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INTRODUCTION TO ELECTROMAGNETIC FIELDS

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Introduction to

ELECTROMAGNETIC FIELDS

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CASE INSTITUTE OF TECHNOLOGY

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PREFACE

There has been a growing awareness that the undergraduate student in electrical engineering requires a broad and deep understanding of the principles of electric and magnetic fields as a background for the rapidly developing areas referred to as electromagnetic engineering by some and as energy conversion by others. Thus, in addition to the general aspects of electromagnetic fields, a discussion of the sources and points of development of forces and torques in such fields, and the general features of energy storage, energy flow, energy transfer, and energy conversion require added emphasis. Since many of these requirements are different from those of the past, it is not surprising that most previous texts on electric and magnetic fields do not quite satisfy the needs, and new books in this classical field are justified. This book has been written to meet the introductory phase of the needs of the electrical engineering student.

As a background to the text, it is assumed that the reader is familiar with the calculus and that he has an understanding of the important concepts of classical mechanics. Where any special concepts are required, these are introduced within the context of the development of the book.

A number of special features have been adopted in the development of the text material. These include pedagogical ideas as well as a changed emphasis from the technical content of the past. Among these are the following:

Since the presentation of electric and magnetic fields has long been a challenge to educators owing to the mathematical complexity inherent in such a study and the rather elusive character of fields, this text presents the material in order of increasing complexity of field concepts. It begins with a consideration of scalar fields and then leads into vector fields. The standard treatment of electric and magnetic fields has been abandoned in favor of that suggested by K. Küpfmüller's book, "Einfuhrung in die theoretische Elektrotechnik." This order of presentation has been classroom tested with very satisfactory results.

To emphasize the fact that the lumped circuit parameters, resistance, capacitance, and inductance, are quantities defined from static field considerations, these and related concepts proceed from the systematic exploitation of static fields before dynamic fields are introduced.

Considerable use has been made of the principles of virtual displace-

ment and virtual work in order to explore the general ideas of energy storage in fields and the forces at boundaries (dielectric boundaries in electric fields, and magnetic boundaries in magnetic fields). In dynamic problems, the seat of the forces and torques and the flow and storage of energy are carefully considered.

The development of the Maxwell field equations, the relationships of the static field equations to these, and the relationship of the fundamental mathematical problems to the solution of the field equations are emphasized. Frequent opportunity is taken to show that the mathematical problem is substantially the same in various phases of the work, that Poisson's or Laplace's equation is the controlling equation for most static field problems.

The use of vector notation and vector mathematics has been introduced early as an integral part of the text development.

An extensive number of illustrative examples have been included throughout the text to relate the results to practical problems.

The text material divides itself into five major areas: current flow fields, static electric fields, static magnetic fields, dynamic fields, Maxwell's equations, and electromagnetic phenomena. Every effort is made, however, to show the interrelationship among these areas, rather than to emphasize the differences. Such an interrelationship is readily possible since the electric charge, whether stationary, in uniform motion, or in accelerated motion, is fundamental to a discussion of most of these phenomena.

The author wishes to acknowledge his indebtedness to the many fine books covering the broad aspects of electromagnetic theory which have been written during the past seventy-five years. He owes a particular debt of gratitude to the late Prof. W. W. Hansen of Stanford University who, during the war, did so much to make field theory a living and vital study. It is a pleasure also to acknowledge the help of former colleagues, Dr. Edward Erdelyi and Dr. Richard E. Gildersleeve, in the early stages of the book's development, and to thank Dr. Robert Plonsey for his assistance in proofreading the entire text.

Samuel Seely

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CHAPTER 1

THE ELECTRIC FIELD IN MULTIDIMENSIONAL CONDUCTORS

This chapter will introduce the basic field concepts through a study of the experimental results which may be obtained with the aid of an electrolytic tank, or equivalently through the use of a flux plotter provided with Teledeltos paper. The field plots so obtained for any specified field configuration will be examined physically, and from this physical field distribution, which will be described in mathematical terms, many of the important field quantities will be developed.

1-1. The Current Flow-field. Our study will begin with an experiment. This experiment may be performed by the student with equipment which is available commercially or which may be constructed locally. The equipment is generally referred to as *flux-plotting equipment.* In one form, it consists of a sheet of conducting Teledeltos paper. This paper may be cut to any desired shape, to represent a specified configuration. Electrodes of prescribed shape may be attached to the paper in specified positions. By applying a known source of potential between the electrodes, a current, known as a *conduction* current, will flow between the electrodes. By means of a high-impedance voltmeter, it is possible to explore the potential distribution over the surface of the conducting paper. Of course, the equipment may consist of a sheet of iron on an insulated support. It might also consist of a nonconducting shallow tank which contains a uniform layer of conducting liquid. A typical flux-plotting equipment is illustrated in Fig. 1-1.

Suppose that such a typical flux-plotting equipment consists of a large rectangular conducting surface, with small circular electrodes which have been attached to it. If the conducting sheet is sufficiently large, then the system approximates a pair of conducting electrodes immersed in an infinite conducting plane. This system is closely analogous to a pair of wires immersed in an infinite conducting medium, a cross section of which is being explored. The situation is substantially that illustrated in Fig. 1-2. The electrode terminals are connected to the source of power, and the voltmeter will be used to explore the field. That a potential variation should be expected follows from the fact that current will flow between

the terminals attached to the conducting sheet. It is this current flow-field that is to be carefully investigated.

Suppose that it is possible to adjust the potential of the source so that the indicating voltmeter reads 100 scale divisions when connected between

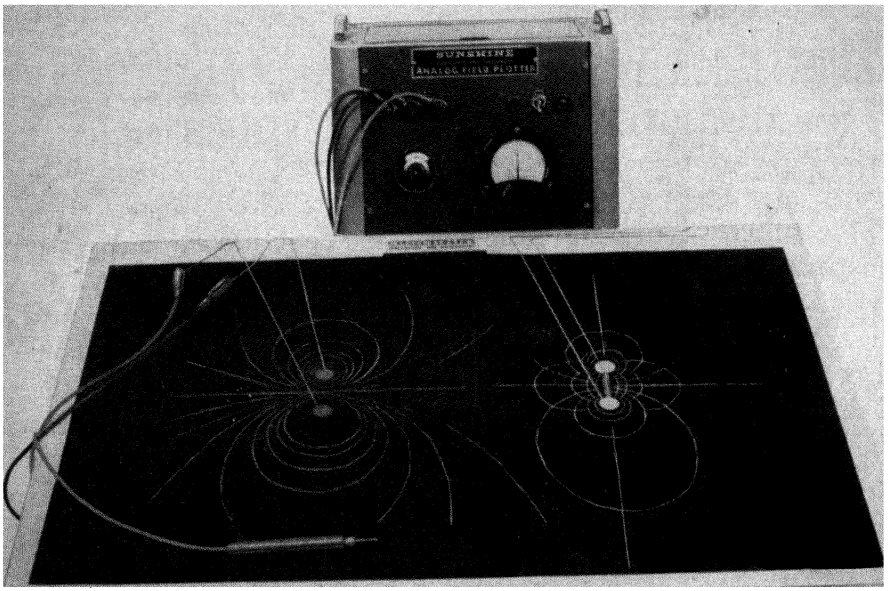


FIG. 1-1. Photograph of a commercial flux-plotting equipment. (*Courtesy Sunshine Scientific Instrument.*)

