UNIVERSAL
LIBRARY

OU_156360

UNIVERSAL
LIBRARY

OSMANIA UNIVERSITY LIBRARY

Call No. Accession No. 6 1141

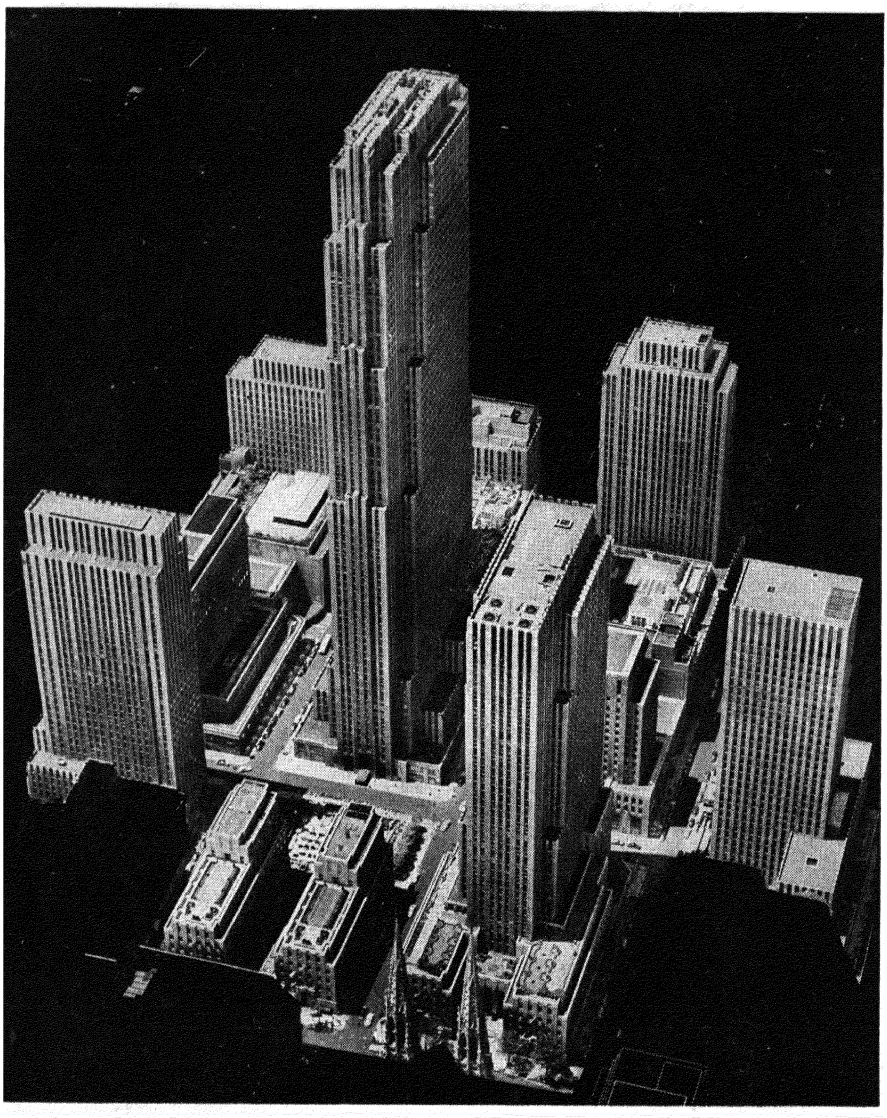
Author Schnell, Leroy H., and Crawford Mildred R.

Title Solid geometry. 1953.

This book should be returned on or before the date last marked below.

SOLID GEOMETRY

A Clear Thinking Approach



ROCKEFELLER CENTER, NEW YORK CITY

SOLID GEOMETRY

A Clear Thinking Approach

LEROY H. SCHNELL
State Teachers College
Indiana, Pennsylvania

MILDRED G. CRAWFORD
Roosevelt School, Michigan State Normal College
Ypsilanti, Michigan

McGRAW-HILL BOOK COMPANY, INC.
NEW YORK TORONTO LONDON

SOLID GEOMETRY: *A Clear Thinking Approach*

Copyright, 1953, by the McGraw-Hill Book Company, Inc.
Printed in the United States of America. All rights reserved.
This book, or parts thereof, may not be reproduced in any form
without permission of the publishers.

Library of Congress Catalog Card Number: 52:10339

Contents

		PREFACE	**ix**
Chapter	**1.**	**INTRODUCTION**	**1**
		Basic Concepts: Surfaces and Solids · Planes · Drawing Figures · Assumptions · Chapter Test · Keep the Iron Hot	
Chapter	**2.**	**PERPENDICULAR LINES AND PLANES**	**13**
		Theorem · "One, and Only One" · Indirect Proof · Summary of Indirect Proof · Theorems · Chapter Review · Chapter Test · Keep the Iron Hot	
Chapter	**3.**	**PARALLEL LINES AND PLANES**	**30**
		Theorems · Chapter Review · Analyzing Geometric Problems · Chapter Test · Keep the Iron Hot	
Chapter	**4.**	**DIHEDRAL AND POLYHEDRAL ANGLES**	**47**
		SECTION I: Dihedral Angles · Preliminary Theorems · Theorems · SECTION II: Polyhedral Angles · Theorems · Symmetric Figures · Theorem · Chapter Review · Chapter Test · Keep the Iron Hot	
Chapter	**5.**	**LOCI AND PROJECTIONS**	**67**
		SECTION I: Loci · Theorems · SECTION II: Projection · Theorems · Chapter Review · Chapter Test · Keep the Iron Hot	

Chapter 6. POLYHEDRONS 77

General Polyhedrons: Introduction · SECTION I: Prisms · Preliminary Theorems · Theorems · Parallelepipeds · Preliminary Theorems · Theorem · Volume · SECTION II: Cavalieri's Principle · Theorems · Crystals · SECTION III: Pyramids · Preliminary Theorems · Theorems · SECTION IV: Special Polyhedrons · Similar Polyhedrons · Theorems · The Prismatoid Formula · Regular Polyhedrons · Chapter Review · Chapter Test · Keep the Iron Hot

Chapter 7. CYLINDERS AND CONES 106

SECTION I: Cylinders · Preliminary Theorem · Theorem · Measurement of Cylinders · Limits—A Basic Theory of Mathematics · Theorems · SECTION II: Cones · Preliminary Theorems · Theorems · Conic Sections · Chapter Review · Chapter Test · Keep the Iron Hot

Chapter 8. SPHERES 126

Theorem · Definitions of New Terms · Theorems · Chapter Review · Chapter Test · Keep the Iron Hot

Chapter 9. SPHERICAL POLYGONS 141

Preliminary Theorems · Theorems · Polar Triangles and Symmetric Spherical Polygons · Two New Assumptions · Theorems · Chapter Review · Chapter Test · Keep the Iron Hot

Chapter 10. SPHERICAL AREAS AND VOLUMES 159

SECTION I: Areas · Theorems · A New Unit of Measure · Preliminary Theorems · Spherical Excess · Theorem · SECTION II: Volumes · Preliminary Discussion for Theorem 67 · Theorem · Special Volumes · Chapter Review · Chapter Test

Chapter 11. REFERENCE 176

SECTION I: Vocabulary for Solid Geometry · SECTION II: Symbols for Plane and Solid Geometry · SECTION III: Vocabulary for Plane Geometry · SECTION IV: Axioms · SECTION V: Assumptions and Theorems of Plane Geometry · Lines and Points · Triangles · Parallel Lines and Perpendicular Lines · Parallelograms · Angles · Circles · Circle Measurement · Locus · Concurrent Lines · Inequalities · Ratio and Proportion · Similarity · Areas and Regular Polygons · Constructions in Plane Geometry · SECTION VI: Assumptions in Solid Geometry · SECTION VII: Formulas · Notation Used in Formulas · Formulas in Plane Geometry · Formulas in Solid Geometry

INDEX 197

Preface

The authors do not present this book as a sequel to *Plane Geometry: A Clear Thinking Approach*, in the usual sense. However, the essential pedagogical fundamentals upon which the plane geometry is based have been retained. Since solid geometry is a one-semester course and since there are approximately the same number of theorems to present as there are in plane geometry, it is obvious that the pattern of organization of the two books cannot be exactly parallel unless both are formal and quite rigidly subject-matter centered.

All the required theorems listed in the National Committee Report and by the College Entrance Examination Board have been included, as well as those listed in the Syllabus of the University of the State of New York.

Great care has been exercised in developing the basic concepts so that they will take on complete meaning for the students. To the greatest extent possible a maximum number of opportunities for pupil-discovery have been provided. Whenever feasible, the students are expected to develop proofs independently. However, in difficult situations, adequate guidance is given. Since the authors wish to discourage memorization of proofs, complete defenses for steps of formal proofs are rarely given. Only when the student is required to contribute his best thinking will the total situation assume maximum meaning. When the "key defenses" in a proof are stated or referred to in the traditional manner, memorization is encouraged.

Many modern plane geometry textbooks give considerable attention to reasoning in everyday situations, in an effort to "force the transfer" of the kind of assumptive, logical reasoning done in geometry, to daily living. Users of *Plane Geometry: A Clear Thinking Approach* know that this is a distinctive feature of that book. The authors do not believe that this activity should be compartmentalized and assigned only to a course in plane geometry. Although the same amount of attention given in plane geometry to applying geometric reasoning to everyday situations cannot be given in a course in solid geometry, it is important to keep the iron hot and to give further practice to this worth-while educative experience. At the

end of several chapters, everyday reasoning exercises are given in the "Keep the Iron Hot" sections. Since these sections are not made a part of the main sequence of development, they can be studied or omitted at the discretion of the teacher. However, it is the sincere hope of the authors that these sections will be given ample attention.

Not all "Keep the Iron Hot" sections are devoted to everyday reasoning. Some exercises are offered merely to "keep alive" some of the important skills learned in previous mathematics courses, while in some instances, the review called forth is to prepare the student for material to be presented in the next chapter.

Following are other features of this book:

1. A Chapter Review is given at the end of each chapter (except Chapters 1 and 11).
2. Following the exercises at the end of each chapter (except Chapter 11), a carefully worked out Chapter Test is presented.
3. Students are encouraged to devise a solid geometry Tool Box as an aid in analyzing geometric problem situations.
4. Careful attention is given to helping the student develop skill in making *three-dimensional drawings*, and in using easily accessible materials for visual aids.
5. The assumptions, theorems, and corollaries of plane geometry are listed in the Reference Chapter as well as a summary of the assumptions and formulas of solid geometry. In the Vocabulary section of the Reference chapter the important definitions of plane and solid geometry are accessible.
6. Illustrative pictures have been carefully selected in order to help the student see the many applications of solid geometry concepts in the world about him.
7. Some material is presented at greater length than is usual, primarily to stimulate an interest in students who have the ability to study more mathematics. The discussion of the conic sections in Chapter 7 illustrates this point.

The authors wish to express gratitude to many friends who offered suggestions and encouragement while this book was being written. Particular appreciation is due to Dr. I. L. Stright, State Teachers College, Indiana, Pennsylvania, for numerous suggestions and for his critical reading of the final manuscript.

LEROY H. SCHNELL

MILDRED G. CRAWFORD

CHAPTER 1 *Introduction*

Since you have *elected* to study solid geometry, it is reasonable to assume that you are convinced of the importance of mathematics in general and geometry in particular. Therefore, you need not review the reasons why you should study geometry or what you may expect to gain from your efforts. However, it might be profitable to spend some time in class discussing what is meant by the following terms: *proof*, *assumption*, *axiom*, *postulate*, *sequence*, *logical reasoning*, *theorem*, *corollary*, *indirect proof*, and *converse*.

In your present study you will observe that plane and solid geometry are similar in many respects but different in others. Did it occur to you that in plane geometry you were dealing with figures that existed *only in your imagination*? You and I have actually never seen a triangle, rectangle, or parallelogram. These figures belong to the *two-dimensional* world of length and width. We are *three-dimensional* creatures, and all the things about us that we can see, touch, and move around are three-dimensional objects. Of course, we can represent a rectangle by drawing four lines in a plane. The sheet of paper on which we draw the rectangle merely *represents* a plane. The paper has thickness as well as length and width. The lines we draw have thickness, formed by particles of carbon from the drawing pencil. Hence the rectangle we draw has thickness as well as length and width. Since, by definition, a rectangle has only length and width, the figure we draw to represent it is really not a true rectangle.

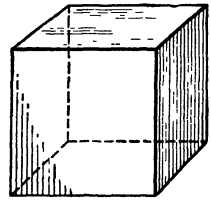
Ours is a world of reality, a three-dimensional one, in which objects are solids that have *length*, *width*, and *thickness*. The "world of plane geometry" is a two-dimensional world in which no object has thickness. If this two-dimensional world were peopled with reasoning creatures, they would find it tremendously difficult to imagine what exists in a world beyond the boun-

daries of their own.* Even a two-dimensional genius could not build or fly an airplane. He would find it almost impossible to imagine any type of locomotion that would take him out of his world of length and width. Such a creature would have no difficulty in understanding the properties of such figures as rectangle, square, parallelogram, or trapezoid, but it would tax his intellect even to imagine the appearance of a cube, a sphere, or any other solid figure. We shall have no such difficulty since we are three-dimensional creatures living in a three-dimensional world.

Reversing matters, we three-dimensional creatures are confronted with a problem when we deal with figures of two dimensions. However, our problem is quite easily solved since we can *represent* two-dimensional figures with three-dimensional objects, the thickness of which is negligible, or we can *represent* the desired figures on some three-dimensional object. Nevertheless, we should always remember that these representations of two-dimensional figures actually have some thickness, as well as length and width.

In solid geometry we shall continue to deal with points, lines, and plane figures, but we shall not confine our study to a **single plane** as we did in plane geometry. Our main concern will be with the relationships and properties of these figures **in space** and also with the properties of *figures of three dimensions*.

In representing three-dimensional objects you may encounter some difficulties since you will frequently need to draw such objects on a plane surface, representing the third dimension by various devices. The accompanying figure is not actually a cube, but you have no difficulty in visualizing it as a cube even if the sketch is drawn on the plane surface of this page. Sketches for some of the figures you will study will be somewhat more complicated than this one. However, at the proper time, you will study certain basic principles that will enable you to represent objects of three dimensions in a plane surface.



Less time will be given to studying logical procedures and methods of proof than was given in plane geometry. We assume that you will apply these procedures automatically because they have become fixed in your thinking. Continue to challenge every step in the proofs of theorems and exercises; be sure that every argument is defended by a previously proved theorem or corollary, an assumption, a definition, or a fact given in the hypothesis. Slovenly thinking in solid geometry is just as inappropriate as in plane geometry.

*For a "running broad-jump into the fourth dimension" see *Plane Geometry: A Clear Thinking Approach*, by Schnell and Crawford, Chapter 16, McGraw-Hill Book Company, Inc., New York, 1953.

The assumptions, theorems, and corollaries of plane geometry are listed in Chapter 11 on "Reference," pages 176–196. These accepted and proved facts may be used, when appropriate, as defenses in proving theorems in solid geometry. Obviously, if we are logically consistent in plane geometry, follow the same rules of logic in solid geometry, and accept new assumptions for three-dimensional figures that do not contradict the assumptions and proved facts of plane geometry, solid geometry becomes merely an extension of our previous system of sequence and logic.

In some modern plane geometry textbooks, considerable attention is given to applying logical reasoning techniques to everyday, non-mathematical problem situations. This activity is exceedingly important, and it is hoped that you will continue to use sound reasoning in *all* your thinking. Do not reserve it for classroom problems only, and then let prejudice, ignorance, or emotions dictate the solution of the non-mathematical problems you meet. In this present one-semester course there are so many important geometric relationships to be considered that we will be kept busy discussing them. However, in the "Keep the Iron Hot" sections which appear frequently throughout this book, we will include some everyday, non-mathematical problems in order that you may have more practice in trying to apply the principles of **clear thinking** to the type of problems you will meet continually for the rest of your life.

BASIC CONCEPTS: SURFACES AND SOLIDS*

Just as in plane geometry we gave no precise definitions for *point*, *line*, and *plane*, in solid geometry we will not give a formal definition for *surface* or *space*. However, the terms are familiar. Everywhere we look in our world of physical objects we see surfaces. The smooth top of your desk is a *plane surface*. A baseball has a *spherical surface*. The ordinary tin can on the grocer's shelf has a *cylindrical surface* as well as two parallel *plane circular surfaces*. Notice that the surfaces of these physical objects can be seen and touched. The surfaces are the "boundaries" of the solids.

It is easy to see that *if* **any** *two points on a surface can be joined by a straight line, all points of which lie in the surface, the surface is a* **plane surface.** This leads to the observation that *a* **curved surface** *is one, no part of which is a plane.* Thus a *cylindrical surface* and a *spherical surface* are both *curved surfaces.*

Can you draw a straight line on a cylindrical surface so that it will lie entirely in the surface? If so, does this illustration argue that a cylindrical surface is also a plane surface? Why or why not? Try the same experiment with a sphere.

*Definitions of the important terms of plane and solid geometry are listed in the vocabulary sections of Chapter 11. See pages 176–184 *now*. Get the habit of referring to this chapter whenever you are not sure of the definition in geometry.

A **geometric solid** is defined as a *closed surface which encloses a portion of space.* Properties of the various geometric solids will be studied later. For the present, it is assumed that you are familiar with such common solids as *cylinder, cone, sphere, rectangular solid,* and *pyramid.* See the drawings above and compare them with the photographs of the "real life" solids on page 5.

PLANES

Although we shall offer no formal definition for a plane, a brief discussion about the properties of a plane should be helpful. The word comes from a Latin word which means "level throughout." As we know from plane geometry, a plane may be thought of as extending indefinitely instead of being confined to a specific number of units of length and width. It is important to remember that a plane has *no thickness.* The thinnest sheet of paper ever made has still some thickness and therefore would not be an actual plane, although it could be used to *represent* one.

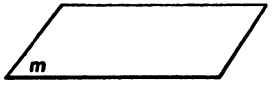


FIG. 1.

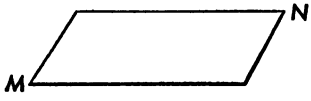


FIG. 2.

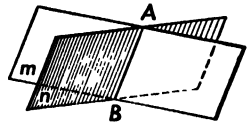


FIG. 3.

In referring to a plane, one small letter written in the plane is sometimes used, as plane *m* in Fig. 1. Two capital letters written at two opposite vertices can also be used to name a plane, as in Fig. 2. Figure 3 represents planes ? and ?, intersecting each other in line* ?.

PRELIMINARY EXERCISES: DETERMINING A PLANE

1. Imagine a straight line in space. How many *different* planes do you think can be passed through it? Use stiff wire and cardboard to illustrate. (The phrase "passed through it" means that the plane will contain *all* points which lie on the line.)
2. With the help of a classmate place three sharpened pencils of *different lengths* upright on your desk. Be sure that the three are *not* in a straight

*Unless otherwise stated, "line" shall be understood to mean *straight line.*

line. How many points in space will the sharpened ends represent? How many different planes can be passed through all three points so that each plane will contain the three points? Illustrate with cardboard.

3. Repeat Ex. 2 using four pencils.
4. Hold a stiff wire in a fixed position in space to represent a straight line. Have someone hold a sharpened pencil to represent a fixed point that does not lie on the line. Show how many planes can be passed through the straight line and the external point.
5. Illustrate with wires and cardboard how many different planes can be passed through two intersecting straight lines. (The phrase "passed through two intersecting straight lines" means that the plane, or planes, will contain *both* lines.)
6. Repeat Ex. 5 for two lines in space drawn so that they run in the *same* direction and will never meet.
7. Represent two straight lines in space drawn so that they run in *different* directions and will never meet. Demonstrate how many planes can be passed to contain *both* lines.
8. Which of the two preceding exercises (6 and 7) obviously involved *parallel* lines and which involved *skew* lines? In what respect are parallel and skew lines alike? Different? What would be a good definition for skew lines?
9. What two conditions must be established before you can say that two lines are parallel?
10. Decide which is always steady, a three-legged or a four-legged stool, even when its legs are not of equal length. Support your conclusion by citing one or more of the foregoing exercises.

Before generalizing on the above exercises, consider the following statement. **A plane is determined** *by certain given points, or lines, or points and lines, when it is the* **only** *plane that can contain the given parts.*

On the basis of the above exercises, complete the following statements.

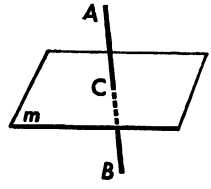
11. ?_ points, not in the same straight line, determine a plane.
12. A line and a ?_ not on the line determine a plane.
13. Two ?_ straight lines determine a plane.
14. Two ?_ lines determine a plane.

Do not write in your book.

PRELIMINARY EXERCISES: LINES AND PLANES

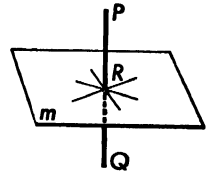
In Exs. 1–5, use stiff wire to represent a line and cardboards to represent planes. Experiment in order to complete the statements given.

1. The intersection of a straight line and a plane is a ? .
2. The intersection of two planes is ? ? ? .
3. If two planes have one point in common they .... (Remember that a plane extends indefinitely.)
4. If a line lies entirely in *each of two planes*, it is ? ? .
5. Through a fixed line in space, ? ? number of planes can be passed.
6. Show whether or not the following statement is sensible: A straight line joining *any* two points in a plane lies entirely in the plane.
7. Discuss how the statements in Exs. 2, 3 and 4 relate to each other.
8. *The intersection of two surfaces, or of a line and a surface, contains* **all** *points common to both.* In other words, the intersection is the *locus* of their common points. Select the statement or statements from the foregoing exercises that relate to the intersection of a plane and line; of two planes.



The point of intersection of *a straight line and a plane* is called **the foot of the line.** In the accompanying drawing, C is the *foot* of line AB since AB intersects m at point C.

If a straight line is perpendicular to every line in a plane that can be drawn through its foot, it is said to be **perpendicular to the plane.** The *distance from a point to a plane* is measured *along the perpendicular* drawn from the point to the plane.



In the drawing, R is the foot of line PQ, and the three lines through R are a few of the infinite number that can be drawn through R in plane m. If all the lines that can be drawn through R are perpendicular to PQ, then $PQ \perp m$. Since definitions can be reversed, it is also true that *if a line is perpendicular to a plane, it is perpendicular to* **every** *line in the plane drawn through its foot.* The plane is also said to be perpendicular to the line.

When a straight line *intersects a plane* but is *not perpendicular to it,* it is said to be **oblique to the plane.**

A line and a plane, *or* **two planes, are parallel** *if they do not meet* no matter how far extended.

Two straight lines that *do not lie in the same plane and which will never intersect,* no matter how far extended, are called **skew lines.** (See Exs. 6, 7, and 8 on page 5.)

DRAWING FIGURES

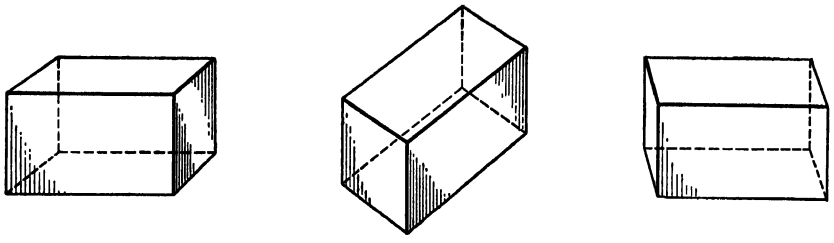
As previously emphasized, the figures with which we deal in solid geometry are three dimensional. Since it is not always convenient to make models for all such figures, we often must draw them on paper or on a blackboard. Representing a three-dimensional figure on a two-dimensional surface is often somewhat complicated. However, with practice, and by following a few simple suggestions, you should develop the necessary skill quite easily. The trick lies in learning to *represent objects* **as they appear to an observer.** Almost everyone has observed how the parallel rails of a railroad track *seem* to converge and eventually to meet in the distance. We know that they actually do not meet, but if we want to represent two parallel rails so that a person looking at our drawing can get a *realistic picture*, we must make the parallel lines seem to converge as they approach the vanishing point at eye level.

If you stand at a roadside near the foot of a telephone pole and compare the height of the pole with the height of another pole several hundred yards down the road, the one near you *appears* to be taller than the one farther away. In fact, as you look down such a line of poles which are all approximately the same height, each succeeding one seems to be a little shorter than the preceding ones. Therefore, in representing a series of parallel objects of equal height, each succeeding one, as they recede from the point of observation, should be drawn a little bit shorter in order to give the drawing *reality* and **perspective.**

Cut a circular disk from cardboard. Have someone hold it so that the plane of the disk is perpendicular to your line of vision. Then have that person recede from you, holding the disk in the same position. You will note that the disk seems to decrease in diameter the farther away it is taken but that its shape (circle) does not change.

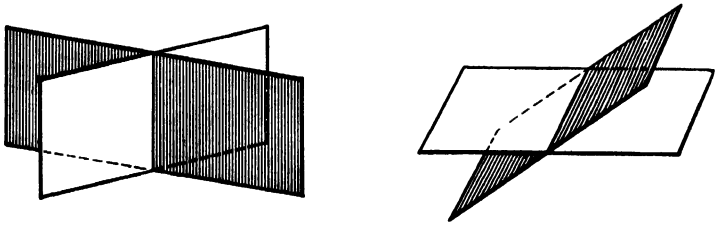
Next hold the disk in the same position as before. Slowly rotate it so that the upper edge recedes from your eye. You will observe that the shape of the disk seems to change. It no longer appears to be an exact circle but seems to become somewhat elongated into an ellipse. Eventually, when the plane of the disk coincides with your line of vision, the disk *seems* to have become almost a straight line.

The foregoing illustrations emphasize the fact that when a three-dimensional figure is drawn *in perspective*, two factors must be taken into consideration: first, the distance of the parts of the object from the observer, and second, the apparent shape that the object assumes because of its position in space in relation to the observer. Compensation must be made for both factors.



Above are three different views of the *same* rectangular solid in which the unseen edges are represented by dotted lines.

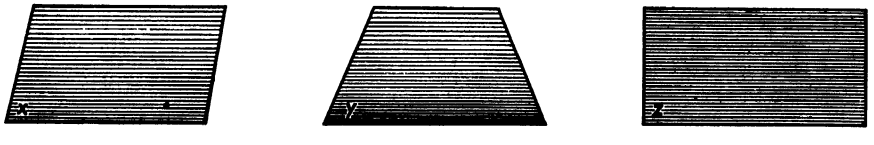
Below are different views of two intersecting planes taken from different distances and positions with respect to the observer. After studying them see if *you* can make a drawing for each view.



The following suggestions will help you in making perspective drawings. Refer to them when you make drawings for theorems and exercises.

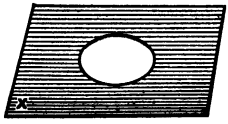
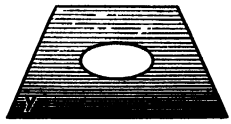
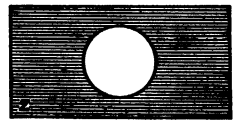
1. *Use a rectangle to represent a plane that does not deviate from the perpendicular to the line of vision.* The position of such a plane not only will be approximately perpendicular to the line of vision but also perpendicular to the plane on which the observer stands.

2. *Let a parallelogram represent a plane that deviates only slightly from the perpendicular to the line of vision. Use a trapezoid if the plane deviates sharply*



3. *Make lines nearer the observer darker and slightly heavier than those farther away.*

4. *Represent a horizontal line that does not recede from the eye, by a horizontal line. Represent one that does recede by a slanting line.*

5. *Represent vertical lines by vertical lines.*

6. *Represent parallel lines by parallel lines if they do not recede. Otherwise, make them converge slightly.*

7. *Let an ellipse represent a circle whose plane surface is not perpendicular to the observer's line of vision. Otherwise, use a circle.*

8. *Use dotted lines to represent the lines which are actually part of the figure but which cannot be seen by the observer.*

ASSUMPTIONS

On the basis of the experimental exercises in this chapter, you should be willing to accept the following assumptions. Once accepted, they can be used as defenses in proofs, along with definitions, assumptions from plane geometry, and the theorems and corollaries proved in plane geometry.

1. *A straight line joining any two points in a plane lies entirely in the plane.*
2. *The intersection of a straight line and a plane is a point.*
3. *The intersection of two planes is a straight line.*
4. *Through a given straight line an unlimited number of planes can be passed.*
5. *Three points not in a straight line determine a plane.*
6. *A straight line and a point not on the line determine a plane.*
7. *Two intersecting straight lines determine a plane.*
8. *Two parallel lines determine a plane.*
9. *If two planes have one point in common, they either intersect or coincide.*
10. *If a straight line lies in each of two planes, it is their intersection.**

*Assumptions, in a sense, are the "rules of the game" which are to be adhered to. Therefore, all the "rules" should be agreed upon at the beginning. In addition to the ten assumptions stated above, twelve more will be needed in this course. To give them here would require the introduction and explanation of many new ideas and delay developing proofs unnecessarily. Therefore, the rest of the assumptions will be introduced later nearer to the time when they will be used. All twenty-two assumptions are listed on pages 194–195. It is not wise for you to study them now, but there will be no harm in scanning them briefly.

CHAPTER TEST

1. What is meant by determining a plane?
2. What are the conditions for determining a straight line?
3. List the conditions under which a plane is determined.
4. When is a line perpendicular to a plane?
5. How is the distance of a point from a plane measured?
6. What is an oblique to a plane?
7. What is the foot of a line that is perpendicular to a plane?
8. How many different planes are determined by four points, no three of which lie in a straight line? By five such points? Hint: The answers to these questions can be found experimentally or by applying the principle of *combinations* if you studied that topic in algebra.
9. If four lines pass through a given point, will they necessarily lie in the same plane?
10. What is the greatest number of planes that can be determined by four lines passing through a given point X, but drawn so that no three of them lie in the same plane?
11. Point out an example of skew lines in your room, of parallel lines, of an oblique to a plane, of a line perpendicular to a plane, of a line parallel to a plane.
12. If a paper is folded and creased, why does it form a straight edge along the crease?
13. What are the two conditions that must be met for two lines to be parallel?
14. Why is a three-legged stool always "steady" while a four-legged one may not be?
15. Represent each of the following situations with wire and cardboard and then make good drawings of them.
    (a) A line perpendicular to a plane.
    (b) Two intersecting planes.
    (c) A line that is oblique to a plane but perpendicular to a line in the plane.
    (d) A line drawn to a plane from a point in a perpendicular to a plane.
    (e) A line perpendicular to each of two intersecting lines.
16. Through a given line an ? number of ? can be passed.
17. The intersection of ? planes is a ? ?.
18. If two planes have ? point in common, they either ? ? ?.
19. A straight line joining any ? points in a plane lies entirely ? ? ?.
20. If a straight line lies in ? of two planes, it is ? ?.
21. The intersection of a straight line and a plane is a ?.

*Keep the Iron Hot**

Set your goal at 100 *percent accuracy.*

Write all answers on separate paper. Copy exercises when necessary.

1. Mr. Smith has an annual taxable income of \$4,150. Besides his wife he has two children, aged 12 and 15. If he is allowed \$600 for each dependent as well as for himself, upon what amount will his tax computation be based? If he pays 18 percent of that amount in income tax, how much does he have to send to the Collector of Internal Revenue?

2. Add:

(a) \$7.08 / .98 / 4.56 / 9.89

(b) 3973 / 1605 / 7968 / 4327

(c) $15\frac{1}{2}$ / $7\frac{1}{4}$ / 9 / $5\frac{3}{4}$

(d) $23\frac{1}{3}$ / $64\frac{5}{6}$ / $6\frac{1}{3}$ / 31

(e) $6\frac{5}{8}$ / $2\frac{1}{4}$ / $3\frac{3}{8}$ / $12\frac{3}{16}$

(f) $2\frac{7}{12}$ / 15 / $1\frac{3}{8}$ / $4\frac{1}{2}$

3. Subtract:

(a) 230 − 76

(b) 1406 − 897

(c) $6\frac{3}{4} - 5\frac{7}{8}$

(d) $3\frac{5}{8} - 1\frac{1}{4}$

(e) $20 - 9\frac{5}{8}$

(f) $19\frac{2}{3} - 4\frac{7}{12}$

4. Multiply:

(a) 736×80

(b) 24.9×3.07

(c) 690×0.015

5. Divide:

(a) $8\overline{)720}$

(b) $7\overline{)4214}$

(c) $86\overline{)3698}$

*We tend to forget facts and skills that we learn if we do not use them. It would seem unwise for geometry students to permit the information and skills acquired in earlier mathematics classes to disappear or to become weakened by disuse. It is therefore essential to KEEP THE IRON HOT by frequent practice and drill. The practice exercises given frequently throughout this book are to help you keep alive the important skills of arithmetic, algebra, and plane geometry. Also, many of the drills will help you to review some of the processes you will need to use in this course.

It is hoped that you will make the most of these drills. If you find that you have forgotten how to solve some of the exercises, ask your teacher to suggest further "outside" review. Use these drills to locate your difficulties and make a serious effort to improve.

Perform the indicated operations:

6. (a) $9 \times \frac{2}{3}$ (b) $2\frac{1}{2} \times 3\frac{1}{2}$ (c) $21 \div \frac{1}{3}$
(d) $3\frac{3}{4} \div \frac{5}{8}$ (e) $12\frac{1}{2} \div 3\frac{1}{4}$

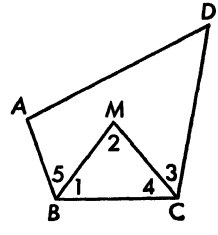
7. (a) $26.8 \div 0.4$ (b) $\frac{3}{4}$ of 48 (c) $288 \div 0.09$
(d) $3\frac{1}{4} \div 12$ (e) $57.6 \div 3.6$

8. (a) 1.7×10 (b) 0.0268×100 (c) $6.3 \div 10$
(d) 0.014×100 (e) $17.04 \div 10$

9. Mary has three times as many dimes as quarters in her purse.
If x represents the number of quarters,
Then ? represents the number of dimes, and
Then ? represents the value, in cents, of the quarters, and
Then ? represents the value, in cents, of the dimes.

10. In a triangle, one angle is three times as large as the smallest angle, and another is 40° less than four times the smallest. How large is each angle?

11. In quadrilateral $ABCD$, BM and MC bisect $\measuredangle B$ and C respectively. Select from the conclusions listed below the ones you think are correct, and defend each one.



(a) $\angle 2$ is a rt. $\angle$.
(b) $\angle CBM = \angle MBA$.
(c) $\angle MCB = \angle MBA$.
(d) $\angle 5 + \angle 2 + \angle 3 = 180°$.
(e) $\angle B + \angle D = 180°$.
(f) $\angle 1 = \angle 4$.
(g) $\angle 2 = \angle 5$.
(h) $\angle A + \angle D = 180°$.

12. Identify the type of reasoning employed in the following geometric and non-geometric situations. If the conclusions are not valid, show why.

(a) **Geometric.** Since two triangles are congruent when the three sides of one are equal respectively to the three sides of the other, two quadrilaterals are congruent when the four sides of one are equal respectively to the four sides of the other.

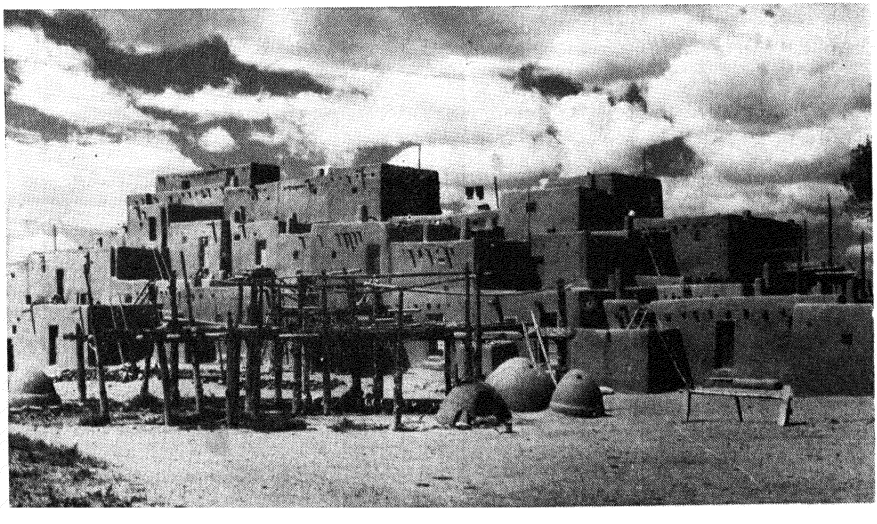
(b) **Non-geometric.** Beautiful Miss Movie Starr uses Sparko Shampoo. You too can be as beautiful as Miss Starr. Use Sparko Shampoo!

13. It was recently reported that $125,000,000 is spent annually in this country on fortune tellers. Over ten million rabbits' feet have been purchased in the United States at a cost of ten cents to five dollars each. Millions of dollars are spent annually for horoscope readings. What are some of the assumptions accepted by people who spend these huge sums of money? Discuss in class the validity of the assumptions.

CHAPTER 2 *Perpendicular Lines and Planes*

In the picture below, there are many examples of plane surfaces and lines. Indicate some parallel planes, intersecting planes, planes perpendicular to each other, lines perpendicular to planes, lines oblique to a plane, and skew lines. Then test your understanding by completing the following statements:

1. Two parallel lines lie in the ? ? and will never ? no matter how ? ?.
2. If any two ? on a surface can be joined by a ? line, all points of which lie in the ? , it is a plane surface.



Taos Indian Pueblo in New Mexico. (Frasher's Photos, Pomona, California)

3. All physical objects have _?_ dimensions.
4. The point of intersection of a straight line and a plane is called the _?_ of the line.
5. A line is said to be perpendicular to a plane if it is _?_ to every _?_ that can be drawn in the plane passing through _?_ _?_.
6. When a line intersects a plane but is not perpendicular to it, it is said to be _?_ to the plane.
7. Two planes intersect in a _?_ _?_.
8. The line of intersection of two _?_ contains all the _?_ which are common to the two planes.
9. If a line is _?_ to a plane, the plane is said to be perpendicular to the _?_.
10. Two lines which are not in the same plane and will never meet no matter how far they are extended are called _?_ lines.
11. A line and a plane, or two planes, are _?_ if they do not _?_, however far extended.
12. If two planes, or a line and a plane, are not _?_, they will intersect or coincide.
13. A plane is said to be _?_ by certain given points or lines when it is the _?_ plane that can contain the given parts.
14. The distance from a point to a plane is measured along _?_ _?_ from the point to the _?_.

For the first few theorems in solid geometry you may encounter some difficulty in making and interpreting the drawings. While any difficulty you may have will clear up rapidly as you gain experience in visualizing geometric figures in three-dimensional space, you are urged to make models from cardboard, wire, and string. You should do this for at least the first several theorems, and also later whenever the conditions given in a theorem are not clearly understood, or whenever you have difficulty in visualizing the three-dimensional aspects of a drawing given in the textbook. *Whenever you construct a model, compare it with the textbook drawing.* In doing so, you should grow rapidly in your ability to draw and interpret perspective drawings.

The pattern for developing proofs in solid geometry is the same as you employed in plane geometry. Every statement in the argument must be defended by one of the following: (1) a previously proved theorem or corollary from *plane* or solid* geometry; (2) an assumption from plane* or solid geometry; (3) a definition; or (4) a fact that has been given in the hypothesis.

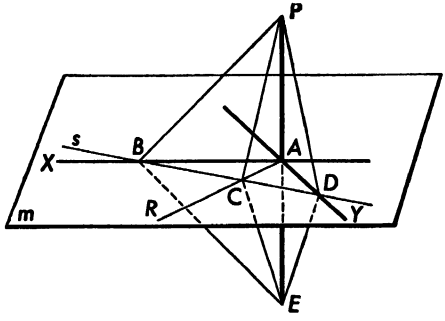
*The assumptions and theorems of plane geometry are given in a syllabus on pages 185–194.

PRELIMINARY EXERCISES

1. Experiment to see how many planes can be drawn perpendicular to a line at a given point on the line.
2. Experiment to see how many lines can be drawn perpendicular to a plane at a given point in the plane.
3. Hold a pencil to represent a line perpendicular to the plane of your desk top. How many lines can be drawn *in the plane* that are also perpendicular to the line?
4. Repeat Ex. 3 when your pencil is oblique to the desk top.
5. Hold a pencil so that it is perpendicular to the plane of your desk top. Hold another pencil so that it intersects the first one at a point which is not in the plane of the desk top and so that the second pencil is not parallel to the desk top. (*Note:* The second pencil will be oblique to the desk top.) Determine how many other obliques can be drawn to the plane *from the same point* on the perpendicular.
6. In Ex. 5, *where* do the obliques meet the plane if the obliques are equal line segments? Where if they are unequal?
7. Read Th. 1 on page 16. Study the "Given" and the "To Prove" statements. Make a model which shows the lines and planes indicated. Using the model, show what is to be proved in the theorem.
8. Study the arguments for steps 2, 3, and 4 in the proof for Th. 1. Add the indicated lines to your model. Use string to represent the **auxiliary** lines that do not lie in plane m. Bore small holes in the cardboard as necessary.
9. Compare your model with the textbook drawing and shift the model with respect to your eye level until the perspectives of the drawing and of the model are the same.
10. Using a piece of cardboard and a piece of wire to represent a plane and a straight line respectively, bore a hole in the cardboard so that the wire can be inserted but so that it will not fit too snugly. With chalk, mark a point on the line (wire).
11. Read Th. 2 on page 19. Using the materials prepared in Ex. 10 demonstrate *experimentally* the validity of the theorem. (Check against the conclusion you drew in Ex. 1 above.)
12. Read Th. 3 and the "given" statement on page 20. With wire and cardboard, construct a model. Hold the model so that it appears to be in the same perspective as the drawing in the book.
13. Reread Exs. 5 and 6 above. Then with cardboard, wire, and string, construct a model for Th. 7 on page 24.

THEOREM

Theorem 1. *If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane determined by the two lines.*



Given: Plane m containing lines AX and AY which intersect at A. $PA \perp AX$ and $PA \perp AY$.
To prove: $PA \perp$ plane m.

PROOF

Argument	*Defense*
1. $PA \perp AX$ and AY at their point of intersection A, and plane m is determined by AX and AY.	1. Given.
2. (a) Draw s to represent *any* line in plane m which does not pass through A. Let it intersect AX and AY in points B and D respectively.	2. (a) Defend.
(b) Through A draw *any* line AR to meet BD in some point such as C.	(b) Two lines can intersect . . ..
3. Extend PA to E making $PA = AE$.	3. A line may be extended . . ..
4. Draw $PB, PC, PD, BE, CE,$ and DE.	4. Defend.
5. AB and AD are $\perp$ bisectors of PE.	5. Given $\perp$ and definition of bisect.
6. $BP = BE$ and $DP = DE$	6. The $\perp$ bisector of a line is the locus of points equidistant . . ..
7. $\triangle PBD \cong \triangle BDE$	7. When three sides . . ..
8. $\angle CBP = \angle EBC$.	8. Why?
9. $\triangle CBP \cong \triangle CBE$	9. Why?

10. $CP = CE$	10. Why?
11. $AC \perp PE$ or $AP \perp AC$	11. Two points each equidistant from the extremities . . ..
12. $\therefore$ since AC is *any* line in the plane passing through A, $PA \perp$ plane m.	12. Why?

"ONE, AND ONLY ONE"

From your experience in plane geometry, you should be thoroughly aware that whenever a theorem involves the phrase "*one, and only one,*" two conditions are established, **each of which must be proved.** When you proved that *through three points, not in the same straight line, one, and only one, circle can be drawn*, you not only proved that *one* circle could be drawn under the conditions of the hypothesis, but you also proved that it was the *only* circle that could be drawn under those conditions. You had no right to *assume* that since you could prove that one circle was possible, it was the *only* circle. A proof was required to establish the validity of the second condition.

Perhaps the importance of the phrase "one, and only one" will be more apparent if you will recall the statement in plane geometry that *at a given point on a line one, and only one, perpendicular can be drawn.* A person who carelessly applied the foregoing statement to a line in three-dimensional space instead of restricting the line to a given plane, would be entirely incorrect since an infinite number of perpendiculars can be drawn in three-dimensional space to a point on a straight line. Consequently, the phrase "*and only one*" is of tremendous importance and cannot be assumed.

Since the phrase "*one, and only one*" appears in the statements of Ths. 2, 3, 5, and 6 in this chapter, the proof for each must establish the validity for both conditions and, therefore, there must be two distinct parts for each proof.

INDIRECT PROOF

Many times in plane geometry a satisfactory direct proof for a theorem could not be developed and it was necessary to resort to an indirect proof. It will be necessary to employ this type of proof occasionally in solid geometry. The following discussion will help you to refresh your memory.

In some problem situations *only two* possible conclusions exist, and *both* of them cannot be valid. For example, if Mr. Jones is being tried for the theft of $1,000, it must be true that he either **did** or **did not** steal the money. He cannot be both innocent and guilty at the same time! He might be

proved innocent by demonstrating that he *could not be guilty*. This type of indirect reasoning is based on the assumption that a statement must either be true or false and that it cannot be both true and false at the same time.

Part of Th. 2 states that, *through a given point on a given line, there can be only one plane perpendicular to the line.* This statement is either true or false. No other possibility exists. If it is true, there is *only one* such plane, but if it is false, there will be *more than one.* Therefore, if we **assume** that there is more than one such plane and by logical reasoning, show that our assumption is false because it leads to an absurdity, or a contradiction of the hypothesis of the theorem, or to a contradiction of an assumption or previously proved fact, we have thereby eliminated the possibility that there can be more than one such plane, and consequently, we have *proved* that there can be *only one* plane perpendicular to the line.

You will remember that in this type of indirect proof we *assume the opposite* of the fact we wish to prove and then show, logically, that our assumption *cannot* be valid. This leaves only one other possibility, namely, that the fact we wished to prove in the first place is valid.

Another type of indirect reasoning is based on the assumption which states that "*a quantity is equal to, greater than, or less than any other quantity.*" Therefore, if we wished to prove that $a = b$, we could do so by: (1) assuming that $a > b$ and demonstrating that this assumption is not valid; and (2) assuming that $a < b$ and demonstrating that this assumption is also invalid. Consequently, since two of the three possibilities are invalid, the remaining one must be accepted as valid.

SUMMARY OF INDIRECT PROOF

In general we try to develop a direct proof of propositions in geometry. When this seems impossible, or if we are asked to develop an indirect proof, proceed as follows:

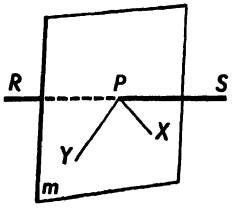
1. Set up "*Given*" and "*To prove*" as usual.
2. List *all* possibilities of the "*To prove*" statement. Be sure that no two of them can be true *at the same time*, and also that one of them *must* be true.
3. Leave the one you are asked to prove *absolutely alone.*
4. *Assume* the other possibility *to be true.* (If there are more than two possibilities to begin with, you may choose each one in turn.)
5. Begin a *direct proof based on this assumption.* Continue this *direct* proof until you come to a statement which contradicts one of the given facts *or* a previously proved fact, assumption, or definition.

6. When you arrive at such a contradiction, you can be sure that the proof based on your assumption is invalid, and so must be discarded. If the *only* possibility left is the one you want to prove, it will now automatically stand as valid.*

THEOREMS

Theorem 2. *Through a given point on a given line there can be one, and only one, plane perpendicular to the line.*

Part 1

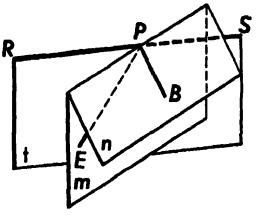


Given: Point P on line RS.
To prove: One plane can be passed through $P \perp RS$.

Suggestions for proof: Draw two lines such as PY and PX each $\perp RS$ at P. Show that two such lines can be drawn and that they determine a plane.

Proof left to you.

Part 2



Given: Point P on line RS.
To prove: Only one plane can be drawn $\perp RS$ at P.

Suggestions: *Assume* two planes m and n each $\perp RS$ at P. Let E be a point in m but not in n. Pass a plane through E and RS. Call the plane t. Plane t intersects m in line EP and n in some line such as PB. Therefore, EP and PB both lie in plane t and would then both be $\perp RS$ at P.

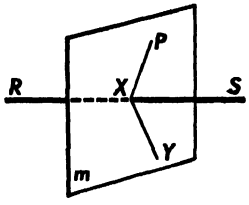
Is this possible? If this assumption that two planes m and n can be drawn $\perp RS$ at P leads us to a contradiction of an accepted fact, what about the validity of the assumption? Then what have we proved indirectly?

Proof left to you.

