O, (Fig. 154), and let a *unit mass* be situated at P. Then, the force of attraction on the unit mass due to m is clearly equal to $\frac{m \times 1}{x^2} \cdot G = \frac{m}{x^2} \cdot G$,

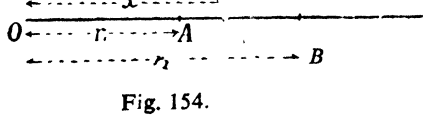


Fig. 154.

where x is the distance of P from O, the force being directed towards O. Therefore, work done when the unit mass moves through a small distance dx, towards O, is equal to $m.G.dx/x^2$.

And, therefore, work done when it moves from B to A

$$= \int_B^A \frac{m}{x^2}.G.dx = G.m \int_B^A \frac{1}{x^2}.dx.$$

$$= -Gm\left[\frac{1}{x}\right]_B^A = -Gm\left[\frac{1}{r} - \frac{1}{r_1}\right] = G.m\left[\frac{1}{r_1} - \frac{1}{r}\right],$$

where r and r_1 are the distances of A and B from O. This, obviously, is the **potential difference** between the points A and B.

If B be at infinity, *i.e.*, if $r_1 = \infty$, we have

$$\textit{potential difference between } A \textit{ and } \infty = G.m\left(\frac{1}{\infty} - \frac{1}{r}\right) = -\frac{m}{r}.G.$$

But potential difference between A and ∞ is equal to the potential at A, because the gravitational force at ∞, due to m, is equal to *zero* and, therefore, the work done in moving a mass about, at ∞, is also *zero*. In other words, the potential at infinity is *zero*. Therefore, the gravitational potential at A due to the mass m is equal to $-G.m/r$.

Or, denoting the gravitational potential at a point, distant r from a body of mass m by V, we have $V = -\frac{m}{r}.G.$

It will be noticed that whereas the value of gravitational potential at an infinite distance from a mass is *zero*, it goes on decreasing as we approach that attracting mass,—*i.e., it is an essentially negative quantity*, its maximum value being zero at infinity, where, at all points, therefore, the potential will be the same.*

92. Velocity of Escape. We know that, ordinarily, a body say, a rifle bullet, projected upwards comes down to the earth due to the gravitational pull of the earth on it. Let us see if it is possible to project it with a velocity such that it will never come back. Obviously, it will be so if it can be given a velocity that will take it beyond the gravitational field of the earth This velocity of the bullet is called the *velocity of escape*.

Thus, if m be the mass of the bullet and M, that of the earth, the force acting on the bullet at a distance x from the centre of the earth is clearly $= m.M.G./x^2$,

$\therefore$ work done by the bullet *against the gravitational field*, when the body moves a distance dx upwards $= \frac{m.M.G}{x^2}.dx.$

$\therefore$ total work done by the bullet escaping

$$= \int_R^\infty \frac{m\,M.G}{x^2}.dx = \frac{m\,M.\,G.}{R}$$

[where R = radius of the earth.

This, as will readily be seen, is m times (*i.e., the mass of the bullet times*) the gravitational potential on the surface of the earth.

If v be the *initial velocity of the bullet*, (*i.e.*, its velocity of escape), its initial kinetic energy (*i.e.*, at the time it starts) will be $\frac{1}{2}mv^2$. And this must, therefore, be equal to the work done by the bullet during escape. So that,

$$\tfrac{1}{2}mv^2 = \frac{m.M.G}{R}, \quad \text{or,} \quad v^2 = 2MG/R,$$

which, on substituting the value of M, G and R, works out to

$$v = 11{\cdot}19 \times 10^5 \textit{ cms./sec.}$$

Thus, *velocity of escape* $= 11{\cdot}19 \times 10^5$ *cms./sec.*

93. Equipotential Surface. A surface, at points of which the gravitational potential is the same, is called an *equipotential surface.*

Thus, if we imagine a hollow sphere, of radius r, with a particle of mass m at its centre, the potential at each point on it will be the

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*The *negative sign* is often omitted in writing, but it must always be understood to be there.

same, *viz.*, $-G.m/r$. The surface of the sphere is thus an equipotential surface.

Now, since the difference of potential between any two points on an equipotential surface is *zero*, no work is done against the gravitational force in moving a *unit* (or any other) mass along it. In other words, in moving a mass along an equipotential surface, we must be moving it in a direction perpendicular to the gravitational field at every point on it. Or, the direction of the gravitational field is, at every point, perpendicular to an equipotential surface, being directed towards the nearest equipotential surface, having a potential lower than it.

94. Potential at a Point Outside and Inside a Spherical Shell. (*a*) *At a point outside the shell.*—Let P be a point, distant d from the centre O of a spherical shell of *radius a*, (Fig. 155), and *surface density*, (*i.e.*, *mass per unit area* of the surface), ρ.

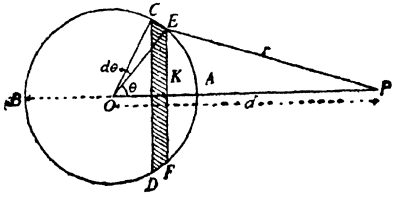


Fig. 155

Join OP and cut out a slice $CEFD$, in the form of a ring, by two planes close to each other and perpendicular to the radius OA, meeting the shell in C and D, and in E and F respectively. Let $\angle EOP = \theta$, and let the small $\angle COE = d\theta$.

Clearly, the radius of the ring is $EK = OE \sin\theta = a \sin\theta$; so that, its circumference $= 2\pi a. \sin\theta$, and its *width* $= CE = a.d\theta$.

$\therefore$ *area* of the ring or slice = its circumference $\times$ its width.

$$= 2\pi a. \sin\theta \times a.d\theta$$

And $\therefore$ its *mass* $= 2\pi a. \sin\theta \times a.d\theta \times \rho = 2\pi a^2. \sin\theta.d\theta.\rho$.

If $EP = r$, every point of the slice is at a distance r from P', and, therefore, the potential at P due to this small slice is given by

$$dV = -\frac{\text{mass of slice}}{r}.G = -\frac{2\pi a^2.\sin\theta.d\theta.\rho}{r}.G, \quad ..(i)$$ [See § 91.

Now, in the triangle OEP,

$$EP^2 = OE^2 + OP^2 - 2OE.OP.\cos\theta.$$ [Appendix I, 7 (2)

Or, $$r^2 = a^2 + d^2 - 2\,a.d.\cos\theta.$$ [$\because$ $OE = a$, the *radius* of the shell.

Differentiating the above expression, we have

$$2r.dr = 0 + 0 + 2a.d.\sin\theta.d\theta = 2a,d.\sin\theta.d\theta.$$ [a and d being constants.

Hence $$r = \frac{2a.d.\sin\theta.d\theta}{2.dr} = \frac{a.d.\sin\theta.d\theta}{dr}.$$

Substituting this value of r in expression (i) above, we have

$$dV = -\frac{2\pi a^2.\sin\theta.d\theta.\rho}{a.d\sin\theta.d\theta}.G.dr = \frac{2\pi\, a.\rho.G\, dr}{d}.$$

Integrating this between the limits, $r = AP = (d-a)$, and $r = BP = (d+a)$, we get V, the potential due to the whole shell at the point P.

Thus, $V = \int_{(d-a)}^{(d+a)} -\frac{2\pi.a.\rho.G.dr}{d} = -\frac{2\pi.a.\rho.G}{d}\int_{(d-a)}^{(d+a)} dr.$

$$= -\frac{2\pi.a.\rho.G}{d}\Big[\, r \,\Big]_{(d-a)}^{(d+a)} = -\frac{2\pi.a.\rho.G}{d}\Big[\, (d+a)-(d-a) \,\Big]$$

$$= -\frac{2\pi.a.\rho.G}{d}\,(d+a-d+a) = -\frac{2.\pi.a.\rho.G}{d}.2a,$$

i.e.,
$$V = -\frac{4\pi.a^2.\rho\, G}{d}.$$

Now, $4\pi a^2$ is the surface area of the whole shell, and, therefore, $4\pi.a^2\rho$ is equal to its mass M.

We thus, have
$$V = -\frac{M}{d}.G.$$

Or, *the potential at the point P due to the whole shell is equal to* $-M.G/d$, *i.e., the same as it would be due to a mass M at O.*

The mass of the whole shell thus behaves as though it were concentrated at its centre.

In the above case, if we imagine the point P to be at A, *i.e.*, on the circumference of the shell itself, we get the potential there by integrating the expression for dV, between the limits $r = 0$ and $r = 2a$. So that, in this case,

$$V = \int_0^{2a} \frac{2\pi.a.\rho.G}{d}\, dr = -\frac{2\pi.a.\rho.G}{d}\Big[\, r \,\Big]_0^{2a} = \frac{4\pi.a^2\rho\, G}{d}.$$

Or,
$$= -\frac{M.G}{d} = -\frac{M.G}{a} \qquad [\because \text{here, } d = a.$$

Hence
$$V = -\frac{M}{a}G.$$

(*b*) *At a point inside the shell.* Imagine now the point P to lie inside the spherical shell, (Fig. 156).

Proceeding as above, we have

potential at P due to the slice, or ring $CEFD$, *i.e.*,

$$dV = -\frac{2\pi.a.\rho.G}{d}.\, dr.$$

In this case, the limits of r are $(a-d)$ and $(a+d)$. So that, we have

$$V = \int_{(a-d)}^{(a+d)} -\frac{2\pi.a.\rho.G}{d}\cdot dr.$$

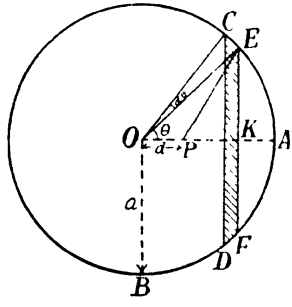


Fig. 156.

$$= -\frac{2\pi.a.\rho.G}{d}\Big[\, r \,\Big]_{(a-d)}^{(a+d)} = -\frac{2\pi.a.\rho.G}{d}\Big[\, (a+d)-(a-d) \,\Big]$$

$$= -\frac{2\pi.a.\rho.G}{d}\cdot 2d = -4\pi.a.\rho.G = -\frac{4\pi.a^2.\rho.G}{a}$$ [Multiplying and dividing by a.

Now, $4\pi.a^2.\rho = M$, the mass of the shell.

Hence $$V = -\frac{M}{a}.G,$$

i.e., the *same* as at a point *on* the shell. [See case (*a*) above]

Since the above value for V has been obtained for a point P, *anywhere* inside the shell, it follows that *the potential at all points inside a spherical shell is the same, and is numerically equal to the value of the potential on the surface of the shell itself.*

95. Gravitational Field inside a Spherical Shell or a Hollow Sphere. We have seen above that the gravitational potential at all points inside a spherical shell is the same.

Now, the field at a point is given by the *potential gradient* (*i.e.*, *the rate of change of potential with distance*), at that point.

Or, $$I = -dV/dx.$$

Since V is constant for all points inside the shell, $dV/dx = 0$, *i.e.*, *the field in the interior of the shell, due to the shell, is zero; in other words, there is no gravitational field inside a spherical shell.*

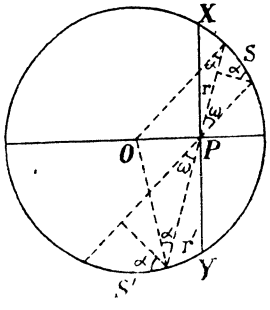


Fig. 157.

Alternative proof. Let P be any point inside a spherical shell, or a hollow sphere, of surface density, (*i.e.*, mass per unit area), ρ. Through P, draw straight lines, so as to form two small cones, with their apices at P, and intercepting small areas S and S' on the shell, as shown, (Fig. 157), on the opposite sides of the plane XY, drawn perpendicular to the diameter passing through P. Let S and S' be at distances r and r' from P, and let the *solid angles** at P be equal to ω, each.

Then, clearly, the area of the right section of the cone S is equal to $r^2.\omega$ and of that at S', equal to $r'^2.\omega$.

If, therefore, α be the angle that the right sections of the cones S and S' make with S and S' respectively, we have

$$S \cos\alpha = r^2.\omega \quad \text{and} \quad S' \cos\alpha = r'^2.\omega.$$

And $\therefore$ $$S = \frac{r^2.\omega}{\cos\alpha} \text{ and } S' = \frac{r'^2.\omega}{\cos\alpha}.$$

So that, mass of area $S = \frac{r^2.\omega.\rho}{\cos\alpha}$, and mass of area $S' = \frac{r'^2\omega.\rho}{\cos\alpha}$

$\therefore$ intensity at P due to $S = \frac{r^2.\omega\,\rho}{\cos\alpha\, r^2}.G = \frac{\omega.\rho.G}{\cos\alpha}$ in the direction PS.

And intensity at P due to $S' = \frac{r'^2.\omega.\rho}{\cos\alpha.r'^2}.G = \frac{\omega.\rho.G}{\cos\alpha}$ in the direction PS'

These two intensities at P, *being equal and opposite*, their resultant is *zero*. Similar is the case for all other pairs of cones on opposite sides of XY, into which the shell may be divided; so that, the resultant intensity or field at P due to the whole shell is *zero*. And, the same is true for any other point inside the shell. In other words, *there is no gravitational field inside a spherical shell.*

*See page 259.

96. Potential and Field Intensity due to a Solid Sphere at a point, (*i*) Inside the Sphere and (*ii*) Outside the Sphere.

(*a*) *Potential at a point P, inside the solid sphere.*—Let the point P lie inside a solid sphere of radius a, and of volume density σ, at a distance d from the centre O of the sphere, (Fig. 158),

The solid sphere may be imagined to be made up of an inner solid sphere of radius d surrounded by a number of hollow spheres, concentric with it, and with their radii ranging from d to a. The potential at P due to the solid sphere is, therefore, equal to the sum of the potentials at P due to the inner solid sphere and all such spherical shells outside it.

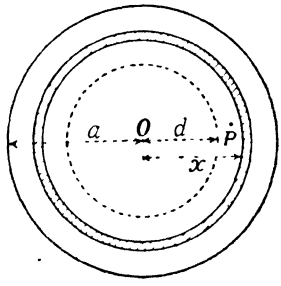


Fig. 158.

Clearly, P lies on the surface of the solid sphere of radius d, and inside all the shells of radii greater than d.

$\therefore$ potential at P due to the sphere of radius d

$$= -\frac{\text{mass of the sphere}}{d} \times G. \qquad [\S 94\ (a)$$

$$= -\frac{4}{3}\pi.d^3\sigma.G/d = -\frac{4}{3}.\pi d^2\sigma.G \quad ...(i) \quad \left[\because \text{mass of the sphere} = \tfrac{4}{3}\pi d^3.\sigma.\right.$$

To determine the potential at P due to the outer shells, imagine a shell of radius x, and thickness dx.

Clearly, its *volume* = *area* × *thickness* = $4\pi x^2.dx$,

and $\therefore$ its *mass* = $4\pi x^2.dx.\sigma$.

Now, the potential at any point within a shell is the same as at any other point on its surface.

So that, *potential at* $P = -\dfrac{4\pi.x^2.dx.\sigma.G}{x} = -4\pi\, x\, dx.\sigma.G.$

Integrating this for the limits, $x = d$ and $x = a$, we get the potential at P due to all the shells.

Thus, potential at P due to all the shells

$$= \int_d^a -4\pi.\sigma.G.x.dx = -4\pi.\sigma.G\int_d^a x.dx.$$

$$= -4\pi.\sigma.G\left[\frac{x^2}{2}\right]_d^a = -4\pi.\sigma.G\left[\frac{a^2}{2} - \frac{d^2}{2}\right].$$

$$= -4\pi.\sigma.G.\frac{(a^2-d^2)}{2}* = -\frac{4}{3}.\pi.\sigma.G.\frac{3(a^2-d^2)}{2}.$$

$$= -\frac{4}{3}\pi.\sigma.G.\frac{(3a^2-3d^2)}{2}. \qquad ...\ (ii)$$

Now, the total potential at P due to the whole solid sphere is equal to the potential at P due to the inner sphere of radius d, *plus* potential at P due to all the outer shells.

*This expression is equal to $-2\pi\,\sigma.G\,(a^2-d^2)$; so that, the potential a P due to all the outer shell $= -2\pi.\sigma.G(a^2-d^2)$, and, the potential at P due to the whole solid sphere is, therefore, also

$$= -\frac{4}{3}.\pi.d^2.\sigma.G - 2\pi.\sigma.G(a^2-d^2) = -2\pi.\sigma.G\,(\tfrac{2}{3}d^2 + a^2 - d^2):$$

$$= -2.\pi\sigma.G\left(a^2 + \frac{2d^2-3d^2}{3}\right) = -2\pi.\sigma.G\left(a^2 - \frac{d^2}{3}\right).$$

$\therefore$ *total potential at P*

$$= -\frac{4}{3}\pi.d^2.\sigma.G - \frac{4}{3}\pi.\sigma.G.\frac{(3a^2-3d^2)}{2}.$$

$$= -\frac{4}{3}\pi.\sigma.G.\left(d^2+\frac{3a^2}{3}-\frac{3d^2}{2}\right) = -\frac{4}{3}\pi.\sigma.G\left(\frac{2d^2+3a^2-3d^2}{2}\right).$$

$$= -\frac{4}{3}\pi.\sigma.G.\left(\frac{3a^2-d^2}{2}\right) = -\frac{4}{3}\pi.a^3.\sigma\,G\frac{(3a^2-d^2)}{2a^3}.$$ [Multiplying and dividing by a^2.

But $\frac{4}{3}.\pi.a^3.\sigma$ is the mass of the sphere $= M$.

$\therefore$ *potential at P due to the sphere* $= MG.\frac{(3a^2-d^2)}{2a^3}$.

(*b*) *Potential at a point P, outside a solid sphere.* Imagine the sphere to be broken up into a number of thin spherical shells, concentric with the sphere, and of masses m_1, m_2, m_3, etc., (Fig. 159). Then, as we have seen before, the potential at P due to each spherical shell will be equal to its *mas* $\times$ G/d, where d is the distance of the point P from the centre of the sphere—*i.e.*, the same as though the mass of each shell were concentrated at its centre O. So that, the potential at P due to the different shells will be $-m_1.G/d$, $-m_2.G/d$, $-m_3.G/d$, and so on.

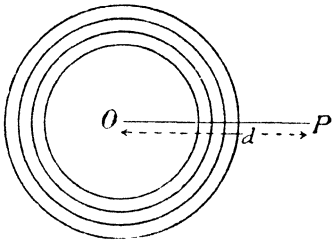


Fig. 159

Therefore, the potential at P due to all such shells, *i.e.*, due to the whole solid sphere

$$= -\left[\frac{m_1}{d}.G + \frac{m_2}{d}.G + \frac{m_3}{d}.G + \ldots\ldots\ldots\right]$$

$$= -(m_1+m_2+m_3+\ldots\ldots\ldots).\frac{G}{d} = -\frac{M}{d}.G,$$

where $m_1+m_2+m_3+\ldots = M$, the mass of whole solid sphere.

Hence *potential at P due to the solid sphere* $= -\frac{M}{d}.G$.

(*c*) *Gravitational field due to a solid sphere at a point inside the sphere.* We have seen that the potential at a point inside a solid sphere, distant x from the centre of the sphere, is given by

$$V = -M.G.\frac{(3a^2-x^2)}{2a^3}.$$ [See above.]

Now, intensity, or gravitational field, at a point is equal to the potential gradient, (or the rate of change of potential with distance), at the point.

Therefore, intensity or gravitational field at a point distant x from the centre $= -\frac{dV}{dx} = \frac{d}{dx}\left[M.G.\frac{(3a^2-x^2)}{2a^3}\right]$

$$= -M.G.\frac{2x}{2a^3} = -\frac{M.G.x}{a^3}.$$

---

* Taking the value of V at distance x from O to be $-2\pi\sigma G.\left(a^2-\frac{x^2}{3}\right)$, (see foot-note, page 255), we have intensity or field at distance x from O

$$= -\frac{dV}{dx} = \frac{d}{dx}\left[2\pi.\sigma.G\left(a^2-\frac{x^2}{3}\right)\right] = -2\pi\sigma.G.\frac{2x}{3} = -\frac{4}{3}\pi\sigma.G.x.$$

This shows that the force decreases as x decreases, being zero at the centre of the sphere. The negative sign only indicates the attractive nature of the field.

(*d*) *Gravitational field due to a solid sphere at a point outside the sphere.* We know that the potential at a point outside a sphere, distant x from the centre of the sphere is $= -\frac{M}{x} \cdot G$, as explained above in § 96 (*b*), (page 256).

And, since intensity or gravitational field at a distance x is equal to the potential gradient at x, we have

$$\text{gravitational field at distance } x = \frac{d}{dx}\left[\frac{M.G}{x}\right] = \frac{d}{dx}\left[M.G.\ (x^{-1})\right]$$

$$= M.G. \times (-1 \times x^{-2}) = -\frac{M.G}{x^2}. \quad \text{Or,} \quad = -\frac{MG}{x^2}.$$

Alternatively, we may get the same result by applying, in this case also, the assumption, found valid in the case of potential outside a sphere, *viz.*, that *the potential is the same as though the whole mass of the sphere were concentrated at the centre of the sphere* ; so that gravitational intensity or field at a point, distant x from the centre, and *outside* the sphere $= -M.G/x^2$, the negative sign merely indicating, as before, the nature of the field.

N B. This result is of great historical importance in that it enabled Newton to apply his law of gravitation to the motion of the moon. For, the radius of the earth not being negligible compared with that of the moon's orbit around it, there would have been no means, in the absence of the above result, of determining what correction terms, if any, would be necessary in the equation of the moon's motion, in view of the distribution of matter inside the earth and the finite value of its radius.

97. Intensity and Potential of the Gravitational Field at a Point due to a Circular Disc. Let MN, (Fig. 160), represent a circular disc of radius R, with its plane perpendicular to the plane of the paper and let O be a point on its axis, distant x from its centre C, where the intensity and potential due to its gravitational field are to be determined.

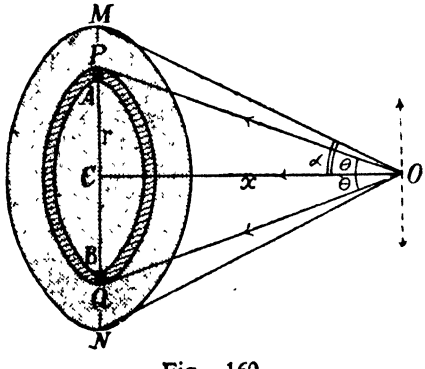


Fig. 160.

Imagine the disc to consist of an infinite number of concentric rings, with C as their common centre.

Consider one such ring PQ of radius r and thickness dr.

Join O to the extremity A of the radius CA of the ring. Then, if angle AOC be equal to θ, we have

$$r = CA = x \tan \theta.$$

And, therefore, differentiating it with respect to θ, we have

$$\frac{dr}{d\theta} = x.\sec^2\theta. \quad \text{Or,} \quad dr = x.\sec^2\theta.d\theta.$$

And, $AO = x.\sec\theta.$

Now, *area of the ring* = *circumference* × *thickness* $= 2\pi r.dr$,
And, therefore, *mass of the ring* $= 2\pi r.dr.\rho$,
where ρ is the *mass per unit area* of the disc.

Considering a very small element A of the ring, we have intensity at O due to *this* element = *mass of the element* $\times G/AO^2$, along the direction OA.

This can be resolved into two rectangular components, (*i*) along OC and (*ii*) at right angles to it. So that,

the former component $= \frac{\text{mass of the element}}{AO^2}.G\cos\theta$, *along* OC,

and, the latter component $= \frac{\text{mass of the element}}{AO^2}.G\sin\theta$, *vertically upwards.*

Similarly, for an equal element at B, *diametrically opposite to* A, we have

intensity at O, resolved into the same two components, *viz.*,

(*i*) $\frac{\text{mass of the element}}{AO^2}.G\cos\theta$, *along* OC ; *and*

(*ii*) $\frac{\text{mass of the element}}{AO^2}.G\sin\theta$, *vertically downwards*, as shown.

Thus, the two vertical components, *being equal and opposite*, cancel out, and the components along OC alone being *effective*, are added up, *both acting in the same direction* OC ; the same will be the case with other elements into which the ring may be supposed to be broken up ; and, therefore, the intensity at O due to the whole ring is equal to the sum of the components along OC due to the different elements, *i.e.*,

$$= \frac{\text{mass of the ring}}{AO^2}.G\cos\theta.$$

Or, intensity at O due to the ring $= \frac{2\pi r.dr.\rho}{AO^2}.G\cos\theta.$

Or, substituting the values of r, dr, and AO, we have

intensity at O due to the ring $= \frac{2\pi\rho\, x\tan\theta.x\sec^2\theta.d\theta}{x^2.\sec^2\theta}.\cos\theta.G.$

$$= 2\pi\rho.G.\sin\theta.d\theta, \text{ along } OC.$$

Therefore, intensity at O due to all the rings into which the disc is supposed to be divided up, *i.e.*, intensity at O due to the whole disc, is obtained by integrating the above expression for the intensity due to the ring, between the limits, $r = 0$ and $r = R$, or $\theta = 0$ and $\theta = \alpha$, where α is the angle between OC and the line joining O with the extremity M of the radius CM of the disc.

Thus, intensity at O due to the disc is given by the expression,

$$\int_0^\alpha 2\pi.\rho.G\sin\theta.d\theta = 2\pi.\rho.G\int_0^\alpha \sin\theta.d\theta,$$

$$= 2\pi.\rho.G\Big[-\cos\theta\Big]_0^{\alpha} = 2\pi.\rho.G.\Big[-\cos\alpha-(-\cos 0)\Big]$$

$$= 2\pi.\rho.G.(-\cos\alpha+1)$$

Or, *intensity* at O, *due to the disc* $= 2\pi.\rho.G.(1-\cos\alpha)$. ... ... (*i*)

And, since $\cos\alpha = \frac{CO}{MO} = \frac{x}{\sqrt{x^2+R^2}}$, we have

intensity at O, *due to the disc* $= 2\pi\rho.G.\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$. ... (*ii*)

Or, again, because $2\pi(1-\cos\alpha)$ is the *solid angle*, ω, say, (see *Note on solid angle* below), subtended by the disc at the point O, we have, from relation (*i*), above,

intensity at O, *due to the disc* $= \rho.G.\omega$. ... ... (*iii*)

Now, potential at P due to the ring of radius r is equal to the intensity at P due to the ring $\times x = 2\pi\rho.G\sin\theta.d\theta \times x$, because field at P is equal to the *potential gradient* at P.

Hence, potential at P due to the whole disc is

$$= \int_0^{\alpha} 2\pi.\rho.G.x\ \sin\theta.d\theta = 2\pi.\rho.G.x\int_0^{\alpha} \sin\theta.d\theta,$$

Or, potential at P due to the disc

$$= -2\pi\rho.G.x(1-\cos\alpha). \quad \ldots \quad \ldots \ (iv)$$

Or,
$$= -2\pi\rho.G.x\left(1-\frac{x}{\sqrt{x^2+R^2}}\right) \quad \ldots \quad \ldots \ (v)$$

Or,
$$= -\rho.G.\omega.x,$$

to give it its proper negative sign.

Note on Solid Angle. Suppose, we have an area *PS*, (Fig. 161), as the base of a cone, with its apex at point O. Then, if we draw a sphere, with centre O and any radius R, so that a surface of area pq of it is cut off by the cone, then pq is proportional to r^2, where r is the radius of the spherical surface pq ; and, therefore, *area* pq/r^2 is constant for any given cone. This quantity, *area* pq/r^2, is called the *solid angle* of the cone, or the *solid angle* subtended by the *area* pq at O, and is usually denoted by the letter ω. Obviously, it is also equal to *area* PS/OP^2, or the solid angle subtended by *area* PS at O, and its numerical value is equal to *area* pq, if $r = 1$ *cm.*

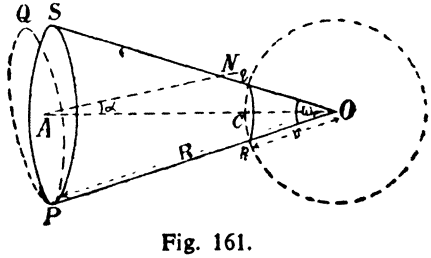


Fig. 161.

Now, suppose the given *area* be PQ and not PS. Then, if AN be the normal to it at its centre A, we have *area* PS = *area* $PQ\cos\alpha$, where α is the angle between AO and the normal AN to the surface PQ at A.

Thus,
$$\omega = \frac{\text{area } PQ.\cos\alpha}{OP^2}.$$

Now, to determine the solid angle ω, subtended by a circular shell or disc MN*, of radius R, at a point O, distant x from its centre C, (Fig. 162), we draw a

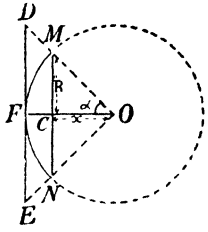


Fig. 162.

sphere, with O as its centre and $(R^2+x^2)^{\frac{1}{2}}$ as its radius, such that its circular face lies on this sphere. Then, we have, (from above),

$$\omega = \frac{\textit{surface area of slice } MFN}{(R^2+x^2)}.$$

Since the area of a sphere, lying between two parallel planes, is equal to the area of the circumscribing cylinder in between these planes, with its axis perpendicular to them, we have

$$\textit{area } MFN = 2\pi(R^2+x^2)^{\frac{1}{2}} \times FC = 2\pi\,(R^2+x^2)^{\frac{1}{2}} \times (FO-CO).$$

$$= 2\pi(R^2+x^2)^{\frac{1}{2}} \times (MO-x). \qquad [\because FO = MO, \text{ and } CO = x,$$

$$= 2\pi(R^2+x^2)^{\frac{1}{2}} \times [(R^2+x^2)^{\frac{1}{2}} - x)]$$

Hence $$\omega = \frac{\pi(R^2+x^2)^{\frac{1}{2}} \times [(R^2+x^2)^{\frac{1}{2}} - x)]}{R^2+x^2} = 2\pi\left(1 - \frac{x}{(R^2+x^2)^{\frac{1}{2}}}\right).$$

Or, $$\omega = 2\pi(1 - \cos\alpha),$$

where α is the *semi-vertical* angle, subtended by the shell or disc at O.

98. Intensity and Potential of the Gravitational Field at a Point due to an Infinite Plane. In the case of the disc, above, (§ 97), if its radius R becomes infinite, the disc becomes an infinite plane. In this case, obviously, a becomes $\pi/2$, so that $\cos\alpha = 0$, and $\omega = 2\pi$.

Thus, if we put $R = \infty$ in expression (*ii*), or $\cos\alpha = 0$, in expression (*i*), or $\omega = 2\pi$ in expression (*iii*) above, we have

$$\textit{intensity at } O \textit{ due to an infinite plane} = 2\pi\rho G.$$

which is, clearly, quite independent of the distance x from it.

Similarly, putting these values of R, $\cos\alpha$ and ω in relations, (*iv*), (*v*) and (*vi*) above, we have

$$\textit{potential at } O \textit{ due to an infinite plane} = 2\pi\rho.G.x.$$

99. Inertial and Gravitational Mass. We ordinarily define the mass of a body *by the acceleration produced in it by a known force.* This is known as its *inertial mass.*

But since, as we have seen, the gravitational field due to body is proportional to its mass, it is also possible to define the mass of a body as proportional to *the gravitational force of attraction it exerts on a standard test body at unit distance away from it.* Thus defined, the mass of the body is called its *gravitational mass.*

Now, *Galileo* showed that the acceleration of a falling body was quite independent of its mass and the same is found to be true in the case of pendulums used for the determination of the value of g, showing that the gravitational force between a given mass and the earth is proportional to the inertia of the mass.

There appears to be no *a priori* reason, however, why this should be so; for, in the case of an inclined plane, for example, we have seen how the acceleration of a body rolling down the plane depends not only on the mass of the body but also on the distribution of its mass, (§ 39, page 87). The above may thus be regarded to be only an experimental law.

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*MN, here, represents the side-view of the shell or disc.

It is confidently claimed by some, but equally hotly disputed by others, that the equality of the inertial and gravitational mass can be '*predicted*' from the general theory of relativity, so that nothing very definite can yet be said on the point. Their equality*, however, is of great consequence in astronomy. For example, due to the proportionality of gravitational force to inertial mass, *the orbit of a satellite round a planet is quite independent of its mass*, and we can thus '*weigh*', the planet from a mere observation of the orbit of its satellite. And, again, we can determine the mass of *one component* of a *double star*, by observing the orbit of the other round their common centre of gravity,—the required value of G being obtained from terrestrial experiments, dealt with above.

100. Earthquakes—Seismic Waves and Seismograph. An earthquake is caused by a portion of the rigid crust† of the earth giving way or getting fractured, some distance below its surface and the consequent sudden slipping of the resulting portion, or due to '*fault slipping*', as it technically called. So to speak, it is just a *landslide* on a large scale, or a re-adjustment of the earth's crust, in response to a change of forces, or more precisely, to changes of pressure deep in the earth's crust, down to a distance of 100 *miles* or so, brought about by a variety of causes like erosion, deposition, tidal forces, centrifugal forces, etc etc. An earthquake thus represents the energy released by this 'relative motion of portions of the earth's crust'.

The place where the actual fracture occurs is called the **focus** of the earthquake, and it not a geometrical point, but an extended region. The point nearest to the focus, on the surface o the earth, is called the **epicentre.**

From the focus, (which we may, for our purposes here, regard as just a point), originate a number of different types of waves, collectively called **seismic waves**, which spread on to different points on the surface of the earth and which we *feel* as **'earthquake tremors'**. The general pattern of these seismic waves is as shown in Fig. 163, and they consist of the following different types of waves :

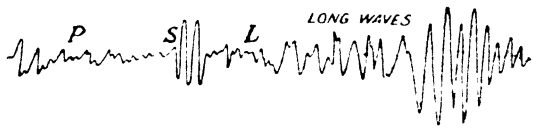


Fig. 163.

(*a*) **The Primary or *P* Waves.** The first to arrive at the Observing Station, these are *longitudinal waves*, in which the particles of the earth vibrate about their mean position, along the direction of the waves themselves.

If the earth be regarded to be a homogeneous sphere, these waves, starting from the focus, travel along the chord of a huge circle of the earth, with a velocity equal to $\sqrt{j/\triangle}$, where j is what is called the '*elongational elasticity*'‡ of the earth and $\triangle$, its density. These waves are also variously called as '*condensational*', '*irrotational*' and '*push*' waves and their velocity is found to be about 5 *miles per second.*

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*Since, as we have seen, they are proportional to each other, a proper choice of units can make them equal.

†See foot note on page 230.

‡The *elongational elasticity* $j = \gamma(1-\sigma)/(1+\sigma)(1-2\sigma)$, where γ and σ stand for *Young's modulus* and *Poisson's ratio* respectively.

(*b*) **The Secondary or *S* Waves.** These are *transverse waves*, in which the particles of the earth vibrate at right angles to the direction of propagation of the waves, thus having no component along this direction. Starting from the focus, these waves also travel along a chord of a huge circle of the earth and are the next to arrive at the Observing Station, with a velocity equal to $\sqrt{n/\triangle}$, where n and $\triangle$ represent the modulus of rigidity and the density of the earth respectively. The other names given to these waves are *'distortional'*, *'equivoluminal'* and *'shake'* waves, their velocity being about 3 *miles per second.*

(*c*) **Rayleigh Waves.** Discovered by *Lord Rayleigh*, these waves are found to remain confined to a comparatively thin layer in the close vicinity of the earth's surface. Unlike the *P* and *S* waves, they start from the epicentre and arrive at the Observing Station, along a huge circle of the earth,—the displacement of the particles at any point on the earth's surface, due to them, being in the vertical plane containing their direction of propagation. Resolving this displacement, we have (*i*) *a vertical component* and (*ii*) a *horizontal component, along the direction of propagation, there being no horizontal component at right angles to it.* These waves thus persist over long distances along the surface of the earth, and are almost unique in this respect. If the earth were a homogeneous sphere, these waves also would travel with a constant velocity, but, due to its heterogeneous character, each single wave, starting from the epicentre, gets split up into a number of different sets of waves, each set having a different wave-length, velocity etc. ; so that, what we receive at Observing Station is a series of oscillations, instead of one single *'kick'* or *'throw'* as would be the case if there were no such splitting up of the original wave, *i.e.*, if the earth were really homogeneous in composition.

(*d*) **Love Waves.** The heterogeneity of the layers of the earth is responsible for yet another type of surface waves, known as *Love Waves*, in which the displacement of the earth is horizontal, but transverse to the direction of their progapation. The velocity of these waves is less in the earth's crust than in the matter below. Immediately after an earthquake, oscillations, corresponding to these waves, can be detected at almost any place on the surface of the earth.

Unlike *P* and *S* waves, which are separately and distinctly received and recorded at the Observing Station, these waves get intermingled with Rayleigh waves to form a somewhat complicated system of waves, (not yet properly understood), called *long* or *L* waves, or the *main shock*, registering themselves as a long series of oscillations.

101. Seismology. The study of the seismic waves constitutes what is called the science of *Seismology*, and it owes a great deal to **Prof. John Milne,** who did almost the whole of the initial pioneering work on the subject. As early as the year 1883, when he was residing in Japan, he predicted that *"every large earthquake might*

be, with proper appliances, recorded at any point on the land surface of the globe". And, then, in the year 1889, a curious incident confirmed his prophetic words. For, a delicate horizontal pendulum, set up for the measurement of the gravitation-action of the moon, gave recordings, which turned out to be due to an earthquake, with its origin somewhere in Japan. This started a new era of intensive researches on the subject, with Prof. Milne in the very forefront ; and, in 1895, he set up his own observatory at Shide, in the Isle of Wight, which became the centre of a world-wide seismic survey.

By the year 1901, the main facts as to how the tremors travelled through and round the earth were fully established, again, due in main, to the labours of the eminent Professor himself.

His reports to the British Association on Earthquake Phenomena in Japan from 1881 to 1895, together with those on Seismological Investigation from 1895 to 1913, (the year of his death), form a fascinating and a detailed study of the growth and development of the present-day science of Seismology.

102. Seismographs. A *seismograph* (or a *seismometer*), is an instrument used to record the *earth tremors* or the *seismic waves*, to some dynamical function of which, (like displacement, velocity, acceleration, etc.), they respond or react. The record of the vibrations so obtained is called the *seismogram*. The instruments, responding to displacement, are of the mechanical type and we are, therefore, concerned here only with these. The following is, in brief, the theory underlying the mechanical type of seismographs.

All vibrations of the earth may ultimately be resolved into (*i*) *vertical* and (*ii*) *horizontal* components*. The problem thus reduces itself to merely recording these vertical and horizontal vibrations. We shall confine our attention here only to the measurement of the *horizontal displacements*, accompanying these latter vibrations. There are two types of instruments in use for the purpose, *viz.*, (*a*) the *vertical pendulum* and (*b*) the *horizontal pendulum type*.

(*a*) **The Vertical Pendulum Seismographs.** A vertical pendulum is just a rigid body, suspended from a stand resting firmly on the ground ; so that, with the horizontal displacement of the ground and the stand with it, the point of support of the pendulum also gets displaced horizontally.

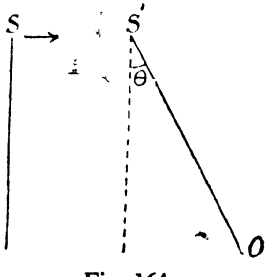


Fig. 164.

Thus, if the point of support S of the vertical pendulum, (Fig. 164), is displaced horizontally to S', due to the horizontal displacement of the ground, it can be shown that a style or pen, attached to its lower end, reproduces faithfully the movements of the support, with precisely the same frequency, (though on a different scale), it being assumed that the support moves with a definite frequency and amplitude.

*These components may be along East and West or along North and South but will be *horizontal*, nevertheless. These can also be used to measure the horizontal velocity and acceleration of the earth, or rather of the earth's crust.

These vertical pendulum seismographs, however, suffer from two defects, *viz.*, (*i*) they have to be very heavy,—as much as 20 *tons* or more,—if a good magnification of the vibrations be desired, and (*ii*) their period of vibration is rather small.

(*b*) **The Horizontal Pendulum Seismographs.** We are already familiar with the horizontal pendulum, [see § 78 (*ii*), page 214]*. Only some slight additions to it convert it into a sensitive and a reliable seismograph. With the horizontal movement of the earth, the supports of the pendulum, which are firmly fixed on to it, also share its movement, thus setting its stem or '*boom*' into motion, which can then be magnified mechanically or electrically by various devices.

The best known seismograph of its class is that due to **Prince Boris Galitzin,** in which the greatest care has been taken to see that its indications correspond exactly to the actual movements or vibrations of the earth. We shall, therefore, discuss in some detail only this one instrument here.

103. Galitzin's Seismograph. This seismograph measures the horizontal velocity of the earth's crust, and consists of a horizontal pendulum, having a *boom* or stem, 28 *cms.* long, carrying a cylindrical brass bob, weighing 7 *k.gms.* and having its centre at a distance of 14 *cms.* from the inner end of the *boom.* The suspension of the pendulum is of the *Zollner type,* (as shown in Fig. 135, page 214), with a very small inclination of the axes, so that the period of oscillation of the pendulum is about 24 *seconds.* The whole pendulum is built up on a rigid frame-work, firmly secured to the ground, and consisting of four metal pillars, braced together, and arranged rectangularly on four points on a metal base or plate, provided with levelling screws.

The recording of the vibrations or *tremors* at the Observing Station is done electromagnetically, and, for this purpose, a flat copper coil is wrapped round a portion of the stem or *boom* of the pendulum, extending beyond its cylindrical bob, and connected to a sensitive moving-coil mirror galvanometer.

With the motion of the stem, (caused by the motion of the ground), the coil moves in the strong magnetic field of a pair of permanent horse-shoe magnets of tungsten-steel. *A current, which is proportional to the angular velocity of the stem,* is thus induced in the coil and produces a deflection in the galvanometer.

A beam of light, reflected from the mirror fixed on to the suspension of the galvanometer coil, is passed through a semi-cylindrical lens and allowed to fall on a sensitized (*i.e.*, photographic) paper, wrapped round a rotating drum, worked by a clock-work arrangement and moving uniformly along its axis, with a peripheral speed of 3 *cms. per minute.* Time-signals are also similarly recorded on the paper by cutting off, by means of an accurately-timed shutter, the beam of light for two seconds at the beginning of each successive minute. A permanent record of a series of curves, (*i.e.*, the *seismo-*

*The student would do well to refresh his memory by going over this article once again before proceeding further.

gram), is thus obtained on the sensitized paper, from which the time of occurrence of any seismic phenomenon can be determined accurately to within *one* second.

In order that the horizontal displacement of the earth may be correctly calculated from the seismogram thus obtained, it must faithfully correspond to the movements of the earth. To achieve this, damping of both the pendulum and the galvanometer is necessary. Or, else, if the period of oscillation of the pendulum agrees, or nearly agrees, with that of the seismic wave, resonance will occur, producing large deflections, which would give an utterly deceptive picture of the actual movement of the ground. And, if the damping be made *critical*, (*i.e.*, *dead-beat*), the calculations become greatly simplified.

This damping is produced by attaching to the outer end of the boom, a horizontal brass plate, which moves in another strong magnetic field, due to a separate pair of horse-shoe magnets, arranged above and below it. The eddy currents, thus induced in the plate, then produce, with proper adjustments, the desired damping effect on the pendulum.

This seismograph has the additional advantage of great magnification*, as also of enabling the recording apparatus to be arranged in a separate compartment, away from the pendulum.

N.B. It will be readily understood that for a large or severe earthquake, less sensitive seismographs are more suitable, while, for smaller, local or nearby earthquakes, the more sensitive ones or the short period ones, are the more desirable.

104. Determination of the Epicentre and the Focus.

The Epicentre. To determine the epicentre of an earthquake, we determine what are called the *epicentral distances* of it from a network of Observing Stations or Observatories,—*the epicentral distance of an earthquake from a given station being the shortest distance of its epicentre from the station, measured along the surface of the earth, in terms of the angle it subtends at the centre of the earth.* This is done with the help of the *Tables*, compiled by *Zoppritz*, *Turner* and others, which give the relation between the epicentral distances of past earthquakes and the interval between the first arrivals of the Primary (*P*) and the Secondary (*S*) waves at a station, *i.e.*, which express the epicentral distances as functions of the corresponding time-intervals *S—P*.

Thus, from the seismogram of an earthquake, obtained at an Observing Station, we can determine the time-interval *S—P* for it at that station, and the *Tables* then give the epicentral distance of the earthquake from it. This is done at as many stations as possible. Circles are then drawn on a globe, with these different stations as their respective centres and their epicentral distances as the radii. The point of intersection of these circles then gives the most probable position of the epicentre of the earthquake in question. Or, the same may be obtained from the method of least squares.

*Out of a set of 8 seismographs at the Fordham University in New York city, there are three which magnify the motion of the ground about 2,000 times.

Usually, three seismographs are used for the purpose, one responding to motions of the ground along the East-West direction, the other along the North-South direction, and the third, having a pendulum suspended by a coiled spring, to respond to the vertical displacements of the ground. The information supplied by the three, when pieced together, enables not only the epicentre, but also the character, of the earthquake to be determined fairly accurately.

(*ii*) *The focus.* To determine the position of the focus, immediately below the epicentre, we use what is known as **Seebach's method**, explained below :

Let F, (Fig. 165), be the focus of an earthquake, (taken to be a point here), a distance h vertically below the epicentre E, and let O be the position of the Observing Station, a horizontal distance d from the epicentre.

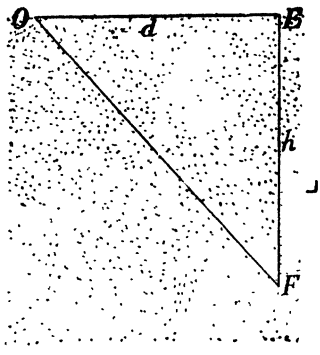


Fig. 165.

Then, assuming the homogeneity of the medium in-between the earth's surface and the surface of a sphere, concentric with it and passing through F, the time t taken by the P waves to travel from F to O is clearly given by

$$t = FO/v = \sqrt{d^2+h^2}/v,$$

whence, $(d^2+h^2) = v^2.t^2$, ...(*i*)

where v is the velocity of P waves in the medium and can be determined independently by other methods.

Now, if the time at which the earthquake occurs be τ_0*, and the time at which the first P waves arrive at O be τ, we have $t = (\tau-\tau_0)$, where τ naturally varies with distance d.

Thus, relation (*i*) above may be put in the form,

$$(d^2+h^2) = v^2(\tau-\tau_0)^2.$$

Obtaining the corresponding values of d and t from a number of different Observing Stations, we plot a graph between d and vt, which gives a hyperbola, from which h can be easily calculated out, and hence the position of the focus determined.

A better method, however, is to calculate, by the method of least squares, the most probable values of h and τ_0.

N.B. It may be of interest to know that the severest earthquakes have their foci about a hundred kilometres below the earth's surface.

105. Modern Applications of Seismology. The development of the modern science of seismology has led to its application in four important fields, *viz.*, (*i*) *investigation of the nature of the interior of the earth,* (*ii*) *prospecting for oils and minerals,* (*iii*) *construction of quake-proof buildings, and* (*iv*) *forecasting of the occurrence of earthquakes.*

(*i*) It is now almost fully established (according to *Jeffreys*) that the earth consists of a *dense core* of a molten mass, mostly of

*This, though not known to us, is certainly constant.

iron, together with some nickel, of a density of about 12·0 *gms./c.c.*, (at the pressure existing there), surrounded by a *solid outer shell or crust*, about 3,000 kilometres thick, the density of which decreases from about 5·0 *gms./c.c.* at its innermost layers to about 2·7 *gms./c.c.* at the outermost layers, or at the surface of the earth.

The existence of the dense core is deduced from the observed refraction of the seismic waves, as they pass through the earth, and is further confirmed by the production and propagation of the secondary or shake waves (*S*) through the core. These waves, as we know, are *transverse* in nature and, as such, can only be produced and propagated in media, possessing elasticity of shape or rigidity, *viz.*, in solids.

(*ii*) Prospecting for oil, coal and other minerals is now being increasingly done with the help of seismic waves*, the process being technically known as '*seismic prospecting*'.

Artificial earthquakes as set up in the ground-region to be surveyed for the purpose, by detonating an explosive, like gun-cotton

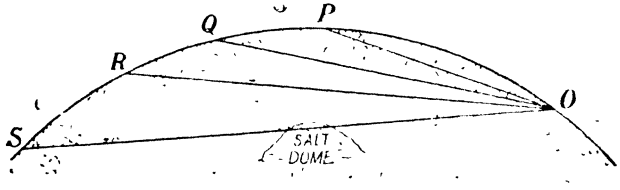


Fig. 166.

or gelignite, at a point *O* on the earth's surface, (Fig. 166), and the time of explosion noted. The time of arrival of the first low frequency longitudinal waves, or the primary waves, thus produced, is noted, with the help of seismographs, at different stations *P*, *Q*, *R*, *S*, etc., all lying in the same plane. The distances from *O*, covered along the chords *OP*, *OQ*, *OR*, *OS* etc. of the earth are carefully measured and the mean velocities of the waves calculated along these different paths or chords.

If one of the paths or chords, say, *OS*, happens to pass through a mineral deposit, like a salt dome, the value of the mean velocity along this particular chord will be different from that along the other chords. The experiment is then repeated along a direction, perpendicular to the first, by exploding a fresh charge of explosives. And, if this confirms the results of the first experiment, a more elaborate survey determines the positions of the top and the sides† of the salt dome.

(*iii*) It has now been found possible to erect '*quake-proof*' buildings in California, Japan and other places, frequently visited by earthquakes, at a surprisingly low additional cost of just 15%. For, it has been shown by *Prof. Suyehiro* that the severest earthquakes of

*We have already studied the gravitational methods of prospecting, by means of the Eotvos balance etc. (sees § 80, on page 216).

† The locating of the sides of the dome is equally important, because some mineral oil is almost always found to be there.

Japan can do but little damage to buildings, *designed to resist a horizontal force, equal to one-tenth of their total weight.* The day is thus not far off when damage to buildings due to earthquakes will just become a memory of a dreadful past.

(*iv*) And, finally, the prediction of the occurrence of an earthquake, a good time in advance, is also fast coming into the realm of practical possibility. For, it has now been established that the region, where an earthquake occurs, exhibits, for quite a few years before, a *'tilt'*, or a *gradual rise*, very much like the rubber tube of a pneumatic tyre or a football bladder swelling up before it actually bursts.

There seems to be but little doubt that much sooner than we can imagine at the moment, an earthquake forecast will become as general and universal an affair as the weather forecast is today.

But even as it is, the loss in buildings etc., due to the severest earthquakes, seldom exceeds 5%, due to their being confined to a very small area, and, quite often, an uninhabited one. The disastrous effects of earthquakes have thus been unduly magnified ; and, for all we know, they may be for our own good, designed by a benign Providence, by way of safety devices to save us from being blown up, all in a heap.

SOLVED EXAMPLES

1. Given $G = 6{\cdot}7\times10^{-8}$ c.g.s. units, the radius of the earth $= 6{\cdot}4\times10^{8}$ cms. and its mean density, 5·5 gms./c.c., calculate the acceleration due to gravity at the earth's surface.

Imagining the earth to be a perfect sphere, we have

volume of the earth $= \frac{4}{3}.\pi.(6{\cdot}4\times10^{8})^{3}$ *c.cs.*,

And $\therefore$ mass „ „ „ $= \frac{4}{3}.\pi.(6{\cdot}4\times10^{8})^{3}\times5{\cdot}5$ *gms.*

Consider a mass *m gms.* on the surface of the earth. Obviously, the force with which it is being attracted by the earth towards its centre is, according to the *Law of Gravitation*,

$$= \frac{m.\frac{4}{3}.\pi.(6{\cdot}4\times10^{8})^{3}\times5{\cdot}5}{(6{\cdot}4\times10^{8})^{2}}.G. = \frac{4}{3}m.\pi.(6{\cdot}4\times10^{8})\times5{\cdot}5\times6{\cdot}7\times10^{-8}.$$

$$= \frac{4}{3}m.\pi\times6{\cdot}4\times5{\cdot}5\times6{\cdot}7 \text{ dynes.}$$

$\therefore$ this must be equal to the *weight* of the mass, *i.e.*, mg.

Thus, $mg = \frac{4}{3}.\pi.m\times6{\cdot}4\times5{\cdot}5\times6{\cdot}7.$

whence, $g = \frac{4}{3}.m\times6{\cdot}4\times5{\cdot}5\times6{\cdot}7 = 988{\cdot}3 \text{ cm/sec}^{2}.$

Or, the acceleration due to gravity at the earth's surface is 988·3 *cms./sec*^{2}.

2. Two lead spheres of 20 cms. and 2 cms. diameter respectively are placed with their centres 100 cms. apart. Calculate the force of attraction between the spheres, given the radius of the earth as $6{\cdot}67\times10^{8}$ cms. and its mean density as 5·33 gms./c.c. ; (Sp. gr. of lead = 11·5).

If the lead spheres be replaced by brass spheres of the same radii, would the force of attraction be the same ?

Clearly, force of attraction between the masses $= \frac{\text{product of the masses}}{(\text{distance})^{2}}.G.$

$$= \tfrac{4}{3}.\pi.(10)^{3}\times\tfrac{4}{3}.\pi.(1)^{3}\times(11{\cdot}5)^{2}.G/100^{2} = 16\pi^{2}\times10^{3}\times(11{\cdot}5)^{2}.G/9\times10^{4}.$$

$$= 16\pi^{2}\times(11{\cdot}5)^{2}\times G/90 = 8\pi^{2}\times(11{\cdot}5)^{2}.G/45.$$

Now, force on a mass of *one gram* on the earth's surface $= G.M./R^{2} = g.$

Or, $G.M/R^2 = 980$, taking $g = 980$ *cms./sec*^{2}.

Or, $G.\frac{4}{3}.\frac{\pi R^3}{R^2} \times 5{\cdot}33 = 980$. Or, $G.4.\pi.R \times 5{\cdot}33 = 980 \times 3$.

Or, $G = 980 \times 3/4.\pi.R \times 5{\cdot}33$.

∴ force of attraction between the lead spheres is given by

$$F = \frac{8\pi^2 \times (11{\cdot}5)^2}{45} \times \frac{980 \times 3}{4.\pi.R \times 5{\cdot}33}.$$

$$= \frac{2\pi \times (11{\cdot}5)^2 \times 196}{3 \times 6{\cdot}67 \times 10^8 \times 5{\cdot}33} = 15{\cdot}28 \times 10^{-6} \text{ dynes.}$$

Since the force of attraction between the spheres depends upon their masses, it will naturally be different in the case of brass spheres whose mass will be much less than that of the lead spheres, (the density of brass being much less than that of lead).

3. Calculate the mass of the earth from the following data : Radius of the earth $= 6 \times 10^8$ cms. ; Acceleration due to gravity = 980 cm./sec^2. and Gravitational Constant $= 6{\cdot}6 \times 10^{-8}$ cm.^{3} gm.$^{-1}$.sec.$^{-2}$

We know that the force with which the earth attracts a *unit mass* towards itself, (*i e.*, towards its centre) $= 1 \times g = 980$ *dynes.*

Also, the force of attraction between the mass and the earth is given by $M \times 1 \times G/R^2$, where M is the mass of the earth, R, its radius and G, the Gravitational Constant. Clearly, therefore, $M.G/R^2 = g$.

Or, $M = g.R^2/G$. ... ... (*i*)

∴ substituting the values of g, R and G in relation (*i*), we have

$M = 980 \times (6 \times 10^8)^2/6{\cdot}6 \times 10^{-8} = 53{\cdot}47 \times 10^{26}$ *gms.*

Or, the mass of the earth is equal to $53{\cdot}47 \times 10^{26}$ *gms.*

4. Calculate the mass of the Sun, given that the distance between the Sun, and the Earth is $1{\cdot}49 \times 10^{13}$ cms., and G $= 6{\cdot}66 \times 10^{-8}$ c.g.s. units. Take the year to consist of 365 days. (*Punjab, 1942*)

Let the mass of the Sun = M *gms.* and that of the Earth = m *gms.*

Distance between the two, or *the radius of Earth's orbit round the Sun, i.e.*,

$$r = 149 \times 10^{11} \text{ cms.}$$

Time of one revolution of the Earth round the Sun = 365 *days.*

$$= 365 \times 24 \times 60 \times 60 \text{ secs.}$$

Clearly, *force of attraction* between the Sun and the Earth

$$= G.M.m/r^2 = \frac{M\,m}{(1{\cdot}49 \times 10^{13})^2} \times 6{\cdot}66 \times 10^{-8} \text{ dynes.}$$

Now, centripetal force acting on the Earth in its orbit $= mv^2/r$.

And, the distance covered by the Earth in 1 revolution, *i.e.*, in 365 *days*, clearly $= 2\pi r = 2\pi \times 1{\cdot}49 \times 10^{13}$ *cms.*

∴ distance covered by it in 1 *sec.*, or, $v = \frac{2\pi \times 1{\cdot}49 \times 10^{13}}{365 \times 24 \times 60 \times 60}$.

Hence, centripetal force on the Earth $= mv^2/r$.

$$= m\left(\frac{2\pi \times 1{\cdot}49 \times 10^{13}}{365 \times 24 \times 3600}\right)^2 \times \frac{1}{1{\cdot}49 \times 10^{13}} \text{ dynes.}$$

This must, clearly, be equal to the force of attraction between the Sun and the Earth ; and, therefore,

$$\frac{m \times M \times 6{\cdot}66 \times 10^{-8}}{(1{\cdot}49 \times 10^{13})^2} = m.\frac{4\pi^2 \times 1{\cdot}49 \times 10^{13}}{(365 \times 24 \times 3600)^2}.$$

$$\therefore M = \frac{4\pi^2 \times 1{\cdot}49 \times 10^{13})^2}{(365 \times 24 \times 3600)^2} \times \frac{(1{\cdot}49 \times 10^{13})^2}{6{\cdot}66 \times 10^{-8}} = \frac{4\pi^2 \times (1{\cdot}49 \times 10^{13})^3}{(365 \times 24 \times 3600)^2 \times 6{\cdot}66 \times 10^{-8}}.$$

Or, $M = 19{\cdot}72 \times 10^{32}$ *gms.*

Or, the mass of the Sun = $19{\cdot}72 \times 10^{32}$ *gms.*

Assuming that a sphere of mass 40 kilograms is attracted by a second sphere of mass 80 kilograms, when their centres are 30 cms. apart with a force equal to the weight of $\frac{1}{4}$ mg. ; calculate the Constant of Gravitation.

Here, force of attraction between the two masses

$$= \frac{40\times1000\times80\times1000}{30^2}\cdot G = \frac{32\times10^6}{9\times10^2}\cdot G = \frac{32\times10^6}{9}\cdot G.$$

But this is equal to $\frac{1}{4}$ *mg. wt. i.e.*,

$$= \frac{1}{4}\times\frac{1}{1000}\ gms.\ wt. = \frac{1}{4}\times\frac{1}{1000}\times981\ dynes. \therefore \frac{32\times10^6}{9}\cdot G = \frac{981}{4000}.$$

Or, $$G = \frac{981\times9}{4000\times32\times10^6} = \frac{981\times9}{4\times32\times10^9} = 6{\cdot}898\times10^{-8}.$$

Or, the value of $G = 6{\cdot}898\times10^{-8}$ *C.G.S. units.*

6. Two small balls of mass m each, are suspended side by side by two equal threads of length l. If the distance between the upper ends of the threads be a, find through what angle the threads are pulled out of the vertical by the attraction of the balls.

Let the upper ends of the threads be at A and B, (Fig. 167), such that the distance $AB = a$.

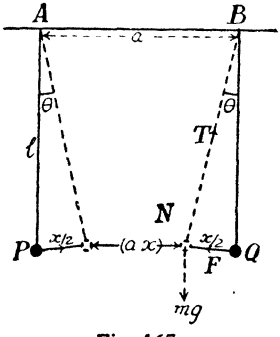


Fig. 167.

Due to mutual attraction, the balls are drawn towards each other, say, through a distance x each, from their original positions.

Considering the forces acting on the ball Q, which keep it in equilibrium, we have

(*i*) the *weight* of ball = mg, acting vertically downwards.

(*ii*) the *force of attraction*,

$$F = \frac{m\times m}{(a-x)^2}\cdot G = \frac{m^2.G}{(a-x)^2}.$$

(*iii*) the *tension* of the thread T.

Since the ball is in equilibrium, the three forces can be represented by the three sides of a triangle, taken in order.

If the dotted lines show the positions of the threads when the balls are in equilibrium, the three forces can be represented in magnitude as well as direction by the sides of the triangle BQN,—the side BQ, representing the *weight* mg, the side QN, representing F and the side NB, representing the *tension* T of the string, *in cylic order.*

So that, clearly, $$\frac{F}{mg} = \frac{QN}{BQ} = \tan\theta. \quad \text{Or,} \quad \frac{m^2G}{(a-x)^2.mg} = \tan\theta.$$

whence, $$\tan\theta = \frac{m.G}{(a-x^2)\,g}. \quad \text{Or,} \quad \theta = \tan^{-1}\frac{m\,G}{(a-x)^2.g}.$$

Thus, the threads will be pulled out of the vertical through an angle $\theta = \tan^{-1} mG/(a-x).^2g$.

7. The radius of the Moon's orbit, r, is 240,000 miles, and the period of revolution is 27 days ; the diameter of the Earth is 8,000 miles and the value of gravity on its surface is 32 ft./sec^2. Verify the statement that the gravitational force varies inversely as the square of the distance.

Here, distance covered by the Moon in 27 *days* = $2\pi\times240000$ *miles.*

$\therefore$ velocity of the Moon, $v = \frac{2\pi\times240000\times1760\times3}{27\times24\times60\times60}$ *ft./sec.*

Now, centripetal acceleration of a body moving in a circle = v^2/r. Hence the centripetal acceleration of the Moon towards the centre of the Earth, is given by

$$f_m = \left[\frac{2\pi\times24\times176\times3\times10^5}{27\times24\times60\times60}\right]^2\times\frac{1}{240000\times1760\times3}.$$

$$= \frac{4\pi^2 \times 24 \times 176 \times 3 \times 10^5}{(27 \times 24 \times 36)^2 \times 10^4} = \frac{4\pi^2 \times 176 \times 3 \times 10}{(27 \times 36)^2 \times 24}.$$

$$= 0{\cdot}009189 \text{ ft./sec}^2.$$

Then, denoting the acceleration due to gravity on the surface of the Earth by g, (= 32 *ft. per sec*^{2}.), and supposing it to be inversely proportional to the n^{th} power of the distance, we have

$$\frac{g}{g_e} = \left(\frac{R_e}{R_m}\right)^n.$$

where R_e is the radius of the earth, and R_m, the distance of the Moon from the Earth.

Or, $$\frac{0{\cdot}009189}{32} = \left(\frac{4000 \times 1760 \times 3}{240000 \times 1760 \times 3}\right)^n = \left(\frac{1}{60}\right)^n.$$

Or, taking logarithms, we have

$$\bar{4}{\cdot}4582 = n(\bar{2}{\cdot}2218). \quad \text{Or, } n = \frac{4{\cdot}4582}{2{\cdot}2218} = 2, \textit{ nearly.}$$

Thus, g varies inversely as the *second power* of the distance and hence *the gravitational force varies inversely as the square of the distance.*

8. The radius of the earth is $6{\cdot}37 \times 10^{-8}$ cms., its mean density, 5·5 gms./c.c. and the gravitational constant, $6{\cdot}66 \times 10^{-8}$ c g s. units. Calculate the earth's surface potential.

We know that potential, $V = GM/x$. [Taking the earth to be a perfect sphere.

Now, mass of the earth, M = volume × density $= \frac{4}{3}.\pi(6{\cdot}37 \times 10^8)^3 \times 5{\cdot}5$,

distance, $x = r$, in this case, $= 6{\cdot}37 \times 10^8$ *cms.*

$G = 6{\cdot}66 \times 10^{-8}$ *C.G.S. units* (given).

$$\therefore \quad V = \frac{6{\cdot}66 \times 10^{-8} \times 4\pi(6{\cdot}37 \times 10^8)^3 \times 5{\cdot}5}{3 \times 6{\cdot}37 \times 10^8} = \frac{6{\cdot}66 \times 10^8 \times 4\pi(6{\cdot}37)^2 \times 5{\cdot}5}{3}.$$

$$= 2{\cdot}22 \times 10^8 \times 4\pi(6{\cdot}37)^2 \times 5{\cdot}5 = 62{\cdot}27 \times 10^{10} \textit{ ergs lgm.}$$

9. Calculate the intensity at a point due to an infinitely long straight wire of line density ρ.

Let AC be a portion of the wire, (Fig. 168), of line density ρ, and consider an element AB of the wire, of length dl. Let AO be a length l of the wire ; and let P be a point at a distance x from O.

Join PA, and let $\angle APO = \theta$.

Then, clearly, $\tan\theta = l/x$. Or, $l = x.\tan\theta$.

Differentiating it with respect to θ, we have

$$dl = x.\sec^2\theta.d\theta.$$

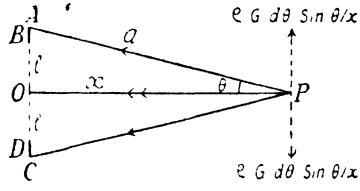


Fig. 168.

Therefore, mass of this element $AB = \rho.dl = x.\sec^2\theta.d\theta.\rho$.

And, $\therefore$ intensity at P due to the element $= \frac{x.\sec^2\theta.d\theta.\rho}{AP^2}.G$, in the direction PA.

$\therefore$ intensity at P due to the element

$$= \frac{x.\sec^2\theta.d\theta.\rho}{x^2.\sec^2\theta}.G = \frac{\rho.G.d\theta}{x}, \text{ along } PA. \quad \left[\because AP = x\sec\theta.\right.$$

Resolving it into two rectangular components, along PO and perpendicular to it, we have (*i*) the *component along PO*, equal to $\rho.G.d\theta.\cos\theta/x$, and (*ii*) the component at right angles to it, equal to $\rho.G.d\theta.\sin\theta/x$.

Similarly, the intensity at P due to an equal element dl and the other end, at a distance l from O will be $\rho\ G.d\theta/x$, in the direction PC. Resolving it into two rectangular components, the component $\rho.G.d\theta.\sin\theta/x$, being equal and opposite to that due to the element AB will cancel out and the component

$\rho.G\,d\theta\cos\theta/x$ will act along PO, as before, and the two will, therefore, be added up. The same is true for any two similarly situated equal elements of the wire.

Therefore, considering the whole wire, we have

$$\text{intensity at } P \text{ due to the whole wire} = 2\int_0^{\pi/2}\frac{\rho.G.d\theta}{x}.\cos\theta = \left[\frac{\rho.G}{x}.\sin\theta\right]_0^{\pi/2}$$

$$= .\frac{\rho.G}{x}\cdot\sin\frac{\pi}{2} = 2\ \frac{\rho.G}{x}.1 = 2\cdot\frac{\rho\,G}{x}.$$

Or, intensity at a distance x due to an infinitely long straight wire is $2\rho.G/x$.

EXERCISES VII

1. Mention different methods for determining the Constant of universal gravitation, and describe one which you consider to be the most accurate. (*Punjab, 1940 and 1944*)

2. What is meant by 'gravitation constant'? What are its dimensions? Give an account of the experiments of Cavendish and Boys to determine this constant. (*Banaras, 1945*)

3 If $G = 6{\cdot}66\times10^{-8}$ *c.g.s. units*, what is the force between two small spheres weighing 2 *k.gms*, placed 30 *cms.* apart. **Ans.** $2{\cdot}931\times10^{4}$ *dynes.*

4. State and explain Newton's law of gravitation and describe an accurate method of measuring the gravitation constant. What celestial evidence led to the formulation of the law? Is this law universally correct? Explain your statement. (*Calcutta, 1945*)

5. If the earth were a solid sphere of iron, of radius 6 37 million *metres*, and of density 7·86 *gms /cms*^{3}., what would be the value of gravity at its surface, taking the *gravitational constant* to be $6\ 658\times10^{-8}$ *c.g.s. units*?

Ans. 1396 *cms./sec*^{2}.

6. Give the theory of Cavendish experiment, explaining how the density of the earth is determined. Explain why and how Boys modified the Cavendish method. (*Madras, 1950*)

7. Explain how Cavendish determined the value of gravitation constant.

Indicate how, from the knowledge of the value of the gravitation constant, it is a possible to calculate the mass of the earth. (*Saugar, 1948*)

8. If $G = 6{\cdot}66\times10^{-8}$ *c g.s units*, and the radius of the earth equal to $6{\cdot}37\times10^{8}$ *cms.*, what is the density of the earth? **Ans.** 5 62 *gms./cms*^{3}.

9. The earth moves round the Sun in a circle of radius $9{\cdot}288\times10^{7}$ *miles*, and completes a revolution in 365 *days* A satellite of Jupiter moves about the Jupiter in a circle of radius $1{\cdot}161\times10^{6}$ *miles*, completing one revolution in 16·6 *days*. Calculate the mass of Jupiter in terms of the mass of the Sun.

Ans. 945×10^{-6}.

10. Assuming the law of gravitation, find an expression for the period of revolution of a planet.

The moon describes a circular orbit of radius $3{\cdot}8\times10^{5}$ *km.* about the earth in 27 days and the earth describes a circular orbit of radius $1{\cdot}5\times10^{8}$ *km.* round the Sun in 365 days. Determine the mass of the Sun in terms of that of the earth. (*Bombay, 1935*)

Ans. $3{\cdot}366\times10^{5}$.

11. Define the gravitational constant and describe a laboratory method for measuring it accurately.

A small satellite revolves round a planet of mean density 10 *gms./c.c.*, the radius of its orbit being slightly greater than the radius of the planet. Calculate the time of revolution of the satellite. ($G = 6{\cdot}66\times10^{-8}$ *c g.s. units*) (*Bombay, 1940*)

Ans. 1·044 *hours.*

12. Define 'Potential' and 'Potential Energy' of a gravitational field. Derive an expression for the potential due to a sphere of uniform density at an external point.

The radius of the earth is $6{\cdot}37\times10^{8}$ *cms.*, its mean density 5·5 *gms./cm*^{3}. and the gravitation constant, $6{\cdot}66\times10^{-8}$. Calculate the earth's surface potential. (*Agra, 1940*)

Ans. $6{\cdot}22\times10^{11}$ *ergs./gm.*

13. What is meant by the gravitational potential ? How does it vary with the distance from the centre of the earth ? What initial velocity would be required to project a body beyond the attractive force of the earth ? (Radius of earth is $6{\cdot}4\times10^8$ *cms.*) *(Cambridge Scholarship)*

Ans. $1{\cdot}12\times10^6$ *cms./sec.*

14. Explain what you mean by gravitational potential at a point. How does it differ from other kinds of potential with which you are familiar ?

Find an expression for gravitational potential due to a thin hollow sphere of uniform density at a point outside it. *(Calcutta, 1947)*

15. Two balls, each weighing 10 *gms*, are hung side by side by threads, 10 *metres* long. If the threads are 1 *cm.* apart at the upper ends, by how much is the distance between the centres of the balls less than 1 *cm.*

Ans. $1{\cdot}5\times10^{-6}$ *cms.*

16. Describe one of the most accurate methods of measuring the constant of gravitation.

The star Sirius has a mass of $6{\cdot}9\times10^{33}$ *gms.* and its distance is 8×10^{13} *km.* The mass of the earth is 6×10^{27} *gms.* The tensile strength* of steel is about 20,000 *kg./cm*^{2}. Calculate the cross-section of a steel bar which could just withstand the gravitational pull between Sirius and the earth. ($G = 6{\cdot}67\times10^{-8}$ *dyne-cm*^{2}*./gm*$^{-2}$. *(Bombay, 1951)*

Ans. $2{\cdot}169\times10^6$ *sq. cm.*

17. Prove that the least velocity with which a particle must be projected from the surface of a planet of radius R and density ρ in order that it may escape completely is $R\sqrt{8\pi G\rho/3}$, where G is the gravitational constant. Calculate the velocity in the case of the moon from the following data :
mean density of earth = 5·52 *gms./c.c.* ; *mean density of moon* = 3·36 *gms./c.c.* ; *mean radius of earth* = 638 *km.* ; *mean radius of moon* = 1740 *km.* ;
Acceleration of gravity at earth's surface = 980 *cms. per sec. per sec.*
(Oxford Scholarship)

Ans. $2{\cdot}38\times10^5$ *cm. sec*$^{-1}$.

18. Describe an accurate '*balance-method*' for the determination of the value of G, and write a short note on the '*qualities of gravitation*'.

19. What are *seismic waves* ? Give a brief description of their characteristics. How may they be detected ? Also mention some of the applications of *seismology*.

20. What is an earthquake ? How is it caused ? Describe in brief the principle underlying *seismographs*. Why are they so called ?

21. Describe in detail Galitzin's seismograph and explain how the *epicentre* and the *focus* of an earthquake may be determined with its help.

22. What is *geophysical prospecting* ? Write a short descriptive note on (*i*) the *gravitational* and (*ii*) the *seismic* methods used for the purpose.

*See next chapter.

CHAPTER VIII

ELASTICITY

106. Introductory. All bodies can, more or less, be *deformed* by suitably applied forces. The simplest cases of deformation are those (*i*) in which a wire, fixed at its upper end, is pulled down by a weight at its lower end, *bringing about a change in its length* and (*ii*) in which an equal compression is applied in all directions, so that *there is a change of volume but no change in shape*, or (*iii*) in which a system of forces may be applied to a body such that, although there is no motion of the body as a whole, there is relative displacement of its continuous layers causing a *change in the shape or 'form' of the body with no change in its volume.* In all these cases, the body is said to be *strained* or *deformed.*

When the deforming forces are removed, the body tends to recover its original condition For example, the wire, in the case above, tends to come back to its original length when the force due to the suspended weight is removed from it, or, a compressed volume of air or gas throws back the piston when it is released, in an attempt to recover its original volume. This property of a material body to regain its original condition, on the removal of the deforming forces, is called *elasticity*. Bodies, which can recover completely their original condition, on the removal of the deforming forces, are said to be *perfectly elastic*. On the other hand, bodies, which do not show any tendency to recover their original condition, are said to be *plastic.* There are, however, no *perfectly* elastic or plastic bodies. The nearest approach to a perfectly elastic body is a *quartz fibre* and, to a perfectly plastic body, is *putty*. But even the former yields to large deforming forces and, similarly, the latter recovers from small deformations. Thus, there are *only differences of degree*, and a body is more elastic or plastic when compared to another.

We shall consider here only bodies or substances, which are (*i*) *homogeneous* and (*ii*) *isotropic, i.e.*, which have the *same properties at all points* and *in all directions*. For, these alone have similar elastic properties in every direction, (together with other physical properties like linear expansion, conductivity for heat and electricity, refractive index etc.). Fluids (*i.e.*, liquids and gases), as a rule, belong to this class, but not necessarily all solids, some of which may exhibit different properties at different points and in different directions, *i.e.*, may be *heterogeneous* (or *non-homogeneous*) and *anisotropic* (or *non-isotropic*). Examples of this class of solids are wood, and crystals in general, including those metals, which are crystalline in structure. As a class, however, *metals*,—particularly in the form of *rods* and *wires*,—may be regarded to be more or less wholly isotropic, in so far as their elastic behaviour is concerned.

107. Stress and Strain. As a result of the deforming forces applied to a body, *forces of reaction* come into play *internally* in it,

due to the relative displacement of its molecules, tending to restore it to its original condition. *The restoring or recovering force per unit area set up inside the body is called stress,* and is measured by the deforming force applied per unit area of the body, being *equal in magnitude* but *opposite in direction* to it, until a permanent change has been brought about in the body, *i.e.*, until its elastic limit has been reached, (see § 108, below). If the force be inclined to the surface, its component, perpendicular to the surface, measured per unit area, is called *normal stress** and the component acting along the surface, per unit area, is called *tangential* or *shearing stress.* Further, the former may be *compressive* or *expansive* (*i.e.*, tensile) according as a decrease or increase in volume is involved. Obviously, being force per unit area, the units and dimensions of stress are the same as those of pressure, *viz.*, $ML^{-1}T^{-2}$, (see page 5).

The change produced in the dimensions of a body under a system of forces or couples, in equilibrium, is called strain, and is measured by the *change per unit length* (*linear strain*), per unit volume, (*volume strain*), or the angular deformation, (*shear strain*, or simply, *shear*)† according as the change takes place in *length, volume* or *shape* of the body. Thus, being just a *ratio,* (or *an angle*) it is a *dimensionless* quantity, having no units.

It will be readily seen that *for a perfectly elastic body* (*i*) *the strain is always the same for a given stress* ; (*ii*) *the strain vanishes completely when the deforming force is removed and* (*iii*) *for maintaining the strain, the stress is constant.*

108. Hooke's Law. Hooke's law is the fundamental law of elasticity and states that, *provided the strain is small, the stress is proportional to the strain* ; *so that, in such a case, the ratio stress/strain is a constant E, called the* **modulus of elasticity,** (a term first introduced by *Thomas Young*), *or the* **coefficient of elasticity.**

Since stress is just pressure, (or tension per unit area), and strain is just a ratio, the units and dimensions of the modulus of elasticity are the same as those of stress or pressure.

When the stress is continually increased in the case of a solid, a point is reached at which the strain increases more rapidly than is warranted by Hooke's Law. This point is called the **elastic limit,** and if the body happens to be a wire under stretch, it will not regain its original length on being unloaded, *if the elastic limit be passed,* as it acquires what is called a **'permanent set'.** On loading it further, a point is reached when the extension begins to increase still more rapidly and the wire begins to *'flow down'* in spite of the same constant load. This point is called the **'yield point'** ; and, after a large extension, it reaches the **'breaking point'** and the wire snaps. In the case of plastic substances, like lead, there is a long range between the yield point and the breaking point.

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*The stress is always normal in the case of a change in the length of the wire, or in the case of a change in the volume of a body, but is tangential in the case of a change in the shape of a body.

†This will be dealt with more fully later in § 109 (3).

Thus, if we were to plot a graph between the load suspended from a wire, fixed to a rigid support at its upper end, and the extension produced thereby, we obtain, in general, a curve of the form shown in Fig 169,—the straight part OA of the curve showing that the extension produced is directly proportional to the load applied, or that Hooke's law is obeyed perfectly up to A, and that, therefore, on being unloaded at any point between O and A, the wire will come back to its original condition, (represented by O). In other words, the wire is perfectly elastic up to A, which thus measures the **elastic limit*** of the specimen in question,—the extension here being of the order of 10^{-3} of the original length.

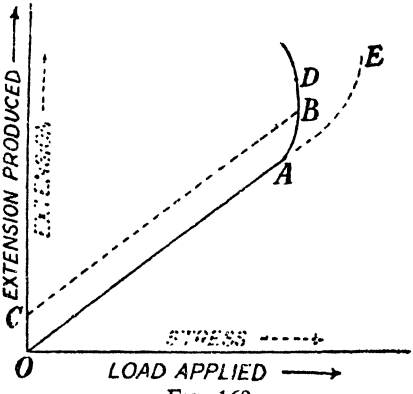


Fig. 169.

On loading the wire beyond the elastic limit, say, up to B, the curve takes a bend almost vertically upwards, as shown, and, on being unloaded at any point here, (at B, say), it *does not come back to its original condition* but takes the dotted path BC, thus acquiring a **'permanent set'** OC.

On increasing the load still further, a point D is reached, where the extension is much greater even for a small increase in the load, *i.e.*, Hooke's law is obeyed no longer ; and, beyond D, the extension increases continuously, *with no addition to the load*, the wire starting *'flowing down'*, as it were. For, due to its thinning down, the *stress* (or the load per unit area) increases considerably and it cannot support the same load as before ; and, if the wire is to be prevented from *'snapping'*, the load applied to it must be decreased. That is why the curve starts turning towards the extension-axis beyond this point D, which thus represents the **yield point** of the wire. And, once the yield point is crossed, the thinning of the wire no longer remains *uniform* or *even*, its cross-section decreasing more rapidly at some points than at others, resulting in its developing small *'necks'* or *'waists'* at the former points, so that the stress is greater there than at the latter points ; and the wire ultimately *'snaps'* at one of these. This point on the curve, at which the snapping or the breaking of the wire actually occurs, is called its **breaking point**,—the corresponding stress and strain there being referred to as the **breaking stress** (or *tensile strength*) and the **breaking strain,** respectively.

Note. The *elastic limit* of a material is also sometimes defined as *the force producing the maximum reversible or recoverable deformation in it*, and may,

*In quite a few cases, Hooke's law is obeyed only up to a point a little below the elastic limit, represented by A. The portion of the curve from O to *this point* (below A), is then said to indicate the *limit of proportionality*, to distinguish it from the *elastic limit*. The two are thus not always identical, though they are generally regarded to be so, in view of the very small difference between them.

for a given specimen, be determined by loading and unloading it with a number of different loads and measuring its length after *each unloading*, until it acquires a *permanent set*. The latter is then plotted against the load, and from the curve thus obtained, the particular load at which the permanent set *just* starts, can be easily estimated.

Even within the elastic limit, however, few solids come back to their original condition, directly the deforming force is removed. Almost all of them only '*creep*' back to it, (*i.e.*, take some time to do so), though they all do so, ultimately. *This delay in recovering back the original condition, on the cessation of the deforming force, is called* **elastic-after effect.** Glass exhibits this effect to a marked degree, the few *exceptions* to this almost general rule being *quartz, phosphor-bronze, silver* and *gold*, which regain their original condition as soon as the deforming force ceases to operate. Hence their use in Cavendish's and Boys' experiments for the determination of G, in quadrant electrometers and moving-coil galvanometers etc. etc.

As a natural consequence of the elastic after-effect *the strain in a material, (in glass, for example), tends to persist or lag behind the stress to which it is subjected*, with the result that during a rapidly changing stress, the strain is greater for the same value of stress, when it is decreasing than when it is increasing, as is clear from the curve in Fig. 170. *This lag between stress and strain is called* **elastic hysterisis,** (the term '*hysterisis*', meaning '*lagging behind*'). The phenomenon is similar in its implications to the familiar *magnetic hysterisis*, where the magnetic effects tend to persist or lag behind even after the magnetising influence is removed,—the curve referred to above may thus be called the *elastic hysterisis loop*. And, exactly in the same manner the energy, dissipated as *heat*, during a cycle of loading and unloading is given by the *area enclosed by the loop*. There is, however, very little hysterisis in the case of metals or of quartz.

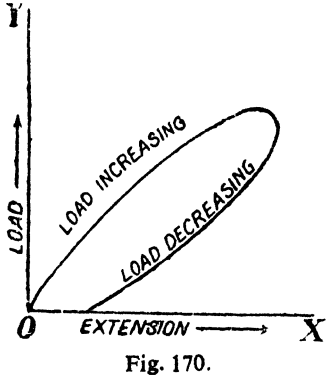


Fig. 170.

Further, it was shown by *Lord Kelvin*, during his investigation of the rate of decay of torsional vibrations of wires, that the vibrations died away much faster in the case of a wire kept vibrating continuously for some time than in that of a fresh wire. The same happens to any elastic body, subjected to an alternating strain. The continuously vibrating wire got '*tired*' or '*fatigued*', as it were, and found it difficult to continue vibrating. Lord Kelvin fittingly expressed this by the term **'elastic fatigue'**.

A body, thus subjected to repeated strains beyond its elastic limit, has its elastic properties greatly impaired, and may break under a stress, less than its normal breaking stress even within its elastic limit. This phenomenon is, obviously, of great importance in cases like those of the piston and the connecting rods in a locomotive, which, as we know, are subjected to repeated tensions and compressions during each revolution of the crank shaft.

It may be mentioned here that all these elastic properties of a material are linked up with the fine mass of its structure. It is now finally established by careful microscopic examination, that metals are just an aggregation of a large number of fine crystals, in most cases, arranged in a random or a chaotic fashion, *i.e.*, their *cleavage planes* (or the planes along which their constituent atoms can easily slide over each other), being distributed haphazardly, in all possible directions. Now, single crystals, when subjected to deformation, show a remarkable increase in their hardness. Thus, for example, a single crystal of silver, on being stretched to a little more than twice its length, is known to increase to as much as ninety-two times its original strength or stiffness. So that, operations like hammering and rolling, which help this sort of distribution, *i.e.*, which break up the crystal grains into smaller units, result in an increase or extension of their elastic properties ; whereas, operations like annealing (or heating and then cooling gradually) etc., which tend to produce a uniform pattern of orientation of the constituent crystals, by orienting them all in one particular direction and thus forming larger crystal grains, result in a decrease in their elastic properties or an increase in the softness or plasticity of the material.

This is because in the latter case, slipping (or sliding between cleavage planes), starting at a weak spot proceeds all through the crystal and, in the former, the slipping is confined to one crystal grain and stops at its boundary with the adjoining crystal. Indeed, the former may be compared to a small cut, developing into a regular tear all along a fabric and the latter to the tear stopping as it reaches a seam in the fabric. Thus, '*paradoxically*', *as Sir Lawrence Bragg* puts it, '*in order to be strong, a metal must be weak*,' meaning thereby that metals with smaller grains are stronger than those with larger ones.

A change in the temperature also affects the elastic properties of a material,—a rise in temperature usually decreasing its elasticity and *vice versa*, except in certain rare cases, like that of *invar steel*, whose elasticity remains practically unaffected by any changes in temperature. Thus, for example, lead becomes quite elastic and rings like steel when struck by a wooden mallet, if it be cooled in liquid air. And, again, a carbon filament, which is highly elastic at the ordinary temperature, becomes plastic when heated by the current through it, so much so that it can be easily distorted by a magnet brought near to it.

109. Three Types of Elasticity. Corresponding to the three types of strain, we have three types of elasticity, *viz.*,

(*i*) *linear elasticity*, or *elasticity of length, called* **Young's Modulus,** corresponding to linear (or tensile) strain ;

(*ii*) *elasticity of volume* or **Bulk Modulus**, corresponding to volume strain ; and

(*iii*) *elasticity of shape, shear modulus*, or **Modulus of Rigidity,** corresponding to shear strain.

(1) **Young's Modulus.** When the deforming force is applied to the body *only along a particular direction*, the change per unit length in *that* direction is called *longitudinal, linear or elongation strain*, and the force applied per unit area of cross-section is called *longitudinal or linear stress. The ratio of longitudinal stress to linear strain, within the elastic limit, is called Young's Modulus, and is usually denoted by the letter Y.*

Thus, if F be the force applied normally to a cross-sectional area a, the *stress* is F/a. And, if there be change l produced in the original length L, the *strain* is given by l/L. So that,

$$\textit{Young's Modulus, } Y = \frac{F/a}{l/L} = \frac{F.L}{a.l}.$$

Now, if $L = 1$, $a = 1$ and $l = 1$, we have $Y = F$.

In other words, if a material of unit length and unit area of cross-

section could be pulled so as to increase in length by unity, *i.e.*, to double its length, the force applied would measure the value of Young's Modulus for it.

Since, however, the elastic limit is exceeded when the extension produced is 10^{-3} *cm./cm.*, the material will snap before *this much* extension is produced.

In cases, where, elongation produced is not proportional to the force applied, we can still determine Young's Modulus from the ratio $L.dF/a.dL$, where dF/a is the infinitesimal increase in the longitudinal stress and dL/L, the corresponding increase in strain.

Or,
$$Y = \frac{L}{a} \cdot \frac{dF}{dL}.$$

N.B. The particular case of rubber may, with advantage, be mentioned here, which the beginner finds so confusing, when, in ordinary conversational language, we refer to it as being '*elastic*'. For, he knows well enough that it requires a much smaller force than steel to stretch it, (and that, therefore, its elasticity is much less than that of steel). In fact, the value of Young's Modulus for rubber is about *one-fiftieth* of that of steel. What we mean when we say that it is elastic, therefore, is just that *it has a very large range of elasticity*, for, whereas a crystalline body can be stretched to less than even one per cent of its original length before reaching its elastic limit, rubber can be stretched to about eight times (or 80%) of its original length.

This high extensibility of rubber is due to its molecule containing, on an average, some 4,000 molecules of *isoprene* (C_5H_8), whose 20,000 carbon atoms, spreading out in a chain, make it very *long* and *thin*,—about 1/4000 *mm.* in length.

Rubber, in bulk, has thus been rightly compared to an intertwined mass of long, wriggling snakes,—its molecules, like the snakes, tending to uncoil when stretched and getting coiled up again when the stretching force is removed.

(2) **Bulk Modulus.** Here, the force is applied *normally* and *uniformly* to the whole surface of the body ; so that, *while there is a change of volume, there is no change of shape.* Geometrically speaking, therefore, we have here *a change in the scale of the coordinates* of the system or the body. *The force applied per unit area, (or pressure), gives the* **Stress**, *and the change per unit volume, the* **Strain**, *their ratio giving the Bulk Modulus for the body. It is usually denoted by the letter K.*

Thus, if F be the force applied *uniformly and normally* on a surface area a, the stress, or pressure, is F/a or P ; and, if v be the change in volume produced in an original volume V, the strain is v/V. and, therefore,

$$\textit{Bulk Modulus}, K = \frac{F/a}{v/V} = \frac{F.V}{a.v} = \frac{P.V}{v} \qquad [\because F/a = P.$$

If, however, the change in volume be not proportional to the stress or the pressure applied, we consider the infinitesimal change in volume dV, for the corresponding change in pressure dP ; so that, we have $K = dP.V/dV.$

The Bulk Modulus is sometimes referred to as **incompressibility** and hence its reciprocal is called **compressibility** ; so that, *compressibility* of a body is equal to $1/K$, where K is its *Bulk Modulus.* It must thus be quite clear that whereas bulk modulus is stress per unit strain, *compressibility represents strain per unit stress.*

Since fluids (*i.e.*, liquids and gases) can permanently withstand or sustain only a hydrostatic pressure, the only elasticity they possess is Bulk Modulus (K), which is, therefore, all that is meant when we refer to their *elasticity*. This, however, is of two types : *isothermal* and *adiabatic*.

For, when a fluid is compressed, there is always some heat produced. If this heat be removed as fast as it is produced, the temperature of the fluid remains constant and the change is said to be *isothermal* ; but if the heat be allowed to remain in the fluid, its temperature naturally rises and the change is then said to be *adiabatic.*

It can be easily shown that the *isothermal elasticity of a gas* (*i.e.*, when its temperature remains constant) *is equal to its pressure P, and its adiabatic elasticity equal to* γP, where γ is the ratio between C_p* and C_v* for the gas in question,—its value being 1·41 for air, [see solved Example 1 (*b*) at the end of the Chapter.]

It will thus be readily seen that the Bulk Modulus of a gas (whether isothermal or adiabatic) is *not a constant quantity*, unlike that of a solid or a liquid.

(3) **Modulus of Rigidity**. In this case, while there is a change in the shape of the body, there is no change in its volume. As indicated already, it takes place by the movement of contiguous layers of the body, one over the other, very much in the manner that the cards would do when a pack of them, placed on the table, is pressed with the hand and pushed horizontally. Again, speaking geometrically, we have, in this case, *a change in the inclinations of the coordinate axes* of the system or the body.

Consider a rectangular solid cube, whose lower face *aDCc*, (Fig. 171), is fixed, and to whose upper face a tangential force F is applied in the direction shown. The couple so produced by this force and an equal and opposite force coming into play on the lower fixed face, makes the layers, parallel to the two faces, move over one another, such that the point A shifts to A', B to B', d to d' and b to b', *i.e.*, *the lines joining the two faces turn through an angle* θ†.

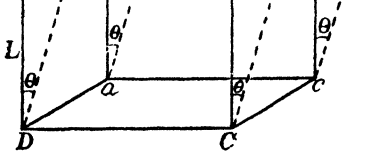


Fig. 171.

The face ABCD is then said to be **sheared** through *an angle* θ. *This angle* θ *(in radians), through which a line originally perpendicular to the fixed face is turned, gives the strain or* **the shear strain**, *or the* **angle of shear**, *as it is often called.* As will

*The symbols C_p and C_v, stand for the specific heats of a gas at constant pressure and at constant volume respectively,—their ratio $\gamma = C_p/C_v$, being the highest (1·67) for a mono-atomic gas, like helium, goes on decreasing with increasing atomicity of the gas but is always greater than 1.

†As a matter of fact, if this were the only couple acting on the body, it would result in the rotation of the body. This is prevented by another equal and opposite couple, formed by the weight of the body (plus any vertical force applied) and the reaction of the surface on which the body rests.

be readily seen, $\theta = AA'/DA = l/L$, where l is the *displacement* AA' and L, the length of the side AD or the *height* of the cube ; or θ = *relative displacement of plane ABba/distance from the fixed plane aDCc.* So that, if the distance from the fixed plane, *i.e.*, $L = 1$, we have $\theta = l$ = *relative displacement* of plane *ABba.*

Thus, *shear strain (or shear) may also be defined as the relative displacement between two planes unit distance apart.*

And, *stress or* **tangential stress** *is clearly equal to the force F divided by the area of the face ABbd, i.e., equal to F/a. The ratio of the tangential stress to the shear strain gives the* **co-efficient of rigidity** *of the material of the body, denoted by n.*

Thus, *tangential stress* $= F/a$, and *shear strain* $= \theta = l/L$.

And, therefore, *Co-efficient of Rigidity*, or *Modulus of Rigidity* of the material of the cube is given by

$$n = \frac{F/a}{\theta} = \frac{F/a}{l/L} = \frac{F.L}{a.l} \quad \ldots \quad \ldots \quad \ldots \ (i)$$

This is a relation exactly similar to the one for Young's Modulus, with the only difference that, here, F is the tangential stress, not a linear one, and l, a displacement at right angles to L, instead of along it.

Again, if the shearing strain, or shear, be not proportional to the shearing stress applied, we have

$$n = \frac{dF/a}{d\theta},$$

where $d\theta$ is the increase in the angle of shear for an infinitesimal increase dF/a in the shearing stress.

Further, it is clear from relation (i) above, that if $a = 1$, and $\theta = 1$ *radian* (or 57° 18′), we have $n = F$.

We may thus define modulus of rigidity of a material as the shearing stress per unit shear, i.e., a shear of 1 *radian, taking Hooke's law to be valid even for such a large strain*.*

110. Equivalence of a shear to a compression and an extension at right angles to each other. Consider a cube $ABCD$, (Fig. 172), with the face DC fixed, and let the face $ABCD$ be *sheared* by a force, applied in the direction shown, through an angle θ, into the position $A'B'CD$. Then, clearly, the diagonal DB is *increased in length* to DB', and the diagonal AC is *shortened* to $A'C$.

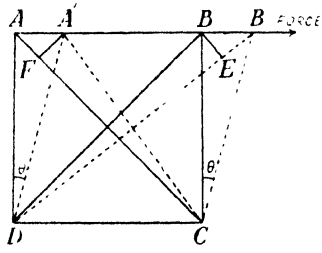


Fig. 172.

The shear is really very small in actual practice, and, therefore, triangles AFA' and BEB' are *isosceles right-angled triangles, (i.e., right-angled* 45° *triangles).*

*In the case of metals, however, Hooke's law no longer holds even if the shear exceeds 1/200 *radian*, or ·33°.

And, therefore, $EB' = BB'.\ cos\ BB'E = BB'.1/\sqrt{2} = BB'/\sqrt{2}$.

$\because\ \angle BB'E = 45°$ and $cos\ 45° = 1/\sqrt{2}$.

If $AB = l$, then, clearly, $DB = DE = l\sqrt{2}$.

$\therefore$ *extension strain* along diagonal DB

$$= \frac{EB'}{DB} = \frac{BB'}{\sqrt{2}} \cdot \frac{1}{l\sqrt{2}} = \frac{BB'}{2l} = \frac{\theta}{2}. \quad [\because BB'/l = \theta$$

Similarly, the *compression strain* along the diagonal AC is given by

$$\frac{AF}{AC} = \frac{AA'.\ cos\ A'AF}{AC} = \frac{AA'.\ cos\ 45°}{l\sqrt{2}}.$$

$$= \frac{AA'}{l\sqrt{2}.\sqrt{2}} = \frac{AA'}{2l} = \frac{\theta}{2}. \quad [\because AA'/l = \theta.$$

Thus, we see that *a simple shear θ is equivalent to two equal strains, an extension and a compression, at right angles to each other.*

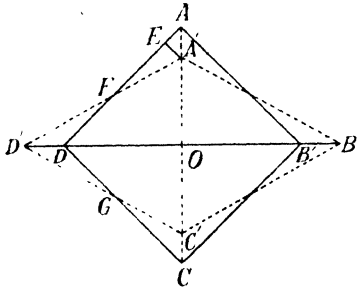


Fig. 173.

Corollary. *The converse of the above follows as a corollary, viz., that simultaneous equal compression and extension at right angles to each other are equivalent to a shear*, as will be seen from the following :

Let the cube $ABCD$, of side l, be compressed along the diagonal AC, so that the new diagonals become $A'C'$ and $B'D'$, (Fig. 173).

Let $AA' = BB' = a$.

And since $OA = AB\ cos\ BAO$

$= AB.\ cos\ 45° = AB/\sqrt{2}$

$= l/\sqrt{2}$,

we have $$OA' = OA - AA' = \left(\frac{l}{\sqrt{2}} - a\right),$$

and $$OB' = OB + BB' = \left(\frac{l}{\sqrt{2}} + a\right).$$

Clearly, $\therefore$ $$(A'B')^2 = (OA')^2 + (OB')^2$$

$$= \left(\frac{l}{\sqrt{2}} - a\right)^2 + \left(\frac{l}{\sqrt{2}} + a\right)^2.$$

$$= \frac{l^2}{2} - \frac{2al}{\sqrt{2}} + a^2 + \frac{l^2}{2} + \frac{2al}{\sqrt{2}} + a^2 = l^2 + 2a^2.$$

In practice, $2a^2$ is very small as compared with l^2, and may, therefore, be neglected.

So that, $(A'B')^2 = l^2$. Or, $A'B' = l = AB$.

Thus, $A'B'C'D'$ may be rotated through the angle DGD' = angle AFA', so that $D'C'$ coincides with DC. Then, it is obvious that $A'D'$ would make an angle $2AFA'$ with AD, so that the *angle of shear* is equal to *twice* the angle AFA', *i.e.*, is equal to $2\ \angle AFA'$.

Or, *angle of shear* $= 2A'E/EF$, [$\because$ the angle is small.

where $A'E$ is the perpendicular from A', to AF.

Now, $A'E = a/\sqrt{2}$ and $EF = l/2$. $\left[\because A'E = AA' \cos EA'A = a \cos 45° = a/\sqrt{2}\right.$

$$\therefore 2\angle AFA' = \frac{2a}{\sqrt{2}} \div \frac{l}{2} = \frac{2a}{\sqrt{2}} \times \frac{2}{l} = \frac{2a\sqrt{2}}{l}.$$

Denoting this *angle of shear* by θ, we have $\theta = 2a\sqrt{2}/l$.

Now, *compression strain* along the diagonal AC is

$$\frac{AA'}{AO} = \frac{a}{l/\sqrt{2}} = \frac{a\sqrt{2}}{l} = \frac{\theta}{2}.$$

Or, *the compression strain is half the angle of shear, i.e., the angle of shear is twice the angle of compression.*

Similarly, it can be shown that the extension strain is also half the angle of shear.

Thus, we see that *simultaneous and equal compression and extension at right angles to each other are equivalent to a shear*, the direction of each strain being at an angle of 45° to the direction of shear.

111. Shearing stress equivalent to an equal linear tensile stress and an equal compression stress at right angles to each other. In the case of the cube above, if F were the only force acting on its upper face it would move bodily in the direction of this force. Since, however, the cube is fixed at its lower face DC, an equal and opposite force comes into play in the plane of this face, giving rise to a couple $F.l.$*, tending to rotate the cube in the *clockwise* direction, (Fig. 174).

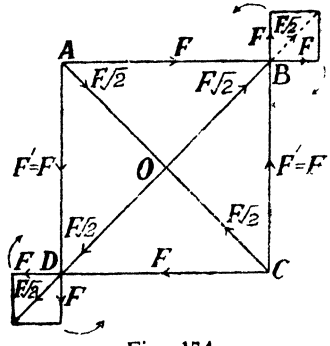


Fig. 174.

Again, since the cube does not rotate, it is obvious that the plane of DC applies an *equal* and *opposite couple* $F'.l$, say, by exerting forces F' and F' along the faces AD and CB, tending to rotate it in the *anticlockwise* direction, as shown. Thus, because the cube is in equilibrium under the two couples, we have

$$F.l = F'.l \quad \text{Or,} \quad F = F',$$

i.e., a tangential force F applied to the face AB results in an equal tangential force acting along all the other faces of the cube in the directions shown.

Clearly the resultant of the two forces F and F' or F and F along AB and CB respectively is $F\sqrt{2}$ along OB, and of those acting along AD and CD is also $F\sqrt{2}$ along OD. And, thus, an *outward* pull acts on the diagonal DB of the cube at B and D resulting in its extension, as we have just seen above, (§110). Precisely similarly, an *inward* pull acts on the diagonal AC at A and C, thereby bringing about its compression.

*l being the length of each edge of the cube and hence the perpendicular distance between the two forces F and F.

Thus, a tangential force F applied to one face of a cube gives rise to a force $F\sqrt{2}$ *outward along one diagonal* (*BD*, in the case shown) and an equal force $F\sqrt{2}$ *inward along the other diagonal* (*AC*) of the cube, resulting in an extension of the former and a compression of the latter.

Now, if the cube be cut up into two halves, by a plane passing through *AC* and perpendicular to the plane of the paper, each face, parallel to the plane, will have an area $l \times l\sqrt{2} = l^2\sqrt{2}$, and, clearly, the outward force $F\sqrt{2}$ along *BD* will be acting perpendicularly to it. So that, we have

$$\textit{tensile stress along } BD = F\sqrt{2}/l^2\sqrt{2} = F/l^2.$$

Similarly, if we cut the cube into its two halves by a plane passing through *BD* and perpendicular to the plane of the paper, we shall have an *inward* force $F\sqrt{2}$ along *AC* acting perpendicularly to a face on an area $l^2\sqrt{2}$. So that, we have

$$\textit{compression stress along } AC = F\sqrt{2}/l^2\sqrt{2} = F/l^2.$$

Obviously, F/l^2 is the shearing stress over the face *AB* of the cube, which produces the shear θ in it, (see page 281).

Thus, it is clear that *a shearing stress is equivalent to an equal tensile stress and an equal compression stress at right angles to each other.*

112. Work done per unit volume in a strain. In order to *deform* a body, work must be done by the applied force. The energy so spent is stored up in the body and is called the *energy strain.* When the applied forces are removed, the stress disappears and *the energy of strain appears as heat.*

Let us consider the work done during the three cases of strain.

(*i*) **Elongation Strain**—(*stretch of a wire*). Let *F* be the force applied to a wire, fixed at the upper end. Then, clearly, for a small increase in length *dl* of the wire, the work done will be equal to *F.dl.* And, therefore, during the whole stretch of the wire from 0 to *l.*

$$\textit{work done} = \int_0^l F.dl.$$

Now, Young's modulus for the material of the wire, *i.e.*,

$$Y = F.L/a.l.,$$

where *L* is the original length, *l*, the increase in length, *a*, the cross-sectional area of the wire, and *F*, the force applied.

And $\therefore$ $$F = Y.a.l/L.$$

Therefore, work done during the stretch of the wire from 0 to *l* is given by

$$W = \int_0^l \frac{Y.a}{L} . l.dl = \frac{Y.a}{L} \int_0^l l.dl.$$

$$= \frac{Y.a}{L} \cdot \frac{l^2}{2} = \frac{1}{2} \cdot \frac{Y.a.l}{L} . l.$$

But $Y.a.l/L = F$, the *force* applied.

Hence $W = \frac{1}{2} F.l = \frac{1}{2} \times$ *stretching force* $\times$ *stretch.*

$\therefore$ work done per *unit* volume $= \frac{1}{2} Fl \times \frac{1}{L.a}$. [$\because$ *volume* of the wire $= L \times a$.

$= \frac{1}{2} \cdot \frac{F}{a} \cdot \frac{l}{L} = \frac{1}{2}$ *stress* $\times$ *strain.* [$\because F/a =$ *stress.* and $l/L =$ strain.

Alternatively, the same result may also be obtained graphically as follows :

Let a graph OP be plotted between the *streitching force* applied to the wire and the *extension produced* in it, *within the elastic limit*, as shown in Fig. 175.

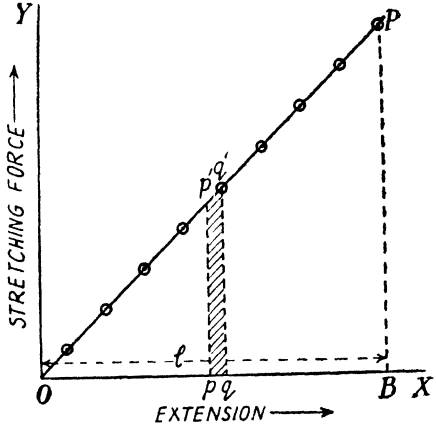


Fig. 175.

Consider a small extension pq of the wire and erect ordinates at p and q to meet the graph in p' and q' respectively, where pp' is very nearly equal to qq', (the extension pq being really small).

Then, clearly, *work done upon the wire or energy stored up in it*

$=$ *stretching force* $pp' \times$ *extension* pq.

$= pp' \times pq =$ *area of strip* $pp'q'q$.

So that, imagining the whole extension $OB = l$, of the wire, to be broken up into small bits like pq and erecting ordinates at their extremities, we have

total work done upon the wire or total energy stored up in it

$=$ *sum of the areas of all such strips formed*

$=$ *area of the triangle* $OBP = \frac{1}{2} OB \times BP = \frac{1}{2} l \times F$,

where the total extension $OB = l$ and the stretching force corresponding to it is $BP = F$.

Now, if L be the original length of the wire and a, its area of cross-section, clearly, *volume of the wire* $= L \times a$.

$\therefore$ *work done*, or *strain energy, per unit volume of the wire*

$= \frac{1}{2} lF/L.a = \frac{1}{2} \cdot \frac{F}{a} \cdot \frac{l}{L} = \frac{1}{2}$ *stress* $\times$ *strain.*

(*ii*) **Volume Strain.** Let p be the stress applied. Then, over an area a the force applied is $p.a$, and, therefore, the work done for a small movement dx, in the direction of p, is equal to $p.a.dx$. Now, $a.dx$ is equal to dv, the small change produced in volume. Thus, work done for a change dv is equal to $p.dv$.

And, therefore, total work done for the *whole change in volume*, from 0 to v, is given by

$$W = \int_0^v p.dv.$$

Now, $K = p.V/v$; so that, $p = K.v/V$,

where V is the original volume, and K, the *Bulk Modulus*.

And $\therefore W = \int_0^v \frac{K.v}{V} . dv = \frac{K}{V}\int_0^v v.dv = \frac{K}{V} \cdot \frac{1}{2}v^2.$

$= \frac{1}{2} \cdot \frac{Kv}{V} . v = \frac{1}{2} p.v =$ *stress* $\times$ *change in volume.*

Or, work done per *unit* volume $= \frac{1}{2} p.v/V = \frac{1}{2}$ *stress* $\times$ *strain.*

(*iii*) **Shearing Strain.** Consider a cube (Fig. 176), with its lower face DC fixed ; and let F be the *tangential force* applied to its upper face in the plane of AB, so that the face $ABCD$ is distorted into the position $A'B'CD$, or *sheared* through an angle θ. Let the distance AA' be equal to $BB' = x$. Then, work done during a small displacement dx is equal to $F.dx$. And, therefore, work done for the *whole of the displacement*, from 0 to x, is given by

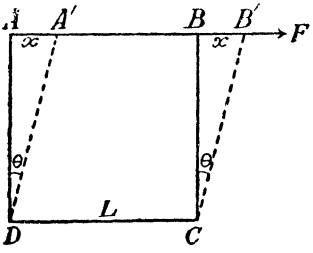


Fig. 176.

$$W = \int_0^x F.dx.$$

Now, $n = F/a\theta$, or $F = n.a.\theta$, and $a = L^2$; also $\theta = x/L$, where L is the length of each edge of the cube.

So that, $F = n.L^2.x/L = n.L.x.$

$\therefore$ work done during the *wole stretch* from 0 to x, *i.e.*,

$$W = \int_0^x n.L.x.dx = \frac{1}{2} n.Lx^2.$$

$\therefore$ work done per *unit* volume $= \frac{1}{2} \cdot \frac{n.L.x^2}{L^3} \cdot$ [$\because$ *volume of the cube* $= L^3$]

$$= \frac{1}{2} \cdot \frac{n.x.L}{L^2} \times \frac{x}{L} = \frac{1}{2} \cdot \frac{F}{a} \cdot \frac{x}{L} = \frac{1}{2}$$ *stress* $\times$ *strain.*

Thus, we see that, in any kind of strain, *work done per unit volume is equal to* $\frac{1}{2}$ *stress* $\times$ *strain.*

113. Deformation of a Cube—Bulk Modulus. Let $ABDCGHEFA$ be a *unit cube* and let forces T_x, T_y and T_z act perpendicularly to the faces $BEHD$ and $AFGC$, $ABDC$ and $EFGH$, and $ABEF$ and $DHGC$ respectively, as shown, (Fig. 177). Then, if α be the increase per *unit* length per *unit* tension *along the direction of the force* and β, the contraction produced per *unit* length per *unit* tension, *in a direction perpendicular to the force*, the elongation produced in the edges AB, BE and BD, will, obviously, be $T_x.\alpha$, $T_y.\alpha$ and $T_z.\alpha$, respectively, and the contractions produced *perpendicular to them* will be $T_x.\beta$, $T_y.\beta$, and $T_z.\beta$. The lengths of the edges thus become the following :

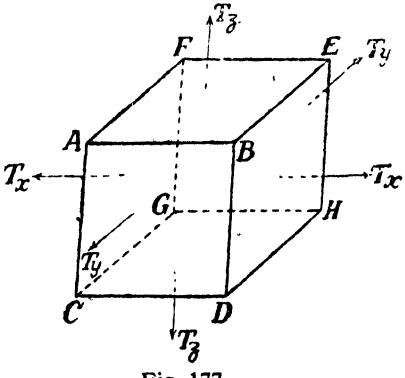


Fig. 177.

$$AB = 1+T_x.\alpha-T_y.\beta-T_z.\beta.$$
$$BE = 1+T_y.\alpha-T_x.\beta-T_z.\beta.$$
$$BD = 1+T_z.\alpha-T_x.\beta-T_y.\beta.$$

Hence the volume of the cube now becomes

$$(1+T_x.\alpha-T_y\,\beta-T_z.\beta)\,(1+T_y.\alpha-T_x.\beta-T_z.\beta)\,(1+T_z.\alpha-T_x\,\beta-T_y.\beta)$$
$$=1+(\alpha-2\beta)\,(T_x+T_y+T_z),$$

neglecting squares and products of α and β, which are very small compared with the other quantities involved.

If $$T_x = T_y = T_z = T,$$

the volume of the cube becomes $1+(\alpha-2\beta).3T$.

And, therefore, *increase in the volume of the cube*

$$= 1+3T(\alpha-2\beta)-1 = 3T(\alpha-2\beta).$$

If, instead of the tension T outwards, we apply a pressure P, compressing the cube, *the reduction in its volume* will similarly be $3P(\alpha-2\beta)$, and, therefore, *volume strain* is equal to $3P(\alpha-2\beta)/1$, or equal to $3P(\alpha-2\beta)$. [$\because$ *original volume* of the cube = 1.

Hence *Bulk Modulus*, $K = \dfrac{\text{stress}}{\text{volume strain}} = \dfrac{P}{3P(\alpha-2\beta)}$.

Or, $$K = \frac{1}{3(\alpha-2\beta)}. \qquad \ldots(I)$$

And, *Compressibility*, which is the *reciprocal of Bulk Modulus*, is, therefore, equal to $3(\alpha-2\beta)$.

114. Modulus of Rigidity. Let the top face $ABHG$, of a cube (Fig. 178), be '*sheared*' by a shearing force F, *relative to the bottom face*, such that A takes up the position A' and B, the position B', the angle ADA' being equal to the angle $BCB' = \theta$. Then,

$$\text{stress} = \frac{F}{\text{area of the face } ABHG}$$
$$= \frac{F}{L^2} = T, \text{ say,}$$

where L is the length of each edge of the cube.

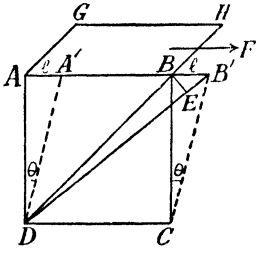


Fig. 178.

Let the displacement $AA' = BB' = l$.

Then, *shear strain* $= l/L = \theta$.

And $\therefore$ *coefficient of rigidity*, $n = T/\theta$.

Now, extension of the diagonal DB, due to extension along AB is $DB.T.\alpha$, and that due to contraction* along CA is $DB.T.\beta$. Therefore, *total extension in length of the diagonal* DB now becomes

$$DB\times(T.\alpha+T.\beta) = DB\times T(\alpha+\beta).$$
$$= L\sqrt{2}.T(\alpha+\beta). \quad [\because DB = \sqrt{L^2+L^2} = L\sqrt{2}$$

Drop a perpendicular BE from B on to DB'.

* See § 117, page 288.

Then, *increase in length of DB* is practically equal to EB'.

And, clearly, $EB' = BB'.\ cos\ BB'E.$

$= l\ cos\ 45° = l/\sqrt{2}$ [$\because$ $< BB'E = 45°$, very nearly, and $cos\ 45° = 1/\sqrt{2}$.

$\therefore$ $L\sqrt{2}.T(\alpha+\beta) = l/\sqrt{2}.$

Or, $\frac{L.T}{l} = \frac{1}{2(\alpha+\beta)}$. Or, $\frac{T}{l/L} = \frac{1}{2(\alpha+\beta)}$.

Or, $\frac{T}{\theta} = \frac{1}{2(\alpha+\beta)}$. [$\because$ $l/L = \theta$.

And, since $T/\theta = n$, the *coefficient of rigidity* of the material of the cube, we have

$$n = \frac{1}{2(\alpha+\beta)}. \qquad ...(II)$$

115. Young's Modulus. If we now imagine a *cube of unit edge*, acted upon by *unit tension* along one edge, the extension produced is α. Then, clearly,

stress $= 1$, and *linear strain* $= \alpha/1 = \alpha$.

Therefore, *Young's Modulus*, $Y = 1/\alpha$. ...(*III*)

116. Relation connecting the Elastic Constants. We have from relation (*I*), above,

$$\alpha - 2\beta = 1/3K. \qquad ...(i)$$

And, from relation (*II*), $\alpha+\beta = 1/2n.$ .. (*ii*)

$\therefore$ subtracting (*i*) from (*ii*), we have

$$3\beta = \frac{1}{2n} - \frac{1}{3K} = \frac{3K-2n}{6nK},$$

whence, $$\beta = \frac{3K-2n}{18nK}.$$

Again, multiplying (*ii*) by 2 and adding to (*i*), we have

$$3\alpha = \frac{1}{n} + \frac{1}{3K} = \frac{3K+n}{3Kn}. \quad \text{Or, } \alpha = \frac{3K+n}{9Kn}.$$

$\therefore$ $\frac{1}{Y} = \frac{3K+n}{9Kn}$. Or, $Y = \frac{9Kn}{3K+n}$. .. (*a*)

[$\because$ $\alpha = 1/Y$ from (III), above.

Or, $$\frac{9}{Y} = \frac{3K+n}{Kn} = \frac{3K}{Kn} + \frac{n}{Kn},$$

whence, $$\frac{9}{Y} = \frac{3}{n} + \frac{1}{K}. \qquad ...(b)$$

This, then, is the relation connecting the three elastic constants.

117. Poisson's Ratio. It is a commonly observed fact that when we stretch a string or a wire, *it becomes longer but thinner, i.e.*, the increase in its length is always accompanied by a decrease in its cross-section (though not sufficient enough to prevent a slight increase in its volume). In other words, *a longitudinal or tangential strain produced in the wire is accompanied by a transverse or a lateral strain in it.* And, of course, what is a true of a wire, is true of all other bodies under strain. Thus, for example, when a cube is subject-

ed to an outward force perpendicular to one pair of its faces, there is elongation produced along this face, but a contraction in a direction perpendicular to it, (as we have seen already in §113).

The ratio between lateral strain (β) *to the tangential strain* (α) *is constant* for a body of a given material and is called the Poisson's ratio for that material. It is usually denoted by the letter* σ.
Thus, *Poisson's ratio = lateral strain/tangential strain ;* or, $\sigma = \beta/\alpha$.

It follows, therefore, that if a body under tension suffers no lateral contraction, the *Poisson's ratio* (σ) for it is *zero* ; and, because its volume increases, is density decreases.

The relations for K and n above may now be put in terms of *Poisson's ratio, as follows* :

We have, from relation *I*, above,

$$K = \frac{1}{3\alpha\left(1-\frac{2\beta}{\alpha}\right)} = \frac{1}{3\alpha(1-2\sigma)} = \frac{Y}{3(1-2\sigma)}. \quad \left[\because \frac{1}{\alpha} = Y \text{ [see (III) above.}\right.$$

whence, $$Y = 3K(1-2\sigma),\dagger \qquad \ldots \qquad \ldots \quad (iv)$$

Similarly, from relation (II) above, we have

$$n = \frac{1}{2\alpha\left(1+\frac{\beta}{\alpha}\right)} = \frac{1}{2\alpha(1+\sigma)} = \frac{Y}{2(1+\sigma)},$$

whence, $$Y = 2n(1+\sigma)\dagger \qquad \ldots \qquad \ldots \quad (v)$$

Now, from relations (*iv*) and (*v*) we have

$$3K(1-2\sigma) = 2n(1+\sigma),$$

whence, $$\sigma = \frac{3K-2n}{6K+2n},$$

which gives the value of *Poisson's ratio* in terms of K and n.

Similarly, if we eliminate σ from (*iv*) and (*v*), we have

$$Y = \frac{9Kn}{3K+n}, \quad \text{[Same as relation } (a), \text{ above.}$$

whence, $$\frac{9}{Y} = \frac{3}{n} + \frac{1}{K}. \quad \text{[Same as relation } (b), \text{ above.}$$

Limiting values of σ. We have seen above how

$$3K(1-2\sigma) = 2n(1+\sigma),$$

where K and n are essentially *positive* quantities. Therefore,

(*i*) *if* σ *be a positive quantity*, the right hand expression, and hence also the left hand expression, must be positive, and for this to be so, $2\sigma < 1$, or $\sigma < \frac{1}{2}$ or $\cdot 5$. And,

---

**i.e.*, the lateral strain is proportional to the longitudinal strain. This is, however, so only when the latter is small.

† These relations would not be found to apply in the case of wire specimens of materials for the simple reason that the process of wire-drawing brings about at least a partial alignment of the minute crystals of the substance, which thus no longer remain oriented at random, with the result that the substance loses its isotropic character.

(*ii*) *if* σ *be a negative quantity*, the left hand expression, and hence also the right hand expression, must be positive, and this is possible only when σ be not less than -1.

Thus, the limiting values of σ are -1 and ·5. Or, else, as will be readily seen from relations (*iv*) and (*v*) above, either the bulk modulus or the modulus of rigidity would become *infinite*. Further, a negative value of σ would mean that, on being extended, a body should also expand laterally, and one can hardly expect this to happen, ordinarily. At least, we know of no such substance so far. Similarly, a value of $\sigma = 0{\cdot}5$ would mean that the substance *is perfectly incompressible*, and, frankly, we do not know of any such substance either.

In actual practice, the value of σ is found to lie between ·2 and ·4, although Poisson had a theory that the value of σ for all elastic bodies should be ·25, but this is not borne out by any experimental facts.

118. Determination of Young's Modulus. Young's modulus, as we know, is the ratio between *tensil stress* (or tangential force applied per unit area) and *elongation strain* (or extension per unit length). The extension produced is rather small and it is difficult to measure it with any great degree of accuracy. The different methods used are thus merely attempts at measuring this extension accurately. We shall consider here only two methods, *viz.*, one for a wire, and the other for a thick bar.

(*i*) **For a Wire—Searle's Method.** Two wires, *A* and *B*, of the *same* material, length and area of cross-section, are suspended from a rigid support and carry, at their lower ends, two metal frames, *C* and *D*, as shown in Fig. 178, one carrying a constant weight *W* to keep the wire stretched or *taut*, and the other, a hanger *H*, to which slotted weights can be slipped on, as and when desired.

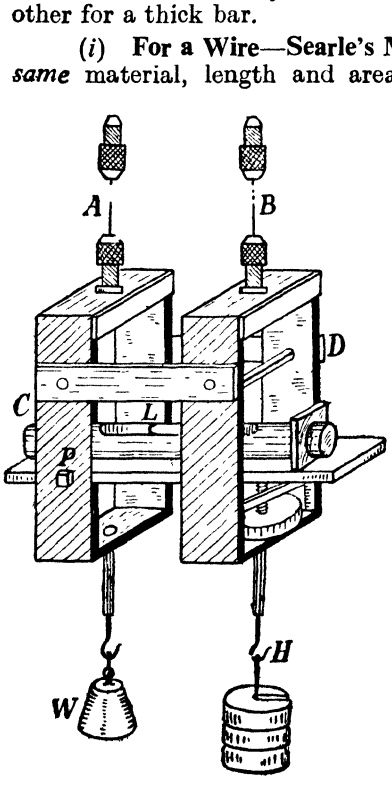


Fig. 178

A spirit-level *L* rests *horizontally* at a point *P* in frame *C*, and on the tip of a *micrometer screw* (or spherometer) *S*, working through a nut in frame *D*.

The screw is worked up or down, until the air bubble in the spirit-level is just in the centre. Weights are now slippedo n to the hanger, so that the frame *D* moves down a little due to the extension of wire *B*, and the air bubble shifts towards *P*. The screw is now worked up to restore the bubble back to its central position. The distance through which the screw is moved up is read on the vertical scale, graduated in half-millimetres.

and fixed alongside the disc of the screw. This gives the increase in length of wire *B*. A number of observations are taken by increasing the weight in the hanger by the *same equal steps* and making the adjustment for the level for each additional weight. The mean of all these readings of the screw gives the mean increase in the length of the wire, for the stretching force due to the given weight. Thus, if *l cms.* be the increase in the length of wire *B*, and *L cms.*, its original length, we have

$$\textit{elongation strain} = l/L.$$

And if *W k.gms.* be the weight added each time to the hanger, the stretching force is equal to $W \times 1000$ *gms. wt.* $= W \times 1000 \times 981$ *dynes*, or equal to *F dynes*, say.

So that, if *a sq. cms.* be the area of cross-section (πr^2) of the wire, we have $\textit{tensile stress} = F/a$.

And, $\therefore$ *Young Modulus* for the material of the wire, *i.e.*,

$$Y = \frac{F}{a} \div \frac{l}{L} = \frac{F \times L}{a \times l}.$$

The other wire *A* merely acts as a *reference wire*, its length remaining constant throughout, due to the constant weight suspended from it (which need not be known). Any yielding of the support or change in temperature during the experiment affects both the wires equally, and the *relative increase* in the length of *B* (with respect to *A*) thus remains unaffected by either change.

If a graph be now plotted between the load suspended and the extension produced, it would be found to be a straight line (just like *OA* in Fig. 169), passing through the origin, showing that the extension produced is directly proportional to the load. Hooke's law also can thus be easily verified.

(*ii*) **For a thick Bar—Ewing's Extensometer Method.** *Ewing's Extensometer* is merely a device to magnify the small extension of the bar under test and consists of two metal arms, *APS* and *CQD*, (Fig. 179), pivoted at *P* and *Q*, by means of pointed screws, on the vertical bar *B* itself, (the *Young's modulus* for the material of which is to be determined), so that they are free to rotate about *P* and *Q*. The arm *APS* is bent at right angles, as shown, and carries a *micrometer screw S* at its lower end, and a microscope *M*, fitted with a *micrometer scale*, at the end of an arm, pivoted at its upper end *A*.

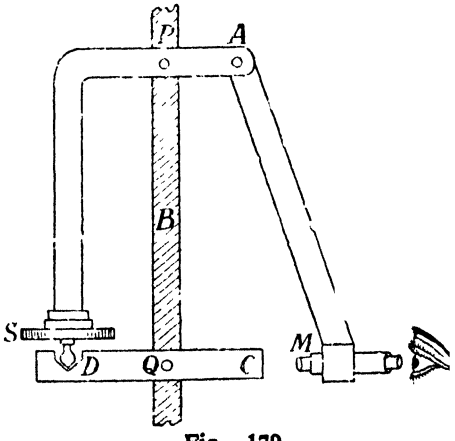


Fig. 179.

The other horizontal arm CQD, has a V-shaped groove at D for the micrometer screw to rest in, and a fine *horizontal line* marked on the end C.

The bar B is fixed at its upper end, the two metal arms are adjusted to be horizontal, by means of the micrometer screw S, and the microscope focused on the horizontal line on C. The bar is now stretched downwards (by means of a *testing machine*), so that *the horizontal arm CQD gets tilted a little about D as its fulcrum*,—the end C, with the fine mark on it, moving down to C', (Fig. 180), and the point Q to Q'. The microscope is again focused on the mark and the distance CC' through which it has shifted downwards is measured accurately on the micrometer scale of the eye-piece. Let it be equal to h.

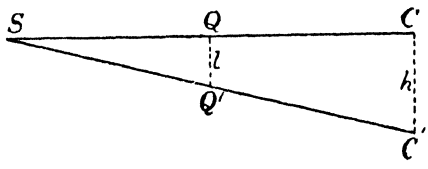


Fig. 180.

Now, obviously, the increase in the length PQ of the rod is $QQ' = l$, say.

Then, clearly, in the two similar triangles SQQ' and SCC',

we have $QQ'/CC' = SQ/SC$. Or, $l/h = SQ/SC$,

whence, $l = SQ.h./SC$.

Thus, knowing SQ, SC and h, we can determine l to quite a high degree of accuracy.

Then, from the length PQ of the bar, its area of cross-section and the stretching force applied to it, we can easily calculate the value of *Young's Modulus* for its material.

N.B. A modification of Ewing's Extensometer, as shown in Fig. 181, is called the *Cambridge Extensometer*, in which there is a vibrating reed R arranged, as shown, the arrangement being such that as the bar B is stretched by the testing machine, *that part of the reed which touches the micrometer screw M, moves downwards through a distance five times the extension of the rod.* Thus, by noting the micrometer screw readings, when the vibrating reed just touches the micrometer screw-point both before and after the rod B has been stretched, we can *directly* obtain the increase l in the length of the rod.

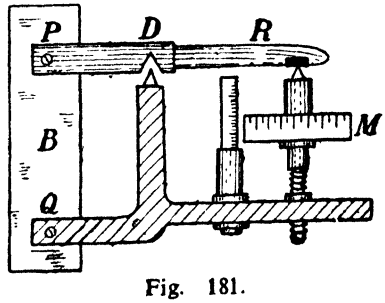


Fig. 181.

119. Determination of Poisson's Ratio for Rubber. To determine the value of σ for rubber, we take about a metre-long tube AB of it, (Fig. 182), such, for example, as the tube of an ordinary cycle tyre, and suspend it vertically, as shown, with its two ends properly stoppered with rubber bungs and seccotine*. A glass tube C, open at both ends, about half a metre long and about 1 *cm.* in diameter, graduated in cubic centimetres, is fitted vertically into it through

* a type of liquid glue.

a suitable hole in the stopper at the upper end A, so that a major part of it projects out.

The rubber tube is completely filled with water until the water rises up in the glass tube to a height of about 30 *cms.* from A. A suitable weight W is now suspended from the lower end B of the tube. This naturally increases the length as well as the internal volume of the tube. The increase in length is read conveniently on a vertical metre scale M, with the help of a pointer P, attached to the suspension of W, and the increase in volume, from the change in the position of the water column in C.

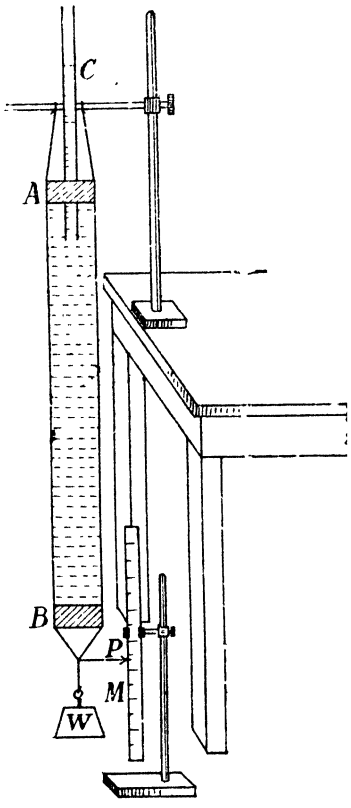


Fig. 182.

Let the original *length*, *diameter* and *volume* of the rubber tube be L, D and V respectively.

Then, its *area of cross-section*, $A = \pi(D/2)^2 = \pi D^2/4$, ... (*i*)

differentiating which, we have

$$dA = \frac{\pi D}{2} \cdot dD,$$

whence, $dA = 2A.dD/D$. ... (*ii*)

[From (*i*) above, by eliminating π.

Now, if corresponding to a small increase dV in the volume of the rubber tube, the increase in its length be dL, and the decrease in its area of cross-section be dA, we have

$$V+dV = (A-dA)(L+dL). \quad \left[\because \text{volume} = \text{area of cross-section} \times \text{length}.\right.$$
$$= AL+A.dL-dA.L-dA.dL$$

Or, $V+dV = V+A\,dL-dA.L$, [where $A.L = V$, the original volume of the tube.

neglecting $dA.dL$, as a very small quantity, compared with the other terms in the expression.

So that, $dV = A.dL - dA.L = A.dL - \frac{2AL}{D} dD$. [Substituting the value of dA from (*i*), above.

Or, dividing both sides by dL, we have

$$\frac{dV}{dL} = A - \frac{2AL}{D} \cdot \frac{dD}{dL}. \quad \text{Or,} \quad \frac{2AL}{D} \cdot \frac{dD}{dL} = A - \frac{dV}{dL}$$

whence, $\frac{dD}{dL} = \left(A - \frac{dV}{dL}\right) \Big/ \frac{2AL}{D} = \frac{AD}{2AL} - \frac{dV}{dL} \cdot \frac{D}{2AL}$

$$= \frac{D}{2L} - \frac{dV}{dL} \cdot \frac{D}{2AL}.$$

Or, $$\frac{dD}{dL} = \frac{D}{2L}\left(1 - \frac{1}{A}\cdot\frac{dV}{dL}\right) \qquad \ldots(iii)$$

Now, *Poisson's ratio*, $\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{dD/D}{dL/L} = \frac{dD}{D} \times \frac{L}{dL}$.

Or, $$\sigma = \frac{L}{D}\cdot\frac{dD}{dL}.$$

Or, substituting the value of dD/dL from relation (*iii*) above, we have

$$\sigma = \frac{L}{D}\cdot\frac{D}{2L}\left(1 - \frac{1}{A}\cdot\frac{dV}{dL}\right) = \frac{1}{2}\left(1 - \frac{1}{A}\cdot\frac{dV}{dL}\right).$$

Thus, knowing the *area of cross-section* (A) of the tube, the *change in its volume* (dV) and the *change in its length* (dL), we can easily calculate the value of σ for its material.

N.B.—An identical method may be used for the determination of the value of σ for glass, but since the change in its volume is comparatively much too small, we have to use a capillary tube, instead of an ordinary glass tube, to measure it to an adequate degree of accuracy.

120. Resilience. By the *resilience* of an elastic body we understand its *capacity for resisting a blow or a mechanical shock, without acquiring a permanent set,** and we measure it by the *amount of work done in straining the body up to the elastic limit.* Let us consider it for a uniform bar of length L and area of cross-section a.