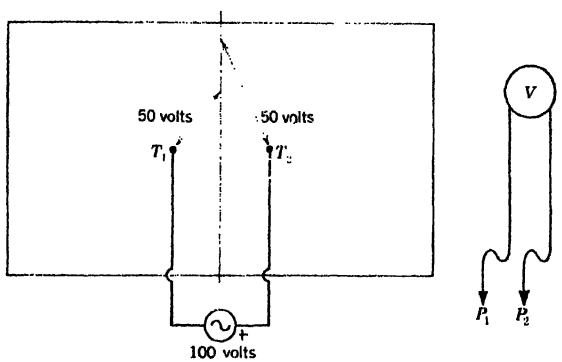


FIG. 1-2. Flux-plotting equipment for the system under survey.

the two electrodes. For convenience, suppose that each scale division is 1 volt, so that the maximum scale reading is 100 volts. Suppose, now, that one of the voltmeter probes, say, $P1$, is attached to terminal $T1$ and it is desired to search for all points on the sheet which will indicate 5C

volts on the voltmeter V. By actually performing this experiment, it will be found that the 50-volt level of potential occurs on the perpendicular bisector of the line between the two terminals $T1$ and $T2$. Such a result would be expected from the symmetry of the arrangement. Clearly, if probe $P1$ is placed anywhere along this line and probe $P2$ is placed on terminal $T2$, a 50-volt reading will also result.

Now a systematic exploration of the field is to be undertaken. Connect probe $P1$ to terminal $T1$, and search the field for the curves of 10 volts potential difference by moving probe $P2$ over the field. When this has been completed and the curve of the 10-volt potential difference

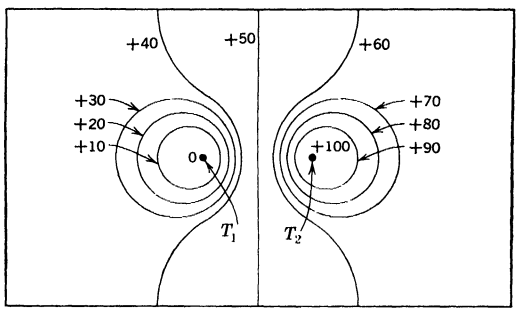


FIG. 1-3. Equipotential lines of the current field.

has been drawn, the process will be repeated for the 20-volt curve, then the 30-volt curve, etc., until the entire field has been systematically explored. The curves of equal potential difference are referred to as *equipotential lines*, or *curves*. A system of curves will result, as illustrated in Fig. 1-3. Clearly, if both probes are applied to two points of the same equipotential, no deflection will be noted on the voltmeter. This is, in fact, the real meaning of the term *equipotential*.

If the conducting material were to be of substantial thickness, then points of equal potential difference could be found within the material. These points constitute surfaces whose base lines are the equipotentials on the surface of the sheet. The surfaces are referred to as *equipotential surfaces*. As an extension of the above, there is no potential difference between any two points of the same equipotential surface.

1-2. Potential and Potential Difference. Each equipotential curve of Fig. 1-3 has been labeled with a number which specifies the potential difference relative to $T1$, which has been assigned the value zero and is being chosen as the reference, or datum, of potential. Also, the + sign applied to the numerical value of potential difference has been chosen to be consistent with the designated reference positive potential. Similarly, the reference current direction is here being chosen to coincide with the requirement that the direction of conventional current be from a

region of higher to a region of lower potential. That is, the current in the conducting sheet will be from $T2$ to $T1$ when the applied potential has the reference polarity. Clearly, if the terminal connections are interchanged, the direction of the current will reverse although the same distribution of equipotentials will result. Now, however, the equipotentials will carry $-$ signs in order to denote that they are the negatives of the previous data.

The reference point, or datum, of the potential is entirely arbitrary, as all effects depend only on differences of potential. That is, $T1$ may be chosen at any desired reference level without in any way affecting the equipotentials in the field. All that happens is that the equipotentials in the field will be either increased or decreased, depending upon the specified reference level of $T1$. Conversely, any point of the field may be chosen as the reference point, or datum, of potential. For example, suppose that a point on the bisector (shown as the $+50$ in Fig. 1-3) is chosen as the zero reference level of potential. In this case, all potential values of Fig. 1-3 will be reduced by the constant value of 50 volts. $T1$ will now have associated with it the designation -50 volts, and $T2$ will be designated as $+50$ volts. All other values will be correspondingly changed. The potential difference between any two points in the field is independent of the choice of reference point of potential. Clearly, therefore, it follows that the potential at any point of the field is equal to the potential difference between this point and a reference point. Conversely, it follows that the potential difference between two points of the field is equal to the difference of potential between these two points.

Consider, therefore, that the potential of point a of the field is given by the symbol ϕ_a relative to some arbitrary point as reference. Correspondingly, the potential of some other point b of the field relative to the same arbitrary reference point is given as ϕ_b. The potential difference between these two points is

$$V_{ab} = \phi_a - \phi_b \tag{1-1}$$

The numerical value given by Eq. (1-1) may be positive or negative, depending upon whether ϕ_a is greater or less than ϕ_b. To avoid confusion, in what follows V_{ab} will denote the *potential drop* from point a to point b. If, therefore, V_{ab} is negative, this merely means that the potential of point a is less than the potential of point b, relative to the same reference points.

1-3. Vectors and Two Vector Operations. To continue with the description of current-flow fields, it will be found desirable to do so in terms of vectors and certain vector operations. Consequently, this section will serve as a mathematical digression to introduce these concepts.

Physical quantities may be of two general classes. Those quantities

which can be described by a single number, as, for example, temperature and humidity, are known as *scalar* quantities. Those quantities, such as force and velocity, which have a direction in space as well as a magnitude are known as *vector* quantities. Vector quantities are represented geometrically by means of straight lines with arrowheads, the arrow pointing in the direction of the vector, the length being proportional to its magnitude. They are represented symbolically by boldface type. Attention is specifically directed to the difference between the complex number (phasor, sinor) of a-c circuit theory and the space vector here being considered.

The sum of two vectors **A** and **B** is the vector **C**, or

$$\mathbf{A} + \mathbf{B} = \mathbf{C} \tag{1-2}$$

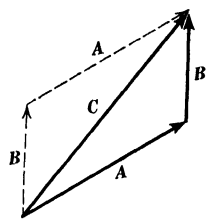


FIG. 1-4. Addition of vectors.