*For a discussion of a third type of indirect reasoning, one often employed in everyday reasoning situations, see *Plane Geometry: A Clear Thinking Approach*, by Schnell and Crawford, Chapter 9, McGraw-Hill Book Company, Inc., New York, 1953.

Theorem 3. *Through a given point outside a given line there can be one, and only one, plane perpendicular to the line.*

Part 1



Given: Line RS and external point P.
To prove: One plane can be passed through $P \perp RS$.

PROOF

Argument	*Defense*
1. From P drop a $\perp$ to RS and call point of intersection X.	1. Why possible?
2. At X erect another line $\perp RS$ and call it XY.	2. Why possible?
3. PX and XY determine a plane.	3. Why?
4. Plane $m \perp RS$.	4. If a line $\perp$ each of two . . ..
5. $\therefore$ one plane can be drawn $\perp$ a line through an external point.	

Part 2

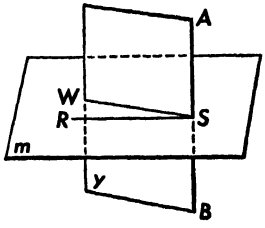
Given: Line RS and external point P.
To prove: *Only* one plane can be drawn $\perp$ a line through a given external point.

Suggestion: *Assume* that another plane can be drawn through $P \perp RS$. Suppose this plane intersects RS in a point called A. Draw PA. PX and PA would then both be $\perp RS$ by the definition of a line $\perp$ a plane. PX and PA would determine a plane. In that plane we would have how many $\perp$s from a point to a line? Since our assumption led us to a contradiction of a known fact, what about the validity of the assumption?

Proof left to you.

Experiment: With a wire, represent *a point* (P) *on a given line.* Show that an unlimited number of lines can be drawn perpendicular to the given line at P. Any two of the perpendiculars will determine a plane which will be $\perp$ to the given line at P. Why? Will each pair of perpendiculars determine a *new* plane? Where will all the possible perpendiculars lie? Why? Now study Th. 4.

Theorem 4. *If a plane is perpendicular to a line at a given point, then it contains all the lines perpendicular to the given line at that given point.**



Given: Plane $m \perp$ line AB at point S, and any line such as $RS \perp AB$ at S.
To prove: RS lies in plane m.

PROOF

Argument	*Defense*
1. Plane $m \perp AB$ at S, and $RS \perp AB$ at S.	1. Given.
2. RS and AB determine a plane y that intersects m in some such line as WS.	2. Why?
3. $WS \perp AB$.	3. If a line $\perp$ a plane, it $\perp$ any line in . . ..
4. Since RS is given $\perp AB$ at S, it will coincide with WS and lie in plane m.	4. At a point on a line in a plane, one and only one $\perp$ . . ..

5. Since RS represents any line $\perp AB$ at S, plane m must contain all lines $\perp AB$ at S.

Constructions in solid geometry: In plane geometry, to *devise a method of construction* means restriction to the use of compasses and straightedge. Review the method in plane geometry for constructing a perpendicular to a line at a point on the line. Then draw a line in the plane of your desk top. Select a point (P) *outside the plane.* Try to *construct* a line through P perpendicular to the given line. The **physical** impossibility of doing this with the tools of construction is obvious.

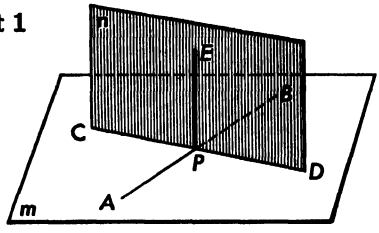
While we cannot construct, in the usual sense, in three-dimensional space, we can often use plane geometry constructions to help us. In solid geometry "construct" means to make or build a model in conformity with the conditions.

On cardboard construct a perpendicular to a line at a point on the line. Cut along the lines so that the edges of the cutout are perpendicular to each other. Use it like a carpenter's square to show how to construct a line perpendicular to a plane. Make cutouts of other constructions. Use them to make models of a line meeting a plane at an angle of 45°; perpendicular planes; parallel planes.

*This may also be stated: *All the perpendiculars that can be drawn to a straight line at a given point lie in a plane which is perpendicular to the line at the given point*

Theorem 5. *Through a given point in a plane there can be one, and only one, line perpendicular to the plane.*

Part 1

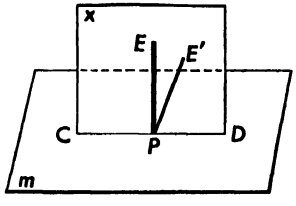


Given: Point P in plane m.
To prove: One line such as EP can be drawn through $P \perp$ plane m.

PROOF

Argument	*Defense*
1. In plane m draw AB through P.	1. Why possible?
2. Draw plane $n \perp AB$ at P.	2. Through a given point on . . ..
3. Plane n intersects plane m in some line such as CPD.	3. Why?
4. In plane n draw $EP \perp CD$.	4. Why possible?
5. $EP \perp AB$.	5. If a line $\perp$ a plane, it $\perp$ . . ..
6. $\therefore EP \perp$ plane m.	6. Why?

Part 2



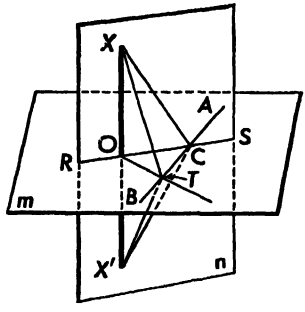
Given: Point P in plane m.
To prove: Only one line can be drawn $\perp$ plane m at point P.

PROOF

Argument	*Defense*
1. At point P in m, $EP \perp$ plane m.	1. Part 1 above.
2. Assume that PE' also $\perp$ plane m at P. Then PE and PE' would determine a plane such as x.	2. Why?
3. Plane x would intersect m in some line such as CPD.	3. Why?
4. Then PE and PE' would be $\perp$ CD at P.	4. If a line $\perp$ a plane, it $\perp$ every line . . ..
5. PE and PE' cannot both be $\perp$ CD at P.	5. Why?
6. $\therefore$ only one $\perp$ . . . since the assumption leads	

Theorem 6. *Through a given external point there can be one, and only one, line perpendicular to a given plane.*

Part 1

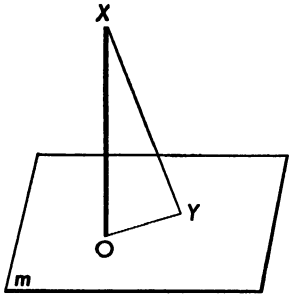


Given: Plane m with external point X.
To prove: One line can be drawn from $X \perp m$, or $XO \perp m$.

PROOF

Argument	*Defense*
1. In plane m draw any line such as AB.	1. Why possible?
2. Through X draw a plane $n \perp AB$ and intersecting m in line RS. (RS intersects AB at C.)	2. Through an external point one plane can be drawn . . ..
3. In plane n draw $XO \perp RS$.	3. Why possible?
4. In plane m draw any other line through O to intersect AB at T.	4. Why possible?
5. Extend XO its own length to X'.	5. Why possible?
6. Draw XC, XT, $X'C$, and $X'T$.	6. Why possible?
7. $\measuredangle XOC$ and $X'OC$ are rt. $\measuredangle$.	7. Why?
8. $\triangle XOC \cong \triangle X'OC$.	8. Why?
9. $\therefore XC = X'C$.	9. Why?
10. $\angle XCT$ and $\angle X'CT$ are rt. $\measuredangle$.	10. If a line $\perp$ a plane, it $\perp$ every line that passes . . ..
11. $\triangle XCT$ and $X'CT$ are rt. $\triangle$.	11. Why?
12. $CT = CT$.	12. Why?
13. $\triangle XCT \cong \triangle X'CT$.	13. Why?
14. $\therefore XT = X'T$.	14. Why?
15. $\triangle XOT \cong \triangle X'OT$.	15. Why?
16. $\angle XOT$ is a rt. $\angle$.	16. Why?
17. $XO \perp OT$.	17. Why?
18. $XO \perp$ plane m.	18. If a line $\perp$ each of two intersecting . . ..

Part 2



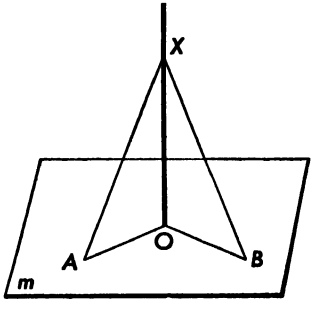
Given: Plane m with external point X.
To prove: Only one line can be drawn from $X \perp$ plane m.

PROOF

Argument	*Defense*
1. $XO \perp m$.	1. By Part 1.
2. Let XY be any other line from X to the plane, and draw OY.	2. Why possible?
3. $XO \perp OY$.	3. A line $\perp$ a plane is $\perp$ every line in the plane which . . ..
4. OX and OY would determine a plane.	4. Why?
5. XY is not $\perp OY$.	5. Why?
6. Since OY lies in m, XY is not $\perp$ plane m.	6. Why?
7. $\therefore XO$ is the *only* line from $X \perp$ plane m.	7. Why?

The proofs of Ths. 7 through 10 are relatively easy and will be left to you.

Theorem 7. *If oblique lines are drawn from a point in a perpendicular to a plane so that they meet the plane at equal distances from the foot of the perpendicular, they are equal.*



Given: $XO \perp$ plane m. XA and XB oblique lines drawn from X to the plane making $AO = OB$.
To prove: $XA = XB$.

Proof left to you.

Theorem 8. *If two oblique lines are drawn from a point in a perpendicular to a plane so that they meet the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.*

Hint: Draw a line from the point to meet the plane at the same distance as does the oblique which is nearer the foot of the perpendicular.

Proof left to you.

Corollary 8a. *The perpendicular is the shortest line from a point to a plane.*

Theorem 9. *If two equal oblique lines are drawn from a point in a perpendicular to a plane, they will meet the plane at equal distances from the foot of the perpendicular.*

Proof left to you.

Theorem 10. *If two unequal oblique lines are drawn from a point in a perpendicular to the plane, they will meet the plane at unequal distances from the foot of the perpendicular and the longer line will be the more remote.**

Proof left to you.

CHAPTER REVIEW

As you learned from your study of plane geometry each theorem proved can be used as a defense in later proofs. The statements of theorems, corollaries, and definitions are your "stock in trade." The more of them you understand and can quote the more power you will have in attacking new theorems.

1. Complete the following statements:

(a) Through a given external point there can be one line and . . ..
(b) If a line is perpendicular to each of two intersecting . . ..
(c) Through a given point outside a given line there can . . ..
(d) If oblique lines are drawn from a point in a perpendicular to a plane so that they . . ..
(e) Quote three other theorems about oblique lines which you had in this chapter.
(f) All the perpendiculars that can be drawn to a straight line at a given . . ..
(g) The perpendicular is the . . ..
(h) At a given point in a plane there can be . . ..
(i) Through a given point on a given line there can be . . ..
(j) If a plane is perpendicular to a given line at a given point, it contains all . . ..

*How are Ths. 9 and 10 related to Ths. 7 and 8?

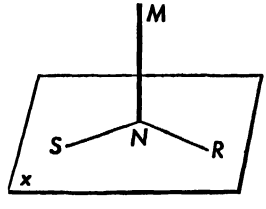
2. Quote the main assumption, definition, or theorem needed to defend your answers to the following questions.

(a) Open your geometry book and stand it upright on your desk. Is the back edge of the book perpendicular to the desk?
(b) If the spokes of a wheel are each perpendicular to the axle, do all the spokes lie in the same plane?
(c) Show why a carpenter's square may be used to test whether or not a post is perpendicular to the ground.
(d) How does a workman laying a cement walk make sure that the surface is a plane?
(e) Can a line be perpendicular to each of two lines in a plane?
(f) List several examples in your room or school of lines perpendicular to a plane and of perpendicular planes.

EXERCISES

1. Given: A line through the center of a circle and perpendicular to the plane of the circle.
To prove: Any point in this perpendicular is equidistant from any two points on the circle.

2. Given: $MN \perp$ plane x and NR and NS lie in x.

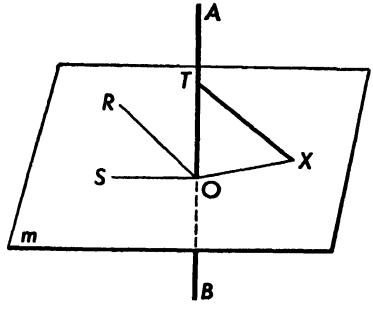


(a) What is true of $\angle RNM$? Of $\angle MNS$? Why?
(b) If $SN = NR$, what would be true of MR and MS?
(c) If MN is 10 inches long and NR is 6 inches, how long is MR?
(d) If MN is 10 inches and NS is 4 inches, how long is MS?
(e) If MN is 10 inches and $\angle RMN$ is 30°, what would be the length of MR?
(f) If MN is 10 inches and NS is 7 inches, what would be the area of $\triangle SNM$?
(g) If NR is twice as long as NS, how would MR and MS compare?
(h) If MR were 12 inches and MS 16 inches, how would NR and NS compare?
(i) If $\angle SNR$ is 60° and SN and NR are each 6 inches, how long would SR be? What would be the area of $\triangle SNR$?
(j) What would you know about any other line drawn from M to the plane as compared with MN? Why?

3. If equal oblique lines are drawn from external point M to plane x, what is the locus (in plane x) of their points of intersection with the plane?

4. Three guy wires of equal length are used to brace a 40-foot steel tower. If the wires are 50 feet long and are attached to the top of the tower, how far from the foot of the tower will each wire touch the ground?

5. **Given:** Plane $m \perp AB$ at O.
RO and SO are each $\perp AB$ at O.
OX lies in plane m and T lies on AB.
Which of the following statements are valid? Be able to defend your answers.



(a) RO and SO lie in plane m.
(b) XT lies in plane m.
(c) $AB \perp OX$.
(d) $XT < XO$.
(e) $XT \perp AB$.
(f) $\angle BOR = \angle BOS$.
(g) R is equidistant from A and B.
(h) If $OS = OR$, then $BS = BR$.
(i) m is the only plane that can be drawn perpendicular to AB at point O.
(j) If $BX = BR$, then $XO = RO$.
(k) At O an unlimited number of lines can be drawn perpendicular to m.
(l) At point O an unlimited number of lines can be drawn perpendicular to AB.
(m) $XO = RO$.
(n) Any line drawn perpendicular to AB at O will lie in plane m.

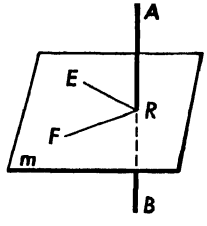
CHAPTER TEST*

1. If equal oblique lines are drawn from a given external point X to plane m, prove that they make equal angles with lines drawn from their points of intersection with plane m to the foot of the perpendicular dropped from X.
2. Given line segment XY, 10 inches long, with points A, B, and C each 7 inches from both X and Y. Prove that A, B, and C all lie in a plane which is perpendicular to XY at its mid-point.
3. Study the following statements from plane geometry and decide which ones can be extended to give valid relationships in solid geometry. Substitute the word *line* for *point* and the word *plane* for *line* or *straight line*. Give the hypothesis and conclusion of each valid statement which results.

***To the Teacher.** At the discretion of the teacher, chapter tests may be used in part or in toto for purposes of testing, or they may be used merely to provide additional exercise material.

(a) Through a given point an unlimited number of straight lines can be drawn.
(b) Through two given points it is possible to draw one and only one straight line.
(c) Two straight lines can intersect in only one point.
(d) At a given point on a line one and only one perpendicular can be drawn to a given line.
(e) From an external point one and only one perpendicular can be drawn to a given line.
(f) A perpendicular is the shortest distance from a point to a line.

4. Given: $AB \perp$ plane m at R. RE and RF both lie in m.



(a) If $RE = RF$, how would lines BE and BF compare? Why?
(b) If $RE = 8$ inches and $RB = 6$ inches, how long is BE?
(c) Where would a line RG lie if it were $\perp AB$? Why?
(d) What would be the relationship of a line RW, not in plane m, to line AB? Why?

5. Explain how you could draw a circle with a radius of 8 inches by using a pair of compasses which are kept set at 10 inches.

6. If two points lie on opposite sides of a plane and are equidistant from it, prove that the plane bisects the line segment joining the two points.

Keep the Iron Hot: RULE OF PYTHAGORAS—RATIO—EQUATIONS

Set your goal at 100 *percent accuracy.*

1. Helen was making a cake. The recipe said to use a rectangular pan 8 inches x 11 inches. However, the only cake pan she had was a round one with a 9-inch diameter. Would she be able to use the circular pan and have the cake the thickness the recipe suggested? Why or why not?

2. (a) Represent the sum of the squares of a and b.
(b) Represent the square of the sum of a and b.

3. A man travels y miles in h hours. Represent his average rate of travel.

4. What will represent the perimeter of an equilateral triangle a side of which is $(2x + 5)$ units? Of a rectangle whose width is 3 units less than its length?

5. Represent the complement of an angle of $m°$; of $(x + 16)°$; of $(3n - 1)°$.

6. How long is the side of a square whose diagonal is 12 units?

7. A diagonal drawn in a certain rectangle divides one of the angles in the ratio of 2 to 1. (a) Find the size of all the angles formed in the figure. (b) How does the diagonal compare in length with one of the sides? (c) If the length of the rectangle is 15 inches, find its width and the length of the diagonal.

8. Complete: (a) A line which bisects one side of a triangle and is parallel to a second side will . . ..

(b) If a line is parallel to one side of a triangle and intersects the other two sides, it divides those sides ?.

(c) A series of parallel lines cut off ? . ? on two transversals.

9. If a line parallel to the base of a triangle divides one side into segments of 2 inches and 5 inches, what will be true of the segments cut off on the third side? If the shorter of those segments is 3 inches, how long is the longer segment?

10. A transversal is cut into three segments by a series of parallel lines. The segments are 3, 4, and 5 inches respectively, and the distance between the first pair of parallels is 1.5 inches. What are the distances between the other pairs?

Solve the following equations and check:

11. $5x - 3 = 2x - 6$

12. $37 - 5m = 29 - 4m$

13. $7(a - 3) = x(3 - a)$

14. $\frac{2x}{3} + \frac{x}{4} = 22$

15. $(2a + 5)(a - 6) = 6a - a(3 - 2a)$

16. $3(y + 9) = 3y - 9(3y + 4)$

17. $(x + 2)^2 - (x - 3)^2 = 30$

18. $\frac{m}{5} + \frac{m + 5}{2} = 6$

19. The sum of three numbers is 180. Two of them are equal and the third is 6 more than the sum of the others. Find the numbers.

20. A purse contains 20 coins. There are nickels and dimes only. How many of each are there if the total value is \$1.60?

21. Find two numbers in the ratio of 2 to 3 if their sum is 120.

22. Two supplementary angles are in the ratio of 5 to 1. How many degrees are there in each?

23. The angles of a triangle are in the ratio of 1 to 3 to 5. How large is each angle?

24. Find four consecutive numbers whose sum is 130.

25. How would \$82.60 be divided among three people so that the first one would get twice the amount of the second and the second three times the amount of the third?

CHAPTER 3 *Parallel Lines and Planes*

As you already know, a *line* and a *plane*, or *two planes*, are **parallel** if they do not meet, no matter how far extended. There are innumerable examples of parallel planes all around you. Observe the planes determined by the top of your desk, the seat of your chair, and the floor; the covers of this book when it is closed; shelves in a bookcase; a pile of papers on your teacher's desk; and the opposite walls in a rectangular room. Point out other examples. Show how the picture on page 39 illustrates the use of parallel planes in architecture.

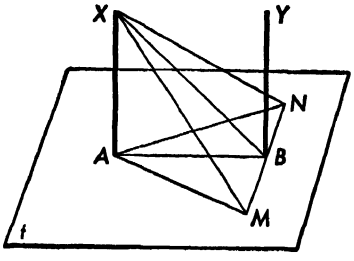
It was previously pointed out that we have never really seen a straight line. Even though the legs of a table are solids, imagine them to be straight lines. How do they meet the plane determined by the top of the table? The plane of the floor? How do they relate to each other? How does the flagpole in the picture mentioned above *intersect* the plane of the roof? Cite illustrations of two or more lines that are perpendicular to the same plane. Assume that the playing field and end zones of a football field represent a plane. How does each upright of the goal posts meet this plane? Since the uprights are perpendicular to the same plane, what is their relationship to each other?

Quote as many postulates or theorems as you can from plane geometry that deal with parallel lines. Substitute the word *line* for *point* and the word *plane* for *line*. Decide which of the resulting statements seem to be valid in three-dimensional geometry.

Read the statement of Th. 11, study the "Given" and "To prove" and the *first four* steps of the "Argument." Without looking at steps 5 through 13, see if you can develop a proof independently. Compare your proof with the one in the book. The procedure suggested here is a good one to follow frequently. Try it and see how much you will grow in ability to work independently.

THEOREMS

Theorem 11. *Two lines perpendicular to the same plane are parallel.*

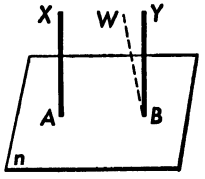


Given: XA and YB each $\perp$ plane t at A and B respectively.
To prove: $XA \parallel YB$.

PROOF

Argument	*Defense*
1. XA and YB are $\perp$ plane t.	1. Given.
2. Connect A and B, and connect X and B.	2. Why possible?
3. Draw MN in the plane $t \perp AB$, making $MB = BN$.	3. Why possible?
4. Draw XM, XN, AM, and AN.	4. Same as 2.
5. $AM = AN$.	5. The $\perp$ bisector of a line segment is the locus of points equidistant from the extremities of the line.
6. $XM = XN$.	6. If two oblique lines are drawn from a point in a $\perp$ a plane so that they meet the planes at equal . . ..
7. XM, XB, and XN lie in a plane determined by X and MN.	7. Why?
8. $\therefore XB \perp MN$.	8. Two points equidistant from the extremities of a line . . ..
9. $YB \perp MN$.	9. A line $\perp$ a plane is $\perp$ every . . ..
10. AB, XB, and YB lie in same plane.	10. All the $\perp$s that can be drawn to a straight line at a given point lie in a plane which is $\perp$ the line at the given point.
11. XA lies in this plane.	11. A line joining two points in a plane . . ..
12. $XA \perp AB$ and $YB \perp AB$.	12. Why?
13. $\therefore XA \parallel YB$.	13. Why?

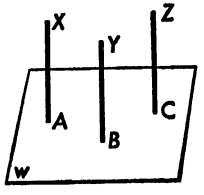
Corollary 11a. *If one of two parallel lines is perpendicular to a plane, the other one is also perpendicular to the plane.*



Given: $XA \parallel YB$, and $XA \perp$ plane n.
To prove: $YB \perp$ plane n.

Suggestions: Draw $BW \perp$ plane n. Prove that it coincides with BY and lies in the plane determined by XA and BY.

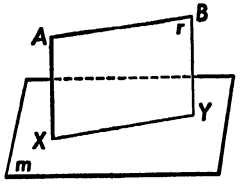
Corollary 11b. *If two lines are parallel to a third line, they are parallel to each other.*



Given: $XA \parallel YB$ and $ZC \parallel YB$.
To prove: $XA \parallel ZC$.

Suggestions: Draw a plane such as $w \perp YB$ at B. Then XA and ZC will both be $\perp$ plane w and therefore parallel to each other.

Theorem 12. *If two lines are parallel, every plane containing one of the lines, and only one, is parallel to the other line.*

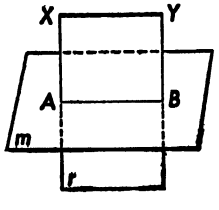


Given: $AB \parallel XY$. XY lies in plane m. AB lies outside m.
To prove: $m \parallel AB$.

Suggestions: AB and XY determine a plane, r, and r and m intersect in line XY. If AB, in plane r, were to meet plane m it would be in line XY. Why? Why is this impossible? Then $AB \parallel m$.

Proof left to you.

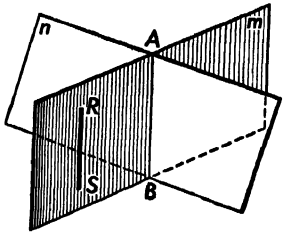
Corollary 12a. *If a line is parallel to a plane, it is parallel to the intersection of that plane with any plane containing the line.*



Given: $XY \parallel$ plane m, and r is *any* plane containing XY and intersecting m in a line such as AB.
To prove: $XY \parallel AB$.

Suggestions: XY and *any* point in m such as A determine a plane such as r. What line contains all the points common to the two planes? Does XY have any points in common with m? Then what is the relationship between XY and AB?

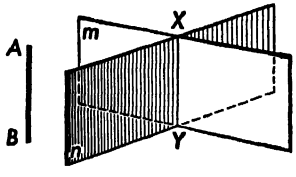
Corollary 12b. *If a line is parallel to the line of intersection of two planes and lies in one of them, it is parallel to the other.*



Given: Plane m intersects plane n in line AB, and RS lies in plane m and $\parallel AB$.
To prove: $RS \parallel$ plane n.

Suggestions: AB contains all the points common to the two planes so if RS meets plane n it will coincide with line AB. Is that possible?

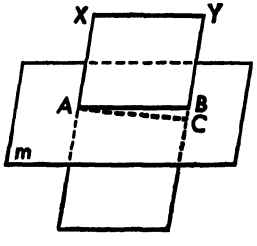
Corollary 12c. *If a line is parallel to each of two intersecting planes, it is parallel to their intersection.*



Given: $AB \parallel$ plane m and also $\parallel$ plane n. Planes n and m intersect in line XY.
To prove: $AB \parallel XY$.

Suggestions: Let AB and X determine a plane. This plane must intersect m. The line of intersection would be $\parallel AB$. Why? Since AB is parallel to a line through X, which lies in both m and n, what conclusion is necessary?

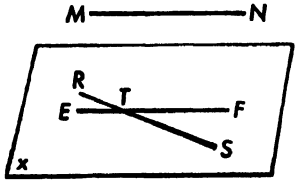
Corollary 12d. *If a line and a plane are parallel, a line through a point in the given plane parallel to the given line lies in the plane.*



Given: $XY \parallel$ plane m and A a point in the plane m and $AB \parallel XY$.
To prove: AB lies in plane m.

Suggestions: Draw the plane determined by XY and A and call the line of intersection of the two planes AC. What would be true of XY and AC? Why? Of AC and AB? Why?

Corollary 12e. *Through either of two skew* lines one plane, and only one, can be passed parallel to the other.*

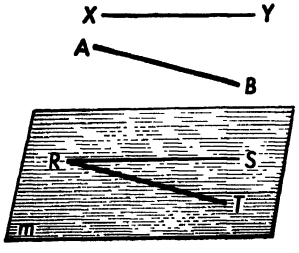


Given: Two skew lines, MN and RS.
To prove: One plane, and only one, can be passed through $RS \parallel MN$.

Suggestions: Through T on RS draw $EF \parallel MN$. Draw the plane x determined by EF and RS. Why possible? Then how is MN related to x?

If another plane could be passed through $RS \parallel MN$, then RS would be the line of intersection of the two planes and MN would be $\parallel RS$. Why? Is this possible?

Corollary 12f. *Through any given point one plane, and only one, can be passed parallel to each of two skew lines.*



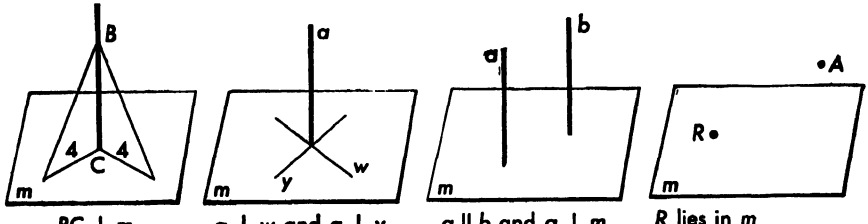
Given: Skew lines XY and AB and point R.
To prove: One, and only one, plane can be passed $\parallel$ to XY and AB through R.

Suggestions: Through R draw $RS \parallel XY$ and $RT \parallel AB$. Why is the plane determined by RS and $RT \parallel XY$ and to AB?

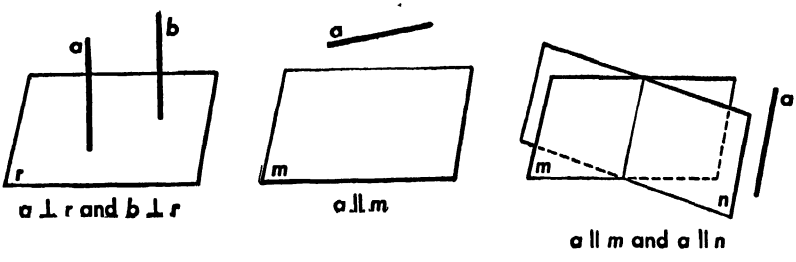
Suppose another plane could be passed through R and $\parallel$ to both XY and AB. Why would it have to contain RS and RT? Is this possible?

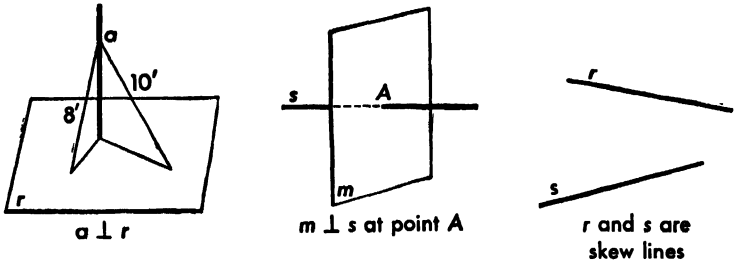
EXERCISES: QUOTING THEOREMS SUGGESTED BY DRAWINGS

Quote the theorems or corollaries suggested by the drawings given below.

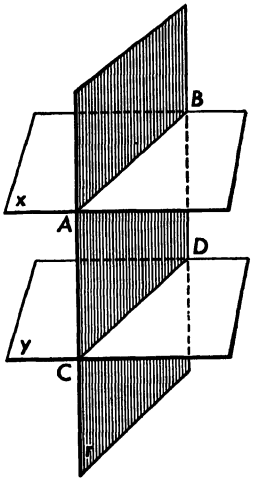


*Review Exs. 6, 7, and 8 on page 5 and see the definition of skew lines on page 6





Theorem 13. *If two parallel planes are cut by a third plane, the lines of intersection are parallel.*



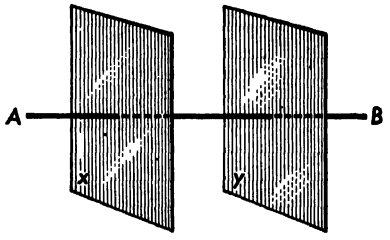
Given: Plane x || plane y, and plane r intersects x and y in lines AB and CD respectively.
To prove: $AB \parallel CD$.

Suggestions: Do AB and CD lie wholly within their respective planes? What is the definition of parallel planes?

Proof left to you.

Corollary 13a. *Parallel line segments included between parallel planes are equal.*

Theorem 14. *Two planes perpendicular to the same line are parallel.*

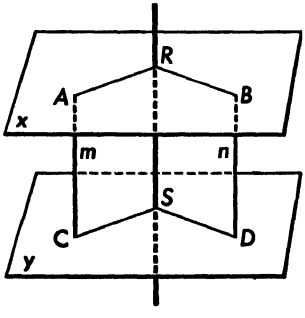


Given: Planes x and y each $\perp AB$.
To prove: $x \parallel y$.

Suggestion: Assume that x and y are not $\parallel$. They would then intersect in a line. Through *any* point on the line of intersection how many planes could be passed $\perp AB$?

Proof left to you.

Theorem 15. *A line perpendicular to one of two parallel planes is perpendicular to the other one also.*



Given: Plane $x \parallel$ plane y, and $RS \perp y$.
To prove: $RS \perp x$.

PROOF

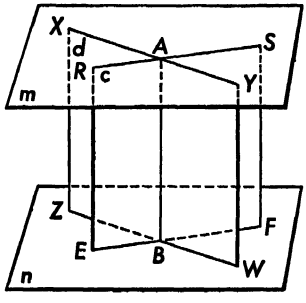
Argument	*Defense*
1. Through RS pass any two planes m and n intersecting x and y in lines AR, CS, RB, and SD.	1. Why possible?
2. $AR \parallel CS$ and $RB \parallel SD$.	2. Why?
3. $RS \perp CS$ and $RS \perp SD$.	3. Why?
4. $RS \perp AR$ and $RS \perp RB$.	4. Why?
5. $\therefore RS \perp$ plane x.	5. If a line $\perp$ each of two . . ..

Corollary 15a. *Through a given point outside a plane there can be one plane, and only one, parallel to the given plane.*

Suggestion: From external point X drop a $\perp$ to the given plane. Then at X pass a plane $\perp$ the line just drawn. How many planes can be drawn at this point $\perp$ the line?

Corollary 15b. *Two parallel planes are everywhere equidistant.*

Theorem 16. *If two intersecting lines are each parallel to a plane, the plane of those lines is parallel to the given plane.*

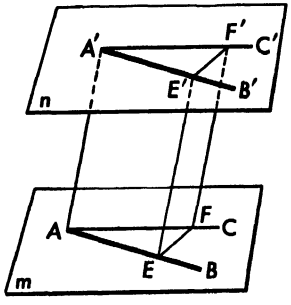


Given: XY and RS intersecting at A, XY || plane n, and RS || plane n.
To prove: Plane m || plane n.

PROOF

Argument	*Defense*
1. XY and RS intersect at A, and each line || plane n.	1. Given.
2. Draw $AB \perp n$.	2. Why possible?
3. AB and RS determine plane c which will intersect n in some line such as EF. AB and XY determine plane d which will intersect n in some line such as ZW.	3. Why?
4. RS || EF and XY || ZW.	4. If a line || a plane, it || the intersection of that plane with . . ..
5. $AB \perp EF$ and $AB \perp ZW$.	5. Why?
6. $AB \perp XY$ and $AB \perp RS$.	6. Why?
7. $AB \perp m$.	7. Why?
8. $\therefore m$ || n.	8. Why?

Theorem 17. *If two angles not in the same plane have their sides respectively parallel and lying on the same side of the line joining their vertices, then they are equal and their planes are parallel.*

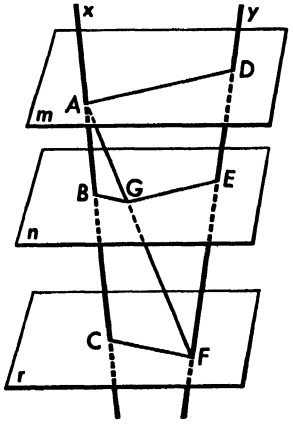


Given: $\angle A$ and $\angle A'$ with $AB \parallel A'B'$ and $AC \parallel A'C'$.
To prove: $\angle A = \angle A'$ and plane $m \parallel$ plane n.

Suggestions: Make $AE = A'E'$ and $AF = A'F'$. Connect A and A', E and E', and F and F'. Draw EF and $E'F'$. Show that three parallelograms result and that $\angle A = \angle A'$. Why is $m \parallel A'B'$ and also $\parallel A'C'$? Why is m then $\parallel n$?

Proof left to you.

Theorem 18. *If two lines are cut by three parallel planes, the corresponding segments are proportional.*



Given: Lines x and y cut by the parallel planes m, n, and r.
To prove: $\dfrac{AB}{BC} = \dfrac{DE}{EF}$.

Suggestions: Draw AF. AC and AF determine a plane intersecting n in BG and r in CF. AF and DF determine a plane intersecting n in GE and m in AD. Why is $BG \parallel CF$ and $GE \parallel AD$? What proportion from $\triangle ACF$ can you defend? From $\triangle AFD$?

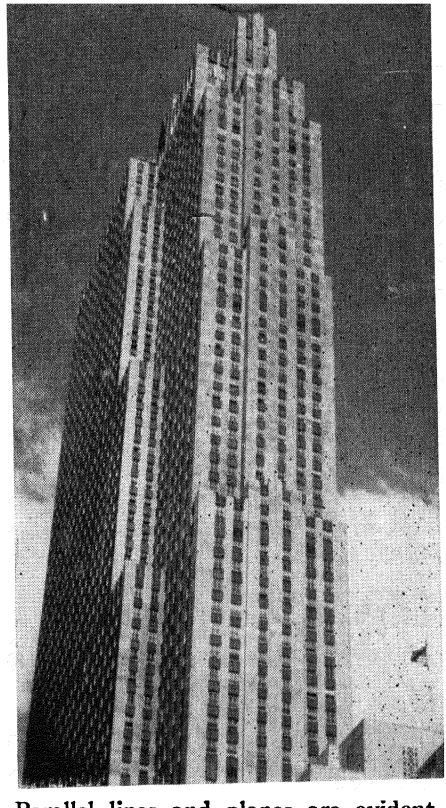
Proof left to you.

Corollary 18a. *If three parallel planes cut off equal segments on one transversal, they cut off equal segments on any transversal.*

CHAPTER REVIEW

1. Quote as many theorems as you can about parallel planes.
2. Define: skew lines, parallel lines, parallel planes, a line parallel to a plane, a line perpendicular to a plane.
3. Complete the following to make each statement valid.

(a) If one of two parallel lines is ? to a plane, the other line is . . ..
(b) If two lines are ? to a third line, they are ? to each other.
(c) If two lines are parallel, every plane containing one of the lines, and only one, is . . ..
(d) If two parallel planes are cut by a ? ?, the lines of ? are parallel.
(e) Two lines which are ? to the same plane are ?.
(f) Two planes ? to the same line are ?.
(g) If a line is parallel to a plane, it is ? to the intersection of that plane with any plane . . ..
(h) If a line is parallel to two intersecting planes, it is ? to their ?.
(i) A line ? to one of two parallel ? is perpendicular to the other plane also.
(j) If two intersecting lines are each parallel to a plane, the ? of these lines is ? to the given plane.
(k) If two lines are cut by three parallel planes, the . . ..
(l) If a line is parallel to the line of intersection of two ?, and lies in one of them, it is ? to the other.
(m) If two angles not in the same plane have their sides respectively . . ..



Parallel lines and planes are evident in this view of one of the buildings of the Rockefeller Center group in New York City.

(n) If a line and a plane are parallel, a line through a point in the given plane and parallel to the given line, ...
(o) Through a given external ? there can be ? plane, and ? one, parallel to a given ?.

In Exs. 4 through 7, complete the statements to make each one valid in terms of the stated conditions.

4. To prove that two planes are parallel, show that:

(a) They will never ? no matter . . ..
(b) They are both ? to the same ?.
(c) One of them is the plane of two ? ?, each one of which is ? to the given plane.
(d) They are the planes of two ? which have their sides respectively ? and lying on the ? ? of the line joining their ?.

5. To prove that a line is parallel to a plane, show that:

(a) It is one of two ? lines, and that a plane which is passed through the other line can contain only ? of the two lines.
(b) It is parallel to the line of ? of two planes and lies in ? of the planes.
(c) It can never ? the plane.
(d) It coincides with a line which is ? to the ?.

6. To prove that two lines are parallel, show that:

(a) They can never ? and they lie in the ? ?.
(b) They are the same ? ?.
(c) They are both ? to the same plane.
(d) They are both ? to a third line.
(e) They are the lines of ? of two ? planes cut by a third ?.
(f) One of the lines is the line of ? of two planes that are both parallel to the other ?.

7. To prove that a line is perpendicular to a plane, show that:

(a) It is ? to each of two ? lines at their point of ?.
(b) It is perpendicular to ? line in the plane which passes ? ? ?.
(c) It is the ? line that can be drawn from a ? to the plane.

(d) It is one of two parallel ? and the other line is ⊥ to the plane.
(e) It is ⊥ to one of ? parallel planes.
(f) It coincides with ? ? which is perpendicular to the ?.

ANALYZING GEOMETRIC PROBLEMS

When you try to prove a theorem or an exercise in geometry, you undoubtedly employ a *method of approach* in which you try to do one of two things.

1. You recall as many theorems as possible which mention the same kind of angles, lines, or planes as those which the new problem involves. Then you try to use the facts proved in these theorems to develop a proof which will lead to the conclusion required in the new problem.

2. You recall as many theorems as possible in each one of which the conclusion matches the conclusion of the new problem. Then you try to make the conditions given in the new problem fit the hypothesis given in one of the previously proved theorems.

It will be helpful if you will take time now to analyze the assumptions, theorems, and corollaries studied thus far and develop a **Tool Box** to use whenever you are trying to decide on a *method of approach* to solve new problems. You may classify them with reference to their hypotheses or their conclusions. Exercise 4 on page 40 may serve to help you get started. Are the parts of Ex. 4 based on the hypotheses or conclusions of the statements?

After completing your **Tool Box** through Chapter 3, keep it up to date by classifying theorems and corollaries for each succeeding chapter. All exercises in italic type are **subsidiary** (minor) **theorems** and, when proved, can be used later as defenses. (See Exs. 5, 7, 15, and 16 on pages 42 and 43.) They may be included in your **Tool Box.**

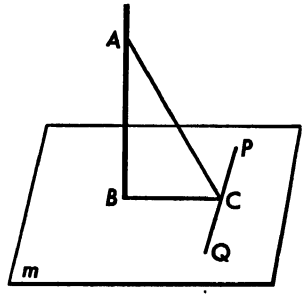
You may have developed some sort of "tool box" when you studied plane geometry. If so, and if you have saved it, such a tool box may be put to good use frequently in solid geometry. Check rapidly to see how many statements from plane geometry have been used as defenses for the first eighteen theorems in solid geometry.

EXERCISES

1. **Given:** $AB \perp$ plane m, and $AC \perp PQ$.
   Prove: $BC \perp PQ$.

2. **Given:** $AB \perp$ plane m, and $BC \perp PQ$.
   Prove: $AC \perp PQ$.

3. **Given:** $AB \perp$ plane m and line BC lies in m.
   Prove: All lines $\perp m$ at any points in BC lie in a plane.



Exs. 1–3.

4. If two parallel planes intersect each of two other parallel planes, prove that the four lines of intersection are parallel.

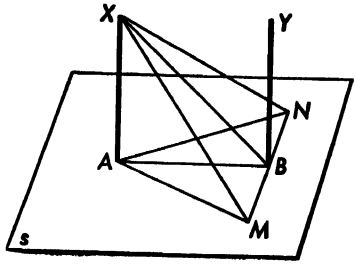
5. *If two planes are perpendicular to the same line, any plane containing the line will cut the two planes in parallel lines.*

6. Prove: If a plane is passed through one diagonal of a parallelogram, then two perpendiculars drawn to this plane from the ends of the *other* diagonal are equal.

7. *If two planes are parallel to a third plane, they are parallel to each other.*

8. If two parallel lines are cut by an unlimited number of transversals, prove that all such transversals lie in the same plane as the parallel lines.

9. **Given:** XA and YB each $\perp$ plane s, and MN lies in s.
   Is:
   (a) $XA \parallel YB$?
   (b) A plane containing XA, and not YB, $\parallel YB$?
   (c) $XB \perp MN$?
   (d) $MB = BN$?
   (e) $AB \perp XA$?
   (f) $XM = XN$ if $MB = BN$?
   (g) XN in the same plane as XB and XM?
   (h) $AB \perp MN$?



Ex. 9

10. Lines x and y are transversals intersecting planes m, n, and r in points A, B, C and D, E, and F respectively.

In each of these exercises consider, as given, *only* the facts stated in that particular exercise. Do not depend upon the appearance of the figure. Give reasons for your answers.

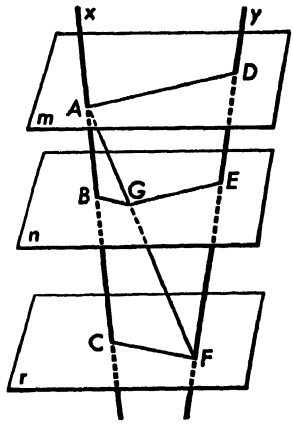
(a) If planes n, m, and r are parallel, and if $AB = BC$, then:
Does $\frac{AB}{BC} = \frac{DE}{EF}$?
Is $BG \parallel CF$?
Is $AD \parallel CF$?
If AG is 8 units, GF is 12 units, and DF is 30 units, find the length of DE and EF.

(b) If $GE \parallel AD$ and plane n contains GE, but not AD, then . . ..

(c) If BG and GE are each parallel to r, then plane n is . . ..

(d) If $n \parallel r$ and $BC \perp r$, then . . ..

(e) If $BG \parallel r$ and plane ACF contains BG, then . . ..



Ex. 10

11. Where would all the points be located which are 4 feet from the floor of your classroom?

12. Where would all the points be located, in space, which are equidistant from two parallel planes?

13. What is the possible location of a point 3 feet from the back wall of your room and 3 feet above the floor?

14. If a plane and a line are both perpendicular to the same line, prove they are parallel to each other.

15. *Two planes perpendicular to two intersecting lines will intersect.**

Suggestion: Develop an indirect proof.

16. *If from the foot of a perpendicular to a plane a line is drawn at right angles to any line in the plane, the line drawn from the point of their intersection to any point in the perpendicular is perpendicular to the line of the plane.*

*Exercises 5, 7, 15, and 16 are **subsidiary** (minor) **theorems.** Throughout this book, all exercises printed in *italic* type are subsidiary theorems and, **when proved** can be added to your **Tool Box** and used as defenses.

CHAPTER TEST

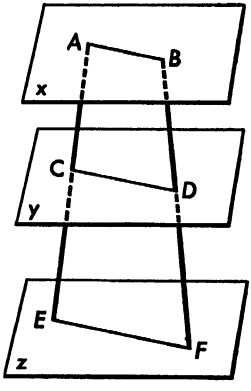
1. Quote two theorems which conclude that two planes are parallel.

2. Quote two theorems which conclude that two lines are parallel.

3. Quote two theorems or corollaries which conclude that a line is perpendicular to a plane.

4. If two straight lines are perpendicular to the same line, are they parallel? Explain. What is true of two lines parallel to the same line? Of two lines perpendicular to the same plane?

5. **Given:** $x \parallel y \parallel z$, and AE and BF intersect the three planes in points A, B, C, D, E, and F.



   (a) If $AE \perp z$, then . . . .
   (b) If AE and BF intersect when extended, CD is $\parallel$ to EF because . . . .
   (c) If BF and AE are both $\perp z$, then . . . .
   (d) If $x \parallel y$, then through point A there can be . . . .
   (e) If $BD = DF$, then . . . .
   (f) If $AE \parallel BF$, and $BF \perp z$, then . . . .
   (g) If AC is one-half the length of CE, how do BD and DF compare? Why?
   (h) If DF is four times as long as BD and $AE = 30$ units, find the lengths of AC and CE.
   (i) Does $\frac{AE}{AC} = \frac{BF}{BD}$? Explain.
   (j) If EF is twice the length of CD, is CD twice the length of AB? Explain.
   (k) If $AE \parallel BF$, how do CE and DF compare in length? Why?

6. **Given:** Line $w \parallel$ line y.
   (a) If line $z \parallel$ line y, then . . . .
   (b) A plane containing w and not y would be . . . .

7. From a line parallel to a plane, parallel segments are drawn to the plane. Prove that these segments are equal.

8. If a plane is passed through the mid-points of two sides of a triangle, it is parallel to the third side unless it also passes through the third side. Prove.

Keep the Iron Hot: Everyday Reasoning—
NECESSARY CONCLUSIONS

Very often certain statements *necessarily imply* others. For example, consider the two statements:

All automobiles have wheels.
This vehicle is an automobile.

Certainly a third statement is implied, namely:

Then this vehicle has wheels.

If a person accepts the first two statements, would he be likely to challenge the third one?

What conclusion do the following two statements imply?

All football players must learn signals.
Tom is a football player.

If the second statement read, "*John* has learned the signals for the football plays," what conclusion would you draw? Why?

Rate the conclusion in each of the following as *good* if it follows as a *necessary result* of the first two statements, or as *poor* if it does not *necessarily follow.* The question of whether or not the first two statements are truth or nonsense is unimportant. For this exercise we *accept* them without argument and then decide what *must* follow.

1. Students must be passing in all their subjects to have a part in the Junior play.
   Mary has a part in the Junior play.
   Therefore Mary is passing in all her subjects.
2. All camp counselors must be good swimmers.
   Jack is a camp counselor.
   Therefore Jack is a good swimmer.
3. All camp counselors must be good swimmers.
   Terry is a good swimmer.
   Therefore Terry is a camp counselor.
4. Only people with a teacher's certificate can teach school.
   Mr. Stone is a history teacher.
   Therefore Mr. Stone has a teacher's certificate.
5. Seniors take a course in American History.
   Lucille is taking American History.
   Lucille is a senior.

6. Only Freshmen take English at 9 o'clock.
Harry is a Freshman.
Harry takes English at 9 o'clock.

7. All school buildings are made of brick.
This building is made of brick.
This building is a school.

8. School buildings are made of brick.
This building is a school.
This building is made of brick.

In each of the following exercises, if there is a *necessary conclusion*, state it.