We know that when the bar is subjected to a stretching force W, so that it increases in length by l, we have

Young's modulus for the material of the bar, $Y = \frac{\text{stress}}{\text{strain}}$

$$= \frac{W/a}{l/L} = \frac{W}{a}\cdot\frac{L}{l} = \frac{F}{\text{strain}},$$

where F denotes the stress W/a.

Now, *work done per unit volume in elongation strain* $= \frac{1}{2}$ *stress* $\times$ *strain.*

$\therefore$ *work done in producing extension* $l = \frac{1}{2}$ (*stress* $\times$ *strains*) $\times$ *volume.*

$$= \tfrac{1}{2} F \cdot \frac{F}{Y} \times \text{volume } (V) = \frac{1}{2}\cdot\frac{F^2}{Y}\cdot V = \frac{VF^2}{2Y} \quad \left[\because \text{strain} = F/Y.\right.$$

Thus, *work done*, or *resilience of the bar*,

$$= \frac{VF^2}{2Y} = \frac{\text{volume} \times (\text{stress})^2}{2 \times \text{Young's modulus}}.$$

And $\therefore$ *resilience per unit volume of the bar*

$$= \frac{F^2}{2Y} = \frac{(\text{stress})^2}{2 \times \text{Young's modulus}}.$$

Height from which the bar can be dropped without acquiring a permanent set.—Since resilience is a measure of the power to resist a

*The meaning attached to the word '*resilience*' in our common everyday parlance is different, *viz.*, that the body comes back to its normal condition when the applied forces are removed.

blow or shock without acquiring a permanent set, let us see from what height the bar can be dropped without taking on a permanent set.

This height must obviously be one in falling through which the bar acquires energy equal to its own resilience. Let it be h. Then, if w be the *weight per unit volume* of the bar, clearly,

energy acquired by the bar in falling through height $h = Vw.h.$

Equating this against the resilience of the bar, therefore, we have

$$Vw.h. = VF^2/2Y, \text{ whence, } h = F^2/2wY.$$

Thus, the bar can absorb a blow or a shock due to fall from this much height.

Proof Resilience. The maximum amount of energy per unit volume that can be stored in a body or a piece of material, without its acquiring a permanent set, *i.e.*, without its undergoing a permanent strain, is called its *proof resilience*. Thus, if F_m be the *maximum stress* to which a material, in the form of a wire, can be subjected *i.e.*, if F_m be its *elastic limit*, we have

$$\textit{proof resilience of the material} = F_m^2/2Y.$$

121. Effect of a suddenly applied load. Suppose we have a uniform bar of length L and area of cross-section a, suspended vertically from one end with a collar C provided at the other and with a weight W, in the form of a ring, threaded on to it at a height h from the collar, as shown in Fig. 183.

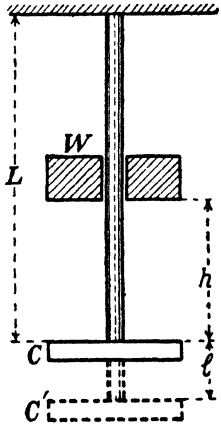


Fig. 183.

If we now allow the weight to fall freely so as to hit the collar, so that the length of the bar is increased by a small amount l, with the collar taking up the position C', clearly, the total height through which the weight has fallen is $(h+l)$.

$\therefore$ *potential energy lost by the weight* $= W(h+l)$.

This has obviously been utilised in stretching the bar through l and must, therefore, be equal to the work done in so stretching it.

If F_m be the *maximum stress* in the bar, the *resistance offered by the bar, or the restoring force set up in it* $= F_m.a.$

And, since *work done during stretch* $= \frac{1}{2}$ *stretching force* $\times$ *stretch*, we have [see page 285.

$$\textit{work done in stretching the bar} = \tfrac{1}{2}.F_m.a.l$$

And, therefore, $$W(h+l) = \tfrac{1}{2}F_m.a.l \qquad \ldots(i)$$

Now, as we know, $Y = \dfrac{\textit{tensile stress}}{\textit{tensile strain}} = \dfrac{F_m}{l/L}$, and $\therefore\ l = F_m.L/Y.$

Substituting this value of l in expression (i) above, we have

$$W\left(h+\frac{F_m.L}{Y}\right) = \frac{1}{2}F_m.a.\frac{F_m.L}{Y} = \frac{1}{2}\frac{F_m^2.a.L}{Y}$$

Or, $$2W\left(h+\frac{F_m.L}{Y}\right)=\frac{F_m^2.aL}{Y}.$$

Solving this quadratic equation, we have

$$F_m=\frac{2W\frac{L}{Y}\pm\sqrt{4W^2.\frac{L^2}{Y^2}+8\frac{aL}{Y}.Wh}}{2aL/Y}$$

$$=\frac{W}{a}\pm\frac{\sqrt{W^2L^2+2aL.W.h.Y}}{aL}.$$

So that, if $h=0$, we have $F_m=0$ or $2W/a$.

Since the zero value of F_m has no physical significance, we have

$$F_m=2W/a.$$

This clearly shows that when the full load W is applied to a bar *all at once*, the maximum stress is $2W/a$, which is clearly twice the value of the maximum stress W/a, which is set up in the bar when the load (W) is applied gradually to it, as for example, when the bar is stretched in a testing machine.

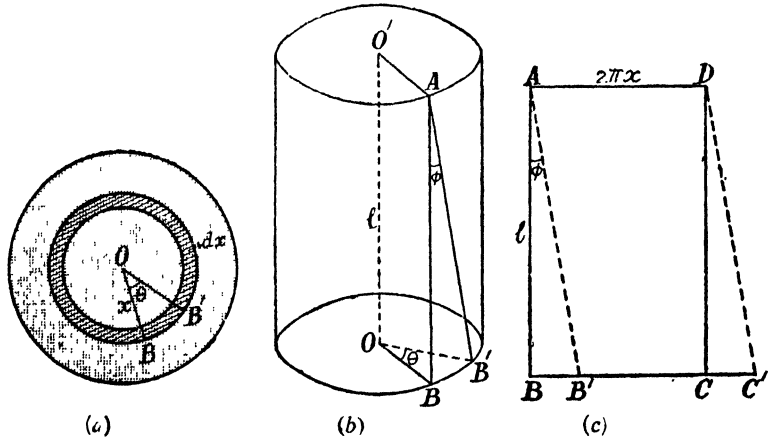
In other words, *the effect of a suddenly applied load is to produce a stress double that produced by a gradually applied one.*

122. Twisting Couple on a Cylinder (or Wire). If we have a cylinder or a wire, clamped at one end, and *twist it through an angle* θ about its axis, it is said to be *under tension.* Due to the elasticity of the material of the cylinder or the wire, *a restoring couple is set up in it, equal and opposite to the twisting couple.*

Consider a cylindrical rod of length l and radius r, of a material of coefficient of rigidity n.

Let its upper end be fixed and let a *couple* be applied, in a plane *perpendicular to its length* (with its axis coinciding with that of the cylinder) twisting it through an angle θ (*radians*).

This, incidentally, is an example of what is called a '*pure*' shear,



(a) (b) (c)

Fig. 184.

for the twist produces a change neither in the length nor the radius of the cylinder,—the value of the twist for any cross-section of the

cylinder being quite independent of the direction of the couple, a reversal of which also reverses the direction of twist.

Now, *in the position of equilibrium, the twisting couple is equal and opposite to the restoring couple.* Let us calculate the value of this couple.

Imagine the cylinder to consist of a large number of *co-axial, hollow* cylinders, and consider one such hollow cylinder of *radius* x, and *radial thickness* dx, [Fig. 184 (*a*)]. Each radius of the lower end is turned through the *same angle* θ, but *the displacement is the greatest at the rim, decreasing as the centre is approached, where it is reduced to zero.*

Let AB, [Fig. 184 (*b*)], be a line, parallel to the axis, before the cylinder is twisted. On twisting, since the point B shifts to B', the line AB takes up the position AB', such that, before twisting, if this hollow cylinder were to be cut along AB and flattened out, it will form the rectangular plate, $ABCD$, but, after twisting, it takes the shape of a parallelogram, $AB'C'D$, [Fig. 184 (*c*)]. The angle through which this hollow cylinder is *sheared* is, therefore, $BAB' = \phi$, say. Then, clearly,

$$BB' = l.\phi. \quad \text{Also } BB' = x.\theta. \quad \therefore \phi = x.\theta/l. \quad \text{[See Fig. 184 (a).}$$

Obviously, ϕ will have the maximum value where x is the greatest, *i.e., the maximum strain is on the outermost part of the cylinder, and the least, on the innermost.* In other words, *the shearing stress is not uniform all through.*

Thus, although the angle of shear is the *same* for any *one* hollow cylinder, it is different for different cylinders, —being the *greatest for the outermost* and the *least for the innermost cylinder.*

Since
$$n = \frac{\text{shearing stress}}{\text{strain or angle of shear}} = \frac{F}{\phi},$$

we have
$$F = n.\phi = n.x.\theta/l.$$

Now, *face area* of *this* hollow cylinder $= 2\pi x.dx.$

And $\therefore$ *total shearing force on this area*

$$= 2\pi x.dx \times \frac{n.x\theta}{l} = 2\pi n.\frac{\theta}{l}.x^2.dx.$$

Therefore, *moment of this force about the axis* OO', [Fig. 184 (*b*)], of the cylinder is equal to $2\pi n.\theta.x^2.dx.x/l = 2\pi n.\theta\ x^3.dx/l.$

Intergrating this expression between the limits, $x = 0$ and $x = r$, we have

$$\text{total twisting couple on the cylinder} = \int_0^r 2\pi n.\frac{\theta}{l}.x^3.dx.$$

$$= \frac{2\pi n.\theta}{l}\int_0^r x^3.dx = \frac{2\pi n.\theta}{l}\left[\frac{x^4}{4}\right]_0^r$$

$$= \frac{2\pi n.\theta.r^4}{4l} = \frac{\pi.n.\theta.r^4}{2l}.$$

If $\theta = 1$ *radian*, we have

twisting couple per unit twist of the cylinder (or wire) $= \pi nr^4/2l$.

This twisting couple, per unit twist of the wire, is also called the **torsional rigidity** of the cylinder or wire.

Note. If the cylinder be a *hollow* one, of inner and outer radii, equal to r_1 and r_2 respectively, we have

$$\textit{twisting couple on the cylinder} = \int_{r_1}^{r_2} 2\pi n.\frac{\theta}{l}.x^3.dx = \frac{\pi n\theta}{2l}.(r_2^4 - r_1^4).$$

123. Variation of stress in a twisted cylinder (or wire). Let us again imagine a cylinder or wire, of length l and radius n, to consist of a large number of co-axial, hollow cylinders and consider both a cylinder of radius x and the outermost cylinder of radius r, (Fig. 185), in which the lines AB and CD respectively are parallel to the axis OO', before the cylinder is twisted, and shift into the positions AB' and CD' after it has been twisted, as explained above. Then, clearly,

$$\angle BOB' = DOD' = \theta,$$

and the angles through which the two cylinders are sheared are $BAB' = \phi$ and $DCD' = \phi_m$ respectively, where $\phi = BB'/l$ and $\phi_m = DD'/l$, this latter being the strain on the surface of the cylinder and, therefore, the maximum on it. [$\because$ $DD' = r\theta$.

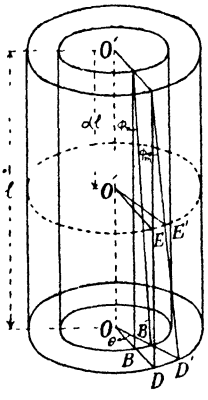


Fig. 185.

Since $OB = OB' = x$ and

$$OD = OD' = r, \text{ we have } \theta = BB'/x = DD'/r.$$

So that, $$\phi_m = \frac{DD'}{l} = \frac{r\theta}{l} = \frac{r}{l}\cdot\frac{BB'}{x} = \frac{r}{x}\cdot\frac{BB'}{l} = \frac{r}{x}.\phi,$$

whence, $$\phi = \frac{x}{r}\cdot\phi_m.$$

Or, *strain in the cylinder or wire at distance x from the axis*

$$= \frac{x}{\textit{radius of cylinder or wire}} \times \textit{maximum strain.}$$

Now, $n = F/\phi$ and, therefore, $F = n\phi = n.\frac{x}{r}.\phi_m$,

where F is the shearing stress at distance x from the axis.

And, since ϕ_m is the *maximum strain*, we have $n.\phi_m = F_m$, the *maximum shearing stress on the wire, i.e., on its surface.*

We, therefore, have $$F = \frac{x}{r}.F_m,$$

Or, *shearing stress at distance x from the axis*

$$= \frac{x}{\textit{radius}} \times \textit{maximum stress.}$$

In other words, both *shearing strain and stress, go on increasing as we proceed away from the axis of the cylinder or wire and acquire their maximum values on its surface.*

Let us now see whether there is any variation of shearing strain and stress along the length of the cylinder or wire also.

Let us, therefore, consider the stress in the plane $EO''E'$ of the cylinder at a distance al from its upper fixed end, where $a>0$ and <1. Here, clearly, $\angle EO''E'$ between the radii $O''E$ and $O''E' = a\theta$.

$= EE'/r$ and $\therefore$ $EE' = r.a\theta$. [$\because$ $O''E = O''E' = r$.

$\therefore$ shear strain on the surface of the wire, *in this plane*

$$= EE'/CE = r.a\theta/al = r\theta/l.$$

Now, $$\theta = \frac{l}{r}.\phi_m,$$ as we have seen above.

So that, *shear strain on the surface of the wire in plane* $E''OE'$

$$= \frac{r}{l} \cdot \frac{l}{r}.\phi_m = \phi_m,$$

i.e., the same as in plane DOD', as discussed above. Clearly, therefore, shearing stress in this plane is also the same as in plane DOD', namely, $n\phi_m = F_m$.

Thus, we see that *the shearing stress at a point in a cylinder, or a wire, depends only on the distance of the point from the axis, and not its vertical distance from either end, of the cylinder or the wire.*

124. Strain energy in a twisted cylinder (or wire). Let C be the couple applied to the lower end of a cylinder of length l and radius r, with its upper end fixed and let θ be the angle of twist produced at the former (*i.e.*, lower) end. Then, if the limit of elasticity is not exceeded, the relation between C and Q is a linear one and we obtain a *straight line graph* OP between the two, as shown in Fig 186.

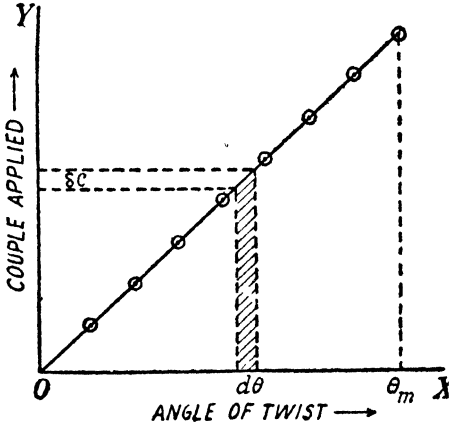


Fig. 186.

So that, for a small increase δC in the value of the couple, the increase in the angle of twist is $d\theta$, and the *work done* on the cylinder, or the *energy stored up* in it, is, therefore, $C.d\theta$, where C is the *average* value of the couple. This is represented by the area of the shaded strip in the Figure. And, therefore, the *total work done* on the wire, or *the total energy stored up in it* for the maximum twist θ_m (represented by OA), to which its lower end is subjected, is represented by the whole area OAP.

This strain energy is obviously equal to $E = \int_0^{\theta_m} C.d\theta$.

Now, C **for a twist** θ **in the wire is, as we know, equal to** $\pi n\theta r^4/2l$,

And, therefore, $$E = \int_0^{\theta_m} \frac{\pi n \theta r^4}{2l}.\theta.d\theta = \frac{\pi n r^4}{2l} \cdot \frac{\theta_m^2}{2} = \frac{\pi n r^4 \theta_m^2}{4l}$$

$$= \frac{1}{2} \cdot \frac{\pi n r^4 \theta_m}{2l}.\theta_m = \tfrac{1}{2} C_m \theta_m,$$

where $\pi n r^4 \theta_m / 2l$ is the value of the maximum couple C_m, corresponding to the maximum twist θ_m ; *i.e.*, $C_m = \pi n r^4 \theta_m / 2l$.

Or, substituting from this the value of θ_m in the expression for E above, we have

strain energy in the twisted cylinder, $E = \frac{1}{2} C_m \theta_m = C_m^2 l / \pi n r^4$.

As will be readily seen, this is half the energy ($C_m.\theta_m$) that would be stored up in the cylinder if the stress in it were to have the same value throughout, equal to its maximum value on the surface of the wire, which, as we know, is not actually the case,—the stress increasing from *zero at the axis* to a *maximum on its surface*, (see § 122 above).

125. Alternative expression for strain energy in terms of stress. We know that *strain energy per unit volume*

$$= \tfrac{1}{2} \text{ stress} \times \text{strain} = \tfrac{1}{2}.F.\phi = \tfrac{1}{2} F. \frac{F}{n} = \tfrac{1}{2} \frac{F^2}{n}. \qquad \ldots (i)$$

Now, if we consider an element of the cylinder or wire, defined by radii x and $x+dx$, the stress will, as we have seen in § 123, be constant at all points in it and its value will be $x.F_m/r$, where F_m is the maximum value of the stress in the wire on its surface ; *i.e.*, $F = x.F_m/r$.

Since the volume of the cylindrical element we are considering is $2\pi x.l.dx$, we have, from relation (i) above, *energy of the cylindrical element*

$$= dE = \tfrac{1}{2} \frac{x^2}{r^2} \cdot \frac{F_m^2}{n} 2\pi x.l.dx.$$

Or, $$dE = \frac{\pi l F_m^2}{n r^2} x^3.dx.$$

And $\therefore$ $$E = \frac{\pi.l F_m^2}{n r^2} \int_r^0 x^3.dx = \frac{\pi l F_m^2}{n r^2} \cdot \frac{r^4}{4} = \frac{\pi r^2 l}{4n}.F_m^2.$$

If ϕ_m be the maximum shear strain corresponding to the maximum stress F_m, we have $F_m = n\phi_m = n.r.\theta_m/l$, (see Fig. 123), where θ_m is the angle of twist for the maximum value of the couple.

So that, $$E = \frac{\pi r^2 l}{4n} \cdot \frac{n^2 r^2 \theta^2}{l^2} = \frac{1}{2} \cdot \frac{\pi r^4 n \theta_m}{2l}.\theta_m = \frac{1}{2} C_m \theta_m,$$

$$\left[\because \quad \frac{\pi n r^4 \theta_m}{2l} = C_m. \right.$$

i.e., the strain energy is again half the value it would have if all the elements of the cylinder (or wire) were subjected to the same maximum stress F_m.

126. Torsional Pendulum. A heavy cylindrical rod or disc, suspended from one end of a fine wire, (attached to its centre), whose

upper end is fixed, constitutes what is called a *torsional pendulum*, (Fig. 187). The rod or disc is turned in its own (*i.e.*, in the horizontal) plane to twist the wire, so that, on being released, it executes *torsional vibrations* about the wire as axis.

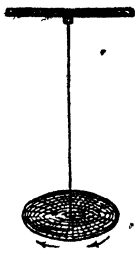


Fig. 187.

Let θ be the angle through which the wire is twisted. Then, the restoring couple set up in it is equal to $\pi.nr^4.\theta/2l = C.\theta$,

where $\pi nr^4/2l$ is the *twisting couple per unit (radian) twist* of the wire, usually denoted by the letter C.

This produces an angular acceleration $d\omega/dt$, in the rod or the disc.

$\therefore$ If I be the moment of inertia of the rod about the wire, we have

$$I.d\omega/dt = -C.\theta \qquad \text{Or,} \qquad d\omega/dt = -C.\theta/I,$$

i.e., the angular acceleration ($d\omega/dt$), of the disc or the rod is proportional to its angular displacement (θ), and, therefore, its motion is simple harmonic. Hence, its time-period is given by

$$t = 2\pi\sqrt{\frac{I}{C/I}} = 2\pi\sqrt{\frac{I}{C}}.$$

Or, $$t = \sqrt{\frac{\textit{moment of inertia of the disc or rod about the wire}}{\textit{restoring couple per unit twist of the wire}}}.$$

127. Determination of the Coefficient of Rigidity (n) for a Wire.

(1) **Statical Method.** This method is based on a direct application of the expression for the twisting couple on a wire deduced in § 122. There are two different types of apparatus used for the purpose, according as the specimen under test is a rod or a wire. We shall now consider these in detail.

(*a*) **Horizontal Twisting apparatus for a Rod.** Here, a couple, which can be measured directly, is applied to a horizontal rod and equated against the expression for the torsional or twisting couple, $\pi nr^4\theta/2l$, whence the value of n for the rod can be easily calculated.

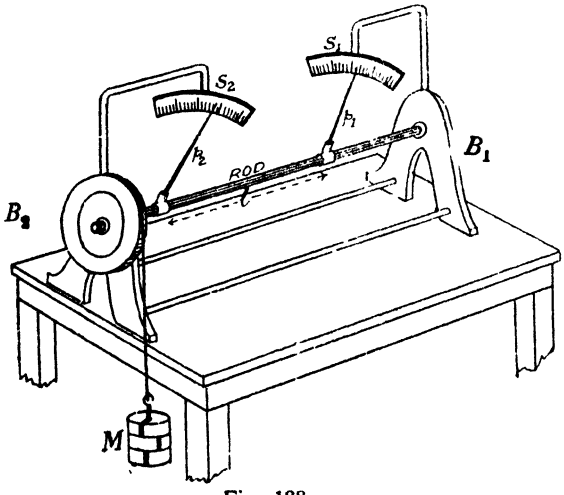


Fig. 188.

The arrangement of the apparatus is as shown in Fig. 188, where one end of the rod, under test, about 50 *cms.* in length and of radius about ·25 *cm.*, is secured firmly to a block B_1, with its other end attached to a steel axle of a large pulley B_2.

A cord is wound round the pulley and has a mass M suspended from its lower free end. Thus, a couple acts on the rod, tending to twist it about its own axis.

Two pointers p_1 and p_2, are clamped on to the rod, at two points, a known distance l apart, so as to move freely over the circular scales S_1 and S_2, graduated in degrees, on which the twist produced in the rod at those two points can be read directly.

Now, if R be the radius of the pulley, the couple acting on the rod, due to the suspended mass (M) is, clearly, equal to $Mg.R$.

This couple is balanced by the couple due to the torsional reaction of the rod, equal to $n\pi r^4(\theta_2-\theta_1)/2l$, where r is the *radius* of the rod and θ_1 and θ_2, the *angles of twist* (*in radians**) produced at the two chosen points, as indicated by the two pointers.

So that, $n\pi r^4(\theta_2-\theta_1)/2l = Mg.R$. Or, $n = \dfrac{2Mg\ R.l.}{\pi r^4(\theta_2-\theta_1)}$,

whence, the value of n for the material of the rod can be easily determined.

The apparatus, though quite simple in manipulation, suffers from two serious drawbacks, *viz.*,

(i) there being one single pointer moving over the circular scale, an error is caused due to eccentricity of the axis of the rod with respect to it ;

(ii) there being just one pulley, only one single force is applied to the end of the rod, attached to it, thus exerting a side-pull on it. This results in friction between the rod and the bearings, thus appreciably hindering the rod from twisting freely.

(*b*) **Vertical Twisting apparatus for a Wire.** This was designed by **Barton,** and here also a couple, which is measured directly, is applied to the lower end of the vertically suspended wire, and the twist produced in it is noted. Then, equating this couple against the expression $n\pi r^4\theta/2l$ for it, the value of n for the wire can be easily calculated.

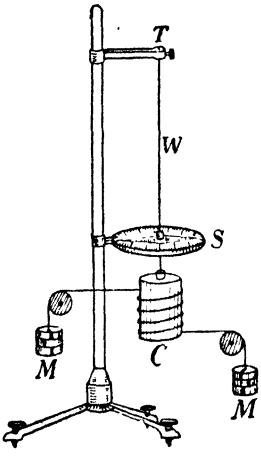


Fig. 189.

The wire W, whose *coefficient of rigidity* is to be determined, is clamped at its upper end T, (Fig. 189), and has a *heavy* cylinder C attached to it, at its lower end.

Two pieces of cords are wound round the cylinder and, leaving it *tangentially* at either end, pass over two frictionless pulleys, as shown, with equal masses M and M, suspended from their free ends.

The *couple*, formed by two masses, rotates the cylinder about the wire as axis, and thus twists the wire through an angle

*To convert *degrees* into *radians*, multiply by $\pi/180$, because $180° = \pi$ radians.

θ (*radians*), say, which is read directly on the horizontal circular scale S by the movement over it of the pointer, attached to the wire, or by the more sensitive lamp and scale method, with the help of a telescope.

Then, if d be the *diameter of the cylinder*, we have

$$\textit{twisting couple applied to the wire} = Mg.d.$$

But, the twisting couple for a twist of θ radians is also

$$= n\pi r^4\theta/2l,$$

where r is the radius of the wire, l, its length and n, the coefficient of rigidity for its material.

Clearly, therefore, $\pi nr^4\theta/2l = Mg.d$, whence, $n = \dfrac{2lMg.d}{\pi r^4\theta}$,

from which n for the material of the wire can be easily obtained.

The two sources of error present in the first method are eliminated here. For, (*i*) due to the very nature of the arrangement of the pointer and the scale, the error due to eccentricity of the axis of the wire does not arise and (*ii*) due to the use of two pulleys, the side-pull on the wire is also avoided.

N B.—It will be noted that the weak point in the above two methods is the radius r of the rod or or wire, the *fourth power* of which occurs in the expression for n. It must, therefore, be measured most carefully.

(2) **Dynamical Method — Maxwell's Vibrating Needle.** The dynamical method of determining n for the material of a wire consists in determining, by direct observation, the time period t of a body, like a disc or a rod, suspended from the wire and executing torsional vibrations about the wire as axis, *i.e.*, of a *torsional pendulum.*

Then, since $t = 2\pi\sqrt{I/C}$, where I is the *moment of inertia* of the body about the wire, and C, the *couple per unit* (*radian*) *twist of the wire*, we can easily obtain from it the value of C. Equating this against the expression $\pi nr^4/2l$ for it, the *coefficient of rigidity* (n) for the wire can be easily calculated.

It is not, however, easy to determine the moment of inertia (I) of the body accurately. *Maxwell*, therefore, devised a method in which the necessity of determining it was altogether obviated. Let us study his method in detail.

Maxwell's Vibrating Needle Method. A hollow tube or cylinder, open at both ends, is *rigidly* fastened in the middle to the wire, the coefficient of rigidity of the material of which is to be determined, and which is suspended vertically from a support, and has a small piece of mirror attached to it, as shown, (Fig. 190), to enable the vibrations of the tube to be observed by the telescope and scale method.

Two *hollow* and two *solid* metal cylinders, *of equal lengths and diameters*, can be fitted into the tube such that, put end to end, they *just fill it completely.*

The solid cylinders are first put into the inner positions and the hollow ones in the outer positions, as shown in Fig. 190 (*a*), and the tube, so loaded, is then given a torsional vibration, and its time-period determined. Let it be t_1. Then,

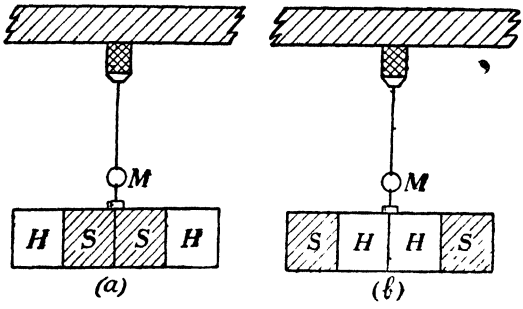


Fig. 190.

$$t_1 = 2\pi\sqrt{I_1/C}, \quad ...(i)$$

where C is the *twisting couple per unit deflection or twist of the wire* and is equal to $n\pi r^4/2l$, and I_1, *the moment of inertia of the loaded tube about the suspension wire as axis.*

The solid and hollow cylinders are then interchanged in position, *i.e., the hollow cylinders are now put in the inner positions and the solid ones in the outer positions*, as shown in Fig. 19) (*b*), and the time-period of the torsional vibration of the loaded tube determined again. Let it be t_2.

Then, $$t_2 = 2\pi\sqrt{I/C}, \quad ... (ii)$$

where I_2 is *now* the *M.I.* of the loaded tube about the suspension wire.

Squaring and subtracting (*i*) from (*ii*), we have

$$(t_2^2 - t_1^2) = \frac{4\pi^2 I_2}{C} - \frac{4\pi^2 I_1}{C} = \frac{4\pi^2}{C}.(I_2 - I_1). \quad .. (iii)$$

Now, let the mass of each hollow cylinder be m_1 and that of each solid cylinder, m_2, and let the length of the hollow tube be $2a$, so that the length of each solid or hollow cylinder is $2a/4$, or $a/2$.

Clearly, then, the *centres of mass* of the inner and outer cylinders are at distances $a/4$ and $3a/4$ respectively from the axis of oscillation.

Therefore, *the change from the first adjustment, when the solid cylinders occupy the inner positions to the second adjustment, when they occupy the outer positions, consists in transferring an extra or excess mass* $(m_2 - m_1)$ *from a distance* $a/4$ *to a distance* $3a/4$ *from the axis of oscillation, on either side of it.* The moment of inertia of the loaded tube, therefore, *increases*, and, by the application of the *principle of parallel axes*, we have

$$I_2 = I_1 + 2(m_2 - m_1)\left[\left(\frac{3a}{4}\right)^2 - \left(\frac{a}{4}\right)^2\right]$$

Here, we multiply the mass by 2, because the change takes place on both the sides.

Or, $I_2 = I_1 + 2(m_2 - m_1) \times \left(\frac{9a^2}{16} - \frac{a^2}{16}\right) = I_1 + 2(m_2 - m_1) \cdot \frac{a^2}{2}$.

i.e., $I_2 = I_1 + (m_2 - m_1).a^2$ And $\therefore$ $(I_2 - I_1) = (m_2 - m_1).a^2$.*

Substituting this value of $(I_2 - I_1)$ in relation (*iii*) above, we have

$$(t_2^2 - t_1^2) = \frac{4\pi^2}{C} \cdot (m_2 - m_1).a^2.$$

Further, substituting the value of C, we have

$$(t_2^2 - t_1^2) = \frac{4\pi^2}{n\pi r^4/2l} \cdot (m_2 - m_1).a^2 = \frac{4\pi^2 \times 2l}{n\pi r^4} \cdot (m_2 - m_1).a^2.$$

Or, $(t_2^2 - t_1^2) = \frac{8\pi.l.a^2}{n.r^4} \cdot (m_2 - m_1)$, whence, $n = \frac{8\pi.l.a^2.(m_2 - m_1)}{(t_2^2 - t_1^2).r^4}$.

Thus, knowing l, a, m_2, m_1, t_2, t_1 and r, the value of n for the material of the given wire can be easily determined.

N B—The value of n obtained by the dynamical method is slightly higher than that obtained by the statical method, because, in most cases, the twist produced by a torsional couple depends, to some slight extent, upon the *time* for which the couple is applied and so, in the dynamical method, where the time of vibration is rather short, the twist (θ) is smaller for the same value of the couple than in the statical method.

Further, since wires are made by squeezing the molten metal through holes, (as in a sieve), their outer layers are invariably tougher than the inner ones, and hence the value of n for a thinner wire needs must be higher than for a thicker wire of the *same* material.

128. (*a*) Determination of Moment of Inertia with the help of a Torsional Pendulum. The moment of inertia of a body of a *regular geometrical shape* can be easily calculated from its *mass and dimensions*. But, if it be of an *irregular shape*, it is not possible to do so. In either case, however, it may be determined by using a torsional pendulum with a disc or a rod of *known* moment of inertia I about the suspension wire and noting the time period (t) for its torsional vibration. Then, mounting on it the given body, such that the axis of the moment of inertia of the two together is again the *same* wire, the time-period (t_1) for the torsional vibration of the combination is determined. Then, if I_1 be the moment of inertia of the body about the wire as axis, the moment of inertia of the combination, in the second case, is clearly equal to $I + I_1$. So that, if C be the torsional couple per unit twist of the suspension wire, we have

$$t = 2\pi\sqrt{I/C} \quad \ldots (i) \quad \text{and} \quad t_1 = 2\pi\sqrt{I + I_1/C}. \quad \ldots (ii)$$

$\therefore$ squaring and dividing relation (*ii*) by (*i*), we have

$$\frac{4\pi^2(I + I_1)/C}{4\pi^2 I/C} = \frac{t_1^2}{t^2}, \text{ whence, } \frac{I + I_1}{I} = \frac{t_1^2}{t^2}.$$

---

*Or, this may easily be deduced as follows :

If I be the moment of inertia of the *hollow tube* about the suspension wire, and I_s and I_h, those of the solid and the hollow cylinders about the vertical axes through their respective centres of mass, we have, *by the principle of paralle axes*,

$$I_1 = I + 2[I_s + m_2.(a/4)^2] + 2[I_h + m_1.(3a/4)^2],$$

and $$I_2 = I + 2[I_h + m_1.(a/4)^2] + 2[I_s + m_2.(3a/4)^2].$$

So that, $$(I_2 - I_1) = (m_2 - m_1).a^2.$$

So that, $$\frac{(I+I_1)-I}{I} = \frac{t_1^2-t^2}{t^2}.$$ [Subtracting the denominator from the numerator on either side.

Or, $$\frac{I_1}{I} = \frac{t_1^2-t^2}{t^2}, \text{ whence, } I_1 = \left[\frac{t_1^2-t^2}{t^2}\right] \times I.$$

Thus, knowing I, t and t_1, we can easily calculate I_1, the moment of inertia of the given body.

(*b*) **Comparison of Moments of Inertia.** If, however, it is simply desired to *compare* the moments of inertia of two bodies, we first use one and then the other, *as the disc or rod of the torsional pendulum*, and determine the time-periods t_1 and t_2 respectively for their torsional vibration about the wire as axis. Then, if I_1 and I_2, be their respective moments of inertia about this axis and C, the torsional couple per unit twist of the wire, we have

$$t_1 = 2\pi\sqrt{I_1/C} \text{ and } t_2 = 2\pi\sqrt{I_2/C}.$$

So that, squaring and dividing one by the other, we have

$$I_1/I_2 = t_1^2/t_2^2,$$

and thus, knowing t_1 and t_2, the moments of inertia of the two bodies may be easily compared.

Note. In the above cases, the amplitude of vibration need not be small, because it is found that the restoring couple continues to be proportional to the twist θ in the wire, up to fairly large values of θ. The assumption made, however, that even with different bodies suspended from the wire, resulting in a change in its longitudinal tension, the value of C (or the twisting couple per unit twist of the wire) remains the same is found to be only approximately true.

129. Bending of Beams—Bending Moment. We must first be clear about the terms, *beam* and *bending moment*.

Beam. A beam is a rod of uniform cross-section, circular or rectangular, whose length is very great compared with its thickness, so that the shearing stresses over any section are small and may be neglected.

Bending Moment. When a beam is fixed at one end and loaded at the other, it bends due to the moment of the load, *the plane of bending* being the same as that of the couple applied.* Restoring forces are called into play by this *deformation* of the beam and, *in the equilibrium state, the restoring or resisting couple is equal and opposite to the bending couple, both being in the plane of bending.*

Irrespective of the manner in which the beam is bent by the couple applied, its filaments on the inner or the concave side get shortened or compressed, and those on the outer or the convex side get lengthened or extended, as shown in Fig. 191. Along a section, in between these two portions, there is a layer or surface in which the filaments are neither compressed

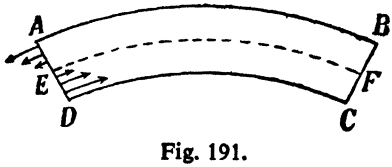


Fig. 191.

*In the case of uniform bending, the longitudinal filaments all get bent into circular arcs in planes parallel to the plane of symmetry, which is then known as the *plane of bending*. And, the straight line, *perpendicular to this plane*, on which lie the centres of curvature of all these bent filaments, is called the *axis of bending*.

nor extended. This surface is called the **neutral surface** and its section (EF) by the plane of bending which is perpendicular to it, is called the **neutral axis.**

In the unstrained condition of the beam, the neutral surface becomes a plane surface, and the filament of this unstrained or unstretched layer or surface, lying in the plane of symmetry of the bent beam, is referred to as the **neutral filament**. It passes through the *c.g.* (or the *centroid*) of every transverse section of the beam.

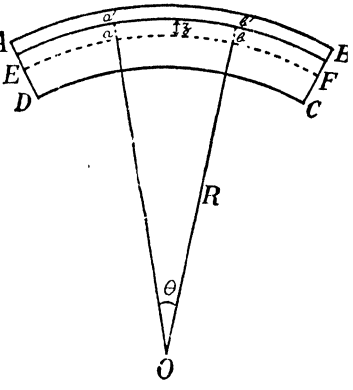


Fig. 192.

The change in length of any filament is proportional to its distance from the *neutral surface.*

Let a small part of the beam be bent, as shown in Fig. 192, in the form of a circular arc, subtending an angle θ at the centre of curvature O. Let R be the radius of curvature of this part of the *neutral axis*, and let $a'b'$ be an element at a distance z from the *neutral axis.*

Then, $$a'b' = (R+z).\theta,$$

and its original length $$ab = R\theta.$$

$\therefore$ *increase in length of the filament* $= a'b' - ab.$

$$= (R+z).\theta - R.\theta = z.\theta.$$

And, since the original length of the filament $= R.\theta$, we have

$$strain = z.\theta/R.\theta = z/R,$$

i.e., the strain is proportional to the distance from the neutral axis.

Since there are no shearing stresses, nor any change of volume, *the contractions and extensions of the filaments are purely due to forces acting along the length of the filaments.*

If $PQRS$ (Fig. 193), be a section of the beam* *at right angles to its length and the plane of bending*, then, clearly, the forces acting on the filaments are perpendicular to this section, and the line MN lies on the *neutral surface.*

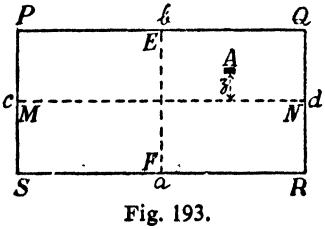


Fig. 193.

Let the *breadth* of the section be $PQ = b$, and its *depth*, $QR = d$.

The forces producing elongations and contractions in filaments act perpendicularly to the upper and the lower halves, $PQNM$ and $MNRS$ respectively, of the rectangular section $PQRS$, their directions being *opposite* to each other.

*The section is shown rectangular purely for the sake of convenience.

Consider a small area δa about a point A, distant z from the *neutral surface*. The strain produced in a filament passing through this area will be z/R, (see above).

Now, $Y = stress/strain$ and $\therefore$ $stress = Y \times strain$.

Therefore, *stress* about the point $A = Y \times z/R$, where Y is the value of *Young's Modulus* for the material of the beam.

And, therefore, *force on the area* $\delta a = \delta a.Y.z/R$

and, *moment of this force about the line* $MN = Y.z \times \delta a \times z/R.$

$$= Y.\delta a.z^2/R.$$

Since the moments of the forces acting on both the upper and the lower halves of the section are in the same direction, the *total moment of the forces acting on the filaments in the section PQRS* is given by

$$\Sigma \frac{Y.\delta a.z^2}{R} = \frac{Y}{R} \Sigma \delta a.z^2.$$

Now, $\Sigma \delta a.z^2$ is the *geometrical moment of inertia* (I_g)* of the section about MN, and, therefore, equal to ak^2, where a is the *whole area* of the surface $PQRS$ and k, its *radius of gyration* about MN.

Hence, the *moment of the forces about* $MN = \frac{Y}{R} \cdot ak^2 = \frac{YI_g}{R}$.

This, then, balances the couple of moment M, say, called the *bending moment*, acting on the beam due to the load, when the beam is in equilibrium ; for, there is no resultant force acting on the area $PQRS$, and the resultant moment about EF, perpendicular to MN, is also *zero*. In other words, it is the moment of the stress set up in the beam or the **moment of resistance to bending**, as it is usually called in engineering practice, and is also *of the nature of a couple*, for only a couple can balance a couple. Obviously, it *acts in the plane of bending* and is *equal to* the *bending moment at the section due to the load*, though, quite frequently, (but, not strictly correctly) it is itself referred to as the *bending moment*. This forms the very basis of the theory† regarding the bending of beams and is, therefore, a relation of fundamental importance.

---

*It is so called because it is proportional to the mechanical moment of inertia of a plane lamina of the same shape as the cross-section. It is denoted, here, by the symbol I_g, so that, the student may not confuse it with the ordinary mechanical moment of inertia, denoted by I.

†The theory is subject to the limitations mentioned in §131, (page 313), which the student would do well to keep in mind.

Imagine the section as a rectangular plate of *unit mass per unit area*, (Fig. 194).

Then, area of the strip AB, of length b and breadth dz, is equal to $b.dz$. And, therefore, its mass $= b.dz.1 = b.dz$.

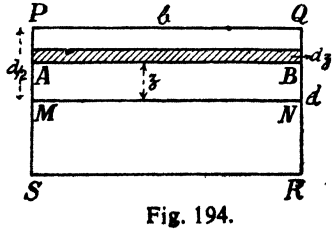


Fig. 194.

Hence, geometric moment of inertia of the strip about $MN = b.dz\,z^2$, and, therefore, moment of inertia of the whole plate or section about MN

$$= 2\int_0^{d/2} b.z^2.dz = 2b\left[\frac{z^3}{3}\right]_0^{d/2}$$

$$= 2b\left[\frac{d^3}{24}\right] = \frac{bd^3}{12}.$$

The quantity $Y.I_g = Y.ak^2$ is called the **flexural rigidity** of the beam.

$\therefore$ *bending moment* $= (Y/R) \times$ *geometric moment of inertia of the section.*

$=$ *flexural rigidity*$/R$,

whatever the shape of the cross-section of the beam.

For a *rectangular cross-section*, $a = b \times d$, and $k^2 = d^2/12$.

$\therefore \quad I_g = ak^2 = b.d^3/12.$

Hence, *bending moment for a rectangular cross-section* $= Y.b.d^3/12R$.

For a *circular section*, $a = \pi r^2$ and $k^2 = r^2/4$.

$\therefore \quad I_g = ak^2 = \pi r^4/4,$

i.e., the same as the moment of inertia of a disc about a diameter.

$\therefore$ *bending moment for a circular cross-section* $= Y.\pi r^4/4R$.

Note. We have seen above how strain in a beam is proportional to the distance z from its neutral axis, and is equal to z/R, where R is the radius of curvature of the portion of the neutral axis under consideration. So that, if F be the stress corresponding to the strain z/R, we have

$$Y = \frac{F}{z/R}. \quad \text{Or,} \quad \frac{F}{z} = \frac{Y}{R}.$$

If, therefore, F_1, F_2 e.c. be the values of stress at distances z_1, z_2 . from the neutral axis, we have

$$F_1/z_1 = F_2/z_2 = Y/R.$$

And $\therefore$ *bending moment* $M = Y\, I_g/R$.

$$= \frac{F_1}{z_1}.\, I_g = \frac{F_2}{z_2}.I_g \text{ etc.} = \frac{I_g}{z_1}.F_1 = \frac{I_g}{z_2}.F_2 \text{ etc.}$$
$$= Z_1 F_1 = Z_2 F_2 \text{ etc.},$$

where $Z_1 = I_g/z_1$ and $Z_2 = I_g/z_2$ are called the **moduli of the section** under consideration.

Thus, **modulus of a section** $=$ $\dfrac{\text{geometrical moment of inertia}}{\text{distance from the neutral axis}}$.

Now, in the case of a flat bar or beam, *of rectangular cross-section*, if the bending be small, there is brought about a change in the shape of the section, such that all lines in it, originally perpendicular to the plane of bending, get bent into arcs, which are all *concentric* and *convex to the axis of bending*. In other words, the layer of the beam, which was originally plane and perpendicular to the plane of bending, and which contained the neutral filament, *now* gets changed into what is called an *anticlastic* surface (Fig. 195), of radius R in the plane of bending (which, here, coincides with the plane of the paper), and, of radius R' in the plane perpendicular to it, the two centres of curvature lying on either side of the beam. This is what is to be expected, because a transverse bending must, of necessity, be associated with a longitudinal bending of the beam, with the curvature of the former opposite to that of the latter. For, the filaments above the neutral axis, which get extended, must obviously suffer a lateral contraction σ times as great and, similarly, the filaments below the neutral axis, which get compressed, must suffer a lateral extension.

Thus, by way of illustration, if a rectangular piece of India-rubber be bent longitudinally in the form of an arc, it takes up the form shown in Fig. 196, with its longitudinal fibres bent so as to be concave with respect to a point *below*, and the transverse fibres, so as to be concave with respect to a point *above*, the rubber piece, in the case shown. *It is this bending, which occurs in a plane normal to the longitudinal plane, that gives the rubber piece (or the beam) an anticlastic curvature.*

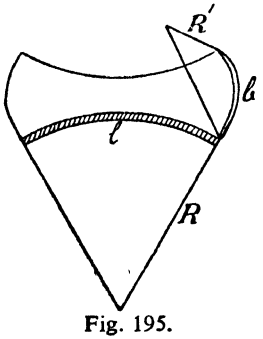
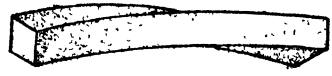


Fig. 195.

And, therefore, as we have seen before, (page 307), the longitudinal and lateral strains in a filament, distant z from the neutral axis will be given by z/R and z/R' respectively. So that, *Poisson's ratio* σ, for the material of the beam, is given by the expression


Fig. 196.

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{z/R'}{z/R} = \frac{R}{R'}.$$

This, then, gives us a method for the determination of σ for the material of a given beam or bar, the two radii being determined directly by attaching suitable pointers to the rod and noting the distances and angles traversed by them, when a known couple is applied to the beam.

130. The Cantilever. A *cantilever* is a beam fixed horizontally at one end and loaded at the other.

(*i*) Cantilever loaded at the free end. Here, two cases arise, *viz.*, (*a*) when the weight of the beam itself produces no bending, and (*b*) when it does so. Let us consider both the cases.

(*a*) *When the weight of the beam is ineffective.* Let AB, (Fig. 197) represent the neutral axis of a cantilever, of length L fixed at the end A, and loaded at B with a weight W, such that the end B is deflected or depressed into the position B' and the neutral axis takes up the position AB, it being assumed that the weight of the beam itself produces no bending.

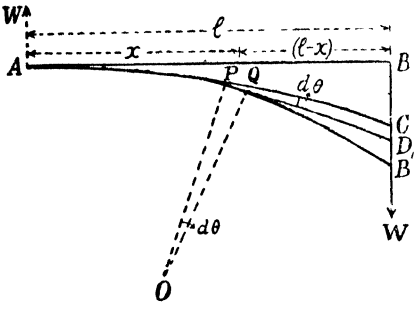


Fig. 197.

Consider a section P of the beam at a distance x from the fixed end A.

The moment of the external couple at *this* section, due to the load W, or the *bending moment* acting on it

$$= W \times PB' = W(L-x).$$

Since the beam is in equilibrium, this must be equal to $YI_g/R = Y.ak^2/R$, where R is the radius of curvature of the *neutral axis* at P.

Therefore, $$W.(L-x) = Y.I_g/R = Y.ak^2/R. \qquad ...(i)$$

Since the moment of the load increases as we proceed towards the fixed end A, the radius of curvature is different at different points and decreases as we approach the point A. For a point Q, however, at a small distance dx from P, it is practically the same as at P. So that,

$$PQ = R.d\theta. \quad \text{Or,} \quad dx = R.d\theta, \quad [\text{where } d\theta \text{ is the } \angle POQ.$$

whence, $$R = dx/d\theta.*$$

Substituting the value of R in (i) above, we have

$$W.(L-x) = \frac{Y.ak^2.d\theta}{dx}.$$

Or, $$d\theta = \frac{W.(L-x).dx}{Y.ak^2}. \qquad ...(ii)$$

Draw tangents to the neutral axis at P and Q, meeting the vertical line through BB' in C and D respectively. Then, the angle subtended by them is also equal to $d\theta$, the radii at P and Q being perpendicular to the tangents there.

Now, clearly, the depression of Q below P is equal to CD, equal to dy, say.

Then, $$dy = (L-x).d\theta = \frac{(L-x)\,W.(L-x).dx}{Y\,ak^2}. \qquad [\text{From } (ii) \text{ above.}$$

$$= \frac{W(L-x)^2.dx}{Y.ak^2}. \quad \ldots \quad \ldots \qquad ...(iii)$$

Therefore, the depression $y = BB'$ of the loaded end B below the fixed end A, is obtained by integratmg the expression for dy between the limits, $x = 0$ and $x = l$.

Or, $$y = \int_0^L \frac{W.(L-x)^2}{Y.ak^2}.dx = \frac{W}{Y.ak^2}\int_0^L (L^2-2xL+x^2).dx.$$

$$= \frac{W}{Y.ak^2}\int_0^L (L^2x^0-2xL+x^2).dx = \frac{W}{Y.ak^2}\left[L^2x-\frac{2x^2L}{2}+\frac{x^3}{3}\right]_0^L$$

$$= \frac{W}{Y.ak^2}\left[L^3-L^3+\frac{L^3}{3}\right] = \frac{W}{Y.ak^2}.\frac{L^3}{3}.$$

Or, $$y = \frac{W.L^3}{3Y.ak^2} = \frac{W.L^3}{3YI_g}. \qquad [\text{Putting back } I_g \text{ for } ak^2.$$

Thus, the free end of the cantilever is depressed by

$$\frac{W\,L^3}{3Y.ak^2} = \frac{W.L^3}{3YI_g}.$$

(*b*) *When the weight of the beam is effective.* In this case, in addition to the weight W at B, the weight of the portion $(L-x)$ of the beam is also acting at the mid-point or the *c.g.* of this portion ; so that, if w be weight per unit length of the beam, a weight $w(L-x)$ is acting at a distance $(L-x)/2$ from the section PQ. And, therefore,

*See solved example 4, page 332, where it is shown that $1/R = d^2y/dx^2$, the *rate of change of slope*. A mathematical minded student will find the solution given there with this value R, much neater and also perhaps a trifle easier.

the *total bending moment on the beam*

$$= W(L-x)+w(L-x).(L-x)/2 = W(L-x)+\frac{w}{2}(L-x)^2.$$

Since the beam is in equilibrium, this must be equal to YI_g/R or Yak^2/R.

Thus, $$W(L-x)+\frac{w}{2}(L-x)^2 = \frac{Y.ak^2}{R} = \frac{Yak^2.d\theta}{dx}.$$

And, $$\therefore\ d\theta = \frac{W(L-x).dx+\frac{w}{2}.(L-x)^2.dx}{Yak^2}.$$

Then, $$dy = (L-x).d\theta = \frac{W(L-x)^2.dx+\frac{w}{2}(L-x)^3.dx}{Yak^2}.$$

And $$\therefore\ y = \frac{W}{Yak^2}\int_0^l (L-x)^2dx+\frac{w}{2Yak^2}\int_0^l (L-x)^3.dx.$$

$$= \frac{WL^3}{3Yak^2}+\frac{wL^4}{8Yak^2}.$$

Now $w.L = W_1$, say, the weight of the beam.

So that, $$y = \frac{WL^3}{3YI_g}+\frac{W_1L^3}{8yI_g}.$$

Or, $$y = \left(W+\frac{3}{8}W_1\right)L^3/3YI_g,$$

i.e., the beam now behaves as though it is loaded at the end B with a weight W plus 3/8ths of the weight of the beam.

(*ii*) **Cantilever loaded uniformly**. Let the uniform load on the cantilever be w per unit length. Then, the weight of the portion of the beam $(L-x)$, *i.e.*, $w(L-x)$ alone produces a bending moment about the section PQ, there being no weight suspended from the end B. And, since this weight $w(L-x)$ acts at a distance $(L-x)/2$ from the section PQ, we have

bending moment due to it $= w(L-x).(L-x)/2.$

$$= \frac{w}{2}.(L-x)^2.$$

For equilibrium of the beam, therefore,

$$\frac{w}{2}(L-x)^2 = \frac{YI_g}{R} = \frac{YI_g.d\theta}{dx},$$

whence, $$d\theta = \frac{w(L-x)^2}{2YI_g}.dx.$$

Substituting this value of $d\theta$, in the relation $dy=(L-x)d\theta$, we have

$$dy = \frac{w}{2YI_g}.(L-x)^3.dx.$$

And $$\therefore\ y = \frac{w}{2YI_g}\int_0^L (L-x)^3.dx = \frac{wL^4}{8YI_g}.$$

Clearly, $w.L = W$, the total load on the beam

So that, $$y = \frac{WL^3}{8YI_g}.$$

It will be seen that this expression could be obtained directly from the result in case (*i*) *b*, above, by putting W, the load at the end B equal to zero.

131. Limitations of the simple theory of bending. In discussing the above simple cases of bending, we have tacitly made the following assumptions :

(*i*) *That the cross-section of the beam remains unaltered during bending.* This, as we have seen, is not strictly true ; for, the extension of the filaments above the neutral surface brings about their lateral contraction, σ times as great, and the contraction of the filaments below the neutral surface brings about their lateral extension. So that, the cross-section of a rectangular beam, bent so as to be *concave downwards along its length is convex downwards across its length.* Similarly, a circular cross-section may change into an oval form. This change in the shape of cross-section of the beam, due to bending, results in a change in the value of the cross-section and hence in that of I_g for it. Usually, however, it is much too small to be of any practical consequence and may safely be ignored.

(*ii*) *That the radius of curvature of the bent beam, or rather that of its neutral surface, is large compared with its thickness.* This is almost always true for all cases of elastic bending.

(*iii*) *That the minimum deflection of the beam is small compared with its length.* This, while more or less true for ordinary engineering problems, is not strictly so in quite a number of cases. Thus, for example, in the case of a clock spring, the deflection produced is very large even within the elastic limit. We shall, therefore, do well to discuss this particular case here, as representing the more general case of strongly bent beams.

132. Strongly bent beams. When a beam bends very strongly, its inclination to its original, unbent or unstrained position, and hence the tangent of this inclination, is no longer small. Consequently, its curvature $(1/R)$ can no longer be taken to be equal to the rate of change of slope, d^2y/dx^2, as is done in the ordinary cases of small curvatures (see solved example 4, page 332) but it is now given by

$$\frac{d^2y}{dx^2} \Big/ \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}.$$

The problem thus becomes quite complicated in many a case, with the differential equations obtained not being amenable to easy solution.

However, there are also some cases which can be investigated in a much simpler manner and we shall here consider only one of these, *viz.*, that of a *flexible cantilever*, like a clock-spring, clamped at one point and loaded at its free end. As we pass or 'pay out' more and more of the spring through the clamp, *keeping the load constant*, its free end drops further and further down, as a result of the large amount of bending, *until finally it becomes quite vertical.* The horizontal distance between the clamp and the loaded end of the spring is now the *maximum* and any more of the spring 'paid out' through the clamp merely hangs vertically.

Thus, let OAB, (Fig.198), be the bent position of the spring, changed at O and loaded with a weight W at the end B, such that the tangent at A, (coordinate $x = x$), makes an angle θ, and that at B (coordinate $x = a$), an angle ϕ, with the axis of x. Then, the *bending moment at A* is clearly equal to $W(a-x)$. So that,

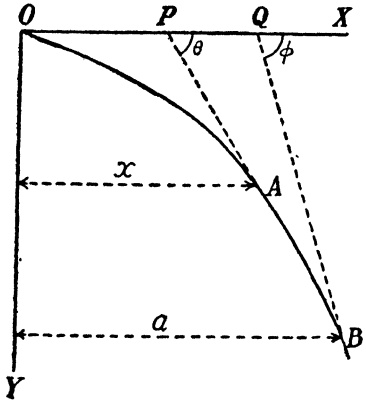


Fig. 198.

$$W(a-x) = Y.I_g.\frac{1}{R},$$

where R is the radius of curvature of portion OA of the spring.

Now, $\frac{1}{R} = \frac{d\theta}{ds}$. [See solved Example 4, page 332.

And $\therefore \quad W(a-x) = Y.I_g.\frac{d\theta}{ds}$.

$$= Y.I_g.\frac{d\theta}{dx}.\frac{dx}{ds}.$$

$$= Y.I_g.\cos\theta.\frac{d\theta}{dx}. \qquad [\because\ dx/ds = \cos\theta.$$

Or, $\quad W(a-x).dx = Y.I_g.\cos\theta.d\theta.$

$\therefore$ integrating this expression between the limits $x = 0$ and $x = a$, the horizontal distance of the loaded end from O, we have

$$\int_0^a W(a-x).dx = Y.I_g\int_0^{\phi} \cos\theta.d\theta.$$

Or, $$W\left[ax - \frac{x^2}{2}\right]_0^a = Y.I_g\left[\sin\theta\right]_0^{\phi}$$

Or, $$\frac{Wa^2}{2} = Y.I_g.\sin\theta. \qquad ...(I)$$

Now, when the loaded end becomes vertical, $\phi = 90°$ and the horizontal distance a becomes the maximum, say, a_m. So that, substituting $a = a_m$ and $\phi = 90°$ in expression I above, therefore, we have

$$wa_m^2/2 = Y.I_g, \quad \text{whence,} \quad a_m^2 = 2YI_g/W.$$

Or, $$a_m = \sqrt{2YI_g/W}. \qquad ...(II)$$

It will be seen that the value of Y for the material of the flexible beam can easily be determined from either of the relations I or II above, if we know the angle of inclination ϕ of the loaded end of the beam with the horizontal, or the maximum horizontal distance, (*i.e.*, a or a_m) of the loaded end from the clamp.

133. Transverse Vibrations of a Loaded Cantilever. If the loaded free end of a cantilever be depressed a little and then released, it starts moving up and down its original position, *i.e.*, executes *transverse vibrations*. Let us calculate the time-period of these vibrations.

We have seen above, (§ 130, pages 311), how

$$y = \frac{WL^3}{3YI_g}, \quad \text{whence,} \quad W = \frac{3YyI_g}{L^3},$$

which gives the load or the force W, required to maintain the cantilever in equilibrium, with its free end depressed or *displaced* through y. This, thus, also measures the *elastic reaction* of the cantilever, which is oppositely directed to it.

Now, if $a = d^2y/dt^2$ be the *acceleration* of the mass M suspended from the free end of the cantilever, (*i.e.*, if M be the mass of the load W), the force of *inertial reaction* on it is equal to $M.a$.

Hence, since the cantilever is in equilibrium and there is no other external force acting on it, we have

$$M.a = -\frac{3.YyI_g}{L^3}, \text{ whence, } a = -\frac{3.YI_g}{ML^3}.y.$$

Or,
$$a = -\mu.y,$$

where $3YI_g/ML^3 = \mu$, a constant for the given cantilever, with the given load.

Thus,
$$a \propto y,$$

i.e., the acceleration of the mass (or of the free end of the cantilever) is directly proportional to its displacement. It is thus a case of *simple harmonic motion*, and its *time-period* is, therefore, given by

$$t = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{1}{3YI_g/ML^3}}.$$

Or,
$$t = 2\pi\sqrt{\frac{ML^3}{3YI_g}}.$$

As can be seen at once, this relation for t gives a good dynamical method for the determination of the value of Young's modulus (Y) for the material of a given beam or rod. It is particularly suitable for beams like a metre stick etc.

134 Depression of a Beam supported at the ends.

(*i*) **When the beam is loaded at the centre.** Let a beam be supported on two knife edges at its two ends A and B, as shown in Fig. 199, and let it be loaded in the middle at C with a weight W.

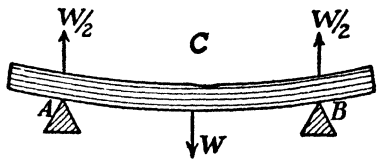


Fig. 199.

The reaction at each knife-edge will clearly be $W/2$, in the *upward* direction.

Since the middle part of the beam is horizontal, *the beam may be considered as equivalent to two inverted cantilevers, fixed at C*, the bending being produced by the loads $W/2$, acting upwards, at A and B.

If, therefore, L be the length of the beam AB, the length of each cantilever (AC and BC) is $L/2$, and the elevation of A or B

above C or, what is the same thing, the depression of C below A and B is given by

$$y = \frac{W/2.(L/2)^3}{3.Y.ak^2}, \qquad \text{[See §130, above.}$$

whence, $$y = \frac{W.L^3}{48\ Y.ak^2} = \frac{WL^3}{48 Y I_g}. \qquad [\because ak^2 = I_g.$$

If the beam be of a *circular* cross-section, we have $ak^2 = \pi r^4/4$, where r is the radius of the cross-section ; so that, for such a beam,

$$y = \frac{WL^3}{48\ Y} \times \frac{4}{\pi r^4} = \frac{WL^3}{12.Y\pi r^4}. \quad \ldots \quad \ldots \quad \ldots (i)$$

And, if the beam be of a *rectangular* cross section, of breadth b and depth d, we have $ak^2 = bd^3/12$, and, therefore, for such a beam,

$$y = \frac{WL^3}{48.Y} \times \frac{12}{bd^3} = \frac{WL^3}{4\ Ybd^3} \quad \ldots \quad \ldots \quad \ldots (ii)$$

(*ii*) **When the beam is loaded uniformly.** Here, let w be the load per unit length of the beam, so that the total weight acting downwards at the c.g. is $wL = W$, where L is the length of the beam. The reaction at each knife-edge is thus obviously $\frac{1}{2}wL$, acting *upwards*, (Fig 200) with the beam behaving as a system of two cantilevers, fixed at C.

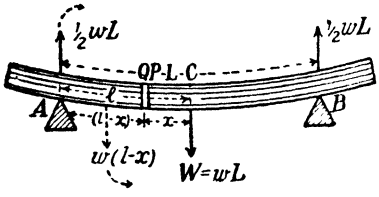


Fig. 200.

Considering again, a section PQ distance x from the mid-point C of the beam and taking half-length of the beam equal to l, the weight of the portion $(l-x)$ of the beam, *i.e.*, a weight $w(l-x)$ acts *downwards* at a distance $(l-x)/2$ from the section PQ.

Thus, the *bending moment about* the section

$$= \tfrac{1}{2}wL\ (l-x) - w(l-x)\ \tfrac{1}{2}(l-x).$$
$$= \tfrac{1}{2}wL\ (l-x) - \tfrac{1}{2}w(l-x)^2.$$

For equilibrium, therefore, this must be equal to the *moment of the resistance to bending viz.*, $Y I_g/R$, where R is the radius of curvature of the neutral axis at PQ.

i.e., $$\tfrac{1}{2}wL(l-x) - \tfrac{1}{2}w(l-x)^2 = Y I_g/R = Y.I_g \frac{d\theta}{dx}, \quad \left[\begin{matrix}\textit{See } 130, \text{ page} \\ 310.\end{matrix}\right.$$

whence, $$d\theta = \frac{\tfrac{1}{2}\ wL(l-x)dx - \tfrac{1}{2}w(l-x)^2.dx}{Y.I_g}.$$

$\therefore$ $$dy = (l-x)\ d\theta = \frac{\tfrac{1}{2}\ wL(l-x)^2.dx - \tfrac{1}{2}\ w(l-x)^3.dx}{Y.I_g}.$$

And hence $$y = \int_0^l \frac{\tfrac{1}{2}wL(l-x)^2.dx - \tfrac{1}{2}w(l-x)^3.dx}{YI_g}$$

$$= \frac{1}{2} \cdot \frac{wL}{YI_g} \int_0^l (l^2 - 2xl + x^2).dx - \frac{1}{2}\frac{w}{YI_g} \int_0^l (l^3 - 3l^2x + 3lx^2 - x^3)dx.$$

$$= \frac{1}{2} \cdot \frac{wL}{YI_g} \left[l^2x - 2l \cdot \frac{x^2}{2} + \frac{x^3}{3} \right]_0^l - \frac{1}{2} \frac{w}{YI_g}$$

$$\left[l^2x - 3l^2 \cdot \frac{x^2}{2} + 3 \cdot \frac{lx^3}{3} - \frac{x^4}{4} \right]_0^l$$

$$= \frac{1}{2} \cdot \frac{wL}{YI_g} \cdot \frac{l^3}{3} - \frac{1}{2} \cdot \frac{w}{YI_g} \cdot \frac{3l^4}{12} = \frac{wL}{YI_g} \cdot \frac{l^3}{6} - \frac{wl^4}{8YI_g}.$$

Now, substituting $L/2$ for l, we have

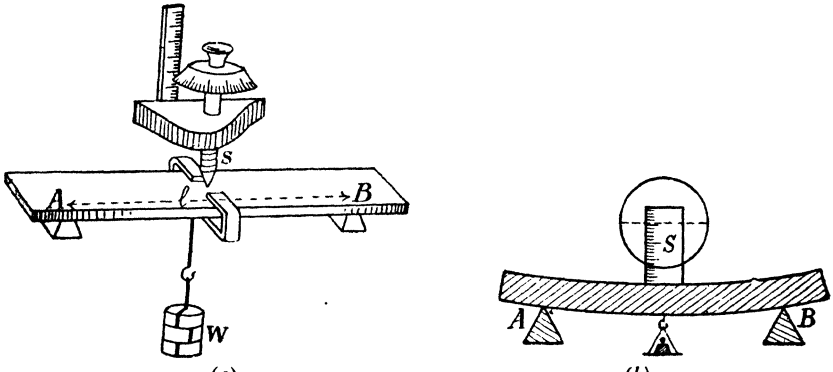
$$y = \frac{w}{YI_g} \left(\frac{L^4}{48} - \frac{L^4}{128} \right) = \frac{w}{YI_g} \cdot \frac{5wL^4}{384} = \frac{1}{YI_g} \cdot \frac{wL}{384} \cdot 5L^3.$$

But $wL = W$, the total weight on the beam.

Hence
$$y = \frac{5WL^3}{384YI_g}.$$

Determination of Y by bending of a beam. It will be easily seen that if we measure the depression (y) of a beam of known dimensions, supported at the ends and loaded at the centre, as in case (*i*) above, we can easily determine the value of Y for its material, by applying relation (*i*) or (*ii*), as the case may be. In practice, it is convenient to use a beam of rectangular cross-section ; so that, knowing W, L, b, d and y, Y can be easily calculated from relation (*ii*) above.

The arrangement is as shown in Figs. 201 (*a*) and (*b*). The beam is supported horizontally and symmetrically, on two parallel knife-



(*a*) (*b*)

Fig. 201.

edges, a known distance L apart and the load is applied by placing weights in a scale pan, also supported on a knife-edge, midway between them, as shown. The depression y of the mid-point, thus produced, is noted directly with the help of a micrometer screw, [Fig. 201 (*a*)], or, more accurately, with the help of a microscope, the eye-piece of which is fitted with crosswires, [Fig. 201 (*b*)].

Readings are taken, first with the load increasing, in *equal steps*, and then with the load decreasing, in the *same equal steps*, and their mean taken. This gives y. Then, if the load were increased (or decreased) in regular steps of W each, we have, as explained above,

$$y = \frac{WL^3}{4Yhd^3},$$

where b and d are the *breadth* and the *depth* of the beam, and Y, the *Young's modulus* for its material.

Hence $$Y = \frac{WL^3}{4y\,b.d^3}.$$

Now, since the depression of the beam is given by the relation $y = WL^3/4Y.b.d^3$, it is clear that, *for a given load, the depression of the beam is*

(*i*) *directly proportional to the cube of its length,*

(*ii*) *inversely proportional to its breadth,*

(*iii*) *inversely proportional to the cube of its depth, and*

(*iv*) *inversely proportional to the Young's modulus for its material.*

It follows, therefore, that in order that the depression of a beam may be small for a given load, its length should be small, *i.e* , its *span* should be small, its breadth and depth should be large and the Young's modulus for its material should also be large.

When a girder is supported at its two ends, its middle part is depressed, and the surfaces above and below the *neutral surface* are respectively compressed and extended, the compression being the utmost at the upper face, and the extension, the maximum at the lower face, the stresses being the maximum there and decreasing as we proceed towards the neutral surface from either side. It follows, therefore, that the upper and the lower faces of the beam should be much stronger than its middle portions In other words, the middle portions may be made of a much smaller breadth than the upper and the lower faces, thus affecting a good deal of saving in the material. It is for this reason that girders are usually manufactured with their cross-section in the form of the letter I.

Stiffness of a beam. The *ratio between the maximum deflection of a beam and its span* measures what is called the *stiffness* of the beam. It is usually denoted by the symbol $1/n$. For *steel girders* of large span, n should lie between 1000 and 2000 and for those of shorter spans, between 500 and 700. And for *beams of timber*, the value of n should in no case be less than 360.

135. Searle's Method for the Comparison of Young's Modulus and Coefficient of Rigidity for a given material. A short length of the wire, the values of Y and n for the material of which are to be compared, is fastened to the middle points of two similar and equal metal bars AB and CD, (Fig. 202), of circular or rectangular cross-section. The bars are then suspended from a rigid support by means of two small vertical lengths of threads, so that, when the wire is straight, the bars are parallel to each other, as shown.

On slightly pulling together the ends A and C of the two bars *symmetrically and through equal distances*, the wire is bent into a circular arc, (Fig. 203). On releasing the bars, they begin to vibrate in a horizontal plane from a circular arc on one side to a similar arc on the other, due to the *torque* exerted on them by the wire, the mid-point of the bars remaining almost at rest.

If l be the length of the wire and θ, the angle of deflection of each bar from its normal position, the angle subtended by the wire at

the centre of curvature of the circular arc into which it is bent, is clearly 2θ, such that

$$l = R.2\theta, \text{ where } R \text{ is the } \textit{radius} \text{ of the arc.}$$

Or,
$$R = l/2\theta.$$

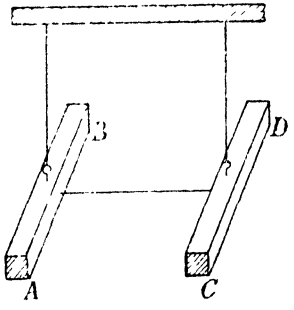


Fig. 202.

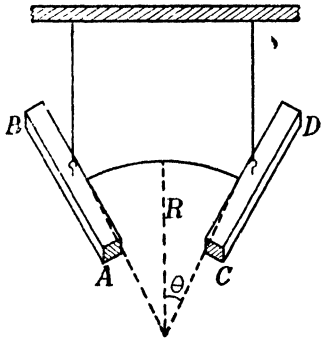


Fig. 203.

Now, the bending moment of the wire and the couple exerted by it on each bar is, as we know,

$$\frac{YI_g}{R} = \frac{Y.\pi r^4}{4R} = \frac{Y.\pi r^4.2\theta}{4l} = \frac{Y.\pi r^4.\theta}{2l}. \qquad \left[\because I = \pi r^4/4 \text{ for the wire.}\right.$$

This couple produces an angular acceleration $d\omega/dt$ in each bar, and, therefore,

$$I.d\omega/dt = -\frac{Y.\pi.r^4.\theta}{2l},$$

where I is the *M.I.* of each bar about an axis through its mid-point and perpendicular to its length, *i.e.*, about the thread from which it is suspended.

And $\therefore$ $\quad d\omega/dt = -\dfrac{Y.\pi r^4}{2l.I}.\theta.$ $\quad$ Or, $\quad d\omega/dt \propto \theta.$

i.e., the acceleration is proportional to the (angular) displacement.

Therefore, the motion is a *simple harmonic* one, and hence the time-period of each bar is given by

$$t_1 = 2\pi\sqrt{\frac{2lI}{Y.\pi r^4}}, \quad \text{whence,} \quad Y = \frac{8\pi lI}{r^4.t_1^2}. \qquad \ldots \text{(I)}$$

The suspension threads are then removed, and one of the bars is clamped horizontally, so that the other bar hangs vertically below it at the other end of the wire. The suspended bar is then turned about the wire in the horizontal plane, so as to twist the wire when, on being released, it begins to vibrate *torsionally*. Its time-period t_2 is noted.

Now, $t_2 = 2\pi\sqrt{I/C}$ where C is the twisting couple set up in the wire per unit deflection or twist, and is equal to $n\pi r^4/2l$. (§ 122)

$\therefore$ $\quad t_2 = 2\pi\sqrt{\dfrac{I.2l}{n\pi r^4}}, \quad$ whence, $\quad n = \dfrac{8\pi l.I}{r^4.t_2^2}.$ $\quad \ldots$(II)

Dividing relation (I) by relation (II), we have

$$\frac{Y}{n} = \frac{8\pi.l.I}{r^4.t_1^2} \cdot \frac{r^4.t_2^2}{8\pi.l.I} = \frac{t_2^2}{t_1^2}.$$

This gives the ratio of *Young's modulus* and the *co-efficient of rigidity* of the material of the wire in terms of t_1 and t_2.

Now, *Poisson's ratio*, $\sigma = \frac{Y}{2n} - 1.$

So that, substituting the value of $Y/2n$, we have

$$\sigma = \frac{t_2^2}{2t_1^2} - 1 = \frac{t_2^2 - 2t_1^2}{2t_1^2}.$$

Thus, *Poisson's ratio* for the material can also be easily determined.

N.B. As will be readily seen, the radius (r) of the wire, the measurement of which is the chief source of error (see page 303) has been eliminated altogether.

136. Strain energy in a bent beam We have seen before, (page 307), how when a small portion AB of a beam is bent into the form of a circular arc (Fig. 204), subtending an angle $\delta\theta$ at its centre of curvature, the strain produced in an element of it at a distance z from the neutral axis is given by z/R, where R is the radius of curvature of the neutral axis.

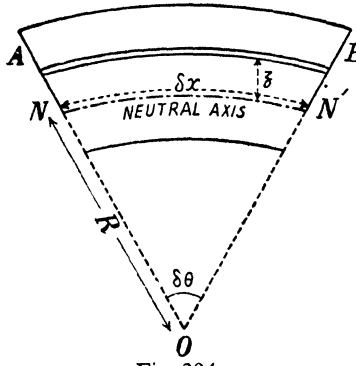


Fig. 204.

Now, *energy per unit volume in any type of strain* = ½ *stress* × *strain.*

So that, *energy associated with the element in question* = ½ *stress* × *strain* × *volume of the element.*

Then, if the stress be F, the strain, e, the area of cross section of the element normal to the plane of the diagram equal to δA, and the length of the neutral axis, δx, clearly,

$$\text{energy in the element} = \tfrac{1}{2}F.e.\delta A.\delta x.$$

Now, $Y = \text{stress}/\text{strain} = F/e$ and $\therefore \quad F = Ye = Y.z/r.$

So that, $$\text{energy in the element} = \tfrac{1}{2}.Y.\frac{z^2}{R^2}.\delta A.\delta x.$$

Integrating this expression over the whole cross-section, we have

strain energy in the entire portion AB of the beam

$$= \tfrac{1}{2}.\delta x.\frac{Y}{R^2}\int z^2.\delta A = \tfrac{1}{2}\delta x.\frac{YI_g}{R^2}.$$

Since, as we know, $\int z^2.dA = I_g$, the geometrical moment of inertia of the section considered.

Or, *strain energy in portion AB of the beam* $= \frac{1}{2}\left[\frac{YI_g}{R}\right]^2.\frac{1}{YI_g}.\delta x.$

But $YI_g/R = M$, *the bending moment of the section.*

Hence $$\text{strain energy in portion } AB \text{ of the beam} = \frac{1}{2}\cdot\frac{M^2}{YI_g}\cdot\delta x.$$

And $\therefore$ *strain energy E in the whole beam of length L* $= \int_0^L \frac{M^2}{YI_g}.dx.$

Or, $$E = \frac{1}{2YI_g}\int_0^L M^2.dx.$$

So that, substituting the relevant values of the bending moment, we can easily determine the *strain energy* of the beam in different cases.

Thus, for example, in the *case of a light cantilever with a load W suspended from its free end*, we have $M = W(L-x)$, (see page 310), and, therefore,

$$\textit{strain energy, in this case} = \frac{1}{2YI}\int_0^L W^2(L-x)^2.dx = \frac{1}{6YI_g}.W^2L^3.$$

137. Resilience of bent beams. The work done in deflecting a horizontal beam of whatever type, loaded with a weight W is equal to $\frac{1}{2}W \times$ (*deflection at the loaded point*). Since this work done measures the resilience of the beam, as explained in § 120, we have

$$\textit{resilience of the beam} = \tfrac{1}{2}W.y,$$

where y is the deflection of the loaded point.

Thus, in the case of a light beam, of *length L*, *breadth b*, and *depth d*, supported at the two ends and loaded with a weight W at the centre, we have $y = WL^3/48YI_g$, (page 316) and I_g, in this case, is given by $bd^3/12$, (page 308). So that, we have

$$\textit{resilience of such a beam} = \frac{1}{2}W.\frac{WL^3}{48YI_g} = \frac{W^2L^3}{96Y}.\frac{12}{bd^3} = \frac{W^2L^3}{8Ybd^3}.$$

138. Colums, Pillars and Struts. A long beam of an *isotropic* material used for supporting loads is called a *column*, a *pillar*, or a *strut*. Now, whereas a column or a pillar must always be *vertical* and generally fixed rigidly at its ends, a strut may be *vertical, horizontal* or *inclined* and may either have both its ends fixed rigidly or both connected to the surrounding structure through flexible joints, or it may have one end rigidly fixed and the other connected to a joint. The theory underlying the two is, however, the same, the commonest case being that in which the load applied is a compressive one, *i.e.*, acts at one end of the column or strut, along its axis, tending to compress or shorten it lengthwise, though, in some cases, there may as well be a lateral load, in addition.

Let us take the case of a column or a pillar first.

139. Critical load for a long column. Let us take a *long* and *straight* strip AB, [Fig. 205 (*a*)], of wood or metal, arranged in a vertical position, representing a column, with both its ends rounded and fitted into metal sockets, as shown magnified in Fig. 205 (*b*), so as to allow it freedom to bend all along its length, and let a load be applied to it at the top in the form of a metal cylinder, containing lead shot or mercury, so that its magnitude may be varied at will, with the cylinder moving between two parallel guides GG to ensure its vertical descent.

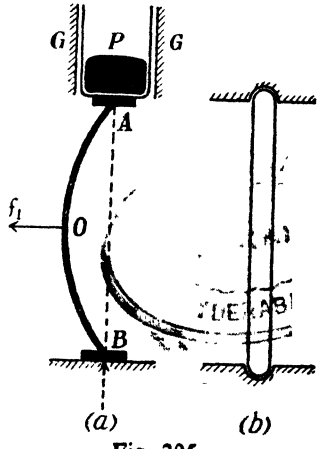


Fig. 205.

Now, first, with the load insufficient to bend the strip or the column, we apply a lateral force f_1 at its mid-point O, as shown, to make it bend a little and we find that, on removing the lateral force, the column straightens itself out. We increase the load at the top, apply the lateral force, as before, and then remove it. Perhaps, the column

again bends and then straightens itself. In this manner, we go on repeating the experiment with successively increasing loads at the top until we find that when the lateral force is applied and then removed, the column remains bent. At this stage, we find that whatever the deflection we produce in the column by the lateral force, the column continues to retain the same on the removal of this force, provided, of course, that the elastic limit has not been exceeded. *This load which just keeps the column bent, but does not bend it further, is called the* **critical load** *for it.*

If we increase the load beyond the critical value and give a slight bend to the column, as before, we find that the load now increases the bending further and the column either acquires a permanent set or collapses due to buckling.*

Let us see how we may account for this critical load. Let us, therefore, consider the equilibrium of the column AB, under a vertical load P_1 and a lateral force f_1 at its mid-point O, with the deflection of the column equal to y_1, as indicated in Fig. 206 (a).

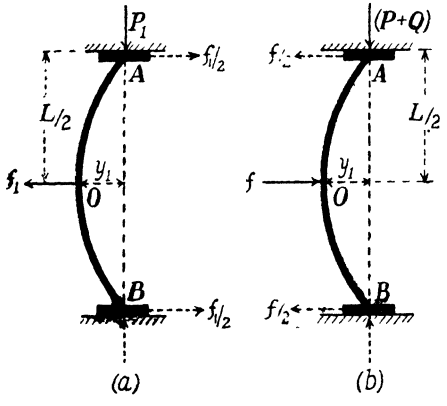


Fig. 206.

Since, for equilibrium, the lateral force f_1 must be balanced by two horizontal forces, each equal to $f_1/2$, acting at A and B, in the opposite direction to that of f_1, we have

total bending moment about O

$$= P_1y_1 + (f_1/2).L/2.$$

$$= P_1y_1 + f_1L/4.$$

And, if F_1 be the *maximum stress* due to bending and Z, the *modulus of cross-section* at O (see *note* on page 309), *the moment about O due to stress or the moment of resistance to bending* $= F_1Z$. So that, for equilibrium,

$$P_1.y_1 + f_1L/4 = F_1.Z.$$

If now we decrease the lateral force f_1 to 0 and increase load P_1 to P (the critical load) so that the deflection of the column remains the same y_1, clearly, the condition for equilibrium demands that

$$P.y_1 = F_1Z, \text{ whence, } P = F_1Z/y_1 \qquad [\because f_1 \text{ is now } 0.$$

Again, if the column be in equilibrium when subjected to a vertical load P_2 and a lateral force f_2, with its deflection now equal to y_2 and we reduce f_2 to 0 and increase P_2 to P', with the deflection remaining unaltered at y_2, we have, proceeding as before, $P' = F_2Z/y_2$, where F_2 is *now* the maximum stress due to bending.

Since the bending is proportional to stress, we have $F_1/y_1 = F_2/y_2$. And, therefore, $P' = P$ (the critical load), thus clearly showing that *the column will remain in equilibrium under the same critical load P for*

*i.e. by bending or bulging out.

any value of the deflection we choose to give it within the elastic limit, as we have seen already in the experiment discussed above.

If, however, we increase the vertical load to a value beyond its critical value P, say, to $(P+Q)$, as shown in Fig. 206 (*b*), the bending moment will clearly increase to $(P+Q)\,y_1$ or to $P_1y_1+Qy_1$. And, since the moment of resistance to bending, *viz.*, F_1Z balances only the portion P_1y_1 of it (as we have just seen above), the portion Qy_1 remains unbalanced, resulting in an increase in the bending or deflection of the beam beyond y_1. In order to keep the deflection at the same value y_1, therefore, we shall have to apply a force f, say, at the mid-point O of the column, in the opposite direction this time, so as to balance the portion Qy_1 of the bending moment. Since f is supposed to be balanced by two equal forces $f/2$ and $f/2$ acting at A and B, as shown, *its moment about* $O = (f/2).(L/2) = f.L/4$. So that, to prevent the column from bending further (beyond y_1) we must have $f.L/4 = Q.y_1$.

Now, within the elastic limit, the moment of resistance to bending is proportional to the stress, *i.e.*, $FZ \propto F$, and hence also to the deflection of the column (because, then, $y \propto F$). But, once the elastic limit is exceeded, the column acquires a permanent set, though it is also possible that, due to the moment of resistance due to bending now increasing more rapidly (as it always does beyond the elastic limit), the column may acquire a new position of equilibrium under the additional load Q. But, if this does not happen, the column will continue to bend further and further and finally collapse.

140. Euler's Theory of Long Columns.

(*i*) **When the two ends of the column are rounded or hinged.** Let AB represent a long and initially straight column of an isotropic material, of length L and of a *uniform cross-section* and *uniform elasticity*, with rounded or hinged ends *so as to be free to bend throughout its length.* Further, let the critical load P act *axially* upon it, *i.e.*, in a line with its axis in its straight unloaded position, and let it be given a slight bend by the application of a lateral force for an instant, (Fig. 207).

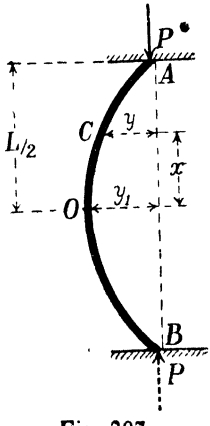


Fig. 207.

Now, consider a point C in the column, at a distance x from its mid-point O. If the deflection here be y, clearly, the *bending moment here due to* $P = P.y$. But, if the radius of curvature of the bend at C be R, the *moment of the resistance to bending* there is $Y.I_g/R$. And, clearly, therefore,

$$Y.I_g/R = P.y, \text{ whence, } 1/R = P.y/Y.I_g.$$

But, as we know, $1/R = -d^2y/dx^2$, (the $-$ve sign being given to make R *positive*, for dy/dx decreases as y increases). We thus have the differential equation

$$\frac{P.y}{Y\,I_g} = -\frac{d^2y}{dx^2}$$

The solution of this equation gives

$$y = A \sin \lambda x + B \cos \lambda x, \qquad \ldots(I)$$

where $\lambda = \sqrt{P/Y.I_g}$ and A and B are constants to be determined.

Differentiating this, we have

$$dy/dx = A\lambda \cos \lambda x - B\lambda \sin \lambda x.$$

(*a*) Now, when $x = 0$, $dy/dx = 0$ and, obviously, $\sin \lambda x = 0$; so that, $A = 0$. From equation I, therefore, we have $y = B \cos \lambda x$.

But when $x = 0$, we have $y = y_1$, $\lambda x = 0$ and $\therefore \cos \lambda x = 1$ and $\sin \lambda x = 0$. So that, from equation I, we have $y = B$.

(*b*) When $x = L/2$, (*i.e.*, at A), $y = 0$. And $\therefore B \cos \lambda L/2 = 0$. [From (*a*) above.

This means that either $B = 0$ or $\cos \lambda L/2 = 0$. But, as we have seen [in (*a*) above], $B = y_1$. It follows, therefore, that

$$\cos \lambda L/2 = 0, \text{ or. that } \lambda L/2 = \pi/2,$$

i.e., $\sqrt{P/YI_g} \cdot L/2 = \pi/2$, [for, $\lambda = \sqrt{P/Y.I_g}$.

Or, $\sqrt{P/Y.I_g} \cdot L = \pi$. Or, $P/YI_g = \pi^2/L^2$,

whence, $$P = \frac{\pi^2 YI_g}{L^2} = \frac{9{\cdot}87\ YI_g}{L^2}.$$

This is, therefore, the value of the *critical load*, or the load which can just keep the column bent at the initial curvature given to it An addition to this makes the column collapse.

It will be clear from this expression for the critical load that for the same values of Y and I_g, *the smaller the length of the column, the greater the critical load for it.*

(*ii*) **When the two ends of the column are fixed.** Let the ends A and B of the column be now fixed, as shown in Fig. 208 (*a*), so that when it gets bent or deflected, the tangents to it at points A, O and B are all vertical, with the line of action of the resultant load now no longer passing through the centres of its end-points. It passes, instead, between the initial unbent position AB of the column, and its mid-point O in the bent position, cutting the bent column in points C and D. At these points, therefore, there is no bending moment, so that they are points of opposite flexure.

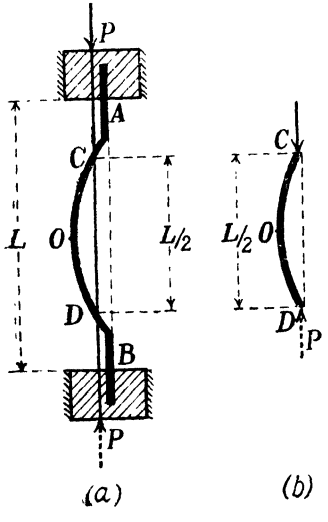


Fig. 208.

Now, considering the portions CA and CO of the bent column, we observe that the deflections at certain points in the two curves (as measured from the vertical line through C) are equal. At all such points in the two curves, therefore, the bending moments must be equal, and hence also the radii of curvature there must be the same, in view of

the fact that the column is of a uniform cross-section. Further, the two curves (CA and CO) have clearly the same slope at C and also at A and O, the tangents at all these points being vertical. Obviously, therefore, the two curves are *equal* and *similar*. The same is also true of the curves DB and DO. The points C, O and D thus divide the whole column into four equal parts and the length of the portion COD of the column is, therefore, equal to half its total length, *i.e.*, $COD = L/2$.

Clearly, this portion COD of the column, the whole of which is bent, behaves exactly as the cases considered above, *i.e.*, like a column of length $L/2$, with its ends rounded or hinged and carrying an axial load P at C So that, proceeding as in case (*i*) above, we have

$$P = \frac{\pi^2 YI_g}{(L/2)^2} = \frac{4\pi^2 YI_g}{L^2},$$

i.e., the critical load (P) for the column, in this case, is *four times* that in case (*i*).

Thus, *a column with its ends fixed, has four times the strength (to resist thrust) that it will have with its ends rounded or hinged.* Or, putting it differently, a column, with its ends fixed, can support, without bending, the same load as one of half the length, with its ends rounded or hinged, would do.

(*iii*) **When one end of the column is fixed and the other loaded.** This is an easy deduction from case (*i*) above. For, suppose we have a column AB, with rounded or hinged ends, and of length L', as shown in Fig. 209 (*a*), with P as the critical load on it. Then, the tangent to it at its mid-point O is vertical. If, therefore, we clamp it tightly at O, without disturbing the direction of the tangent at that point, the lower half OB of the column might as well be removed, without in any way affecting the upper half OA. So that, the upper half then behaves as an independent column, of length $L = L'/2$, fixed at its lower ends and loaded at the top, as indicated in Fig. 209 (*b*).

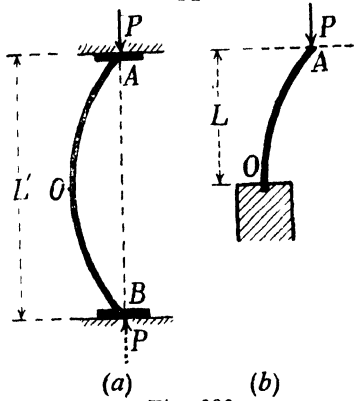


Fig. 209.

All that we have to do to calculate the critical load, in this case, is to consider the column of length L and fixed at one end, as equivalent to a column of length $L' = 2L$, with both its ends rounded or hinged. Therefore, proceeding in the same manner as in case (*i*), we have the critical load given by

$$P = \frac{\pi^2 YI_g}{(L')^2} = \frac{\pi^2 YI_g}{(2L)^2} = \frac{\pi^2 YI_g}{4L^2},$$

which is clearly one-fourth of the critical load for a column of length L, with its ends rounded or hinged.

Thus, we find that *a column, hinged at one end and loaded at the other, has only one-fourth the strength of the same column when hinged at both ends.*

N.B. Exactly a similar treatment to that in cases (*i*) and (*ii*) applies to the corresponding cases of a *strut*, arranged horizontally, so long as the strut is

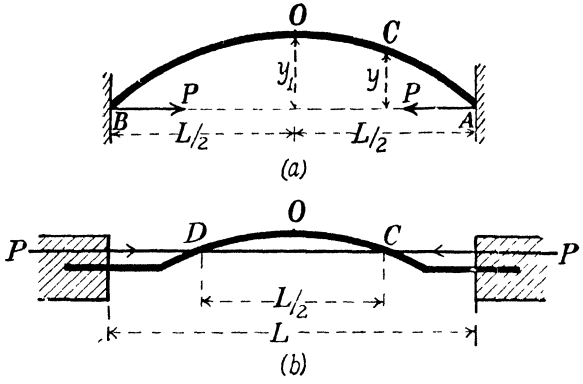


Fig. 210.

loaded axially, or along its axis, like the columns in cases (*i*) and (*ii*). Thus, if the two ends of the strut be rounded or hinged, so that the whole of it can bend, we can represent its behaviour as in Fig. 210 (*a*), and, when its two ends are fixed, as in Fig. 210 (*b*). It will be noted that these are essentially the same figures as in cases (*i*) and (*ii*), respectively, but are rotated, as it were, through an angle of 90°, so that instead of a vertical load we now have a horizontal load. The method of calculation for the critical load, therefore, remains the same.

141. Elastic Waves. When a system of stresses, to which a body or a medium is being subjected, is suddenly altered, we have (*i*) *a corresponding motion of the body or the medium* itself and (*ii*) *propagation through it of the changes in stress*,—the two occurring simultaneously and constituting what is called the *propagation of an elastic or a stress wave.*

Now, as we know, even in the case of an isotropic medium, a deformation in one direction is invariably accompanied by deformations in two other directions, at right angles to the first, (the familiar case of the deformation of a cube), so that the theory of elastic waves is really quite a complicated one ; and this complication is further aggravated in the case of bodies like the earth, for example, where the elastic properties vary with depth,—which explains at once the complicated pattern of the seismic waves (§ 100).

In general, however, we have three types of elementary elastic waves in the case of a uniform, isotropic medium, *viz.*, (*i*) *compressional*, (*ii*) *shear* and (*iii*) *flexural waves.* We shall only briefly touch upon them here.

(*i*) **Compressional Waves.** These waves are produced when we give an *axial blow* to a long bar, *i.e.*, strike it along its axis, and, assuming the sides of the bar to have freedom of movement, their velocity is given by the relation $v = \sqrt{Y/\rho}$, where Y is the value of *Young's modulus* for the material of the bar and ρ, its *density*. But, in case the sides of the bar too are fixed, Y is to be replaced by $Y(1-\sigma)/(1+\sigma)(1-2\sigma)$, where σ is the value of Poisson's ratio for the material of the bar. This expression takes many forms, the simplest among them being $K+4n/3$, where K and n are the coefficients of bulk and rigidity modulii for the material of the bar. Thus, we have different types of compressional waves, all of which, however, have the common features that (*i*) *the vibrations occur along the direction of propagation of the wave, i.e., the wave is of the longitudinal type*, and (*ii*) *the velocity of the wave is given by*

$$v = \frac{\textit{appropriate modulus of elasticity}}{\textit{density}}.$$

Thus, in the case of a liquid or a gas, $n = 0$ and we, therefore, have the *wave-*

Pascal discussed this result in his *'Treatise on the equilibrium of liquids'*, published in the year 1663, with reference to vessels of different shapes, known as *Pascal's vases*, (Fig. 213), all having *equal bases* and *containing water upto the same vertical height h*, so that the *pressure on the base of each vessel was equal to h gms. wt./cm².*, and, therefore, the *thrust* on it was *h.a. gms. wt.*, where *a* is its *area*.

He was perhaps the first person to have pointed out the paradoxical truth that even if vessel (*i*) contains 100 *lbs.* of water and vessel (*v*) only 1 *oz.* of it, the thrusts on the bases of both is the *same*. Aptly, therefore, it is called the *hydrostatic paradox*.

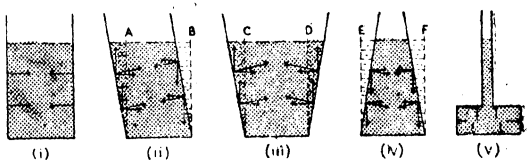


Fig. 213.

Strange as it may seem, but if the water in vessel (*v*) be frozen into ice and detached from its sides, the thrust exerted by *this* ice on its base will be only 1 *oz. wt.*, but once this ice is melted back into water, the thrust again increases to 100 *lbs. wt.* The explanation of this seeming paradox is, however, simple. The ice does not exert any upward thrust on the part of the vessel opposite to the base and the latter, therefore, exerts no equal and opposite thrust on it. But the water does exert an upward thrust on it and hence receives back an equal and opposite downward thrust from it.

In case of vessel (*i*), the thrust on the base is equal to the entire weight of the water on it.

In vessel (*ii*), the upward component of the thrust due to the left side of the vessel supports the weight of water in it, between the left side and the dotted line *A*, while the downward component of the thrust due to the right side of the vessel exerts a downward thrust on it, equal to the weight of the water in-between the right side of the vessel and the dotted line *B* ; so that, the thrust on the base is the *same as due to a vertical column h of water.*

In vessel (*iii*), the upward components of the thrusts due to both the left and the right sides of the vessel support the extra weight of the water, in-between the two sides and the dotted lines *C* and *D*, and, again, therefore, *the thrust on the base is equal to that due to the cylindrical column h of water in-between the dotted lines C and D.*

And, similarly, in vessel (*iv*), the downward components of the thrust due to the two sides of the vessel exert an *extra* thrust on the base, equal to the weights of the water contained between either side and the dotted lines *E* and *F*; so that, once again, *the total thrust on the base is the same as that due to a cylindrical water column of height h.*

Pascal experimentally verified the above fact by supporting, by means of a separate stand, bottomless vases of the above shapes, one by one, on a large disc, *D* (Fig. 214), suspended from the shorter pan of a hydrostatic balance and kept pressed against their bases by placing a heavy weight in the longer pan, and pouring water into the vessel. *The disc just got detached from its base as the water reached the same level in each case*, thus clearly demonstrating the equivalence of the thrust on the disc in *each* case and fully vindicating his deductions.

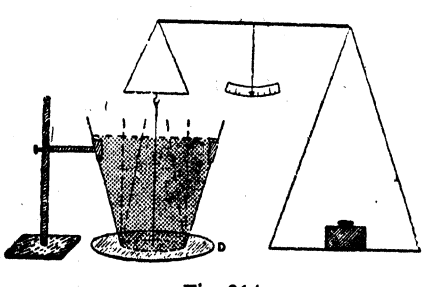


Fig. 214.

mum values being respectively 1, *in the case of perfectly elastic bodies, and zero, in the case of perfectly plastic ones.*

Thus, if u_1 and u_2 be the velocities of two bodies before the impact and v_1 and v_2, after the impact, we have

$$\frac{(v_1-v_2)}{(u_1-u_2)} = e, \qquad \left[\begin{array}{l}\text{neglecting the } -ve \text{ sign of the}\\ \text{relative velocity after the im-}\\ \text{pact.}\end{array}\right.$$

Or, $$(v_1-v_2) = e(u_1-u_2), \qquad \ldots (i)$$

where (u_1-u_2) and (v_1-v_2) are their relative velocities, before and after impact.

It will thus be seen that if $e = 1$, *i.e., if the bodies be perfectly elastic,* $(v_1-v_2) = (u_1-u_2)$, *i.e., the relative velocities of the bodies are the same before and after the impact,* (suffering only a reversal of direction, in the latter case).

But if $e = 0$, *i.e., if the bodies be perfectly plastic,* $(v_1-v_2) = 0$, *or,* $v_1 = v_2$, *i.e., the two bodies move with the same common velocity, after the impact.*

What happens is that when one elastic body, of mass m_1, moving with a velocity u_1, collides against another elastic body, of mass m_2, moving with velocity u_2, (where $u_1 > u_2$), the surface between them gets compressed and when this compression or pressure reaches its maximum value, their relative velocity becomes zero. Thereafter, the elastic stress between them makes them recede from each other, the compression is released, and the two bodies move away with different velocities, say v_1 and v_2.

Clearly, then, in accordance with the *law of conservation of momentum,* we have

sum total of momenta after the impact
$$= \textit{sum total of momenta before the impact.}$$

Now, sum total of momenta after the impact $= m_1v_1+m_2v_2$,
and sum total of momenta before the impact $= m_1u_1+m_2u_2$.

So that, $$m_1v_1+m_2v_2 = m_1u_1+m_2u_2. \qquad \ldots (ii)$$

From relations (i) and (ii), we can easily calculate the values of v_1 and v_2. For, multiplying relation (i) by m_2, we have

$$m_2v_1-m_2v_2 = m_2e(u_1-u_2). \qquad \ldots (iii)$$

And, adding relations (ii) and (iii), we have

$$m_1v_1+m_2v_1+m_2v_2-m_2v_2 = m_1u_1+m_2u_2+m_2e.(u_1-u_2).$$

Or, $$v_1(m_1+m_2) = m_1u_1+m_2u_2+m_2.e(u_1-u_2),$$

whence, $$v_1 = \frac{m_1u_1+m_2u_2+m_2.e(u_1-u_2)}{(m_1+m_2)}. \qquad \ldots (iv)$$

Similarly, multiplying relation (i) by m_1, and subtracting from relation (ii), we have

$$v_2 = \frac{m_1u_1+m_2u_2-m_1.e(u_1-u_2)}{(m_1+m_2)}. \qquad \ldots (v)$$

Further, it can also be shown that

$$e = \frac{\textit{Impulse during restitution}}{\textit{Impulse during compression}}.$$

The value of e, nowever, is found to diminish with increase in the velocities of the approaching bodies, and *vice versa*, and it has been shown by **Sir C.V. Raman**, that its value is very nearly equal to 1, if the collision be very weak.

143. Loss of Kinetic Energy on Impact. It can be shown from the above relations, that

$$\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2-\tfrac{1}{2}.\frac{(1-e^2).m_1m_2.(u_1-u_2)^2}{m_1+m_2},$$

where $\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2$ is the total kinetic energy E_2 of the colliding bodies, *after* the impact, and $\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2$, their total kinetic energy E_1 *before* the impact.

Thus, $$E_2 = E_1-\frac{1}{2}.\frac{(1-e^2).(m_1.m_2).(u_1-u_2)^2}{m_1+m_2}.$$

Or, $$E_1-E_2 = \frac{1}{2}.\frac{(1-e^2)(m_1.m_2).(u_1-u_2)^2}{m_1+m_2}.$$

i.e., loss of energy on impact $= \frac{1}{2}\frac{(1-e^3).(m_1.m_2)(u_1-u_2)^2}{m_1+m_2}$,

an expression, with always a *positive* value, showing that *there is always a loss of energy* on impact between two bodies.

Now, the following special cases arise :

(*i*) *When* $e = 1$, *i.e., when the colliding bodies are perfectly elastic.* In this case, $(E_1-E_2) = 0$, *i.e., there is no loss of energy on impact of perfectly elastic bodies.*

(*ii*) *When* $e = 0$, *i.e., when the colliding bodies are perfectly plastic.* Here (E_1-E_2) has the maximum value, *viz.*,

$$\frac{1}{2}.\frac{(m_1m_2).(u_1-u_2)^2}{(m_1+m_2)},$$

i.e., there is maximum loss of energy on impact of plastic bodies.

(*iii*) When $u_1 = u_2$, *i.e., when the bodies have the same velocity, (in magnitude as well as direction).* In this case, the relative velocity of one body, with respect to the other, is *zero*, so that *no impact takes place at all* between the two bodies, and, therefore, $(E_1-E_2) = 0$, or again, *there is no loss of energy.*

The question now arises as to what happens to this loss of energy on impact ? Until very recently, it was supposed that the energy lost during impact was converted into (*i*) sound, (*ii*) heat, or (*iii*) vibration or rotation of the colliding bodies.

Sir C. V. Raman's experiments have shown, however, that the production of sound is in no way related to the energy of impact, being solely due to the impulse set up in the air during the reversal of the motion of the colliding bodies, after impact.

The change in temperature too is almost always very small and hence it appears that an appreciable portion of this energy lost during impact is used up in bringing about a re-distribution of the molecules in the surface layers of the colliding bodies. Indeed, it has been shown by *Hertz* that impact produces a definite flattening of the point of contact of the colliding bodies, with a finite common area between

them, each body being compressed in its neighbourhood,—the compression increasing at first to a maximum, (which is proportional to the *two-fifth* power of the velocity), then diminishing and finally vanishing altogether, when the bodies get separated from each other.

Prof. Deodhar has also verified this molecular displacement in the surface layers of the impinging bodies, by making different bodies (*'similar and dissimilar'*) impinge upon each other, with *'extremely low velocities'* and measuring their velocities before and after the impact. From his experiments, he has come to the following conclusions :

(*i*) With great velocities of the colliding or impinging bodies, the changes in their surface are *'vivid'*, and a greater portion of their energy of impact is used up in producing these deformations.

(*ii*) With very small velocities of the colliding bodies, on the other hand, the value of e increases, in the limit, to unity—the increase of e with the *'minimal velocities'* being independent of the nature of their material.

(*iii*) The rate of change of e is quite independent of the medium in which the impact takes place.

(*iv*) The duration of impact is observed to be greater in water than in air, *i.e.*, it depends upon the density of the medium.

(*v*) A distinct change in the structure of the impinging bodies is noticeable under the microscope, though no trace of it is visible to the naked eye.

He estimates from this that energy, of the order of 1000 *k.gms./cm*^{2}. is used up *'in displacing the molecular aggregates'*. Further, bodies, when strained, take time to recover their original condition, and a rapid rise and fall in the stress may result in the dissipation of some energy, provided the elastic limit of the bodies, for gradually varying forces, is not exceeded.*

144. Relative masses of colliding bodies. If, in the above example of two colliding bodies or balls, the second one be at rest, so as to have no kinetic energy, we have

$$\textit{total kinetic energy before impact} = \tfrac{1}{2}m_1.u_1^2.$$

And, $$\textit{loss of energy during impact} = \frac{1}{2}.\frac{(1-e^2)}{(m_1+m_2)}.m_2.u_1^2 .$$

So that, $$\frac{\textit{loss of energy}}{\textit{total energy}} = \frac{1}{2}.\frac{-e}{(m_1+m_2)}.m_1.m_2.u_1^2 \Big/ \tfrac{1}{2}m_1u_1^2.$$

$$= \frac{(1-e^2).m_2}{m_1+m_2} = \frac{(1-e^2)}{(m_1+m_2)/m_2} = \frac{(1-e^2)}{\frac{m_1}{m_2}+1}.$$

Or, $$\textit{loss of energy} = \left\{\frac{(1-e)^2}{\frac{m_1}{m_2}+1}\right\} \times \textit{total energy}.$$

Clearly, therefore, the loss of energy will be small if m_1/m_2 be large, and *vice versa*.

Thus, *in order to minimise loss of energy, the ratio m_1/m_2 must be made large, i.e., the mass of the striking body must be much greater than that of the body struck.* Hence it is that a slow-moving heavy hammer is more suitable for imparting momentum to a body than a

*This should not, however, be understood to mean that an exceeding or *'overlapping'* of the elastic limit is necessary for a loss of energy to occur.

quick-moving lighter one, even though the two may possess the same momentum.

On the other hand, *if the loss o energy is to be converted into useful work, the ratio m_1/m_2 must be small, i.e., m_2 must be much greater than m_1.* That is why while forging instruments etc., we must have heavy anvils underneath them.

SOLVED EXAMPLES

1. Show that (*a*) a small and uniform strain v is equivalent to three linear strains v/3, in any three perpendicular directions;

(*b*) the bulk modulus for a gas (*i*) at constant temperature (*i.e.*, under isothermal conditions) is equal to its pressure and (*ii*) when the temperature is not constant, (*i.e* , when the conditions are adiabatic), it is equal to γ times its pressure, where γ is the ratio c_p/c_v for it.

(*a*) Imagine a *unit cube* to be compressed equally and uniformly from all sides, so that the length of each edge is decreased by a length l, *i.e.*, becomes $(1-l)$.

Then, clearly, *decrease in volume of the cube, i.e.*,

$$v = 1-(1-l)^3 = 1-1+3l-3l^2+l^3 = 3l, \text{ i.e., } l = v/3,$$

neglecting l^2 and l^3, the value of l being small.

Thus, a small uniform *volume strain* is equal to three *linear strains*, each equal to $v/3$, in three perpendicular directions.

(*b*) (*i*) Let P be the pressure and V, the volume of a gas, and let it be compressed *isothermally** by increasing the pressure to $(P+dp)$, so that the volume is reduced by dv and becomes $(V-dv)$.

Then, clearly, *stress* = force per *unit* area = pressure applied = dp, and *volume strain* = change in volume/original volume = dv/V.

$\therefore$ *Bulk modulus* for the gas, *i.e.*, $K = \dfrac{dp}{dv/V} = \dfrac{dp}{dv}.V.$

Since the temperature of the gas remains constant, *Boyle's law* holds good, and we, therefore, have

$$PV = (P+dp)\times(V-dv] = PV-P.dv+dp.V-dp.dv.$$

Or, $PV = PV-P.dv+dp.V$. Or, $P.dv = dp.V$. [neglecting $dp.dv$.

whence, $V.dp/dv = P.$

Since $V.dp/dv = K$, we have $K = P.$

Or, the *Bulk Modulus* for a gas, at constant temperature, *i.e.*, its *isothermal elasticity*, is equal to its *pressure*.

(*ii*) If, on the other hand, the change in the volume is brought about *ically*†, we have

$$PV^\gamma = a \text{ constant}.$$

Diffrentiating this, we have

$$P\gamma V^{\gamma-1}dv+V^\gamma\,dp = 0. \text{ Or, } -V\frac{dp}{dv} = \gamma P.$$

The $-ve$ sign merely indicates that dv and dp are of opposite signs.

Or, $K = \gamma P.$

thus, the *adiabatic elasticity* of a gas is equal to γ times its pressure, *i.e., is γ times its isothermal elasticity.*

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*This may be done by using a cylinder and a piston of a perfectly conducting material, so that the heat is conducted out into the surrounding air as soon as it is generated and the temperature of the gas remains the same as before.

†In this case, the cylinder and piston are of a perfectly non-conducting material or the cylinder is placed on a perfect insulator, so that the heat generated on compression of the gas cannot escape out but remains inside the gas itself, thus raising its temperature a little.

2. Show that the Bulk modulus K, Young's modulus Y and the Poisson's ratio σ are connected together by the relation, K = Y/3(1−2σ). *(Punjab, 1940 ; Delhi, 1947)*

We have $K = \dfrac{1}{3(\alpha-2\beta)} = \dfrac{1}{3\alpha(1-2\beta/\alpha)}$. [See page 287.

Now, $1/\alpha = Y$, and $\beta/\alpha = \sigma$. [See pages 288-89.

Therefore, $K = Y/3(1-2\sigma)$.

3. Show that the rigidity *n*, and Young's modulus Y are connected by the relation, $n = Y/2(1+\sigma)$, where σ is Poisson's ratio. *(Punjab, 1938)*

We have $n = 1/2(\alpha+\beta) = \dfrac{1}{2\alpha(1+\beta/\alpha)}$. [See page 289.

But $1/\alpha = Y$, and $\beta/\alpha = \sigma$.

$\therefore$ $n = Y/2(1+\sigma)$.

4. Obtain an expression for the radius of curvature of a flat curve in terms of the slope of the curve, and use the result to find the value of deflection in the case of a bar fixed horizontally at one end and loaded at the other. *(Bombay, 1928)*

Let *APQ* be a flat curve (Fig. 211), and let *P* and *Q* be two points on it small distance δx apart. Draw tangents to the curve at *P* and *Q*, and let O be the centre and *R*, the radius of curvature of the portion *PQ* of the curve.

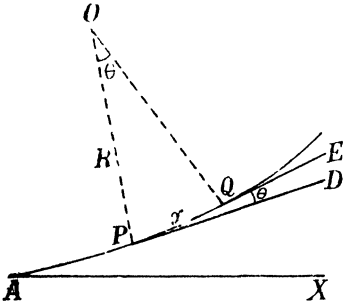


Fig. 211.

Then, if $\angle POQ = \theta$, we have $PQ = R.\theta$.

Or, $\delta x = R.\theta$. ...(*i*)

Now, θ = *difference in slope of the tangents at P and Q.*

And since slope of the tangents at a point is measured by dy/dx at the point, we have

$$\theta = \left[\frac{dy}{dx} \text{ at } Q - \frac{dy}{dx} \text{ at } P\right]$$

Now, the *rate of change of slope* is given by the *second differential coefficient*, d^2y/dx^2.

$\therefore$ Change in slope from *P* to *Q* = $\delta x\, d^2y/dx^2$.

Or, $\theta = \delta x.d^2y/dx^2$. And, $\therefore$ $\delta x = \delta x.R.d^2y/dx^2$. [From (*i*) above.

i.e., $R.d^2y/dx^2 = 1$.

Or, $1/R = d^2y/dx^2$ = *rate of change of slope at P.*

Since in the case of bent rods, or beams, the curve of the neutral axis is very slight, the relation $1/R = d^2y/dx^2$ gives the *radius of curvature* of the axis at any given point.

Now, for a bar fixed horizontally at one end and loaded at the other, (*i.e.*, in the case of a *cantilever*), we have

$$W.(L-x) = Y.I_g/R,$$ [See page 310.

the axis of *x* being taken along the horizontal and the axis of *y*, vertically downwards.

Here, *L* is the length of the bar from the fixed to the free end, *x*, the distance of the section *PQ* from the fixed end, and *W*, the weight applied at the free end.

Therefore, substituting the value of $1/R$, from above, we have

$$W.(L-x) = Y.I_g.\frac{d^2y}{dx^2}. \quad \text{Or,} \quad \frac{Y.I_g}{W}.\frac{d^2y}{dx^2} = (L-x).$$

Integrating this, we have

$$\int \frac{Y.I_g}{W}.\frac{d^2y}{dx^2}.dx = \int (L-x).dx.$$

Or, $$\frac{Y.I_g}{W}.\frac{dy}{dx} = (Lx-x^2/2)+C_1,$$ ...(*ii*)

where C_1 is a *constant of integration*, to be determined from the conditions of the experiment.

Clearly, dy/dx is *zero* at A, *i.e.*, $dy/dx = 0$, when $x=0$.

Substituting these values of $\frac{dy}{dx}$ and x in (*ii*), we have $C_1 = 0$.

$$\therefore \quad \frac{Y.I_g}{W}.\frac{dy}{dx} = Lx - \frac{x^2}{2}, \qquad \ldots(iii)$$

Again, integrating this expression (*ii*), we have

$$\int \frac{Y.I_g}{W}\frac{dy}{dx}.dx = \int\left(Lx - \frac{x^2}{2}\right).dx. \quad \text{Or,} \quad \int \frac{YI_g}{W}.dy = \int\left(Lx - \frac{x^2}{2}\right).dx.$$

Or,
$$\frac{YI_g}{W}.y = \frac{Lx^2}{2} - \frac{x^3}{6} + C_2, \qquad \ldots(iv)$$

where C_2 is another *constant of integration*.

To determine this constant, we observe that the depression y of the rod is *zero* at the end A ; so that, $y=0$, when $x=0$.
Putting these values of y and x in (*iv*), we have $C_2 = 0$.

Hence
$$\frac{Y.I_g}{W}.y = \frac{Lx^2}{2} - \frac{x^3}{6}.$$

Now, to obtain the deflection of the loaded end, let us put $x = L$.

Then, clearly,
$$\frac{Y.I_g}{W}.y = \frac{Lx^2}{2} - \frac{L^3}{6} = \frac{L^3}{3}.$$

Or,
$$y = W L^3/3Y.I_g.$$

If the rod be of *rectangular* cross-section, $I_g = bd^3/12$, where b and d are its breadth and depth respectively ; so that,

for a rectangular rod, $\quad y = \frac{W.l^3}{3Y.b.d^3/12} = \frac{4W.l^3}{Y.bd^3}.$

And, if the rod be of a *circular* cross-section, (*i.e.*, if it be *cylindrical*), $I_g = \pi r^4/4$, where r is its *radius* ; so that,

for a cylindrical rod, $\quad y = \frac{W.l^3}{3Y.\pi r^4/4} = \frac{4W.l^3}{3Y.\pi r^4}.$

5. A brass bar 1 cm. square in cross-section is supported on two knife-edges 100 cms. apart. A load of 1 k.gm. at the centre of the bar depresses that point by 2·51 mm. What is Young's modulus for brass ?

We know that the depression of the mid-point of the bar is given by

$$y = Wl^3/48YI_g. \qquad \text{[See page 316.}$$

Now, for a bar of rectangular cross-section,

$$I_g = b.d^3/12. \qquad \text{[See foot-note page 308.}$$

Here, $b = d = 1$ *cm.*, because the bar is 1 *cm. square* in cross-section.

$\therefore \quad b.d^3 = 1\times1 = 1$; $W = 1$ *k. gm. wt.* $= 1000\times981$ *dynes.*

$l = 100$ *cms.* and $y = 2{\cdot}51$ *mm.* $= {\cdot}251$ *cm.*

Therefore, $y = \frac{Wl^3}{48\frac{Y.b\,d^3}{12}} = \frac{12Wl^3}{48\,Y.b.d^3}$. Or, $Y = \frac{Wl^3}{4ybd^3}$.

Or,
$$Y = \frac{1000\times981\times100^3}{4\times{\cdot}251\times1} = \frac{981\times10^9}{4\times{\cdot}251} = 9{\cdot}77\times10^{11}\ dynes/cm^2.$$

Or, the value of Young's Modulus for brass is $9\,77\times10^{11}$ *dynes/cm*^{2}.

6. Establish an expression for the work done in stretching a wire through *l* cms. assuming Hooke's law to hold.

Find the work done in Joules in stretching a wire of cross-section 1 sq. mm. and length 2 metres through 0·1 mm., if Young's modulus for the material of the wire is 2×10^{12} dynes/cm^2. *(London Inter. Science)*

For answer to *first part*, see § 112 (*i*), where it is shown that *work done in stretching the wire* $=$ ½ *stretching force* $\times$ *the stretch.*

r, $\text{work done} = \frac{1}{2} F.l = \frac{1}{2} \cdot \frac{Y.a.l.}{L}$., where F is the stretching force.

Numerical. Here, $Y = 2\times10^{12}$ *dynes/cm².*, $a = 1$ *sq. mm.* $= 1/100 = \cdot01$ *sq. cm.*, $l = \cdot1$ *mm.* $= 01$ *cm.* ; and $L = 2$ *metres* $= 200$ *cms*

Therefore, $\text{work done} = \frac{1}{2} \cdot \frac{Y.a\,l}{L} \cdot l = \frac{1}{2} \cdot \frac{2\times10^{12}\times\cdot01\times 01\times\cdot01}{200}$.

$= \frac{10^{12}\times\cdot01^3}{200} = \frac{10^9}{200} = \frac{10^4}{2} = 5000$ *ergs* $= \frac{5000}{10^7} = \frac{5}{10^4} = 5\times10^{-4}$ *Joules.*

Thus, work done in stretching the wire is 5×10^{-4} *Joules.*

7. Show that for a homogeneous isotropic substance, $Y/N = 2(\sigma+1)$, where Y is the Young's modulus, N, the simple rigidity, and σ, the Poisson's ratio.

A gold wire 0·32 mm. in diameter, elongates by 1 mm., when stretched by a force of 330 gm. wt., and twists through 1 radian, when equal and opposite torques of 145 dyne-cm. are applied at its ends. Find the value of Poisson's ratio for gold.

An isotropic substance is such that two equal, similar portions cut from it, with any orientation, are exactly like and indistinguishable.

For proof of the relation, $Y/N = 2(\sigma+1)$, see solved example 3, above, (page 332), where it is shown that $N = Y/2(\sigma+1)$, whence, $Y/N = 2(\sigma+1)$.

Now, $$Y = \frac{F\times L}{a\times l}.$$

Here, $F = 330\times981$ *dynes*; $l = 1$ *mm.* $= \cdot1$ *cm.* ; and $a = \pi\times(\cdot016)^2$ *sq. cms.*, [because radius $= 032/2 = \cdot016$ *cm.*, and $a = \pi r^2$].

$\therefore$ $$Y = \frac{330\times981\times L}{\pi\times\cdot016^2\times1} \text{ dynes/cm}^2.$$

Since *couple* acting on the wire $= 145$ *dynes-cm.*, and *angle of twist* $= 1$ *radian*, we have, couple per unit twist $= 145/1 = 145$ *dynes-cm.*

This must be equal to $N.\pi r^4/2L$, where r is the radius of the wire.

Thus, $N\pi r^4/2L = 145$, whence, $N = \frac{145\times2L}{\pi\times(\cdot016)^4}$.

Or, $$\frac{Y}{N} = \frac{330\times981\times L}{\pi\times(\cdot016)^2\times\cdot1} \times \frac{\pi\times(\cdot016)^4}{145\times2L}.$$

$$= \frac{330\times981\times(\cdot016)^2}{2\times\cdot1\times145} = \frac{330\times911\times(\cdot016)^2}{29} = 2\cdot858.$$

Since $\frac{Y}{N} = 2(\sigma+1)$, we have $2(\sigma+1) = 2\cdot858$.

Or, $(\sigma+1) = 2\cdot858/2 = 1\cdot429$. And, $\therefore$ $\sigma = 1\cdot429 - 1 = \cdot429$.
Hence, *Poisson's ratio* for gold $= 0\cdot429$.

8. A square metal bar of 2·51 cms. side, 37·95 cms. long, and weighing 826 gms. is suspended by a wire 37 85 cms long and 0·0501 cm. radius. It is observed to make 50 complete swings in 335 7 secs. What is the rigidity coefficient of the wire ?

Here, time-period of the bar, *i.e.*, $t = 335\cdot7/50 = 6\cdot714$ *secs.*

Now, time-period of a body executing a torsional vibration is given by $t = 2\pi\sqrt{I/C}$, where I is its *moment of inertia about the suspension wire as axis* and C, the *twisting couple per unit deflection or twist of the wire.*

Here, $I = \text{mass}\left(\frac{\text{length}^2+\text{breadth}^2}{12}\right) = 826.\frac{(37\cdot95)^2+(2\cdot51)^2}{12}$.

$= 826.\left(\frac{1440+6\,301}{12}\right) = 826\times\frac{1446\,3}{12} = 826\times120\cdot5.$

$= 99540$ *gm. cm².* $\therefore$ $6\cdot714 = 2\pi\sqrt{\frac{99540}{C}}$,

squaring which, we have

$$(6{\cdot}714)^2 = 4\pi^2 \times \frac{99540}{C}. \qquad \text{Or,} \qquad C = \frac{4\pi^2 \times 99540}{(6{\cdot}714)^2}.$$

Now, C is also $= n.\pi r^4/2l$, where the *symbols* have their usual meanings.

$$\therefore \quad \frac{n \times \pi \times (\cdot 0501)^4}{2 \times 37{\cdot}85} = \frac{4\pi^2 \times 99540}{(6{\cdot}714)^2},$$

$$\text{whence, } n = \frac{8\pi \times 99540 \times 37{\cdot}85}{(\cdot 0501)^4 \times (6{\cdot}714)^2} = 3{\cdot}357 \times 10^{11} \textit{ dynes./cm}^2.$$

Or, the rigidity coefficient of the wire is $3{\cdot}357 \times 10^{11}$ *dynes/cm*^{2}.

9. A disc of 10 cms. radius and mass 1 k. gm. is suspended in a horizontal plane by a vertical wire attached to its centre. If the diameter of the wire is 1 mm. and its length is 1·5 metres and the period of torsional vibration of the disc is 5 secs., find the rigidity of the material of the wire. Prove the equation you employ. (*Bombay, 1931*)

For proof of the formula, see § 126, (pages 300-301), which gives the relation,

$$t = 2\pi\sqrt{I/C}.$$

Here, in this case, $t = 5$ *sec.*, and I, for the disc, about the axis of suspension, is given by $Mr^2/2$, where M is the mass of the disc, and r, its radius. Since $M = 1$ *k.gm.* $= 1000$ *gms* ; and $r = 10$ *cms*, we have

$$I = \frac{1000 \times 10^2}{2} = 500 \times 100 = 5 \times 10^4 \textit{ gms. cm}^2.$$

$$\text{Therefore, } \quad t = 5 = 2\pi\sqrt{5 \times 10^4/C}. \quad \text{Or,} \quad 25 = \frac{4\pi^2 \times 5 \times 10^4}{C}.$$

$$\text{whence, } \quad C = \frac{4\pi^2 \times 5 \times 10^4}{25} = \frac{4\pi^3 \times 10^4}{5}.$$

Since C is also equal to $n\pi r^4/2l$, we have $\frac{n\pi r^4}{2l} = \frac{4\pi^2 \times 10^4}{5}$.

$\therefore$ substituting the values of r and l, we have

$$\frac{n \times \pi \times (\cdot 05)^4}{2 \times 150} = \frac{4\pi^2 \times 10^4}{5},$$

($\because$ $r = \cdot 5$ *mm.* $= \cdot 05$ *cm.* and $l = 1{\cdot}5$ *metres* $= 150$ *cms.*)

$$\text{whence, } \quad n = \frac{4\pi^2 \times 10^4 \times 2 \times 150}{\pi \times (\cdot 05)^4} = \frac{4\pi \times 10^4 \times 60}{(\cdot 05)^4}.$$

$$= 1{\cdot}206 \times 10^{12} \textit{ dynes/cm}^2.$$

$\therefore$ the rigidity of the material of the wire is $1{\cdot}206 \times 10^{12}$ *dynes/cm*^{2}.

10. An elastic string has a mass M suspended at its lower end, the upper being fixed to a support. The mass is pulled down over a short distance and let go. Explain the motion that ensues and find an expression for the time of oscillation.

If a mass m is added to the mass M, the time is altered in the ratio of 5 : 4. Compare the masses m and M. (*Bombay, 1936*)

For first part, see solved example 10, Chapter IV, (page 143), where it is shown that if l be the extension produced in the string in the equilibrium position due to the mass, the string executes a *S.H.M.*, its time period being given by

$$t = 2\pi\sqrt{l/g}.$$

Let the time-period in the first case, when the mass M is suspended from the end of the string be t_1, and, in the second case, when the mass $(m+M)$ is suspended, be t_2. Then, if l_1 and l_2 be the respective extensions produced by the two masses in the equilibrium position, we have

$$t_1 = 2\pi\sqrt{l_1/g} \quad \text{and} \quad t_2 = 2\pi\sqrt{l_2/g}.$$

$$\text{So that, } \quad t_1^2/t_2^2 = l_1/l_2.$$

$\therefore$ if $t_1 = 4$ *secs.* and $t_2 = 5$ *secs.*, we have $16/25 = l_1/l_2$.

Now, l_1 and l_2 are directly proportional to the stretching force applied *viz.*, Mg and $(M+m)g$ respectively ; for,

in the *1st case*, $Y = \frac{Mg.L}{a.l_1}$, and in the *2nd case* $Y = \frac{(M+m)g.L}{a.l_2}$,

whence, $l_1 = Mg.L/ya$ and $l_2 = (M+m)g.L/Y.a$,

where L is the *original length of the string*, a, its *area of cross-section* and Y, the *Young's modulus* for its material.

$\therefore$ $l_1/l_2 = mg/(M+m)\,g = M/(M+m)$.

Hence, $\frac{M+m}{M} = \frac{25}{16}$, whence, $\frac{M+m-M}{M} = \frac{25-16}{16}$ *i.e.*, $\frac{m}{M} = \frac{9}{16}$.

Thus, the two masses, m, and M, are in the ratio 9 : 16.

11. The breaking stress of Aluminium is $7{\cdot}5\times10^8$ dynes cm.$^{-2}$ and of Copper, 22×10^8 dynes/cm.$^{-2}$. Find the greatest lengths of the two wires that could hang vertically without breaking. Density of Aluminium = 2·7 gms./c.c. and of Copper = 8·9 gms /c.c.

Let *a sq. cms.* be the *area of cross-section* of the wires and l_1 and l_2 *cms.*, their *lengths* respectively that could hang vertically without breaking.

(*i*) *Case of the Aluminium wire.*—

Here, volume of aluminium wire $= l_1\times a$ *c cs.*, and $\therefore$ its mass $= l_1\times a\times 2\,7$ *gms.*,

and its weight $= l_1\times a\times 2\,7$ *gms. wt.* $= l_1\times a\times 2{\cdot}7\times 981$ *dynes.*

This must be equal to the maximum force the wire can withstand.

Now, breaking stress $= 7{\cdot}5\times10^8$ *dynes* /*cm*^{2}.

Therefore, total force that can be applied to the wire, without breaking it, is equal to $7\,5\times10^8\times a$ *dynes.*

Or, $l_1\times a\times 2\,7\times 981 = 7\,5\times10^8\times a$,

whence, $l_1 = \frac{7{\cdot}5\times10^8}{2\,7\times981} = 283{,}100$ *cms.* $= 2{\cdot}831$ *kilometres.*

(*ii*) *Case of copper wire.*—

Proceeding as above, we have, in this case,

$$l_2\times a\times 8{\cdot}9\times 981 = 22\times10^8\times a,$$

whence, $l_2 = \frac{22\times10^8}{8\,9\times981} = 252{,}000$ *cms.* $= 2\,52$ *kilometres.*

Thus the required lengths of the aluminium and copper wires are 2·831 and 2 52 *kilometres* respectively.

12. A copper wire 3 metres long for which Young's modulus is $12{\cdot}5\times10^{11}$ dynes per. sq. cm., has a diameter of 1 mm. If a weight of 10 k. gms. is attached to one end, what extension is produced ? If Poisson's ratio is 0·26, what lateral compression is produced ?

Here, original length of the wire (L) $= 3$ *metres* $= 300$ *cms.*,

Young's modulus for the wire (Y) $= 12{\cdot}5\times10^{11}$ *dynes. cm.*$^{-2}$,

radius of the wire (r) $= \frac{1}{2}$ *mm.* $= {\cdot}5$ *mm.* $= {\cdot}05$ *cm.*

and $\therefore$ its area of cross-section $= \pi r^2 = \pi\times({\cdot}05)^2$ *sq. cms.*

And force applied (F) $= 10$ *k. gm. wt.* $= 10\times1000^2$ *gms. wt.*

$= 10\times100\times981$ *dynes* $= 981\times10^4$ *dynes.*

Now, we have the relation,

$$Y = \frac{F\times L}{a\times l}, \text{ whence, } l = \frac{F.L}{a.Y}.$$

$$= \frac{981\times10^4\times300}{\pi\times({\cdot}05)^2\times12{\cdot}5\times10^{11}}.$$

$$= \frac{981\times3}{\pi\times({\cdot}05)^2\times12{\cdot}5\times10^5}. = {\cdot}2997\ cm.$$

Or, extension produced = ·2997 *cm.*

Again, we have Poisson's ratio, $\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}}$.

$\therefore$ $\cdot 26 = \frac{\text{lateral strain}}{l/L}$.

Or, lateral strain $= \cdot 26 \times l/L = \cdot 26 \times \cdot 2997/300 = 2 \cdot 598 \times 10^{-4}$.

This, therefore, gives the value of lateral strain, *i.e.*, d/D, where d is the decrease in diameter and D, the original diameter of the wire.

Hence $d/D = 2 \cdot 598 \times 10^{-4}$.

Since $D = 1$ *mm.* $= \cdot 1$ *cm.*, we have

$d/\cdot 1 = 2 \cdot 598 \times 10^{-4}$. Or, $d = 2 \cdot 598 \times 10^{-4} \times \cdot 1 = 2 \cdot 598 \times 10^{-5}$.

Hence, lateral compression produced $= 2 \cdot 598 \times 10^{-5}$ *cms*

13. A uniform glass tube is hung from a support and stretched by a weight. It is found that one metre of the tube stretches by ·08 cm , but that a column of water 1 metre long contained within the tube lengthens by only 0·4 cm. Find Poisson's ratio for glass.

We know that *Poisson's ratio*, $\sigma = \beta/\alpha$.

Now, let the stress be $= P$ *dynes/cm*^{2}.

and internal radius of the tube $= r$ *cms.*

Then, increase per unit length of the tube $= P.\alpha$ *cms.*

And, decrease per unit radius of the tube $= P.\beta$ *cms.*

$\therefore$ increase in 1 *metre* or 100 *cms* length of the tube $= 100.P\,\alpha$ *cms.*

and decrease in the radius of the tube $= P.\beta\, r$. *cms.*

So that, the radius of the tube is now $= (r - P.\beta.r)$ *cms.* $= r(1 - P\beta)$ *cms.*

Now, increase in length of the tube $= \cdot 06$ *cm.*

$\therefore$ $100.P.\alpha = \cdot 06$, whence, $\alpha = 06/100\ P$.

And clearly, initial cross-section of the tube $= \pi r^2$ *sq. cms.*,

and, final cross-section of the tube $= \pi[r(1 - P\beta)]^2$ *sq. cms.*

$= \pi r^2 \times [1 - 2P\beta + (P\beta)^2]$ *sq. cms.*

$= \pi r^2 \times [1 - 2P.\beta]$ *sq. cms.*,

neglecting $(P.\beta)^2$ as a very small quantity.

Therefore, volume of water column *initially* $= 100\ \pi r^2\, c$ *cs.*

And volume of water column *finally* $= 100\ 4 \times \pi r^2(1 - 2P.\beta)\, c$ *cs.*,

$\because$ length of the water column is now $100 + \cdot 04$ *cms.*

Since *volume of the water column remains the same*, we have

$100 \cdot 04 \times \pi r^2(1 - 2P\beta) = 100\pi r^2$, Or, $100 \cdot 04(1 - 2P\beta) = 100$.