Observe carefully that, since **A** and **B** each possesses a magnitude and direction in space, **C** will also possess both magnitude and direction. Geometrically, the mathematical process is that illustrated in Fig. 1-4. Thus, the vector **C** is obtained from **A** and **B** by a parallelogram process, in which the origin of **B** is made to coincide with the terminus of **A**, and with **C** having the origin of **A** and the terminus **B** as its origin and terminus, respectively. Observe from the diagram that interchanging **A** and **B** does not affect the resultant **C**, whence

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1-3}$$

It follows from this that addition follows the commutative law. The addition of more than two vectors is readily accomplished geometrically by continuing the process defined in Eq. (1-2), adding successive vectors to the result of the previous operations. Thus, it readily follows that such an addition follows the associate law, namely,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{1-4}$$

It is noted that vectors, and the operation of vector arithmetic, have been defined without reference to any system of coordinates. In fact, it will be found that all the subsequent vector operations will be expressed in general terms. Often, however, it is convenient to refer the vector and its operations to a particular system of coordinates, but this will be the secondary and not the primary approach to our study of vector analysis. In fact, in this text, we shall limit consideration to three different orthogonal systems, the rectangular, cylindrical, and spherical systems of axes, with primary consideration given to the rectangular system of axes.

Refer to Fig. 1-5, which shows a vector **A** oriented with respect to a cartesian system of coordinates. The unit vectors **i**, **j**, **k** (vectors of unit magnitude) are chosen in the X, Y, Z directions, respectively, as illustrated. But, based on Eq. (1-2), it is possible to define a vector in terms of a number of other vectors. It is convenient, therefore, to express the vector **A** in terms of the sum of three vectors parallel to the rectangular (orthogonal) axes, thus,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{1-5}$$

In this expression, there appears the product of a vector and a scalar. Such a product is defined as a vector having a magnitude equal to the

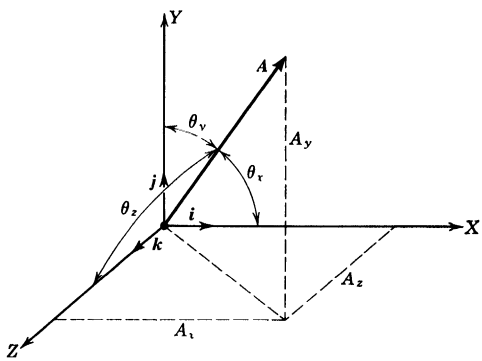


FIG. 1-5. Rectangular components of vector **A**.

product of the scalar and the magnitude of the vector (in this case, unity), and having a direction that is the direction of the original vector. The scalars A_x, A_y, A_z are the components of **A** and are given (see Fig. 1-5) by

$$\begin{aligned} A_x &= A\cos\theta_x \\ A_y &= A\cos\theta_y \\ A_z &= A\cos\theta_z \end{aligned} \tag{1-6}$$

where θ_x, θ_y, θ_z are the angles between **A** and the positive directions of the axes and where A is the magnitude of **A**.

To find the vector sum of two vectors **A** and **B** requires that each vector be referred to the same set of axes. Thus, for two vectors having the three components,

$$\begin{aligned} &\mathbf{A} + \mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) + (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ \text{or}\quad &\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k} \end{aligned} \tag{1-7}$$

This equation follows directly from the associative property of vector addition, as in Eq. (1-4).

Consider now the product of two vectors. There are two types of

product, the *scalar* product and the *vector* product. These names serve to indicate that the result of the respective multiplication in the first case is a scalar and in the second case is a vector. Since our present needs involve only the scalar product of two vectors, we shall limit our discussion to these.

By definition, the *scalar*, or *dot*, product of two vectors **A** and **B** is a scalar quantity and is the product of the magnitudes of the vectors and the cosine of the angle between them, thus,

$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos (\mathbf{A},\mathbf{B}) \tag{1-8}$$

Note that the dot between the vectors is essential to the expression. Clearly, from this definition,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{1-9}$$

and the scalar product obeys the commutative law. Also, the scalar product of a vector with itself is simply

$$\mathbf{A} \cdot \mathbf{A} = A^2 \tag{1-10}$$

since the angle between the two vectors is zero. Of course, the scalar product of two vectors which are perpendicular to each other will be zero, owing to the presence of the cos 90° that would appear in the expression for the dot product.

The scalar product of **A** and **B**, given by Eq. (1-8), may be expressed in terms of their rectangular components, thus,

$$\mathbf{A} \cdot \mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

Note, however, that

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

and

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

Therefore, it follows that

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z \tag{1-11}$$

This shows that the scalar product of two vectors is equal to the sum of the products of the corresponding components.

1-4. Potential Gradient and Line Integral. In addition to consideration of the potential at all points of the field, it is often convenient to consider the *gradient* of the potential at all points of the field. In order to understand the meaning of this term, consider two adjacent equipotential surfaces, one of which is denoted ϕ and the second denoted as $\phi + d\phi$, as illustrated in Fig. 1-6. Consider the point a on one equipotential and two points b and c on the adjacent equipotential. Point b is chosen to lie along the normal drawn to the equipotential at point a.

The distance between the adjacent equipotentials along the direction of the chosen normal, that is, the distance between points a and b, is designated as dn. Moreover, as illustrated in Fig. 1-6, the unit normal $\mathbf{n}$ is a space-vector quantity, and its direction would change, in general, if the point a were chosen elsewhere on the equipotential surface except when the surface is a plane. The magnitude of the normal $\mathbf{n}$ will be chosen by definition to be unity, and in the present case the positive direction of the normal is taken in the direction of increasing potential.

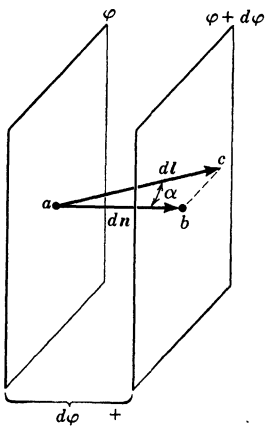


FIG. 1-6. Adjacent equipotentials in a potential field.

Consider now the quantity $d\phi/dn$, which specifies the greatest space rate of change of potential, and which is the slope, or gradient, of the potential field at the point a. Mathematically, the quantity discussed, when written with its associated direction $\mathbf{n}$, is called the *gradient of the potential*, or

$$\text{grad } \phi = \frac{d\phi}{dn}\mathbf{n} \tag{1-12}$$

Physically, therefore, the potential gradient, grad ϕ, is a measure of the maximum slope or the maximum rate of change of potential with distance in the direction of the increasing potential. For example, if one were to stand on the side of a hill, the gradient would be a measure of the slope in the direction of steepest ascent. In this case, the gradient is the tangent of the angle between the horizontal and the side of the hill. Often, in practice, the slope is specified as, say, $\frac{1}{100}$. This means that the ground rises 1 ft in each 100 of horizontal distance in the specified direction. Note particularly that, since the direction of steepest ascent is specified, the gradient has both magnitude and direction and is therefore a vector quantity. In vector analysis, it is often customary to write the symbol ∇ instead of the letters grad, where the symbol ∇ is called the *del* operator. Using this notation, Eq. (1-12) may be written as

$$\text{grad } \phi = \nabla\phi = \frac{d\phi}{dn}\mathbf{n} \tag{1-13}$$

Note from Fig. 1-6 that

$$dn = dl \cos \alpha$$

whence Eq. (1-13) becomes

$$\frac{d\phi}{dl} = |\text{grad } \phi| \cos \alpha$$

which is the *directional derivative* of the potential ϕ. This is the rate of change of ϕ in a particular direction and depends on the direction selected. If a rectangular system of axes were to be specified which was fixed at point a, then the directional derivative of ϕ in the direction of the X axis would be

$$\frac{\partial \phi}{\partial x} = (\text{grad } \phi) \cos \alpha_x = \mathbf{i} \cdot \text{grad } \phi$$

with corresponding terms for the Y and Z directions. Thus, for a general direction which possesses components along the three axes, it follows that

$$\text{grad } \phi = \nabla \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} \tag{1-14}$$

The corresponding forms for the gradient in cylindrical and spherical coordinates are written down, without proof.