9. All nurses study chemistry.
Helen is studying chemistry.

10. Horses have four legs.
This animal is a horse.

11. Good watches are expensive.
This watch is a good one.

12. Only Juniors study solid geometry.
George is studying solid geometry.

13. Each angle of an equilateral triangle contains 60°.
This angle contains 60°.

14. Cars made by the FRX company are good cars.
Mr. Smith's car is an FRX car.

15. Teen age drivers have many accidents with cars.
Jerry is 17 years old.

16. People with curly hair have bad tempers.
Paul has curly hair.

17. All girls with gray eyes are good in music.
Ruth has gray eyes.

18. All boys 6 feet or more in height are good basketball players.
Henry is 6 feet 1 inch tall.

19. All ducks have web feet.
This fowl does not have web feet.

20. All equilateral triangles are similar.
These two triangles are not equilateral.

CHAPTER 4 *Dihedral and Polyhedral Angles*

SECTION 1. DIHEDRAL ANGLES

Just as we have extended our ideas about points and lines from plane geometry to three-dimensional space, we can also extend our concepts of angle. In plane geometry, when two straight lines meet at a point, we say that an angle is formed. Extending this idea to three-dimensional space, *when two planes meet,* **a dihedral angle** is formed. In Fig. 1, planes AC and BD meet in line CB. The *two planes* which form the dihedral angle are called the **faces** *of the dihedral angle* and *the line of intersection* (CB) is called the **edge** *of the dihedral angle.*

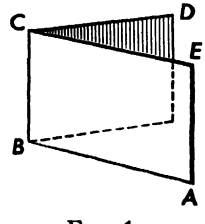


FIG. 1.

Open the door of a room and point out the dihedral angle formed by the door and a wall. What are the *faces* and the *edge* of this dihedral angle? Show how two facing pages of this book illustrate a dihedral angle. To what "parts" of a plane angle do the faces of a dihedral angle compare? The edge?

In plane geometry, if a straight line, lying in a plane, is permitted to rotate about a fixed point on the line, an angle is generated. This same general procedure can be applied to three-dimensional space. If a rectangular plane is permitted to rotate about one edge, which remains in a fixed position, a *dihedral angle is generated.* To illustrate this concept, you would need to use two rectangular planes, one to show the *initial* position of the plane before rotation began and the other to show the plane rotating and its *terminal* position. Since the upper edge of the rectangular plane is a straight line, it is easy to see that as the plane rotates to generate a dihedral angle, the upper edge of the plane is also generating a *plane angle.* Notice that

during this procedure, the plane rotates as far in determining a given dihedral angle as the upper edge in determining a *related* plane angle.

A third approach to the idea of a dihedral angle and its relation to a plane angle requires a little more use of the imagination than usual. An angle lies in a given plane. At its vertex, erect a line perpendicular to the given plane. Then permit the given angle to move along the perpendicular so that its vertex is always on the perpendicular and so that its sides are always parallel to their original position in the given plane. What does each side of the plane angle generate? What does the plane angle generate? What would seem to be true about the *size* of the plane *angle* and the *size* of the dihedral *angle* generated?

The dihedral angle in Fig. 1 on page 47 is referred to as $\angle CB$, or as $\angle D\text{-}CB\text{-}E$, or as $\angle D\text{-}CB\text{-}A$. A dihedral angle is named either by naming the edge, or by using four letters, a point in one face - the edge - a point in the other face. The letters representing the edge are always placed in the middle. Show how this method of naming dihedral angles is consistent with the method of naming plane angles.

Obviously, when several dihedral angles have a common edge, as in Fig. 2, four letters must be used in naming a specific dihedral angle. Name as many dihedral angles in Fig. 2 as possible. Two dihedral angles having the *same edge* and a *common face* between them are said to be **adjacent** dihedral angles. Which dihedral angles in Fig. 2 are adjacent?

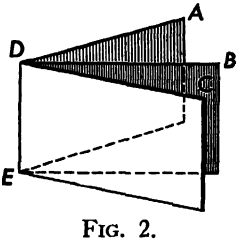


FIG. 2.

The plane angle *of a dihedral angle* is formed by two lines, *one in each* face, drawn *perpendicular to the edge* at the same point. In Fig. 3, $\angle GFE$ is a plane angle of dihedral $\angle A\text{-}BC\text{-}D$ since EF and FG are perpendicular to BC at F, which lies in the edge. EF lies in face AC and GF lies in face BD. Does a dihedral angle have only one plane angle? Why or why not?

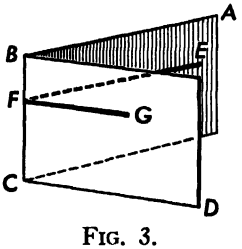


FIG. 3.

Based on the above definition of plane angle and the introductory discussion about dihedral angles, it should be obvious that a dihedral angle is **acute, right,** or **obtuse** according to whether its *plane angle* is acute, right, or obtuse, and similarly that two dihedral angles are **complementary** or **supplementary** if their *plane angles* are complementary or supplementary.

Two dihedral angles are said to be **congruent** if they can be made to coincide, and conversely. If the faces of a dihedral angle are the *prolongations* of the faces of another, *two pairs of* **vertical** *dihedral angles* are formed.

Two **planes are perpendicular** if they meet forming dihedral angles whose *plane angles* are *right angles.*

PRELIMINARY THEOREMS*

1. *The size of a dihedral angle is not dependent upon the extent of its faces.*
2. *All plane angles of a dihedral angle are equal.*
3. *A dihedral angle is measured by its plane angle.*
4. *Two dihedral angles are equal if their plane angles are equal, and conversely.*
5. *A dihedral angle can be bisected by one, and only one, plane.*

Dihedral angles have an important application in aeronautics. A front view cross section of a monoplane is shown in the accompanying sketch. Each wing meeting the horizontal plane forms a dihedral angle. In flight the dihedral angle tends to bring the plane back to horizontal flight from a bank and thus helps to prevent side-slipping.

During a bank, if a pilot released the controls entirely, the plane would *tend* to right itself and assume horizontal flight. This tendency is brought about by the dihedral angle. The greater the dihedral angle, the greater the tendency for the lowered wing to right itself. It is readily seen that the dihedral angle could be so large that it would force the lower wing up with such momentum that the higher wing would then be forced down below the horizontal plane. Thus a seesaw (oscillation) would result. Furthermore, if the dihedral angle is too great, a sudden side gust of wind, when landing, could cause a ground loop. Aeronautical engineers have determined that, for purposes of safety, the dihedral angle should be between one and three degrees.†

EXERCISES

1. Sketch two complementary adjacent dihedral angles.

2. How many dihedral angles are formed by the walls, ceiling, and floor of an ordinary rectangular room?

3. If two parallel planes are cut by a third plane to form corresponding‡ dihedral angles of 72°, find the size of the other dihedral angles that are formed.

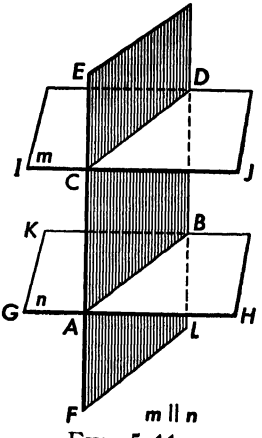
*These five statements are subsidiary theorems and are for informal class discussion. No formal proofs need be required, but the statements may be used as defenses in future proofs. They should be added to the **Tool Box.**

†A more comprehensive discussion can be found in the *Civil Pilot Training Manual*, C. A. A. Bulletin No. 23 (U.S. Department of Commerce), obtainable from the U.S. Government Printing Office.

‡*Corresponding* is used here in the same sense as in plane geometry.

4. Given line AB perpendicular to plane m at B. How many planes can contain AB? What will be their relationship to plane m?

5. Name two pairs of equal dihedral angles in the accompanying figure. How many more pairs of equal angles are there?

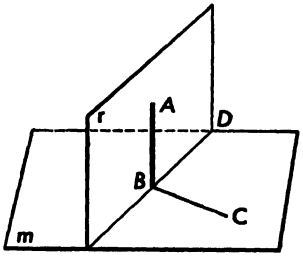


Exs. 5–11.

6. Name two supplementary adjacent dihedral angles.
7. Name two non-adjacent supplementary dihedral angles.
8. Name two corresponding dihedral angles.
9. If dihedral $\angle E\text{-}CD\text{-}J$ is 80°, how large is $\angle I\text{-}CD\text{-}B$? $\angle C\text{-}AB\text{-}K$?
10. What would need to be known before you could say that plane $EL \perp$ plane m?
11. If plane $EL \perp$ plane m, what is the size of $\angle C\text{-}AB\text{-}K$? Why?
12. Does the size of a dihedral angle depend upon the extent of its faces? What is a corresponding question about a plane angle?

THEOREMS

Theorem 19. *If a line is perpendicular to a given plane, every plane which contains this line is perpendicular to the given plane.*

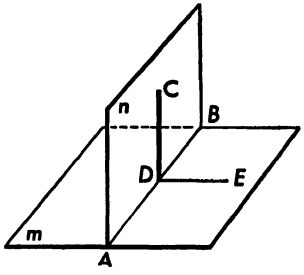


Given: $AB \perp$ plane m, and plane r any plane containing AB.
To prove: $r \perp m$.

Suggestions: In plane m draw $BC \perp BD$. Then how are AB and BC related? Why? Is $\angle CBA$ a plane $\angle$ of a dihedral $\angle$? Why? Will plane r be $\perp$ plane m?

Proof is left to you.

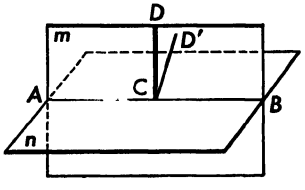
Theorem 20. *If two planes are perpendicular to each other, any line in one of them that is perpendicular to their intersection is perpendicular to the other.*



Given: Plane $n \perp$ plane m and intersecting in line AB. CD, in plane n, $\perp AB$.
To prove: $CD \perp$ plane m.

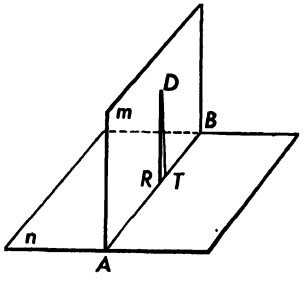
Suggestions: Draw DE in plane $m \perp AB$. Why is $\angle EDC$ the plane $\angle$ of the dihedral angle? Why is $\angle EDC$ a rt. $\angle$? Why is $CD \perp m$?
Proof left to you.

Corollary 20a. *If two planes are perpendicular to each other, a line perpendicular to one of them at a point of their intersection also lies in the other plane.*



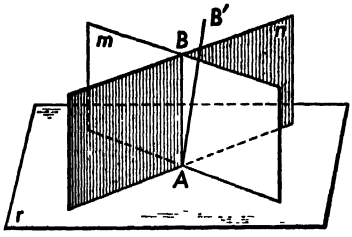
Suggestions: $DC \perp$ plane n at point C in the line of intersection AB. Draw $D'C$ in $m \perp AB$ at C. Why will $D'C$ be $\perp n$? At point C how many $\perp$s can be drawn to the plane? Why?

Corollary 20b. *If two planes are perpendicular to each other, a line perpendicular to one of them from any point of the second will lie in the second plane.*



Suggestions: DR is given $\perp n$ and D lies in m. In plane m draw $DT \perp AB$ which is the intersection of the two $\perp$ planes. Why is $DT \perp$ plane n? How many $\perp$s can be drawn from D to plane n? Why?

Theorem 21. *If each of two intersecting planes is perpendicular to a third plane, their intersection is also perpendicular to that plane.*

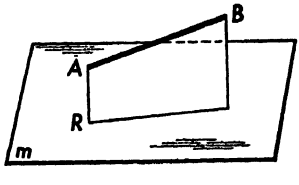


Given: Planes m and n each $\perp$ to plane r. m and n intersect in line AB.
To prove: $AB \perp r$.

PROOF

Argument	*Defense*
1. Planes n and m intersect in AB. m and n are each $\perp r$.	1. Given.
2. Assume AB is not $\perp r$ and draw a line such as AB' which will be $\perp r$.	2. Why possible?
3. AB' lies in both m and n.	3. If two planes are $\perp$, a line $\perp$ one of them at a point in their intersection . . ..
4. AB' is then the intersection of m and n.	4. Why?
5. But AB is *given* the intersection of m and n, so AB and AB' must coincide.	5. Why?
6. $\therefore AB \perp r$.	6. Why?

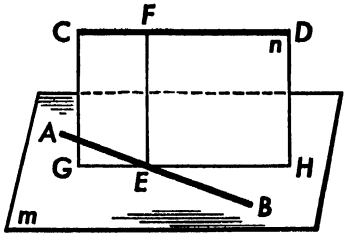
Corollary 21a. *Through a given line which is not perpendicular to a given plane one, and only one, plane can be drawn perpendicular to the given plane.*



Suggestions: From A, any point in AB, draw $AR \perp m$. How will the plane determined by AB and AR be related to m? If another plane could be drawn through $AB \perp m$, why would AB be $\perp m$? Develop a proof. *Two parts are required.*

Theorem 22. *Between any two skew lines there is one, and only one, common perpendicular.*

Part 1

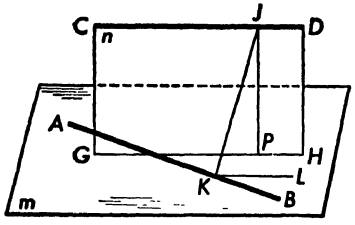


Given: Skew lines AB and CD.
To prove: One common $\perp$ can be drawn between AB and CD.

PROOF

Argument	*Defense*
1. Through AB pass plane $m \parallel CD$.	1. Why possible?
2. Through CD pass plane $n \perp m$ and intersecting m in line GH.	2. Why possible?
3. $GH \parallel CD$.	3. If a line $\parallel$ a plane, it is $\parallel$ the intersection of that plane with . . ..
4. GH will intersect AB in some point such as E.	4. GH cannot be $\parallel AB$ for, if so, AB would be $\parallel CD$ which is contrary to the given facts.
5. In plane n draw $FE \perp GH$ at E.	5. Why possible?
6. $FE \perp m$.	6. Why?
7. $FE \perp CD$.	7. Why?
8. $FE \perp AB$.	8. Why?
9. $\therefore$ FE is a common perpendicular to CD and AB.	9. Why?

Part 2



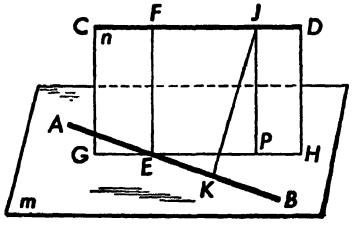
Given: Skew lines AB and CD.
To prove: FE is the only perpendicular common to AB and CD.

PROOF

Argument	*Defense*
1. Assume there is another common $\perp$ such as JK. In plane m draw $KL \parallel GH$ and in plane n draw $JP \perp GH$.	1. Why possible?
2. $KL \parallel CD$.	2. Why?
3. $JK \perp KL$.	3. Why?
4. $JK \perp$ plane m.	4. Why?
5. But this is impossible because $JP \perp$ plane m.	5. If two planes $\perp$ each other, any line in one of them $\perp$ their . . .. and from an external point there can be one and only one . . ..

6. $\therefore$ only one common perpendicular (FE) can be drawn between AB and CD.

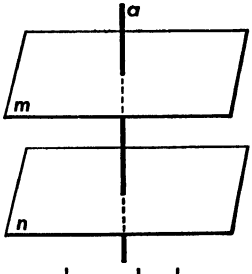
Corollary 22a. *The shortest line that can be drawn between two skew lines is the common perpendicular.*



Suggestions: Draw plane n through $CD \perp m$ and intersecting m in line GH. FE is the common perpendicular. Let JK be any other line between AB and CD. Draw $JP \perp GH$. Compare length of JP and EF with JK.

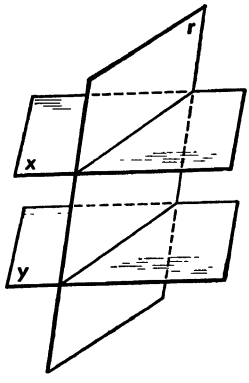
EXERCISES: QUOTING THEOREMS SUGGESTED BY DRAWINGS

Quote the theorems or corollaries suggested by the drawings given below.



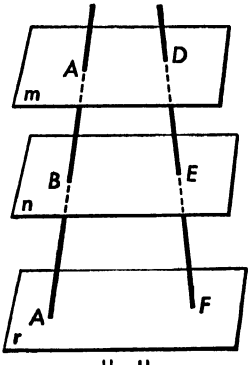
$m \perp a$ and $n \perp a$

(1)



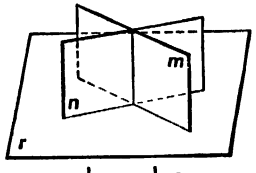
$x \parallel y$

(2)



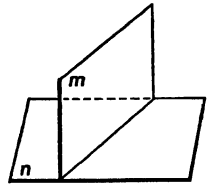
$m \parallel n \parallel r$

(3)



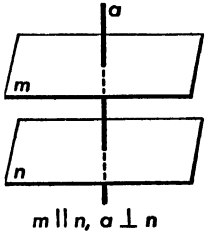
$m \perp r$, $n \perp r$

(4)



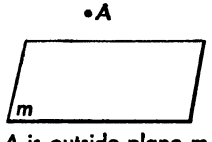
$m \perp n$

(5)



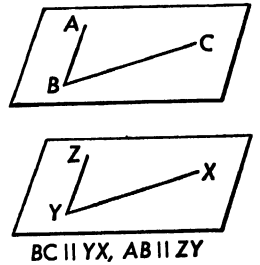
$m \parallel n$, $a \perp n$

(6)



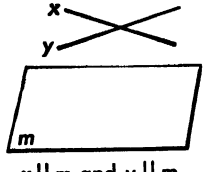
A is outside plane m

(7)



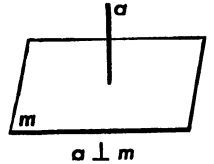
$BC \parallel YX$, $AB \parallel ZY$

(8)



$x \parallel m$ and $y \parallel m$

(9)



$a \perp m$

(10)

SECTION II. POLYHEDRAL ANGLES

The angle formed by *three or more* planes meeting at a point is called a **polyhedral angle.** The portions of the planes which form the polyhedral angle are called the **faces** and the point at which they meet is called the **vertex.** The polyhedral angle in Fig. 4 can be referred to as $\angle V$ or as $\angle V\text{-}ABCDE$.

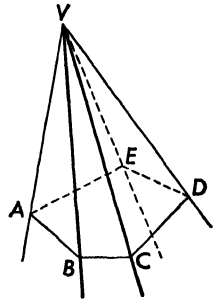


FIG. 4.

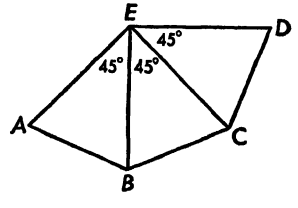
The dihedral angles formed by the consecutive faces, *taken in pairs*, are the **dihedral angles** *of the polyhedral angle.* The edges of the dihedral angles are the **edges** *of the polyhedral angle.* The angles formed by the consecutive edges, taken in pairs, are the **face angles** *of the polyhedral angle.* Name a few dihedral angles of the polyhedral angle in Fig. 4. Name a few face angles. How many edges does this polyhedral angle have? Name them.

If the *polygon formed by a plane* intersecting **all** the *faces* of a polyhedral angle is convex, the *polyhedral angle* is **convex.** Only convex polyhedral angles will be studied in this course.

Some polyhedral angles have special names. One with *three* faces is called a *trihedral angle*, one with *four* faces is a *tetrahedral angle*, one with *five* faces is a *pentahedral angle*, and so on. What kind of polyhedral angle is formed at one corner of a room by two walls and the ceiling? How many right angles does it have?

If the edges of one polyhedral angle are *prolongations* of the edges of another, the two are called **vertical polyhedral angles.**

PRELIMINARY EXERCISES



1. Construct three congruent isosceles triangles with the three vertex angles each equal to 45° and placed so that the three triangles are in the same relative positions as in the accompanying figure. Cut out the resulting figure and crease along the lines BE and CE. Then fold to make AE and DE coincide to form a *trihedral angle.*

2. Repeat Ex. 1, making each vertex of the *three* isosceles triangles equal to 90°.

3. Construct *four* isosceles triangles, each of whose vertex angles is 45° and arranged in fanshape as in Ex. 1. Fold to form a *tetrahedral angle.*

4. Repeat Ex. 3 making each of the vertex angles of the *four* isosceles triangles 90° each. Fold into a *tetrahedral angle.*

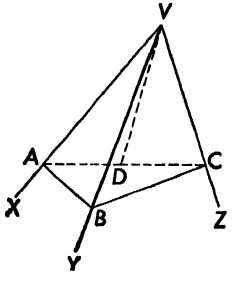
5. In Ex. 4, you found it impossible to obtain a tetrahedral angle with four

face angles of 90° each. The figure "flattened out." Can you tell why Exs. 1, 2, and 3 were possible and Ex. 4 was impossible?

6. Could a polyhedral angle with *six* face angles of 50° each be constructed? With *four* face angles of 85° each? With *seven* face angles of 45° each? With *eight* face angles of 45° each?
7. Generalize and complete: *The sum of the face angles of any polyhedral angle is* ? ? ?.

THEOREMS

Theorem 23. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*

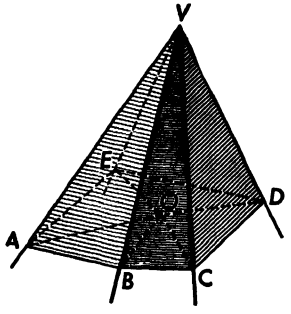


Given: Trihedral $\angle V\text{-}XYZ$ with $\angle XVZ$ the largest face angle.
To prove: $\angle XVY + \angle YVZ > \angle XVZ$.

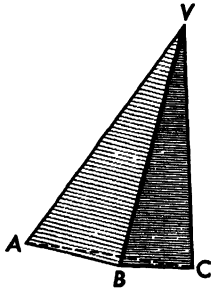
PROOF

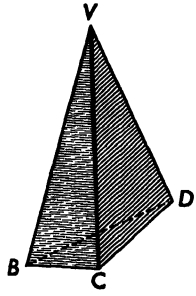
Argument	*Defense*
1. $\angle V\text{-}XYZ$ is a trihedral $\angle$ with XVZ the largest face $\angle$.	1. Given.
2. Draw AC any line to cut XV and VZ.	2. Why possible?
3. Make $\angle AVD = \angle AVY$.	3. Why possible?
4. Make $VB = VD$.	4. Why possible?
5. Draw AB and BC.	5. Why possible?
6. $\triangle AVD \cong \triangle AVB$.	6. Why?
7. $AB + BC > AC$ or $AB + BC > AD + DC$.	7. Why?
8. $\therefore BC > DC$.	8. Why?
9. In $\triangle BVC$ and DVC, since $BV = VD$ and $VC = VC$, $\angle BVC > \angle DVC$.	9. Why?
10. $\angle AVB + \angle BVC > \angle AVB + \angle DVC$ or $> \angle AVD + \angle DVC$.	10. Why?
11. $\therefore \angle AVB + \angle BVC > \angle AVC$. or $\angle XVY + \angle YVZ > \angle XVZ$.	11. Why?

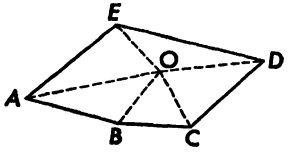
Theorem 24. *The sum of the face angles of any convex polyhedral angle is less than* 360.°



Given: A convex polyhedral angle V.
To prove: The sum of all the face angles $< 360°$.







The three smaller figures show three different parts of the larger figure above at the left. They will help you to understand the proof.

PROOF

Argument	*Defense*
1. Pass a plane which cuts the edges of $\angle V$ in points A, B, C, etc., and from any point such as O in the plane of the polygon $ABCDE$ draw OA, OB, OC, etc.	1. Why possible?
2. By observation it can be seen that there are as many ▲ with V as a vertex as there are ▲ with O as a vertex, so the sum of all ∡ of the ▲ with vertex V = the sum of all ∡ of the ▲ with vertex O.	2. Why?
3. Using trihedral ∡ with vertices A, B, C, etc., $\angle VBA + \angle CBV > \angle CBA$, $\angle VCB + \angle DCV > \angle DCB$, etc.	3. Quote the theorem in full.
4. The sum of the ∡ at vertices A, B, C, etc. (in ▲ having V as a vertex) is greater than the sum of the ∡ at vertices A, B, C, etc. (in ▲ having O as a vertex).	4. If unequals are added to unequals, the results are . . ..
5. The sum of the ∡ at vertex V is less than the sum of the ∡ at O.	5. If unequals are subtracted from equals, the results are . . ..

6. $\therefore$ since the sum of the $\measuredangle$ at vertex $O = 360°$, $\angle BVC + \angle CVD + \angle DVE +$ etc., $< 360°$. 6. Why?

SYMMETRIC FIGURES

In plane geometry you probably were introduced to the concepts of *line symmetry*, *plane symmetry*, and *point symmetry*. The accompanying picture is an ink blot. It was made by folding a piece of paper along a line, putting a drop of ink in the fold, and pressing the two leaves together. The leaves were then opened and flattened into the position of the original plane. The *axis of symmetry* (the original fold) was then drawn. The result was the *plane symmetric figure* reproduced on this page. Observe that it is in balance with respect to its *axis of symmetry*. For every point in the figure on the right, there is a corresponding point in the figure on the left. The two figures appear to be congruent but *their parts are arranged in reverse order*. Keeping the figure on the left in the same plane, you could slide it into every position conceivable and yet you could not make it coincide, part for part, with the other figure. However, by permitting one to remain in its plane and rotating the other *about the axis of symmetry*, it would return to the plane after one-half of a complete rotation. The two figures would then coincide and hence would be *congruent*.

The *corresponding parts* of any two symmetric figures are arranged *in reverse order*. One of two *symmetric plane* figures can always be rotated out of its plane, thus reversing the arrangement of its parts, and be shown to coincide with the other one. Thus, *any two symmetric plane figures are also congruent*.

We have talked about symmetry thus far somewhat informally. As a formal definition we say that **two symmetric figures** are those whose *corresponding parts* are *respectively equal* **but are arranged** *in reverse order*.

In Fig. 5, $AB = NO$, $BC = MO$, $AC = MN$, $\angle A = \angle N$, $\angle C = \angle M$, and $\angle B = \angle O$. In terms of the above definition, show why these triangles are symmetric. From the given facts, we know that the triangles are also congruent. Why? Notice that they could be made to coincide by rotating $\triangle ABC$ through 180° about AC so that vertex B would again lie in its original plane but on the opposite side of AC. Thus the order of its parts would be reversed and either triangle

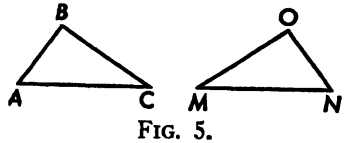


FIG. 5.

could then be moved in the plane until the triangle coincided with the other.

Looking at an automobile from the front, imagine a *vertical plane* passed through the middle of the car. Obviously the two halves are symmetrical. The corresponding parts are equal but are *arranged in the opposite (reverse) order.* Testing for the congruence of these two solids, it is easily seen that it is impossible to turn one of the solids in any position so that it could be slid into the other and thus be made to coincide with it. In fact, the only possibility would be to turn one of these solids "inside out." Even if it were physically possible to do this, the solid, so altered, would become a different solid with its parts entirely rearranged. The same observations can be made with a pair of shoes or with a pair of gloves. Each pair can be arranged symmetrically but cannot be made to coincide (one fit into the other) without changing one of the objects into a different one by turning *one of the pair* inside out. This leads to the observation that the idea of *congruence* for some *solids* must be restricted when compared to congruence of plane figures. Even when the corresponding parts of some solid figures are equal, they cannot be made to coincide *if their parts are arranged in reverse order.* The foregoing leads to the definition that **two polyhedral angles are congruent** if the *face angles* and *dihedral angles* of one are respectively equal to the *face angles* and *dihedral angles* of the other, and are arranged **in the same order,** but **symmetric** if arranged **in reverse order.**

In Fig. 6, polyhedral angles V and V' are symmetric. What two conclusions can be drawn? Are these polyhedral angles congruent? Why or why not?

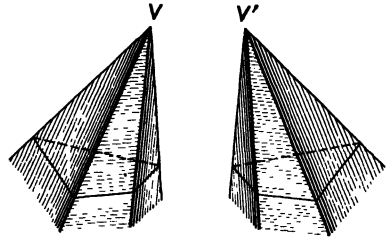
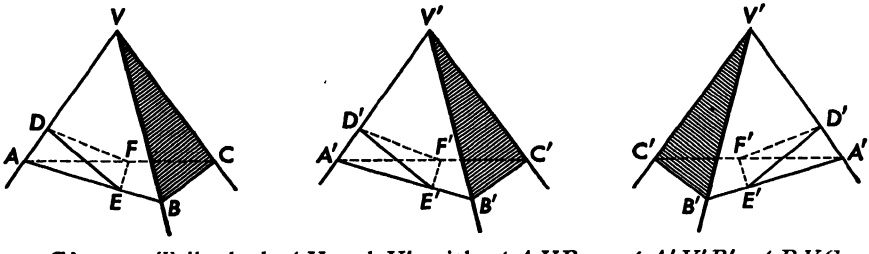


FIG. 6.

THEOREM

Theorem 25. *Two trihedral angles are either congruent or symmetric if the three face angles of one are equal respectively to the three face angles of the other.*



Given: Trihedral $\measuredangle V$ and V', with $\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, $\angle CVA = \angle C'V'A'$.

To prove: Trihedral $\measuredangle V$ and V' are $\cong$ or symmetric.

In the *first two* drawings the equal parts are arranged in the *same order*, and in the *last two* the equal parts are arranged in *reverse order*.

PROOF

Argument	*Defense*
1. Choose points A, B, C, A', B', and C' so that $VA = V'A' = VB = V'B' = VC = V'C'$. Draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.	1. Why possible?
2. $\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, $\angle AVC = \angle A'V'C'$.	2. Given.
3. $\triangle ABV \cong \triangle A'B'V'$, $\triangle BVC \cong \triangle B'V'C'$, and $\triangle AVC \cong \triangle A'V'C'$.	3. Why?
4. Then $AB = A'B'$, $BC = B'C'$, and $CA = C'A'$.	4. Why?
5. $\triangle ABC \cong \triangle A'B'C'$.	5. Why?
6. $\angle BAC = \angle B'A'C'$.	6. Why?
7. On VA and $V'A'$ make $AD = A'D'$, and draw $DE \perp VA$ in plane VAB and $DF \perp VA$ in plane VAC. Draw $D'E'$ and $D'F'$ in the same manner. Draw EF and $E'F'$.	7. Why possible?
8. $\angle BAV = \angle B'A'V'$.	8. Why? (See step 3.)
9. $\triangle DAE \cong \triangle D'A'E'$.	9. Why?
10. $AE = A'E'$ and $DE = D'E'$.	10. Why?
11. $\angle VAC = \angle V'A'C'$.	11. Why? (See step 3.)
12. Prove $\triangle DAF \cong \triangle D'A'F'$ to get $AF = A'F'$ and $DF = D'F'$.	12. Explain.
13. Prove $\triangle AEF \cong \triangle A'E'F'$ to get $EF = E'F'$.	13. Explain.
14. Prove $\triangle DEF \cong \triangle D'E'F'$.	14. Explain.
15. Then $\angle EDF = \angle E'D'F'$ and they are plane $\measuredangle$ of dihedral $\measuredangle B$-AV-C and B'-$A'V'$-C'.	15. Why?
16. $\therefore \angle B$-AV-$C = \angle B'$-$A'V'$-C'.	16. Why?
17. Similarly, $\angle A$-VC-$B = \angle A'$-$V'C'$-B' and $\angle C$-BV-$A = \angle C'$-$B'V'$-A'.	17. See steps 8–17.
18. $\therefore$ trihedral $\angle V \cong$ trihedral $\angle V'$ or is symmetric to trihedral $\angle V'$.	18. Definition.

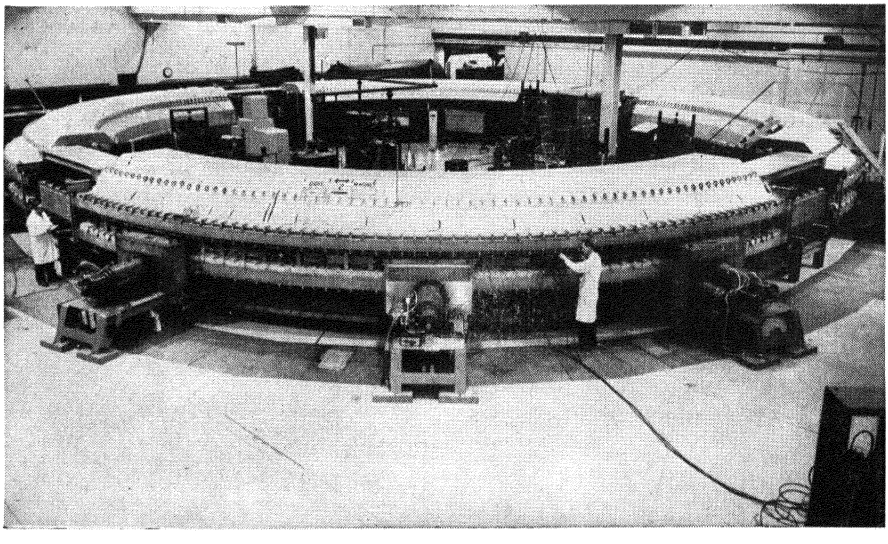
Corollary 25a. *If two trihedral angles have the face angles of the one equal respectively to the three face angles of the other, the dihedral angles opposite the equal face angles are equal.*

Suggestion: This is proved in the body of the proof for Th. 25.

CHAPTER REVIEW: JUDGING RELATIONSHIPS

Complete the following statements using *always*, *sometimes*, or *never*, to make each one valid.

1. If a line is perpendicular to the intersection of two planes, it is ? perpendicular to the planes.
2. Two lines perpendicular to the same plane are ? parallel.
3. Two lines parallel to the same plane are ? parallel.
4. Through a given external point, one line perpendicular to a given plane can ? be drawn.
5. Through a given external point, more than one line parallel to a given plane can ? be drawn.
6. At a given point in a plane more than one line perpendicular to the plane can ? be drawn.
7. If two lines are parallel to a third line, they are ? parallel.
8. If two lines will never meet, no matter how far extended, they are ? parallel.



The cosmotron, a circular magnet with an inside diameter of over sixty feet and a weight of 2,200 tons. (Wide World Photos)

9. If two planes are cut by a third plane, the lines of intersection are $\perp^?$ parallel.
10. If a line is perpendicular to a plane it is $\perp^?$ perpendicular to lines which pass through its foot and lie in the plane.
11. Parallel lines drawn to a plane from points in a line which is parallel to the plane are $\perp^?$ equal.
12. When two lines are parallel but lie in different planes, the two planes are $\perp^?$ parallel.
13. If a line intersects one of two parallel planes, it will $\perp^?$ intersect the other.
14. If a line intersects one of two perpendicular planes, it will $\perp^?$ intersect the other.
15. If a line intersects two skew lines, all three lines will $\perp^?$ lie in the same plane.
16. A plane can $\perp^?$ be perpendicular both to a given plane and to a given line.
17. Two lines which lie in parallel planes are $\perp^?$ parallel.
18. Two planes which are perpendicular to the same plane are $\perp^?$ parallel.
19. A line perpendicular to one of two parallel planes is $\perp^?$ perpendicular to the other plane.
20. If there is a common perpendicular between two lines, it is $\perp^?$ the shortest line joining them.

Bring your **Tool Box** *up to date.*

EXERCISES

Develop proofs for Exs. 1–9.

1. *If a line is parallel to the intersection of two planes, it is parallel to each of the planes.*
2. If two planes bisect two supplementary adjacent dihedral angles they are perpendicular to each other.
3. *If a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.*
4. If two planes intersect, the adjacent dihedral angles are supplementary.
5. If a line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.
6. The equal sides of an isosceles triangle make equal angles with any plane which contains the base.
7. A plane that is perpendicular to the edge of a dihedral angle is perpendicular to both of its faces.

8. If three lines are perpendicular to each other at some common point, each line is perpendicular to the plane of the other two, and the three planes are perpendicular.
9. If two points lie on opposite sides of a plane and are equidistant from the plane, the segment joining them is bisected by the plane.
10. Find the length of the longest diagonal that can be drawn in a rectangular room whose dimensions are 18 feet, 12 feet, and 30 feet.
11. The plane angle of a dihedral angle is 60°. Point X lies in the bisector of the dihedral angle 12 inches from the edge of the dihedral angle. A perpendicular is drawn from X to one face of the dihedral angle. How far is X from the face of the angle?
12. A swimming pool is filled with water to within 1 foot of the top. A 20-foot grappling pole is placed obliquely in the pool so that one end is on the bottom of the pool and the other end just reaches to the top edge of the pool. If the pool is 10 feet deep at that end, how much of the pole is under water?
13. Prove: A plane perpendicular to one of two parallel planes is also perpendicular to the other plane.
14. Prove: If perpendiculars are drawn from a given external point to the faces of a dihedral angle, the angle formed between the two perpendiculars is supplementary to the dihedral angle.

CHAPTER TEST

1. Define: perpendicular planes, dihedral angle, polyhedral angle, faces of a trihedral angle, and plane angle of a dihedral angle.
2. How is a dihedral angle measured?
3. What different kinds of dihedral angles are there?
4. How is a dihedral angle read?
5. Quote the theorem or corollary which applies most precisely to each of the following statements or situations.

   (a) The intersection of the front wall and a side wall of your classroom is perpendicular to the floor.
   (b) In an ordinary classroom the floor is parallel to the ceiling.
   (c) A door jamb is perpendicular to the plane of the floor.
   (d) Considering the leg of an ordinary classroom table to be a line, any plane containing the leg will be perpendicular to the floor.

6. Point out two skew lines in your room and show how Th. 22 applies.
7. Complete the following statements:

   (a) Between two skew lines . . ..
   (b) The sum of any two face angles of a trihedral angle . . ..

(c) Two trihedral angles are either congruent or ? if the three . . ..
(d) If each of two intersecting planes is perpendicular to a third plane, then their . . ..
(e) If two trihedral angles have the three face angles of one equal . . ..
(f) If a line is perpendicular to a plane, then every plane which . . ..
(g) All plane angles of a ? angle are ?.
(h) If two planes are perpendicular to each other, a line perpendicular to one of them at a point in . . ..
(i) Two dihedral angles are equal if their ? angles are ?.
(j) A dihedral angle is right, acute, or obtuse according to whether its ? angle is ?, ?, or ?.
(k) If two planes intersect the ? dihedral angles are equal.
(l) Two planes are perpendicular if the ? angles formed are ? dihedral angles.

8. Is it possible to have a trihedral angle whose face angles are:

(a) 70°, 60°, and 120°?
(b) 40°, 100°, and 60°?
(c) 200°, 140°, and 70°?

9. Prove that any face angle of a trihedral angle is greater than the difference of the other two.

Keep the Iron Hot: LOCUS—SPECIAL PRODUCTS—INEQUALITIES

Set your goal at 100 *percent accuracy.*

1. What is the locus of the center of a circle which rolls about a given circle and is always tangent to it?
2. What is the locus of the vertex of the vertex angles of all the possible isosceles triangles constructed on a given base?
3. What is the locus of the vertex of the right angle of the right triangles constructed with a given segment for a hypotenuse?
4. Draw the graph or *locus* of the following equations:

(a) $2x + y = 16$ (b) $x - 3y = 1$
(c) $x - y + 4 = 0$ (d) $y = 4x - 3$

5. Illustrate each of the following facts:

(a) There are an indefinite number of pairs of values which will satisfy a linear equation.

(b) Points whose coordinates satisfy a linear equation will lie on the locus (graph) of the equation, but points whose coordinates *do not* satisfy the equation, will *not* lie on the locus.

(c) The coordinates of any point which lies on the locus of an equation will satisfy the equation, but the coordinates of any point which does *not* lie on the locus of an equation will *not* satisfy the equation.

6. The locus of a point is the path determined by a point moving according to certain restrictions. The graph of each equation you plotted in Ex. 4 can be regarded as the path determined by a moving point. Thus each graph is a locus. What imposed the restrictions in each of the exercises you graphed?

Give the products in each of the following exercises *at sight:*

7. $(x + 3)(x - 1)$ **8.** $(2a + 5)(3a - 1)$ **9.** $(6c - d)(d - 5c)$

10. $(4x - 7)(4x + 7)$ **11.** $(3x - y)^2$ **12.** $(4a + 3)^2$

13. $(6m + y^2)^2$ **14.** $(5a - 2b)^2$ **15.** $(2m + 9)(2m + 9)$

16. $(a^3 - 5)(a^3 + 3)$ **17.** $(x^2y + 6z)(x^2y - 6z)$ **18.** $(5 - 4x)(5 + 4x)$

19. Quote as many statements as possible from plane geometry which refer to inequalities.

20. If a square and an equilateral triangle are inscribed in the same circle, prove that a side of the square is shorter than a side of the triangle.

CHAPTER 5 *Loci and Projections*

SECTION 1. LOCI

You should remember two facts about loci from plane geometry.

1. *Every point which is on a locus satisfies the given conditions.*
2. *Every point which satisfies the given conditions lies on the locus.*

If you read the above two conditions carefully you will observe that each is the converse of the other. You realize that we cannot assume the validity of a statement because we *prove* the validity of its converse. Therefore, in proving a locus theorem, both of the above conditions must be established.

There are five basic locus problems generally studied in plane geometry. See if you can recall them and complete the following statements. (All conditions and loci are restricted to a plane.)

1. The locus of points at a given distance from a *given point* is . . ..
2. The locus of points at a given distance from a *given line* is . . ..
3. The locus of points equidistant from *two parallel lines* is . . ..
4. The locus of points equidistant from *two given points* is . . ..
5. The locus of points equidistant from the *sides of an angle* is . . ..

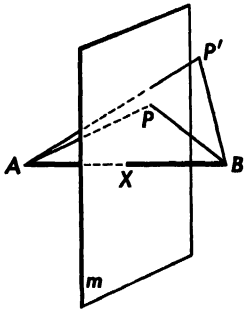
What is the locus, *in space*, in each of the above five statements?

PRELIMINARY EXERCISES

1. What is the locus of points, *in a plane*, equidistant from three given points?
2. What is the locus of points in space equidistant from three given points?
3. What is the locus of points equidistant from a given plane?
4. Find the locus of points equidistant from the three faces of a trihedral angle.
5. What is the locus of points, in space, equidistant from the vertices of a rectangle?

THEOREMS

Theorem 26. *The locus of points equidistant from two given points is the plane perpendicular to the line joining the points and drawn at the mid-point of the line.*



Given: Points A and B; X, the mid-point of the line joining A and B; plane $m \perp AB$ at X; point P is on plane m; and point P' is located so that $AP' = P'B$.
To prove: Plane m is the locus of points equidistant from A and B.

Part 1

Given: Point P lying in m.
To prove: P is equidistant from A and B.

Suggestions: Draw PX and prove $\triangle APX \cong \triangle PXB$.

Proof left to you.

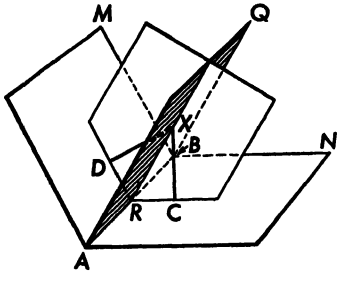
Part 2

Given: Any point P' such that $AP' = P'B$.
To prove: P' lies in m.

Proof left to you.

Corollary 26a. *The locus of points equidistant from the vertices of a triangle is the line through the center of the circumscribed circle perpendicular to the plane of the triangle.*

Theorem 27. *The locus of points equidistant from the faces of a dihedral angle is the plane bisecting the dihedral angle.*



Given: Plane QA bisecting dihedral $\angle M\text{-}AB\text{-}N$.
To prove: QA is the locus of points equidistant from the sides of dihedral $\angle M\text{-}AB\text{-}N$.

Part 1

Given: Any point X in QA.
To prove: X is equidistant from planes AM and AN.

PROOF

Argument	*Defense*
1. Draw $XC \perp$ plane AN and $XD \perp$ plane AM.	1. Why possible?
2. Let plane XCD intersect plane AM in DR and plane AN in RC. Draw XR.	2. Why?
3. Plane $XCD \perp$ both planes AM and AN.	3. If a line $\perp$ a plane, every plane which contains this . . ..
4. Plane $XCD \perp AB$.	4. If each of two intersecting planes $\perp$ a third plane, their intersection . . ..
5. $AR \perp DR$, XR, and CR.	5. Why?
6. $\angle XRC$ is a plane $\angle$ of dihedral $\angle Q\text{-}AB\text{-}N$, and $\angle XRD$ is a plane $\angle$ of dihedral $\angle M\text{-}AB\text{-}Q$.	6. Why?
7. $\angle CRX = \angle XRD$, or XR bisects $\angle CRD$.	7. Given.
8. $\triangle XRC \cong \triangle XRD$.	8. If the hypotenuse and acute $\angle$ of . . ..
9. $XC = XD$.	9. Why?
10. $\therefore X$ is equidistant from planes AM and AN.	10. Why?

Part 2

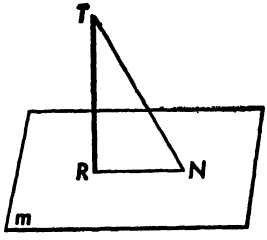
Given: Let X' represent any point equidistant from planes AM and AN.

To prove: X' lies in plane QA.

Suggestions: Drop $\perp$s from X' to the planes of the given angle and proceed as in Part 1.

Proof left to you.

EXERCISES



1. **Given:** $TR \perp$ plane m.
   What is the locus in plane m of points 10 inches from T? If TN is 10 inches and $\angle TNR$ is 60°, find the length of the locus.
2. Describe or sketch the locus for each of these conditions:
   (a) Points equidistant from two given parallel lines and also equidistant from two given points.
   (b) Points 6 inches from a given plane.
   (c) Points equidistant from three given points not in a straight line.
   (d) Points within a trihedral angle which are equidistant from the three faces of the angle.
   (e) Same as Ex. (d) above but equidistant from the three edges.
3. In a given plane find the locus of points equidistant from each of two given points which are not in the given plane.
4. If an isosceles triangle revolves about its base, what is the locus of its vertex?
5. Prove that the planes bisecting the dihedral angles of a trihedral angle meet in a line and that every point on the line is equidistant from the faces of the trihedral angle.
6. What is the locus in your classroom of:
   (a) All points equidistant from the ceiling and floor, and also equidistant from the two side walls?
   (b) Points equidistant from the front and back walls, and also equidistant from the side walls?
   (c) Points equidistant from the two side walls, equidistant from the front and back walls, and also 4 feet from the floor?
   (d) Points equidistant from the eight corners of an ordinary rectangular room?
7. A right triangle is rotated about its shorter leg as an axis. What is the locus of the mid-point of the hypotenuse? If the triangle is a 3-4-5 triangle, find the length of the locus.

8. The locus of points equidistant from two given intersecting planes is the pair of planes bisecting the dihedral angles formed by the given planes.

9. The locus of points equidistant from all points on a circle is a line through the center of the circle perpendicular to the plane of the circle.

10. **Given:** A triangle with planes perpendicular to each side at the midpoints of the sides.
Prove that the three planes intersect in a common line.

11. Find the locus of points a units from a circle.

12. Given that point P lies $2m$ units from plane n. What is the locus of points equidistant from n and P?

13. A and B are fixed points in a plane. C is a third point such that AC equals a and BC equals b. What is the locus of C which moves so that $a + b$ is always constant?

SECTION II. PROJECTION

Many times you have seen your shadow *projected* on the ground. In general, projection means to "throw" the image of a figure forward and reproduce it on a surface or another figure by means of straight lines or rays which pass through all points of the object. Thus, when you see your shadow on the ground, your image has been "thrown forward" by the parallel rays of the sun and *projected on a plane.* The projection may be larger or smaller than the actual object. Lengthening or foreshortening is effected by the directness or obliqueness of the sun's rays. Even though the projection thus may appear to be distorted, it still bears a definite relationship in terms of measurements of the actual object.

In Fig. 1, assume AB to represent a man standing on plane m and perpendicular to it. The sun's rays, represented by a series of parallel lines, make an angle of 45° with the plane. Thus, the shadow CA is actually the projection of AB on plane m. Under the given conditions, how does the length of AB compare with the length of CA? If $\angle ACB$ were 30°, how would the two lines compare in length? As $\angle ACB$ grows smaller and smaller, how do the two lengths compare? As the angle grows larger, approaching 90°? What is the projection of AB on m when the rays are perpendicular to m?