Or, $100 \cdot 4 - 100 \cdot 04 \times 2P\beta = 100$. Or, $100\ 04 \times 2P\beta = 100 \cdot 04 - 100 \cdot 00 = \cdot 04$,

whence, $\beta = \frac{\cdot 04}{2P \times 100 \cdot 04}$.

$\therefore$ $\alpha = \frac{\beta}{\alpha} = \frac{\cdot 04}{2P \times 100 \cdot 04} \times \frac{100\ P}{\cdot 06} = \frac{\cdot 04 \times 100}{2 \times 100 \cdot 04 \times \cdot 06}$.

$= \frac{4}{2 \times 100 \cdot 04 \times \cdot 06} = \frac{1}{50 \cdot 02 \times \cdot 06} = 0 \cdot 3332$.

Hence, Poisson's ratio for glass $= 0 \cdot 3332$.

14. A steel wire 2 mm. in diameter is just stretched between two fixed points at a temperature of 20°C. Determine its tension when the temperature falls to 10°C. [Coefficient of linear expansion of steel is ·000011 and Young's modulus for steel is $2 \cdot 1 \times 10^{12}$ dynes per sq. cm.]

Let the length of the wire be l *cms.*

Then, on a fall in its temperature, from 20°*C* to 10°*C*, its length will decrease by an amount $= l \times \cdot 000011 \times 10$ *cms.*

And $\therefore$ *strain* produced in it $= l \times \cdot 000011 \times 10/l = 000011 \times 10 = 11 \times 10^{-5}$.

Let T *dynes* be the tension in the wire. Then,

$$stress = T/\pi r^2 = T/\pi(\cdot 1)^2 = T/\pi \times \cdot 01 \text{ sq. cm.}$$

$\because$ radius of the wire, $r = 1$ *mm.* $= \cdot 1$ *cm.*, and $\therefore$ its area of cross-section

$$= \pi r^2 = \pi \times (\cdot 1)^2 \text{ sq. cms.}$$

Now, Young's modulus, (Y) = stress/strain. So that,

$$Y = \frac{T}{\pi \times \cdot 01} \times \frac{1}{11 \times 10^{-5}}. \quad \text{Or,} \quad \frac{T}{\pi \times \cdot 11 \times 10^{-5}} = 2 \cdot 1 \times 10^{12},$$

whence, $T = 2 \cdot 1 \times 10^{12} \times \pi \times \cdot 11 \times 10^{-5} = 2 \cdot 1 \times 10^{7} \times \pi \times \cdot 11 = 7 \cdot 257 \times 10^{6}$ *dynes.*

Therefore, tension of the wire $= 7 \cdot 257 \times 10^{6}$ *dynes.*

15. If one body impinges on another which is at rest, find the relation between (*a*) momenta, (*b*) the kinetic energies of the system before and after impact.

A steel ball is let fall through a height of 64 cms. on a plate of steel. The height through which it rebounds is 36 cms. Calculate the coefficient of restitution.

We know that $e = \dfrac{\textit{relative velocity after impact}}{\textit{relative velocity before impact}}$.

Here, relative velocity *after* impact, say, $v = \sqrt{2.g.36}$, [$\because$ $h = 36$ *cms.*

and „ „ *before* impact, say, $u = \sqrt{2.g.64}$. [$\because$ here, $h = 64$ *cms.*

Therefore, $e = \sqrt{2.g.36}/2\ g\ 64 = \sqrt{36}/64 = \sqrt{9/16} = 3/4 = \cdot 75$.

Thus, the coefficient of restitution $= \cdot 75$.

EXERCISE VIII

1. Define Young's modulus. Show that a shear is equivalent to a compression and an extension.

Find an expression for the work done in stretching a wire and hence deduce an expression for the energy per unit volume of the wire.

(*Madras B.A., 1947*)

2. A wire 300 *cms.* long and 0·625 *sq. cm.* in cross-section is found to stretch 0·3 *cm.* under a tension of 1200 kilogrammes. What is the Young's modulus of the material of the wire ? (*A.M I.E., 1961*)

Ans. $2 \cdot 3 \times 10^{12}$ *dynes/sq cm.*

3. Explain the terms : stress, strain, Young's modulus, Poisson's ratio, bulk and rigidity moduli. Show that the value of Poisson's ratio must lie between -1 and $+1/2$. (*Calcutta*)

4. Define Young's modulus, Bulk modulus and modulus of Rigidity. If E, K and n represent these moduli respectively, prove the relation $E = 9nK/3K + n$. (*Allahabad, 1943*)

5. A solid ball 330 *cms.* in diameter is submerged in a lake at such a depth that the pressure exerted by water is 1·00 *k. gm. wt /cm*^{2}. Find the change in volume of the ball. (K for the material $= 1 \cdot 00 \times 10^{7}$ *dynes/cm*^{2}.)

(*Bombay, 1959*)

Ans. 1 386 *c.cs.*

6. While at 0°*C.*, a square steel bar of 1 *cm.* side is rigidly fixed at both ends so that it cannot expand. Its temperature is then raised to 20°*C.* What force does it exert on the clamps ? (Young's modulus for steel $= 2 \times 10^{12}$ *dynes/sq. cm.* and coefficient of expansion of steel $= 0\ 000011$).

Ans. 448 *k. g wt.*

7. Find the formula for the work done in stretching a wire, and apply it to find the elastic energy stored up in a wire, originally 5 *metres* long and 1 *mm.* in diameter, which has been stretched by 3/10 *mm.* due to a load of 10 *k. gm.* Take $g = 300\pi$. (*Bombay*)

Ans. $4 \cdot 5\pi \times 10^{4}$ *ergs.*

8. A bar of iron, 0·4 *sq. in.* in cross-section is heated to 100°*C.* It is then fixed at both ends and cooled to 15°*C.* Calculate the force exerted by the bar on the

fixings. Young's modulus for iron is 30,00,000 *lb./sq. in.* The coefficient of linear expansion of iron is 0·0000121°*C.*

(*Institute of Civil and Electrical Engineering*)

Ans. 5·464 *tons.*

9. A wire of length 50 *cms.*, and diameter 9 *mms.* was fixed at the upper end while a wheel of 10 *cms* diameter was fastened to the lower end. Two threads were wrapped round the wheel and passed horizontally over pulleys ; each thread supported a scale pan. On placing a weight of 230 *gms.* on each pan the lower end of the wire was twisted through 45°*C.* What is the rigidity coefficient of the material of the wire ? **Ans.** $7{\cdot}96\times10^{11}$ *dynes cm.*$^{-2}$

[**Hint.**—*Convert degrees into radians* — π *radians* = *180°.*]

10. Explain what you understand by 'shearing strain'. What are its dimensions ? Deduce an expression for the moment of the couple required to twist the lower end of a rod of circular cross-section by 90°, the upper end being clamped. (*Agra, 1945*)

[**Hint.** $\theta = 90° = \pi/2$ *radians.*] **Ans.** Couple $= \pi^2.nr^4/4l$.

11. What couple must be applied to a wire, 1 *metre* long, 1 *mm.* diameter, in order to twist one end of it through 90°, the other end remaining fixed ? The rigidity modulus is $2\,8\times10^{11}$ *dynes cm.*$^{-2}$

[**Hint.**—$90° = \pi/2$ *radians.*] **Ans.** $4{\cdot}3\times10^{6}$ *dynes cm.*$^{-2}$

12. Explain what is meant by 'modulus of rigidity' and find out its dimensions. Describe one method of finding experimentally the modulus of rigidity of a wire and give the theory of the method. Find the force necessary to stretch by 1 *mm.* a rod of iron 1 *metre* long and 2 *mms* in diameter. Also calculate the energy stored in the stretched rod, [Young's Modulus for Iron $=2\times12^{14}$ *C.G.S. units*] (*Patna, 1949*)

Ans. (*i*) 64 *k gm wt* (*ii*) $\pi\times10^{6}$ *ergs.*

13. Find the relation between the bending moment and the curvature of the neutral axis at any point in a bar.

A vertical rod of circular section of radius 1 *cm.* is rigidly fixed in the earth and its upper end is 3 *metres* from the ground level. A thick string which can stand a maximum tension of 2 *k gm.* is tied at the upper end of the rod and pulled horizontally. Find how much will the top be deflected before the string snaps. (Y for steel $= 2\times10^{12}$ *C.G S. units*, $g = 1000$ *C.G.S. units*).

(*Saugar. 1948*)

Ans. 11·47 *cms.*

14. If a brass bar, 1 *cm. square* in section, is clamped firmly in a horizontal position at a point, 100 *cms.* from one end, and a weight of one *k. gm.* is applied at the end, what depression would be produced ? (Y for brass $= 9{\cdot}78\times10^{11}$ *dynes cm.*$^{-2}$). **Ans.** 4·01 *cms.*

15. A uniform beam is clamped horizontally at one end and loaded at the other. Obtain the relation between the load and the depression at the loaded end.

Compare loads required to produce equal depression for two beams, made of the same material and having the same length and weight, with the only difference that while one has a circular cross-section, the cross-section of the other is a square. (*Saugar, 1950*)

Ans. $3 : \pi$.

16. A weight is suspended from the free end of a uniform cantilever. Find the equation of the curve into which the cantilever is bent. The weight of the cantilever may be neglected.

A uniform rigid rod 120 *cms.* long is clamped horizontally at one end. A weight of 100 *gms.* is attached to the free end. Calculate the depression of a point 90 *cms* distant from the clamped end. The diameter of the rod is 2 *cms.* Young's modulus of the material of the rod is $1{\cdot}013\times10^{11}$ *dynes per sq. cm.* and $g = 980$ *cm./sec*^{2}. (*Bombay, 1940*)

Ans. 2·834 *mms.*

17. A light beam of circular cross-section is clamped horizontally at one end and a heavy mass is attached at the other end. Find the depression at the loaded end.

If the mass is pressed down a little and then released, show that it will form simple harmonic motion. Explain how from a knowlege of the period

of oscillation, the mass and the dimensions of the bar, the value of Young's modulus for the material of the bar may be determined. (*Madras*)

18. A vertical wire is loaded (within the limits of Hooke's Law) by weights, which, produce a total extension of 3 *mms.* and 5 *mms.* respectively. Compare the amounts of work necessary to produce these extensions.

Ans. 9 : 25.

19. A sphere of mass 800 *gms.* and radius 3 *cms.* is suspended from a wire of length 100 *cms.* and radius 0·5 *mm.* If the period of torsional vibration is 1·23 *secs*, calculate the rigidity of the material of the wire. (*Bombay*)

Ans. $7{\cdot}654\times10^{11}$ *dynes cm.*$^{-2}$

20. A bar, one metre long, 5 *mms.* square in section, supported horizontally at its ends and loaded at the middle, is depressed 1·96 *mm.* by a load of 100 *gms.* Calculate Young's modulus for the material of the bar. (Take $g = 980$ *cm./sec*^{2}.)

Ans. $19{\cdot}99\times10^{11}$ *dynes cm.*$^{-2}$

21. Calculate the time of vertical oscillation of a mass of 1 *k. gm.* hanging by a steel wire 3 *metres* long and ·5 *mm.* in diameter. (Y for steel $= 2\times10^{12}$ *C.G.S. units*). **Ans.** ·05 *sec.*

[**Hint.** *Find extension l produced Then,* $t = 2\pi\sqrt{l/g}$. (*See solved example* 10, *page* 143].

22. Prove that Young's modulus Y, the bulk modulus K, the modulus of rigidity n and Poisson's ratio σ satisfy the relations :

$$(i)\ \frac{Y}{2n} = 1+\sigma\ ;\quad (ii)\ \frac{Y}{3K} = 1-2\sigma\ ;\text{ and }(iii)\ \frac{3}{n}+\frac{1}{K} = \frac{9}{Y}.$$

23. Define Poisson's ratio, and show mathematically, from first principles, that it must be less than 0·5 and cannot be less than -1. Calculate Poisson's ratio for, and the rigidity of silver from the following data :

Young's modulus for silver wire $= 7{\cdot}25\times10^{11}$ *dynes cm.*$^{-2}$

Bulk modulus for silver wire $= 11\times10^{11}$ *dynes cm.*$^{-2}$

Ans. $n = 2\,607\times10^{11}$ *dynes cm.*$^{-2}$, and $\sigma = 0{\cdot}39$.

[**Hint.** (*i*) *From* (*i*) *and* (*ii*) *above* (*Ex. 22*), *we have* $3K(1-2\sigma) = 2n(1+\sigma)$. *Since K and n are both +ive,* σ *cannot be more than* ·5 *and less than* -1.

(*ii*) See § 116, page 288, whence, it can be shown that

$$n = \frac{3YK}{(9K-Y)},\quad K = \frac{nY}{(9n-3Y)} \text{ and } Y = \frac{9Kn}{n+3K}.$$

Also $$3\beta = \frac{2}{2n} - \frac{1}{3K}\text{, whence, } \beta = \frac{3Y-2n}{18nK},$$

and $$a = \frac{1}{Y}.\quad \text{Or, } Y = \frac{1}{a} = \frac{9Kn}{n+3K}.$$

$\therefore$ $$\sigma = \frac{\beta}{a} = \frac{3K-2n}{18nK}\times\frac{9Kn}{n+3K} = \frac{3K-2n}{2(n+3K)},$$

Or, directly from (*i*) (Ex. 22), $\sigma = \dfrac{Y}{2n}-1$.

24. A wire 1 *metre* in length and 1 *mm.* in diameter is stretched by 0·*mm.* by a load of 10 *k. gms wt.* and is twisted through 70° by a force of 5 *gms. wt.* applied to each end of a 20 *cm.* rod soldered at its mid-point to the end of the wire. Calculate (1) Young's modulus, (2) Shear modulus, and (3) Bulk modulus of the wire.

Ans $Y = 20{\cdot}81\times10^{11}$ *dynes cm.*$^{-2}$; $n = 8{\cdot}268\times10^{11}$ *dynes cm.*$^{-2}$ and $K = 14\ 35 = 10^{11}$ *dynes cm.*$^{-2}$

[**Hint.** (*i*) *Twisting couple* $= n\pi\theta r^4/2l = 5\times981\times20$ *dyne-cm.*,

and $$\theta = 70\times\pi/180 \text{ radians.}$$

(*ii*) $$K = nY/(9n-3Y).$$ [see *Ex.* 23, **Hint** (*ii*)]

25. A block of soft rubber, 5″ square, has one face fixed, while the opposite face is sheared through a distance ·5″ parallel to the fixed face by a tangential force of 39 *lbs. wt.* How much work is done per unit volume of the cube to do this ? **Ans.** 8·64 *ft. lbs.*

26. Calculate the depression at the free end of a thin light beam, clamped horizontally at one end and loaded at the other.

For the same mass per unit length, show that a beam of square section is stiffer than one of circular section, the deflections being in the ratio $3/\pi$. (*Bombay, 1949*)

27. A rectangular bar of iron is supported at its two ends on knife-edges and a load is applied at the middle point. Calculate the depression of the middle point.

How can this be utilized to determine Young's modulus of iron ? (*Allahabad, 1947*)

28. Find the value of Young's modulus for copper. In an experiment, the diameter of the rod was 1·26 *cms.* and the distance between the knife-edges 70 *cms.* On putting a load of 900 *gms.* at the middle point, the depression was 0·025 *cm.* Calculate the Young's modulus of the substance. (*Agra, 1948*)

Ans. $20{\cdot}42 \times 10^{11}$ *dynes/cm*^{2}.

29. Define Poisson's ratio and describe a method for its determination. Derive the formula used. (*Agra, 1947*)

30. Derive the expression for the bending of a tube supported at the two ends and loaded in the middle. (*Banaras, 1947*)

31. How do you differentiate between a *column* and a *strut* ? Obtain an expression for the critical load for a long column with its ends rounded or hinged.

32. Discuss *Euler's theory of long columns* for the case (*i*) *when both ends of a column are rounded or hinged,* (*ii*) *when both ends of the column are fixed.*

33. Show that (*i*) *a column, with its ends fixed, has four times the strength to resist a thrust than a similar column, with its ends rounded or hinged* and (*ii*) *a column, hinged at one end and loaded at the other has only one-fourth the strength of the same column when hinged at both ends.*

34. Two steel balls of masses 1 and 10 *k.g.* respectively are moving each towards the other with a relative velocity of 4 *metres* per second. Find the loss of energy after impact and state the reason thereof. (*Bombay, 1932*)

Ans. 50290 *ergs.*

35. A sphere of mass 3 *lbs.*, moving with a velocity of 7 *ft./sec.*, impinges directly on another sphere, of mass 5 *lbs.*, at rest ; after the impact, the velocities of the spheres are in the ratio of 2 : 3. Find the velocities after impact and the loss of kinetic energy. (*London University*)

Ans. (*i*) 2 *ft./sec.* and 3 *ft./sec.* (*ii*) 45 *ft. poundals.*

36. Explain briefly the terms : resilience and stiffness of a beam. What is proof resilience ?

37. Write a brief note on elastic waves.

CHAPTER IX

HYDROSTATICS

145. Fluids—Liquids and Gases. Hydrostatics deals with the mechanics of fluids in equilibrium and our first step, therefore, is to understand clearly as to what exactly do we mean by a '*fluid*'.

Unlike a solid, in which, as we have seen, the strain set up under a shearing stress lasts throughout the period of application of the stress, *a fluid may be defined as that state of matter which cannot indefinitely or permanently oppose or resist a shearing stress.* In fact, it constantly and continuously yields to it, though the yield may be rapid in some cases and slow in others. In the former case, the liquid is said to be *mobile,* (like water, alcohol etc.) and in the latter, *viscous*, (like honey, treacle etc.). In either case, however, a *fluid has no definite shape of its own* and assumes ultimately the shape of the containing vessel.

And yet, with all this seemingly clear-cut distinction between a solid and a fluid, it is not quite so easy to distinguish between the two in many a border-line case. Thus, for example, pitch, which looks so much like a solid that it has to be hammered in order to be broken, is *essentially* a fluid ; for, when subjected to the shearing stress of its own weight, by putting a piece of it in a funnel, or by putting a barrel of it on its side, it does begin to yield or flow, although infinitely slowly. On the other hand, metal wires, which are obviously solids, when subjected to an excessive tension, begin to flow in the manner of fluids, and, indeed, may be considered to be so, for the duration of the yield Once the yield is over, however, they behave like solids they in fact are.

Then, again, we have, on the one hand, highly elastic solids, like *quartz*, in which no change of shape is discernible even in millions of years, as is evidenced by the sharpness of its crystals which look as though they had *just* been formed, and, on the other, fluids, like water, the rapid flow of which almost instantaneously does away with any sharpness of its edges and which, in small quantities, assumes a spherical form, with no sharp edges or corners whatever.

The fundamental distinction between the two nevertheless remains, and we declare a substance to be a fluid or a solid according as it does or does not yield to a shearing stress applied to it over a long enough period.

Now, fluids too are further divided into two classes, *viz.*, (*i*) *liquids* and (*ii*) *gases*.

A **liquid** *is a fluid which, although it has no shape of its own, occupies a definite volume, which cannot be altered, however great the*

*force applied to it**. In other words, *a liquid is a fluid which is quite incompressible* and *has a free surface of its own*; as, for example, water, alcohol, ether, honey, treacle etc.

A gas, *on the other hand, is a fluid, which cannot only be easily compressed when subjected to pressure but, which, with a progressive reduction of the pressure on it, can also be made to expand indefinitely, occupying all the space made available to it.* Thus, the whole of the gas will escape out from a vessel, if there be the tiniest aperture in it somewhere.

Summarizing then, *a gas is a fluid which has neither a shape nor a free surface of its own*; as, for example, oxygen, hydrogen, carbon dioxide, air (a mixture of gases) etc. We shall consider first the case of liquids.

146. Hydrostatic Pressure. Since a liquid possesses *weight*, it exerts force on all bodies in contact with it, *e.g.*, on the bottom and the walls of the vessel containing it,—the force due to it being always spread over an area†. And, if this force be uniformly distributed over the whole area, *i.e.*, be the *same* on each small equal element of the surface, its value per unit area is called **pressure** or **hydrostatic pressure** of the liquid,—meaning *pressure due to the liquid at rest*‡. And, if the force be not uniform, the ratio between the small force δF and the area δA on which it acts gives the pressure.

Thus, $$pressure = \delta F/\delta A.$$

So that, when δA is progressively diminishing, we have

$$pressure\ at\ the\ point = \underset{\delta A \to 0}{\text{Limit}} \frac{force}{area}.$$

Or, denoting pressure at the point by p, we have, in mathematical notation,

$$p = dF/dA.$$

The *total force* exerted by a liquid column on the whole of the area in contact with it is called *thrust*.

Thus, $$thrust = pressure \times area.$$

That a liquid, at rest, always exerts a thrust *normally* to the surface in contact with it is obvious. For, if it were not so, there would be a component of the thrust along, or parallel to, the bounding surface, and an equal and opposite thrust on it due to the reaction of the surface would cause it to flow, since it must, by its very nature, yield to a tangential force. It follows, therefore, that since the liquid is at rest, the thrust due to it must be perpendicular to the bounding surface at every point.

*Strictly speaking, all liquids do get compressed a little, when subjected to very high pressures. The compression is, however, almost negligible. Thus, water, when subjected to a pressure of about 200 atmospheres, undergoes a reduction of only a hundredth part of its original volume.

†The same is true of the force exerted *on* a liquid, as, for example, when we press the piston down in a cylinder containing the liquid.

‡It is also sometimes called the *pressure in a liquid due to gravity.*

In other words, the free surface of a liquid at rest must be at right angles to the forces acting on it. Thus, when the only force acting on it is due to gravity, its surface remains *horizontal**, being perpendicular to the force of gravity, but, with a steady wind blowing, it is slightly inclined, again, however, at right angles to the resultant of the forces due to gravity and the wind.

Further, since every layer of a liquid at rest is in equilibrium, it follows that the *downward* thrust on it, due to the liquid column above it, is just balanced by an equal *upward* thrust due to the liquid column below it. *In other words, at any given level, in a liquid at rest, the downward thrust due the liquid column is equal to the upward thrust on it.*

147. Hydrostatic Pressure due to a Liquid Column. Let us now calculate the *hydrostatic pressure* due to a liquid column h. Imagine a narrow metal cylinder, of area of cross-section a and fitted with a frictionless piston, of negligible weight, to be supported in a liquid of density ρ, (Fig. 212).

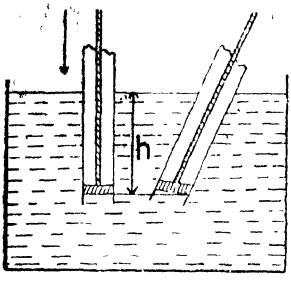


Fig. 212.

Then, if the *upthrust* on the piston due to the water below it be F, obviously an equal and opposite force F has to be exerted on the piston to keep it in position.

Hence, if the piston be moved down through a distance x, work done on it is clearly equal to $F.x$.

This downward motion of the piston will obviously expel *a volume* $x.a$ of the liquid out of the tube, its *mass* being $x.a.\rho$ and its *weight*, equal to $x.a.\rho.g$.

Since the level of the liquid in the containing vessel is thus slightly raised, *it is tantamount to this weight of the liquid* $x.a.\rho.g$. *rising up through a vertical distance h up to the liquid surface.*

In other words, *increase in potential energy of the liquid* $= x.a.\rho.g.h$. And, this gain in potential energy will obviously be equal to the work done on the piston. So that, we have

$$F.x = x.a.\rho.g.h. \quad \text{Or,} \quad F/a = h.\rho.g.$$

i.e., the hydrostatic pressure due to a liquid column at depth h from the surface is equal to $h.\rho.g.$, ($\because F/a = force/area = pressure$).

N.B. The argument remains the same even if the metal tube is inclined and not vertical, so long as the *vertical depth* of the piston remains the same. It will thus be seen that the *pressure due to a liquid column depends only upon its depth and density*, and not to any other factor like the surface area of the containing vessel etc.

148. The Hydrostatic Paradox. A remarkable fact follows from the above, *viz.*, that so long as the vertical height of the column of a liquid remains the same, the *pressure* exerted by it remains the same, irrespective of its actual mass or weight.

*In the case of a large expanse of water, the surface is spherical and is thus again perpendicular to the direction of gravity at every point.

Pascal discussed this result in his *'Treatise on the equilibrium of liquids'*, published in the year 1663, with reference to vessels of different shapes, known as *Pascal's vases*, (Fig. 213), all having *equal bases* and *containing water upto the same vertical height h*, so that the *pressure on the base of each vessel was equal to h gms. wt./cm².*, and, therefore, the *thrust* on it was *h.a. gms. wt.*, where *a* is its *area*.

He was perhaps the first person to have pointed out the paradoxical truth that even if vessel (*i*) contains 100 *lbs.* of water and vessel (*v*) only 1 *oz.* of it, the thrusts on the bases of both is the *same*. Aptly, therefore, it is called the *hydrostatic paradox*.

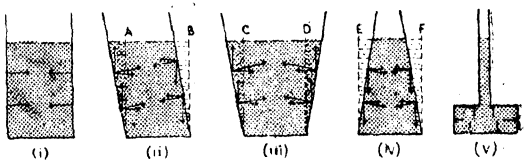


Fig. 213.

Strange as it may seem, but if the water in vessel (*v*) be frozen into ice and detached from its sides, the thrust exerted by *this* ice on its base will be only 1 *oz. wt.*, but once this ice is melted back into water, the thrust again increases to 100 *lbs. wt.* The explanation of this seeming paradox is, however, simple. The ice does not exert any upward thrust on the part of the vessel opposite to the base and the latter, therefore, exerts no equal and opposite thrust on it. But the water does exert an upward thrust on it and hence receives back an equal and opposite downward thrust from it.

In case of vessel (*i*), the thrust on the base is equal to the entire weight of the water on it.

In vessel (*ii*), the upward component of the thrust due to the left side of the vessel supports the weight of water in it, between the left side and the dotted line *A*, while the downward component of the thrust due to the right side of the vessel exerts a downward thrust on it, equal to the weight of the water in-between the right side of the vessel and the dotted line *B* ; so that, the thrust on the base is the *same as due to a vertical column h of water.*

In vessel (*iii*), the upward components of the thrusts due to both the left and the right sides of the vessel support the extra weight of the water, in-between the two sides and the dotted lines *C* and *D*, and, again, therefore, *the thrust on the base is equal to that due to the cylindrical column h of water in-between the dotted lines C and D.*

And, similarly, in vessel (*iv*), the downward components of the thrust due to the two sides of the vessel exert an *extra* thrust on the base, equal to the weights of the water contained between either side and the dotted lines *E* and *F*; so that, once again, *the total thrust on the base is the same as that due to a cylindrical water column of height h.*

Pascal experimentally verified the above fact by supporting, by means of a separate stand, bottomless vases of the above shapes, one by one, on a large disc, *D* (Fig. 214), suspended from the shorter pan of a hydrostatic balance and kept pressed against their bases by placing a heavy weight in the longer pan, and pouring water into the vessel. *The disc just got detached from its base as the water reached the same level in each case*, thus clearly demonstrating the equivalence of the thrust on the disc in *each* case and fully vindicating his deductions.

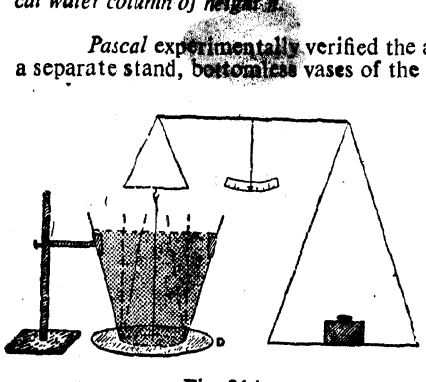
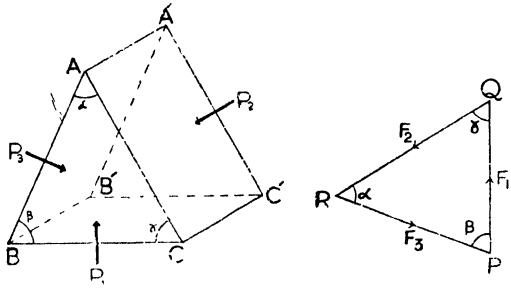


Fig. 214.

149. A liquid transmits pressure equally in all directions—Pascal's Law. Since we do not have any boundary demarcated in the interior of a liquid, *we may define pressure there as the force exerted per unit area across any plane in it*, and it can be easily shown that this pressure is exerted *equally in all directions* in the liquid.



(*i*) Fig. 21 (*ii*)

Thus, let us consider a portion of the liquid, in the form of a triangular prism, [Fig. 215 (*i*)] with its faces ABC and $A'B'C'$ vertical and its edges AA', BB' and CC' horizontal.

This triangular liquid prism is obviously in equilibrium under the action of the forces acting on its different faces. Let us study the inter-relation of all these forces.

It is clear that due to its small size, every part of this prism can be taken to be at the *same depth* from the liquid surface and also the pressure on each face of it to be uniform*.

Now, the forces on the two end-faces are equal and opposite, thus neutralising each other's effect and may, therefore, be ignored in our discussion. Hence, if P_1, P_2 and P_3 be the pressures on the faces $BCC'B'$, $CAA'C'$ and $ABB'A'$ respectively, and l, the *length* of the prism, we have

force F_1 on face $BCC'B' = P_1 \times$ area $BCC'B' = P_1.l.BC$,

force F_2 ,, ,, $CAA'C' = P_2 \times$ area $CAA'C' = P_2.l\,CA$,

and force F_3 ,, ,, $ABB'A' = P_3 \times$ area $ABB'A' = P_3.l.AB$.

Since these three forces keep the liquid-prism in equilibrium, they can be represented by the three sides of a triangle, taken in order. Let PQR, [Fig. 215 (*ii*)], be this triangle of forces, with its sides PQ, QR and RP representing F_1, F_2 and F_3 respectively. Then, clearly,

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}. \qquad \text{[Lame's theorem.}$$

Or, $$\frac{P_1.l.BC}{\sin\alpha} = \frac{P_2.l.CA}{\sin\beta} = \frac{P_3.l.AB}{\sin\gamma}.$$

Or, $$\frac{P_1.BC}{\sin\alpha} = \frac{P_2.CA}{\sin B} = \frac{P_3.AB}{\sin\gamma} \qquad ..(i)$$

Now, since angles A, B and C, of the triangle ABC, are respectively equal to angles α, β and γ, (the sides PQ, QR and RP being perpendicular to BC, CA and AB respectively, we have

*This is so, because of the small dimensions of its faces.

$$\frac{BC}{\sin\alpha} = \frac{CA}{\sin\beta} = \frac{BA}{\sin\gamma} \quad ..(ii)$$

[The sides of a triangle being proportion to the sines of the angles opposite to them.

From relations (*i*) and (*ii*), therefore, we have

$$P_1 = P_2 = P_3,$$

i.e., the pressures on the three faces are equal.

Further, since the same relation holds good in whatever position the prism may be rotated, it follows that *a liquid exerts pressure equally in all directions within itself.*

N.B. It also follows from the above that if we replace the liquid prism by a solid one, of the *same size and weight*, the forces acting on the latter would also be the *same* and hence it would also be in equilibrium.

150. Thrust on an Immersed Plane. If we have a plane horizontal surface of area A, immersed in a liquid of density ρ, the pressure P on it is uniform, since all points on it are at the same depth h, say, from the liquid surface, which, as we know, is also horizontal and, therefore, parallel to the immersed surface.

So that, $$P = h.\rho.g.$$

Now, *force* or *thrust*, *i.e.*, $F = pressure \times area = P.A = h.\rho.g.A.$

If, on the other hand, the plane of the immersed surface be inclined at an angle θ to the liquid surface, (Fig. 216), we must first determine the thrust on a small area dA of the surface and integrate its value over the entire surface.

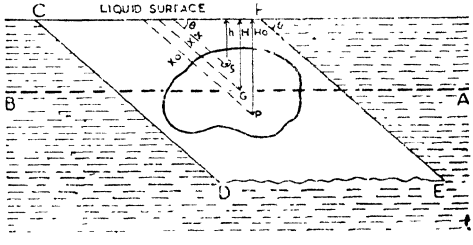


Fig. 216.

Let h be the *depth* of this element of area dA. Then, the *thrust* dF on this area is clearly equal to $h\,\rho.g.d.A$.

Now, if x be the distance of this element from the line CF, in which the plane of the immersed surface meets the liquid surface, we have

$$\sin\theta = h/x. \quad \text{Or,} \quad h = x\sin\theta.$$

So that, $$dF = \rho.g.\sin\theta.x.dA.$$

And, therefore, the thrust on entire area A of the surface is given by

$$F = \int \rho.g.\sin\theta.x.d\,A = \rho.g.\sin\theta \int x.dA. \quad ..(i)$$

Now, $x.dA$ is clearly the moment of the element dA of the surface about the line CF, and, therefore, $\int x.dA$ is the moment of the whole area A about CF, *i.e.*,

$$\int x.dA = A.X,$$

where X is the distance of the centroid* G of the area from CF.

*The term '*Centroid*' or '*Centre of mass*' is ordinarily used synonymously with '*Centre of gravity*', and, in a uniform gravitational field, the two are one and the same point. But, in a non-uniform field, the weights of the particles are not proportional to their masses. In such a case, therefore, the weights may not form a system of parallel forces, reducing to a single resultant force, but may form a couple, instead, varying with the different orientations of the body, whereas the centre of mass is quite independent of the gravitational field.

So that, substituting the value of $\int x.dA$ in relation (*i*) for F above, we have $F = \rho.g.\sin\theta.A.X.$

Again, if H be the depth of the centroid G of the immersed surface from the liquid surface, we have $\sin\theta = H/X$.

And, therefore, $F = \rho.g.\frac{H}{X}.A.X. = H.\rho.g.A.$ .. (*ii*)

Clearly, $H.\rho.g$ is equal to the pressure at the centroid or the centre of area. So that, we have

resultant thrust on the immersed plane = pressure at centroid or centre of area × area of the plane.

It should be carefully noted that the thrust on the immersed plane is quite independent of its angle of inclination (θ) with the liquid surface.

151. Centre of Pressure. Having obtained the value of the resultant thrust on the immersed plane, our next step obviously is to determine the point of the plane through which this resultant acts,—this point being known as the *centre of pressure*.

We know that the liquid pressure acts *normally* at every point of the immersed plane. So that, the forces $h.\rho.g.d.A$ acting on elementary areas of the plane, (like dA), are so many *like parallel forces*. We may, therefore, determine the centre of these parallel forces (*i.e.*, the point through which their resultant acts) by an application of the *principle of moments, viz.*, that the *algebraic sum of the moments of a system of parallel forces, about a given axis, is equal to the moment of the resultant about the same axis.*

Now, clearly, *moments of the thrust (or force) $h.\rho.g.dA$ acting on area dA* about CF, (Fig. 216) $= h.\rho.g.dA.x = \rho.g.x\sin\theta.dA.x.$ [$\because h = x\sin\theta$.

$= \rho.g.\sin\theta.x^2.dA.$

Therefore, *total moment of the forces on such like elementary areas dA of the plane about CF* $= \int \rho.g.\sin\theta.x^2.dA,$

$= \rho.g.\sin\theta \int x^2.dA,$ .. (*iii*)

where the integration extends over the entire surface of the plane.

Now, $\int x^2.dA$ is the *geometrical moment of inertia I_g of the area A of the plane about CF*. So that,

total moment about $CF = \rho.g.\sin\theta.I_g$.

And, since $I_g = Ak^2$, where k is the *radius of gyration* of the area A about CF, we have

total moment about $CF = \rho.g.\sin\theta.Ak^2$.

Again, if X_0 be the distance of the point P through which the resultant thrust F acts on the plane, *i.e., the distance of the centre of pressure from CF*, we have

moment of the resultant thrust about $CF = F.X_0$.

And, therefore, by the principle of moments, we have

$$F.X_0 = \rho.g.\sin\theta.Ak^2,$$

whence, $X_0 = \frac{\rho.g.\sin\theta.Ak^2}{F} = \frac{\rho.g.\sin\theta.Ak^2}{H.\rho.g.A} = \frac{\sin\theta}{H}.k^2.$

Or, $$X_0 = \frac{k^2}{X}. \qquad ..(iv) \qquad [\because H = X \sin\theta.$$

If k_0 be the *radius of gyration of the area about a parallel axis through its centroid G*, we have, by the principle of parallel axes

$$I = I_0 + A.X^2, \qquad \left[\begin{array}{l}\text{where } I_0 \text{ is its moment}\\ \text{of inertia about } \textit{this} \text{ axis}\\ \text{through } G.\end{array}\right.$$

Or, $$Ak^2 = Ak_0^2 + AX^2, \text{ whence, } k^2 = k_0^2 + X^2.$$

And, therefore, $$X_0 = \frac{k_0^2 + X^2}{X}, \qquad .. (v)$$

whence, X_0 may be easily determined.

Alternatively, equating $F.X_0$ against expression (*iii*) above, for total moment about CF due to the thrusts on elementary area dA, we have $$F.X_0 = \rho.g.\sin\theta \int x^2.dA \qquad .. (vi)$$

And, clearly, if H_0 be the *distance of the centre of pressure from the liquid surface*, we have $H_0/X_0 = \sin\theta$; and, therefore,

$$X_0 = H_0/\sin\theta.$$

Putting this value of X_0 in relation (*vi*) above, we have

$$\frac{F.H_0}{\sin\theta} = \rho.g.\sin\theta \int x^2.dA.$$

Or, $$\frac{F.H_0}{\sin\theta} = \rho.g.\sin\theta \int \frac{h^2}{\sin^2\theta}.dA. \qquad \left[\begin{array}{l}\because\ h = x\sin\theta.\\ \text{and } \therefore\ x = \frac{h}{\sin\theta}.\end{array}\right.$$

Or, $$H_0 = \frac{\rho.g.\sin^2\theta \int \frac{h^2}{\sin^2\theta}.dA}{F} = \frac{\rho.g \int h^2.dA}{F}.$$

$$= \frac{\rho.g \int h^2.dA}{H.\rho.g.A} = \frac{\int h^2.dA}{H.A}, \qquad [\because F = H.\rho.g.A.$$

the value of the integral $\int h^2.dA$, like the expression $\int x^2.dA$, depending upon the *shape* of the immersed plane.

We thus see that whereas the distance of the centre of pressure from the liquid surface is quite independent of the density of the liquid, it depends upon the shape of the plane.

152. Particular Cases of Centre of Pressure. Let us now consider some simple cases of centre of pressure on surface of a definite geometrical shape.

(*i*) **Centre of Pressure on a Rectangular Lamina.** Suppose we have a rectangular lamina of *length l* and *breadth b*, immersed vertically in a liquid to a depth *h* below its free surface, (Fig. 217).

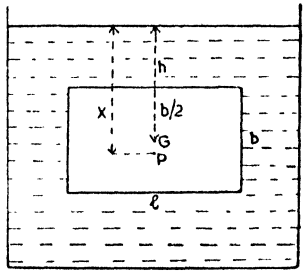


Fig. 217.

Then, since for a rectangular lamina,

$I_0 = A.k_0^2 = l.b^3/12$, (see page 308),

where I_0 is its *geometrical moment of inertia* about an axis through its centre of area (or centroid) and parallel to its length, and k_0 its *radius of gyration* about this axis, we have

$$A.k_0^2 = l.b.b^2/12 = Ab^2/12, \qquad \left[\because l \times b = A, \text{ the area of the lamina.}\right.$$
$$k_0^2 = b^2/12.$$

And, clearly, $X = h + \frac{b}{2}$.

So that, $$X_0 = \frac{k_0^2 + X^2}{X} = \frac{\frac{b^2}{12} + \left(h + \frac{b}{2}\right)^2}{h + \frac{b}{2}} = \frac{(b^2 + 3hb + 3h^2)}{2h + b}$$

whence, the position of the centre of pressure for the rectangular lamina may be easily calculated for any depth h.

Clearly, therefore, *if the lamina be just submerged in the liquid, i.e., with its upper edge just lying in the liquid surface,* $h = 0$, and we therefore, have $X_0 = \frac{2}{3}b^2/b = \frac{2}{3}b$,

i.e., the centre of pressure, in this particular case, lies two-thirds below the top of the lamina.

(*ii*) **Centre of Pressure on a Circular Lamina.** Let the centroid of a circular lamina, of radius r, lie at a depth X from the free surface of the liquid.

Then, since k_0^2 for a circular lamina is $r^2/4$, we have

$$X_0 = \frac{k_0^2 + X^2}{X} = \frac{\frac{r^2}{4} + X^2}{X} = \frac{r^2}{4X} + X.$$

Or, the centre of pressure lies at $\frac{r^2}{4X} + X$ below the liquid surface.

If, however, the lamina be submerged in the liquid, with its edge just touching the liquid surface, we have $X = r$, and, therefore,

$$X_0 = \frac{r^2}{4r} + r = \frac{r}{4} + r = \frac{5}{4}r.$$

Or, the centre of pressure, in this case, lies at $5r/4$ from the liquid surface.

(*iii*) **Centre of Pressure on a Triangular Lamina.** Here, two cases arise *viz.*, (*a*) *when the lamina is immersed upright into the liquid, i.e., with its vertex up and base down and* (*b*) *when it is immersed upside down, with its base up and vertex down.*

(*a*) **When the Lamina is immersed Upright.** Let h be the *height* of the lamina and let its apex be at a depth d below the free surface of the liquid. Then, clearly,

$$k_0^2 = h^2/18 \text{ and } X = d + \tfrac{2}{3}h. \qquad \left[\text{Its centroid lying } 2h/3 \text{ below the apex.}\right.$$

And, therefore, $$X_0 = \frac{\frac{h}{18} + (d + \frac{2}{3}h)^2}{d + \frac{2}{3}h} = \frac{h^2}{18(d + \frac{2}{3}h)} + d + \tfrac{2}{3}h.$$

$$= \frac{h^2}{18d + 12h} + d + \frac{2}{3}h = \frac{h^2 + 18d^2 + 12hd + 12hd + 8h^2}{18d + 12h}$$

$$= \frac{9h^2 + 18d^2 + 24hd}{18d + 12h} = \frac{3h^2 + 6d^2 + 8hd}{6d + 4h}$$

If, however, the vertex of the lamina just touches the free surface of the liquid, we have $d = 0$. In *this* case, therefore,

$$X_0 = 3h^2/4h = \tfrac{3}{4}h.$$

i.e., the centre of pressure *now* lies $\frac{3}{4}$*ths* down the height of the lamina.

(*b*) **When the Lamina is immersed Upside Down.** Again, let the depth of the base of the lamina be d from the free surface of the liquid.

Then, we have $k_0^2 = h^2/18$ and $X = (d+\frac{1}{3}h)$.

And, therefore, $$X = \frac{\frac{h^2}{18}+(d+\frac{1}{3}h)^2}{(d+\frac{1}{3}h)} = \frac{\frac{h^2}{6}+3(d+\frac{1}{3}h)^2}{3d+h}.$$

$$= \frac{3h^2+18d^2+12hd}{6(3d+h)} = \frac{\frac{1}{2}(h^2+6d^2+4hd)}{3d+h}.$$

And, *if the lamina be just submerged in the liquid, with its base in the surface of the liquid,* we have, $d = 0$. In that case, therefore,

$$X_0 = h^2/2h = \tfrac{1}{2}h,$$

i.e., the centre of pressure lies a distance $\frac{1}{2}h$ below the liquid surface.

153. Change of Depth of Centre of Pressure. Let a plane lamina of area A be immersed in a liquid such that its centroid G is at a depth X from the liquid surface CD, (Fig. 218).

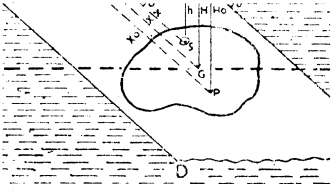


Fig.

Then, if k_0 be the radius of gyration of the lamina about the axis AB passing through G, in the plane of the lamina, and parallel to the liquid surface CD, its radius of gyration about a parallel axis, lying in the liquid surface, will clearly be k_1, such that $k_1^2 = k_0^2+X^2$.

Now, let the level of the liquid surface be raised through a distance h by adding some more liquid to that already present. Then, clearly, radius of gyration k_2 of the lamina about a parallel axis to AB, and lying in this elevated liquid surface $C'D'$, is given by $k_2^2 = k_0^2+(X+h)^2$.

Subtracting one from the other, we have

$$k_2^2-k_1^2 = [k_0^2+(X+h)^2]-(k_0^2+X^2)$$
$$= k_0^2+X^2+2Xh+h^2-k_0^2-X^2 = 2Xh+h^2.$$

Or, $$k_2^2-k_1^2 = h(2X+h). \qquad \ldots(i)$$

If X_0 be the depth of the centre of pressure of the original amount of the liquid from its free surface, we have

$$X_0 = k_1^2/X,$$ [See relation (*iv*), Page 349.

So that, after the addition of a liquid layer of thickness h, its depth becomes

$$X'_0 = X_0+h = \frac{k_1^2}{X}+h = \frac{k_1^2+Xh}{X},$$

whence, $k_1^2+Xh = X.X_0'$. Or, $k_1^2 = X.X_0'-Xh = X(X_0'-h)$.
And, if X_0'' be the depth of the *new* centre of pressure from the raised surface of the liquid, after the addition of the liquid layer, we have

$$X_0'' = \frac{k_2^2}{(X+h)}. \qquad \text{Or,} \qquad k_2^2 = X_0''(X+h).$$

So that, $k_2^2-k_1^2 = X_0''(X+h)-X(X_0'-h)$. ...(*ii*)

From relations (*i*) and (*ii*), therefore, we have

$$X_0''(X+h)-X(X_0'-h) = h(2X+h).$$

Or, $$X_0''(X+h)-X.X_0'+Xh = 2Xh+h^2.$$

Or, $$X_0''(X+h) = 2Xh+h^2-Xh+X.X_0' = h^2+Xh+X.X_0'.$$
$$= h(h+X)+X.X_0'.$$

Or, $$X_0''(X+h) = h(h+X)+X(X_0+h). \qquad [\because X_0' = (X_0+h).$$

Dividing throughout by $(X+h)$. we have $X_0'' = h+\frac{X.(X_0+h)}{(X+h)}$.

And, therefore, *the shift in the position of the centre of pressure is given by*

$$X_0'-X_0'' = (X_0+h)-\left[h+\frac{X(X_0+h)}{(X+h)}\right].$$
$$= X_0-\frac{X(X_0+h)}{X+h} = \frac{X_0h-Xh}{X+h}.$$

Or, $$X_0'-X_0'' = \frac{h(X_0-X)}{X+h}.$$

It will also be easily seen that the distance between the depths of the new centre of pressure and the centroid of the lamina is given by

$$X_0''-(X+h) = h+\frac{X(X_0+h)}{X+h}-(X+h) = \frac{X.X_0-X^2}{X+h}.$$

Or, $$X_0''-(X+h) = \frac{X(X_0-X)}{X+h},$$

which approaches *zero* as h approaches infinity.

Thus, *the greater the depth of the liquid, the nearer does the centre of pressure come to the centroid of the lamina ; so that, at an infinite distance, the two must just coincide with each other.*

154. Principle of Archimedes. Imagine a body *ABCD* to be immersed in a liquid, (Fig. 219), and a vertical line *GA* to travel round it, touching it along the line *AECF* and meeting the liquid surface in the curve *GHKL*.

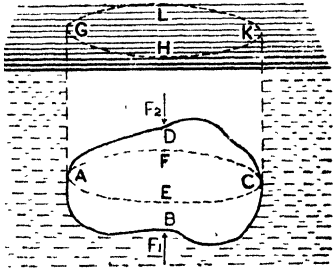


Fig. 219.

Then, clearly, the resultant upward thrust F_1 on the surface *ABCEA* and the resultant downward thrust F_2 on the surface *ADCEA* are given by the weights of the liquid that would occupy the spaces *ABCKG* and *ADCKG* respectively, acting through their respective centres of gravity. So that, the resultant thrust on the whole body is given by

$F_1 - F_2$ = *weight of liquid occupying space ABCKG minus weight of liquid occupying space ADCKG.*

= *weight of liquid occupying space ABCD.*

= *weight of liquid equal in volume to that of the immersed body.*

= *weight of liquid displaced by the body.*

In other words, when a body is immersed in a liquid, it experiences an upward thrust equal to the weight of the liquid displaced by it. It can easily be shown that the same is also true for a body which is only partially immersed in the liquid, the upward thrust on it being equal to the weight of the liquid displaced by its *immersed part*. We may, therefore, generalise and state that

when a body is wholly or partly immersed in a liquid, it experiences an upthrust equal to the weight of the liquid displaced by it, (i.e., by its immersed part).

This is known as the *Principle of Archimedes**, for it was he who first enunciated it.

The point where this upthrust acts is obviously the *c.g.* of the displaced liquid, which is called the *centre of buoyancy*, the upthrust being referred to as the *force of buoyancy*.

N. B. The applications of *Archimedes Principle* are many and various It gives us the method of determining specific gravities or densities of liquids as well as the instruments, known as *Hydrometers*, with which the Degree students are no doubt already familiar.

155. Equilibrium of Floating bodies. A body, immersed wholly or partly in a liquid, is subject to two forces, *viz.*, (*i*) *its own weight W, acting vertically downwards at its c.g., G,* and (*ii*) *the upthrust W', acting vertically upwards at its centre of buoyancy B,* (Fig. 220).

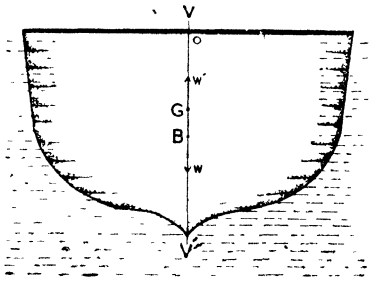


Fig. 220.

If these two points of application, (*G* and *B*), of the two forces respectively, coincide or lie in the same vertical line, called the **centre line**, *the body sinks, just remains suspended (or floating), or rises up, according as W is greater than, equal to, or less than W'.*

For, obviously, if $W > W'$, the body sinks further down, displacing more and more of the liquid and thereby increasing the upthrust until the two balance each other, and the body just stays there, *i.e.*, is in equilibrium.

If $W = W'$, then obviously, the two forces on it are equal and opposite and their line of action being the same, they just neutralise

*Archimedes, (287—212 *B.C.*), was a Greek philosopher. He was asked by *King Heiro*, at *Syracuse*, to test the gold-content of a crown. Engaged on this problem, he suddenly discovered the law of upthrust, while taking a bath, which enabled him to determine the specific gravity, and hence the quantity of gold in the crown, without in any way damaging it Overjoyed at his success, he ran home, with the triumphant cry 'Eureka', 'Eureka'.—'I have found it, I have found it.'

each other and the body remains suspended or floating in the liquid. And finally, if $W<W'$, the body rises up, so that a lesser volume of it is under the liquid, *i.e.*, it displaces a smaller volume of the liquid, and the upthrust on it is now less. This rise of the body continues until the upthrust is just equal to the weight of the body, and the body then continues to float in that very position.

Thus, two conditions are necessary for the equilibrium of a floating body, *viz.*, (*i*) *its weight must be equal to the weight of the displaced liquid**, and (*ii*) *the c.g. of the body and the centre of buoyancy of the displaced liquid must either coincide with each other*† *or lie in the same vertical line.*

156. Stability of Equilibrium. If the floating body be tilted a little to one side or the other from its original equilibrium position, through a small angle θ, (Fig. 221), *so that the weight of the displaced liquid, or the upthrust on it, remains the same*, then, since *the shape of the displaced liquid changes*, its centre of buoyancy shifts a little, say, into the position B' ; so that, the vertical line drawn through the *new* centre of buoyancy B', meets the old centre line in M. This point M is called the **metacentre** of the body and the distance MG', where G' is its centre of gravity, is called its **metacentric height**.

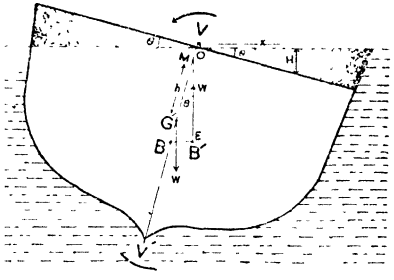
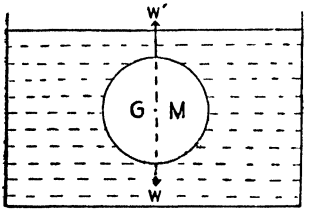


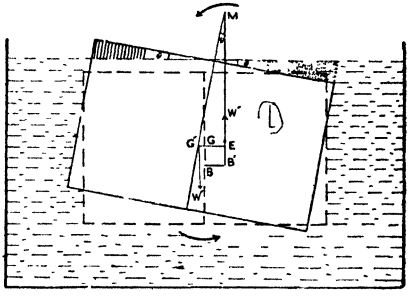
Fig. 221.

Now, whether the *meta-centre* (M) coincides with, lies above, or below, the *shifted position of the* centre of gravity (G') of the body, depends upon the shape of the body and determines whether the body will be in *neutral*, *stable* or *unstable* equilibrium.

(*i*) Thus, in the case of a *sphere*, [Fig. 222 (*i*)], a tilt this way or that brings about no change in the shape of the displaced liquid,



(*i*)



(*ii*)

Fig 222.

the metacentre coinciding with the *c.g.* of the body all the time.

*It is for this reason that the weight of a ship or boat is often referred to as its *displacement*,—the weight of the water displaced by it being equal to its own weight

†As happens in the case of a spherical body.

It is, therefore, in *neutral equilibrium* and continues to float in all positions.

(*ii*) In the case of a *rectangular body*, floating in a liquid, as shown in Fig. 222 (*ii*), M lies above G', the position of the *c.g.* of the body in its tilted position, and a couple is formed by W and W', the two *equal*, *opposite*, *parallel* and *non-collinear* forces acting at G' and B', which represent the shifted positions of the *c.g.* of the body and the centre of buoyancy of the displaced liquid respectively. This couple tends to rotate the body back into its original position, thus making its equilibrium a *stable* one.

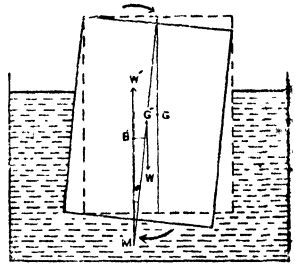


Fig. 222. (*iii*)

Obviously, the moment of this *restoring couple* $= W.G'M \sin \theta$. It has been appropriately called the **'righting moment'**, (particularly in the case of ships and other floating vessels), because it tends to bring the body, or to *'right it'* into its original position.

(*iii*) In the case of a rectangular body, floating in the manner, shown in Fig. 222 (*iii*), M lies below G', and the body is, therefore, in unstable equilibrium. For, the couple formed by W and W' tends here to rotate the body further in the *same* direction in which it has been tilted already. There is thus no prospect of its coming back (or being righted) into its original position.

Let us now discuss the problem in a little more detail, with particular reference to a floating vessel or ship.

Identical consideration to the above applies also to a floating ship ; so that, when the ship is *'on an even keel'*, its centre of gravity G and the centre of buoyancy B, of the displaced water, lie in the same vertical line, *i.e.*, its plane of symmetry (VV') is vertical (Fig. 220).

If, however, the ship *rolls* or gets tilted through an angle θ, (Fig. 221), its plane of symmetry (VV') is no longer vertical, and, although this rolling or tilting does not alter its *c.g.* with respect to the ship, the centre of buoyancy shifts to B', giving rise to the *righting moment* $W.G'M \sin \theta$ or $W.h \sin \theta$, where h denotes the metacentric height $G'M$. If θ be small, so that $\sin \theta = \theta$, this righting moment is equal to $W.h\,\theta$.

It will thus be clear that *the greater the value of h, the metacentric height of a ship (or a floating body, in general), the greater the stability of its equilibrium.* It is for this reason that heavy cargo is stowed as low as possible in a ship or that it is provided with a *leaden keel*, to lower its *c.g.* or to increase h*.

N.B The lowering of the *c.g.* is, however, not quite so desirable beyond a certain point. For, due to the waves in the sea, the ship is subject to lateral forces in different directions and the moment of these forces can be quite considerable if the *c.g.* of the ship is very low down, resulting in its being tossed about

*The stability may also be increased by making the ship quite broad at the water line, *i.e.*, the line along which it is *just* touched by the water-surface.

this way and that, which is obviously most unpleasant and annoying to those on board. Judicious care must, therefore, be taken to lower the *c.g.* of the ship within reasonable limits.

157. Rolling and Pitching of a Ship—The righting moment $W\,h.\theta$, acting on the ship when it is tilted through a small angle θ results in its oscillation (*or rolling*), as we have seen above; and if T be its natural period of oscillation, we have

$$T = 2\pi\sqrt{I/M}$$ [See pages 300-301.

where I is the *moment of inertia* of the ship and M, the *turning moment* on it per unit (radian) deflection.

Obviously, the turning moment per unit deflection is also equal to $W.h$; for, if $\theta = 1$, the value of the righting or turning moment becomes $W.h$. So that, substituting $W.h$ for M, we have

$$T = 2\pi\sqrt{I/Wh}.$$

It is thus clear that *the period of rolling* (T) *of a ship is inversely proportional to the square root of its metacentric height* (h). A ship, with a small metacentric height, is, therefore, less liable to rolling. It is for this reason that large ocean liners are designed to have a comparatively small metacentric height of just a few *metres, for small displacements*, which obviously makes them much steadier. At the same time, however, to avoid the danger of the ship turning over or capsizing, if the deflections be large, the designing is such that the metacentric height increases for large deflections*.

Similarly, to avoid '*pitching*', or tilting of a ship in the direction of its length, its metacentric height in this direction also is suitably adjusted.

158. Determination of Metacentric Height. The displacement of the ship through an angle θ causes a wedge-shaped portion of the ship, (shown shaded in Fig. 221), to be immersed on the right hand side and an equal wedge-shaped portion of it to rise out of the water, on the left hand side. Let these wedge-shaped portions be divided into a number of elementary vertical prisms, by planes perpendicular to the water surface, on either side, and consider one such prism, of height H, at a distance x from O, where the plane of symmetry meets the water-surface.

Then, clearly, $H = x \tan\theta = x.\theta$. [Since θ is supposed small.

If the *base area* of the prism be dA, we have

volume of the prism $= x.\theta.dA$, an its *mass* $= x.\theta.dA.\rho$,

where ρ is the *density* of water.

Clearly, therefore, *weight of the prism*, or *the weight of the water displaced, i.e.*,

the upthrust on the prism $= x.\theta.dA.\rho.g.$,

and *its moment about* $O = x.\theta\ dA.\rho.g.x = \rho.g.\theta.x^2.dA.$

Similarly, considering the equal wedge-shaped portion on the left-hand side, we find that there is a loss of upthrust due to its rising out of the water, whose moment about O is, obviously, also equal to $\rho.g.\theta.x^2.dA$, the direction being *anticlockwise* in either case. Hence, the moment of the couple acting on the ship (or a floating body, in general, due to its displacement θ, is given by

$$\int \rho.g.\theta.x^2.dA = \rho.g.\theta.\int x^2.dA = \rho\, g.\theta A k^2, \quad \ldots \quad \ldots (i)$$

*The rolling motion of a ship can be greately minimised by the use of a Gyrocompass (See page 98).

where $\int x^2.dA = I_g = Ak^2$, the *geometrical moment of inertia* of the surface-plane of the liquid about the axis through O, $-k$ being its *radius of gyration* about this axis.

This displacement (θ) of the ship being small, the volume V of water displaced remains unaffected by it, and the upthrust $\rho\, g.V$, due to this displaced water, acts through its new centre of buoyancy after the displacement θ. The floating body or ship is thus acted upon by a couple equal to

$$\rho.g.V.BM \sin\theta = \rho.g.V.BM.\theta. \qquad ...(ii) \quad [\theta \text{ being small.}$$

Equating the two values of the couple, we have

$$\rho.g.V.BM.\theta = \rho.g.\theta.Ak^2.$$

Or, $$V.BM = Ak^2. \text{ Or, } BM = \frac{Ak^2}{V}.$$

So that, the *metacentric height* h of the ship or the vessel is given by $G'M = BM - BG' = \frac{Ak^2}{V} - BG'$ and may thus be easily determined.

Alternatively, in the case of a ship, its metacentric height may be easily determined by moving a known weight w from one point A to another point B across the deck, say through a distance d, *i.e.*, $AB = d$.

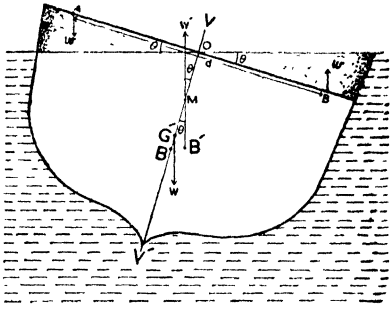


Fig. 224.

Now, this shift of weight w from A to B is equivalent to an *upward* force w at B and a *downward* force w at A, (Fig. 224), thus constituting a *couple*, of moment $w.d\cos\theta$. For equilibrium, therefore, this must be equal to the couple due to the weight W of the ship and an equivalent upward thrust at the new centre of buoyancy B', *i.e.*, equal to couple of moment $W.G'M\sin\theta$. [See above]

Or, $$W.G'M\sin\theta = w.d\cos\theta.$$

Or, $$G'M = \frac{w.d}{W}\cdot\frac{\cos\theta}{\sin\theta} = \frac{w.d}{W}\cdot\cot\theta = \frac{w.d}{W}\cdot\frac{1}{\tan\theta}.$$

Or, $$G'M = \frac{w.d}{W.\theta}. \qquad \left[\begin{array}{l}\theta \text{ being small,}\\ \tan\theta = \theta.\end{array}\right.$$

Thus knowing w, W, d and θ, we can easily calculate the metacentric height of the ship.

159. Pressure due to a Compressible Fluid or a Gas. A gas differs from a liquid in that, unlike the letter, it is highly compressible* and, therefore, also highly expansible, tending to expand perpetually and indefinitely.

*An idea of the high compressibility of a gas, compared with that of a liquid, can be had from the fact that whereas the density of sea-water at a depth of 5 miles is about the same as that of the surface layer, the density of the atmosphere at the same height above sea-level is reduced to one quarter of that at the latter.

The pressure exerted by a gas, is thus fundamentally different in the nature of its cause from that of a liquid and cannot be taken to be proportional to the height of the gaseous column, for the simple reason that, except in the case of a small volume of a gas, the density goes on progressively increasing as we go further down the column, due to the layers above pressing down upon the layers below, thus giving rise to a well defined *pressure gradient* all along the column.

So that, whereas a liquid exerts pressure only under the action of gravity, *i.e.*, due to its weight, or due to an external force applied to it, as, for example, when it is pressed down by a piston, *the pressure due to a gas is entirely a consequence of the incessant mobility and the kinetic energy of its molecules, or due to what* **Boyle** *called the 'spring of the gas.*

The mad and random motion of the gaseous molecules results in their colliding not only against each other but also against the walls of the containing vessel, and it is this bombardment of the walls by the fast and haphazardly moving molecules that causes the pressure. (See Chapter XV).

We are here concerned mainly with the pressure exerted by the gaseous mantle or envelope, surrounding us over land and sea alike and in all latitudes, which we call *'air'* or *'atmosphere'* and which, as we know, is a mixture of a number of gases and vapours. In pursuance of its inherent property of indefinite expansion, this air or atmosphere should expand to an infinite distance above the earth, but the earth's gravitational attraction on a huge mass like it sets a limit to its expansion. Even so, it has been known to exist up to a height of 300 *miles* from the surface of the earth, although even at 25 *miles* or so, its density and pressure start falling off so rapidly that at altitudes above 300 miles, it may be said to be as good as non-existent, with just a void or a vacuum beyond.

Now the atmosphere can be divided into two very distinct regions, *viz.*, (*i*) a lower region, called the *troposphere* or the *convective zone*, and (*ii*) an upper region, called the *stratosphere* or the *advective zone*, the surface of separation of the two being known as the **tropopause,** which varies with the latitude and falls from a height of about 14 *kilometres* at the equator to about 8 or 10 *kilometres* at the poles, and is found to be higher in summer than in winter.

(*i*) **The Troposphere** This extends to a height of about 6 *miles* at the poles and about 10 *miles* at the equator, *with a vertical distribution of temperature as its chief characteristic*, the temperature falling off rapidly with altitude, there being a vertical temperature gradient or a *lapse rate** of 1°*C* per 500 *feet* rise in altitude.

This *temperature gradient* is probably due to a variety of causes. *Lord Kelvin* attributes it to the atmosphere being in a state of convective equilibrium, which is brought about, on the one hand, by the earth getting heated by the *solar radiation* falling on it†, and then warming up the layers above it, by direct

*Usually, a vertical *temperature gradient* is taken to be the fall in temperature per 100 *metres* rise in altitude and the *lapse rate*, as the fall in temperature per one *kilometre* rise in altitude.

†Little or no heat is absorbed by the air during the passage of the solar radiation to the earth through it, and whatever little is, is distributed over too large a mass to be able to produce any appreciable rise in its temperature,—this absorption being the *same* at all altitudes.

contact and by emitting out radiations which are absorbed by them, and, on the other, by the lower atmosphere getting cooled by radiation due to its emitting out more energy than it absorbs at the ordinary temperature. The two processes, going on side by side, produce changes in the density of the air, conducive to the setting up of vertical convection currents, the lower warmer air rising up and getting cooled by adiabatic expansion and the upper colder air coming down and getting heated up by adiabatic compression. A vertical temperature gradient is thus established and maintained throughout this region of vertical convection. Hence the name, '*convective zone*,' also given to it.

This seems to be amply borne out by the fact that the lapse rate for dry air, calculated on this assumption, comes out to be 3°*C per* 1000 *ft.*, which, though appreciably higher than the observed value, is quite understandable, considering that the air is really never '*dry*' and the moisture present in it inevitably tends to lower the lapse rate.

(*ii*) **The Stratosphere.** Also known as the *adiactive zone*, it is the region above the troposphere, where the vertical convection, referred to above, becomes much too feeble, with the temperature falling to such an extent that the heat radiated out is equal to the heat absorbed from radiations from the earth and the solar radiation passing through it, there being set up a *radiative equilibrium* in the region, the temperature remaining constant at about 55°*C*,—hence the name, '*isothermal layer*', also given to it.

It will thus be readily seen that the stratosphere is a direct consequence of, and is characterised by, the cessation of vertical convection and the setting up of a radiative equilibrium, with the temperature constant at 55°*C* up to a height of 300 *miles* or so, after which it probably shoots up to 700°*C* or thereabouts.

160. Measurement of Atmospheric Pressure. The instruments used to measure the atmospheric pressure are known as **barometers,** one of the best forms of which is the cistern-type **Fortin's barometer.** Another handy and portable type of barometer is the **Aneroid barometer,**—(from '*a*'—without, and '*neros*'—liquid)—as no mercury or any other liquid is used in it. We have already studied these in good detail in the junior classes and shall not, therefore, repeat them here. Instead, we shall pass on to a consideration of the corrections that must be applied to the readings obtained from them.

161. Correction of Barometric Reading. Although the *Fortin's barometer* is quite an efficient instrument, a few corrections have to be applied to its readings for greater accuracy. We shall consider here only two important ones of them, *viz.*,

(*i*) *correction for the expansion of the brass scale*, on which the reading is taken, and which is usually calibrated at 0°*C* ;

(*ii*) *correction for expansion of mercury*, and consequent lowering of its density.

(*i*) Let the temperature, at which the reading is taken, be t°*C*, and let H_t be the observed reading at this temperature. Then, H_t is in fact just the value of the divisions of the scale, *correct only at* 0°*C*. If, therefore, α be the coefficient of linear expansion of brass, the correct length at t°*C*, is given by $H = H_t(1+\alpha t)$ *cms.*

(*ii*) Again if v_0 and ρ_0 be the volume and density of a certain mass m of mercury at 0°*C*, and v_t and ρ_t, its volume and density respectively, at t°*C*, we have $m = v_0.\rho_0 = v_t\rho_t$.

Or, $\dfrac{v_t}{v_0} = \dfrac{\rho_0}{\rho_t}$. Or, $\dfrac{v_0(1+\gamma t)}{v_0} = \dfrac{\rho_0}{\rho_t}$ [Where γ is the cofficient of cubical expanssion of mercury

Or, $\rho_0/\rho_t = 1+\gamma t$, whence, $\rho_0 = \rho_t(1+\gamma t)$.

Now, clearly, $H_0.\rho_0.g = H.\rho_t.g$,
where H_0 is the true barometric height at $0°C$.
Or, $H_0.\rho_0 = H.\rho_t$. Or, $H_0.\rho_t(1+\gamma t) = H.\rho_t = H_t(1+\alpha t).\rho_t$,
whence, $H_0 = \dfrac{H_t(1+\alpha t).\rho_t}{\rho_t(1+\gamma t)} = H_t(1+\alpha t)/(1+\gamma t) = H_t[1+(\alpha-\gamma)t]$,
neglecting squares and higher powers of α and γ.
Or, $H_0 = H_t[1-(\gamma-\alpha)t]$.

For *mercury*, $\gamma = \cdot00018$ and for *brass*, $\alpha = \cdot000019$.

$\therefore$ $H_0 = H_t[1-(\cdot00018-\cdot000019)t]$. Or, $H_t = H_t(1-\cdot000161t)$,
whence, the barometric height at $0°C$ can be easily calculated.

Other errors, due to pressure of mercury vapour and capillarity etc., are much too small for tubes of reasonably wide bores, and are, therefore, usually neglected.

162. Change of Pressure with Altitude. Consider two points A and B, distance dx apart, vertically below each other, in air, (Fig. 225). If A be at a height x above the ground, the height of B from the ground is obviously $(x+dx)$. Since the density of air and, therefore, its pressure, decreases with altitude, for a pressure p at A, that at B will be $p-dp$, say. If, therefore, ρ be the density of air between A and B, and g, the acceleration due to gravity, we have $-dp = \rho.g.dx$, ...(*i*)
the $-$ve sign being used, because the pressure decreases with height.

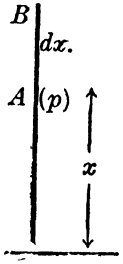


Fig. 225.

If the temperature of the air be constant, $\rho \propto p$.
Or, $\rho = K.p$, $\left[\because 1/V \propto p$, (Boyle's law) and $1/V \propto \rho.\right]$
where K is a constant, equal to ρ/p.
Substituting this value of ρ in relation (*i*) above, we have

$$-dp = K.p.g.dx. \quad \text{Or,} \quad -\frac{dp}{p} = K.g.dx.$$

Or, $dp/p + K.g.dx = 0$.
Integrating this, we have

$$\log_e p + K.g.x = \text{a constant } C. \qquad ...(ii)$$

Now, if the pressure at heights h and H be p and P respectively, we have, from (*ii*) above,

$\log_e p + K.g.h. = C$, ...(*iii*) and $\log_e P + K.g.H = C$ ...(*iv*)
$\therefore$ subtracting (*iv*) from (*iii*), we have

$$\log_e p - \log_e P = K.g.H - K.g.h = K.g.(H-h).$$

Or, $$\log_e\left(\frac{p}{P}\right) = K.g.(H-h). \qquad ...(v)$$

Thus, $$(H-h) = \frac{\log_e\left(\frac{p}{P}\right)}{K.g}. \qquad ...(vi)$$

Or, substituting the value of K, *i.e.*, ρ/p, in (*vi*), we have

$$(H-h) = \frac{\log_e\left(\frac{p}{P}\right).P}{\rho.g} = \frac{(\log_e p - \log_e P)p}{\rho.g}.$$

Thus, knowing p, P, ρ and g, the altitude $(H-h)$ can be easily determined.

In the above treatment, it has been assumed that the temperature of the air, or the atmosphere, remains the *same* throughout. This, as we know, is far from being the case. Nevertheless, the result is accurate enough for the determination of small heights.

If now we have a number of heights, h_1, h_2, h_3 etc., in *arithmetical progression* then $(h_2-h_1) = (h_3-h_2)$ and so on.

And, if p_1, p_2, p_3 etc., be the pressures at these heights, we have from (*v*) above,

$$\log_e (p_1/p_2) = K.g.(h_2-h_1), \text{ and } \log_e (p_2/p_3) = K.g.(h_3-h_2).$$

Since $(h_2-h_1) = (h_3-h_2)$, we have

$$\log_e (p_1/p_2) = \log_e (p_2/p_3).$$

Or, $$p_1/p_2 = p_2/p_3.$$

i.e., p_1, p_2, p_3 etc. are in *geometrical progression.*

Thus, we see that *as the height or altitude increases in arithmetical progression, the pressure decreases in geometrical progression.*

Note. To convert logarithms to the base *e* into common logarithms, (*i.e.*, to the base 10), multiply the former by 2·302.

SOLVED EXAMPLES

1. A rectangular cistern 2 ft. long and 5 ft. wide is filled with water to a depth of 3 ft. Taking one cu. ft. of water to weigh 62·5 lbs., find the magnitude and position of the resultant fluid thrust on each side.

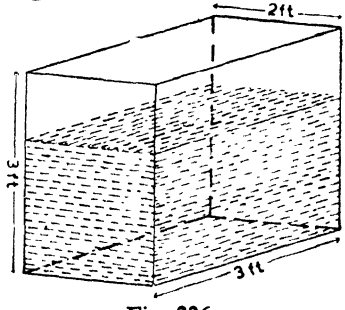


Fig. 226.

(*a*) Here, clearly, (Fig 226),

depth of water = 3 *ft.*

∴ centre of area for each side of the cistern
$= 3/2 = 1{\cdot}5$ *ft.*

∴ *pressure at centre of area* = *h.ρ.g.*

$= 1{\cdot}5 \times 62{\cdot}5 \times 32$ *poundals.*

$= 1{\cdot}5 \times 62{\cdot}5$ *lb. wt.*

Now, area of each longer side in contact with water $= 3 \times 3 = 9$ *sq. ft.*

and area of each smaller side in contact with water $= 2 \times 3 = 6$ *sq. ft.*

Since *thrust = pressure at centroid or centre of area × area,*

we have *pressure on each longer side* $= 1{\cdot}5 \times 62{\cdot}5 \times 9 = 843{\cdot}7$ *lbs. wt.*

and, *pressure on each smaller side* $= 1{\cdot}5 \times 62{\cdot}5 \times 6 = 562{\cdot}6$ *lbs. wt.*

And, *centre of pressure* $= \frac{2}{3} \times$ *depth* $= \frac{2}{3} \times 3 = 2$ *ft.*

2. Find the position of the centre of pressure of a triangular plate immersed in a liquid with its plane vertical and one side in the surface.

ABC is a vertical triangular door in the side of a ship, AB is horizontal, C below AB, and the triangle equilateral of side 5 ft. The door is hinged along AB, and kept shut against the pressure of the water by a fastening at C. If the

water rises to the level of AB, find the force on the fastening. (One cu. ft. of water weighs 62·5 lbs.) (*Liv. Inter.*)

The *centre of pressure* of the triangular plate, with one of its sides in the plane of the liquid surface will be at a depth $h/2$, from the liquid surface, where h is the height of the plate, (see page 351).

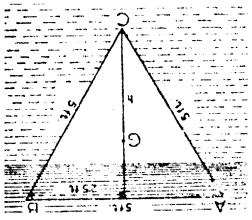


Fig. 227.

Let ABC be the triangular door hinged along AB and having a fastening at C, where AB lies in the plane of the water surface, (Fig. 227).

Obviously, height h of the triangular door $= \sqrt{5^2-2\cdot5^2} = \sqrt{25-6\cdot25} = \sqrt{18\cdot75} = 4\cdot329$ *ft.*

Since the centre of a triangular lamina is $\frac{2}{3}rds$ of its height below the vertex, its depth below AB, or the water surface, is $\frac{1}{3}rd$ of its height, *i.e.*, equal to $\frac{1}{3}\times 4\cdot329 = 1\cdot443$ *ft.*

Therefore, pressure at the centroid $= h.\rho$ *lbs wt.* $= 1\cdot443\times 62\cdot5$ *lbs. wt.*

And $\therefore$ *thrust on the door* $= 1\cdot443\times 62\cdot5$ *lbs.* $\times$ *area of the door.*

$= 1\cdot443\times 62\cdot5\times \frac{1}{2}$ *base* $\times$ *altitude* $= 1\cdot443\times 62\cdot5\times\frac{1}{2}\times 5\times 4\cdot329$ *lbs. wt.*

$= 1\cdot443\times 62\cdot5\times 2\cdot5\times 4\cdot329$ *lbs. wt.*

Now, centre of pressure of the triangular door lies at $\frac{1}{2}\,h$, *i.e.*, at

$\frac{1}{2}\times 4\cdot329 = 2\cdot164$ *ft.* [See above.

$\therefore$ moment of the thrust about AB

= thrust $\times$ depth of centre of pressure.

$= 1\cdot443\times 62\cdot5\times 2\cdot5\times 4\cdot329\times 2\cdot164$ *lbs wt.*

And, if F be the magnitude of the force on the hinge, its moment about $AB = F\times h = F\times 4\cdot329$ *lbs. wt.*

Clearly, therefore, $F\times 4\cdot329 = 1\cdot443\times 62\cdot5\times 2\cdot5\times 4\cdot329\times 2\cdot164$.

Or, $F = 1\cdot443\times 62\cdot5\times 2\cdot5\times 2\cdot164 = 487\cdot5$ *lbs. wt.*

Thus, the force on the fastening at $C = 487\cdot5$ *lbs wt.*

3. Find the centre of pressure of a rectangular sheet 'a' in. long and 'b' in. wide, of uniform thickness, immersed in a liquid of uniform density, with one side of length 'b' in. in the surface, the plane of the rectangle being inclined at an angle θ to the vertical.

If the rectangular sheet remains in the same position with respect to the vessel containing the liquid, and the depth of the liquid be increased by h in., find the new position of the centre of pressure. (*London Higher School Certificate*)

Let $ABCD$ be the rectangle, immersed in the liquid, of density ρ, with its side $AB = b$ in the liquid surface EF, and its plane inclined to the vertical at an angle θ, (Fig. 228)

Then, since the *vertical depth* of the rectangle is clearly $BK = BC\cos\theta = a\cos\theta$, its centroid lies at a vertical depth $\frac{a}{2}\cos\theta$, from the liquid surface.

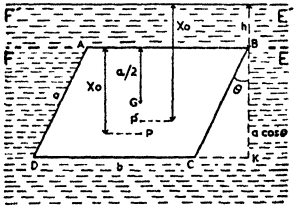


Fig. 228.

And, hence, proceeding as in § 152 [case (*i*), page 349], we have

depth X_0, of the centre of pressure P from the surface of the liquid, clearly given by

$X_0 = \frac{2}{3}\times$ *vertical depth* $= \frac{2}{3}\,a\cos\theta$.

Now, let a column of liquid $EE'F'F$, h *in.* thick, be added on to the top of the liquid surface to increase its depth by h *in.*, and let P' be the *new centre of pressure* of the rectangle, whose position is otherwise unchanged with respect to the vessel, at a distance X_0' from the new surface $E'F'$.

Then, clearly, *thrust on the rectangle due to the original liquid column*
$= $*area of the rectangle* $\times$ *depth of centroid* $G \times$ *density of the liquid* $\times g$

$$= ab.\frac{a}{2}\,.\,\cos\theta \times \rho \times g = \frac{1}{2}\,a^2 b \cos\theta \times \rho \times g\ ;$$

and *increase in thrust* on the rectangle due to the new layer h of the liquid added
$= $*area of rectangle* $\times$ *depth of new liquid column added* $\times$ *density of liquid* $\times g$.

$$= ab \times h \times \rho \times g.$$

So that, the *total thrust on the rectangle* $= \frac{1}{2}a^2 b \cos\theta \times \rho \times g + ab \times h \times \rho \times g.$

$$= ab\rho g(\tfrac{1}{2}a\cos\theta + h).$$

Clearly, therefore, the moment of this thrust about the new liquid surface $E'F'$

$$= ab\rho g(\tfrac{1}{2}a\cos\theta + h).X_0' \qquad \ldots \quad \ldots \quad \ldots \quad (i)$$

Again, the distances of the new centre of pressure P' and the centroid of the rectangle from the new liquid surface $E'F' = (X_0 + h) = (\frac{2}{3}\,a\cos\theta + h)$ and $(\frac{1}{2}\,a\cos\theta + h)$ respectively.

And, therefore, the moment about the new liquid surface $E'F'$ is also equal to

$$\tfrac{1}{2}a^2 b.\cos\theta.\rho.g(\tfrac{2}{3}\,a\cos\theta + h) + ab.h\,\rho.g.\,(\tfrac{1}{2}\,a\cos\theta + h),$$

$$= ab\rho g[\tfrac{1}{2}\,a\cos\theta\,(\tfrac{2}{3}\,a\cos\theta + h) + h(\tfrac{1}{2}\,a\cos\theta + h)]. \qquad \ldots (ii)$$

Equating (i) and (ii), therefore, we have

$$a.b.\rho.g\,(\tfrac{1}{2}\cos\theta + h)\,X_0' = a.b.\rho.g[\tfrac{1}{2}\,a\cos\theta(\tfrac{2}{3}\,a\cos\theta + h) + h(\tfrac{1}{2}\,a\cos\theta + h)]$$

Or, $$\left(\frac{a\cos\theta + 2h}{2}\right).X_0' = \tfrac{1}{3}a^2\cos^2\theta + \tfrac{1}{2}ah\cos\theta + \tfrac{1}{2}ah\cos\theta + h^2.$$

Or, X_0', or the depth of the new centre of pressure of the rectangle from the new liquid surface

$$= \frac{h^2 + a\,h\cos\theta + \frac{1}{3}a^2\cos^2\theta}{\left(\frac{a\cos\theta + 2h}{2}\right)} = \frac{2(h^2 + ah\cos\theta + \frac{1}{3}a^2\cos^2\theta)}{2h + a\cos\theta}\ \textit{inches.}$$

4. Neglecting atmospheric pressure, find the depth of the centre of pressure of a circular lamina just completely immersed with its plane vertical in an incompressible liquid.

A circular door in the vertical side of a tank is 'hinged' at the top and opens inwards, and the tank contains water to a height just sufficient to cover the door. If the diameter of the door is 2 ft , find the magnitude of the force that must be applied normally to the centre of the door in order just to open the door. Find also the reaction at the hinge when this force is being applied.

(*Cambridge Higher School Certificate*)

The *centre of pressure* of the vertical circular lamina, *just* immersed in an incompressible liquid, is equal to $5r/4$, as explained in § 152 (ii), (page 350).

Here, obviously, the centroid of the circular lamina is at its centre, at a depth equal to its radius from the water surface, *i e.*, at a depth 2 *ft.*/2 or 1 *ft.* below it, (the lamina being *just immersed* in water).

$\therefore$ *pressure at the lamina* $= h\,\rho.g = 1 \times 62\,5 \times g$ *poundals* $= 62{\cdot}5$ *lbs. wt.*

and *area of the circular lamina* $= \pi r^2 = \pi \times 1 = \pi$ *sq. ft.*

So that, *thrust on the lamina* $=$ *pressure on the centroid* $\times$ *area of the lamina*

$= 62{\cdot}5 \times \pi$ *lbs. wt.* $= 196{\cdot}3$ *lbs. wt.*

This thrust acts at the centre of pressure of the lamina, whose depth from the water surface, as we know,

$= 5r/4 = 5 \times 1/4 = 5/4$ *ft.*

$\therefore$ moment of this thrust about the liquid surface, or about the hinge

$= 62{\cdot}5 \times \pi \times 5/4 = 245{\cdot}5$ *lbs. wt.*

If F *lb. wt.* be the force applied to the centre of the door, *just to open it*, we have

moment of F above the hinge $= F \times r = F \times 1$ *lb. ft.*

And $\therefore$ $F \times 1 = 245{\cdot}5$. Or, $F = 245{\cdot}5$ *lbs. wt.*

The force required to be applied to the centre of the door to *just* open it is thus equal to 245·5 *lbs. wt.*

Thus, the forces acting on the two sides of the door are respectively 245·5 *lbs. wt.* and 196·3 *lbs. wt.* ; and, therefore, reaction at the hinge

$$= 245\ 5 - 196{\cdot}3 = 49{\cdot}2\ \textit{lbs. wt.}$$

5. If a load of 20 tons, moved 50 ft. across the deck of a ship of 20,000 tons displacement, causes the ship to tilt through ½°, what is its metacentric height ?

Let a load $w = 20$ *tons* be moved across the deck of a ship from A to B through a distance of 50 *ft.* (Fig. 229), and let the ship be tilted through an angle ½° into the position shown. This, as we know, is equivalent to the application of a *couple*, of moment 20×50 *ton-feet*, tending to turn the ship clockwise, [see § 158 (alternative treatment), page 357].

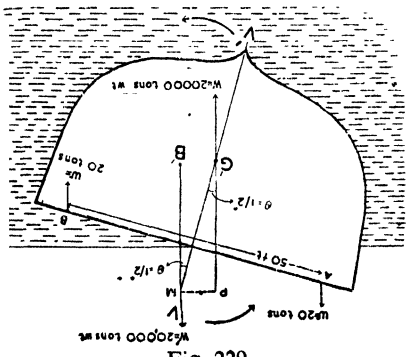


Fig. 229.

And, the couple, formed by the weight of the ship acting vertically at its *c g.*, G' and the equal weight of displaced water at its *centre of buoyancy* B', tends to restore the ship back into its original position, the moment of the couple being $W \times MP$, where MP is the perpendicular from the *metacentre* M on to the vertical line through G'.

Since the ship is *in equilibrium* under the action of these two opposite couples, they must obviously be balancing each other. So that,

$$W \times MP = 20 \times 50. \quad \text{Or,} \quad 20{,}000 \times MP = 20 \times 50.$$

Or, $$MP = 20 \times 50/20{,}000 = 1/20\ ft. = {\cdot}05\ ft.$$

Now, $$MP = MG' \sin ½° = MG' \times {\cdot}0087,$$

whence, $$MG' = \frac{MP}{{\cdot}0087} = \frac{{\cdot}05}{{\cdot}0887} = 5{\cdot}748\ ft.$$

Thus, the *metacentric height* of the ship = 5·748 *ft.*

6. State the theorem of Archimedes, and explain what you understand by the terms "force of buoyancy", "centre of buoyancy". A cylinder of radius 1 cm. and length 4 cms., made of material of specific gravity 0·75 is floated in water with its axis vertical. It is then pushed vertically downwards so as to be just immersed. Find (*a*) the work done, (*b*) the reduction in the force on the bottom of the containing vessel when the cylinder is subsequently taken out of the water.

(Oxford and Cambridge Higher School Certificate)

Here, the volume of the cylinder is clearly equal to $\pi r^2 l = \pi .\ 1.\ 4$ *c.cs.*

And ∴ its *weight* $= \pi \times 4 \times 0{\cdot}75 = 3\pi$ *gms. wt.*

Since it floats vertically in the water, we have

weight of displaced water also $= 3\ \pi$ *gms. wt.*

Let length of the cylinder inside water be $= x$ *cms.*

Then, the volume of the immersed part of the cylinder $= \pi .1.x = \pi x$ *c.cs.*

i.e., volume of water displaced by the cylinder $= \pi x$ *c.cs.*

And ∴ the weight of this displaced volume of water $= \pi .x .1$ *gms wt.*

Hence, $\pi x = 3\pi$. Or, $x = 3$ *cms.*

i.e., length of the cylinder inside water = 3 *cms.*

And, therefore, length of the cylinder outside water $= 4 - 3 = 1$ *cm.*

Thus, to immerse the whole of the cylinder just inside the water, we have to simply push it down through 1 *cm.*

Obviously, the volume of the displaced water or upthrust on the cylinder will be $\pi \times 1 \times 4 \times 1 = 4\pi$ *gms. wt.*

Hence, increase in the upthrust on the cylinder will be $= 4\pi - 3\pi = \pi$ *gms. wt.*

Since this increase in the upthrust takes place gradually from 0 to π, as the cylinder is pushed steadily down into the water from its initial position, we may take the average value of the upthrust against which we work in pushing the cylinder down through 1 *cm.* to be $(0+\pi)/2$ or $\pi/2$ *gms. wt.*

And, therefore, work done in pushing the cylinder down through 1 *cm*, against this average increase in the upward thrust, will clearly be equal to average thrust × the distance through which the cylinder is pushed down.

i.e, work done $= (\pi/2) \times 1 = \pi/2 = 1{\cdot}571$ *gms. wt. cm.*

Now, with the whole of the cylinder immersed in the water, the weight of water displaced

$$= 4\pi \text{ gm. wt.} = 12{\cdot}57 \text{ gm. wt.}$$

This must also, therefore, be the downward thrust on the bottom of the containing vessel. So that, when the cylinder is removed out of the water, the reduction in the thrust on the bottom of the containing vessel will also be the same, *viz.*, 12·57 *gm. wt.*

7. Calculate the metacentric height and determine the necessary condition for the stable equilibrium of a cylinder of length l, radius r, and density ρ, floating vertically in water.

Let a portion x of the cylinder be *inside* water.

Then, volume of water displaced by the cylinder, *i.e.*, $v = \pi r^2.x$ *c.cs.*

And $\therefore$ weight of water displaced or upthrust on the cylinder $\pi r^2.x.1 = \pi r^2 x$.

This must, for equilibrium, be equal to the weight of the cylinder, *i.e.*, equal to $\pi.r^2.l.\rho$ *gms. wt.*

Or, $\pi r^2.x = \pi r^2.l.\rho$ *gms. wt.* Therefore, $x = l\rho$.

The centre of buoyancy of the displaced liquid must, therefore, be at a height $x/2 = l\rho/2$ from the bottom of the cylinder.

Now, as we know, the distance between the centre of buoyancy of the displaced liquid and the metacentre is Ak^2/v, (see page 357), where Ak^2 is the moment of inertia of the surface-plane of the cylinder about its diameter. So that, $k^2 = r^2/4$,

k being the *radius of gyration* of the plane about the surface-line or the diameter of the cylinder.

Substituting the value of $v = \pi r^2 x$, therefore, we have

distance between the centres of buoyancy and metacentre

$$= \frac{A\,r^2/4}{\pi r^2 x} = \frac{\pi r^2.r^2/4}{\pi r^2 l\rho} = \frac{r^2}{4l\rho}. \qquad \left[\because A=\pi r^2 \text{ and } x=l\rho.\right.$$

Now, distance of the *c.g.* of the cylinder from the centre of buoyancy

$$= \frac{l}{2} - \frac{x}{2} = \frac{l}{2} - \frac{l\rho}{2} = \frac{l(1-\rho)}{2}.$$

And, therefore, distance between the *c.g.* of the body and the metacentre or the *metacentric height*, h, of the cylinder = distance between the centre of buoyancy and the metacentre *minus* distance between the *c.g.* of the body and the centre of buoyancy.

Or, $$h = \frac{r^2}{4l\rho} - \frac{l(1-\rho)}{2} = \frac{r^2 - 2l\rho l(1-\rho)}{4l\rho} = \frac{r^2 - 2l^2\rho(1-\rho)}{4l\rho}.$$

Now, for stable equilibrium of the cylinder, the metacentre should be above the *c.g.* of the body, *i.e.*, h should have a *positive* value.

And, obviously, this is possible only when $r^2 > 2l^2\rho(1-\rho)$. This is, therefore, the necessary condition required.

EXERCISE IX

1. Define pressure at a point in a fluid. Find the total thrust on the sides and vertical ends of a V-shaped trough, 1 *ft.* deep, 2 *ft.* wide at the top and 4 *ft.* long, when filled with water, density 62·5 *lb./cu. ft.*

(*Oxford and Cambridge Higher School Certificate*)

Ans 177 *lbs. wt.*; 20·8 *lbs wt.*

2. Define clearly the term '*Centre of pressure*'. Determine its position for (*i*) *a circular lamina of radius r*, just immersed vertically, and (*ii*) a *triangular*

lamina of height h, immersed vertically with (*a*) its apex, and (*b*) its base in the surface. Ans. (*i*) $5h/4$; (*ii*) (*a*) $3h/4$; (*b*) $h/2$.

3. A square lamina with its sides 3 *ft.* long is just immersed vertically n water with an edge in the surface and is then lowered 10 *ft.* Find the distance of the centre of pressure in the new position from the centre of the square Neglect the pressure of the atmosphere in each case).

(*Joint Matriculation Board and H.S. Certificate*)
Ans. 0·0303 *ft.*

4. A circular area of radius a *ft* is immersed in water, with its plane vertical. The surface of water rises from $2a$ *ft* above the centre of the circle to $\frac{1}{2}a$ *ft.* above it. Neglecting atmospheric pressure, prove that the centre of pressure rises through a distance $a/16$ *ft* (*London Higher School Certificate*)

5 State the *Principle of Archimedes* and define clearly the terms (*i*) *Centre of buoyancy*, (*ii*) *Metacentre*, and (*iii*) *Metacentric Height*. Discuss in detail the conditions for the stable equilibrium of a floating body, with particular reference to a floating ship.

6. Show that if a floating body be given a small rotational displacement in its plane of symmetry, the distance between the centre of buoyancy of the displaced liquid and the metacentre is Ak^2/V, where A is the area of the surface plane of the body, k, the radius of gyration about the surface-line and V, the volume of the displaced liquid

Discuss the conditions necessary for a hollow cylinder of height h, and density ρ, open at both ends, with r_1 and r_2 as its internal and external radii, to float vertically in a liquid in stable equilibrium. Ans. $r_1^2+r_2^2>2h^2.\rho(1-\rho)$

7. Discuss how the atmospheric pressure changes with altitude above the surface of the earth, the temperature remaining constant, and show how if the altitude increases in arithmetical progression, the pressure decreases in geometrical progression.

8. A mercury barometer is known to be defective and to contain a small quantity of air in the space above the mercury. When an accurate barometer reads 770 *mm*, the defective one reads 760 *mm.* and when the accurate one reads 750 *mm.*, the defective one reads 742 *mm* What is the true atmospheric pressure, when the defective barometer reads 750 *mm.* ?

(*Cambridge Scholarship*) Ans. 758·8 *mm.*

9. A simple barometer has the glass tube attached to a spring balance. What weight does the balance record when the open end of the tube is just dipping under the surface of the mercury in the reservoir, and what changes occur when the tube is lowered so that more of it dips under the mercury ?

(*Oxford Higher School Certificate*)

Ans. (*i*) *The balance records the weight of the tube and the mercury column.*

(*ii*) *A progressive decrease in the weights, due to buoyancy of the tube, until the latter is full, when finally it decreases to zero.*

10. A sealed spherical cellophane balloon has a diameter of 5 metres and weighs, with the apparatus it carries, 1 *k.gm.* It contains one-tenth of the volume of hydrogen required to fill it at atmospheric pressure. The balloon is allowed to ascend ; if the cellophane does not expand and if the temperature of the atmosphere is assumed to be constant at 0°*C* at all heights, calculate at what height the envelope becomes full and the height to which the balloon rises. The pressure p at height h (*km.*) is related to that at the ground (p_0) by the relation $h = 20 \log_{10} (p_0/p)$. (Densities of air and hydrogen at 0°*C* and atmospheric pressure are 1·29 and 0·09 *gm* per litre respectively).

(*Oxford and Cambridge Higher School Certificate*)
Ans (*i*) 20 *km.* and (*ii*) 34·5 *km.*

CHAPTER X

FLYING MACHINES

Jet Planes, Rockets and Satellites

163 Flying Machines. There are two types of *flying-machines, viz.*, (*i*) *lighter-than-air machines*, or **Air ships**, (*ii*) *heavier-than-air machines*, or **Airplanes.**

The Airship. An airship is based on the *principle of Archimedes.* The weight of the air displaced by it is greater than its own weight, *i.e.*, the upward thrust on it, due to the displaced air, is greater than the downward thrust, (due to its weight), and hence it rises up.

An airship is in fact a big cigar-shaped balloon of a light material, like aluminium or its alloy, covered with a specially treated water-proof linen or silk and divided up into a number of compartments containing bags filled with a light gas like *hydrogen* or *helium* (preferably the latter, due to its non-inflammable nature) from which it derives its buoyancy, *i.e.*, which makes the total weight of the airship less than the weight of the air displaced by it, or the upward thrust on it greater than its weight. This excess of upward thrust that it possesses over its weight is called its *lifting power*, and gives the maximum extra load it can be made to carry.

For steering purposes, it is fitted with rudders or other suitable devices and for horizontal motion, it is provided with propellers, worked by light and powerful engines.

164. The Kite. Before dealing with the airplane, it will be helpful to study the principle underlying the ordinary *kite.* This will be understood from the following :

Let *AB*, (Fig. 230) represent the mid-line of the kite. Then, the different forces on it are (*i*) *its weight W*, acting *vertically downwards* at its *c.g.*, *G*. (*ii*) the *tension T of the string*, acting *along the string*, as shown, (*iii*) *the pressure due to the wind, acting along the direction of the wind*, all along the undersurface of the kite.

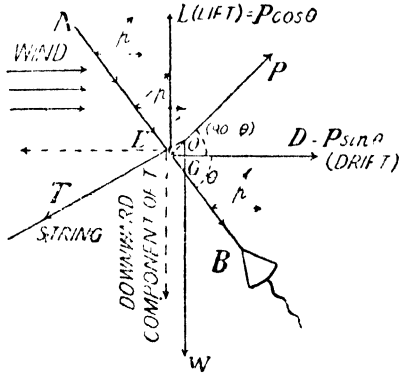


Fig. 230.

Now, this pressure of the wind may be resolved into two rectangular components at all points, (*i*) *perpendicular to the plane of the kite*, and (*b*) *along the plane of the kite.* These latter components play no part in supporting the kite and may thus be ignored ; and the former components *p*, *p*... being so many *like parallel forces*, have a resultant *P*, equal to their sum, called the *effective pressure of the wind*, acting at the point *C*. Thus, the three forces acting on the kite are

(*i*) *W*, acting *vertically downwards at G*, the *c.g.* of the kite.

(*ii*) T, acting *along the string* at E, and

(*iii*) P, acting *perpendicularly to the plane of the kite* at C.

Condition for the Kite to be in Equilibrium. In order that the kite may be *in equilibrium*, these three forces acting on it must be represented by the three sides of a triangle, *taken in order*. And in order that this may be so, they must all meet in a point, say, at O. It will easily be seen that this can be possible only when G, the *c.g.* of the kite, lies below the point C, where the *effective pressure* (P) of the wind acts. *In other words*, the *c.g.* of *the kite must be pretty low down for it to be in equilibrium.* It is for this reason that the lower part of the kite is made slightly heavier, and that a small paper tail is sometimes attached to it, which, in addition to bringing its *c.g.* down, also makes it look more attractive.

Condition for the Kite to rise up. The effective pressure P of the wind may also be resolved into two rectangular components, *viz.*,

(*i*) D, *along the direction of the wind*, called the **drift**, or the **drag.**

(*ii*) L, *upwards, perpendicularly to it*, called the **lift.**

If θ be the angle that the kite makes with the direction of the wind, we have

$$Drift\ (D) = P \cos (90-\theta) = P \sin \theta,$$

and

$$Lift\ (L) = P \sin (90-\theta) = P \cos \theta.$$

Similarly the tension (T) of the string may be resolved into two rectangular components, (*i*) *along the horizontal* and (*ii*) *downwards, along the vertical*, (shown dotted).

Now, clearly, the only force tending to make the kite rise upwards is the lift $L = P \cos \theta$, and the forces tending to make it fall downwards are (*i*) its *weight* W and (*ii*) the *downward component of the tension of the string. The moment, therefore, that the lift* (L) *is greater than* W + *the downward component of* T, *the kite rises upwards.*

Thus, to make the kite rise up, we must increase the lift, *i.e.*, $P \cos \theta$. This can be done by increasing P, *i e.*, by running *against the wind*, and by decreasing θ, by giving small *jerks*, (*Tanka*) to the string. If, however, the *drift* ($P \sin \theta$) be greater than the *lift* ($P \cos \theta$), the kite drifts along in the direction of the wind.

165. The Airplane. An airplane is a heavier-than-air machine and the principle underlying it is in main the same as that of the kite. Obviously, however, there is no tension of the string, here, pulling it downwards, so that the only force tending to take it up is the *lift* and the force tending to take it down is its *weight.*

Further, as the propeller-blades rotate rapidly, they throw the air backwards from in front of the plane, and its reaction is a thrust R, *forwards.*

Let us now consider the relation between these different forces on the plane in the different phases of its flight, *viz.*, (*i*) *when it flies level*, (*ii*) *when it climbs*, (*iii*) *when it dives* and (*iv*) *when it glides.*

(*i*) **Level Flight.** An aeroplane flying along the horizontal is said to be *flying level.*

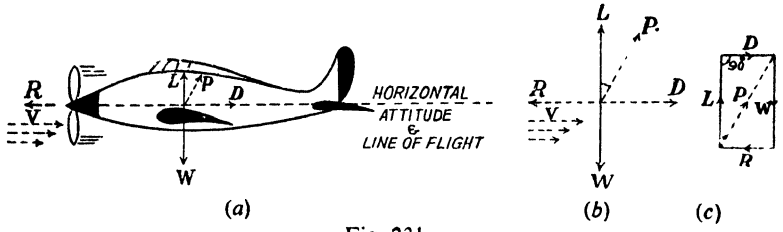


Fig. 231.

Fig. 231 (*a*)* shows an aeroplane in *level flight, from right to left,* with a constant speed V. This is tantamount to wind blowing *from left to right* with velocity V and striking the undersurface of the plane ; so that, proceeding, as explained above, we have

Reaction or *Thrust* R, *forwards* = the *drift* or *drag* D, *backwards,* both acting practically *along the horizontal* ;

and, *weight* W of the plane *downwards* = the *lift* L†, *upwards,* both acting *along the vertical,* [Fig. 231 (*b*)]

Or, $L = W$, ...(*i*)

and, $R = D$, ...(*ii*)

which, represented *vectorially,* form a closed polygon, [Fig. 231 (*c*)], the plane being in equilibrium.

It will at once be clear from relation (*ii*) that, *for level flight, the forward thrust R must just be balanced by* (*i.e., must be equal to*) *the backward drag D, at that particular speed of flight.*

Further, if the speed falls, the lift decreases and the 'plane starts losing altitude ; so that, a *minimum speed* (*about* 50 *m./hr.*) is *essential to keep the plane at a certain height.*

(*ii*) **Climbing.** If an aeroplane flies obliquely upwards, it is said to be *doing a 'climb'.*

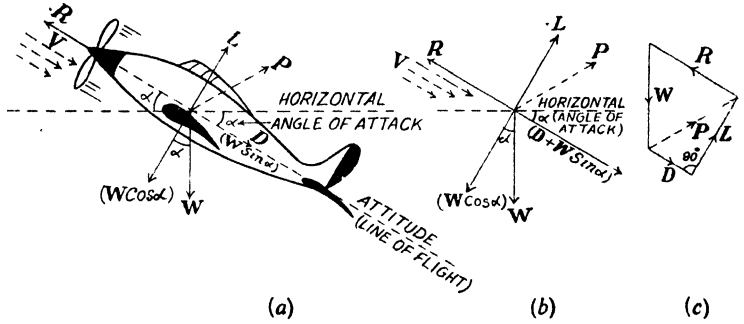


Fig. 232.

We shall, for the sake of simplicity, take the line of flight dur-

*For simplicity, the student may simply show these forces acting on the wing or the aerofoil, instead of sketching the whole plane.

†The *lift* is not necessarily vertical. It is just the component *perpendicular to the current of air.*

ing the climb to coincide with the direction of thrust R due to the propeller, or with the *attitude* of the plane.*

Since the relative velocity of the wind now makes an angle α with the horizontal, the lift (L) no longer acts in a vertical line with the weight (W) of the plane and is, therefore, balanced by the component $W \cos \alpha$ of the weight [Fig 232 (*a*) and (*b*)] ; and, similarly, the thrust R of the propeller, by the drift or drag (D) along the line of flight *plus* the component $W \sin \alpha$ of the weight, *i.e.*, now

$$L = W \cos \alpha \qquad \ldots(iii)$$

and

$$R = D + W \sin \alpha. \qquad \ldots(iv)$$

It will thus be clear that *for a climb, the thrust R must be greater than the drag (D) by the factor W sin* α and that it increases with the angle α or the steepness of the climb.

It follows, therefore, that if $\alpha = 0$, $\sin \alpha = 0$ and $\cos \alpha = 1$; so that, equations (*iii*) and (*iv*) reduce to (*i*) and (*ii*) respectively. In other words, the plane then flies level with a constant velocity, without a climb.

The forces in equilibrium, during the climb of the 'plane, represented *vectorially*, give a closed polygon [Fig. 232 (*c*)], which, in the case of level flight, reduces to a rectangle, [Fig. 231 (*c*)], with R and D *equal and horizontal* and L and W, *equal* and *vertical*.

(*iii*) **Diving.** When a plane flies *obliquely downwards*, it is said to be *making a dive*.

Again, taking the speed of the plane to be constant and its line of flight coincident with its *attitude*, the different forces on the 'plane are as shown in Fig. 233 (*a*) and (*b*).

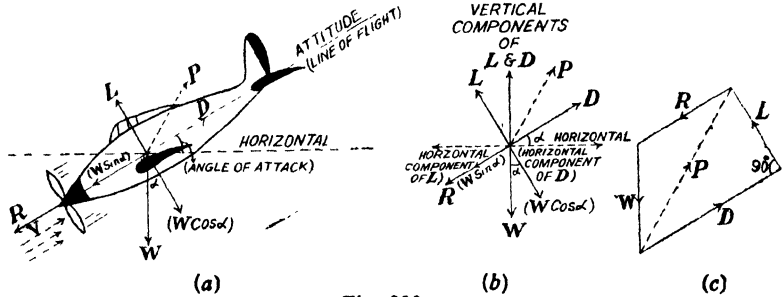


Fig. 233.

Since the relative wind velocity (V) makes an angle α with the horizontal, the lift here also does not act along the vertical line with W; and, since it makes an acute angle with the downward vertical line, the vertical components of both the lift and the drag act upwards, thus opposing W. As will be readily seen from Fig. 233 (*a*) and (*b*), the lift is balanced here by the component $W \cos \alpha$ of the weight (W) of the 'plane, the other component $W \sin \alpha$ of which acts along the *same* direction as the thrust (R) ; so that, for equilibrium, we now have

*It is by no means necessary that it should always be so. The line of flight can—in fact it often does—make an angle with the thrust (R), which, of course, acts along the *attitude* of the 'plane.

$$L = W \cos \alpha, \qquad \ldots(v)$$

and $$R = D - W \sin \alpha, \qquad \ldots(vi)$$

showing that, *for a dive, the thrust R must be less than the drag (D) by the factor W sin* α and, therefore, it decreases with α or the steepness of the dive, necessitating the throttling down of the engine.

And, once again, if $\alpha = 0$, $\sin \alpha = 0$ and $\cos \alpha = 1$, so that relations (*v*) and (*vi*) reduce to (*i*) and (*ii*) respectively, the 'plane flying level with a constant velocity, without a dive.

Representing the different forces *vectorially*, we again obtain a closed polygon. [Fig. 233 (*c*)], which with $\alpha = 0$, becomes a rectangle, with *R* and *D* equal and opposite, and acting along the horizontal, and *L* and *W*, equal and opposite, and acting along the vertical.

(*iv*) **Gliding.** With the engine not functioning, *i.e.*, with $R = 0$, as the 'plane descends down, it is said to be *gliding*.

In this case, obviously, the lift and the drag are balanced by the components $W \cos \alpha$ and $W \sin \alpha$ of the weight (*W*) of the 'plane respectively, and we have, for equilibrium,

$$L = W \cos \alpha, \qquad \ldots(vii)$$

and $$D = W \sin \alpha\ ; \qquad \ldots(viii)$$

so that, with the increase of α, the drag increases and the lift decreases.

166. Different parts of an Airplane and their functions.

The following are the different parts of an airplane and their respective functions.

(*i*) **The Wings.** The *wings* or the *aerofoils*, as they are technically called, are, appropriately, the most important part of an airplane (*a flying machine*) and much research has gone into perfecting their design, in order to obtain the maximum *lift* for the 'plane. In fact, the lift due to them accounts for as much as about *two-thirds* of the total available lift.

To minimise the fractional force to its motion and to ensure a smooth air-flow along its surfaces, the wing is *shaped to the streamlines of the air* through which it passes, with a gradual tapering off of its thickness from its **front or leading edge** to its **rear or trailing edge**, with the upper surface more curved than the lower, as shown in Fig. 234. The axis of wing (shown dotted) is called the **chord** and the angle that the chord makes with the direction of the wind is called the **angle of attack.**

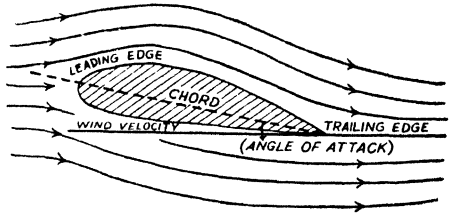


Fig. 234.

The air, moving more rapidly over the upper than the lower surface, brings about a difference of pressure on the two surfaces, in accordance with *Bernoulli's principle*, (see Chapter XII) and *this gives the wing an upward lift.* This lift on the wing really consists of (*i*) *an upward thrust on its lower surface* and (*ii*) *a suction effect on its upper surface.* For, as the leading edge of the wing

moves through air, it parts the air stream into two parts, which tend to flow as close to the two surfaces as possible. The upper stream is, however, deflected upwards by the curved shoulder of its upper surface and its return back to that surface is retarded due to its inertia, resulting in an area of partial vacuum above it and a consequent upward pull on it due to suction.

For a given wind-speed, the lift increases with the angle of attack up to a certain limit, beyond which it begins to decrease and the 'plane begins to sink. This *limiting value of the angle of attack* is called the **stalling angle** and its value varies from about 15° to 20°. The ratio *lift/drag* is, however, the maximum when the angle of attack is about 4°. Hence we have the maximum efficiency in flight at *this* angle of attack.

An important consideration in the structure of the *wing* is to combine *lightness with strength*, and it, therefore usually consists of two main *spars* of wood or metal, running all along its length, with light girders of the same material, set perpendicularly to them at suitable intervals, the whole framework being covered with a '*skin*' of sheet metal or thin plywood, having a tightly stretched fabric over it, well coated with a liquid solution, called '*dope*', which not only shrinks the fabric and makes it *taut* like the skin of a drum, but also serves to increase its strength and to make it *water and air-proof*.

It is found that the force or *effective pressure* (P) due to the wind, as it is called, depends (*i*) *directly upon the area A of the aerofoil*, (*ii*) *directly upon the square of the velocity* (V) *of the wind*, and (*iii*) *directly upon the density* (ρ) *of the air at the height of the 'plane*. Thus,

$$P \propto A.\rho.V^2. \qquad \text{Or,} \qquad P = K.A.\rho.V^2,$$

where K is a constant, depending upon the shape of the aerofoil and the angle of attack.

Since the lift, $L = P \cos\theta$, and the drag $D = P \sin\theta$, we have

$$L = K.A.\rho V^2.\cos\theta \quad \text{and} \quad D = K.A.\rho.V^2.\sin\theta.$$

Or, multiplying and dividing each expression by 2, we have

$$L = 2K\cos\theta \cdot \tfrac{1}{2}A\rho V^2 \qquad \text{and} \qquad D = 2K\sin\theta \cdot \tfrac{1}{2}A\rho V^2.$$

The factor $2K\cos\theta$ is called the **Lift coefficient** and the factor $2K\sin\theta$, the **Drag coefficient,** usually denoted by the symbols C_L and C_D respectively. So that,

$$L = C_L \cdot \tfrac{1}{2}A\rho V^2 \qquad \text{and} \qquad D = C_D \cdot \tfrac{1}{2}A\rho V^2,$$

where L and D are expressed in *lb. wt.* ; A, in *sq. ft.* ; ρ, in *slugs* per c. ft.* and V, in *ft. per sec.*

The Lift and Drag coefficients increase with the angle of attack, the former having its maximum value 1·2 at about 16°, when the value of C_D is about 20. The ratio of the two coefficients *i.e.*, C_L/C_D or the ratio *Lift/Drag* also varies with the angle of attack, and has its maximum value (12) at about 4°, at which value of the angle of attack, therefore, we have the maximum efficiency in flight. Normally, the angle of attack is arranged to lie between 3° to 6°.

Further, it will be clear from the expression for L above that a certain minimum wind speed is essential for the *lift* to be large enough to make the 'plane rise up against the force of gravity. It is for this

*Mass in *slugs* is equal to weight in pounds weight, divided by 32.

reason that the plane must first be made to run on the ground before it can take off.

(*ii*) **The Propeller or the Air-Screw.** It is a large *fan-like* structure, carried right in front* of the plane and rotated rapidly about a horizontal axis by an internal combustion engine. Its two (or more) blades are set at an angle in a central hub, a shown in Fig. 235, and may be made of wood or metal, consisting, in the former case, of a number of layers firmly glued together, with their edges tipped with metal, and their surface provided with a suitable protective covering of fabric or cellulose.



Fig. 235.

A propeller blade is in fact a small wing and functions precisely as such. For, just as a moving wing, meeting the air at an angle, experiences an upward thrust in a direction almost at right angles to that of its motion, so also does a revolving propeller blade experience a thrust at right angles to its direction of motion, *i.e.*, along the horizontal, for the very air which it sweeps from in front of it and throws backwards, pushes it forwards.

Thus, because the propeller cuts its way through air, much in the manner of a screw cutting its way through wood or metal, it is, on that analogy, also referred to as the **air-screw.**

Apart from the two most important parts of an airplane, discussed above, there are others which make for its *stability* and easy *manageability* in any desired position and direction in the air. These together constitute what are called the **surface controls** of the airplane and we shall now deal briefly with these.

(*iii*) **The Tail Unit.** Carried at the rear end of the airplane, it consists of two sets of surfaces, (*i*) *vertical* and (*ii*) *horizontal*, each being made up of two parts, *one fixed and the other movable, viz.*, the *fin* and the *rudder* ; and the *tail plane* (or *stabilizer*) and the *elevator* respectively.

(*a*) **The Fin.** It is the fixed or the front part of the vertical surface of the tail unit and takes the form of a vertical plate, *arranged at a small angle with*

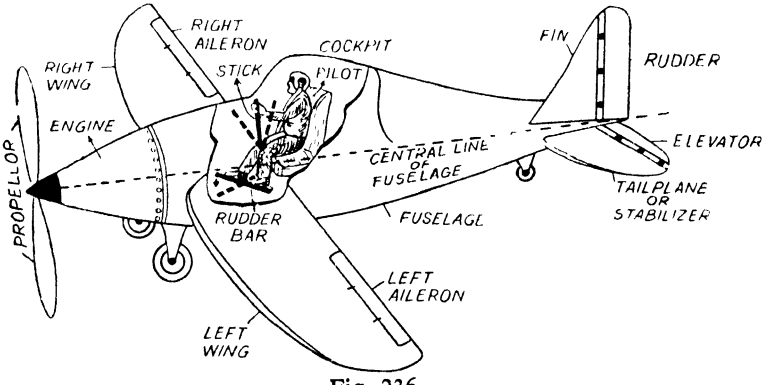


Fig. 236.

the central line of the fuselage or the body of the 'plane, (Fig. 236). Its function is

*This is the most usual position of the propeller in most 'planes, such planes belonging to the **tractor type**, because of their being pulled through air by the action of the propeller. In what are called the **pusher type** of 'planes, the propeller is carried behind the line of the wings, so that it exerts a pushing action on them.

to give *directional stability* to the 'plane, (very much in the manner of the feathers tipping a dart), making for its straight-line flight in the horizontal plane and tending to bring it back to its original course, should a disturbance cause it to turn to one side or the other.

Thus, if the 'plane be thrown to the left, the fin will deflect the air to the right, which would then push it back to the left, to resume its original straight course and *vice versa*.

The student may perhaps wonder as to why the *fin*, with its avowed purpose of keeping the 'plane along its straight-line course, should be offset a few degrees from the central line of the fuselage. The reason is that the air stream, blown back by the propeller, (and called the *slip stream*), shares with the latter its corkscrew motion and would strike the fin at an angle, were it set along the central line, producing precisely the opposite of the desired effect; for, it would result in turning the plane rather than keeping it along its straight course. *The small inclination of the fin to the central line just counteracts this turning effect due to the slip stream.*

(*b*) **The Rudder.** It is the rear portion of the vertical surface, (Fig. 236), hinged on to the front portion or the fin, and has *freedom of lateral movement in the vertical plane*. Its function is very much similar to the rudder on a boat and it enables the 'plane, in level flight, to be steered to the right or to the left in the horizontal plane.

Connected by means of cables to the *rudder bar*, pivoted horizontally on a central vertical pin in the *cockpit**, it is operated by the pressure of the pilot's feet, (see Fig. 236)—a pressure with the right foot (*i.e.*, on the right-hand end of the bar) makes it swing out of the central line and turns the plane to the right, and a pressure with the left foot similarly turns the plane to the left.

(*c*) **The Tail Plane or the Stabilizer.** This is the *fixed part* of the *horizontal surface* of the tail unit, (Fig. 236), and its function is identical with that of the fin, but in the up and down direction, *i.e.*, *it serves to give the airplane stability in the vertical plane*, or the *'fore and aft' stability*, as it is called.

(*d*) **The Elevator.** It is the *movable part* of the *horizontal surface* of the tail unit and controls the vertical motion of the 'plane, *i.e.*, its climbing and gliding movements.

Lying normally in level with the tail plane or the stabilizer, its up and down movement is controlled by the *central column*, or the *central stick*, (or, simply the **stick**, as it is usually called), which is connected to it by cables and is arranged conveniently in front of the pilot's seat, (see Fig. 236, above). A backward or inward pull on the stick raises the elevator up above the level of the tail plane and the air, rushing over the surface of the plane, strikes against it, tending to blow it down to its original position, in level with the tail plane, thus exerting a downward pressure on the tail of the 'plane, as a whole, with the result that its nose is pushed upwards and it *climbs* up.

Similarly, a forward or outward push on the stick lowers the elevator below the level of the tail plan and the air thrust on it now pushes the tail up, which is the same thing as pushing the nose down, and the 'plane, therefore, now *glides* down.

(*iv*) **The Ailerons.** These are *hinged flaps*, free to move up and down at the rear or the trailing edges of the two wings, extending from the tip of each wing to almost its mid-point, (Fig. 236), their up and down movement being controlled by the side-ways pull on the *stick*†, to which they are connected by means

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*The Cockpit is a closed or open well, in the front portion of the aeroplane in which the pilot takes his seat, (Fig. 236), with different controls and instruments arranged in front of him.

†In the larger type of aircraft, the aileron is controlled not by the stick but by what looks like an incomplete steering wheel of a motor car, fitted on to the top of the stick.

of cables. The arrangement is such that as the stick is pulled to one side, it simultaneously causes one aileron to be raised above, and the other to be lowered below, the undersurface of the corresponding wing, with the result that the lift on one wing increases and that on the other decreases, making the plane *'bank'* or heel over to one side,—a pull on the stick towards the left making the 'plane bank to the left and a pull towards the right, making it bank to the right.

It will thus be seen that the stick and the rudder bar, between themselves—either singly or in combination with each other—enable the 'plane to be manoeuvred into any desired position and to perform all sorts of aerobatics.

(*v*) **The Tail Trim.** If an airplane continues to fly level, even when the pilot releases his hold on the stick for a while, it is said to be *'flying trimmed'*. This ideal state of affairs may however be easily disturbed by the entry or exit of a passenger or two, the plane becoming *'nose heavy'* or *'tail heavy'* and thus starting to fall down or to rise up. This puts an undue strain on the pilot, always alert to exert an inward or an outward pull on the stick.

The *tail trim* is just the device to prevent all this and to enable the 'plane to fly trimmed even with different loads in it, by automatically adjusting the inward or outward pressure on the stick, to suit the load. Of immense help to the pilot during *'take offs'* and *'landings'*, it just consists of a lever on one side of the cock pit which, working on a quadrant, suitably alters the tension of a spring attached to the lower end of the stick*, always exerting the requisite pull, commensurate with the load in the 'plane.

(*vi*) **The Undercarriage or the Chassis.** It is that part of the airplane behind the engine and at the base of the fuselage, which serves as a carriage for the 'plane to run on the ground and includes the wheels and a shock-absorbing mechanism (the *oleomechanism*) to take up the unavoidable impact on landing or the bumps on uneven ground, which may otherwise severely strain the fuselage even to the extent of damaging it.

To minimise the air resistance to the flight of the airplane, the undercarriage is now almost universally made retractable (except perhaps in the case of very small aircraft) ; so that, it can be drawn up into the fuselage once the 'plane is up in the air, and lowered again when about to land, there being a suitable device to warn the pilot in time, when preparing to land, in case his undercarriage remains retracted.

(*vii*) **The Wheels.** The undercarriage is supported on two† wheels (excluding the one at the tail end), fitted with wide-track pneumatic rubber tyres, inflated at low pressure. These, besides enabling the 'plane to run on the ground before a take off also absorb part of the shock of impact, on landing, passing on the rest to the *oleomechanism*.

In modern aircraft, we have also wheel brakes fitted more or less in the manner of our motor car brakes, which (*a*) keep the plane stationary during the running of the engine on the ground and (*b*) also shorten its run on landing. In addition, they enable more pressure to be applied to one wheel than to the other, thereby greatly facilitating the steering and the manoeuvring of the plane, while still on the ground.

(*viii*) **The Tail Skid.** The rear of an airplane is supported either on a small wheel or a spar-like structure, called the *tail skid*. When the two front wheels and this spar, or small wheel, touch the ground simultaneously on landing, the plane is supposed to have made a perfect *'three point landing'*.

(*ix*) **The Slot.** Oftentimes, when an aeroplane climbs too steeply, or when it is about to land, and in fact, when for any reason, the speed of the 'plane falls below a certain minimum, the lift on the wings becomes insufficient to keep the 'plane flying and there is every possibility of its *'stalling'*. Not only that, but with an insufficient air-flow, the other controls, and particularly the ailerons cease to function properly and the 'plane starts dropping in a dive.

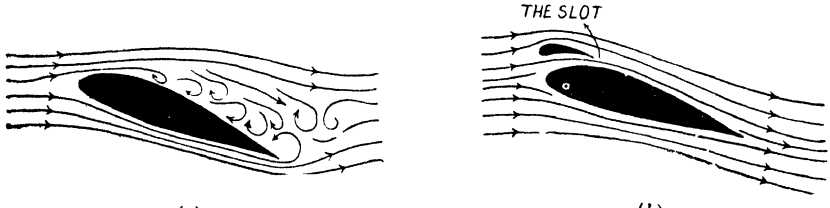
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*Sometimes, the lever is replaced by a wheel, whose movement suitably adjusts the position of the tail plane instead of acting on the stick.

†In some cases, we have a three-wheeled or a *'tricycle'* undercarriage, the third wheel being arranged well ahead of the other two. This not only prevents the 'plane tipping on its nose, thus greatly reducing the possibility of accidents on landing or manoeuvring the plane on the ground, but also greatly simplifies both take-offs and landings.

An ingenious saftey device, known as the *Handley Page Slot*, or, simply the *Slot*, is, therefore, used to avert this danger of a *'stall'*. It is just a small gap between the upper surface of the wing and another miniature wing-like structure the *'slat'* arranged over its leading edge*.

Without the *slat*, if the airplane were to stall, the air-flow ceases along the upper surface of the wing and breaks up to form a series of eddies, as shown



(*a*) (*b*)

Fig. 237.

in Fig. 237 *a*), thus depriving the 'plane of about 60% of its lifting power When, however, the slat is fitted to the 'plane, it opens up as shown in Fig. 237 (*b*), and forms a small passage or slot between itself and the wings and the air stream is directed through it on to the wing surface, instead of breaking up into eddies. The lift on the wing is thus avoided and the danger of a stall averted.

If, however, the wing be tilted too steeply, a stall may eventually occur, but the 'plane recovers from it much sooner than would be possible without the *slat*.

(*x*) **Engine Controls and Other Instruments.** Among these, the main of the important ones are the following :

The Throttle. This corresponds to the accelerator of the car and controls the speed of the 'plane. Operated by the throttle lever on one side of the cockpit, it differs from the car accelerator in that it stays in the position in which it is set, without springing back when the pressure is released on it, thus enabling 'plane to fly at the desired constant speed. There is no gear changing or slowing down for negotiating corners, for which, indeed, it must fly a little faster.

(*xi*) **The Altimeter.** As its very name indicates, it is an instrument to indicate the altitude of the airplane. It is, in fact, a modification of the *aneroid barometer* and is calibrated to indicate height or altitude in terms of *'thousands'* of feet. Thus, if the pointer be at 5, it indicates a height of 5000 *ft* and so on. Since, however, the altimeter really measures variations of pressure at ground level, which can occur due to changes of weather, it may indicate different height even at one fixed point on the ground, and its readings may thus be highly misleading and may prove dangerous. To obviate this risk, therefore, it is so arranged that the pilot sets it at *zero* altitude before taking off, so that its readings later indicate the heights *above this starting point*, and *not the absolute height above the ground at any given moment*. Thus, even if it indicates a height of 5000 *ft*., it may well be within a couple of hundred feet from a mountain top. Improved instruments to indicate the absolute height of the 'plane above the ground at a given moment (instead of from the starting point) are however well in the offing and would greatly reduce the hazard of an airplane flight in foggy weather.

(*xii*) **The 'Engine' Revolution Counter** This enables the pilot to feel the pulse of the engine, as it were, telling him all about the condition of the engine, including its undue vibrations and uneven running etc. Further, should there be an unexpected or unaccountable drop in the revolutions of the engine, it is a warning to the pilot that trouble is imminent. *The revolutions are measured in terms of hundreds per minute.*

(*xiii*) **The Oil Pressure Gauge.** It is a small but vitally important instrument and indicates the pressure (*in pounds*) under which the oil is pumped round to the different parts of the aeroengine,—an operation about just as essential to

*Sometimes the *slot* is also arranged close to the aileron flap, when it helps to maintain the requisite air flow over the aileron surface, thus enabling it to function *effectively* even at low speeds of the 'plane.

its life as the blood supply to the various parts of our body. A sudden drop in this pressure forewarns the pilot of a coming serious trouble and alerts him to take remedial measures in time.

Airship	Airplane
1. It is a 'lighter-than-air' flying machine.	1. It is a 'heavier-than-air' flying machine.
2. It is based on the principle of floatation and its lifting power is provided by the buoyancy of the air displaced by it.	2. Here, the lifting power is due to the thrust produced by a strong artificially created wind and the characteristic shape of its wings.
3. It rises vertically upwards, directly from the ground.	3. It must first be made to run on the ground before it can 'take off'.
4. It is very much bigger in size than an airplane.	4. It is comparatively smaller in size.

Thus, though the air-ship and the airplane are based on entirely different principles, they have in common (*i*) *an upward motion against the action of gravity* and (*ii*) *propulsion through air.*

167. Jet Propulsion. We are all familiar with the meaning of the word '**jet**' which is just the term applied to a *high velocity stream of fluid* (liquid or gas) *issuing out of a nozzle*, as for example, a *'jet of water'* or a *'jet of steam'* etc. And, therefore, *jet-propulsion* is obviously the method of driving or propelling a body or a machine forwards through the agency of a jet,—the body or the machine thus driven being said to be *jet propelled.*

That a jet possesses such a motive or tractive force can be easily seen from a number of facts of every day life, if only we care to stop a while and analyse them. Thus, for example, when a bullet is forced out of the barrel of a rifle by the exploding mixture of gases inside it, the rifle suddenly moves or *'kicks'* back in a direction opposite to that of the bullet and the exploded gases.

So that, if we continuously fire a rifle fastened to the rear of a boat, with its barrel facing outwards, we shall find that the boat continues to move forwards with a jerky motion so long as the firing continues,—each bullet fired producing a push forwards. We, therefore, have here a jet-propelled boat ! In fact, even when we ply the oars, the action is similar. For, what we do is simply to push some volume of water backwards and the boat, as a consequence, moves forwards.

Indeed, if we did nothing else but simply sit quietly in the boat and throw stones into the water, with our face towards the stern of the boat, the boat will still move forwards (*i.e.*, opposite to the direction in which the stones are thrown).

All these examples are, as the student is no doubt already aware, a consequence of the well known *Newton's third law of motion*, according to which *action and reaction are equal and opposite*, or what follows from it, *viz., the law of conservation of momentum*, which

demands the equality of the momentum lost with the momentum gained. So that, the boat, in the cases above, moves oppositely as a result of the reaction to the motion of the bullet (and the gases) or the stones, or because the momentum lost by the bullets or the stones is equal to the momentum gained by the boat. A force such as the one experienced by the boat is called the *reactive force*; and, in the case of a jet, sometimes also the *jet-force*.

Now, does it surprise the student when he is told that even the usual type of airplane, in which we use the ordinary reciprocating (*i.e.*, the piston-type) engine makes use of a jet for its propulsion forwards* ? For, the propeller, as it whirls round at a high speed, throws a jet of air (or in the case of a ship, a jet of water) backwards, as a reaction to which the plane (or the ship) is pushed forwards against the viscous resistance of the air (or water). The question, therefore, naturally arises as to why then do we not call them jet-propelled planes. The answer is that, technically speaking, the narrower the cross-section of the high-velocity fluid stream, the more nearly does it come up to the definition of a jet, and the term *jet-propelled planes* is, therefore, reserved for planes in which the jet is a narrow one, about one foot in diameter, as compared with ten feet or more in the case of the ordinary airplane.

Again, it must not be inferred from what has been said above that a jet must necessarily consist of hot gases. No, it may just as well be of cold air, as in the case of what are called the *ducted-fan type* of planes†, or as was the case with perhaps the earliest jet-propelled plane, constructed by *Campini* in Italy,—the jet in this latter case, being produced by a compressor, driven by the ordinary reciprocating type of engine.

168. Thrust supplied by the jet. Let us now calculate the thrust supplied to an aircraft by the jet produced by the power unit inside it.

Suppose we have an aircraft travelling with a speed V and fitted with a power-unit which produces a jet of fluid, of velocity u, relative to the aircraft, where u is higher than V,—the velocity u of the jet being measured at a point in it a little away from the nozzle, where the static pressure is the same as that in the surrounding air. Then, if we impose a velocity V on the aircraft in the opposite direction to its own, the aircraft comes to rest, with the air streaming past it with velocity V. So that, if a be the area of cross-section of the jet at the point where its velocity is u, the *volume of the fluid flowing per second in the jet is clearly* $a.u$. If, therefore, ρ be the *density* of the fluid, we have

$$\textit{mass-flow of the fluid per second in the jet} = a.u.\rho = m, \text{ say}$$

And, therefore, $\textit{momentum of this fluid in the jet} = m.u$.

If this mass (m) of the fluid finally emerges out from the aircraft

*The same being the case with a ship.

†In these planes, air is sucked in through two holes or ducts, by two fans, rotating in opposite directions,—the latter thrusting the air away with considerable force, propelling the plane forward. This sucking action also helps to buoy the plane up. Further, the gyroscopic effect, produced by the oppositely rotating fans greatly helps in enhancing and assuring the stability of the plane.

with a velocity V, its momentum is clearly reduced to $m.V$. It thus suffers a loss of momentum, equal to $(mu-mV)$ or $m(u-V)$ *per second, i.e., its rate of change of momentum* $= m(u-V)$.

And this, therefore, in accordance with Newton's second law of motion, must be the *force*, or the *thrust*, F supplied to the aircraft by the jet in the direction opposite. So that,

$$F = m(u-V) = au\rho\ (u-V)$$

169. Efficiency of the jet. If we consider the exact state of affairs in the case above, *viz.*, that the aircraft is really not at rest, as we had imagined, but is moving with velocity V, then, the velocity of the final jet, moving in the opposite direction clearly becomes $(u-V)$ and, therefore,

$$K.E.\ of\ the\ final\ jet = \tfrac{1}{2}m(u-V)^2 = \tfrac{1}{2}F(u-V) \quad [\because m(u-V)=F]$$

Also, the aircraft does FV amount of work per second against the air resistance as it moves forwards. So that,

total energy that must be supplied by the jet propulsion unit,

(i.e., by the power-unit) $= FV+\tfrac{1}{2}F(u-V)$.