Cylindrical coordinates

$$\text{grad } \phi = \frac{\partial \phi}{\partial r} \mathbf{a}_r + \frac{\partial \phi}{r \partial \theta} \mathbf{a}_\theta + \frac{\partial \phi}{\partial z} \mathbf{a}_z \tag{1-15a}$$

Spherical coordinates

$$\text{grad } \phi = \frac{\partial \phi}{\partial r} \mathbf{a}_r + \frac{\partial \phi}{r \partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \psi} \mathbf{a}_\psi \tag{1-15b}$$

where the $\mathbf{a}$'s are unit vectors in the direction specified by the subscript, with positive direction in the direction of increasing variable.

It is now possible to reverse the foregoing considerations and consider the question of the difference of potential between the two points a and b. By definition, the difference of potential between these two points is the potential difference that exists between these two points. According to Eq. (1-13), this potential difference is $d\phi$, so that

$$d\phi = (\text{grad } \phi)_n \, dn$$

where the subscript n denotes that the component of grad ϕ is chosen in the direction n. This is, of course, just grad ϕ itself. The total potential difference between the points a and b is thus given by the expression

$$V_{ba} = \phi_b - \phi_a = \int_a^b d\phi = \int_a^b (\text{grad } \phi)_n \, dn \tag{1-16}$$

Some simple mathematical manipulations of this expression lead to useful forms. Suppose, therefore, that the right-hand side of the expression is multiplied and divided by cos α. This yields

$$V_{ba} = \phi_b - \phi_a = \int_a^b (\text{grad } \phi)_n \cos \alpha \frac{dn}{\cos \alpha} \tag{1-17}$$

This equation may therefore be written in the form

$$V_{ba} = \phi_b - \phi_a = \int_a^b (\text{grad } \phi)_l \, dl \tag{1-18}$$

In the light of Eq. (1-8) for the scalar multiplication of two vectors, Eq. (1-18) may now be written in the following form:

$$V_{ba} = \phi_b - \phi_a = \int_a^b d\mathbf{l} \cdot \text{grad } \phi$$

or

$$V_{ba} = \phi_b - \phi_a = \int_a^b d\mathbf{l} \cdot \nabla\phi \tag{1-19}$$

In these expressions, $d\mathbf{l}$ has now been written as a vector element of path length and has the meaning indicated in Fig. 1-6.

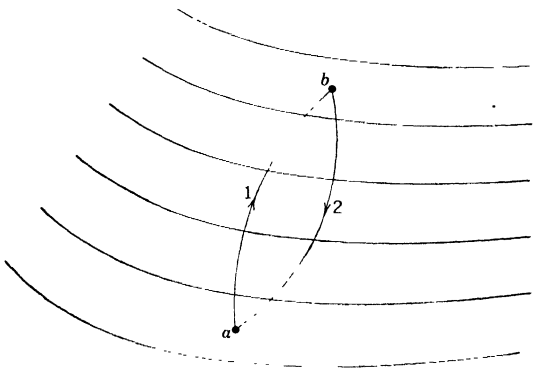


FIG. 1-7. Line integral of the potential gradient over a closed path.

Suppose that the two points a and b now denote any two points of the field, as illustrated in Fig. 1-7. The system of lines is to denote equipotentials of the field. The total potential difference between the points a and b along path 1 is simply the summation of the potential differences along all elements of this path. That is,

$$V_{ba} = \phi_b - \phi_a = \int_{a,\text{Path 1}}^{b} d\phi$$

In a similar way, the total potential difference between the points b and a along path 2 is given by the expression

$$V_{ab} = \phi_a - \phi_b = \int_{b,\text{Path 2}}^{a} d\phi$$

These two quantities are the negatives of each other, so that

$$V_{ba} = -V_{ab} \tag{1-20}$$

If in Eq. (1-18) $(\text{grad } \phi)_l$ is positive if its direction coincides with the

path of integration and is negative if its direction is in the reverse sense, then it follows that

$$\int_a^b (\text{grad } \phi)_l \, dl = -\int_b^a (\text{grad } \phi)_l \, dl \tag{1-21}$$

The integral given in Eq. (1-18) or in Eqs. (1-19) is referred to as the *line integral* of the potential gradient. In a potential field in which this relation is valid, the line integral is independent of the path of integration and is equal to the difference of the potentials at the two end points of integration. Note, therefore, that the indicating voltmeter in Fig. 1-2 actually measures the line integral of the potential gradient between the two points where the probes are applied.

Attention is called to the fact that the physical fields herein being discussed satisfy the condition

$$\oint d\mathbf{l} \cdot \nabla\phi = 0 \tag{1-22}$$

where the symbol $\oint$ represents an integration taken around a complete path. Fields of the present type which can be expressed as the gradient of a scalar potential are classed as *conservative* fields. Another designation is that they are *curl-free* fields, a terminology that will be explained at a later point in the discussion. However, not all fields are conservative, as we shall later discuss.

1-5. Current and Current Density. Return, now, to consideration of the current-flow field and the equipotential-line (now considered in the more general case as an equipotential-surface) distribution of Fig. 1-3. It is desired to examine the current flow itself, rather than the resulting equipotentials that are produced thereby in the conducting plate. While nothing has yet been said about the current, it will be postulated here that the current itself is comprised of the drift of charged carriers, here chosen to be electrons. Actually, a current results from the drift of either positively or negatively charged carriers, or both, if they exist simultaneously. Only drift electrons can exist in simple conductors, and our discussions in what follows will be confined to these, unless otherwise clearly specified.

Another feature of the current flow-field is that the equipotential surfaces in space determine the current direction at each point of the space. Moreover, the current direction must be perpendicular to the equipotential surfaces. It should be noted that these two statements of fact are *not* obvious, but the validity can be established from force considerations, a topic that will not be discussed until Chap. 5.

It is possible to represent the current direction by means of flow lines which mark the path of the current. These flow lines will everywhere

pierce the equipotential surfaces perpendicularly. Thus, for the experimental system of Fig. 1-3, the flow lines will appear as illustrated in Fig. 1-8. Since the equipotentials and the flow lines are everywhere normal to each other, the mathematical functions representing each set bear a known relationship to each other and are known as *conjugate* functions.