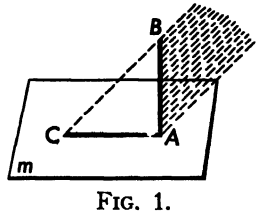


FIG. 1.

In Fig. 1, the projecting lines (rays) were parallel to each other, but were not perpendicular to the plane. In Fig. 2, the projecting lines *are perpendicular* to the plane. This method is known as orthogonal projection and will be the only type of projection used in this book. The word orthogonal is derived from two Greek words meaning right-angled.

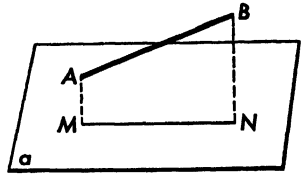


FIG. 2.

In Fig. 2, the projection of point A is the foot of the perpendicular from A to plane a. In other words, the projection of point A on plane a is point M. What is the projection of point B on a? What is the projection of line segment AB on plane a? Does the length of the projection of this line segment on plane a equal the length of the line segment? Under what conditions will a line segment and its projection on a plane be equal? When will the orthogonal projection of a line segment on a plane be a point?

The foregoing discussion can be formalized with the following definitions.

The projection of a given point *on a given plane* is the foot of the perpendicular drawn from the point to the plane.

The projection of any line *on a given plane* is the locus of the projections of *all* of its points on the given plane.

The acute angle that a line segment makes with its projection (both extended if necessary) is called **the inclination of the line to the plane.** In Fig. 3, what angle represents the inclination of ST to n? SB is drawn parallel to RW. What other angle in Fig. 3 equals the inclination? What kind of triangle is STB? In plane geometry you learned that the cosine of an angle in a right triangle is the ratio of the adjacent side to the hypotenuse. What is the cosine of $\angle BST$?

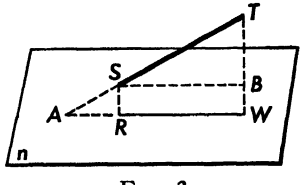


FIG. 3.

THEOREMS

The following theorems are not part of the sequence in this book and are numbered independently. Study the statements. Whether or not you develop proofs is *optional*, but the proofs are not difficult.

Theorem 1. *The projection on a plane of a straight line not perpendicular to the plane, is a straight line.*

Theorem 2. *The acute angle which a straight line makes with its projection on a plane is the least angle which it makes with any line in the plane.*

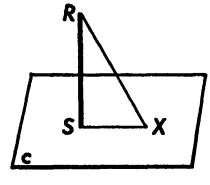
Theorem 3. *The projection of a straight line segment on a plane equals the product of the segment and the cosine of the inclination of the line segment to the plane.*

EXERCISES

1. Prove that a line segment parallel to a plane and its projection on the plane are equal.
2. Hold a cube above a plane so that no face of the cube is parallel to the plane. Project each vertex on the plane and join the respective points of projection. The resulting figure is the ? of the cube on the plane.
3. Prove that two equal line segments drawn to a plane from a point outside the plane make equal angles with the plane.
4. In each of the following find the length of the projection of the given line segment. Leave answers in radical form. (Use $\cos 45° = \frac{\sqrt{2}}{2}$; $\cos 30° = \frac{\sqrt{3}}{2}$; and $\cos 60° = \frac{1}{2}$.)
   (a) Line segment AB inclines 30° with plane m and is 12.8 units long.
   (b) Line segment CD inclines 45° with plane n and is 28 inches long.
   (c) Line segment EF inclines 60° with plane r and is 15.4 feet long.
5. Prove that a line segment that inclines 60° with a plane is twice as long as its projection on that plane.
6. Show how the projection of a plane curve can be a straight line.
7. Describe the projection of a sphere.
8. Describe the projection of two parallel line segments. (Four cases.)
9. Describe the projection of two intersecting lines. (Three cases.)
10. Show how the projection of a circle can be an ellipse.

CHAPTER REVIEW: MISCELLANEOUS EXERCISES

1. M is the mid-point of XY. Plane s is perpendicular to XY at M. Prove that any point in plane s is equidistant from the ends of the line.
2. One given edge of a rectangular chalk box touches how many faces of the box? Prove that it is perpendicular to two faces and is parallel to two other faces.
3. Prove: A plane which is perpendicular to a line in a second plane is also perpendicular to that second plane.
4. If RX is 20 inches long, $\angle SRX = 30°$, and $RS \perp c$, how long is SX? RS? What is the length of the locus of all points 20 inches from R?



5. If a plane is perpendicular to a line segment at its mid-point, any point in the plane 3 inches from one end of the segment is . . ..
6. If each of three lines is perpendicular to the other two and if the three

lines are concurrent, what can be proved about the planes determined by the lines?

7. If two planes are perpendicular to each other, prove that a line perpendicular to either plane through any given external point is parallel to the other plane.
8. In Ex. 7 change "external point" to "internal point but not on their line of intersection" and prove that the same conclusion is valid.
9. A plane perpendicular to the edge of a dihedral angle has what relation to the faces of the angle?
10. If perpendiculars are drawn to a plane from the vertices of a parallelogram lying outside their plane, their sum is equal to four times the length of the perpendicular drawn to the plane from the point of intersection of the diagonals of the parallelogram.
11. *The projection on a plane of a straight line not perpendicular to it is a straight line.*

CHAPTER TEST

1. Point M in the bisector of a dihedral angle is 8 inches from the edge of the angle. If the plane angle of the dihedral angle is 60°, how far is M from the faces of the angle?
2. Describe the loci called for in the following statements:
   (a) Points at a given distance from a given point; equidistant from two points; equidistant from three points.
   (b) Points at a given distance from a given line; equidistant from two lines; equidistant from three lines.
   (c) Points at a given distance from one given plane; equidistant from two planes; equidistant from three planes.
3. If a rectangle is rotated around one of its shorter sides as an axis, what is the locus of each of its longer sides?
4. The legs of a right triangle are 8 units and 11 units. If it is rotated about the longer leg as an axis what is the *length* of the locus of the mid-point of the hypotenuse?
5. A point X is 12 inches from plane m. What is the locus in plane m of all points 15 inches from X?
6. What is the locus of a point equidistant from the four corners of a 9′ x 12′ rug and also 4 feet from the floor?
7. A line 18 units long inclines 45° with plane y. How long is its projection on plane y? Explain your answer.
8. At what angle would the line in Ex. 7 be inclined to the plane to make its projection half as long as the line itself? Explain.
9. What is the projection of a point on a plane? Of a line on a plane?
10. What method of projection is discussed in this unit?

Keep the Iron Hot: SPECIAL TRIANGLES—EVERYDAY REASONING

Set your goal at 100 *percent accuracy.*

1. Find the altitude of an equilateral triangle whose side is 6 inches.
2. Find the area of the equilateral triangle in Ex. 1.
3. What relationship exists between the sides of a right triangle if one acute angle is 60°?
4. The hypotenuse of a right triangle is 18 units. If one angle is 30°, find the length of each leg.
5. Find the length of the hypotenuse of a right triangle if one leg is 7 inches and one angle is 45°.
6. If a represents the side of an equilateral triangle, find both the altitude and the area in terms of a.
7. Is an altitude of an equilateral triangle also a median of the triangle? Why or why not?
8. Where do the medians of a triangle meet? What other lines are concurrent?
9. Quote as many conclusions as possible about the diagonals of a square; of a rhombus.
10. If a side of a rhombus is 12 units long and two of its angles are in the ratio of 2 to 1, find the length of the diagonals.

Generalizations

In Exs. 16, 17, and 18 on page 46 there is a necessary conclusion which follows *logically* from the first two statements. However, it is obvious that each of the three conclusions is untrue and does not make good sense. You can easily see what the difficulty is. In each case the generalization upon which the reasoning was based was not sensible. It is therefore evident that the generalizations upon which we are to base deductive reasoning must be reasonable and sensible, if we want conclusions which are to be reasonable and sensible.

Not until a physician has observed many, many cases and done much experimentation in the laboratory does he venture to announce that a particular drug or a special treatment is effective for a certain disease. Even then he often qualifies his statement by saying that it is the result of the studies made so far. Any careful thinker does the same thing and avoids hasty generalizations.

Determine how valid the following statements are:

Most people do not like spinach.
Musicians are always temperamental.

Miss Smith is a poor teacher. (After a pupil makes a low grade on the first test in Miss Smith's class.)

That new boy is no help to our class. (After he didn't go out for football.)

World history is easy. (After a pupil made a good grade on the first test in world history.)

I'll never eat in that restaurant again. (After a poor lunch in the restaurant.)

Generalizations made on the basis of too few cases or without complete and accurate data are often untrue and misleading. Give several examples of poor generalizations that you have heard or read.

Discuss briefly the value of sound generalizations. Is the rule you had in algebra for squaring a binomial a generalization? For cubing a binomial? What statements in geometry are really generalizations? What are several generalizations you have made about how to act in certain social situations? Did you base them on your own and other people's observation of *many* specific cases?

CHAPTER 6 *Polyhedrons*

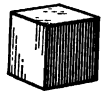
GENERAL POLYHEDRONS: INTRODUCTION

A **polyhedron** is a *solid bounded by planes.* These planes are called the **faces,** the intersections of the planes are the **edges,** and the intersections of the edges are the **vertices** of the polyhedron. A line joining any two vertices not in the same face is a **diagonal** of the figure.

A **section** of a polyhedron is the polygon formed by the intersection of a plane passing through it. If *every* polygon made by a cutting plane is convex, the polyhedron is said to be *convex.* Only convex polyhedrons will be considered in this book.

Polyhedrons are named according to the number of faces they have. What is the *least* number of faces possible? Why? Five **regular** polyhedrons are shown at the foot of this page. The one with **four** faces is a **tetrahedron;** with **six** faces a **hexahedron** (cube); with **eight** faces an **octahedron;** with **twelve** faces a **dodecahedron;** and with **twenty** faces an **icosahedron.**

Tetrahedron

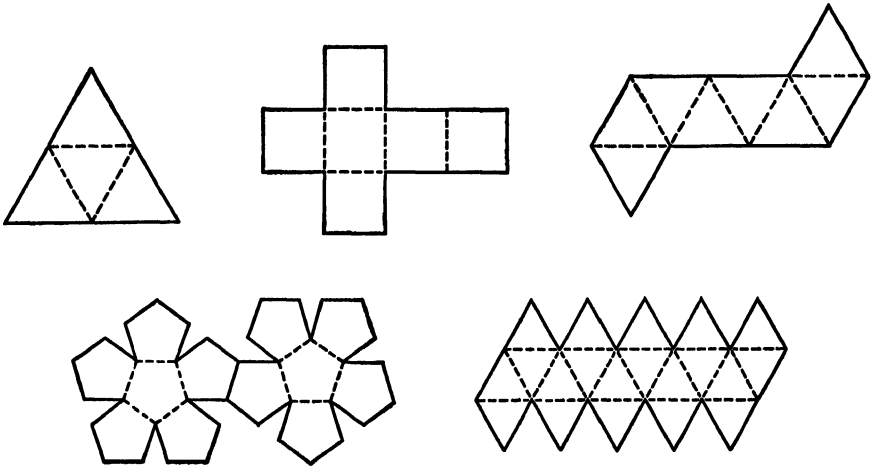


Cube

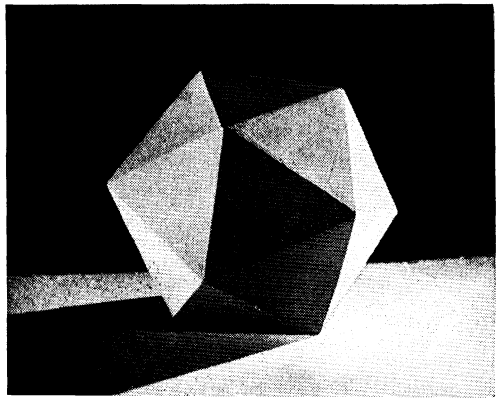
Octahedron

Dodecahedron

Icosahedron



The sketches given above show patterns for each of the five regular polyhedrons. If you construct carefully the plane figures indicated, cut them out, and fold along the dotted lines, the solid figures can be completed. Having a model of a tetrahedron and of a rectangular solid to manipulate as you study some of the proofs and problems in this chapter would be helpful.



"It" tells how often you will telephone. (Courtesy of the Bell Telephone Laboratories)

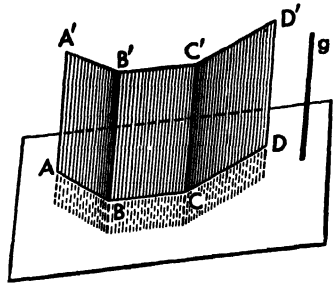
" 'It' is an icosahedron—a solid with twenty regular faces. The laws of probability say that if you roll a hundred icosahedrons on a table, eleven or more will come to rest with any one side on top only once in a hundred throws.

"Identical laws of probability rule the calls coming into your local Bell Telephone exchange. Suppose you are one of a group of a hundred telephone subscribers whose practice is to make one three-minute call each during the busiest hour of the day. The chance that eleven or more of you will be talking at once is also only one in a hundred. Thus it would be wasteful for the Bell System to supply your group with a hundred trunk circuits. Eleven trunks will suffice to give you good service."

The foregoing quotation from a publication of the Bell Telephone Laboratories tells what the telephone laboratories are doing in applying the *theory of probability*. The application of the theory is interesting in the light of our desire to have good telephone service at reasonable cost. If every subscriber had a separate trunk circuit, the cost of telephone service would be prohibitive. Moreover, it is to the advantage of everyone to use the costly telephone equipment wisely.

SECTION I. PRISMS

A **prismatic surface** is generated by a straight line which moves so that it **always** *intersects a broken line in a plane* and is *always parallel to a fixed line* **not** *in the plane*. Thus AA' moving always parallel to g and always intersecting line $ABCD$ generates the prismatic surface $A'D$. The moving line AA' that generates the surface is the **generatrix,** and in any of its positions is called an **element** of the surface. The broken line $ABCD$ that serves to direct the generation of the surface is called the **directrix.** If the broken line is a closed line, the surface is a **closed surface.**



A **prism** is a *polyhedron* formed by *two parallel planes* intersecting a *closed prismatic surface* and *cutting* **all** *the elements*. The various parts of a prism have names consistent with the general meanings of the words. The polygons made by the two parallel planes are the **bases.** The other faces are called the **lateral faces** of the prism, and the intersections of the lateral faces with each other are the **lateral edges.** The **altitude** of a prism is the *perpendicular distance between the bases*. The **lateral area** of a prism is the *sum* of the *areas* of the *lateral faces*, and the **total area** is the *sum* of the *lateral area and the areas of the two bases*.

Detroit skyline.

EXERCISES

1. From the definition of a prism, what relationship exists between the lateral edges? The corresponding sides of the bases?
2. Describe the shape of the lateral faces of a prism.
3. Using a triangle for a base, sketch two prisms, so that in one of them a lateral edge is *longer* than its altitude, and in the other, a lateral edge is *equal* to its altitude.
4. Is an ordinary rectangular box a prism? Why or why not?
5. How many faces does a prism have whose base is a square? How many edges? If the lateral faces are squares, what name does the solid have?
6. What is the limit of the number of sides for the base of a prism? Explain your answer.

Prisms are often classified according to their bases. A **triangular prism** has a *triangle* for a *base*; a **quadrangular prism** has a *four-sided base*; etc.

If the *base* of a prism is *perpendicular to an edge*, it is a **right prism.** Otherwise it is an **oblique prism.**

A **regular prism** is a *right prism* whose base is a *regular polygon*.

A **right section** of a prism is a section formed by a *plane cutting all the lateral edges* and *perpendicular to one of them*.

PRELIMINARY THEOREMS*

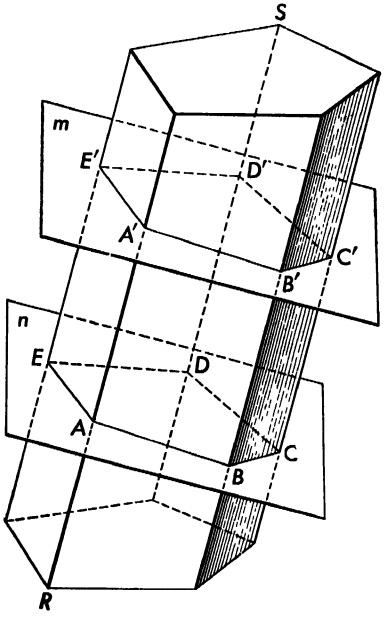
1. *The lateral edges of a prism are equal and parallel.*
2. *The lateral faces of a prism are parallelograms.*
3. *A right section of a prism is perpendicular to each of the lateral edges.*

*See footnote on page 49.

4. *The lateral faces of a right prism are rectangles.*
5. *The lateral edges of a right prism are altitudes.*
6. *The lateral edges of a right prism are perpendicular to both bases.*

THEOREMS

Theorem 28. *Sections of a prism formed by parallel planes cutting all the lateral edges are congruent polygons.*



Given: Prism RS with parallel planes m and n cutting all its lateral edges.
To prove: Section $ABCDE \cong$ section $A'B'C'D'E'$.

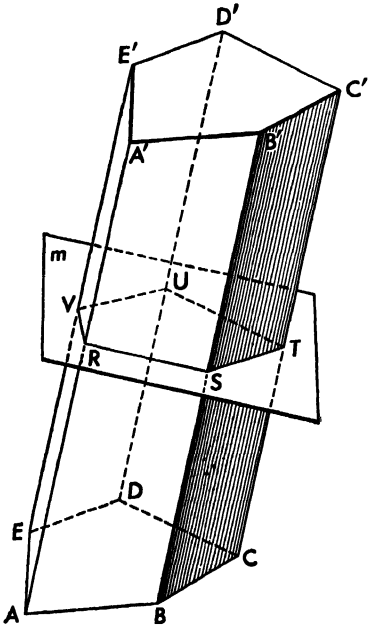
PROOF

Argument	*Defense*
1. m and n are $\parallel$ planes cutting all lateral edges of prism RS.	1. Given.
2. $AB \parallel A'B'$, $BC \parallel B'C'$, etc.	2. If two $\parallel$ planes are cut by a third plane, the lines of intersection are parallel.
3. $AA' \parallel BB' \parallel CC'$, etc.	3. Lateral edges of a prism are parallel.
4. $\angle CBA = \angle C'B'A'$, $\angle DCB = \angle D'C'B'$, etc.	4. Why?
5. $ABB'A'$, $BCC'B'$, etc., are $\square$.	5. Why?
6. $AB = A'B'$, $BC = B'C'$, etc.	6. Why?
7. $\therefore ABCDE \cong A'B'C'D'E'$	7. Definition.

Corollary 28a. *The bases of a prism are congruent polygons.*

Corollary 28b. *Every section of a prism made by a plane parallel to the base is congruent to the base.*

Theorem 29. *The lateral area of any prism is equal to the product of a lateral edge and the perimeter of a right section.*



Given: Prism AD' with right section $RSTUV$. Represent the lateral area of the prism by S, the length of an edge by e, and the perimeter of $RSTUV$ by p.
To prove: $S = ep$.

PROOF

Argument	*Defense*
1. Prism AD' with right section $RSTUV$.	1. Given.
2. $AA' = BB' = CC' = e$.	2. Why?
3. $ABB'A'$, $BCC'B'$, etc. are ▱.	3. Why?
4. $RS \perp AA'$, $ST \perp BB'$, etc.	4. Why?
5. Area of $ABB'A' = AA' \times RS$ or $e \times RS$. Area of $BCC'B' = BB' \times ST$ or $e \times ST$, etc.	5. Why?
6. $S = e \times RS + e \times ST + e \times TU + \ldots$, or $S = e(RS + ST + TU + \ldots)$	6. Why?
7. $p = RS + ST + TU + \ldots$.	7. Why?
8. $\therefore S = ep$.	8. Why?

Corollary 29a. *The lateral area (S) of a right prism equals the product of its altitude (h or e) and the perimeter (p) of its base.*

EXERCISES

1. Prove: A section of a prism made by a plane parallel to a lateral edge is a parallelogram.
2. The perimeter of the base of a certain right prism is 3 feet, and its altitude is 16 inches. Find the lateral area.
3. The right section of a 10-foot steel bar is a square 8 inches on a side. Find the lateral area.
4. Prove: Any lateral edge of a prism is parallel to the plane determined by any other two lateral edges.
5. Find the lateral area of a regular hexagonal prism each side of whose base is 6 inches, and whose altitude is 10 inches. Find its total area.

PARALLELEPIPEDS

A prism whose *bases are* **parallelograms** is called a **parallelepiped.** If its lateral edges are perpendicular to a base, it is a **right parallelepiped.**

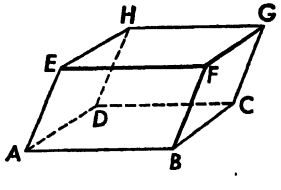
Visualize a right parallelepiped in which the *bases* are *rectangles.* It is generally called a *rectangular solid,* but may be called a **rectangular parallelepiped.** If all faces of a rectangular parallelepiped are squares, it is called a **cube.**

PRELIMINARY THEOREMS

1. *All faces of a parallelepiped are parallelograms.*
2. *All faces of a rectangular parallelepiped are rectangles.*
3. *All faces of a cube are squares.*
4. *All edges of a cube are equal.*

THEOREM

Theorem 30. *The opposite faces of a parallelepiped are congruent and parallel.*



Given: Parallelepiped AG.
To prove: $ABFE \cong DCGH$, $ABFE \parallel DCGH$, $BCGF \cong ADHE$, $BCGF \parallel ADHE$.

Proof left to you.

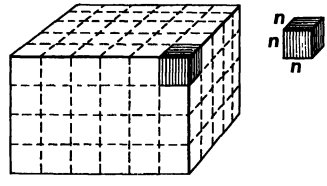
VOLUME

A **solid** is defined as a *closed surface enclosing a portion of space*, and the **volume** of a solid is the *measure* of that space. Therefore the volume of any solid is a *number* which tells how many units of a solid measure the figure contains. The unit we are accustomed to use is a *cube whose edge is one linear unit.*

Two solids are **congruent** if they *can be made to coincide* throughout all their parts.

Two solids are **equal, or equivalent**, if they have *equal volumes.*

The dimensions of a rectangular solid are the lengths of the three edges meeting at any vertex and are generally called *length*, *width*, and *height.*



If there is some number such as n, a whole number or a fraction, which is contained *evenly* in the length of *each* of the dimensions, it is a simple matter, as you learned in junior high school, to find the number of tiny cubes n units on a side which can be fitted into the solid. The *number* of units in the length multiplied by the *number* of units in the width multiplied by the *number* of units in the height will give the *number* of cubic units in the volume of the solid. Thus, you should be willing to accept the following assumptions:

Assumption 11. *The volume of a rectangular solid equals the product of its three dimensions (measured in the same unit)*, or $V = lwh$.

This assumption can be expressed a little differently to fit two special cases because obviously the same principle is involved. You may therefore use any one of the three statements which is most appropriate to the problem at hand.

Assumption 11a. *The volume of a rectangular solid equals the product of its base (area) and its height (altitude)*, or $V = Bh$.

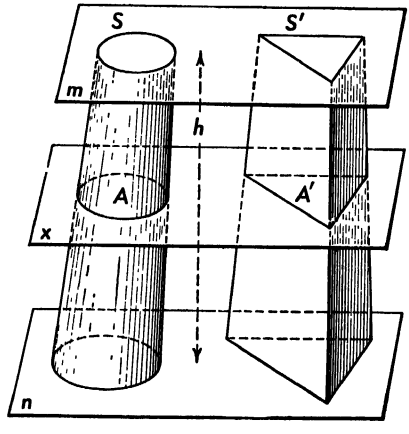
Assumption 11b. *The volume of a cube equals the cube of an edge*, or $V = e^3$.

SECTION II. CAVALIERI'S PRINCIPLE

Preparation for Theorem 31

Many great names stand out in the history of mathematical discovery during the fourteenth, fifteenth, and sixteenth centuries. During this period many of the separate discoveries of basic mathematical fact were synthesized and the fundamentals of analytic geometry, differential calculus, and integral calculus were developed into related but independent systems of

mathematical thought. While Sir Isaac Newton (1642–1727) was far more than an arranger of the ideas of other men, nevertheless, many of his discoveries were founded on the basic ideas developed by others. Even though there is still some controversy as to whether Newton or Gottfried Wilhelm Leibnitz (1646–1716) should be given the greater credit for developing the *calculus*, there is honor enough for both to share. The important point in this discussion is that Bonaventura Cavalieri (1598–1647), a Jesuit priest and pupil of Galileo, preceded both Newton and Leibnitz and propounded a principle that is basic to the calculus. Since proof of Cavalieri's theorem requires knowledge of the calculus, it will be treated informally here. Cavalieri's theorem states that *if two solids have their bases in the same two parallel planes and if equal sections are always formed by any plane passed parallel to the bases and between them, the solids have equal volumes.*



In the accompanying figure, two solids S and S' have their bases in parallel planes m and n. Note that neither the upper bases nor the lower bases of the two solids have the same shape, but their areas are given *respectively* equal. Cavalieri's theorem indicates that if any plane, such as x, is passed parallel to and between m and n to determine sections of the two solids, such as A and A', and if all such sections possible *have equal areas* respectively, the solids will then have equal volumes.

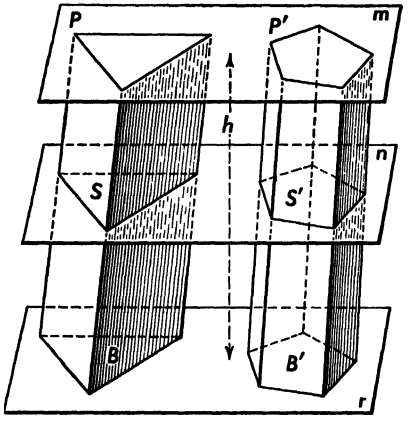
If S and S' are divided by plane x into small solids, obviously the sum of their volumes in either given solid will equal the volume of that solid.

Suppose x is permitted to cut sections of the two solids at such tiny intervals along h that a series of thinner than wafer-like volumes is determined in each solid. The altitudes of each of these tiny volumes will, *for all practical purposes*, be negligible. When the altitudes are so small that, for practical purposes, they can be ignored, each tiny solid can be regarded as a *section* of its given solid. Since corresponding sections of the two given solids are equal in area, the two corresponding wafer-like solids must be equal in volume. Thus, if the areas of *any* two sections of the given solids, as determined by the foregoing procedure, are equal, the volumes of the solids are equal. Therefore we may assume:

Assumption 12. *If two solids have their bases in the same two parallel planes and if* **equal** *sections are always formed by any plane passed parallel to the bases and between them, the solids have equal volumes.*

THEOREMS

Theorem 31. *If two prisms have equal bases and equal altitudes, they are equal in volume.*

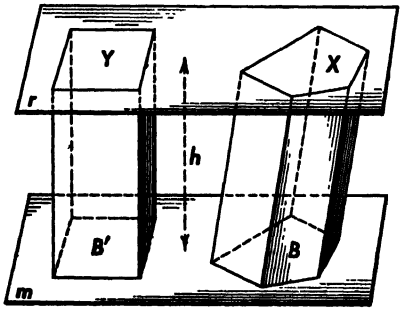


Given: Any two prisms, P and P', with altitude h, and with equal bases B and B' lying in plane r.
To prove: $P = P'$.

PROOF

Argument	*Defense*
1. Pass plane $m \parallel r$ so it will contain the upper bases of P and P'.	1. P and P' have the same altitude.
2. Pass *any* plane $n \parallel r$ forming sections S and S'.	2. Why possible?
3. $S = B$ and $S' = B'$.	3. Why?
4. $B = B'$.	4. Given.
5. $S = S'$.	5. Why?
6. $P = P'$.	6. See Assumption 12.

Theorem 32. *The volume of any prism equals the product of its base and its altitude.*



Given: Prism X with base B, altitude h, and volume V. Planes m and r are the planes of the bases.
To prove: $V = Bh$.

PROOF

Argument	*Defense*
1. Construct a rectangular solid Y having its bases in planes m and r and its base B' equivalent to B.	1. B can be reduced to an equivalent Δ and then a rectangle can be made equal to the Δ.
2. The altitude of each prism is h.	2. Definition of $\parallel$ planes.
3. Volume of Y = Vol. of X.	3. Why?
4. Volume of $Y = B'h$.	4. Why?
5. $\therefore$ volume of $X = B'h$, or $V = Bh$.	5. A quantity may be ....

Corollary 32a.* *The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equal triangular prisms.*

Suggestion: Compare the bases and the altitudes of the two prisms formed.

Corollary 32b. *Two prisms with equal bases have the same ratio as their altitudes.*

Corollary 32c. *Two prisms with equal altitudes have the same ratio as their bases.*

Corollary 32d. *Two prisms have the same ratio as the product of their bases and altitudes.*

*In the sequence of theorems followed in some textbooks, Cavalieri's theorem (Assumption 12) is not used. Cor. 32a is stated as a theorem and proved. It is followed by a theorem stating: *The volume of any parallelepiped is equal to the product of its base and altitude.* This theorem about a *special* kind of prism is later used to prove the more *general* theorem which you just proved as Th. 32. The proof of the theorem quoted above is not difficult, but it is long and involves a series of projections in such a way that a rectangular solid, whose volume has been assumed to be the product of its base and altitude, results. The rectangular solid is shown to be equivalent to the given parallelepiped. The resulting sequence of theorems is no more valid than the sequence followed in this text. However, your teacher may wish to introduce proofs for the following theorems in the order suggested. If this is done, our Assumption 12 cannot be used. Also note that corollaries for the following theorems are not stated.

1. *Two prisms are congruent if three faces which include a trihedral angle of one are congruent respectively to the three faces which include a trihedral angle of the other, and are similarly placed.*
2. *An oblique prism equals a right prism whose base is a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*
3. *If a plane is passed through two diagonally opposite edges of a parallelepiped, it divides the prism into two equal triangular prisms.*
4. See theorem stated above in this footnote.
5. *The volume of a triangular prism is equal to the product of its base and altitude.*
6. *The volume of any prism equals the product of its base and altitude.*

Corollary 32e. *The volume of any parallelepiped is equal to the product of its base and altitude.*

CRYSTALS

When studying plane geometry your attention was called to the various geometric forms found in nature such as the hexagon in the honey-comb, the five-pointed star in the cross section of an apple, and the many circular designs in flowers. In solid geometry it is interesting to note some of the regular polyhedral solids as they appear in the formation of crystals.

Water crystallizes into six-sided crystals which determine the shape of the snowflakes. Common salt usually forms in cubes, while quartz is found in hexagonal crystals, usually prisms. Sometimes the prism is capped at either end with a six-sided pyramid. Diamonds and graphite (pencil lead) are crystalline forms of ordinary carbon. The graphite crystallizes in hexagonal shapes, but the diamond in octahedral shapes.

Many substances in crystal form, when they are broken into smaller pieces, have a definite tendency to split along special lines so that almost all the new particles are in the same form. It is interesting to examine some of these very tiny crystals under a microscope to see the "family" shape. These "cleavage lines" are helpful in working with diamonds when flawed portions must be removed in preparing the stones for final cutting and polishing.

SECTION III. PYRAMIDS

A straight line which moves so that it always intersects *a fixed broken line in a plane*, and *also* **always** *passes through a fixed point* **not** *in the plane* generates a surface called a **pyramidal surface.** In this book we will consider only closed pyramidal surfaces, those formed when the directrix is a *polygon*.

In Fig. 1, AA' moves so that it always intersects polygon ABC and also passes through the *fixed point* P, called the **vertex.** The surface formed consists of two parts, one on each side of the vertex, each of which is called a **nappe** (năp). The moving line (generatrix) in any position is called an **element.**

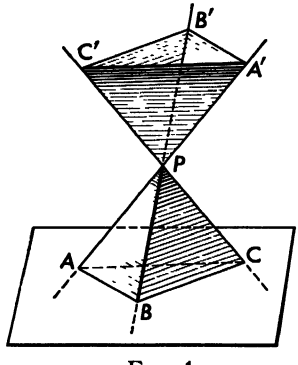


FIG. 1.

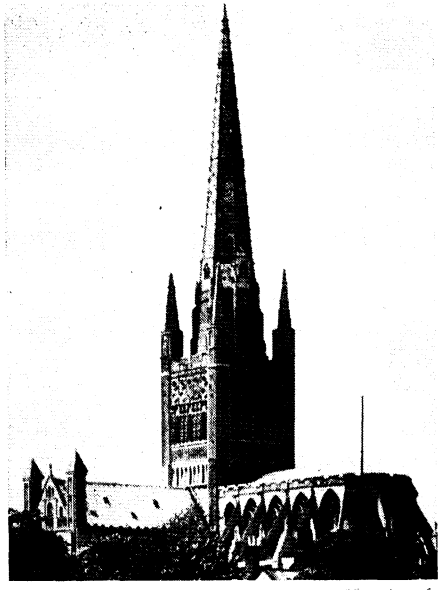
A **pyramid** is the solid formed by *one nappe* of a closed pyramidal surface cut off by a *plane* cutting **all** the *elements*. The *section* made by the cutting plane is the **base** of the pyramid, and the triangular faces are the

lateral faces of the pyramid. The intersections of the lateral faces with each other are the **lateral edges.**

The **altitude** of the pyramid is the *perpendicular distance* from the *vertex to the base.*

Pyramids, like prisms, are called *triangular, quadrangular, pentagonal,* and so on, according as their bases are triangles, quadrilaterals, pentagons, etc. A *triangular pyramid* is also called a **tetrahedron.** Each of its four faces is a triangle and any one of them can be the base.

A **regular pyramid** is one whose *base* is a *regular polygon,* and whose *altitude* meets the *center* point of the base. The **slant height** of a regular pyramid is the *altitude of one of the lateral faces.*



Cathedral, Norwich, Norfolk, England. (Courtesy of British Information Services)

In pyramid *P-ABCDEF,* in Fig. 2, name the altitude and a lateral edge. The base is what plane figure? How is the pyramid classified? What conditions would have to be met for it to be a *regular* pyramid? What would be the slant height? How many lateral faces are there?

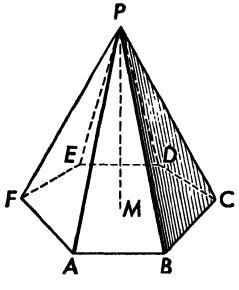


FIG. 2.

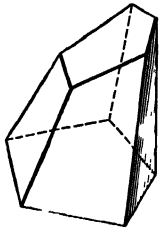


FIG. 3.

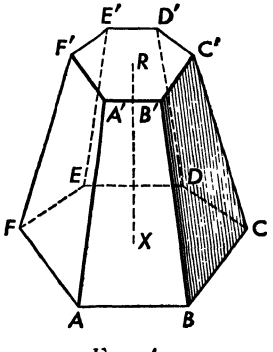


FIG. 4.

The part of a pyramid included *between the base* and **any** *plane* which cuts **all** the *lateral edges* is called a **truncated pyramid.** A *truncated prism* can be defined in the same way. If the cutting plane is parallel to the base, the solid is a **frustum of a pyramid.** Fig. 3 illustrates a truncated pyramid, and Fig. 4 shows a frustum of a pyramid. The **altitude** of a frustum of a pyramid

is the *perpendicular distance between the bases*, and its **slant height** is the *altitude of one of the lateral faces*. What are the bases of the frustum of the pyramid in Fig. 4? The altitude?

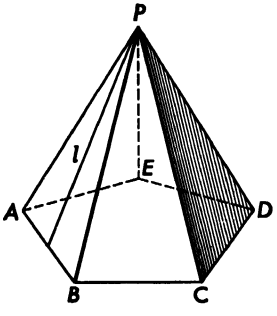
Lateral area and **total area** of pyramids have the same meanings as for prisms.

PRELIMINARY THEOREMS

1. *The lateral edges of a regular pyramid, or frustum of a regular pyramid, are equal.*
2. *The lateral faces of a regular pyramid are congruent isosceles triangles.*
3. *The slant heights of a regular pyramid, or frustum of a regular pyramid, are equal.*
4. *The lateral faces of a frustum of a pyramid are trapezoids.*
5. *The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.*

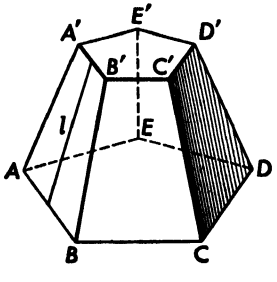
THEOREMS

Theorem 33. *The lateral area of a regular pyramid is equal to one-half the product of its slant height and the perimeter of its base.*



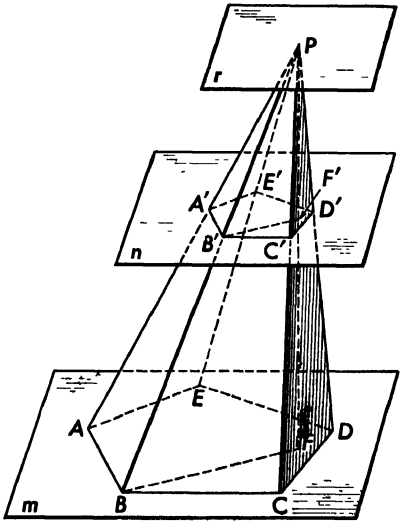
Given: Regular pyramid $P\text{-}ABCDE$ with slant height l, perimeter of the base p, and lateral area S.
To prove: $S = \frac{1}{2}(lp)$
Proof left to you.

Theorem 34. *The lateral area of a frustum of a regular pyramid is equal to one-half the product of its slant height and the sum of the perimeters of the bases.*



Given: The frustum of a regular pyramid with bases $ABCDE$ and $A'B'C'D'E'$ and slant height l.
To prove: The lateral area $S = \frac{1}{2}l(p + p')$ when p and p' represent the perimeters of the bases.
Proof left to you.

Theorem 35. *If a pyramid is cut by a plane parallel to the base, (a) the lateral edges and the altitude are divided proportionally, and (b) the section formed is similar to the base.*



Given: Pyramid $P\text{-}ABCDE$ with plane n parallel to plane m of the base and cutting the lateral edges in points A', B', C', D', and E'. PF is the altitude.

To prove: (a) $\frac{PA'}{PA} = \frac{PB'}{PB} = \frac{PC'}{PC} = \frac{PD'}{PD} = \frac{PE'}{PE} = \frac{PF'}{PF}$

(b) $A'B'C'D'E' \sim ABCDE$.

Suggestions: (a) Pass a plane r through $P \parallel m$. What is true if two lines are cut by three $\parallel$ planes? (b) Show that $\triangle PA'B' \sim \triangle PAB$, that $\triangle PB'C' \sim \triangle PBC$, $\triangle PB'F' \sim \triangle PBF$, etc., and that $\frac{A'B'}{AB} = \frac{PB'}{PB} = \frac{B'C'}{BC}$, and so forth.

Proof left to you.

Corollary 35a. *The area of a section of a pyramid parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

Suggestions: In $\triangle PF'B'$ and PFB,

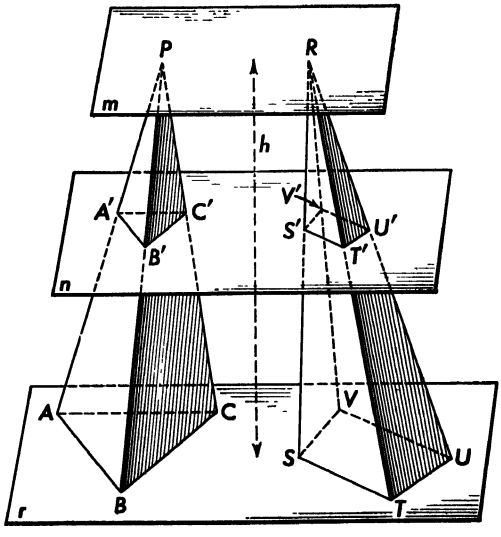
$$\frac{PF'}{PF} = \frac{PB'}{PB}, \text{ but } \frac{PB'}{PB} = \frac{A'B'}{AB}$$

$$\frac{\overline{PF'}^2}{\overline{PF}^2} = \frac{\overline{A'B'}^2}{\overline{AB}^2}$$

$$\therefore \frac{A'B'C'D'E'}{ABCDE} = \frac{\overline{PF'}^2}{\overline{PF}^2}$$

Corollary 35b. *If two pyramids have equal altitudes and equal bases, sections of the pyramids parallel to the bases and equidistant from the vertices are equal.*

Theorem 36. *If two pyramids have equal altitudes and equal bases, their volumes are equal.*



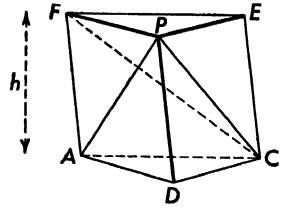
Given: Pyramids $P\text{-}ABC$ and $R\text{-}STUV$ with equal altitudes and with $ABC = STUV$.

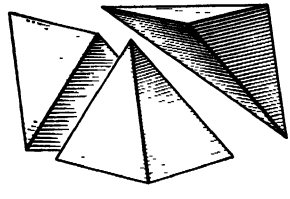
To prove: Volume of $P\text{-}ABC$ = volume of $R\text{-}STUV$.

Suggestions: Since the altitudes of the two pyramids are equal, they can be placed so that their vertices lie in a plane m parallel to a plane r containing their bases. Pass a plane n parallel to m and it will intersect the pyramids in sections $A'B'C'$ and $S'T'U'V'$ respectively. How will these sections compare? Since n represents *any* plane that could be passed through the pyramids parallel to the bases, what is true of the volumes?

Proof left to you.

Theorem 37. *The volume of a triangular pyramid equals one-third the product of its base and altitude.*





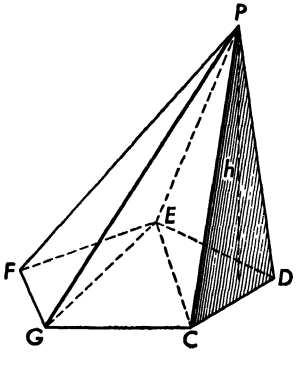
Given: Triangular pyramid $P\text{-}ADC$ with V its volume, h its altitude, and base B.

To prove: $V = \frac{1}{3}Bh$.

PROOF

Argument	*Defense*
1. Use ACD as a base and PD as an edge to construct a prism $PFE\text{-}ACD$.	1. Why possible?
2. Planes PAC and FPC divide the prism into three pyramids $P\text{-}ACD$, $P\text{-}FAC$, and $P\text{-}CEF$, whose sum equals the whole prism.	2. Explain.
3. $ACEF$ is a $\square$.	3. Why?
4. $\triangle AFC \cong \triangle FCE$.	4. Why?
5. $P\text{-}ACF$ and $P\text{-}FCE$ have $=$ bases, and $=$ altitudes, and therefore $=$ volumes.	5. Why?
6. $P\text{-}ACD$ and $C\text{-}FPE$ have $=$ bases, and $=$ altitudes, and therefore $=$ volumes.	6. Why?
7. But $P\text{-}FCE$ is $C\text{-}FPE$.	7. By identity.
8. $\therefore P\text{-}ACD = P\text{-}ACF = P\text{-}FCE$.	8. Why?
9. The volume of prism $PFE\text{-}ACD = Bh$.	9. Why?
10. $\therefore V = \frac{1}{3}Bh$.	10. Why?

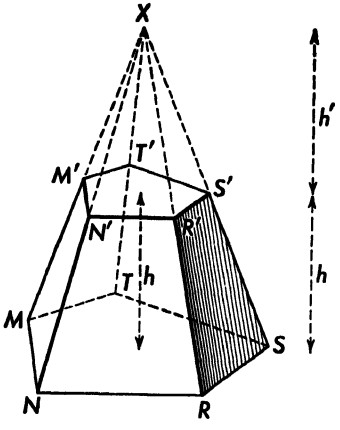
Theorem 38. *The volume of any pyramid equals one-third the product of its base and altitude.*



Given: Pyramid $P\text{-}CDEFG$ with base B, altitude h, and volume V.
To prove: $V = \frac{1}{3}Bh$.

Suggestion: Divide the pyramid into triangular pyramids.
Proof left to you.

Corollary 38a. *The volume of a frustum of a pyramid is given by the formula*



$V = \frac{1}{3}h(B + b + \sqrt{Bb})$ *when* B *and* b *are the areas of the bases and* h *is the altitude.*

Suggestions: Extend the lateral edges of the frustum to form a pyramid. Subtract the volume of the small pyramid $X\text{-}M'N'R'S'T'$ from the volume of the large pyramid $X\text{-}MNRST$. By Cor. 35a, $\frac{b}{B} = \frac{h'^2}{(h + h')^2}$. Solve for h' and make the appropriate substitution.

Corollaries 38b, 38c, and 38d. State and prove three corollaries that are related to Th. 38 in the same way as Cors. 32b, 32c, and 32d are related to Th. 32.

SECTION IV. SPECIAL POLYHEDRONS

SIMILAR POLYHEDRONS

In plane geometry similar polygons were defined as two polygons whose corresponding angles are equal and whose corresponding sides are proportional. In this course we say that two polyhedrons which have the *same number of faces similar each to each and similarly placed* and that have *their corresponding polyhedral angles equal* are **similar polyhedrons.** Explain how these are parallel definitions.

Which of the following plane figures are always similar or are similar under certain special conditions: two squares, two rectangles, two equilateral triangles, and two regular hexagons?

Are any of these polyhedrons similar: two cubes, two rectangular solids, two tetrahedrons, two hexagonal prisms, two pentagonal pyramids?

How do the altitudes of two similar triangles compare? How do the areas of two circles compare? How do the areas of two similar triangles compare?

THEOREMS*

Theorem 1. *Two similar polyhedrons can be separated into the same number of tetrahedrons similar each to each and similarly placed.*

Corollary 1a. *Any two corresponding line segments of two similar polyhedrons have the same ratio as any other two corresponding lines.*

Corollary 1b. *The total areas of two similar polyhedrons have the same ratio as the squares of any two corresponding lines (or line segments).*

Corollary 1c. *The volumes of any two similar polyhedrons have the same ratio as the cubes of any two corresponding line segments.*

Corollary 1d. *The squares of the volumes of two similar polyhedrons have the same ratio as the cubes of any two corresponding areas.*

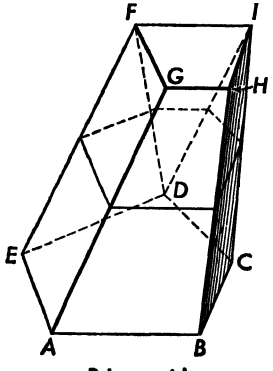
EXERCISES

1. If it costs $42 to paint a certain rectangular solid, how much would it cost to paint a similar solid if its length were twice the length of the first one?
2. How will the total areas of two similar polyhedrons compare if the edges are in the ratio of 2 to 3? How will their volumes compare?
3. An edge of a regular polyhedron is 4 inches. What will be the length of an edge of another similar regular polyhedron if their volumes are in the ratio of 1 to 8?
4. The total area of one polyhedron is 256 square inches and one edge is 12 inches. What is the total area of a similar polyhedron whose corresponding edge is 8 inches?
5. It is desired to have a polyhedron with a total area four times as great as that of a given similar polyhedron. If an edge of the first polyhedron is 3 units, what must be the length of a corresponding edge of the second?

*Since the following two theorems are not used in our main sequence, they are assumed here and presented for class discussion. No formal proofs need be attempted.

THE PRISMATOID FORMULA

A polyhedron all of whose *vertices* lie *in two parallel planes*, and whose *lateral faces* are *triangles*, *trapezoids*, or *parallelograms* is called a **prismatoid.** Its **altitude** is the *perpendicular distance* between the two parallel planes, and its **mid-section** is a section made by a *plane parallel to and halfway between* the bases. On the basis of the definition, must the bases of a prismatoid have the same number of sides?



Prismatoid

Theorem 2. *The volume of a prismatoid is given by the formula* $V = \frac{1}{6}h(B + b + 4M)$ *when* B *and* b *are the areas of the bases, and* M *is the area of the mid-section.*

EXERCISES

1. According to the definition, which of the following solids can be thought of as special cases of the prismatoid: a prism, a truncated prism, a pyramid, a frustum of a pyramid, a truncated pyramid?
2. How do the bases of a prism compare? How does the area of each base compare with the area of the mid-section? What, then, will the prismatoid formula become for a prism?
3. What will the prismatoid formula become for a pyramid?

REGULAR POLYHEDRONS

At the beginning of this chapter five regular polyhedrons were pictured and patterns suggested for constructed models. If you did not construct the models, it is urged that you do so now and examine them as you study the following discussion.