Of this, obviously, *the portion usefully employed is only* FV, the rest being simply a waste, creating a disturbance behind the aircraft. So that,

efficiency of the jet, or the *Froude efficiency*, as it is commonly known $= \dfrac{energy\ converted\ into\ useful\ work}{total\ energy\ supplied} = \dfrac{FV}{FV+\tfrac{1}{2}F(u-V)}$.

$$= \frac{2FV}{Fu+FV} = \frac{2V}{u+V}.$$

Note. Clearly, the efficiency will have the maximum value 1, when $u = V$, *i.e.*, when the initial jet velocity is equal to the flight velocity of the aircraft, for, then, the energy wasted in the form of *K.E.* of the final jet $[\tfrac{1}{2}F(u-V)]$ will also become zero. But, then, the thrust on the aircraft $[m(u-V)]$ will also become zero. This condition of maximum efficiency is, therefore, not a practicable proposition, just as it is not in any other type of machine also.

170. Effect of smaller cross-section of the jet. As indicated earlier, the cross-section of the jet in a jet-propelled plane should be narrow. Let us see what advantage is to be gained by it.

Apparently, from the relation $F = m(u-V)$ for the thrust supplied to the aircraft by the jet, we find that a reduction in its cross-section will mean a diminution in the value of the mass flow of the fluid, m, so that, to obtain the *same* thrust F, as before, $(u-V)$ will have to be correspondingly greater. This will naturally mean a higher value of $\tfrac{1}{2}(u-V)^2$, the *K.E.* of the final jet, which, as we have seen, is a mere waste of energy. Not only that, but, as a natural consequence, the efficiency of the jet $FV/FV+\tfrac{1}{2}F(u-V)$, will also fall below its previous value. It would thus appear that a decrease in the cross-section of the jet, far from improving matters, does just the reverse, *viz.*, increases the loss of energy and decreases the efficiency for propulsion. In what manner, then, is jet-propulsion a better mode of propulsion?

The answer is manifold:

(*i*) Initially, when jet-propulsion was just introduced round about the year 1940, it was intended to render auxiliary support to

the then prevalent *gas turbine* engine. The materials of the gas turbine could not function satisfactorily at the temperatures obtaining in the earlier reciprocating type of engines and the products of combustion required to be diluted with a large excess of air. This seeming difficulty was actually turned into an advantage by the engine-designers, who used this necessary excess of air as a narrow jet to supply the entire thrust required to be given to the aircraft, thus eliminating the necessity of the propeller and quite a few other accessories. The jet was made to escape through a small turbine which then supplied the necessary power to the generator, the fuel pumps and the compressor etc. Thus, although the introduction of the jet inevitably entailed a loss in efficiency, with the fuel-consumption rate rising higher, it gave the *distinct advantage of reducing the weight of the whole unit for the same value of power.* In view of this smaller weight but higher rate of fuel-consumption, the *turbo-jet* engines, as these engines were aptly christened, came to be considered more suitable for flights of shorter durations, say, of less than 2 hours in those early days when the highest speed was only 400 miles per hour.

(*ii*) It was found that although the efficiency of a narrow jet is rather low at moderate flight speeds, it increases rapidly with the flight speed. In fact, if we take into consideration also the other advantages that go with high speed, (*e.g.*, assistance given to the compression-process in the engine, etc.), the over-all result is that the power output (FV) increases directly with flight speed with only a comparatively very small increase in fuel consumption, *i.e.*, $FV \propto V$. Clearly, therefore, F remains practically constant for varying flight speeds.*

This linear increase in power (FV) with speed (V), with practically a constant fuel-consumption rate, necessarily implies that if the flight speed be high, the turbo-jet unit will also be about as economical as the ordinary propeller-engine and will, in addition, possess the advantage of (*a*) having *less weight* and (*b*) *capacity of packing large power in a smaller space.*

In fact, both the turbo-jet and the propeller engine will have the *same* efficiency, *i.e.*, their power output for the same fuel-consumption will be the same, at a speed of 700 *m.p.h.*, *provided the propeller engine had a constant power-output upto this speed.* And this is the point where the jet-unit scores over the propeller unit. For, the power output of the propeller engine does not really remain constant with speed but falls steeply as the flight speed approaches the speed of sound, *viz.*, 762 *m.p.h.* at ground level and 660 *m.p.h.* at altitudes above 3600 *ft.* This is so, because a propeller may be regarded essentially as a wing, with the difference that whereas the latter provides a lifting force to the aircraft against the force of gravity, the former supplies a similar force in the form of a thrust in the direction of its motion, for which purpose it is rotated in a plane perpendicular to the direction of flight,—the lifting force in the case of the wing and the forward thrust in the case of the propeller being always roughly perpendicular to the direction of their respective motions through air, both experiencing an air-resistance or 'drag'

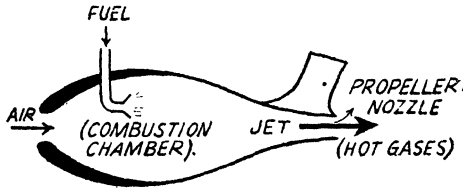
*That is why, in the case of a turbo-jet unit, *only its thrust* (F) *is indicated and not its power* (FV).

opposing their motion, with some power used up in overcoming the same.* So that, despite all improvements made in the designs of propellers (such as making their sections near the tips very thin, etc.) their propulsive power falls greatly at high flight speeds, whereas that of a jet-unit rises equally greatly. This comes about because the actual velocity of the propeller blade is the resultant of its velocity of rotation and translation and, as such, is higher than the flight speed of the aircraft itself,—even a non-rotating wing experiencing a large increase in the drag on it much before the speed of sound is attained.

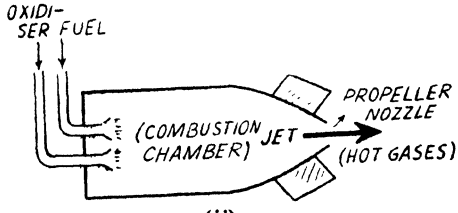
Thus, from the point of view of over-all efficiency, a jet unit is certainly superior to a propeller-unit at speeds of 600 *m.p.h.* and above.

Then, again, another advantage that a jet unit possesses over the propeller-unit is that there being no accessories and protubrances like radiators, oil-cookers etc., the drag is comparatively less. And the absence of the propeller which makes for a smoother flow over the entire surface of the plane, cuts down the drag over these surfaces by as much as 20 to 30 per cent.

171. Rocket Planes. The small fire-work rockets, rising pretty high up in the air, to the amusement of on-lookers, are a common enough sight everywhere and they are obviously jet-propelled on their own small scale. A rocket plane is merely a large scale version of the same phenomenon. It possesses a higher speed and can rise to a much greater height than even a turbo-jet plane. In fact, it is a turbo-jet plane,



(*i*)



(*ii*)

Fig. 238.

in which the technique of jet-propulsion is carried a stage further, with the jet still narrower in cross-section and its velocity higher.

Is this difference in degree, then, the only factor that distinguishes it from a jet-plane? Of course not; for, the essential difference between the two lies in the method of production of the jet. *In a jet-propulsion unit, the fuel alone is carried on the aircraft, with the oxygen necessary for its combustion being drawn from the surrounding air,* [Fig. 238 (*i*)]—only a fraction of which is usually consumed, the rest, together with the considerable larger quantity of nitrogen 'swallowed', merely serving to keep the temperature down throughout the jet-propulsion

*The resistance of the compressed air in front of the aircraft is of little consequence at speeds below that of sound, for it simply moves away with the speed of sound. But when the speed of the aircraft is higher than that of sound, the condensed air in front can only move sideways but not forward with the result that the nose of the aircraft has to carry along a bulk of compressed air, with a consequent large increase in the drag on it.

unit. *In the case of a rocket-propulsion unit, on the other hand, the fuel as well as the oxidising agent, required for its combustion, are together carried on the aircraft and no air has to be drawn in from the surroundings. It is thus a self-contained unit in itself,* [Fig. 238 (*ii*)]. This is a point of great importance in that it makes the working of the motor quite independent of the presence or absence of any surrounding air. Thus, whereas a jet plane can only attain a height of 50 to 80 thousand feet up to which it can have its supply of air from the surroundings, there is no such limit to the height of a rocket-plane, which alone is capable of rising up to higher altitudes beyond the earth's atmosphere, where there is obviously no air to be drawn in. So that, its being self-contained, with its own supply of the oxidising agent, while it may be a comparative disadvantage at lower altitudes, is clearly a tremendous advantage at higher altitudes

Calculating the thrust (F) given to the aircraft by the issuing jet in the same manner as in the case of the jet propulsion unit (§ 168, page 378), we have $F=m.u$, where m is the *mass-flow* of the fluid per second in the jet and u, its *final* velocity, its initial velocity here being *zero*, since all the constituents are carried on the aircraft itself. Its *efficiency* thus works out to $2Vu/(u^2+V^2)$, which again, as in the case of jet propulsion, will have the maximum value 1, when $u = V$.

Further, the power-unit, in the case of the rocket plane, is also much lighter. Thus, while, for supplying a thrust of 1 *lb.* at ground level, or of 0·2 *lb.* at an altitude of 50,000 ft. a turbo-jet unit weighs about 0·3 *lb.*, a rocket-unit, weighing only 0·1 *lb.* can supply the same thrust of 1 *lb. at all altitudes.* It follows, therefore, that a rocket plane is the more suitable for use only for flights of *short duration* or at *very high speeds.*

172. Rocket Fuel. The rocket-unit, in which both the fuel and the required amount of oxygen for its consumption are carried on the aircraft itself, is much simpler than that other one which requires the compression of large quantities of air. For in this case, the only problems with which we are concerned are those of the combustion chamber and the propelling jet.

The oxygen may be carried either in the liquid form, or in the form of oxidisers rich in oxygen, like hydrogen peroxide, (H_2O_2) or nitric acid (HNO_3). In the latter case, the remaining part of the oxidiser, going into the propelling jet, merely serves to cool the jet and the combustion chamber.

Now, if the fuel and the oxidiser are carried in separate containers, the system is known as a *bipropellant rocket*, but if the fuel contains its own oxidiser with it and is carried in a simple container, it is known as a *monopropellant* and its decomposition is brought about either by the application of heat or through the agency of a catalyst. Obviously, a monopropellant must be some sort of an explosive and, therefore, requires careful handling. Quite a common one being hydrogen peroxide, which decomposes as shown by the equation

$$2H_2O_2 \longrightarrow 2H_2O+O_2+690 \ C.H.U. \ lb.*$$

*1 *C.H.U.* (Centigrade heat unit) is the heat required to raise the temperature of 1 *lb.* of water through 1°*C.*

The products of decomposition of a monopropellant substance are sometimes themselves rich in oxygen, as we can see in this case of H_2O_2, and the substance can, therefore, also be used as one of the components of a bipropellant. Thus, for example, H_2O_2, used with methyl alcohol (CH_3OH) would react as shown :

$$CH_3OH+3H_2O_2 \longrightarrow CO_2+5H_2O+1702 \text{ C.H.U./lb.},$$

releasing heat at a much higher rate and hence resulting in a much higher exit velocity of the gases through the exhaust nozzle or the *venturi*, as it is called, and consequently a *much higher thrust*.

The propellant may be injected into the combustion chamber in one of the two ways : (*i*) by exerting pressure by a compressed gas, like nitrogen or air on the propellant tanks or (*ii*) by means of a pumping mechanism, usually a turbine. The former method admits of no variation or control of thrust and is, therefore, suitable only for short-duration flights or remotely controlled missiles, and the latter is the one commonly used for rocket-propelled air-crafts.

173. Specific Impulse. The performance of a rocket motor is measured in terms of what is called the *specific impulse* or the *specific pull, I*, which is *the thrust generated by unit rate of fuel-consumption, i.e.*,

$$I = \frac{\text{thrust (lbs.)}}{\text{rate of fuel consumption (lbs./sec.)}} = \frac{F}{mg}.$$

So that, the dimensions of I are the same as those of time. Physically, therefore, it is *the time for which a unit thrust can be generated by a unit weight of fuel.*

Now, as we have seen, the thrust in the case of a rocket is equal to mu, where m is the *mass-flow* through the nozzle and u, the *exhaust velocity* of the gases. So that,

$$I = mu/mg = u/g.$$

And, therefore, *the higher the jet-velocity, the higher the specific impulse and the smaller the fuel-consumption for a given thrust.*

Besides fuel-consumption and thrust, there are quite or few other factors which determine the suitability of various fuels, *e.g.*, the weight of the engine, the temperature in the combustion chamber etc., etc. In the modern rocket motors, the total weight of the pump, control and installation etc. must be about one-tenth of the maximum thrust developed. In short, the performance of a rocket depends chiefly upon three factors, (*i*) *jet velocity*, (*ii*) *density of the propellant* and (*iii*) *weight of the power plant*, which includes that of the propellant tanks and the fuel-supply system etc., into details of which we need not enter in an elementary discussion of the type we are concerned with here.

174. Shape of the Rocket. During an upward flight, particularly, through the denser layers of the atmosphere, the components of the rocket are subjected to intense air pressure, and also a lot of heat is produced due to viscous friction of the air. Both these factors are taken into account while designing a rocket. Its frame is accordingly made of a heat-resisting material and its velocity during the first part of its flight, through the denser layers of the air, kept sufficiently low. Further, it is so designed as to reduce the air pressure

on its each individual part to the very minimum, its over-all shape being more or less like that of a cigar.

175. The Multi-stage Rocket. If a rocket is hurled into space beyond the earth's gravitational field then, supposing that its acceleration takes place in the latter region, where the value of g is 32 *ft./sec*^{2}., the velocity V_0 that it must acquire to escape from the earth's gravitational field or the '*escape velocity*', as it is called, is given by the relation $V_0^2 = 2MG/R$, (*see* § 92, page 251) from which the value of V_0 works out to about $11{\cdot}19 \times 10^5$ *cms./sec.* or about 36000 *ft./sec.*

Now, at the present stage of rocket development, no single rocket can achieve this velocity. To tide over the difficulty, therefore, we make use of what is called a *multi-stage rocket*, which is just a combination of rockets, either (*i*) *joined consecutively or in series*, as it were, or (*ii*) *one inside the other* or (*iii*) *with the rear part of one inside the nozzle of the other*, as indicated diagrammatically in Fig. 239. In all these three types, the first stage rocket is the largest in both dimensions and weight, and the last stage one, the smallest.

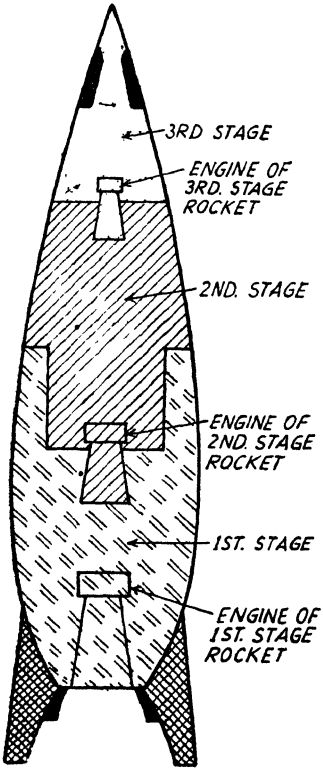


Fig. 239.

Naturally, the first stage rocket is used first and when it has done its job, it gets detached and is discarded, with the second stage rocket taking over the task of producing further acceleration. Then, this too is discarded and the third stage rocket takes over and so on. The velocity thus goes on increasing at each stage *by the same amount* as it does in a single stage rocket and each stage has its own propulsion and control system. Obviously enough, the fuel-consumption and the thrust for the first stage rocket are the highest of all, say about a hundred times the corresponding values for the third stage rocket ; and the fuel-stock too in the first stage is about sixty times that in the third stage, the same being the ratio of the total weights carried by the former to that by the latter.

Considerations of both weight and cost demand that the number of stages should not be large and that, therefore, the pay-load of each stage (which includes, in addition to the useful load of the final stage, the weights of the intervening stage rockets to be discarded later) be limited to about 20% of its own weight. Clearly, the useful pay load of the final stage thus works out to be a very small fraction of the initial over-all weight. Thus, for example, if there be n stages in all, this fraction is just $1/5^n$ of the initial total weight. Or,

to give a more concrete example, a space ship of the size of the well-known *V*2. designed by Dr. Verner Von Braun, would be about just sufficient to land a match box or a packet of cigarettes, by means of a purachute, on the planet Mars.

Each individual rocket of the multi-stage rocket, has its own independent design and basic characteristics, with its function correlated with those of the others. These characteristics include the following :

(*i*) **Net weight.** The net weight for a single stage rocket includes also the weight of the instruments and appliances or the weight of ammunition, if any, etc. And, in the case of a multi-stage rocket, obviously, the total weight of the second stage is the net weight of the first stage and the total weight of the third stage, the net weight of the second and so on, the ratio between the two being usually for 3 : 1 for each stage.

(*ii*) **Steering Equipment.** This is necessary to steer the course of the rocket during its flight during the other stages except the first which only serves as a sort of runway for the rocket, as it were.

(*iii*) **Design.** This includes the frame of the rocket or of the individual rockets, in the case of a multi-stage rocket, with its fortifications and fastenings etc.

(*iv*) **Rocket-length.** This obviously means the height of the rocket or that of the individual rockets of the multi-stage one. This is an important factor in as much as *the very stability of the rocket in its trajectory depends upon the ratio between its length and its mean diameter* (*i e.*, the mean diameter of the whole rocket or of each one of the stage-rockets)

(*v*) **Number of Motors.** Each stage rocket has its own separate motors. The first stage rocket, naturally, in view of the highest total weight it has to carry and the greatest resistance of the lower denser layers of air it has to overcome, has more than one motor and the last stage rocket, because of its lightest load and the least resistance to be overcome, is provided with only one motor.

Apart from these, there are also other characteristics of a rocket, like its *fuel-consumption, thrust, specific pull or impulse, time of combustion* (in seconds), *acceleration, lift* or *range* etc.

176. Take off of the rocket. This is perhaps the most important part in the flight of a rocket and must be fully ensured to be correct. The slightest error in the timing or the accuracy of firing makes all the difference between the rocket returning back in this generation or the next or perhaps not at all.

177 Salvaging the various stage rockets. Let us wind up our elementary study of a rocket flight with a word about salvaging the various stage rockets which are discarded after they have performed their respective functions. This problem cannot yet be said to have been satisfactorily solved. Experiments are, however, being made with various systems of parachutes and other devices and, if they succeed, it will mean a tremendous economy in cost. And for all one knows, the ideal solution may turn out to be the utilisation of the material of the stage used up as fuel for the next stage.

178. Satellites. Among celestial bodies, a satellite is what may be called a minor or a junior member of the solar system revolving round one of the major planets in its own prescribed orbit. Till recently, it was not thought possible that anything man-made could also be so placed round the earth or any other major planet to revolve in a given orbit. But, then, with the development of jet-propulsion (in the year 1940), followed by that of high speed rockets, man began to dream of

flight into space and of inter-planetary travel, when, all of a sudden, on October 4, 1957, the Russian scientists made the whole world gasp with wonder and surprise by launching their first *'sputnik'* or *artificial satellite*. This *Sputnik I*, of the form of a ball, 58 *cms.* in diameter and weighing 83·6 kilogrammes (roughly 185 *lbs.*) was placed into an elliptical trajectory round the earth—exactly like a celestial satellite —making, in its initial phase, one full revolution in 96·2 minutes and attaining a speed of 8 *km.* or nearly 5 *miles/sec.* at a distance of 950 *km.* from the earth.

The progress of this latest wonder was watched with dumb admiration by scientists all over the globe and the radio signals sent out by it listened to attentively as long as its source of power lasted. It existed *as a satellite* for full 58 days, during which it made 1400 revolutions of the earth, thus covering a distance of 39 million kilometres. Its existence, however, continued for 92 days and the entire distance covered by it totalled up to the enormous figure of 60 million kilometres, when, finally, on January 4, 1958, it entered the denser layers of the atmosphere and got burnt out due to the intense heat produced by friction.

This artificial satellite was obviously an *automatic rocket*, hurled into its pre-determined and well-calculated orbit by a multi-stage rocket. Indeed, the rocket carrier too continued to revolve round the earth at about the same height as the sputnik but at a distance of about a thousand kilometres from it ; and, then, while descending through the denser layers of the atmosphere, it also began to burn out, with fragments from it falling somewhere in *Alaska* and *North America.*

After almost exactly a month, on November 3, 1957, the Russians put their second artificial satellite *'Sputnik II'* into orbit round the earth, containing scientific equipment for exploratory purposes, as well as the first space traveller, the dog *'Laika'*, in a sealed cabin, which they successfully retrieved back, safe and sound. The total weight of the Sputnik was this time much greater, being 508·3 *kgms.* or 1126 *lbs.* (including the dog). Its distance from the earth was also greater, 1700 *kms.*, its period of revolution, 102 *secs.*, with the angle of tilt of its orbit roughly 65° from the equatorial plane.

This was followed by the first American artificial satellite, *'The Explorer'*, on January 31, 1958, though of a comparatively much smaller weight and size.

These sensational events brought still more sensational and breath-taking ones in their wake, with the Russians putting the first cosmonaut of the immortalised name, *Major Yuri Gagarin*, into space in a much larger space-vehicle or space-ship and retrieving him back, with the Americans later repeating the performance. The race in still on in right earnest and who knows what greater wonders yet are in store for us.

Let us try to understand the basic principles underlying this phenomenon.

179. Conditions for a satellite to be placed in orbit. It is obvious that an artificial satellite goes round the earth exactly as a celestial

satellite goes round a planet, as the moon, which, for all practical purposes, is a satellite of the earth, goes round it, or as the earth and the other planets go round the sun, *i.e.*, in accordance with the laws, first enunciated by *Kepler*, leading to Newton's celebrated Law of Gravitation, which forms the basis of the entire celestial mechanics.

The student is quite familiar with the whirling motion of a stone, tied to one end of a string, the other end of which is held in the hand. Precisely similar is the case with a planet going round the sun or an artificial satellite going round the earth, with the force of gravitational attraction replacing the tension in the string. There is, however, one fundamental difference between the two, *viz.*, that whereas the tension in the string is, within limits, a variable quantity, permitting a lower or a higher velocity of the stone, *the attractive force of the earth on the satellite in a specific quantity and thus permits only a specific velocity for the satellite, if it is to remain in orbit*, —this velocity for a satellite close to the earth being, as mentioned already, about 8 *kms.* or 5 miles per second. Since, however, the gravitational force decreases with increase of distance from the centre of the earth, a satellite further away from the earth will need a smaller velocity to remain in its orbit than the one nearer to the earth, though up to about a 1000 *kms.* above the earth's surface, this reduction in velocity is only nominal. This is clear from the fact that the moon, which is roughly 38000 *kms.* away from the earth and, therefore, moves in a much larger orbit, has only a velocity of about 1 *km./sec.*, which is about one-eighth of a satellite close to the earth, so that, whereas the moon makes only one revolution of the earth in one month, the satellite makes as many as 15 revolutions in one day.

Now, the question is how to have the satellite with such a high velocity away from the earth, to enable it to go into orbit around it. As can be seen, not only has the opposing gravitational force to be overcome but also the very considerable air resistance, particularly in the lower denser part of it. As we have seen above, the least velocity for the purpose is about 8 *km.* or 5 *miles/sec.*, called the *first cosmic velocity*. But, if the velocity rises to about 11·2 *km./sec.*, called the *second cosmic velocity* or the *velocity of escape*, the satellite passes right out of the earth's gravitational field and flies away into the cosmos, within the range of the solar system.

This formidable problem, can, as mentioned earlier, be easily solved by carrying the satellite on a multi-stage rocket, for no single rocket can possibly (at any rate, not yet) achieve the requisite velocity all by itself alone. We have already discussed the essential features of such a rocket in § 175. Let us now see how exactly to launch the rocket, carrying the satellite, into the required orbit.

180. Launching of the Satellite. Apparently, the shortest route for the satellite to take from the launching base to its assigned orbit would be the vertical one. This, however, is not feasible in actual practice, for the simple reason that the gravitational pull of the earth will then be in the directly opposite direction to its motion and counteract the pull of the engines. So that, before it can gather the necessary speed, its limited fuel-stock may get exhausted, resulting in its first coming to a stop and then starting falling down. Vertical launching of the satellite is, therefore, not a practicable proposition.

To ensure that the satellite does not fall back to the earth, it is essential to give it a sufficient horizontal velocity. Its upward flight is, therefore, so arranged that it is brought into its orbit in the shortest possible time, acquiring meanwhile the requisite horizontal velocity. It is thus clear that the particular trajectory that will take the satellite to its assigned orbit has first to be most carefully calculated.

It is usual to arrange the first portion of the flight of the rocket to be vertical, so that it may pass through the first 20 *kms.* of the denser portion of the atmosphere the earliest. Thereafter, as it enters the rarefied portions of the atmosphere, it is given a gradual tilt by means of a mechanical pilot, so that it emerges into its orbit with a horizontal velocity large enough for the centrifugal force coming into play on it, (on account of its circular motion), to just balance the force due to the gravitational pull. And, the trajectory of its path is so chosen that the loss of velocity entailed, due to air-resistance and the earth's pull, is a small percentage of its required or characteristic velocity. In fact, to make up for this loss, the actual velocity given to it is a little higher than the computed value of its characteristic velocity. When launched laterally to the earth's rotation, however, an increase in its velocity is automatically obtained at the expense of the velocity of the earth's rotation, depending upon the latitude of the launching site. Thus, for example, this increase is the maximum at the equator, being as much as 400 *metres/sec.*, which is higher than that of the fastest fighter planes of the day.

If it be desired to give the satellite an elliptical orbit, instead of a circular one, the rocket carrying it must either be given a higher velocity than the perepheral one or its velocity, immediately after completion of the motor's performance, must not be directed along the tangent to the circular orbit. In the elliptical orbit, the point nearest to the earth is called '*perihe*' and the farthest from it, the '*acme*'. And it is quite possible that the satellite at the former point may be nearer to, and at the latter, farther from, the earth than at any point in its circular path.

In any case, the accuracy demanded in the firing of the rocket into its correct orbital path is really exacting. For, even an error of 1% in the direction of velocity may produce a height variation of the *perihe* and the *acme* which may be as much as 120 *kms.* or more. This firing accuracy is secured by means of proper steering devices, directing the course of the rocket at every stage of its flight. And, clearly, rudders of the type used in the ordinary jet air-craft, are hardly suitable for the purpose, since they cannot possibly function equally effectively both in the denser and the rarefied regions of the atmosphere.

The necessary steering control can, however, be effected in a number of ways but the one usually resorted to is to so design the rocket as to enable it to change the direction of the escaping jet by a mere tilt of the longitudinal axis of its motor with respect to its own. This is actually the device adopted in most of the present-day long-range rockets.

The manner in which the angle of inclination of the longitudi-

nal axis of the rocket with the horizon is varied, will be clear from Fig. 240. As will be readily seen, the trajectory of the rocket from

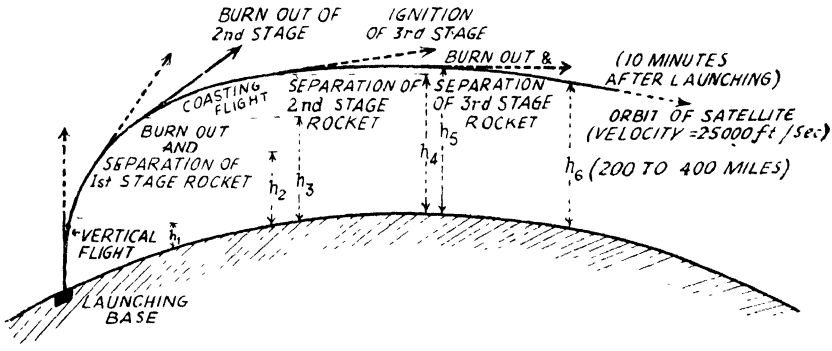


Fig. 240.

its very start until its longitudinal axis takes up the horizontal position, (*i.e.*, until its outward motion towards its orbit) is split up into a number of stages, indicated by h_1, h_2, h_3 etc., depending upon the height of the orbit. The angles that its horizontal axis makes with the horizon at each stage is carefully, calculated before hand and the control instruments set accordingly, to ensure that the rocket takes its assigned trajectory. And, this very setting of the instruments also regulates the fuel-supply in keeping with the predetermined requirement at the time.

Now, it will be easily understood that, while going round in its allotted orbit, the satellite passes over different parts of the globe in its successive rounds. For, by the time it has completed one round, the earth has also rotated about its axis and hence, in its next round, it naturally passes over other parts that now fall below its orbit. This will always be so except when the satellite goes round an orbit coinciding with the equatorial plane, in which case, obviously, it will always pass over the same parts or countries situated at the equator. It does not mean that we can launch the satellite in any orbit we choose. For, *the orbit must be one such that its plane passes through the centre of the earth and it will, therefore, clearly depend upon the site of launching.*

Not only that, but *even the time of the day and the season at the time of launching matter a great deal.* For, a satellite receives energy direct from the sun through special type of solar batteries fitted into it, a particular side of which must all along be illuminated by the sun. The satellite must, therefore, be launched in an orbit, the plane of which is perpendicular to the rays of the sun, and this is possible only at a particular hour of the day, *viz.*, when the radius of the earth connecting the starting point of the satellite with its centre is perpendicular to the sun's rays. And, the season is important because, with the satellite launched in its orbit, as explained, the earth which also moves round the sun, comes in-between it and the sun at a particular time, thus preventing the rays of the sun from reaching it. Account has, therefore, to be taken of this occurrence and the season of launching chosen such that the satellite

can get the maximum time to store up enough energy from the sun to suffice for the period when the sun will remain hidden from it later during its flight.

And, finally, it must also be clearly understood that in view of the uneven distribution of the mass of the earth and, therefore, with its *c.g.* some 500 *km.* away from its geometric centre, the satellite in its orbit is subjected to varying forces of attraction at different intervals, with the result that its real course is neither circular nor elliptical. It does not even lie in either of the two planes and is, in fact, a curve of a complicated pattern. For the same reason, there are variations in the velocity of the satellite at different points along its path.

181. Stability of the rocket during flight. It is imperative that throughout its flight, along its allotted trajectory, the rocket should not get tilted. This is achieved by means of an auto-pilot (see § 46) and a suitable gyroscopic arrangement.

182. Form of the Satellite. In designing a satellite, attention is naturally paid to the geometrical shape it should be given to ensure its smooth motion in its orbit. The present view is that this form should be spherical, for, then, it will always have the same area of resistance and thus help calculation of the air resistance to its motion at higher altitudes and hence in the assessment of the density of air at those altitudes. Further, with a spherical shape, there will be less chances of its getting overturned than if it were cylindrical or of any other shape. At the same time, a spherical shape is also a drawback, since it does not make for an easy setting of the various instruments and other equipment inside it For, as will be easily realised, the instruments must be set, not haphazardly but in a definite order so as to ensure both an equitable distribution of the total weight inside the satellite and a specific position of its *c.g.* This *'balancing'* of the satellite, as it is called, is obviously important and must be done with great precision.

183. Weight and size of the Satellite. The weight of a satellite clearly depends essentially on the potentialities of the rocket carrier, and its dimensions, upon those of the last stage rocket, which is usually the third stage one.

The satellite which gets detached from the last stage rocket may not necessarily be included as part of the rocket itself and may simply be arranged to lie inside a cavity in the nose-part of it. In such a case, it is possible to give the satellite a bigger diameter than the mean diameter of the rocket, as a whole, but only slightly so, or else it will mean a change in the ballistic characteristics of the rocket as also an increase in the air-resistance encountered. The satellite in the cavity is sometimes covered by a protective streamline cone, during the course of the flight of the rocket, which is later discarded and the satellite pushed out by means of a spring or a compressed gas, when the rocket has actually reached the orbit in which the satellite is intended to move. This was exactly the case with Russian *Sputnik I,* whereas *Sputnik II* formed part of the third stage rocket itself and did not get detached from it.

184. Material of the frame of the satellite. Obviously, the material

of the satellite frame must be both *light* and *strong*,—the former from considerations of its weight and the latter, to make sure that the instruments etc. inside it are securely attached and that it can withstand the onslaught of micrometeorites to which it is subjected during its orbital motion in space. Then, the material must also be less sensitive to changes of temperature and must be able to properly reflect radio-waves. It must, therefore, be either *aluminium, magnesium* or one of their alloys, with, in some cases, a suitable outer covering.

If, however, it is desired to study the electric currents in the ionosphere, the frame of the satellite should neither be a conductor of electricity nor should it possess any magnetic properties. So that, in this case, a metallic frame is clearly ruled out in favour of one of a plastic material, some of the modern varieties of which are just as tough and durable as steel.

185. Duration of satellite's existence. It is only natural to enquire as to how long can a satellite be expected to stay in its orbit. Well, if the space in which it moves along its orbital path were completely devoid of air, there would be nothing to stop it and it could go on perpetually, like the moon, for instance. But there being air even at a height of 1000 *kms.* and above, it has to encounter resistance due to it, however small,—this resistance being greater for orbital paths nearer the earth than further away from it. So that, when its velocity is thus sufficiently retarded, it cannot possibly remain in its orbit and starts falling down along a spiral path. In doing so, it either gets burnt up due to the heat produced by friction in the denser atmosphere or drops down to the earth with the help of parachutes.

The actual calculation for its 'life' is rather a complicated one, but it basically depends upon the density of the upper regions of the atmosphere, *i.e.*, on the height of its orbit from the earth.

186. Other Essentials. In case a man is to be placed in the artificial satellite or the sputnik, there are quite a few other problems to be tackled, as, for example, provision of an hermetically sealed cabin, with requisite conditions for the sustenance of life, and with windows fitted with the type of glass that absorbs ultraviolet and X-rays, a prolonged exposure to which is harmful in its effects. It is, however, almost impossible to afford any protection to the cosmonaut inside the cabin against cosmic rays which, as we know, can penetrate even through a block of lead, one metre thick. Luckily, although their effect on human or animal life is yet not quite clear, they do not appear to produce any baneful effects. Then, there are other problems, like those of weightlessness etc. All four have now been more or less overcome, as is evidenced by four Astronauts, —two Russian and two American—having made orbital flights and come safely back to the earth.

Another very essential item is the special type of dress that an astronaut must wear during his voyage in the cosmos. This is fittingly called the **Astrosuit** and must at once be *air-tight* and *loose-fit* to allow free respiration. In fact, the astronaut needs one type of dress during the take-off of the rocket from the earth, which

must be so designed as to free him from the feeling of overload, experienced during a vertical ascent. Then, he should have another lighter dress for free locomotion inside the cabin. A small cylinder, provided in the girdle of this dress, creates an artificial pressure on him to increase his blood pressure (which falls appreciably at greater heights) as also to counteract weightlessness. The dress should have provision to ensure normal respiration and the requisite body temperature and must not restrict movement.

187 Return of Artificial Satellite. For the return of the satellite back to the earth, the main problem is of sufficiently slowing down its motion or braking it. There are two devices for it, *viz.*, (*i*) *utilising air as the resisting medium and* (*ii*) *using rockets.*

If air-resistance is to be used for slowing down the motion of the satellite on its journey back, it must be given the form of a rocket. For, then, as it enters the denser layers of the air, its velocity falls but it rebounds back into the cosmos ; it then re-enters the air with a reduced velocity and goes a little deeper than before and there is a further reduction in its velocity. This process is repeated a few times and the velocity of the satellite is thus sufficiently reduced to enable it to continue falling on specially provided slide-wings and sliding planes.

On the other hand, if a rocket is to be used for its downward journey, an automatically-controlled rocket-motor is necessary, the reaction of which is in the opposite direction to that of the motion of the satellite, *i.e.*, it produces an effect opposite to that of the rocket carrier during upward flight. It is, therefore, called a *retro-rocket*, (*i.e.*, a rocket, taking the satellite back). The velocity of the satellite is thus reduced and can be controlled by regulating the fuel-supply to the rocket motor, the distance it thus has to cover up to the landing strip being carefully estimated with the help of a radar or other similar appliances. And, an automatic guidance system is provided to control and manoeuvre the downward descent of the satellite.

Now, the first method is certainly the simpler of the two, from the technical point of view, but its great handicap is that it is extremely difficult to design a landing strip to receive the landing satellite. The second method, although more complicated technically, ensures a smooth and an accurate landing on a properly constructed landing strip.

188. Uses of an Artificial Satellite. Ignoring the military uses to which a satellite may be put, we shall concern ourselves here only with its uses for strictly scientific purposes, among the more important ones of which may be mentioned the following :

(*i*) *Proper study of the upper regions of the atmosphere.* Despite the fact that the atmosphere is being studied for a long enough time, our present knowledge of it is still much too meagre and superficial, particularly about the region, called the *Ionosphere*, as also about cosmic rays. The artificial satellites will, it is hoped, help to improve this.

(*ii*) *Weather forecasting.* This can be made much more accurate and dependable with a number of satellites around the earth in various orbits. Meteorological observations over various countries could then be made simultaneously, thereby greatly improving the reliability of weather forecasts.

(*iii*) *Determination of the exact shape and dimensions of the earth.* This is the task that scientists all over the earth have set for themselves during the third International Geo-physical year.

(*iv*) *Detailed study of the solar radiation.*

(*v*) *Study of meteorites.*

(*vi*) *Experimental verification of the theory of relativity.*

(*vii*) *Use of a system of three artificial satellites for universal telecasting.*

(*viii*) *Study of propagation-characteristics of radio waves in the upper regions of the atmosphere.*

(*ix*) *Astronomical observations, without atmospheric and other disturbances etc., etc.,*

EXERCISE X

1. Explain clearly the principle underlying an airplane. How does it remain in equilibrium in air and how does it rise up ?

2. Differentiate between *climbing*, *diving* and *gliding* of an airplane and explain the co-relation of forces in each case.

3. Name the principal parts of an airplane and mention concisely but clearly their respective functions.

4. What do you understand by the term '*jet-propulsion*' ? Give, in brief, an account of jet-propelled planes.

5. What is a *rocket* ? How do *rocket-planes* differ from *et-planes* ? Explain the principle underlying a multi-stage rocket.

6. What is an *artificial satellite* ? Explain as clearly as you can how a satellite may be placed in its orbit around the earth.

7. Mention the essential pre-requisites and conditions for a satellite to be placed in its orbit and its return back to the earth.

Also mention some scientific uses of an artificial satellite.

CHAPTER XI

FRICTION AND LUBRICATION—PRINCIPLE OF VIRTUAL WORK AND ITS SIMPLE APPLICATIONS

189. Static Friction—Laws of Friction. In all cases of motion of material bodies, counter forces come into play in-between their surfaces which tend to nullify or neutralise the effect of the driving force applied. These counter forces are called *resistances*, the most important among which is *friction*. As we have already learnt in our junior classes, when one solid body is sought to be moved over the surface of another on which it rests, an opposing force, called the *'force of friction'*, comes into play in-between the two surfaces, tending to destroy the relative motion between them,* and which is usually measured by the force required to produce *uniform* relative motion between the two surfaces. It is this force which always acts in a direction opposite to that in which the motion is desired and which is called the *force of friction* or rather *static friction*†. Experiment shows that friction roughly obeys the following laws, called the *'laws of friction'*, discovered by **Amontons** (1699) and **Coulomb** (1779) and hence sometimes referred to as **Coulomb's laws** :

(*i*) *The frictional force is a self-adjusting force and increases with the applied force, so as to be equal and opposite to it, until motion is just about to ensue ;—this maximum frictional force is called the* **'limiting friction'** *and its value is different for different pairs of surfaces.*

Before this limiting value of friction is reached, its magnitude is just enough to preserve equilibrium.‡

(*ii*) *The limiting friction between the surfaces of two bodies is directly proportional to the normal reaction of the supporting surface.*

Thus, if R be the *normal reaction* of the supporting surface and F, the *limiting friction* set up between the two surfaces, we have

$$F \propto R, \qquad \text{Or,} \qquad F/R = \mu,$$

where μ is a constant, called the **'Static Coefficient of Friction'** or simply the *coefficient of friction* for the given pair of surfaces.

i.e., $$\textit{static coefficient of friction} = \frac{\textit{limiting friction}}{\textit{normal reaction}}.$$

*Strictly speaking, this is not the only force that opposes the relative motion between the two surfaces. There is also another force, called the *'force of adhesion'*, (from the Latin word *'adhaerere'—to stick* or *to cling*, which is molecular in origin and which tends to make the bodies cling together.

†*'Static'*, because the two surfaces are initially at rest with respect to each other.

‡It will be readily seen that this really follows from *Newton's third law of motion.*

Thus, obviously, $F = \mu R$; so that, if $R = 1$, we have $F = \mu$, *i.e.*, the *coefficient of friction for a given pair of surfaces may be defined as the limiting friction coming into play in-between them, for unit normal force applied to them, or, as the fraction of the normal force that is required to keep the two surfaces in uniform relative motion.*

(*iii*) *The frictional force is independent of the surface areas in contact with each other* and of their relative velocities.*

(*iv*) *The frictional force is independent of the relative velocities of the two surfaces.*

It may be pointed out here that these laws apply only in the case of clean and smooth or well-polished surfaces.

190. Sliding Friction. We have already seen how the force of friction between the surfaces of two bodies, one resting over the other, continues to increase with the applied force and is always equal and opposite to it until its maximum or limiting value is reached for that particular pair of surfaces.

If at this stage, *viz.*, when the friction is about to attain its limiting value, we apply a force in the form of a gentle push to the body resting over the other, such that it is maintained in *uniform motion* over the latter, *then, this force* measures what is called the *sliding friction* between their surfaces, *i.e.*, *the frictional force in-between them when motion ensues* It is also spoken of as *kinetic* or *dynamic* friction, to distinguish it from *static* friction (that comes into play before motion actually takes place) and is found to be somewhat less than the *limiting friction* for the same pair of surfaces. That is why we find it easier to maintain a body in uniform motion over the surface of another than to start it moving.

The ratio between *this* sliding friction and the normal reaction then gives the **coefficient of sliding friction** for the given pair of surfaces and is also obviously less than the coefficient of static friction for them. The difference between the two is however quite small and we usually assume them to be the same for all practical purposes.

191. Angle of Friction—Cone of Friction. If we place a body

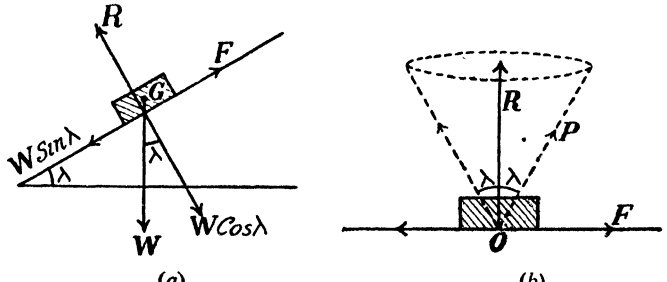


Fig. 241.

*This is no longer so, if what are called *lubricants*, like grease, graphite, talc etc., are introduced in-between the two surfaces. For, the normal force applied is more likely to squeeze out the lubricant from in-between the two surfaces, when applied to a small area than when applied to a larger area.

on an inclined plane, [Fig. 241 (*a*)], it is clear that it is under the simultaneous action of three forces, *viz.*,

(*i*) *its weight W, acting vertically downwards at its c.g., G* ;

(*ii*) *the normal reaction R of the plane, acting perpendicularly to the plane* ; *and*

(*iii*) *the limiting friction F, acting upwards along the plane.*

If the angle of inclination of the plane be so adjusted that the body is *just* on the point of sliding down, it is clear that the three forces are in equilibrium and, therefore, concurrent, so as to be represented by the three sides of a triangle, taken in order. This angle of inclination of the plane, λ, at which the body is just on the verge of sliding down, is called the *angle of friction.*

Resolving W into its two rectangular components, we have (*i*) component $W \sin \lambda$, along the plane, tending to move the body down the plane and component $W \cos \lambda$, at right angles to the plane.

Since the body is in equilibrium, we have

$$F = W \sin \lambda \quad \text{and} \quad R = W \cos \lambda.$$

So that,
$$\mu = \frac{F}{R} = \frac{W \sin \lambda}{W \cos \lambda} = \tan \lambda.$$

Or, *the coefficient of friction for a given pair of surfaces is equal to the tangent of the angle of friction for them.*

From the above relation for μ, we have $F = R \tan \lambda$, from which it follows that the resultant of F and R must lie along the surface of a *cone*, with λ as its semi-vertical angle, and the direction of the normal reaction, as its axis.

The same is true in the case of two horizontal surfaces, where the frictional force F acts along the supporting surface and the normal reaction R, perpendicular to it, [Fig. 241 (*b*)]. Their resultant P then makes with the latter the angle of friction λ and, again, therefore, the resultant (P) lies along a cone of semi-vertical angle λ, such that $\tan \lambda = F/R$.

This *cone* is called the **cone of friction**, and it is obvious that no force, whatever its magnitude, with its line of action lying *within* the cone, can possibly produce motion in the body, its component along the surface of contact being less than the limiting friction (F) between them.

192. Acceleration down an Inclined Plane. We have just seen in § 191, above, that a body placed on an inclined plane will not start sliding down along it until the angle of inclination θ of the plane is equal to the angle of friction λ for the surfaces of the body and the plane ; for, at a smaller angle of inclination than this, $\tan \theta < F/R$ or μ, where F is the limiting friction and μ, the *coefficient of friction* for the two surfaces in question. And, when $\theta = \lambda$, clearly $\tan \theta = \tan \lambda = F/R$ and sliding *just* commences.

But when $\theta > \lambda$, so that $\tan \theta > \tan \lambda$ and hence greater than F/R, the body slides down the plane with an accelerated motion. Let us calculate this acceleration of the body.

As before, resolving the weight of the body $W = mg$ (where m is

the mass of the body) into two rectangular components, along and perpendicular to the plane, we have

$$\textit{component along the plane} = mg \sin \theta.$$

and $\textit{component perpendicular to the plane} = mg \cos \theta.$

Since there is no motion perpendicular to the plane, the normal reaction R of the plane is equal to $mg \cos \theta$ and the two, *being equal and opposite*, neutralise each other. And, thus, the only two forces effective on the body are (*i*) $mg \sin \theta$, *downwards along the plane*, and (*ii*) *the sliding frictional force* F *upwards along it*. So that, the *resultant force acting on the body downwards along the plane* is equal to $mg \sin \theta - F$.

Now, if μ be the coefficient of sliding friction for the given pair of surfaces, we clearly have

$$F = \mu R = \mu\, mg \cos \theta.$$

So that, *the resultant force on the body downwards along the plane*

$$= mg \sin \theta - \mu\, mg \cos \theta = mg\,(\sin \theta - \mu \cos \theta).$$

And since *acceleration* = *force*/*mass*, we have

acceleration of the body downwards along the plane

$$= mg\,(\sin \theta - \mu \cos \theta)/m = g\,(\sin \theta - \mu \cos \theta).$$

If the plane be perfectly smooth, the acceleration of the body sliding down the plane would be $g \sin \theta$. Clearly, therefore, the acceleration of the body down the plane is reduced by $\mu g \cos \theta$ due to the frictional force between them.

193. Rolling Friction. The frictional forces between two surfaces when one *rolls* over the other is called *rolling friction* and is found to be much less than when sliding occurs between the same two surfaces. That is why vehicles are provided with wheels and their axles, with ball-bearings*, the latter converting the chief frictional loss of the wheel that occurs at the axle or the *journal* in the form of sliding friction,—here, called *journal friction*—into rolling friction.

It was shown by **Osborne Reynolds** that in rolling an appreciable amount of slipping or sliding of one surface over the other occurs and that *the frictional resistance to this slipping, or sliding, really constitutes rolling friction.* As extreme cases of this slipping between two rolling surfaces may be mentioned (*i*) an iron cylinder rolling over a plane rubber surface or (*ii*) a rubber cylinder rolling over a plane iron surface. In the former case, the cylinder covers a distance equal to only *nine-tenths* of its circumference in its one full turn and, in the latter case, a distance equal to eleven-tenths of its circumference ; so that, in either case, there is a slip of one-tenth of its circumference In ordinary cases too, some *slip* always occurs between two rolling surfaces, even when the two surfaces are of the same material.

It follows as a natural consequence that rolling friction between two *surfaces* would be *zero*, (*a*) when either the sliding friction between them is *zero, i.e , when* μ *for them is zero*, (*b*) or when no

*Here, there is a ring of small balls between the wheel, (pulley or disc etc.) and the axle, so that when the former rotates, the balls all roll also.

slipping occurs between them during rolling, *i.e., when μ for them is infinite.* For all other values of μ (between 0 and ∞) there must be friction and, for a particular value of μ, it must have its maximum value.

It may as well be mentioned here that *while lubrication of the surfaces always reduces the value of the coefficient of sliding friction (μ) for them, it may or may not reduce the rolling friction between them.* Thus, as is so well and so generally known, lubricating ball-bearings only results in increasing friction*.

194. Friction and Stability. When a body, say, a block of wood, rests on a plane horizontal surface, it does so because the weight

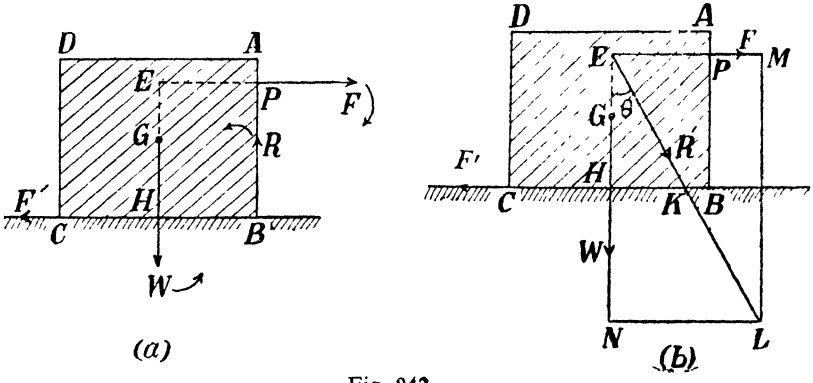


Fig. 242.

of the block W, acting vertically downwards at its *c.g.*, is just neutralised by the *equal and opposite* normal reaction R acting there. And, when a horizontal force F is applied to the block to move it forwards on the plane, it does not move or slide along it so long as $F < \mu R$, where μ is the coefficient of friction between the surfaces of the block and the plane, (see page 396). The possibility is, however, there that the block may topple over ; for, the moment the horizontal force F is applied, at a point P, say, a frictional force F', *equal and opposite to* F, comes into play in-between the surfaces of the block and the plane, [Fig. 242(*a*)], thus constituting a *couple* $= F \times PB$, tending to rotate the block (in the clockwise direction, in the case shown), and thus making it topple over.

Now, as F is gradually increased, this couple formed by F and F' makes the centre of reaction of the plane shift from H towards B†, with the force at C progressively decreasing and that at B, increasing, until, in the limiting case, the whole reaction R acts at B (that at C being zero). We thus have another couple, formed by W and R, equal to $W \times HB$, tending to rotate the block in the opposite direction to that due to the first couple (in this case, anti-clockwise) which thus tends to restore the block back to its original position. So long, therefore, as this restoring couple $W \times HB$, is greater than the couple

*This might raise a question in the mind of the student as to why then are they lubricated at all ? The simple answer is that it is done only with a view to reducing wear.

†For, with no force F acting on the block, its weight is uniformly distributed over its base B.

$F \times PB$, due to F and F', the block remains at rest and upright, *i.e., in stable equilibrium*, but if $W \times HB < F \times PB$, it topples over.

Let us now investigate the conditions under which the block will remain upright but slide along the plane. For this, let us first determine the resultant of F and W by the ordinary application of the parallelogram law of forces and then take the moments of this resultant R' and the frictional force F' about B. Let R' be represented by the diagonal EL of the parallelogram $EMLN$, with its adjacent sides EM and EN representing F and W respectively, and let it cut the plane in K, [Fig. 242(b)].

Now, clearly, moment of F' about B is *zero*, since its line of action passes through B. So that, so long as $HK < HB$, *i.e.*, so long as the resultant R' passes through the contour of the base of the block, it will have a restoring moment about B and the block will remain upright. But, if the angle θ made by R' with the vertical, or the direction of the normal reaction R, be greater than λ, the *angle of friction* for the surfaces of the block and the plane, (see § 191), it will *slide* along the surface. If however, $HK > HB$, *i.e.*, if the resultant R', of F and W, passes outside the contour of the base of the block, the block will topple over.

Another important case of stability due to friction is that of a fast moving vehicle on a curved track, discussed already in § 18 (4).

195. Friction, a Necessity. Taking most of our daily activities in life as a matter of course, we seldom care to pause and think as to how much they are dependent on the existence of friction. Thus, for example, in the absence of friction, we would find it impossible to walk or to drive on a road, and if we just start moving, we shall not be able to stop ; again, it would be impossible to climb a tree, tie a knot or even fix a nail in the wall. Brick would not stand on brick and buildings would tumble down like a house of cards and so on. Indeed, we find it so much of a necessity that we deliberately increase it for many of our purposes, as, for instance, when we apply brakes to our bicycles or cars.

In many other cases, on the other hand, we find friction so irksome ; as, for example, in the various parts of our machines, making their speed slower and their output lower and bringing about a greater wear and tear in them. And, yet, we know that friction is necessary even for them. What we do, therefore, is just to adopt ways and means of minimising it in such cases by means of oils and other lubricants, and ball-bearings etc. etc., (see page 404).

196. Simple Practical Applications of Friction—Rope Machines. Apart from the ordinary uses of friction, some of which have been mentioned above, there are various types of useful machines based on it. We shall consider here a couple of them by way of illustration of the principle underlying them.

1. **The Prony Brake.** This is a simple appliance to measure the power of machines, which we owe to *Baron G. C. F. Prony*, a French Mathematician, (1750–1839). It is in fact a *brake dynamometer* and consists of two wooden arms or, '*cheeks*', A and B (Fig. 243) in between which can be clamped the shaft of the machine whose power (*i.e.*, rate of working) it is desired to measure. The frictional force between the shaft and the cheeks is regulated by tightening or loosening the screws S and S, provided on the upper cheek A, to which is also attached a small rod R, about ·5 to 1·0 *metre* in length, carrying a scale pan at its other end.

Suppose the shaft of the machine turns in the anti-clockwise direction, as shown. Then, if there be no weights placed in the scale pan, *i.e.*, if the brake

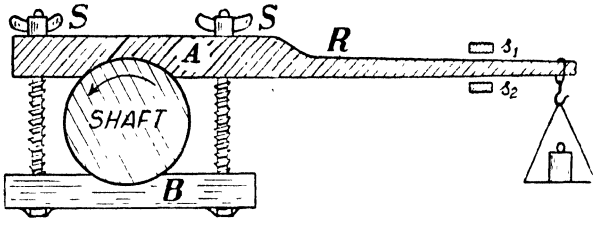


Fig. 243.

be unloaded, it will tend to be carried around with the shaft due to friction between itself and the shaft. But, if the brake be sufficiently loaded, before the shaft starts rotating, *i.e.*, before starting the machine, the moment due to the load may be enough to overcome the moment due to friction between the brake and the shaft and the brake may turn in the clockwise direction ; so that, it is quite possible to so adjust the weights W in the scale pan, *i.e.*, to so load the brake, that the rod R remains quite horizontal in-between and equidistant from the stops s_1 and s_2, placed on its two sides, a little distance away from it. When this is so, obviously, the frictional resistance between the brake and the shaft is equal to the force F exerted by the machine on the periphery of the shaft and is exactly balanced by the weights or the *load* W placed in the scale pan. So that, we have

moment of force F *(due to machine) on the shaft = moment of weight* W *on it.*

If r be the radius of the cross-section of the shaft and l, the length of the rod R, we clearly have

moment of F on the shaft $= F \times r$,

and moment of W ,, ,, ,, $= W \times l$.

Thus, $F \times r = W \times l$, whence, $F = W.l/r$.

If the shaft makes one full rotation in time T, we have

work done by the shaft, *i.e.*, by the machine, in time T

$= F \times$ *circumference of the shaft* $= F \times 2\pi r$.

And ∴ work done by the machine per unit time, *i.e.*, the *power of the machine*

$$= \frac{F \times 2\pi r}{T}.$$

Or, substituting the value of F, obtained above, we have

$$\textit{power of the machine} = \frac{W.l}{r} \times \frac{2\pi r}{T} = \frac{2\pi}{T}.W.l = \frac{2\pi}{T} \times \textit{moment of the load.}$$

Thus, knowing W, l and T, we can easily determine the power of the machine, which, as we can see, is clearly proportional to the moment of the load.

N.B. If W be taken in *dynes*, l in *cms.* and T in *secs.*, we get the power of the machine in *ergs per second.*

2. **The Rope Brake.** Before discussing any rope-machines, we must be clear in our minds as to what exactly is meant by a rope and what, if any, are the peculiar properties possessed by it that make for its usefulness.

A rope, then, is any flexible body or combination of flexible bodies, capable of transmitting tension. Thus, *the string of a violin, a piece of spring, a strap, a band* and *a chain* all come under the definition of a rope.

When one end of a rope is connected to a body and its other end pulled, stresses are caused at every cross-section of it,

in exactly the same manner as in a rigid rod. The stresses in neighbouring cross-sections neutralise each other and the pull is thus directly exerted on the body at the other end. *A rope can thus transmit a tension, undiminished in magnitude, from one end to the other.* Due to its flexibility, however, *it cannot transmit any compressional forces along it* ; but, at the same time, its flexibility is of great advantage in many ways and makes it particularly suitable as a means of transmitting tensional force, unaffected even by a change of direction*, as in the case of a pulley etc.

On the other hand, when coiled round a cylindrical body, a rope can exert a very large couple on the body, due to friction between itself and the body, so much so indeed, that a man pulling at one end may even hold a ship, fastened to its other end. Let us see how this comes about.

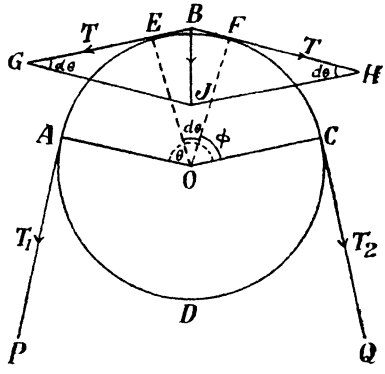


Fig. 244.

Let *ABCD* be the cross-section of the cylindrical surface, with its centre at *O* and let a rope *PABCQ*, coiled round it, leave its surface at points *A* and *C*, (Fig. 244).

Consider an infinitesimal portion *EF* of the rope at *B* and let the mean tension over this portion be *T*, with the angle subtended by it at *O* equal to $d\theta$.

Then, representing the tensions *T*, at *E* and *F*, by the tangents *EG* and *FH* respectively, we have their resultant force represented by *BJ* (by the simple application of the parallelogram law of forces), where *BJ* is clearly equal to $BG.d\theta = T.d\theta$, [$\because$ *BG* represents *T* and $\angle BGJ = d\theta$], in the direction *BJ*, normal to the section *EF* at *B*.

If, therefore, μ be the coefficient of friction for the rope and the cylindrical surface, we have

frictional force between the rope and the surface $= \mu.T.d\theta$†, in the direction of the rope at *B*.

Due to this frictional force, there comes about a change in the tension at the two ends of the rope, which, in the absence of any friction, would have been the same, (see above). In fact, the difference in the tensions at the two ends is just equal to this frictional force. So that, if *dT* be the difference in tensions at *E* and *F*, we have

$$dT = \mu.T.d\theta. \quad \text{Or,} \quad \frac{1}{T}.dT = \mu.d\theta,$$

*This is possible only so long as the body, (*e.g.*, the pulley or the ring etc.) over which the rope is passed does not interfere with its freedom of motion *i.e.*, is perfectly smooth and round.

†Because frictional force $= \mu \times$ *normal reaction R*, and here, $R = T.d\theta$.

integrating which, we have

$$\int \frac{1}{T}.dT = \int \mu.d\theta. \quad \text{Or,} \quad \log_e T = \mu\theta + K, \quad \ldots \quad \ldots(i)$$

where θ is the angle EOA and K, a constant of integration.

Clearly, if $\theta = 0$, $T = T_1$, the tension at A; so that, we have

$$\log_e T_1 = \mu \times 0 + K = K.$$

Substituting this value of K in expression (i) above, we have

$$\log_e T = \mu\theta + \log_e T_1. \quad \text{Or,} \quad \log_e T - \log_e T_1 = \mu\theta.$$

Or, $$\log_e T/T_1 = \mu\theta. \quad \text{Or,} \quad T/T_1 = e^{\mu\theta},$$

whence it is clear that the tension T increases as θ increases.

Obviously, therefore, if ϕ be the total angle AOC, and T_2, the tension at C, we have

$$T_2/T_1 = e^{\mu\phi}.$$

It is thus clear that if ϕ be large, *i.e*, if the rope be coiled many times round the cylinder, T_2/T_1 is also very large and a small tension applied at P may be made to exert a large pull at Q.

(3) **The Band Brake.** The principle, discussed in § 196, (2) above, is made use of in many forms of *brakes*, for the measurement of the power (or the rate of doing work) of machines, one of the simplest of which is the *Band brake*.

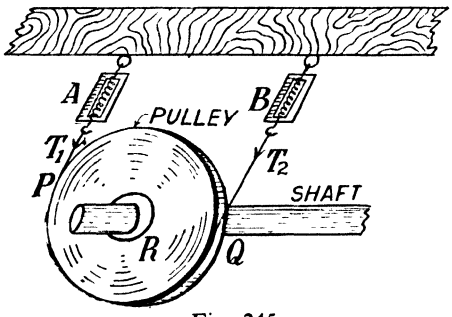


Fig. 245.

It is a simple device, consisting of a pulley fixed on to the rotating shaft of the machine whose power is to be determined. A band (or a cord) is passed round the pulley, once, twice or thrice etc., and has its two ends attached to two spring balances A and B suspended from a rigid support, as shown in Fig. 245.

Then, if T_1 and T_2 be the readings in the two spring balances respectively, we have

couple exerted on the pulley due to friction $= (T_2 - T_1).R$,

where R is the radius of the pulley.

Now, *work done by or against a couple = couple × angle of rotation* (in radians).

And, therefore, *work done against a couple per second*

$=$ *couple × angle turned through in one second.*

Or, *rate of doing work, i.e., power = couple × angular velocity.*

Thus, *work done per second by the machine against the couple due to friction* $= (T_2 - T_1).R \times 2\pi n$,

where n is the number of rotations made by the *shaft* or the axle of the pulley per second and hence $2\pi n$, the *angular velocity of the shaft.*

This, then, gives the work done per second by the machine, or the power of the machine. Thus, we have

$$\textit{power of the machine} = (T_2 - T_1).R \times 2\pi n.$$

Thus, knowing T_1 and T_2 (from the two spring balances), R and n, we can easily calculate out the *power* of the machine.

Further, we have the relation $T_2/T_1 = e^{\mu\theta}$,

where μ is the coefficient of friction for the cord or band in question and the surface of the pulley and θ, equal to π, 3π, or 5π, etc. according to the number of times the cord or the band passes round the pulley. So that, knowing T_1, T_2 and θ, we can also easily calculate out the value of μ.

197. Mechanism of Friction. Since the laws of solid friction were enunciated by *Amontons* and *Coulomb*, much has come to be known as to the how and why of friction, thanks to the work of **Hardy, Bowden** and others.

According to *Dr. Bowden's* lucid exposition, the mechanism of friction may be summed up as follows :

The smoothest or even the most polished surfaces are really not smooth enough, having projections and depressions of larger than molecular dimensions on them ; so that, the area of the two surfaces, *in actual contact* is much smaller than the *apparent* one we see to be so,—being less than ten-thousandth part of the apparent area of contact, in the case of plane steel surfaces.

It is possible *now* to form a fairly correct estimate of the actual area of contact between two metals by measuring their electrical resistance, and it is found to be *practically independent of the size and roughness of the surfaces and to depend only upon the load* The obvious inference is that the projections on the two surfaces get shorn off or crushed down under the load, until an area, enough to support that load, is cleared up and comes into actual contact.

Viewed in *this* light, *Amonton's* law, regarding the independence of the force of friction of the surface area, means no more than that, for a given load, a change in the apparent area makes little or no difference to the area in *actual* contact.

A logical consequence of this smallness of the area of true or actual contact is that the pressure at the points of contact must be enormously high. Indeed, it is estimated to be of the order of 100 *tons per square inch* in the case of mild steel. And this inevitably leads to the result that during sliding between the two surfaces, the temperature at these points of actual contact must rise to equally enormous proportions All this now stands amply verified by actual experiment. For, it has been found possible to measure the temperature of the points of actual contact in the case of two dissimilar metals by using them as a *thermocouple* and by amplifying and applying the thermo-electromotive force thus generated to the deflecting plates of a *cathode-ray oscillograph*. Temperatures, as high as 1000°*C**, have been found to obtain at these actual contact points or *'hot spots'* although they last only for just a split second,—the temperatures thus reached depending upon (*i*) *the magnitude of the load*, (*ii*) *the speed at which sliding takes place*, (*iii*) *the thermal conductivity* and (*iv*) *the melting points of the metals in question.*

In most cases, therefore, there is welding at these points during sliding, due to melting and consequent flow of metal. The surfaces of both metals thus get damaged by this sliding occurring between them. The surface of the softer metal gets torn and 'ploughed' by the projections on the surface of the harder metal and the latter has the softer metal welded on to it,—the maximum damage occurring in the case of two *similar* metals sliding one over the other, the coefficient of friction being the highest in their case.

*Even higher temperatures are reached in the case of poor conductors of heat. Thus, for example, glass rubbed against glass, or quartz against quartz, gives tiny sparks of light, changing from reddish to white as the vigour and speed of rubbing is increased.

The frictional force between two metal surfaces may thus be attributed to the shearing of these temporarily-welded points as well as the 'ploughing' resulting from the projections of the harder metal,—the former being by far the greater of the two forces, depending upon (*a*) the actual contact area of the metals and the shear-strength of the softer metal*. To minimise friction, therefore, both these factors should be made as small as possible. In almost all cases, however, if one is less, the other is high ; as, for example, if a hard metal is made to slide on steel, factor (*a*) *i.e* , the actual area of contact, is small, but factor (*b*), *i e.*, the 'ploughing' is greater, and if a softer metal is made to slide on steel, the ploughing, [factor (*b*)] is small, but the contact area [factor (*a*)] becomes greater. As a consequence, for most pairs of metals, the coefficient of friction comes to lie within the same common limits, ·6 to 1·0.

198. Lubricants. A thin film of a soft metal, like lead or iridium†, smeared on a harder surface, like steel, often reduces friction, because of its *reducing the shear-strength without unduly increasing the contact area.* This explains why we frequently use copper or silver bearings, smeared with lead or iridium in our modern aero-engines. A substance so used to minimise friction is called a **lubricant.**

Even apparently clean metal surfaces in air have a thin smearing of some sort of a lubricant between them, *e g.*, a thin film of air or a thin film of oxide or some such other contaminant, like moisture etc., which greatly reduces their shear-strength and hence the friction between them. Consequently, the friction between two metal surfaces is found to be greater in vacuo than in air,—the cause of this enhanced friction in the former case being the *force of adhesion* between the two surfaces, which grip or seize each other under the smallest load. Thus, as *Dr. Bowden* so aptly puts it, it is really fortunate that all metal surfaces are more or less contaminated in air, or else the world would be a very sticky place indeed.

All cases of friction fall into one or the other of the following three categories, *viz.*, (*i*) *dry friction between solid surfaces, i e* , the frictional force encountered when no lubricant is present between them ; (*ii*) *boundary friction*, when only a thin film of a lubricating material is present between the two given solid surfaces ; and (*iii*) *fluid or floatation friction*, when there is plenty of a liquid lubricant present in-between the two surfaces ; so that, the friction, in this case, depends almost wholly upon the properties of the lubricant, in particular, upon its viscosity, (see Chapter XII).

We have already studied briefly the essential details of the first two categories, *viz* , friction between solid surfaces in the absence of any lubricating substance, and 'boundary friction', with a thin film of lubricant present in-between them. The ordinary cases of friction between solid surfaces, which, as we have already seen, are almost always contaminated, to some slight extent, with oxide or moisture etc., strictly belong to this second category of boundary friction. **Hardy** thoroughly studied this type of friction and, according to him, it is the chemical natures of the metal and the lubricant which determine the strength of the boundary film,—a metal attracting more strongly the polar group (*COOH*) at one end of the molecules of an oil than the non-polar ones at the other. This results in the layer of molecules nearest the metal surface being orientated so as to stand upright (*i.e.*, normal to the surface), *with their polar ends inwards*, with possibly further double layers, having their polar ends adjacent, being formed on them ; so that, sliding actually occurs between pairs of these molecular layers over their non-polar ends Obviously, therefore, the more strongly does the lubricant adhere to the surface, the more effective will it be in minimising friction.

In the case of *fluid or floatation friction*, the thickness of the film makes most of the molecules lie outside the range of the adhesive force of the solid surface, so that the only force to be overcome now is that due to the viscosity of the lubricant.

**Coulomb* had originally suggested that the frictional force between two surfaces was purely a consequence of their roughness and consequent interlocking of their projections and depressions, so that there should be practically no frictional force between perfectly polished surfaces. This is, however, not found to be so, as we shall presently see.

†Iridium is even softer than lead.

Beuchamp Tower, in 1883, experimented on a revolving shaft, resting in a well lubricated bearing and showed that not only was the friction between the two greatly reduced but that the laws obeyed by frictional force were entirely different from those in the case of dry friction. For, (*i*) *while in the case of dry surfaces, the frictional force is proportional to the normal reaction between them, it is practically quite independent of the normal reaction in the case of lubricated surfaces,* (*ii*) *while in the case of dry surfaces, the magnitude of the frictional force depends upon the nature of the two surfaces, it is quite independent of the nature of the lubricated surfaces, depending only upon the nature of the lubricant,* and (*iii*) *whereas in the case of dry surfaces, the frictional force is quite independent of their relative velocity, it varies directly with the relative velocity in the case of lubricated surfaces.* Three years later, in 1886, **Osborne Reynolds** put forward his fundamental theory of lubrication. *Tower* had shown that a film of the lubricant was formed in between the shaft and the bearing and *Reynolds* showed that the film, which he thought must be several molecules thick, could be maintained in spite of the enormous pressure between the two surfaces, provided that the diameter of the bearing were a wee bit, (*about one-thousandth part*) greater than the diameter of the shaft. This makes the two surfaces slightly eccentric, with a *wedge-shaped* clearance between them, filled in with the lubricating oil, its pressure increasing with the narrowing down of the wedge and being equal to that developed between the sliding surfaces.

The oil is carried round *by the adhesive forces on layers, nearer the shaft, and by viscous forces on those away from it,* and the two surfaces are thus kept reasonably apart for the quantity of the labricant in this wedge to be sufficient to have its normal bulk-properties and thus to be able to resist the shear, purely by virtue of its viscosity. *The shearing of the temporarily welded points between the two surfaces is thus replaced by this inter-liquid shear.*

Obviously, therefore, *liquids, possessing the two essential properties of viscosity and of adherence to the solid surfaces will be the best lubricants. But, from the practical stand-point, perhaps by far the most important property required of a lubricant is that of chemical stability,* and mineral oils undoubtedly claim a definite superiority over vegetable oils in this respect as well as in supplying the necessary minimum viscosity* required to maintain a multi-molecular film between the two surfaces.

On the other hand, vegetable oils excel over mineral oils in their property of forming strongly adhering films on the solid surface, (*i e.*, the boundary films), in view of the polar groups in their molecule, due to the presence of free fatty acids in them.

To take full advantage of both these properties, therefore, the modern lubricants used are a *mixture of mineral oils with vegetable oils* (like *castor oil*) in proper proportion,—*the former supplying film lubrication and the latter, boundary lubrication.* In addition, it is also quite usual to add to the above mixture of lubricating oils, a measure of *colloidal graphite,* a thin film of which is formed on the solid surfaces, so that even if contact occurs between the two metal surfaces, seizure takes place between the graphite layers, offering little or no resistance to any relative motion between them.

In addition to all this, other factors, such as change of viscosity with increase of temperature and pressure† between the two surfaces have also to be taken into consideration.

199. Principle of Virtual Work. The principle of *virtual work* was first enunciated by **John Bernoulli,** in the year 1717, and later, in 1788, quite independently by **Lagrange,** and *applies to all cases of equilibrium,* thus leading to the deduction of all theorems relating to mechanical transmission of force, (of course, neglecting

*A high viscosity of the lubricant being also not quite desirable, in view of the resistance it would offer to the sliding motion between the two surfaces. Further, since the resistance to motion, due to viscosity, increases with speed, a higher viscosity at lower speeds, and a lower viscosity at higher speeds, is deemed to be more desirable

†As we know, viscosity decreases with temperature and increases with pressure.

friction). Let us, before coming to the *Principle* itself or its applications, try to understand clearly as to what is meant by the term *virtual work*.

If we imagine a system of forces, acting on a particle, to suffer a slight displacement, consistent with the geometrical conditions, *i.e., compatible with the constraints to which the particle is subjected*, the displacement so imagined is called a *virtual displacement*, for it may never really occur in actual practice, and is thus purely arbitrary, existing only in our imagination. *The product of each force of the system and the virtual displacement along its line of action is called* **virtual work***.

The **principle of virtual work** may now be stated as follows :

A system of forces, acting at points, connected by any mechanism, will be in equilibrium, if the total work, accompanying a virtual displacement, is zero.

Let us consider a few simple cases, illustrative of this principle :

(*i*) **Case of a body in equilibrium on a smooth inclined plane under the action of a force.** Let P be a body of mass m, on a smooth incline, of angle θ, (Fig. 246) such that a force F, acting on it at an angle ϕ with the plane, keeps it in equilibrium. The normal reaction R of the plane, obviously, acts perpendicularly to the plane, as shown. In order to eliminate any reference to it and thus to simplify our calculations, let us imagine a *virtual displacement* dr of the body, along the plane, so as to be perpendicular to R, so that $R.dr$ is *zero*. Then, clearly,

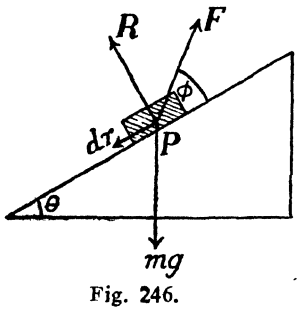


Fig. 246.

virtual work done *by* the weight mg of the body $= mg \sin \theta.dr$,

and *virtual work* done *against* force $F = -F \cos \phi.dr$,

where $mg \sin \theta$ and $F \cos \phi$ are the components of mg and F along the plane, the *negative sign*, in the latter case, merely indicating that the work is done *against* the force.

Hence, according to the *principle of virtual work*, the condition that must be satisfied for the equilibrium of the body is that

$$mg \sin \theta.dr - F \cos \phi.\ dr = 0.$$

Or, that $$(mg \sin \theta - F \cos \phi).dr = 0.$$

Since, obviously, dr is not *zero*, we have

$$mg \sin \theta - F \cos \phi = 0. \quad \text{Or,} \quad F \cos \phi = mg \sin \theta,$$

which gives the condition for the equilibrium of the body, and which we know, is the one we obtain by the ordinary *conventional* method, in which we also take into consideration the normal reaction of the plane.

(*ii*) **Case of equilibrium of a body on a rough inclined Plane.** Let a body, of mass m, be *just in equilibrium i.e., just on the point*

*Very aptly so, since the displacement is only virtual or imagined.

of sliding down a rough incline of angle θ, when a horizontal force F acts upon it, (Fig. 247).

Here, obviously, the reaction R of the plane will make an angle, α, say, (the *angle of repose*) with the normal PN to the plane.

Again, considering the body to have a *virtual displacement* dr, perpendicular to the direction of R, we have

$R.dr = 0$. And, therefore,

virtual work done *by* the weight mg of the body $= mg \sin(\theta-\alpha).dr$,

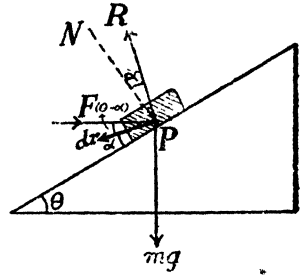


Fig. 247.

and virtual work done *against* force $F = -F\cos(\theta-\alpha)\,dr$.

So that, by the *principle of virtual work*, we have, for equilibrium,

$$mg \sin(\theta-\alpha).dr - F\cos(\theta-\alpha).dr = 0.$$

Again, since dr is *not* zero, we have $F\cos(\theta-\alpha) = mg\sin(\theta-\alpha)$.

Or, $$F = mg.\tan(\theta-\alpha),$$

which thus gives the condition for the equilibrium of the body.

(*iii*) **Case of equilibrium of a system of two or more connected bodies.** Here, a slight complication arises in that, if we imagine the bodies to undergo *virtual displacements* dr_1, dr_2, etc., under the action of forces F_1, F_2 etc., acting on them respectively, these displacements, *since they must be compatible with the constraints of the system*, cannot possibly be completely arbitrary, but will rather be inter related to each other, although they will satisfy the relation

$$F_1.dr_1 + F_2.dr_2 + \ldots\ldots F_n.dr_n = 0. \qquad \ldots I$$

We can, however, tide over the difficulty by first writing down the above general relation for the different impressed forces F_1, F_2, etc., and then deducing, from the geometry of the system, the inter-relation between the different virtual displacements. This will then enable us to obtain a set of completely arbitrary displacements, and hence the necessary conditions that the different forces must satisfy for equilibrium of the given system of bodies.

Let us, for example, consider the equilibrium of two bodies, m_1 and m_2, tied to the two ends of a string, passing round a smooth peg or pulley, (Fig. 248), and subjected to two impressed forces F_1 and F_2 respectively, as shown.

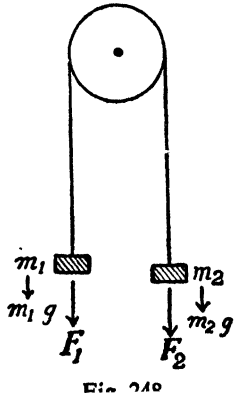


Fig. 248

Taking, for convenience, the virtual displacements dr_1 and dr_2 of the two bodies in the downward direction (which is quite compatible with the restraints imposed upon the system), we have, for equilibrium, on the *principle of virtual work*,

$$(F_1+m_1g).dr_1 + (F_2+m_2g)dr_2 = 0.$$

Now, an examination of the situation at once reveals the relationship between dr_1 and dr_2, *viz.*, that *the string being inextensible*,

So that, the above relation reduces to

$$(F_1+m_1g).dr_1-(F_2+m_2g).dr_1 = 0,$$

whence,

$$F_1+m_1g = F_2+m_2g,$$

which thus gives the necessary condition for the equilibrium of the two bodies.

It must be noted here that if we were to use the old conventional method to determine the equilibrium condition of the bodies we should have had to take into account the tension in the string. Since it is *not an impressed force*, we can clean ignore it in our calculations, on the principle of virtual work.

(*iv*) **Relation between Equilibrium and Potential Energy.** We may express relation *I*, of case (*iii*) above, in *Cartesian form*, and write for the pth particle, in a system of particles,

$$\mathbf{dr}_p = \mathbf{i}dx_p+\mathbf{j}.dy_p+\mathbf{k}.dz_p,$$

and

$$\mathbf{F}_p = \mathbf{i}Fx_p+\mathbf{j}Fy_p+\mathbf{k}.Fz_p*,$$

where dx_p, dy_p and dz_p denote the virtual displacements of the particle in the x, y and z directions respectively, and Fx_p, Fy_p and Fz_p, the components of the applied force $\mathbf{F}_p$ in these directions.

The principle of virtual work may now be expressed in the form

$$\sum_{p=1}^{n} (Fx_p.dx_p+Fy_p.dy_p+Fz_p.dz_p) = 0.$$

If the forces be *conservative* (*i.e.*, for which the total mechanical energy remains constant or '*conserved*'), so that there exists a potential energy function $V_p(x_p.y_p.z_p)$, such that

$$Fx_p = -\frac{\partial V_p}{\partial x_p}, \quad Fy_p = -\frac{\partial V_p}{\partial y_p} \text{ and } Fz_p = -\frac{\partial V_p}{\partial z_p}.$$

Then, the total potential energy of the whole system of particles is given by

$$\sum_{p=1}^{n} V_p = V.$$

The principle of virtual work then takes the form

$$\sum_{p=1}^{n} dV_p = 0,$$

(*i.e., for the virtual displacements considered, the change in the total potential energy is zero.*

In other words, *for the equilibrium position, the potential energy must either be a maximum or a minimum* ; for only then, can the variation due to a small displacement be zero†. In the latter case, the equilibrium is said to be *stable*, and in the former, *unstable*.

---

*In vector notation, **i**, **j** and **k** denote *vectors of unit magnitude*, in x, y and z directions respectively.

†Strictly, mathematically speaking, it is not quite correct to say so, and all we can reasonably assert, therefore, is that the potential energy will have a stationary value. In most mechanical problems, however, it is quite enough to confine our attention to *maxima* and *minima*.

Thus, taking the simplest case of a single particle, acted upon by the force of gravity and resting on a smooth surface, we have potential energy given by $V = mg.z$, taking the z-axis vertical, $-z$ being *positive* upward.

Now, if the particle be in equilibrium on a *concave* surface, the total force acts on it normally to the surface, so that z and, therefore, V is a *minimum*, and hence the equilibrium, *stable* ; for, any finite displacement of the particle on the surface tends to increase its potential energy. But if it be in equilibrium on a convex surface, we have z and, therefore, V, a *maximum*, and hence the equilibrium *unstable* ; for, a finite displacement of the particle on the surface *now* tends to decrease its potential energy.

(*v*) **Tension in a Flywheel.** Let us now consider the application of the *principle of virtual work* in a problem on the mechanical transmission of force and calculate the tension in a rotating flywheel.

If the radius of the ring-shaped flywheel be r and its angular velocity, ω, it is clear that an element δm of it will experience a *centrifugal force* $\delta m.r\omega^2$, *outwards*, at right angles to its circumference, and *the ring will thus tend to stretch itself, i.e., will be in a state of stress.*

If we imagine each element of the ring to suffer a *radial virtual displacement* dr, the virtual work done *by* it would be $\delta m.r\omega^2.dr$. And, therefore,

total virtual work done *by* the ring, as a whole $= \Sigma(\delta m.r\omega^2.dr)$.

Obviously, if the ring be cut at any point on its circumference, its two ends will, under *this* centrifugal force, fly apart through a distance

$$= 2\pi(r+dr)-2\pi r = 2\pi.dr.$$

In order to hold the two ends of the ring together, obviously, an external force F, *equal to the tension in the ring*, will have to be applied to it *inwards*. And, therefore,

virtual work done *against* the force $F = -F.2\pi.dr$.

For equilibrium, therefore, we have, in accordance with the principle of virtual work,

$$\Sigma(\delta m.r\omega^2.dr)-F.2\pi.dr = 0. \quad \text{Or,} \quad \Sigma(\delta m.r\omega^2.dr) = F.2\pi.dr.$$

Or, $$r\omega^2.dr\Sigma\delta m = F.2\pi.dr.$$

Now, $\Sigma\delta m = M$, the mass of the ring or the flywheel.

So that, $r\omega^2.dr.\ M = F.2\pi.dr.$ Or, $F = \dfrac{r\omega^2.dr.M}{2\pi.dr} = \dfrac{r\omega^2.M}{2\pi}$,

which enables the tangential tension in the flywheel to be easily calculated.

The above examples will, it is hoped, suffice to bring home to the student the great utility and importance of this rightly celebrated principle of virtual work.

SOLVED EXAMPLES

1. A gramophone disc is set revolving in a horizontal plane and reaches a steady state of motion of two revolutions per second. It is found that a small

coin placed on the disc will remain there if its centre is not more than 5 cms. from the axis of rotation. Explain this and calculate the coefficient of friction between the coin and the disc. *(Oxford and Cambridge Higher School Certificate)*

When the small coin is placed on the rotating gramophone disc, it is subjected to a centrifugal force tending to pull it outwards, away from the axis of rotation of the disc. Its motion is, however, opposed by the force of friction coming into play in-between the surfaces of the disc and its own. But, as the distance of the coin from the axis of rotation increases, the centrifugal force pulling it outwards also increases, (being equal to $mr\omega^2$, where m is its *mass*, ω, its angular velocity and r, its distance from the axis of rotation) and is just balanced by the limiting friction between itself and the disc, *when it is at a distance of 5 cms. from the axis of rotation.* Beyond this distance, the centrifugal force on it is greater than the limiting friction between it and the disc and it, therefore, moves outwards.

Here, $r = 5$ *cms.* and $\omega = 4\pi$. [$\because$ in 1 rotation, the disc describes an angle 2π.

So that, if m be the *mass* of the coin, we have

centrifugal force on the coin $= mr\omega^2 = m \times 5 \times (4\pi)^2 = 80\,\pi^2.m$ *dynes.*

And, clearly, the *normal reaction* (R) *of the disc on it* = its own weight mg.

And, therefore, the *limiting friction between the coin and the disc* $= \mu R = \mu.mg$, where μ is the *coefficient of friction* for them.

Since the coin is *just in equilibrium here*, we have $\mu.mg = 80\,\pi^2 m$.

Or, $\mu g = 80\pi^2$,

whence, $\mu = 80\pi^2/g = 80\,\pi^2/980 = 4\pi^2/49$, [Taking $g = 980$ *cms./sec.*^{2}.

Or, $\mu = \cdot 8054$,

Thus, the coefficient of friction for the surfaces of the coin and the disc is equal to ·8054.

2. Define the coefficient of sliding friction.

A uniform ladder of length $2l$ and weight W rests against a vertical wall with its foot on the ground at a distance l from the wall. If the coefficient of friction between the wall and the ladder and between the ground and the ladder is 0·4, find how far up the ladder a man of weight $2W$ can ascend without disturbing equilibrium. *(Joint Matriculation Board)*

For definition of *coefficient of sliding friction*, see § 190, page 395.

Here, let PQ be the ladder resting against the wall AB, (Fig. 249), such that it makes an angle θ with the ground, the ladder being in the vertical plane, perpendicular to the wall, with its weight W acting vertically downwards at its mid-point O.

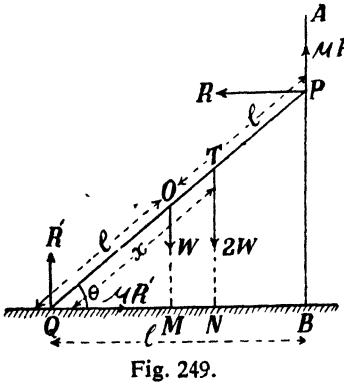


Fig. 249.

Let the man climb up a distance x on the ladder, up to T, before the ladder starts slipping. Then, clearly, the weight $2W$ of the man acts vertically down at T. And since, when the ladder is just on the point of slipping, its upper end P moves downwards towards B and its foot Q away from B, the frictional forces at P and Q act *away from and towards B* respectively. If R and R' be the normal reactions of the wall and the ground at P and Q, and μ, the *coefficient of friction* in either case, we have

limiting frictional force at P $= \mu R$, *along the wall, away from B*

and *limiting frictional force at Q* $= \mu R'$, *along the ground, towards B.*

In the equilibrium position of the ladder, clearly.

$R = \mu R'$, ...(*i*) and $R' + \mu R = W + 2W = 3W$. ..(*ii*)

So that, substituting the value of R in relation (*ii*) we have

$R' + \mu^2 R' = 3W$ ..(*iii*) Or, $R'(1+\mu^2) = 3W$, ...(*iv*

whence, $R' = 3W/(1+\mu^2)$; ...(v) And, $\therefore \quad = 3\mu W/(1+\mu^2)$. ...(vi)

Now, taking moments about Q, we have

$$W \times QM + 2W \times QN = R \times PB + \mu R \times QB.$$

Or, $$W \times l\cos\theta + 2W \times x\cos\theta = R \times 2l\sin\theta + \mu R \times 2l\cos\theta,$$

where, $QM = l\cos\theta$, $QN = x\cos\theta$, $PB = 2l\sin\theta$ and $QB = 2l\cos\theta$,

Or, $$Wl\cos\theta + 2Wx\cos\theta - 2\mu Rl\cos\theta = 2Rl\sin\theta.$$

Or, $$\cos\theta\,(Wl + 2Wx - 2\mu Rl) = 2Rl\sin\theta.$$

Or, $$\frac{Wl + 2Wx - 2\mu Rl}{2Rl} = \tan\theta = \frac{\sqrt{3}l}{\quad} = \sqrt{3}. \quad \left[\because \tan\theta = PB/QB \text{ and } PB = \sqrt{(2l)^2 - l^2} = \sqrt{3l^2} = \sqrt{3}l.\right]$$

So that, $$Wl + 2Wx - 2\mu Rl = 2\sqrt{3}\,Rl.$$

Or, $$Wl + 2Wx = 2\mu Rl + 2\sqrt{3}\,Rl = 2Rl(\mu + \sqrt{3}).$$

Or, substituting the value of R from relation (*vi*) above, we have

$$W(l+2x) = \frac{2\times 3\mu W}{1+\mu^2} \times l(\mu + \sqrt{3}).$$

Or, $$l + 2x = \frac{6\mu l}{1+\mu^2}(\mu + \sqrt{\quad} = \frac{6\times\cdot4l}{+\cdot16}(\cdot4+\sqrt{3}). \quad [\because \mu = \cdot4.$$

Or, $$l + 2x = \frac{\cdot \times \cdot4 + 1\cdot732)}{1\cdot16} = \frac{2\cdot4l \times 2\cdot132}{1\cdot16}.$$

Or, $$2x = \frac{2\cdot4\times2\cdot132l}{1\cdot16} - l = l\left(\frac{2\cdot4\times2\cdot132}{1\cdot16} - 1\right).$$

Or, $$2x = (4\cdot411 - 1)l = 3\cdot411\,l, \quad \text{whence,} \quad x = \frac{3\cdot411\,l}{2} = 1\cdot7055\,l.$$

Thus, the man can ascend up a distance $1\cdot7055l$ without disturbing the equilibrium of the ladder.

3. A rough plane is inclined to the horizontal at an angle θ, where θ is less than the angle of friction λ, and a body of weight W is placed on the plane. Calculate the minimum horizontal force required to make the body move (*i*) down the plane and (*ii*) up the plane.

(*i*) Let AB be the rough plane, inclined to the horizontal at an angle θ, and let the body of weight W be placed on it, as shown, (Fig. 250).

Then, since θ is less than the angle of friction λ for the body and the plane, it will not by itself slip down, (see page 398).

Let P be the least horizontal force required to make the body just move down the plane.

Then, clearly, resolving P into its two rectangular components, we have its component $P\cos\theta$, acting downwards, *along the plane* and its component $P\sin\theta$, acting upwards, *perpendicular to the plane*.

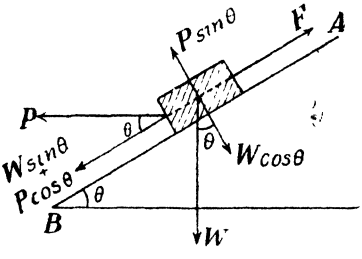


Fig. 250.

Similarly, resolving W into its two rectangular components, we have its component $W\sin\theta$ acting downwards, *along the plane* and its component $W\cos\theta$ acting *perpendicular to the plane*.

Thus, since there is no motion of the body in a direction perpendicular to the plane, we have

normal reaction of the plane $R = W\cos\theta - P\sin\theta$,

and the *downward force along the plane* $= W\sin\theta + P\cos\theta$.

Since the body is just on the point of moving *down* the plane, it is clear that

$$W\sin\theta + P\cos\theta = F, \quad \ldots (i)$$

where F is the *limiting friction* between the body and the plane, acting *upwards*, along the plane.

Now, $F = \mu R$, where μ is the *coefficient of friction* for the surfaces of the body and the plane.

Again, $\mu = tan\ \lambda$, where λ is the *angle of friction* for the given pair of surfaces. So that,

$$F = \mu R = tan\ \lambda\ (W\ cos\ \theta - P\ sin\ \theta).$$

Substituting this value of F in relation (i) above, we have

$$W\ sin\ \theta + P\ cos\ \theta = tan\ \lambda\ (W\ cos\ \theta - P\ sin\ \theta).$$

Or, $$P\ cos\ \theta = W\ tan\ \lambda\ cos\ \theta - P\ tan\ \lambda\ sin\ \theta - W\ sin\ \theta.$$

Or, $$P = W\ tan\ \lambda - P\ tan\ \lambda\ tan\ \theta - W\ tan\ \theta.$$ [*i.e.*, dividing both sides by $cos\ \theta$.

Or, $$P + P\ tan\ \lambda\ tan\ \theta = W\ tan\ \lambda - W\ tan\ \theta.$$

Or, $$P(1 + tan\ \lambda\ tan\ \theta) = W\ (tan\ \lambda - tan\ \theta).$$

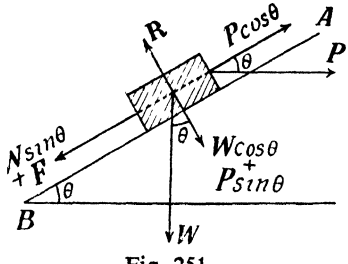
Or, $$P = W \cdot \frac{tan\ \lambda - tan\ \theta}{1 + tan\ \lambda\ tan\ \theta}.$$

Clearly, $(tan\ \lambda - tan\ \theta)/1 + tan\ \lambda\ tan\ \theta = tan\ (\lambda - \theta)$.

So that, $$P = W\ tan\ (\lambda - \theta)$$

Thus, the least horizontal force required to make the body move *down* the plane is $W\ tan\ (\lambda - \theta)$.

(*ii*) In this case, again, let P be the least force required to make the body *just* move *up* the plane. Obviously, it will now have to be applied in the opposite direction to that in the first case, as shown in Fig 251, and the frictional force F will now act *downwards* along the plane.



Fig, 251.

Resolving P, as before, into its two rectangular components, we have

component $P\ cos\ \theta$, *acting upwards, along the plane,*

and *component* $P\ sin\ \theta$ *acting downwards, perpendicular to the plane.*

So that, the *normal reaction on the* plane is now $R = W\ cos\ \theta + P\ sin\ \theta$.

$\therefore$ for equilibrium of the body, we have

$$P\ cos\ \theta = W\ sin\ \theta + F = W\ sin\ \theta + \mu R.$$

Or, $$P\ cos\ \theta = W\ sin\ \theta + tan\ \lambda\ (W\ cos\ \theta + P\ sin\ \theta).$$ [$\because\ \mu = tan\ \lambda$ and $R = W\ cos\ \theta + P\ sin\ \theta$,

Or, $$P\ cos\ \theta = W\ sin\ \theta + W\ tan\ \lambda\ cos\ \theta + P\ tan\ \lambda\ sin\ \theta.$$

Or, $$P = W\ tan\ \theta + W\ tan\ \lambda + P\ tan\ \lambda\ tan\ \theta.$$

Or, $$P - P\ tan\ \lambda\ tan\ \theta = W\ (tan\ \lambda + tan\ \theta).$$

Or, $$P(1 - tan\ \lambda\ tan\ \theta) = W\ (tan\ \lambda + tan\ \theta).$$

whence, $$P = W \frac{tan\ \lambda + tan\ \theta}{1 - tan\ \lambda}.$$

Since $(tan\ \lambda + tan\ \theta)/1 - tan\ \lambda\ tan\ \theta = tan\ (\lambda + \theta)$, we have

$$P = W\ tan\ (\lambda + \theta).$$

Or, the minimum horizontal force required to make the body just move *up* the plane is $W\ tan\ (\lambda + \theta)$.

4. (*a*) Define the angle of friction.

A uniform rod rests in limiting equilibrium in contact with a horizontal floor and a vertical wall, the rod being in a vertical plane which is perpendicular to the wall. If the wall and the floor be equally rough, prove that the angle between the rod and the wall is twice the angle of friction.

(*London Higher Secondary Certificate*)

(*b*) What would be the value of the angle in the question above, (*i*) if the wall and the floor be unequally rough, (*ii*) if the wall be smooth, (*iii*) if the floor be smooth, and (*iv*) if both be smooth ?

(*a*) Let PQ be the rod, of length $2l$, resting in limiting equilibrium, with its upper end P in contact with a vertical wall AB and its foot with a horizontal floor, (Fig. 252), such that it lies in a vertical plane perpendicular to the wall and makes an angle θ with it.

Then, its weight W acts vertically downwards at its mid-point O, with the normal reactions R and R' of the wall and the floor acting at P and Q respectively, as shown.

Since the wall and the floor are *equally rough*, the coefficient of friction for either of them and the rod must be the *same*. Let it be μ. Then, the frictional forces at P and Q are clearly μR and $\mu R'$, acting in the directions shown, as explained in answer to Ex. 2 above.

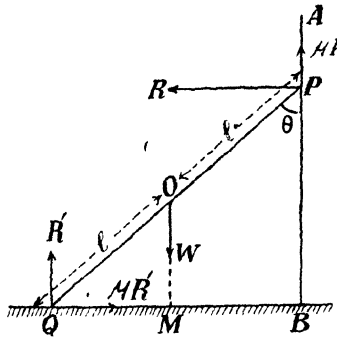


Fig. 252.

The rod being in limiting equilibrium, we clearly have

$$R = \mu R' \quad \ldots \quad \ldots (i) \text{ and } \mu R + R' = W. \quad \ldots \quad \ldots \quad \ldots (ii)$$

So that, substituting the value of R from expression (i) in expression (ii), we have

$$R' + \mu^2 R' = W. \quad \text{Or,} \quad R'(1+\mu^2) = W. \quad \ldots \quad \ldots (iii)$$

And $\therefore$

$$\mu R'(1+\mu^2) = \mu W = R(1+\mu^2) = \mu W, \quad \ldots (iv) \quad [\because \mu R' = R.$$

whence,

$$R = \mu W/(1+\mu^2), \quad \ldots \quad \ldots \quad \ldots \quad (v)$$

Now, taking moments about Q, we have

$$W \times QM = R \times PB + \mu R \times QB.$$

Or,

$$W \times l \sin\theta = R \times 2l \cos\theta + \mu R \times 2l \sin\theta. \quad \ldots (vi)$$

Or,

$$Wl \sin\theta - \mu R.2l \sin\theta = 2Rl \cos\theta.$$

Or,

$$l \sin\theta(W - 2\mu R) = 2Rl \cos\theta.$$

Or,

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{2Rl}{l(W-2\mu R)} = \frac{2R}{W-2\mu R}.$$

So that, substituting the value of R from relation (v) above, we have

$$\tan\theta = \frac{2\mu W}{(1+\mu^2)\left(W - 2\mu.\frac{\mu W}{1+\mu^2}\right)} = \frac{2\mu W}{W(1+\mu^2) - 2\mu^2 W}.$$

Or,

$$\tan\theta = \frac{2\mu}{(1+\mu^2)-2\mu^2} = \frac{2\mu}{1-\mu^2}. \quad \ldots \quad \ldots (vii)$$

Now,

$$\mu = \tan\lambda, \text{ where } \lambda \text{ is the } \textit{angle of friction.}$$

And, therefore,

$$\tan\theta = \frac{2\tan\lambda}{1-\tan^2\lambda} = \tan 2\lambda, \text{ whence, } \theta = 2\lambda.$$

Thus, *the angle* θ *that the rod makes with the vertical wall is twice the angle of friction.*

(*b*),—(*i*) *If the wall and the floor be unequally rough,* the coefficient of friction for the rod and the wall will be different from that between the rod and the floor. Let these be μ and μ' respectively. Then, clearly,

$$R = \mu' R' \quad \text{and} \quad R' + \mu R = W.$$

Or,

$$R' + \mu\mu' R' = W. \quad \text{Or,} \quad R'(1+\mu\mu') = W.$$

So that,

$$\mu' R'(1+\mu\mu') = \mu' W. \quad \text{Or,} \quad R(1+\mu\mu') = \mu' W. \quad [\because \mu' R' = R.$$

Or,

$$R = \mu' W/(1+\mu\mu').$$

And, $\therefore$

$$\tan\theta = \frac{2\mu' W}{(1+\mu\mu') - 2\mu\mu' W} = \frac{2\mu'}{1-\mu\mu'} \quad [\text{See above.} \quad \ldots (viii)$$

Or,

$$\theta = \tan^{-1}[2\mu'/(1-\mu\mu')].$$

Thus, *the rod now makes with the wall an angle* $\theta = \tan^{-1}[2\mu'/(1-\mu\mu')]$.

(*ii*) *If the wall be smooth,* we shall have the co-efficient of friction for the rod and the wall equal to *zero, i.e.,* $\mu = 0$.

And, therefore, expression (*viii*) above becomes

$$\tan\theta = 2\mu' = 2\tan\lambda.$$

i.e., the tangent of the angle that the rod makes with the wall is twice the tangent of the angle of friction.

(*iii*) *If the floor be smooth,* the coefficient of friction for the rod and the floor is equal to *zero, i e.,* $\mu' = 0$. So that,

$$\tan\theta = 0. \quad \text{Or,} \quad \theta = 0.$$

Thus, in this case, *the rod can rest only in the vertical position.*

(*iv*) *If the wall and the floor be both smooth,* the coefficient of friction for the rod and the wall as well as that for the rod and the floor is equal to *zero, i.e.,* $\mu = 0$ and also $\mu' = 0$.

Again, therefore, we have $\tan\theta = 0$, Or, $\theta = 0$.

Thus, here also the rod cannot possibly rest in any inclined position*.

5. If a ladder rests in the limiting position against a vertical wall and a horizontal floor, how far can a man climb up the ladder before the ladder starts slipping ?

Let the ladder PQ, of length l, rest in the limiting position against a vertical wall AB and a horizontal floor, as shown in Fig. 252, making an angle θ with the wall. Then, the weight W of the ladder acts vertically downwards at its mid-point, as shown. So that, using the same symbols as before, in Ex. 4 (*b*), above, we *now* have

$$R' = \mu' R' \quad \text{and} \quad R' + \mu R = (W + w),$$

where w is the weight of the man, also acting vertically downwards.

And, therefore, $R' + \mu\mu' R = (W + w)$. Or, $R'(1 + \mu\mu') = (W + w)$.

Or, $$\mu' R'(1 + \mu\mu') = \mu'(W + w).$$

Or, $R(1 + \mu\mu') = \mu'(W + w)$, whence, $R = \dfrac{\mu'(W+w)}{1+\mu\mu'}$. $[\because \mu R' = R.$

Now, if the man can climb a maximum distance x up the ladder before slipping just occurs, we have, using the same symbols as in Ex. 4 (*b*) above, and taking moments about Q,

$$Wl \sin\theta + wx \sin\theta = R\,2l\cos\theta + \mu R.2l\sin\theta.$$

Or, $$Wl\sin\theta + wx\sin\theta - \mu R.2l\sin\theta = R.2l\cos\theta.$$

Or, $$\sin\theta\,(Wl + wx + 2\mu Rl) = 2Rl\cos\theta.$$

Or, $$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{2Rl}{Wl + wx - 2\mu Rl}.$$

But, $\tan\theta = 2\mu'/1 - \mu\mu'$, [See Ex. 4 (*b*), above.

And, therefore, $$\frac{2\mu'}{1-\mu\mu'} = \frac{2Rl}{Wl + wx - 2\mu Rl}.$$

Or, $$2\mu' Wl + 2\mu' wx - 4\mu\mu' Rl = 2Rl - 2\mu\mu' Rl.$$

Or, $$2\mu' Wl + 2\mu' wx = 2Rl - 2\mu\mu' Rl + 4\mu\mu' Rl = 2Rl + 2\mu\mu' Rl.$$
$$= 2Rl(1 + \mu\mu').$$

Substituting the value of R from above, therefore, we have

$$2\mu' Wl + 2\mu' wx = 2l(1 + \mu\mu') \times \frac{\mu'(W+w)}{1+\mu\mu'} = 2\mu' l(W + w).$$

Or, $$2\mu'.Wl + 2\mu' wx = 2\mu' Wl + 2\mu' wl.$$

Or, $$2\mu' wx = 2\mu' wl.$$

whence, $$x = l.$$

Thus, the man (irrespective of his weight) can climb up the ladder to only a maximum distance equal to half its length before slipping occurs.

6. A framework ABCD, consisting of four uniform, freely jointed rods, each of the same length and weight w, is hung from A, and the corners A, C are connected by a string. Find the tension in the string.

Let AC be the string, connecting the opposite corners of the framework $ABCD$, formed by the four equal and uniform rods, AB, BC, CD and DA, (Fig. 253), in which the tension is to be determined.

*This is obvious otherwise also ; for, in this case, the three forces acting on the rod do not all meet in one point.

Clearly, the weight w, of each rod acts vertically downwards at its *c.g.*, *i.e.*, at its midpoint, (the rod being uniform).

Then, if $2l$ be the length of each rod and θ, the angle it makes with AC, we have

$\cos\theta = AP/AD = a/2l$, whence, $a = 2l\cos\theta$.

(P being the mid-point of AC, and $AP = a$).

So that, $AC = 2AP = 2a = 2.2l\cos\theta = 4l\cos\theta$.

Now, let x be the depth below A, of the *c.g.*, of each of the two upper rods. Then, clearly, $x = l\cos\theta$.

And, therefore, $AC = 4l\cos\theta = 4x$.

And since the *c.g.*, of each of the lower pair of rods lies a distance x above C, its depth below A is $4x - x = 3x$.

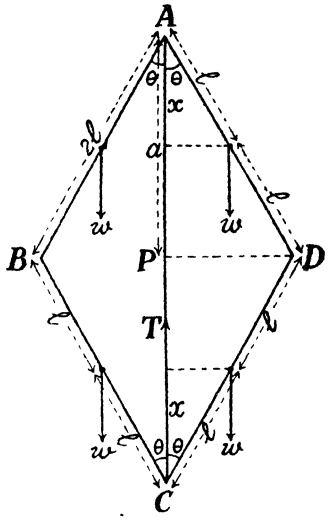


Fig. 253.

If we, therefore, imagine the whole system to be displaced downwards a little, so that the *c.g.* of the upper rods is shifted through a distance δx, we have

virtual work done by the weights of the two upper rods $= 2w.\delta x$.

Similarly, *virtual work done by the weights of the two lower rods* $= 2w.\delta(3x) = 6w.\delta x$.

And, therefore, *total virtual work done by the weights* $= 2w\,\delta x + 6w.\delta x = 8w.\delta x$.

And, if T be the tension in the string, we have

virtual work of the string $= -T\,\delta(4x) = -4T.\delta x$,

the negative sign indicating that T acts in the opposite direction to the displacement of C.

Since the system is in equilibrium, we have *total virtual work* $= 0$.

Or, $8w.\delta x - 4T.\delta x = 0$ Or, $4T.\delta x = 8w.\delta x$,

whence, $T = 2w$.

Thus, the tension in the string is $2w$, *i.e.*, equal to twice the weight of each rod.

7. **A uniform rod of length $2l$ lies in equilibrium over a smooth peg, with its lower end resting against a smooth vertical wall. If the peg be at a distance d from the wall, show that the rod is inclined to the wall at an angle $\sin^{-1}(d/l)^{\frac{1}{2}}$.**

Let PQ be the rod, resting *in equilibrium* over the smooth peg K and against the smooth vertical wall AB, such that it makes an angle θ with the wall at P, (Fig. 254).

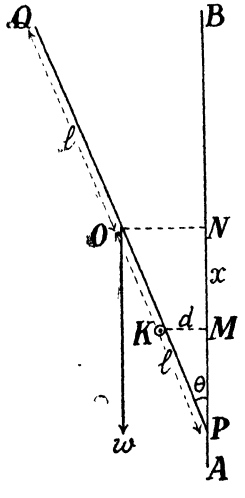


Fig. 254.

Obviously, the weight w of the rod acts vertically downwards at its *c.g.*, *i.e.*, at its mid-point O, at a vertical height $MN = x$, above the peg K, where

$x = MN = PN - PM.$ [$\because$ $PN = OP\cos\theta$,
$= OP\cos\theta - KM\cot\theta$ and $PM = KM\cot\theta$.

Or, $x = l\cos\theta - d\cot\theta$. [$\because$ $OP = l$ and $KM = d$.

Differentiating this with respect to θ, we have

$$dx = (-l\sin\theta + d\,\mathrm{cosec}^2\theta)\,.\,d\theta.$$

Since the rod is in equilibrium, we have $dx = 0$.

And, therefore, $d\,\mathrm{cosec}^2\theta - l\sin\theta = 0$.

Or, $d/\sin^2\theta = l\sin\theta$, whence, $\sin^3\theta = d/l$.

Or, $$\theta = \sin^{-1}(d/l)^{\frac{1}{3}}$$

Thus, the rod makes an angle $\sin^{-1}(d/l)^{\frac{1}{3}}$ with the wall, in its position of equilibrium.

8. An elastic ring of weight W rests on a smooth cone of semi-vertical angle θ. Find the tension in the ring.

Let the elastic ring be in the position shown in Fig. 255, with its centre O at a depth h below the vertex A of the cone, so that its radius $r = h \tan \theta$.

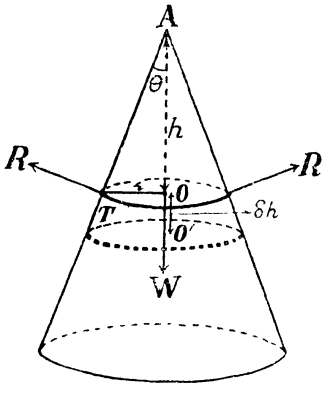


Fig 255.

Clearly, the forces operating on the ring are (*i*) *its weight W, acting vertically downwards at its centre of gravity or centre O*, (*ii*) *the tension T of the ring acting along the circumference of the ring and* (*iii*) *the normal reaction R of the cone, acting at right angles to the surface of the cone*, its surface being smooth (and hence there being no frictional component along its surface .

Now, imagine the ring to be given a small downward displacement δh, so that its distance from the apex A of the cone now becomes $h + \delta h$ and hence its radius becomes

$$(h + \delta h) \tan \theta.$$

Its radius thus increases by

$$(h + \delta h) \tan \theta - h \tan \theta = \delta h. \tan \theta,$$

and its circumference, by $2\pi\delta h \tan \theta$.

Clearly, then, we have

virtual work done by the weight W of the ring $= W.\delta h.$

and ,, ,, ,, *against tension T of the ring* $= -T.2\pi\delta h. \tan \theta$,

the negative sign, indicating that the displacement here takes place in the direction *opposite to that of T*.

And, because the ring moves at right angles to the direction of R, *no work is done by or against R.*

Since the ring is in equilibrium, the total virtual work done must be *zero*. We, therefore, have

$$W.\delta h - T\,2\pi\delta h.\tan\theta = 0. \quad \text{Or,} \quad T\,2\pi\delta h \tan\theta = W.\delta h,$$

whence,
$$T = \frac{W.\delta h}{2\pi\delta h. \tan\theta} = \frac{W}{2\pi} \cdot \frac{1}{\tan\theta} = W \cot\theta/2\pi.$$

This, therefore, is the tension in the elastic ring.

EXERCISE X

1. A man weighing 140 *lbs.* climbs up a uniform ladder, 20 *ft.* long and 70 *lbs.* in weight, which rests against a rough vertical wall at an angle of 45°. If the coefficient of friction at each end of the ladder is 0·5, how far will the man be able to climb up the ladder before it begins to slip.

(Northern Universities Higher School Certificate)

Ans. 13·0 *ft.*

2. A uniform rod is in limiting equilibrium, one end resting on a rough horizontal plane and the other on on equally rough plane inclined at an angle α to the horizon. If λ be the angle of friction and the rod be in a vertical plane, show that the inclination θ of the rod to the horizon is given by

$$\tan\theta = \frac{\sin(\alpha - 2\lambda)}{2\sin\lambda\sin(\alpha - \lambda)}. \qquad \textit{(Delhi, 1939)}$$

3. Distinguish between *static* and *sliding* (kinetic) friction and define the *coefficient of sliding friction.*

How would you investigate the laws of sliding friction between wood and iron ?

An iron block, mass 10 *lbs.*, rests on a wooden plane inclined at 30° to the horizontal. It is found that the least force parallel to the plane which causes the block to slide *up* the plane is 10 *lb wt.* Calculate the coefficient of sliding friction between wood and iron.

(Northern Universities Higher School Certificate)

Ans. 0·58.

4. A uniform ladder, 13 *ft.* long and weighing 60 *lbs.*, rests with its upper end against a smooth vertical wall and with its lower end 5 *ft* from the wall on rough ground. By a graphical method find the magnitude and direction of the reaction at the foot of the ladder. Check your result by calculation. What is the least coefficient of friction between the ground and the ladder necessary to maintain equilibrium? (*Cambridge Local Higher School Certificate*)

Ans. 61·3 *lb wt.*; at 78° 10′ to the ground; 0·21.

[**Hint.** To determine the *magnitude* and *direction* of the reaction at the foot of the ladder, we must remember that the resultant reaction of the ground passes through the point of intersection of the lines of action of R and the 50 *lb. wt.*, since, as we know, three forces in equilibrium must all be concurrent. The angle λ (*i.e.*, the angle between the resultant ground reaction and the vertical) is then measured and $\tan \lambda = \mu$ determined.]

5. Explain the meaning of the term *coefficient of friction* and describe an experimental method of measuring it.

A rectangular block with a square base of side 10 *cm.* rests on a horizontal surface. If a horizontal force is applied near the bottom of one vertical face, the block slides. If the force is applied near the top, the block topples over. When the force is applied at 20 *cm.* from the bottom, the block sometimes slides and sometimes topples. Find the coefficient of friction between the block and the surface. (*Cambridge Higher School Certificate*)

Ans. 0·25.

6. A uniform rod of weight W rests with its one end against a rough inclined plane AB, of inclination α, and the other end against a smooth vertical wall BD, B being higher than A. If θ be the inclination of the rod to the vertical in the limiting position of equilibrium, show that $\tan \theta = 2 \tan (\lambda - \alpha)$, where λ is the angle of friction for the plane.

7. Define the *terms*: *friction*, *limiting friction*, *angle of friction* (λ) and *coefficient of friction* (μ), and show that the coefficient of friction (μ) = $\tan \lambda$.

What is meant by the *cone of friction*?

8. A square framework formed of uniform heavy rods of equal weights W, is hung by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find its tension? (*Allahabad* and *Delhi*)

Ans 4 W.

9. Three equal smooth pencils, each of weight W, are tied together by a single loop of fine inextensible cotton so that each touches the other two. Prove that the bundle can rest in stable equilibrium on a smooth table only if the breaking tension of the cotton is not less than $W\sqrt{3}/6$.

(*Oxford Scholarship and Higher School Certificate*)

10 A cylinder of radius 1 *cm.* and length 4 *cms.* is standing on end on an inclined plane, the angle of which is gradually increased. If the coefficient of friction between the cylinder and the plane is 0·3, find whether the cylinder will slide or topple first. **Ans.** *Slides* first.

CHAPTER XII

FLOW OF LIQUIDS—VISCOSITY

200. Rate of Flow of a Liquid. A liquid, for our present purpose, is taken to be *perfectly mobile* and practically *incompressible* and, therefore, the same amount of it flows across every section of a tube in a given time. *The rate of flow of a liquid is, therefore, defined as the volume of it that flows across any section in unit time.*

If the velocity of flow of a liquid be v, in a direction perpendicular to two sections A and B, (Fig. 256), of area a, and distance l apart, and if t be the time taken by the liquid to flow from A to B, we have

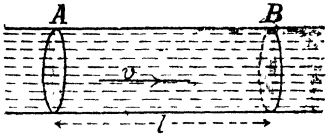


Fig. 256.

$$vt = l.$$

Obviously, the volume of liquid flowing through the section AB, in *this* time, is equal to the cylindrical column $AB = l \times a$, or $= vt \times a$. This, therefore, is the volume of the liquid flowing across the section in time t.

$$\therefore \quad \textit{rate of flow of liquid} = \frac{vt \times a}{t} = v \times a.$$

$$= \textit{velocity of liquid} \times \textit{area of cross-section of the tube.}$$

Sometimes, the rate of flow of a liquid is also expressed in terms of the *mass of the liquid flowing across any section in unit time*; so that, in this case,

$$\textit{rate of flow of liquid} = \textit{mass of liquid flowing across any section per unit time.}$$

$$= \textit{velocity of liquid} \times \textit{area of cross-section} \times \textit{density of liquid.}$$

$$= v \times a \times \rho.$$

201. Lines and Tubes of Flow. In a simple flow of liquid, *i.e.*, *when it is not turbulent*, but *steady*, the velocity at every point in the liquid remains constant, (in magnitude, as well as direction), the energy needed to drive the liquid being used up in overcoming the '*viscous drag*' between its layers. In other words, each particle follows exactly the *same path* and has the *same velocity* as its predecessor and the liquid is said to have an *orderly* or a **stream-line flow.** In such a case, if we consider a line along which a particle of the liquid moves, the direction of the line at any point is the direction of the velocity of the liquid at that point. Such a line is called a **stream-line.** More correctly, *a stream-line may be defined as a curve the tangent to which at any point gives the direction of flow of the liquid at that point*; for, *it may be straight or curved, according as the lateral pressure on it is the same throughout or different,—in the latter case the pressure being greater on the convex side than on the concave one.*

This holds good, however, only so long as the velocity of the liquid does not exceed a particular limiting value, called its *critical velocity*, beyond which the flow of the liquid loses all its steadiness or orderliness, and becomes *zig zag* or *sinuous*, acquiring what is called a *turbulent motion*. This may be easily seen by introducing a small jet of colouring matter into a tube through which a liquid may be made to flow with a gradually increasing velocity ; when, as long as the velocity remains below its critical value, we see only a thin streak of the colouring matter along the axis of the tube. [Fig. 257, (*a*)], representing a stream-line motion, but when the velocity reaches *this* value, the colouring matter takes a *zig zag* path, [Fig. 257, (*b*)], and later, when this value is exceeded, the colouring matter spreads out in all directions, filling the entire tube, showing that the motion is no longer steady or orderly but has become '*turbulent*'. The energy needed to drive the liquid is here dissipated, for the most part, in setting up eddy currents in the liquid.

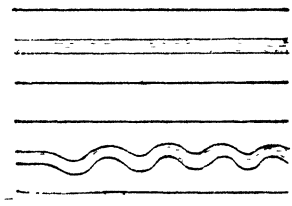


Fig. 257.

Consider two areas, *A* and *B*, at right angles to the direction of flow of the liquid, (Fig. 258), and draw *stream lines* through their boundaries ; then, a *tube AB of the liquid* is obtained. This is known as a **tube of flow**.



Fig 258.

As explained above, the volume of liquid passing through section *A* is equal to that passing through section *B*. For, the sides of the tube being everywhere in the direction of flow of the liquid, no liquid can cross the sides but must enter or leave through the ends. Since the velocity is constant over a section, *i.e., the motion is steady*, (if the tube be narrow), the volume of the liquid entering section *A* is equal to $a_1.v_1$ *per sec.*, and the volume of the liquid leaving section *B* is equal to $a_2.v_2$ *per sec.*, where a_1, a_2, and v_1, v_2, are the areas of cross-section and velocities at sections *A* and *B* respectively.

$\therefore$ we have $$a_1.v_1.\rho_1 = a_2.v_2.\rho_2,$$

where ρ_1 and ρ_2 are the densities of the liquid at the two sections respectively.

The liquid being incompressible, $\rho_1 = \rho_2$, and so we have

$$a_1.v_1 = a_2.v_2,$$

i.e., the volume of the liquid entering section A is equal to that leaving section B.

202. Energy of the Liquid. Since a liquid has *inertia*, it possesses *kinetic energy*, when in motion. It is also subject to *pressure*, and may also have *potential energy*, due to its position. We have thus three types of energy possessed by a liquid in flow, *viz.*, (*i*) *kinetic energy*, (*ii*) *potential energy, and* (*iii*) *pressure energy.*

(*i*) **Kinetic Energy.** We know that $K.E. = \frac{1}{2} mv^2$, so that the kinetic energy of a mass *m* of a liquid, flowing with a velocity *v*, is

given by $\frac{1}{2} mv^2$. If we consider *unit volume* of the liquid, $m = \rho$, the *density* of the liquid, and, therefore, we have

kinetic energy per unit volume of the liquid $= \frac{1}{2} \rho v^2$.

And, if we consider *unit mass* of the liquid, $m = 1$, and, therefore,

kinetic energy per unit mass of the liquid $= \frac{1}{2} v^2$.

(*ii*) **Potential Energy.** We have $P.E. = mgh$; so that, the potential energy of a liquid of mass m at a height h above the earth's surface (*i.e.*, in its gravitational field) is equal to mgh. Again, if we consider *unit volume* of the liquid, $m = \rho$, the *density* of the liquid, and, therefore,

P.E. per unit volume of the liquid $= \rho.g.h$,

But, if we consider *unit mass* of the liquid, $m = 1$ and we have

P.E. per unit mass of the liquid $= gh$.

(*iii*) **Pressure Energy.** Consider a tank A, containing a liquid of density ρ, provided with a narrrow side tube T, of cross-sectional area a, properly fitted with a piston P that can be smoothly moved in and out, (Fig. 259). Let the *hydrostatic pressure* due to the liquid, at the level of the axis of the side tube, be p, so that the force on the piston is $= p.a$. If, therefore, more liquid is to be introduced into the tank, this much force has to be applied to the piston in moving it inwards. Let the piston be moving slowly inwards through a distance x, so that the velocity of the liquid be very small and *there may be no kinetic energy acquired by it*. Then, clearly, a volume of the liquid $a.x.$, or a mass $a.x.\rho$ of it, is forced into the tank, and an amount of work $p.a.x$ is performed to do so. This work, (or energy), $p.a.x$, required to make the liquid move against pressure p, *without imparting any velocity to it*, thus becomes the energy of the mass $a.x.\rho$ of the liquid in the tank, for it can do the same amount of work in pushing the piston back, when escaping from the tank. It is referred to as the *pressure energy* of the liquid.

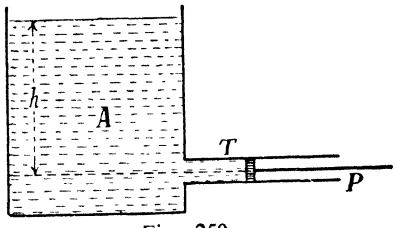


Fig. 259.

Thus pressure energy of a mass $a.x.\rho$ of the liquid is equal to $p.a.x$, and, therefore.

pressure energy per unit mass of the liquid $= \frac{p.a.x}{a.x.\rho} = \frac{p}{\rho} = \frac{pressure}{density}$.

Now, if we consider *unit volume* of the liquid, we have

pressure energy of volume $a.x$ of the liquid $= p.a.x$,

and $\therefore$ *pressure energy per unit volume of the liquid* $= \frac{p.a.x}{a.x} = p$,

the pressure of the liquid.

The three types of energy possessed by a liquid under flow are mutually convertible, one into the other. For, consider a liquid of density ρ contained in a vessel, and let its depth be h, (Fig. 260). Then, pressure due to the liquid column h at the bottom of the vessel is $p = h\rho.g$. If we take unit mass of the liquid from the bottom B to the surface A, clearly, $h.g$. units of work has to be done against gravity, and, therefore, the potential energy of the liquid increases by this much amount ; or this much work is done by gravity if unit mass of the liquid comes down through a depth h. Hence, potential energy of unit mass of the liquid is equal to $h.g$. And, since pressure at a depth h, is given by $p = h.\rho.g$, and pressure energy per unit mass of the liquid = pressure/density, we have *pressure energy per unit mass of the liquid* $= h.\rho.g/\rho = h.g =$ *potential energy lost by the liquid in descending through* h.

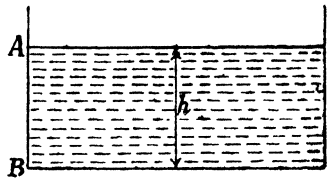


Fig. 260.

Thus, we see that pressure energy and potential energy are convertible, one into the other, and, therefore, their sum for a liquid at rest is constant.

Again, consider the flow of liquid through a tube, (Fig. 261). If the liquid has a constant velocity, there is no resultant force acting upon it. But, if the flow is accelerated, there must be a *pressure gradient* along the tube of flow. Let the change of pressure for a distance dx be dp, *i.e.*, let the pressure gradient be dp/dx, which may be taken to be constant for a short length of the tube.

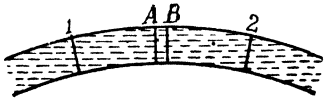


Fig. 261.

If the direction of flow be from A to B, the pressure decreases from A to B. If, therefore, p be the pressure at the cross-section B, that at A will be greater by $\delta x.dp/dx$, if the small distance AB be δx, *i.e.*, the pressure at A will be $p - \delta x.dp/dx$. The resultant force on the slice AB of the liquid will, therefore, be $a.\delta x.dp/dx$, where a is the cross-section of the tube, (*force being equal to pressure* $\times$ *area*).

Let the *velocity gradient* along the tube of flow be dv/dx ; then, if v be the velocity at A, the velocity at B will be $v + \delta x.dv/dx$, because *the velocity increases in the direction A to B*, and, therefore, increase in velocity through the distance δx will be $\delta x.dv/dx$.
If the liquid covers this distance in time δt, we have

$$\delta t = \delta x/v, \qquad \text{whence,} \qquad v = \delta x/\delta t.$$

Or, *in the limit*, $v = dx/dt$.

Now, *acceleration = rate of change of velocity* and, therefore, acceleration at the section $AB = dv/dt$, and mass of liquid in the section $= a.\delta x.\rho$; so that, force on it $= a.\delta x\ \rho\ dv/dt$, (because *force = mass* $\times$ *acceleration*).

But force on this slice of the liquid is also equal to $a.\delta x.dp/dx$.

$$\therefore \qquad -a.\frac{dp}{dx}.\delta x = a.\delta x.\rho.\frac{dv}{dt},$$

the *−ve sign* merely indicating that the pressure and velocity gradients are opposite in sign, *i.e., whereas the pressure decreases,* the *velocity increases along AB.*

Or, $$-\frac{dp}{dx} = \rho.\frac{dv}{dt} = \rho.\frac{dv}{dx}\frac{dx}{dt} = \rho.v.\frac{dv}{dx}. \qquad [\because dx/dt = v.$$

And $\therefore$ $-dp = \rho.v.dv.$ Or, $-dp/\rho = v.dv.$

Or, $$-\frac{1}{\rho}\int_{p_1}^{p_2} dp = \int_{v_1}^{v_2} v.dv.$$ { where p_1, p_2 and v_1, v_2 are pressures and velocities at sections 1 and 2, respectively.

Or, $$-\frac{p_2}{\rho} - \left(-\frac{p_1}{\rho}\right) = \frac{v_2^2}{2} - \frac{v_1^2}{2}.$$

Or, $$\frac{p_1}{\rho} - \frac{p_2}{\rho} = \frac{1}{2}(v_2^2 - v_1^2),$$

i.e., pressure energy and kinetic energy are convertible, one into the other.

Since pressure energy is also convertible into potential energy, it follows that the *three types of energy are mutually convertible into each other.*

203. Bernoulli's Theorem and its Important Applications. *Bernoulli's theorem states that the total energy of a small amount of liquid flowing from one point to another, without any friction, remains constant throughout the displacement.*

We have seen that pressure energy and potential energy of a liquid are convertible, one into the other, and so are its pressure energy and kinetic energy. It follows, therefore, that in any *stream-line flow* of liquid, the loss of energy in one form is equal to the gain of energy in another, or that the sum total of its energy, *viz.,*

potential energy + pressure energy + kinetic energy = a constant.

Or, $$hg + p/\rho + \tfrac{1}{2}v^2 = C, \text{ a constant} \quad \ldots \quad \ldots \quad (i)$$

This relation is known as **Bernoulli's Equation.**

If we divide relation (*i*) by g, we have

$$h + \frac{p}{\rho g} + \frac{1}{2}\cdot\frac{v^2}{g} = C', \text{ another constant.} \quad \ldots \quad \ldots \quad (ii)$$

Now, h is what is called the *gravitational head*, $p/\rho g$, *the pressure head* and $\frac{1}{2}v^2/g$, the *velocity head**. Thus,

gravitational head + pressure head + velocity head = a constant.

We may, therefore, also state Bernoulli's theorem in another way, *viz.,* that *at all points, in the stream-line flow of a liquid, the sum of the gravitational head, the pressure head and the velocity head remains constant throughout.*

It follows at once from relation (*ii*) that if the flow of the liquid be horizontal, the gravitational head h is a constant ; so that, here,

$$\frac{p}{\rho.g} + \frac{1}{2}\cdot\frac{v^2}{g} = a \text{ constant.}$$

Similarly, from relation (*i*), we would have $p/\rho + \frac{1}{2}v^2 = a$ *constant,* since the potential or gravitational energy hg would be a constant.

Or, $$p + \tfrac{1}{2}\rho v^2 = a \text{ constant.} \quad \ldots (iii)$$

*For, the liquid must fall through this much height to attain the velocity v.

Here, p is referred to as the *static pressure* of the liquid and $\frac{1}{2}\rho v^2$ as its *dynamic or velocity pressure* So that, we may express this result by saying that *for a horizontal motion of the liquid, the sum of its static and dynamic pressures remains a constant.*

Thus, if in a liquid, flowing herizontally, the pressure and velocity at one point be p_1 and v_1 and at another, p_2 and v_2 respectively, we have

$$\frac{p_1}{\rho} + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + \frac{1}{2} v_2^2,$$

which shows that pressure and velocity (and, therefore, kinetic energy) can only increase at the expense of one another, *i.e., points of maximum pressure correspond to those of minimum velocity*, and *vice versa**. This principle is made use of in various important practical applications, (see § 204).

204. Important Applications of Bernoulli's Equation

(*i*) **Velocity of Efflux of a Liquid.** Let the surface of the liquid be at a height h above the level of the orifice O in a tank, (Fig. 262). If the tank be sufficiently wide, the velocity at the liquid surface may be taken to be *zero*, the pressure there being, clearly, atmospheric. Since the pressure is also atmospheric at the orifice, where the liquid emerges, it plays no part in the flow of the liquid. If v be the velocity at the level of the orifice, we have, considering a tube of flow beginning at A and ending at O,

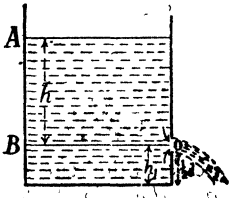


Fig. 262.

total energy at A = pressure energy + potential energy + kinetic energy

$$= 0 + hg + 0,$$

because *pressure* at $A = 0$, $P.E. = gh$ and $K.E. = 0$. [$\because v = 0$.

And, *total energy at O*, the level of the orifice

$$= 0 + 0 + \tfrac{1}{2}v^2,$$

because pressure at $O = 0$, $P.E. = 0$, and $K.E. = \frac{1}{2}v^2$.

Since total energy remains the *same*, we have

$$\tfrac{1}{2}v^2 = hg. \quad \text{Or,} \quad v^2 = 2gh,$$

whence,

$$v = \sqrt{2gh}.$$

This, then, is the *velocity of efflux* of the liquid at the orifice O.

This result was first obtained by *Torricelli* (in the year 1644) and hence is known as **Torricelli's Theorem,** or *the Law of Efflux*, and may be stated as follows :

The velocity of efflux of a liquid through an orifice is equal to that which a body attains in falling freely from the surface of the liquid to the orifice.

For, clearly, if the liquid had fallen freely through this height h, its velocity would be given by the relation, $v^2 = 2gh$, to be equal to $v = \sqrt{2gh}$, the same as obtained above.