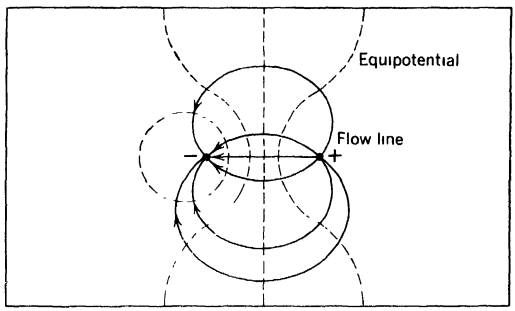


FIG. 1-8. Current flow-lines of the current field.

Consider a small area dS on an equipotential surface. A certain part of the total current passes through this area. If this definite portion of the whole is denoted as dI, then the ratio

$$J = \frac{dI}{dS} \tag{1-23}$$

which is referred to as the *current density*, denotes the current intensity per unit area at any point of the space. The current density can be represented as the density of so-called *current lines*, if one chooses the arbitrary convention that the number of the current lines passing through the element of area dS of the equipotential surface is proportional to the current intensity per unit area. If the current lines converge, this signifies that the current is increasing. Should the current lines diverge, this signifies that the current is decreasing. This general representation of flow densities by line densities will often be used in describing field phenomena.

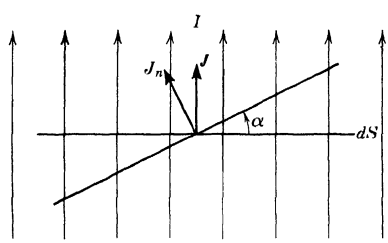


FIG. 1-9. Current lines through a surface element.

From an inspection of Fig. 1-8, it is apparent that the current density may have a different direction at every point in space. It is convenient to interpret the current density as a vector which indicates the direction of the current at any point in space. Refer to Fig. 1-9, which represents

a bundle of current lines passing through an element of area dS of an equipotential surface. Since dS denotes a very minute area, the current passing through it may be assumed to be homogeneous. The current lines will be perpendicular to dS.

Suppose that the area dS is now turned through a small angle α relative to its original position. In this case, the element of area is not confined to an equipotential surface. The number of current lines, and so the current through the element of area dS, will be reduced. The current will be proportional to the projection $dS \cos \alpha$ of the area dS on its original position. Hence, in this case,

$$dI = J \cos \alpha \, dS \tag{1-24}$$

This expression may be interpreted as follows: If the current density $\mathbf{J}$ is regarded as a vector whose direction coincides with the current direction, then the quantity $\mathbf{J} \cos \alpha$ represents the component of the vector which is perpendicular to dS. If this component of $\mathbf{J}$ is denoted as J_n, then Eq. (1-24) may be written as

$$dI = J_n \, dS \tag{1-25}$$

The total current is given by the summation of the currents through the elements of the total surface of which dS is an element, and is expressed by the relation

$$I = \int_S J_n \, dS \tag{1-26}$$

The integration must be carried out over the entire surfaces. The reference positive direction of the current will be assumed arbitrarily, and J_n is taken as positive if this component is parallel to the assumed positive direction. J_n is taken as negative if this component is opposite to the assumed reference positive direction of the current.

Vector equations corresponding to Eqs. (1-25) and (1-26) are readily possible. The vector character of the current density has already been noted, and $\mathbf{J}$ has been shown in Fig. 1-9 as a vector quantity. In vector analysis, elements of area are also considered as vectors. The magnitude of such a vector is equal to the area of the element, and the direction of this vector is perpendicular to the area. The situation is illustrated in Fig. 1-10. Later, the results will be shown to be related to an operation in vector analysis known as the *vector*, or *cross*, *product* of two vectors. Clearly, from this diagram, the vector element of area may be written in two forms, namely, $d\mathbf{S}$ and $\mathbf{n} \, dS$, where

$$d\mathbf{S} = \mathbf{n} \, dS \tag{1-27}$$

where $\mathbf{n}$ is the unit vector giving the direction of the area. Note that it is now possible to write Eq. (1-24) in the vector form

$$dI = J \cos \alpha \, dS = \mathbf{J} \cdot d\mathbf{S} = \mathbf{J} \cdot \mathbf{n} \, dS \tag{1-28}$$

where the scalar, or dot, product, defined in Sec. 1-4, is here given. The total current through the surface S is then given by the equation

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot \mathbf{n} \, dS \tag{1-29}$$

which is the surface integral corresponding to Eq. (1-26) when given in vector terms.

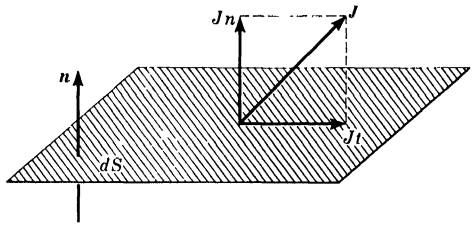


FIG. 1-10. Vector representation of current density and of area.

1-6. Electric Field Intensity. Attention is called to the fact that the current lines indicate the direction of the current-density vector and, moreover, the current-density vector is normal to the equipotential surfaces and point from the equipotential surfaces of higher potential to the equipotential surfaces of lower potential. If reference is made to Sec. 1-4, it will be noted that the current density is in the direction opposite to the potential gradient. It is convenient to define the vector function $\mathbf{E}$, which is called the *electric field intensity*, as the negative of the potential gradient, or

$$\mathbf{E} = -\operatorname{grad} \phi = -\nabla\phi \tag{1-30}$$

Thus, it is noted that the current-density vector $\mathbf{J}$ and the electric field intensity $\mathbf{E}$ are in the same direction. By Eq. (1-12),

$$\mathbf{E} = -\frac{d\phi}{dn}\mathbf{n} \tag{1-31}$$

Some of the results of Sec. 1-4 are conveniently expressed in terms of $\mathbf{E}$ rather than in terms of grad ϕ. The results follow, of course, simply by replacing grad ϕ by $-\mathbf{E}$ in the appropriate equations. In the interests of demonstrating vector manipulations, the important results will be

deduced. Refer to Fig. 1-11 which shows the equipotential distribution in a certain region of a scalar field. Suppose that both terms of Eq. (1-31) are dot- or scalar-multiplied with the element of length $d\mathbf{l}$. Thus,

$$\mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi}{dn}\mathbf{n} \cdot d\mathbf{l} \tag{1-32}$$

From an inspection of Fig. 1-11, it is seen that the quantity $\mathbf{n} \cdot d\mathbf{l}$, which is the product of the magnitude of the two terms and the cosine of the

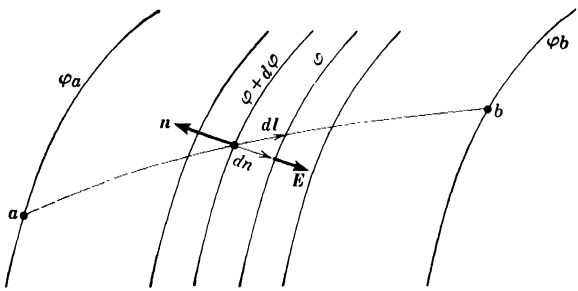


FIG. 1-11. For determining the potential from the field distribution.

angle between them, and since the magnitude of $\mathbf{n}$ is unity, then

$$\mathbf{n} \cdot d\mathbf{l} = dn$$

By combining this result with Eq. (1-32), it follows that

$$d\phi = -\mathbf{E} \cdot d\mathbf{l} \tag{1-33}$$

If one now integrates both sides between the limits a and b, then

$$\int_a^b d\phi = -\int_a^b \mathbf{E} \cdot d\mathbf{l}$$

which is

$$\phi_b - \phi_a = V_{ba} = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \tag{1-34}$$

This is just the line integral of the electric field intensity. This is just Eq. (1-19) but in this form is an expression of fundamental importance in field theory.