A **regular polyhedron** is one whose *faces are congruent regular polygons*, and whose *polyhedral angles are equal.* Therefore, the equal polyhedral angles formed at each vertex of a regular polyhedron will be formed by *three or more* equal face angles. If the faces of a regular polyhedron are triangular, each face is an _?_ _?_, each angle of which is _?_ degrees. If the faces are quadrilaterals, each is a _?_, each angle of which is _?_ degrees.

There are ? degrees in each angle of a regular pentagon; of a regular hexagon. The formula to find the number of degrees in an interior angle of *any* regular polygon is . . .. As the number of sides of a regular polygon increases, the size of each of its interior angles ?.

In Th. 24 (page 58) you proved that the sum of the face angles of any convex polyhedral angle is less than 360°. With this fact and the foregoing discussion in mind, work the following exercises.

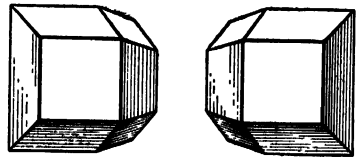
EXERCISES

1. If the face angles meeting at one vertex of a regular polyhedron are each 60°, how many face angles *could* there be at the vertex? How many different polyhedral angles with face angles of 60° are *possible*? Does one of the regular polyhedrons represent each of these possible cases? (Check your models.) Describe the polyhedron or polyhedrons possible.
2. If the face angles meeting at one vertex are each 90°, how many face angles *could* there be at each vertex? Describe the resulting polyhedron or polyhedrons.
3. Repeat Ex. 2 for face angles of 108° and 120°.
4. In trying to determine the number and kind of regular polyhedrons possible, only face angles of 60°, 90°, 108°, and 120° were tested. Would it be contrary to known facts, and therefore ridiculous, to test other angles between 0° and 120°? Angles greater than 120°? Explain.
5. On the basis of the above exercises, are you willing to agree that *there cannot be* **more than** *five regular polyhedrons*? Does the above inductive reasoning and experimentation convince you that there actually *are* five regular polyhedrons possible?
6. Explain why a regular polyhedron cannot be constructed with hexagonal faces.

Examine the model of each of the regular polyhedrons and complete the following table. See if you can find a relationship between the *sum of* the number of *vertices and faces* and the number of *edges*. Express the relationship as a formula. Check to see that the formula holds good for each of the five regular polyhedrons. Does it apply to *any* polyhedron? Test several cases.

Name	*Number of Edges (E)*	*Number of Vertices (V)*	*Number of Faces (F)*
Tetrahedron			
Cube			
Octahedron		**Copy this form on paper**	
Dodecahedron			
Icosahedron			

The crystals of tartaric acid have an unusual shape. Some of them have their parts arranged in opposite order and the faces of two such polyhedrons would be symmetric rather than congruent. Does the formula* $E + 2 = V + F$, which you should have discovered above, hold for this type of polyhedron?



Tartaric Acid Crystals.

CHAPTER REVIEW

Complete the statements in Exs. 1 -15.

1. A prism whose lateral edges are ? to its bases is called a ? ?.
2. The polygon formed by a plane cutting all the lateral edges of a prism and perpendicular to the edges is called a ? ?.
3. A prism whose ? are ? is called a parallelepiped.
4. A right parallelepiped is . . ..
5. A regular prism is a ? prism whose base is a ? polygon.
6. If two prisms have equal ? and ? ?, their volumes are equal.
7. Two prisms with ? altitudes have the same ? as their bases.
8. The plane passed through two diagonally opposite edges of a parallelepiped . . ..
9. The area of a section of a pyramid parallel to the base is to the area of the base as the . . ..
10. If a pyramid is cut by a plane parallel to the base, then the lateral edges and the . . ..
11. If two or more parallel planes cut all the lateral edges of a given prism, the sections formed are ? ?.
12. If two pyramids have ? altitudes and ? bases, their ? are equal.
13. Two pyramids with equal ? have the same ratio as their ?.
14. The volume of a parallelepiped is . . ..
15. The opposite faces of a parallelepiped are . . ..

Do not write in your book.

*This relationship is known as Euler's theorem (or formula) and is named for Léonard Euler who lived in the eighteenth century. However, historians point out that René Descartes, who lived a century earlier, used the relationship in his writings.

Some of you have met Euler before in connection with "Euler's circles" used to analyze syllogisms in plane geometry.

16. Write formulas for:

(a) Lateral area of a right prism
(b) Volume of any prism
(c) Lateral area of a regular pyramid
(d) Volume of any pyramid
(e) Volume of a cube
(f) Volume of a rectangular solid
(g) Total area of a cube
(h) Total area of a rectangular solid

17. The edge of one cube is twice that of a second cube. Compare their volumes, their total areas.

18. Describe a regular pyramid.

19. Compare a truncated pyramid and a frustum of a pyramid.

20. How do you find the lateral area of a frustum of a regular pyramid?

21. If a pyramid is cut by a plane parallel to the base, describe the section formed. What does the plane do to the lateral edges and the altitude?

22. If two prisms have equal bases, but the altitude of the first one is one-third the altitude of the second, how do the volumes compare?

23. Give the most precise name possible for a polyhedron whose faces are:

(a) Six squares
(b) Four equilateral triangles
(c) Six rectangles
(d) Three rectangles and two triangles
(e) Six triangles and a hexagon
(f) Four rectangles and two parallelograms
(g) Six parallelograms
(h) Five trapezoids and two pentagons
(i) Six parallelograms and two hexagons
(j) Four isosceles triangles and a square

24. Find the volume of a cube if the diagonal of a face is 12 inches.

25. Find the total surface of a rectangular block of steel whose dimensions are 12 feet by 1 foot by 8 inches.

26. (a) How many square feet of metal are needed to make a rectangular tank 6 feet long, $2\frac{1}{2}$ feet wide, and 15 inches deep, allowing 8 percent for waste?

(b) How many gallons of water will the tank hold? (One cubic foot holds $7\frac{1}{2}$ gallons.)

27. All right sections of the same prism are congruent polygons.

28. Bring your **Tool Box** up to date.

EXERCISES

Prove the first ten exercises.

1. The diagonals of a parallelepiped bisect each other.
2. The area of the mid-section (a section halfway between the vertex and the base) of a pyramid equals one-fourth the area of the base.
3. The diagonals of a rectangular parallelepiped are equal.
4. The square of a diagonal of a rectangular parallelepiped is equal to the sum of the squares of the three edges that meet at any one vertex.
5. In any regular tetrahedron the sum of the face angles at any vertex is equal to two right angles.
6. The diagonals of a cube divide the cube into six congruent pyramids.
7. The lines joining the mid-points of the opposite edges of a tetrahedron meet in a point.
8. If from any point within a regular tetrahedron perpendiculars are drawn to the faces, their sum is equal to an altitude of the tetrahedron.
9. The plane which passes through an edge of a tetrahedron and the mid-point of the opposite edge divides the tetrahedron into two equivalent tetrahedrons.
10. In any parallelepiped the sum of the squares of the four diagonals is equal to the sum of the squares of the twelve edges.
11. The volume of a regular pyramid with a square base is 1,296 cubic feet and the altitude is 12 feet. Find the lateral area.
12. The base of a pyramid is a triangle whose sides are 6 inches, 8 inches, and 10 inches. If the volume is 360 cubic inches, find the height.
13. The base of a right prism is a rhombus 10 inches on a side. If the height is 18 inches and the shorter diagonal of the base is 12 inches, find the total surface of the prism and its volume.
14. In a certain cube the number of square inches in the total surface equals the number of cubic inches in the volume. Find the length of an edge of the cube.
15. The volumes of two rectangular solids are 20 cubic feet and 45 cubic feet. If their altitudes are equal and the base of the second solid is 9 square feet, find the base of the first.
16. The ordinary building brick is 8 inches long, 4 inches wide, and 2 inches thick. Find the number of bricks in a stack 4 feet by 8 feet by 12 feet, assuming there is no space between the bricks.
17. How much earth was removed in an excavation which is 6 feet deep, 44 feet square at the top, and 36 feet square at the bottom?
18. Small boxes of note paper are 6 inches long, 4 inches wide, and $1\frac{1}{4}$ inches deep. How many of them will fit in a large carton 3 feet by 2 feet and 20 inches high?

19. How many cubic yards of dirt will be required to fill in a lot 75 feet wide and 120 feet deep if the fill is to be approximately 3 feet deep at the back and 1 foot at the front, and the slope is to be uniform?
20. A covered box made of steel $\frac{1}{8}$ inch thick has the dimensions 18 inches by 8 inches by 6 inches. If these are *inside* dimensions and the steel weighs 490 pounds per cubic foot, how much does the box weigh?
21. A covered box made of wood $\frac{3}{4}$ inch thick is 3 feet long, 2 feet wide, and $1\frac{1}{2}$ feet deep, inside measurements. If it is to be painted inside and out, how many square feet of surface must be covered?
22. Find the lateral area of a lampshade in the form of the frustum of a hexagonal pyramid, if the sides of the bases are 7 inches and 5 inches respectively and the altitude is 9 inches. (First find the slant height or altitude of one side.)
23. If the edge of a cube is x units, find the length of a diagonal in terms of x.
24. If the diagonal of a cube is d units, find the length of an edge in terms of d.
25. Find the lateral edge of a regular tetrahedron whose altitude is 10 units.
26. In each of the following, two of the three unknowns of Euler's formula are given. Find the third and describe the polyhedron that would have the parts indicated.

    (a) $V = 8, F = 6$
    (b) $E = 6, V = 4$
    (c) $E = 9, F = 5$
    (d) $V = 10, E = 15$
    (e) $F = 5, V = 5$

EXERCISES: JUDGING RELATIONSHIPS

Read the following statements, filling the blanks with *always*, *sometimes*, or *never*, whichever word best applies.

1. Any two faces of a pyramid are ? congruent triangles.
2. Any two lateral edges of a prism are ? parallel.
3. The altitude and a lateral edge of a prism are ? parallel.
4. A plane which intersects a prism and is parallel to its base ? forms a section congruent to the base.
5. A prism and a pyramid with equal altitudes and congruent bases are ? equal in volume.
6. A plane perpendicular to an edge of a prism ? forms a section congruent to the base.
7. A plane which intersects a pyramid and is parallel to the base ? forms a section similar to the base.

8. The volume of a cube with a base equivalent to the base of a given pyramid is ? greater than the volume of the pyramid.
9. The volume of a prism with a square base 6 inches on a side is ? three times as great as the volume of a pyramid which has a rectangular base 12 inches by 3 inches.
10. Any two lateral edges of a regular pyramid are ? equal.
11. The altitude and a lateral edge of a regular prism are ? parallel.
12. If a plane parallel to the base of a regular pyramid bisects the altitude, the area of the section formed is ? one-half the area of the base.
13. The lateral faces of a pyramid with a square base are ? triangles.
14. If the altitudes of a prism and a pyramid are equal and the bases are similar polygons, the volumes are ? equal.
15. The faces of a parallelepiped are ? congruent.
16. Any two sections of a prism made by parallel planes are ? congruent.
17. Any two sections of a pyramid made by parallel planes are ? congruent.
18. Any two sections of a pyramid made by parallel planes are ? similar.
19. The lateral faces of a frustum of a pyramid are ? isosceles trapezoids.
20. The altitude and a lateral edge of a prism are ? equal.
21. The lateral faces of a prism are ? rectangles.
22. If two pyramids have equal bases, but the altitudes are in the ratio of 1 to 3, then the volumes are ? in the ratio of 1 to 3.
23. If a pyramid and a prism have equal altitudes, the volume of the pyramid is ? one-third the volume of the prism.
24. The volume of an oblique prism is ? equal to the volume of a right prism with the same base.
25. If the base of one prism is twice the base of another prism but the altitude of the first prism is one-half the altitude of the second, the volume of the first is ? equal to the volume of the second.
26. A regular tetrahedron is ? a regular pyramid.
27. A regular pyramid whose base is an equilateral triangle is ? a regular tetrahedron.

CHAPTER TEST

1. Find the volume of a rectangular parallelepiped if its dimensions are 6, 8, and 10 units. Find its total surface.
2. Two prisms with equal altitudes have volumes of 64 cubic inches and 16 cubic inches. If the area of the base of one is 8 square inches, what is the base of the other?
3. Find the diagonal of a cube with an 8-inch edge.
4. The volume of a prism is 480 cubic inches. The altitude is 10 inches, and the base is an equilateral triangle. Find the length of one side of the base.

5. Find the edge of a cube whose diagonal is 24 units.

6. Find the lateral area of a regular pyramid whose base is a hexagon 5 inches on a side, and whose slant height is 8 inches.

7. Find the total area of the pyramid in Ex. 6.

8. Two pyramids with equal bases have volumes of 35 and 63 cubic feet respectively. The altitude of the first one is 3 feet. Find the altitude of the second pyramid.

9. Find the lateral area of the frustum of a regular octagonal pyramid whose slant height is 12 feet and the sides of whose bases are 16 and 28 feet respectively.

10. The base of a regular pyramid is a square 8 inches on a side. The altitude is 12 inches and a plane parallel to the base cuts it 4 inches below the vertex. Find the area of the section formed.

11. Find the volume of a regular triangular prism whose altitude is 18 inches and one side of whose base is 10 inches.

12. Find the volume of a prism whose altitude is 8.6 feet, whose slant height is 11 feet, and the area of whose base is 30 square feet.

13. An open tank is 8 feet long, 3 feet wide, and $2\frac{1}{2}$ feet deep. How much water will it hold? (One cubic foot holds $7\frac{1}{2}$ gallons.) Find the weight of the water if a cubic foot of water weighs $62\frac{1}{2}$ pounds.

14. Find the number of cubic yards of dirt which would have to be removed in digging a cellar 30 feet long, 22 feet wide, and 7 feet deep.

15. Given: X-$ABCDEF$ is a regular hexagonal pyramid cut by plane r which is parallel to the base. Altitude XM is 12 inches and AB is 6 inches.

(a) Find the area of the base.

(b) Find the area of section $A'B'C'D'E'F'$ if plane r bisects the altitude.

(c) What is the area of the section if plane r is one-third of the way from X to M?

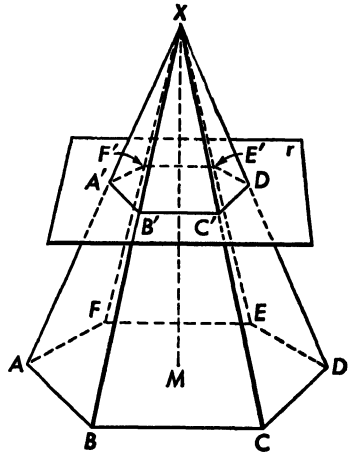
(d) Find the length of an edge of the pyramid.

(e) Find the slant height of the pyramid.

(f) Using the value of the slant height secured in Ex. 15 (e), compute the lateral area of the pyramid.

(g) Find the volume of the pyramid.

(h) What would be the volume of a regular prism with the same base and the same altitude?



(i) What would be the volume of a second pyramid whose base was twice the base of pyramid X-$ABCDEF$ and whose altitude was twice the altitude of pyramid X-$ABCDEF$?

(j) At what distance, approximately, from the vertex X would plane r cut the altitude so that the area of the section would be exactly one-half the area of the base?

(k) If $\angle CBA$ is 120°, how large is $\angle C'B'A'$? Why?

(l) If $A'B'$ is 4 inches, how long is $B'C'$? Why?

(m) If the altitude is divided by plane r into segments having the ratio of 2 to 3, what will be true of the segments of an edge? Why?

(n) If plane r cuts off segments of 3 inches and 2.5 inches on a lateral edge and the altitude respectively, find the length of the edge. (Use: altitude = 12 inches.)

(o) Is XM perpendicular to the section? Why or why not?

Keep the Iron Hot

Types of accidents resulting in deaths and injuries on highways and streets in the United States in a recent year

	Number of Persons Killed	*Number of Persons Injured*
Collision with:		
Pedestrian	10,940	184,440
Automobile	10,410	838,650
Horse-drawn vehicle	40	2,810
Railroad train	1,170	6,410
Street car	100	17,970
Other vehicle	190	5,730
Fixed object	3,940	125,880
Bicycle	630	21,790
Non-collision	4,620	154,590
Miscellaneous	460	6,730
Total	32,500	1,365,000

Answer the following questions from the data in the chart.

1. What was the average number of people injured per month? Per day? Per hour?

2. What was the average number of deaths per month? Per day? Per hour?

3. What percent of the deaths were caused by colliding with a pedestrian?

4. What percent of the injuries were caused by collision with an automobile?
5. What other conclusions can be drawn from these data?
6. What is the ratio of the number of deaths to the number of injuries?
7. If 7 percent of the pedestrians killed or hurt were children playing in the street, how many children were killed or hurt?
8. Of the 10,940 pedestrians killed, 3,190 of them were killed while crossing a street between intersections. Find the percent.
9. Nineteen percent of the pedestrians were killed while walking on a rural highway. How many pedestrians were killed on rural highways?
10. Try to get information so that you can compare the number of injuries arising from automobile accidents with those resulting from accidents in the homes of the people in the U. S.

In the year preceding the one upon which these figures are based, drivers under the age of twenty-five were involved in a total of 298,970 automobile accidents. In the following year they were involved in 360,420 automobile accidents. What is the percent of increase?

CHAPTER 7 *Cylinders and Cones*

SECTION I. CYLINDERS

In Fig. 1, *curve* ABC lies in plane s. FX is a fixed line not in the plane. MN is parallel to FX and is permitted to move so that it will always intersect the curve ABC and also so that it will always remain parallel to FX. The figure that is generated by this procedure is called a *cylindrical surface.* The moving line (MN) that *generates* the surface is called the **generatrix,** and the curve (ABC) that serves to *direct* the path of the generatrix is called the **directrix.** *Any* position that MN (the generatrix) assumes during the process is called an **element** of the surface. Thus MN, OP, QR, or *any* line in the cylindrical surface parallel to them, is an *element.* Summarizing, when *a moving straight line* **always** *intersects a plane curve* and *remains parallel to a fixed line not lying in the plane of the curve,* a **cylindrical surface** is generated.

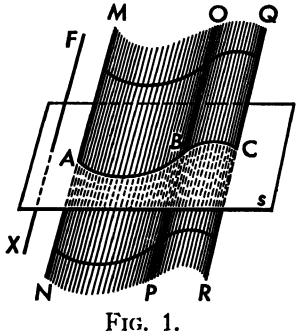
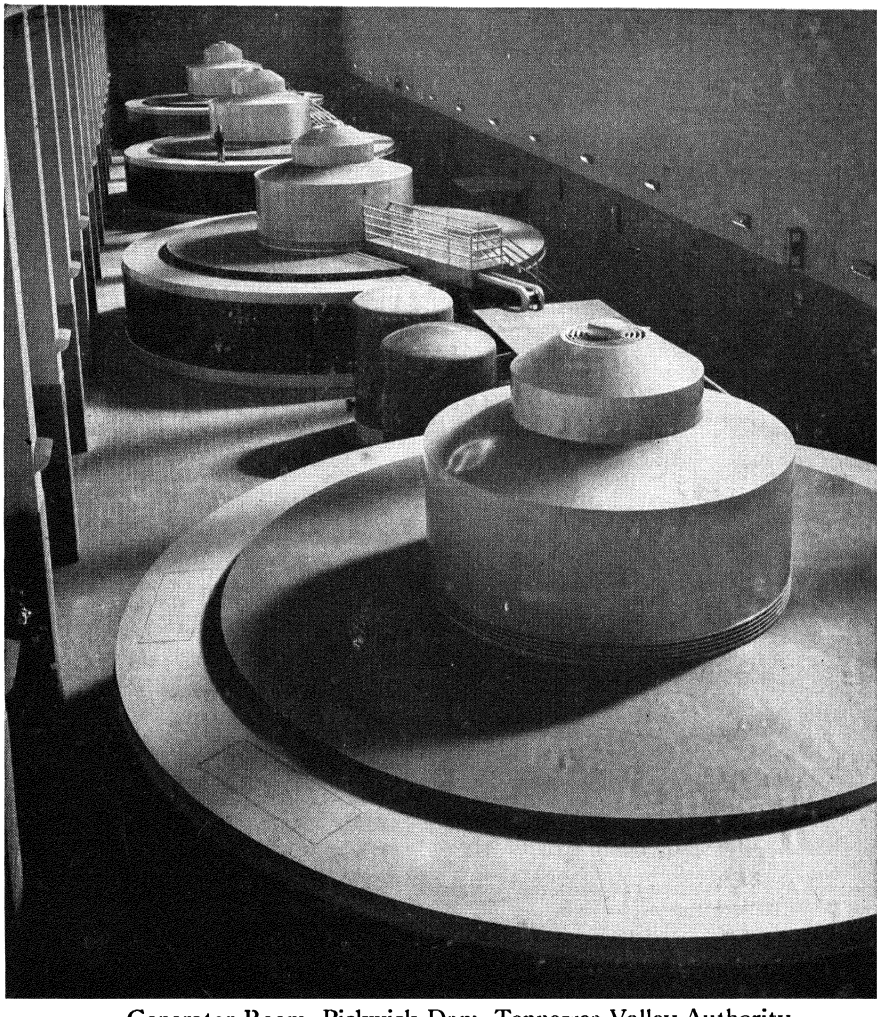


FIG. 1.

If the directrix of a cylindrical surface is a *closed curve,* the surface generated will be a **closed cylindrical surface.**

A **cylinder** is *a solid* determined by *a closed cylindrical surface* and *two parallel planes* that *cut* **all** *the elements of the surface.* The sections determined by the two parallel planes are the **bases** *of the cylinder,* and the cylindrical surface is the **lateral surface** *of the cylinder. The lateral area of the cylinder* is the *area of its lateral surface.* The *perpendicular distance between the two* bases is the **altitude** *of the cylinder.*



Generator Room, Pickwick Dam, Tennessee Valley Authority.

A **right section** *of a cylinder* is a section formed by a *plane intersecting all elements* and *perpendicular to one of them.*

If the bases of a cylinder *are perpendicular to an element,* it is a **right cylinder.** Otherwise it is an **oblique cylinder.**

If the bases are circles the cylinder is a **circular cylinder.** *The line joining the centers of the bases* is the **axis** *of the cylinder.*

On the basis of the foregoing definitions, what is a *right circular cylinder*? Hereafter, unless otherwise indicated, understand the word *cylinder* to mean *circular cylinder.*

A right circular cylinder may also be regarded as a *cylinder of revolution*. In Fig. 2, $ACDB$ is given as a rectangle. If AC, DB, and CD are permitted to revolve (rotate) about AB as an axis, AC and BD will generate parallel circular bases and CD will generate a cylindrical surface.

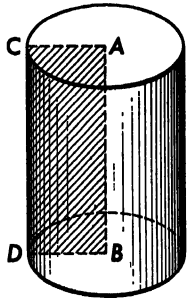


FIG. 2.

PRELIMINARY THEOREM

The elements of a cylinder are parallel and equal.
Develop an informal proof.

EXERCISES

1. Review items 3 and 7 on suggestions for making perspective drawings on pages 8–9.

2. In the following, draw figures as required:

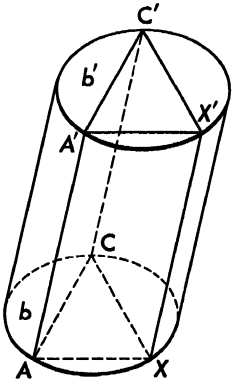
(a) A cylinder whose bases are irregular.
(b) An oblique circular cylinder.
(c) A right section of the figure obtained in (b) above. Indicate the altitude.
(d) A right circular cylinder and show a right section. Indicate the altitude *and* the axis. What relationship exists between these two line segments?
(e) Describe the lateral surface in the figures obtained in (b) and (d) above.

3. Using cardboard to represent a 3-inch by 6-inch rectangle, are identical cylinders of revolution obtained when *each side* of the rectangle is used, *in succession*, as the axis?

4. In the right circular cylinders obtained in Ex. 3, are their bases equal to each other? Their altitudes? Their elements? Their lateral surfaces?

THEOREM

Theorem 39. *The bases of a cylinder are congruent.*



Given: Cylinder AC' with bases b and b'.
To prove: $b \cong b'$.

PROOF

Argument	*Defense*
1. Let A and C be two fixed points in the circumference of base b, and X *any* other point. Draw elements AA', CC', and XX', and connect A and C, A and X, C and X, A' and C', A' and X', and C' and X'.	1. Why possible?
2. $AA'C'C$, $AA'X'X$, and $CC'X'X$ are ▱.	2. Why?
3. $\triangle ACX \cong \triangle A'C'X'$.	3. Why?
4. Place b on b' so that ◬ ACX and $A'C'X'$ will coincide. Since X is *any* point on the circumference of b, *every* point on the circumference of b will coincide with the corresponding point on the circumference of b'.	4. Explain.
5. $\therefore b \cong b'$.	5. Why?

Corollary 39a. *Sections of a cylinder made by parallel planes cutting all the elements are congruent.*

Corollary 39b. *A section of a circular cylinder made by a plane parallel to the plane of one base is a circle.*

MEASUREMENT OF CYLINDERS

If a **plane,** no matter how far extended, *contains one and only one element* of a cylinder, it is **tangent to the cylinder.** (Fig. 3)

If *each lateral edge* of a prism is an *element* of a cylinder and if *the bases* of the prism are *inscribed* in the bases of the cylinder, the **prism is inscribed in the cylinder** (Fig. 4). How would you describe the cylinder with respect to the prism?

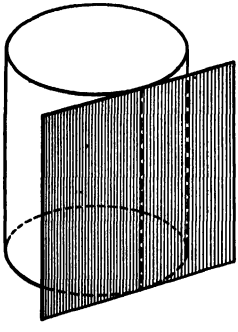


FIG. 3.

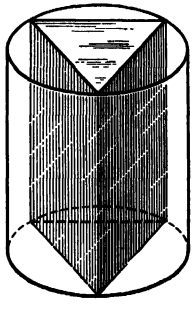


FIG. 4.

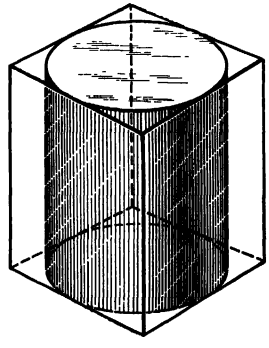


FIG. 5.

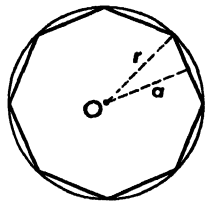
If the *lateral faces* of a prism are *tangent* to a cylinder and if *the bases* of the prism are *circumscribed* about the bases of the cylinder, the **prism is circumscribed about the cylinder** (Fig. 5). How would you refer to the cylinder with respect to the prism?

LIMITS—A BASIC THEORY OF MATHEMATICS

The *theory of limits** is not new to you but a review and extension of some of the basic ideas involved are necessary to understand several of the proofs in solid geometry.

In plane geometry the formulas for the circumference and area of a circle could not be developed directly in the same manner as the development of formulas for perimeter and area of polygons. The difficulty encountered was due to the fact that we were trying to measure curves and surfaces bounded by curves with linear and square units of measurement. In solid geometry we will encounter the same type of difficulty but complicated by finding volumes of solids bounded by curved surfaces, when we try to apply the cubic units of measurement.

*In advanced mathematics theorems on limits are proved. For our purpose an informal discussion of the basic principles involved is sufficient.



In the accompanying figure a *regular* polygon of eight sides is inscribed in $\odot O$. Is the perimeter of the polygon $>$, $=$, or $<$, the circumference of the circle? How do their areas compare? Now inscribe a regular polygon of sixteen sides in $\odot O$. Repeat the process, each time inscribing a regular polygon with double the number of sides as in the preceding step. Eventually the number of sides will become, 128, 256, 512, etc. With each succeeding step, what does the apothem of the *polygon approach as its limit*? What is the *limiting value* of the perimeter? Of the area of the polygon? Obviously the perimeter of the inscribed regular polygon and the circumference of the circle will never be *exactly* equal. Under what circumstances can you say that, *for all practical purposes*, the perimeter and circumference will be equal?

Before reading farther, circumscribe a square about a circle. Which are respectively greater, the perimeter and area of the square or the circumference and area of the circle? Follow the same procedure as with the inscribed regular polygon. What happens to the perimeter of the circumscribed polygon each time a new one is constructed with twice the number of sides as the preceding one? What *limit* will the perimeter of the polygon *approach*? The area?

The foregoing discussion will perhaps take on a more vivid meaning by assuming that the radius of $\odot O$ is 1 inch and by finding the circumference and area of the circle and the perimeters and areas of regular circumscribed polygons of 4, 32, 128, and 256 sides.

If you will imagine that the drawings for a square inscribed in a circle and for the square circumscribed about a circle represent the "top" view of a rectangular prism inscribed in a cylinder and of a rectangular prism circumscribed about a cylinder, you will be able to see how the foregoing discussion of limits can be applied to the lateral areas of a prism and of a cylinder. You should also be able to see that the volume of the inscribed prism with regular bases **approaches** the _?_ of the _?_ **as a limit.**

To be more specific, in Fig. 4 on page 110, a prism with regular bases is inscribed in a cylinder while in Fig. 5 a prism with regular bases is circumscribed about a cylinder. Apply the same "line of reasoning" to these solids as to the plane figures, doubling, redoubling, etc., the lateral faces of each prism. What *limit* do the perimeters of the bases of the two prisms *approach*? The areas of their bases? Their lateral surfaces? Their volumes?

On the basis of the above discussion of "limits" it should be obvious that Assumption 13, quoted below, is acceptable.

Assumption 13. *Any theorem about a prism whose bases are regular polygons and* **which does not depend upon a specific number of faces** *is valid for a cylinder.*

THEOREMS

Theorem 40. *The lateral area of a circular cylinder is equal to the product of an element and the circumference of a right section.*

Suggestions: Theorem 29 (lateral area of any prism) was developed for the general case of *any* prism and hence applies to prisms with regular bases. Since the relationship determined in Th. 29 did not depend upon the number of lateral faces of the prism, Assumption 13 makes Th. 40 valid. Thus no formal proof is required.

Corollary 40a. *The lateral area S of a right circular cylinder equals the product of the altitude h and the circumference of the base,* or $S = 2\pi rh$.

Suggestion: Show how Assumption 13 applies.

Corollary 40b. *The formula for the total area T of a right circular cylinder is $T = 2\pi r(r + h)$ when h is the altitude and r is the radius of the base.*

Theorem 41. *The volume of a circular cylinder equals the product of its base and altitude.*

Suggestion: Apply Assumption 13.

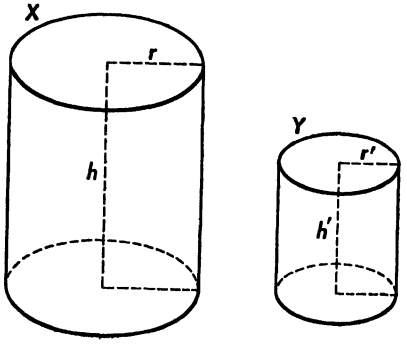
Proof left to you.

Corollary 41a. *The formula for the volume of any cylinder is $V = Bh$ when V represents the volume, B the area of the base, and h the altitude.*

Corollary 41b. *The formula for the volume of a circular cylinder is $V = \pi r^2 h$ when V represents the volume, r the radius of the base, and h the altitude.*

Similar cylinders of revolution are cylinders *generated by similar rectangles* revolving about corresponding sides as axes.

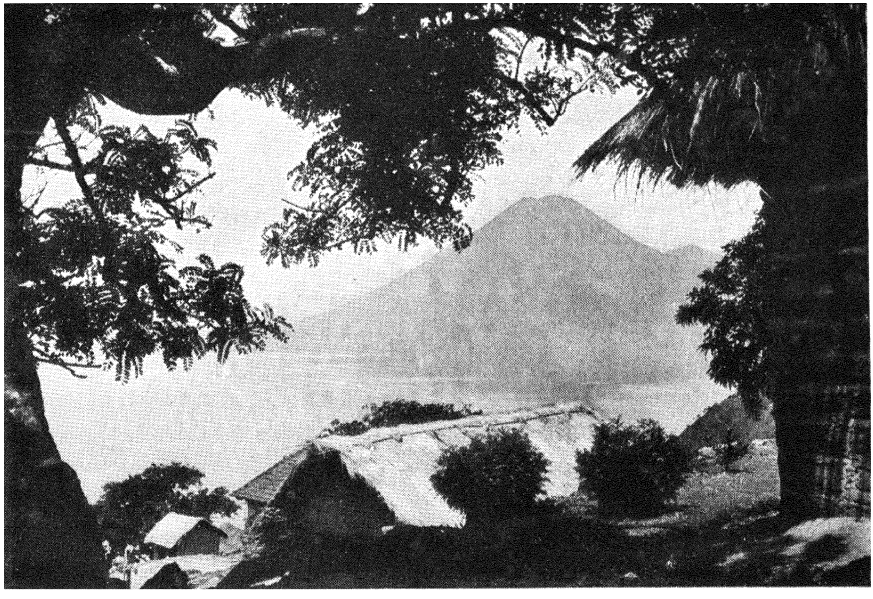
Theorem 42. *The lateral areas or total areas of similar cylinders of revolution (right circular cylinders) have the same ratio as the squares of their radii or as the squares of their altitudes; and their volumes have the same ratio as the cubes of their radii or as the cubes of their altitudes.*



Given: X and Y are similar cylinders of revolution with r and r' their radii, h and h' their altitudes, S and S' their lateral areas, T and T' their total areas, V and V' their volumes.

To prove: $\frac{S}{S'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}$; $\frac{T}{T'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}$; $\frac{V}{V'} = \frac{r^3}{r'^3} = \frac{h^3}{h'^3}$.

Proof left to you.



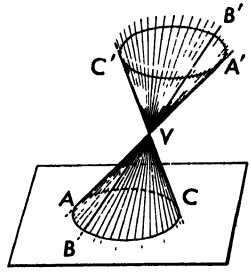
Cone-shaped mountain overlooking Lake Atitlan, Guatemala.
(Courtesy of the United Fruit Company)

SECTION II. CONES

A **conical surface** is generated by a line which moves so that it *always intersects a plane curve*, and *always passes through a fixed point*. The moving line is the **generatrix**, the plane curve is the **directrix**, and the fixed point is the **vertex**. The generatrix in any of its positions is an **element** of the surface.

When the directrix is a closed curve, the surface is a *closed conical surface*.

The two parts of the conical surface on either side of the vertex are called the **nappes** of the conical surface.



A solid formed by one nappe of a closed conical surface and a plane cutting all the elements is called a **cone.** The section made by the cutting plane is the *base* of the cone, the conical surface is its **lateral surface,** and the vertex of the conical surface is the **vertex** *of the cone.* The lateral area of a cone is the area of its lateral surface. A perpendicular from the vertex to the base is the **altitude** *of the cone.*

A **circular cone** is a cone whose *base is a circle.* Only circular cones will be considered in this book. The **axis** of a cone is the line from the vertex to the center of the base. If the *axis is perpendicular to the base,* the cone is a **right circular cone,** as in Fig. 6. Otherwise the cone is said to be **oblique.** The cone in Fig. 7 is oblique.

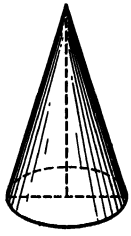


FIG. 6.

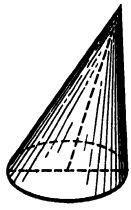
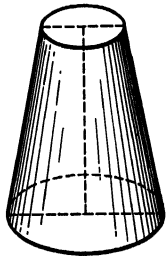


FIG. 7.

If a right triangle is revolved about one of its legs as an axis, the surface of a right circular cone is generated. Therefore a right circular cone is sometimes called a *cone of revolution.*

Any element of a right circular cone is called its **slant height.**

A **frustum of a cone** is the part of a cone included *between the base and a plane parallel to the base.* The base of the cone and the section made by the plane parallel to the base are called the **bases** *of the frustum*; the curved surface is the **lateral surface** *of the frustum*; and the perpendicular distance between the bases is the **altitude** *of the frustum.* The **slant height** *of the frustum* of a right circular cone is the part of an element of the cone which is included between the two bases of the frustum.



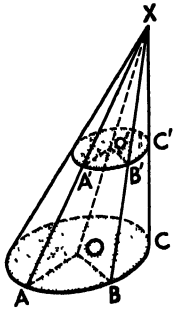
PRELIMINARY THEOREMS

1. *The axis of a right circular cone is also its altitude.*
2. *The elements of a right circular cone are equal.*
3. *The elements of a frustum of a right circular cone are equal.*

Develop informal proofs for these preliminary theorems.

THEOREMS

Theorem 43. *The section of a circular cone made by a plane parallel to the base is a circle.*



Given: $A'B'C'$ is a section of circular cone $X\text{-}ABC$ made by a plane parallel to the base ABC.
To prove: $A'B'C'$ is a circle.

PROOF

Argument	*Defense*
1. Draw axis XO intersecting plane $A'B'C'$ at some point O'.	1. Why possible?
2. Let A' and B' be any two points on curve $A'B'C'$ and draw $XA'A$, $XB'B$, $O'A'$, $O'B'$, OA, and OB.	2. Why possible?
3. XA and XO, and XB and XO determine two planes XAO and XBO.	3. Why?
4. $O'A' \parallel OA$, and $O'B' \parallel OB$.	4. Why?
5. XAO, $XA'O'$, XBO, and $XB'O'$ are $\measuredangle$.	5. Why?
6. $\triangle XAO \sim \triangle XA'O'$, and $\triangle XBO \sim \triangle XB'O'$.	6. Why?
7. $\frac{XO}{XO'} = \frac{OA}{O'A'}$ and $\frac{XO}{XO'} = \frac{OB}{O'B'}$.	7. Why?
8. $\frac{OA}{O'A'} = \frac{OB}{O'B'}$.	8. Why?
9. $OA = OB$.	9. Why?
10. $\therefore O'A' = O'B'$.	10. Why?
11. $A'B'C'$ is a $\odot$.	11. Why?

Cylindrical storage tanks, Goshen, Indiana.

Corollary 43a. *The area of a section of a circular cone made by a plane parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the cone.*

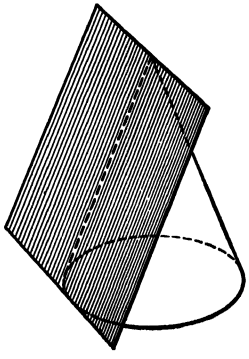


FIG. 8.

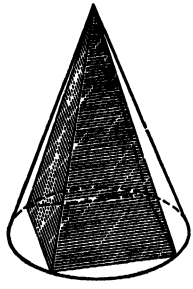


FIG. 9.

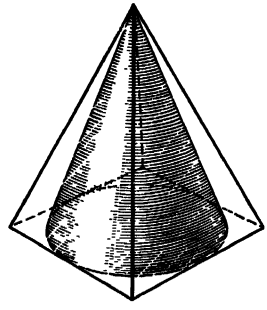


FIG. 10.

A **plane** is **tangent to a cone** if it *contains one and only one element* of the cone (Fig. 8).

A **pyramid** is **inscribed in a cone** when its lateral edges are elements of the cone and its base is inscribed in the base of the cone. The **cone** is said to be **circumscribed about the pyramid** (Fig. 9).

If the *lateral faces* of a pyramid are *tangent* to the lateral surface of the cone and if *the base* of the pyramid is *circumscribed* about the base of the cone, the **pyramid** is a **circumscribed pyramid,** and the **cone** is said to be **inscribed in the pyramid** (Fig. 10).

Read again the discussion on the measurement of cylinders and on limits on pages 110–111. The same line of reasoning can be applied to cones and Assumption 14, quoted below, should be acceptable.

Assumption 14. *Any theorem about a pyramid whose bases are regular polygons which does not depend upon a specific number of faces is valid for a cone.*

On the basis of Assumption 14, Ths. 44 and 45 can be *accepted without further proof.*

Theorem 44. *The lateral area of a right circular cone is equal to one-half the product of its slant height and the circumference of its base.*

Corollary 44a. *The lateral area of a right circular cone is given by the formula* $S = \pi rl$, *when* S *represents the lateral area,* l *the slant height, and* r *the radius of the base.*

Corollary 44b. *The total area of a right circular cone is given by the formula* $T = \pi r(r + l)$, *when* T *represents the total area,* l *the slant height, and* r *the radius of the base.*

Corollary 44c. *The lateral area of a frustum of a right circular cone is given by the formula* $S = \pi l(r + r')$, *when* S *represents the lateral area,* l *the slant height, and* r *and* r' *the radii of the two bases.*

Theorem 45. *The volume of a circular cone is equal to one-third the product of its base and altitude.*

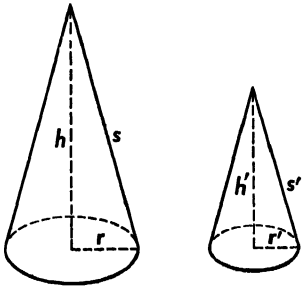
Proof left to you.

Corollary 45a. *The volume of a circular cone is given by the formula* $V = \frac{1}{3}\pi r^2 h$, *when* V *represents the volume,* h *the altitude, and* r *the radius of the base.*

Corollary 45b. *The volume of a frustum of a circular cone is given by the formula* $V = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$, *when* h *represents the altitude of the frustum and* r and r' *are the radii of the bases.*

Similar cones of revolution are cones generated by *two similar right triangles* revolving about corresponding legs as axes.

Theorem 46. *The lateral area or the total areas of two similar right circular cones have the same ratio as the squares of their altitudes, or as the squares of their slant heights, or as the squares of the radii of their bases; and their volumes have the same ratio as the cubes of their altitudes, or as the cubes of their slant heights, or as the cubes of the radii of their bases.*



Given: Two similar right circular cones with S and S' representing the lateral areas, T and T' the total areas, and V and V' the volumes, while h and h' denote the altitudes, r and r' the radii of the bases, and s and s' the slant heights.

To prove:

$$\frac{S}{S'} = \frac{h^2}{h'^2} = \frac{s^2}{s'^2} = \frac{r^2}{r'^2}; \quad \frac{T}{T'} = \frac{h^2}{h'^2} = \frac{s^2}{s'^2} = \frac{r^2}{r'^2};$$

$$\frac{V}{V'} = \frac{h^3}{h'^3} = \frac{s^3}{s'^3} = \frac{r^3}{r'^3}.$$

Proof left to you.

CONIC SECTIONS

Any section of a *right circular cone* is called a *conic section.* Certain conic sections have special significance in mathematics. The curves determined by the four conic sections in the panels on the following two pages are studied extensively in analytic geometry, where they are treated algebraically as special loci. Note that each *conic section* is identified by a figure number only. In each case, the drawing which follows immediately

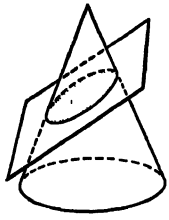
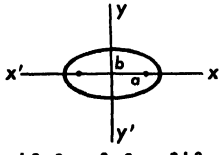


FIG. 11.



$b^2x^2 + a^2y^2 = a^2b^2$

Ellipse

FIG. 11a.

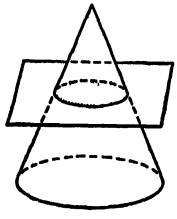
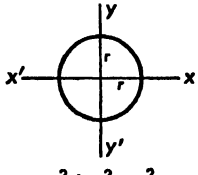


FIG. 12.



$x^2 + y^2 = r^2$

Circle

FIG. 12a.

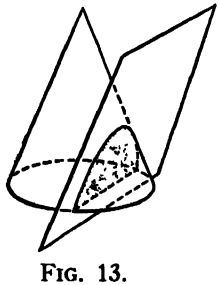


FIG. 13.

shows the *graph* of the preceding conic section as well as the name of the curve (which is also the name of the conic section) and one form of its general equation. You are not expected to understand all of the details now, but your experience in algebra is sufficient for you to grasp the ideas presented here.

In Fig. 11, the conic section is formed by a plane oblique to the base which cuts all elements of the cone. The equation of the curve shown in Fig. 11a, is an "algebraic description" of the following locus problem.

The locus of a point that moves so that the sum of its distances from two fixed points is constant is an **ellipse.**

The **circle** in Fig. 12 is the conic section determined by a plane passed through the cone parallel to its base. It is really a special ellipse whose *major* and *minor* axes are equal.

When a plane cuts a cone so that it is parallel to one, and only one, element of the cone, the conic section is a **parabola** (Fig. 13). The locus of points *equidistant* from a fixed point and a fixed line is a parabola. In Fig. 13a, what is the fixed point? Where is the fixed line?

The **hyperbola** in Fig. 14 is a conic section determined when a plane, parallel to the axis, cuts both nappes of a right circular cone. It always has two branches.

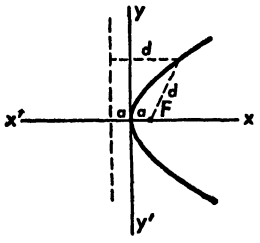
The coordinate graph of each conic section will generate a curved surface if permitted to revolve. What kind of surface would be generated if the circle in Fig. 12a were rotated about the x-axis?

An *ellipsoid of revolution* will result if the ellipse in Fig. 11a is rotated about the x-axis. If the parabola in Fig. 13a is rotated about the x-axis, a *paraboloid of revolution* is generated, and a *hyperboloid of revolution* will be the result of rotating the hyperbola in Fig. 14a around the y-axis.

Obviously the equations given for the plane conic sections will not hold for the *surfaces of revolution* suggested above. A new axis (z) would need to be introduced, perpendicular to the plane determined by the x- and y-axes, and the equations would thus involve a new variable.

These curves and surfaces have important applications, only one of which will be mentioned here. After rotating the parabola in Fig. 13a to produce a paraboloid of revolution, imagine a plane, beyond F, passed so that it is perpendicular to the x-axis. The solid thus formed could be used as a pattern for a reflector which would be similar to many headlight reflectors on automobiles or even the reflectors of ordinary flashlights. The point F is called the *focus* of both the plane curve and the solid. When the source of light is at F, the "lamp" will be in focus since the reflector will send out the rays in parallel lines. When the source of light is not at F, the rays of light will not be evenly diffused, and the "lamp" will be out of focus.

It is hoped that the foregoing discussion will convince you that a study of analytic geometry can be fascinating as well as practical.



$y^2 = 4ax$
Parabola

FIG. 13a.

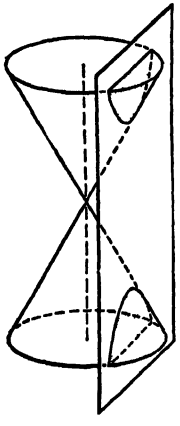
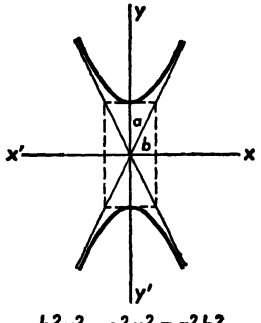


FIG. 14.



$b^2y^2 - a^2x^2 = a^2b^2$
Hyperbola

FIG. 14a.

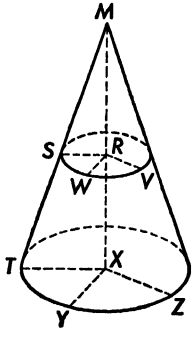
CHAPTER REVIEW

Part 1

Choose from Column B the letter appearing before the phrase which fits each formula in Column A. (Not all statements in Column B may match a formula in Column A, and possibly some will match more than one.)

Column A	*Column B*
1. $S = \pi rl$	(a) Volume of a right circular cylinder
2. $V = Bh$	(b) Volume of a prism
3. $S = 2\pi rh$	(c) Lateral area of a cone
4. $T = 2\pi r(r + h)$	(d) Total area of a cone
5. $V = \pi r^2 h$	(e) Lateral area of the frustum of a right circular cone
6. $V = \frac{1}{3}Bh$	(f) Volume of a right circular cone
7. $S = \pi l(r + r')$	(g) Volume of a pyramid
8. $V = \frac{1}{3}\pi r^2 h$	(h) Volume of a frustum of a cone
9. $S = ep$	(i) Lateral area of a right circular cylinder
10. $T = \pi r(l + r)$	(j) Total area of a right circular cylinder

Part 2



Given: A right circular cone with altitude MX, and a plane parallel to the base which cuts the altitude at point R and forms section SWV.

Complete each of the following:

1. Section SWV is a ?.
2. XY and XZ are ?.
3. If $MR = RX$, then:
   (a) Section SWV is ? of the base.
   (b) MS is ? ? ST. Why?
   (c) XT is ? RS.
   (d) The volume of M-SWV is ? of the volume of M-TYZ.

4. If MR is one-half RX, then the lateral area of cone M-SWV is ? the lateral area of M-TYZ, and the volumes are in the ratio of ? to ?.
5. If MX is 12 units and XY is 9 units, the total area of M-TYZ is ?.
6. If MS is 8 inches and RS is 2 inches, the lateral area of M-SWV is ?.
7. A cylinder with the same base and altitude as the cone in Ex. 5 would have a volume of ?.
8. If RS is 5 units, XT is 8 units, and ST is 6 units, the lateral area of the frustum is ?.
9. A right circular cone is also called a ? of ?.
10. An element of a right circular cone is called its ? ?.