This *ideal* velocity is, however, seldom reached, for no liquid is perfectly free from friction (or viscosity).

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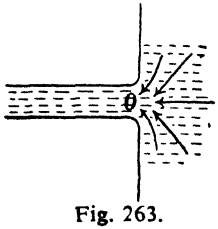
* This result is also true for compressible fluids and is sometimes referred to as **Hawksbee's law.**

Now, the liquid-jet flows out in the form of a *parabola*, and takes time equal to $\sqrt{2h_1/g}$ to fall through a height h_1 to a plane, in level with the bottom of the vessel, striking the plane at a distance d, called its **range**, such that

$$d = v \times \sqrt{\frac{2h_1}{g}} = \sqrt{2gh \times \frac{2h_1}{g}} = 2\sqrt{h \times h_1}.$$

For a given height $(h+h_1)$, of the liquid column, this range will be a maximum when $h=h_1$. And, obviously, if the jet were directed upwards, it should theoretically rise to the level A of the free surface of the liquid. But, again, due to air-resistance and viscosity, the height attained is actually less than this ideal one.

(*ii*) **Vena Contracta**. The *whole* of the liquid entering the orifice does not move perpendicularly to it, but comes from all directions, as shown in Fig. 263, the stream-lines near the edges being curved. The liquid coming from the sides of the vessel, as it enters the orifice, has still a lateral velocity due to inertia and continues to move inwards towards the centre of cross-section of the jet, until the increasing outward pressure is balanced by the atmospheric pressure at the jet. The liquid jet thus *contracts* at C, a little outside the orifice, to a *neck*, called the *Vena Contracta*. *It is here that the jet becomes uniform and the velocity becomes the same throughout*, and it is *this* velocity which is given by Torricelli's equation, (see § 204, (*i*), above, page 423).


Fig. 263.

Obviously, the area of the jet at the *Vena Contracta* is smaller than the area of the orifice and is found to be about ·62 times the latter. The volume of the liquid passing out through the orifice in unit time is, therefore, equal to $\cdot 62a\sqrt{2gh}$. This ratio between the area of the *Vena Contracta* and the orifice is called the *coefficient of contraction*.

N.B. If *outflow tubes* of suitable shapes be used, the *Vena Contracta* may be almost completely avoided, but the velocity of efflux always suffers a diminution in its value due to a loss in the kinetic energy of the liquid, caused by its internal friction or viscosity, —this diminution being quite independent of the *Vena Contracta*.

(*iii*) **Venturimeter**. It is an arrangement to measure the amount of flow of a liquid in a pipe,—usually water, when it is called a *venturi water-meter*.

The principle underlying it is that *when a liquid flows through a tube of a varying bore or cross-section, the velocity and pressure vary along the tube, the pressure being the least where the velocity is the greatest, and vice versa.*

For, if we have a tube KLM, with a constriction at L, (Fig. 264), the velocity of the liquid will be greater at L, the narrowest part of the tube, than that at K or M. Let the velocity at L be v_l and that at K be v_k. Then, $v_l > v_k$.

Applying *Bernoulli's theorem*, we have

(*potential energy+pressure energy+kinetic energy*) at L
$=$ (*potential energy+pressure energy + kinetic energy*) at K.

Or, $h_l.g+\frac{p_l}{\rho}+\frac{1}{2}v_l^2$
$=h_k.g+\frac{p_k}{\rho}+\frac{1}{2}v_k^2$

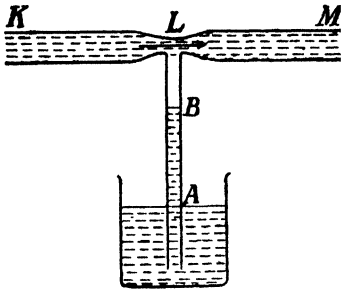


Fig. 264.

where h_l, p_l and v_l are the *height*, *pressure* and *velocity* of the liquid at L and k_k, p_k and v_k, their corresponding values at K, ρ being the *density* of the liquid, supposed constant, because the liquid is taken to be incompressible.

If the tube be horizontal, $h_l = h_k$, so that the above relation becomes

$$\frac{p_l}{\rho}+\tfrac{1}{2}v_l^2=\frac{p_k}{\rho}+\tfrac{1}{2}v_k^2. \quad \text{Or,} \quad \tfrac{1}{2}v_l^2-\tfrac{1}{2}v_k^2=\frac{p_k}{\rho}-\frac{p_l}{\rho}. \quad \ldots(i)$$

Since $v_l > v_k$, it is clear that $p_k > p_l$, *i.e.*, the pressure at L is less than at K. This can be shown by attaching a vertical tube, connected to KLM at L and dipping it into a liquid, not miscible with the one in KLM, when the liquid rises up in the vertical tube, as shown at AB, and it will be seen that the *narrower the bore at L, the greater the rise of the liquid in the vertical tube.*

Let us now consider a pipe through which water is flowing, such that it has a cross-section a_1 at K and a_2 at L, (Fig, 265). Then, if

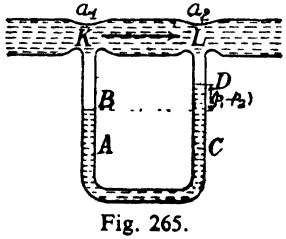


Fig. 265.

v_1 and v_2 be the velocities of water at K and L respectively, we have

$$a_1v_1 = a_2v_2, \text{ [see § 201, (page 419).}$$

whence, $v_2 = a_1v_1/a_2$.

And, since ρ is 1 for water, relation (i) above becomes,

$$\tfrac{1}{2}v_2^2-\tfrac{1}{2}v_1^2 = p_1-p_2,$$

where p_1 and p_2 are the pressures at K and L, respectively.

Or, substituting the value of v_2, we have

$$\frac{1}{2}\frac{a_1^2v_1^2}{a_2^2}-\frac{1}{2}v_1^2=(p_1-p_2). \quad \text{Or,} \quad \frac{1}{2}v_1^2\left(\frac{a_1^2}{a_2^2}-1\right)=(p_1-p_2).$$

Or, $\frac{1}{2}v_1^2\left(\frac{a_1^2-a_2^2}{a_2^2}\right)=(p_1-p_2)$. Or, $v_1^2=\frac{2a_2^2(p_1-p_2)}{a_1^2-a_2^2}$,

whence, $v_1=a_2\sqrt{\frac{2(p_1-p_2)}{a_1^2-a_2^2}}$. So that, $a_1v_1=a_1a_2\sqrt{\frac{2(p_1-p_2)}{a_1^2-a_2^2}}$.

Thus, if we know a_1, a_2 and (p_1-p_2), we can determine a_1v_1, the volume of the liquid flowing across the section K *per second.* The difference of pressure (p_1-p_2) at K and L is read directly on the vertical tubes AB and CD joined together to form a *manometer,* as shown.

(*iv*) **Pitot Tube.** This arrangement is also used to measure the amount of flow of water through a pipe and is based on the same principle as the venturimeter.

It consists of two vertical tubes, with small apertures at their lower ends, —the plane of the aperture of one tube PQ (Fig. 266), being parallel to the direction of flow of water and the aperture of the other tube RS, facing the flow. The rise of the liquid column in the tube PQ, therefore, measures the pressure at Q. And since the water is stopped in the plane of the aperture S of the tube RS, its velocity *there* becomes *zero*. Therefore, its kinetic energy is reduced from $\frac{1}{2}v^2$ to *zero*, where v is the velocity of flow of water. Its pressure, therefore, increases by an amount $\frac{1}{2}v^2$, and the water consequently rises to a higher level in the tube RS than in PQ. If h be the difference of level in the two tubes, we have

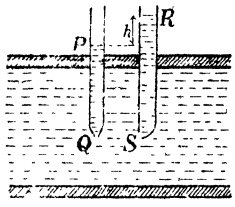


Fig. 266.

$$\frac{1}{2}v^2 = hg. \quad \text{Or,} \quad v^2 = 2gh,$$

whence,

$$v = \sqrt{2gh}.$$

This multiplied by a, the cross-section of the pipe, where the tubes are placed, gives the volume of water flowing *per second* past that section and the amount of flow of water is thus easily measured.

(v) Other Common Applications of Bernoulli's Theorem.

1. The Steam Injector. It is a simple device to accelerate the ejection of the exhaust steam from the cylinder of a steam engine, and consists of a tube A, (Fig. 267), narrowing down into a nozzle N at its lower end, inside another tube B, having a side-tube C, which is connected to the cylinder of the engine.

A jet of steam is introduced into A, and as it issues out of the nozzle N, its velocity is considerably increased, resulting in a corresponding fall in pressure there, and the steam from the engine-cylinder thus rushes into this region of reduced pressure, whence, it is ejected out through the lower end of B.

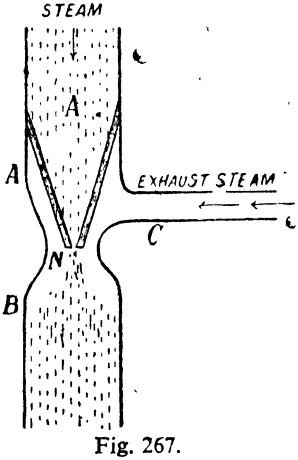


Fig. 267.

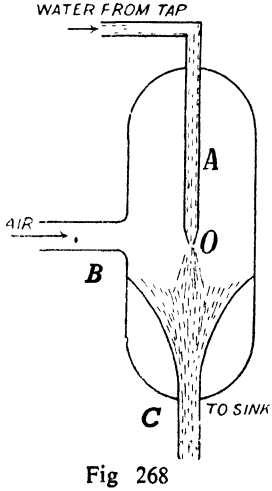


Fig 268

2. The Filter Pump. It is also based on the same principle and is used to reduce the pressure in a vessel. Here, a stream of water from a tap, flowing through a tube A, (Fig. 268), issues out in the form of a jet from its narrow orifice O, which results in a great rise in its velocity and proportionate fall in its pressure, which is thus soon reduced to a value, below that of the atmosphere The air from the vessel, connected through a side-tube B to this region of reduced pressure, then rushes into it, and is carried away by the stream of water as it flows down through C.

In this way, the pressure in the vessel is ultimately reduced to just a little above the vapour pressure of water, in a comparatively very short time.

If the inlet water tube be a twisted, instead of a straight, one, the exhaustion proceeds more rapidly, due to the rotating water-jet in the tube breaking up more readily and mixing up easily with the incoming air from the vessel.

3. The Atomizer. The *atomizer* or *sprayer*, used for spraying scents etc. is yet another example of a fall in pressure due to an increase in velocity.

Here, (Fig. 269), air is blown through a tube T, (usually by compressing a rubber bulb) fitted on the tube at one end, which, when it rushes out of the aperture O, where the tube narrows down, acquires high velocity. The pressure in the vicinity of O is thus greatly reduced, and since O lies directly above the vertical tube, dipping in the liquid in vessel V, the liquid rises up through it, when, on issuing out of the aperture at the top, it is blown into a fine spray by the air stream from T.

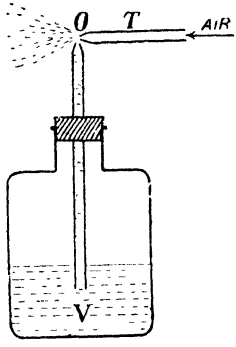


Fig. 269.

4. The Attracted-disc Paradox. The following is a simple and interesting experiment, which the student may well try for amusement at a small gathering at home.

DE is a flat card-board disc (Fig. 270), over which is placed another flat disc BC, fitted with a tube A, the opening of which is in *flush* with BC.

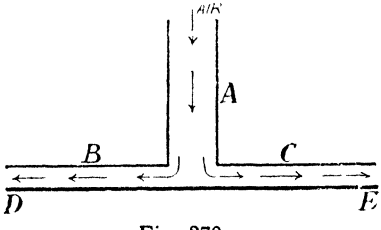


Fig. 270.

On blowing air down through A, on to DE, the latter, instead of being blown away from BC, as one might ordinarily expect, sticks on to it more and more closely, and might even be lifted up a little.

This seeming paradox is, however, easily explained. For, as the air from A rushes through the narrow space in between BC and DE, its velocity increases and consequently the pressure there decreases, so that it soon falls below the atmospheric pressure on DE, which thus pushes it up towards BC.

5. The Bunsen-Burner. This too is a familiar example of a fall of pressure due to increased velocity. For, as the gas issues out with a great velocity from the fine nozzle, down below, the pressure falls in its immediate neighbourhood, and the air is thus sucked in through the hole O, (Fig. 271), and gets mixed up with the gas.

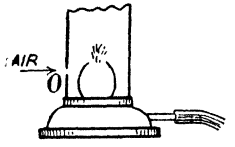


Fig. 271.

6. The Magnus Effect. If a ball, or a sphere be rotated about an axis through it, perpendicular to the plane of the paper, the air surrounding it is also set into motion, the streamlines taking the form of concentric circles in planes, parallel to the plane of the paper, their direction being the same as that of the rotation of the ball, shown in Fig. 272 (*a*). And, obviously, the rougher the surface of the ball, the thicker the layer of air thus set into motion.

If, however, the ball be given only a linear forward motion, it pushes aside the air in front of it, to make room for itself, and this displaced air then flows along its sides on to its back or the rear end, the form of the streamlines being as shown in Fig. 272 (*b*).

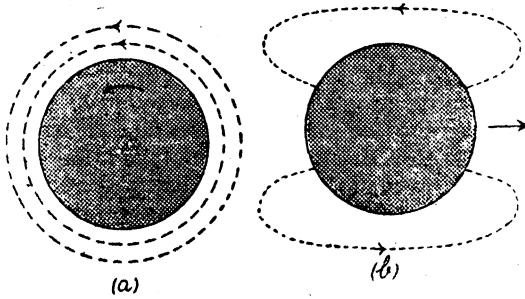


Fig. 272.

And, finally, if the ball be given both, a rotatory and a circular motion simultaneously, it is clear from Figs. 272 (*a*) and (*b*) that the streamlines due to the two motions run in opposite directions on the underside of the ball, but in the same direction on its upper side. Thus, there is a *decrease of velocity* or an *increase of pres-*

sure, on the lower, and an *increase of velocity* or a *decrease of pressure* on its upper, side. The ball, as a result of this difference in the lateral pressure on it, takes a *curved path* which is convex towards the greater pressure side. This is what is called the *Magnus Effect* and is easily observed when a tennis or a golf ball is given a spin.

7. **The Cylindrical Shape of a Bullet.** We have already seen [§ 46, (*ii*), page 98], that for giving directional stability, it is desirable to give the shot or bullet a rapid '*spin*' about an axis, along its direction of motion, and how this object is achieved by '*rifling*' the barrel of a gun or rifle, (*i.e.*, by cutting spiral grooves inside it).

Now, if the bullet were spherical in shape, there will come about, as explained above in case 6, a difference in the lateral pressure on it during its passage through air, on account of its simultaneously possessing a rotatory and a linear motion and the bullet will thus be deflected from its straight path To avoid this, the bullet is made cylindrical in shape, so that the lateral pressure on it remains uniform and it flies undeflected along its path.

8. **Streamline Bodies.** The student has no doubt heard of streamline bodies of automobiles, particularly of racing cars etc. We shall discuss in brief here as to what this *streamlining* of a body really connotes in the language of Science.

As a body moves through air, or through a fluid, in general, it carries a part of it along with itself, pushing the rest on to either side. The steamlines of the fluid, directed towards the body, open out to either side to make way for it, as it were, and meet some distance behind it. This fluid at the rear of the body, enclosed by the streamlines meeting there is thus carried by the body as a sort of a '*tail*'. Some extra work has thus to be done by the body, in carrying this extra burden, resulting in an appreciable decrease in its kinetic energy and velocity. In fact, the body has to encounter a double opposition to its forward motion, *viz.*, (*i*) *an increased pressure in front*, called the **head pressure**, and (*ii*) *a decreased pressure* or the **tail suction** behind, (which exerts a backward pull on it).

Naturally, the surrounding fluid flows into this rear region of decreased pressure or *tail suction*, and is thus thrown up into vortices (*i.e.*, whirls and eddies) there, which results in a further fall in pressure in this region. These vortices are thus responsible for dissipating away a fairly good part of the energy of the body, thus decreasing its velocity or offering resistance to its motion.

If, therefore, the resistance to the forward motion of the body is to be minimised, *it should be given a shape similar to that of the fluid forming its 'tail', so that there is no tailsuction region formed at all at its rear*, and no energy is thus dissipated in the formation of whirls and eddies. The body is thus made with a gradually decreasing cross-section, tapering towards the rear, and having no sharp corners or edges anywhere. The body is then said to have a *streamline shape* and the resistance to its forward motion is considerably decreased. This explains the shape of the bodies of big airliners and of most of the modern cars.

205. Viscosity. When a liquid flows slowly and steadily over a fixed horizontal surface, *i.e*, when its flow is streamline, its layer in contact with the fixed surface is stationary and the velocity of the layers increases with the distance from the fixed surface, *i.e., the greater the distance of a layer from the fixed surface, the greater its velocity.*

Considering any particular layer of the liquid, we have the layer immediately *below* it moving *slower* than it, and the one immediately *above* it moving *faster* than it, so that the former tends to retard its motion and the latter tends to accelerate it. The two layers thus tend to destroy their relative motion, *as though there were a backward dragging force, acting tangentially on the layers.* If, therefore the relative velocity between the two layers is to be maintained, an external force must be applied to overcome this *backward drag.* In the absence of any such outside force, the relative motion between the layers is destroyed and the flow of the liquid ceases. *This property of a liquid by virtue of which it opposes relative motion between its different layers is known as viscosity or internal friction of the liquid.*

206. Coefficient of Viscosity (η). *Newton* showed that the backward dragging, or *viscous*, force, acting tangentially on any liquid layer, is directly proportional to its surface area A, and velocity v, and inversely proportional to its distance x from the stationary layer. Denoting this force by F, therefore, we have

$$F \propto A\ ;\quad F \propto -v\ ;\quad F \propto \frac{1}{x}\ ;$$

the $-ve$ *sign* of v merely indicates that the direction of the force is opposite to that of velocity.

Or, $$F \propto -\frac{A.v}{x}, \qquad i.e., \qquad F = -\eta.\frac{A.v}{x},$$

where η is a constant, depending upon the nature of the liquid, and is called its *coefficient of viscosity.**

Now, v/x may be put as dv/dx, which gives the *rate of change of velocity with distance*, and is called the *velocity gradient* ; so that, we have

$$F = -\eta.A.\frac{dv}{dx}. \qquad \ldots I.$$

This is know as *Newton's law of viscous flow in streamline motion.*

If, $A = 1$ *sq. cm.*, and $dv/dx = 1$, we have $F = \eta$.

Thus, the **coefficient of viscosity** *of a liquid may be defined as the tangential force required per unit area to maintain a unit velocity gradient, i.e., to maintain unit relative velocity between two layers unit distance apart.* And, clearly, if this tangential force be unity, the *coefficient of viscosity* of the liquid is unity, and is called *Poise*, after **Poiseuille,** whose work on viscosity is important.

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* This coefficient n is sometimes referred to as the *dynamic viscosity of the liquid,* with *Poise* as its *C.G.S.* unit (see below). On the other hand, the ratio n/ρ (where ρ is the density of the liquid) is called its *kinematic viscosity* (k), and the corresponding *C.G.S.* unit for it is the *stokes.*

Dimensions of η It is clear from relation *I* above that

$$\eta = \frac{F}{A.dv/dx}.$$

So that, the dimensions of η are those of $\frac{\text{force}}{\text{area} \times \text{velocity gradient}}$.

$$= \frac{MLT^{-2}}{[L^2].\left[\frac{L/T}{L}\right]} = \frac{MLT^{-2}}{L^2.T^{-1}}. \qquad \left\{\because \frac{dv}{dx} = \left[\frac{L/T}{L}\right]\right.$$

Or, $$\eta = ML^{-1}T^{-1}.$$

Viscosity in liquids corresponds to solid friction in so far as, like the latter, it also opposes relative motion between two layers. It, however, differs from solid friction in that, *unlike solid friction, it depends upon (i) the surface area of the liquid layer, (ii) its distance from the stationary layer end (iii) its velocity with respect to the stationary layer.*

207. Fugitive Elasticity. The expression for *F*, above, may be re-arranged and put as

$$\eta = \frac{F/A}{dv/dx}, \qquad \ldots(a)$$

i.e., $$\textit{coefficient of viscosity} = \frac{\textit{tangential stress}^*}{\textit{velocity gradient}}.$$

This is an expression similar to the one for the coefficient of rigidity, *viz.*,

$$n = \frac{F/A}{\theta} = \frac{F/A}{dy/dx}. \qquad \ldots(b) \left[\because \theta = \frac{dy}{dx}\right.$$

$$= \frac{\textit{tangential stress}^*}{\textit{displacement gradient}}.$$

Maxwell, therefore, considered a liquid to possess a certain amount of rigidity, breaking down continually under a shearing stress. Very fittingly, he imagined viscosity of a liquid to be the limiting case of the rigidity of a solid, when the latter breaks down under the shear applied. A liquid is thus regarded as capable of exerting and sustaining an amount of shearing stress for a short time, after which it breaks down and the shear is formed over again. In other words, a liquid offers a *fugitive resistance* to shearing stress, which is continually breaking down, and it may thus be said to possess a fugitive rigidity.

Now, if the rate at which the shear (θ) breaks down be taken to be proportional to shear, we have

rate of the breakdown of shear $\propto \theta$.

Or, ,, ,, ,, ,, ,, $= \lambda.\theta$, where λ is a constant.

And, clearly, the rate of formation of the shear

$$= \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{dv}{dx}.$$

[*v* being the velocity in the *same plane.*

*It will be noted from expressions (*a*) and (*b*) that whereas in a fluid, the viscous drag is proportional to the velocity gradient, perpendicular to the direction of motion, the shearing stress, in a solid, is proporlional to the displacement gradient, perpendicular to the direction of shear.

Thus, when the motion of the fluid becomes quite steady, the rate of formation of the shear must be the same as that of its breakdown. So that,

$$\lambda.\theta = \frac{dv}{dx}. \quad \text{Or,} \quad \theta \Big/ \frac{dv}{dx} = \frac{1}{\lambda}. \qquad \ldots(c)$$

Now, dividing relation (*a*) by (*b*), above, we have $\frac{\eta}{n} = \frac{\theta}{dv/dx}$.

$\therefore$ substituting $\frac{\eta}{n}$ for $\theta \Big/ \frac{dv}{dx}$ in relation (*c*) above, we have

$$\eta/n = 1/\lambda.$$

This quantity $1/\lambda$ is called the *'time of relaxation of the medium'* and gives the time taken by the shear to disappear, provided no fresh shear is applied.

208. Critical Velocity. It was *Osborne Reynolds* who first showed by direct experiment that the *critical velocity* v_c of a liquid is given by the relation, $v_c = k.\eta/\rho r$, called *Osborne Renyold's formula*, where η is its coefficient of viscosity, ρ, its density and r, the radius of the tube, the constant k being called *Reynold's number*, its value being about 1000 for narrow tubes.

The expression for v_c may, however, be easily deduced by the method of dimensions, as explained below :

Since v_c is found to depend upon (*i*) η, (*ii*) ρ, and (*iii*) r, we have

$$v_c = k.\eta^a\rho^b r^c, \text{ say.} \qquad [k \text{ being a constant.}$$

So that, putting the dimensions of the quantities involved, we have

$$\text{Or,} \quad \begin{matrix} [LT^{-1}] = [ML^{-1}T^{-1}]^a[ML^{-3}]^b[L]^c \\ [LT^{-1}] = [M^{a+b}L^{-a-3b+c}T^{-a}] \end{matrix} \quad \left[\begin{matrix} k \text{ having no dimensions being a constant.} \end{matrix} \right.$$

Since the dimensions on the two sides of the equation must be the same, (by the *principle of homogeneity of dimensions*), we have

$$a+b = 0 \quad \ldots(i) \; ; \quad -a-3b+c = 1 \quad \ldots(ii) \; ; \quad -a = -1 \quad \ldots(iii)$$

So that, adding (*i*) and (*iii*), we have $b = -1$.

Substituting this value of (*b*) in (*i*), we have $a = 1$,

[or, directly from (*iii*).

And, substituting the values of a and b in (*ii*), we have $c = -1$.

Hence $v_c = k.\eta/\rho r$,

where k (*Reynold's number*) is, as mentioned above, near about 1000 for narrow tubes. Thus, *for narrow tubes*, $v_c = 1000.\eta/\rho r$.

It must be emphasized again that this relation applies only to narrow tubes. For tubes of wide bores, the value of v_c is very much greater, and may be even a thousand times greater than that given by the above relation.

Now, a mere glance at the expression for v_c, deduced above, will show that

(*i*) $v_c \propto \eta$; (*ii*) $v_c \propto 1/\rho$; and (*iii*) $v_c \propto 1/r$,

i.e., the critical velocity of a liquid is (i) directly proportional to its viscosity, (ii) inversely proportional to its density and (iii) inversely proportional to the radius of the tube through which it flows.

It follows, therefore, that *narrow tubes, and liquids of high viscosity, and low density tend to promote orderly motion,* whereas tubes of wide bores, and liquids of low viscosity and high density lead to turbulence.

Again, if we have a perfectly *mobile* or *inviscid* liquid, *i.e.*, a liquid. for which $\eta = 0$, then, obviously, $v_c = 0$; so that, its flow would be turbulent and not orderly, even for the smallest velocity and in the narrowest of tubes.

Thus, we see that *it is the viscosity of a liquid alone, due to which its flow may possibly be orderly and thus approximate to that of a perfect fluid.*

209. Poiseuille's Equation for flow of liquid through a tube. Imagine a cylindrical layer, or shell of liquid, of radius x, flowing through a capillary tube of radius r. Then, the velocity of flow *at all points* on this cylindrical shell will be the *same*. Let it be v. As the velocity of the layers in contact with the walls of the tube is *zero* and goes on increasing towards the axis, it is obvious that the liquid inside the imaginary cylinder is moving faster than that outside it, and the *backward tangential force* due to the outer slower, moving liquid on the inner faster moving liquid is, in accordance with relation *I* above, given by $\eta.2\pi x.l.dv/dx$, where η is the coefficient of viscosity of the liquid, [because, here, *surface area* (A) of the cylindrical shell of radius x is equal to $2\pi x.l$, where l is the length of the capillary tube, and dv/dx is the *velocity gradient* there].

Let the difference of pressure at the two ends of the capillary tube be P Then, the *forward force* on the cylindrical liquid shell, in the direction of flow, is clearly equal to $P \times \pi x^2$, and tends to accelerate the motion of the liquid. If, therefore, the motion of the liquid be steady, we have

$$\eta.2\pi x.l\frac{dv}{dx} = -P.\pi x^2,$$

the $-ve$ *sign* showing that the two forces act in opposite directions.

And $\therefore$ $$dv = \frac{-P.\pi x^2 dx}{\eta.2\pi x.l.} = \frac{-P.x.dx}{2\eta l}.$$

Integrating this expression for dv, we have $v = -\frac{P}{2\eta l}\int x.dx$.

Or, $$v = \frac{-P}{2\eta l}\cdot\frac{x^2}{2} + C_1 = \frac{-Px^2}{4\eta l} + C_1,$$

where C_1 is a constant of integration.

Now, $v = 0$, when $x = r$, because the layers in contact with the sides of the tube are stationary.

$\therefore$ $$0 = \frac{-Pr^2}{4\eta l} + C_1, \text{ whence, } C_1 = \frac{Pr^2}{4\eta l}.$$

$\therefore$ $$v = \frac{Pr^2}{4\eta l} - \frac{Px^2}{4\eta l} = \frac{P}{4\eta l}(r^2 - x^2).$$

This, therefore, is the *velocity of flow of the liquid* at a distance x from the axis of the tube, and a glance at the expression for v will show that the *profile* or the *velocity distribution curve* of the advancing liquid in the tube is a *parabola*, (Fig. 273),—the velocity increasing from 0 at the walls of the tube to a maximum at its centre.

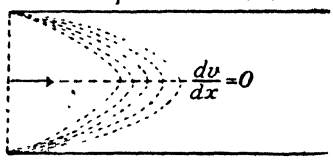


Fig. 273.

Now, imagine another *co-axial* cylindrical shell of the liquid, of radius $(x+dx)$. The cross-sectional area between the two shells is clearly $2\pi x.dx$ and, since v is the velocity of the flow of liquid in-between the two shells, the volume of liquid flowing *per second* through the cross-sectional area is given by $dV = 2\pi x.dx\, v$. If we imagine the whole of the tube to be made up of such like concentric cylindrical shells, the volume V of the liquid flowing through all of them, *i.e.*, through the capillary tube, *in unit time*, will be obtained by integrating the expression for dV between the limits, $x = 0$ and $x = r$.

$$\text{Or,} \qquad V = \int_0^r 2\pi x.dx.v = \int_0^r 2\pi x.\frac{P}{4\eta l}.(r^2-x^2)dx.$$

$$= \frac{\pi P}{2\eta l}\int_0^r (xr^2-x^3).dx = \frac{\pi P}{2\eta l}\left[\frac{x^2r^2}{2}-\frac{x^4}{4}\right]_0^r$$

$$= \frac{\pi P}{2\eta l}\left(\frac{r^4}{2}-\frac{r^4}{4}\right) = \frac{\pi P.r^4}{8\eta l},$$

whence,
$$\eta = \frac{\pi P r^4}{8Vl}. \qquad \ldots II$$

Thus, if we know P, r, V and l, the coefficient of viscosity of the liquid (η) can be easily determined.

The above relation holds good only when

(*i*) *the flow is steady and streamline, i.e.*, when its average velocity is less than its critical velocity ;

(*ii*) *the pressure is constant over every cross-section, i.e., there is no radial flow* ; and

(*iii*) *the liquid in contact with the sides of the tube is stationary.*

When the velocity of flow is small, and the tube is a narrow one, these assumptions are more or less valid. It is clear, therefore, that for tubes of wide bores, the relation breaks down ; for, in their case, the value of the critical velocity is much smaller ($\because v_c \propto 1/r$) and the flow of the liquid becomes turbulent. Thus, if we were to plot a graph between the pressure difference P between the two ends of the outflow tube, and the rate of flow V of the liquid, (*i.e.*, the volume of the liquid flowing out of it per second), we get a curve, as shown (Fig. 274), where the portion OA of the curve corresponds to the velocities less than the critical velocity and the portion AB, to those above its.

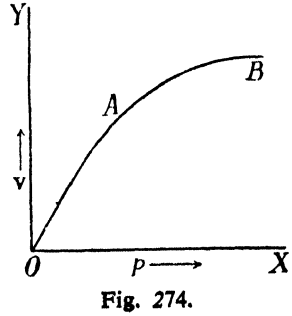


Fig. 274.

It is found that when the velocity of the liquid is below the critical value, the rate of flow V is proportional to P, the pressure difference, (as indicated by the straight part OA of the curve). Thus, within this range of velocity, the rate of flow of a liquid depends chiefly on its viscosity (η), quite in accordance with Poiseuille's formula.

Beyond the critical velocity, however, the pressure difference (P) is almost wholly utilized in combating the turbulence set up in the liquid, and in imparting kinetic energy to it ; so that, its rate of flow is now no longer proportional to P, and hence no longer depends upon its viscosity. In fact, it now depends mainly on the density of the liquid (ρ) and is approximately proportional to $\sqrt{P}$.

The following interesting consequences follow from the above :

(*i*) Since in turbulent motion, the rate of flow of a liquid is quite independent of its viscosity, it obviously follows that *all liquids, irrespective of their different viscosities, would require the same pressure difference to be driven through a tube at velocities higher than their critical velocities*. Thus, for example, a viscous liquid, like treacle, would require the same pressure difference to be driven through a tube, at a velocity greater than its critical velocity, as would be needed to drive water through it at the *same* velocity.

(*ii*) Since the critical velocity of a liquid is inversely proportional to the radius of the tube through which it flows, it is clear that liquids of all viscosities would flow equally readily through tubes of sufficiently wide bores. Thus, in a wide tube, treacle will flow just as freely as water. A typical natural example of this is the free flow of the highly viscous *lava* down the sides of an erupting volcano,—its rate of flow being about the same as we would expect in the case of water.

210. Experimental determination of η for a liquid—Poiseuille's method*. A capillary tube T, of known length l and radius r, is fixed *horizontally* near to the bottom of a vessel A, (Fig. 275), the liquid level in which can be kept constant at any desired height by means of an over-flow arrangement O. A clean and dry beaker, of known weight, is placed below the outer end of tube T to collect the liquid flowing out through it. The liquid is allowed to flow out in a slow trickle and collected in the beaker for a known time, and the beaker is then weighed again. The difference of the two weights gives the *mass* of the liquid flowing out in that time. Then, knowing the density of the liquid its volume can be determined and, dividing it by the time for which the liquid was allowed to flow, its volume V flowing out *per second* is known. Substituting the value of V, so obtained, in relation *II*,

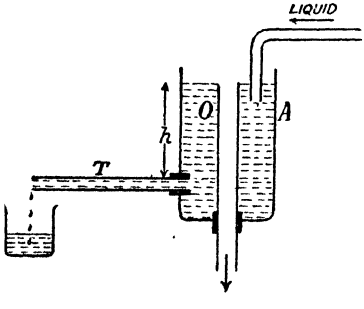


Fig. 275.

*The method is suitable only for comparatively less viscous liquids, like water.

above, the coefficient of viscosity (η) of the liquid can be easily calculated.

There are two important sources of error in the above experiment, *viz.*, (*i*) part of the thrust, due to the difference of pressure between the two ends of the flow-tube, imparts kinetic energy to the liquid and the whole of it, therefore, is not used *simply* in overcoming the viscous resistance of the liquid. This may be corrected for by taking the effective value of the pressure difference to be $P - \frac{V^2.\rho}{\pi^2 r^4}$, instead of P; (*ii*) the motion of the liquid, where it enters the flow-tube, is accelerated, with the result that the velocity of flow is not uniform for the first short length of the tube. This is eliminated by taking the effective length of the flow-tube to be $(l+1{\cdot}64r)$, instead of l. Thus, the corrected relation for η becomes

$$\eta = \frac{\pi P r^4}{8V.(l+1{\cdot}64\,r)} - \frac{V.\rho}{8\pi(l+1{\cdot}64\,r)}.$$

A much better apparatus, however, is the following, in which the flow-tube F is a *long* one, and of a uniform circular cross-section, and the difference of pressure for a length AB of it is given directly by means of a manometer M, whose limbs are arranged over two fine holes at A and B, as shown, (Fig. 276), where A and B lie at a distance of at least 10 *cms.* from the two ends of the flow-tube respectively, so that the velocity of the out-flowing liquid becomes uniform near about them. This very much minimises the two sources of error referred to above,—the second one, almost completely. So that, with a slow rate of flow of the liquid and a fairly small size of the holes at A and B, no further corrections are necessary.

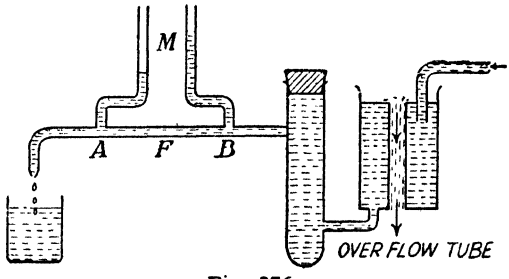


Fig. 276.

Note. In either of these apparatus, it is essential that the outflow tube should have a *perfectly uniform* bore. The uniformity of the bore may be tested in a manner similar to that employed in the construction of a mercury thermometer, *i.e.*, by introducing a small thread of mercury into the tube and measuring its length in the different parts of the tube. In no part should the length vary by more than 5%.

And, since the 4*th* power of the radius occurs in the formula for η, it should be determined most accurately. The tube is, therefore, properly dried and filled with mercury, and the length of the mercury thread measured most carefully by means of a vernier microscope, making the necessary correction for the curvature of the ends of the thread. The mercury is then taken out in a clean, dry and weighed watch glass and its mass determined as accurately as possible. Then, if m be its mass, ρ, its density at the then-temperature t', the length of its thread in the tube, and r, the radius of the tube, we clearly have

$$\pi.r^2.l'.\rho = m, \quad \text{whence,} \quad r = \sqrt{\frac{m}{\pi.l'.\rho}}.$$

So that, knowing m, l' and ρ, the accurate value of the radius r of the tube can be calculated.

211. Motion in a Viscous Medium. When a body falls through a viscous medium, its motion is opposed by a force, *frictional in nature*, due to the fact that whereas the layer of the liquid medium in immediate contact with it is carried along with it, that at an *infinite* distance from it is at rest. Energy is being continually absorbed by the medium and is converted into heat. Possibly also, eddy currents and waves are set up in the medium—particularly when the body moving is a fast one, like high speed cars and airplanes or projectiles--and these absorb still more energy. That is why cars etc., are streamlined these days to minimise the absorption of energy in this way. Even if the body be moving so slowly that no eddy currents or waves are set up, energy is still wasted due to the *viscous drag* it has to overcome.

Now, this opposing force, increases with the velocity of the body, until, in the case of small bodies, it becomes just equal to the *motive* or the *driving force*, and the body then attains a constant velocity, called its *terminal velocity*.

Stokes showed that the *retardation F*, due to to the *viscous drag*, for a spherical body of radius r, moving with velocity v, in a medium whose coefficient of viscosity is η, is given by

$$F = 6\pi v r \eta.$$

This relation, known as **Stokes' law**, may be deduced as follows, by the method of dimensions :

For slow moving bodies,

$F \propto$ *velocity* v ; $F \propto$ *radius* r *of the body* ;

$F \propto$ *coefficient of viscosity* η *of the medium* ;

$F \propto$ *density* σ *of the medium.*

Or, $$F = K.v r^a \eta^b \sigma^c,$$

where K is a constant and a, b and c, the dimensional coefficients of r, η and σ respectively.

Now, putting the proper dimensions of the different terms, we have

$$[MLT^{-2}] = [LT^{-1}]\,[L^a][M^b L^{-b} T^{-b}]\,[M^c L^{-3c}]$$
$$= [M^{b+c} L^{1+a-b-3c} T^{-1-b}]$$

whence (*i*) $b+c = 1$, (*ii*) $1+a-b-3c = 1$, and (*iii*) $-1-b = -2$.

Therefore, from relation (*iii*), $b = 1$; and hence from (*i*) we have $c = 0$; so that, from (*ii*), $a = 1$.

$\therefore$ $F = K.v.r.\eta$; and the value of K was found by *Stokes* to be 6π ; so that, $F = 6\pi v r \eta$, as stated above.

If the density of the spherical body be ρ,

its *weight* = *volume* $\times \rho \times g = \frac{4}{3}.\pi r^3 \times \rho \times g$,

and the *upthrust on it due to the displaced medium* $= \frac{4}{3}\pi r^3.\sigma.g$.

$\therefore$ *resultant downward force on the body* $= \frac{4}{3}\pi r^3.\rho.g. - \frac{4}{3}\pi r^2 \sigma.g$
$= \frac{4}{3}\pi r^3.g.(\rho - \sigma)$.

Equating this against the value of F, we have

$$6\pi v r \eta = \tfrac{4}{3}\pi r^3.g(\rho - \sigma), \quad \text{whence,} \quad v = \tfrac{4}{3}\pi r^3.g(\rho - \sigma)/6\pi r \eta.$$

Or, $$v = \frac{2}{9}\frac{r^2.g(\rho - \sigma)}{\eta} \qquad \ldots (i)$$

Thus, the *terminal velocity* of a body, (of course, of a small size), falling through a viscous medium, is (*i*) *directly proportional to the square of its radius* (r^2), (*ii*) *directly porporti.nal to the difference in the densities of the body and the medium*, $(\rho - \sigma)$, *and* (*iii*) *inversely proportional to the coefficient of viscosity of the medium* (η).

In arriving at the above result, *Stokes* made the following assumptions :

(*a*) That the medium through which the body falls is *infinite in extent.*

(*b*) That the spherical body is *perfectly rigid and smooth.*

(*c*) That there is *no slip between the spherical body and the medium.*

(*d*) That the medium is *homogeneous*, so far as the spherical body is concerned, *i.e.*, the diameter of the spherical body is large compared with the spaces between the molecules of the medium.

(*e*) That there are *no eddy currents or waves set up in the medium due to the motion of the body through it* ;—in other words, that the body is moving very slowly through it, or that the motion of the medium is smooth and not turbulent. *Stokes* found that the relation holds good only when v is smaller than $\eta/\sigma r$*, called the *critical velocity*.

A striking example of a body falling through a viscous medium is that of the tiny rain drops that form what we call *clouds*. These tiny drops of water have a radius as small as ·001 *cm.* and their terminal velocity, as they fall through air, for which $\eta = \cdot 00018$, comes to about 1·2 *cms./sec.*, [from relation (*i*) above]. That is why they remain suspended in the air and appear to us to be floating about as clouds.

Bigger rain drops, on the other hand, have a radius about 10 times as great (*i.e.*, = ·01 *cm.*) and their terminal velocity, therefore, comes to about 120 *cms./sec.* ; so that, they *fall* through the air, instead of floating in it, (v being proportional to r^2).

Also, if the density of the medium in which the body falls be greater than that of the body itself, *i.e.*, if $\sigma > \rho$, it is clear that the terminal velocity v will have a *negative* value. In such a case, therefore, the body will have an upward terminal velocity. That is why bubbles of air or gas can rise up through water or any other liquid,—the smaller the bubble, the smaller its velocity.

212. Determination of coefficient of viscosity of a liquid—Stokes' method. The relation for v obtained above, (§ 211), has been used to determine the viscosity of a liquid. The method consists in finding the time of fall of small spheres, such as ball-bearings etc., in the liquid, and then to apply *Stokes'* relation,

$$v = \frac{2}{9} \cdot \frac{r^2 . g(\rho - \sigma)}{\eta}, \quad \text{whence,} \quad \eta = \frac{2}{9} \cdot \frac{r^2 g(\rho - \sigma)}{v} \quad \ldots(ii)$$

*It was shown by Arnold, however, that in actual practice v should be less than $\cdot 6\eta/\sigma r$.

Care must, however, be taken to see that the velocity of fall does not exceed $\cdot 6\eta/\sigma r$, or else the above relation cannot be used. Thus, for thin machine oils, for which $\eta = 1$ *c.g.s. unit* nearly, spheres of 1 *mm.* radius are the maximum size that can be used. For thick oils, or liquids like glycerine, whose coefficient of viscosity is greater than 10 *c.g.s. units*, steel ball-bearings of about $\frac{1}{4}''$ diameter, are the largest that can be employed. For less viscous liquids, much smaller spheres are required, and these may be obtained, in any desired size, from radii of 0002 to 1 *cm.* by flowing melted *Wood's metal* into cold water through capillary tubes of different bores. The diameters of these spheres, and hence their radii, are measured accurately by means of a microscope.

The liquid is taken in a tall jar, of a large diameter, and the time taken by a sphere of suitable size, dropped *centrally* into it, to pass three marks at different levels is noted and the velocity calculated for each of the two distances between the three marks. If the velocity over each path is the *same*, it means that it has acquired its constant velocity, or terminal velocity, v. If, however, the velocities be different, being greater over the lower track than over the upper one, a smaller ball-bearing must be tried, until the velocities are the *same*, within experimental error.

In actual practice, we have two marks A, and B, (Fig. 277), some distance (say 10 to 12 *cms.*) below the top and above the bottom of the jar respectively, and allow small spheres of different known radii to fall through the liquid *centrally*, noting the time taken by each to cross the distance S between the two marks A and B, it being assumed that due to the small size of the sphere, it has already acquired its terminal velocity before crossing the mark A. Thus, since it moves with a constant velocity over this distance S, its terminal velocity, $v = S/t$.

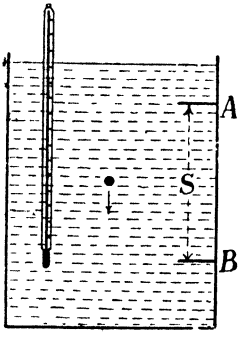


Fig. 277.

Then, putting $v = S/t$ in the above expression for η, we have

$$\eta = \frac{2}{9} \cdot \frac{r^2 g(\rho - \sigma).t}{S}.$$

Now, since r^2t is constant in the case of a given sphere and a given liquid, provided its temperature remains the same, we plot r^2 for a number of spheres against their corresponding values of $1/t$. The slope of the straight-line curve, thus obtained, gives the mean value of $r^2.t$, and this value is then substituted in the above expression for η.

Further, since viscosity of a liquid depends upon its temperature, (see § 214), and the rate of change is fairly rapid in the case of liquids like oils, a sensitive thermometer must also be put into the jar of the liquid to indicate its temperature, and the experiments with the different spheres performed in quick succession.

For greater accuracy, a proper correction must be applied for the finite size of the containing vessel, as the relation for η above was deduced by *Stokes* on the assumption that the medium is infinite.

Ladenburg has shown that if the liquid column in the tall jar be supposed to be divided up into three equal compartments, *lengthwise*, and the sphere be dropped into the central compartment, the velocity v_∞ of the sphere, as it would be in the liquid medium, if it were unbounded by the walls of the containing jar, is given by the relation $v_\infty = v(1+2{\cdot}4r/R)$, where v is the observed velocity of the sphere and R, the radius of the jar. This is called the *correction* for the '*wall-effect*.'*

Similarly, he has shown that if the liquid medium were unhindered by the bottom or the end of the containing jar, the velocity of the sphere would be given by

$$v = v(1+3{\cdot}3r/\mu),$$

where μ is the full depth of the liquid column in the jar. This is called the correction for the '*end-effect*'.

Combining the two corrections, therefore, we have the following relation for η, *viz.*,

$$\eta = \frac{2}{9} \cdot \frac{r^2 g(\rho-\sigma)}{v\left(1+2{\cdot}4\frac{r}{R}\right)\left(1+3{\cdot}3\frac{r}{\mu}\right)}. \qquad \ldots(iii)$$

N.B. Obviously, this method may also be used to determine the radius of a small drop falling through air, if we know the coefficient of viscosity of air, the method being applicable only to drops, bigger in size than the distance between the air molecules, for otherwise *Stokes' law* no loger remains valid.

213. Rotation Viscometer. If we have two coaxial cylinders, with the space in-between them filled up with a fluid, and then rotate the outer one with a constant velocity Ω about their common axis, a *torque* will naturally be communicated to the inner cylinder also through this intervening fluid, and its magnitude will depend upon the viscosity of the fluid.

Let the full line circles in Fig. 278 represent a transverse section of the two cylinders, of radii a and b, with their common axis perpendicular to the plane of the paper, and passing through O.

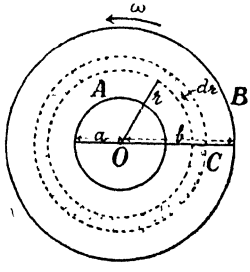


Fig. 278.

Then, as the outer cylinder rotates with its small angular velocity Ω, (*within the limits of a streamline flow*), about the inner cylinder, the layers of the fluid in contact with the former also rotate with it, with its own velocity Ω, the velocity of the other layers progressively decreasing as we move inwards, until, for the layers in contact with the inner cylinder, it is reduced to *zero*. There is thus brought about a relative motion between the different layers of the fluid.

Let us consider a co-axial *cylindrical layer* C of the fluid, at a distance r from O, and of thickness dr. If it be rotating with an angular velocity ω, its *linear velocity* is clearly equal to $r\omega$. And,

*A still more accurate correction of this effect is given by *Faxen* in the form $v = v_\infty\left[1-2{\cdot}104\left(\frac{r}{R}\right)+2{\cdot}1\left(\frac{r}{R}\right)^3-0{\cdot}95\left(\frac{r}{R}\right)^5\right]$, which has been verified by *Bacon*

therefore, if its length or height be l, the *viscous force* F acting on it, in accordance with Newton's Law, (see page 429), is given by the relation,

$$F = \textit{its area} \times \eta \times dv/dr = 2\pi rl.\eta\ dv/dr,$$

where η is the *coefficient of viscosity* of the fluid and dv/dr, the *velocity gradient* at distance r from O.

Now, $$dv/dr = d(r\omega)/dr = \omega + r.d\omega/dr.$$

And, since ω is a constant quantity and would represent the velocity of the layer in the absence of any viscous slip, *i.e.*, if it were to rotate like a rigid body, it does not contribute at all towards the velocity gradient ; and we may, therefore, take the velocity gradient, responsible for the viscous drag on the imaginary cylindrical layer C, to be simply equal to $r\ d\omega/dr$. Hence, this viscous drag or force $F = 2\pi r.l.\eta.r.d\omega/dr$. And, therefore, the *moment of this force*, or the *torque* T, acting on the layer C, is clearly equal to $r.2\pi r.l.\eta.r.d\omega/dr$.

Or, $T = 2\pi r^3.l.\eta.d\omega/dr$, which may be put as $2\pi\ l.\eta\ d\omega/T = dr/r^3$.

So that, the effect of the whole fluid in-between A and B is obtained by integrating this expression for the limits $\omega = 0$ and $\omega = \Omega$ and $r = a$ and $r = b$. Thus, we have

$$\int_0^{\Omega} \frac{2\pi.l.\eta}{T}.d\omega = \int_b^a \frac{dr}{r^3}. \quad \text{Or,} \quad \frac{2\pi.\eta.\Omega}{T} = \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right),$$

whence, $$T = \frac{4\pi l\eta.\Omega}{(1/a^2 - 1/b^2)} = \frac{4\pi l.\eta.\Omega.a^2.b^2}{b^2 - a^2}. \qquad \ldots(i)$$

The torque T tends to accelerate the motion of the fluid in-between C and A. But this fluid being in a state of *steady motion*, the inner cylinder A must also be exerting an equal retarding torque on the fluid in contact with it. And, since action and reaction are equal and opposite, it follows that an equal and opposite torque T' is also exerted on the inner cylinder A, tending to rotate it through an angle θ, say, until it is just balanced by a *restoring torsional torque* equal to T, set up in the suspension wire, carrying it.

Now, if C be the torsional couple, set up in the suspension wire per unit twist of it, we have $T = C\theta$. So that,

$$C\theta = \frac{4\pi.l.\eta\Omega.a^2.b^2}{(b^2 - a^2)}, \text{ whence, } \eta = \frac{C.(b^2 - a^2).\theta}{4\pi.l.\Omega.a^2.b^2}. \qquad (ii)$$

In the above discussion, we have not taken into account the torque on the base of the inner cylinder*, so that if this torque be T_b, we have

total torque on the fluid between A and B given by

$$T_1 = C\theta = \frac{4\pi.\eta.\Omega.a^2.b^2.l}{b^2 - a^2} + T_b. \qquad \ldots(iii)$$

In actual practice, we eliminate T_b altogether, by repeating the experiment with a different length or height l' of the fluid in-between A and B. So that, if now the total torque be T_2 and the angle of rotation of the inner cylinder be θ', we have

$$T_2 = C\theta' = \frac{4\pi.\eta\ \Omega\ a^2.b^2.l'}{(b^2 - a^2)} + T_b. \qquad \ldots(iv)$$

*This torque on the base of the inner cylinder depends not only upon the radii a and b of the two coaxial cylinders, but also upon the distance between their bases.

$\therefore$ subtracting relation (*iv*) from (*iii*), we have

$$(C\theta - C\theta') = C(\theta - \theta') = \frac{4\pi.\eta.\Omega.a^2.b^2}{(b^2-a^2)}.(l-l'),$$

whence, $$\eta = \frac{C.(b^2-a^2)(\theta-\theta')}{4\pi.\Omega.a^2.b^2.(l-l')}. \qquad \ldots(v)$$

The essential parts of the apparatus used are shown in Fig. 279, where the outer cylinder B is clamped on to a table T, so as to be coaxial with its spindle, which can be rotated by a small electric motor at a speed of 20 to 60 rotations *per minute*.

The inner cylinder A is suspended inside B by means of a *long* and *thin* suspension wire, which carries a small mirror m, to enable its angle of rotation, (caused by the constant rotation of B), to be noted accurately by the usual scale and telescope arrangement.

The value of C, the torsional couple per unit twist of the suspension wire, is determined by first setting the inner cylinder alone into torsional vibrations about it, and then with a hollow, metallic disc, of a known moment of inertia I', placed centrally upon it, and noting their time-periods t and t' respectively.

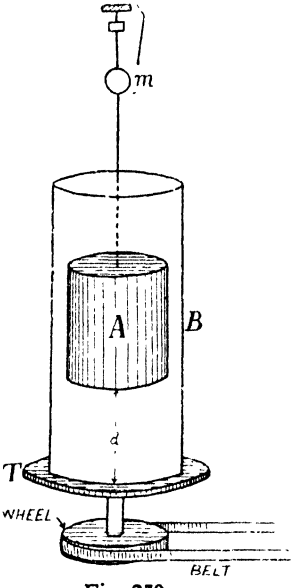


Fig 279.

Then, as we know,

$$t = 2\pi\sqrt{I/C} \text{ and } t' = 2\pi\sqrt{(I+I')/C},$$

where I is the *M. I.* of the inner cylinder alone about the suspension wire.

So that, $$(t'^2 - t^2) = \frac{4\pi^2(I+I'-I)}{C} = \frac{4\pi^2 I'}{C},$$

whence, $$C = 4\pi^2 I'/(t'^2 - t^2).$$

This value of C is then substituted in relation (v) for η, above.

It must be emphasized again, that the speed of rotation of the outer cylinder must be kept low, or else the fluid-flow becomes turbulent and T is then no longer proportional to Ω. This may be clearly seen from the accompanying curve between Ω and T/Ω, (Fig. 280). It will be noted that T/Ω remains constant up to Q, beyond which it varies, as shown by the dotted curve, in a somewhat uncertain and irregular manner, and at higher values of Ω, the relation takes the form $(\alpha.\Omega + \beta.\Omega^2)$, where α and β are constants.

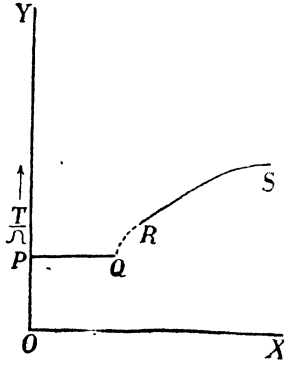


Fig. 280.

Note The rotation viscometer may be used for the determination of η for both liquids and gases. Only, whereas for liquids, we take two observations with different heights l and l' of the liquid in the outer cylinder, in the case of gases, we use two different inner cylinders, of lengths l

and l', the distance between their bases and that of the outer cylinder being kept the *same* in either case.

214. Variation of viscosity of a liquid with Temperature. As we have seen above, to determine the viscosity of a liquid we have to measure its rate of flow through a capillary tube. (Figs. 275 276).

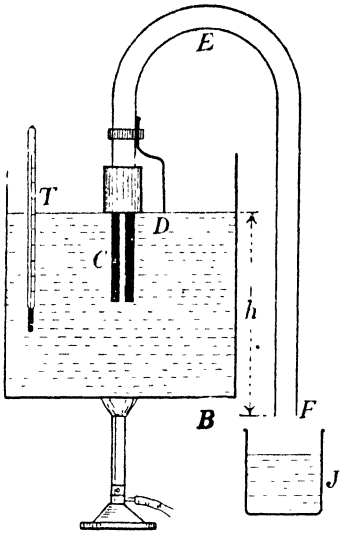


Fig 281.

If, therefore, we wish to see how the viscosity of a liquid is affected by a change in its temperature, the capillary tube must obviously be immersed in a bath of known temperature. This is not possible in the case of the horizontal tubes of the apparatus shown in Figs. 275 and 276. A simple modification of it, as shown in Fig 281, is, therefore, used for the purpose.

Here, the liquid, whose coefficient of viscosity is to be determined, is taken in a beaker *B* and maintained at any desired temperature, which can be noted on the thermometer *T*. A capillary tube *C* is then arranged vertically, completely immersed in the liquid, and is connected to a wider tube *DEF*, bent as shown, to siphon the liquid over *through it* into a weighed vessel *J*.

The method of procedure is the same as in the case of the horizontal capillary tube, with the difference that here the flow of the liquid through *C* is caused by the liquid column *DF* or the head of liquid *h* ; for, clearly, no liquid will flow if the end *D* of the capillary tube were open to the atmosphere. The pressure due to the column *h* is, therefore, the *effective* pressure difference *P* responsible for the flow of the liquid through *C*.

Now, as the liquid flows through the tube, the liquid level falls in *B*, thus altering the head of liquid *h* To maintain it constant throughout the experiment, *B* is gradually raised as the liquid flows through C and *DEF*, so that the liquid level in *B* is always maintained at *D*. This is easily done by attaching a bent pin in the tube, as shown, and making the liquid level in *B* always touch its tip at *D*.

The liquid is thus allowed to flow into *J* for a known time *t* and its mass determined ; then, dividing it by its density and the time *t* (in *seconds*), the volume flowing out *per second, i.e.*, its rate of flow *V* is determined. Substituting the values of *V*, *P*, *r* (radius of the capillary tube *C*) and *l* (length of the tube *C*) in Poiseuille's formula $\eta = P\pi r^4/8Vl$, we can easily calculate the value of η for the liquid, at the temperature of the bath.

The experiment is repeated with the liquid at different constant temperatures and a graph plotted between *t* and η. It will be found that η *usually* falls with a rise in temperature of the liquid, though there is no definite or universal relationship between the two,—the variation being more pronounced in some cases than in others.

215. Comparison of Viscosities—Ostwald Viscometer. The viscosities of two liquids, or the same liquid at different temperatures, may be conveniently compared with the help of the simple apparatus, shown in Fig. 282, and known as the *Ostwald Viscometer*, a well known form of the so called *commercial viscometers*, which avoid the exact measurement of the dimension of the viscometer, so tedious, yet so essential, for the absolute determination of viscosities.

As will be readily seen, this viscometer consists of a U-shaped tube *OGDEQT* with two bulbs *A* and *B*, a capillary portion *CDE*, and a side-tube *T*, fitted with a tap *S*, three marks being engraved on

it at M, N and Q respectively. The whole apparatus can be kept immersed in a suitable bath, to keep its temperature constant.

Each liquid is introduced into the bulb A through the open end O of the tube, until, with the tap S closed, it occupies the space QDM of its bent portion, thus ensuring that the same volume of the liquid is taken in each case.

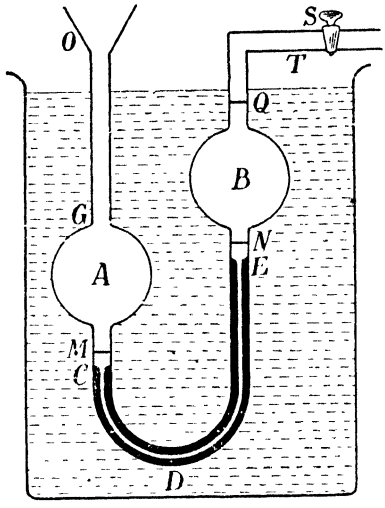


Fig. 282.

Now, first with one liquid thus taken into the tube, some air is abstracted from it by connecting T to a suitable pump through the tap S, until the liquid column rises above the level of Q, and the tap S is then duly closed. The tap is now opened again, when, due to the increased pressure on it, the liquid flows from the portion QE of the tube into the portion CG. The time when the liquid meniscus *just* passes the mark Q downwards is carefully noted, and again when it *just* passes the mark N. The difference gives the time t, say, taken by the liquid to flow from the position QM into the position NG, *its rate of flow being determined by the capillary portion* EC of the tube, the wider parts of the tube hardly affecting it, because, as we know, the retarding force due to viscosity varies inversely as the fourth power of the radius of the tube. Thus, if V be its *rate of flow*, we have

$$V = \pi Pr^4/8\eta l,$$

where l is the length of the capillary portion EC; P, the pressure difference across it and η, the coefficient of viscosity of the liquid.

The experiment is then repeated with the second liquid, of coefficient of viscosity η', (*at that very temperature*). Then, if V' be its rate of flow, determined as above, we have

$$V' = \pi Pr^4/8\eta' l.$$

Now, although the pressure difference keeps on changing during the flow of the liquid it is proportional to the density of the liquid for every position of it And, therefore, if in the corresponding positions of the two liquids, of densities ρ and ρ' respectively, the pressure differences be P and P', we have

$$P \propto \rho \quad \text{and} \quad P' \propto \rho'.$$

And $\therefore$ *for the first liquid*, $V \propto \pi\rho r^4/8\eta l$,

and *for the second liquid*, $V' \propto \pi\rho' r^4/8\eta' l$.

So that, $$\frac{V}{V'} = \frac{\pi\rho r^4/8\eta l}{\pi\rho' r^4/8\eta' l} = \frac{\rho}{\rho'}\cdot\frac{\eta'}{\eta}. \qquad \ldots (i)$$

Again, the rates of flow, V and V', also vary (because of the variation in the pressure difference P and P') during the flow of the liquid, but

the times for the flow of *unit volume* of the two liquids in *corresponding positions*, vary inversely as V and V', and it follows, therefore, that the total times t and t', for the flow of the two liquids respectively must likewise vary inversely as V and V'. In other words,

$$t/t' = V'/V. \quad \ldots \quad \ldots \quad \ldots (ii)$$

From relations (*i*) and (*ii*), therefore, we have

$$\frac{\rho'.\eta}{\rho.\eta'} = \frac{t}{t'}, \quad \text{whence,} \quad \frac{\eta}{\eta'} = \frac{\rho}{\rho'}.\frac{t}{t'}. \quad \ldots \quad \ldots (iii)$$

Thus, knowing ρ and ρ', the densities of the two liquids, (or of the same liquid at different temperatures), and t and t', the times for their corresponding flows, we can easily compare their coefficients of viscosities, η and η'.

N.B. It will be seen that we may not be merely able to compare the viscosities of the two liquids with the help of this viscometer, but may also determine the viscosity of a given liquid. For, as have seen above, $\eta \propto \rho.t$, where ρ is the density of the liquid and t, the time for its flow through the distance QN. And, therefore, $\eta = \alpha \rho.t$,

where α is a constant of the viscometer and may be determined once for all, for a given viscometer, by noting t for a liquid of known density (ρ) and viscosity (η). Or, from relation (*iii*) above, we have

$$\frac{\eta/\rho}{\eta'/\rho'} = \frac{t}{t'}, \quad \text{or,} \quad \frac{k}{k'} = \frac{t}{t'},$$

where k and k' are the *kinematic viscosities* of the two liquids (see foot note on page 429), which can thus be compared easily In fact, this was the relation actually used by *Ostwald* In practice, it is found to be more satisfactory, however, to plot a curve between k and t for a number of liquids, so that, from the noted values of t, the corresponding values of k can be obtained straightway.

Among other commercial viscometers, based on a similar capillary principle, may be mentioned the *Redwood viscometer*, used in England, the *Engler viscometer*, used all over the continent and the *Saybolt viscometer*, designed by the Standard Oil Company, and used in America.

216. Determination of Viscosity of Gases. Since the definition of viscosity of a liquid deduced in §205, applies equally well to a gas, it might at first sight appear that Poiseuille's formula for the rate of flow of a liquid should also apply in the case of a gas. But the snag is that whereas the density of a liquid is practically independent of the pressure on it, (liquids being almost incompressible), that of a gas varies directly with it ; and hence whereas in the case of a liquid, the *volume* (as well as the mass) of it flowing through any section of the tube in a given time can be taken to be constant, in the case of a gas, it is the *mass* of it (and *not the volume*) flowing across a section of the tube in a given time that alone can be taken to be constant.

Thus, if V be the volume of a gas flowing across a section *per second*, at a distance x from the inlet-end of the tube, ρ, its density at the uniform pressure P over that section, we have

$$\rho V = \text{a constant.}$$

Since $\rho \propto P$, we have $PV =$ a constant.

Now, if we consider a section dx of the tube at distance x from the inlet-end, with a pressure difference dP across it, we have, in accordance with Poiseuille's formula, the volume of the gas flowing per second through the section given by

$$V = -\frac{\pi.dP.r^4}{8dx.\eta} = -\frac{\pi r^4}{8\eta}.\frac{dP}{dx},$$

the *negative sign* merely indicating that the pressure P decreases as distance x increases.

Or, multiplying both sides of the expression by P, we have

$$PV = \frac{\pi r^4}{8\eta} \cdot \frac{P.dP}{dx} = a\ constant.$$

So that, if V_1 be the volume of the gas entering the tube at pressure P_1, we have $P_1V_1 = PV$. [the cross-section of the tube being constant.

Or, $$P_1V_1 = -\frac{\pi r^4}{8\eta} \cdot \frac{P.dP}{dx}. \qquad \text{Or,} \qquad P_1V_1\,dx = -\frac{\pi r^4}{8\eta}.P.dP.$$

Integrating this expression for the limits $x = 0$ and $x = l$ and $P = P_1$ and $P = P_2$, where l is the length of the tube and P_1 and P_2, the pressures at its inlet and outlet ends, we have

$$P_1V_1 \int_0^l dx = -\frac{\pi r^4}{8\eta} \int_{P_1}^{P_2} P\,dP. \quad \text{Or,} \quad P_1V_1.l = \frac{\pi r^4}{8\eta} \cdot \frac{(P_1^2 - P_2^2)}{2}.$$

Or, $$P_1V_1 = \frac{\pi r^4 (P_1^2 - P_2^2)}{16\eta l}, \qquad \ldots \qquad \ldots \qquad \ldots I$$

whence η, the coefficient of viscosity of the gas, may be easily determined.

This was the method actually used by *Grindley* and *Gibson*, who noted the difference of pressure between the two ends of a flow tube through which the gas was made to flow from one container to another by forcing water into the former. The volume of the gas passing through per second could thus be easily determined and η for the gas evaluated from relation I above.

217. Rankine's method for the determination of the viscosity of a gas. The simple apparatus used by *Rankine* for determining the coefficient of viscosity of a gas is shown in Fig. 283, where $ABCD$ is a glass tube, *one metre* in length, having a *capillary section* AB (of about 2 *mm.* bore), and fitted with two stop cocks I and O, which serve as inlet and outlet for the gas respectively. A small mercury pellet M is introduced into the part of the tube opposite to the capillary section AB.

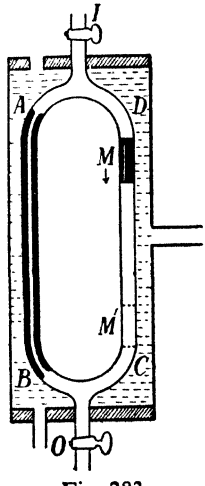


Fig. 283.

When the tube is held vertically in the position shown, the mercury pellet starts falling down under its own weight, forcing some of the gas into the capillary BA, and its rate of flow is observed by noting the time taken by the pellet to fall through a measured distance. This is then equated against its calculated rate of flow, whence the value of η for the gas can be easily obtained.

To start with, suppose the tube is laid horizontally on the table, so that the gas acquires a uniform pressure P all along the tube. Then, if ρ be the density of the gas *at unit pressure*, and if the total volume of the gas enclosed in the tube be V, we have total *mass* of the gas enclosed $= \rho.P.V$.

Let the tube be now held vertically, so that the mercury pellet starts falling down. Let it be in the position M at any given instant, and let the volume of the gas *above it be* v_1. Then, if p_a be the pressure of the gas *above* M, it is clear that the pressure of the gas *below* M will be p_a+mg/a, where mg is the weight of the mercury pellet and a, the area of cross-section of the tube DC. Thus,

$$\textit{mass of the gas above } M = \rho.p_a.v_1.$$

and $$\textit{mass of the gas below } M = \rho(p_a + mg/a)(V-v_1),$$

where $(V-v_1)$ is the volume of the gas below M.

$\therefore$ $\textit{total mass of the gas in the tube} = \rho.p_a.v_1+\rho(p_a+mg/a)(V-v_1).$

Or, $$\rho.P.V = \rho.p_a v_1 + \rho(p_a+mg/a)(V-v_1),$$

which solves out to $$p_a = P - \frac{mg}{a} + \frac{mg}{a}\cdot\frac{v_1}{V}. \qquad ...(i)$$

And, if p_b be the pressure of the gas below M, we have

$$p_b = p_a+\frac{mg}{a} = \left(P-\frac{mg}{a}+\frac{mg}{a}\cdot\frac{v_1}{V}\right)+\frac{mg}{a}.$$

Or, $$p_b = P+\frac{mg}{a}\cdot\frac{v_1}{V}. \qquad ...(ii)$$

Let the mercury pellet fall down to the position M', so that the volume of the gas above M now becomes v_2. Then, if the pressure of the gas now becomes p'_a here, we have, as before

$$p'_a = P- \frac{mg}{a}+\frac{mg}{a}\cdot\frac{v_2}{V}. \qquad ...(iii)$$

And, if p'_b be now the pressure of the gas below the mercury pellet, we have

$$p'_b = P+\frac{mg}{a}\cdot\frac{v_2}{V}. \qquad ...(iv)$$

Now, with the mercury pellet at M, the mass of the gas below it up to B is equal to $\rho.p_b(V-v_1)$ and that of the gas below it, when it takes up the position M', is equal to $\rho\, p'_b.(V-v_2)$. So that, the difference of these two gives the mass of the gas forced into the capillary tube by the fall of the mercury pellet from M to M'.

Thus, *mass of the gas forced through the capillary tube*

$$= \rho.p_b.(V-v_1) - \rho.p'_b(V-v_2).$$

If the position of the mercury pellet be so arranged that $(v_1+v_2)=V$, we have

$$\textit{mass of the gas forced through the capillary} = \rho.P.(v_2-v_1).$$

Hence, if t be the time taken by the mercury pellet to traverse the distance MM', we have

mass of the gas flowing per unit time through the capillary tube

$$= \rho.P.(v_2-v_1)/t. \qquad ...(I)$$

Now, if v be the volume of the gas flowing per unit time when the pellet is at M, we have

$$\frac{(p_b^2-p_a^2).\pi r^4}{16\eta l} = p_b.v, \qquad \left[\begin{array}{l}\text{See relation}\\ I, \text{(page 445)}.\end{array}\right.$$

where r is the radius and l, the length of the capillary section.

And, similarly, if v' be the volume of the gas flowing per unit time when the pellet is at M', we have

$$\frac{(p'_b{}^2-p'_a{}^2).\pi r^4}{16\eta l}=p'_b.v'.$$

Hence, the *average rate of flow of the gas is proportional to* $(p_b^2-p_a^2)+(p'_b{}^2-p'_a{}^2)/2$, which works out to be equal to $2mg.P/a$. So that, *mass* of the gas flowing per unit time through the capillary tube $=\rho\times 2mg.\frac{P}{a}.\pi r^4/16\eta l.$ ..(*II*)

Thus, from relations (*I*) and (*II*), we have

$$\rho\times\frac{2mg.P}{a}\frac{\pi r^4}{16\eta l}=\frac{\rho.P(v_2-v_1)}{t},$$

whence, $$\eta=\frac{mg.\pi r^4.t}{8a.l(v_2-v_1)},$$

from which the coefficient of viscosity of the given gas may be easily determined.

As can be readily seen, the apparatus may also be used to study the effect of pressure on the viscosity of a gas. Rankine determined the viscosities of many gases with the help of this apparatus and showed that viscosity is quite independent of pressure, as predicted by the kinetic theory of gases.

For extreme accuracy, however, Rankine's method is far from suitable, with its inherent defect of a capillary tube experiment. For such purposes, therefore, a rotating cylinder apparatus is found to be more satisfactory and the one form of it almost universally used is that due to *Bearden*, who gave the value $(1{\cdot}82462\pm 0{\cdot}00006)\times 10^{-4}$ *poise* for air at 23°*C*.

SOLVED EXAMPLES

1, A flat plate of area 10 sq. cms. is separated from a large plate by a layer of glycerine 1 mm. thick. If the viscous coefficient of glycerine is 20 gms. per cm. per sec., what force is required to keep the plate moving with a velocity of 1 cm. per second ?

We know that the *viscous force* is given by $F=\eta.A.v/x$.

Here, $\eta=20$ *gm./cm. sec.*, $A=10$ *sq. cms.*, $v=1$ *cm./sec.*, and $x=1$ *mm.* $=\cdot 1$ *cm.*

$$\therefore\quad F=\frac{0\times 10\times 1}{\cdot 1}=20\times 10\times 10=2000\ \textit{dynes}.$$

Hence, the force required is equal to 2000 *dynes*.

2. Water flows along a horizontal pipe, of which the cross-section is not constant. The pressure is 1 cm. of mercury, where the velocity is 35 cms./sec. Find the pressure at a point where the velocity is 65 cms. sec.$^{-1}$.

Here, $p_1=1$ *cm.* $=1\times 13\,6\times 981$ *dynes/cm*^{2}.; $v_1=35$ *cms./sec.*

$v_2=65$ *cms./sec.*; $\rho=1$ *gm./c.c.*; $p_2=$?

Applying *Bernoulli's relation*, we have

$\frac{1}{2}v_2^2-\frac{1}{2}v_1^2=p_1-p_2$. Or, $\frac{1}{2}(65^2-35^2)=(13{\cdot}6\times 981)-p_2$.

Or, $\frac{1}{2}(65+35)\times(65-35)=(13{\cdot}6\times 981)-p_2$.

Or, $\frac{1}{2}\times 100\times 30=13350-p_2$. Or, $1500=13350-p_2$.

Or, $p_2=13350-1500=11850$ *dynes/cm*^{2}.

$$=\frac{11850}{13{\cdot}6\times 981}=\cdot 8884\ \textit{cm. of mercury}.$$

Thus, *the pressure at the point is* ·8884 *cm. of mercury*.

3. If the diameters of a pipe are 10 cms. and 6 cms. at the points where a venturimeter is connected and the pressures at the points are shown to differ by 5 cms. of water column, find the volume of water flowing through the pipe per second.

We know that the volume of the water flowing across any section of the pipe is the same, *i.e.*, $a_1v_1 = a_2v_2$ where a_1 and a_2 are the areas of cross-section at two different points, and v_1 and v_2, the respective velocities of flow at these sections.

Further, we know that $a_1v_1 = a_1a_2\sqrt{\frac{2(p_1-p_2)}{a_1^2-a_2^2}}$, (see page 425), where p and p_2 are the pressures at the two points of cross-sectional areas a_1 and a respectively.

Hence, $a_1 = \pi r_1^2 = \pi.\left(\frac{10}{2}\right)^2 = 25\pi$ *sq. cms.* ; $a_2 = \pi r_2^2 = \pi.\left(\frac{6}{2}\right)^2 = 9\pi$ *sq. cms.*

and $(p_1-p_2) = 5$ *cms of water column* $= 5\times1\times981$ *dynes/cm*^{2}.

$\therefore$ substituting these values in the relation above, we have

$$a_1v_1 = 25\pi\times9\pi\sqrt{\frac{2\times5\times1\times981}{(25\pi)^2-(9\pi)^2}} \quad \text{Or, } a_1v_1 = 225\pi^2\sqrt{\frac{9810}{(25\pi+9\pi)(25\pi-9\pi)}}$$

$$= 225\pi^2\sqrt{\frac{9810}{34\pi\times16\pi}} = 225\pi\sqrt{\frac{9810}{34\times16}} = 3002 \text{ } c.cs.$$

Thus, the volume of water flowing through the pipe *per second* is 3002 *c.cs.*, or 3·002 *litres.*

4. A tube having its two limbs bent at right angles to each other is held with one end dipping in a stream and opposite to the direction of flow. If the speed of the stream be 6 miles/hr , find the height to which water rises in the vertical limb of the tube.

Clearly, the flow of water will be stopped by the tube dipping in the stream and facing the flow, so that the loss of *K E.* per unit mass of water is $\frac{1}{2}v^2$. This much must, therefore, be the gain in the pressure energy, *i.e.*, p/ρ.

$\therefore$ $p/\rho = \frac{1}{2}v^2$.

Since ρ for water $= 1$, we have $p = \frac{1}{2}v^2$, or, $hg = \frac{1}{2}v^2$,

[$\because$ $p = h.\rho.g = hg$, as $\rho = 1$

Now, $v = 6$ *miles/hr.* $= \frac{6\times1760\times3}{60\times60} = \frac{88}{10} = 8\cdot8$ *ft./sec.*

$\therefore$ $hg = \frac{1}{2}\times(8\cdot8)^2$. Or, $h = \frac{(8\cdot8)^2}{2\times g} = \frac{(8\cdot8)^2}{2\times32} = \frac{(8\cdot8)^2}{64} = 1\cdot21$ *ft.*

Therefore, water rises to a height of 1·21 *ft.* in the tube.

5 Calculate the velocity of efflux of kerosene oil from a tank in which the pressure is 50 lbs. wt. per sq. inch above the atmospheric pressure. The density of kerosene is 48 lbs. per cubic foot. (*Bombay*

Let h be the height of the level of kerosene oil in the tank above the axis of the orifice.

Then, pressure due to it at the level of the axis of the orifice $= h.\rho$., *poundals/ft*^{2}., *i.e.*, $= h.\rho$. *lbs. wt./ft.*^{2}

But, pressure $= 50$ *lbs. wt. per sq. inch* $= 50\times144$ *lbs. wt./ft*^{2}.

$\therefore$ $h.\rho = 50\times144$ *lbs. wt./ft*^{2}., whence, $h = 50\times144/\rho = 50\times144/48$ *ft.*

[$\because$ $\rho = 48$ *lbs./c. ft*

Now, velocity of efflux is given by $v = \sqrt{2gh}$

$$= \sqrt{2\times32\times\frac{50\times144}{48}} = \sqrt{9600}.$$

$$= 97\cdot97 \text{ } ft./sec.$$

The velocity of efflux of the oil is, therefore, 97·97 *ft./sec.*

6. A vertical tube of 4 mm. diameter at the bottom has water passing through it. If the pressure be atmospheric at the bottom, where the water emerges

at the rate of 800 gms. per minute, what is the pressure at a point in the tube 25 cms. above the bottom, where the diameter is 3 mms. ?

Here, rate of emergence of water $= 800$ *gms./mt.* $= \frac{800}{60} = \frac{40}{3}$ *gms./sec.*

$= 40/3$ *c.cs. per sec.*

because, density of water $= 1$ *gm./c.c.*

This will be the *same* across any section of the tube.

Now, diameter of the tube at the bottom $= 4$ *mm.* $= \cdot 4$ *cm.*

$\therefore$ radius ,, ,, $= \cdot 4/2 = \cdot 2$ *cms.*,

and $\therefore$ area of cross-section of the tube, at the bottom

$= \pi \times (\cdot 2)^2 = \cdot 04\pi$ *sq. cm.*

Since, volume of water passing through any cross-section

= cross-sectional area × velocity, we have

velocity of water passing through the bottom

$$= \frac{\text{volume flowing per sec.}}{\text{cross-sectional area}} = \frac{40}{3 \times \cdot 04\pi} \text{ cm./sec.} = v_1, \text{ say.}$$

$\therefore$ *K.E.* per unit mass of water at the bottom

$$= \tfrac{1}{2}v_1^2 = \frac{1}{2}\left(\frac{40}{3 \times \cdot 04\pi}\right)^2 \text{ ergs.}$$

and *pressure energy* per unit mass at the bottom

$= p_1/\rho = p_1 = 76 \times 13{\cdot}6 \times 981$ *ergs.* [$\because \rho = 1$ *gm. c c*

$\therefore$ *total energy* $=$ *K.E.* + *pressure energy.*

$$= \frac{1}{2} \cdot \left(\frac{40}{3 \times \cdot 04\pi}\right)^2 + (76 \times 13{\cdot}6 \times 981) \text{ ergs.}$$

Again, diameter of the tube, 25 *cms.* above the bottom $= 3$ *mms.* $= \cdot 3$ *cm.*

$\therefore$ radius of the tube, 25 *cms.* above the bottom $= \cdot 3/2 = \cdot 15$ *cm.*

And $\therefore$ area of cross section of the tube $= \pi \times (\cdot 15)^2$ *sq. cms.*

So that, velocity of water, here, $= \frac{40}{3 \times \pi \times (\cdot 15)^2}$ *cms./sec.*

Hence, *K.E.* of water per unit mass, here,

$$= \frac{1}{2} \times \left(\frac{40}{3 \times \pi \times (1{\cdot}15)^2}\right)^2 \text{ ergs.}$$

Let the pressure here be p_2,

Then, pressure energy per unit mass, in terms of mercury column

$= p_2 \times 13{\cdot}6 \times 981$ *ergs.*

Also *P.E.* per unit mass of water, here, $= hg = 25 \times 981$ *ergs.*

And, $\therefore$ *total energy*, here, $=$ *K.E.* + *pressure energy* + *P.E.*

$$= \frac{1}{2} \cdot \left(\frac{40}{3 \times \pi \times (\cdot 15)^2}\right)^2 + (p_2 \times 13{\cdot}6 \times 981) + (25 \times 981).$$

Now, in accordance with *Bernoulli's theorem*, the total energy must be the same at the two places ; so that, we have

$$\frac{1}{2}\left(\frac{40}{3 \times \cdot 04\pi}\right)^2 + (76 \times 13{\cdot}6 \times 981) = \frac{1}{2}\left(\frac{40}{3\pi \times \cdot 0225}\right)^2 + (p_2 \times 13{\cdot}6 \times 981) + (25 \times 981)$$

Or, $$= \frac{800}{(\cdot 12\pi)^2} + (76 \times 13{\cdot}6 \times 981).$$

Or, $$= \frac{80}{(\cdot 0675\,\pi)^2} + (p_2 \times 13{\cdot}6 \times 981) + (25 \times 981).$$

Or, $5627 + 1014000 = 17780 + 13350p_2 + 24520.$

Or, $13350p_2 = 5627 + 1014000 - 17780 - 24520.$

$= 977327 = 977300$, say*.

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*The figure is rounded off thus, to be able to use the ordinary four-figure logarithmic table.

Or, $$p_2 = \frac{977300}{13350} = 73{\cdot}24 \text{ cms.}$$

Thus, the pressure at the point where the diameter of the tube is 3 *mms.* is 73·24 *cms.* in terms of mercury column.

7. A pitot tube is fixed in a main of diameter 15 cms., and the difference of pressure indicated by the gauge is 4 cms. of water column ; find the volume of water passing through the main in a minute.

Here, radius of the main $= 15/2 = 7{\cdot}5$ *cms*

$\therefore$ area of cross-section of the main $= \pi \times (7{\cdot}5)^2$ *sq. cms.*

Loss of *K.E* per unit mass of water due to the stoppage of flow by the tube $= \frac{1}{2}v^2$ *ergs.*

And, gain of pressure energy $= p/\rho = p = h.g = 4 \times 981$ *ergs.* [$\because$ $\rho = 1$ *gm./c.c.*

Therefore, $\frac{1}{2}v^2 = 4 \times 981$. Or, $v^2 = 8 \times 981 = 7848$,

whence, $$v = \sqrt{7848} = 88{\cdot}57 \text{ cms./sec.}$$

$\therefore$ *volume of water flowing per second across the section*

$= \text{area of section} \times \text{velocity} = \pi \times (7{\cdot}5)^2 \times 88{\cdot}57$ *c.cs.*

And $\therefore$ volume flowing through the section per minute

$= \pi \times (7{\cdot}5)^2 \times 88{\cdot}57 \times 60 = 9{\cdot}396 \times 10^5$ *c.cs.*

$\therefore$ the volume of water passing through the main is $9{\cdot}396 \times 10^5$ *c.cs. per minute.*

8. A water main of internal diameter 8″ is fed by a pipe of internal diameter 2″, which delivers water at the rate of 1 c. ft per second. If the pressure of water in the pipe be 50 lbs. wt. per sq inch, calculate the pressure in the main.

Here, radius of the main $= 8/2 = 4'' = \frac{1}{3}$ *ft.*

and radius of pipe $= \frac{2}{2} = 1'' = \frac{1}{12}$ *ft.*

$\therefore$ cross-sectional area of the main $= \pi \times (\frac{1}{3})^2 = \pi/9$ *sq. ft.*

and ,, ,, pipe $= \pi \times (\frac{1}{12})^2 = \pi/144$ *sq ft.*

Rate of flow of water $= 1$ *c.ft./sec.*

Now, $$\text{rate of flow} = \text{cross-sectional area} \times \text{velocity of flow.}$$

$\therefore$ $$\text{velocity of flow} = \frac{\text{rate of flow}}{\text{cross-sectional area}}.$$

So that, velocity of flow of water (v_1) in the main $= \frac{1}{\pi/9} = \frac{9}{\pi}$,

and velocity of flow of water (v_2) in the pipe $= \frac{1}{\pi/144} = \frac{144}{\pi}$.

$\therefore$ *K.E.* per unit volume (*i e.*, per 1 *c. ft.* of water) in the main $= \frac{1}{2}\rho v_1^2$, where is the density of water, (equal to 62·5 *lbs. per c.ft.*)

$$= \frac{1}{2} \times 62{\cdot}5 \times \left(\frac{9}{\pi}\right)^2 = \frac{62{\cdot}5 \times 81}{2\pi^2} = 256{\cdot}4 \text{ ft. poundals.}$$

And *K.E* per unit volume in the pipe $= \frac{1}{2}\rho v_2^2$.

$$= \frac{1}{2} \times 62{\cdot}5 \times \left(\frac{144}{\pi}\right)^2 = \frac{62{\cdot}5 \times (144)^2}{2\pi^2} = 65660 \text{ ft. poundals.}$$

Let pressure in the main be $= p_1$ *lbs. wt /sq. inch.* $= p_1 \times 32 \times 144$ *poundals/ft*^{2}

And, pressure in the pipe, as we know, $= p_2 = 50$ *lbs. wt./sq. inch.*

$= 50 \times 32 \times 144 = 230400$ *poundals/ft*^{2}.

$\therefore$ applying *Bernoulli's equation*, we have $p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$.

Or, $p_1 \times 32 \times 144 + 256{\cdot}4 = 230400 + 65660 = 296060$

Or, $p_1 \times 32 \times 144 = 296060 - 256{\cdot}4 = 295803\ 6$,

whence, $$p_1 = 295803{\cdot}6/32 \times 144 = 64{\cdot}19 \text{ lbs. wt./inch}^2.$$

Therefore, the pressure in the main

$= 64{\cdot}19$ *lbs. wt. per sq. inch.*

9. Water is escaping from a cistern by way of a horizontal capillary tube, 10 cms. long and 0·4 mm. in diameter, at a distance of 50 cms. below the free surface of water in the cistern. Calculate the rate at which the water is escaping. *(Bombay)*

We know that the rate at which a liquid escapes out from a capillary tubing, *i.e.*, the volume of the liquid flowing out *per second*, is given by $V = \pi.Pr^4/8\eta.l$, where P is the difference of pressure between the two ends of the capillary tube, r its radius, l, its length and η, the coefficient of viscosity of the liquid flowing through it.

Here, $P = 50 \times 1 \times 981$ *dynes/cm²*., $l = 10$ *cms.*,

and $r = \cdot 4/2 = \cdot 2$ *mm.* $= \cdot 02$ *cm.*

Hence $V = \dfrac{\pi \times 50 \times 981 \times (02)^4}{8\eta \times 10} = \dfrac{3 \cdot 082 \times 10^{-4}}{\eta}$ *c.cs.*,

where η is the coefficient of viscosity of water.

Thus, the rate at which the water is escaping from the capillary tube is $3 \cdot 082 \times 10^{-4}/\eta$ *c.cs. per second.*

10. A gas bubble of diameter 2 cms., rises steadily through a solution of density 1·75 gms./c.c. at the rate of ·35 cms /sec. Calculate the coefficient of viscosity of the solution. (Neglect the density of the gas).

We have the relation, $\eta = \dfrac{2}{9} \cdot \dfrac{r^2.g(\rho - \sigma)}{v}$, (see page 436), so that, neglecting ρ, (the density of the gas bubble), we have $\eta = \dfrac{2r^2\, g\sigma}{9v}$.

Here, $r = 2/2 = 1$ *cm.*, $g = 981$ *cms./sec²*., $\sigma = 1 \cdot 75$ *gm./c c.* and $v = \cdot 35$ *cms./sec*

Hence, $\eta = \dfrac{2 \times 1 \times 981 \times 1 \cdot 75}{9 \times \cdot 35} = \dfrac{2 \times 981 \times 1 \cdot 75}{3 \cdot 15} = 1 \cdot 09 \times 10^3$ *poise.*

Thus, coefficient of viscosity of the solution is equal to $1 \cdot 09 \times 10^3$ *poise.*

EXERCISE XII

1. Derive the formula for the flow of a liquid through a capillary tube. Why does the formula fail in the case of a wide bore ? *(Agra)*

2. What is meant by the term 'coefficient of viscosity' ? Obtain an expression for the rate of flow of a liquid through a capillary tube of circular cross-section. Note the precautions to be adopted in the experimental determination of this coefficient, using this expression.

3. Define 'coefficient of viscosity' for a liquid. Describe how the coefficient of viscosity is determined for water.

Water is conveyed through a horizontal tube 8 *cm.* in diameter and 4 *kilometres* in length, at the rate at 20 *litres per sec.* Assuming only viscous resistance, calculate the pressure required to maintain the flow. (Coefficient of viscosity of water is 0·01 *C G S.* units). *(Bancras)*

Ans. $1 \cdot 274 \times 10^7$ *dynes/cm².*

4. Define coefficient of viscosity of a liquid and find its dimensions.

Describe the way in which the different parts of a viscous liquid move when flowing through a fine tube. What changes take place if the motion is increased ? In an experiment with *Poiseuille's apparatus*, the following figures were obtained :

Volume of water issuing *per minute* = 7·08 *c.cs.* ; Head of water = 34·1 *cms.* Length of the tube = 56 45 *cms.* Radius of the tube = ·0514 *cm.*

Find the coefficient of viscosity. *(Calcutta)*

Ans 0·01377 *poise.*

5. Calculate the mass of water flowing in 10 minutes through a tube 0 1 *cm.* in diameter, 40 *cm.* long, if there is a constant pressure head of 20 *cms.* of water. The coefficient of viscosity of water is 0·0089 *c g.s.* units. *(A.M.I E., 1960)*

Ans. 81·19 *gms.*

6. Define coefficient of viscosity. Give examples of some viscous substances. How would you determine the coefficient of viscosity of a liquid ?

Why do you find cloud particles floating in the sky ? (*Agra, 1944*)

7. Define coefficient of viscosity.

Describe fully what you would do to compare the viscosities of two liquids. (*Madras, 1949*)

8. Explain what is meant by the 'viscosity' of a liquid ? How will you study experimentally the variation of viscosity with temperature ?

What are the 'dimensions' of viscosity in terms of length, mass and time ? (*Allahabad, 1946*)

9. Define the coefficient of viscosity of a liquid. What is the effect of temperature upon it ? How would you determine the viscosity of water at different temperatures ? Derive the formula you use. (*Punjab, 1947*)

10. A square plate of 10 *cms.* side moves parallel to another plate with a velocity of 10 *cms. per sec.*, both plates being immersed in water. If the viscous force between them is 200 *dynes*, and the viscosity of water is 0·01 *gm./cm. sec.*, what is their distance apart ? Ans. 0·44 *cm.*

11. Enunciate and prove *Bernoulli's theorem*, and mention some of its important applications.

12. Two tubes, with small apertures at their lower ends, are held vertically with their lower ends dipping in a pipe carrying water, such that the aperture of one faces the flow and that of the other has its plane parallel to the direction of flow of water, which rises in the former to a height 10 *cms.* above that in the latter. Determine the velocity of flow of water in the pipe. If the pipe has a diameter 20 *cms.*, what is the volume of water flowing along the pipe per minute ? Ans. (*i*) 140·1 *cms./sec.* (*ii*) $26{\cdot}4 \times 10^5$ *c.cs.*

13. Calculate the velocity of efflux of alcohol (*sp. gr.* ·80) from a cylinder in which the total pressure is 2 *atmospheres*.

[Hint. Here, pressure due to the alcohol is one atmosphere, equal to 76 *cms.* of mercury column. Ans. 15·92 *metres./sec*

14. A venturimeter is connected to two points in a main where its radii are 20 *cms.* and 15 *cms.* respectively, and the levels of the water columns in the two limbs differ by 10 *cms.* How much water flows through the pipe per hour ? Ans. $43{\cdot}11 \times 10^4$ *litres.*

15. Water flowing in a horizontal main, of a non-uniform bore, has a velocity 100 *cms./sec.* at a point where the pressure is 7/19*th* of the atmospheric pressure. What will be the velocity at a point where the pressure is one half of that at the first point ? Ans. 251·7 *cms /sec.*

16. Deduce the expression for the rate of steady flow of a liquid through a capillary tube of circular section.

A vessel of cross section 20 *sq. cm.* has at the bottom a horizontal capillary tube of length 10 *cms.* and internal radius 0·5 *mm.* It is initially filled with water to a height of 20 *cms.* above the capillary tube. Find the time taken by the vessel to empty one-half of its contents, given that the viscosity of water is 0·01 *C.G.S.* unit. (*Madras, 1947*)

Ans. 9 *min.* 36 *secs.*

17. A capillary tube, 1 *mm.* in diameter and 20 *cms.* in length, is fitted horizontally to a vessel kept full of alcohol, of density ·8 *gm./c.c.* The depth of the centre of the capillary tube below the surface of alcohol is 30 *cms.* If the viscosity of alcohol is 0·012 *c.g.s. unit*, find the amount that will flow out in 5 *minutes.* Prove the formula you use. (*Bombay, 1933*)

Ans. 57·74 *gms.*

18. Water at 20°*C* ($\eta = 0{\cdot}010$ *gm.* $cm.^{-1}$. $sec.^{-1}$.) is escaping from a tank by a horizontal capillary tube, 20 *cms.* long and 1·2 *mm.* diameter. The water stands 100 *cms.* above the tube. At what rate is the water escaping ?

Ans. 2·5 *c.cs./sec.*

19. If in question 18, the area of the tank be 10^4 *sq. cms.*, how long will it take for the water level to fall to 50 *cms.* above the tube ? Ans. 91 *hours.*

20. A funnel is attached to a vertical capillary tube of length 49·6 *cms.* Water is poured into the funnel and kept at a height of 68·3 *cms.* above the lower end of the tube, whilst 78 *c cs.* are collected in 10 *minutes.* The tube was then removed and entirely filled with mercury ; 3·695 *gms.* were required. If the density of mercury is 13·56 *gms. per c.c.,* find the viscosity of water at the temperature of the experiment. **Ans.** 0·01247 *gm./cm. sec,*

[**Hint.** *The radius of the tube can be determined from the amount of mercury filling the tube. For, mass = volume × density.*

∴ volume of mercury filling the tube = mass of mercury/its density.

Or, *volume of tube* = 3·695/13 56 *c.cs.*

Now, volume = $\pi r^2.l$, *where r and l are the radius and length of the tube respectively.*

$$\therefore \quad \pi r^2 = \text{volume}/l = 3{\cdot}695/13{\cdot}56 \times 49{\cdot}6 \text{ sq. cms.}$$

Hence $r^2 = \dfrac{3{\cdot}695}{\pi \times 13{\cdot}56 \times 49{\cdot}6}$, or, $r = \sqrt{\dfrac{3{\cdot}695}{\pi \times 13{\cdot}56 \times 49{\cdot}6}}$ *cms.*

This value of r is then substituted in the relation $\eta = P.\pi r^4/8\ vl.$]

21. Give Poiseuille's method of measuring the viscosity of liquids. Derive the formula used. Indicate a method which could be employed to measure the viscosity of liquids at different temperatures. (*Banaras*)

22. What kind or kinds of energy result from the work done by a fluid against viscosity ? How can the viscosity of a liquid be determined ? (*Bombay*)

23. Define "coefficient of viscosity".

Derive *Stoke's* formula for the velocity of a small sphere falling through a viscous liquid. Explain how this is utilised to determine the viscosity of a liquid like castor oil. Mention one more application of *Stoke's* formula. (*Madras*)

24. Determine the radius of the drop of water falling through air, if the terminal velocity of the drop is 1·2 *cms./sec.* Assume the coefficient of viscosity for air = $1{\cdot}8 \times 10^{-4}$ and the density of air = $1{\cdot}21 \times 10^{-3}$ *gm./c.c.* (*A.M.I.E., 1961*)

Ans. 0·001 *cm.* (approximately)

25. What is a rotation viscometer ? Explain clearly its construction and working.

26. How may the viscosities of two liquids, or the viscosities of the *same* liquid at two *different* temperatures, be compared with the help of *Ostwald viscometer ?*

27. Explain the limitations of *Poiseuille's formula* for the rate of flow of a liquid through a capillary tube. Why does it fail in the case of a gas ?

Explain, with necessary theory, *Rankine's* method for the determination of the viscosity of a gas.

CHAPTER XIII

DIFFUSION AND OSMOSIS

216. Diffusion. If we place a solution of, say, copper sulphate, (or even solid crystals of potassium permanganate or potassium dichromate), in a beaker, and cover it with pure water, *disturbing it the least,* so that the line of demarcation between the two is clearly seen, we shall find that on standing for some time,—may be, even days or months,—the characteristic colour of the salt *ascends upwards* and fills the entire space occupied by the liquid, the concentration of the entire solution becoming the *same* all over.

The same is true for gases also. For, if we divide a tall glass jar into two compartments by a movable horizontal glass plate and put a heavier, or a denser gas, like carbon dioxide, on the lower compartment and a lighter or a rarer gas like hydrogen, in the upper one, we find that, on removing the glass plate, the molecules of the two gases intermix with each other, until we get a mixture of the two, of a uniform density throughout.

This process, by virtue of which the molecules of a solute, (in this case, copper sulphate), move upward from the lower portions of greater concentration to the upper ones of lower concentration, or, by virtue of which the molecules of one gas mix with those of another, even against the force of gravity, is called **diffusion.**

The solute is pushed up as though under some pressure, until equilibrium is attained, and the concentration and pressure of the solution become uniform throughout.

Solids too, although they possess a definite crystalline structure, have been known to exhibit the phenomenon of diffusion, if placed in good contact with each other over a long enough time, the diffusion taking place more readily between two different solids than between two portions of the same solid. Thus, for example, the diffusion of gold into lead has been clearly shown by **Robert Austen,** by fusing a small lead cylinder (about 7 *cms.* long and 1·4 *cms.* in diameter) into a thin gold plate and subjecting it to pressure for well over a month, in a constant temperature enclosure,—the temperature being kept below their lowest melting point. And, more recently, it has been clearly established by **Groh** and **Hevesy** that *radioactive* lead can diffuse into ordinary lead, if the two be kept in contact for some reasonably good time, *i e.,* well over a year. This then, is a case of what is aptly called *self-diffusion,* for we have, here, the case of a substance (lead) diffusing into its own self, (*i.e.,* into lead), for although the atomic weights of the two types of lead are different, they are otherwise identical in their chemical properties.

The phenomenon of diffusion of liquids was first investigated by **Graham,** in the year 1851. It would, therefore, be of interest to give a brief account of his experiments here.

He filled a large-mouthed bottle with the salt solution under investigation, to a level very near the top, floated a cork-disc centrally on the surface of the solution and carefully filled the rest of the bottle with pure water, (the solvent), by lightly squeezing it on the discs from the sponge. Then, covering the mouth of the bottle with a glass disc, lowered it lightly into a larger jar, containing water, so that the mouth of the bottle was some distance below the water surface in the jar. Then, allowing the motion of the two liquids to subside, he carefully uncovered the mouth of the bottle, causing the least possible disturbance to the liquids, and finally covered the jar with a glass plate to minimise evaporation, maintaining the jar at a constant temperature to avoid convection currents.

Due to diffusion of the salt into the pure water above, the concentration of the solution began to change, and this was determined from time to time by withdrawing portions of the solution by means of a pipette. From the change in concentration of the solution, *Graham* estimated the amount of the solute diffusing into the solvent or the water above.

Graham repeated his experiment with different salt solutions, and, although he was unable to formulate any exact law for the rate of diffusion, he arrived at the following conclusions :

(*i*) that solutions of different salts, of the *same strength or concentration*, diffuse at *different rates*, and thus, *the rate of diffusion depends upon the nature of the solute* ;

(*ii*) that solutions, of the *same salt, of different strengths or concentrations*, diffuse at rates proportional to their concentrations ;

(*iii*) that, in general, diffusion alters the proportion of two salts in a mixture ;

(*iv*) that the rate of diffusion inceases with temperature ;

(*v*) that solutions may be divided into two distinct categories *viz.*, (*a*) **crystalloids,** consisting of mineral acids, salt solutions, sugar, etc., *which diffuse comparatively quickly*, and (*b*) **colloids**, consisting of albumen, gum, caramel and gelatine, etc., *which diffuse more slowly than the crystalloids.*

Diffusion, in the case of liquids, may easily be explained on the basis of the kinetic theory of liquids. For, according to this theory, the molecules of the solvent possess kinetic energy of translation, which is directly proportional to their absolute temperature. This, they share with the molecules of the solute present and the latter thus roam about in the solvent, so much so that they even rise up against the force of gravity.

217. Fick's Law. Four years later, in 1855, **Fick** formulated a *law of diffusion*, based on *Fourrier's law* of conduction of heat, due to a close analogy between the process of diffusion of a solute and the flow of heat through a solid. He established the law *that the rate of diffusion in any direction is proportional to the concentration gradient of the solute in that direction.* The law, after him, is known **as Fick's law.**

Thus, if we imagine a rectangular slab of a solution, of thickness x, such that the concentrations of the solute all over its two

opposite faces are C_1 and C_2 respectively, $-C_1$ being greater than C_2 —then, the amount of solute passing into one face and out at the other, *i.e.*, passing any cross-section of it in a given time, is proportional to the *concentration gradient*, $(C_1-C_2)/x$.

If we denote the change of concentration for a small distance dx by dc, the *concentration gradient* is dc/dx, and the quantity (Q) of the solute diffusing *in unit time* through an area A is given by the relation, $$Q = K.A.dc/dx,$$
where K is a constant, called *coefficient of diffusion, or the diffusivity of the solute*, and depends not only on the nature of the solute in a given solvent but also on its concentration, *i.e.*, mass (*in grams*) of the solute *per c.c.* And, since the dimensions of Q are $[MT^{-1}]$, of A, $[L^2]$ and of dc/dx, $[ML^{-4}]$, we have
$$[MT^{-1}] = [K]\,[L^2][ML^{-4}],$$
whence, $$[K] = [L^2T^{-1}].$$

Now, if the concentration at the first layer be c, the concentration at a layer opposite, distance δx apart, will be $c-\delta x.dc/dx$, because the concentration gradient is dc/dx and, therefore, a change of concentration in a distance δx will be $\delta x.dc/dx$. Thus, the quantity of the solute entering the first layer in *unit time* is $K.A.dc/dx$, and that leaving the second layer in the same time, (*i.e.*, *unit time*), is given by $K.A.\frac{d}{x}\left(c-\frac{dc}{x}.\delta x\right) = K.A.\frac{dc}{dx} - K.A.\frac{d^2c}{dx^2}.\delta x$, *i.e.*, the rate at which the solute is leaving the second layer is lower than that at which it enters the first layer by $K.A.\delta x.d^2c/dx^2$. This amount of the solute is, therefore, added in unit time to the volume of the solution between the first and the second layers, *i.e.*, to a volume $A.\delta x$.

Thus, the *rate of change of concentration* $= K.A.\delta x.d^2c/dx^2/A.\delta x = K.d^2c/dx^2$. Denoting this rate of change of concentration by dc/dt, we, therefore, have
$$dc/dt = K.d^2c/dx^2,$$
a relation of fundamental importance in diffusion of liquids.

The analogy between *Fick's law* of diffusion and *Fourrier's law* of conduction of heat on the one hand, and *Ohm's law* of conduction of electricity on the other, will at once be apparent. For, just as heat always flows from a point at a higher temperature to that at a lower temperature, and just as electricity always flows from a point at a higher potential to that at a lower potential, so also does a solute flow from regions of higher concentration to those of lower concentration. Not only that, but like conduction of heat and electricity, diffusion too is, to all intents and purposes, an *irreversible process*, because, if the solvent, or a portion of it, gets impregnated with the solute, it can never divest itself of it without external aid.

218. Relation between Time of Diffusion and Length of Column. Suppose we have two columns of solution, of lengths l_1 and l_2 respectively, having *identical* concentrations at different points along them at a given instant. Let the two columns again acquire similar concentrations at corresponding points after intervals t_1 and t_2 respectively.

Quite obviously, these time-intervals must be proportional (*i*) directly to the distances covered by the solute and (*ii*) inversely to the absolute velocity of the solute at each layer. And, therefore, since the distances covered are in proportion to these lengths of the columns, we have

$$t_1/t_2 = (l_1/l_2)(v_2/v_1) \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

where v_1 and v_2 are the solute-velocities at corresponding layers of the two columns.

Now, by *Fick's law*, $Q = K.A.dc/dx$.

But $Q = A.v.c$, if v be the velocity of the solute at a given layer, of concentration c.

$$\therefore \quad A.v.c. = K.A.\frac{dc}{dx}. \quad \text{Or,} \quad v = \frac{K}{c}.\frac{dc}{dx}.$$

i.e., $v \propto dc/dx$, [K/c being a constant for the given layers

But dc/dx will be inversely proportional to the length of the column, if the two columns have similar distribution of concentrations. And, therefore,

$$v \propto 1/l, \quad \text{[where } l \text{ is the length of the column.}$$

Hence, $v_2/v_1 = l_1/l_2$.

So that, substituting the value of v_2/v_1 in relation (*i*) above, we have

$$t_1/t_2 = l_1^2/l_2^2 \quad i.e., \quad t \propto l^2.$$

Or, the time of diffusion from one distribution of concentration to another, in a given solution, is directly proportional to the square of the length of its column.

***221. Experimental measurement of Diffusivity.** The diffusivity of a solution may be determined by measuring its concentration at a chosen point, from time to time, and applying *Fourrier's theorem* to the relation,

$$dc/dx = K.d^2c/dx^2.$$

Different methods have been used for the purpose. Thus, for example, **Kelvin** determined the densities of the different layers of the solution by introducing into it, a series of beads of different but known densities and noting their equilibrium positions. This method, however, is open to two very valid objections, *viz.*, (*i*) some salt from the solution crystallizes on the beads and (*ii*) air bubbles are formed on their surfaces. Both these factors tend to alter the buoyancy of the beads and the results obtained are thus far from reliable.

Other methods used to measure concentration are based on the measurement of (*a*) the refractive index of the solution, (*b*) the contact potential difference, and (*c*) in the case of optically active solutions, like that of sugar, the rotation of the plane of polarisation etc., etc. **Littlewood** has succeeded in measuring the concentration of the solution at a given layer within ·05 *gm. per litre* by an optical method, based on the bending of light rays.

For aqueous solutions of salts like *NaCl*, *KCl*, KNO_3 etc., however, the method devised by **Clack**, (1942), is perhaps the best. We shall, therefore, study this in some detail.

Clack's Method. He took a '*diffusion cell*', which was just a rectangular tube, about 5 *cms.* long, 1 *cm.* wide and with a horizontal thickness 4 *cms.*, made *up of glass plates*, and fitted it vertically into the bottom of a glass box containing an air-free, saturated solution of the salt in question, to maintain which at its saturation point, crystals of the salt, or the solute, were also placed in the solution. Outside the tube, and above the glass box, were arranged compartments carrying pure distilled water, at a level about 5 *cms.* higher than the tube. An inlet-tube for the water to enter and an outlet-tube for its exit were also provided, so that *there was a constant slow stream of the distilled water flowing across the upper free end of the cell*, resulting in the incoming water being carried away as a feeble solution of constant concentration, due to the upward diffusing solute from the cell getting mixed up with it. The whole system was allowed to come to a steady state, which took as many as 12 days or more and the flow of water was maintained at about 50 *c.cs.* per day, the temperature being kept constant by means of a suitable *thermostat*. The theory underlying the method will be clear from the following :

Consider a layer P, say, distant l from the top of the cell, and let Q be the ***mass*** of the solute crossing upwards *here* per second. Then, if c be the concentra-

tion of the solution at this layer and A, the area of cross-section of the cube, we have, by *Fick's law*,

$$K = \frac{Q}{A} \cdot \frac{dl}{dc},$$ [See page 455.

where K is the *diffusivity* of the solution.

Now, $$\frac{dl}{dc} = \frac{dl}{d\mu} \cdot \frac{d\mu}{dc},$$

where μ is the optical refractive index of the solution at P.

$$\therefore \quad K = \frac{Q}{A} \cdot \frac{dl}{d\mu} \cdot \frac{d\mu}{dc} \quad \ldots \quad \ldots \quad \ldots I$$

Now, apart from diffusion, the motion of the solute is also affected by the *mass motion* or *bulk-motion* of the solvent, (*i.e.*, water) downwards into the tube. If v be the *uniform, downward* velocity of this mass-motion of the solvent, it is obvious that mass $A.v.c.$ of the solute and a mass $A.v.(\rho - c)$ of the solvent flow downwards at P, where ρ is the density of the solution at P and, therefore, $(\rho - c)$, the concentration of the solvent there.

Hence, if M be the mass of the solvent entering the cell *per second*, at the top, and m, that of the solute leaving the cell *per second* there, we have

$$M = A.v.(\rho - c). \quad \ldots \quad \ldots \quad \ldots (i)$$

and, $$m = Q - A.v.c. \quad \text{Or,} \quad Q = m + A.v.c. \quad \ldots \quad \ldots (ii)$$

But, since $A.v. = M/(\rho - c)$, [from relation (*i*) above.

we have, from relation (*ii*),

$$Q = m + \left[\frac{M}{(\rho - c)}\right].c. \quad \ldots \quad \ldots \quad \ldots (iii)$$

So that, putting $M/m = \delta$, a constant, we have

$$Q = m + \left(\frac{\delta m}{\rho - c}\right).c = \frac{m(\rho - c) + \delta.mc}{\rho - c}.$$

Or, $$Q = \frac{m(\rho - c + \delta.c)}{\rho - c}. \quad \ldots \quad \ldots \quad \ldots (iv)$$

Now, the net loss in the mass of the system *per second* is clearly equal to *loss in mass of the solute per second minus gain in mass of the solvent per second.* Denoting this net loss by i, therefore, we have

$$i = m - M = m - \delta.m = m(1 - \delta), \quad \text{whence,} \quad m = i/(1 - \delta).$$

Substituting this value m, in relation (*iv*) above, we have

$$Q = \frac{i.(\rho - c + \delta.c)}{(1 - \delta).(\rho - c)}.$$

And, putting this value of Q, in relation I above, we have

$$K = \frac{i(\rho - c + \delta.c)}{A(1 - \delta)(\rho - c)} \cdot \frac{dl}{d\mu} \cdot \frac{d\mu}{dc}. \quad \ldots \quad \ldots \quad \ldots II$$

The various quantities involved in this relation for K are obtained as indicated below :

(*i*) i and δ are determined either by *chemical analysis* or by drying and weighing the solution.

(*ii*) $dl/d\mu$ is measured by making a narrow horizontal beam of monochromatic light incident on layers of increasing densities, when the beam gets refracted in the vertical plane and emerges out downwards at an angle α with the horizontal, such that $$\frac{d\mu}{dl} = \frac{\sin \alpha}{t},$$ [t being the horizontal thickness of the cell.

where α is measured by noting the vertical displacement of the central fringe in the interference pattern produced by two narrow horizontal slits, close to each other, and illuminated by the green light from a mercury arc, when this light is allowed to pass through the cell.

(*iii*) $d\mu/dc$ is determined by means of *Rayleigh's interferometer*, for solutions of different concentrations c, and

(*iv*) the concentration c at P is computed from the curves (l, $d\mu/dl$) and (c, $d\mu/dc$).

Thus, all quantities in relation *II* above for *K* being known, we can easily calculate *K*, the diffusivity for the given solution.

219. Graham's Law for Diffusion of Gases. Gases diffuse much more rapidly than liquids, due to their high molecular speed, their diffusion being governed by the following law, known as *Graham's law of diffusion*:

"*The rates of diffusion of two gases are inversely proportional to the square roots of their densities.*"

Thus, if r_1, r_2 and ρ_1, ρ_2 be the rates of diffusion and densities respectively of two gases, we have

$$\frac{r_1}{r_2} = \sqrt{\frac{\rho_2}{\rho_1}}.$$

The difference in the rates of diffusion of gases was used by *Graham* to separate gaseous mixtures, to which process he gave the name **'atmolysis'.**

220. Effusion—Transpiration and Transfusion.

(*i*) When a gas escapes from a vessel into a vacuum, through a small hole in a thin plate, such that the width of the hole is greater in comparison with its length, the gas is said to *effuse*, and the process is called **effusion.** Graham showed that *the rate of effusion varies directly as the square root of the difference of pressure on the two sides of the hole, and inversely as the square root of its density, and is quite independent of the passage of any other gas at the same time.*

Thus, $$\textit{velocity of effusion} \propto \sqrt{\frac{\textit{pressure difference}}{\textit{density}}}.$$

It must be noted that in this process, the gas flows, as a whole, through the plate, there being no separation of a mixture of gases into its constituents.

(*ii*) If, on the other hand, the hole in the plate be not too fine, and the thickness of the plate be greater than the diameter of the hole, the process of escape of the gas through it is called *'transpiration'*. Here, the flow of the gas is controlled by viscosity alone, which is subject to the same laws as are applicable to the flow of a gas through a long tube.

Here, too, the gaseous mixture, as a whole, passes through the plate, there being no separation of it into its various constituents.

(*iii*) If, however, the holes be so fine that their diameters are comparable with those of the molecules of the gas, the passage of the gas is known as *diffusion* or *'inter-diffusion.'*

Here, partial separation of a mixture of gases, *i.e.*, *atmolysis*, takes place and its constituents may thus be easily separated.

221. Osmosis and Osmotic Pressure. If two liquids, which can diffuse into one another, be initially separated by a membrane, they inter-diffuse, one into the other. *This process of inter-diffusion of two liquids through a membrane is called* **osmosis.**

To demonstrate the phenomenon of osmosis we use what are called *semipermeable membranes*, which have the property of *selective transmission*, allowing some liquids to pass through them and not others. There are various membranes of animal or vegetable origin,

which have this property of selective transmission and are, therefore, semipermeable. Examples of such membranes are : ordinary animal bladder, cell-walls of plants, and certain inorganic precipitates etc.

For instance, if we fill an animal bladder with alcohol, tie its mouth tightly and place it in a vessel of water, the bladder begins to swell, and may ultimately burst If, on the other hand, we fill it with water and place it in alcohol, it shrinks. This clearly shows that *the bladder is more permeable to water than to alcohol.* For, water can pass through it easily whereas alcohol cannot. *Graham* made use of the semipermeable properties of pig's bladder to separate crystalloids from colloids,—a process he termed '**dialysis**'. The mixture of crystalloids and colloids is placed in a tray with a bottom of parchment paper and the tray is then floated in water, when the crystalloids pass through the parchment into the water, leaving the colloids behind in the tray. Poisons in the viscera of poisoned animals can be detected in this manner.

The systematic study of the laws of osmosis was first made by **Pfeffer**, who used a semipermeable membrane of *precipitated copper ferrocyanide*, supported by the pores of an ordinary battery-pot or porous biscuit-ware, as ordinary parchment membranes are too feeble to withstand high pressures. The pot is first soaked in water to drive out all the air from its pores. It is then filled with a solution of copper sulphate and placed inside a solution of potassium ferrocyanide. Both these salts diffuse into the walls of the pot, where they meet to form a brown semipermeable precipitate of *cupric cyanide.* The pot is then washed with distilled water.

If such a pot be now filled with a salt or a sugar solution, closed tightly with a rubber bung, carrying a mercury manometer, as shown, (Fig. 273), and then placed in a vessel containing pure water, (the solvent), it is found that water passes into the pot and an increase of pressure is recorded by the manometer, *showing thereby that more water is entering the pot from the outside than is escaping out from the inside.* More and more water thus gets into the pot and the pressure goes on steadily rising until a certain definite pressure is reached. This will happen when equilibrium is attained, *i.e., when water passes in at just the same rate at which it passes out.* This increase of pressure, or the excess pressure inside over the atmospheric pressure on the water outside, now prevents any further passage of water into the pot and measures what is called the **osmotic pressure** of the solution. *Osmotic pressure is thus the pressure which must be applied on the solution-side of the membrane to prevent any flowing in of the solvent through the membrane.* Its value depends upon the *nature, concentration and temperature* of the solution.

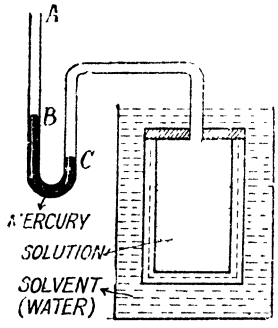


Fig. 273.

If the manometer be open at the top (*A*), the travel of mercury in it is inconveniently large. To prevent this, the upper end is

closed, and if the volume of air in AB be known, when the pressure is atmospheric, any subsequent reading of the volume of air in it enables the pressure to be calculated.

This method for the measurement of osmotic pressure, however, is not quite a suitable one ; for

(*i*) due to the solvent entering the pot, *the concentration* is *lowered*, and hence the osmotic pressure measured is for this final *lowered* concentration of the solution, and not for the original one ;

(*ii*) *the final concentration of the solution is not uniform throughout*, and the pressure measured is thus for the concentration of the solution in the immediate neighbourhood of the membrane, which it is not quite so easy to determine.

A better method is due to **Berkeley and Hartley,** who, in 1906, measured the osmotic pressure by determining the external pressure necessary to be applied to the solution to *just* prevent any solvent passing into it.

Their apparatus, (Fig. 274), consists of a porous tube MN, (with a membrane of cupric cyanide formed inside its walls), which is filled with the solvent and placed inside a metallic vessel B, which is filled with the solution, through the tube E. The porous tube MN is connected to two tubes at its two ends, as shown,—one fitted with a stop-cock S, and the other being a graduated capillary tube of glass, D

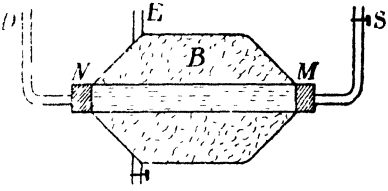


Fig. 274.

Pressure is applied to the solution through the tube E, so that no solvent can enter into it from the tube MN, *i.e.*, the meniscus of the liquid column in D remains stationary, or the condition of equilibrium is attained. *The external pressure applied through E thus gives the osmotic pressure of the solution in B (plus the small and almost negligible hydrostatic pressure of the solvent in the tube MN).*

Osmotic pressures, as high as 130 atmospheres, may thus be easily measured with the help of this apparatus.

222. Laws of Osmotic Pressure. Pfeffer's results led to the establishment of the following laws of osmotic pressure *for dilute solutions of non-electrolytes.*

1. The osmotic pressure (P) of the solution of a given solute is proportional to its concentration (c), provided the temperature remains constant ; or, $P \propto c$.

Since concentration is inversely proportional to volume (V), this corresponds to *Boyle's law for gases, viz.*, that $P \propto 1/V$.

2. The osmotic pressure of the solution of a given concentration is proportional to its absolute temperature ; or $P \propto T$. This, obviously, corresponds to *Charles' law for gases.*

Combining the two laws, therefore, we have $P \propto c.T$.

Or, $P = K.c.T$, [where K is a constant.

Or, putting $c = 1/V$, where c is the concentration of the

solution, in *gram-molecules* of the solute per c.c., and V, its volume containing 1 *gram-moelcule* of the solute, we have $P = K.(1/V).T$.

Or, $$PV = KT.$$

This constant K for very dilute solutions, is found to be identical with the *gas constant* R ; so that, *for dilute solutions*, we have

$$PV = RT.$$ [*i.e.*, the same as the standard equation for a gas.

3. Solutions of non-electrolytes exerting equal osmotic pressures contain the same number of gram-molecules *per c c.*, *i.e.*, are of concentrations proportional to the molecular weights of the solute. This corresponds to *Avogadro's hypothesis for gases. Solutions exerting the same osmotic pressure are called* **isotonic** or **isomotic solutions.**

4. Solutions of *electrolytes* exert a greater osmotic pressure than those of non-electrolytes, due to their splitting up into ions. In the case of electrolytes, the relation between P, V, R, and T is given by $PV = iRT$, where i is a factor depending upon its *degree of dissociation*, and is equal to the ratio of the observed osmotic pressure to that calculated as above ; so that, knowing i, the percentage dissociation of an electrolyte can be calculated *

223. Kinetic Theory of Solutions. Van't Hoff propounded the kinetic theory of solutions, similar to the one for gases, and deduced, for infinitely dilute solutions, the relation $PV = RT$, from purely thermodynamical considerations. This close similarity with the behaviour of gases led him to suggest that *the osmotic pressure of a dilute solution is the same as would be exerted by the solute, if it could exist as a gas, and if it occupied a volume equal to that occupied by the solution, at the same temperature.* This is known as **Van't Hoff law.**

This parallelism between dilute solutions and gases has been amply verified by the accurate results obtained by Berkeley and Hartley.

224. Osmosis and Vapour Pressure of a Solution. The vapour pressure of a solution is always less than that of the pure solvent, as will be clear from the following :

If we take a solvent enclosed in an evacuated chamber (Fig. 275), and place vertically in it a long glass tube, containing a solution, with its lower end closed by means of a semi permeable membrane which allows the solvent to pass through it, but not the solute, we find that the liquid rises in the tube up to H, say, until equilibrium is attained. Now, clearly, the vapour at H is in contact with the solution, and at J, with the solvent. Let the maximum pressures at J and H be p and p' respectively. Then, obviously, p is greater than p' by an amount $h.\sigma.g$, where σ is the density of the vapour of the solvent and is practically the same as that of the vapour of the solution at H, the column h being small.

Thus, $$p - p' = h.\sigma.g. \quad \ldots (i$$

*See solved example 9.

If P be the osmotic pressure of the solution, we have

P = *excess pressure inside the tube over that outside* at J on the solvent.

$= h.\rho.g - h.\sigma.g,$

where ρ is the density of the solution.

Or, $P = h.g.(\rho - \sigma)$, ... (*ii*)

whence, $h.g = \left(\frac{P}{\rho - \sigma}\right)$.

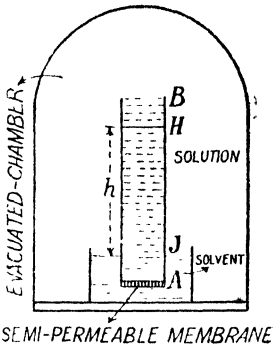


Fig. 275.

Substituing this value of $h.g.$ in relation (*i*) above, we have

$$p - p' = \sigma\left(\frac{P}{\rho - \sigma}\right).$$

Or, $p - p' = P\sigma/\rho$, [neglecting σ in comparison with ρ.

i.e., the lowering of the vapour pressure $(p - p')$, *is directly proportional to the osmotic pressure* (P).

Now, $\sigma = 1/v$, where v is the volume of 1 *gm.* of vapour at pressure p

So that, $p - p' = P/v.\rho$. ...(*iii*)

If P_0 be the atmospheric pressure, v_0, the volume of the vapour at this pressure and σ_0, its density, we have

$$p.v. = p_0 v_0 = P_0/v_0. \qquad [\because v_0 = 1/\sigma_0$$

Thus, $p.\sigma_0 = P_0/v = P_0.\sigma$. Or, $\sigma = p.\sigma_0/P_0$. ... (*iv*)

And, clearly, $v = P_0.v_0/p$.

So that, substituting this value of v in relation (*iii*) above, we have

$$p - p' = \frac{P}{\rho} \times \frac{p}{P_0.v_0} = \frac{P.p.\sigma_0}{P_0.\rho}. \qquad [\text{Putting } \sigma_0 \text{ for } 1/v_0$$

Or, $$\frac{p - p'}{p} = \frac{P.\sigma_0}{P_0\rho}.$$

Or, putting $p - p' = dp$, we have

$$\frac{dp}{p} = \frac{P}{P_0} \cdot \frac{\sigma_0}{\rho}, \qquad \text{... } (v)$$

where P/P_0 is the osmotic pressure in atmospheres, and σ_0/ρ, the ratio of the densities of the solvent in the gaseous and the liquid states at *N.T.P.**

This relation thus gives the ratio between lowering of the vapour pressure and the vapour pressure of the solvent. It is true only, however, if h be small, which will be the case, when the concentration, and, therefore, the osmotic pressure of the solution is small. If these be large, and, therefore, h also fairly large, the relation between p and p' is deduced as follows :

*N.T.P. stands for normal temperature and pressure, *i.e.*, for 0°C temperature and 76 *cms.* pressure, or a pressure of one atmosphere.

Considering a very small length, dh, of the column, we have

$$dp = -dh.\sigma.g,$$

(the negative sign merely indicating that pressure decreases as h increases)

Now, $\sigma = p.\sigma_0/P_0$. [See (*iv*) above.

$$\therefore \quad dp = -g.\frac{p.\sigma_0}{P_0}.dh. \quad \text{Or,} \quad -dh = \frac{P_0}{g.\sigma_0}.\frac{dp}{p}.$$

Integrating this expression for dh between the limits, p and p', we have

$$\int dh = -\frac{P_0}{g.\sigma_0}\int_p^{p'}\frac{1}{p}.dp.$$

Or, $$h = -\frac{P_0}{g\,\sigma_0}.\Big[\log p\Big]_p^{p'} = -\frac{P_0}{g.\sigma_0}.(\log p' - \log p).$$

Or, $$h = \frac{P_0}{g.\sigma_0}.(\log p - \log p') = \frac{P_0}{g.\sigma_0}.\log\frac{p}{p'}.$$

Or, $$\log\frac{p}{p'} = \frac{h.g.\sigma_0}{P_0}.$$

But $P = h.\rho.g$, whence, $h = P/\rho.g$.

$$\therefore \quad \log\frac{p}{p'} = \frac{P}{\rho.g}.\frac{g\sigma_0}{P_0} = \frac{P}{P_0}.\frac{\sigma_0}{\rho}. \qquad \ldots(vi)$$

This gives the relation between p and p' for large values of concentration and osmotic pressure of a solution.

Since a dilute solution behaves as a gas, the volume occupied at *N.T.P.* by one gram molecule of a substance in solution would be the same as it would occupy under the same conditions, (*i.e.*, at *N.T.P.*) in the gaseous state, *viz.*, 22·4 *litres* ; for, we know that one *gram-molecule* of a gas at *N.T.P.* occupies 22·4 *litres.*

It follows, therefore, that the osmotic pressure for a solution of one gram-molecule of a substance in a litre would be 22·4 atmospheres.

Now, the volume occupied by 1 *gram-molecule* of water vapour, *i.e.*, 18 *gms.* of it, is 22·4 *litres* or 22,400 *c.cs.*, at *N.T.P.*

$\therefore$ density of the vapour σ_0 at *N.T.P.* = 18/22400 *gms./c.c.*

And, density of water, ρ = 1 *gm./c.c.* ; so that, *for a molar solution, i.e.*, a solution containing 1 *gram molecule* of the solute, we have, applying the above relation,

$$\frac{dp}{p} = \frac{P}{P_0}.\frac{\sigma_0}{p}.$$ [See relation (*v*) above.

Here, P/P_0 = 22·4 *atmospheres,*

σ_0 = 18/22400 *gms./c.c.*, and ρ = 1 *gm./c.c.*

$$\therefore \quad \frac{dp}{p} = \frac{22{\cdot}4\times 18}{22400\times 1}\times 100\% = 1{\cdot}8\%.$$

Thus, the lowering of the vapour pressure of the molar solution of a non-electrolyte in water is 1·8%.

As already indicated, the lowering will be greater in the case of an electrolyte, due to its molecules dissociating into ions.

225. Osmosis and Boiling Point of a Solution. We know that a liquid begins to boil when its maximum or saturation vapour pres-

sure is equal to the external atmospheric pressure on its surface, *i.e.*, *the boiling point of a liquid is the temperature at which its maximum vapour pressure is equal to the atmospheric pressure.* This value of the maximum vapour pressure for a liquid increases with increase of temperature.

Now, as we have seen, the vapour pressure of a solution, at a given temperature, is lower than that of the pure solvent ; so that, at the temperature at which the solvent begins to boil, its vapour pressure is equal to the atmospheric pressure, but that of the solution will be lower than it, and the solution will, therefore, have to be heated further in order that its vapour pressure may become equal to the atmospheric pressure, *i.e.*, it may begin to boil.

In other words, the boiling point of a solution is higher than that of the pure solvent.

Let us see how this rise or *elevation of the boiling point* of a solution is related to its osmotic pressure.

The elevation of the boiling point of a solution may be obtained with the help of **Claussius-Clapeyron's latent heat equation,** *viz.*,

$$\frac{dT}{T} = \frac{(V_{vap} - V_{liq})}{L} dp,$$

where V_{vap} and V_{liq} are the volumes of the vapour and the liquid respectively, L, the latent heat of the liquid, T, the boiling point of the solvent, dp, the lowering of the vapour pressure of the solution, and dT, the elevation of the boiling point produced.

The value of dp to be used in the above equation is obtained from relation (*v*), § 227, above, *viz.*, $dp/p = P.\sigma_0/P_0.\rho$.

Or, it may be determined directly as follows :

Let A and B be the solution and the solvent, separated by a semipermeable membrane and enclosed inside a bell jar, (Fig. 276), such that both are at their boiling points $(T+dT)^0$ and T^0 Absolute, respectively, so that their vapour pressures are equal, and their vapours are in free communication with each other above the membrane, the system being in equilibrium.

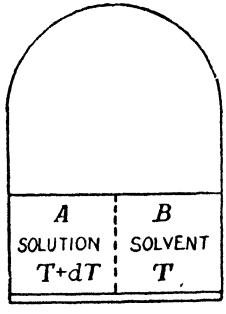


Fig. 276.

Treating the arrangement as a reversible heat engine, working between the solution and the solvent as source and sink respectively, consider the following cycle of operations :

(*i*) Let a small quantity *v c.cs.* of the solvent pass through the membrane into the solution and dilute it. Then, work done in this *expansion* of the solute is equal to the osmotic pressure of the solution × change in volume, *i.e.*,

$$= P \times v \text{ ergs},$$

where P is the osmotic pressure for the solution in *dynes per sq. cm.*

(*ii*) Let these *v c.cs.*, or $v.\rho$ *gms.*, of the solvent be evaporated from the solution in A, where ρ is the density of the solvent.

If L be the latent heat of vaporisation of the solvent, in *ergs*

per gm., the total heat absorbed by the mass $v.\rho$ *gms.* of the solvent will be equal to $(v.\rho L+P.v)$ *ergs.*

(*iii*) Let this vapour pass on from the space above *A* to that above *B*. Obviously, *no work is done during this passage of the vapour*, as the pressure is the same above *A* and *B*.

(*iv*) And, finally, let this vapour condense back into liquid in *B*, giving up $v.\rho.L$ *ergs* of heat, thus completing the cycle.

Then, we have, from the theory of the reversible heat engine,

$$\frac{\textit{heat absorbed from the source } (A)}{\textit{heat given to the sink } (B)} = \frac{\textit{temperature of source}}{\textit{temperature of sink}}.$$

Or, $$\frac{v.\rho.L+P.v}{v.\rho.L} = \frac{T+dT}{T}.$$

Or, $$\frac{v.\rho.L+Pv-v.\rho.L}{v.\rho.L} = \frac{T+dT-T}{T}. \quad \text{Or,} \quad \frac{P.v}{v.\rho.L} = \frac{dT}{T},$$

whence, $$dT = T.\frac{P}{\rho L}.$$

Thus, the elevation of the boiling point of a solution is directly proportional to its osmotic pressure. It follows, therefore, that the boiling points of isotonic solutions, in the same solvent, will be the same.

Let us now calculate the elevation of the boiling point for a solution of *one gram-molecule in one litre of water* at 100°*C*, under the normal atmospheric pressure.