1-7. Ohm's Law. As pointed out in the discussion of Sec. 1-6 leading to the introduction of the electric field intensity $\mathbf{E}$, both $\mathbf{E}$ and $\mathbf{J}$ are in the same direction. But two related vectors which have the same direction are related through a constant factor, giving the ratio of their mag-

nitudes. That is, one may write in this case that

$$\mathbf{J} = \gamma \mathbf{E} \tag{1-35}$$

This is a very important relationship and is an expression of Ohm's law in field terms. The proportionality factor γ is the *electrical conductivity* of the medium.

Another viewpoint of Ohm's law is that γ is a constant, independent of **E**. The experiments conducted by Ohm proved to his satisfaction that a change in **E** produced no change in any of the factors which determine the flow of current in conductors. All so-called departures from Ohm's law can be explained in terms of other physical factors. This matter will receive further attention in Chap. 5.

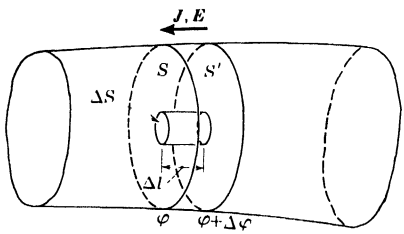


FIG. 1-12. Ohm's law in current flow-fields.

The customary statement of Ohm's law, namely,

$$V = IR \tag{1-36}$$

is obtained through the introduction of the resistance R of the medium. An expression for the resistance is readily derived. Refer to Fig. 1-12. In this figure, S and S' are two equipotential surfaces, which have the potentials ϕ and $\phi + \Delta\phi$, respectively. Let ΔS be an element of area S and the corresponding points on S' be a distance of Δl apart. Since the element of area ΔS is chosen in an equipotential, it is perpendicular to the lines of current flow, J, and

$$J = \gamma E$$

To find the total current through the section of the conductor, multiply both sides by the element of area ΔS, and then sum over all areas making up the section. The result is

$$I = \sum_{\Delta S} J\,\Delta S = \sum_{\Delta S} \gamma E\,\Delta S \tag{1-37}$$

Now, multiply numerator and denominator on the right by Δl, to get

$$I = \sum_{\Delta S} \gamma E\,\Delta S \frac{\Delta l}{\Delta l} = \sum_{\Delta S} \gamma E\,\Delta l \frac{\Delta S}{\Delta l}$$

But since $E\,\Delta l = \Delta\phi$ (drop) expresses the potential difference between the two equipotential surfaces S and S' and is independent of the path, then this expression may be written

$$I = \Delta\phi \sum_{\Delta S} \gamma \frac{\Delta S}{\Delta l}$$

It is convenient to write this result as

$$I = \frac{\Delta\phi}{\Delta R}$$

where

$$\Delta R = \frac{1}{\sum_{\Delta S} \gamma \frac{\Delta S}{\Delta l}} \tag{1-38}$$

To convert this latter expression from incremental to differential form, consider the expressions in the limit as $\Delta l \to 0$ and $\Delta S \to 0$. Under these circumstances, $\Delta\phi/\Delta l \to d\phi/dl$, and $\sum_{\Delta S} \gamma\, dS \to \int_S \gamma\, dS$, and Eqs. (1-38) become

$$I = \frac{d\phi}{dR}$$

where

$$dR = \int_S \frac{dl}{\gamma\, dS} \tag{1-39}$$

To extend these results to the entire conductor, consider the first of Eqs. (1-39), which is written as

$$d\phi = I\, dR$$

But since I is a constant, then between any equipotential surfaces

$$\int_a^b d\phi = I \int_a^b dR \tag{1-40}$$

which gives, upon integrating,

$$V_{ba} = I \int_a^b dR = IR_{ba} = IR_{ab} \tag{1-41a}$$

where, from the second of Eqs. (1-39),

$$R_{ab} = \int_a^b \frac{1}{\int_S \gamma\, dS/dl} = \int_a^b \frac{dl}{\int_S \gamma\, dS} \tag{1-41b}$$

This is a general expression for the resistance.

The value of the resistance may be evaluated readily for several simple cases.

Example 1-7.1. Find an expression for the resistance of a long cylindrical wire of uniform cross section, as illustrated in Fig. 1-13.

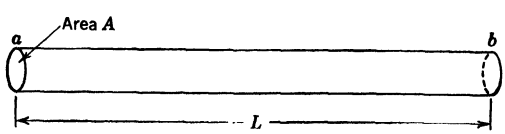


FIG. 1-13. A long cylindrical wire of uniform cross section.

Solution. For homogeneous media, γ = const, and Eq. (1-41) becomes

$$R_{ab} = \frac{1}{\gamma} \int_a^b \frac{dl}{\int_S dS}$$

Note also that

$$\int_S dS = \text{area of cross section, which is uniform and equal to } A$$

Hence, the expression becomes

$$R_{ab} = \frac{1}{\gamma} \int_a^b \frac{dl}{A} = \frac{1}{\gamma} \frac{L}{A} \tag{1-42}$$

Ordinarily, one defines the resistivity of the material, ρ, where $\rho = 1/\gamma$, so that

$$R = \frac{\rho L}{A} \tag{1-43}$$

In this expression, L is the total length of the wire.

Example 1-7.2. Find an expression for the resistance between the core and the sheath of a coaxial cable. This is often referred to as the *leakage*, or *sheath resistance*, of the cable.