Part 3

1. Two similar cylinders of revolution have altitudes of 6 inches and 3 inches. How do their lateral areas compare? Their volumes? The areas of their bases?
2. How many gallons of gasoline will a cylindrical tank hold that is 8 feet long and whose diameter is 30 inches? (One cubic foot = $7\frac{1}{2}$ gallons.)
3. How many square inches of metal would be required to make a cone-shaped funnel 7 inches long and with a 2-inch radius?
4. If the diameter of a wastepaper basket is 16 inches at the top and 12 inches at the bottom, how many square feet of material were required to construct it if its slant height is 18 inches?
5. How many cubic yards of sand are in a conical pile 14 feet high if the diameter of the base is 30 feet?
6. A silo at a certain farm is in the shape of a cylinder with a cone-shaped top. Find the area to be painted if the height of the cylinder is 28 feet, the diameter of the base is 10 feet, and its conical top is 5 feet high.
7. How much silage will be stored in the silo mentioned above if it is filled to a depth of 25 feet?
8. If both the altitude and the radius of a right circular cylinder are doubled, what change would be made in the volume? In the lateral area?
9. (a) If the altitude of a right circular cone were trebled while the radius remained the same, what change would be made in its volume? Its lateral area?
   (b) If the radius of a right circular cone were trebled while the altitude remained the same, what change would be made in its volume? Its lateral area?
10. Find the total area of a right circular cylinder with a radius of $2x$ units and an altitude of $7x$ units.
11. Bring your **Tool Box** up to date.

Do not write in your book.

EXERCISES

1. Find the volume of a conical pile of sand 12 feet high if the diameter of the base is 24 feet.
2. The diameter of a well is 5 feet. How many gallons of water are there in the well if it is filled to a depth of 12 feet? (One cubic foot = $7\frac{1}{2}$ gallons.)
3. A hollow steel shaft 12 feet long is 16 inches in diameter. The inside diameter is 9 inches. Find the weight of the shaft if the steel weighs 0.29 pound per cubic inch.
4. Find the lateral area, total area, and the volume of a right cylinder whose radius is 10 inches and whose height is 3 feet.
5. A cone is made from a semicircular piece of tin with a 4-inch radius. Find the lateral area of the cone.
6. A right circular cylinder 4 feet in diameter has the same volume as a right circular cone with the same size base. If the cone is 6 feet high, what is the height of the cylinder?
7. The diameter of the top of a water pail is 12 inches, the diameter of the bottom is $9\frac{1}{2}$ inches, and the altitude is 10 inches. How many quarts does it hold? (One cubic foot = $7\frac{1}{2}$ gallons.)
8. Find the number of square inches of tin in the pail described in Ex. 7 if its slant height is 11 inches.
9. Find the volume of a right circular cone the radius of whose base is 4 feet and whose slant height is 12 feet.
10. Find the total area of the frustum of a right circular cone if the radii of the bases are 4 feet and 6 feet and the slant height is 5 feet.
11. How many gallons of water will a cylindrical tank hold which is 8 feet long and 3 feet in diameter?
12. How many feet of copper wire 0.08 inch in radius can be made from a cubic foot of copper?
13. What is the capacity, in gallons, of a cylindrical water tank 56 inches in diameter and 12 feet long?
14. Some cardboard milk containers are in the shape of a frustum of a cone. The diameters of the bases are approximately $1\frac{3}{4}$ inches and 4 inches while the slant height is $9\frac{1}{2}$ inches. Find the total area of one of the containers.
15. The slant height of a right circular cone is equal to the diameter of its base. Compare the area of its base with its lateral area.
16. (a) Where would a plane parallel to the base of a right circular cone have to be passed to bisect the lateral surface of the cone?
    (b) Where would the plane have to be passed to form a section equal in area to one-fourth the area of the base?

17. If the number of cubic inches in the volume of a given right circular cylinder equals the number of square inches in its lateral area, what is the radius of the base?

18. What is the cost of digging a well 4 feet in diameter and 15 feet deep if the price is \$3.10 per cubic yard?

19. The altitudes of two similar cones of revolution are in the ratio of 3 to 2. Compare their lateral areas, and also compare their volumes.

20. (a) How many gallons of fuel oil can be stored in a cylindrical tank 24 feet high and 10 feet in diameter?
(b) What would it cost to paint the tank at 3 cents a square foot?

21. (a) Find the lateral area and volume of the two cones generated by revolving a rt. $\triangle$ about its hypotenuse as an axis if the sides are 6, 8, and 10 inches respectively.
(b) Suppose the hypotenuse is 16 inches, one acute angle is 30°, and the triangle is revolved using the shortest leg as an axis. Find the lateral area and the volume of the cone of revolution.

22. A cylindrical tank with a 75-gallon capacity has a diameter of 3 feet. What is its height? (One cubic foot = $7\frac{1}{2}$ gallons.)

23. Prove that every section of a cylinder made by a plane passing through only two elements is a parallelogram.

24. Prove that the lateral area of a cone formed by revolving an equilateral triangle about an altitude is twice as large as the area of its base.

CHAPTER TEST

1. What are the properties of a right circular cone? Of a frustum of a right circular cone?

2. Why is a right circular cone sometimes called a cone of revolution?

3. Describe a circular cylinder and quote as many statements as possible pertaining to one.

4. What is meant by an inscribed prism? A circumscribed prism?

5. Are all cylinders of revolution similar? Why or why not?

6. Write the formulas for the lateral area and the volume of: a cylinder of revolution, a right circular cone, the frustum of a right circular cone.

7. (a) Sketch two similar cones of revolution whose total areas are in the ratio of 1 to 4. (What data were necessary?)
(b) Sketch two similar cylinders of revolution whose volumes are in the ratio of 1 to 8.

8. Find the area and the volume of a cylinder of revolution whose altitude is 15 centimeters and whose radius is 4 centimeters.

9. Find the area and volume of a right circular cone whose altitude is 12 inches, slant height is 13 inches, and radius is 5 inches.

10. If the cone in Ex. 9 is cut at a point 3 inches below the vertex by a plane parallel to the base, the section formed has an area equal to what fractional part of the area of the base?
11. A certain right circular cylinder 8 inches high has the same volume as a right circular cone of equal height. If the base of the cylinder has a 3-inch diameter, what is the diameter of the cone?
12. The altitude of a cylinder of revolution is four times the altitude of a similar cylinder. How do their areas compare? Their volumes? Their total areas? Their radii?
13. The lateral areas of two similar right circular cones are 324 square inches and 144 square inches respectively. If the slant height of the smaller cone is 8 inches, what is the slant height of the larger one?
14. Find the number of gallons of water a cylindrical tank will hold if its height is 5 feet and the diameter of its base is 30 inches.

Keep the Iron Hot: EVERYDAY REASONING

The following comments were overheard by a member of the Community Chest Board as he waited for a meeting of townspeople to begin. In which of the statements is a conclusion drawn? If any reason is given, or implied, to back up the conclusion, indicate whether or not you think the conclusion follows logically from the reason. Then put the statement in **syllogistic* form** so that the conclusion will be a *necessary* one. Also tell whether or not you would accept the first statement (major premise), in each syllogism as valid.

1. The professional Rovers are a good football team because they have exceptionally good guards.
2. Our bridge club is planning a picnic for next Wednesday so it doubtless will rain.
3. She didn't call me this afternoon so I know she isn't coming.
4. I knew that it hadn't rained at Cross-Road Heights because the side roads were dry.
5. Five people arranged plans for our meeting tonight.
6. Since chicken is so expensive, we'd better serve meat loaf for the victory dinner.
7. My son is a miserable speller, but the schools don't teach spelling the way I was taught.
8. Mrs. Lablah arranged for the speakers so I'm afraid tonight's meeting will be long and boring.

*See exercises on page 45.

9. He is a certified accountant so his audit of the Community Chest funds will be acceptable.
10. If the Bob Cats orchestra plays, the music will be good.

Follow the same directions for these miscellaneous statements:

1. Mrs. Snow is a good teacher so she must be a teacher at the Larur School.
2. This cat has long hair so it is a Persian cat.
3. We are going on a picnic because Dad brought home some wieners and buns.
4. Helen will get a good grade in Miss Jones' class because she invited Miss Jones to her house for dinner.
5. Herbert does excellent work in Chemistry lab so he will get a high grade in Chemistry.

If a person wishes to have his conclusions respected, he should be very careful to draw conclusions only on the basis of facts and statements which are generally accepted as valid. Make a list of statements for class discussion which *you* hear people make. Select some examples of *good* reasoning and some of *poor* reasoning. Explain why you think they are sound or otherwise.

Find several advertisements, each of which states or implies conclusions about a particular article. Then, for each *conclusion*, write the preliminary statements (major and minor premises) required to make it a *necessary* one. Decide whether you are willing to accept the preliminary statements as reasonable. If you reject them, can you accept the conclusion as reasonable?

CHAPTER 8 *Spheres*

A **sphere** is a *closed curved surface all points* of which are *equidistant from a point within* called the **center.** A single letter naming the center is generally used to designate a sphere. The terms **radius, diameter,** and **chord** have the same meanings as properties of a sphere as when they are properties of a circle. In the drawings on the opposite page, name a radius of sphere O, a diameter, a chord.

The meanings of many other terms we shall use in connection with a sphere are very similar to the meanings of corresponding terms relating to circles. Complete the following statements. Then check them with the definitions that you will find given in the Vocabulary section on pages 176–181.

1. A line is tangent to a sphere if it has . . ..

2. A plane is tangent to a sphere if . . ..

3. Two spheres are tangent to each other if . . ..

4. A sphere is said to be inscribed in a polyhedron if it is . . ..

5. A sphere is said to be circumscribed about a polyhedron if . . ..

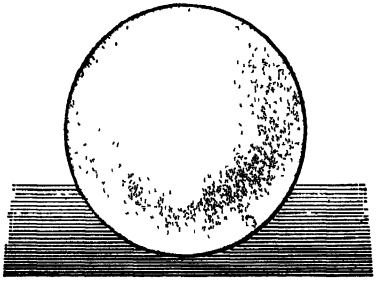
6. Two spheres which have the same center are called $\frac{?}{}$ spheres.

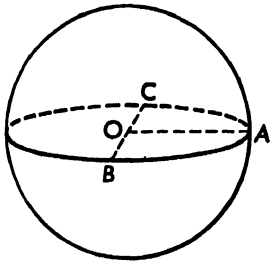
7. A hemisphere is . . ..

8. The radius of a sphere is . . ..

9. A chord of a sphere is . . ..

10. The diameter of a sphere is . . ..





It is necessary in beginning a study of spherical surfaces to agree upon a few basic ideas. From the definition of a sphere, what *plane* figure do you think would be formed if a plane were passed through the center of a sphere?

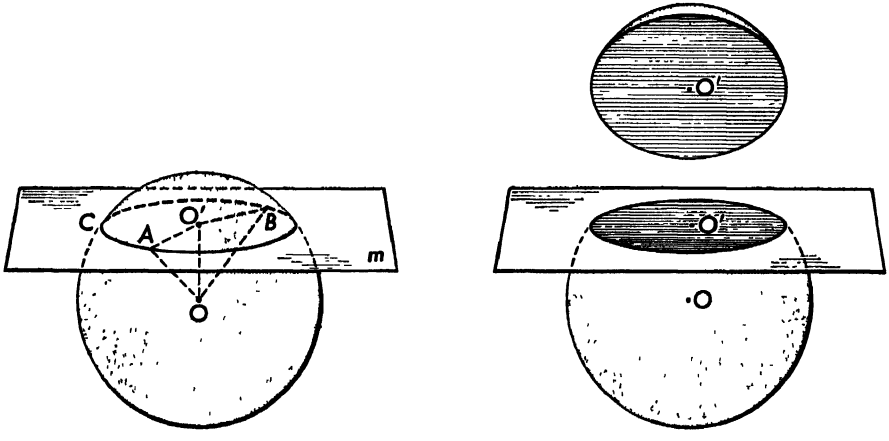
Read the following sentences filling in the proper words:

1. With a given center and a given radius one and ....
2. Radii of the same sphere or of ? spheres are ?.
3. Spheres with ? radii are ?.
4. A point is within, on, or outside a sphere according to whether its ....
5. A section made by a plane passing through the center of a sphere is a ?.

Since Theorem 47 describes the section made by a plane passed through *any* point in a sphere, it will be well to consider its proof before discussing other definitions.

THEOREM

Theorem 47. *If a plane intersects a sphere, the intersection is a circle.*



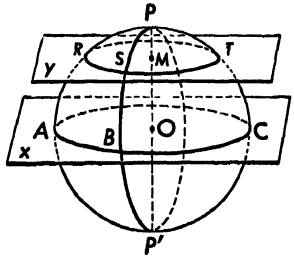
Given: Plane m intersecting sphere O and determining section CAB.
To prove: CAB is a circle.

PROOF

Argument	*Defense*
1. From center O of the sphere draw a $\perp$ plane m. Call the intersection O'.	1. Why possible?
2. Let A and B represent any two points on the intersection of the plane and the sphere. Draw $O'A$, $O'B$, and also radii OA and OB.	2. Why possible?
3. $OO' \perp AO'$, $OO' \perp BO'$.	3. Why?
4. $\measuredangle AO'O$ and $BO'O$ are rt. $\measuredangle$.	4. Why?
5. $\triangle AO'O \cong \triangle BO'O$.	5. Why?
6. $AO' = BO'$.	6. Why?
7. Since A and B represent *any* two points on the intersection of sphere O and plane m, CAB is a circle.	7. Why?

DEFINITIONS OF NEW TERMS

The circle formed on a sphere by a plane passing through the *center* of the sphere is called a **great circle** *of the sphere.* In the accompanying figure, plane x passes through center O. Therefore by definition, $\odot ABC$ is a great circle of sphere O.



A circle formed by a plane cutting a sphere but *not* passing through the center of the sphere, is a **small circle** *of a sphere.* $\odot RST$ is a small circle of sphere O.

One-fourth of a great circle of a sphere is called a **quadrant.** PB and PC are quadrants. Name several others.

The **axis** *of a circle of a sphere* is the *diameter* of the sphere which is *perpendicular to the plane of the circle.* If $POP' \perp x$ then it is the axis of circle ABC. If $POP' \perp y$, then it is the axis of circle RST.

The extremities of the axis of a circle of a sphere are called the **poles** of the circle.

The **spherical distance** between two points on a sphere is the *minor arc of the great circle* connecting them. $\overset{\frown}{AB}$ is the spherical distance from A to B. If $PSBP'$ is a great circle, $\overset{\frown}{BP'}$ is the spherical distance between B and P'.

The spherical distance from the *nearer pole* of a circle of a sphere to *any* point on the circle is called the **polar distance** of the circle. $\overset{\frown}{PS}$ is the polar distance of $\odot RST$ and $\overset{\frown}{PB}$ is the polar distance of $\odot ABC$, provided P is a pole of each circle.

No assumptions concerning spheres were included in the list* in Chapter 1, but we must consider several now in order to develop formal geometric proofs about spheres. Study the following set of six assumptions to see if they are acceptable. Remember that a conclusion can be no more reasonable than the assumptions upon which it is based.

Assumption 15. *With a given center and a given radius one and only one sphere can be drawn.*

Assumption 16. *Radii of the same or equal spheres are equal.*

*Read again the footnote on page 9.

Assumption 17. *Spheres with equal radii are equal.*

Assumption 18. *A point is within, on, or outside a sphere according to whether its distance from the center is less than, equal to, or greater than the radius of the sphere.*

Assumption 19. *A sphere can be generated by revolving a semicircle about its diameter as an axis.*

Assumption 20. *The shortest distance between two points on a sphere is the minor arc of the great circle which passes through the two points.*

PRELIMINARY EXERCISES

1. If the earth is considered to be a sphere, what geometric term can be used to designate the following:

(a) The equator?
(b) The parallels of latitude?
(c) The meridians?
(d) The distance from a point on the equator to the south pole?

2. Make a sketch of each of the new terms defined on page 129.

3. Secure two rubber balls of the same size (or any two objects to represent equal spheres) and perform the activities suggested below:

(a) Bisect a sphere.
(b) Compare the size of two great circles of the same sphere; of equal spheres.
(c) How significant are the points of intersection of two great circles of a sphere?
(d) Pass a great circle through any two points on a sphere; through any three points.

4. Complete:

(a) A fourth of a great circle is a ?.
(b) A diameter perpendicular to the plane of a circle of a sphere is a ? of the circle.
(c) The minor arc of a great circle which is drawn between two points on a sphere is the ? ? between the points.
(d) The poles of a circle are the . . ..
(e) The distance from the nearer pole of a circle of a sphere to any point on the circle is the ? ? of the circle.
(f) If a plane cutting a sphere does not pass ? the ? of the sphere, it forms a ? circle of the sphere.

5. The polar distance of a great circle is a quadrant.

6. If an airliner is to take the shortest route between two cities, what path would it have to follow? In the world of reality name two major factors which might necessitate taking a longer route.

Review the statement and proof of Th. 47 and then prove its corollaries which follow.

Corollary 47a. *The axis of a circle of a sphere passes through the center of the circle.*

Corollary 47b. *The center of a great circle of a sphere and the center of the sphere are the same.*

Corollary 47c. *All great circles of the same or equal spheres are equal.*

Corollary 47d. *Any two great circles of a sphere bisect each other, and the line joining the points of intersection is a common diameter of the circles and the sphere.*

Corollary 47e. *Through any three points on a sphere one, and only one, circle of the sphere can be drawn.*

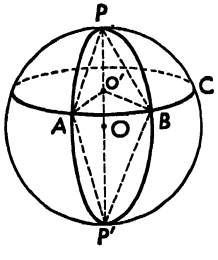
Corollary 47f. *Through any two points on a sphere that are not the end-points of a diameter one, and only one, great circle of the sphere can be drawn.*

Corollary 47g. *Parallel circles of a sphere (circles lying in parallel planes) have the same axis and the same poles.*

Corollary 47h. *If the planes of two great circles are perpendicular to each other, each circle passes through the poles of the other.*

THEOREMS

Theorem 48. *The spherical distances of all points on a circle of a sphere from a pole of the circle are equal.*



Given: $\odot ABC$ of a sphere with P and P' the poles of the $\odot$. PAP' and PBP' are arcs of great Ⓢ drawn through any two points A and B of $\odot ABC$.

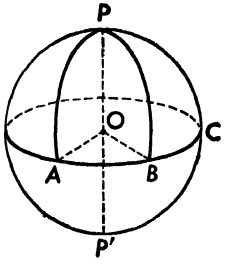
To prove: $\overset{\frown}{PA} = \overset{\frown}{PB}$, and $\overset{\frown}{AP'} = \overset{\frown}{BP'}$.

Suggestions: Draw PA, PB, $P'A$, $P'B$, $O'A$, and $O'B$. Why is $PP' \perp$ plane of $\odot ABC$? $O'A = O'B$ because they are radii and so $PA = PB$. Why? All great Ⓢ of a sphere are equal. $\therefore$ how will $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$ compare?

Proof left to you.

Corollary 48a. *The polar distance of a great circle is a quadrant.*

Theorem 49. *If a point on a sphere is at the distance of a quadrant from each of two other points on the sphere which are not the extremities of a diameter, then the point is a pole of the great circle passing through these two points.*

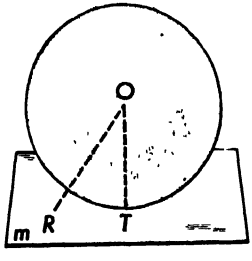


Given: A and B are two points, not extremities of a diameter, on sphere O each at a quadrant's distance from point P. ABC is a great circle of the sphere.
To prove: P is a pole of great $\odot ABC$.

Suggestions: Draw diameter POP' and radii OA and OB. What kind of $\measuredangle$s are POA and POB? Show that $PO \perp$ plane ABC.

Proof left to you.

Theorem 50. *If a plane is perpendicular to a radius of a sphere at its outer extremity, it is tangent to the sphere.*



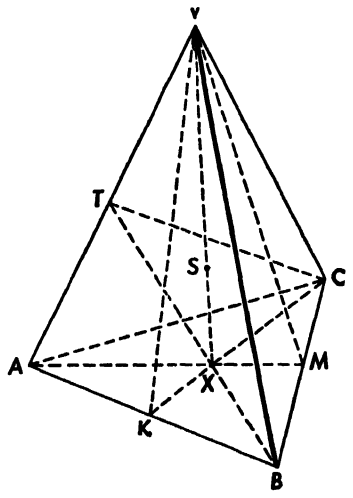
Given: Plane $m \perp$ radius OT at point T.
To prove: Plane m is tangent to the sphere O.

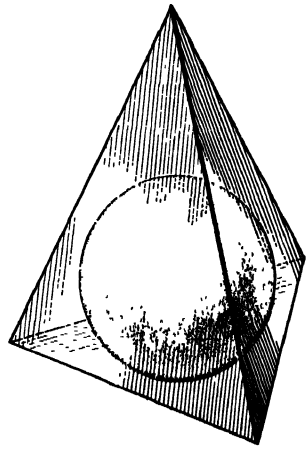
Suggestions: Choose any point R, other than T, in plane m, and draw OR. Is $OR \perp m$? How does it compare in length with OT?

Proof left to you.

Corollary 50a. *If a plane is tangent to a sphere it is perpendicular to the radius drawn to the point of contact.*

Theorem 51. *A sphere can be inscribed in any given tetrahedron.*





Given: Tetrahedron V–ABC.

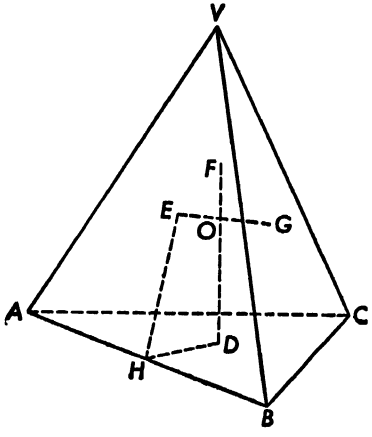
To prove: A sphere can be inscribed in tetrahedron V-ABC.

PROOF

Argument	*Defense*
1. Bisect dihedral $\measuredangle AV$ and VC with planes AMV and VCK.	1. Why possible?
2. These planes will intersect in some line such as VX.	2. Why?
3. VX is the locus of points equidistant from planes BAV, BCV, and ACV.	3. Why?
4. Bisect dihedral $\angle BC$ with a plane (BCT) and it will intersect VX in some point such as S.	4. Why?
5. S is equidistant from all four faces of the tetrahedron.	5. Why?
6. A sphere with center S and a radius = the distance from S to any side of the tetrahedron will be tangent to each face of V–ABC.	6. If a plane $\perp$ a radius of a sphere at its . . ..
7. $\therefore$ a sphere can be inscribed in the tetrahedron V–ABC.	7. Definition.

Corollary 51a. *A sphere can be inscribed in any given cube.*

Theorem 52. *A sphere can be circumscribed about any given tetrahedron.*



Given: Tetrahedron V–ABC.
To prove: A sphere can be circumscribed about V–ABC.

PROOF

Argument	*Defense*
1. Find D the center of a $\odot$ circumscribed about $\triangle ABC$, and E the center of a $\odot$ circumscribed about $\triangle VAB$. Find the mid-point (H) of AB.	1. Why possible?
2. Draw $FD \perp$ plane ABC and $GE \perp$ plane VAB.	2. Why possible?
3. Any point on GE is equidistant from V, A, and B. Any point on FD is equidistant fron A, B, and C.	3. The locus of points equidistant from the vertices of a $\triangle$ is the line through the center of the . . ..
4. Since any point on either GE or FD is equidistant from A and B, GE and FD both lie in a plane which is $\perp AB$ at its mid-point, H.	4. The locus of points equidistant from two given points is the plane . . ..
5. The plane $\perp AB$ at H will intersect plane ABC in line HD and plane VAB in line EH.	5. Why?
6. $EG \perp EH$ and $DF \perp DH$.	6. If a line $\perp$ a plane, it is $\perp$ . . ..
7. GE will intersect FD at some point O.	7. If two lines intersect, $\perp$s to them will . . ..

8. O is equidistant from V, A, B, and C.	8. Why?
9. $\therefore$ a sphere can be circumscribed about tetrahedron V–ABC using O as a center and VO as a radius.	9. Definition.

Corollary 52a. *Through any four points not in the same plane one, and only one, sphere can be passed.*

Corollary 52b. *A sphere can be circumscribed about any given cube.*

CHAPTER REVIEW

Complete Exs. 1–21.

1. Through three points on a sphere one and . . ..
2. The axis of a circle of a sphere is the . . ..
3. The poles of a circle of a sphere are the . . ..
4. All great circles of a sphere ? ?.
5. The planes of two great circles of a sphere . . ..
6. The shortest distance between two points on a sphere . . ..
7. A sphere can be generated by revolving a . . ..
8. The polar distance of a ? circle is a ?.
9. The polar distances of equal circles on the same or ? spheres ? ?.
10. If a plane intersects a sphere, the line of . . ..
11. If a plane is perpendicular to a radius of a sphere . . ..
12. Through any two points on a sphere that are not the end-points of . . ..
13. A sphere can be inscribed in . . ..
14. The spherical distances of all points on a circle . . ..
15. A plane or a line is tangent to a sphere if it . . ..
16. Two spheres are tangent to each other if . . ..
17. A sphere can be circumscribed . . ..
18. Radii of the same sphere or of . . ..
19. If a point on a sphere is at the distance of a quadrant from each of two other points on the sphere which are not the extremities . . ..
20. If the planes of two great circles are perpendicular to each other . . ..
21. The axis of any circle of a sphere passes . . ..
22. Find the area of a circle formed by a plane cutting a sphere at a point on the diameter 4 inches from the center, if the radius of the sphere is 5 inches.

23. Prove: Two lines drawn from a point on a sphere to the extremities of a diameter of the sphere are perpendicular to each other.

24. Find the circumference of the parallel of latitude which lies 45° north of the equator. (Use 4,000 miles as the radius of the earth.)

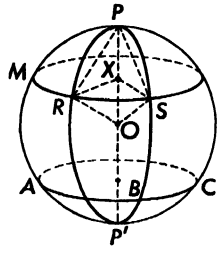
25. Prove: A radius of a sphere perpendicular to a chord of the sphere bisects the chord.

26. Prove: In the same or equal spheres equal chords are equidistant from the center, and conversely.

27. How many spheres can be drawn through two points? Where will their centers lie?

28. How many spheres can be drawn through three points? Where will their centers lie?

29. **Given:** Sphere O with X the center of a small circle whose axis is PP'. $\odot ABC$ is parallel to $\odot X$.



Decide which statements are correct. Be able to defend each answer. Is:

(a) $\overset{\frown}{PR} = \overset{\frown}{PS}$?
(b) $\overset{\frown}{PRP'} = \overset{\frown}{PSP'}$?
(c) $PR \perp$ a line from R to P'?
(d) $XS \perp PP'$?
(e) $XR = XS$?
(f) $PS = RX$?
(g) distance RP' = distance SP'?
(h) the center of $\odot ABC$ on PP'?
(i) $RX = \sqrt{PX \times XP'}$?
(j) a $\triangle PRP' \cong$ a $\triangle PSP'$?
(k) P a pole of $\odot ABC$?
(l) $\odot X = \odot ABC$?
(m) $\overset{\frown}{PM} = \overset{\frown}{PS}$?
(n) $OR = OS$?
(o) $\overline{PR}^2 = PP' \times PX$?
(p) it possible to pass a circle through points M, R, and A?

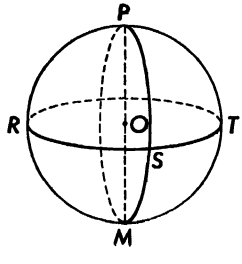
30. Bring your **Tool Box** up to date.

EXERCISES

1. If a plane is tangent to a sphere, prove that any line in that plane drawn through the point of contact is tangent to the sphere at that point.

2. Prove: All small circles at equal distances from the center of the sphere are equal.

3. If a great circle such as PSM is drawn through the poles of a great circle such as RST, the plane of PSM is perpendicular to the plane of RST. Prove.



4. Explain how Ex. 3 is related to Cor. 47h.
5. Prove that the intersection of two spheres is a circle whose plane is perpendicular to the line which joins the centers of the spheres and whose center is on that line.
6. If a line is tangent to a great circle of a sphere, prove that it lies in the plane tangent to the sphere at the point of contact.
7. Four perpendiculars are erected at the centers of the circles circumscribed about the faces of a tetrahedron. Prove that they are concurrent.
8. Prove: Of two unequal circles on a sphere the smaller is at the greater distance from the center of the sphere, and conversely.
9. A sphere has a diameter of 12 inches. How far from its center is the plane of a small circle if its polar distance is $\overset{\frown}{60°}$?
10. Prove: Circles on the same sphere which have equal polar distances are equal.
11. Two lines are tangent to a sphere at the same point. Prove that they determine a plane tangent to the sphere at that point.
12. Prove that all the lines drawn from an external point to a sphere and tangent to the sphere are equal and their points of tangency determine a circle.
13. Find the radius of a small circle 4 inches distant from the center of a sphere whose radius is 9 inches.
14. Find the circumference of the earth. (Use 4,000 miles as the radius.)
15. A plane cuts a sphere at a distance of 3 inches from the center. Find the diameter of the small circle if the radius of the sphere is 5 inches.
16. If the radius of a sphere is 15 inches, find the circumference and area of a circle which is 12 inches from the center of the sphere.
17. What is the circumference of the parallel of latitude which lies 60° south?
18. The radius of a sphere is 21 inches. Find the area of a section made by a plane 12 inches from the center.
19. Find the radius of a circle which has a polar distance of 45°, if the radius of the sphere is 20 inches.
20. What is the length of the radius of a sphere inscribed in a regular tetrahedron whose edges are x units long?

CHAPTER TEST

Complete Exs. 1–9.

1. A sphere is ? about a polyhedron if all ? of the polyhedron lie on the sphere.
2. The spherical distance between two points on a sphere is the length of the ? arc of a ? ? passing through the ?.

3. The ? distance is the distance from any point of a circle to the ? pole of the circle.
4. Every great circle bisects the ?.
5. Any two great circles ? each other.
6. One-fourth of a great circle is called a ?.
7. The axis of any circle of a sphere is the ? of the sphere ? to the plane of the circle.
8. Parallel circles of a sphere have the same ? and the same ?.
9. Prove: Two planes tangent to a sphere at the ends of the same diameter are parallel.
10. If we consider the earth as a sphere, what kind of circles are the parallels of latitude? The meridians? The equator? What are the poles of the equator? Of the parallels of latitude?
11. If a sphere with a 14-inch radius is cut by a plane 7 inches from the center, is the area of the circle formed one-half the area of a great circle of the sphere? Give data to support your answer.
12. How many lines can be tangent to a sphere at a given point on the sphere? Explain.
13. How many lines can be drawn from an external point so that they will be tangent to a given sphere? Describe the length of these tangents and the points at which they touch the sphere. What kind of surface do these tangent lines determine?
14. Find the circumference of a circle which has a polar distance of $\overset{\frown}{45}°$ on the sphere, if the radius of the sphere is 10 inches.
15. The diameter of a certain sphere is 30 feet. Find the area of a section made by a plane 9 feet from the center.
16. A radius of a sphere which bisects a chord of the sphere is ? perpendicular to the chord. (Choose the word *always*, *sometimes*, or *never*.)

Keep the Iron Hot

Section 1: Everyday Reasoning

Drawing Conclusions Based on Tabular Data

Passenger revenues of Class I railways for a decade are given below.

Year	*Amount*	*Year*	*Amount*
A.....	$ 405,883,269	F.....	$1,652,867,962
B.....	416,903,154	G.....	1,790,305,288
C.....	417,268,960	H.....	1,716,379,470
D.....	514,687,032	I.....	1,259,168.733
E.....	1,028,185,616	J.....	963,322,179

On the basis of the data given, judge the validity of each conclusion stated below. On a sheet of paper list the numbers from 1 to 6. If a conclusion is valid, mark a plus sign after the proper number. If a conclusion is *not* valid, put a minus sign. Give reasons for each rating.

1. The railways received more revenue from this source in year G than in any other year.
2. Fewer people left their homes for the purpose of traveling in year A than in any other year.
3. The cost of traveling for train passengers increased from year F to year H.
4. Less money accrued to the railways from passenger travel in year A than in any other year.
5. More people traveled during year F than during year I.
6. Fewer passenger trains were running in year F than in year G.

The following table indicates weather conditions prevailing in automobile accidents over a given period.

Kind of Weather	*Number of Fatal Accidents*
Clear	24,560
Fog	700
Rain	2,960
Snow	780

On the basis of the data given above, which of the following conclusions are valid?

1. More than 8 out of every 10 accidents occurred in clear weather.
2. The speeder and the reckless driver are to blame for most accidents.
3. A higher percentage of accidents occurred with cars driven in the rain and in snow.
4. Snow is more dangerous for driving than fog.
5. Driving in rain is more dangerous than driving in snow.

Section 2: Formulas

A. Solve for x in each of the following.

1. $ax + b = c$
2. $n = 2x$
3. $ax + bx = m$
4. $\frac{x}{a} + \frac{y}{b} = 1$
5. $y = mx + b$
6. $2x - a = x - 3b$
7. $\frac{y}{x} = c$
8. $nx + \frac{b}{a} = c$

B. Solve these formulas for the unknown indicated.

9. $T = 2(r^2 + rh)$	$h =$	**10.** $x = vt + 16t^2$	$v =$
11. $X = 4r^2$	$r =$	**12.** $A = \frac{1}{2}h(b + B)$	$B =$
13. $s = \frac{1}{2}gt^2$	$g =$	**14.** $F = \frac{9}{5}C + 32$	$C =$
15. $Y = mp + \frac{n}{m}$	$p =$		

Section 3: Converses, Inverses, and Contrapositives

A. You already know that a **converse** of a theorem is, in general, formed by interchanging one condition of the hypothesis and the conclusion. Thus every theorem has at least one converse. If there are several items in the hypothesis or conclusion, several converses can be formed. Also you know that not all converses are valid. Each one must be proved.

An **inverse** of a proposition is formed by denying *both* the hypothesis and the conclusion, and the contrapositive of the original statement results from denying both the hypothesis and the conclusion of the converse. Study the following illustrations.

Statement: If a vehicle is a car, then it has four wheels.

Inverse: If a vehicle is not a car, then it does not have four wheels.

Converse: If a vehicle has four wheels, then it is a car.

Contrapositive: If a vehicle does not have four wheels, then it is not a car.

B. Write the converse, inverse, and contrapositive of each of the following:

1. If three sides of a $\triangle$ are equal then the $\triangle$ is equilateral.
2. The diagonals of a rectangle are equal.
3. If two lines are $\perp$ same plane then they are parallel.
4. If two $\parallel$ planes are cut by a third plane the lines of intersection are parallel.
5. If a person steals a car he is breaking the law.
6. In which of the cases above is the converse always true? The inverse always true? The contrapositive always true?
7. Choose a few other geometric and non-geometric statements to study. Write the converse, inverse, and contrapositive for each and answer the questions raised in Ex. 6.
8. On the basis of the foregoing, would you accept the generalization: "**Both** *the converse and inverse of a statement are true, or* **both** *of them are false*?" Is it possible for a statement which is itself valid to have a converse and an inverse which are *not* valid? What conclusion can you draw about the validity of contrapositive statements?

CHAPTER 9 *Spherical Polygons*

When the arcs of two great circles of a sphere meet, a **spherical angle** is formed, the *vertex* of which is the point of intersection of the arcs. In Fig. 1, $\overset{\frown}{BX}$ and $\overset{\frown}{AX}$ are arcs of two great circles of sphere O. They meet at X to form spherical $\angle BXA$. If a tangent is drawn to each arc at their point of intersection, a plane angle is formed, which obviously is related to the spherical angle. The plane angle thus formed is the *measure* of the spherical angle. In Fig. 1, $\angle RXT$ is formed by tangents RX and TX drawn respectively tangent to $\overset{\frown}{BX}$ and $\overset{\frown}{AX}$ at X. Thus if $\angle RXT = 30°$, spherical $\angle BXA = 30°$.

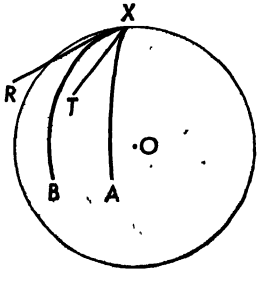


FIG. 1

A **spherical polygon** is a closed figure on a sphere formed by the arcs of three or more of its great circles. The *arcs* are the **sides** of the polygon and their respective points of intersection are the **vertices** of the spherical polygon. In Fig. 2, $ABCD$ is a spherical polygon of four sides, formed by $\overset{\frown}{AD}$, $\overset{\frown}{DC}$, $\overset{\frown}{CB}$ and $\overset{\frown}{BA}$ meeting at D, C, B, and A respectively. These four *arcs are arcs of great circles of sphere O*. Name the four spherical angles of the polygon. How are these angles measured?

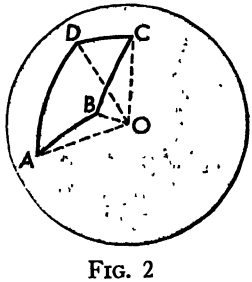


FIG. 2

It must be kept in mind that a great circle of a sphere is the intersection of the sphere and a plane which passes through its center. Therefore in Fig. 2, $\overset{\frown}{AD}$ is "part of the intersection" of a plane through the center of O. If straight lines AO and DO are drawn, the plane

determined by them obviously is the same one that determines the great circle of which $\overset{\frown}{AD}$ is an arc. Thus, if we say, "*the plane of* $\overset{\frown}{AD}$," we mean the plane through the center of the sphere which contains $\overset{\frown}{AD}$. Why will it *always* be possible to pass a plane through the center of a sphere so that the plane will contain a side of a spherical polygon lying on the sphere's surface? Will such a plane exist for *each* side of a spherical polygon? Why?

In Fig. 2, the *plane of AD* and *the plane of DC* form $\angle A\text{-}DO\text{-}C$. What kind of angle is it? Name three other such angles in Fig. 2 and tell how each one is formed. How are such angles measured? If tangents were drawn to $\overset{\frown}{AD}$ and $\overset{\frown}{DC}$ at D, would the *plane angle* formed have any relation to the measure of $\angle A\text{-}DO\text{-}C$? Be specific.

What kind of angle is $\angle O\text{-}ABCD$ in Fig. 2? What kind of angle is $\angle BOA$ with respect to $\angle O\text{-}ABCD$? $\angle COD$? Name other such angles.

Note that the polyhedral angle formed by joining the vertices of spherical polygon $ABCD$ with the center of sphere O is convex. The *polygon* is thus said to be *convex*. Only convex spherical polygons will be studied in this book.

PRELIMINARY THEOREMS

1. *The sides of a spherical polygon, when measured in degrees, are equal to the corresponding face angles of the polyhedral angle formed by the planes of the arcs.*

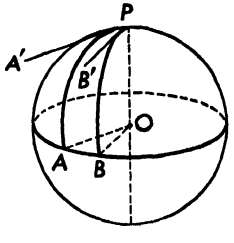
2. *The angles of a spherical polygon are equal in degrees to the corresponding dihedral angles of the polyhedral angle whose vertex is at the center of the sphere.*

A spherical polygon of *three* sides is called a **spherical triangle.** The words *right, acute, obtuse, isosceles,* and *congruent* are used in the same sense with spherical triangles as with plane triangles.

A method for measuring spherical angles was given on page 141. Theorem 53 establishes another method. Study its proof and develop proofs for its two corollaries.

THEOREMS

Theorem 53. *A spherical angle is measured by the arc of the great circle which has the vertex of the angle as a pole and is included between the sides of the angle, extended if necessary.*



Given: Sphere O on which spherical $\angle APB$ is formed by $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$. $\overset{\frown}{AB}$ is an arc of a great $\odot$ with P as its pole. $A'P$ and $B'P$ are tangents to $\overset{\frown}{AP}$ and $\overset{\frown}{BP}$ respectively.

To prove: Spherical $\angle APB \overset{m}{=} \overset{\frown}{AB}$.

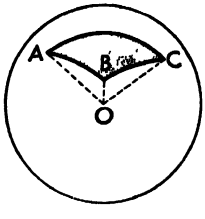
PROOF

Argument	*Defense*
1. Draw radii OA and OB.	1. Why possible?
2. $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$ are quadrants.	2. Why?
3. $\measuredangle POA$ and POB are rt. $\measuredangle$.	3. Why?
4. $PO \perp AO$ and BO.	4. Why?
5. $A'P \perp OP$ and $B'P \perp OP$.	5. A line tangent to a $\odot$ is $\perp$ the radius . . ..
6. $A'P$ and AO lie in the same plane. $B'P$ and BO lie in the same plane.	6. (a) A great $\odot$ is formed by a plane . . .. (b) A tangent to a $\odot$ is a line in the same plane which has . . ..
7. $A'P \parallel AO$ and $B'P \parallel BO$.	7. Why?
8. $\angle AOB = \angle A'PB'$.	8. If two $\measuredangle$ not in the same plane have their . . ..
9. Spherical $\angle APB \overset{m}{=} \angle A'PB'$.	9. Definition.
10. $\therefore$ spherical $\angle APB \overset{m}{=} \angle AOB$.	10. Why?
11. But $\angle AOB \overset{m}{=} \overset{\frown}{AB}$.	11. Why?
12. $\therefore$ spherical $\angle APB \overset{m}{=} \overset{\frown}{AB}$.	12. Why?

Corollary 53a. *A spherical angle has the same measure as the dihedral angle formed by the planes of the two circles.*

Corollary 53b. *If an arc of a great circle passes through the pole of another great circle, the angle formed is a right angle.*

Theorem 54. *Any side of a spherical triangle is less than the sum of the other two sides.*

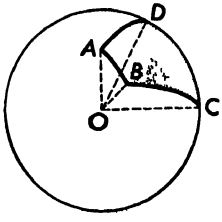


Given: Sphere O and spherical $\triangle ABC$.
To prove: $AC < AB + BC$.

Suggestions: Draw radii OA, OB, and OC. $\measuredangle AOB$, BOC, and COA are face $\measuredangle$ of the trihedral $\angle$ formed. How does one face $\angle$ of a trihedral $\angle$ compare with the sum of the other two? How is $\overset{\frown}{AB}$ measured?

Proof left to you.

Theorem 55. *The sum of the sides of any convex spherical polygon is less than* 360°.



Given: $ABCD$ is a convex spherical polygon.
To prove: $\overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{CD} + \overset{\frown}{DA} < 360°$.

Suggestions: Draw radii AO, BO, CO, and DO. $\angle DOA$, $\angle COD$, etc., are face angles of the polyhedral $\angle$ with O as a vertex.

Proof left to you.

POLAR TRIANGLES AND SYMMETRIC SPHERICAL POLYGONS

The following ten items are not exercises in the usual sense. All of them should be studied carefully. Two new ideas are developed and definitions are given. The "exercise approach" is used to help you *discover* the meaning of the new ideas better than you would if they were stated more formally.

EXERCISES

1. Given spherical $\triangle ABC$. Draw a diameter of the sphere from each vertex of the triangle calling the other ends of the diameters A', B', and C' respectively. Why can arcs of great circles be drawn from A' to B', B' to C', and C' to A'? Would the great circle containing $A'B'$ also contain $\overset{\frown}{AB}$? Why or why not? How do the sides of $\triangle A'B'C'$ compare with the sides of $\triangle ABC$? The $\measuredangle$?

   If these two triangles were drawn on an orange (to represent a sphere) and could be "cut out" and moved around, would it be possible to place one triangle on top of the other one so they would coincide? The foregoing leads to the following definition. Two **spherical polygons** on the same sphere or on equal spheres are **symmetric** if their *corresponding parts* are respectively *equal*, but are *arranged in reverse order.**
2. Explain how to draw two symmetric spherical quadrilaterals on a sphere.
3. Would you accept the following statement as valid: *Spherical triangles which correspond to vertical trihedral angles at the center of a given sphere are themselves symmetric.* Explain.

*See discussion of symmetric polyhedral angles on pages 59–60.

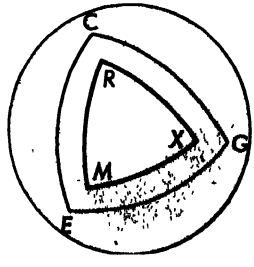
4. Write a definition of *congruent spherical polygons*. Check your definition with the one for congruent figures on page 177.
5. Given spherical $\triangle ABC$ on sphere O. (Use an orange or rubber ball for a sphere.) Sketch the arc of a great circle such that A will be its pole. This arc will be a ? length from A. Sketch an arc such that B will be its pole. Make it intersect the first arc drawn. Draw a third arc such that C will be *its* pole, and make it intersect each of the other two. Call the new spherical $\triangle MNR$. $\triangle MNR$ is called the *polar triangle* of $\triangle ABC$. Read: If the *vertices* of a given spherical triangle *are the poles* of the ? of a second spherical triangle, then *the second triangle is called the* **polar triangle** of the given one.

6. **Given:** $\triangle RMX$ is the polar triangle of spherical $\triangle CGE$.

   Complete: (a) ? is the pole of $\overset{\frown}{RX}$.

   (b) ? is the pole of $\overset{\frown}{RM}$.

   (c) $\overset{\frown}{MX}$ is a ? distance from C.



7. Will the sides of a polar triangle ever intersect the sides of the given triangle? Explain.
8. Given spherical $\triangle MRT$. Draw its polar triangle, but extend each arc of the polar triangle to be a great circle. Into how many parts is the surface of the sphere divided? What kinds of figures are formed? What other conclusions can be drawn?
9. Given an arc of a great circle. Tell how it could be bisected. Hint: In the plane of the great circle draw the central angle determined by the arc.
10. Describe a method for bisecting a spherical angle.

TWO NEW ASSUMPTIONS*

After a study of the previous set of exercises you should be ready to accept Assumptions 21 and 22 as reasonable ones.

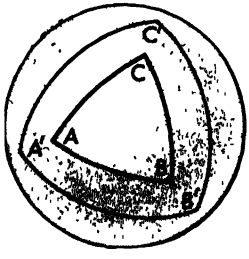
Assumption 21. *If two spherical triangles on the same or equal spheres have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent if the parts are arranged in the same order, but they are symmetric if the parts are arranged in reverse order.*

*Read again the footnote on page 9.

Assumption 22. *If two spherical triangles on the same or equal spheres have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent if the parts are arranged in the same order, but they are symmetric if the parts are arranged in reverse order.*

THEOREMS

Theorem 56. *If one spherical triangle is the polar triangle of another, then the second is the polar triangle of the first.*

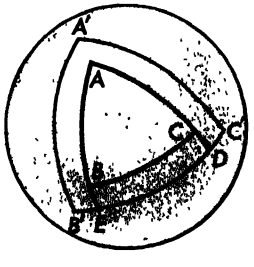


Given: $\triangle A'B'C'$ is the polar $\triangle$ of $\triangle ABC$.
To prove: $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$.

PROOF

Argument	*Defense*
1. A is the pole of $\overset{\frown}{B'C'}$, and B is the pole of $\overset{\frown}{A'C'}$.	1. Given.
2. The distance from A to C' and from B to C' is a quadrant.	2. Why?
3. $\therefore$ C' is a pole of $\overset{\frown}{AB}$.	3. Why?
4. In like manner, B' is a pole of $\overset{\frown}{AC}$ and A' is a pole of $\overset{\frown}{BC}$.	4. Why?
5. $\therefore$ $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$.	5. Definition.

Theorem 57. *In two polar triangles each angle of one is the supplement of the opposite side of the other.*



Given: $\triangle A'B'C'$ is the polar triangle of $\triangle ABC$.
To prove:

$\angle A + \overset{\frown}{B'C'} = 180°$, $\angle A' + \overset{\frown}{BC} = 180°$;

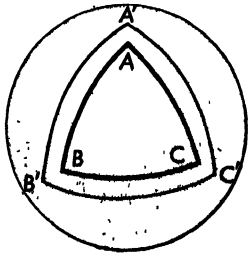
$\angle B + \overset{\frown}{A'C'} = 180°$, $\angle B' + \overset{\frown}{AC} = 180°$;

$\angle C + \overset{\frown}{A'B'} = 180°$, $\angle C' + \overset{\frown}{AB} = 180°$.