We know that the osmotic pressure of a solution of 1 *gram-molecule in one litre is* 22·4 *atmospheres*, at *N. T. P.*, (see page 464). Therefore, osmotic pressure at 100°*C* and normal atmospheric pressure, (76 *cms.*), will be

$$= 22{\cdot}4\times 374/273 \textit{ atmospheres.} \qquad [\because P \propto T.$$

$$= \frac{22{\cdot}4\times 373}{273}\times 76\times 13{\cdot}59\times 981 \textit{ dynes/cm}^2.$$

Now, $L = 540$ *calories.*

$$= 540\times 4{\cdot}19\times 10^7 \textit{ ergs/gm.}, \qquad [\because J = 4{\cdot}19\times 10^7 \textit{ ergs/cal}$$

$\rho = 0{\cdot}958$ *gms./c.c.*, at *the boiling point* of water.

and $T = 100°C = 100+273 = 373°$ *Abs.*

Substituting these values in the relation for dT above, we have

$$dT = \frac{373\times 22{\cdot}4\times 373\times 76\times 13{\cdot}59\times 981}{273\times 0{\cdot}958\times 540\times 4{\cdot}19\times 10^7} = 0{\cdot}534°C.$$

The elevation of the boiling point of water per 1 gram-molecule in 100 c.cs. of it is known as the **molecular elevation of the boiling point,** *and will clearly be* $0{\cdot}534\times 10$ or 5·34°*C*, (because $dT \propto c$ and c, here, will be 10 times as great).

229. Osmosis and Freezing Point of a Solution. A liquid will freeze, as we know, when its temperature is reduced until its vapour pressure equals that of its solid. Therefore, since a solution has a lower vapour pressure than the solvent, it will freeze at a lower temperature than the pure solvent. In other words, ***the freezing point of a solvent is lowered or depressed, when a solute forms a solution***

with it. Let us calculate this depression of the freezing point of a solution.

Consider, again, a solvent B and a solution A, separated by a semipermeable membrane, and enclosed under a bell jar. Let both be at their freezing points, T and $T-dT$ respectively, the system being in equilibrium. Treating the arrangement, again, as a *reversible heat engine*, we take the solvent here to be the source, and the solution to be the sink, because the former is at a higher temperature than the latter. Now, let the following cycle of operations be performed :

(*i*) Let v *c.cs.*, or $v\rho$ *gms.*, of the solvent pass through the membrane into the solution. Then, if P be the osmotic pressure for the solution, work done is equal to $P.v$ *ergs.*

(*ii*) Let the solution be frozen, so that v *c.cs.* of pure solvent separates out, giving up heat, equal to $(v.\rho.L-P.v)$ *ergs*, where ρ. is the density of the solvent at temperature T^0 and L, its latent heat of liquefaction, or latent heat of fusion of its solid, (*in ergs per gram*).

(*iii*) Let this frozen solid be now transferred to the solvent B.

(*iv*) And, finally let this solid be melted back to liquid in B, taking up heat equal to $v.\rho.L$ *ergs*, during the process.

Then, as before, we have

$$\frac{\text{heat absorbed from source } (B)}{\text{heat rejected to sink } (A)} = \frac{\text{temperature of source}}{\text{temperature of sink}}.$$

Or, $$\frac{v.\rho.L}{v\rho.L-P.v} = \frac{T}{T-dT}.$$

Or, $$\frac{(T-dT)-T}{T} = \frac{(v.\rho.L-P.v)-v\rho.L}{v.\rho.L}.$$

Or, $$-\frac{dT}{T} = -\frac{P.v}{v.\rho.L} = -\frac{P}{\rho.L}. \quad \text{Or,} \quad \frac{dT}{T} = \frac{P}{\rho.L}.$$

Or, the *depression of the freezing point*, $dT = T.P/\rho.L$,

a result similar to that for the elevation of the boiling point of a solution.

Since the value of L is smaller in this case than in the first case, (because the latent heat of solidification of a liquid is always smaller than its latent heat of vaporisation), the value of dT will be greater here. Let us calculate the depression of the freezing point of water, (*i.e.*, dT), when 1 *gram-molecule* of a solute is dissolved in a *litre* of it at $0°C$ and under normal atmospheric pressure, (76 *cms.*). We have

$$T = 0°C = 273° \text{ } Abs., \quad P = 22{\cdot}4\times76\times13{\cdot}59\times981 \text{ } dynes/cm^2$$

$$L = 80 \text{ } calories/gm. = 80\times4{\cdot}19\times10^7 \text{ } ergs/gm., \text{ and}$$

$$\rho = 1 \text{ } gm./c.c.$$

Substituting these values in the relation for dT, above, we have

$$dT = \frac{273\times22{\cdot}4\times76\times13{\cdot}59\times981}{1\times80\times4{\cdot}19\times10^7} = 1{\cdot}85°C.$$

***The depression of the freezing point of water produced by 1 gram-molecule of a solute dissolved in 100 c.cs. of it is called the* molecular depression of the freezing point,** *and will clearly be equal to* $1{\cdot}85\times10 = 18{\cdot}5°C$.

230. Determination of Percentage of Dissociation of an Electrolyte. It must be emphasized again that the above relations are true only for non-electrolytes, *i.e.*, for substances like sugar, where the solute exists in the form of single molecules. A greater effect, (*i.e.*, elevation of boiling point or depression of freezing point) than that obtained by the above relations is observed in the case of an electrolyte, because of its dissociation into ions The deviation of the observed effect enables the percentage dissociation of the solute to be determined. For example, if we take a substance, like potassium chloride, sodium chloride, or sodium nitrate, each molecule of which breaks up into two ions, we can calculate its *percentage ionisation* as follows :

Suppose there are α molecules of the solute, and β of them dissociate into two ions each, so that the total number of particles is $(\alpha-\beta+2\beta) = (\alpha+\beta)$. Then,

$$\frac{\textit{observed effect } dT'}{\textit{calculated effect } dT} = \frac{\alpha+\beta}{\alpha},$$

where dT' and dT are the observed and calculated elevations of boiling point, or depressions of freezing point, respectively.

Or, $$dT' = dT\left(\frac{\alpha+\beta}{\alpha}\right) = dT\left(1+\frac{\beta}{\alpha}\right)$$

Or, $$dT.\beta/\alpha = dT'-dT, \text{ whence, } \frac{\beta}{\alpha} = \frac{dT'-dT}{dT}.$$

Or, $$\textit{percentage of ionisation} = \frac{dT'-dT}{dT}\times 100.$$

Similar treatment will also apply to substances whose molecules dissociate into more than two ions each.

A perfectly dilute solution is supposed to dissociate completely into its ions, and the degree of dissociation decreases with concentration, so that a concentrated solution of an electrolyte behaves, more or less like a solution of a non-electrolyte.

The reverse of the above happens in the case of colloids. The substance in a solution, here, forms clusters or aggregations of molecules, having different sizes in different cases. In such cases, the osmotic pressure, and, therefore, the elevation of the boiling point, or the depression of the freezing point, is very much smaller than in the normal case of a non-electrolyte, where the whole solute exists in the form of single molecules.

231. Determination of molecular weight of a substance from elevation of boiling-point, or depression of freezing-point of a solution of the substance.

(*a*) **From elevation of the boiling point.** Suppose we have x *gms.* of the given substance dissolved in 100 *c.cs.* of water, and the elevation of the boiling point observed is dT. Then, since an elevation of 5·34°*C* is due to 1 *gram-molecule* in 100 *c.cs.*, the number of *gram-molecules* in the solution must be $dT/5{\cdot}34$. This number of *gram-molecules* is obviously present in x *grams* of the substance taken.

So that, $dT/5{\cdot}34$ *gram-molecules* of the substance weigh

x grams. And, therefore, 1 *gram-molecule* of the substance would weigh $5{\cdot}34 \times x/dT$ *grams.*

Or, the *molecular weight of the substance*, (M), $= 5{\cdot}34 \times x/dT$ *grams.*

(*b*) **From depression of the freezing point.** Again, if *x grams* of the substance dissolved in 100 *c.cs.* of water lower, or depress, its freezing point by dT, we have

number of *gram-molecules* in *x grams* of the substance $= dT/18{\cdot}5$, because a lowering of the *freezing point* by $18{\cdot}5°C$ indicates the presence of 1 *gram-molecule* in 100 *c.cs.*

Or, $dT/18{\cdot}5$ *gram-molecules* of the substance weigh *x grams.*

And, $\therefore$ 1 *gram-molecule* of it would weigh $18{\cdot}5 \times x/dT$ *grams.*

Or, *molecule weight of the substance* $= 18{\cdot}5x/dT$ *grams.*

SOLVED EXAMPLES

1. One gram of sugar is dissolved in water, making a solution occupying 1000 c.cs. Find the osmotic pressure for the solution at 0°C and 1000°C. Molecular weight of sugar = 342 gms ; R for 1 gm molecule = $8{\cdot}4 \times 10^7$.

We know that the relation connecting osmotic pressure, volume and temperature of a dilute solution, is $PV = RT$. And, therefore,

$$P = RT/V.$$

(*i*) *At* 0°*C.* Here, $R = 8{\cdot}4 \times 10^7$ *ergs/deg.C.*, $T = 0+273 = 273°$ *Abs.*,

and $V = 342 \times 1000 = 342 \times 10^3$ *c.cs.*,

because a mass of 1 *gm.* of sugar occupies 1000 *c.cs.*, a mass of 342 *gms.* would occupy $342 \times 1000 = 342 \times 10^3$ *c.cs.*

$$\therefore \quad P = \frac{8{\cdot}4 \times 10^7 \times 273}{342 \times 10^3} = \frac{8{\cdot}4 \times 10^4 \times 273}{342}.$$

$$= 6{\cdot}707 \times 10^4 \text{ dynes/cm}^2.$$

Or, the osmotic pressure for the solution, at 0°*C*, is equal to $6{\cdot}707 \times 10^4$ *dynes/cm*^{2}.

(*ii*) *At* 100°*C.*

Again, $R = 8{\cdot}4 \times 10^7$ *ergs/deg.C.*, $T = 100+273 = 373°$ *Abs.*

and $V = 342 \times 1000 = 342 \times 10^3$ *c.cs.*

So that, substituting these values in the relation,

$P = RT/V$, we have

$$P = \frac{8{\cdot}4 \times 10^7 \times 373}{342 \times 10^3} = \frac{8{\cdot}4 \times 10^4 \times 373}{342}.$$

$$= 9{\cdot}162 \times 10^4 \text{ dynes/cm}^2.$$

$\therefore$ the osmotic pressure for the solution, at $0°C = 6{\cdot}707 \times 10^4$ *dynes/cm*^{2}.

and ,, ,, ,, ,, at $100°C = 9{\cdot}162 \times 10^4$ *dynes/cm*^{2}.

2. One gram of salt is dissolved in water to make a solution of 100 c.cs. Assuming the salt to be entirely dissociated, find, in atmospheres, the osmotic pressure of the solution at 20°C.

Atomic weights.—Na = 23, Cl = 35·5. Density of mercury = 13·6 gms/c.c. and R = $8{\cdot}4 \times 10^7$ for 1 gm.-molecule.

Each molecule of the salt (*NaCl*) dissociates into two ions, *i.e.*, *Na* and *Cl*. And, since the salt is completely dissociated, *there will be twice as many particles present in 1 c.c. of the solution as the number of whole molecules.* It follows, therefore, that the osmotic pressure will be double of that obtained by the relation, $P = RT/V$, *i.e.*, this osmotic pressure, in this case, is given by

$$P = 2 \times RT/V.$$

Now, $R = 8{\cdot}4 \times 10^7$ *ergs/deg. C.* ; $T = 20+273 = 293°$ *Abs.*

and $V =$ *mol. wt.* of salt $\times 100$ *c.cs.*,

because 1 *gm.* is contained in 100 *c.cs.*, and, therefore, its *mol. wt.* is contained in *mol. wt.* $\times$ 100 *c.cs.*

Or, $V = (23+35{\cdot}5)\times 100$ *c.cs.* $= 58{\cdot}5\times 10^2$ *c.cs.*

$\therefore$ $P = \dfrac{2\times 8{\cdot}4\times 10^7\times 293}{58{\cdot}5\times 10^2} = \dfrac{2\times 8{\cdot}4\times 10^5\times 293}{58{\cdot}5}$ *dynes/cm*^{2}.

$= \dfrac{2\times 8{\cdot}4\times 10^5\times 293}{58{\cdot}5\times 76\times 13{\cdot}6\times 981}$ *atmospheres.*

($\because$ 1 *atmosphere* $= 76\times 13{\cdot}6\times 981$ *dynes/cm*^{2}.)

Or, $P = 8{\cdot}299$ *atmospheres.*

The osmotic pressure for the solution is, therefore, 8·299 *atmospheres.*

3. When 1 gm. of iodine is dissolved in ether making 40 c.cs. of solution, the osmotic pressure at 20°C is 2·4 atmospheres. Show that the molecule of dissolved iodine consists of two atoms. (At. wt. of Iodine = 127).

Here, observed osmotic pressure = 2·4 *atmospheres.*

Now, let us calculate the osmotic pressure from the given data.

We have $P = RT/V$.

Since, $R = 8{\cdot}4\times 10^7$ *ergs/deg. C.* ; $T = 20+273 = 293°$ *Abs.*

and $V = 127\times 40$ *c.cs.*,

$\because$ iodine being moncatomic, its *mol. wt.* is equal to its *atomic weight.*

$\therefore$ $P = \dfrac{8{\cdot}4\times 10^7\times 293}{127\times 40}$ *dynes/cm*^{2}. $= 4{\cdot}778$ *atm.*

$= 4{\cdot}8$ *atmospheres* (nearly),

which is equal to $2\times 2{\cdot}4$ atmospheres.

Thus, the observed osmotic pressure is half of the calculated osmotic pressure and, therefore, it follows that the two atoms of iodine in solution combine to form one molecule, or that *the molecule of dissolved iodine consists of two atoms.*

4. Calculate the strength of a cane-sugar solution whose osmotic pressure at 270°C is one atmosphere. Mol. wt. of sugar = 342 gms., and R = $8{\cdot}4\times 10^7$ ergs./deg. C.

We have the relation $PV = RT$. Or, $V = RT/P$,

where V is the the volume of the solution, containing 1 *gram-molecule* of the substance,—in this case, 342 *gms.* of sugar.

Thus, V *c.cs.* of the solution contain 342 *gms.* of sugar,

$\therefore$ 1000 *c.cs.* ,, ,, ,, $\dfrac{342\times 1000}{V}$ *gms.* of sugar.

Or, strength of the sugar solution $= 342\times 1000/V$.

$= 342\times 1000\times P/RT$ *grams/litre.* [$\because V = RT/P$.

Here, $P = 1$ *atmosphere.* $= 76\times 13{\cdot}6\times 981$ *dynes/cm*^{2}.

$R = 8{\cdot}4\times 10^7$ *ergs/deg. C.*, and $T = 27+273 = 300°$ *Abs.*

$\therefore$ strength of the sugar solution $= \dfrac{342\times 1000\times 76\times 13{\cdot}6\times 981}{8{\cdot}4\times 10^7\times 300}$.

$= \dfrac{342\times 76\times 13{\cdot}6\times 981}{8{\cdot}4\times 10^4\times 300} = 13{\cdot}76$ *gms./litre.*

Thus, the strength of the given cane sugar solution = 13·76 *gms./litre.*

5. The osmotic pressure of a solution containing 6 gms. of cane sugar in 100 c.cs. of water is 307 cms. of Hg. at 13°C. What is the osmotic pressure of a solution of one-sixth this concentration at 50°C ? *(London Inter-Science)*

Let the *molecular weight* of sugar be = m *gms.*

Then, the volume containing 1 *gm. mol.* in the first case, say, V_1, where

$V_1 = \dfrac{m\times 100}{6}$ *c.cs.*, [$\because$ 6 *gms.* are contained in 100 *c.cs.*
and $\therefore$ *m.gms.* ,, $\dfrac{100\times m}{6}$ *c.cs.*

and the volume in the second case, say, $V_2 = m\times 100$ *c.cs.*,

because 1 *gm.* is contained in 100 *c.cs.*

And temperature T_1 in the *1st case* = 13+273 = 286° *Abs.*
,, T_2 ,, ,, *2nd case* = 50+273 = 323° ,,

If P_1 and P_2 be the osmotic pressures in the two cases, we have

$$P_1 = RT_1/V_1, \quad \text{and} \quad P_2 = RT_2/V_2,$$

where R is the constant for the solution,—the same as the gas constant.

$$\therefore \quad \frac{P_2}{P_1} = \frac{RT_2}{V_2} \cdot \frac{V_1}{RT_1} = \frac{T_2}{T_1} \cdot \frac{V_1}{V_2}. \quad \text{And} \quad \therefore P_2 = \frac{T_2}{T_1} \cdot \frac{V_1}{V_2} \cdot P_1.$$

Substituting the values of T_1, T_2, V_1, V_2 and P_1, we have

$$P_2 = \frac{323 \times m \times 100 \times 307 \times 13{\cdot}6 \times 981}{286 \times 6 \times m \times 100},$$

because P_1 = 307 *cms.* of mercury = $307 \times 13{\cdot}6 \times 98$ *dynes/cm*^{2}.

Or,

$$P_2 = \frac{323 \times 307 \times 13{\cdot}6 \times 981}{286 \times 6} \text{ dynes/cm}^2.$$

$$= \frac{323 \times 307 \times 13{\cdot}6 \times 981}{286 \times 6 \times 13{\cdot}6 \times 98} = \frac{323 \times 307}{28 \times 66} = 57{\cdot}77 \text{ cms. of Hg.}$$

Thus, the osmotic pressure in the second case = 57·77 *cms.* of mercury.

6. Find the percentage lowering of the vapour pressure of water produced by dissolving 1 gm. of sugar in 100 c.cs. Mol. wt. of sugar = 342 gms.

We have the relation for lowering of vapour pressure,

$$\frac{dp}{p} = \frac{P}{P_0} \cdot \frac{\sigma_0}{\rho}.$$ [See page 463, relation (*v*).

∴ percentage lowering of vapour pressure

$$= \frac{dp}{p} \times 100 = \frac{P}{P_0} \times \frac{\sigma_0}{\rho} \times 100,$$

where P/P_0 is the osmotic pressure of the solution *in atmospheres*, and σ_0/ρ, the ratio of the densities of the solvent in the *gaseous* and the *liquid* states, at *N.T.P.*

Now, $P = RT/V$,

and, here, T = 273° *Abs.*, V = 342×100 *c.cs.* [taking the solution at 0°*C*.

$R = 8{\cdot}4 \times 10^7$ *ergs/deg.C.*

$$\therefore \quad P = \frac{8{\cdot}4 \times 10^7 \times 273}{342 \times 10^2} \text{ dynes/cm}^2.$$

And, hence,

$$\frac{P}{P_0} = \frac{8{\cdot}4 \times 10^5 \times 273}{342 \times 76 \times 13{\cdot}6 \times 981} \text{ atmospheres},$$

because P_0 = 76 *cms.* of *Hg.* = $76 \times 13{\cdot}6 \times 981$ *dynes/cm*^{2}.

$$\sigma_0 = \frac{18}{22400},$$ [∵ 1 *gm. mol.* of water vapour, or 18 *gms.* of water vapour, at *N.T.P.* occupies 22·4 *litres*, or 22400 *c.cs.*

and ρ or water at *N T.P.* = 1 *gm./c.c.*

$$\therefore \quad \frac{dp}{p} \times 100 = \frac{8{\cdot}4 \times 10^5 \times 273}{342 \times 76 \times 13{\cdot}6 \times 981} \times \frac{18}{22400} \times 100\%$$

$$= 0{\cdot}05316\%.$$

∴ lowering of vapour pressure of water is equal to 0·05316%.

7. By how much will the freezing point of the sugar solution in question 6, above, be depressed ?

Latent heat of ice = 80 cals./gm., and atmospheric pressure = 10^6 dynes./cm^2.

We have, here, 1 *gm.* of sugar in 100 *c.cs.* of water, or 1/342 *gm. molecules* in 100 *c.cs.* of water, and, therefore, 10/342 *gm. molecules* in 1000 *c.cs.*, or *one litre* of water.

Now, 1 *gm. molecule* in 1 *litre* of water exerts an *osmotic pressure* = 22·4 *atmospheres* at *N.T.P.*

∴ 10/342 *gms.-molecules* in 1 *litre* of water will exert an osmotic pressure

$$P = \frac{22{\cdot}4 \times 10}{342} = \frac{224}{342} \text{ atmospheres} = \frac{224}{342} \times 10^6 \text{ dynes/cm}^2.$$ [1 *atm.* = 10^6 *dynes./cm*^{2}.

Now, $dT = T.\frac{P}{\rho.L}$. [See page 467

Here, $T = 0°C = 273°$ *Abs.*, $P = \frac{224\times10^6}{342}$ *dynes/cm²*.,

$\rho = 1$ *gm./c.c.* and $L = 80$ *cals./gm.* $= 80\times4{\cdot}2\times10^7$ *ergs per gm.*

∴ the depression of the freezing point, *i.e.*,

$$dT = \frac{273\times224\times10^6}{342\times1\times80\times4{\cdot}2\times10^7} = \frac{273\times22\,4}{342\times80\times4{\cdot}2} = 0{\cdot}05322°C.$$

The freezing point of the sugar solution will, therefore, be depressed by 0·053°*C*

8. Determine the elevation of the boiling point of ether produced by dissolving 10 gms. of carbon hexachloride (C_2Cl_6) in 100 c.c. of ether.

The boiling point of ether is 35°C, its density at 35°C = 0·695 gm. per c.c. and its latent heat of vaporisation is 81·5 cals./gm. (Take atmospheric pressure $=10^6$ dynes/cm².

Here, molecular weight of carbon hexachloride (C_2Cl_6) = 24+213 = 237 *gms.*

Hence, 100 *c.cs* of ether contain 10 *gms.* of carbon hexachloride, or 10/237 of 237, or 10/237 *gram-molecules* of carbon hexachloride.

∴ 1000 *c cs.* or 1 *litre* of ether will contain $\frac{10}{237}\times\frac{1000}{100}$, or $\frac{100}{237}$ *gram-molecules* of the hexachloride.

Now, osmotic pressure for a solution of 1 *gram-molecule* in 1 *litre* is 22·4 *atmospheres at N.T.P.*

∴ osmotic pressure for a solution of 100/237 *gram-molecules* in 1 *litre* at *N.T.P.*

$$= \frac{100}{237}\times22{\cdot}4 \text{ atmospheres.}$$

And ∴ osmotic pressure for a solution of 100/237 *gram-molecules* in 1 *litre*, at 35°*C*, or (273+35) or 308° *Abs.* is given by

$$P = \frac{100\times22{\cdot}4}{237}\times\frac{308}{273} \text{ atmospheres} = \frac{2240\times308\times10^6}{237\times273} \text{ dynes/cm}^2.$$

Now, elevation of the boiling point $dT = T.P/\rho.L$. [See page 467.

Here, $T = 35+273 = 308°$ *Abs.*, $P = \frac{2240\times308\times10^6}{237\times273}$ *dynes/cm²*.

$\rho = 0{\cdot}695$ *gms/c.c.*, and $L = 81{\cdot}5$ *cals./gm.* $= 81{\cdot}5\times4{\cdot}2\times10^7$ *ergs./gm.*, (taking $J = 4{\cdot}2\times10^7$ *ergs/cal.*).

$$\therefore\ dT = \frac{308\times2240\times308\times10^6}{237\times273\times0{\cdot}695\times81\,5\times4{\cdot}2\times10^7}$$

$$= \frac{308^2\times224}{237\times273\times{\cdot}695\times81\,5\times4{\cdot}2} = 1{\cdot}38°C.$$

Or, the elevation of the boiling point of ether = 1·38°*C*.

9. A one per cent solution of potassium chloride (KCl) has an osmotic pressure 6·604 atmospheres at 27 0°C. Determine its degree of dissociation. (At. wts. — K = 39, Cl = 35 5).

Here, molecular weight of potassium chloride is equal to 39 +35 5 = 74·5 *gms.*

∴ 74·5 *gms.* of potassium chloride dissolved in 1000 *c.cs.* of water would exert a pressure of 22 4 *atmospheres at N.T.P., if undissociated.*

The solution given, however, contains 1 *gm.* of *KCl* in 100 *c.cs.* or 1/74·5 *gm.-molecules* in 100 *c.cs.*, or 10/74·5 *gm.-molecules* in 1000 *c.cs.*

∴ osmotic pressure of the solution = 22·4×10/74·5 *atmospheres* at *N.T.P.*

Hence, osmotic pressure of the solution, at 27°*C*.

$$= 22{\cdot}4\times\frac{10}{74{\cdot}5}\times\left(\frac{273+27}{273}\right) = \frac{224\times300}{74{\cdot}5\times273} = 3{\cdot}303 \text{ atmospheres.}$$

Or, osmotic pressure of the solution, if no dissociation takes place = 3·303 *atmospheres.*

But the observed osmotic pressure = 6·604.

$\therefore \quad i = \frac{6\,604}{3{\cdot}303} = 1{\cdot}999, \quad$ because $\quad i = \frac{observed\ effect}{calculated\ effect}$.

This means that instead of there being 1000 particles, 1999 particles are actually exerting pressure.

Thus, 1000 particles dissociate into 1999 particles.

And $\therefore$ 100 „ „ „ $\frac{1999}{1000} \times 100 = 199{\cdot}9$ particles.

Or, dissociation = 199·9 − 100 = 99·9%.

Thus, degree of dissociation of the *KCl* solution = 99·9%.

10. 31·88 gms. of a substance is dissolved in 2 litres of water. The boiling point of the solution is 0·266°C higher than that of pure water, under normal atmospheric pressure. Calculate the molecular weight of the substance.

Here, 2000 *c.cs.* of water contain 31·88 *gms.* of the substance.

$\therefore$ 100 *c.cs.* of water contain $\frac{31\,88}{2000} \times 100 = \frac{31\,88}{20}$ *gms* of the substance.

Now, we know that the *molecular weight* of a substance, *i.e.*,

$$M = \frac{5{\cdot}34x}{dT} \text{ gms.}$$ [See page 469.

Here, $x = 31{\cdot}88/20$ *gms.* and $dT = 0{\cdot}266°C$.

$\therefore$ *molecular weight* of the substance, *i.e.*,

$$M = \frac{5{\cdot}34 \times 31\,88}{0{\cdot}266 \times 20} = 32{\cdot}00 \text{ gms.}$$

Or, the *molecular weight* of the substance is equal to 32·00 *gms.*

EXERCISE XIII

1. Enunciate Fick's law of diffusion and find the dimensions of the coefficient of diffusion.

Starting with the same concentration of a given solute at the bottom of columns of the same liquid, show that the times required to set up a given concentration at the top are proportional to the squares of the heights of the columns. (*Madras, 1947*)

2. State Fick's law of diffusion in liquids. Compare diffusion in liquids with heat conduction.

Indicate a method of determining experimentally the coefficient of diffusion of a salt in a solution. (*Madras, 1949*)

3. What do you understand by osmosis, dialysis and diffusion ? State the laws of osmosis and describe an arrangement for measuring osmotic pressure. (*Calcutta, 1944*)

4. State the laws of osmotic pressure.

Give an account of the relation between osmotic pressure and other properties of a solution. (*Madras, 1947*)

5. Calculate the osmotic pressure of a sugar solution, (*mol. wt.* = 342 *gms.*), containing 1·5 *gms.* of sugar *per litre*, at 37·0°*C*. Ans. 8·478 *cms. of Hg.*

6 What is the strength of a sugar solution whose osmotic pressure at 17°*C* is 1·5 *atmospheres.* (1 *atmosphere pressure* = 10^6 *dynes/cm*^{2}., and $R = 8{\cdot}4 \times 10^7$ *ergs/gm. mol. per degree Centigrade*). Ans. 21·06 *gms./litre.*

7. Define osmotic pressure and state its laws. How will you demonstrate this pressure ? Deduce an expression for the elevation of the boiling point of a liquid by a non-volatile substance dissolved in it. (*Madras, 1949*)

8. Calculate the osmotic pressure of a one per cent solution of sodium chloride (*NaCl*) at 27°*C*, assuming the dissociation of the molecule to be 99·5%. (*At. wts.*—*Na* = 23·0, *Cl* = 35·5) Ans. 8·013 *atmospheres.*

9 Determine the percentage lowering of the vapour pressure of an aqueous solution of a non-electrolyte, of *mol. wt.* 100 *gms.*, when 1 *gm.* of it is dissolved in 100 *c.cs.* of water at *N.T.P.* ($R = 8{\cdot}4 \times 10^7$/*gm. mol. per degree C*).
Ans. 0·1818%.

10. Deduce the relationship between osmotic pressure and relative lowering of vapour pressure. How is osmotic pressure measured ? (*Madras, 1949*)

11. A solution of 5 *gms.* of an electrolyte in water, total volume 1000 *c.cs.*, freezes at $-0{\cdot}279°C$. What is the molecular weight of the substance in solution ? (Molecular lowering of the freezing point $= 18{\cdot}6°C$).
Ans. 33·3 *gms*.

12. What will be the boiling point of a solution of cane sugar, (*mol. wt* = 342), containing 1 *gram* of sugar per 100 *c cs.* of it, under the normal atmospheric pressure ? Density of water at $100°C$ may be taken to be 0·9580 *gm./c.c.*
Ans. 100 0156°*C*.

13. 12·5 *gms.* of sugar (*mol wt.* = 342) dissolved in 1000 *c.cs.* of water lowers the freezing point by 0 068°*C*. Calculate the molecular lowering of the freezing point for water. Ans. 18·6°*C*.

14. In terms of the molecular theory of matter, discuss the phenomena associated with (*a*) diffusion, (*b*) osmosis. (*Bombay, 1944*)

15. Calculate the freezing point of a salt solution containing 1 *gm.* of salt in 1000 *c cs.* of water, assuming that the salt is completely dissociated into sodium and chlorine ions. (*At. wts.—Na* = 23, *Cl* = 35 5) Ans. $-0{\cdot}064°C$.

16. The osmotic pressure of a solution of 2 *gms.* of acetone dissolved in water to make 100 *c.cs.* is 7·75 *atmospheres* at $10°C$. Calculate the *molecular weight* of acetone. ($R = 8\ 4 \times 10^7$ *ergs/gm. mol per deg. C*). Ans. 60.

17. 1·2 *gms.* of a substance dissolved in 24·5 *gms.* of water caused a freezing point depression of $1{\cdot}05°C$. What is the molecular weight of the substance ? Molecular lowering of freezing point for water may be taken to be 18·6°*C*. Ans. 86·8 *gms.*

CHAPTER XIV

SURFACE TENSION—CAPILLARITY

232. Molecular Forces. Since surface tension is essentially a molecular phenomenon, we had better have first a clear idea as to what forces operate between molecules.

There are two types of molecular forces : (*i*) **forces of adhesion**, or **adhesive forces,** and (*iii*) **forces of cohesion** or **cohesive forces.**

(*i*) *Adhesion is the force of attraction between molecules of different substances*, and is different for different pairs of substances, *e.g.*, gum has a greater adhesive force than water or alcohol.

(*ii*) *Cohesion, on the other hand, is the force of attraction between molecules of the same substance.* This force is different from the ordinary gravitational force and does not obey the ordinary inverse square law, the force varying inversely probably as the eighth power of the distance between two molecules and thus decreases rapidly with distance, —in fact it is '*appreciable when the distance between two molecules is inappreciable and becomes inappreciable when the distance is appreciable.*'* It is the greatest, in the case of solids, less in the case of liquids and the least in the case of gases, almost negligible at ordinary temperature and pressure, when the molecules lie very much further apart for it to be appreciable. This explains at once why *a solid has a definite shape, a liquid has a definite free surface* and a *gas has neither.*

233. Molecular Range—Sphere of Influence. The maximum distance up to which the force of cohesion between two molecules can act is called their *molecular range*, and is generally of the order of 10^{-7} *cms.* in the case of solids and liquids, being different for different substances. *A sphere drawn around a molecule as centre, with a radius equal to its molecular range is called the sphere of influence*† *of the molecule.* Obviously, the molecule is affected only by the molecules inside this sphere, *i.e.*, it attracts and is, in turn, attracted by them, remaining unaffected by the molecules outside it, as they lie beyond its range of attraction. **Laplace** (1806) and **Gauss** (1830) were the first to have evolved this theory of cohesive force between molecules in order to satisfactorily explain the various effects of surface tension, like capillarity etc.

234. Surface Tension. It is a general experience that a liquid in small quantity, free from any external force, like that due to

*As *N K. Adam* remarked in his '*Physics and Chemistry of Surfaces,*' if the force were gravitational in character and obeyed the inverse square law, 'the surface tensions of the ocean would be far greater than that of a cup full of water, because the distant parts would act with sensible effect '

†It is also sometimes referred to as the *sphere of molecular attraction*, or, *molecular activity.*

gravity, will always assume the form of a spherical drop — *e.g.*, rain drops, small quantities of mercury placed on a clean glass plate etc.

Now, *for a given volume, a sphere has the least surface area.* Thus, *a liquid always tends to have the least surface area.* The following experiments beautifully illustrate this tendency of a liquid to decrease its surface area.

1. If we take a wire-ring and dip it in a soap solution, we find that, on taking it out, a thin film is formed across the ring. If we place a moistened cotton loop lightly on the film, it will remain in any form or position in which it is placed, as shown in Fig. 277 (*a*); for, the soap-film lies both inside and outside the loop and at every point on the loop, therefore, there are equal and opposite forces, tending to pull it *outward* (due to the outside film) and *inwards* (due to the inside film), thus cancelling each other out. But if the film inside the loop be pricked, so that the film there disappears, we find that the loop at once gets stretched into the form of a circular ring, as shown in Fig. 277 (*b*), because the inward forces having all vanished, only the outward forces are left acting perpendicularly to it at every point. Now, *for a given perimeter, a circle encloses the greatest area,* so that the loop now encloses the *maximum area.* In other words, the area of the film left between the loop and the wire ring is now reduced to a *minimum,* clearly showing that the film has a tendency to contract or shrink, or that there is *tension* in it.

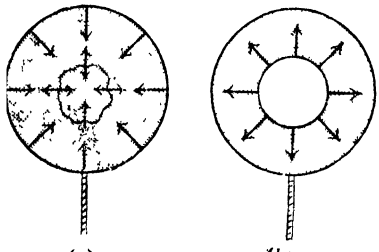


Fig 277.

2. If we place a greased needle on a piece of blotting paper and put the paper lightly on the surface of water, the blotting paper will soon sink to the bottom but the *needle will remain floating* on the surface Careful observation will show that there is a small depression formed below and around the needle, and that the free surface of water is slightly extended *The weight of the needle is here supported by the tension in the depression.* If one end of the needle be made to pierce the surface of water, it rapidly goes slantingly down to the bottom

3. If we immerse an ordinary camel hair paint brush in water, its hair all spread out, presenting a sort of a bushy appearance, but the moment it is withdrawn, they all come closer together in a more or less compact mass, as though bound down by some sort of a contracting membrane.

4. Yet another beautiful experiment, often performed for fun by junior students, is the rapid movement of a *camphor scorpion* on water. What they do is simply to arrange pieces of camphor together, in the shape of a *scorpion,* and put it on water, when, due to the reduction in the surface tension of water, on account of the camphor gradually dissolving into it, the camphor is drawn or pulled aside by the surrounding uncontaminated water of a higher surface tension. And, since we have camphor dissolving more rapidly at some points than at others, this force due to surface tension is not uniform all round, with the result that the 'scorpion' scampers about haphazardly in different directions.

If, however, the water be already contaminated with some grease etc., its surface tension may be reduced to an extent that the camphor has no further possibility of reducing it. In such a case, therefore, the movement of the camphor may altogether stop.

The above experiments clearly show that the surface of a liquid behaves as though it were covered with an elastic skin or membrane, having a natural tendency to contract, with the important difference, however, that whereas in the case of the membrane or skin, the tension increases as the skin is stretched, or its surface area is increased, in accordance with Hooke's Law, it is quite independent of the area of the surface in the case of a liquid, unless the liquid film is reduced in thick-

ness to less than 10^{-7} cms., when the tension in it decreases rapidly. This tension or pull in the surface of a liquid is called its **surface tension,** and may be defined *as the force per unit length of a line drawn in the liquid surface, acting perpendicularly to it at every point, and tending to pull the surface apart along the line.*

235. Explanation of Surface Tension. An extension of the kinetic theory of gases to the case of liquids easily explains the phenomenon of surface tension and the credit for it must go to **Laplace,** who first attempted it.

Consider four molecules, *A*, *B*, *C* and *D* of a liquid, with their spheres of influence drawn around them, as shown in Fig. 278,—sphere *A* being *well inside* the liquid *B*, *near to the free surface* of the liquid, *C*, *just on the free surface* and *D*, *above the free surface.*

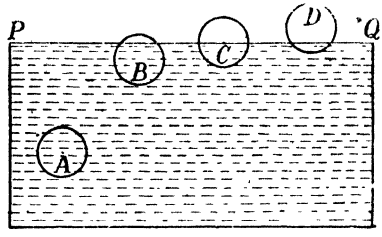


Fig. 278.

Since the sphere of influence of *A* lies wholly inside the liquid, it is attracted equally in all directions by the other molecules lying within its sphere of influence, so that there is no resultant cohesive force on it one way or the other, and it, therefore, merely possesses its *thermal velocity.*

The sphere of influence of molecule *B*, on the other hand, lies partly outside the liquid, and this part contains only a comparatively few molecules of the gas or vapour above the liquid, so that the upper half of the sphere contains fewer molecules attracting it upwards, than the lower half, attracting it downwards, and so there is a *resultant downward force* acting on *B*.

The molecule *C* lies on the surface of the liquid, so that full one-half of its sphere of influence lies above the surface of the liquid, containing only a few molecules of the gas or vapour, whereas there are liquid molecules in its entire lower half, and thus, the resultant downward force in this case is the *maximum This downward or inward force exerted per unit area of a liquid surface is called its internal, intrinsic* or *cohesion pressure*, and is the cause of cohesion It is this pressure which is represented by the term a/v^2 in **van de Waal's** modified gas-equation.

In the case of the molecule *D*, which has passed out of the liquid surface, only a part of the sphere of influence lies inside the liquid, so that the downward force on it decreases, and when the sphere of influence passes entirely outside the liquid surface, there is no downward force on the molecule at all, and it is free to wander about as a molecule of the vapour or gas. We thus see that *all over the surface of the liquid there is a downward pull due to the attraction between the molecules.*

236. Surface Film and Surface Energy. If a plane be drawn parallel to the free surface layer and *at a distance equal to the molecular range* from it, the layer of the liquid, lying in-between the free

surface and this plane is called the *surface-film.* Clearly, all the molecules in the film are being acted upon by a resultant cohesive force in the downward direction, the magnitude of which increases as the free surface of the liquid is approached.

When a molecule is brought from the interior of the liquid to the surface-film, work has to be done against the downward cohesive force acting on it, and its potential energy is thus increased. *It follows, therefore, that the potential energy of the molecules in the surface-film is greater than that of those inside the liquid.*

Since, however, *the potential energy of a system tends towards a minimum,* (*i.e* , a system tends to have the lowest possible potential energy), *the film tends to have the least surface area,* (its thickness being fixed) *in order that the number of molecules in it may be a minimum.* The potential energy per unit area of the surface-film is called its *surface energy.*

Now, before a molecule can be brought to the surface of the liquid, room has to be made for it, and this is done by separating the adjacent molecules on the surface. *Work is, therefore, done against the cohesive forces between the molecules in the surface, and not against the internal or downward cohesive force,* for the molecules are moved in a direction perpendicular to this latter force. It thus appears that *in increasing the surface area of a liquid, work is done as though the surface under a state of tension were being extended, very much like a rubber sheet being extended.* The analogy, however, does not go far; for, as already indicated, whereas in the case of a rubber sheet, the tension increases with extension or increases in its surface area, it is quite independent of the surface area in the case of a surface-film, and is the same at all points in it.

237. Free Energy of a Surface and Surface Tension. Take a rectangular framework of wire *PQRS*, (Fig. 279), with a horizontal wire *AB* placed across it, free to move up and down, and form a soap-film across *AQRB*, by dipping it in a soap solution. The wire *AB* is pulled upwards by the surface tension of the film, acting perpendicularly to the wire and in the plane of the film. To keep the wire in position, therefore, a force has to be applied downwards, equal and opposite to the upward force due to surface tension. Let this downward force be equal to *F* including the weight of the wire *AB*, which is also acting downwards. Then, if *T* be the surface tension of the film, *i.e.*, if *T* be the *force per unit length of the film* and *l*, the length of the wire *AB*, we have upward force acting on the wire $AB = 2\,l.T.$, because the film has *two surfaces* and *each has a surface tension T.* Since the film is in equilibrium, it is clear that

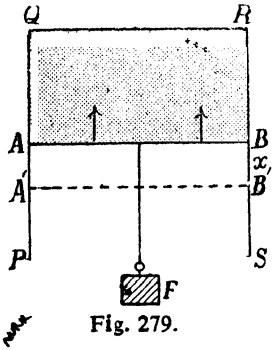


Fig. 279.

$$2.l.T = F.$$

Now, if the wire *AB* be pulled downwards through a small

distance x into the position $A'B'$, *i.e.*, if the film be extended by an area $l.x$ on each side, we have

$$work\ done = x = F \times 2.l.T\ x.$$

The film gets cooled on being stretched, because the drawing out of the molecules from the interior against the attractive force results in a retardation of their thermal agitation, with a consequent lowering of temperature*. It, therefore, takes up heat from the atmosphere to come to its original temperature. *This heat absorbed together with the mechanical work done, forms the energy of the new surface area 2lx of the film formed.*

If, therefore, E be the *surface energy* of the film and Q *ergs* of heat be absorbed per unit area of the *new* surface formed†, we have.

$$E \times 2l.x = 2.l\ T.x + Q.2.l.x.$$

Or, $E = T + Q.$ [Dividing throughout by $2lx$.

Or, $T = (E - Q) =$ (*surface energy—heat energy per unit area*)

i.e., $=$ *potential energy per unit area.*

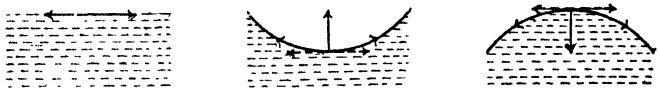
$T =$ *work done in creating unit area of the film.*

Thus, the surface tension of a liquid may be defined as the amount of work done in increasing the surface area of the liquid-film by unity, or as the mechanical part of the surface energy of the liquid film. This mechanical part of the surface energy of a liquid-film is free energy, so that, the surface tension of a liquid is equal to the free energy of the liquid film or surface.‡

238. Pressure Difference Across a Liquid Surface—Drops and Bubbles. (*i*) Suppose the free surface of a liquid is *plane*, as shown in Fig. 280 (*i*), then, the resultant force due to surface tension on a molecule on its surface is *zero*, and the cohesion-pressure is, therefore, just *nominal.*

(*ii*) If the free surface of the liquid be *concave*, [Fig. 280 (*ii*)], the resultant force on a molecule on the surface would be upwards, and the cohesion pressure is, therefore, *decreased.*

(*iii*) And, finally, if the liquid surface be *convex*, as shown in Fig. 280 (*iii*), the resultant force due to surface tension on a molecule



(*i*) (*ii*) (*iii*)

Fig. 280.

*This is clearly an example of *Le Chatelier's principle, viz., that 'if one of the factors of any system in equilibrium is changed, thus disturbing the equilibrium, the effect produced tends to restore that factor to its original value.'* Thus, a lowering of the temperature of the surface results in a rise in its tension, which increases the force opposing enlargement of its surface.

†It can be shown that the value of Q is equal to $\theta.dT/d\theta$, where θ is the absolute temperature and $dT/d\theta$, the rate of change of surface tension with temperature.

‡Surface tension of a liquid is generally, but *erroneously*, defined as the surface energy of the liquid surface. But, obviously, the surface energy can be equal to the surface tension *only if* the heat absorbed by the film be ignored.

on the surface will be directed downwards, so that the cohesion-pressure is, in this case, *increased.*

239. Excess Pressure inside a Liquid Drop. It must be clear from the above that the molecules near the surface of a drop, (which is a *convex* surface), experience a resultant pull *inwards.* The pressure inside it must, therefore, be greater than the pressure outside it. Let this excess pressure inside over the pressure outside the drop be p. Then, if r be the radius of the drop, and T, its surface tension, we have, considering the equilibrium of one-half of the drop—say, the upper half, or the upper hemisphere,—the upward thrust on the plane face $ABCD$, (Fig. 281), due to the excess pressure p is equal to $p.\pi r^2$.

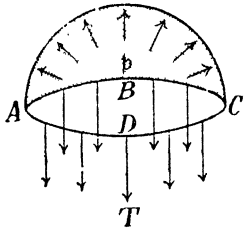


Fig. 281.

And, force due to surface tension, acting downwards on it and round its edge, is equal to $T.\ 2\pi r$. Since the hemisphere is in equilibrium, we have $p.\pi r^2 = T.2\pi r$,

whence, $$p = \frac{T.2\pi r}{\pi r^2} = \frac{2T}{r}.$$

240. Excess Pressure inside a Soap Bubble. If, instead of a drop of liquid, we consider a bubble, there are *two surfaces* to be considered, and *not one*, because it is like a spherical shell or a hollow cylinder, having an inner and an outer surface ; so that, the force due to surface tension in this case is $2 \times 2\pi r.T$, (*i.e.*, $2\pi r.T$ due to each surface). Therefore, for equilibrium of the hemisphere, we have, in this case, $p.\pi r^2 = 2 \times 2\pi r.T = 4\pi rT$,

whence, $$p = \frac{4\pi r.T}{\pi r^2} = \frac{4T}{r}.$$

It will thus be seen that the excess pressure inside a drop or a bubble is inversely proportional to its radius (i.e., $p \propto 1/r$) ; so that, the smaller the bubble, the greater the excess pressure inside it.

This can be beautifully shown by blowing two soap bubbles, of *unequal* sizes, at the two ends of a tube of the form of the letter T, as shown in Fig. 282, and then putting them into communication with each other. The air passes from the smaller one into the bigger one, (because the pressure inside the smaller one is greater than that inside the bigger one), so that the smaller bubble goes on shrinking, and the bigger one goes on swelling, until the smaller one is reduced to a *hemisphere* ; for, then, any further shrinkage would mean an increase in its radius and, therefore, a decrease in the pressure inside it. Equilibrium between the two bubbles is attained when the curvatures of the two become the *same.*

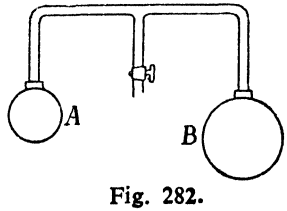


Fig. 282.

241. Determination of the Surface Tension of a Bubble. The expression for the excess pressure inside a bubble, deduced in §240, affords us a simple method of determining the surface tension of the bubble. For, if we have a vertical glass tube AB, (Fig. 283), with a *fine* orifice or aperture at its lower end B, connected to a manometer M, and a stop-cock S, then, on dipping AB into the experimental liquid, a liquid film is formed at the orifice. This film is then blown into a bubble at B, by opening the stop-cock for a while and allowing some air to come into AB. The difference h in the heights of the liquid columns in M is noted. Then, if ρ be the density of the *manometer-liquid*, the excess pressure inside the bubble is clearly equal to $h.\rho.g$. But, as we know, the excess pressure inside the bubble is also equal to $4T/r$, where T is its surface tension. Hence, equating the two, we have

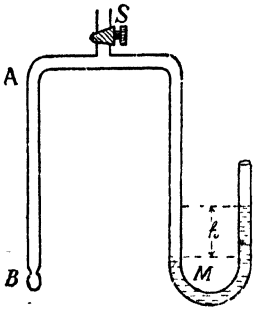


Fig. 283.

$$4T/r = h.\rho.g, \qquad \text{whence,} \qquad T = r.h.\rho.g/4,$$

where r is the radius of curvature of the bubble which is obtained from its diameter, measured accurately by means of a travelling microscope.

It will easily be seen that for accuracy of the result, the value of h should be large, *i.e.*, the excess pressure inside the bubble should be large. This would be so, if the bubble be small or the size of the aperture at B be small.

242. Work done in Blowing a Bubble. If, for the sake of simplicity, we neglect the cooling produced when a film is stretched, the work done in blowing a bubble is easily calculated out as follows :

We know, from § 237 above, that

work done in creating a film = surface tension × area of the film formed.

If, therefore, the radius of the bubble blown be r, the area of the film forming the bubble $= 2 \times 4\pi r^2$, for it has two surfaces, an inner and an outer one, each of surface area $4\pi r^2$.

Therefore, *work done in blowing the bubble* $= T \times 8\pi r^2 = 8\pi r^2 T$.

243. Curvature, Pressure and Surface Tension. If we have a curved liquid surface at rest, the inward pressure on it due to surface tension must be balanced by an equal pressure acting outwards.

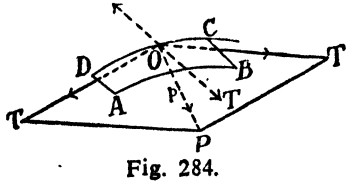


Fig. 284.

Consider a portion $ABCD$ of a liquid surface, (Fig 284), *cylindrical in form, i.e.*, curved only in one direction. Then, the force of surface tension T acts at right angles to every unit length of its boundary. The forces over AB and CD are equal and opposite and hence cancel out, but those over AD and BC, though equal in magnitude are inclined to each other, and have thus a resultant p, normal to the surface, as shown.

To determine the value of p, consider unit length of the surface at O, (Fig. 285), taken along the direction of DA. Forces T and T act at the point O, as shown, their resultant being $OP = T.d\theta$.

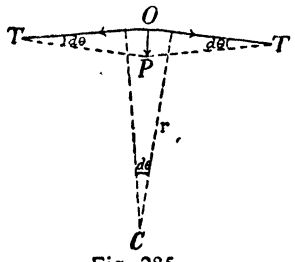


Fig. 285.

Thus, a force $T.d\theta$ acts inwards on an area equal to $1 \times r.d\theta$, where r is the radius of curvature of the surface, (because we are considering unit length of the surface of breadth $r.d\theta$). If a resultant pressure p acts outward on this area, the total force directed outward on the area is equal to $p.r.d\theta$.

For this element of the liquid surface to be at rest, therefore, we have $p.r.d\theta = T.d\theta$,

whence, $p = T/r$ *dynes/cm*^{2}.

This resultant pressure is, therefore, the difference of pressure on the two sides of the surface, which is required to balance the effect due to surface tension. *It follows, therefore, that the pressure must be greater on the concave side than on the convex side, to balance the effect of surface tension on the surfaces.*

If the surface be curved in two directions, as shown in Fig. 286, and the radii of the two curvatures be r_1 and r_2 respectively, the pressure due to the curvature of AB and CD will be T/r_1 and that due to the curvature of AQ and BC will be T/r_2 ; and, therefore, the total difference of pressure on the two sides of the surface will be given by

$$p = \left(\frac{T}{r_1} + \frac{T}{r_2}\right) = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

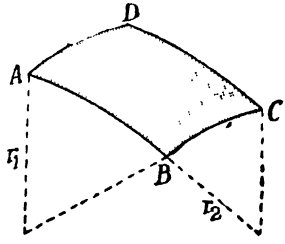


Fig. 286.

And, if one of the surfaces be convex and the other, concave, the radii r_1 and r_2 of the two surfaces will have opposite signs. So that, in such a case, we shall have

$$p = T(1/r_1 - 1/r_2).$$

Combining the cases, therefore, we may put the general relation as

$$p = T(1/r_1 \pm 1/r_2).$$

Now, let us consider a few special cases :

(*i*) **Case of a spherical surface.** In the case of a spherical surface, like that of a liquid drop, or an air bubble inside a liquid, we have $r_1 = r_2 = r$, say ; so that, excess pressure inside it is given by

$$p = T\left(\frac{1}{r} + \frac{1}{r}\right) = \frac{2T}{r}.$$ [See §239

In the case of an air or a soap bubble, because there are two surfaces, an *inner* and *an outer one*, we have

$$p = 2 \times \frac{2T}{r} = \frac{4T}{r}.$$ [See § 240

(*ii*) **Case of a cylindrical surface.** In this case, one of the radii is infinite, whereas the other is the same as the radius (r) of the cylinder.

So that, *for one single surface*, $p = T/r$,

and *for two surfaces*, $p = 2T/r$.

(iii) **Case of a catenoid.** Here, the surface being one of revolution, with no difference of pressure, we have $p = 0$. And, therefore,

$$\left(\frac{1}{r_1} \pm \frac{1}{r_2}\right) = 0.$$

An example of such a surface is that of a soap-film, supported in-between two parallel rings with its two ends burst, to make the pressure difference zero.

244. Layer of liquid between two plates. It is common experience that if two plates of glass be placed one over the other, there is no difficulty in separating them, but if a drop of liquid, say, water, be placed in between them and squeezed into a thin layer, it may, require considerable force to pull them apart. The reason will be clear from the following :

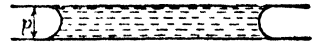


Fig. 287.

The thin layer or film of water wets the plates over an almost circular area, which is concave outwards, as shown in Fig. 287.

If d be the thickness of the water layer in-between the plates, the radius of curvature of the two concave edges of the liquid layer is *nearly* $d/2$. Thus, if r be the radius of the circular area or film of water, the excess pressure p *inside* the liquid-film over the *outside* atmospheric pressure is given by

$$p = T\left(\frac{1}{r} - \frac{2}{d}\right),$$

where T is the surface tension of the water-film.

Since r is very large compared with d, $1/r$ is almost negligible in comparison with $2/d$; so that, we have

$$p = T\left(0 - \frac{2}{d}\right). \quad \text{Or,} \quad p = -\frac{2T}{d}.$$

Thus, the pressure inside the film is less than the outside atmospheric pressure, (as is clear from the $-ve$ sign of p), by $2T/d$ and, therefore, the excess pressure of the atmosphere on the two plates pushes them closer together, making d still smaller and r larger, thus further increasing p.

Now, if A be the circular area over which the water wets the plates, the *total force* which squeezes them together is equal to

$$\frac{2T}{d} \times A = \frac{2AT}{d} = \frac{2\pi r^2 . T}{d}.$$

This much force, perpendicular to their surface, will, therefore, be needed to pull them apart. Obviously, the smaller the value of d, *i.e., the thinner the layer of water, the greater the force required to separate them.*

245. Shape of Liquid Meniscus in a Capillary Tube. We know that a liquid meniscus in a capillary tube is concave for a liquid, like water and convex for a liquid, like mercury. Let us see what determines its shape.

Let a capillary tube of glass be dipped vertically in a liquid, meeting its surface at P, (Fig. 288). Then, a liquid molecule at P, in contact with the tube there, will be attracted outward by the solid molecules of the tube near to it, due to the *force of adhesion*,

and attracted inwards by the near molecules of the liquid, due to the *force of cohesion*; the resultant force on it will, therefore, be the resultant of these two forces of adhesion and cohesion.

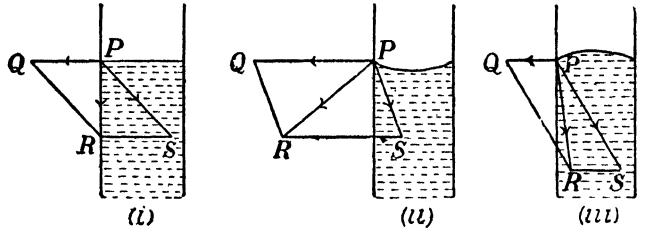


Fig. 288.

Now, the resultant force of adhesion acts at right angles to the tube at the point *P*. Let it be represented by the straight line *PQ*. And, the resultant force of cohesion acts at an angle of 45° to the vertical, and is represented by the straight line *PS*, so that, the two forces acting on the molecule are inclined to each other at an angle of 135°. Their resultant, obtained by the ordinary law of parallelogram of forces, is represented by the diagonal *PR* of the parallelogram *PQRS*, with *PQ* and *PS* as adjacent sides. Its direction will naturally depend upon the relative magnitudes of the two forces *PQ* and *PS*, and the following different cases arise :

(*i*) *If PQ/PS be equal to* $1/\sqrt{2}$, *i.e., if* $PS = \sqrt{2}.PQ$, *the resultant, (PR) will lie along the vertical, as shown in Fig.* 288 (*i*).

(*ii*) *If PS be smaller than* $\sqrt{2}.PQ$, *the resultant (PR) will lie outside the liquid, as shown in Fig.* 288 (*ii*).

(*iii*) *If PS be greater than* $\sqrt{2}.PQ$, *then, PQ will lie inside the liquid, as shown in Fig.* 288 (*iii*).

What happens to molecule *P*, happens to all other molecules in contact with the glass of the tube. And since a liquid cannot permanently withstand a shearing stress, its surface at every point will be at right angles to the resultant force there, when the liquid attains the position of equilibrium.

Thus, in the *first case*, when the resultant force *PR* acts along the vertical, *i.e.*, when $PS = \sqrt{2}.PQ$, or, *the cohesive force is* $\sqrt{2}$ *times the adhesive force*, the molecules of the liquid are neither raised nor lowered and *the liquid surface remains flat or plane.*

In the *second case*, when *PR* lies outside the liquid, *i.e.*, when $PS < \sqrt{2}.PQ$, or *the cohesive force is less than* $\sqrt{2}$ *times the adhesive force*, the molecules of the liquid near the walls of the tube are raised up against the tube, those in the middle remaining practically unaffected, thus making the liquid surface *concave upwards*, as in the case of water and other liquids which wet the walls of the tube.

And, when *PR* lies inside the liquid, *i.e.*, when $PS > \sqrt{2}.PQ$, or, *the cohesive force is greater than* $\sqrt{2}$ *times the adhesive force*, the liquid molecules near the walls of the tube are depressed there, making the surface *convex upwards*, as in the case of mercury and other liquids, which do not wet the walls of the tube.

246. Angle of Contact.

(*i*) **Case of a liquid in contact with a solid.** When a liquid meets a solid, its surface near its plane of contact with the solid is, in general, curved. *The angle between the tangent to the liquid surface at the point of contact and the solid surface, inside the liquid, is called the* **angle of contact** *for that pair of solid and liquid.*

The angle may have any value between 0° and 180°. For most liquids and glass, it is less than 90°; for mercury and glass it is about 140°. *It really depends upon the nature of the liquid and the solid, and is not altered by a change in the inclination of the solid.*

In the figures shown, [Fig. 289 (*i*) and (*ii*)] ; $\angle PQR$ is the *angle of contact*. It is *acute* in (*i*) and *obtuse* in (*ii*) ; for, in the former case, the liquid rises up a little alongside the glass plate, dipped in the lquid, and the angle between QR, the tangent to the liquid surface and the part QP of the plate, *inside the liquid*, is acute, whereas, in the latter case, the liquid is depressed a little where it comes into contact with the glass plate, and the angle between the tangent PR to the liquid surface on the part QP of the plate, *inside the liquid*, is obtuse.

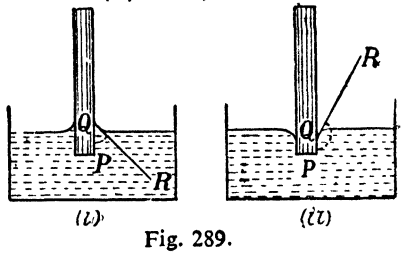


Fig. 289.

For *pure water* and *clean glass*, the angle of contact is 0°. For ordinary water and glass it is about 18° ; and if the surface of the glass be *contaminated with grease*, its value may be as much as 35°.

(*ii*) **Case of two liquids in contact with each other and with air.** If two liquids, *not miscible with each other*, be brought into contact as at O, (Fig. 290), both being in contact with air, three surface tensions are to be taken into consideration, (*a*) that of the surface between air and liquid *I*, *viz.*, T_1 ; (*b*) that of the surface between air and liquid *II viz.*, T_2 ; and (*c*) that between liquid *I* and liquid *II*, *viz.*, T_3.

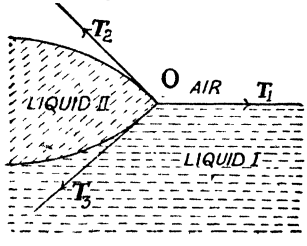


Fig. 290.

For equilibrium T_1, T_2 and T_3 should be represented by the three sides of a triangle, taken in order. This triangle of forces is known as **Neumann's triangle**. In actual practice, we come across no two pure liquids for which the *Neumann's triangle* may be constructed,—one of the surface tensions being always greater than the other two ; so that, the equilibrium condition shown in the figure is never attained. Thus, for example, in the case of water, mercury and air, the water drop, when placed over mercury, spreads all over its surface, provided both water and mercury are pure. This is so, because the surface tension of mercury is about 550 *dynes/cm.*, and that of water, only 75 *dynes/cm*. But, if the mercury surface be contaminated with grease, its surface tension

decreases and some water drops may stay on it, so that, in that case, the construction of *Neumann's triangle* can be possible.

(iii) **Case of a solid, liquid and air in contact.** This is a more important case than the previous one, for we have to consider three surface tensions, *viz.*, T_1 for air-liquid, T_2 for air-solid and T_3 for liquid-solid surfaces respectively, (Fig. 291).

Let θ be the angle of contact of the liquid with the solid,—*acute*, in case (*a*) and *obtuse* in case (*b*). For equilibrium, therefore, T_3 *plus* the component $T_1 \cos\theta$ of T_1, in the direction of T_3, must be balanced by T_2, *i.e.*,

$$T_3 + T_1 \cos\theta = T_2.$$

Or, $T_1 \cos\theta = T_2 - T_3$, whence, $\cos\theta = (T_2 - T_3)/T_1$.

Clearly, therefore, if T_2 is greater than T_3, $\cos\theta$ will be positive, *i.e.*, θ will be less than 90° ; and if T_2 is less than T_3, $\cos\theta$ will be

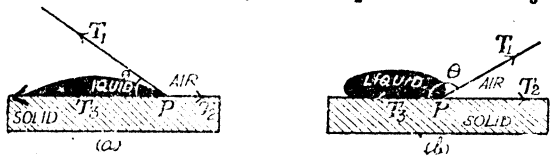
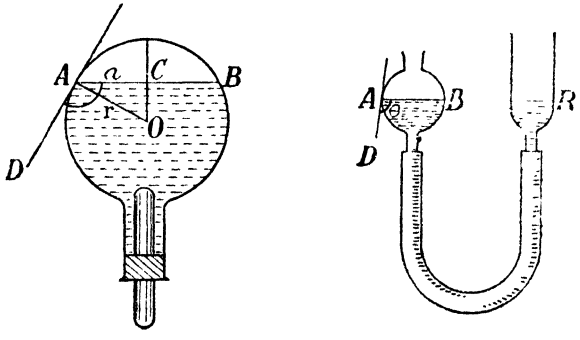


Fig. 291.

negative, and θ will lie between 90° and 180°. If, however, $T_2 > T_1 + T_3$, there will be no equilibrium, and the liquid will spread over the solid, as happens when a water drop is placed over a perfectly clean plate of glass, or a grease-free mercury surface.

247. Measurement of the Angle of Contact. For mercury and glass, the angle of contact may be determined by the following simple method due to *Gay Lussac.*

A small round-bottomed glass flask is nearly filled with mercury (more than $\frac{2}{3}$*ths* of it), and its mouth closed tightly by a rubber bung, through which passes a *glass rod R* to adjust the level of mercury in it. The flask is then clamped in the inverted position, as shown in Fig. 292 (*a*), and the rod (*R*) is moved in or out, *until the*



(*a*) Fig. 292. (*b*)

surface of mercury is plane where it meets the glass, i.e., no meniscus or curved portion (or *capillary curve*, as it is sometimes called), *appears there.* This can be tested by observing the image of a printed sheet of paper held against the flask, by the light reflected

at *grazing incidence* from the mercury surface. An undistorted image indicates that the surface of mercury is *plane* where it meets the glass.

The mercury surface in the flask forms a circular sheet of diameter AB, which is measured by means of a pair of calipers, whence the radius AC of the sheet is known. Let it be denoted by a. Then, if r be the radius of the *bulb* of the flask, and $\angle BAO = \phi$, we have

$$\cos\phi = \frac{a}{r}. \quad \text{Or,} \quad \phi = \cos^{-1}\frac{a}{r}. \qquad \ldots (i)$$

Now, if a tangent AD be drawn to the spherical bulb at A, the angle of contact θ, for mercury and glass, is clearly equal to the angle BAD.

Or, $$\theta = \angle BAD = \angle BAO + \angle OAD.$$

Or, $$\theta = (\phi + 90°). \qquad \ldots (ii)$$

So that, knowing ϕ, from relation (i) above, the value of θ can be easily determined.

A better arrangement for making the surface of mercury plane is the one shown in Fig. 292 (b). Here, the level of mercury in the spherical bulb, which is open at both ends, is adjusted by raising or lowering the reservoir of mercury (R), connected to it by an India-rubber tubing. The proceure otherwise is the same as in the first experiment.

Another simple method to determine the angle of contact for mercury and glass is to insert a small slanting glass plate AB into mercury, as shown in Fig. 293, and to adjust its inclination until the mercury meets the glass at P without curvature, *i.e.*, the surface of mercury is horizontal there. Then, the angle of contact for mercury and glass is equal to $(180-\phi)$ where $\angle APC = \phi$.

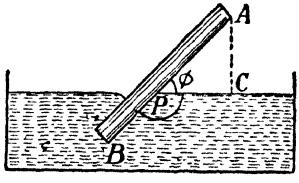


Fig. 293.

To measure the angle ϕ, a plumb line is dropped from A.

Then, clearly, $$\tan\phi = \frac{AC}{PC}. \text{ Or, } \phi = \tan^{-1}\frac{AC}{PC}.$$

For measuring the angle of contact between water and glass, *coated with wax*, a similar method was used by Adam. A glass trough is coated on the inside with wax, so that it may be filled with water above the level of the sides, and is supported by two screws, S and S, (Fig. 294), such that, by working them up or down, the top of the trough is made *perfectly horizontal*. It is then filled with water and any impurities on its surface skimmed off by moving lightly across it a wax-coated glass plate.

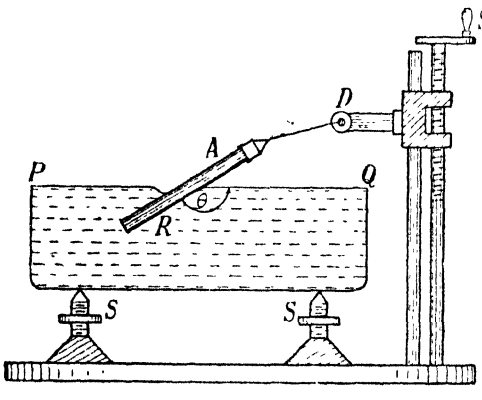


Fig. 294.

A glass plate A, also coated with wax, is clamped, as shown, so that it can be moved up or down in the vertical plane, about a horizontal axis at D, by means of a screw S'. It is then dipped into the water, and its angle of inclination adjusted until the water surface touching it is perfectly horizontal. Then, the angle θ between the lower part of the plate and the horizontal water surface is the angle of contact for water and the wax-coated glass plate, and is measured as in the above case for mercury and glass.

The angle θ, in this case, is found to depend on whether the plate is lowered into the water or raised up and hence two readings are taken—one, when the plate is lowered and the other, when it is raised, and their mean is taken as the correct value of θ, or the required *angle of contact*.

248. Rise of Liquid in a Capillary Tube. One of the most striking effects of surface tension is to raise a liquid in a capillary tube dipped into it,—a capillary tube being just a tube of a very fine bore (from the Latin word, *capillus*—a hair). It is for this reason that surface tension is also sometimes called *capillarity*.

When a capillary tube is dipped in a liquid like water, which wets it and for which the angle of contact may be taken to be *zero*, the liquid immediately rises up into it, and if the tube be a *fine* one, the shape of the liquid meniscus is spherical and concave upwards, as shown at B, (Fig. 295).*

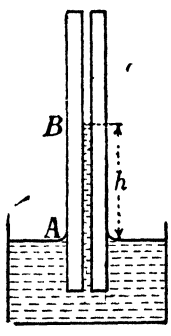


Fig. 295.

Let r be the radius of the tube at B, the point up to which the liquid rises into it. Then, it will be practically the same as the radius of the concave meniscus, so that the excess pressure above the meniscus over that immediately below it is $2T/r$, *i.e.*, the pressure in the liquid, just below the meniscus, is less than the atmospheric pressure above it by $2T/r$. And, since the pressure on the liquid surface, outside the tube, is atmospheric, the liquid will be forced up into the tube, until the hydrostatic pressure of the liquid column in the tube equals this excess pressure $2T/r$. If the liquid rises to a height h, the hydrostatic pressure due to the liquid column in the tube on the surface of the liquid will clearly be $h.\rho.g$, where ρ is the density of the liquid.†

$$\therefore \qquad 2T/r = h.\rho.g. \quad \text{Or,} \quad 2T = r.h.\rho.g,$$

whence,
$$T = \frac{r.h.\rho.g}{2}. \qquad \ldots (i)$$

*The rise of a liquid in a capillary tube was first explained by *Hawksbee* in a book of his, published in the year 1709, as being due to the attraction of the glass for the liquid. Observing that a liquid does not rise higher in a thick-walled tube than in a thin-walled one, he naturally concluded that only the molecules close to the surface of the glass must be concerned in this attraction. As mentioned already, we now know that the forces between molecules causing surface tension are precisely the same as those operating in the cases of cohesion, solution or chemical reaction.

†It will thus be seen that the capillary tube acts like a *manometer*, giving the difference of pressure above and immediately below the liquid meniscus.

Thus, knowing r, h, ρ and g, the surface tension T of the liquid can be easily determined.

In case, however, the angle of contact for the liquid be not zero and the tube be not narrow, the surface tension T of the liquid acts *inwards* along the tangent to the liquid meniscus at every point of its contact with the inner surface of the tube, making an angle θ with the wall of the tube, (Fig. 296), where θ is the *angle of contact* for the liquid and the glass of the tube.

There is thus exerted an *inward* pull on the glass in *this* direction at all these points.

Since, in accordance with Newton's third law of motion, action and reaction are equal and opposite, there is an *equal* and opposite reaction R exerted by the glass on the liquid. This reaction R (equal to T) may be resolved into two rectangular components—(*i*) $R \cos \theta/cm. = T \sin \theta/cm.$, *along the vertical, in the upward direction and* (*ii*) $R \sin \theta/cm. = T \sin \theta/cm.$, *at right angles to it, in the outward direction, as shown.* Taking the whole meniscus into consideration, the horizontal or outward components all cancel each other out, and only the vertical components are effective, which are thus added up.

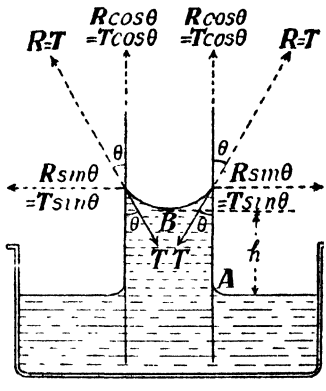


Fig 296.

Now, if r be the *radius of curvature of the tube* at the height of the meniscus, then, obviously, the meniscus touches it along a length $2\pi r$, the *circumference of the circle* of radius r; so that, the *total upward force on the liquid in the tube is* $2\pi r.T \cos \theta$. It is *this force* which supports the weight of the column h of the liquid in the tube, (where h is the length of the column from the horizontal surface A of the liquid, outside the tube, to the bottom of the meniscus at B) *plus* the weight of a volume v of the liquid, *in the meniscus itself,—i.e.*, the weight of a total volume of the liquid, $\pi r^2.h+v$, (where $\pi r^2.h$ is the volume of the liquid column h).

$$\therefore \qquad 2\pi r.T \cos \theta = (\pi r^2.h+v).\rho.g,$$

where ρ is the density of the liquid.

Or,
$$T = \left(\frac{\pi r^2.h+v}{2\pi r.\cos \theta}\right).\rho.g.$$

If the volume of the liquid in the meniscus be negligible, in comparison with that in the column h, *i.e., if the tube of a very fine bore,* we have

$$T = \frac{\pi r^2.h.\rho.g}{2\pi r \cos \theta} = \frac{r.h.\rho.g.}{2 \cos \theta}, \qquad \ldots (ii)$$

the relation being known as *Jurin's Equation.*

N.B. In case the capillary tube is not vertical, but inclined at an angle α to it, we take into consideration only the *vertical height* (h) of the liquid

*For experimental details of the method, see page 504.

column in it, which will obviously be equal to $l \cos \alpha$, where l is the length of the tube occupied by the liquid. So that, here,

$$T = r.\rho.g.l \cos \alpha / 2 \cos \theta.$$

If, however, v be not negligible, its value may be determined by taking the meniscus to be *hemispherical* in shape, (since the tube is narrow), of radius nearly equal to r, *i.e.*, the *same* as that of the tube at that place, so that the volume of the liquid in the meniscus is equal to the difference between the volumes of *a cylinder of radius r and length r and a hemisphere of radius r.*

Or, $v = \pi r^2.r - \frac{2}{3}.\pi r^3 = \pi r^3 - \frac{2}{3}.\pi r^3.$ { volume of a *cylinder* $= \pi r^3 l$, of a sphere, $4\pi r^3/3$, and of a hemisphere $= 2\pi r^3/3$.

Or, $$v = \tfrac{1}{3}\pi r^3.$$

In this case, therefore,

$$2\pi r.\, T \cos \theta = (\pi r^2.h + \tfrac{1}{3}\pi r^3)\rho\, g = \pi r^2.(h + \tfrac{1}{3}r).\rho.g.$$

Or, $$T = \frac{\pi r^2.(h+\frac{1}{3}r).\rho.g}{2\pi r.\, \cos \theta}.$$ { *i.e., the effective height of the liquid column is now* $h + \frac{1}{3}r$, *and not* h.

And, in case $\theta = 0$, $\cos \theta = 1$; so that, we then have

$$T = \frac{r(h+\frac{1}{3}r).\rho.g}{2} \text{ dynes/cm.}$$

But if θ be greater than 90°, cos θ is *negative* and the liquid column in the tube is depressed below the liquid level outside it, *i.e.*, h is *negative*. Hence it is that we find it so difficult to introduce mercury, (for which $\theta = 140°$ nearly), into a fine capillary tube.

Again, if we introduce into water, a capillary tube, *with its inside coated with paraffin wax* to make $\theta > 90°$, the surface of the water column also, inside the tube, lies below that in the outer vessel.

And, further, we may come across two liquids for which the values of T are the same but whose angles of contact (θ) are different. They will naturally rise to different heights when the same capillary tube is introduced into them.

249. Rise of Liquid in a Tube of Insufficient Length. We have seen above, how a liquid rises up into the capillary tube, dipped into it, until the weight of the liquid in the tube is just balanced by the force due to its surface tension. If θ be the angle of contact between the liquid and the tube, and R, the radius of the *liquid meniscus* in the tube, we have $r = R \cos \theta$, (Fig. 297), where r is the radius of the tube ; so that, relation (*ii*), above, now becomes

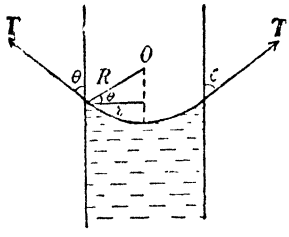


Fig. 297.

$$T = \frac{R \cos \theta.h.\rho.g}{2 \cos \theta} = \frac{R\, h.\rho.g}{2},$$

where h is the height of the liquid column in the tube.

Hence, clearly, $$R.h = 2T/\rho.g, \text{ a constant.}$$

Now, with the tube sufficiently longer than h, it is the value of h alone that changes to satisfy the above relation for T. But, if the tube be smaller than the calculated value of h, the only variable in the above relation is R, because now $h = l$, the length of the tube (a

constant) and so is θ a constant for the given liquid and the tube. The liquid thus just spreads over the walls of the tube at the top and its meniscus acquires a new radius of curvature R', such that $R.'l=2T/\rho.g$, or, that $R'.l=R.h=$a constant.

250. Rise of Liquid in a Conical Capillary Tube. Suppose we take a concial capillary tube, (*i.e., a capillary tube with a fine conical bore*) ABC, with a hole at its upper end, (Fig. 298), and dip its in a liquid, like water, which wets it walls. We shall find that the liquid rises up into it to a height h, above the liquid surface, in accordance with the relation,

$$h=2T\cos\alpha/R.\rho.g, \quad \ldots \quad \ldots (i)$$

where α is the semi-vertical angle of the tube, and R, the radius of the tube at the liquid meniscus FG.

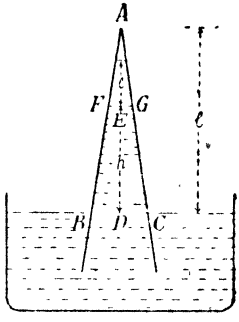


Fig. 298

Let the vertical height AD of the apex of the cone from the liquid surface be denoted by l, and the radius of the tube, where it meets the liquid surface, (*i.e.*, of the portion BC of it), by r. Then, from the similar right-angled triangles AEF and ADB, we have

$$\frac{FE}{BD}=\frac{AE}{AD}.$$

Or, $$\frac{R}{r}=\frac{l-h}{l}, \text{ whence, } R=\left(\frac{l-h}{l}\right).r.$$

Putting this value of R in relation (i), above, we have $h=\dfrac{2.l.T.\cos\alpha}{(l-h).r.\rho.g}$

So that, $$h(l-h)=\frac{2l.T.\cos\alpha}{r.\rho.g}.$$

Or, $$h^2-hl+\frac{2l.T\cos\alpha}{r\rho g}=0,$$

which is a quadratic equation in h.

So that, $$h=\frac{+l\pm\sqrt{l^2-4\times\dfrac{2l.T.\cos\alpha}{r.\rho.g}}}{2}$$

Or, $$h=\frac{l}{2}\pm\sqrt{\frac{l^2}{4}-\frac{2l.T.\cos\alpha}{r.\rho\,g.}}, \quad \ldots \quad \ldots (ii)$$

whence the height h, up to which the liquid rises into the tube, can be easily determined.

And, if the tube be only slightly conical, we have $\cos\alpha=1$, very nearly. In that case, therefore,

$$h=\frac{l}{2}+\sqrt{\frac{l^2}{4}-\frac{2lT}{r.\rho.g}}.$$

N.B. The $-ve$ sign is not usually indicated.

Further, in such a tube, R is practically equal to r, *i.e.*, it is practically a uniform capillary tube, as shown in Fig. 295, and, therefore, for the liquid to rise up to the top into the tube, the minimum value of T should be greater than $r.l.\rho.g/2$.

251. Energy required to raise a liquid in a capillary tube. We have seen above how when a capillary tube is dipped vertically into a liquid which wets the walls of the tube, there is a rise of the liquid inside the tube. *The rise, obviously, takes place against the action of gravity and the liquid, therefore, must gain in potential energy.* The question, therefore, arises as to where does it get this increase in its potential energy from. For, according to the law of conservation of energy, energy can only be converted from one form into another, but cannot be created. The explanation is, however, simple.

We have three surfaces of separation to consider when a capillary tube is immersed in a liquid, *viz.*, (*i*) *an air-liquid surface* (*ii*) *an air-glass surface* and (*iii*) *a glass-liquid surface,* each having its own surface tension, different from the others, and *equal to its free surface energy per unit area.*

Now, as the *plane* liquid surface in the tube acquires a curvature, (*i.e.*, becomes concave), the air-liquid surface increases and, as the liquid rises in the tube, the glass-liquid surface increases, the air-glass surface decreasing by an equal amount. Thus, the surface energy of the air-liquid and the glass-liquid surfaces increases while that of the air-glass surface decreases by the same amount. *In other words, the energy required to raise the liquid in the capillary tube is obtained from the surface energy of the air-glass surface.*

On the other hand, a liquid, which does not wet the walls of the tube, gets depressed inside it, below its level outside the tube. In this case, obviously, the glass-liquid surface decreases, whereas the air-glass surface increases by an equal amount, resulting in a *net increase* in the surface energy of the whole system. This energy is derived from the depression of the liquid inside the tube, whose gravitational potential energy is thus decreased by an equal amount.

252. Rise of a Liquid Between Two Parallel Plates. An almost similar case to the above is that of two vertical plates kept parallel and close to each other in a liquid, when the liquid rises in between them, (if it wets the plates). Let us calculate the height to which it rises.

If d be the distance between the two plates and, r, the radius of the meniscus, (which is *cylindrical*, in this case), we have $\frac{d/2}{r} = \cos\theta$, whence, $2r\cos\theta = d$, or $r = d/2\cos\theta$, where θ is the angle of contact for the liquid and the plates.

Now, we know that the pressure just below the meniscus is less than the pressure just above it by an amount equal to $T\left(\frac{1}{r}+\frac{1}{r'}\right)$, where r and r' are the radii of the two curvatures, at right angles to each other. Since the meniscus is cylindrical, one of the curvatures has a radius equal to that of the cylinder and the other, a radius equal to ∞, (the surface being plane), so that $r' = \infty$, and, therefore, $1/r' = 0$, and hence the excess pressure just above the meniscus over that just below it is equal to T/r. Or, substituting the value of r, we have *excess pressure above the meniscus* $= \frac{T}{d/2\cos\theta} = \frac{2T.\cos\theta}{d}$.

Therefore, the liquid column will rise up in-between the two plates, until the hydrostatic pressure due to it becomes equal to $2T.\ cos\ \theta/d$. Thus, if h be the height up to which the liquid rises, and ρ, its density, we have $2T\ cos\ \theta/d = h.\rho.g$.

Or,
$$h = \frac{2T\ .\ cos\ \theta}{d.\rho.g}.$$

Or, h is inversely proportional to d, *i.e.*, *the rise* of the liquid is inversely proportional to the distance between the two plates.*

In case, however, the two plates be not parallel but inclined at a small angle to each other, *meeting along a common vertical edge*, the liquid does not rise uniformly in-between them. But, at any point on the liquid column, the distance (d) between the plates is proportional to the distance x of the point from their common edge. And, since, as we have seen, $h \propto 1/d$, it follows at once that $h \propto 1/x$.

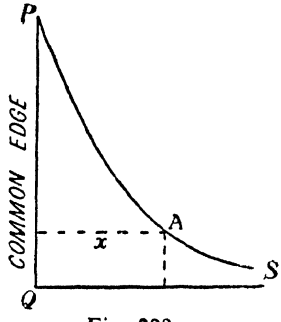


Fig. 299.

Now, from the relation for h, deduced above, we have

$h.d = 2T\ cos\ \theta/\rho.g =$ a constant.

It is, therefore, clear that

$h.x = a$ constant, also. [$\because d \propto x$.

In other words, *the liquid surface in-between the two plates will, in this case, be a part of a rectangular hyperbola*, (Fig. 299).

253. Force Between Bodies Partly Immersed in a liquid. It is a common observation that pieces of cork and such other light bodies, when floating in water, cling and collect together into clusters. This is due to capillary action, *i.e.*, the rise of water into the small spaces in-between the pieces.

There are three cases to be considered, *viz.*, (*i*) *when the liquid wets the two bodies*, (*ii*) *when it does not, and* (*iii*) *when it wets one and not the other.*

In the first case, say, for example, two glass plates, partly immersed in a liquid, like water, [Fig. 300 (*i*)] the liquid rises in between them to a level higher than that outside them. And, since the pressure in the liquid just below the meniscus in-between them is smaller than the pressure due to the atmosphere at the same level outside them, they get pushed towards each other, *i.e.*, *they seem to attract each other.*

In the second case, as for example, when the two plates are partly immersed in a liquid, like mercury, (*which does not wet them*), the liquid is depressed between them below the level outside them, [Fig. 300 (*ii*), and the atmospheric pressure above the liquid meniscus, in-between them, is smaller than the pressure in the liquid at

*In case $\theta > 90°$, $cos\ \theta$ will be *negative*, and the liquid will get depressed in-between the two plates instead of rising up.

the same level outside them, and so this excess pressure on their out side again pushes them towards each other, and again, therefore, *they seem to be attracting each other.*

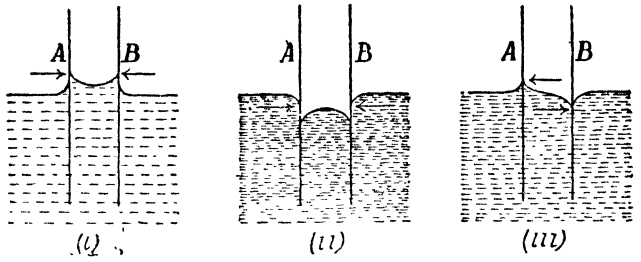


Fig. 300.

In the third case, as for example, when one of the glass plates, say, *A*, be clean so that it can get wet with water, and the other *B*, be coated with wax, (so as not to get wet with water), and both be partly immersed in water, the liquid meniscus is concave near to the plate *A*, *wetted* by the liquid, and convex, near to the other plate *B*, *not wetted* by it. Due to the pull of one meniscus on the other, therefore, the concave meniscus on the inside of *A* is a little lower than the concave meniscus on its outside, and the convex meniscus, on the inside of *B* is a little above the convex meniscus on its outside, [Fig. 300 (*iii*)]. Since the pressure just below the concave meniscus outside of *A* is lower than the atmospheric pressure at the same level and above the concave meniscus on its inside, it is pushed outward, away from *B*, as shown by the arrow-head. Again, since the atmospheric pressure above the convex meniscus outside *B* is lower than the pressure in the liquid below the convex meniscus at the same level on its inside, it is also pushed outwards, away from *A*, as indicated by the arrow head, and thus both *A* and *B* move away from each other, *i.e., they seem to repel each other.*

This explains why a small piece of wood, (wetted by water) and a needle (not so wetted), when floated close together in water, steadily get apart,—the former drifting towards the edge and the latter moving towards the centre.

It should be noted that the above is true only when the two floating bodies are near together. For, if the distance between them be large, the portion of the meniscus where it changes from convex to concave, or *vice versa*, will become quite straight or horizontal, *i.e.*, in a level with the rest of the liquid outside the plates, and there will, therefore, be no resultant force acting on them, one way or the other. On the other hand, if they be very close to each other, the point of inflexion of the meniscus disappears altogether, resulting in the rise of liquid in-between them, and they get pushed towards each other.

254. Shape of Liquid Drop on a Horizontal Plate. There are two forces acting on the drop of a liquid, placed on a horizontal plate,—*viz.*, (*i*) *the force due to gravity* and (*ii*) *the force due to surface tension.*

Whereas the former tends to flatten it and to spread it out until its surface becomes horizontal, *i.e.*, tends to *'squash'* the drop, as it were, the latter, by trying to *decrease* its surface area, tends to gather it up into the form of a sphere. *It is, therefore, a case of gravity versus surface tension.* Now, the gravitational force depends upon the mass of the drop, which is proportional to its volume, *i.e.*, to the *cube* of its linear dimensions, and the surface tension depends upon the surface area, *i.e.*, upon the *square* of its dimensions. *Therefore, the effect of gravitation is more pronounced on a large drop and it gets flattened out, whereas the surface tension has the upper hand, in the case of a small drop, and gives it a spherical shape.* That is why the *small* dew and rain drops, or those of mercury or oil etc., are all spherical in shape*. This can also be readily seen by placing small and large drops on a plate of glass or water drops on paraffin wax, when it will be found that small drops assume a spherical shape, but the large ones get flattened out, until their upper surface becomes horizontal. In the case of mercury on a glass plate, a large drop will assume a long elliptical sort of shape, its upper surface, in the middle, being plane, with the edges protruding on either side, the angle of contact being about 140°.

It is possible to calculate the surface tension of mercury or the angle of contact, by considering the various forces that keep it in equilibrium. The method was developed by **Quincke**, and later improved by **Edser**, [see § 255 (4)].

255. Experimental Determination of Surface Tension. We shall now consider some of the usual methods employed to determine the surface tension of liquids.

1. Searle's Torsion Balance Method. This is perhaps the simplest, most direct and the quickest method for the determination of surface tension of liquids, particularly of those for which the angle of contact is zero.

The essentials of the apparatus used are as indicated in Fig. 301 where R is a rigid rod, fixed to a fine torsion wire w, (about 1·25 *mm.* in radius), stretched horizontally across a frame work, as shown. The rod terminates in a pointer, moving over a vertical scale S at one end, and carrying a sliding weight

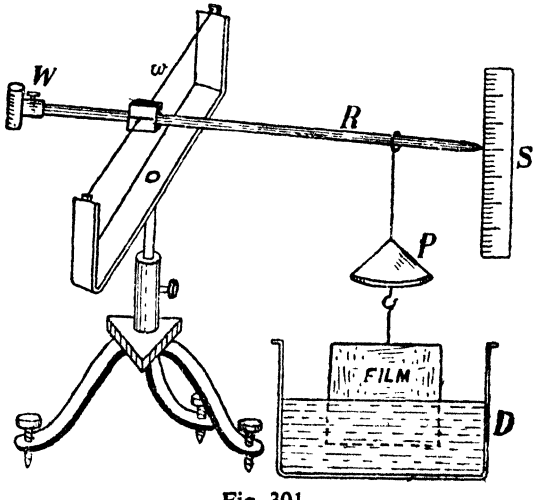


Fig. 301.

*A striking proof of the rain drops being perfect spheres is the natural phenomena of the *rainbow* and the *halos*, whose arrangement of colours and shape can only be explained on this basis. The slightest deviation from the spherical shape of the drops would materially affect both these.

W at the other, with the help of which it is counterpoised, so as to remain horizontal.

From a point, near the front end of the rod, is suspended a scale pan, which, in its turn, carries a rectangular wire-frame or a glass plate, (about 10 *cms.* long and ·02 to ·05 *cm.* in thickness), fitted in a metal socket or clip, with its lower edge *perfectly horizontal.*

A vessel D, containing the experimental liquid, is placed on an adjustable table, which can be raised up or lowered down, as desired.

To start with, the vessel D is raised up until the wire-frame dips into it or until the lower edge of the glass plate just dips a little into the liquid. It is then lowered down until this lower edge of the wire-frame or the plate lies exactly in a level with the liquid surface. A film of the liquid is thus formed in the wire-frame, or in-between the plate and the liquid surface, exerting a downward pull on the frame or the plate due to the surface tension. This results in the pointer-end of the rod also deflecting downwards and its position is read on the scale. The vessel is removed from under the frame or the glass plate. The film is now punctured or the plate allowed to dry up. The liquid film thus disappears and with it also the downward pull on the frame or the plate, and hence on the rod, which, therefore, returns back to its initial position.

Weights are now placed in the scale pan until the pointer-end of the rod is again deflected downward to the same extent as before. Clearly, then, the downward force on the rod due to this weight, mg. say, is the same as the downward pull on it due to surface tension, in the first case.

Now, if l be the length of the wire-frame or the plate and t, its thickness, it is in contact with the liquid along a total length equal to $2(l+t)\,.\,\cos\theta$, where θ is the angle of contact for the liquid. Hence, if T be the surface tension of the liquid, the downward pull on the frame or the plate due to surface tension is equal to $2(l+t).\ \cos\theta.T$.

Hence
$$2(l+t).\ \theta.T=mg.$$
$$T = mg/2(l+t).\ \cos\theta.$$

In the case of the wire frame, t is negligible, and, therefore,
$$T = mg/2l\ \cos\theta.$$

And, if θ be *zero* for the given liquid, we have $\cos\theta = 1$; so that, $T = m.g/2(l+t)$, *in the case of the plate and* $T = m.g/2l$, *in the case of wire frame.*

And thus, knowing mg, l and t, the value of surface tension (T) for the liquid can be easily calculated.

As will be readily seen, the method may also be used to compare the surface tensions of two given liquids, θ for which is zero.

N.B. It happens sometimes that, despite all care the lower edge of the plate, instead of being just in a level with the liquid surface, dips a little, say, to a depth h inside it. If this be so, it is obvious that downward pull on the plate, due to the surface tension, is reduced by an amount equal to the upward force on it, due to the buoyancy of the displaced liquid, which as we know, is equal to the weight of the liquid displaced by the plate, *i.e.*, equal to $(l\times t\times h\times\rho\times g)$, where ρ is the density of the liquid and g, the acceleration due to gravity at the place.

Thus, we have $2(l+t).\cos\theta.T.-(l\times t\times h\times\rho\times g) = mg.$

Or, $2(l+t).\cos\theta.T = mg+l.t.h.\rho.g$, whence, $T = \dfrac{mg+l.t.h.\rho.g}{2(l+t)\cos\theta}$.

2. Jaeger's Method. We know that excess pressure inside an air bubble in a liquid is equal to $2T/r$. [§239], where T is the surface tension of the liquid and r, the radius of the bubble. *Jaeger* has, by a simple and ingenious method measured this excess pressure p, necessary to produce such a bubble ; so that, knowing p and r, the surface tension (T) of the liquid can be easily determined.

The apparatus simply consists of a long thin glass tube AB, (Fig. 302), with its lower portion ending in a fine jet of about ·2 to ·5 *mm.* in diameter, and with its tip cut quite smooth and square*, so as to be perpendicular to its axis. This dips in the experimental liquid, contained in a beaker, with about 4 to 5 *cms.* of its length inside the liquid. It is then connected to a manometer M and a Woulff's bottle, fitted with a dropping funnel F, containing water, as shown. The liquid used in the manometer M is *Xylol* (a liquid hydrocarbon) in preference to water,—because of its lower density,—so that the difference of level in the two limbs may be large.

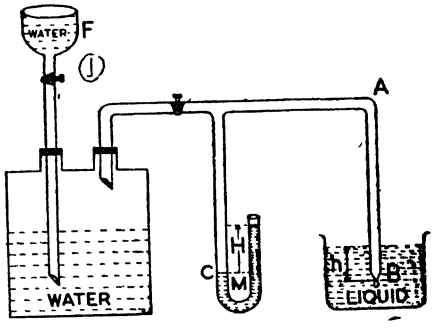


Fig. 302.

Due to capillary action, some liquid rises up into the tube AB, the shape of its meniscus being nearly *hemispherical*. Some air is now forced into the tube by dropping water into the Woulff's bottle, which displaces its own volume of air from it. The liquid column in AB thus slowly moves down until it reaches B, when a bubble is formed there The process must be regulated to be so slow that about 10 *secs.* are needed for the bubble to form. The radius of curvature of the bubble gradually decreases with increasing pressure inside it, until it reaches the *minimum* value, and the bubble acquires a more or less hemispherical shape, with a radius r, equal to that of the aperture at B, *the pressure inside being now the maximum*, as indicated by the difference of levels (H) in the two limbs of the manometer.

The bubble now becomes unstable ; for, any further growth of it tends to increase its radius, which results in a corresponding decrease in the pressure inside it due to surface tension, thus destroying the equilibrium between its internal and the constant external pressure. It, therefore, now gets detached from the tube, and the whole process starts all over again.

Now, clearly, just before the bubble breaks away from B, the *pressure inside it* is equal to that at C, *i.e.*, equal to $P+H.\rho.g$, where

* This must be so, not only to the naked eye but even under a microscope with no trace of any roughness or raggedness at its inner or outer edges.

P is the atmospheric pressure and $H.\rho.g$, the pressure due to the liquid column H in the manometer. (ρ being its density).

And, when the bubble just breaks away from B, the *pressure on it* is equal to that at the level of B in the beaker, *i.e.*, equal to $P+h.d\,g$, where h is the depth of the tip B from the surface of the liquid in the beaker and d, its density.

And, therefore, the excess pressure inside the bubble

$$= (P+H.\rho.g)-(P+h.d.g) = g(H.\rho-h.d).$$

But the excess pressure inside the bubble, we know, must be equal to $2T/r$, where T is the surface tension of the liquid in the beaker. Hence, $2T/r = g(H.\rho-h.d)$.

Or, $$T = r.g\,(H.\rho-h.d)/2.$$

Thus, noting H, h, ρ and d, and determining r with the help of a microscope, fitted with a micrometer eye-piece, we can easily calculate the value of T for the given liquid.

Despite all care, however, there is no absolute certainty as to the radius of the bubble, when it gets detached from the tube, and it may not be hemispherical and of quite the same radius as the aperture at B. In fact, quite the contrary. For, it can be hemispherical only in the case of *extremely narrow* tubes, and its radius is found to be always a function of the radius of the aperture. For a greater accuracy in the result, therefore, the following relation is often used :

$$T = \frac{rg}{2}\cdot[H\rho-d(h+2r/3)].$$

This method is of importance in that it can be used

(*i*) for determination of the surface tension of molten metals ;

(*ii*) for determining the variation of surface tension of a solution, with different concentrations of the solute ;

(*iii*) for comparison of surface tensions of different liquids ;

(*iv*) for determining the variation of surface tension of a liquid with temperature, (as the temperature of the containing vessel can be easily controlled and the bubble is formed inside the liquid itself) ; and

(*v*) for studying the molecular aggregation of the liquid, (*i.e.*, the number of atoms in its molecule), from the slope of the curve between temperature and surface tension.

3. **The Drop-Weight Method.** This is a simple, though perhaps not quite so accurate a method for determining the surface tension of a liquid by considering the vertical forces that keep a small drop of liquid in equilibrium, *just before it gets detached* from the end of a *vertical* glass tube of *circular aperture*.



Fig. 303.

At the instant the drop gets detached, it assumes a cylindrical shape at the orifice of the tube, (Fig. 303), so that, if T be the surface tension of the liquid and r, the radius of the orifice, we have

excess pressure (p) *inside the drop over the outside atmospheric pressure* $= T/r$. [see § 243, page 481.

Hence the *downward force* on the drop due to this

$$= \pi r^2.T/r.$$

And, since the *weight mg* of the drop also acts vertically downwards on it, we have

total downward force on the drop $= \pi r^2.T/r + mg.$

Now, the liquid touches the tube all along its circumference $2\pi r$, and hence the *upward force* acting on the drop *due to surface tension* is equal to $2\pi r.T.$

It is clear, therefore, that while the drop is yet in equilibrium, immediately before its detachment from the tube, the two sets of opposing vertical forces acting on it must just be balancing each other. Hence,

$$2\pi r.T = \pi r^2.T/r + mg = \pi r.T + mg.$$

Or, $2\pi r.T - \pi r.T = mg.$ Or, $\pi r.T = mg,$

whence, $T = mg/\pi r.$

Thus, knowing the mass of the falling drop and the radius of the lower end of the tube, we can easily calculate the value of T, the surface tension of the liquid in question.

The actual experiment is carried out as follows :

A burette is fitted vertically in a suitable clamp and a thin, clean and dry tube of glass (of about 4 *mm.* bore) is attached to its nozzle by a piece of India rubber tubing, carrying a pinch-cock on it. The burette is filled with the experimental liquid and its flow through the glass tube regulated by the pinch-cock, so that small drops* form slowly at its lower end,—their rate of detachment from it being about *one per minute*, when an accuracy of about ·2% may possibly be attained.

The drops, as they fall, are received in a clean, dry and an accurately weighed beaker and the *average mass* (*m*) of a drop determined by weighing the beaker again with its contents. The diameter of the orifice of the tube is also determined carefully by means of a travelling microscope.

As indicated above, the method is far from accurate, for the simple reason that the liquid drop seldom gets detached from the end of the tube, under the ideal statical conditions assumed,—the radius of the '*neck*' of the drop as well as the *amount* of it that actually gets detached being more or less uncertain quantities, even if the end of the glass tube be smeared with wax, thus making the whole problem a complicated dynamical one.

N.B.—*Lord Rayleigh* has suggested the relation $T = mg/3{\cdot}8r$, which yields better results, and *Harkins* and *Brown* have experimentally shown by using liquids of known values of T, that the relation $T = mg.F/r$ holds true, where F is a factor related to v/r^3†—the symbols v and r standing for the volume of the drop and the radius of the tube respectively.

4. Quincke's Method. This method is applicable in the case of liquids which do not wet the surface in contact with them, as for example, mercury. Not only can it be used to determine the surface tension of such a liquid but also its angle of contact with the solid

*The drops formed at the narrow orifice of the tube needs must be small, for, as they grow in size, the *liquid skin* there is unable to support their weight.

†This relation too is found to be true only up to the limit where the maximum value of v/r^3 does not exceed 10.

surface on which it rests. We shall consider here the case of mercury, for which the method is so frequently used.

It has been indicated already that in the matter of the shape of a liquid drop resting on a solid surface, which it does not wet, it is a tussle between surface tension and gravity, the former having the upper hand in the case of small drops, which thus assume a spherical shape and the latter holding sway in the case of the larger ones, tending to spread them out until their surface becomes horizontal. This explains the gradual flattening up of a drop as it grows in size, until its top becomes quite horizontal, beyond which no more flattening occurs due to the limit imposed by the angle of contact of the liquid and the solid in question. So that, the shape ultimately acquired by a large drop of the liquid, (in our case, mercury), is as shown, (Fig. 304), with the central part of the top surface horizontal and the two ends protruding outwards, such that the tangents to them make an angle α each, with the horizontal solid surface, where α is the supplement of the angle of contact (θ) for the liquid and the solid, —its value being about 40° in the case of mercury and glass, (the angle of contact for them being nearly 140°). The actual shape of these protruding edges of the drop is of little or no concern to us for our present purpose, except that, at their most protruding part, such as at D, (Fig. 305), they are more or less *vertical*.

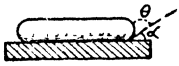


Fig. 304.

Let us consider the equilibrium of the drop in two ways (*i*) *without involving the angle of contact, or, when it is not known*, and (*ii*) *involving the angle of contact*, or, *when it is known*.

Without involving the angle of contact. (*i*) Imagine a large drop with its central part horizontal, to be cut into two halves by a vertical plane perpendicular to its length and passing *centrally* through it, and consider the equilibrium of a thin slice *ABDEFG*, of one-half of the drop, thus obtained, with the vertical faces cut parallel, (by two other planes, perpendicular to the first) and a horizontal width $GA = FJ = b$

As is evident from the very symmetry of the drop, (*a*) *the forces acting on the two sides, GCEF and the one opposite to it at the back, must be equal and opposite, and* (*b*) *so also must be the pressure immediately above and below the horizontal part of it at the top.*

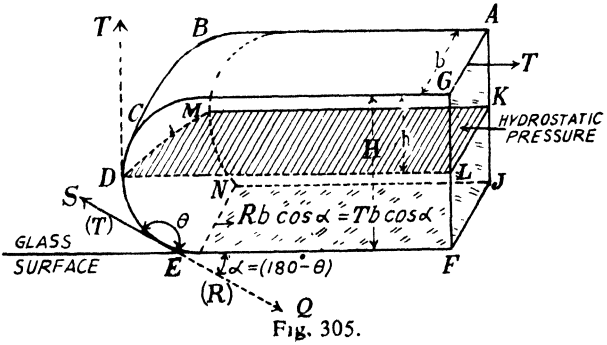


Fig. 305.

Let us consider the portion or the *slice*, lying above the horizontal plane *DLKM*, a distance h below the top and passing through D, *where the*

drop protrudes most and which, therefore, represents *the greatest horizontal surface, i.e., the surface having the maximum area in the section of the drop.*

The forces acting on this slice are the following :

(*i*) **the surface tension at D,** *which acts* **vertically** *upwards and has, therefore,* **no component along the horizontal ;**

(*ii*) *the* **lateral hydrostatic pressure (P) on the plane face GLKA** *due to the neighbouring part of the liquid on its right,* **acting towards the left ;**

(*iii*) the **pull due to surface tension over GA,** *acting perpendicularly to it,* **towards the right.**

Thus, the only horizontal forces acting on the slice are *the latter two, in opposite directions,* and since the slice is in equilibrium, they must be *equal in magnitude.*

Now, *lateral hydrostatic pressure due to a liquid being equal to the pressure half-way down,* we have

hydrostatic *pressure* P over $GLKA = \frac{1}{2}GL.\rho.g = \frac{1}{2}h\,\rho\,g,$

where $GL = h$; ρ, the *density* of the liquid and g, the *acceleration due to gravity.*

$\therefore$ *hydrostatic thrust* over $GLKA = P \times$ area $GLKA.$

$$= \tfrac{1}{2}\, h.\rho.g \times h \times b = h^2.\rho.g.b/2.$$

And, the *pull on GA due to surface tension* $= T.b.$, where T is *the* surface tension of the liquid ; so that,

$$T.b = h^2.\rho.g.b/2. \qquad \text{Or,} \qquad T = h^2.\rho.g/2 \qquad \ldots(A)$$

whence, T, the *surface tension* of the liquid may be easily determined, *if* h be measured (see below), *without involving the angle of contact for the liquid and the solid in question.*

(*ii*) **Involving also the angle of contact.** In this case, we consider the equilibrium of the *whole* slice. So that, the surface-tension pull $T.b.$, due to mercury on glass, at E, *i.e.*, on the line of contact EF of glass and mercury, acts tangentially to the slice in the direction ES, with an equal reaction of the glass on the slice, *in the opposite direction* EQ. The horizontal component of this *reactional force due to glass,* along EF, is clearly equal to $T.b.\ cos\ \alpha$, where α is the supplement of the angle of contact for the given liquid and solid surfaces, and acts *towards the right.* And, so does the pull due to surface tension over GA, *viz.*, $T\,b.$, acting perpendicularly to GA ; so that, the total horizontal force towards the right $= T.b.\ cos\ a + T.b.$

This is opposed by the horizontal hydrostatic thrust over the face $GFHA$, acting *towards the left,*—this thrust being now equal to

$$\tfrac{1}{2}\, H.\rho.\ g \times H \times b = \tfrac{1}{2}\, H^2.\rho.b.g,$$

where, $GF = H$, the *total height* of the slice.

So that, there being no other horizontal forces to be considered, we have, for the equilibrium of the slice,

$$T.b\ cos\ a + T.b = \tfrac{1}{2}\, H^2.\rho.b.g. \qquad \text{Or,} \qquad T\ cos\ a + T = \tfrac{1}{2}\, H^2.\rho.g.$$

$$\text{Or,} \qquad 2T + 2T\ cos\ \alpha = H^2.\rho.g. \qquad \text{Or,} \qquad 2T\,(1 + cos\ \alpha) = H^2\rho.g.$$

Or, $$T = \frac{H^2.\rho.g}{2(1+\cos\alpha)} \qquad \ldots(B)$$

whence, T may be calculated, if H and α be known.

The quantities h, H and α, involved in relations (A) and (B) above may be measured as follows :

(i) Measurement of h. For the measurement of h, the distance between the top of the drop and the horizontal surface of maximum area, we must determine the point D, where the free surface of the drop is *perfectly* vertical. This has been beautifully done by *Edser*, by means of the arrangement, shown in Fig. 306.

A special vernier microscope, having a plane glass plate P, cemented on to its objective at an angle of 45°, is arranged horizontally, (*i.e.*, with its axis horizontal), right in front of the most protruding part of the drop. Light from an incandescent lamp S is focused by a lens L and the plate glass P, acting as a mirror, on to the edge of the drop, when a *bright, thin, horizontal line* is seen at D, where the drop protrudes out most and the surface is vertical. The aperture of the microscope is adjusted until this bright horizontal line coincides with the horizontal crosswire of the eye-piece, and its reading is noted. The microscope, which is capable of both a vertical and a horizontal movement, is then raised up and moved towards the drop until the image of the top *flat* surface of the drop coincides with the horizontal crosswire of the instrument. This latter adjustment is greatly facilitated by sprinkling a little lycopodium powder over the top of the drop and making the image of a speck or two of the powder coincide with the horizontal crosswire of the eye-piece. The distance through which the instrument has had to be raised up directly gives the required distance $GL = h$.

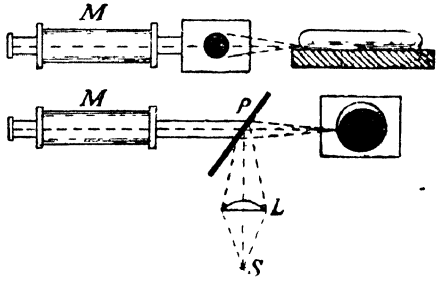


Fig. 306.

(ii) Measurement of H. This may be easily done by means of a spherometer, or a microscope, the latter being focused on the top, (*i.e.*, on a speck of the lycopodium powder on its surface, as before), and then on the point of the edge at the bottom, which is in contact with the surface of the glass plate.

(iii) Measurement of α, and, therefore, also the angle of contact θ. We have, from relations A and B above,

$$\frac{h^2.\rho.g}{2} = \frac{H^2.\rho.g}{2(1+\cos\alpha)}. \quad \text{Or,} \quad h^2 = \frac{H^2}{(1+\cos\alpha)}.$$

Or, $1+\cos\alpha = H^2/h^2$, whence, $\cos\alpha = (H^2/h^2)-1$.

So that, knowing H and h, we can easily calculate the value of $\cos\alpha$, and hence that of α.

Clearly, *this also gives us a method of measuring θ, the angle of contact for the given liquid and solid ; for, θ = (180°−α), the two angles θ and α being supplementary angles.*

Corollaries. (*i*) Relation (*A*) above tells us that h, the distance of the top flat surface of the drop from the horizontal surface of maximum area, is quite independent of the size of the drop, but depends only upon the nature of the liquid and the solid in contact, (*i.e.*, upon the angle of contact for them).

(*ii*) We have, from relation (*B*) above,

$$H^2 = \frac{2T(1+\cos\alpha)}{\rho g} = \frac{4T\cos^2(\alpha/2)}{\rho.g} \qquad [\because\ 1+\cos\alpha = 2\cos^2(\alpha/2).$$

Or,
$$H = 2\cos\frac{\alpha}{2}\sqrt{\frac{T}{\rho.g}},$$

whence, it is clear that

(*a*) *all flat drops of a liquid, resting on a horizontal solid plate, which they do not wet, must have the same height.*

(*b*) *all liquids that wet the solid surface spread out indefinitely on it for, in their case,* $\theta = 0°$ *and, therefore,* $\alpha = \pi$; so that

$$\cos\alpha/2 = 0, \text{ and consequently, } H = 0.$$

5. Rayleigh's Jet-Method. A liquid jet, issuing horizontally out of an orifice, shows a strange recurrence of forms in its surface—a phenomenon, in which surface tension plays its own part. This has been utilized by *Rayleigh*, *Pedersen*, *Bohr* and *Stocker* for the determination of the surface tension of a liquid. We shall consider here only the simpler Rayleigh's method, by way of an illustration as to how this may be done.

We know that, under ordinary conditions, a liquid jet is quite '*steady*', in the hydro-dynamical sense of the term ; for, although made up of drops *in motion* (*i*), its surface is *fixed in space* and (*ii*) the velocity at any point remains constant. But, on making a closer and careful examination of it, we find that, in its initial formative stage, *its various parts are in motion with respect to each other*, its '*form*' oscillating this way and that about mean spherical one, with its lateral dimensions, measured along a particular direction, exhibiting a cyclic change. Thus, for example, points P and Q in the jet, (Fig. 307), include one such cyclic change in its cross-section.

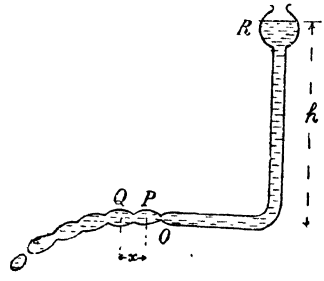


Fig. 307.

The *time-period*, t, of the oscillation of such a drop, may be easily determined by the method of dimensions. For, obviously, it will depend upon (*i*) r, the *radius* of the orifice (*O*) for the horizontal tube, whence the jet emerges, (*ii*) T, the *surface tension* and (*iii*) ρ, the *density* of the liquid. Thus, let

$$t = K.r^a.\rho^b.T^c,$$

where K is a *dimensionless constant*.

Now, the dimensions of t, (*time*) $= [T]$; of r (*radius*) $= [L]$;
of *density*, or *mass/volume* $= [ML^{-3}]$;

and, of *surface tension* T or *force/length* $= [MLT^{-2}].[L^{-1}] = MT^{-2}$.

Hence $[T] = K[L]^a\,[ML^{-3}]^b\,[MT^{-2}]^c = K.L^{a-3b}.M^{b+c}T.^{-2c}$

So that, $a-3b = 0$, $b+c=0$ and $-2c = 1$.

whence, $c = -\frac{1}{2}$, $b = \frac{1}{2}$ and $a = \frac{3}{2}$.

And $\therefore$ $t = K.r.^{\frac{3}{2}}.\rho.^{\frac{1}{2}}T.^{-\frac{1}{2}}$

Or, $t = K\sqrt{r^3/\rho T}$

The value of constant K was experimentally found out by *Rayleigh* to be $\pi/\sqrt{2}$.

So that, $t = \frac{\pi}{\sqrt{2}}\sqrt{\frac{r^3.\rho}{T}}$. Or, $t = \pi\sqrt{\frac{r^3\rho}{2T}}$ ... (*I*)

Now, the jet takes a *parabolic* path, its horizontal velocity v being given by $\sqrt{2gh}$, where h is the vertical height of the liquid surface above the orifice, *i.e.*, the height of the *liquid head* above the horizontal.

And, if x be the distance between P and Q, (Fig. 307), we have $x = v.t$. Or, $x^2 = v^2.t^2$. Or, $t^2 = x^2/v^2$.

But $t^2 = \pi^2.r^3.\rho/2T$. [From I, above.

So that, $x^2/v^2 = \pi^2r^3.\rho/2T$. Or, $x^2/2gh = \pi^2r^3\,\rho/2T$. [Substituting $2gh$ for v^2.

Or, $x^2/gh = \pi^2.r^3.\rho/T$, whence, $T = \pi^2r^3.\rho\ gh/x^2$.

So that, measuring x with the help of a travelling microscope, (by properly illuminating the jet) we can easily calculate the value of T, the surface tension of the given liquid.

6. The Capillary Rise Method. A direct application of the relation, $T = r(h+r/3)\rho.g/2$, (§ 248 page 488), connecting the rise of a liquid in capillary tube with its surface tension, it is a simple laboratory method for the determination of surface tension of liquids which do not wet the walls of the tube, *i.e.*, for which the angle of contact is *zero*.

A capillary tube is taken and the uniformity of its bore carefully tested by introducing a thread of mercury inside it and measuring its length, by jerking it into different positions along the tube. If the thread measures the same everywhere, the tube has a uniform bore. When a proper tube has thus been selected, it is thoroughly cleaned by rinsing it first with caustic soda, then, with nitric acid and finally with distilled water. If the liquid, whose surface tension is to be determined, be water, the tube may be used straightaway, but if it be some other liquid than water, the tube must be properly dried by passing a current of warm air through it, and then fixed *vertically*, alongside a plumb line, with its lower end immersed in the experimental liquid,

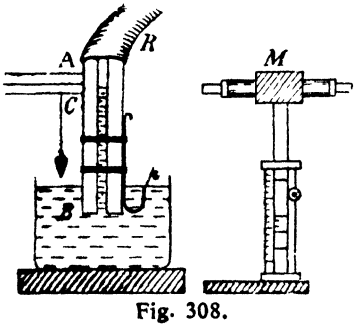


Fig. 308.

And, if the liquid be one which *wets* the tube, *i.e.*, for which $\theta = 0°$; so that, $\cos\theta = 1$, the expression for T reduces to

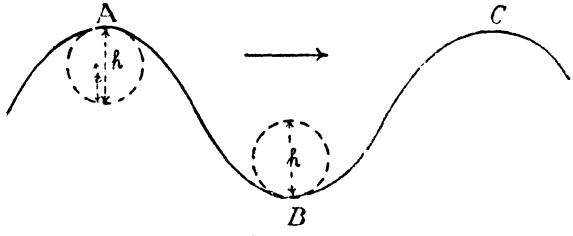
$$T = r.h.\rho\, g/2,$$

from which the surface tension T of the given liquid can be easily calculated.

There is only one slight error possible in this method, *viz.*, the shape of the liquid meniscus at Q may not be really perfectly spherical (as assumed), due to the distortion effects on account of gravitational forces. But *Ferguson* and *Kennedy* have shown that they can be safely neglected, if the bore of the tube be of a really small diameter, about 1 *cm.* or less.

8. Ripples Method. Before proceeding with the method proper, we must first try to understand the difference between waves and ripples. We are all familiar with the waves travelling over the surface of liquids. Their velocity of propagation depends on both the force of gravity as well as surface tension For an amplitude smaller than the wave length, *the wave-curve is given by the successive positions of a point fixed to a circle rolling along a straight line,* the amplitude (a) and the wave-length (λ) being equal to the distance of the point from the centre of the circle, and the circumference of the circle respectively. In other words, *each particle of the liquid in the wave describes a circle in the vertical plane,* the wave itself advancing forwards through a distance λ, (its wave length), during the time that a particle takes to complete its one full round along the circle, —the direction of motion being *anti-clockwise*, for a wave travelling from *left to right.*

Let us imagine a vertical section of the liquid by a plane, parallel to the direction of propagation of the wave. (Fig. 311), and consider first the effect of gravity alone. Then, if V be the velocity of the wave along the horizontal ; r, the radius of the circular path of the particle ; and t, the time it takes to describe it, we have



Fi . 311.

velocity (v_1) of the particle at the *crest* A, at any given instant, given by the relation, $v_1 = V - 2\pi r/t$. ... ...(*i*)

And, velocity (v_2) of the particle at the *trough* B, given by the relation, $v_2 = V + 2\pi r/t$. ... ... (*ii*)

Taking this increase in the velocity of the particle to be solely due to its having fallen from A to B, through a vertical distance $h = 2r$, under the action of gravity, we have

$$v_2^2 - v_1^2 = 2g.2r = 4g.r \ldots (iii)$$

{ From the kinematic relation $v^2 - u^2 = 2as$, where $v = v_2$, $u = u_1$, $a = g$ and $s = 2r$.

And, squaring and subtracting relation (*i*) from (*ii*), we have

$$v_2^2 - v_1^2 = (V + 2\pi r/t)^2 - (V - 2\pi r/t)^2,$$
$$= 4 \times V.2\pi r/t = 8\pi.Vr/t. \quad \ldots \quad \ldots \quad (iv)$$

So that, from relation (*iii*) and (*iv*), we have

$$8\pi\, Vr/t = 4g.r, \text{ whence, } V + 4g.r.t/8\pi r = gt/2\pi.$$

Now, obviously, $t = \lambda/V$. And, $\therefore V = g\lambda/2\pi V$. Or, $V^2 = g\lambda/2\pi$, whence,

$$V = \sqrt{g\lambda/2\pi} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (v)$$

This, then, is the expression for the velocity of the wave due to the action of gravity alone.

Let us now see how the surface tension of the liquid affects this velocity of the wave.

Let Fig. 312 represent a vertical section of the liquid, with *ABCDE* as the *harmonic wave* travelling over it, and *AE*, its undisturbed surface.

Let the displacement $PQ = y$, of a particle Q of the liquid, at any instant, at a distance x from an arbitrary origin A, be given by the relation,

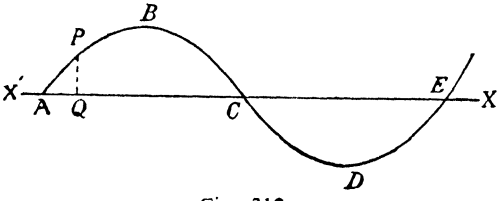


Fig. 312.

$$y = a \sin\left(\frac{2\pi x}{\lambda} + C\right), \quad \ldots \quad \ldots \quad \ldots \quad (vi)$$

where a is the amplitude of the particle and C, the *phase constant*.

Now, due to the action of gravity alone, the vertical pressure at Q would increase by $y.\rho\, g$, where ρ is the density of the liquid and g, the acceleration due to gravity.

Owing to the surface tension (T) of the liquid, however, there will also be an excess pressure equal to $T.(1/R + 1/R')$, acting normally at P, from the concave towards the convex side, where R and R' are the respective radii of curvature of the liquid surface *in the plane*, and *perpendicular to the plane of the paper*. But since the wave system is a cylindrical one, one surface of which is plane, R' is infinite and hence $1/R' = 0$. The pressure at P due to surface tension thus reduces to $T(1/R)$ or T/R. Since the amplitude of the wave is small, compared with its wave-length, the normal at P almost coincides with the vertical through P. The net increase in the vertical pressure at P is, therefore, given by

$$p = y.\rho\, g - T/R.$$

Now, $\frac{1}{R} = \frac{d^2y}{dx^2}$, if $a << 1$ and $\therefore \frac{dy}{dx} << 1$. See page 332, Ex. 4 (solved)

Hence $$p = y.\rho.g - T.d^2y/dx^2. \quad \ldots \quad \ldots \quad \ldots \quad (vii)$$

To determine the value of d^2y/dx^2, we differentiate relation (*vi*) twice, when we get $d^2y/dx^2 = -4\pi^2 y/\lambda^2$.

Putting this value of d^2y/dx^2 in relation (*vii*) above, we have

$$p = y.\rho.g + \frac{4\pi^2 yT}{\lambda^2} = y.\rho\left(g + \frac{4\pi^2 T}{\lambda^2 \rho}\right).$$

It is thus clear that the effect of surface tension is tantamount to increasing the value of g by $4\pi^2T/\lambda^2.\rho$.

Clearly, therefo e, if V be the velocity of the wave under the action of both gravity and surface tension, we obtain its value by substituting $(g+4\pi^2T/\lambda^2.\rho)$ for g in relation (v) above ; *i.e.*,

$$V=\sqrt{\frac{\lambda}{2\pi}\left(g+\frac{4\pi^2T}{\lambda^2.\rho}\right)}=\sqrt{\frac{\lambda g}{2\pi}+\frac{2\pi T}{\lambda.\rho}}. \quad \ldots \quad (viii)$$

A mere glance at this relation tells us that $V=\infty$, both when $\lambda=0$ and when $\lambda=\infty$. In-between these two extreme values of λ, there must be a certain value of it for which V has the *minimum* value. Clearly, the product of the two terms $\lambda g/2\pi$ and $2\pi T/\lambda.\rho$, *viz.*, gT/ρ is a constant, and it follows, therefore, that *their sum will be a minimum when they are equal* ; *i.e.*, when

$$\lambda g/2\pi = 2\pi T/\lambda.\rho. \text{ Or, when } \lambda^2.g.\rho = 4\pi^2T.$$

Or, when $\lambda^2=4\pi^2T/g.\rho$. Or, when $\lambda=2\pi\sqrt{T/g.\rho}$.

This value of λ, for which the velocity of the wave is the *minimum*, is called the *critical wave-length*, and may be denoted by the symbol λ_c. Thus,

$$\lambda_c = 2\pi\sqrt{T/g.\rho}. \quad \ldots \quad \ldots \quad \ldots \quad (ix)$$

Substituting this value of λ, therefore, in expression (*viii*) for V, above, we get the *minimum velocity* (V_m) of the wave given by the relation,

$$V_m = \sqrt{2}(T.g/\rho)^{\frac{1}{4}}*.$$

Now, examining relation (*viii*) again, we find that

(*i*) If $\lambda>\lambda_c$, the first term $\lambda g/2\pi$ becomes more important as λ increases and, therefore, neglecting the second term in comparison with it, we have

$$V=\sqrt{\lambda g/2\pi}.$$

Disturbances of this type, whose wave-length is greater than the critical value, are known as **waves**. Their propagation is mainly due to the force of gravity and, as can be readily seen, *their velocity increases as λ increases*.

(*ii*) If $<\lambda_c$, the second term becomes more predominant, and the first term may, therefore, be neglected. So that, in this case,

$$V=\sqrt{2\pi T/\lambda\rho}.$$

Waves of this type for which the wave-length is less than the critical wave length, are called **ripples** or **capillarity waves**. Their propagation is, in main, due to surface tension, and, as can be easily seen, *their velocity decreases as λ increases*.

---

*For, substituting the value of $T/\rho=\lambda^3g/4\pi^2$ from (*ix*) in relation (*viii*), above, we have

$$V_m=\sqrt{\frac{\lambda g}{2\pi}+\frac{2\pi}{\lambda}\cdot\frac{\lambda^3 g}{4\pi^2}}=\sqrt{\frac{\lambda g}{2\pi}+\frac{\lambda g}{2\pi}}=\sqrt{\frac{2\lambda g}{2\pi}}=\sqrt{\frac{\lambda g}{\pi}}.$$

Now, substituting $2\pi\sqrt{T/g\rho}$ for λ, we have

$$V_m=\sqrt{\frac{\pi\sqrt{\frac{T}{g\rho}}\rho.g}{\pi}}=\sqrt{2}\left(\frac{Tg^2}{g\rho}\right)^{\frac{1}{4}}=\sqrt{2}\left(\frac{T.\rho}{\rho}\right)^{\frac{1}{4}}.$$

N.B. Taking the case of water, for which $T = 75$ *dynes/cm.* and $\rho = 1$ *gm./c.c.*, we get $\lambda_c = 1{\cdot}7$ *cms.* and $V_m = 23$ *cms./sec.*

The Method Proper. *Lord Rayleigh* (1890) was the first to have used ripples, excited on the surface of a liquid, to measure its surface tension, by the direct application of relation (*viii*) above, although the method has subsequently been greatly improved by a host of other workers, including *Dorsey*, *Ghosh*, *Banerji* and *Datta*. And, still more recent improvements have been effected by *Tyler* and *Brown*, enabling the ripples to be photographed and their images thrown on a screen.

We shall deal only in brief outline, here, with the essentials of the method employed. The experimental liquid is taken in a shallow rectangular porcelain trough, about 10″ long, 6″ wide and 1·5″ deep, (Fig. 313), and above it is arranged an electrically maintained tuning fork *F*, of a large frequency, with its prongs horizontal and one above the other in their position of rest, so as to vibrate in the vertical plane. A light *style* or *'dipper' P*, of polished silver or aluminium, about 3″ long, is attached to the lower prong of the fork, with its plane also vertical but perpendicular to the plane of vibration of the fork, such that it just touches the liquid surface in the trough.

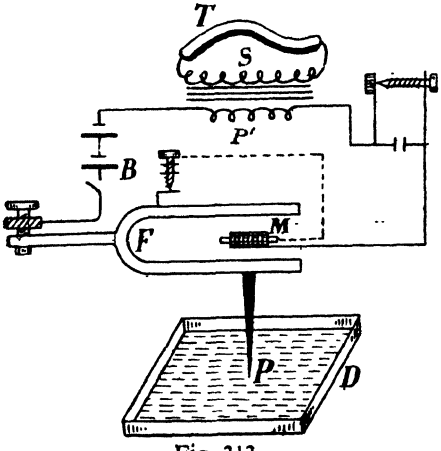


Fig. 313.

Now, as the fork is set vibrating, the style or the dipper alternately moves in and out of the liquid, thus exciting trains of ripples on its surface. These, on reflection from the walls of the trough, give rise to stationary ripples.

To enable these ripples to be observed and their wave-length measured, the liquid surface must be properly illuminated. This is done by completing the tuning fork circuit through the primary *P'* of a small *induction coil*, to the secondary *S* of which is connected a *neon-discharge-tube* ; so that, every time the tuning fork circuit is made, an electric discharge passes through the *neon-tube*, the light from which then brilliantly illuminates the liquid surface in the trough, thus enabling the wave-form of the ripples to be easily seen and photographed.

Now, *during one vibration of the fork, there is one alternate make and break of the tuning fork circuit, giving us one view of the ripples on the liquid surface.* The same *recurs*, when the fork has made one vibration, and when, therefore, the ripples have advanced through a distance equal to their one full wave-length ; so that, as the liquid surface is lit up again, as before, we get a *second* view of it, *identical with the first*, the ripples *appearing to be stationary* in their earlier position. This is so, because we do not see the liquid surface continuously but only at intervals equal to the time-period of the

ripples, the frequency with which the liquid surface is lit up being the *same* as that of the fork and, therefore, of the ripples themselves. The succsesive views we have of this liquid surface are thus only those in which one ripple has just replaced another *similar* ripple and hence, *due to persistence of vision*, (the frequency of the fork being large), the surface of the liquid *appears* to us to be exactly the *same* as before, with the ripples in their original position.

The distance between as large a number of ripples as possible is measured with the help of a travelling microscope and their mean wave-length λ thus determined. Then, if n be the frequency of the fork (which is known to us), we have $V = n\lambda$, where V is the velocity of the ripples. Substituting this value of V in relation (*viii*) above, we have

$$n\lambda = \left(\frac{\lambda g}{2\pi} + \frac{2\pi T}{\lambda \rho}\right)^{\frac{1}{2}}. \quad \text{Or,} \quad n^2\lambda^2 = \frac{\lambda g}{2\pi} + \frac{2\pi T}{\lambda.\rho}.$$

$$\text{Or,} \qquad n^2\lambda^2 - \frac{\lambda g}{2\pi} = \frac{2\pi T}{\lambda.\rho}, \text{ whence, } T = \frac{\lambda.\rho}{2\pi}.n^2\lambda^2 - \frac{\lambda.\rho}{2\pi}.\frac{\lambda g}{2\pi}.$$

$$\text{Or,} \qquad T = \frac{n^2.\lambda^3.\rho}{2\pi} - \frac{\lambda^2.\rho.g}{4\pi^2},$$

from which, λ being known, the value of T, the surface tension of the given liquid, may be easily calculated.

9. The Ring Method. This method derives its importance from the fact that it enables us to study the changes that come about in the surface tensions of different liquids with the passage of time. It has been used with great accuracy by *Harkins, Young* and *Cheng*, and is being increasingly employed in Applied Physics.

A metal ring of a wire of circular cross-section is suspended in the experimental liquid, with its plane horizontal. It is then raised gradually out of the liquid, when, in addition to its own weight, the extra downward pull on it due to surface tension passes through a *maximum value*. If the wire, constituting the ring, be a thin one, this maximum pull is approximately given by $P = 4\pi RT$, where R is the radius of the ring and T, the surface tension of the liquid. More correctly, however, P is only proportional to $4\pi RT$, so that $\alpha P = 4\pi RT$, where α is *a non-dimensional* factor. To avoid calculating the value of α, *Harkins* and his co-workers assumed that

$$\frac{4\pi RT}{P} = \phi_1\left(\frac{R}{r}\right)\ldots(i) \quad \text{Or,} \quad \frac{4\pi RT}{P} = \phi_2\left(\frac{R^3}{V}\right), \quad \ldots(ii)$$

where r is the radius of the wire, V, the volume of the liquid held up by the maximum pull of the ring, equal to $P/\rho.g$, (ρ being the density of the liquid).

They used three liquids of known surface tensions, *viz.*, *water*, *benzene* and *bromobenzene* and measured the maximum pull on three different rings, with different values of R and r, but the *same constant ratio* R/r, and then plotted a graph between R^3/V, (along the x axis), and $4\pi RT/P$, (along the y-axis), when the points for all the three liquids were found to lie on the same smooth curve. From this, they came to the legitimate conclusion that, if the *same* three rings

were to be used with other liquids, the points corresponding to the values of R^3/V and $4\pi RT/P$ for them too would lie on the same curve.

Thus, the procedure to determine the surface tension of a given liquid by this method becomes the following :

We take a ring, identical with one of the rings used by Harkins, Young and Cheng, (*i e.*, of the same material and the same value of R/r), measure, with its help, the maximum pull as the ring is raised out of the given liquid and then calculate the value of R^3/V for it, which is equal to $R^2.g.\rho/P$. Next, we locate the point corresponding to *this* value of R^3/V as abscissa, on the graph, plotted by them for the ring of the same value of R/r. The ordinate y, corresponding to it, then gives the value of $4\pi RT/P$ for the liquid and is carefully noted.

Thus, $y = 4\pi RT/P$, and, therefore, $T = yP/4\pi R$,

whence the value of T for the liquid can be easily calculated.

The method has the merit of *quickness*, *ease* and *accuracy*, but the following precautions are necessary for its success :

(*i*) *The liquid surface must be perfectly clean.* This is ensured by sweeping the surface of the liquid by means of what are called 'barriers', before performing the actual experiment.

(*ii*) *The liquid surface must be kept properly covered by an inverted glass funnel, to avoid evaporation.*

(*iii*) *The dish containing the liquid must be wide enough,* so that errors due to curvature of the meniscus etc. are almost altogether eliminated.

(*iv*) *The whole apparatus must be arranged inside a thermostat,* to ensure that the temperature of the liquid remains *constant* throughout, and the thermostat must be supported independently of the ring and other apparatus, to avoid disturbance or agitation of the experimental liquid.