Solution. Refer to Fig. 1-14, which illustrates the geometry of the cable. Also illustrated in Fig. 1-14 are the equipotential surfaces, which are coaxial

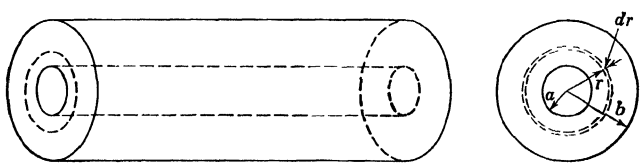


FIG. 1-14. The coaxial cable.

cylinders also. In this case, for a homogeneous material between the core and sheath, Eq. (1-41) becomes

$$R_{ab} = \frac{1}{\gamma} \int_a^b \frac{dr}{dS}$$

For length L of the cable,

$$\int_S dS = 2\pi r \cdot L$$

Therefore,

$$R_{ab} = \frac{1}{2\pi\gamma L} \int_a^b \frac{dr}{r}$$

from which it follows that the resistance of length L of the cable is

$$R_{ab} = \frac{1}{2\pi\gamma L} \ln \frac{b}{a} \quad \text{ohms} \tag{1-44}$$

and the resistance of a unit length is

$$R = \frac{1}{2\pi\gamma} \ln \frac{b}{a} \quad \text{ohms-meters}$$

Example 1-7.3. Discuss the problems involved in finding the resistance between the two ends of the irregular conductor illustrated in Fig. 1-15.

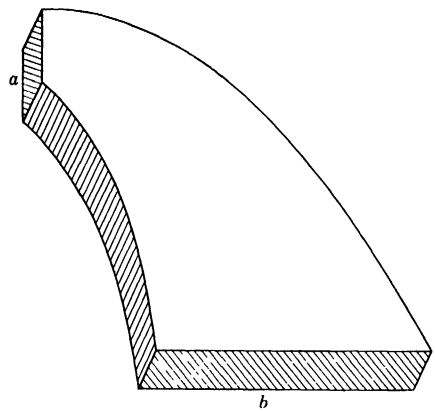


FIG. 1-15. An irregular-shaped conductor of uniform thickness.

Solution. The procedure to be followed in finding the resistance of the specified conductor is substantially that leading to Eq. (1-41). However, before this can be done, it is necessary that the equipotential surfaces be determined. This cannot be done analytically except for shapes of relatively simple geometry. Examples 1-7.1 and 1-7.2 are examples of two such shapes that can be handled analytically. Experimental techniques do exist which will permit the equipotential surfaces to be determined. The flux-plotting techniques discussed in Sec. 1-1 would be one such method. A second method of field plotting known as the *method of curvilinear squares* will be discussed below.

With the equipotential distribution known, it will be noted that the effective length of the conductor varies quite markedly, in passing from the equipotential surface a to the equipotential surface b, along different paths. Clearly, the choice of a single effective length will introduce an error. What must be done, therefore, is to divide the equipotential surfaces into elementary areas which must be appropriately chosen and evaluate the resistance of each elemental strip. Then, by combining the effects of the several elemental strips in parallel, the total effective resistance of the conductor is found.

Details of the method of curvilinear squares and the subsequent calculations of resistance therefrom will now be discussed.

1-8. The Method of Curvilinear Squares. Several fundamental features of conservative fields have already been established. These are noted again for convenience:

1. Current-flow lines and equipotential surfaces must everywhere intersect at right angles.

2. The application of a potential source to an electrode surface will cause this surface to be an equipotential, and thus all flow lines must meet these at right angles. This follows from the fact that the material is assumed to have an infinite conductivity, and if the electrode were not equipotential, an infinite current would circulate.

If, in addition to satisfying the boundary conditions, the field lines and the equipotentials can be made to form *curvilinear squares*, then the result can be shown to be a true picture of the field distribution. A

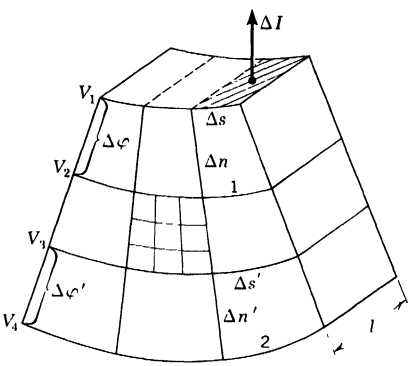


FIG. 1-16 A representative field distribution.

curvilinear square is a four-sided figure usually, the sides of which intersect at right angles. When properly subdivided into smaller units, by the use of additional flow lines and equipotentials, the resulting curvilinear squares approach true squares.

In order to examine the technique in greater detail, consider Fig. 1-16, which represents a portion of a two-dimensional field distribution and can, in fact, be a portion of the coaxial cable of Fig. 1-14. In this figure, the curvilinear squares are large, and there is considerable curvature to the sides of the square. The center section shows the effect of using more equipotentials and flow lines. It is clear that the squares in this center section approach more nearly perfect squares. If this subdivision were carried to the limit, the resulting curvilinear squares would become perfect squares.

It is desired, now, to calculate the total current ΔI through an elementary area or tube of the current-flow field in the region marked 1. From Fig. 1-16, it is clear that the total current through the area Δs wide and l deep is, since the current-density vector is normal to this area,

$$\Delta I = Jl\,\Delta s$$

If, for simplicity, l is chosen equal to unity, then

$$\Delta I = J\,\Delta s/\text{unit depth} \tag{1-45}$$

Also, the *magnitude* of the electric field strength E is given by

$$E = \frac{d\phi}{dn}$$

Hence, from Ohm's law, it follows that

$$J = \gamma E = \gamma \frac{d\phi}{dn}$$

which may be written in the approximate form

$$J = \gamma \frac{\Delta\phi}{\Delta n} \tag{1-46}$$

Now, combine Eq. (1-46) with Eq. (1-45), which yields

$$\Delta I = \gamma \frac{\Delta\phi}{\Delta n}\,\Delta s = \gamma\,\Delta\phi \left(\frac{\Delta s}{\Delta n}\right) \tag{1-47}$$

A similar analysis may be carried out on some other square, say, that labeled 2 in Fig. 1-16. Suppose that the values of all quantities in this region are denoted by primes. Then, following the same development as above, it follows that

$$\begin{aligned} \Delta I' &= J'\,\Delta s' \\ J' &= \gamma E' = \gamma \frac{\Delta\phi'}{\Delta n'} \\ \Delta I' &= \gamma\,\Delta\phi' \left(\frac{\Delta s'}{\Delta n}\right) \end{aligned} \tag{1-48}$$

From the definition or significance of a current line, the tube of the current-flow field, and Ohm's law, the currents ΔI and $\Delta I'$ must be equal, since no sources exist in the region being considered. Then

$$\gamma\,\Delta\phi \left(\frac{\Delta s}{\Delta n}\right) = \gamma\,\Delta\phi' \left(\frac{\Delta s'}{\Delta n'}\right) \tag{1-49}$$

or

$$\Delta\phi = \frac{\Delta s'/\Delta n'}{\Delta s/\Delta n}\,\Delta\phi'$$

In general, Δs, Δn and $\Delta s'$, $\Delta n'$ vary with the convergence or divergence of the current lines and the equipotentials. But, by the method of construction here contemplated, their ratio is everywhere constant and chosen equal to unity for convenience in drawing. Thus,

$$\Delta\phi = \Delta\phi'$$

As a result, the potential difference between adjacent equipotentials is constant, whence

$$V_4 - V_3 = V_3 - V_2 = V_2 - V_1 = \Delta V = \Delta\phi$$

Moreover, since by construction the ratio $\Delta s/\Delta n$ is unity, it follows that the total current through each tube of the current field is the same and from Eq. (1-47)

$$\Delta I = \gamma\,\Delta\phi\left(\frac{\Delta s}{\Delta n}\right) = \gamma\,\Delta\phi \tag{1-50}$$

This is a very important result, since it implies that only one field distribution exists for a given set of boundary conditions, and this field satisfies Ohm's law. If, therefore, the current lines and equipotentials are laid out so that they form curvilinear squares in the region of the specified boundaries, the resulting field distribution is the true field distribution, since it satisfies the boundary conditions and Ohm's law. (The real significance of this and its relation to Laplace's equation will be discussed later.)