PROOF

Argument	*Defense*
1. Extend $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ to meet $\overset{\frown}{B'C'}$ in points E and D.	1. Why possible?
2. B' is a pole of $\overset{\frown}{AC}$ and C' is a pole of $\overset{\frown}{AB}$.	2. Why?
3. $\overset{\frown}{B'D} = 90°$ and $\overset{\frown}{C'E} = 90°$.	3. Why?
4. $\overset{\frown}{B'E} + \overset{\frown}{ED} = 90°$, and $\overset{\frown}{DC'} + \overset{\frown}{ED} = 90°$.	4. Why?
5. $\overset{\frown}{B'E} + \overset{\frown}{ED} + \overset{\frown}{DC'} + \overset{\frown}{ED} = 180°$.	5. Why?
6. $\overset{\frown}{B'C'} + \overset{\frown}{ED} = 180°$.	6. Why?
7. $\angle A \overset{m}{\cong} \overset{\frown}{DE}$.	7. Why?
8. $\therefore \angle A + \overset{\frown}{B'C'} = 180°$.	8. Why?

Theorem 58. *The sum of the angles of a spherical triangle is greater than* 180° *and less than* 540°.



Given: Spherical $\triangle ABC$.
To prove: $\angle A + \angle B + \angle C > 180°$ and $< 540°$.

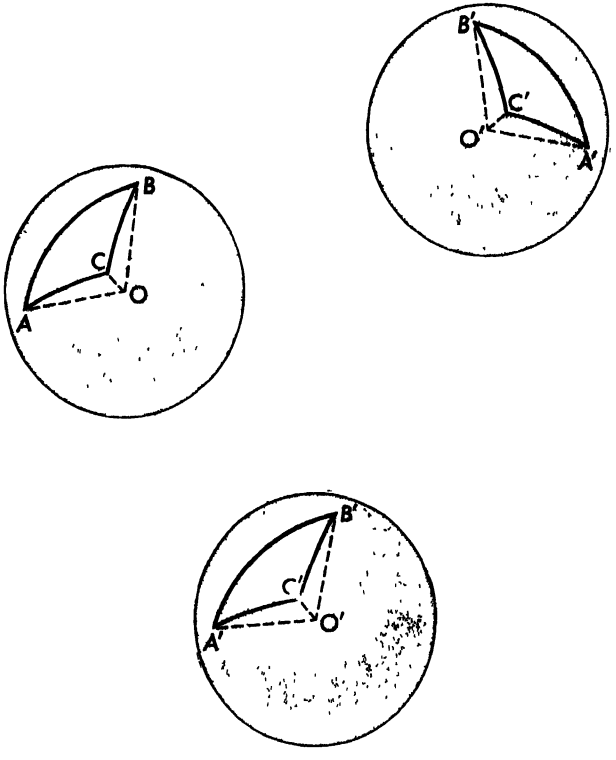
PROOF

Argument	*Defense*
1. Draw $A'B'C'$, polar $\triangle$ of ABC.	1. Why possible?
2. $\angle A + \overset{\frown}{B'C'} = 180°$, $\angle B + \overset{\frown}{A'C'} = 180°$, and $\angle C + \overset{\frown}{A'B'} = 180°$.	2. Why?
3. $\angle A + \angle B + \angle C + \overset{\frown}{B'C'} + \overset{\frown}{A'C'} + \overset{\frown}{A'B'} = 540°$.	3. Why?
4. $\overset{\frown}{B'C'} + \overset{\frown}{A'C'} + \overset{\frown}{A'B'} < 360°$.	4. Why?
5. $\angle A + \angle B + \angle C > 180°$.	5. Unequals subtracted from equals.
6. $\overset{\frown}{B'C'} + \overset{\frown}{A'C'} + \overset{\frown}{B'A'} > 0°$.	6. Why?
7. $\therefore \angle A + \angle B + \angle C < 540°$.	7. Why? (See steps 3 and 6.)
8. $\therefore \angle A + \angle B + \angle C > 180°$ and $< 540°$.	8. Why? (See steps 5 and 7.)

What is the greatest number of right angles possible in a *plane* triangle? In a *spherical* triangle? Use Th. 58 to defend your answer to the last question.

When a *spherical triangle* has two right angles it is **birectangular.** One with three right angles is **trirectangular.** Make a sketch to illustrate each case.

Theorem 59. *If two spherical triangles on the same or equal spheres have the three sides of one equal respectively to the three sides of the other, then the triangles are either congruent or symmetric.*



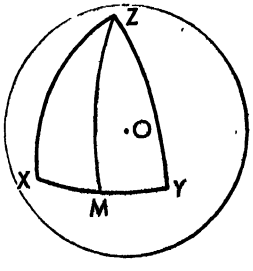
Given: Spherical $\triangle ABC$ and $A'B'C'$ with $\overset{\frown}{AB} = \overset{\frown}{A'B'}$, $\overset{\frown}{BC} = \overset{\frown}{B'C'}$, and $\overset{\frown}{AC} = \overset{\frown}{A'C'}$. Spheres O and O' are equal.

To prove: $\triangle ABC$ is either $\cong$ or symmetric to $\triangle A'B'C'$.

PROOF

Argument	*Defense*
1. Draw radii AO, BO, CO, $A'O'$, $B'O'$, and $C'O'$.	1. Why possible?
2. $\angle BOA = \angle B'O'A'$, $\angle BOC = \angle B'O'C'$, and $\angle COA = \angle C'O'A'$.	2. Why?
3. Trihedral $\angle O\text{-}ABC$ is equal to or is symmetric to trihedral $\angle O'\text{-}A'B'C'$.	3. Why?
4. Dihedral $\measuredangle C\text{-}OA\text{-}B$, $A\text{-}OB\text{-}C$, and $B\text{-}OC\text{-}A$ are equal respectively to dihedral $\measuredangle C'\text{-}O'A'\text{-}B'$, $A'\text{-}O'B'\text{-}C'$, and $B'\text{-}O'C'\text{-}A'$.	4. If two trihedral $\measuredangle$ have the three face $\measuredangle$ of one equal . . ..
5. Spherical $\measuredangle A$, B, and C equal respectively spherical $\measuredangle A'$, B', and C'.	5. A spherical $\angle$ has the same measure as . . ..
6. $\therefore$ $\triangle ABC$ and $A'B'C'$ are either $\cong$ or symmetric.	6. Definition.

Corollary 59a. *If two sides of a spherical triangle are equal, the angles opposite those sides are equal.*



Given: $\overset{\frown}{XZ} = \overset{\frown}{ZY}$ in spherical $\triangle XYZ$.
To prove: $\angle X = \angle Y$.

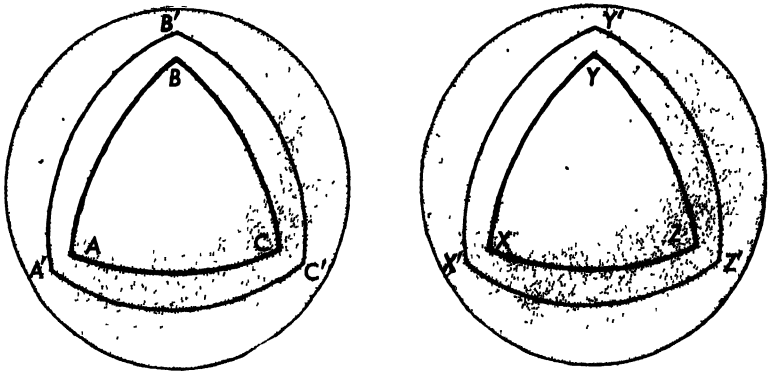
Suggestion: Draw a line from Z to the mid-point of $\overset{\frown}{XY}$.

Corollary 59b. *An equilateral spherical triangle is also equiangular.*

Corollary 59c. *Two symmetric isosceles triangles are congruent.*

Suggestions: Since the $\triangle$ are symmetric, the parts are equal and are arranged in reverse order. However, since they are isosceles, show that the parts may be considered to be arranged in the same order.

Theorem 60. *If two spherical triangles on the same or equal spheres have the three angles of one equal respectively to the three angles of the other, the triangles are either congruent or symmetric.*



Given: Spherical $\triangle ABC$ and XYZ with $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$.

To prove: $\triangle ABC$ and XYZ are either $\cong$ or symmetric.

PROOF

Argument	*Defense*
1. Draw the polar $\triangle$ of $\triangle ABC$ and of $\triangle XYZ$.	1. Why possible?
2. $\angle A + \overset{\frown}{B'C'} = 180°$ and $\angle X + \overset{\frown}{Y'Z'} = 180°$.	2. Why?
3. But $\angle A = \angle X$ and so $\overset{\frown}{B'C'} = \overset{\frown}{Y'Z'}$.	3. Why?
4. In the same way $\overset{\frown}{A'B'} = \overset{\frown}{X'Y'}$, and $\overset{\frown}{A'C'} = \overset{\frown}{X'Z'}$.	4. Steps 2 and 3.
5. $\therefore \triangle A'B'C'$ and $X'Y'Z'$ are $\cong$ or symmetric.	5. Why?
6. $\angle A' = \angle X'$, $\angle B' = \angle Y'$, and $\angle C' = \angle Z'$.	6. Why?
7. $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$, and $\triangle XYZ$ is the polar $\triangle$ of $\triangle X'Y'Z'$.	7. Why?

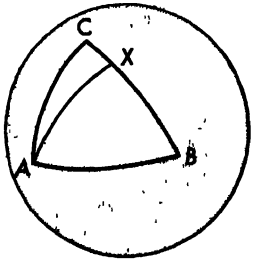
8. $\angle A' + \widehat{BC} = 180°$ and $\angle X' + \widehat{YZ} = 180°$.	8. Why?
9. $\widehat{BC} = \widehat{YZ}$.	9. Why?
10. In the same way $\widehat{AB} = \widehat{XY}$ and $\widehat{AC} = \widehat{XZ}$.	10. Why?
11. $\therefore$ $\triangle ABC$ and XYZ are $\cong$ or symmetric.	11. Why?

Corollary 60a. *If two angles of a spherical triangle are equal, the sides opposite these angles are equal.*

Suggestions: Draw the polar triangle and show that it is isosceles.

Corollary 60b. *An equiangular spherical triangle is also equilateral.*

Theorem 61. *If two angles of a spherical triangle are unequal, the side opposite the greater angle is the greater.*

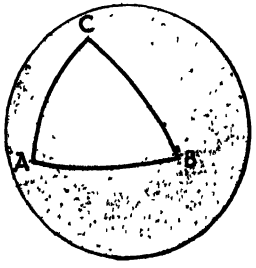


Given: Spherical $\triangle ABC$ with $\angle A > \angle B$.
To prove: $\widehat{BC} > \widehat{AC}$.

PROOF

Argument	*Defense*
1. Draw $\widehat{AX}$ an arc of a great circle such that $\angle BAX = \angle B$.	1. Why possible?
2. $\widehat{AX} = \widehat{XB}$.	2. Why?
3. $\widehat{CX} + \widehat{AX} > \widehat{AC}$.	3. Why?
4. $\widehat{CX} + \widehat{XB} > \widehat{AC}$.	4. Why?
5. $\therefore \widehat{BC} > \widehat{AC}$.	5. Why?

Theorem 62. *If two sides of a spherical triangle are unequal, the angle opposite the greater side is the greater.*

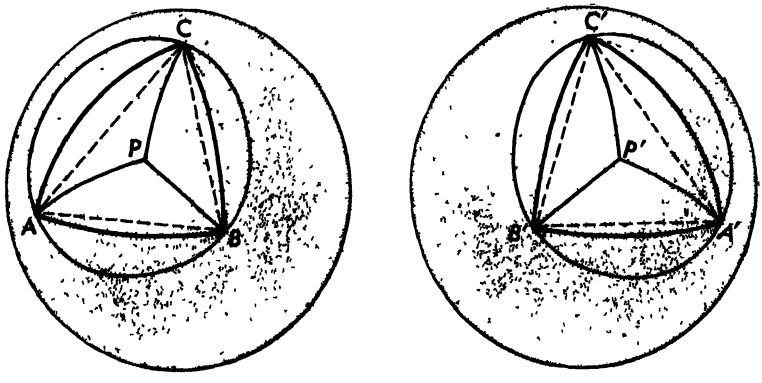


Given: Spherical $\triangle ABC$ with $\overset{\frown}{CB} > \overset{\frown}{AC}$.
To prove: $\angle A > \angle B$.

Suggestions: One of the following statements *must* be true: $\angle A = \angle B$, $\angle A < \angle B$, or $\angle A > \angle B$.

Develop an indirect proof.

Theorem 63. *Two symmetric spherical triangles are equal in area.*



Given: Symmetric spherical $\triangle ABC$ and $A'B'C'$ on equal spheres.
To prove: $\triangle ABC = \triangle A'B'C'$.

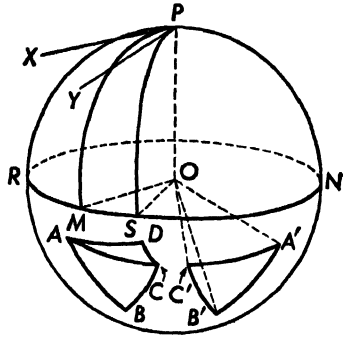
PROOF

Argument	*Defense*
1. Pass planes through points A, B, and C, and through A', B', and C' to form two small circles.	1. Why possible?
2. $\overset{\frown}{AC} = \overset{\frown}{A'C'}$, $\overset{\frown}{AB} = \overset{\frown}{A'B'}$, and $\overset{\frown}{BC} = \overset{\frown}{B'C'}$.	2. Why?

3. Chords AC and $A'C'$, BC and $B'C'$, and BA and $B'A'$ are respectively equal.	3. Why?
4. Plane $\triangle ABC$ = plane $\triangle A'B'C'$.	4. Why?
5. $\odot ABC = \odot A'B'C'$.	5. Why?
6. Let P and P' be the poles of the two circles. Then $\widehat{PA} = \widehat{PB} = \widehat{PC} = \widehat{P'A'} = \widehat{P'B'} = \widehat{P'C'}$.	6. Why?
7. $\triangle APB \cong \triangle A'P'B'$, $\triangle BPC \cong \triangle B'P'C'$, and $\triangle CPA \cong \triangle C'P'A'$.	7. Two symmetric isosceles . . ..
8. $\triangle APB + \triangle BPC + \triangle CPA = \triangle A'P'B' + \triangle B'P'C' + \triangle C'P'A'$.	8. Why?
9. $\therefore \triangle ABC = \triangle A'B'C'$.	9. Why?

CHAPTER REVIEW

1. Given: Sphere O with spherical $\triangle ABC$, ACD, and $A'B'C'$. $AB = A'B'$, $BC = B'C'$, and $AC = A'C'$. Also $A'B' = A'C'$. RMN is a great circle, and lines XP and YP are tangent to $\widehat{MP}$ and $\widehat{SP}$ respectively. Is



(a) $\widehat{AB} < \widehat{BC} + \widehat{CA}$?
(b) $A'C' = AC$?
(c) $\angle A' + \angle B' + \angle C' = 180°$?
(d) $\triangle ABC \cong \triangle A'B'C'$?
(e) $AB + BC + CD + DA > AB + BC + AC$?
(f) $\triangle ABC$ symmetric to $\triangle A'B'C'$?
(g) $\angle A' \neq \angle C'$?
(h) $AB + BC + CD + DA < 360°$?
(i) $\angle A' + \angle B' + \angle C' < 540°$?
(j) $\angle B = \angle ACB$?
(k) the measure of dihedral $\angle A'$-$C'O$-B' the same as the measure of $\angle B'C'A'$?
(l) $\angle B' = \angle C'$?
(m) $\angle D = \angle DCA$?
(n) $\angle MPS \overset{m}{\cong} \angle XPY$?

(o) $PO \perp MO$?
(p) $XP \parallel OS$?
(q) spherical $\angle MPS \overset{m}{=} \overset{\frown}{MS}$?
(r) $PO \perp PY$?
(s) $\overset{\frown}{RS}$ a quadrant?
(t) $\overset{\frown}{PS}$ a quadrant?
(u) $\overset{\frown}{AC}$ the same size, in degrees, as $\angle C'OA'$?

2. Define polar triangles.
3. Quote as many theorems as possible pertaining to polar triangles.
4. Quote theorems from plane geometry about sides and angles of triangles, and then quote a corresponding statement about spherical triangles.
5. *The straight lines joining any two points in a circle of a sphere to a pole of the circle are equal.*
6. Bring your **Tool Box** up to date.

EXERCISES

1. If the angles of a spherical triangle are 75°, 80°, and 95°, how many degrees are in the sides of its polar triangle?
2. (a) Sphere O has a radius of 12 inches. What is the polar distance of any point on a great circle of the sphere?
   (b) What is the length of the chord of the arc which represents the polar distance?
3. *If an arc of a great circle passes through the pole of another great circle, the two circles are perpendicular to each other.*
4. *The base angles of an isosceles spherical triangle are equal.*
5. *If two angles of a spherical triangle are equal, the triangle is isosceles.*
6. Find the size of the angles of a triangle if the sides of its polar triangle are $\overset{\frown}{80}°$, $\overset{\frown}{65}°$, and $\overset{\frown}{100}°$.
7. Prove: If two spherical triangles on the same sphere are mutually equiangular their polar triangles are equilateral.
8. If three mutually perpendicular planes are passed through the center of a given sphere, prove that eight congruent, trirectangular triangles are formed.
9. *The sum of all the spherical angles about a point on a sphere is equal to* 360°.
10. Prove: In a birectangular spherical triangle the sides opposite the right angles are quadrants, and the third angle is measured by the side opposite it.

11. Prove: The arc drawn from the vertex of an isosceles spherical triangle to the mid-point of the base bisects the vertex angle and is perpendicular to the base.

12. If arcs of great circles are drawn from the extremities of one side of a spherical triangle to a point within the triangle, prove that the sum of these arcs is less than the sum of the other two sides of the spherical triangle.

13. State and prove a theorem about the relationship of an exterior angle of a spherical triangle and the sum of the two opposite interior angles of the spherical triangle.

14. Find the radius of a sphere inscribed in a regular tetrahedron which is 12 inches on an edge.

15. What is the locus of the centers of all spheres which are tangent to the faces of a trihedral angle?

16. Will the sides of a spherical triangle ever intersect the sides of its polar triangle? Explain.

17. Complete: Each side of a trirectangular spherical triangle is a ?.

18. Complete: The angle of two curves passing through a common point is the angle formed by the two ? to the two curves at ? .?.

In Exs. 19–23, show *how to draw:*

19. A great circle on a sphere.

20. A small circle on a sphere.

21. A circle with a given radius on a sphere.

22. A great circle through two given points on a sphere.

23. Two symmetric spherical polygons of any given number of sides.

In Exs. 24–29, show how to:

24. Find the pole of a great circle of a sphere.

25. Find the radius of a given sphere.

26. Bisect a spherical angle.

27. Bisect the arc of a great circle.

28. Inscribe a sphere in a cube.

29. Circumscribe a sphere about a cube.

CHAPTER TEST

1. How many right angles can a spherical triangle have? How many obtuse angles? Explain.

2. What is the measure of a spherical angle?

3. If the sides of a spherical triangle are $\overset{\frown}{120}°$ and $\overset{\frown}{70}°$, what is the upper limit for the size of the third side of the triangle?

4. Three sides of a spherical quadrilateral are $\overset{\frown}{88}°$, $\overset{\frown}{75}°$, and $\overset{\frown}{70}°$. What is known about the number of degrees in the fourth side?
5. Prove: If two sides of a spherical triangle are quadrants, then the third side is the measure of the opposite angle.
6. Find the number of degrees in the sides of a spherical triangle if the angles of its polar triangle are 82°, 77°, and 65°.
7. Find the number of degrees in the angles of a spherical triangle if the sides of its polar triangle are $\overset{\frown}{68}° \overset{\frown}{35}'$, $\overset{\frown}{83}° \overset{\frown}{48}'$, and $\overset{\frown}{39}° \overset{\frown}{36}'$.
8. Tell whether or not it is possible to have spherical triangles with the following sets of angles. Explain your answers.

   (a) 75°, 98°, 12°. (b) 60°, 85°, 35°.
   (c) 190°, 168°, 185°. (d) 110°, 5°, 80°.
   (e) 150°, 100°, 210°. (f) 126°, 210°, 204°.

9. If a spherical triangle has one right angle, can the sum of the other two angles be equal to or less than a right angle? Explain.
10. Prove: Two great circles on a sphere intersect and the vertical angles formed are equal.
11. Quote as many statements as possible which conclude "then the spherical triangles are congruent or symmetric."

Complete:

12. The sum of the sides of a spherical polygon . . ..
13. The sum of the angles of a spherical triangle . . ..
14. If two angles of a spherical triangle are unequal . . ..
15. Quote a converse of the statement in Ex. 14.
16. If two sides of a spherical triangle are equal . . ..

Keep the Iron Hot

Set your goal at 100 *percent accuracy.*

Section 1: Everyday Reasoning

What conclusions can be drawn from the data given in each situation described below? Try not to "jump at conclusions."

1. In High School A the successful completion of a course in plane geometry is required for graduation in a college preparatory course. Of the twelve boys in one of the geometry classes, eight plan to attend college. A grade of A, B, or C is necessary for college entrance credit, although D is a

passing grade. Miss Day requires a certain amount of homework to be done by pupils in her geometry class. Mr. Wilkins requires more homework than Miss Day does.

2. A certain high school does not offer third year French or third year Latin. However, students who wish to do so may go across the street to the college and take advanced French or Latin with the college students. Last year eight high school pupils took French III and four of them made A and four made B. Three pupils took Latin III with the college class and all three made A. This year six other high school pupils are taking Latin III at the college and four are taking French III.
3. In a certain high school the Juniors were making plans for their prom. In a class meeting Bill argued that they should hire the orchestra he recommended which would cost \$150. Some students agreed with him, but another group said the class should look further for an orchestra which charged less because the social regulations stated: "A class may contract any orchestra which charges a reasonable amount." Which group was right?

Section 2: Equations and Algebraic Problems

Solve and check:

1. $y^2 - 11y + 30 = 0$
2. $2x^2 + 7x = 4$
3. $a^2 + 36 = 13a$
4. $\frac{1}{3}(4x - 2) = 6 + \frac{x}{2}$
5. $x - (5x - 12) = 2x + (6x - 30)$
6. $\frac{1}{2}(8m - 6) = m + 3$
7. $3m^2 + 12m = 135$
8. $\frac{4 - x}{5} - \frac{x}{2} = \frac{2x - 5}{5}$

9. Solve each of the following quadratic equations both algebraically *and* graphically:

   (a) $3x^2 + 5x - 2 = 0$.
   (b) $x^2 - 2x = 15$.
   (c) $2x^2 - 13x + 6 = 0$.

10. The altitude of a triangle is five units more than twice the base. If the area of the triangle is 26 square units, find the length of the base.
11. The length of a rectangle is 3 feet more than the width. If the length is doubled and the width is increased by 2 feet, the area will then be 8 square feet less than three times the original area. Find the dimensions of the original rectangle.

12. One side of a right triangle is 4 inches less than the hypotenuse and another side is 2 inches less than the hypotenuse. Find the length of the hypotenuse.
13. Mr. Brown had to make a business trip to a city 150 miles away. His average speed on the return trip was 5 miles per hour greater than going. The return trip took an hour less time. Find his average rate of speed both going and returning.
14. The distance an object, falling from rest, will drop in a given length of time is given by the formula $d = 16t^2$ (d represents distance in feet and t represents time in seconds). If a rock falling over a cliff takes 3.2 seconds to reach the bottom, how many feet does it fall vertically?

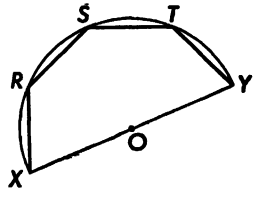
Section 3: Experimentation

1. Given a semicircle. Permit it to rotate about its diameter as an axis, keeping the diameter fixed in its plane. Describe the solid which is generated, as well as its properties as they relate to the similar properties of the semicircle.
2. Draw three equal chords in a semicircle. Then repeat Ex. 1.
3. Draw line XY of indefinite length in a plane. In the same plane, draw *line segment* $AB \parallel XY$. With XY as an axis, remaining fixed in the plane, permit AB to rotate so that it is always $\parallel XY$. Describe the surface generated and its properties.
4. Repeat Ex. 3, but draw AB so that it intersects XY.
5. Repeat Ex. 3, but draw AB so that it is oblique to XY but does not intersect it.

CHAPTER 10 *Spherical Areas and Volumes*

SECTION I. AREAS

Do not begin this chapter until you have completed the five exercises in Section III on page 158. When we developed principles relating to areas and volumes of cylinders and cones, it was necessary to apply the theory of limits.* This theory will be used again in developing formulas for the area and the volume of a sphere.



As a preliminary to developing proofs establishing these formulas, study the following procedure. Inscribe a half-polygon of an even number of equal sides in a semicircle, as in the accompanying figure. Revolve the figure about the diameter XY as an axis, keeping XY fixed in its plane. Obviously, the semicircle will generate a sphere and the half-polygon will generate an inscribed polyhedron. What kind of *surface* will the chord XR generate? Chord RS? How is the area of each one of these *surfaces* found?

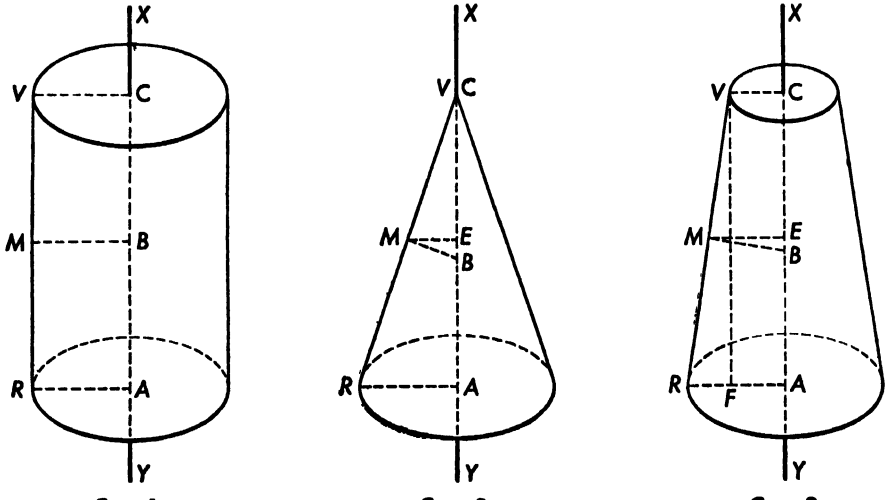
If the number of equal sides of the original half-polygon is doubled, redoubled, and so on indefinitely, it is obvious that the total area of the polyhedron generated by the semi-polygon approaches, *as a limit*, the area of the sphere generated by the semicircle.

Theorem 64, and the formula developed, may not seem to be directly related to the foregoing discussion or to the area of a sphere. However, the proof of Th. 65, which develops the formula for the area of a sphere, depends upon the validity of Th. 64.

*See discussion on pages 110–111.

THEOREMS

Theorem 64. *If a straight line segment revolves about an axis in its plane, the area of the surface generated equals the product of the projection of the line segment on the axis and the circumference of the circle whose radius is a perpendicular from the axis to the line segment at its mid-point.*



Given: Line segment RV, AC the *projection* of RV on XY, and BM the $\perp$ bisector of RV. S represents the area generated by RV revolving about XY as XY remains fixed in its plane.
To prove: $S = AC \times 2\pi(BM)$.

Case 1. When $RV \parallel XY$. The area S is the lateral area of a right circular cylinder.

PROOF

Argument	*Defense*
1. $S = RV \times 2\pi(RA)$.	1. Why?
2. $RV = AC$.	2. Why?
3. $BM = RA$.	3. Why?
4. $\therefore S = AC \times 2\pi(BM)$.	4. Why?

Case 2. When RV is not $\|$ AC and V lies on XY. The area generated is the lateral area of a right circular cone.

PROOF

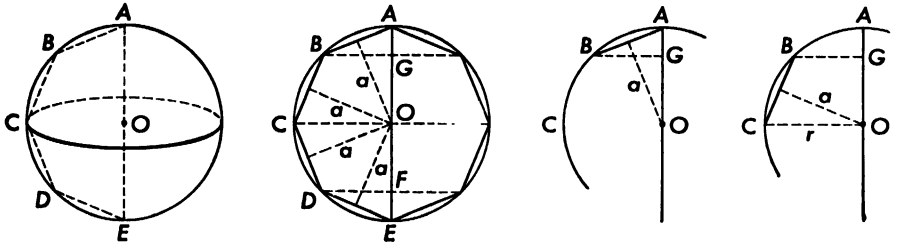
Argument	*Defense*
1. Draw $ME \perp AC$.	1. Why possible?
2. $S = \frac{1}{2}RV \times 2\pi(RA)$.	2. Why?
3. In $\sim$ $\triangle RAV$ and MEV, $\frac{VM}{VR} = \frac{ME}{RA} = \frac{1}{2}$, and $RA = 2(ME)$.	3. Why?
4. $S = RV \times 2\pi(ME)$.	4. By substitution.
5. $\triangle MEB \sim \triangle RAV$.	5. If two $\measuredangle$ of one $\triangle$ = two ....
6. $\frac{MB}{RV} = \frac{ME}{AC}$ or $(MB) \times (AC) = (RV) \times (ME)$.	6. Explain.
7. $\therefore S = AC \times 2\pi(MB)$.	7. Substitution in step 4.

Case 3. When RV is not $\|$ XY but does not extend to AC. The area generated is the lateral area of the frustum of a right circular cone.

PROOF

Argument	*Defense*
1. Drop $VF \perp RA$, and $ME \perp AC$.	1. Why possible?
2. $S = \pi \times RV(RA + VC)$.	2. Why?
3. $ME = \frac{1}{2}(RA + VC)$, or $2ME = RA + VC$.	3. The median of a trapezoid ....
4. $S = \pi \times RV \times 2(ME)$.	4. By substitution.
5. $\triangle RFV \sim \triangle MEB$.	5. Why? ($\angle BME = \angle RVF$).
6. $\frac{RV}{MB} = \frac{FV}{ME}$ or $RV \times ME = (MB) \times (FV)$.	6. Explain.
7. $FV = AC$.	7. Why?
8. $S = 2\pi \times (MB) \times (AC)$.	8. Why?

Theorem 65. *The area of a sphere with radius r is* $4\pi r^2$.



Given: Sphere O with radius r and area S.
To prove: $S = 4\pi r^2$.

PROOF

Argument	*Defense*
1. Let ACE be a semicircle of a great circle of sphere O. $\therefore$ sphere O could be generated by revolving the semicircle about its diameter AE.	1. Why?
2. Construct half of a regular polygon with an even number of sides in semicircle ACE.	2. Defend.
3. From the vertices B, C, D of the semi-polygon drop $\perp$s to AE.	3. Defend.
4. AG, GO, OF, and FE are the *projections* of AB, BC, CD, and DE respectively on AE.	4. Why?
5. Draw the apothems a to the sides of the semi-polygon.	5. Defend.
6. Apothems a are equal and are the $\perp$ bisectors respectively of AB, BC, CD, and DE.	6. Why?
7. As each side of the semi-polygon revolves about axis AE, the area of the surfaces generated are:	7.
(a) $S_1 = (AG)\cdot 2\pi a$	(a) Th. 64, Case 2.
(b) $S_2 = (GO)\cdot 2\pi a$	(b) Th. 64, Case 3.
(c) $S_3 = (OF)\cdot 2\pi a$	(c) Why?
(d) $S_4 = (FE)\cdot 2\pi a$	(d) Why?

8. $\therefore$ the area of the surface of the polyhedron obtained by generating semi-polygon $ABCDE$ is: $S_p = 2\pi a(AG + GO + OF + FE)$. — 8. Why?
9. But $AG + GO + OF + FE = AE = 2r$. — 9. Why?
10. $\therefore$ the surface of the polyhedron is: $S_p = 4\pi r \cdot a$. — 10. Why?
11. As the number of sides of the inscribed semi-polygon is doubled, redoubled, etc., the figure will approach a semicircle as a limit, the apothems will approach the radius of the semicircle as a limit, and the polyhedron generated will approach sphere O as a limit. Therefore, eventually, *for all practical purposes* $a = r$, and,
12. The surface of sphere O is: $S = 4\pi r^2$. — 12. Why?

Before defending the validity of the five corollaries of Th. 65, study the following paragraphs.

A **zone** is that portion of the *surface of a sphere* included *between two parallel planes* which cut the sphere. The *circles* formed by the parallel

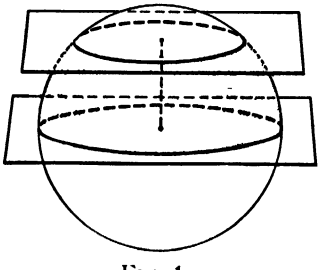


FIG. 1.

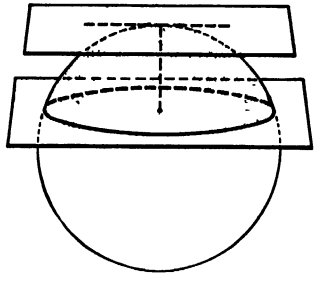


FIG. 2.

planes cutting the sphere are the **bases** of the zone and the perpendicular distance between the parallel planes is the **altitude** of the zone. When one of the parallel planes is *tangent* to the sphere, as in Fig. 2, a **zone of one base** is formed. Does a zone of one base have an altitude?

In geography you have referred to certain portions of the earth's surface as zones. The meaning of the term "zone" in geography and in geometry is the same, although in geography, certain zones are given the special names, Torrid Zone, Temperate Zone and Frigid Zone. (There are two each of the two last-named zones. Why?) Which of the zones of the earth's surface are one-base zones?

There is a second method for obtaining a zone. If a *great circle* of a sphere revolves about its diameter as an axis, *any arc* of the great circle generates a *zone.* Use Fig. 1 to show how the zone determined by the two parallel planes could be regarded as the result of the second method described above.

Corollary 65a. *The area of a sphere equals the product of its diameter and the circumference of a great circle.*

Corollary 65b. *The area of a sphere equals four times the area of one of the great circles of the sphere.*

Corollary 65c. *The areas of two spheres have the same ratio as the squares of the radii or as the squares of the diameters.*

Corollary 65d. *The area of a zone equals the product of its altitude and the circumference of a great circle of the sphere.*

Corollary 65e. *The area of a zone is given by the formula* $S = 2\pi rh$, *when* S *represents the area,* r *the radius of the sphere, and* h *the altitude of the zone.*

A NEW UNIT OF MEASURE

To measure a *plane angle*, we use a small plane angle which is $\frac{1}{360}$ of the complete rotation of a straight line about a point on the line. To measure an *arc*, we may use a standard unit of linear measure to find its length *or* we may use an *arc degree* which is $\frac{1}{360}$ of the circle of which the arc is a part. To measure a *spherical surface* we obviously must use a spherical surface as a unit. To standardize a unit for measuring a spherical surface, mathematicians have agreed that the unit shall be a tiny spherical triangle, *two sides of which are* **quadrants** of the sphere with *the third side an arc of* 1°. This unit is called a **spherical degree.** In spherical $\triangle ABC$ in Fig. 3, AB and AC are quadrants of sphere O and hence are 90°, and $\angle BAC = 1°$. How many degrees are in $\measuredangle CBA$ and ACB? Show why a spherical degree could be defined as *a birectangular spherical triangle, the third angle of which is* 1°.

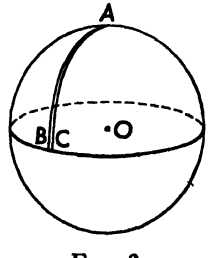


FIG. 3.

A **lune** is a portion of a sphere bounded by the **halves of two great circles** of the sphere. The shaded portion of the surface of sphere O in Fig. 4 is a lune. Use four letters to name it. The angle at which the two half-circles meet, at A or C, is the **angle of the lune.** Since $\angle BAD$ (or $\angle DCB$) is a spherical angle, how will it be measured? A lune in which the *angle of the lune* is 1°, will contain how many *spherical degrees*? Explain. If the angle of the lune of a certain lune is 32°, how many spherical degrees will there be in its area?

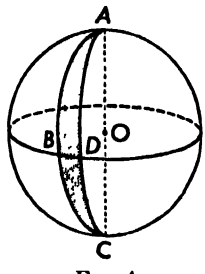


FIG 4.

PRELIMINARY THEOREMS

1. *The number of spherical degrees in the area of a lune equals twice the number of degrees in the angle of the lune.*
2. *The area of a sphere equals* 720 *spherical degrees.*
3. *The area of a lune is to the area of its sphere as the angle of the lune is to* 360°.
4. *One-half the area of a lune of n degrees is equal to the area of a lune of* $\frac{n}{2}$ *degrees.*

Develop informal proofs for each of these four theorems.

SPHERICAL EXCESS

To find the area of a spherical triangle in terms of the new unit of measure, the *spherical degree*, another idea must be understood. In plane geometry it was proved that the sum of the angles of a triangle is 180°, while for a spherical triangle the sum is greater than 180° and less than 540°. *The difference between the sum of the degrees of the angles of a spherical triangle and* 180° is called the **spherical excess** *of the triangle. Spherical excess* is a measure of the **area** *of spherical polygons*, as you will prove in Th. 66 and its corollaries.

The principle of spherical excess holds for spherical polygons of more than three sides. The spherical excess of a spherical quadrilateral is equal to the difference between the sum of the degrees of the angles of the quadrilateral and 360°. The spherical excess of *any* spherical polygon is the difference between the sum of the angles of the polygon and $(n - 2)180°$, where n represents the number of sides of the polygon. Generalizing, the spherical **excess** of a *spherical polygon* means the number of spherical degrees by which the sum of its angles **exceeds** the sum of the angles of a *plane polygon* of the same number of sides.

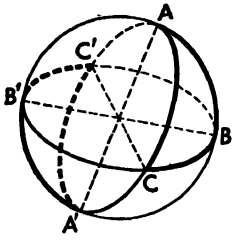
What is the spherical excess of a trirectangular spherical triangle? Of a spherical triangle whose angles are 58°, 120°, and 92°? Of a spherical quadrilateral whose angles are 99°, 76°, 112°, and 89°?

While *spherical excess* is a measure of the area of a spherical polygon, it is relative rather than exact as in the case of area measured in square units. It only helps determine what proportion of the total area of the sphere is covered by the spherical polygon. A spherical polygon whose spherical excess is 60 spherical degrees has an area $\frac{60}{720}$ (or $\frac{1}{12}$) of the total area of its sphere. Why? Therefore, if a spherical polygon of five sides has a spherical excess of 60 spherical degrees and lies on a sphere whose radius is 3 inches, its area will be different from a polygon of the *same* number of sides which lies on a sphere with a 5-inch radius even if its spherical excess is also 60 spherical degrees. Thus the spherical excess of a spherical polygon, divided by 720, gives the fractional part of the *total area of the sphere* which is covered by the polygon.

Can a spherical pentagon have angles of 75°, 80°, 90°, 100°, 70°? Can a spherical quadrilateral have angles of 90°, 90°, 90°, 90°? Why or why not?

THEOREM

Theorem 66. *The number of spherical degrees in the area of a spherical triangle is equal to the number of degrees in its spherical excess.*



Given: Spherical $\triangle ABC$.
To prove: Area of $\triangle ABC = (A + B + C - 180)$ spherical degrees.

PROOF

Argument	*Defense*
1. Draw diameters AA', BB', and CC'. Extend sides of $\triangle ABC$ to complete the great ⑤.	1. Why possible?
2. $\triangle ABC'$ is symmetric to $\triangle A'B'C$.	2. Explain.

3. $\triangle ABC'$ is equal in area to $\triangle A'B'C$.	3. Why?
4. $\triangle ABC + \triangle AC'B$ = lune $CAC'B$ or $2C$ spherical degrees.	4. Why?
5. $\triangle ABC + \triangle A'CB$ = lune $ACA'B$ or $2A$ spherical degrees.	5. Why?
6. $\triangle ABC + \triangle CB'A$ = lune $BCB'A$ or $2B$ spherical degrees.	6. Why?
7. $\triangle ABC + \triangle A'B'C = 2C$ spherical degrees.	7. Substituting $\triangle A'B'C$ in step 4.
8. $\triangle ABC + \triangle A'CB + \triangle ABC + \triangle CB'A + \triangle ABC + \triangle A'B'C = 2(A + B + C)$ spherical degrees or $3\triangle ABC + \triangle A'CB + \triangle CB'A + \triangle A'B'C = 2(A + B + C)$ spherical degrees.	8. Why?
9. But $\triangle ABC + \triangle A'CB + \triangle CB'A' + \triangle AB'C$ = a hemisphere or 360 spherical degrees.	9. Why?
10. From step 8, $2\triangle ABC + (\triangle ABC + \triangle A'CB + \triangle CB'A' + \triangle A'B'C) = 2(A + B + C)$ spherical degrees.	10. Why?
11. $\triangle ABC + 180° = (A + B + C)$ spherical degrees or $\triangle ABC = (A + B + C - 180)$ spherical degrees.	11. Why?

Corollary 66a. *The number of spherical degrees in the area of a spherical polygon is equal to the number of degrees in its spherical excess.*

Corollary 66b. *The area of a spherical polygon, in square units, is found by the formula* $S = \frac{E\pi r^2}{180}$, *where* E *represents the spherical excess and* r *represents the radius of the sphere.*

SECTION II. VOLUMES

PRELIMINARY DISCUSSION FOR THEOREM 67

To develop a proof for Th. 67, *the volume of a sphere equals its area multiplied by one-third its radius*, it will be necessary to apply the theory of limits. Therefore, it will be helpful to reread the discussion on this topic which has appeared on pages 110–111.

In Fig. 5, a cube is circumscribed about sphere *O*. Obviously the volume of the cube is greater than the volume of the sphere. However, if the excess volume of the cube can be gradually "trimmed away" by increasing the number of faces of the *circumscribing* polyhedron, the volumes of these successive solids will **approach** the volume of the sphere *as a* **limit.**

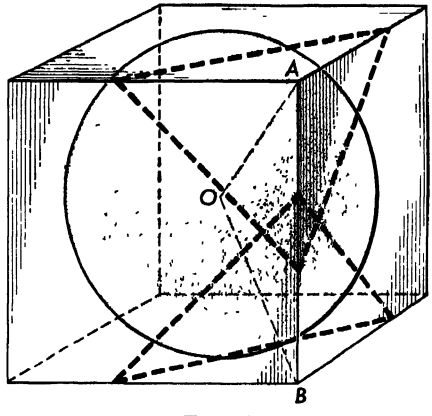


FIG. 5.

To increase the number of faces of the circumscribing solid, thus reducing its volume, first, connect each vertex of the polyhedron with the center of the sphere. (Only two of these lines are shown in Fig. 5.) Then at the points where these lines intersect the sphere, pass planes *tangent* to the sphere. Note that each plane passed tangent to the sphere cuts off a pyramid whose vertex is a vertex of the "old" circumscribing polyhedron and whose base is a face of the resulting "new" polyhedron. Furthermore, all faces of each successive polyhedron are not equal nor do all have the same shape. Nevertheless, each step increases the number of faces of the circumscribing polyhedron, each successive polyhedron will *automatically* circumscribe sphere *O* (why?), and the volume of each successive solid will **approach** the volume of the sphere *as a* **limit.**

If this procedure is continued indefinitely at each vertex of each successive polyhedron, eventually, *for all practical purposes*, the surface of a polyhedron of an infinitely large number of faces will equal the surface of the sphere and the volume of the polyhedron will equal the volume of the sphere.

To make the foregoing more meaningful, apply the procedure to the original cube. In Fig. 5, two of the pyramids that would be "trimmed

away" are indicated by dotted lines at vertices A and B. Notice that in the case of the cube, the planes passed tangent to the sphere will overlap.*

If planes are passed tangent to the sphere to cut off pyramids at the other six vertices of the cube, a new circumscribing polyhedron of fourteen faces results. Figure 6 shows, in perspective, one view of this solid. Eight of its faces are alike. What shape are they? What shape are the other six?

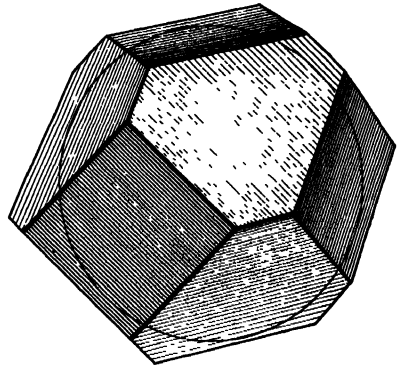


FIG. 6.

Since the volume of *any pyramid* equals one-third its base times its altitude, the volume of any *polyhedron* can be determined if it can be divided into pyramids. In Fig. 6, if lines are drawn from the center of the sphere to each vertex of the circumscribing polyhedron, it will be divided into fourteen pyramids with vertices at the center of the sphere. Each face of the polyhedron will be the base of one of the pyramids and the altitude of *each* pyramid will equal the radius of the inscribed sphere. (Why?) Consequently, the volume of this polyhedron equals the sum of the volumes of the fourteen pyramids and,

$$V = \tfrac{1}{3}r(B_1 + B_2 + B_3 + \cdots B_{14})$$

If the general procedure discussed above is applied to the solid in Fig. 6, and again to the resulting polyhedron, and so on indefinitely, it should be obvious that the volume of any of the successive circumscribing polyhedrons more nearly **approaches** the volume of the sphere than the preced-

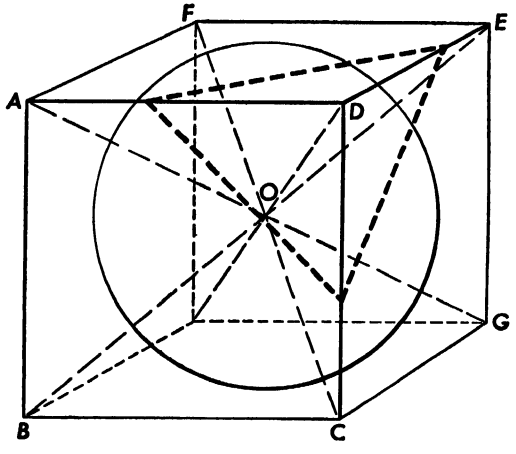
*It can be shown by trigonometric methods that the planes tangent to the sphere, and cutting off pyramids at each vertex of the original *cube*, will cut each edge of the cube 0.63396 of the distance between its vertices. If you wish to construct the polyhedron shown in Fig. 6, you might "construct" a cube from a potato or modeling clay. Along *each* edge, locate points 0.6 of the distance between each of the adjacent vertices, and connect *each* set of three points as is shown by the dotted lines at vertices A and B in Fig. 5. Note that each set of three points determines a plane that will be tangent to the inscribed sphere and that these eight planes overlap in pairs. Cut along the dotted lines with a knife. The resulting polyhedron will *automatically* circumscribe the sphere that was inscribed in the original cube. Observe the "remnants" and you will see that the surface and volume of the cube have been substantially reduced. Select one face of this polyhedron. Do the faces adjacent to it form any sort of pattern?

ing one, and also that the volume of each polyhedron in the series can be determined by the same method used for finding the volume of the solid in Fig. 6.

Write the formula that will represent the volume of a polyhedron of n sides if the polyhedron circumscribes a sphere.

THEOREM

Theorem 67. *The volume of a sphere equals its area multiplied by one-third its radius.*



Given: Sphere O with radius r, volume V, and area S.
To prove: $V = \frac{1}{3}rS$.

PROOF

Argument	*Defense*
1. Circumscribe a cube about sphere O.	1. A cube can be . . ..
2. Connect O with each vertex of the cube, to form six regular pyramids whose bases are faces of the cube.	2. Defend.
3. The altitude of each pyramid is equal to the radius (r) of the sphere.	3. Why?
4. If S_1, S_2, S_3 $\cdots$ S_6 represent the respective areas of the bases of the pyramids, then the respective volumes are $\frac{1}{3}S_1r$, $\frac{1}{3}S_2r$, $\frac{1}{3}S_3r$ $\cdots$ $\frac{1}{3}S_6r$.	4. The volume of any pyramid equals . . ..

5. Therefore the volume, V, of the cube is represented by the formula: $V = \frac{1}{3}r(S_1 + S_2 + S_3 + S_4 + S_5 + S_6)$. 5. Why?

6. If a pyramid is cut off at each vertex of the cube by passing a plane tangent to the sphere at the point where the line joining the vertex and the center of the sphere intersects the sphere, it is obvious that a new circumscribing polyhedron will be formed and that the volume of this polyhedron will be less than the volume of the cube. By continuing this process indefinitely, it is obvious that the surface of each successive polyhedron **approaches** the surface of the sphere *as a* **limit**, and that the volume of each successive polyhedron **approaches** the volume of the sphere *as a* **limit.** Therefore, when the polyhedron has an *infinitely* large number of faces, we may assume that $S_1 + S_2 + S_3 + \cdots S_n = S$ and that $V = \frac{1}{3}rS$.