256. Surface Tension of Liquid Interfaces. If we consider a system of immiscible liquids in contact, we naturally expect a new phase to develop at their *interface,* with a definite energy of its own, depending upon the nature of the two liquids.

Antonow gave a rule that the interfacial tension between two liquids, in equilibrium, is equal to the difference between their individual surface tensions. Thus, if T_1 and T_2 be the surface tensions of two liquids separately, their interfacial tension T_{12}, when they are in equilibrium, is given by $T_{12} = T_1 - T_2$.

This rule would, however, apply only to mutually saturated solutions, for then alone would they be in equilibrium, or else the addition of one to the other would reduce its surface tension. Being the difference of the individual surface tensions of the two liquids, the surface tension of their interface is obviously a small quantity, in general, increasing with the decrease in the solubility of one liquid into the other, and decreasing with a rise in temperature.

Since a knowledge of interfacial tension is of importance while considering problems like spreading of one liquid over another or those relating to chemical constitution, we shall now proceed to see how it may be determined for a given liquid-liquid interface.

Most of the methods discussed above, for the measurement of surface tension of pure liquids would, in principle, be equally applicable in this case also. Thus, for example, the *drop-weight* method can be easily used for the purpose and probably gives the best results. We shall discuss this method in proper detail.

(*i*) **The Drop-weight Method.** Here, we proceed in the same manner as in determining the surface tension of a pure liquid, (see page 498), with the difference that the tube, with a fine orifice, of about 3 to 5 *mms.* radius, containing one liquid, dips inside the other, *so that the drop of the first liquid is formed inside the second, instead of in air*, the weight of the drop being obtained by collecting and weighing a known number of them.

Then, if ρ_1 and ρ_2 be the densities of the two liquids respectively, we have *volume of the drop* $= m/\rho_1$. This is obviously also the *volume of the second liquid displaced by it.* So that,

weight of the second liquid displaced or, upthrust on the drop

$$= \frac{m}{\rho_1}.\rho_2.g. \qquad \left[\begin{array}{l} m \text{ being the } \textit{mass} \\ \text{of the drop.} \end{array}\right.$$

And, therefore, *apparent or effective weight of the drop*

$$= mg - \frac{m}{\rho_1}\rho_2.g. = mg\left(1 - \frac{\rho_2}{\rho_1}\right) = mg\left(\frac{\rho_1 - \rho_2}{\rho_1}\right).$$

Now, assuming, as before, that, just at the time of being detached from the orifice, the drop is *cylindrical* in shape and has the same cross-sectional area as that of the orifice, the excess pressure inside it is given by $p = T_{12}/r^*$,
where T_{12} is the surface tension of the interface.
The downward force on the drop due to it is thus

$$= p \times \pi r^2 = \frac{T_{12}}{r}.\pi r^2 = \pi.T_{12}.r.$$

And, therefore, *total downward force on the drop*

$$= \text{effective weight of the drop} + \pi.T_{12}.r,$$

$$= \frac{mg(\rho_1 - \rho_2)}{\rho_1} + \pi\, T_{12}.r.$$

And, the upward force on it due to surface tension $= T_{12}.2\pi r$ $= 2\pi T_{12}r$, where $2\pi r$ is the circumference of the cross-section of the drop.

In the equilibrium position of the drop, therefore, we have

$$\frac{mg(\rho_1 - \rho_2)}{\rho_1} + \pi.T_{12}r = 2\pi T_{12}r. \text{ Or, } \frac{mg(\rho_1 - \rho_2)}{\rho_1} = \pi.T_{12}.r.$$

Or,
$$T_{12} = \frac{mg(\rho_1 - \rho_2)}{\rho_1.\pi\, r}.$$

In the above treatment, we have assumed that the drop attains static equilibrium when, having attained its cylindrical form, it is just about to be detached. This is, however, not so, the detachment of the drop being essentially a dynamical process, for which we obtain the following relation, by dimensional analysis :

$$\frac{mg(\rho_1 - \rho_2)}{\rho_1} = T_{12}.r\phi\left[\frac{T_{12}}{gr^2(\rho_1 - \rho_2)}\right],$$

*The radius of the other face of the cylinder being infinite.

where ϕ is an arbitrary function of the non-dimensional variable $T_{12}/gr^2(\rho_1-\rho_2)$.

Lord Rayleigh has, however, shown from his work on water drops in air that the value of $\phi\left[\frac{T_{12}}{gr^2(\rho_1-\rho_2)}\right]$ comes to 3·8 for a large variation in the value of r.

So that, we have $$\frac{mg(\rho_1-\rho_2)}{\rho_1} = T_{12}.r.3{\cdot}8.$$

Or, $$T_{12} = \frac{mg(\rho_1-\rho_2)}{3{\cdot}8.\rho.r},$$

whence, the surface tension T_{12} for the given liquid interface can be easily calculated.

***(ii)* Wilhelmy Method.** *Wilhelmy* had suggested a straightforward method of determining the surface tension of a liquid by measuring, with the help of a balance, the additional force necessary to counteract the vertical pull of surface tension on a verticle metal plate, suspended from the balance arm, when it is dragged away from the surface. Thus, if the lower edge of the metal plate be in level with the undisturbed liquid surface and if l be the length of its line of contact with it, the vertical pull on it due to surface tension is equal to $T.l\cos\theta$, where θ is the angle of contact of the liquid-plate surface. And, therefore, if mg be the additional weight required in the scale-pan to balance this additional force, we have $mg = T.l\cos\theta$.

It we use a *torsion balance*, in place of the ordinary balance, the method can be easily used for interfacial tension also, as will be clear from the following :

The two liquids are taken in a beaker, one above the other, a vertical plate suspended from the arm of the torsion balance and adjustment made for equilibrium with the plate wholly immersed but well above the interface of the two liquids. The beaker is now gradually raised and the equilibrium continues to be maintained, until the plate *just* approaches the *interface*, when the additional downward pull on it, due to the interfacial tension T_{12} of the two liquids, disturbs the balance. The torsion head has thus to be turned through an additional angle α, say, to restore the plate back to its equilibrium position. This angle of twist (α) measures the force $l\,T_{12}$ on the plate, where l is the *horizontal perimeter*, the angle of contact (θ) being assumed to be *zero* (or $\cos\theta$ very nearly equal to 1). The value of T_{12} for the given interface can thus be easily determined.

***(iii)* Mack and Bartell Method.** *Mack* and *Bartell* used the following simple method for the measurement of interfacial tension of water and organic liquids. Besides its simplicity, it has the great merit of precision and of requiring only a very small quantity of the liquid, just 2 *c.cs.* of it or so.

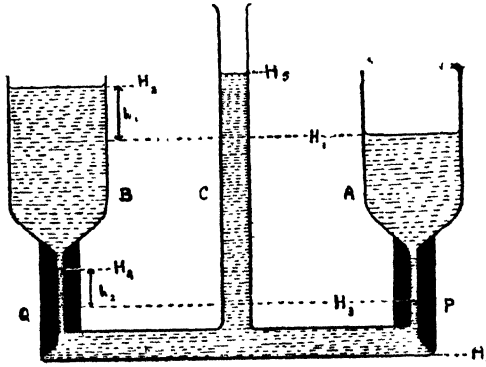
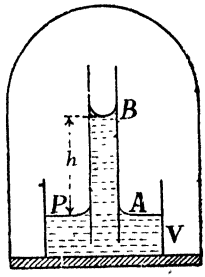


Fig. 314.

The apparatus used by them is as shown in Fig. 314, where A and B are two wide glass cups, communicating with a central wide tube C, through two *capillary tubes* P and Q of slightly differing radii R_1 and R_2, sealed on to them, where R_2, the larger of the two, is less than 1 *mm.*

Water is first poured into the cup A, connected to the narrow capillary P, to ensure that no air bubbles get entrapped, and a larger quantity of it is then poured into cup B, a little over and above that required to fill the two capillaries.

259. Surface Tension and Vapour Pressure over a Liquid Surface. Let a capillary tube be dipped vertically in a liquid, (which *wets* the tube), of surface tension T, and density ρ, and let the whole arrangement be enclosed in an exhausted bell jar or chamber, (Fig. 316), so that the effect of the atmosphere may be neglected. Obviously, then, there is nothing but its own vapour over the liquid, and when equilibrium is attained, the space above the liquid becomes saturated with its vapour.



Suppose the liquid rises into the tube up to a height h above the horizontal liquid surface A in the vessel.

Then, if P be the vapour pressure at the horizontal surface A, and σ, the density of the vapour at this pressure, the vapour pressure above the concave surface B of the liquid, in the tube, will clearly be less than P by an amount equal to that of a column h of the vapour, *i.e.*, by an amount $h.\sigma.g = p$, say ; so that vapour pressure *just above the meniscus* at $B = P - h.\sigma.g$. And, clearly, the pressure in the liquid just below the meniscus is equal to $(P - h.\rho.g.)$, where ρ is the density of the liquid.

If the tube be narrow, the meniscus may be regarded as spherical of radius r, *nearly the same as that of the tube*, and so the excess pressure, just above the meniscus over that just below it, will be equal to $2T/r$.

Clearly, *this pressure just above the meniscus is greater than the pressure just below it by*

$$(P - h.\sigma.g) - (P - h.\rho.g) = P - h.\sigma.g - P + h.\rho.g.$$
$$= h.\rho.g - h.\sigma.g = h.g.(\rho - \sigma).$$
$$\therefore \quad h.g.(\rho - \sigma) = 2T/r.$$

Now, $h.\sigma.g = p$; so that, $h.g = p/\sigma$. [See above.

And, $\therefore \quad \frac{p}{\sigma}.(\rho - \sigma) = \frac{2T}{r}$, whence, $p = \frac{2T}{r}\left(\frac{\sigma}{\rho - \sigma}\right)$.

Since ρ is greater than σ, the expression for p is positive, and, therefore, *the vapour pressure above the concave surface of a liquid is less than the vapour pressure at the horizontal surface of the same liquid by an amount* $2T\,\sigma/r\,(\rho - \sigma)$, or as a near approximation, we may say that $p = 2T.\sigma/r.\rho$, because σ is very small compared with ρ.

Now, if we treat the vapour as a *gas*, for which the relation $PV = R.\theta$, is applicable, (denoting the *absolute temperature* by θ, instead of by T, which we have used for surface tension here), we have $V = R.\theta/P$. [Where V is the volume of the vapour.

Now, $\sigma = 1/V$. And $\therefore \quad 1/V$ or $\sigma = P/R\theta$.

Hence, substituting this value of σ in the expression for excess pressure p above, we have

excess vapour pressure on the horizontal surface of a liquid over that on its concave surface given by

$$p = \frac{2T}{r}.\frac{\sigma}{\rho}. \text{ Or, } = \frac{2T}{r} \div \frac{P}{R.\theta.\rho}.$$

Thus, *the relative lowering of vapour pressure above a concave surface is given by the expression* $\frac{p}{P} = \frac{2T}{rR.\theta.\rho}$.

Similarly, if we have a capillary tube, dipping in a liquid which *does not wet* the tube, and enclose it, as before, in an exhausted chamber, the liquid column in it will be depressed below the horizontal surface of the liquid in the vessel, as shown, (Fig. 317), through a distance h, say.

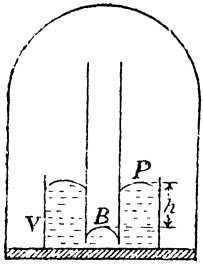


Fig. 317.

Then, if P be the vapour pressure at the horizontal liquid surface, outside the tube, the vapour pressure just above the convex meniscus in the tube will be equal to $(P+h.\sigma.g)$, where σ is the density of the vapour at pressure P; or, putting $h.\sigma\, g. = p$, as before, we have $h.g = p/\sigma$.

Now, pressure in the liquid just below the convex meniscus is clearly equal to P+*the hydrostatic pressure due to the liquid column* h, or equal to $(P+h.\rho.g)$, where ρ is the density of the liquid.

Since ρ is greater than σ, clearly, $(P+h.\sigma.g)>(P+h.\sigma.g)$, and, therefore, *excess pressure just below the meniscus over that just above it is equal to* $(P+h.\rho.g)-(P+h.\sigma.g)$ *i.e.*,

$$h.\rho.g-h.\sigma\, g = h.g.(\rho-\sigma) = \frac{p}{\sigma}.(\rho-\sigma).$$

But we know that *excess pressure just below the meniscus over that just above it* is also equal to $2T/r$, where T is the surface tension of the liquid and r, the radius of the meniscus, (supposed to be *spherical*).

$$\therefore \qquad \frac{p}{\sigma}.(\rho-\sigma) = \frac{2T}{r}, \text{ whence, } p = \frac{2T}{r}\left(\frac{\sigma}{\rho-\sigma}\right).$$

Thus, *the vapour pressure above the convex surface of a liquid is greater than that on a plane or horizontal surface of the same liquid by an amount equal to* $2T.\sigma/r.(\rho-\sigma)$ or $2T.\sigma/r.\rho$, as a near approximation.

Or, as before, $p=(2T/r).\ P/R\theta.\rho$, whence, $p/P = 2T/r.R.\theta.\rho$.

It should be noted that the *excess pressure* p *is inversely proportional to* r, *the radius of curvature of the surface*; so that, the smaller the value of r, or the greater the curvature, the higher the value of the saturation vapour pressure at the curved surface. *It thus follows that the saturation vapour pressure over a small drop of a liquid will be greater than over a large drop of it.*

Let us take a *large drop of water of diameter* 1 *mm.*, or $r = \cdot 5$ *mm.* $= \cdot 05$ *cm.*

Then, putting $\theta=273°$ *Abs.*, $\rho=1$ *gm./c.c.*, $\sigma=\cdot 61\times 10^{-3}$ *gm./c.c.*, and $T = 75$ *dynes/cm.*, we have

$$p = \frac{2.T.\sigma}{r.\rho} = \frac{2\times 75\times \cdot 61\times 10^{-3}}{\cdot 05\times 1} = 1\cdot 83 \text{ dynes/cm}^2.$$

For a drop of water, of diameter 1/1000 *mm.*,

$$p = 1\cdot 83\times 1000 = 1\cdot 83\times 10 \text{ dynes/cm}^2.$$

And for a drop of water, of diameter equal to *one-millionth of a mm.* as is the case when it just starts forming, we have

$$p = 1{\cdot}83 \times 10^3 \times 10^3 = 1{\cdot}83 \times 10^6 \text{ dynes/cm.}^2,$$

which is greater than a pressure of one atmosphere.

Thus, we see how the saturation vapour pressure rises with the diminution in the size of the drop.

260. Effect on Evaporation and Condensation. We have seen above how the maximum vapour pressure is less for a concave liquid surface than that for a plane or horizontal surface and also how the vapour pressure for a convex liquid surface is greater than that for a plane surface. If, therefore, we place a drop of water in a space in which the vapour is at the saturation value for a plane surface, the drop will begin to evaporate ; for the vapour pressure in the space will be less than the saturation vapour pressure for the drop, and it will, therefore, be converted into vapour, in order to increase the vapour pressure to its own saturation value.

This will result in a further decrease in the radius of the drop or an increase in its curvature (*i.e.*, convexity) and a consequent rise in the saturation value of its vapour pressure, and it will, therefore, evaporate more and more rapidly. That is why a saturated vapour does not condense into drops ; for, as soon as a tiny drop is formed, it begin to evaporate. Thus, condensation may not take place even when the vapour becomes supersaturated.

If, however, dust particles or charged ions be introduced into the saturated vapour, they offer a flatter surface to it and condensation at once starts on them ; for, the radius of curvature of the drop, so formed, is not very small, even in the beginning, and hence it has little tendency to evaporate And, as its radius increases, and, therefore, its curvature or convexity decreases, its tendency to evaporate becomes smaller still. For, the saturation value of the vapour pressure for it goes on decreasing, and it continues to grow in size.

Thus, dust particles or charged ions play an important part in the condensation of vapours. And it is precisely because of the absence of these dust particles, (which act as *nuclei* for the vapour to condense on), that dust-free vapour does not condense, even if its temperature be lowered below its normal temperature of condensation.

Again, because the saturation vapour pressure over a concave surface is less than on a plane or horizontal surface, *the vapour will condense more readily on a concave surface than on a plane surface.* This might be clearly seen by closing the bottom of the tube in the last experiment and removing some liquid from it, when condensation would set in on the liquid inside the tube and will go on, until the liquid column in the tube attains its previous height.

SOLVED EXAMPLES

1. A sphere of water, of radius 1 mm, is sprayed into a million drops of equal size. Find the work expended in doing so.

Breaking the liquid drop means an increase in surface area and, therefore, work is required to be done for the purpose, which is equal to the product of the surface tension and the *new* surface area formed. This work done becomes the *surface energy* of the new surface area. We shall, therefore, first calculate

the *initial surface energy of the sphere of water*, when it is intact, and then *the surface energy of the million drops* into which it is broken up and sprayed. The *difference* of the two will give the work expended in the process

Let the surface tension of water be taken to be 72 *dynes/cm* Then, this is equal to the *free surface energy* of the sphere of water, (see page 478).

Or, free surface energy = 72 *ergs/cm*^{2}.

Radius of the sphere = 1 *mm.* = ·1 *cm.*

$\therefore$ surface area of the sphere $= 4\pi r^2 = 4\pi \times (1)^2$ *sq.cm.*

And $\therefore$ *initial surface energy of the sphere* $= 72 \times 4\pi \times (\cdot 1)^2 = 9\,048$ *ergs.*

Now, volume of the sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3} \times \pi \times (\cdot 1)^3$ *c.cs.*

And, number of drops into which it is broken $= 10^6$.

$\therefore$ volume of 10^6 drops $= \frac{4}{3} \times \pi \times (\cdot 1)^3$ *c.cs.*

And $\therefore$ volume of *one* drop $= \frac{4}{3} \times \pi \times (\cdot 1)^3 / 10^6 = \frac{4}{3} \times \pi \times (\cdot 1/10^2)^3$ *c.c.*

Hence, radius of *one* drop $= \cdot 1/10^2$ *cm.* $= \cdot 001$ *cm.*

$\therefore$ surface area of *one* drop $= 4\pi \times (\cdot 001)^2$ *sq. cm.*

And surface area of *all* drops $= 4\pi(\cdot 001)^2 \times 10^6 = 4\pi \times 1 = 4\pi$ *sq. cms.*

$\therefore$ surface energy of *all* drops, or, *final surface energy*

$= 4\pi \times 72 = 904 \cdot 8$ *ergs.*

Hence, *increase in surface energy* $= (904 \cdot 8 - 9 \cdot 048)$ *ergs.*

$= 895 \cdot 752$ *ergs.*

Or, *work expended in the process* $= 895 \cdot 752$ *ergs.*

2. A glass plate, of length 10 cms., breadth 1·54 cms. and thickness ·20 cm, weighs 8·2 gms. in air. If it is held vertically, with its long side horizontal, and its lower half immersed in water, what will be its apparent weight? (Surface tension of water = 73 dynes per cm.)

(Cambridge Higher School Certificate)

Here, apart from its weight, there will be two other forces acting on the plate, *viz.*, (*i*) the upthrust of water, and (*ii*) downward force due to surface tension along its edges inside the water.

Now, upthrust of water = weight of water displaced by it

$=$ *half the volume of the plate* $\times$ *density of water* $\times g$,

because half the plate is immersed in water.

Or, $= \frac{10 \times 1 \cdot 54 \times \cdot 20}{2} \times 1 \times 981$ *dynes* [$\because$ volume of the plate $= 10 \times 1 \cdot 54 \times \cdot 20$ *c cs.*

$= 1 \cdot 54 \times 1 \times 981$ *dynes* $= 1 \cdot 54$ *gms. wt.*

And, the force due to surface tension acts downwards along a distance equal to twice the sum of the length and the thickness of the plate, *i.e.*, along a total length $= 2(10 + \cdot 2) = 20 \cdot 4$ *cms.*

$\therefore$ total downward force on the plate due to surface tension

$= 20 \cdot 4 \times 73$ *dynes.* $= \frac{20 \cdot 4 \times 73}{981}$ *gm. wt.* $= 1 \cdot 518$ *gms. wt.*

$\therefore$ *net upward thrust* on the plate $= 1 \cdot 54 - 1 \cdot 518 = 0 \cdot 022$ *gm. wt.*

Hence, *apparent weight of the plate in water = weight in air − upthrust on it.*

$= 8 \cdot 2 - \cdot 022 = 8 \cdot 178$ *gms. wt.*

3. The pressure of air in a soap bubble of 0·7 cm. diameter is 8 mms. of water above the atmospheric pressure. Calculate the surface tension of the soap solution. *(Delhi 1944)*

We know that *excess pressure* inside a soap bubble over that outside it is given by $p = 4T/r$ where T is the surface tension of the soap solution and r, the radius of the bubble.

Here, the excess pressure inside a soap bubble over the outside atmosphere is given to be equal to 8 *mms.* or ·8 *cm.* of water column.

$\therefore$ $p = \cdot 8$ *cm.* of water column. [Taking density of water $= 1$ *gm./c.c.* and value of $g = 981$ *cm./sec*^{2}.

$= \cdot 8 \times 1 \times 981$ *dynes/cm*^{2}.

$= 784 \cdot 8$ *dynes/cm.*^{2}

and, $r = \cdot 7/2 = \cdot 35$ *cms.*

$\therefore$ substituting the value of p and r in the above relation, we have

$784 \cdot 8 = 4T/35$, whence, $4T = 784 \cdot 8 \times \cdot 35$.

Or, $T = 784 \cdot 8 \times \cdot 35/4 = 68 \cdot 66$ *dynes/cm.*

Therefore, surface tension of the soap solution is 68·66 *dynes/cm.*

4. Why is the upper surface of mercury in a glass capillary tube convex upward, while for water it is concave ? Assuming the surface tension of rain water to be 72 dynes/cm., find the difference of pressure inside and outside a rain drop of diameter ·02 cm. What would this difference of pressure amount to, if the drop were to be decreased by evaporation to a diameter of 0·0002 cm. ? *(Punjab)*

For answer to part one of the question, see § 245, (page 483).

We know that the excess pressure inside a liquid drop over that outside it is given by $p = 2T/r$, where T is the surface tension of the liquid drop, and r, its radius.

Here, $T = 72$ *dynes/cm.*, and $r = \cdot 02/2 = \cdot 01$ *cm.*

$\therefore \quad p = 2T/r = 2 \times 72/\cdot 01 = 144/\cdot 01 = 14400 = 1 \cdot 44 \times 10^4$ *dynes/cm*^{2}.

If the diameter of the drop be reduced to ·00002 *cm.*, its radius becomes ·00002/2 = ·00001 *cm.*, and we, therefore, have

$p = 2 \times 72/\cdot 00001 = 14400000$ *dynes/cm*^{2}. $= 1 \cdot 44 \times 10^7$ *dynes/cm*^{2}

Thus, the excess pressure inside the drop in the two cases will be $1 \cdot 44 \times 10^4$ *dynes/cm*^{2}., and $1 \cdot 44 \times 10^7$ *dynes/cm*^{2}., respectively.

5. What will be the pressure in a spherical cavity within a mass of paraffin oil ? The cavity is at a depth of 20 cms. below the surface and has a diameter of 0·0026 cm. The specific gravity of the oil is 0·85 and its surface tension is 26 c.g.s. units The pressure of the air over the liquid surface is equal to 76 cms. of mercury. *(Bombay)*

Here, pressure on the surface of the oil = 76 *cms.* of mercury column.

$= 76 \times 13 \cdot 6 \times 981 = 1 \cdot 014 \times 10^6$ *dynes/cm*^{2} ,

pressure due to the oil column, 20 *cms.* long

$= 20 \times \cdot 85 \times 981 = 1 \cdot 667 \times 10^4$ *dynes/cm*^{2}., because $P = h \rho g$.

$\therefore$ *total pressure on the spherical cavity, i.e., on the bubble*

$= 1 \cdot 014 \times 10^6 + 1 \cdot 667 \times 10^4 = 10^4 (1 \cdot 014 \times 10^2 + 6 \cdot 667)$.

$= 10^4 (101 \cdot 4 + 1 \cdot 667) = 103 \cdot 067 \times 10^4 = 1030670$ *dynes/cm*^{2}.

Now, the pressure inside the spherical bubble is in excess of the pressure outside it by $p = 2T/r$.

Here, $T = 26$ *dynes/cm.*, and $r = \cdot 0026/2 = \cdot 0013$ *cm.*

$\therefore \quad p = 2 \times 26/\cdot 0013 = 2 \times 2/\cdot 0001 = 40000$ *dynes/cm*^{2}.

Hence *pressure inside the cavity* $= 1030670 + 40000 = 107067$.

$= 10 \cdot 7067 \times 10^5$ *dynes/cm*^{2}.

6. A minute spherical air bubble is rising slowly through a column of mercury contained in a deep jar. If the radius of the bubble at a depth of 100 cms. is 0·1 mm., calculate its depth when its radius is 0·126 mm. ; given that the surface tension of mercury is 567. Assume that the atmospheric pressure is 760 mm. of mercury.

Here, pressure on mercury surface in the jar = 760 *mms.*

$= 76$ *cms.* of mercury $= 76 \times 13 \cdot 6981$ *dynes/cm*^{2}. $\times$

and pressure due to 100 *cms.* of mercury column $= 100 \times 13 \cdot 6 \times 981$ *dynes/cm*^{2}.

$\therefore$ total pressure on air bubble $= 76 \times 13 \cdot 6 \times 981 + 100 \times 13 \cdot 6 \times 981$.

$= (76 + 100) \times 13 \cdot 6 \times 981 = 176 \times 13 \cdot 6 \times 981 = 2348000$ *dynes/cm*^{2}.

And, excess pressure inside the air bubble

$2T/r = 2 \times 567/\cdot 01 = 113400$ *dynes/cm*^{2}.

because $T = 567$ *dynes/cm.*, and $r = \cdot 1$ *mm* $= \cdot 01$ *cm.*

$\therefore$ total pressure inside the bubble at depth 100 *cms.*

$= 2348000 + 113400 = 2461400$ *dynes/cm*^{2}.

And, volume of the bubble $=\frac{4}{3}\pi r^3=\frac{4}{3}\pi(\cdot01)^3$ *c.cs.*

Let depth at which its radius becomes 1·26 *mm.* be *h cms.*

Then, total pressure on the bubble at this depth
$= 76\times13{\cdot}6\times981+h\times13{\cdot}6\times981$;
$= (76+h)\times13{\cdot}6\times981 = 13350(76+h)$ *dynes/cm*^{2}.

And, excess pressure inside the bubble over *this* pressure on its outside
$= \frac{2T}{r}=\frac{2\times567}{\cdot0126}=89990$ *dynes/cm*^{2}. [$\because$ here, $r=\cdot126$ *mm* $=\cdot0126$ *cm.*

$\therefore$ total pressure inside the bubble at depth *h cms.*
$= 13350(76+h)+89990$ *dynes/cm*^{2}.
$= 1014000+13350h+89990 = 1103990+13350h$.

And, volume of the bubble at this depth $=\frac{4}{3}\pi(\cdot0126)^3$ *c.cs.*

Now, in accordance with Boyle's law, the product of pressure and volume of the bubble must be the same in the two cases ; so that,

$$2461400\times\tfrac{4}{3}\pi\cdot(\cdot01)^3 = (1103990+13350h)\times\tfrac{4}{3}\cdot\pi\cdot(\cdot0126)^3.$$

Or, $$2461400\times(\cdot01)^3 = (1103990+13350h)\times(\cdot0126)^3.$$
$$=1103990\times(\cdot0126)^3+13350\times(\cdot0126)^3\times h.$$

Or, $$13350\times(\cdot0126)^3\times h=2461400\times(\cdot01)^3-1103990\times(\cdot0126)^3.$$

Or, $$\cdot02669h = 2{\cdot}461-2{\cdot}209 = \cdot252,$$

whence, $$h = \frac{\cdot252}{\cdot02669} = 9{\cdot}441 \text{ cms.}$$

$\therefore$ the required depth of the bubble $= 9\ 441$ *cms.*

7. A capillary tube of 0·5 mm. bore stands vertically in a wide vessel containing a liquid of surface tension 30 dynes/cm. The liquid wets the tube and has a specific gravity of 0 8. Calculate the rise of the liquid in the tube.

Here, *diameter* of the tube $= 0{\cdot}5$ *mm.* $= \cdot05$ *cm.*

and, therefore, *radius* $r = \cdot05/2 = \cdot025$ *cm.*

surface tension $(T) = 30$ *dynes/cm.*

Sp. gravity $\rho=0{\cdot}8$ and angle of contact $=0$ (because the liquid wets the tube).

Now, $$T=\frac{rh\rho g}{2\cos\theta}=\frac{rh\rho g}{2}. \qquad [\because\ \theta = 0 \text{ and } \therefore \cos\theta=1]$$

Or, $$h=\frac{2T}{r\rho g}=\frac{2\times30}{\cdot025\times\cdot8\times980}=\frac{3}{\cdot02\times49}=3{\cdot}061 \text{ cms.},$$

Thus, the liquid will rise to a height $=3{\cdot}061$ *cms.* in the tube.,

8. A capillary tube of internal diameter 1 mm. and external diameter 5 mms. hangs vertically from the arm of a balance, the lower end of the tube being in a liquid of surface tension 40 dynes/cm. Assuming that the liquid wets the tube, what is the change in the apparent weight of the tube due to surface tension ? ($g=980$ cms. sec.$^{-2}$).

(London Higher School Certificate)

Here, clearly, the force of surface tension will act downward on the capillary tube along the inner as well as the outer circumference of its lower end dipping into the liquid, *i.e.*, along a length $2\pi\times\cdot25+2\pi\times\cdot05=2\pi(\cdot25+\cdot05)$.

$=2\pi\times\cdot30$ *cms.* $=\cdot60\pi$ *cms.*

$\therefore$ force acting downward on the tube due to surface tension $=\cdot60\pi\times T$ dynes $=\cdot60\pi\times40$ dynes.
$=\frac{\cdot60\pi\times40}{980}=\cdot07696$ *gm. wt.*

[$\because$ the external radius $=5/2$ $=2{\cdot}5$ *mms.* $=\cdot25$ *cm.* and the internal radius $=1/2=\cdot5$ *mm.* $=\cdot05$ *cm.*

Or, increase in apparent weight of the tube is equal to $\cdot07696$ *gm. wt.*

9. The stem of a common hydrometer is a circular cylinder of diameter 2 mms. It floats, with its stem wetted, in alcohol, whose specific gravity is ·796, and surface tension, 25·5 dynes/cm. Calculate how much deeper it floats than if alcohol had had zero surface tension.

(*Cambridge Higher School Certificate*)

Because of the downward force due to surface tension, the hydrometer goes deeper down into alcohol than it would otherwise do, such that the weight of the alcohol displaced by this *additional* immersion of the stem, or *the upthrust due to this displaced alcohol*, equal to the force of surface tension acting on it.

Now, diameter of the stem = 2 *mms.* and radius of the stem (r) = 1 *mm.* = ·1 *cm.*

And, $\therefore$ circumference of the stem = $2\pi r = 2\pi \times \cdot 1 = \cdot 2\pi$ *cm.*

So that, the surface tension acts along this length ; and, since its value is 25·5 *dynes/cm.*, the force on the stem due to surface tension

$$= 25 \cdot 5 \times \cdot 2\pi = 16\ 03 \text{ dynes.}$$

Let the stem go further down through a distance x *cms.* than it would do if alcohol had had *zero* surface tension. Then, additional volume of stem immersed, or the additional volume of alcohol displaced is equal to $\pi r^2.x$. *c.cs.*

$\therefore$ *mass* of this alcohol displaced = $\pi.r^2.x.\ \rho$ *gms.*

where ρ is the density of alcohol.

And $\therefore$ *weight* of alcohol displaced, or *upward thrust due to alcohol displaced*

$$= \pi r^2 \times x \times \rho \text{ gm. wt.} = \pi r^2 \times x \times \rho \times g \text{ dynes.}$$

$$= \pi \times (\cdot 1)^2 \times x \times \cdot 796 \times 981 \text{ dynes} = 24 \cdot 53\ x \text{ dynes.}$$

Since upward thrust due to displaced alcohol is equal to downward force due to surface tension, we have

$$24 \cdot 53x = 16 \cdot 03,$$

whence, $$x = 16 \cdot 03/24 \cdot 53 = \cdot 6530 \text{ cms.} = 6 \cdot 53 \text{ mms.}$$

Thus, the hydrometer floats deeper by 6·53 *mms* than it would, if alcohol had had zero surface tension.

10. Water rises to a height of 5·0 cms in a certain capillary tube. In the same tube the level of mercury surface is depressed by 1·54 cms. Compare the surface tensions of water and mercury, (the specific gravity of mercury is 13·6, the angle of contact for water is 0° and for mercury 130°).

(*London Inter-Science*)

We know that surface tension of a liquid is related to the rise or depression of a liquid in a capillary tube, density and radius of the tube, by the relation, $T = \dfrac{r.h\rho.g}{2.\cos\theta}$, where r is the radius of the tube, h, the rise or depression of the liquid, ρ, the density of the liquid and θ, the angle of contact for the liquid and the tube.

Let T_w be the surface tension of water and T_m that of mercury. Then, we have

$$T_m = \frac{r \times (-1 \cdot 54) \times 13 \cdot 6 \times g}{2 \cos 130^\circ},$$

$\because$ $h = 1 \cdot 54$ *cms.* (depression), $\rho = 13 \cdot 6$ *gms./c c.*, and $\theta = 130^\circ$.

Or, $$T_m = \frac{-1 \cdot 54 \times 13 \cdot 6 \times r \times g}{2 \times (-\cdot 6428)} = \frac{1 \cdot 54 \times 13 \cdot 6 \times r \times g}{2 \times \cdot 6428}.$$

And, $$T_w = 5 \times r \times 1 \times g/2 \times 1 = 5 \times r \times g/2,$$

$\because$ $h = 5$ *cm.*, $\rho = 1$ *gm./c.c.* and $\theta = 0^\circ$, so that $\cos\theta = 1$.

$\therefore$ $$\frac{T_m}{T_w} = \frac{1 \cdot 54 \times 1 \cdot 36 \times r \times g}{2 \times \cdot 6428} \times \frac{2}{5 \times r \times g}.$$

$$= \frac{1 \cdot 54 \times 13 \cdot 6}{\cdot 6428 \times 5} = 6 \cdot 516. \quad \text{Or,} \quad \frac{T_w}{T_m} = \frac{1}{6 \cdot 516}.$$

$\therefore$ the surface tensions of water and mercury are in the ratio, 1 : 6·516.

11. A U-tube is made up of two capillaries of bore 1 mm. and 2 mms. respectively. The tube is held vertically and partially filled with a liquid of

surface tension 49 dynes cm^{-1}. and zero contact angle. Calculate the density of the liquid, if the difference in the levels of the menisci is 1·25 cms.

(London Inter-Science)

We have the relation $T = r.h.\rho\, g/2$ for a liquid for which the angle of contact is *zero*, ($cos\ \theta$ being = 1).

Let h_1 and h_2 be the heights of the liquid columns in the two limbs, and r_1 and r_2, their respective radii. Then, we have

$$T = \frac{r_1.h_1\,\rho.g}{2}, \quad \text{Or,} \quad h_1 = \frac{2T}{r_1\rho.g}. \quad [\text{where } \rho = \text{density of the liquid.}$$

And, similarly, $T = \frac{r_2.h_2.\rho.g}{2}$. Or, $h_2 = \frac{2T}{r_2.\rho.g}$.

$$\therefore \quad (h_1 - h_2) = \left(\frac{2T}{r_1.\rho.g} - \frac{2T}{r_2.\rho.g}\right) = \frac{2T}{\rho.g} \times \left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Or, $$1{\cdot}25 = \frac{4\times 49}{\rho \times 981} \times \left(\frac{1}{{\cdot}05} - \frac{1}{0{\cdot}1}\right)$$

$\because$ $h_1 - h_2 = 1{\cdot}25$ *cms.*, $r_1 = {\cdot}5$ *mm.* $= {\cdot}05$ *cm.*

and $r_2 = 1$ *mm.* $= {\cdot}1$ *cm.*

Or, $$1{\cdot}25 = \frac{98}{\rho \times 981} \times (20-10). \quad \text{Or,} \quad 1{\cdot}25 = 980/\rho \times 981,$$

whence, $$\rho = \frac{980}{1{\cdot}25 \times 981} = {\cdot}7991\ gm./c.c.$$

Therefore, the density of the given liquid is ·7991 *gm./c.c.*

12. Find the difference in the levels of mercury in the two limbs of a U-tube, if the diameter of the bore of one limb is 1 mm. and of the other 8 mms. The surface tension of mercury is 440 c.g.s. units, its density 13·6 gms./c.c. and the angle of contact with the walls of the tube 140°.

(Joint Matriculation Board Higher School Certificate)

We have the relation, $T = \frac{r\,h\,\rho\,g}{2\ cos\ \theta}$, whence, $h = \frac{2T\ cos\ \theta}{r.\rho.g}$.

Let the depression of mercury in one limb be h_1 and that in the other, h_2.

Then, since $r_1 = {\cdot}5$ *mm.* $= {\cdot}05$ *cm.* and $r_2 = 4$ *mm.* $= {\cdot}4$ *cm.*, we have

$$h_1 = \frac{2\times 440 \times {\cdot}7660}{{\cdot}05 \times 13{\cdot}6 \times 981}. \qquad [\because\ cos\ 140° = -{\cdot}7660.$$

and $$h_2 = \frac{2\times 440 \times {\cdot}7660}{{\cdot}4 \times 13{\cdot}6 \times 981}.$$

$$\therefore \quad (h_1 - h_2) = \frac{2\times 440\times {\cdot}7660}{13{\cdot}6\times 981} \cdot \left(\frac{1}{{\cdot}05} - \frac{1}{{\cdot}4}\right) = \frac{2\times 440\times {\cdot}7660}{13{\cdot}6\times 981} \times \frac{35}{2}.$$

$$= \frac{440\times {\cdot}7660\times 35}{13{\cdot}6\times 981} = 0{\cdot}8843.$$

$\therefore$ the difference in the levels of the two columns is equal to 0·8843 *cm.*

13. What is the work done in blowing a soap bubble of radius 10 cms. ? (T = 30 dynes per cm.). What additional work will be performed in further blowing it, so that its radius becomes 15 cms ?

(*i*) We know that work done in blowing a soap bubble is equal to its surface area (*inner and outer*) × its free surface energy, *i.e.*, equal to its surface area × its surface tension, ($\because$ free surface energy = surface tension).

Here, surface area of the bubble $= 2\times 4\pi r^2 = 8\pi.10^2 = 800\pi$ *sq. cms.*

and $T = 30$ *dynes/cm.*

$\therefore$ work done in blowing the bubble of 10 *cms.* radius

$$= 800\pi \times 30 = 24000\pi = 7{\cdot}541 \times 10^4\ ergs.$$

(*ii*) Surface area of soap bubble, when its radius becomes 15 *cms.*

$$= 2\times 4\pi \times (15)^2 = 8\pi \times 225 = 1800\pi\ sq.\ cms.$$

$\therefore$ increase in area of the soap film $= (1800\pi - 800\pi) = 1000\pi$ *sq. cms.*

$\therefore$ work done in creating this new area of soap film

$$= 1000\pi \times 30 = 30{,}000\pi = 9{\cdot}426 \times 10^4 \text{ ergs.}$$

Hence, work done in increasing the size of the soap bubble from 10 *cms.* radius to 15 *cms.* radius is equal to $9{\cdot}426 \times 10^4$ *ergs.*

14. There is a pin hole, 0·1 mm. in diameter, at the bottom of a vessel containing mercury. What depth of mercury may be put in the vessel without any leak occurring? (Surface tension of mercury = 550 dynes/cm.$^{-1}$, and density = 13·6 gms./c.c. Neglect the angle of contact.)

We have to calculate, here, the length of mercury column that would be supported by the surface tension acting along the circumference of the hole.

Clearly, $$T = \frac{r.h.\rho.g}{2\cos\theta},$$

where h is the height of mercury column without any leak occurring.

Or, since $\theta = 0°$, and $\therefore \cos\theta = 1$,

we have $$T = \frac{r.h.\rho.g}{2}, \quad \text{whence,} \quad h = \frac{2T}{r.\rho.g}.$$

Here, $T = 550$ *dynes/cm.*, $r = \cdot 1/2 = 0{\cdot}5$ *mm.* $= \cdot 005$ *cm.* and $\rho = 13{\cdot}6$ *gms./c.c.*

$\therefore$ $$h = \frac{2 \times 550}{\cdot 005 \times 13{\cdot}6 \times 981} = 16{\cdot}49 \text{ cms.}$$

$\therefore$ mercury can be poured into the vessel to a depth of 16·49 *cms., without any leak occurring.*

15. The pressure inside a soap bubble of radius 1 cm. balances a 1·4 mm. column of oil, of specific gravity 0·80. Calculate the surface tension of the soap solution.

We know that the excess pressure inside a soap bubble is equal to $4T/r$, where T is the surface tension of the soap solution and r, the radius of the bubble. Since this is balanced by the column of oil, we have

$$4T/r = h.\rho.g,$$

where h is the length of the oil column, ρ, the density of oil, and g, the value of acceleration due to gravity.

$\therefore$ $$\frac{4T}{1} = \cdot 14 \times 80 \times 981.$$

$\left.\begin{array}{l} \because\ r = 1 \text{ cm.} \\ h = 1{\cdot}4 \text{ mm.} \\ \quad = \cdot 14 \text{ cm.} \\ \text{and } \rho = \cdot 80 \text{ gm./c.c.} \end{array}\right\}$

Or, $$T = \frac{\cdot 14 \times 80 \times 981}{4}$$

$$= \cdot 14 \times \cdot 2 \times 981 = 27{\cdot}47 \text{ dynes/cm.}$$

Thus, the surface tension of the soap solution is 27·47 *dynes/cm.*

EXERCISE XIII

1. Show that the surface tension of a liquid is equal to the mechanical part of its surface energy. Calculate the work done on the film in blowing a soap bubble from a diameter of 4 *cms.* to one of 30 *cms.*, if its surface tension be 45 in *c.g.s. units.* **Ans.** $2{\cdot}5 \times 10^5$ *ergs.*

2. Show that the excess pressure inside a soap bubble of radius r over the atmospheric pressure outside it is equal to $4T/r$, where T is the surface tension of the soap solution.

How may the surface tension of a bubble be determined?

3. Define surface tension, and show how it can be regarded as involving potential energy. From consideration of energy show that (*i*) oil will spread on the surface of water, (*ii*) mercury will collect on a clean glass plate in drops of different shapes according to their size, and (*iii*) water will rise in a capillary tube. (*Bombay*)

4. What would be the pressure inside a small air bubble of 1 *mm.* radius, situated just below the surface of water? Surface tension of water may be taken to be 70 *dynes/cm.* and the atmospheric pressure to be $1{\cdot}012 \times 10^6$ *dynes/cm*^{2}. **Ans.** $1\,027 \times 10^6$ *dynes/cm*^{2}

5. Calculate the loss of enerygy if 1000 drops of water, each of diameter ·2 *mm.* coalesce to form one large drop. The surface tension of water is equal to 72 *dynes/cm.* Ans. 8143 *ergs*

6. Calculate the work done in breaking up a drop of petrol, of volume 1 *c.c.* into a thousand million drops. The surface tension of petrol is 26 *dynes/cm* Ans. 125574 *ergs*

7. Calculate the amount of energy needed to break a drop of water 2 *mm.* in diameter, into 10^9 droplets of equal size, taking surface tension of water as 73 *dynes/cm.* (*Madras*)
Ans. 9160 *ergs.*

8. Define surface tension. Show that the excess pressure acting on the curved surface of a curved membrane is given by

$$P = 2S\left[\frac{1}{r_1}+\frac{1}{r_2}\right],$$

where r_1 and r_2 are the radii of curvature and S, the surface tension of the membrane. (*Punjab*)

9. Calculate the amount of energy evolved when eight droplets of water (surface tension 72 *dynes per cm.*), of radius 1/2 *mm.* each, combine into one. (*Punjab*)
Ans. 9·05 *ergs.*

10. A soap bubble is spherical in shape, and has a diameter of 10 *cms.* ; if the surface tension of the surface separating soap solution and air is 40 *c.g.s. units*, what is the excess pressure of the air in the bubble over the atmospheric pressure ? Ans. 32 *dynes/cm*^{2}.

11. Find in the terms of mercury column the excess pressure inside a rain drop 1 *mm.* in diameter, for which $T = 74$ *dynes/cm.* Ans. 2·22 *mms* of mercury.

12. Calculate the force required to separate two plates of glass, of area 10 *sq. cms.* each with a layer of water ·001 *mm.*, thick in-between them. The surface tension of water = 72 *dynes/cm.* Ans. $1{\cdot}44\times10^7$ *dynes.*

13. Describe a method of determining the surface tension of a soap bubble Deduce the formula used.

The equal spherical soap bubbles coalesce ; if V is the consequent change in volume of the contained air and S, the change in the total surface area, show that $3PV = 4ST$, where T is the surface tension of the soap bubble and P, the atmospheric pressure. (*Allahabad*)

14. If a number of little droplets of water, all of the same radius r *cm.*, coalesce to form a single drop of radius R *cm.* show that the rise of temperature of water will be given by $\frac{3S}{J}\left(\frac{1}{r}-\frac{1}{R}\right)$, where S is the surface tension of water and J, the mechanical equivalent of heat. (*Saugar*)

15. Find the relation between the radius of (*a*) a spherical drop, (*b*) a spherical bubble of a liquid, the surface tension and pressure.

Two soap bubbles of radii 2 and 3 *cms.* coalesce into a single bubble of radius R *cms.* If the surface tension of the soap solution is 25 *dynes per cm.* and the atmospheric pressure is 76 *cm.* of mercury, (whose density = 13·6 *gm./c.c.*), find the equation to determine R. (*Madras*)
Ans. $1{\cdot}014\times10^4.(R^3-35)+(R^3-13)=0.$

16. A soap film, ·001 *mm.* thick, and at 0°*C*, is stretched *adiabatically*, until its area is doubled. What is the resulting fall of temperature, if we assume that the specific heat and density of the film are both unity, that $dT/d\theta$ at 0°C $=-{\cdot}15$ *dyne per cm. per* 1°*C*, and that $J = 4{\cdot}2\times10^7$ *ergs per calorie* ?
Ans. 0195°*C*

17. Explain the terms surface tension and angle of contact. Show that the pressure inside a spherical bubble of radius r exceeds that outside it by $4T/r$. If this excess pressure is balanced by that due to a column of oil, (sp. gr. 0·8), 2 *mm.* high, when $r = 1{\cdot}0$ *cm.*, find the surface tension of the soap-bubble. (*Bombay*)
Ans. 39·24 *dynes per cm.*

18. Calculate the difference of pressure between the inside and outside of a spherical bubble blown inside a liquid. (*Agra*)

19. A soap film is formed on a rectangular frame-work of length 7 *cms.*, dipping into a soap solution. This frame-work hangs from the arm of a balance. An extra weight of 0·38 *gm.* must be placed on the opposite pan to balance the pull of the film. What is the surface tension of the soap solution ?
Ans. 26·6 *dynes/cm.*

20. Calculate the density of a liquid, a column 2·09 *cms.* of which balances the excess pressure inside a soap bubble of radius 1 *mm.* Surface tension for the soap solution may be taken to be 40 *dynes/cm.* Ans. ·78 *gm./c.c.*

21. Describe and explain how the surface tension of a liquid may be measured by forcing bubbles of air through it. Discuss whether the result obtained this way should be the same as that given by the capillary tube method.

What would be the pressure inside a small air bubble of 0·1 *mm.* radius, situated just below the surface ? Surface tension of water = 72 *dynes per cm.* and atmospheric pressure = $1{\cdot}013 \times 10^6$ *dynes per sq. cm.* (*Bombay*)
Ans. $1{\cdot}0274 \times 10^6$ *dynes/cm*^{2}.

22. Show that the excess pressure inside a drop of water, of diameter 1/100th of a *mm.* is ·0137 *cm.* of mercury column at 0°*C*. Density of water at *C* = 1 *gm./c.c.*, density of water vapour at 0°*C* and 760 *mm.* pressure = $\cdot 61 \times 10^{-3}$ *m./c.c.*, and surface tension for water = 75 *dynes/cm.*

23. Explain the method of finding the surface tension of a liquid by the method of drops.

A drop of water, 0·5 *cm.* radius, is split into 1000 tiny drops. Find the mechanical work expended. Calculate the pressure inside one of these small drops (surface tension of water = 75 *dynes per cm*) (*Madras*)
Ans. ·675 π *ergs* ; (P + 3000) *dynes/cm*^{2}., (where P is the atmospheric pressure).

24. In a drop-weight determination of the surface tension between water and chloroform, a glass tube of 4 *mm.* external diameter was used and 50 drops of chloroform, density 1·5 *gm. per c.c.*, were allowed to fall in the water. The weight of these drops was 3·43 *gms.* Find the interfacial surface tension.
Ans. 29·5 *dynes per cm.*

25. Define surface tension. Show how it is related to surface energy in a liquid.

Two glass plates are pressed together with a very thin film of water between them. Explain clearly why the two plates firmly adhere to each other.
(*Agra, 1929*)

26. (*a*) Define surface energy. Give the theory and practice of the method of determining the surface tension of a liquid by weighing drops. (*Punjab*)

(*b*) If a globe of water of diameter 2 *cms.* suddenly splits into 100 equal globules under isothermal conditions, determine the gain in surface energy which occurs, given that the surface tension of water is 75 *dynes per cm.* (*A.M.I.E.1950*)
Ans. 3432 *ergs.*

27. Explain clearly from where the energy comes when a liquid rises against gravity in a capillary tube. Derive an expression for the height h through which the liquid of surface tension T will rise in a capillary tube of radius r. What will happen if the length of the tube is smaller than h ?
(*Punjab*)

28. Describe *Jaeger's* maximum bubble pressure method of determining surface tension.

A large soap bubble of radius R contains inside it a smaller soap bubble of radius r. If the smaller bubble now bursts *isothermally*, with no leakage of air from the system as a whole, so that a new bubble of radius R' is formed, show that the radii of the three bubbles are connected by the relation $P(R'^3 - R^3) + 4T(R'^2 - r^2 - R'^2) = 0$, where P stands for the atmospheric pressure, and T, for the surface tension of the soap solution

29. Give the theory and experimental details of a method for determining the surface tension of mercury, and the angle of contact for mercury and glass.

Calculate the work done in spraying a spherical drop of mercury, of one millimetre radius, into a million droplets, of the same size, the surface tension of mercury being 550 *dynes/cm.* (*Agra*)
Ans. 6839 *ergs.*

30. Define surface tension and angle of contact.

Describe Quincke's drop method of measuring these quantities for mercury in contact with glass. Give the theory of the method. (*Banaras*)

31. What forces determine the shape of a liquid drop on a horizontal plate? Show that all large drops of mercury resting on a clean glass plate will have the same height.

32. Describe the laboratory method of finding the surface tension of a liquid by the rise of the liquid in a capillary tube.

A tube of conical bore is dipped into water with apex upwards. The length of the tube is 20 *cms.* and the radii at the upper and lower ends are 0·1 and 0·3 *cm.* Find the height to which the liquid rises in the tube. (Surface tension of water = 80 *dynes/cm.*) (*Allahabad*)

Ans. 0·55 *cm.*

33. Show how the existence of an acute angle of contact and of a pressure difference due to curvature accounts for the rise of liquid in a capillary tube.

A capillary tube of internal diameter 1 *mm.* and external diameter 5 *mm.* hangs vertically from the arm of a balance, the lower end of the tube being in water (angle of contact = 0°). Calculate the change in the apparent weight of the tube due to surface tension. (Surface tension of water = 78 *dynes/cm.* and $g = 980$ *cm./sec*^{2}.). (*Allahabad*)

Ans. 0·15 *gm.*

34. Describe an experiment for determining the surface tension of a liquid. Deduce the formula used. (*Punjab*)

35. Explain the capillary tube method of determining the surface tension of a liquid. Why is this method not suitable for temperatures other than that of the surroundings? Suggest and explain other methods for doing so, in which the surface tension may be calculated by measuring the pressure necessary to force the meniscus back in a level with the surrounding liquid.

36 A capillary tube is dipped in water. Water rises to a height of 4 *cms.* above the surrounding liquid. If the angle of contact is zero and the radius of the tube is 0·1 *mm.*, what is the surface tension of the liquid?

(*A.M.I E., 1961*)

Ans. 19·63 *dynes/cm.*

37. A verticle U-tube containing mercury has one limb of diameter 5 *cms.* and the other, of diameter ·1 *cm.* Calculate the difference in level of the mercury columns in the two limbs. (T for mercury = 550 *dynes/cm.*, density of mercury = 13·6 *gms. per c.c.* and its angle of contact with the walls of the tube = 140°). Ans. 1·04 *cms.*

38. A liquid of density 1 05 *gms. /c.c.* and angle of contact 20° has a vertical capillary tube of 2 *mm.* diameter dipping into it. If the surface tension of the liquid be 23·5 *dynes/cm.*, find the rise of the liquid in the capillary tube.

Ans. 4·29 *cms*

39. The surface tension of water is 72 *dynes/cm.* Calculate how far water will rise up a circular tube, 2 *cms* in diameter. Ans. 1·47 *cms.*

40 A U-tube, whose ends are open and whose limbs are vertical, contains oil of sp. gravity 0·85 and surface tension 28 *dynes/cm.* If one limb has a diameter of 2·2 *mm.* and the other a diameter of 0 8 *mm.*, what is the difference in level of the oil in the two limbs? Assume the angle of contact between the oil and the glass to be zero. (*A.M.I Mech Engineering*)

Ans. 1·07 *cms*

41. Define surface tension. Explain how you can determine the surface tension of a solution with the help of a bubble blown out of it.

The limbs of a capillary U-tube have the internal diameters of 1 *mm.* and 2 *mm.* The tube is held vertically and is partially filled with a liquid of surface tension 50 *dynes/cm* Find the density of the liquid if the difference of levels in the two limbs is 1·25 *cms.* Assume that the angle of contact is zero.

(*Bombay*)

Ans. ·8155 *gm./c c.*

42. Deduce the relation between surface tension and vapour pressure at a curved surface and discuss its effects on evaporation and condensation

(*Bombay, 1946 and 1948*)

43. Calculate the difference in vapour pressure of water for a plane surface and that for a drop of radius 0·1 *mm*. *Density of water vapour* $= 6{\cdot}1 \times 10^{-4}$ *gms./c.c.* and *surface tension* of water = 75 *dynes/cm.* (*Bombay*)
Ans 9·15 *dynes/cm*^{2}

44. Show that the maximum vapour pressure over a curved liquid surface differs from that over a plane surface. Deduce the value of this difference in terms of the densities of the liquid and the vapour, the radius of curvature of the surface and the surface tension of the liquid. Discuss the bearing of the above fact on the formation of rain drops. (*Madras*)

45. Show that the vapour pressure over the curved surface of a liquid drop of radius r exceeds that over a flat surface by an amount equal to $2T/\sigma r.(\rho - \sigma$ where T is the surface tension of the liquid, ρ, its density and σ, the density of its vapour. Discuss the application of this result to the condensation of super-saturated vapour on dust particles and other nuclei. (*Bombay*)

46. If the aqueous vapour tension at 0°*C* be 4·6 *mm*. of mercury, calculate the radius of a water drop, at 0°*C*, which would be in equilibrium with its vapour at twice this pressure. (Surface tension of water = 75 *dynes per cm*., 18 *gms*. of water at *N.T.P.* measure 22·4 *litres*). Ans. $2{\cdot}42 \times 10^{-6}$*cm.*

47. A capillary tube is immersed in water and, by exerting a pressure of 15·65 *cm*. of water, the meniscus in the tube was kept 1·25 *cm*. below the surface of the water outside. The radius of the tube was 0·104 *cm*. Calculate the surface tension of water. Ans. 73·5 *dynes/cm*.

48. Explain concisely, but clearly, the following :

(*i*) pieces of camphor scurry about on the surface of water but their motion slows down if we immerse our figure in the water ;

(*ii*) it is difficult to introduce mercury in a fine glass tube ; see 400

(*iii*) small pieces of cork and straw etc., floating on water, appear to attract each other ;

(*iv*) small drops of mercury on a plane glass sheet are spherical in shape, but large drops of it are flat at the top ;

(*v*) ploughing of fields helps retain moisture in them.

49. Distinguish between a wave and a ripple. How may the surface tension of a liquid be determined by the method of ripples ?

50. What is the effect of temperature on the surface tension of a liquid ? Describe how it may be studied experimentally.

CHAPTER XV

GASES—KINETIC THEORY

261. The Kinetic Theory—Introduction. In order to connect and co-relate the many facts accumulated by experience, some hypothesis or theory became necessary, and the *kinetic theory of matter* is the one that has proved most helpful for the purpose. The distinction of being the founder of the modern Kinetic Theory goes, by common consent, to **Daniel Bernoulli** (1730), as he was the first to explain Boyle's law on its basis, though the theory may be said to have had its beginnings in the speculations and ideas of *Thales of Miletus* (640 to 547 *B.C.*) about the possible structure of matter, and a host of other early workers. And, the credit for having established it on a firm *mathematical* basis is due to **Claussius** and **Maxwell**, in whose hands it attained its present form. The triumph of the theory lies in its success in explaining known results and predicting new ones.

The theory is based on two assumptions :

(1) That *matter is not continuous*, but consists of small aggregations or lumps, called *molecules*, very much like a handful of sand, composed of fine granules ; so that, even when the molecules are in contact with each other, there are *inter-spaces* in-between them.

The molecule of a substance is the smallest part of it that possesses the characteristic properties of that substance, and can have an independent existence of its own. It can be broken up into smaller bits, called *atoms*, by various methods, but then, these no longer exhibit the distinctive properties of that substance. A molecule may consist of one or more atoms, *e.g.*, the molecule of Helium is monoatomic, *i.e.*, consists of only one atom, that of Hydrogen or Oxygen is diatomic, *i.e.*, consists of two atoms, and that of Carbon dioxide is triatomic, *i.e.*, consists of three atoms, and so on.

It should be noted with care that although there may be any number of different kinds of molecules, there can be only a hundred and odd different kinds of atoms (including isotopes), corresponding to the different elements. These atoms are the smallest particles that can take part in chemical reactions.

(2) That the molecules are generally not in contact with each other but are in a *continuous state of agitation*, moving about with great speed, haphazardly, in all directions,—their freedom of movement, however, being different in the three states of matter, *viz.*, the solid, the liquid and the gaseous.

Now, there is a huge mass of evidence in favour of both these assumptions, *e.g.*, (*a*) the phenomena of *diffusion* and *solution*, which clearly suggest the molecular structure of matter and agitation of molecules ; for, we find heavier gases, like Carbon dioxide, diffusing

into lighter ones, like Hydrogen, in defiance of gravity, as it were, and even a gas diffusing into lead, (*b*) the familiar phenomenon of *expansibility* of gases, which is clearly due to the tendency of the molecules to fly away from each other, (*c*) the phenomena of *vapour pressure* and *evaporation*, and (*d*) phenomenon of **Brownian movement**, which is perhaps the one single experimental fact, which enables us to actually see, so vividly, the molecules of a substance moving about hither and thither before our very eyes.

This last phenomenon derives its name from **Robert Brown**, an English Botanist, who first discovered it in 1827 while observing, under a high-power microscope, the suspension of plant pollen in water, when he found the pollen grains dancing about in the wildest manner, and thought that they were perhaps very tiny living creatures. The phenomenon has been fully investigated by **Parrin** (in (1908) and may be readily observed by examining a colloidal suspension under an ultra-microscope, when a strong beam of light is passed through the liquid. The particles, as small in size as 6×10^{-7} *cms.*, which just appear as mere points of light, surrounded by diffraction rings, are visible by the light, they scatter at right angles to the beam, and are seen in a spontaneous and eternal dance of a most irregular and haphazard manner,—now spinning, now resting, and now rising again, and so on and on.

Now, this mad movement of the molecules is the clearest proof of *molecular agitation*. For, the movement of the tiny particles is due to the large number of molecular impacts they receive simultaneously on all sides. Since these impacts are not necessarily uniformly distributed, there is a resultant unbalanced force on the tiny particles which causes them to move. And, because the force varies most haphazardly and irregularly, the motion of the particles also exhibits the same haphazardness and irregularity. Quite obviously, the smaller the particles, the more readily are they subject to these irregular motions ; for, on a large or a heavy particle, the impacts will almost balance and no resultant motion will ensue, *e.g.*, when a large body like a glass bead, or a marble piece, is immersed in water or any other liquid, it is not tossed about in this manner.

262. Kinetic Theory of Gases. Whereas the kinetic theory of solids and liquids is still in a formative stage, the kinetic theory of gases has made rapid strides and can fully explain the various properties of gases. The reason is not far to seek and will be clear if we try to picture to ourselves the structure of a solid, a liquid and a gas.

(*i*) In the case of a *solid*, due to the great force of cohesion, its molecules are all compactly or closely packed and every molecule is more or less *fixed in its position, having only the freedom to vibrate about this position.* It is not free to move over the whole volume of the solid, much less to escape away from it.

(*ii*) In the case of a *liquid*, the cohesive force is still there, but is not so strong, with the result that although it is sufficient to hold the liquid together and to give it a *free surface*, it cannot prevent a molecule from roaming over the whole volume of the liquid, with the result that while *a liquid has a definite volume and a well defined free surface, it has no shape of its own.* The average distance between two

molecules of a liquid is estimated to be more or less of the same order as the size of its molecule.

(*iii*) In the case of a *gas*, on the other hand, these *cohesive forces are almost negligible* under ordinary conditions of temperature and pressure, so that the molecules lie far apart from each other, the average distance between two molecules being about a hundred times the size of a molecule. A molecule is, therefore, free to wander over the entire space available to it, *with the result that a gas has neither a volume nor a free surface of its own.*

It will thus be clear from the above why the problem becomes simpler in the case of a gas, compared with that in the case of a solid or a liquid. It is obviously so, because in the case of a gas, (*i*) *the molecules lie far apart and we can, therefore, neglect, under ordinary conditions, the actual volume occupied by the molecules themselves,* (if compactly arranged), *compared with that occupied by the gas, as a whole.* Clearly, this cannot be done in the case of a liquid or a solid, (*ii*) *the molecular forces* can be neglected because of the large distance between the molecules.*

Now, in order to further simplify our problem, we shall first consider, here, a gas whose molecules have negligible size, *i.e.*, whose molecules are mere *mass-points*, and in which the molecular forces are also negligible. *Such a gas, with zero molecular size and zero molecular forces, is called an ideal or a* **perfect gas**. No such gas, however, exists in reality, and the properties of an actual or a **real** gas only approximate to those of *this* ideal gas.

263. Pressure exerted by a perfect Gas. Due to the constant random motion of the molecules of a gas and their high speed, they *'bombard'* the walls of the containing vessel, and thus exert pressure. To calculate this pressure exerted by a gas, we make the following further assumptions, to simplify matters.

(1) That the molecules of gas are *all alike*, (though different from those of another), and are *perfectly elastic spheres*, and that there is *no force of attraction or repulsion between them, or between them and the walls of the containing vessel.* In other words, that *all their energy is kinetic and that they do not suffer any loss of momentum, or kinetic energy, on a direct or 'head on' impact with the walls of the vessel, only their direction of motion being reversed.*

(2) That *large numbers of molecules exist in the smallest volume of a gas* with which we can deal. Under ordinary conditions of temperature and pressure, the number of molecules present in 1 *c.c.* of a gas is estimated to be of the order of 10^{19}, *their size or diameter being very small compared with the inter-molecular space,*—the actual volume occupied by the molecules being only about 3 *c.cs.* in 10 *litres* of the gas.

(3) That due to their large number and ceaseless haphazard motion in all directions, with velocities, ranging from *zero to infinite, there are frequent collisions against one another and consequent changes in the direction and magnitude of their velocities at each colli-*

* These molecular forces are entirely different from Newton's gravitational forces. They are electrical in nature and *do not obey the ordinary gravitational inverse square law*, as has been indicated already.

sion ; and that this does not affect the molecular density of the gas, i.e., the molecules do not, in the steady state, collect more at one place than at another.

(4) That the molecules, being material bodies, subject to the laws of motion, *move in straight lines with uniform velocity between any two collisions.*

(5) *That the time for which a collision lasts is small compared with the time-interval between two collisions, or, the time taken to cover the distance traversed between two collisions.* This distance is called the *mean free path of the molecule* and depends upon the temperature and pressure of the gas.

We are now in a position to calculate the pressure exerted by an *ideal or a perfect gas* on the walls of the containing vessel.

Let there be a gas, enclosed in a cubical vessel *of unit edge, with its walls perfectly elastic,* (Fig. 318), and let the number of molecules present in it be n, the mass of each molecule being m.

Since the molecules are constantly moving about with different velocities in different directions, they are bombarding the walls of the cube. Consider one molecule m, having a velocity c_1 at a given instant. This velocity of the molecule may be resolved into three *rectangular components,* u_1, v_1 *and* w_1, *along the axes of* x, y, *and* z *parallel to the three edges of the cubical vessel respectively.* Then, clearly,

$$c_1^2 = u_1^2 + v_1^2 + w_1^2.$$

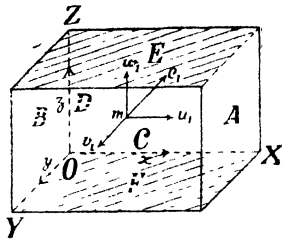


Fig. 318.

Now, consider the motion of this molecule along the axis of x, *i.e.*, perpendicular to the walls A and B of the vessel ; so that, striking the wall A with velocity u_1, it rebounds with the *same* velocity in the opposite direction (the molecule as well as the wall being perfectly elastic), strikes wall B, rebounds back to A and so on.

Clearly, the momentum of the molecule as it approaches A is mu_1 and is obviously directed towards A. When, however, it collides against A, it rebounds *with only its velocity reversed* ; so that, *its momentum still remains the same in magnitude but is now directed oppositely, i.e.,* is now $-mu_1$.

Therefore, *change in its momentum* $= mu_1 - (-mu_1) = 2mu_1$.

This, then, is the change in momentum of the molecule at *each* collision with wall A and, clearly, as it collides against the wall, it imparts *this* momentum ($2mu_1$) to the wall.

Hence, *momentum imparted by the molecule to wall A per second* $= 2mu_1 \times$ *number of collisions it makes with the wall per second.*

Now, for each successive collision or impact with wall A, the molecule must traverse the distance from A to B and back, *i.e.*, a distance equal to twice the length of the *unit* cube, or a distance of 2 *cms.* Since it covers a distance u_1, along this direction, in 1 *second,* it will cover the distance 2 *cms.* (from A to B and back) in $2/u_1$

seconds ; and, therefore, *the number of collisions it will have with the wall in one second is equal to* $u_1/2$.

So that, *momentum imparted to wall A by the molecule, per second*

$$= 2mu_1 \times u_1/2 = mu_1^2.$$

In other words, this is the force f_1, say, exerted by the molecule on wall A ; for, in accordance with Newton's second law of motion, *force is just rate of change of momentum.* Thus,

$$f_1 = mu_1^2.$$

Similarly, forces $f_2, f_3, \ldots\ldots\ldots, f_n$, exerted on wall A by other molecules, having velocities $c_2, c_3 \ldots\ldots\ldots\ldots, c_n$, and rectangular components $(u_1, v_1, w_1), (u_2, v_2, w_2), \ldots\ldots (u_n, v_n, w_n)$, along the three axes respectively, are given by

$$f_2 = mu_2^2, \quad f_3 = mu_3^2 \ldots, \quad f_n = mu_n^2,$$

the mass of each molecule being the same m, in each case.

Thus, *total force on wall A due to all the n molecules* is given by

$$\begin{aligned} F_A &= f_1 + f_2 + f_3 + \ldots\ldots + f_n. \\ &= mu_1^2 + mu_2^2 + mu_3^2 + \ldots\ldots + mu_n^2. \\ &= m(u_1^2 + n_2^2 + u_3^2 + \ldots\ldots u_n^2). \end{aligned}$$

Now, area of wall $A = 1 \times 1 = 1$ *sq. cm.* [$\because$ each edge of the cube $= 1$ *cm.*

So that, F_A is the force exerted by the molecules *on unit area* and is thus equal to pressure P_1 exerted by them on A, because, as we know, *pressure is equal to force per unit area.* Thus, pressure P_1 on wall $A = F_A$.

Or,
$$P_1 = m(u_1^2 + u_2^2 + u_3^2 + \ldots\ldots u_n^2).$$

Exactly in the same manner, considering the motion of the molecules along the axes of y and z, —*i.e.*, perpendicular to walls C and D and walls E and F of the cubical vessel respectively,—we have *pressure exerted by the molecules on wall C* given by

$$P_2 = m(v_1^2 + v_2^2 + v_3^2 + \ldots\ldots + v_n^2),$$

and *pressure exerted by the molecules on wall E* given by

$$P_3 = m(w_1^2 + w_2^2 + w_3^2 + \ldots\ldots + w_n^2).$$

Since the pressure exerted by the gas is the *same in all directions,* we have $P_1 = P_2 = P_3 = P$, say.

And, therefore,
$$P_1 + P_2 + P_3 = 3P.$$

So that,

$$3P = m(u_1^2 + u_2^2 + \ldots u_n^2) + m(v_1^2 + v_2^2 + \ldots v_n^2) + m(w_1^2 + w_2^2 + \ldots w_n^2).$$

Or, $P = \frac{1}{3}m[u_1^2 + u_2^2 + \ldots u_n^2) + (v_1^2 + v_2^2 + \ldots v_n^2) + (w_1^2 + w_2^2 + \ldots w_n^2)$

Or,
$$P = \tfrac{1}{3}m(c_1^2 + c_2^2 + \ldots c_n^2).$$

Now,
$$\frac{(c_1^2 + c_2^2 + \ldots + c_n^2)}{n} = \bar{c}^2,$$

where the symbol $\bar{c}$ stands for the **root mean square velocity** of the molecules*, usually written, for brevity, as *r.m.s. velocity.*

So that,
$$(\bar{c}_1^2 + \bar{c}_2^2 + \ldots + \bar{c}_n^2) = n\bar{c}^2.$$

And, therefore,
$$P = \tfrac{1}{3}mn\bar{c}^2. \qquad \ldots(i)$$

Thus, the pressure exerted by the gas enclosed in the vessel is equal to $\frac{1}{3}mn\bar{c}^2$.

*i.e., the square root of the mean square velocity of the molecules.

264. The Value of $\bar{c}$. Now, in relation (*i*) above, we have

$m.n$ = mass of the molecules in *unit* volume of the gas.

Or, = *mass per unit volume of the gas = density of the gas*, ρ.

Substituting this value of m, n in relation (*i*) for P, we have

$$P = \tfrac{1}{3}\rho.\bar{c}^2, \text{ whence, } \bar{c}^2 = 3P/\rho.$$

Or,
$$\bar{c} = \sqrt{3P/\rho}.$$

This enables us to determine the value of $\bar{c}$ for any particular gas, at a given temperature, if P and ρ for it be known.

Thus, for example, taking P for air, at $0°C$, to be $= 76\times13{\cdot}6\times98$ *dynes/cm²*.
$= 10^6$ *dynes/cm²*. (approx.),

and ρ for air, at $0°C$ = 1/1000*th* of that of water = $1/1000 = 10^{-3}$ *gms./c.c.*,

we have
$$\bar{c} = \sqrt{\frac{3\times10^6}{10^{-3}}} \text{ cms./sec.} = \sqrt{3\times10^6\times10^3}.$$
$$= \sqrt{3\times10^9} = \sqrt{30\times10^8} = 5{\cdot}5\times10^4 \text{ cms./sec.}$$
$$= 5{\cdot}5\times10^4/100 = 5{\cdot}5\times10^2 = 550 \text{ metres/sec.}$$
$$= \tfrac{1}{2} \text{ mile/sec.} = \textit{the velocity of a rifle bullet.}$$

We thus see how tremendous is the speed with which the molecules of the gas move about haphazardly in any given volume of it, and, obviously, *the lighter the molecule, the faster it moves.*

The velocity of the molecule of any other gas may also be calculated in the same manner, at any desired temperature.

265. Relation between $\bar{c}$ and T. We have the experimental *gas equation*, connecting *Boyle's law* and *Charles' law, viz.*,

$$PV = RT,$$

where R is the *gas constant*, T, *the absolute temperature of the gas* and P and V, its *pressure* and *volume*, respectively.

Now,
$$P = \tfrac{1}{3}\, m.n.\bar{c}^2.$$

$\therefore$
$$PV = \tfrac{1}{3}\, m.n.V.\bar{c}^2.$$

Or, putting $nV = N$*, the total number of molecules in the volume V of the gas, we have
$$PV = \tfrac{1}{3}m\ N.\ \bar{c}^2.$$

Or,
$$\tfrac{1}{3}m.N.\bar{c}^2 = RT, \text{ whence, } \bar{c}^2 = 3RT/mN.$$

Or,
$$\bar{c}^2 \propto T/m. \qquad [\because R \text{ and } N \text{ are constants.}$$

Thus, *absolute temperature of a gas is directly proportional to the square of the r.m.s velocity of its molecules*,—the greater the velocity (and, therefore, the greater the kinetic energy) of the molecules, the greater the temperature of the gas.

266. Deduction of Gas Laws on the basis of the Kinetic Theory.—

1. Boyle's Law. We have the relation, $P = \tfrac{1}{3}m.n.\bar{c}^2$.

Multiplying it by V, the volume of the gas, we have

$$PV = \tfrac{1}{3}m.n.V.\bar{c}^2 = \tfrac{1}{3}m.N.\bar{c}^2.$$

*For the sake of simplicity, N is taken to be the total number of molecules in a *gram-molecule* of the gas ; for, at the same temperature and pressure, its value will be the *same* for *all* gases. It is called the **Avogadro Number**, and its value is found to be $6{\cdot}0\times10^{23}$.

where $nV = N$, the total number of molecules in the given volume V of the gas.

Here, m and N being constants, it follows that $PV \propto \bar{c}^2$.

Or, PV = a constant, if $\bar{c}$ be constant.

Now, $\bar{c}$ is constant, if T, (the temperature of the gas) be constant. So that, *at cnstant temperature, PV = a constant*, which is our *Boyle's Law.*

Alternative Deduction of Boyle's Law.—A more rigid derivation of the law from the kinetic theory is the following :

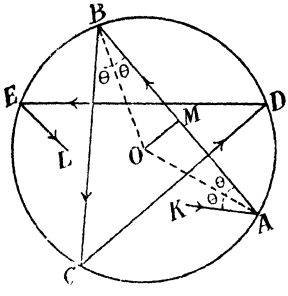


Fig. 319.

Let N molecules of a gas, each of mass m, be enclosed in a spherical vessel of radius r and volume V ; so that, $V = \frac{4}{3}\pi r^3$.

Consider one molecule striking the wall of the vessel at A with a velocity, (*i.e., the root mean square velocity*), $\bar{c}$ at an angle θ with the normal at A, (Fig. 319). Let the molecule then follow the path $ABCDE$. The component of the velocity $\bar{c}$ perpendicular to the wall is clearly $\bar{c} \cos \theta$, and gets *reversed*, the other component $\bar{c} \sin \theta$, at right angles to it, remaining unchanged. The change of momentum at A is, therefore, equal to $2 m \bar{c}. \cos \theta$, *inwards.*

Now, the distance travelled by the molecule between collisions at A and B is equal to $AB = 2AM = 2AO. \cos \theta = 2 r \cos \theta$.

And, therefore, the time taken by the molecule to travel *this* distance is equal to $2r. \cos \theta/\bar{c}$ *secs.* ; so that, the number of collisions in *one* second $= \bar{c}/2r \cos \theta$.

$\therefore$ change of momentum is *one second*; or, the *rate of change of momentum* due to *one* molecule is

$$= \frac{2m. \bar{c} \cos \theta \times \bar{c}}{2r \cos \theta} = \frac{m\bar{c}^2}{r} \text{ per second.}$$

Since there are N molecules in all, the total rate of change of momentum $= m.N.\bar{c}^2/r$, *inwards.*

Again, since the rate of change of momentum is equal to force, the force exerted by the molecules on the walls of the vessel, *outwards*

$$= m.N.\bar{c}^2/r.$$

Now, the area of the vessel $= 4\pi r^2$. [Because it is spherical.

$\therefore$ force exerted by the molecules per unit area of the vessel

$$= \frac{m.N.\bar{c}^2}{r.4\pi r^2} = \frac{m.N.\bar{c}^2}{4\pi r^3}.$$

Or, *pressure P exerted by the molecules, on the wall of the vessel given by* $P = m.N\bar{c}^2/4\pi r^3$. [$\because$ force per unit area = pressure.

Now, volume of the vessel, $V = 4\pi r^3/3$,

whence, $4\pi r^3 = 3V$.

Substituting this value of $4\pi r^3$ in the expression for P above, we have $P = m.N.\bar{c}^2/3V.$

Or, $PV = m.N.\bar{c}^2/3.$ ...(*ii*)

And since m and N, as also $\bar{c}$ (at a given temperature), are all constants, PV = *a constant*, at a given temperature, which is Boyle's law.

N.B.—*Note how the above treatment may be used to show that* $P = \frac{1}{3}m.n.\bar{c}^2$. *For,* $N = nV$, *where n is the number of molecules per unit volume. Substituting this value of N in relation (ii) above, we have* $P = \frac{1}{3}\,m.n\,\bar{c}^2$.

2. Avogadro's Hypothesis. According to this *hypothesis, equal volumes of all gases and vapours, under the same conditions of temperature and pressure, contain the same number of molecules.* Let us see how we can deduce it from the kinetic theory.

Let N_1 and N_2 be the numbers of molecules in two equal volumes of gases respectively, at the same temperature and pressure.

Then, clearly, $PV = \frac{1}{3}m_1N_1\bar{c}_1^2 = \frac{1}{3}m_2N_2\bar{c}_2^2,$ ...(*a*)

where m_1 and m_2, and $\bar{c}_1$ and $\bar{c}_2$ are the masses and the *root mean square velocities* of the molecules of the two gases in the two cases respectively.

Now, *kinetic energy* $= \frac{1}{2} \times$ *mass* $\times$ (*velocity*)^2, and, therefore, the kinetic energy of each molecule, in the *first case* $= \frac{1}{2}m_1\bar{c}_1^2$, and the kinetic energy of each molecule, in the *second case* $= \frac{1}{2}m_2\bar{c}_2^2$.

As shown by **Maxwell**, the average kinetic energy of *any* gas is the *same* at a given temperature ; so that, $\frac{1}{2}m_1\bar{c}_1^2 = \frac{1}{2}m_2\bar{c}_2^2$.

Multiplying both sides by $\frac{2}{3}$, we have $\frac{1}{3}m_1\bar{c}_1^2 = \frac{1}{3}m_2\bar{c}_2^2.$ .. (*b*)

Now, dividing (*a*) by (*b*), we have $N_1 = N_2$,

i.e., the number of molecules in the case of the two gases is the *same*, which is *Avogadro's hypothesis.*

Thus, *since a gram-molecule of every gas contains the same number of molecules under the same conditions of temperature and pressure*, it follows that the greater the density of the gas, the greater its molecular weight ; or, the molecular weights of two gases are proportional to their densities at the same temperature and pressure.

3. Graham's Law of Diffusion. This law states that the *rates of diffusion of two gases* (which depend upon the velocities of the molecules of the two gases) *are inversely proportional to the square roots of their densities.* Let us see how can we arrive at this result from the kinetic theory.

Let M_1 and M_2 be the molecular weights, (*i.e.*, weights, of *one gram-molecules*), of the two gases respectively.

Then, since $PV = \frac{1}{3}m_1.N.\bar{c}_1^2 = \frac{1}{3}m_2.N\bar{c}_2^2$ at the same temperature and pressure, V being the volume of a *gram-molecule* of the gas, N, the *Avogadro number*, and m_1 and m_2 and $\bar{c}_1$ and $\bar{c}_2$, the respective masses and *r.m.s.* velocities of the molecules in the two cases we have

$$m_1N = M_1 \text{ and } m_2N = M_2.$$

And, therefore, $PV = \frac{1}{3}M_1.\bar{c}_1^2 = \frac{1}{3}M_2.\bar{c}_2^2.$

Or, $M_1\bar{c}_1^2 = M_2\bar{c}_2^2$. Or, $\bar{c}_1^2/\bar{c}_2^2 = M_2/M_1$,

whence, $\bar{c}_1/\bar{c}_2 = \sqrt{M_2/M_1}$.

But $M_1 \propto \rho_1$ and $M_2 \propto \rho_2$, where ρ_1 and ρ_2 are the densities of the two gases at the *same* temperature and pressure.

$\therefore \quad \bar{c}_1/\bar{c}_2 = \sqrt{\rho_2/\rho_1}$.

Since $\bar{c}_1 \propto r_1$ and $\bar{c}_2 \propto r_2$ where r_1 and r_2 are the *rates of diffusion* of the two gases respectively, we have

$$r_1/r_2 = \sqrt{\rho_2/\rho_1},$$

which is the mathematical expression of Graham's law.

For deduction of **Charles' law** and **Dalton's law**, see solved example 3, at the end of the chapter.

267. Kinetic Energy of a Molecule. We have seen that the pressure of a gas is given by the relation,

$$P = \tfrac{1}{3}m.n.\bar{c}^2.$$

Multiplying by V, where V is the *volume of a gram-molecule of the gas*, we have $PV = \tfrac{1}{3}m.n.V\bar{c}^2$.

Now, $nV = N$, the *Avogadro number*.

$\therefore \quad PV = \tfrac{1}{3}m.N.\bar{c}^2$.

Also, $PV = R.T$, where R is the *gas constant*.

$\therefore \quad \tfrac{1}{3}m.N.\bar{c}^2 = R.T$.

Multiplying both sides by 3/2, we have $\tfrac{1}{2}m.N.\bar{c}^2 = \tfrac{3}{2}R.T$.

Or, $\tfrac{1}{2}m.\bar{c}^2 = \tfrac{3}{2}.(R/N).T$.

But $\tfrac{1}{2}\ m\bar{c}^2$ is, clearly, the *kinetic energy* of a molecule.

Therefore, kinetic energy of a molecule $= \tfrac{3}{2}.(R/N).T$.

Now, R/N is a constant K, called *Boltzmann's constant*.

$$K.E.\ of\ the\ gram\text{-}molecule\ of\ a\ gas = \tfrac{3}{2}.K.T.$$

Obviously, therefore, the kinetic energy of the *gram-molecule* of a gas is equal to $\tfrac{3}{2}\ K.N.T. = \tfrac{3}{2}.(R/N).N.T$.

Or, $K.E.\ of\ the\ molecule\ of\ a\ gas = \tfrac{3}{2}R.T$.

This tells us at once that the kinetic energy of a molecule (*i*) depends upon T, the absolute temperature of the gas and (*ii*) is quite independent of its mass. Hence a tiny molecule will be more active than a bulky one.

This fact is often referred to as the *kinetic interpretation of temperature*.

Again, examining the above expressions for kinetic energy a little more carefully, we see that the factor $\tfrac{1}{2}$ appears there because of its presence in the expression for kinetic energy, and the factor 3 appears there because the molecule has *three degrees of freedom* of straight line motion, (*i.e.*, along the three axes). Therefore, the kinetic energy per degree of freedom is equal to *one-third* of the above.

Thus, $K.E.\ of\ one\ molecule\ of\ a\ gas,\ per\ degree\ of\ freedom = \tfrac{1}{2}\ K.T.$

And, therefore, $K.E.\ of\ the\ gram\text{-}molecule\ of\ a\ gas,\ per\ degree\ of\ freedom, = \tfrac{1}{2}RT.$

268. Value of the Gas Constant (R). We know that a *gram-molecule* of every gas occupies the same volume at the same temperature and pressure. This volume is 22·4 *litres* at the normal temperature and pressure, (*N.T.P.*) *i.e.*, at 0°*C* and 76 *cms.* pressure. And, since $PV = RT$, it follows that the value of R must be the same for all gases at the same temperature and pressure.

Now, $$PV/T = R.$$

So that, putting the value of P, V and T in the above relation, we have

$$R = 76 \times 13{\cdot}6 \times 981 \times 22400/273$$
$$= 8{\cdot}29 \times 10^7 \text{ ergs per degree } C.$$

[∵ $P = 76$ *cms.* $= 76 \times 13{\cdot}6 \times 981$ *dynes/cm²*. $V = 22{\cdot}4$ *litres.* $= 22{\cdot}4 \times 1000$ $= 22400$ *c.cs.* and 0°*C* = 273° *Abs.*

Now, 1 *calorie* $= 4{\cdot}18 \times 10^7$ *ergs*, (value of J)

$$\therefore \quad R = 8{\cdot}29 \times 10^7/4{\cdot}18 \times 10^7$$
$$= 1{\cdot}98 \text{ calories/degree } C.$$

Or, $= 2$ *calories/degree C*, as a near approximation.

This, then, is the value of R for *one gram-molecule* of a gas. If we consider only *one gram* of a gas, however, the value of R will, naturally, be 1·98/*molecular weight of the gas*, and will, therefore, be different for different gases, depending upon their molecular weights.

Note. In the Constant Tables, (*Kaye and Laby's*), the relation $PV = RT$ is used with a different meaning,—*viz.*, P is taken in *atmospheres*, and V, as the ratio between the volume of the gas at pressure P and temperatute T and its volume at normal atmospheric pressure and 0°*C*; so that, in this case,

$$P = 1 \text{ atm}, V = 1 \quad \text{and} \quad T = 0°C = 273° \text{ Abs.}$$

And ∴ $$R = PV/T = 1 \times 1/273 = 1/273, \text{ for all gases.}$$

269. Van der Waal's Equation. In our discussion so far, we have concerned ourselves only with *ideal* gases, *i.e.*, gases *in which the molecular size is zero, and the molecular force*, (*i.e.*, force of cohesion between the molecules), is *zero*. These assumptions are far from valid in the case of any *real* gas, even at ordinary pressures, and they become absolutely inadmissible at higher pressures.

This fact was recognised as early as 1827, when *Desperetz* discovered that the resulting volumes of originally equal volumes of air and carbon dioxide differed from each other, when subjected to a pressure of 15 *atmospheres*. Surely, then, he argued, they could not both be obeying Boyle's law. The problem of obtaining a relationship between pressure, volume and temperature of a gas, more in conformity with actual facts, therefore, engaged the attention of many investigators and many a different equation were proposed. The one, however, most satisfactory from both the theoretical and the practical stand-points, is the one due to **Van der Waal**, known after him, as *Van der Waal's equation.* He takes into consideration both the factors mentioned above, *viz.*, (*i*) *the size of the gas-molecules*, and *the force of cohesion between them.*

(*i*) **Correction for molecular size.**—A gas molecule has a *finite size* and, therefore, occupies some volume; so that, the actual space in which it is free to roam about, when enclosed in a vessel, is a little less than the volume of the containing vessel, (or the gas), and this is much more so, when the gas is compressed. A correction must,

therefore, be applied for this decrease in the free space available to the gas.

Now, it may be recalled that we have imagined a molecule to be a perfectly hard or elastic sphere, so that when the total volume occupied by a gas is decreased, the volume of the free space available for the movement of its molecules is decreased to a greater extent due to the incompressibility of the latter, which still continue to occupy the *same volume* as before. For precisely the same reason, when the total volume occupied by a gas is increased, the free space available to the molecules increases in a greater ratio. Thus, the volume of the free space that undergoes change with pressure is not V but something less than V, and, at first sight, it might appear that this volume should be V *minus* the volume actually occupied by the molecules, when compactly arranged together. This would be quite a valid assumption, only if the motion of the molecules were orderly, and if only some of them moved and others remained at rest. Actually, however, the molecules fly about in a chaotic manner, in all possible directions, and, therefore, they interfere with, and obstruct, each other to a very much greater extent, thus greatly reducing the space for free movement. For this reason, the volume of the free space is taken to be $(V-b)$, where b is near about four times the total volume of the molecules. We must, therefore, substitute this value of volume $(V-b)$ for V, in the relation $PV = RT$ for a perfect gas.

(*ii*) **Correction for the force of cohesion.**—In an actual gas, the force of cohesion is not zero and, therefore, every molecule in a gas is attracting, and is being attracted by, every other molecule near about it. Now, a molecule, well inside the containing vessel, is being attracted equally in all directions, with the result that the resultant force on it is *zero*, and its speed remains undiminished. But a molecule close to a wall of the vessel is only being attracted backwards by molecules in the body of the gas and hence its speed is somewhat diminished, with the result that the force with which it strikes the wall of the vessel is lessened. In other words, the pressure exerted by the gas on the wall is now smaller.

Now, it is obvious that if we double the number of molecules *per c.c.* of the gas, this reduction of the observed pressure will become *four times as great.* for the simple reason that

(*i*) *there will now be twice as many molecules striking the walls of the containing vessel, and*

(*ii*) *each molecule, near to a wall, will now be attracted inwards or backwards by twice as many molecules.*

Now, both these depend upon the number of molecules per *c.c.* of the gas ; so that,

reduction of observed pressure $\propto$ (*number of molecules of the gas per c.c.*)^2.

Or, *reduction of observed pressure* $\propto \frac{1}{(\text{volume of the gas})^2}$, *i.e., reduction of observed pressure* $\propto 1/V^2$.

Or, *reduction of observed pressure* $= a/V^2$, where a is a constant.

It follows, therefore, that if the observed pressure be P for an *actual* gas, it would be $(P+a/V^2)$ for an *ideal* gas, and, therefore, we must substitute $(P+a/V^2)$ for P in the relation, $PV = RT$.

Thus, substituting the values, $(V-b)$ for V, and $(P+a/V^2)$ for P in the perfect gas equation $PV = RT$, we have

$$(P+a/V^2)(V-b) = RT.$$

This is known as **Van der Waal's equation**, and, as stated above, is more in conformity with experimental results, but is still far from representing the behaviour of gases accurately, and has its own defects.

273. Mean Free Path of a Moleculea (λ). *The mean free path of a molecule is the average distance covered by it between two successive collisions.** The following is a simple way of calculating its approximate value.

Let us assume that only the particular molecule that we are considering is in motion, all the others being at rest. This moving molecule will, then, naturally collide with all those molecules whose centres happen to lie within its *sphere of influence*, (see page 439). If the radius of the sphere of influence of the molecule be r, then, all these molecules lie in a sphere of radius r, described about this molecule as centre. If, therefore, $\bar{c}$ be the *r.m.s.* velocity of the molecule through the gas, it will, in one second, collide with all the molecules lying in the region traversed by its sphere of influence as it covers a distance $\bar{c}$. Now, the region thus traversed in one second is, obviously, a cylinder of length $\bar{c}$, and an area of cross-section πr^2, *i.e.*, of a volume $\pi r^2.\bar{c}$. Therefore, if n be the number of molecules *per c.c.*, there will be $\pi r^2.\bar{c}n$ number of molecules enclosed within this cylinder, and hence, the number of collisions in *one second* will be $\pi r^2.\bar{c}n$. The average time between collisions is, thus, equal to $1/\pi^2\ \bar{c}n$, and, therefore, average distance covered by a molecule between successive collisions is equal to *velocity* $\times$ *this time*, *i.e.*,

$$= \bar{c} \times 1/\pi r^2.\bar{c}n = 1/\pi r^2.n.,$$

Or, *the mean free path of a molecule*, $\lambda = 1/\pi r^2.n$.

Further, if m be the mass of one molecule, the mass per unit volume, or *density* ρ, of the gas, is equal to mass of one molecule *into* number of molecules per unit volume $= m \times n$.

$$\therefore \qquad \lambda = m/mn.\pi r^2 = m/\pi r^2\rho, \qquad [\because\ m.n = \rho.$$

i.e., λ is inversely proportional to ρ, the *density* of the gas; and, since density varies directly as the pressure of the gas, λ varies inversely as the pressure of the gas, and directly as its absolute temperature.

To have an idea as to the magnitude of λ, it might be mentioned that, at ordinary pressures, its value is of the order of 10^{-5} *cms.*, but at low pressures, such as 10^{-4} *cms.* of *Hg column*, (as in electric glow lamps etc.), its value ranges between 5 to 10 *cms.*

In the above discussion, we have made the simplifying assumption that all molecules but the one under consideration, are at rest.

*The exact nature of these collisions is not yet known, and it is not quite clear whether molecules come into actual contact or whether they recede away from each other, when at a distance, close to each other.

Maxwell has taken into consideration the motion of all the molecules and has shown, by a more rigorous treatment that $\lambda = 1/\sqrt{2}\pi r^2.n$, which gives a smaller value of λ than the one obtained above.

271. Viscosity of Gases. The viscosity of a gas is just a mechanical property of it, and its viscosity coefficient may be defined in precisely the same manner as that of a liquid (see Chapter XII, page 42).

Suppose we have a gas, flowing from left to right, over a solid horizontal surface with a velocity which is very small compared with the velocity of its molecules. This velocity will be the same everywhere in a plane XOY, (Fig. 320), parallel to the horizontal surface, but will increase upward in the direction of the *z-axis*, being the least, (*i e., zero*) for the layers in contact with the solid surface, and increasing with the distance of the layer from it, *i.e.*, there will be a *velocity-gradient* dv/dz along the *z-axis*.

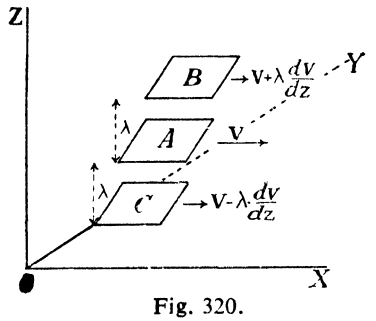


Fig. 320.

Now, we may imagine the molecules of the gas to be divided up into three distinct parts, moving parallel to the three mutually perpendicular axes, x, y, and z, in either direction; so that, the average number of molecules moving in *one* direction along any *one* axis will be *one-sixth* of the total number of molecules in the gas.

Consider an area A, parallel to the plane XOY. Let V be the velocity of the gas in *this* plane. Let there be two other layers B and C, parallel to A, above and below it respectively, *each at a distance from it equal to the mean free path of the molecules*, so that the molecules from it, moving normally upwards to B or downwards to C, do so *without any collision.*

Then, clearly, velocity of the molecules in the layer B is $V+\lambda dv/dz$, and, that of the molecules in the layer C is equal to $V-\lambda dv/dz$.

Now, due to molecules of the gas moving about indiscriminately in all directions, there is a continual interchange of molecules between the two layers B and C. Since the molecules from B, crossing A downwards, have a velocity $(V+\lambda dv/dz)$, the *forward momentum carried by them per unit area of A* is equal to *mass of molecules* $\times (V+\lambda.dv/dz)$.

If n be the number of molecules per unit volume of the gas, and $\bar{c}$ and m, the velocity and mass of each molecule respectively, the number of molecules crossing unit area of A downward, *in unit time* is equal to $n\bar{c}/6$, and their *mass* $= n.\bar{c}\,m/6$.

So that, the forward momentum carried by them *downward*, in unit area of A, per unit time is

$$= \frac{n.\bar{c}}{6} m.\left(V+\lambda.\frac{dv}{dz}\right) = \frac{\rho.c}{6}.\left(V+\lambda.\frac{dv}{dz}\right),$$

because $n.m = \rho$, the density of the gas in *grams per c.c.*

This, then, is the momentum lost per unit area, per unit time, in the direction of flow, by the faster moving layer *B*.

Similarly, the momentum carried *upward* per unit area of *A*, per unit time, in the direction of flow, from the slower moving layer *C*

$= \frac{n.\bar{c}}{6}.m\left(V-\lambda.\frac{dv}{dz}\right) = \frac{\rho.\bar{c}}{6}\left(V-\lambda.\frac{dv}{dz}\right)$, and this is, therefore, the momentum gained per unit time by the layer *B*.

Thus, *the net momentum lost by layer B above A, per second,*

$$= \frac{\rho\,\bar{c}}{6}\left(V+\lambda.\frac{dv}{dz}\right)-\frac{\rho\,\bar{c}}{6}\left(V-\lambda.\frac{dv}{dz}\right).$$

$$= \frac{\rho\,\bar{c}}{6}.2\,\lambda.\frac{dv}{dz} = \frac{1}{3}\,\rho..\bar{c}\,\lambda.\frac{dv}{dz}.$$

And, clearly, the same amount of momentum is being gained by the layer *below A*. Thus, the layer above *A* tends to accelerate its motion, and that below *A* tends to retard it; so that, the *backward-dragging force acting per unit area on it is equal to* $\frac{1}{3}\,\rho.\bar{c}..\lambda\ dv/dz$.

This must be equal to the tangential force acting per unit area of the layer A, i.e., $= \eta.dv/dz$, *where* η *is the coefficient of viscosity of the gas.*

$$\therefore \qquad \eta.\frac{dv}{dz} = \tfrac{1}{3}.\rho.\bar{c}\ \lambda.\frac{dv}{dz}.$$

Or,
$$\eta = \tfrac{1}{3}.\rho.\bar{c}.\lambda.$$

Since, at a constant temperature, ρ increases with pressure and λ decreases in the same ratio, η *is quite independent of the pressure, provided the temperature remains constant.* This fact is amply borne out by experiment and lends powerful support to the kinetic theory of gases Further, η is proportional to $\bar{c}$, the *molecular velocity*, and, therefore, to the square root of the absolute temperature of the gas. This result is not so well borne out by actual experiment and is only approximately true.

272. Production of Low Pressure—Exhaust Pumps. In the present-day staggering development of Science, the technique of producing high vacua is of the utmost importance. Apart from its well known use in radio and X-ray equipment etc, we owe our initiation into the comparatively new realm of atomic physics to the welcome development of exhaust or vacuum pumps and other methods for the production of high vacua, for it has helped us to study the behaviour of atoms and molecules under low-pressure conditions. And this in turn, has had the reciprocal effect of enabling us to make further improvements in our exhaust pumps and high vacuum technique, in general, with the result that we can today produce as low a pressure as 10^{-9} *mm*.

And, simultaneously with the development of these high vacuum pumps must go the development of delicate gauges, to enable us to measure the very low pressures produced by them. We shall now proceed, therefore, to study these twin-devices.

273. Exhaust Pumps—Their Characteristics. Before dealing with any specific types of pumps, it will be worthwhile to understand

the characteristics of a good vacuum pump. These are (*i*) *the exhaust pressure*; (*ii*) *the degree of attainable vacuum*; and (*iii*) *the speed of the pump*. Let us take each, in turn.

(*i*) **The Exhaust Pressure.** In any vacuum pump, there is an *inlet*, *fine* or *intake* side, from which the gas or vapour from the vessel to be exhausted is drawn into the pump, and an *outlet* or *exhaust* side, from which it is expelled out. As its very name indicates, the exhaust pressure is the pressure on the exhaust side of the pump, and may be just atmospheric or much lower than it, varying from pump to pump. But, generally speaking, the higher the vacuum desired to be produced on the *fine* or the *intake* side of the pump, (*i.e.*, in the vessel connected to it), the smaller must be the exhaust pressure or the '*rough vacuum*', as it is alternatively called, on its outlet side. For, the recognised procedure in high-vacuum technique today is to first reduce the pressure from atmospheric to a small fraction of it, say to ·1 *mm.* or so, *i.e.*, to create a '*fore-vacuum*', by means of an ordinary pump, here called the *auxillary*, the *rough* or the '*backing pump*'. This *fore-vacuum* or *backing pressure*, as it is also termed, is then reduced further from 10^{-4} to 10^{-7} *mm.* by means of a suitable *fine* or *high vacuum pump*. For this purpose, the backing and the high-vacuum pumps are arranged *in series* or *tandem*, so that the gas or vapour from the vessel to be exhausted is drawn in at the inlet of the latter and expelled at its outlet into the fore-vacuum of the former, which, then finally expels it out into the atmosphere.

(*ii*) **The Degree of Attainable Vacuum.** By this we understand the lower limit of the pressure that it is possible to obtain in the vessel, connected to the pump. This depends to a very large extent on the exhaust pressure. For, if it be very low, it may result in the passage of the gas or vapour in the reverse direction *i.e.*, in its leakage from the exhaust to the intake side of the pump.

Now, theoretically speaking, there is no lower limit to the attainable pressure in the case of a *diffusion-condensation pump*, but, in a *molecular pump*, a definite limit is set by the constant ratio it bears to the exhaust pressure. The limit may, however, in general, be considerably extended by using connecting tubes of wide bores in-between the vessel to be exhausted and the pump, as it greatly minimises the resistance to the flow of the gas or the vapour from the former to the latter.

(*iii*) **The Speed of the Pump.** The *speed* of a pump may, in a general way, be defined as *the relative rate of reduction of pressure in a given volume*.

Thus, if p_0 be the limiting value of the attainable pressure, with the help of a given pump, p, the pressure at an instant t in the vessel of volume V, connected to it, and S, the *speed* of the pump *at this pressure*, the rate of reduction of pressure in the vessel, *i.e.*, dp/dt, is given by the relation,

$$-dp/dt = S.(p-p_0)/V, \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

whence, we have

$$-dp/(p-p_0) = S.dt/V. \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

So that, if p_1 and p_2 be the values of the pressure in the vessel at instants t_1 and t_2 respectively, we have, by integrating expression (*ii*) for the limits $p = p_1$, $t = t_1$ and $p = p_2$, $t = t_2$,

$$\log_e\left(\frac{p_1-p_0}{p_2-p_0}\right) = \frac{S}{V}.(t_2-t_1),$$

whence,

$$S = \frac{V}{t_2-t_1}.\log_e\left[\frac{p_1-p_0}{p_2-p_0}\right]^*, \quad \ldots \quad \ldots \quad \ldots (iii)$$

which, when p_0 is comparatively negligibly low, reduces to

$$S = V.\log_e(p_1/p_2)/(t_2-t_1).$$

This is known as *Gaede' equation*; and gives us a definition of S, the *intrinsic speed* of the pump. For, if $p_1/p_2 = e$ and $(t_2-t_1) = 1$ *sec.*, we have $S = V$

*As will be readily seen, this equation enables us to determine the interval (t_2-t_1), taken by the vacuum system to recover from the rush or '*surge*' of the gas, which raises the pressure to p_1,—the working pressure required by the system being p_2.

Thus, the *intrinsic speed of a pump may be defined as the volume in which it can reduce the pressure to 1/eth (which comes to about 36·79%) of its instantaneous value in 1 second*, (∵ $e = 2{\cdot}17828$).

Putting $P_0 = 0$, *i.e.*, taking the attainable vacuum to be perfect, we have, from (*i*) above, $-dp/dt = S.p/V$, whence, $S = -\frac{V}{p}\cdot\frac{dp}{dt}$. This gives the *speed of exhaust* of the pump, or the *pumping speed E*, as defined by *Langmuir*

So that, $$E = -\frac{V}{p}\cdot\frac{dp}{dt} \qquad \ldots (iv)$$

Now if dv be the volume of the gas or vapour, measured at pressure p, extracted in the time-interval dt, from the volume V, we have

$$pv = (p-dp)(V+dv), \qquad [\because \textit{pressure decreases}.$$

whence, neglecting the product $dp.dv$, compared with the other terms, we have

$$dv = -V.dp/p,$$

and, therefore, $$\frac{dv}{dt} = -\frac{V}{p}\cdot\frac{dp}{dt} \qquad (v)$$

Or, from relations (*iv*) and (*v*), we have $E = dv/dt$.

Thus, *the speed of exhaust of a pump may be defined as the rate of change of volume of the gas or vapour in the vessel at any given instant, the measurement of volume being effected at the pressure attained by the pump at that very instant.*

Substituting the value of dp/dt from relation (*i*) above in expression (*iv*) for E, we have

$$E = -\frac{V}{p}\cdot\left[-\frac{S}{V}\cdot(p-p_0)\right] = \frac{S.(p-p_0)}{p}$$

Or, $$E = S(1-p_0/p).$$

This relation tells us that, in the beginning when the pump starts working, p is very much greater than p_0, so that p_0/p is practically *zero*, and, therefore, E is almost equal to S, *i.e.*, in the beginning, the pumping speed of the pump is practically equal to its intrinsic speed. But as p is progressively reduced and approaches p_0, E gradually decreases, and finally becomes *zero* when $p = p_0$. In other words, *a pump loses all its pumping speed at the lowest attainable pressure.* It is, therefore, important to design pumps not only with a view to producing high vacuum, but also with a view to having as high a pumping speed as possible, at all pressures.

And, since the pumping speed (E) is found to depend not only upon its intrinsic speed (S) and the lower limit of the attainable pressure (p_0), but also upon the resistance to its flow, it follows that the wide bores of the connecting tubes (referred to above) also help achieve this end.

274. Different Types of Pumps.—The following shows at a glance the classification of the different types of exhaust or vacuum pumps.

Exhaust Pumps

- **Oil Pumps**
  - *Piston type* — Solid piston pump, (see § 276) produces low pressures only up to 10^{-3} *mm.*— used as a *backing pump*. Familiar examples : *Common Air Pump* and *Geryk Pump*.
  - *Rotary type*
- **Mercury Pumps**
  - *Piston type* — Liquid (mercury) piston pump. Exhausts down to 2×10^{-6} *mm.*, but slow and tedious in action. Now, only in limited use. Examples : *Toepler* and *Springel Pumps*
  - *Rotary type* — Designed by *Gaede*, needs a fore-vacuum of about 1 *mm.* for working. Slow in action, but can produce low pressures down to 10^{-6} *mm.* Still extensively used in the laboratory.
- **Molecular Pumps** (see § 277)
- **Diffusion Pumps** (see § 278)

We shall now deal in detail with some of the simple and important types of pumps.

275. The Common Air Pump.—It consists of a *receiver plate* P, connected to a cylinder C, through a tube bent twice at right angles, as shown in Fig. 321. The cylinder is fitted with a piston, and both the cylinder and the piston carry valves, V_1 and V_2, respectively, such that they open only *upwards*. The vessel F to be exhausted is placed over the receiver plate in the manner shown.

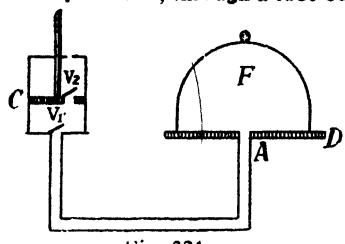


Fig. 321.

To start with, the piston is moved up to the top of the cylinder, from its initial position at the bottom, when the valve V_2 remains closed, due to the atmospheric pressure on it, and the valve V_1 is forced open due to the pressure of the air or gas in the vessel F; so that, it comes and collects in cylinder C. The piston is then moved down. The valve V_1 now remains closed due to the increased pressure on it, and the valve V_2 is thrown open by the gas in the cylinder, which thus escapes out into the atmosphere.

The operation is repeated a number of times; each time the gas comes and collects in the cylinder during the upward stroke of the piston and is forced out during the downward stroke; so that, after some time, there is a fairly good vacuum produced in the vessel F.

This pump is unable to give a high vacuum, because of the pressure of the residual gas or air in the vessel being unable to force the valve V_1 open and get into the cylinder.

That complete vacuum cannot be created by this pump may be shown mathemetically as follows:

Let V *c.cs.* be the volume of the vessel F and the tube up to the bottom of C, and v, that of the cylinder. Then, during the first upward stroke, the volume V of the gas expands to $(V+v)$ *c.cs* And, since during the downward stroke of the piston, a volume v of the gas, (*i.e.*, equal to that of the cylinder), is swept out, the volume of the gas left behind is V *c.cs*, *i.e.*, $V/(V+v)$ of the original volume $(V+v)$ *c.cs.* During the next upward stroke, this volume again expands to $(V+v)$ *c.cs*, and, again, during the downward stroke, v *c.cs.* is forced out, leaving behind $V/(V+v)$ of the volume left after the first stroke, or $V/(V+v)$ of $V/(V+v)$ of the original volume $(V+v)$ *c.cs*, *i.e.*, $(V/V+v)^2$ of the original volume.

Similarly, after the third stroke, the volume of air or gas left behind will be $(V/V+v)^3$ of the original volume, and, therefore, after n strokes, the volume left behind will be $(V/V+v)^n$ of the original volume $(V+v)$.

Now, it is clear that this expression, $(V/V+v)^n$, *can never be zero*, however great the value of n.

In other words, *whatever the number of strokes given, there will always be some air or gas left behind in the vessel*, and thus there can be no perfect vacuum created inside it.

As a matter of fact, the pressure can hardly be reduced below 1 *cm.* of mercury column with the help of this pump, due partly to the inability of the gas to open the valve V_1, and partly to leakage and the presence of moisture in the vessel or receiver to be exhausted. For obtaining low pressures, therefore, other types of pumps are used, the *Rotary Oil Pumps* being the most suitable for the purpose.

276. Rotary Oil Pumps. Originally devised by **Gaede**, these are of two types, *viz.*, (*i*) *the rotary vane oil pump*, and (*ii*) *the stationary-vane oil pump.* The principle underlying both is, however, the same,—a massive cylindrical shaft or '*rotor*', revolving eccentrically inside a hollow stout steel cylinder, or '*stator*' compressing the gas or vapour entering it, and finally ejecting it through a 'non-re-

turn' outlet valve, the whole pump being kept immersed in oil, which serves a three-fold purpose, (*i*) *providing automatic lubrication*, (*ii*) *preventing leakage of gas or vapour into the high vacuum created*, and (*iii*) *making for efficient cooling of the pump*. Let us study each type in a little more detail.

1. The Rotary-vane Oil Pump, or the Gaede Rotary Oil Pump. The main parts of the pump are shown diagrammatically in Fig. 322, where *C* is the hollow, cylindrical steel chamber or '*stator*' and *S*, the stout and massive cylindrical shaft or '*rotor*', rotating *eccentrically* (by means of an electric motor) such that it is always in contact with the stator at some peripheral point, such as *P*.

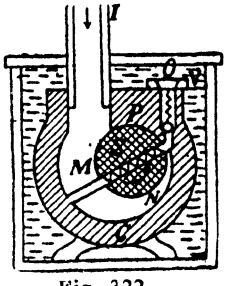


Fig. 322.

A slot, cut *diametrically*, right across the rotor, carries two *vanes*, slid into it, which are not only kept apart from each other, but also pressed against the walls of the stator by means of one or more springs in between them, thus dividing the space between the stator and the rotor into two separate compartments.

On either side of *P*, where the rotor and the stator remain in contact, the stator is provided with an inlet (*I*) and an outlet *port O*, the latter being fitted with a spring-operated valve *V*. The whole pump is kept immersed in oil for the reasons explained above.

As the rotor rotates in the direction shown, the space between the rotor and the stator, on the inlet side, goes on increasing, while that between the rotor and the outlet side of the stator goes on decreasing, so that the gas or vapour from the vessel connected to *I* is continually drawn into the former, and that, in the latter, gets progressively compressed, until when its pressure becomes sufficiently high, it forces open valve *V* and escapes out of the outlet *O*. The process goes on repeating itself until a pressure as low as 10^{-3} *mm.* is produced in the vessel connected to the pump. A special *self-sealing oil-valve* prevents the gas or vapour from being sucked back into the exhausted vessel, even when the pump stops working. No fore-vacuum is required for the working of this pump, and it can, therefore, be used directly from the atmospheric pressure.

2. The Rotary-vane or the Hyvac Rotary Oil Pump. It consists of a stout outer cylinder *C*, inside which is mounted

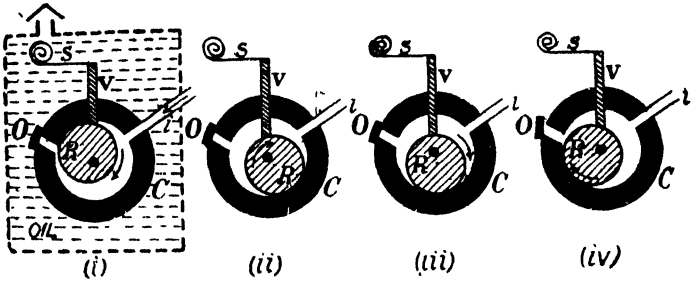


Fig. 323.

eccentrically, a cylinder *R*, called the *rotor*, (Fig. 323). Kept pressed against the rotor, with the help of a spring *S*, is a partition, called the *vane V*, which keeps the gas or air, already inside the cylinder, apart from the fresh in coming gas or air. The outer cylinder is provided with an inlet tube *i*, which is connected to the vessel to be exhausted, and an outlet tube *O*, which is provided with a valve, *opening outwards*. To prevent any leakage, the whole pump is immersed in oil, as shown in Fig. 323 (*i*), but a special type of valve prevents the oil from getting into the vessel being exhausted, when the pump is stopped. The rotor is driven at a very high speed by means of a separate electric motor in the direction shown by the arrow heads.

Fig. (*i*) shows the condition to start with, when the inlet tube is connected to the vessel to be exhausted, and when the gas or air from the vessel has just been admitted into the space in-between the cylinder and the rotor *R*.

Fig. (*ii*) shows the condition when the rotor has started rotating eccentrically, and the gas or air is being compressed. Fresh gas or air comes into the cylinder through the inlet tube *i*, behind the rotor, and is kept apart from that already present by the vane *V*, as explained above.

Fig. (*iii*) shows the process of compression, taken a step further.

Fig. (*iv*) shows the final stage of compression, when, due to increased pressure of the gas in *C*, the valve at the mouth of the outlet tube *O* is forced open, and the gas is expelled out.

The gas or air behind the rotor is similarly compressed and forced out and the cycle is repeated, until a high vacuum is produced in the vessel, connected to it.

In practice, the pumping system consists of two such units, in series with each other and mounted side by side, worked by the same motor. The first unit works directly from the atmosphere, as explained above, and the second then works from the fore-vacuum created by it. The maximum speed of working attainable is about 6 *litres per minute* and the vacuum obtainable, about 10^{-3} *mm.*, as mentioned already.

It will be readily seen that if a vessel be connected to the outlet tube *O*, the gas or air will be compressed into it, and, therefore, this pump can also be used as a compression pump.

277. **Molecular Pumps.** These too are the result of the labours of *Gaede*, together with **Langmuir**, and are based on the principle that if there be a rapidly rotating surface (called the *rotor*) very close or adjacent to a stationary one (called the *stator*), the space or *clearance* between the two being as small as 03 *mm.*, it exerts, due to viscosity, a dragging force, in the direction of its own motion, on the molecules of the gas or vapour in that space. This is so because, at low pressures, the mean free path (λ) of the molecules of the gas or vapour is greater than the linear dimensions of this small annular gap between the rotor and the stator, so that there are too few collisions between molecules and molecules, compared with those between molecules and the walls of this gap, and **Knudsen** has shown that when such is the case, the molecules acquire the 'drift

velocity' of the rotating surface they impinge upon and rebound from, and that this velocity is not altered by any subsequent collisions amongst themselves.

Unlike the oil pumps, which can start working straightaway from the atmospheric pressure, down to a pressure of 10^{-1} to 10^{-3} *mm.*, the molecular pumps (as also the diffusion pumps) operate only from a reduced pressure or a fore-vacuum. They are, therefore, always used in series with a backing pump, connected to the exhaust port, which creates the necessary fore-vacuum or *'rough'* vacuum to receive the gas or vapour driven into it by the rotor, *there being a continuous unbroken communication* between the receiver and the rough vacuum, in contrast with the other types of pumps*, where there is only an intermittent or interrupted communication between the two,—a solid or a liquid piston first separating out a part of the gas or vapour in the receiver and then putting it into communication with the atmosphere or the rough vacuum, into which it is expelled.

Fig. 324 shows a diagrammatic representation of the pump, where *A* represents the *rotor*, revolving round† about an axis through its centre, inside and closely adjacent to the walls of the *hermetically* sealed shell or *stator C*. There are an *inlet* and an *outlet* in the stator at *P* and *Q* respectively, with a *slot* in-between the two, where the annular gap is consequently greater than at all other points.

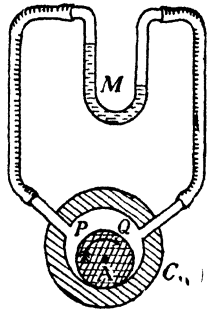


Fig. 324

As the rotor revolves in the direction shown, it drags with it the gas or vapour from the inlet port *P* to the outlet port *Q*, thus creating a pressure difference between the two, as indicated by the manometer *M*. This pressure difference cannot possibly be due to the viscosity of the gas alone ; for, at ordinary pressure, the viscosity of a gas is found to be quite independent of the pressure. Indeed, *Gaede* has shown that the pressure difference (p_1-p_2) between *P* and *Q* is given by the relation,

$$p_1-p_2 = \eta \cdot \frac{6lu}{h^2},$$

where η is the *viscosity* of the gas in question, u, the *speed* of the rotor and l and h, the *length* and *radial depth* of the slot between *P* and *Q*. Thus, at ordinary pressures, since η undergoes no change, the pressure difference (p_1-p_2) remains constant for a constant speed (u) of the rotor.

At low pressures, on the other hand, it has been shown by *Gaede* that it is the *pressure ratio* p_1/p_2 (and *not the pressure difference*) that remains constant for a constant speed of the rotor, and which is also quite independent of the fore vacuum. This pressure ratio is given by the relation $p_1/p_2 = e^{cu}$, where c is a *constant*, depending

*Except the diffusion pumps.

†The speed of its rotation should never be less than 5000 revolutions per minute.

upon both the nature of the gas or vapour and the dimensions o the slot.

In *Gaede's* own form of the pump, *there are a set of twelve slots or grooves along the circumference of the rotor*, their depths decreasing progressively from about ·6 *cm.* in the inner to about ·15 *cm.* in outer section, the sections being all connected in series. Into these rotor-grooves fit projections from the stator, the clearance between the two being ·03 *mm.* and the gas or vapour is swept along this small clearance. The arrangement of the slots is such that the pressure has its lowest value at the centre and goes on gradually increasing as we proceed outwards to the ends, where we have the backing pump connected.

In the *Holweck* type of pump, the working is on similar lines, but *the slots are made in the stator*, with no corresponding projections on the rotor, and the clearance between the two can here be reduced to a figure even lower than ·03 *mm.* The low pressure obtained by these pumps is conditioned by (*i*) the speed of the rotor and (*ii*) the fore-vacuum at which they are worked. Thus, for example, with a fore-vacuum of about 2 *mm.* and with a rotor-speed of 10,000 *revolutions per minute*, the pressure may be reduced by Gaede's pump down to the figure of 10^{-6} *mm.* within a matter of minutes. *Holweck* even succeeded in evacuating nitrogen gas down to a pressure of 10^{-7} *mm.* with the help of his pump, with a rough vacuum of only 15 *mm.* and a rotor-speed of 4500 *rev. per mt.* Such low pressures are, however, well-nigh impossible to attain when there are vapours present.

The one serious drawback of these pumps is the recurrence of mechanical trouble, due to the small clearance between the rotor and the stator. And then, while they can easily deal with gases and vapours, slowly vaporising substances, like mercury, and traces of grease etc., give a lot of trouble.

278 Diffusion-Condensation Pump. The inter-diffusion of one gas into another has been used to create rapid vacuum in vessels. The method was first used by *Gaede* in the year 1815, and *depends upon the principle that, in a mixture of gases, the diffusion of a gas takes place from a region, where its concentration is greater to the one, where its concentration is smaller, or, in other words, from a region where its partial pressure is higher, to the one, where it is lower, irrespective of the total pressure in the two regions.* The action of the pump will be clear from the diagrammatic representation of it in Fig. 325.

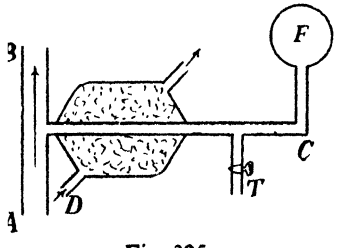


Fig. 325.

The bulb *F* to be exhausted of air or gas is connected by means of a tube *C* to a wider vertical tube *AB*, through which a regular stream of air-free liquid vapour is maintained in a direction shown, the tube *C* being kept at a low temperature, by circulating cold water round it.

Now, clearly, the concentration or the pressure of air, is greater at *C* than

at the other end, because the liquid vapour being air-free, there is little or no air here and, therefore, diffusion of air takes place from *C* into *AB*, where it is swept out by the stream of the liquid vapour, so that, again, the concentration and partial pressure of air are greater at *C* than at the end opposite and hence more air diffuses from *C* into *AB*. This goes on so long as the pressure of air at *C* is greater than its partial pressure in *AB*.

The liquid vapour also tends to diffuse from *AB*, where its concentration (and, therefore, its pressure) is greater, towards *C*, where its concentration and pressure are lower, thus driving the air diffusing from *C* into *AB backwards*, but this is prevented by (*i*) making the aperture of *C*, opening into *AB*, narrow, its dimensions being smaller than the mean free path of the molecules of the gas or air in *AB*, so that there are much fewer vapour-gas or vapour-air molecular collisions and the diffusion velocity of mecury vapour towards *C* is smaller than the diffusion velocity of air from *C* into *B*, and (*ii*) condensing the mercury vapour by cooling the tube *C*, thus not only preventing it from proceeding further and entering the vessel *F*, but also reducing the residual vapour pressure there to less than 10^{-3} *mm.*, thereby ensuring an uninterrupted diffusion of the gas from *C* into *AB*. The condensed mercury is then conveyed back to the boiler, (not shown), to be used over again.

This pump can, however, be used successfully only when the pressure in the receiver or vessel *F*, is about or less than 1 *cm.* of mercury. The pressure in the vessel is, therefore, first reduced to this value, or this *fore-vacuum* created, by means of a rotary vacuum pump, which can be connected to the vessel through the inside-tube *T*, *the maximum effect being obtained when the pressure of the mercury vapour is just above the fore-vacuum, thus produced.*

Such pumps may be used *in series*, each pump, in turn, carrying the evacuation a step further than the last, and thus a pressure as low as *a millionth of a centimetre* of mercury column can be rapidly attained, although, theoretically speaking, there can be no limit to the vacuum produced by them. For, under the ideal conditions, mentioned above, there would be a continuous diffusion of the air or gas from *C* into *AB*, until the air or gas pressure there is reduced to *zero.*

Since diffusion as well as condensation both play their part in the working of these pumps, they are referred to variously as *diffusion-condensation pumps, diffusion pumps* or even *condesation pumps.*

The vacuum obtained with the help of these pumps is of a very high order,—10^{-6} *mm.* or less, with a fore-vacuum of 10^{-1} *mm.*,— and they are, therefore, being increasingly used for the evacuation of X-ray tubes and wireless valves and for such other industrial purposes.

There are only two main *drawbacks* of these pumps, *viz.*,

(*i*) *they have a comparatively slow exhaust speed,* and

(*ii*) *they need a rather vigilant regulation of the vapour-temperature.*

A simple form of diffusion pump, (designed by **Waran**), is shown in Fig. 326. It consists of a conical glass vessel *A*, containing mercury, which is heated, so that the mercury vapour passes up the tube *B*, which ends in a nozzle, into a wider tube *C*, surrounded with a cold water jacket. An inlet tube *i* connects the tube *C* to the vessel to be exhausted, through a *liquid-air-trap*, (not shown). The tube *B* is lagged with asbestos to prevent any condensation of mercury vapour in it.

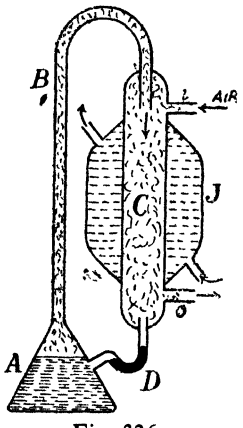


Fig. 326.

As the mercury vapour issues out of the nozzle, it carries along with it, the air, diffusing into *C* through (*i*) from the vessel to be exhausted,—the diffusion of the mercury vapour into the vessel being prevented by its almost immediate condensation on entering *C*. The air is then discharged through the outlet tube and the mercury vapour collected in the bend *D* below *C* and returned back to *A*. The process goes on, until a high vacuum, of the order of about 10^{-5} *mm.* of mercury is created in the vessel, connected to *i*. A still higher vacuum, of the order of about 10^{-7} *mm.* of mercury can be produced with the help of *the process of absorption*, (see § 279, 1) *i.e.*, by placing in the vessel some loose coconut charcoal, recently heated under reduced pressure, which, on being cooled externally, with liquid air, absorbs the gas and the vapour.

N.B. The modern trend is to replace mercury with other liquids, like *Apiezon oils, Butyl Phthalate* or even *ten per cent Paroffin with Butyl Phthalate*, having as low a vapour pressure as 10^{-6} *mm.*; for mercury vapour, if it does once get into the vacuum system, becomes a real source of trouble.

279. Other Methods of Producing Vacua. As we have seen above, the lowest pressure attainable with the help of a pumping device is 10^{-6} *mm.* To produce still lower pressures, other devices have to be used. The following is a very brief description of these :

1 The Sorption Process. This consists in connecting the system to be evacuated to a tube containing some freshly heated coconut charcoal (preferably under reduced pressure) and surrounded by liquid air, when gases, like carbon dioxide, nitrogen and ammonia, are absorbed by the charcoal. To remove Hydrogen, palladium black may be used in place of coconut charcoal. The pressure, thus obtained, is as low as 10^{-7} *mm.*, which may be even further reduced if the method be used in conjunction with a backing pump.

2. The Chemical Process. Known as the *chemical process of 'flashing'*, it consists in suddenly vaporizing a metal, like *magnesium* or *calcium*, in a vessel in communication with the system desired to be evaeuated, when most of the vaporized metal condenses back on the walls of the vessel, the rest forming, with the gases present, a compound of negligibly small vapour pressure, thus, reducing the pressure in the system to a very large extent. Usually, the vessel used is a glass bulb, properly fitted with a tungsten filament, on which the chosen metal is placed. A momentary large current is then passed through the filament, when the metal burns out with a flash and gets vaporized, the condensing part of it forming a bright mirror like deposit on the inner side of the bulb This is the method largely employed in evacuating radio valves.

3. The Thermal Process. This consists in the removal of gases, like Nitrogen, which disappear slowly in the presence of a glowing or incandescent tungsten filament. As can be easily understood this process plays an important

part in maintaining the high degree of evacuation in the ordinary incandescent electric lamps. *Langmuir* is indisputably the pioneer in this branch of high vacuum technique.

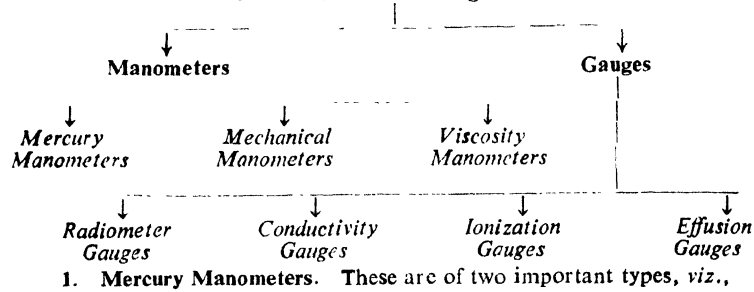
4. The Electrical Process. This consists in ionizing the atoms or molecules of the gas to be removed, either by means of a glow discharge or by bombarding them by means of electrons, obtained by thermionic emmission. The ions, thus produced, get deposited on, or adhere to, the walls of the containing vessel, if it be kept suitably cooled. As can be readily seen, this process finds wide application in the production and maintenance of high order vacuum in incandescent lamps and radio valves.

In actual practice, in most cases, the chemical, the thermal, the electrical processes, although each a completely independent process in itself, operate conjointly and simultaneously, as, for example, in the case of the evacuation of radio valves, where the first one produces the initial vacuum and the latter two augment and maintain it.

And, finally, however effective the method or the means employed, it is imperative for a satisfactory maintenance of a high degree vaccum that the whole evacuated system should be perfectly leak-proof. Hence it is that it should preferably consist wholly of glass or wholly of metal, but, if both needs must be used, they must be directly and carefully sealed together, air-tight.

280. Measurement of Low Pressures.—Manometers and Gauges. As mentioned earlier, the production of low pressures, of necessity led to the development of the proper means to measure them. Such measuring instruments or devices fall under the heading *'manometers and gauges'*. Since we are concerned with the detailed working of only the more important or useful ones, here, we shall content ourselves with only a brief and rapid survey of the rest, not so much with a view to studying their working in detail as to acquainting ourselves with the different principles on which such devices may possibly be based. Here, then, are the different measuring devices at a glance.

Low Pressure Measuring Devices



1. Mercury Manometers. These are of two important types, *viz.*,

(*i*) *Differential and Optical Lever Manometers*, which are improved modifications of the ordinary mercury manometers, with more sensitive methods of observation and measurement,—suitable for measurement of pressures up to 10^{-8} *mm.*

(*ii*) *McLeod Gauge*, which is a standard device of its type, based on the validity of *Boyle's Law* at low pressures,—suitable for accurate measurement of pressures, down to 10^{-5} *mm.* (See pages 558-60).

2 Mechanical Manometers. These are based on the principle of *mechanical deformation* produced in a thin wall or diaphragm, due to pressure. They are calibrated against the McLeod Gauge and their range too does not go below 10^{-3} *mm.* The two well known manometers of this class are

(*i*) the Bourdon Spiral Gauge, (see § 282), and

(*ii*) the Aneroid Barometer type.

3. Viscosity Manometers. The principle underlying these is that at low pressures, the viscous drag between two surfaces in relative motion is proportional to P/M, where P is the pressure of the gas, and M, its molecular weight.

They are of two types, *viz.*,

(*i*) *The Damping or Decrement type*, a good example of which is *Coolidge's Quartz fibre Gauge*, which is suitable for the measurement of pressures, ranging between 10^{-2} *mm.* and 10^{-5} *mm.*

(*ii*) *The Molecular type*, a well known example of which is *Langmuir and Dushman's Molecular Gauge*, suitable for the pressure range 10^{-3} *mm.* to 10^{-7} *mm.*

4. **Radiometer Gauges.** These are based on the measurement of the rate of transference of momentum from a hot to a cold surface due to molecular bombardment. Among gauges of this type may be mentioned

(*i*) *Crooke's Radiometer*, which is suitable only for qualitative work.

(See Foot note – page 568)

(*ii*) *Knudsen's Absolute Gauge*, which is a standard gauge of its type and a close rival of McLeod Gauge in its sensitiveness and accuracy, having a wide range, from 10^{-2} *mm.* to 10^{-7} *mm.* (See § 284).

5. **Conductivity Gauges.** The underlying principle of these is the effect of pressure on the rate of transfer of heat by the process of conduction, their range being comparatively small, from 10^{-1} *mm.* to 10^{-4} *mm.* Among gauges of this type may be mentioned

(*i*) *The thermopile Gauges.*

(*ii*) *Pirani-Hall Resistance Gauge* §285, and

(*iii*) *Gauges, based on linear expansion of metallic wires or strips.*

6. **Ionization Gauges.** These depend for their action on the variation of electrical conductivity of a gas with pressure. Mention may be made here of two gauges of this type, *viz.*, (*i*) *Buckley's type*, and (*ii*) *Dushman and Found type*, this latter one being suitable for measurement of pressures from 10^{-2} *mm.* down to the lowest attainable pressure. (*iii*) *α-ray ionisation gauge* (see §288).

7. **Effusion Gauges.** Used only for the measurement of vapour pressures of metals.

Let us now proceed on with a detailed consideration of some of these.

281. Common Mercury Manometers. These are of two types, *viz.*, (*i*) *Open* and (*ii*) *Closed*

1. The Open Manometer. It consists of a U-tube, with both limbs open, one limb being a little shorter than the other, and bent at right angles, as shown in Fig. 327. A liquid, of suitable density, depending upon the pressure* to be measured, is poured into the tube, so as to be above the bend, and, of course, at the same level in either limb. The shorter limb is then connected to the gas-supply or the vessel, the pressure of which is to be measured. The level of the liquid in the shorter limb then rises above, or falls below, that in the other limb, according as the pressure of the gas is lower or higher than that of the atmosphere, as shown in Figs. (*a*) and (*b*) respectively. The difference of the levels in the two limbs is then read, and the pressure of the gas calculated as follows :

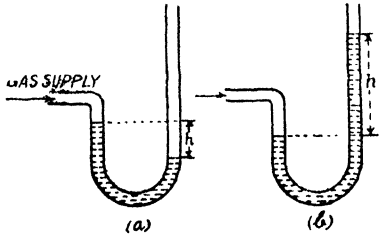


Fig 327.

Let the difference of levels in the two limbs be h, and let the barometric height be H. Then, pressure of the gas in case (*a*) is

*If the pressure to be measured differs from that of the atmosphere by only a small amount, the difference of level in the two limbs will be very small, if a heavy liquid, like mercury, be used. But, with a lighter liquid, like oil, the difference is quite considerable. With mercury as the liquid used, pressures ranging between ½ to 1½ *atmospheres* only can be measured correctly.

$(H-h)$, and in case (b), it is $(H+h)$ *cms.* of mercury column, if the liquid used be mercury. In case, however, the liquid be oil or water, of density ρ, the pressure in the two cases will be $(H-h.\rho/13\,6)$ and $(H+h.\rho/13{\cdot}6)$ respectively, (where 13·6 *gms./c.c.* is the density of mercury).

2. **The Closed Manometer.** It is used for the measurement of high pressures. It is just like the open manometer in construction, but with the longer limb closed at the top, as shown, (Fig. 328), and containing some air, at *atmospheric pressure,* in the closed space above the liquid, with the level of the liquid columns in the two limbs *the same, to start with.*

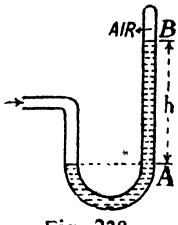


Fig. 238.

When the shorter limb is connected to the gas supply, the level of the liquid column in the shorter limb is pushed down, and that in the other pushed up, so that the air in it gets compressed. The pressure of this enclosed air being inversely proportional to its volume, it can be determined by noting its new volume And, from this pressure and the difference in the levels of the liquid columns in the two limbs, the pressure of the gas-supply, in communication with the shorter limb, can be easily calculated as follows :

Let original volume of the enclosed air be V *c.cs.*, its pressure being *one atmosphere* or 76 *cms.* of mercury column. Then, if v be its volume, after the shorter limb is connected to the gas-supply, we have, by Boyle's law,

$$76 \times V = H \times v,$$

[where H is now the pressure of the enclosed air, in terms of length of mercury column.

whence, $$H = 76 \times V/v.$$

Thus, knowing the original volume V and the new volume v of the enclosed air, H, the pressure of the enclosed air can be known. This, then, is the pressure at B in the longer closed limb. If therefore, the difference of levels in the two limbs be h, and the liquid used in the manometer be mercury, the pressure at A, *i.e.*, the pressure of the gas-supply $=(H+h)$ *cms.* of mercury column. In case the liquid used be oil or water, of density ρ, we have

$$\textit{pressure of the gas-supply} = \left(H + \frac{h.\rho}{13{\cdot}6}\right) \textit{cms.} \text{ of mercury.}$$

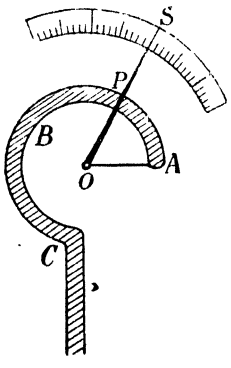


Fig. 329.

282. The Bourdon Gauge. For the measurement of very high pressures, what is called a *Bourdon Gauge* is used. The principle underlying it is the same as that in the case of the *Aneroid barometer,* which is in fact a modification of the gauge.

It consists of a tube ABC, (Fig. 329), *elliptical in section,* with the end A closed and the end C, open, so that it can be put into communication with the gas-supply, the pressure of which is desired to be determined.

Due to the high pressure of the gas entering the tube, it tends to become more circular in section and this results in the end A of the tube

being forced away from C. This movement of A, in its turn, moves the pointer P over a scale, graduated directly *in atmospheres*, by comparing its indications with a standard gauge, as in the Aneroid barometer. The instrument is thus a direct reading one, and can obviously be used to determine low pressures also.

283. McLeod Vacuum Gauge. An ordinary manometer is not quite suitable for the measurement of very low pressures, like those obtaining in incandescent electric lamps and X-ray tubes etc., where the pressure is as low as 10^{-5} *mm.* For such purposes, the *McLeod Vacuum Gauge*, designed by McLeod in the year 1874, is used. The form of the instrument, shown in Fig. 330 is a slight modification of the original, in main due to Gaede, and consists of a cylindrical or spherical bulb B, *of known volume*, ending above in a *graduated capillary tube* CA, and connected to a reservoir of mercury R and a side-tube EF, which can be put into communication with the vessel or the pump in which the pressure is desired to be determined. A side capillary tube G is attached to it, as shown, whose diameter is the same as that of CA. Its use is to counteract the depression of the mercury column in CA due to capillarity, for, being of the *same* diameter as CA, the depression of the mercury column in it is also the same as that in CA.

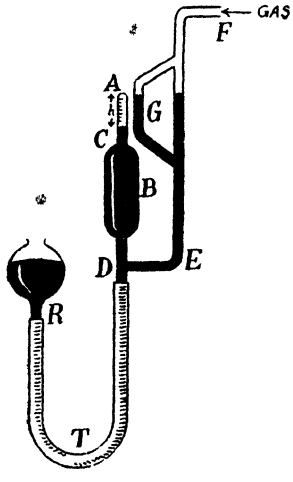


Fig 330

When the reservoir R is lowered until the mercury falls below the bend D, the bulb B and the vessel in which the pressure is to be measured are put into communication with each other and the bulb is filled with the gas, whose pressure P is to be determined. On raising the reservoir, mercury rises into the bulb as well as the side tube, thus cutting off EF from B, and the gas enclosed in the bulb is compressed as mercury rises further and further up into it, until the whole of it is forced into the capillary tube CA. The reservoir is raised still further, until the whole of the bulb B and a part of the capillary tube CA are filled with mercury, and the mercury in the capillary tube G, attached to EF, rises up in a level with the top end A of CA. The depression of the column CA having been compensated for, as explained above, the difference of level between the columns of mercury in the two capillary tubes CA and G, or what is the same thing, between the end A of CA, and the top of mercury column in it (because A is at the same height as the mercury column in G) gives straightaway the pressure of the gas in CA. Let it be h *cms.*

Then, if V be the volume of the capillary tube CA and the bulb B up to the bend D and v, the volume of the gas after the mercury has risen into it, we have $P \times V = h \times v$.

Or, $$P = h.v/V,$$

whence P, the pressure of the gas, can be easily calculated.

It will be readily seen that the greater the value of V and the smaller that of v, the smaller the value of P that can be measured. Thus, the sensitiveness of the gauge depends upon the ratio V/v.

Now, although quite an efficient guage,—so much so that practically all other types of gauges are calibrated with reference to it,—*the performance of the McLeod guage becomes somewhat erratic in the presence of easily condensible vapours.* This can, however, be easily remedied by introducing a liquid air trap in between the guage and the high vacuum side (*i.e.*, the vessel in which the pressure is to be determined). In fact, the liquid-air trap must be used even otherwise to prevent any mercury vapours entering the evacuated vessel.

Apart from this, there are quite a few other drawbacks in the form of the instrument, discussed above. Thus, for example, *it is rather inconvenient to manipulate the reservoir with such a large amount of mercury in it ; and the mercury which remains in contact with the rubber of the flexible tube is likely to get contaminated due to the presence of sulphur in the composition of rubber.* This latter trouble has, in recent years, been sought to be got over by the use of a tube of stainless steel in place of rubber. Even so, a better modification of the gauge is the one described below.

284. Improved modification of McLeod Gauge In this improved version of the gauge, the reservoir, with its attached flexible rubber tube is dispensed with altogether. Instead, tube D is made longer and fitted into a rubber bung in one mouth of a *Woulff's bottle* W, so as to dip inside mercury contained therein, as shown in Fig. 331. Into the other mouth of the bottle is fitted a side-tube N, connected through a stop-cock S, to (*i*) a small *soda lime tower* T, and (*ii*) a tube L, leading to *some simple form* of a backing pump. The tower T has a packing of glass wool at either end to prevent any particles of soda-lime getting into the gauge, and is connected at the top to a long capillary tube J, through a small rubber tubing, provided with a spring-clip, so as to enable it to be put into communication with, or cut off from, the outside air, as desired.

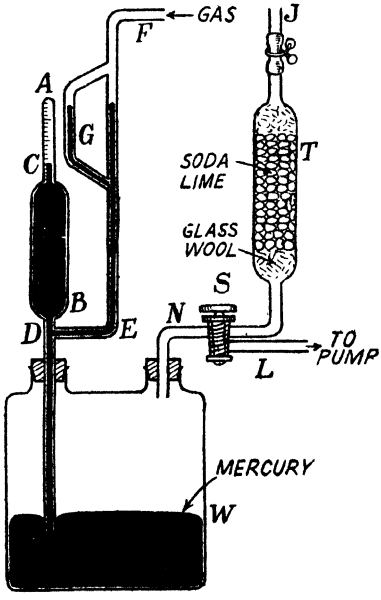


Fig. 331.

The procedure consists in first putting the Woulff's bottle in communication with the pump, through the stop-cock, so that the whole of the mercury in the gauge comes down into the bottle, the pressure throughout being the same as produced by the backing pump, and, of course, very much lower than that of the atmosphere. The communication between the bottle and the pump is now cut off and that between the former and the soda-

lime tower partially established by a slight rotation of S, so that the air from outside gradually enters the bottle, losing its moisture during its passage through soda-lime in the tower. This results in an increase of pressure on the surface of mercury in the bottle and it being forced up into D. The gauge is then used in the manner already explained. It will easily be seen that the labour involved in moving the reservoir up and down for necessary adjustment of the mercury columns in C and G, and the possibility of contamination of mercury are both thus neatly obviated.

When all is said and done, however, the McLeod gauge still continues to have its inherent defects of being rather unwieldy in size and its inability to give a continuous record of pressure changes in the vessel. Then, again, the use of the liquid-air trap adversely affects the rate of pumping and the readings obtained on the gauge may not be truly indicative of the actual pressure inside the vessel at any given instant. In any case, its readings in the last lap of its range, from 10^{-4} to 10^{-5} *mm.* are really never quite so reliable.

285. The Pirani Resistance Gauge. It is fairly well known that whereas at high pressures, the thermal conductivity (K) of a gas is quite independent of pressure, at preasures below 10^{-2} *mm.* of mercury column, when the mean free path of the gas molecules is of the same order of magnitude as the diameter of the containing vessel, it is directly proportional to the pressure (p) *i e.*, K is a linear function of p. Or, $K = \alpha.p$, where α is a constant.

This fact, first made use of by **Warburg,** in the year 1907, for the measurement of low pressures, is really the basis of the *Pirani gauge*, *P.G* in Fig. 332 (*a*), which consists of a tungsten or platinum filament (F), enclosed in a small detachable glass bulb (B),* very much similar in construction to that of the 'cage-type' incandescent lamp and maintained at a temperature, higher than that of the surroundings. The bulb is open at the lower end which can be connected to the vessel in which the pressure is to be determined.

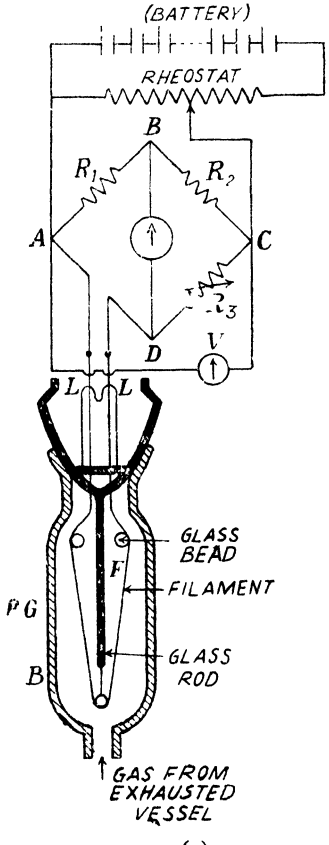
With change in the pressure of the gas in between the filament and the walls of the bulb, the rate of heat conduction across the gas also changes, resulting in a change in the temperature of the filament and hence in its resistance. We measure this change in the resistance of the filament which gives the change in the thermal conductivity and hence, indirectly, the pressure of the gas. A *calibration curve* for the gauge, is, therefore, plotted by measuring simultaneously the resistance of the filament and the pressure of the gas around it,—the former by means of a Wheastone's bridge, to which a constant potential difference is applied, sufficient to heat the filament to a temperature of about 120°C, and the latter, by means of a Mcleod gauge. The pressure, corresponding to any value of the resistance of the filament, can then be read directly on this curve.

Now, for the reason already stated, this calibration curve is a straight line, so long as the pressure of the gas is below 10^{-2} *mm*, but at higher pressures, the relation between K and p no longer remains linear and K varies in a somewhat complicated manner with both the pressure of the gas and the temperature of the surroundings.

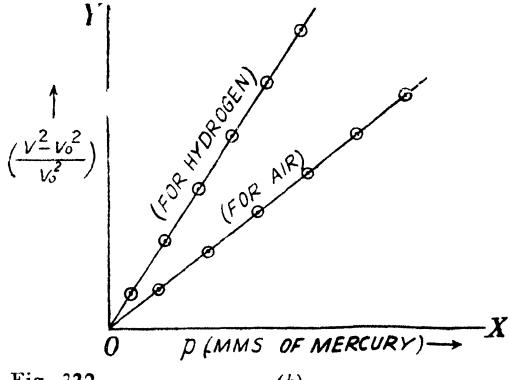
*There are also other forms of the gauge in which the bulb is not detachable.

In fact, the relation between K and p can be represented by no simple formula, with the result that the calibration curve now ceases to be straight. To tide over this difficulty, **Campbell** suggested that instead of keeping the voltage across the bridge constant and measuring the resistance of the filament, the temperature, and hence the resistance, of the filament must be kept constant and the potential difference, required to be applied to the bridge for the purpose, must be measured.

Accordingly, the gauge (*P.G.*) is connected in one arm of the bridge, [Fig. 332 (*a*)], together with fixed resistances R_1 and R_2 and the variable resistance R_3 in the other three arms, as indicated,—all these resistances being made of an alloy like '*manganin*' and '*minalpha*', having an almost zero coefficient of temperature. The potential difference is applied to the bridge at A and C, by means of a *potential-divider*, *i.e.*, through a rheostat included in the battery circuit, so as to be varied at will, and its value read on the voltmeter V connected across the bridge terminals A and C, as shown. To ensure constancy of the temperature of the surroundings of the filament, the bulb of the gauge is placed in a thermostat at 0°*C*.



(*a*)



(*b*)

Fig. 332

Procedure. (*i*) With a known potential difference across A and C, the bridge is balanced in the usual manner by adjusting the variable resistance R_3, so that the deflection in the galvanometer is *zero* and there is sufficient current through the filament of the gauge to raise its temperature to about 100°*C*. Now, every time the pressure changes, the voltage across A and C is adjusted to restore the balance of the bridge and make the galvanometer deflection *zero*. Then, if θ be the excess temperature of the filament over that of the surroundings, the heat-loss along the leads L and L, *assumed small*, will be proportional to θ, say equal to $\beta\theta$, where β is a constant. And, if V be the

voltage applied across the bridge, the heat dissipated per second in the filament is equal to αV^2, where α is another constant And, further, if p be the pressure of the gas around the filament, the heat lost per second by conduction across it is $f(p)$, where $f(p)$ is some function of pressure. We, therefore, have

$$\alpha V^2 = \beta\theta + f(p) \qquad \ldots (i)$$

Now, for the same value of θ, but with $p = 0$, (*i.e.*, with vacuum around the filament), if the voltage required to be applied to the bridge be V_0, clearly, $\alpha V_0^2 = \beta\theta$. $\ldots (ii)$

Dividing relation (*i*) by (*ii*), therefore, we have

$$\frac{V^2}{V_0^2} = 1 + \frac{f(p)}{\beta\theta}. \quad \text{Or,} \quad \frac{V^2 - V_0^2}{\beta\theta} = kf(p),$$

where $1/\beta\theta = k$, a constant, which is almost quite independent of the material and the length of the filament and varies only with the nature of the gas.

But, as we have seen, measurement of the pressure of the gas by means of a McLeod (or any other absolute) gauge shows that, $f(p) = \gamma p$, where γ is also a constant So that

$$(V^2 - V_0^2)/V_0^2 \quad k\gamma p,$$

i.e., $(V^2 - V_0^2)/V_0^2$ is directly proportional to pressure.

If, therefore, we plot a graph between p and $(V^2 - V_0^2)/V_0^2$ for the gas, we get a straight line, its inclination with the axis of p depending upon the nature of the gas in question, as is clear from the two curves drawn by Campbell for hydrogen and air, [Fig. 332, (*b*)], where p was measured by a McLeod gauge.

(*ii*) In actual practice, it becomes rather tedious to use Campbell's method and, therefore, the following simplified procedure is adopted

The bridge is first balanced with only vacuum about the filament and then keeping the voltage across the bridge constant at this very value, the gas or air is allowed into the gauge. The balance of the bridge is thus naturally upset, and a current, corresponding to this upset, or the 'out of the balance current', as it is aptly called, passes through the galvanometer, the deflections of which, in terms of scale divisions are noted for various values of pressure of the gas (as indicated by a McLeod gauge).

Then, if N be the number of scale divisions through which the galvanometer needle is deflected for a pressure p of the gas surrounding the filament, we have $N \propto f(p)$. A graph between the two is, therefore, a straight line and gives the required calibration curve for the gauge, from which the pressure of the gas for any deflection in the galvanometer can be read off straightaway.

Essential points of the gauge. For the success of the gauge as a low-pressure-measurer, the essential points are that (*i*) *the material of the filament must have a high coefficient of temperature*, so that the change in its resistance must be appreciable for a small change in its temperature It is, therefore, made of a tungsten or a platinum wire, of a diameter of about ·06 *mm* ; (*ii*) *the filament must throughout be kept taut*, so that the distance between it and the walls of the enclos-

ing bulb remains unlatered ; (*iii*) *the heat losses along the filament-support must be as small as possible.*

To ensure both these conditions (*ii*) and (*iii*), a poor conductor of heat like a glass rod is used by way of support for the filament and it is taken round glass beads, as shown, (Fig 332), with its longer portions equidistant from the walls of the bulb on either side ; and (*iv*) *the galvanometer should have a high current sensitivity.*

Range, Utility and Drawbacks of the Gauge. The range of the gauge is rather small, *viz.*, between 10^{-2} to 10^{-4} *mm.* of mercury, but because of its almost instantaneous action, it is extremely useful for the measurement of pressure fluctuations.

Its chief drawbacks are that (*i*) *it is wholly unsuitable for use with organic vapours, as its filament gets 'poisoned' by them;* (*ii*) *it is rather much too sensitive to sudden or accidental thermal or mechanical shocks and vibrations*, which must, therefore, be avoided as far as possible. In fact, as a safeguard against the latter disturbance, it is usual to provide a shock-absorbing mounting for it ; (*iii*) *it is not an absolute gauge and has to be calibrated against a McLeod or some other absolute gauge*; (*iv*) *at pressures below* 10^{-3} *mm. of mercury column the heat loss occurs more by radiation than conduction* ; (*v*) *pressures below* 10^{-4} *mm. of mercury cannot be measured with its help, with any reasonable amount of certainty.*

Finally, it may as well be mentioned that, like most other gauges in the pressure range 10^{-3} to 10^{-5} *mm.* of mercury, this too requires some manual adjustments to be made, which are never quite as reliable as mechanical or automatic ones. This has, however, been remedied comparatively recently, (in the year 1939) by **Scott** by his clever introduction of a Triode valve in the gauge circuit, thereby making the working of the gauge at once quick, smooth and automatic.

286. Thermocouple Gauge. It is just a variant of the Pirani Hot Wire or Resistance Gauge, in which instead of measuring the resistance of the filament, we measure the temperature of the hot junction of a thermocouple, attached to the filament, from the thermo-electric *e m.f* developed in it. The temperature of the hot junction will obviously depend upon the thermal conductivity K of the gas in-between itself (or the filament) and the walls of the containing glass bulb the outside of which is maintained at $0°C$ and which is connected at its upper open end (Fig. 333) to the vessel in which the (low) pressure is to be measured. The value of K for the gas, at such low pressures, is as we have seen directly proportional to its pressure.

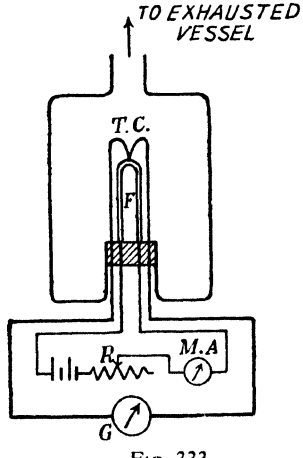


Fig. 333.

The filament F here, is a short ribbon of constantan and is heated by a current of upto 50 *milliamperes* and the hot junction of a suitable thermocouple TC —*ir n-constantan, chromel-alumel antimony bismuth*, or any other—is attached to the mid

point of the filament, which is connected through a rheostat R to a low-voltage battery and a milli-ammeter $M.A.$ The thermocouple circuit includes a sensitive galvanometer, the deflections of which give an indication of the thermo-electric $e.m.f.$ developed and hence that of the pressure of the gas.

The gauge is, as usual, calibrated against a McLeod gauge with the *same* gas in both, if strict accuracy is aimed at

287. Ionisation Gauges. Ionisation, as we know is the process of knocking out an electron from the outer shell of a gas atom. Among other ways, this may also be done by means of a fast moving electron colliding against the gas atom, the process then being spoken of as *ionisation by electron collision.* Before an atom can thus ionise a gas, however, it must possess a certain minimum amount of energy, depending upon the gas in question, and must, therefore, be accelerated through a certain minimum potential difference, called the *ionisation potential*, V_i, say, for that particular gas. The energy thus acquired by the electron is measured in terms of *electron-volts,* where one electron volt is the energy acquired by an electron on being accelerated through a potential difference of 1 volt.

Now, when an electron is thus knocked off from a gas atom, naturally what remains behind is a positive ion, since the atom, as a whole, is neutral. Thus, ionisation produces positive ions and electrons (negative). If these positive ions be collected on another auxiliary electrode (*i.e.*, an electrode in addition to the positive electrode or the anode and the negative electrode or the cathode), we get a positive ion current, or an ionisation current, for a given value V of the accelerating voltage above the ionisation potential V_i. *This ionisation, or the positive ion, current, at low pressures, below* 10^{-3} *mm of mercury, varies linearly with the pressure of the gas.* because at such pressures, an electron is hardly likely to collide with more than one atom on its way from the cathode to the auxiliary electrode. Thus, in the ordinary Triode valve, the grid* may very well act as the auxiliary electrode, if it be given a negative potential with respect to the filament ; so that, any Triode valve may be used as an ionisation gauge. In order to avoid the possibility of electrical leaks between the electrodes, however, the triodes, meant to be used as ionisation gauges, are specially constructed with this end in view.

Since the electrons are emitted on heating the filament or the cathode, such a gauge is called a *hot cathode ionisation gauge* and the first satisfactory form of it was due to *Dushman* and *Found.* A later modification, now in common use is the one shown in Fig. 334. The usual *tungsten filament* F is here supported on a *glass rod* R in the manner indicated, with a co-axial *grid* G, (of tungsten or molybdenum) around it, and a nickel or silver coating on the interior of the glass bulb, enclosing the two, acts as the plate P, with a platinum wire

*In a triode valve, the function of the filament is to emit electrons, when heated by the current from a low voltage battery, that of the plate (which is a metal cylinder around the filament and is ordinarily connected to the positive of a high voltage battery) to attract them. and that of the grid, (which is a spiral, a wire-gauge or a perforated metal cylinder in-between the filament and the plate) to control their flow.

w sealed on to it, to enable it to be connected to the external electrical circuit. In order to prevent deposition of any metal film on it, (from the filament or the plate) and thus cause inter-electrode leaks, the glass rod is provided with loose glass collars *C*, *C*, as shown.

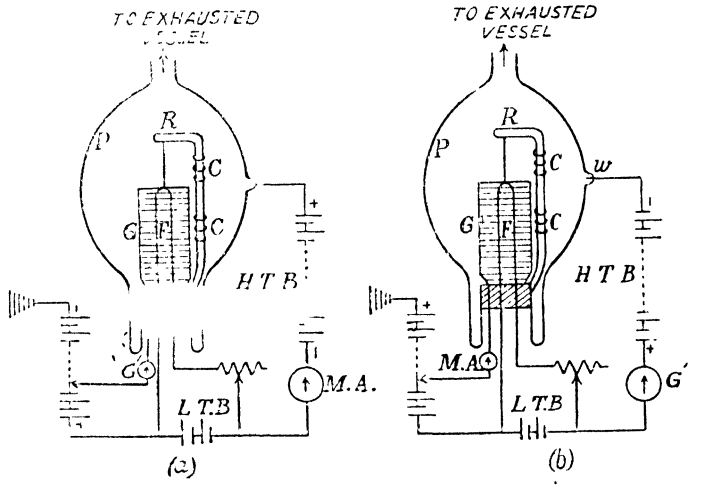


Fig. 334

Now, the positive ions may be collected either on the grid or on the plate, as desired, the electrical connections necessary for the purpose being shown in Fig. 334 (*a*) and (*b*), with the grid being given a negative potential in the former and a positive potential in the latter case, with respect to the filament, and with the plate connected to the positive and the negative poles of the high voltage or high tension battery (*H.T.B.*) of about 120 volts, in the two cases respectively. A milliammeter *M.A.* is included in the plate circuit and a galvanometer *G'* in the grid-circuit, in the first case and *vice versa* in the second.

Working.—(*i*) *First Case.* In this case, the plate being at a positive potential with respect to the filament, electrons emitted by the latter (when heated by the current from the low voltage battery *L.T.B.*) are attracted towards the plate and pass through the meshes or holes in the grid. Thus, on their way to the plate, they bring about ionisation of the gas between the grid and the plate. The positive ions formed are collected by the grid which is at a negative potential with respect to the filament. A small ionisation current, or a positive ion current, thus flows through the grid-filament circuit and can be easily read on the galvanometer *G'*, included in the circuit, the usual electron current being given by the milliammeter *M.A.*

(*ii*) *Second Case.* Here, *since the grid is at a positive and the plate at a negative potential with respect to the filament,* the electrons emitted by the filament or the hot cathode, are strongly attracted by the grid (it being nearer to the filament), but quite a number of them

get through it on account of their momentum and thus ionise the gas in the space between the grid and the plate (in case, of course, the accelerating voltage applied is higher than the ionisation potential for the gas). The positive ions thus released are collected at the plate, any electrons straying into the region being repelled back by it. The positive ion, or the ionisation, current is then read on galvanometer G' included in the plate circuit and the electron current on the milliammeter, as before.

Of the two arrangements discussed above, the latter, *i.e.*, the —ve plate one, is by far the more sensitive, but the first one is simpler to work with.

As already indicated, the relation between the ionisation current and pressure is a linear one only at pressures below 10^{-3} *mm.* of mercury. If, however, the emission of electrons from the filament be relatively small, the relation also becomes linear at higher pressures, but, then, the sensitivity of the gauge is considerably impaired.

This gauge too is not an absolute one and has to be calibrated against a McLeod gauge, with the *same* gas in it as the one, the pressures of which is to be determined by the ionisation gauge.

Once calibrated, the galvanometer G' is replaced by a micro-ammeter, graduated in pressure units.

The gauge can measure much lower pressures, in the range 10^{-3} *mm.* to 10^{-7} *mm* of mercury column. It has also other advantages over the McLeod gauge in that (*i*) *it can be used to measure pressures of both vapours and gases* and (*ii*) being very much smaller in size, *it can be located in close proximity with the vessel being exhausted.*

Among its drawbacks may be mentioned the fact that *its manipulation is somewhat complicated* and that *it requires quite a lot of extra electrical equipment with it.* Also, *its sensitivity* depends upon the nature of the gas, the arrangement of its electrodes and the electric circuit employed. Then, again, organic vapours *'poison'* its filament, *i.e.*, reduce the emission of electrons from it. As a necessary precaution, therefore, a *'cold trap'* of carbon dioxide snow or acetone is arranged in-between the gauge and the exhausted vessel to get rid of them. And, in case an oil diffusion pump is being used to exhaust the vessel, some sort of a *'baffle'* must be used to prevent any oil molecules streaming back into the gauge and thus vitiate its working. And, finally, at higher pressures, the life of the gauge is shortened, as much due to the bombardment of the filament by the ions as to the possibility of the chemical reaction with the gas around it.

288. α-ray Ionisation Gauge. This is the latest form of an ionisation gauge, in which, as indicated by its very name, the ionisation of the gas is brought about by means of α-particles from a radio-active substance. And, since it is not necessary to heat the cathode here, it may be called a *cold cathode ionisation gauge.*

Designed by *Downing* and *Mellen* in the year 1946, it consists of a closed ionisation chamber C, (Fig. 335), inside an outer protective

shell, and perforated at its top and bottom to allow free access to the gas inside it. At the bottom of the chamber is securely held in position a small saucer-shaped plaque P, 1 cm^2. in area, and with its upper surface made of an alloy of gold and radium, which is in equilibrium with its products of decay, *viz.*, *radon*, *radium A* and *radium B*, of which the first one is a gas. To prevent any of this gas escaping out, the upper surface of the plaque is electrolytically coated with a layer of nickel which also serves the additional purpose of preventing contamination by mercury vapour. The losses from the plaque (due to radio-active emission) are so small that the instrument needs to be tested only once in a number of years. This plaque thus forms a highly efficient α-ray emitter, though with a slow emanating power.

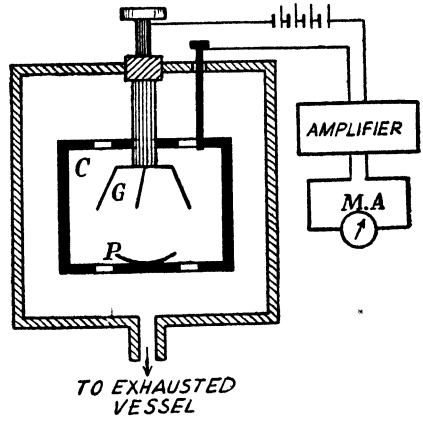


Fig. 335.

The inner electrode or the grid G consists of four wires spread or stretched out, as shown. This limits the distance to be covered by the positive ions produced by the ionisation of the gas (by the α-particles), thus facilitating their 'capture' before they have time to cover longer distances. For, in the latter case, the ions may re-unite and thus the linear relation between the ionisation current and pressure may no longer remain valid and the whole basis of the gauge knocked out The small ionisation current produced is first amplified and then read on the microammeter *M.A.*, the gauge otherwise functioning just like the hot cathode one discussed earlier, and has, likewise, to be calibrated against a McLeod Gauge.

Now, although the relationship between ionisation current and pressure of the gas no longer remains valid beyond a pressure of 10 *mm.* of mercury, *the gauge may be used* (in three successive stages) *to measure pressures within a wide range from* 10^{-3} *mm. to* 1000 *mm.* Further, the gauge is *a continuous reading one.*

The one serious drawback of the gauge is that extra precautions are necessary to work with it, if one is to save oneself from the hazards of exposure to the radio-active substance used.

289. The Knudsen Gauge This simple and efficient gauge, which has been used to measure the lowest pressures yet produced*, depends upon what is called the *radiometric effect*. It therefore, becomes necessary to first understand clearly what this '*effect*' is.

Radiometric Effect. At high and moderately high pressures, a gas behaves like a viscous liquid and its flow through narrow tubes is governed by *Poiseuille's law*, the rate of flow being proportional to

*Up to 3×10^{-8} *mm.*, by *Woodrow*, with a temperature difference of 100°*C* between the two plates. And, up to 5×12^{-9} *mm.*, by *Shrader* and *Sherwood* who used a slight modification of it.

the *fourth power* of the radius of the tube and being limited by the frequency of intra-molecular collisions. At very low pressures, however, the mean free path of the molecules becomes greater than the radius of the tube, the frequency of collisions with the containing walls preponderates over the frequency of intra-molecular collisions and considerations of conductivity and viscosity etc cease to be of any consequence. The flow of a gas, under these conditions is very suggestively called '*molecular flow*' by *Knudsen*, whose extensive work on the behaviour of gases under *vacuum* conditions has earned its deserved recognition in a gas, at such pressures, being known as a '*knudsen gas.*'

He assumes that a typical gas-molecule is accelerated by the pressure gradient along the tube and that all its drift velocity is lost when it collides against a wall ; for, according to him, the smoothest walls are '*molecularly rough*', having, here and there, minute projections of molecular proportions, due to the piling up of one or two atoms above the surrounding ones, so that the directions of motion of the molecules, before and after striking the wall, bear no relation to each other.

Assuming these '*knudsen conditions*', if a molecule acquires extra energy on rebounding from a heated surface, it will collide with the wall a number of times before it reaches the surface, again. It follows, therefore, from the *law of conservation of momentum*, that there is a net force acting normally to the heated surface, which for moderate differences of temperature between the surface and the wall, is directly proportional to these differences of temperature.

This mechanical force exerted between two surfaces, very close to each other and maintained at a difference of temperature, is called **radiometric effect***. Since it becomes manifest only when the mean free path is longer than the distance between the heated surface and the wall, it is a typically *Knudsen effect* and must not be confused with that physical effect of radiation we call *radiation pressure*—the latter being (*i*) quite independent of the presence or absence of any gas, (*ii*) about twice as great on a polished as on a blackened surface, and (*iii*) of a much smaller order of magnitude.

The Gauge Proper. Knudsen very ingeniously exploited and harnessed these radiometric forces at low pressures in devising his *absolute gauge*†, which enables us to measure pressure in an evacuated vessel, from 10^{-3} *mm.* down to 10^{-7} *mm.*, by simply noting the deflection of a cold plate suspended in the vessel, due to its bombardment by the molecules rebounding from a nearby hot plate. If the dimensions of the plates be very large, compared with the distance between them, (as though we were considering only a very small

*A familiar example of this effect is the *Crook's radiometer*, so often used to detect thermal radiations. It consists of a freely pivoted 'spider', fitted with mica vanes with their planes, parallel to the vertical axis, and coated on one side with lamp black, the other being polished, the whole instrument being arranged inside an evacuated glass-bulb. On exposure to thermal radiation; the blackened sides of the vanes become hotter than the polished ones and radiometric forces come into play, making the spider rotate, blackened side moving away from the incident radiation.

†It is termed '*absolute*' because it gives the pressure directly in absolute units, *i.e.*, in *dynes* cm^2.

portion of an infinite area), so that all edge-effects could be neglected, and, further, if this distance be smaller than the mean free path of the molecules, so that the molecules arrive at one, in exactly the same condition in which they leave the other, the deflecting repulsive force on the cold plate is found to be proportional to the gas pressure in the vessel, right up to a pressure of 10^{-7} *mm.*

Fig. 336 shows diagrammatically the essentials of the gauge, where P_1 and P_2 are two fixed plates, kept electrically heated and arranged on opposite sides of a cold plate A, in the form of a rectangular picture-frame, suspended (in the evacuated vessel) by means of a quartz-fibre carrying a mirror M to enable its deflections to be measured by the usual '*lamp and scale*' method.

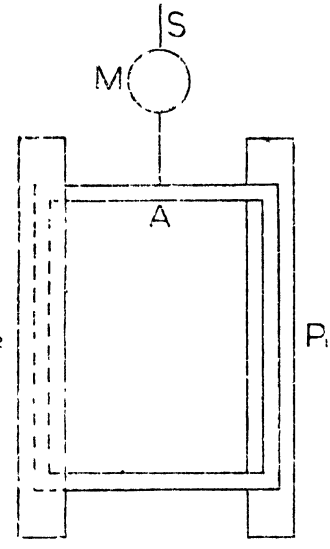


Fig. 336

Let the temperature of P_1 and P_2 be T_1 and that of A (and hence also that of the evacuated vessel, in which it is suspended) be T_2 and let n_1 and n_2 be the *number of molecules per cubic centimetre*, travelling from P_1 to A and from A to P_1, with *root mean square velocities* $\bar{c}_1$ and $\bar{c}_2$ respectively. Then since in the equilibrium state, the rate of molecular collisions *per square centimetre* must be the same, we have

$$n_1\bar{c}_1 = n_2\bar{c}_2. \qquad \ldots (i)$$

And, if a be the *number of molecules per c.c.* in the vessel, outside the space between P_1 and A, then, since the number of molecules flowing out of the latter into the former and *vice versa*, must be the same, we have

$$n\bar{c}_2 = n_1\bar{c}_1 + n_2\bar{c}_2 = 2n_1\bar{c}_1 = 2n_2\bar{c}_2 \quad \ldots \quad (ii)$$ [From relation (*i*) above.

So that, if m be the mass of each molecule, we have

$$\textit{total pressure between the plates} = \frac{mn_1\bar{c}_1^2}{3} + \frac{mn_2\bar{c}_2^2}{3}, \quad [\because P = \tfrac{1}{3}mn\bar{c}^2]$$

whereas *pressure on the sides, remote from* $A = mn\bar{c}_2^2/3$

Obviously, therefore, *excess pressure on the side of A nearer to* P_1, *over that away from it, is given by* $\frac{mn_1\bar{c}_1^2}{6} + \frac{mn_2\bar{c}_2^2}{3} - \frac{mn\bar{c}_2^2}{3}$.

Or, substituting $n\bar{c}_2/2$ for $n_1\bar{c}_1$ and for $n_2\bar{c}_2$, [From relation (*ii*) above. we have

$$\textit{excess pressure on } A = \frac{mn\bar{c}_1\bar{c}_2}{6} + \frac{mn\bar{c}_2^2}{6} - \frac{mn\bar{c}_1\bar{c}_2}{3} = \frac{mn\bar{c}_1\bar{c}_2}{6} - \frac{mn\bar{c}_2^2}{6}.$$

Or, *excess pressure on* $A = \frac{mn\bar{c}_2^2}{6}\left(\frac{\bar{c}_1}{\bar{c}_2} - 1\right)$

Assuming the molecules striking P_1 and A, to take up their respective temperature T_1 and T_2, we have

$$\bar{c}_1/\bar{c}_2 = \sqrt{T_1}/\sqrt{T_2}.$$

[See page 537.]

And, thus, *excess pressure on* $A = \frac{mn\bar{c}_2}{6}\left(\sqrt{\frac{T_1}{T_2}}-1\right)$.

$$= \tfrac{1}{2}.\frac{mn\bar{c}_2^2}{3}\left(\sqrt{\frac{T_1}{T_2}}-1\right)$$

Now, $mn\bar{c}_2^2/3 = p$, the *pressure inside the vessel*

So that, we have *excess pressure on A*, or, *force per sq. cm.* on A, say,

$$F = \frac{p}{2}\left(\sqrt{\frac{T_1}{T_2}}-1\right). \qquad \ldots(iii)$$

If the temperature difference (T_1-T_2) be not very large (*i.e.*, not more than 250°C), we may consider the flow of molecules in-between the space P_1A and the rest of the vessel as a *thermal transpiration** effect between the temperature $\left(\frac{T_1+T_2}{2}\right)$ inside P_1A and T_2, outside ; so that, if p' be the pressure between the plates, we have

$$\frac{p'}{p} = \sqrt{\frac{(T_1+T_2)/2}{T_2}} = \sqrt{\frac{T_1+T_2}{2T_2}} = \sqrt{\frac{2T_2+T_1-T_2}{2T_2}}$$

$$= \left(1+\frac{T_1-T_2}{2T_2}\right)^{\frac{1}{2}} = 1+\tfrac{1}{4}.\frac{(T_1-T_2)}{T_2}.$$

So that, $$p' = p+\frac{p}{4}.\frac{T_1-T_2}{T_2}.$$

And, therefore, $$F = \frac{p}{4}\left(\frac{T_1-T_2}{T_2}\right), \quad \ldots(iv)$$ [From (*iii*), above, since $F = (p'-p)$.

showing that the force exerted on plate A is *quite independent of the nature of the gas in the vessel.*

Now, if a be the area of each vertical strip of A, we have

force acting on each strip $= F.a.$

Since these forces act in *opposite directions* on the two strips, they constitute a couple equal to $F.a.2r$, where r is the *mean* distance of each strip from the suspension wire.

The frame is thus deflected, giving rise to a *restoring torsional couple* in the suspension wire and, therefore, comes to rest, when the two couples just balance each other, say, when it has deflected through an angle θ.

Then, if τ be the torsional couple, set up in the suspension wire per unit twist in it (or per unit deflection of the frame) the total torsional couple tending to restore the frame back to its original position is $\tau\theta$; and we, therefore, have

$$\tau\theta = 2F\,a\,r. \qquad \text{Or,} \quad F = \tau\theta/2ar.$$

*It is the phenomenon of the flow of a gas from a colder to a hotter chamber both containing the gas at a low pressure and connected to each other by means of a capillary tubing, the flow continuing, until a pressure difference, depending upon the temperature difference between two chambers, is established. The ratio between the final pressures attained can be shown to be the same as that between the square roots of the absolute temperatures of the two chambers.

Substituting the value of F in relation (*iv*) above, we have

$$\frac{\tau\theta}{2ar} = \frac{p}{4}\cdot\left(\frac{T_1-T_2}{T_2}\right).$$

Or, $$p = \frac{4T_2.\tau\theta}{2ar(T_1-T_2)} = \frac{2\tau\theta}{ar}\cdot\frac{T_2}{T_1-T_2}. \qquad \ldots (v)$$

Further, if I be the *moment of inertia* of the frame about the *suspension wire* as axis and t, its *time period* of vibration, we have

$$t = 2\pi\sqrt{I/\tau}, \text{ whence, } \tau = 4\pi^2 I/t^2,$$

Hence $$p = \frac{8\pi^2 I.\theta}{t^2\, ar}\cdot\frac{T_2}{T_1-T_2}. \qquad \text{[From (v) above.}$$

whence p, the pressure of the gas in the vessel, can be easily evaluated.

It will be readily seen that apart from the value of p being quite independent of the molecular weight of the gas, the gauge possesses the following advantages :

(*i*) *It gives a continuous indication of pressure in the vessel.*

(*ii*) *It is unaffected by any outside influences.*

(*iii*) *It is stable and yet very sensitive at low pressures, down to* 10^{-7} *mm.*

(*iv*) *It can be used to measure the pressure of all kinds of gases and vapours, irrespective of their mass or condensibility.*

(*v*) *It does not require the use of objectionable liquids, like mercury.*

(*vi*) *It is simple, cheap and easy to construct and work with, though obviously, it cannot be used above a pressure of* 10^{-3} *mm. ; for, then, the mean free path of the molecules becomes comparable with the distance between the plates.*

SOLVED EXAMPLES

1. Given that the gas constant R is $8{\cdot}3\times10^7$ ergs per 1°C, and the atomic weight of chlorine is 35·5, find the root mean square (r.m.s.) velocity of the chlorine molecules at 0°C.

We have the relation, $P = \frac{1}{3}mn\bar{c}^2$. [See page 537]

Multiplying by V, the volume of 1 *gram-molecule* of the gas, we have

$$PV = \tfrac{1}{3}mn\,V\,\bar{c}^2.$$

Now, $nV = N$. [Where N is the *Avogadro number*, or number of molecules in 1 *gram-molecule* of the gas.

$\therefore$ $PV = \frac{1}{3}m.N\,\bar{c}^2$

And, clearly, $mN = M$, the *molecular weight* of the gas.

$\therefore$ $PV = \frac{1}{3}M\bar{c}^2$. Also, $PV = RT$.

So that, $\frac{1}{3}M\bar{c}^2 = RT$. Or, $\bar{c}^2 = 3RT/M$,

whence, $\bar{c} = \sqrt{3RT/M}$.

Here, $R = 8{\cdot}3\times10^7$ *ergs*/°*C*, $T = 0+273 = 273°$ *Abs* and $M = 35{\cdot}5\times2 = 71$ *gms.*, because a chlorine molecule consists of 2 atoms.

$\therefore$ substituting the values of R, T and M, we have

$$\bar{c} = \sqrt{\frac{3\times8{\cdot}3\times10^7\times273}{71}} = 3{\cdot}094\times10^4 \text{ cms./sec.}$$

Or, the root mean square velocity of the chlorine molecule is $3{\cdot}094\times10^4$ *cms./sec.*

2. If the density of nitrogen is 1·25 gms./litre at normal temperature and pressure, calculate the root mean square velocity of its molecule.

We have the relation $P = \frac{1}{3}mn\bar{c}^2$, for a gas.

But $mn = \rho$, the *density* of the gas. Therefore, $P = \frac{1}{3}\rho\bar{c}^2$.

Or, $\rho\bar{c} = 3P$, whence, $\bar{c} = \sqrt{3P/\rho}$

Here $P = 76$ *cms.* $= 76\times13\cdot6\times981$ *dynes/cm².*,

and $\rho = 1\cdot25$ *gms./litre* $= 1\cdot25/1000 = \cdot00125$ *gm./c c.*

$$\therefore \quad \bar{c} = \sqrt{\frac{3\times76\times13\cdot6\times981}{\cdot00125}} = 4\cdot933\times10^4 \text{ cms./sec.}$$

The velocity of the nitrogen molecule is, therefore, equal to $4\cdot933\times10^4$ *cms./sec.*

3. Derive Charles' law and Dalton's law on the basis of the kinetic theory of gases, as also the Standard Gas Equation, PV = RT, and write a brief note on Absolute Zero from the standpoint of the theory.

(*i*) We have $P = \frac{1}{3}mn\bar{c}^2 = \frac{1}{3}\rho\bar{c}^2$. [$\because mn = \rho$, the density of the gas,

Now, $\rho = 1/V$, and $\bar{c}^2 \propto T$ [where V is the volume of the gas.

where T is the *absolute temperature* of the gas

$$\therefore \quad P \propto \frac{1}{3V}.T. \qquad \ldots (1)$$

If V be constant, we have $P \propto T$

Or, at constant volume, the pressure of a given mass of gas varies directly as its absolute temperature, which is *Charles' law.*

Or, we can put relation (1) above, as $V \propto \frac{1}{3P}.T$, so that, if P be constant, $V \propto T$.

Or, *at constant pressure, the volume of a given mass of gas varies directly as its absolute temperature,* which is also *Charles' law.*

(*ii*) Let a number of gases or vapours, having no chemical reaction with each other, and having densities and mean square velocities, ρ_1 ρ_2, $\rho_3\cdots$ and $\bar{c}_1^2$, $\bar{c}_2^2$, $\bar{c}_3^2$ respectively, be mixed together in the *same* volume Then, the resultant pressure P will clearly be given by

$P = \frac{1}{3}\rho_1\bar{c}_1^2 + \frac{1}{3}\rho_2\bar{c}_2^2 + \frac{1}{3}\rho_3\bar{c}_3^2 + \ldots$, considering each set of molecules of the different gases or vapours

Now, $\frac{1}{3}\rho_1\bar{c}_1^2$, $\frac{1}{3}\rho_2\bar{c}_2^2$, $\frac{1}{3}\rho_3\bar{c}_3^2$ are the individual pressures exerted by the different gases or vapours Putting these as $p_1 + p_2 + p_3 \cdots$, we have

$$P = p_1 + p_2 + p_3.$$

i.e., the resultant or total pressure exerted by the mixture of gases is equal to the sum of their individual or partial pressures, which is Dalton's law of partial pressures.

(*iii*) We have $P = \frac{1}{3}\rho\bar{c}^2$.

Multiplying both sides by V, the volume of the gas, we have

$$PV = \tfrac{1}{3}\rho.V.\bar{c}^2.$$

Now, $\rho V = M$, *mass* of the gas. [$\because$ *mass = volume × density*

$$\therefore \quad PV = \tfrac{1}{3}Mc^2 = \tfrac{2}{3}\cdot\tfrac{1}{2}M\bar{c}^2.$$

Now, $T \propto K.E.$ *of the gas,* Or, $T \propto \frac{1}{2}M\bar{c}^2$

Or, $PV \propto T$, Or, $PV =$ a constant $\times T$

Or $PV = RT$,

where R is a constant, called the *gas constant*, whose value, for a given mass of gas, is quite independent of the values of P, V and T.

Thus, the *standard gas equation*, ($PV = RT$), can be deduced from the kinetic theory of gases

(*iv*) According to the kinetic interpretation of temperature, the temperature of a gas is, as, we have seen, proportional to the mean *K. E.* of the molecules, because the *K. E* of a molecule $= \frac{1}{2}m\bar{c}^2$ and $\bar{c}^2$ is proportional to T, the absolute temperature of the gas.

Clearly, $T = 0$, when $\bar{c} = 0$, *i.e.*, when the molecules have zero velocity or are devoid of all motion

Hence, *on the basis of the kinetic theory, the absolute zero of temperature is the temperature at which the gas molecules are devoid of all motion.* This, obviously, is not quite correct, for the above deduction is made on the assumption of the gas being an ideal or a perfect one, and no actual gas approaches this ideal ; so that, even if at the ordinary temperatures and pressures, the actual gases may be taken to approximate, more or less, to the perfect gas condition, these ideal gas conditions do not hold down to the absolute zero

A more satisfactory interpretation of temperature is afforded by **thermodynamics**, which does not require the cessation of all molecular motion at the absolute zero

4. Deduce the relation $\lambda = 1/\pi r^2 n$, for the mean free path of a molecule, and use it to calculate the diameter of a molecule of benzene, if $n = 2{\cdot}79 \times 10^{19}$ mol per c.c and λ for benzene $= 2{\cdot}2 \times 10^{-6}$ cms.

For answer to first part, see page 543.

Substituting the values of λ and n in the above relation, we have

$$2{\cdot}2 \times 10^{-6} = \frac{1}{\pi r^2 \times 2{\cdot}79 \times 10^{19}}.$$

Or, $\pi r^2 \times 2{\cdot}79 \times 10^{19} \times 2{\cdot}2 \times 10^{-6} = 1$ Or, $\pi r^2 \times 2{\cdot}79 \times 10^{13} \times 2{\cdot}2 = 1$

Or, $r^2 = \dfrac{1}{\pi \times 2{\cdot}79 \times 2{\cdot}2 \times 10^{13}}$, whence, $r = \sqrt{\dfrac{1}{\pi \times 2{\cdot}79 \times 2{\cdot}2 \times 10^{13}}}$.

$= 7{\cdot}201 \times 10^{-8}$ *cms.*

Or, the radius of the sphere of influence of the benzene molecule is equal to $7{\cdot}201 \times 10^{-8}$ *cms*

Now, *the radius of the sphere of influence of a molecule is equal to its diameter**

Hence the diameter of the benzene molecule is equal to $7{\cdot}201 \times 10^{8}$ *cms.*

5. Find the mean free molecular path in air, taken as a uniform gas ; given that the density of air $= 1{\cdot}2 \times 10^{-3}$ gms. /cm^3. at 0°C and a pressure of 10^6 dynes/cm^2., and that its coefficient of viscosity $= 1{\cdot}7 \times 10^{-4}$ dynes/cm^2. per unit velocity-gradient.

We have the relation, $\eta = \frac{1}{3}\rho\bar{c}\lambda$, for the coefficient of viscosity of a gas.

Also, $P = \frac{1}{3}\rho\bar{c}^2$,

whence, $\bar{c}^2 = 3P/\rho$, and $\therefore\ \bar{c} = \sqrt{3P/\rho}$

So that, $\eta = \frac{1}{3}\rho.\lambda.\sqrt{3P/\rho} = \lambda\sqrt{3P.\rho^2/9}$

Or, $= \lambda\sqrt{P\rho/3}$

And $\therefore$ $\lambda = \eta\sqrt{\dfrac{3}{P.\rho}}$.

Substituting the value of η, P and ρ, therefore, we have

$$\lambda = 1{\cdot}7 \times 10^{-4}\sqrt{\frac{3}{10^6 \times 1{\cdot}2 \times 10^{-3}}}.$$

$$= 8\ 498 \times 10^{-6}\ cms.$$

Therefore, the mean free molecular path in air $= 8{\cdot}498 \times 10^{-6}$ *cms.*

6. The mean velocity of a molecule of nitrogen gas is $4{\cdot}5 \times 10^4$ cms. per sec. Its density at N.T.P. is $1{\cdot}25 \times 10^{-3}$ gms./c c , and its co-efficient of viscosity $= 166 \times 10^{-6}$ gms /cm. per sec- Calculate (*i*) its molecular mean free path (*ii*) the number of collisions made per second, and (*iii*) its molecular diameter. (Assume $n = 2{\cdot}7 \times 10^{19}$ per c.c.)

---

*Atoms and molecules are really not the simple bodies they were once imagined to be. To speak of the diameter of a molecule in the geometrical sense has, therefore, no meaning *The diameter of a molecule is taken to be the distance between the centres of two molecules beyond which they do not exert any influence over each other, which is clearly equal to the radius of the sphere of influence of the molecule.*

(*i*) We have $\eta = \frac{1}{3}\rho\bar{c}\lambda$, whence, $\lambda = \frac{3\eta}{\rho\bar{c}}$

$$= \frac{3\times 166\times 10^{-6}}{\cdot 125\times 10^{-3}\times 4\cdot 5\times 10^{4}} = \frac{3\times 165\times 10^{-8}}{1\cdot 25\times 10\times 4\cdot 5}$$

$$= 8\cdot 853\times 10^{-3}\ cms.$$

Or, *molecular mean free path, i.e.*, $\lambda = 8\cdot 853\times 10^{-6}\ cms.$

(*ii*) Number of collisions made *per second*

$$= \frac{\bar{c}}{\lambda} = \frac{4\cdot 5\times 10^{4}}{8\ 853\times 10^{-6}} = 5\ 083\times 10^{9}.$$

$\therefore$ number of collisions *per second* $= 5\ 083\times 10^{9}$.

(*iii*) Now, $\lambda = \frac{1}{\sqrt{2}n\pi r^2}$ whence, $r^2 = \frac{1}{\sqrt{2}n\pi\lambda}$.

Or, $$r = \sqrt{\frac{1}{\sqrt{2}n\pi\lambda}} = \sqrt{\frac{1}{\sqrt{2}\times 2\cdot 7\times 10^{10}\times\pi\times 8\cdot 853\times 10^{-6}}}$$

$$= \sqrt{\frac{1}{\sqrt{2}\times 2\cdot 7\times 10^{13}\times\pi\times 8\cdot 853}} = 3\cdot 068\times 10^{-8}\ cms.$$

$\therefore$ the *molecular diameter* of nitrogen $= 0\ 068\times 10^{-8}\ cms$

7. Calculate the molecular K.E. of 1 gm. of Hydrogen (*i*) at 0°C and (*ii*) at 100°C. (R = $8\cdot 3\times 10^{7}$ ergs).

We know that the *K.E.* of 1 *gram molecule* of a gas $= \frac{3}{2}.R.T$

Now, *mol. wt. of Hydrogen* $= 2\ gms.$

$\therefore$ *K.E.* of 1 *gm.* of Hydrogen at 0°C or 273° *Abs.* $= \frac{3}{2}R.T/2$

$$= 3\times 8\cdot 3\times 10^{7}\times 273/4 = 16\cdot 99\times 10^{9}\ ergs$$

And, *K.E* at 100°C or 373° *Abs.*

$$= 16\cdot 99\times 10^{9}\times \frac{373}{273} = 23\cdot 21\times 10^{9}\ ergs$$

Thus, the *K.E.* of 1 *gm.* of Hydrogen is $16\cdot 99\times 10^{9}$ *ergs* at 0°C, and $23\cdot 21\times 10^{9}$ *ergs* at 100°C.

EXERCISE XIV

1. Obtain the expression $\eta = \frac{1}{3}\rho\bar{c}.\lambda$, for the viscosity of a gas, and use it to obtain the mean free path for molecules of benzene vapour, (C_6H_6), at 0°C ; given that η at 0°C = $0\cdot 69\times 10^{4}$ *c.g.s. units* ; *r.m.s* velocity of benzene molecules at 0°C = $2\cdot 95\times 10^{4}$ *cms./sec.*; *atomic wt.* of carbon = 12 ; *c.cs.* in 1 *gm. molecule* = 22,400. **Ans.** $2\ 015\times 10^{-6}$ *cms.*

2. Find the root mean square velocity of the hydrogen molecule at *N.T.P.* the density of hydrogen being 00009 *gm/c.c.*

Ans $1\ 839\times 10^{6}$ *cms./sec.*

3. Calculate the molecular velocity (*i.e.*, *r.m.s.* velocity) in the case of a gas whose density is 1·4 *gms. per litre* at a pressure of 76 *cms* of mercury. Density of mercury = 13·6 *gms./c.c.* ; g = 981 *cms./sec*^{2}. (*Manchester, B.Sc.*)

Ans. $4\ 6\times 10^{4}$ *cms./sec.*

4. Describe the working of a rotary oil pump for producing low pressures. How are these pressures measured ? (*Allahabad, 1950*)

5. Show that the pressure of a gas is equal to two-thirds of the kinetic energy of translation per unit volume. Calculate the kinetic energy of hydrogen *per gm. molecule* at 0°C. (*Allahabad, 1930*)

Ans $5\cdot 67\times 10^{10}\ ergs$

6. Give an account, with a neat sketch, of some form of a gas pump, suitable for the attainment of low pressures. (*Panjab, 1945*)

7. An imperfect barometer tube contained some air above the mercury. When the mercury stood at 18 *inches* above the outside level, the air space was 6 *inches*. On pushing the tube down, so that the air space was only 3 *inches* the mercury stood at 26 *inches*. Find the barometric height. **Ans** 30 *inches*.

8. One hundred litres of a gas at atmospheric pressure are compressed into a cylinder, 100 *cms.* long and 12 *cms.* in diameter. What is the resulting pressure. Ans· 672 *cms. of mercury*

9. Describe, with a neat diagram, the construction and working of (*a*) any modern high vacuum pump and (*b*) a guage which can measure the low pressure this pump produces. (*Madras, 1949*)

10. The reading of a mercury barometer, the brass scale of which was correct, when at 0°C was 76·69 *cms.* the room temperature being 178°C. If the coefficient of cubical expansion of mercury be ·009180 and the coefficient of linear expansion of brass be ·00019 calculate the reading of the barometer reduced to 0°*C*. Ans. 73·54 *cms.*

11. Calculate the number of molecules *per c.c.* of a gas, taking the mean free path as $1{\cdot}83\times10^{-5}$ *cms.* and the molecular diameter equal to $2{\cdot}3\times10^{-8}$ *cms.* [*Hint.* Use the more accurate relation, $\eta = 1/\sqrt{2}\pi r^2.n.$] Ans. $2{\cdot}3\times10^{19}$.

12. Describe in detail, with a neat sketch, the construction and working of a *McLeod Gauge.* What is the recent modification effected in its working ? Enumerate the advantages and disadvantages of the gauge.

13. Explain the principle, construction and working of a good form of diffusion pump with which you are familiar.

14 What are ionisation gauges ? Describe the construction and working of one such gauge, pointing out its advantages and disadvantages over a McLeod Gauge.

15. Describe some form of a *thermocouple gauge.*

16. Describe in detail the *Pirani resistance gauge*, illustrating your answer with a neat diagram and explaining clearly the essential points of the gauge, its range, utility and drawbacks.

17. Explain the principle and working of the *Knudsen gauge.* What are its particular advantages over the other forms of gauges ?

APPENDICES

APPENDIX I

IMPORTANT TRIGONOMETRICAL RELATIONS

1. Trigonometrical Ratios for Compound Angles.

(*i*) $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

(*ii*) $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

(*iii*) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

(*iv*) $\sin(A-B) = \sin A \cos B - \cos A \sin B$.

(*v*) $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

(*vi*) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.

It follows from the above that

(*vii*) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(*viii*) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$.
$= \cos^2 B - \sin^2 A$.

2. Trigonometrical Ratios for Double Angles.

(*i*) $\sin 2A = 2 \sin A \cos A$.

(*ii*) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

(*iii*) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$.

From the above, it follows that

$$\sin 4A = 2 \sin 2A \cos 2A = 4 \sin A \cos A \cos 2A.$$

Also, that $\sin\theta = 2 \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}$

and $\cos\theta = \cos^2 \dfrac{\theta}{2} - \sin^2 \dfrac{\theta}{2}$.

} Putting $2A = \theta$ in (*i*) above.

3. Other Important Relations.

(*i*) $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$. (*ii*) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$.

(*iii*) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$. (*iv*) $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A$.

(*v*) $\dfrac{\sin 2A}{1 - \cos 2A} = \cot A$. (*vi*) $\dfrac{1 - \cos A}{\sin A} = \tan \dfrac{A}{2}$.

(*vii*) $\dfrac{1 + \cos A}{\sin A} = \cot \dfrac{A}{2}$. (*viii*) $\cot A - \tan A = 2 \cot 2A$.

(*ix*) $\cot 2A = \dfrac{\cot^2 A - 1}{2 \cot A}$. (*x*) $\left(\sin \dfrac{A}{2} + \cos \dfrac{A}{2}\right)^2 = 1 + \sin A$

(*xi*) $\left(\sin \dfrac{A}{2} - \cos \dfrac{A}{2}\right)^2 = 1 - \sin A$.

4. Products in Terms of Sums and Differences :

(*i*) $2 \sin A \cos B = \sin(\text{sum}) + \sin(\text{difference})$.
i.e., $= \sin(A+B) + \sin(A-B)$.

(*ii*) $2 \cos A \sin B = \sin(\text{sum}) - \sin(\text{difference})$.
i.e., $= \sin(A+B) - \sin(A-B)$.

(*iii*) $2 \cos A \cos B = \cos$ (*sum*) $+ \cos$ (*difference*)
i.e., $= \cos(A+B) + \cos(A-B)$.

(*iv*) $2 \sin A \sin B = \cos$ (*difference*) $- \cos$ (*sum*)
i.e., $= \cos(A-B) - \cos(A+B)$.

5. Sum or Difference in Terms of Products.

(*i*) *Sum of two sines* = 2 *sin* (*half sum*)· *cos* (*half difference*)

Thus, $\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$.

(*ii*) *Difference of two sines* = 2 *cos* (*half sum*)·*sin*(*half difference*)

Thus, $\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)$.

(*iii*) *Sum of two cosines* = 2 *cos* (*half sum*).*cos* (*half difference*)

Thus, $\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$.

(*iv*) *Difference of two cosines* = 2 *sin* (*half sum*).*sin* (*half difference reversed*).

Thus, $\cos A - \cos B = 2 \sin\left(\frac{A+B}{2}\right).\sin\left(\frac{B-A}{2}\right)$.

6. Trigonometrical Relations for the Three Angles of a Triangle, (*i.e.*, when $A+B+C = 180°$).

Here, because the sum of any two angles is the supplement of the third, we have

(*i*) $\sin(B+C) = \sin A$. (*ii*) $\cos(A+B) = -\cos C$. (*iii*) $\tan(C+A) = -\tan B$.

And, since $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = 90°$, *i.e.*, each half angle is the complement of the sum of the other two half angles ; so that, we have

(*iv*) $\sin\frac{C+A}{2} = \cos\frac{B}{2}$. (*v*) $\cos\frac{A+B}{2} = \sin\frac{C}{2}$. (*vi*) $\tan\frac{B+C}{2} = \cot\frac{A}{2}$.

Also, we have

(*vii*) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$.

(*viii*) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$.

(*ix*) $\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$.

(*x*) $\tan\frac{B}{2}.\tan\frac{C}{2} + \tan\frac{C}{2}.\tan\frac{A}{2} + \tan\frac{A}{2}.\tan\frac{B}{2} = 1$.

7. Relations between Sides and Angles of a Triangle :—

(*i*) *The sides of a triangle are proportional to the sines of the angles opposite to them.* Thus, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

(*ii*) In any triangle. $a^2 = b^2 + c^2 - 2bc \cos A$.

$b^2 = c^2 + a^2 - 2ca \cos B$; and $c^2 = a^2 + b^2 - 2ab \cos C$.

Also, from the above, we have

$\cos A = \frac{b^2+c^2-a^2}{2bc}$; $\cos B = \frac{c^2+a^2-b^2}{2ca}$; and $\cos C = \frac{a^2+b^2-c^2}{2ab}$.

APPENDIX II

LOGARITHMS

1. The *logarithm* of a number to a *given base* is the *index of the power* to which the base must be raised in order to equal that number. Thus, if $a^x = M$, then, x *is the logarithm of the number* M *to the base* a, and we may put it as $\log_a M = x$. So that, both the expressions, $a^x = M$ and $\log_a M = x$, really mean the same thing.

There are two systems of logarithms in use, *viz.*,

(*i*) **Natural or Napierian Logarithms,** invented by **Napier.** *These are to the base e,* where the value of e is 2 17828, and are used in Calculus and other branches of higher mathematics.

(*ii*) **Common Logarithms,** invented by **Briggs** a contemporary of Napier. *These are to the base* 10, and are commonly used in all arithmetical calculations.

The base is usually omitted in writing, in either case, once we know in our minds which system we are using We shall here concern ourselves only with *common logarithms, i.e., to the base* 10. Thus, expressions like log 2 or log π, etc., mean $\log_{10} 2$, $\log_{10} \pi$, *i e* , logarithm of 2 or π to the base 10.

2. Fundamental General Relations.

(*i*) *The logarithm of* 1 *is* 0, or *log* $1 = 0$.

(*ii*) *The logarithm of the base itself is* 1, *or log* $10 = 1$.

(*iii*) *The logarithm of the product of two or more numbers is equal to the sum of the logarithms of the individual numbers.* Thus,

$$\log (M \times N \times K) = \log M + \log N + \log K.$$

(*iv*) *The logarithm of a fraction is equal to the logarithm of the numerator* **minus** *the logarithm of the denominator.* Thus,

$$\log \frac{M}{N} = \log M - \log N.$$

(*v*) *The logarithm of a number, raised to the power n, is equal to n times the logarithm of that number.* Thus,

$$\log M^n = n \log M,$$

and

$$\log \sqrt{M} = \log (M)^{\frac{1}{2}} = \tfrac{1}{2} \log M.$$

3. Characteristic and Mantissa. The integral part of a logarithm is called its *characteristic,* and may be positive or negative And, its fractional part, expressed as a decimal, is called the *Mantissa,* and is *necessarily positive.*

4. Determination of the Characteristic of the Logarithm of a Number.

(*i*) *The characteristic of the logarithm of a number, greater than* 1, *is one less than the number of digits in its integral part, and is positive.* Thus, the characteristics of the logarithms of the numbers, 525, 25 and 5 are 2, 1 and 0 respectively.

(*ii*) *The characteristic of the logarithm of a number, less than* 1, *is one more than the number of zeros immediately after the decimal point, and is negative,—the negative sign being placed above the characteristic, and read as* '*bar*'. Thus, the characteristics of the logarithms of ·254, ·0254 ·00254, and ·000254 are $\bar{1}$, $\bar{2}$, $\bar{3}$ and $\bar{4}$ respectively, and are read as *'bar one', 'bar two', 'bar three',* etc.

5. Determination of the Mantissa of the Logarithm of a Number.

The mantissae for the logarithms of all numbers, with the same significant digits in the same order or sequence, are the same Thus, the mantissa for each of the above numbers, ·254, ·0254, 00254, is the same, and so also for numbers like 7, 70, 700, 70000 etc.

Caution. (*a*) We sometimes come across a logarithm, which is wholly negative, *e.g.*, $-3{\cdot}5661$, so that both the *integral part, viz.*, 3, as well as the *decimal part, viz.*, ·5661, are negative. In such cases, *the fractional part or the Mantissa must be made positive,* by subtracting 1 from the integral part of the logarithm and adding 1 to the fractional or decimal part. Thus,

$$-3{\cdot}5661 = -3-1+1-{\cdot}5661.$$

$$= -4+{\cdot}4339 = \bar{4}{\cdot}4339,$$

where the Characteristic is negative and the Mantissa, positive.

(*b*) It also happens during calculation work that we have to add, subtract, multiply or divide numbers like $\bar{4}{\cdot}4339$, with a −ve characteristic and a +ve Mantissa. In all such cases, the number should be treated as made up of two parts *e.g.*, as −4 and +·4339, in the above case. Thus,

(*i*) $\bar{4}{\cdot}4339+2{\cdot}6371+\bar{3}{\cdot}2567=\bar{4}{\cdot}3277$. (*ii*) $\bar{4}{\cdot}4339-2{\cdot}6780=7\ 7559$.

(*iii*) $\bar{4}{\cdot}4339\times 3=\bar{11}{\cdot}3017$. (*iv*) $\dfrac{4{\cdot}4339}{3}=\dfrac{\bar{4}+2+2\ 4339}{3}=\dfrac{\bar{6}+2{\cdot}4339}{3}=\bar{2}{\cdot}8113$.

i.e., we add to the negative Characteristic the **least** *negative number*, to make it completely divisible by the denominator, and add an equal *positive number* to the Mantissa, so that the logarithm, as a whole, remains unaltered.

6. **Logarithmic Tables.** We have seen above how the Characteristic of the logarithm of a number can be determined by a *mere inspection* of the number. It is, therefore, necessary only to tabulate the '*Mantissae*' or the decimal parts of the logarithms, which is done in what are called *Logarithmic Tables.* Usually, the *Four Figure Logarithm Tables* (given at the end) are used, in which the Mantissae of the logarithms of all numbers from 10 to 9999 are tabulated. Let us see how to use them.

7. **Determination of the Logrithm of a Number.** To determine the logarithm of a number, we proceed as follows ;

(*i*) *First we put down its Characteristic by inspection, in accordance with the rules, mentioned above.*

(*ii*) *Then, we proceed to consult the Tables and, ignoring the decimal point, if any, look for the first two digits of the number in the vertical column on the extreme left, and note the figure in the horizontal column, against these, under the number at the top, corresponding to the third digit of our number.*

(*iii*) *Finally, we add to it the figure given in the same row, in the column of 'mean differences' on the extreme right, under the number corresponding to the fourth digit of the given number.* The sum of the two, with a decimal point prefixed to it, then gives the Mantissa of the logarithm of that given number, and this, in its turn added to the Characteristic, gives the logarithm of the given number.

Thus, for example, if we desire to find out the logarithm of the number 3254, we first note that its Characteristic would be +3, (∵ it is greater than 1, and has four digits). Then, we consult the Tables, and, against 32 on the extreme left, look for the figure *under* 5, (at the top), in the horizontal column, and note that it is 5119. We, then, look for the figure, in the same horizontal row, *under* 4, in the column of *mean differences* (on the extreme right), and note that it is 5. This, when added to 5119 gives 5124. We then prefix a decimal point to this number, and get the required *Mantissa* as ·5124 which, added to the *Characteristic* 3 gives 3·5124 as the *logarithm* of the given number 3254.

If our number were 32·54, its *logarithm* would be $\bar{1}{\cdot}5124$: and similarly, the logarithm of the numbers ·3254 and ·03254, would be $\bar{1}{\cdot}5124$ and $\bar{2}{\cdot}5124$ respectively. It should be noted that the Mantissa remains the same, (*the digits being the same, and in the same sequence*), and only the Characteristic changes in accordance with the position of the decimal point.

8. **Determination of the Antilogarithm of a given Logarithm.** It is the reverse of the above process, and we find out here the *number* from its given logarithm. This number is called the *Antilogarithm* of the given logarithm.

Thus, if $\log m = x$, then x is the *logarithm* of m, and m, the *antilogarithm* of x.

There are separate *Tables* for this purpose, called *Antilogarithmic Tables*, where the Mantissae of logarithms are tabulated, right up to ·9999. These Tables are used as follows :

(*i*) *We ignore the Characteristic of the given logarithm,* for the moment.

(*ii*) *Then, we look for the first two digits of the Mantissa, (with the decimal point prefixed to them), in the vertical column on the extreme left, and note the figure in the horizontal row against these, under the number corresponding to the third digit of our Mantissa.*

(*iii*) *To the figure, so obtained, we add the figure in the column of 'mean differences', on the extreme right, in the same horizontal row as above, under the number corresponding to the fourth digit of the Mantissa.*

(*iv*) *And, lastly, we put the decimal point in the figure, thus determined, in its proper position, knowing, as we do, the Characteristic of its logarithm.* Thus, if the Characteristic be 2, there should be three significant figures before the decimal point, and so we put it down after the third digit of the number obtained

If, on the other hand, the Characteristic be $\bar{1}$, there should be no significant figure before the decimal point, and no *zero* immediately following it, and so, we put the decimal point immediately before the first digit. And, again, if the characteristic be $\bar{2}$, there should be no significant figures before the decimal point, but there should be *one zero* immediately after it, and hence we put a *zero* before the first digit of the number and prefix the decimal point to it. This gives the Antilogarithm of the given logarithm, *i.e.*, the number required.

Thus, if the given logarithm be $\bar{1}$ 5124, we look for ·51 in the vertical column, on the extreme left, and note the figure against it in the horizontal column under 2, at the top. This is found to be 3251. Then, we note the figure in the *same* horizontal row, under mean difference 4, on the right. It is found to be 3 ; so that, adding the two, we have $3251+3 = 3254$ And, since the Characteristic of the given logarithm is $\bar{1}$, there is no significant digit before the decimal point and no *zero* immediately after it , and so we fix the decimal point just before the first digit 3 and thus get ·3254 as the Antilogarithm of the given logarithm $\bar{1}$·5124. In other words, the required number, whose logarithm is $\bar{1}$·5124 is ·3254 or 0·3254

APPENDIX III

DIFFERENTIAL CALCULUS

Introduction. Differential Calculus deals with the way in which one quantity varies with the other, on which it depends, and such other allied topics.

Before trying to understand the actual process of differentiation, however, the meanings of the following mathematical terms must be clearly grasped.

(*i*) **Constants and Variables.** Throughout Calculus, we come across two types of quantities, *viz.*, (*a*) **constants**—*those which retain the same value throughout a set of mathematical investigation, e.g.*, the mass of a body, the value of g at a place, the symbol π, etc. (*b*) **variables**—*those which take on different values or to which any desired values may be given, e.g.*, the radius of a circle, the side of a square or a cube, for they are not fixed quantities, and any values may be given to them. The constants are usually denoted by the earlier letters of the alphabet, a, b, c, d, and the variables by the later ones, x, y, z, u, v, t, etc.

(*ii*) **Continuous Variables.** *A variable which can take every numerical value (or all numerical values from one given number to another)* is called a continuous variable. Thus, if a train, starting from rest, is observed to be moving with a speed of 20 *m./hr.* ten minutes later, *it must have assumed every possible velocity between* 0 *and* 20 *m./hr. during these ten minutes.* Its speed is, therefore, a *continuous variable.* We are concerned here only with continuous variables.

(*iii*) **Dependent and Independent Variables—Functions.** If a quantity x assumes a set of different values and its value does not depend upon that of any other quantity, it is said to be an *independent variable.* On the other hand, if a quantity y bears a certain relation to x, it is said to be a *dependent variable.*

Thus, since the area of a circle depends upon its radius, we say that the *area is a dependent variable, and the radius, an independent variable*, or, mathematically speaking, that the *area of a circle is a function of its radius.* Denoting the area of the circle by y, and its radius by x, we express the relationship by the expression

$$y = \pi . x^2.$$

Here, x is the independent variable, y the dependent variable, and π, a constant.

Thus, all expressions, containing x, *e.g.*, $2x-5$, x^2+2x+3, *sine* x, log x, etc., whose values depend upon the value of x, are *functions of* x.

Actually speaking, therefore, *a variable* y *is said to be a function of the variable* x, *when its value depends on the values that* x *assumes.*

There are, however, certain functions of x, where it is not possible to give *all* values to x. For, a function like $\sqrt{4-x^2}$, can have any meaning *only when* x *is numerically less than, or equal to,* 2. In such a case, we say that the function is defined for values of x, *in the range* -2 *to* $+2$, *both inclusive.* Thus, the range of values of x for which the function is defined must be clearly indicated. In other words, *we must know whether the function is defined for all values of* x, *or only for limited values of* x.

2. Notations. A function of a variable is generally denoted by enclosing the variable in a bracket, and prefixing a letter to it. Thus, $f(x)$, $F(x)$, $\phi(x)$, etc., all stand for functions of x. It must be clearly understood, however, that $f(x)$ *does not mean* f *into* x, *but is only a symbolic way of representing 'some function of* x'.

We can know the value of $f(x)$, if we know the value of x. For example, if $f(x)$ denotes the function, x^2+2x+3, its value is 11, when $x = 2$, and is 3, when $x = 0$, being different for different values of x. Similarly, the value of $f(x)$ is $f(a)$, if x has the value a, and so on.

3. Limits of Functions. (*a*) **Meaning of $x \to \infty$.** Suppose we give *successively increasing* values, 1, 2, 3, 4,...to x. Then, obviously, x will become larger and larger, and there will be no end to the extent to which it may thus be increased. *This continued growth, or increment of x, is expressed by saying that 'x tends to infinity', or that 'x approaches infinity', or, symbolically, as $x \to \infty$.*

This statement means no more than that x goes on continually increasing. To think that x will, at some time, be equal to ∞, is simply absurd, for ∞ is no fixed number. It is something we can only approach but never actually attain.

(*b*) **Meaning of $x \to 0$.** Suppose, now, we give *successively decreasing* values to x. Then, clearly, x will become smaller and smaller, and, if the process be continued, it may become very very small indeed,—*very much near zero, yet not equal to zero. This continued or progressive decrease in the value of 'x' is expressed by saying that x approaches zero, and, symbolically, as $x \to 0$.*

(*c*) **Meaning of $x \to a$.** Again, if we take a finite number a, and assume x to take on values such as $a+\cdot1$, $a+\cdot01$, $a+\cdot001$, etc., so that as x assumes these values, the difference between x and a goes on continually decreasing and x tends to come nearer and nearer to a, and we express it by saying that 'x tends to a', or that 'x approaches a', and, symbolically, as $x \to a$.

1. Let us now consider a function, such as $f(x) = x^2$, and see what its value approaches to, if $x \to 5$.

(*i*) Let us first give to x, continually increasing values, approaching 5. Thus, if $x = 4{\cdot}9$, $x^2 = 24{\cdot}01$.

$x = 4{\cdot}99$, $x^2 = 24{\cdot}90$, and so on.

i.e., as x approaches 5, x^2 approaches 25. Or, symbolically, if $x \to 5$, $x^2 \to 25$.

(*ii*) Let us now give to x continually decreasing values, approaching 5. Thus, if $x = 5{\cdot}1$, $x^2 = 26{\cdot}01$.

$x = 5{\cdot}01$, $x^2 = 25{\cdot}1001$, and so on,

So that, again, as $x \to 5$, $x^2 \to 25$, *i.e.*, the difference between x^2 and 25 becomes smaller and smaller. And, if $x = 5$, $x^2 = 25$.

2. Consider another function, $\frac{x^2-25}{x-5}$, as $x \to 5$.

It will be seen, on giving values to x, such as 4·9, 4·99 etc.$\to 5$; or 5·1, 5·01, etc.,$\to 5$, that as $x \to 5$, $\frac{x^2-25}{x-5} \to 10$. But, if $x = 5$, the expression becomes $\frac{0}{0}$, which is meaningless. Such an expression is said to be '**indeterminate**'.

We can now generalise and say that the limit of the function, $f(x)$ is m as 'x tends to a', if the difference between the function $f(x)$ and m can be made as small as we please, by taking x sufficiently near a.

All that we have to determine in the limit, therefore, is to see what $f(x)$ comes very nearly equal to, as x is made very nearly equal to a.

In the case of the above function, $f(x) = x^2$, we may, therefore, say that the limit of x^2 is 25, as $x \to 5$, or express it symbolically as

$$\underset{x \to 5}{\text{Lt}}\, x^2 = 25.$$

Important Deductions. Suppose we have

$$\underset{x \to a}{\text{Lt}}\, f(x) = m, \quad \text{and} \quad \underset{x \to a}{\text{Lt}}\, \phi(x) = n.$$

Then, it can be shown that (*i*) $\underset{x \to a}{\text{Lt}}\, [f(x) \pm \phi(x)] = m \pm n.$

(*ii*) $\underset{x \to a}{\text{Lt}}\, [f(x) \times \phi(x)] = m \times n.$ (*iii*) $\underset{x \to a}{\text{Lt}} \left[\frac{f(x)}{\phi(x)}\right] = \frac{m}{n}$,

provided $x \neq 0$, (the symbol $\neq$ meaning, '*not equal to*').

5. Differentiation of Functions. We are now in a position to tackle the actual differentiation of a function.

Let $f(x)$ be a function of x, and let x be increased by a small quantity h, (*positive or negative*). Then the function becomes $f(x+h)$, taking h to be *positive*.

Or, the increase in the function is given by $f(x+h)-f(x)$, and *the ratio of the increase in the function to the increase in* x is clearly equal to $\frac{f(x+h)-f(x)}{h}$.

This ratio $\frac{f(x+h)-f(x)}{h}$ *is called the* '**quotient of differences**'.

It is clear that as h becomes smaller and smaller, the increase in the function, *viz.*, $f(x+h)-f(x)$, also becomes smaller and smaller ; so that, as $h\to 0$. $f(x+h)-f(x)$ also$\to 0$.

We have seen above, (example 2, page 582), that although the numerator and the denominator in the case of certain fractions approach the limit *zero*, the limit of the fraction, as a whole, is a finite one. Similarly, here, the limit of the expression, $\frac{f(x+h)-f(x)}{h}$ is, in general, finite as $h\to 0$.

We are concerned in *Differential Calculus* with the determination of the limit of this fraction, $\frac{f(x+h)-f(x)}{h}$, when $h\to 0$. This limit is called the **differential coefficient** of the function $f(x)$, with respect to x. It is also sometimes referred to as the **derivative** of the function.

We may thus define the differential coefficient (or the derivative) of a function, as the limit of the quotient of differences $\frac{f(x+h)-f(x)}{h}$, *when the denominator* h *tends to zero.*

In actual practice, (*i*) it is usual to denote the function $f(x)$ by another variable y, (*ii*) the increment in value of x by the symbol δx, (*read as delta* x), and (*iii*) the increment in the function, by $\delta[(fx)]$ or δy, (*delta* y).

Thus, h of the above expression, corresponds to δx, and $f(x+h)-f(x)$ to δy.

Therefore, the quotient of differences $\frac{f(x+h)-f(x)}{h}$, is given by $\frac{\delta y}{\delta x}$.

Thus, the limit of $\delta y/\delta x$, when $\delta x\to 0$, is the *differential coefficient* of y with respect to x, and is denoted by symbol dy/dx, (read as '*dee y* by *dee x*'). Or, in accordance with our notation, we may say

$$\frac{d}{dx}\left[\, f(x) \,\right] \text{ or } \frac{dy}{dx} = \underset{\delta x\to 0}{\text{Lt}} \frac{\delta y}{\delta x}. \qquad [\because f(x)=y.$$

N. B. Here, $\frac{d}{dx}$ is purely a notation, and *does not mean* d divided by dx. All that it implies is that the expression to which it is appended is to be *differentiated* with respect to x. It is just like so many other symbols we use ; as, for instance, $\surd$, *log*, *sine* etc. ; thus, $\surd x$ does not mean $\surd \times x$, nor does *sine* x mean *sine* $\times x$.

The differentiation of a function is thus merely the process of finding out the differential coefficient of the function, and consists of the following steps :

(*i*) *giving a small increment* δx, *(positive or negative), to* x, *and obtaining* $y+dy$.

(*ii*) *finding* δy, *the increase in the function,*

(*iii*) *obtaining the quotient of differences or* $\delta y/\delta x$, *and*

(*iv*) *finding the limit of the quotient, as* $\delta x\to 0$.

The result, thus obtained, gives the differential coefficient of y or $f(x)$, and is symbolically represented as

$$\frac{dy}{dx} \text{ or } \frac{d}{dx}\left[\, f(x) \,\right].$$

Again, it should be carefully noted that $\delta y/\delta x$ *is the quotient of differences and* dy/dx, *the differential coefficient.*

6. Important General Results.

The following are some of the important general results, with which the student must make himself familiar.

(*i*) *The differential coefficient of a function is equal to the product of the index of the power and the quantity, raised to a power diminished by one. Thus, the differential coefficient of* x^n, *is equal to* nx^{n-1}.

Or, $$\frac{d}{dx}(x^n) = nx^{n-1}.$$

For example, (*i*) the differential coefficient of x^9 is $= 9x^{9-1} = 9x^8$,

(*ii*) ,, ,, ,, ,, ,, x^2 is $= 2x^{2-1} = 2x$,

(*iii*) ,, ,, ,, $1/x$ or x^{-1} is $= -1.x^{-1-1}$
$= -x^{-2} = -1/x^2$,

(*iv*) ,, ,, ,, $\sqrt{x}$ or $x^{-\frac{1}{2}}$ $= \frac{1}{2}x^{\frac{1}{2}-1}$
$= \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

(*ii*) *The differential coefficient of the product of a constant* K *and a function is equal to* K *times the differential coefficient of the function. Thus, the differential coefficient of* $K.u$ *(where* K *is a constant, and* u, *a function of* x*) is equal to* K *times the differential coefficient of* u, *i.e., equal* $K.\,du/dx$.

Or, $$\frac{d}{dx}(K.u) = K.\frac{du}{dx}.$$

For example, the differential coefficient of nx^3, where n is a constant, would be equal to n times the differential coefficient of x^3, *i.e.*, $= n.3x^2 = 3nx^2$.

(*ii*) *The differential coefficient of an added constant is zero.* For, if $y = K$, no matter how x varies, y still remains equal to K. Thus, there is no change dy, corresponding to a change dx, and, therefore, $\frac{dy}{dx}$ is zero, if y be a constant. Or, the *differential coefficient of a constant is zero.*

It follows, therefore, that
the differential coefficient of $y = f(x)+C$ is the same as that of $y = f(x)$, because the differential coefficient of the constant C is zero.

Thus, for example, the differential coefficient of nx^3+7 is the same as that of nx^3, *viz.*, $3nx^2$, because the differential coefficient of the added constant 7 is zero. In other words, *the additive constant disappears during differentiation.*

(*iv*) *The differential coefficient of the algebraic sum of a number of functions is equal to the algebraic sum of their individual differential coefficients, i.e.,*

$$\frac{d}{dx}(u+v-w+v+\ldots.) = \frac{d}{dx}.u+\frac{d}{dx}.v-\frac{d}{dx}.w+\frac{d}{dx}.z+\ldots$$

Thus, for example, if $y = 2x^3+3x^4-4x^5+5x+7$, we have

$$\frac{dy}{dx} = 3\times 2x^{3-1}+4\times 3x^{4-1}-5\times 4x^{5-1}+5x^{1-1}+0.$$

$$= 6x^2+12x^3-20x^4+5.$$

(*v*) *The differential coefficient of the product of two functions,* u *and* v, *is equal to* $v\times$ *the differential coefficient of* u **plus** $u\times$ *the differential coefficient of* v.

In other words, all that we have to do to determine the differential coefficient of the product of two or more functions is to differentiate one function at a time, leaving the others unchanged, and then to add up the resulting expressions together.

Thus, if $y = u.v.w.z.$, we have

$$\frac{dy}{dx} = \frac{du}{dx}.v.w.z.+u.\frac{dv}{dx}.w.z.+u.v.\frac{dw}{dx}.z.+u.v.w.\frac{dz}{dx}.$$

For example, if $v = x^2(5x+3)$, we have $\frac{dy}{dx} = (5x+3).\frac{d}{dx}x^2+x^2\frac{d}{dx}(5x+3)$.

$= (5x+3).2x+x^2.(5+0)$.

$$= 10x^3+6x+5x^2 = 15x^2+6x = 3x(5x+2).$$

Or, $$= \frac{dy}{dx} = 3x(5x+2).$$

(*vi*) *The differential coefficient of a quotient*

$$= \frac{Denom. \times diff.\ coeff.\ of\ numerator - Numerator \times diff.\ coeff.\ of\ Denom.}{(Denominator)^2}$$

Thus, if $y = u/v$, we have

$$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}.$$

For example, if $y = \frac{x^2}{x+5}$, we have

$$\frac{dy}{dx} = \frac{(x+5).\frac{d}{dx}(x^2) - x^2.\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{(x+5)\,2x - x^2(1+0)}{(x+5)^2} = \frac{2x^2+10x-x^2}{(x+5)^2} = \frac{x^2+10x}{x^2+10x+25}$$

(*vii*) *The differential coefficient of the function of a function is equal to the product of the differential coefficient of the inner function and the differential coefficient of the whole function, considering the inner function to be an independent variable.*

Thus, if $y = F(z)$, where $z = f(x)$, we have $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$.

All that we have to do, therefore, is to differentiate the whole function, as if the inner function (in this case), (z) were an independent variable, and then multiply by the differential coefficient of the inner function.

For example, (*i*) if $y = \sqrt{x+a}$, or $y = (x+a)^{\frac{1}{2}}$, we differentiate the whole, as though $(x+a)$ were an independent variable, and get

$$\tfrac{1}{2}(x+a)^{\frac{1}{2}-1} = \tfrac{1}{2}(x+a)^{-\frac{1}{2}}$$

Now, differential coefficient of the inner function $(x+a)$ is equal to $(1 \times x^{1-1}+0) = x^0+0 = 1$.

And, therefore, the differential coefficient of $y = \sqrt{x+a}$ is equal to

$$1 \times \tfrac{1}{2}(x-a)^{-\frac{1}{2}} = \tfrac{1}{2}(x+a)^{-\frac{1}{2}} = \tfrac{1}{2}.\frac{1}{\sqrt{x+a}} = \frac{1}{2\sqrt{x+a}}.$$

(*ii*) If $y = (ax+b)^n$, then, differentiating the whole function as though $(ax+b)$ were an independent variable, we have $n(ax+b)^{n-1}$

And, differentiating the inner function, $(ax+b)$, we have $a \times x^0+0 = a$.

$\therefore$ differential coefficient of the function $y = (ax+b)^n$ is equal to

$$a \times n(ax+b)^{n-1} = na(ax+b)^{n-1}.$$

7. **The Second Differential Coefficient.** We know that if $y = f(x)$, its differential coefficient is $\frac{d}{dx}y$. It is what may be called the *first* differential coefficient of the function $y = f(x)$. For, $\frac{dy}{dx}$ is itself a function of x, and can, therefore, be again differentiated with respect to x, giving us the differential coefficient, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$. It is denoted by the symbol, $\frac{d^2y}{dx^2}$, called *second differential coefficient of* y, and read as *'dee two y by dee x square'*.

We may similarly have the 3*rd*, the 4*th*, or the *nth* differential coefficient of y, denoted by $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ or $\frac{d^ny}{dx^n}$. The successive differential coefficients of y are also denoted by $y_1, y_2, \cdots y_y$, or by D_y, D^2_y etc. Or, if the function be

denoted by $f(x)$, the differential coefficients are represented by $f'(x)$, $f''(x)$, $f'''(x)$, etc.

For example, if $y = 4x^4+3x^3+2x^2+x+1$, we have

the *first differential coefficient* $y_1 = 16x^3+9x^2+4x+1+0$,
the *second* ,, ,, $y_2 = 48x^2+18x+4$,
the *third* ,, ,, $y_3 = 96x+18$,
the *fourth* ,, ,, $y_4 = 96$,
the *fifth* ,, ,, $y_5 = 0$,

and, therefore, also $y_6 = 0$, and $y_7 = 0$, and so on.

We are concerned here only with the second differential coefficient, which finds a wide application in problems in Physics.

A familiar example of the second differential coefficient is the *acceleration of a body*, which, as we know, is the rate of change of velocity ; so that, if we denote acceleration by a, we have $a = du/dt$, *i.e.*, *acceleration is the first differential coefficient of velocity*. But velocity itself is the differential coefficient of distance S, with respect to time, or $v = dS/dt$.

$$\therefore \qquad a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dS}{dt}\right) = \frac{d^2S}{dt^2}.$$

Or, *acceleration is also the second differential coefficient of distance with respect to time.*

8. Differential Coefficients of Logarithmic Functions. Before proceeding with the differentiation of logarithmic functions, a few important cases of limits, given below, must be remembered.

(*i*) *Limit of* $\left(1+\frac{1}{n}\right)^n$ *as* $n\to\infty$.

At the first glance, the beginner may be tempted to conclude that if $n\to\infty$, $\left(1+\frac{1}{\infty}\right)^n = \left(1+\frac{1}{\infty}\right)^\infty = (1+0)^\infty = 1$, *i.e.*, the limit, in this case is equal to 1. But this is *not* true, because, as we know, *unlike other ordinary finite numbers*, ∞ *is not amenable to the laws of algebra.*

Taking n to be a finite number, and expanding the expression by a simple application of the Binomial Theorem, it can be shown that *the value of* $\left(1+\frac{1}{n}\right)^n$ *is more than* 2 *and less than* 3. And, further, that as $n\to\infty$ and $\therefore$ $1/n$, $2/n$ etc.$\to 0$, in the limit, $\left(1+\frac{1}{n}\right)^n = 2{\cdot}71828$. This figure is taken to be the base in the natural, or the Napierian system of logarithms, and is denoted by the letter e. So that, as

$$n\to\infty, \left(1+\frac{1}{n}\right)^n = e.$$

This is a very important limit to remember.

The following are its simple applications :

(*i*) (*a*) $(1+t)^{1/t} = e$, as $t\to 0$. (*b*) $(1+x)^{a/x} = e^a$, as $x\to 0$.

(*ii*) *Limit of* $\frac{e^h-1}{h}$, *as* $h\to 0$ is 1.

(*iii*) *Limit of* $\frac{a^h-1}{h}$, *as* $h\to 0$ *is* $\frac{1}{\log a^e} = \log_e a$.

Now, although we ordinarily use what are called *common logarithms or logarithms to the base* 10, in Calculus, as in all other branches of higher mathematics, *natural logarithms*, or *Napierian logarithms**, (after the name of the inventor of the system), are used, and so we have the following further relations to remember :

1. $\log_a x = \log_e x \times \log_a e$.

*Common logarithms to the base 10 can be converted into Napierian logarithms to the base e by multiplying them by 2·3026.

2. Any quantity, $x = a \log_a x$,

3. ,, ,, $x = e \log_e x$.

The following are the differential coefficients of some important lagarithmic functions :

(*i*) *The differential coefficient of* $\log_e x$ *or* $\log x$, (the base e being understood), *is equal to* $1/x$. Or, $d/dx (\log_e x) = 1/x$.

N.B. If the base were a, and not e, we shall have

$$\frac{dy}{dx} = \frac{1}{x} \cdot \log_a e.$$

Similarly, $d/du (\log u) = 1/u$.

Or, differentiating with respect to x, we have

$$\frac{d}{dx} (\log a) = \frac{d}{du} \log u \times \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx}.$$

(*ii*) *The differential coefficient of* a^x *is equal to* $a^x \log_e a$.

For, if $y = a^x$, we have $\log_e y = \log_e a^x = x \log_e a$.

$\therefore$ differentiating both sides with respect to x, we have

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log_e a. \quad \text{Or,} \quad \frac{dy}{dx} = y \log_e a = a^x \log_e a.$$

From this, a very important result follows, *viz.*,

(*iii*) *that the differential coefficient of* $e^x = e^x$.

For, if $a = e$, we have $y = e^x$,

and $\therefore$ $dy/dx = y \log_e e = e^x \log_e e = e^x \times 1 = e^x$.

9. Differential Coefficients of Circular Functions.

(*i*) *The differential coefficient of sin* x *is cos* x. Or, $d/dx (\sin x) = \cos x$.

(*ii*) *The differential coefficient of cos* x *is* $-\sin x$.

Or, $d/dx (\cos x) = -\sin x$.

(*iii*) *The differential coefficient of tan* x *is* $\sec^2 x$. Or, $d/dx (\tan x) = \sec^2 x$.

(*iv*) *The differential coefficient of cot* x *is* $-\operatorname{cosec}^2 x$.

Or, $d/dx (\cot x) = -\operatorname{cosec}^2 x$.

10. Maxima and Minima.

It can be shown that for maxima and minima, $dy/dx = 0$. To find where the function is a maximum or a minimum, the procedure is the following :

(*i*) Put $dy/dx = 0$, solve the equation $dy/dx = 0$, and obtain several values of x.

(*ii*) See for what values of x, dy/dx changes sign *from positive to negative. For such values of* x, *the function is a minimum.*

(*iii*) See for what values of x, dy/dx changes sign *from negative to positive. For such values of* x, *the function is a maximum.*

N.B. (1) It sometimes happens that for some values of x, obtained from the equation $dy/dx = 0$, dy/dx does not change sign. *The function is neither a maximum nor a minimum at these points. Such points are called* **points of inflexion.**

(2) All points on a curve, where the tangent is parallel to the x-axis, are called **stationary points.**

(3) Points, where the function is a maximum or a minimum are called **turning points,** and the maximum and minimum values are called **turning values.**

Example :

Let it be required to find the maximum and minimum values of

$$x^5 - 5x^4 + 5x^3.$$

Putting $y = x^5 - 5x^4 + 5x^3$, we have

$$dy/dx = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3).$$
$$= 5x^2(x-3)(x-1).$$

And, equating dy/dx to 0, we have $5x^3(x-3)(x-1)=0$.

So that, $x = 3, x = 1, x = 0$.

Taking first, $x = 3$, we have

if x be a little less than 3, $dy/dx = (+)(-)(+)$, *i.e.*, is *negative* ;

and if x be a little greater than 3, $dy/dx = (+)(+)(+)$, *i.e.*, is *positive*.

i.e., at $x = 3$, dy/dx, changes sign from *negative to positive*.

Thus, $x = 3$ gives the *minimum value of the function*.

$\therefore$ the minimum value of the function is $3^5-5\times3^4+5\times3^3=-27$.

Now, if x be little less than 1, we have $dy/dx=(+)(-)(-)$ *i.e.*, *positive*.

and ,, x ,, ,, greater ,, 1 ,, ,, $dy/dx = (+)(-)(+)$, *i.e.*, *negative*.

Thus, at $x = 1$, dy/dx changes sign from *positive* to *negative*, or $x = 1$ gives the *maximum value of the function*.

$\therefore$ maximum value of the function $= 1-5+5=+1$.

And lastly, at $x = 0$, we have

if x be a little less than 0, $dy/dx = (+)(-)(-)$, *i.e.*, *positive*,

and ,, x ,, greater than 0, $dy/dx = (+)(-)(-)$, *i.e.*, *positive*,

i.e., *dy/dx does not change sign* at $x = 0$.

Hence, $x = 0$ gives *a point of inflexion*.

APPENDIX IV

INTEGRAL CALCULUS

1. **Introduction.** The word, "*integral*" simply means '*the whole*', and the process of adding or summing up a large number of little bits of a quantity is called **integration.**

Thus, if x be supposed to be made up of a large number of little bits, each equal to dx, it is obvious that if we add up all these dx's together, we shall get x. Mathematically, we put it as $\int dx = x$, and read it as '*integral of* dx *equals* x', the symbol $\int$ meaning '*integral of*'. It is just a long S, and means the same thing as '*the sum, or summation of*'. In fact, it resembles in its meaning that other familiar symbol, Σ (*sigma*), with the only difference that whereas the latter indicates the summation of a number of *finite quantities*, the integral sign $\int$ indicates the summation of a large number of *indefinitely minute quantities*.

2. **Integration—The Reverse of the Process of Differentiation.** We may look at this process of summing up small bits to get '*the whole*', from another point of view, and say that *integrating is really the reverse of differentiating*; for, whereas in differentiation, we are concerned with the problem of finding dy/dx, given y as a function of x, we have, here, to perform *the reverse process, viz., that of expressing* y *as a function of* x, *when* dy/dx, *is given.* This is the process, called '*integration*', and y *is said to be the integral of* dy/dx.

For example, we know that if $y = x^2$, $dy/dx = 2x$, and so it follows that if we perform the reverse process of determining y, when dy/dx is given to be equal to $2x$, we *must* get $y = x^2$.

Again, if we are given $dy/dx = 3x^2$, and are asked to find out y, *i.e.*, to *integrate* dy/dx, we *shall* have $y = x^3$, because, if $y = x^3$, $dy/dx = 3x^2$.

Similarly, if $\dfrac{dy}{dx} = 4x^3$, $y = x^4$. $\left[\because \text{if } y = x^4, \dfrac{dy}{dx} = dx^3.\right.$

It will thus be seen that the process of integration depends upon the recognition of the differential coefficients of functions.

There are no infallible rules or fool-proof methods here to be guided by, as we have in the case of differentiation. The first essential step, therefore, to be successful in *integration*, is to familiarise oneself well with the results of differentiation of simple functions. A list to help memorise them is appended at the end, for the purpose.

3. **Integration of Simple Functions.** We have seen above that if

$$dy/dx = 3x^2, \text{ we have } y = x^3.$$

We may arrange, $dy/dx = 3x^2$ as $dy = 3x^2.dx$, which is, what is called, *a differential equation*, telling us that a little bit, or, in the language of mathematics, '*an element of* x' is equal to the corresponding '*element of* x' multiplied by $3x^2$. We have now to sum up all these elements dy to get y, or to find out the '*integral of* dy', which will, obviously, be equal to the integral of $3x^2.dx$; so that, putting the proper notations, we have

$$\int dy = \int 3x^2.dx, \text{ (read as 'integral } dy \text{ equals integral } 3x^2.dx\text{')}.$$

Or, $$y = x^3, \text{ as seen above.}$$

The rule to be applied would thus seem to be to increase the power of x *by one, then divide it by this increased power. For, performing this operation, in this*

case, we have

$$y = 3.\frac{x^{2+1}}{2+1} = \frac{3x^3}{3} = x^3.$$

Of course, it is obvious that $\int dy = y$, for the sum of all dy's into which y is broken up, will clearly be y. But we can apply the above rule in this case also.

Let us put $\int dy$ as $\int y^0.dy$, for $y^0 = 1$, and hence the value of the expression remains unchanged.

Now, applying our rule, we have $\int dy = \int y^0.dy = \frac{y^{(0+1)}}{1} = y.$

Or, $$\int dy = y.$$

The student will perhaps feel tempted to ask as to what has happened to the little bits, dx and dy, at the end. It is enough to remember that this dx* or dy is just a part of the symbol of integration, the full notation being $\int dx$ $\int dy$ etc., and when the integration is performed, the symbols, $\int$ and dx, or $\int$ and dy, vanish together.

4. Indefinite Integrals. We may thus generalise the above rule and say that $\int x^n.dx = \frac{x^{n+1}}{n+1}$, so that the process will seem to be wonderfully easy. But a snag soon appears, for we know that if

$$y = x^2, \qquad dy/dx = 2x.$$

Also, if $y = x^2+7$, $dy/dx = 2x$, as before, *because the differential coefficient of an added constant is zero*; so that, performing the reverse process of integration, we get $y = x^2$ in both the cases, which, obviously is far from correct.

Further, the result would still be the same if the added constant were 9 or 11 instead of 7. It would thus appear that integration is not quite so reliable a process, after all, and that *one has to be guided by the results of differentiation.* Therefore, working backwards from dy/dx, *allowance must be kept for there being an added constant.*

Thus, integrating $dy/dx = 2x$, we say $\int dy = \int 2x.dx = 2.\frac{x^2}{2} + C.$

Or, $$y = x^2+C,$$ where C is a constant, called the *constant of integration*,—an abitrary constant, having no particular value; for, as we have seen above, it could very well be 7, or 9, or 11, or any other number, for that matter. The value of the integral is, therefore, not fixed or definite, and such integrals are, therefore, called *indefinite integrals.*

If, however, we are given the value of the function y, for a particular value of x, we can determine the value of this constant C. For example, if $dy/dx = 2x$, and $y = 11$, when $x = 2$,

we have $\int dy = \int 2x.dy$, Or, $y = x^2+C.$

*This dx, originally a part of the differential coefficient, when transferred to the right hand side, as in $3x^2.dx$, only indicates that x is the independent variable with respect to which the operation is to be performed.

Now, putting $y = 11$, and $x = 2$, (as given), we have

$$11 = 4 + C, \text{ whence, } C = 7.$$

So that, $$y = x^2 + 7.$$

Similarly, $\int x^2.dx$ would be equal to $\int \frac{x^3}{2} + C$,

and $\int dx$ would be equal to $x + C$, and so on.

N.B. The constant of integration C need not always be written down, but it is always *supposed* to be there, in the case of indefinite integrals.

5. **Definite Integrals.** When an integral is defined between two limits, it is called a '*definite integral*'. The lower value of the limit is called the *inferior* or, *the lower limit* and the higher value of the limit, the *superior* or *the upper limit*,—the limits themselves being called the '*limits of integration.*'

For example, if the integral of the function, $y = f(x)$ is to be determined between the limits, $x = a$ and $x = b$, we represent it symbolically as

$$\int_{x=a}^{x=b} f(x).dx, \quad \text{or, simply, as} \quad \int_a^b f(x).dx,$$

and read it as '*integral of $f(x).dx$ between the lower limit (x equal to) a, and the upper limit (x equal to) b, or, 'as integral from a to b of the function $f(x)$, with respect to x.*'

Now, if $$\int f(x).dx = \phi(x) + C, \qquad \begin{cases} i.e., \ \frac{dy}{dx} = f(x), \\ \text{and } y = \phi(x). \end{cases}$$

the value of $\int_a^b f(x).dx = \Big[\ \phi(x) + C\ \Big]_a^b = \Big[\ \phi(b) + C\ \Big] - \Big[\ \phi(a) + C\ \Big]^*$

$$= \phi(b) - \phi(a),$$

the constant of integration having disappeared during the process.

Thus, $$\int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = \left[\frac{3^3}{3} - \frac{2^3}{3}\right] = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

And, in general, $$\int_a^b x^n.dx = \left[\frac{x^{n-1}}{n+1}\right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

Corollary. It follow as a necessary corollary from the above that

$$\int_a^b f(x)\,dx = -\int_b^a f(x).dx.$$

Thus, we shall have $$\int_2^3 x^2.dx = -\int_3^2 x^2.dx,$$

6. **Some Illustrative Examples.**

(*i*) **On Indefinite Integrals.—**

(*i*) **The value of** $\int x^{-1}.dx$. Here, $dy/dx = x^{-1}$, or, $dy = x^{-1}.dx$.

Or, $$\int dy = \int x^{-1}.dx.$$

Now, our general rule that $\int x^n.dx$ is $= \frac{x^{n+1}}{n+1}$ breaks down in this case,

for that would give the value of the integral x^{-1} to be $\frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0} = \infty$,

*By substituting first b instead of x, in the expression, and then a, and subtracting the latter from the former.

and, surely, we do not get x^{-1} by differentiating ∞, as we ought to, if our integration is correct. It is thus an exception. Once, again, therefore, we have to go back, searching for the function of x, whose dy/dx is x^{-1}, or $1/x$. We know that it is $y = \log_e x$.

Thus, since the differential coefficient of $y = \log_e x$ is $1/x$, or, x^{-1}, the reverse process of integrating $dy/dx = x^{-1}$, or $dy = x^{-1}.dx$ should give us $y = \log_e x$. But we must, as usual, be careful to add the constant of integration C, so that the final result we obtain is $y = \log_e x + C$.

Or, $$\int x^{-1}.dx = \log_e x + C.$$

This brings into bold relief the fact that it is not possible to integrate an expression unless that expression is known to us to have been obtained as a result of differentiating something else. We must, therefore, learn up the results of differentiating as many general functions of x as possible and make sure this list of ours goes on increasing continually.

(*ii*) **The value of** $\int e^x.dx$ The value of $\int c^x.dx = e^x + C$.

because, if $y = e^x$, $dy/dx = e^x$. [See page 587

(*iii*) **The value of** $\int e^{-x}.dx$. The value of $\int e^{-x}\,dx = -e^{-x} + C$.

because if, $y = -\frac{1}{e^x}$, $\frac{dy}{dx} = \frac{-e^x \times 0 - 1 \times e^x}{e^{2x}} = e^{-x}$.

(2) **On Definite Integrals.**

$$(i)\quad \int_2^3 x^{-2}\,dx = \left[\frac{x^{-2+1}}{-2+1}\right]_2^3 = \left[\frac{x^{-1}}{-1}\right]_2^3 = \left[-\frac{1}{x}\right]_2^3$$
$$= -\tfrac{1}{3} - (-\tfrac{1}{2})] = \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{1}{6}.$$

$$(ii)\quad \int_0^1 x.dx = \left[\frac{x^2}{2}\right]_0^1 = \left[\frac{1}{2} - 0\right] = \frac{1}{2}.$$

$$(iii)\quad \int_a^b x^{-1}\,dx = \left[\log_e x\right]_a^b = \left[\log_e b - \log_e a\right] = \log_e \frac{b}{a}.$$

7. **Integration of Trigonometrical Functions.** We know that if $y = \sin x$, we have $dy/dx = \cos x$.

And, if $y = \cos x$, $dy/dx = -\sin x$.

Hence $$\int \sin x.dx = -\cos x + C, \text{ and } \int \cos x.dx = \sin x + C,$$

where C is the *constant of integration*, as usual.

In the same manner, we can obtain the integrals of their trigonometrical function (see list at the end), if we know them to have been obtained by differentiating other functions.

The value of constant C may be determined in the same manner as in the case of ordinary algebraic functions, discussed above. Further, we may have definite integrals here also, as elsewhere, whose values are obtained in precisely the same manner as those of algebraic functions.

For example, the value of $\int_0^{\pi} \cos x.dx$ can be obtained thus :

$$\int_0^{\pi} \cos x.dx = \left[\sin x\right]_0^{\pi} = \left[\sin \pi - \sin 0\right] = 0 - 0 = 0.$$

8. **Integral of the Differential Coefficient of a Function.**

Since integration is, by its very definition, the reverse of the process of differentiation, it follows that if $\phi(x)$ be the differential coefficient of a function $f(x)$, with respect to x, the integral of $\phi(x)$ with respect to x, will be $f(x)$.

Or, putting it in symbols, if $y = f(x)$,

and $\frac{dy}{dx} = \frac{d}{dx} \cdot f(x) = \phi(x)$.....(*i*) we have $\int \phi(x).dx = f(x)$. ...(*ii*)

Or, substituting the value of $\phi(x)$ in (*ii*), from (*i*) above, we have

$$\int \frac{d}{dx} \cdot f(x).dx = f(x).$$

i. e., the integral of the differential coefficient of a function of x is the function itself.

In other words, *the operative symbols, d/dx and* $\int \ldots dx$, *cancel each other out,*—and quite naturally too, because they represent two inverse operations.

Thus, we can straightaway say that

$$\int \frac{d}{dx} \sin x.\, dx = \sin x.$$

Again, if the integral of a function $\phi(x)$ be $f(x)$, the differential coefficient of $f(x)$ will be $\phi(x)$. Or, putting in symbols, if

$$\int \phi(x)\, dx = f(x). \quad \ldots(iii) \quad \text{we shall have} \quad \frac{d}{dx} f(x) = \phi(x). \ldots(iv)$$

Or, substituting the value of $f(x)$ in (*iv*) from (*iii*) above, we have

$$\frac{d}{dx} \int \phi(x).dx = \phi(x),$$

i. e., the differential coefficient of the integral of a function of x is the function itself.

So that, as before, the symbols $\frac{d}{dx}$ and $\int \ldots dx$ cancel out, and we can say straightaway that $\frac{d}{dx} \int \cos x.dx = \cos x.$

9 Integral of the Product of a Constant and a Function. Just as in Differential Calculus, so also here, we obtain the integral of the product of a constant and a function by multiplying the integral of that function by that constant. Thus,

$$\int K f(x).dx = K \int f(x).dx, \text{ where } K \text{ is a constant.}$$

For example, (*i*) $\int 4x^3.dx = 4 \int x^3.dx = 4\left[\frac{x^4}{4}\right] = x^4$

(*ii*) $\int \frac{3}{5} x^5.dx = \frac{3}{5} \int x^5.dx = \frac{3}{5}\left[\frac{x^6}{6}\right] = \frac{x^6}{10}$.

10 Integral of the Algebraic Sum of any Finite Number of Functions. *The integral of the algebraic sum of any finite number of functions is equal to the algebraic sum of their individual integrals.*

Thus, u, y and z be the functions of x, we have

$$\int (u + y + z)\, dx = \int u\, dx + \int y.dx + \int z.dx.$$

For example,

$$(i) \int (4x^3 + 3x^2 - 2x + 1) = \int 4x^3\, dx + \int 3x^2.dx^2 - \int 2x.dx + \int 1\, dx$$

$$= 4 \int x^3.dx + 3 \int x^2.dx - 2 \int x.dx + \int dx.$$

$$= \frac{4x^4}{4} + \frac{3x^3}{3} - \frac{2x^2}{2} + x = x^4 + x^2 - x^2 + x.$$

$$(ii)\quad \int \cos^2 x\,dx = \tfrac{1}{2}\int(\cos 2x+1)\,dx.$$

$$= \tfrac{1}{2}\int \cos 2x\,dx + \tfrac{1}{2}\int dx = \tfrac{1}{2}\frac{\sin 2x}{2} + \tfrac{1}{2}x.$$

$$= \frac{\sin 2x}{4} + \frac{x}{2}.$$

$$\left[\begin{array}{l} \because \cos 2x \\ = \cos^2 x - \sin^2 x \\ = 2\cos^2 x - (\cos^2 x + \sin^2 x) \\ = 2\cos^2 x - 1. \\ \text{Or,} \\ 2\cos^2 x = \cos 2x + 1. \\ \text{whence, } \cos^2 x \\ \qquad = \tfrac{1}{2}(\cos 2x + 1). \end{array}\right.$$

$$(iii)\quad \int \sin^2\theta\,d\theta = \int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{2}\int d\theta - \tfrac{1}{2}\int \cos 2\theta\,d\theta.$$

$$= \tfrac{1}{2}\theta - \tfrac{1}{2}\frac{\sin 2\theta}{2} = \tfrac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)$$

$$(iv)\quad \int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta \qquad \left[\because 1+\tan^2\theta = \sec^2\theta.\right.$$

$$= \int \sec^2\theta\,d\theta - \int d\theta.$$

$$= \tan\theta - \theta. \qquad \left[\because \frac{d}{d\theta}\tan\theta = \sec^2\theta.\right.$$

Other Fundamental Rules (Theorems).

I. *If we have a fraction, whose numerator is the differential coefficient of the denominator, then, the integral of the fraction is the logarithm of the denominator.*

Thus, if u be a function of x, *i.e.*, $u = f(x)$, such that $du/dx = f'(x)$, then, if $y = \log u$, we have $\frac{dy}{dx} = \frac{d}{dx}(\log u) = \frac{d}{du}(\log u).\frac{du}{dx}$.

$$= \frac{1}{u}\cdot\frac{du}{dx} = \frac{1}{u}\cdot f'(x).$$

Or, $\frac{d}{dx}(\log u) = \frac{f'(x)}{u} = \frac{f'(x)}{f(x)}$. Or, $\frac{d}{dx}\log f(x) = \frac{f'(x)}{f(x)}$.

Or, on integrating, we have $\int \frac{f'(x)}{f(x)}.dx = \log f(x)$,

i.e., the integral of a fraction whose numerator is the differential coefficient of the denominator is equal to the logarithm of the denominator.

Thus, for example,

(*i*) $\int \frac{\cos x}{\sin x}.dx = \log \sin x$, $\because \frac{d}{dx}\sin x = \cos x$.

(*ii*) $\int \frac{2ax+2b}{ax^2+2bx+c}.dx = \log(ax^2+2bx+c)$,

$\because \frac{d}{dx}(ax^2+2bx+c) = 2ax+2b$.

II. *The integral of the product of a function of x, raised to a power n, and the differential coefficient of the function itself (not raised to the power n), is equal to the function, raised to the power (n+1) divided by (n+1).*

For, suppose, we have a function like $y = [f(x)]^{n+1}$; so that,

$$\frac{dy}{dx} = (n+1)[f(x)]^n.f'(x),$$

where $f(x)$ is the differential coefficient of $f(x)$.

Then, $dy = (n+1)[f(x)]^n.f'(x).dx$. And $\therefore \int dy = \int (n+1)[f(x)]^n.f'(x).dx$.

Or, $y = (n+1)\int [f(x)]^n.f'(x)dx$. Or, $\int [f(x)^n.[f'(x)dx] = \frac{y}{n+1}$.

Or, $$\int [f(x)]^n \cdot f'(x).dx = \frac{[f(x)]^{n+1}}{n+1}, \text{ where } n \neq -1$$

For example,

(*i*) $\int sin^2x.cos\ x.dx = \frac{sin^3x}{3}$, $\because$ $cos\ x$ is $= \frac{d}{dx} sin\ x$.

(*ii*) $\int (ax^2+bx+c)^n.(2ax+b).dx = \frac{(ax^2+bx+c)^{n+1}}{n+1}$,

$\because$ $2ax+b = \frac{d}{dx}(ax^2+bx+c)$.

From the above, it follows that $\int \frac{\frac{1}{2}f'(x)}{\sqrt{f(x)}}.dx = \int \frac{1}{2} f'(x)\,[f(x)]^{-\frac{1}{2}}.dx$.

$$= \int \frac{1}{2}[f(x)]^{-\frac{1}{2}}.f'(x).dx^* = \frac{\frac{1}{2}[f(x)]^{1-\frac{1}{2}}}{1-\frac{1}{2}} = \frac{\frac{1}{2}[f(x)]^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= f(x)^{\frac{1}{2}}.$$

Or, $$\int \frac{\frac{1}{2}f'(x)}{\sqrt{f(x)}}.dx = \sqrt{f(x)}$$

Thus, $$\int \frac{(ax+b)}{\sqrt{ax^2+2bx+c}}.dx = \int \frac{\frac{1}{2}(2ax+2b)}{\sqrt{ax^2+2bx+c}}.dx,$$

where $(2ax+2b)$ is the differential coefficient of (ax^2+bx+c).

Hence $$\int \frac{(ax-b)}{(ax^2+2bx+c)}.dx = \sqrt{ax^2+2bx+c}.$$

List of Important Integrals

(1) Algebraic.

(*i*) $\int 1.dx = x$, because $\frac{d}{dx}.x = 1$

(*ii*) $\int a.dx = ax$, ,, $\frac{d}{dx}.ax = a$.

(*iii*) $\int 2x.dx = x^2$, ,, $\frac{d}{dx}.x^2 = 2x$.

(*iv*) $\int x^n\,dx = \frac{x^{n+1}}{n+1}$, (where $n \neq -1$),

because $\frac{d}{dx}.x^{n+1} = (n+1)x^n$.

(*v*) $\int \frac{1}{x}.dx = \log_e x$ because $\frac{d}{dx}(\log_e x) = \frac{1}{x}$.

(2) Exponential and Logarithmic

(*i*) $\int a^x.dx = \frac{a^x}{\log_e a}$

and $\therefore$ $\int a^x.\log_e a.dx = a^x$, because $\frac{d}{dx}.a^x = a^x.\log_e a$

(*ii*) $\int e^x.dx = e^x$, because $\frac{d}{dx}.e^x = e^x$.

*Note that $f'(x)$ is the differential coefficient of $f(x)$, and not of $[f(x)]^{-\frac{1}{2}}$

(3) Trigonometrical.

(*i*) $\int \sin x.dx = -\cos x$, because $\frac{d}{dx}\cos x = -\sin x$.

(*ii*) $\int \cos x.dx = \sin x$, ,, $\frac{d}{dx}\sin x = \cos x$.

(*iii*) $\int \tan x.dx = \log \sec x$, ,, $\frac{d}{dx}\log \sec x = \tan x$.

(*iv*) $\int \cot x.dx = \log \sin x$, ,, $\frac{d}{dx}\log \sin x = \cot x$.

(*v*) $\int \sec^2 x.dx = \tan x$, ,, $\frac{d}{dx}\tan x = \sec^2 x$.

(*vi*) $\int \operatorname{cosec}^2 x.dx = -\cot x$, ,, $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$.

(4) Inverse.

(*i*) $\int \frac{1}{\sqrt{1-x^2}}.dx = \sin^{-1} x$, because $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

(*ii*) $-\int \frac{1}{\sqrt{1-x^2}}.dx = \cos^{-1} x$, , $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$.

(*iii*) $\int \frac{1}{1+x^2}.dx = \tan^{-1} x$, ,, $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$.

(*iv*) $-\int \frac{1}{1+x^2}.dx = \cot^{-1} x$, ,, $\frac{dx}{dx}\cos^{-1} x = -\frac{1}{1+x^2}$.

(5) Miscellaneous.

(*i*) $-\int \frac{1}{(x+a)^2}.dx = \frac{1}{x+a}$ because $\frac{d}{dx}\left(\frac{1}{x+a}\right) = -\frac{1}{(1+a)^2}$.

(*ii*) $-\int \frac{1}{(x^2+a^2)^{\frac{3}{2}}} = \frac{1}{\sqrt{x^2+a^2}}$,, $\frac{d}{dx}\frac{1}{\sqrt{x^2+a^2}} = \frac{-x}{(x^2+a^2)^{\frac{3}{2}}}$.

(*iii*) $\int a\cos ax.dx = \sin ax$, ,, $\frac{d}{dx}\sin ax = a\cos ax$.

(*iv*) $\int a\sin ax.dx = -\cos ax$, ,, $\frac{d}{dx}\cos ax = -a\sin ax$.

(*v*) $\int a\sec^2 ax.dx = \tan ax$, ,, $\frac{d}{dx}\tan ax = a\sec^2 ax$.

(*vi*) $\int \sin 2x\,dx = \sin^2 x$, ,, $\frac{d}{dx}\sin^2 x = \sin 2x$.

(*vii*) $-\int \sin 2x.dx = \cos^2 x$, ,, $\frac{d}{dx}\cos^2 x = -\sin 2x$.

CONSTANT TABLES

I—DENSITIES OF COMMON SUBSTANCES

Solids				Liquids		Gases	
Substance	*Density (gms. per c.c.)*	*Substance*	*Density (gms. per c.c.)*	*Substance*	*Density (gms. per c.c.)*	*Substance*	*Density (gms. per c.c.) at N.T.P.*
Aluminium	2·70	Iron (*Wrought*)	7·8-7·9	Alcohol	·79	Air	1·2928
Brass	8·4-8·7	Ivory	1·8-1·9	Benzene	·88	Carbon dioxide	1·977
Bronze	8·7-8·9	Lead	11·37	Carbon disulphide	1·27	Hydrogen	·084882
Constantan	8·8	Leather	·85-1	Ether	·72	Helium	1·7848
Common Salt	2·15	Nickel	8·9	Glycerine	1·26	Neon	·8999
Copper	8·93	Paper	·71-1·1	Mercury (0°*C*)	13 596	Nitrogen	1·25050
Cork	·22-·26	Platinum	21·50	Methylated Spirit	·83	Oxygen	1·42895
German Silver	8·5-8·9	Porcelain	2·2-2·4	Milk	1·03	Steam (*Saturated at* 100°*C*	·6059
Glass (*Crown*)	2·4-2·6	Quartz	2·66	Naptha	·85		
Glass (*Flint*)	2·9-4·5	Sand	2·63	Oil, *Castor*	·97		
Glass (*Pyrex*)	2·25	Silver	10·5	,, *Olive*	·91-·93		
Gold	19·32	Sugar	1·59	Paraffin	·80		
Gun metal	8·0-8·4	Tar	1·02	Petrol	·68-·72		
Ice (0°*C*)	·9168	Tin	7·29	Turpentine	·87		
India-rubber	·91-·93	Wax (*Bee's*)	·95-·96	Vinegar	1·02		
Iron (*Cast*)	7·1-7·7	Zinc	7·1	Water (4°*C*)	1·00		
Iron (*Pure*)	7·87						

II—ELASTIC CONSTANTS

Substance	*Young's modulus (dynes/cm².)*	*Bulk modulus (dynes/cm².)*	*Rigidity modulus (dynes/cm².)*	*Poisson' Ratio σ*	*Breaking stress (for wires) (dynes/cm².)*
Aluminium	$7{\cdot}05 \times 10^{11}$	$7{\cdot}46 \times 10^{11}$	$2{\cdot}63 \times 10^{11}$	·310	$1{\cdot}7\text{-}2{\cdot}0 \times 10^{9}$
Brass	$9{\cdot}7\text{-}10{\cdot}2 \times 10^{11}$	$10{\cdot}65 \times 10^{11}$	$3{\cdot}5 \times 10^{11}$	·34-·40	$3{\cdot}1\text{-}3{\cdot}9 \times 10^{9}$
Copper	$12{\cdot}4\text{-}12{\cdot}9 \times 10^{11}$	$14{\cdot}3 \times 10^{11}$	$3{\cdot}9\text{-}4{\cdot}0 \times 10^{11}$	·26	$4{\cdot}0\text{-}4{\cdot}6 \times 10^{9}$
Glass (*Flint*)	$5{\cdot}0\text{-}6{\cdot}8 \times 10^{11}$	$3{\cdot}6\text{-}3{\cdot}8 \times 10^{11}$	$2\ 0\text{-}2{\cdot}5 \times 10^{11}$	·24	...
Gold	$8{\cdot}00 \times 10^{11}$	$16{\cdot}6 \times 10^{11}$	$2{\cdot}80 \times 10^{11}$	·495	$2{\cdot}6 \times 10^{9}$
Iron (C)	$10{\cdot}0\text{-}13{\cdot}0 \times 10^{11}$	$9{\cdot}6 \times 10^{11}$	$3{\cdot}5\text{-}5{\cdot}3 \times 10^{11}$	·27	...
Iron (W)	$19{\cdot}0\text{-}20{\cdot}0 \times 10^{11}$	$14{\cdot}3 \times 10^{11}$	$3{\cdot}9\text{-}4{\cdot}0 \times 10^{11}$	·27	$4\ 6 \times 10^{9}$
Mercury	...	$2{\cdot}6 \times 10^{11}$	...	...	...
Silver	$7{\cdot}90 \times 10^{11}$	$10{\cdot}90 \times 10^{11}$	$2{\cdot}86 \times 10^{11}$	·369	$2{\cdot}9 \times 10^{9}$
Steel	$19{\cdot}0\text{-}20{\cdot}6 \times 10^{11}$	$18{\cdot}1 \times 10^{11}$	$7{\cdot}9\text{-}8{\cdot}9 \times 10^{11}$	·25-·33	$18{\cdot}6\text{-}23{\cdot}3 \times 10^{9}$
Water	...	$\cdot205 \times 10^{11}$	...	...	...

III—COEFFICIENTS OF RESTITUTION

1.	Clay	0·20
2.	Cork	0·70
3.	Iron (balls)	0·70
4.	Glass (Brass)	0·90
5.	,, (Iron)	0·80
6.	Ivory	0·80

IV—COEFFICIENTS OF VISCOSITY

Liquids ($20^{\circ}C$)

1.	Alcohol	0·0119
2.	Benzene	0·00649
3.	Carbon disulphide	0·00367
4.	Carbon tetrachloride	0·00969
5.	Chloroform	0·00564
6.	Ether	0·00234
7.	Glycerine	8·500
8.	Mercury	0·016
9.	Turpentine	0·0149
10.	Water	0·01006

Gases ($15^{\circ}C$)

1.	Air	·000181
2.	Carbon dioxide	·000144
3.	Hydrogen	·000089
4.	Nitrogen	·000174
5.	Oxygen	·000198

Vapours

1.	Alcohol	($109^{\circ}C$)	·00010
2.	Benzene	($100^{\circ}C$)	·00009
3.	Ether	($100^{\circ}C$)	·000097
4.	Mercury	($300^{\circ}C$)	·000532
5.	Water	($100^{\circ}C$)	·000120

V—MOLECULAR ELEVATION OF BOILING POINTS OF SOLVENTS

(Elevation per 1 gm.molecule of salt per 100 c.c. of solvent)

Solvent	*Normal Boiling Point (under 1 atm. pressure)*	*Molecular Elevation*
1. Acetone.	56·0°C	17·2°C
2. Alcohol	78·4°C	11·9°C
3. Benzene	80·4°C	26·1°C
4. Chloroform	61·0°C	38·0°C
5. Ether	35·5°C	20·9°C
6. Water	100·0°C	5·15°C

VI—MOLECULAR DEPRESSION OF FREEZING POINTS OF SOLVENTS

(Depression per 1 gm. molecule of salt per 100 c.c. of solvent)

Solvent	*Freezing Points*	*Molecular Depression*
1. Acetic Acid	17·0°C	38·5°C
2. Benzene	5·5°C	51·6°C
3. Phenol	40·0°C	78·6°C
4. Water	0°C	18·6°C

VII—SURFACE TENSIONS OF IMPORTANT LIQUIDS

(in contact with air)

Liquid	*Temp.*	*Surface-tension (dynes) per cm.)*	*Angle of contact with glass & air*	*Liquid*	*Temp.*	*Surface-tension (dynes per cm.)*	*Angle of contact with glass & air*
Alcohol	20·0°C	22·0	0°·0	Paraffin Oil	25·0°C	26·4	26°·0
Benzene	17·5°C	29·2	0°·0	Turpentine	15·0°C	27·3	17°·0
Carbon disulphide	19·4°C	33·6	...	Water	15·0°C	74·0	8°·9·
Chloroform	15·0°C	27·2	0°.0				
Ether	20·0°C	16·5	16°·0				
Mercury	20·0°C	465	127°·20′				
Olive oil	20·0°C	32·0	...				

VIII=MOLECULAR CONSTANTS
(At N.T.P.)

1. *Number of molecules per c.c. of a gas* $=2{\cdot}75\times10^{19}$
2. *Avogadro Number, or number of molecules per gm. molecule of a gas* $=6{\cdot}022\times10^{23}$
3. *Mass of Hydrogen atom* $=1{\cdot}67\times10^{-24}$.

Gas	*Molecular Diameter*	*Molecular Velocity (r.m.s.)*	*Mean Free Path*	*Collision Frequency*
Carbon dioxide	$4{\cdot}32\times10^{-8}$ *cm.*	$3{\cdot}92\times10^{4}$ *cm./sec.*	$6{\cdot}29\times10^{-6}$ *cm.*	$5{\cdot}74\times10^{9}$ *per sec*
Hydrogen	$2{\cdot}40\times10^{-8}$,,	$18{\cdot}39\times10^{4}$,, ,,	$18{\cdot}3\times10^{-6}$,,	$9{\cdot}255\times10^{9}$,, ,,
Nitrogen	$3{\cdot}31\times10^{-8}$,,	$4{\cdot}93\times10^{4}$,, ,,	$9{\cdot}44\times10^{-6}$,,	$4{\cdot}899\times10^{9}$,, ,,
Oxygen	$3{\cdot}11\times10^{-8}$,,	$4{\cdot}61\times10^{4}$,, ,,	$9{\cdot}95\times10^{-6}$,,	$5{\cdot}00\times10^{9}$,, ,,

Mean molecular velocity $=\cdot921$ *r.m.s. velocity.*

Collision Frequency=Mean molecular velocity/Mean free path.

LOGARITHMS

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	5 9 13 4 8 12	17 21 26 16 20 24	30 34 38 28 32 36
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 12 4 7 11	16 20 23 15 18 22	27 31 35 26 29 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 11 3 7 10	14 18 21 14 17 20	25 28 32 24 27 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 6 10 3 7 10	13 16 19 13 16 19	23 26 29 22 25 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9 3 6 9	12 15 19 12 14 17	22 25 28 20 23 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 9 3 6 8	11 14 17 11 14 17	20 23 26 19 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 6 8 3 5 8	11 14 16 10 13 16	19 22 24 18 21 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 5 8 3 5 8	10 13 15 10 12 15	18 20 23 17 20 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7 2 4 7	9 12 14 9 11 14	17 19 21 16 18 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7 2 4 6	9 11 13 8 11 13	16 18 20 15 17 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4	6 7 8	10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 6 8	9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 3 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3	5 6 7	8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3	4 5 6	7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3	4 5 6	7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3	4 5 6	7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3	4 5 6	7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3	4 5 6	7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 5	6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 4 5	6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	123	456	789
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
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98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 4 4

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	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
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·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
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·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
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·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
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·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

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·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
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86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

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