Corollary 67a. *The volume of a sphere is given by the formula* $V = \frac{4}{3}\pi r^3$, *when* V *represents the volume, and* r *the radius of the sphere.*

Corollary 67b. *The volumes of two spheres have the same ratio as the cubes of their radii or as the cubes of their diameters.*

SPECIAL VOLUMES

If two parallel planes are passed through a sphere, the *solid* formed between them is a **spherical segment** *of a sphere*, as shown in Fig. 7. Sketch a spherical segment of *one* base. How are a spherical segment and a zone different? Alike?

If the semicircle $AMNB$ in Fig. 8 is revolved about diameter AOB as an axis, the sector NOM will generate a solid called a **spherical sector** (O-$MNPQ$ in Fig. 9). The *zone* generated by $\overset{\frown}{MN}$ is the **base** of the *spherical sector.* If the zone generated has only *one* base, the spherical sector is called a **spherical cone.** Sector OMA (Fig. 8) will generate the spherical cone shown in Fig. 10, and $\overset{\frown}{MA}$ will generate a zone of one base.

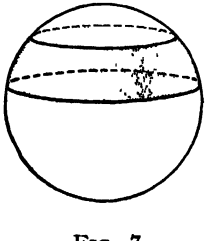


FIG. 7.

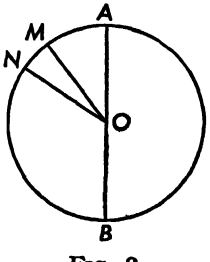


FIG. 8.

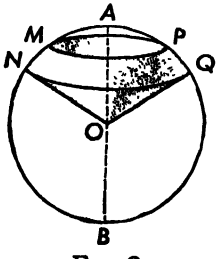


FIG. 9.

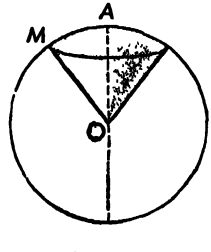


FIG. 10.

The formula for the volume of a **spherical sector** *is* $V = \frac{1}{3}rS$ *when* r *is the radius of the sphere, and* S *is the area of the base of the sector. The volume of a spherical segment* **of two bases** *is* $V = \frac{1}{6}\pi h(3r^2 + 3r'^2 + h^2)$, *where* r *is the radius of the larger base,* r' *is the radius of the smaller base, and* h *is the altitude of the spherical segment.* When a spherical segment has only one base, what is the value of r'? Of $3r'^2$? Write the formula for the volume of a spherical segment of one base. The foregoing formulas are given merely for reference and limited application later in exercises. No formal proofs are offered.

CHAPTER REVIEW

1. Write the formula for:

(a) Surface of a sphere.
(b) Volume of a sphere.
(c) Area of a zone.

Complete the statements in Exs. 2–9.

2. The volumes of two spheres have the same ? as the ? of their radii or as the ? of their ?.

3. The area of a spherical triangle in spherical degrees is . . ..

4. Two symmetric spherical ? are equal in ?.

5. The area of a sphere is equal to four times . . ..

6. The ? of two spheres have the same ratio as the squares of their ?, or as the ? of their ?.

7. The number of spherical degrees in the area of a ? polygon is equal to the number of ? ? in its ? excess.

8. A part of a sphere bounded by the ? of two great ? is called a lune.

9. Define: spherical degree, spherical excess, symmetric spherical triangles, zone, and lune.

10. On a sphere with a 7-inch radius, find the areas of spherical polygons whose angles are:

(a) 95°, 160°, 90°, 120°, 90°. (b) 120°, 110°, 100°, 80°.
(c) 90°, 90°, 90°, 95°. (d) 85°, 102°, 94°, 100°.

11. The diameter of a basketball (spherical) is 10 inches. How many square inches of leather are needed to make the ball, if 30 square inches are allowed for waste?

12. (a) The area of a lune is 45 spherical degrees. What is the angle of the lune?

(b) A lune with an angle of 90° is what part of a sphere? A lune with an angle of 72°?

13. Find the number of square inches in the area of a lune if its angle is 20 degrees and the radius of the sphere is 7 inches.
14. If the area of a certain sphere is 2,160 square units, find the number of square units in a spherical degree of the given sphere.
15. If it costs \$8.20 to paint the surface of a certain sphere, what will be the cost of painting one whose radius is three times as long?
16. Express the volume V of a sphere in terms of the circumference C.
17. If oranges with a 1-inch radius sell at 20 cents per dozen, would one dollar a dozen be out of proportion to pay for oranges with a 2-inch radius? (Assume that the oranges are of the same quality, and disregard the thickness of the skin.)
18. Find the radius of a sphere whose area is 200 square feet.
19. On a certain sphere the area of a trirectangular triangle is 40 square inches. What is the area of the sphere?
20. *The areas of zones of the same or equal spheres are to each other as their altitudes.*

EXERCISES

1. If a sphere is inscribed in a cube, find the ratio of the volume of the sphere to the volume of the cube.
2. Find the radius of a sphere whose area is 350 square inches.
3. A sphere is circumscribed about a given cube. Find the ratio of the volume of the sphere to the volume of the cube.
4. The top of a water tank is a hemisphere with a diameter of 30 feet. Find the number of square feet in its surface.
5. Find the radius of a sphere if its volume is approximately 905 cubic units.
6. The radii of two intersecting spheres are 10 inches and 12 inches respectively, and the distance between their centers is 16 inches. Find the radius of the circle of intersection of the two spheres.
7. Find the radius of a sphere which has the same volume as a regular tetrahedron whose edge is 14.25 inches.
8. If a ball 4 inches in diameter weighs 15 pounds, what is the diameter of a ball of the same material which weighs 30 pounds?
9. Find the amount of the earth's surface which is included in the Torrid Zone; in the North Temperate Zone.
10. The diameter of a sphere is 15 inches. If it is divided into five equal parts by parallel planes, find the area of each zone.
11. If the angle of a lune is 12°, find the area of the lune in spherical degrees. Find the area of the lune in square inches if the radius of the sphere is 8 inches.

12. Find the area, in spherical degrees, of the triangles whose angles are:

(a) 80°, 78°, and 93°.
(b) 76°, 93°, and 85°.
(c) 140°, 88°, and 61°.

13. *The area of a spherical triangle is equal to the area of a lune whose angle is half the spherical excess of the triangle.*

14. *The area of a spherical polygon is equal to a lune whose angle is half the spherical excess of the polygon.*

15. Show how the formula for finding the volume of a prismatoid can be used to find the volume of a sphere. Hint: The areas of the "bases" (two planes *tangent* to the sphere) would be zero. What would M represent on the sphere? What would h represent?

16. Find the volume of a spherical sector if the altitude of its base (a zone) is 8 inches, and the radius of the sphere is 12 inches.

17. Show that the formula for the volume of a spherical sector can be expressed by the following formula: $V = \frac{2}{3}\pi r^2 h$.

18. Find the volume of a spherical segment of one base if its altitude is 3 inches and the radius of the sphere is 12 inches.

19. Find the volume of a spherical segment with altitude 2 inches, if its bases have radii of 3 inches and 4 inches.

20. The diameter of a certain sphere is 14 inches, and the altitude of the base of a spherical sector of the sphere is 3 inches. Find the volume of the sector.

CHAPTER TEST

Given: Sphere O with a great circle TMR. Radius $OM = 12$ units, and $\overset{\frown}{RM} = \overset{\frown}{110}°$.

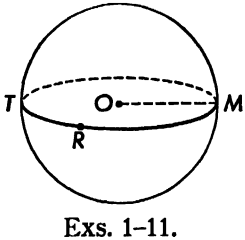
1. Find the area of the sphere.

2. Find the volume of the sphere.

3. Find the number of square units in a spherical degree of the sphere.

4. A spherical triangle with angles of 90°, 112°, and 80° covers what part of the surface of the sphere?

5. If two parallel planes are passed through the sphere so that they divide the radius in three equal parts, find the area of the zone formed.



Exs. 1–11.

6. Find the area of a lune whose angle is 30°.

7. How will the surface of this sphere compare with one whose diameter is 12 units?

8. Compare the volume of this sphere with one whose radius is three times as long.
9. Find the area of a spherical triangle on this sphere if M and R are two of the vertices and the third one is at the pole of circle TMR. Find the area in spherical degrees and also in square units.
10. If the area of a certain zone on this sphere is $452\frac{4}{7}$ square units, what is the altitude of the zone?
11. Find the volume of a regular tetrahedron which could be inscribed in this sphere O.
12. Express the diameter d of a sphere in terms of S, the area of the sphere.

CHAPTER 11 *Reference*

SECTION I. VOCABULARY FOR SOLID GEOMETRY

Altitude. The perpendicular distance between the bases of a prism, cylinder, frustum of a cone or pyramid, or zone. In a cone or pyramid, it is the perpendicular distance from the vertex to the base.

Axis. The line joining the centers of the bases of a circular cylinder, or a line from the vertex to the center of the base of a circular cone.

Axis of a circle of a sphere. The diameter of the sphere which is perpendicular to the plane of the circle.

Base(s). The polygon made by the two parallel planes of a prism, cylinder, frustum of a cone or pyramid, or zone. In a cone or pyramid, the base is the section of the conical surface or pyramidal surface made by the cutting plane.

Base of a spherical sector. The zone generated by the arc of the sector.

Chord of a sphere. A straight line joining any two points on a sphere.

Circle of a sphere. A section made by a plane cutting through a sphere.

Great circle of a sphere. A section made by a plane passing through the center of a sphere. **Small circle of a sphere.** A section made by a plane passing through a sphere, but not passing through its center.

Circumscribe. See pages 110 and 116.

Cone. A solid formed by one nappe of a closed conical surface and a plane cutting all the elements.

Circular cone. A cone whose base is a circle.

Right circular cone. A cone whose axis is perpendicular to the base.

Cone of revolution. A right circular cone.

Similar cones of revolution. Cones generated by similar right triangles revolving about corresponding legs as axes.

Congruent figures. Those whose sides and angles are respectively equal and are arranged in the same order, or that can be made to coincide. (See definition of *congruent figures* in Section III of this chapter. How does it differ from the one for solid figures?)

Conic section. Any section of a right circular cone.

Conical surface. A surface generated by a line which moves so that it always intersects a plane curve, and always passes through a fixed point.

Convex polygon. A polygon in which no side, when extended, will enter the polygon.

Convex polyhedron. A polyhedron in which every polygon made by a cutting plane is convex.

Cube. A rectangular solid with its three dimensions equal.

Cylinder. A solid determined by a closed cylindrical surface and two parallel planes which cut all the elements of the surface.

Circular cylinder. A cylinder whose base is a circle.

Right circular cylinder. A circular cylinder whose bases are perpendicular to an element.

Cylinder of revolution. A right circular cylinder.

Similar cylinders of revolution. Cylinders generated by similar rectangles revolving about corresponding sides.

Cylindrical surface. The surface generated by a line which moves so that it will always intersect a plane curve and always remains parallel to a given fixed line which is not in the plane of the curve.

Degree. See spherical degree.

Diagonal of a polyhedron. A line joining any two vertices not in the same face.

Diameter of a sphere. A chord passing through the center of the sphere.

Dihedral angle. An angle formed by two planes that meet. See page 47.

Directrix. The broken line or curve which directs or guides the generation of a conical, cylindrical, prismatic, or pyramidal surface.

Dodecahedron. A polyhedron with twelve faces.

Element of a cylinder. The part of an element of the cylindrical surface included between the bases of the cylinder.

Element of a surface. The generatrix in any one of its positions.

Excess. See spherical excess on page 180.

Faces of a dihedral angle. The two planes that meet to form the angle.

Faces of a polyhedral angle. The portions of the planes which form the angle.

Faces of a polyhedron. The portions of the planes which bound the solid.

Lateral faces. The faces of a prism and of a pyramid which are not the bases.

Foot of a line. The point of intersection of a straight line and a plane.

Frustum. The portion of a cone or a pyramid which is included between the base and a plane parallel to the base.

Generatrix. The moving line that describes or generates a conical, cylindrical, prismatic, or pyramidal surface.

Geometric solid. A closed surface which encloses a portion of space.

Great circle of a sphere. See circle.

Hemisphere. Either of the two parts of a sphere formed by a plane passing through its center.

Hexahedron. A polyhedron with six faces.

Icosahedron. A polyhedron with twenty faces.

Inscribe. See pages 110 and 116.

Lateral area. The area of all the lateral faces of a prism, pyramid, or frustum of a pyramid. For cones and cylinders it is the area of the lateral surface.

Lateral edges. The intersections of the lateral faces with each other.

Lateral faces. See faces of a polyhedron.

Limits. See pages 110–111.

Locus. See pages 67–69.

Lune. A portion of a sphere bounded by the halves of two great circles.

Angle of a lune. The angle at which the two half-circles meet.

Mid-section of a prismatoid. A section made by a plane parallel to the bases and halfway between them.

Nappes. The two parts of a conical or pyramidal surface separated by the vertex (fixed point).

Octahedron. A polyhedron with eight faces.

Parallel planes. Two planes (or a plane and a line) which will never meet no matter how far they are extended.

Parallelepiped. A prism whose bases are parallelograms.

Right parallelepiped. A parallelepiped with lateral edges perpendicular to the base. (Generally called a rectangular solid.)

Pentahedron. A polyhedron with five faces.

Perspective. See pages 7–9.

Plane. See pages 4–6.

Perpendicular planes. Two planes which meet forming dihedral angles whose plane angles are right angles.

Line perpendicular to a plane. A line which is perpendicular to every line in the plane which passes through its foot.

Plane angle of a dihedral angle. An angle formed by two lines, one in each face, drawn perpendicular to the edge at the same point.

Polar distance of a circle. The spherical distance from the nearer pole of a circle of a sphere to any point on the circle.

Polar triangle. If the vertices of a given spherical triangle are the poles of the sides of a second spherical triangle, then the second triangle is the polar triangle of the first.

Poles of a circle of a sphere. The extremities of the axis of the circle.

Polyhedral angle. The angle formed by three or more planes meeting at a point. See page 56.

Polyhedron. A solid bounded by planes. See convex polyhedron.

Regular polyhedron. A polyhedron whose faces are congruent regular polygons, and whose polyhedral angles are equal. See convex polyhedron.

Similar polyhedrons. Two polyhedrons which have the same number of faces similar each to each and similarly placed, and which have their corresponding polyhedral angles equal.

Prism. A polyhedron formed by two parallel planes intersecting a closed prismatic surface and cutting all the elements.

Right prism. A prism whose base is perpendicular to an edge.

Oblique prism. Any prism not a right prism.

Regular prism. A right prism whose base is a regular polygon.

Prismatic surface. A surface generated by a straight line moving so that it always intersects a broken line in a plane and is always parallel to a fixed line not in the plane.

Prismatoid. A polyhedron all of whose vertices lie in two parallel planes.

Projection of a line on a plane. The locus of the projections of all of its points on the given plane. See pages 71 to 72.

Projection of a point on a plane. The foot of the perpendicular drawn from the point to the plane.

Pyramid. The solid formed by one nappe of a closed pyramidal surface cut off by a plane cutting all the elements.

Regular pyramid. A pyramid whose base is a regular polygon and whose altitude meets the center point of the base.

Pyramidal surface. A surface generated by a straight line moving so that it always intersects a fixed broken line in a plane and also always passes through a fixed point not in the plane of the broken line.

Quadrant. One-fourth of a great circle of a sphere.

Radius of a sphere. A line from the center to any point on a sphere.

Rectangular solid. A right parallelepiped whose bases are rectangles.

Section. The plane figure formed by the intersection of a plane and a solid.

Right section of a cylinder or a prism. A section made by a plane cutting all the elements or lateral edges and perpendicular to one of them.

Skew lines. Lines not in the same plane. (They never meet, no matter how far extended.)

Slant height of a frustum of a regular pyramid. The altitude of any one of its lateral faces.

Slant height of a frustum of a right circular cone. The part of an element included between the two bases.

Slant height of a regular pyramid. The altitude of any one of its lateral faces.

Slant height of a right circular cone. Any element of the cone.

Solid. See geometric solid.

Sphere. A closed curved surface all points of which are equidistant from a point within called the center.

Circumscribed sphere. A sphere is circumscribed about a polyhedron if all the vertices of the polyhedron lie on the sphere.

Inscribed sphere. A sphere is inscribed in a polyhedron if each face of the polyhedron is tangent to the sphere.

Concentric spheres. Spheres with the same center.

Spherical angle. The angle formed by the meeting of the arcs of two great circles.

Spherical cone. See page 171.

Spherical degree. A unit used in measuring a spherical surface, or some part of it. It is a tiny spherical triangle two sides of which are quadrants and the third side is an arc of 1°.

Spherical distance. The distance between two points on a sphere measured along the minor arc of the great circle connecting them.

Spherical excess of a spherical triangle. The difference between the sum of the degrees in the angles of a spherical triangle and 180°. (See page 165 for spherical excess of spherical polygons.)

Spherical polygon. A closed figure on a sphere formed by three or more arcs of great circles.

Spherical sector. A solid generated by a sector of a semicircle as the semicircle rotates about its diameter as an axis.

Spherical segment. If two parallel planes are passed through a sphere, the solid formed between them is a spherical segment.

Spherical triangle. A spherical polygon with three sides.

Surface. See pages 3–4. Also see conical, cylindrical, prismatic, and pyramidal surfaces.

Symmetric figures. See pages 59 and 144.

Tangents.

Plane tangent to a cone or cylinder. A plane which, no matter how far extended, contains one and only one element of the cone or cylinder.

Tangent to a sphere. A line or a plane which has one and only one point in common with the sphere.

Tangent spheres. Two spheres which are tangent to a plane at the same point.

Tetrahedron. A polyhedron with four faces.

Volume. See page 84.

Zone. The portion of a sphere included between two parallel planes.

SECTION II. SYMBOLS FOR PLANE AND SOLID GEOMETRY

$\angle$ *angle*	$\rightarrow$ *approaches as limit*
$\odot$ *circle*	$\perp$ *perpendicular, is perpendicular to*
$\cong$ *is congruent to*	$\sim$ *is (are) similar to*
$=$ *equals, is (are) equal to*	$\square$ *square*
$>$ *is (are) greater*	$\therefore$ *therefore*
$<$ *is (are) less than*	$\triangle$ *triangle*
$\overset{m}{=}$ *is measured by*	$\overset{\frown}{}$ *arc*
$\parallel$ *parallel, is parallel to*	$\overset{\frown}{}^{\circ}$ *arc degree*
$\square$ *parallelogram*	x° *degrees of angle*
$\square$ *rectangle*	

The letter *s* used with any symbol representing a noun forms the plural of the noun. For example, $\triangle$s means triangles.

SECTION III. VOCABULARY FOR PLANE GEOMETRY

Acute angle. An angle which is less than a right angle.

Adjacent angles. Two angles with a common vertex and a common side between them.

Alternate interior angles. Angles which lie on opposite (alternate) sides of a transversal and between the two lines which are cut by the transversal.

Altitude of a triangle or quadrilateral. A line from any vertex perpendicular to the opposite side (extended if necessary).

Apothem of a regular polygon. The radius of the inscribed circle.

Arc. Part of a circle.

Major arc. An arc more than a semicircle.

Minor arc. An arc less than a semicircle.

Area. The number of units of surface enclosed by a geometric figure.

Base. The side of a triangle or quadrilateral to which an altitude is, or is to be, drawn. Any side can thus be considered a base. The "unequal" side of an isosceles triangle is generally considered the base.

Center of a regular polygon. A point equidistant from all the vertices of the polygon.

Central angle. An angle whose vertex is the center of a circle and whose sides are radii of the circle.

Central angle of a regular polygon. The angle formed by connecting any two vertices of the polygon and the center with straight lines.

Chord of a circle. A straight line drawn between any two points on a circle.

Circle. A closed curved line lying in a plane, all points of which are equidistant from a fixed point within, which is called the center.

Circumference. The distance around a circle; the length of a circle.

Circumscribe. To draw around, touching as many points on the given figure as possible. A triangle which circumscribed a circle would touch the circle in three points, but would not pass inside the circle. A circle which circumscribed a triangle would pass through the three vertices of the triangle and lie entirely outside the triangle. The sides of *any* polygon circumscribed about a circle are all tangents of the circle.

Coincide. To occupy the same place in a plane, or in space. When two triangles can be made to coincide they are congruent.

Complementary angles. Two angles whose sum is a right angle.

Concurrent, Concurrence. When three or more lines meet in the same point they are said to be concurrent. The point at which the lines are concurrent is called the point of concurrence.

Congruent figures. Those whose sides and angles are respectively equal. They can be made to coincide and have the same shape and size.

Converse. A statement which is formed by interchanging one given condition and one conclusion of a proposition.

Corresponding angles. Corresponding angles are those which lie on the same side of a transversal and in the same position with respect to the parallel lines which are cut by the transversal.

Corresponding parts. Angles or sides lying opposite equal parts, or included between respectively equal parts, of congruent figures.

Decagon. A polygon of ten sides (and consequently of ten angles).

Degree of angle. A degree of angle is a small angle which is $\frac{1}{360}$ of a complete rotation of a straight line about a fixed point on it.

Degree of arc. A degree of arc is a tiny arc which is $\frac{1}{360}$ of a circle (of the circumference).

Diagonal. A line joining two non-adjacent vertices of a polygon.

Diameter. A chord which passes through the center of a circle.

Equiangular triangle. A triangle having all angles equal.

Equilateral triangle. A triangle having all sides equal.

Exterior angle of a polygon. The angle formed by one side and the extension of an adjacent side.

Hexagon. A polygon of six sides (hence six angles).

Hypotenuse. The side of a right triangle which lies opposite the right angle.

Inscribe. To draw inside so as to touch as many points as possible.

Inscribed angle. An angle whose sides are chords (and whose vertex lies on the circle).

Isosceles trapezoid. A trapezoid having two equal sides. (The non-parallel sides must, obviously, be the equal sides.)

Isosceles triangle. A triangle having two equal sides.

Locus. The path of a point which moves according to certain stated conditions.

Mean proportional. A number that, when repeated as the mean between two given numbers, will make a proportion.

Median of a trapezoid. A line joining the mid-points of the two non-parallel sides.

Median of a triangle. A straight line which joins the mid-point of a side and the opposite vertex.

Obtuse angle. An angle greater than a right angle but less than a straight angle.

Parallel lines. Lines, in the same plane, which do not meet no matter how far extended.

Parallelogram. A quadrilateral whose opposite sides are parallel.

Pentagon. A polygon of five sides (and hence of five angles).

Perimeter. The distance around a polygon.

Perpendicular lines. Two lines which meet forming two equal adjacent angles.

Polygon. A closed geometric figure, the sides of which are straight lines.

Proportion. A true statement of equality of two ratios.

Quadrilateral. A polygon of four sides.

Radius of a circle. A straight line from the center to the circle.

Radius of a polygon. A straight line joining the center and any vertex.

Ratio. The quotient of one quantity divided by another of the same kind. It is generally expressed as a fraction.

Rectangle. A parallelogram with one right angle.

Regular polygon. A polygon whose sides are equal and whose angles are equal.

Rhombus. A parallelogram with two adjacent sides equal.

Right angle. An angle whose sides are perpendicular to each other.

Right triangle. A triangle with one right angle.

Secant of a circle. A straight line which intersects a circle in two points.

Sector of a circle. The area bounded by the sides of a central angle and its

Segment of a circle. The area enclosed by a chord and its arc.

Semicircle. An arc which is one-half a circle.

Similar polygons. Polygons whose corresponding angles are equal and whose corresponding sides are proportional.

Square. A rectangle with two adjacent sides equal.

Supplementary angles. Two angles whose sum is a straight angle.

Tangent. A straight line which touches a circle in only one point.

Trapezoid. A quadrilateral, one, and only one, pair of opposite sides of which are parallel.

Triangle. A polygon of three sides.

Vertical angles. The non-adjacent angles formed by two straight lines intersecting.

SECTION IV. AXIOMS

1. A quantity is equal to, greater than, or less than any other quantity.
2. When equals are added to equals their sums are equal.
3. When equals are subtracted from equals the differences are equal.
4. When equals are multiplied by equals the products are equal.
5. When equals are divided by equals the quotients are equal.
6. Halves of equals are equal.
7. Things equal to the same thing, or to equal things, are equal to each other.
8. A quantity may be substituted for its equal in any process.
9. The whole (a whole thing) is equal to the sum of all its parts.
10. The whole is greater than any of its parts.
11. Like powers or like roots of equal quantities are equal.
12. If equals are added to unequals, the sums are unequal in the same order.
13. If equals are subtracted from unequals, the results are unequal in the same order.
14. If unequals are multiplied by equals, the results are unequal in the same order.
15. If unequals are divided by equals, the quotients are unequal in the same order.
16. If unequals are added to unequals in the same order, then the sums are unequal in that order.
17. If unequals are subtracted from equals, the results are unequal in the reverse order.
18. If the first of three quantities (or magnitudes) is greater than the second and the second is greater than the third, then the first is greater than the third.

SECTION V. ASSUMPTIONS AND THEOREMS OF PLANE GEOMETRY*

LINES AND POINTS

1. A line segment can be produced to any desired length.
2. Through a given point, an unlimited number of straight lines can be drawn.
3. Through two given points it is possible to draw one, and only one, straight line.
4. Two straight lines can intersect in only one point.
5. The shortest path between two given points is the line segment which joins them.
6. Through a given point, one, and only one, perpendicular can be drawn to a given line.
7. The shortest distance from an external point to a line is the perpendicular distance from the point to the line.
8. A line segment can be bisected at one and only one point.
9. An angle can be bisected by one and only one line.
10. Any given figure can be reproduced anywhere (or moved without changing its shape or size).
11. When two straight line segments are equal, their end-points can be made to coincide.
12. If two angles are equal, their sides and vertices can be made to coincide.

TRIANGLES

13. When three sides of one triangle are equal respectively to three sides of another triangle, the triangles are congruent.
14. When two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
15. When two angles and the included side of one triangle are equal respectively to two angles and the included side of another triangle, the triangles are congruent.
16. If two angles and a side of one triangle are equal respectively to two angles and a corresponding side of another, the triangles are congruent.
17. If the hypotenuse and an acute angle of one right triangle are equal respectively to the hypotenuse and an acute angle of another, the triangles are congruent.

*The propositions are not arranged in logical sequence, but are grouped under headings for easy reference.

18. If the hypotenuse and a side of one right triangle are equal respectively to the hypotenuse and a side of another right triangle, the triangles are congruent.
19. If two sides of a triangle are equal, the angles opposite them are equal.
20. An equilateral triangle is also equiangular.
21. If two angles of a triangle are equal, the sides opposite them are equal.
22. An equiangular triangle is also equilateral.
23. The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.
24. If one angle of a right triangle is 30°, the side opposite this angle is one-half the hypotenuse.

PARALLEL LINES AND PERPENDICULAR LINES

25. Through a given external point one, and only one, line can be drawn parallel to a given line.
26. When two parallel lines are cut by a transversal, the alternate interior angles are equal.
27. When two lines are cut by a transversal so that a pair of alternate interior angles are equal, the lines are parallel.
28. When two parallel lines are cut by a transversal, the corresponding angles are equal.
29. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.
30. If two lines intersect, perpendiculars to the lines will themselves intersect.
31. If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.
32. The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.
33. Two points, each equidistant from the extremities of a line, determine the perpendicular bisector of the line.
34. When two straight lines are cut by a transversal so that a pair of corresponding angles are equal, the lines are parallel.
35. If two lines are perpendicular to the same line they are parallel.
36. When two straight lines are cut by a transversal so that a pair of interior angles on the same side of the transversal are supplementary, the lines are parallel.
37. If two lines are each parallel to a third line they are parallel to each other.

38. If a series of parallel lines cut off equal segments on one transversal, they will cut off equal segments on any transversal.
39. If a line joins the mid-points of two sides of a triangle, it is parallel to the third side and equal to one-half the third side.
40. A line parallel to one side of a triangle, and bisecting a second side, will bisect the third side also.
41. The median of a trapezoid is parallel to the bases and equals one-half their sum.

PARALLELOGRAMS

42. The opposite sides of a parallelogram are equal.
43. The opposite angles of a parallelogram are equal.
44. A diagonal divides a parallelogram into two congruent triangles.
45. The diagonals of a parallelogram bisect each other.
46. The consecutive angles of a parallelogram are supplementary.
47. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.
48. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.
49. The lines joining the mid-points of the consecutive sides of a quadrilateral form a parallelogram.

ANGLES

50. When two lines intersect, the vertical angles, formed in pairs, are equal.
51. All right angles are equal.
52. The sum of all angles having a common vertex on a straight line and lying on the same side of the line is 180°.
53. If two angles are adjacent and supplementary, their exterior sides will lie in a straight line.
54. When the exterior sides of two adjacent angles lie in a straight line, the angles are supplementary.
55. Angles which have equal complements, or equal supplements, are equal.
56. The sum of the angles of a triangle is 180°.
57. When two angles of one triangle are equal respectively to two angles of another triangle, the third angles are equal.
58. The acute angles of a right triangle are complementary.
59. Only one angle of a triangle can be a right angle or an obtuse angle.
60. An exterior angle of a triangle equals the sum of the two opposite interior angles.

61. The sum of the angles of an n-sided polygon is $(n - 2)\,180°$.
62. An interior angle of a regular polygon of n sides is $\frac{(n - 2)180°}{n}$.
63. The sum of the exterior angles of a polygon of n sides formed by extending the successive sides in order is 360°.

CIRCLES

64. Through three points, not in a straight line, one and only one circle can be drawn.
65. With a given center and a given radius, one and only one circle can be drawn in a plane.
66. A straight line can intersect a circle in not more than two points.
67. Two circles can intersect in not more than two points.
68. Radii of the same or equal circles are equal.
69. Circles with equal radii are equal.
70. When in the same or equal circles two central angles are equal, the arcs which they determine are equal.
71. If the distance from the center to a point is equal to the radius, the point is on the circle; if less than the radius it is inside the circle, and if greater it is outside.
72. When in the same or equal circles two arcs are equal, the central angles which they determine are equal.
73. When two chords in the same or equal circles are equal, the arcs which they determine are equal.
74. When two arcs in the same or equal circles are equal, the chords which they determine are equal.
75. A diameter of a circle which is perpendicular to a chord bisects the chord and the arcs determined by the chord.
76. A diameter of a circle which bisects a chord that is not itself a diameter is perpendicular to the chord.
77. In the same or in equal circles, equal chords are equidistant from the center.
78. In the same or in equal circles, if chords are equidistant from the center, they are equal.
79. A tangent to a circle is perpendicular to the radius (or diameter) drawn to the point of contact.
80. A line perpendicular to a radius at its outer extremity is tangent to the circle.
81. If two circles intersect, the line of centers is the perpendicular bisector of their common chord.

82. If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.
83. If from an external point two secants are drawn to a circle, the product of one secant and its external segment is equal to the product of the other secant and its external segment.
84. If from an external point a secant and a tangent are drawn to a circle, the tangent is the mean proportional between the whole secant and its external segment.

CIRCLE MEASUREMENT

85. An inscribed angle is measured by one-half its arc.
86. Angles inscribed in the same segment are equal.
87. Angles inscribed in a semicircle are right angles.
88. Parallel lines cut off equal arcs on a circle.
89. An angle formed by a tangent and a chord is measured by one-half its arc.
90. When two tangents intersect, their segments from the point of intersection to the points of contact are equal.
91. Any one of the four angles formed by two intersecting chords is measured by one-half the sum of the arcs which it, and the angle vertical to it, determine.
92. The angle formed by two secants, or two tangents, or a tangent and a secant intersecting outside a circle, is measured by one-half the difference of the arcs which they determine.
93. If a circle is divided into equal arcs, then the chords of these arcs form a regular inscribed polygon, and tangents at the points of division form a regular circumscribed polygon.

LOCUS

94. The locus of a point (or of all points) at a given distance from a given point is a circle with the point as a center and the given distance as a radius.
95. The locus of a point (or of all points) at a given distance from a given line is a pair of lines, one on each side of the given line and parallel to it at the given distance from it.
96. The locus of a point (or of all points) equidistant from two parallel lines is a line parallel to the given lines and halfway between them.
97. The locus of a point (or of all points) equidistant from two given points is the perpendicular bisector of the line joining them.

98. The locus of a point (or of all points) equidistant from the sides of an angle is the bisector of the angle.
99. The locus of points equidistant from two given intersecting straight lines is the pair of lines that bisects the angles formed by them.

CONCURRENT LINES

100. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.
101. The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides.
102. The altitudes of a triangle are concurrent.
103. The medians of a triangle are concurrent in a point which is two-thirds of the distance from any vertex to the mid-point of the opposite side.

INEQUALITIES

104. An exterior angle of a triangle is greater than either of the opposite interior angles.
105. The sum of two sides of a triangle is greater than the third side.
106. If two sides of a triangle are unequal, the angles opposite are unequal and the angle opposite the greater side is the greater.
107. If two angles of a triangle are unequal, the sides opposite are unequal and the side opposite the greater angle is the greater.
108. If two sides of a triangle are equal respectively to two sides of another triangle, but the included angle of the first triangle is greater than the included angle of the second, then the third side of the first triangle is greater than the third side of the second.
109. If two sides of one triangle are equal respectively to two sides of another, but the third side of the first triangle is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.
110. In the same or in equal circles, if two minor arcs are unequal, then their chords are unequal and the greater arc has the greater chord.
111. In the same circle or in equal circles, if two chords are unequal, then their minor arcs are unequal, and the greater chord has the greater minor arc.
112. In the same or in equal circles, the lesser of two unequal chords is at the greater distance from the center of the circle.
113. In the same or in equal circles, if two chords are unequally distant from the center, the more remote is the shorter.

RATIO AND PROPORTION

114. The product of the means equals the product of the extremes.
115. If the product of two numbers equals the product of two other numbers, then either two can be made the extremes of a proportion of which the other two are the means.
116. If four numbers are in proportion, the first term is to the third as the second is to the fourth. (Alternation.)
117. In any proportion the sum of the first two terms is to the second as the sum of the last two terms is to the fourth. (Addition.)
118. In any proportion the first term minus the second is to the second as the third term minus the fourth is to the fourth term. (Subtraction.)
119. If four terms are in proportion, the second term is to the first as the fourth is to the third. (Inversion.)
120. If three terms of one proportion are equal respectively to three terms of another proportion, then the remaining terms in each are equal.
121. If the first and third terms of a proportion are equal, then the second and fourth terms are equal.
122. In a series of equal ratios, the sum of the first, third, fifth (etc.) terms divided by the sum of the second, fourth, sixth (etc.) terms equals any single ratio in the series.
123. If a line is parallel to one side of a triangle and intersects the other two sides, it divides these two sides proportionally.
124. The bisector of an interior angle of a triangle divides the opposite side into segments proportional to the adjacent sides.
125. When a line is parallel to one side of a triangle and intersects the other two, one of these sides is to either of its segments as the other side is to the corresponding segment.
126. The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.
127. If a line divides two sides of a triangle proportionally, it is parallel to the third side.
128. A series of parallel lines cut off proportional segments on two transversals.
129. In any right triangle the altitude drawn to the hypotenuse is the mean proportional between the segments of the hypotenuse.
130. If in any right triangle an altitude is drawn to the hypotenuse, either leg is the mean proportional between the whole hypotenuse and the adjacent segment of the hypotenuse.

SIMILARITY

131. Two triangles are similar if they are mutually equiangular.
132. Two triangles are similar if two angles of one are equal to two angles of the other.
133. Two right triangles are similar if an acute angle of one equals an acute angle of the other.
134. Corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.
135. If an angle of one triangle equals an angle of another and the including sides are proportional, the triangles are similar.
136. Two triangles are similar if their sides are respectively proportional.
137. The perimeters of two similar polygons have the same ratio as any two corresponding sides.
138. In any right triangle, the altitude drawn to the hypotenuse divides the triangle into two triangles each similar to the given triangle and similar to each other.
139. The square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides.

AREAS AND REGULAR POLYGONS

140. The area of a parallelogram is equal to the product of its base and its altitude.
141. Two parallelograms with equal bases and equal altitudes are equivalent.
142. The areas of two parallelograms with equal bases are to each other as their altitudes.
143. The areas of two parallelograms with equal altitudes are to each other as their bases.
144. The areas of any two parallelograms are to each other as the products of their bases and altitudes.
145. The area of a triangle is equal to one-half the product of its base and its altitude.
146. Two triangles with equal bases and equal altitudes are equivalent.
147. The areas of two triangles with equal bases are to each other as their altitudes.
148. The areas of two triangles with equal altitudes are to each other as their bases.
149. The areas of two triangles are to each other as the products of their bases and altitudes.
150. The area of a trapezoid is equal to one-half the product of the sum of the two bases and its altitude.

151. The areas of two similar triangles are to each other as the squares of any two corresponding sides.
152. The area of a regular polygon is equal to one-half the product of its apothem and its perimeter.
153. The central angles of two regular polygons of the same number of sides are equal.
154. Two regular polygons of the same number of sides are similar.
155. The perimeters of two regular polygons of the same number of sides have the same ratio as their radii or as their apothems.
156. The areas of two regular polygons of the same number of sides are to each other as the squares of their radii or apothems.
157. The ratio of the circumference of a circle to its diameter is constant.
158. In any circle, the circumference is $2\pi r$.
159. Two circumferences have the same ratio as their radii.
160. The area of a circle is one-half the product of its circumference and radius.
161. The area of a circle is *pi* times the radius squared.
162. The areas of two circles are to each other as the squares of their radii.

CONSTRUCTIONS IN PLANE GEOMETRY

1. Bisect a line.
2. Bisect an angle.
3. Construct an angle equal to a given angle.
4. From a given external point construct a perpendicular to a line.
5. From a point in a line construct a perpendicular to the line.
6. Construct the perpendicular bisector of a line segment.
7. Through a given external point construct a line parallel to a line.
8. Construct a triangle, given any one of the following sets of parts:
   (a) Three sides.
   (b) Two sides and the included angle.
   (c) Two angles and the included side.
   (d) Two angles and the side opposite one of them.
   (e) Two sides and an angle opposite one of them. (Indeterminate case.)
9. Divide a line into any number of equal parts.
10. Construct a circle which will pass through three points not in a straight line.
11. Construct a tangent to a circle at a given point on the circle.
12. Construct a tangent to a circle from a given external point.
13. Inscribe a circle in a given triangle.
14. Circumscribe a circle about a given triangle.

15. Find the center of any given arc.
16. Construct a fourth proportional to three given line segments.
17. Construct a mean proportional between two given line segments.
18. Upon a given line segment construct a polygon similar to a given polygon.
19. Circumscribe a circle about any regular polygon.
20. Inscribe a circle in any regular polygon.
21. Transform a polygon of any number of sides to an equivalent triangle.
22. Construct a square equivalent to a given rectangle.
23. Inscribe a square in a circle.
24. Inscribe a hexagon in a circle.

SECTION VI. ASSUMPTIONS IN SOLID GEOMETRY

1. *A straight line joining any two points in a plane lies entirely in the plane.*
2. *The intersection of a straight line and a plane is a point.*
3. *The intersection of two planes is a straight line.*
4. *Through a given line an unlimited number of planes can be passed.*
5. *Three points not in a straight line determine a plane.*
6. *A straight line and a point not on the line determine a plane.*
7. *Two intersecting straight lines determine a plane.*
8. *Two parallel lines determine a plane.*
9. *If two planes have one point in common, they either intersect or coincide.*
10. *If a straight line lies in each of two planes, it is their intersection.*
11. *The volume of a rectangular solid equals the product of its three dimensions.*

11a. *The volume of a rectangular solid equals its base times its altitude.*

11b. *The volume of a cube equals the cube of an edge.*

12. *If two solids have their bases in the same two parallel planes and if equal sections are always formed by any plane passed parallel to the bases and between them, the solids have equal volumes.*
13. *Any theorem about a prism whose bases are regular polygons which does not depend upon a specific number of faces is valid for a cylinder.*
14. *Any theorem about a pyramid whose bases are regular polygons which does not depend upon a specific number of faces is valid for a cone.*
15. *With a given center and a given radius one, and only one, sphere can be drawn.*
16. *Radii of the same sphere or of equal spheres are equal.*
17. *Spheres with equal radii are equal.*

18. *A point is within, on, or outside a sphere according to whether its distance from the center is less than, equal to, or greater than the radius of the sphere.*

19. *A sphere can be generated by revolving a semicircle about its diameter as an axis.*

20. *The shortest distance between two points on a sphere is the minor arc of the great circle which passes through the two points.*

21. *If two spherical triangles on the same or equal spheres have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent if the parts are arranged in the same order, but they are symmetric if the parts are arranged in reverse order.*

22. *If two spherical triangles on the same or equal spheres have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent if the parts are arranged in the same order, but they are symmetric if the parts are arranged in reverse order.*

SECTION VII. FORMULAS

NOTATION USED IN FORMULAS

A	area for plane figures	n	number of sides
B, b	area of base	p, p'	perimeter of base (or of right section if indicated)
d	diameter		
e	element *or* lateral edge	r, r'	radius
E	spherical excess	S	lateral area or surface
h, a	altitude (height)	T	total area
l	slant height	V	volume
M	area of mid-section	C	circumference of circle

FORMULAS IN PLANE GEOMETRY

Area of a rectangle	$A = ab$
Area of a parallelogram	$A = ab$
Area of a triangle	$A = \frac{1}{2}ab$
Area of a trapezoid	$A = \frac{1}{2}a(B + b)$
Area of a regular polygon*	$A = \frac{1}{2}ap$
Area of a circle	$A = \pi r^2$
Circumference of a circle	$C = \pi d$
The sum of the interior angles of any convex polygon	$(n - 2)180°$
One angle of any regular polygon	$\frac{(n - 2)}{n} 180°$

*a represents apothem in this formula.

FORMULAS IN SOLID GEOMETRY

	Area	*Volume*
Cone		
General		$V = \frac{1}{3}Bh$
Circular		$V = \frac{1}{3}\pi r^2 h$
Frustum		$V = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$
Right circular	$S = \pi rl$	$V = \frac{1}{3}\pi r^2 h$
Right circular	$T = \pi r(r + l)$	
Frustum	$S = \pi l(r + r')$	
Cylinder		
General*	$S = ep$	$V = Bh$
Circular		$V = \pi r^2 h$
Right circular	$S = 2\pi rh$	$V = \pi r^2 h$
Right circular	$T = 2\pi r(r + h)$	
Cube		$V = e^3$
Parallelepiped		$V = Bh$
Prism		
General*	$S = ep$	$V = Bh$
Right†	$S = ep$	
Prismatoid		$V = \frac{1}{6}h(B + b + 4M)$
Pyramid		
General		$V = \frac{1}{3}Bh$
Frustum		$V = \frac{1}{3}h(B + b + \sqrt{Bb})$
Regular	$S = \frac{1}{2}lp$	$V = \frac{1}{3}Bh$
Frustum	$S = \frac{1}{2}l(p + p')$	$V = \frac{1}{3}h(B + b + \sqrt{Bb})$
Sphere	$S = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$
Spherical Polygon	$S = \frac{E\pi r^2}{180}$	
Spherical Sector‡		$V = \frac{1}{3}rS$
Spherical Segment		
Of two bases		$V = \frac{1}{6}\pi h(3r^2 + 3r'^2 + h^2)$
Zone	$S = 2\pi rh$	

*p represents perimeter of right section for general cylinder or general prism.

†e represents element (or *altitude*) for right prism but represents element only for a general prism.

‡S represents area of *base of sector*.

Index

For terms not included in Index, see pages 176–184.

Angle, dihedral, 47–49
  lune, 165
  pentahedral, 56
  polyhedral, 56
  spherical, 141
  tetrahedral, 56
  trihedral, 56
Assumptions, 9

Birectangular spherical triangle, 148

Cavalieri's Principle, 84–85
Cone, altitude of, 113
  axis of, 114
  base of, 113
  circular, 114
  circumscribed, 116
  frustum of, 114
  lateral surface of, 113
  oblique, 114
  right circular, 114
  slant height of, 114
  tangent plane of, 116
  vertex of, 113
Conic sections, 117–119
Conical surface, definition of, 113
  nappes of, 113
  vertex in, 113
Contrapositive, 140
Converse, 140
Construction in solid geometry, 21
Cube, 2, 77, 83, 97
Cylinder, altitude of, 106
  axis of, 107
  bases of, 106
  circular, 107
  circumscribed, 110
  definition of, 106
  element of surface of, 106
  inscribed, 110
  lateral surface of, 106
  oblique, 107
  revolution of (similar), 112
  right, 107
  right section of, 107
  tangent plane of, 110

Directrix, 79, 88, 106, 113
Dihedral angle, congruent, 48
  corresponding, 49
  definition of, 47
  edge of, 47
  faces of, 47
  kinds of, 48
  naming, 48
  plane angle of, 48
  supplementary, 48

Element, 79, 88, 106, 113
Ellipse, 118
Everyday Reasoning, 3, 12, 45–46, 75–76, 104–105, 124–125, 138–139, 156–157

Foot of line, 6

Generatrix, 79, 88, 106, 113

Hyperbola, 118–119

Inclination (*see* Projection)
Indirect proof, 17–19
Inverse, 140

Keep the Iron Hot, 11–12, 28–29, 45–46, 65–66, 75–76, 104–105, 124–125, 138–140, 156–158

Limits, theory of, 110–111, 168, 171
Lines, oblique, to plane, 6
  parallel to plane, 30
  perpendicular to plane, 6
  skew, 6
Loci, 67
Lune, angle of, 165
  definition of, 165

Models, constructing, 14–15, 21

Oblique, 6
"One and Only One," 17

Parabola, 118–119
Parallel lines, 6, 13
Parallel planes, 6, 30
Parallelepiped, definition of, 83
  rectangular, 83
  right, 83

Perpendicular, 6
Perspective, 7–9
Planes, determining, 4–5
  discussion of, 4
  intersecting, 4
  parallel, 6, 30
  perpendicular, 48
Polar distance, 129
Polar triangle, 145
Poles of circle of sphere, 129
Polyhedral angles, congruent, 60
  convex, 56
  definition of, 56
  edges of, 56
  faces and face angles of, 56
  kinds of, 56
  vertex of, 56
Polyhedrons, convex, 77
  definition of, 77
  diagonal of, 77
  dodecahedron, 77
  edges of, 77
  faces of, 77
  icosahedron, 77, 78
  octahedron, 77
  regular, 77, 96
  section of, 77
  similar, 94
  tetrahedron, 77
  vertices of, 77
Prism, altitude of, 79
  bases of, 79
  definition of, 79
  lateral area of, 79
  lateral edges of, 79
  lateral faces of, 79
  oblique, 80
  quadrangular, 80
  regular, 80
  right, 80
  right section of, 80
  total area of, 79
  triangular, 80
Prismatic surface, 79
Prismatoid, altitude of, 96
  definition of, 96
  mid-section of, 96
  volume, 96
Projection, definition of, 71
  and inclination, 72
  of line on plane, 72
  of point on plane, 72
Proof, pattern for, 14
Pyramid, altitude of, 89
  base of, 88
  frustum of, 89
  lateral area of, 90
  lateral edges of, 89
  lateral faces of, 89
  regular, 89
  slant height of, 89
  tetrahedron, 89
  total area of, 90
  truncated, 89
Pyramidal surface, definition of, 88
  nappe of, 88
  vertex of, 88
Quadrant, 129, 164
Quoting theorems from drawings, 34, 55
Section, conic, 117
  right, 80
Solids, cones as, 4
  congruent, 84
  cylinders as, 4, 106
  definition of, 84
  dodecahedrons as, 77
  equivalent (equal), 84
  geometric, 4
  of icosahedron, 77, 78
  of octahedron, 77
  of polyhedron, 77
Pyramid, 4
  rectangular, 4
  regular, 77
  spherical, 4
  tetrahedron, 77
  volume of, 84
Sphere, center of, 126
  chord of, 126
  definition of, 126
  diameter of, 126
  great circle of a, 129
  radius of, 126
  small circle of a, 129
  zone of, 163
Spherical angle, 141
Spherical cone, 171
Spherical degree, 164
Spherical distance, 129
Spherical excess, 165
Spherical polygon, definition of, 141
  sides of, 141
  symmetric, 144
  triangle, 142
  vertices of, 141
Spherical sector, 171
Spherical segment, 171
Spherical triangle, birectangular, 148
  trirectangular, 148
Surfaces, conical, 113
  cylindrical, 3, 106
  curved, 3
  plane, 3
  prismatic, 79
  pyramidal, 88
  spherical, 3
Symmetric spherical polygons, 144
Symmetry (symmetric figures), axis of, 59
  line, 59
  plane, 59
  point, 59
Tool Box, 41, 49, 63, 121, 136, 154
Trirectangular spherical triangle, 148
Volume, 84
Zone, altitude of, 163
  bases of, 163
  of one base, 163