1-9. Technique of Field Mapping. To develop a facility in field mapping requires a considerable amount of practice and experience. Often

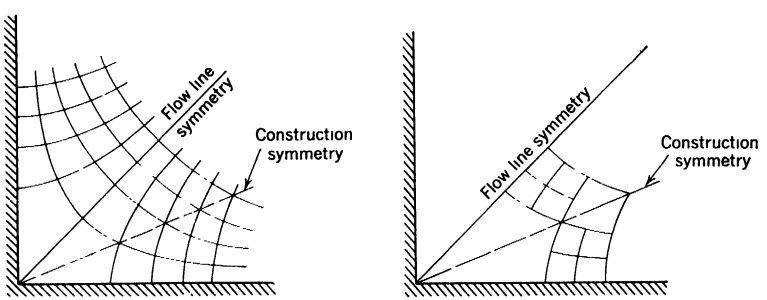


FIG. 1-17. The 90° corner.

a complex field can be reduced to a series of simple special cases which may be properly associated and correlated. This requires experience with a number of special forms and the field distribution associated therewith. Several such special forms are illustrated:

1. The 90° corner. This is illustrated in Fig. 1-17. It should be observed that two symmetries exist, flow-line symmetry and construction symmetry. The second electrode is at an infinite distance from the corner.

2. The 270° corner. This is illustrated in Fig. 1-18. As in *a*, the second electrode is at an infinite distance from the corner.

The mechanics of field plotting are best described in a series of rules of procedure:

1. Begin with several sheets of tracing paper, a soft pencil, and an eraser.

2. Lay out in ink the boundaries around which or within which the field is to be mapped.

3. Examine the geometry carefully to take advantage of any symmetries that may exist.

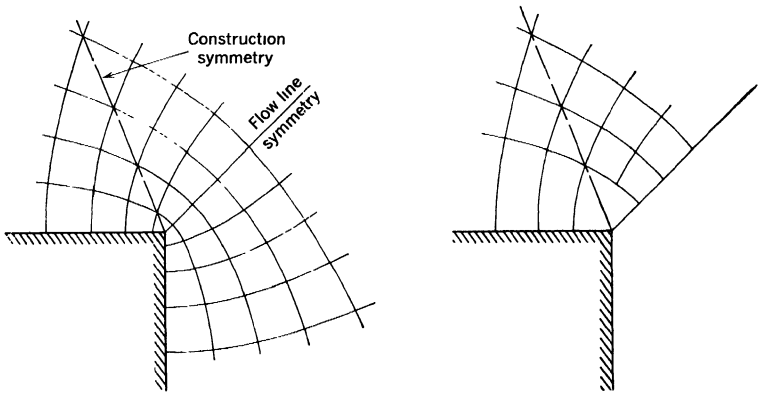


FIG. 1-18. The 270° corner.

4. Cover the entire field with large curvilinear squares, however erroneous they may be. At the outset, it is advisable to concentrate on the orthogonality property rather than on the equilateral property. The number of equipotential lines should be small initially, since subdivision can be effected later, when the general nature of the field distribution has been established.

5. Make successive corrections to the map, maintaining the ultimate curvilinear square objective in mind. It may be necessary, after a number of corrections, to transfer the field plot to a new sheet of paper, for subsequent and continued subdivision. The use of tracing paper facilitates this transfer.

6. Continue the process of revision and subdivision until the orthogonal quadrilaterals become curvilinear squares.

The rules of procedure given above are general guides and may be modified for specific problems. The accuracy of the resulting field map is a function of the artistic skill, the patience and perseverance, and the available time of the sketcher.

1-10. Determination of Resistance from Field Sketch. The essentials of the development of Sec. 1-7 are adapted to a determination of the

resistance of conductors of all plane shapes of constant thickness from graphically determined curvilinear square fields. To examine the situation, refer to a unit depth of a two-dimensional field, as illustrated in Fig. 1-19.

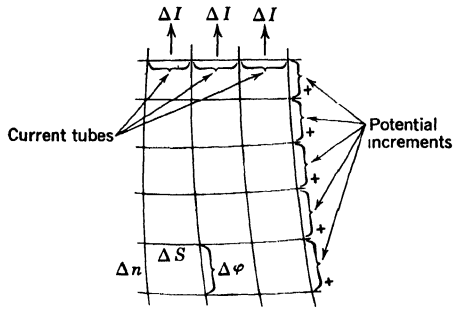


FIG. 1-19. A two-dimensional field.

From Eq. (1-50), the number of current lines per tube (the total current through the specified tube) is given by

$$\Delta I = \gamma \left(\frac{\Delta s}{\Delta n}\right) \Delta \phi$$

But since the field is constructed of curvilinear squares, $\Delta s/\Delta n = 1$. Thus,

$$\Delta I = \gamma \, \Delta \phi$$

But the total number of current lines must equal the current lines per tube times the number of tubes. If there are N_t tubes, then the total current is

$$I = N_t \, \Delta I = N_t \gamma \, \Delta \phi \tag{1-51}$$

Moreover, the potential difference between all equipotential lines is equal, and the total potential difference between the specified boundaries is

$$V = N_v \, \Delta \phi \tag{1-52}$$

where N_v is the total number of potential increments between the specified boundaries. Thus, dividing Eq. (1-52) by (1-51) gives, for the resistance,

$$r = \frac{V}{I} = \frac{N_v}{\gamma N_t} \qquad \text{ohm-m of depth} \tag{1-53}$$

The units are those of $1/\gamma$ since the number ratio is dimensionless, and the depth or thickness has been chosen as 1 m. For some specified length L, the resistance is

$$R = \frac{N_v}{\gamma N_t L} \qquad \text{ohms} \tag{1-54}$$

An alternative approach is the following: Suppose that it is desired to determine the resistance of an individual curvilinear square. This is

$$\Delta R = \frac{\Delta\phi}{\Delta I} = \frac{\Delta\phi}{\gamma\,\Delta\phi} = \frac{1}{\gamma} \tag{1-55}$$

Thus, $1/\gamma$ is the resistance of a prism of unit length whose base is the curvilinear square and whose length is the depth of the plot. The total resistance may be considered as consisting of the resistance of N_s parallel paths each containing N_v resistors in series. Thus,

$$R = \frac{N_v}{N_s} \cdot \frac{\Delta R}{L} = \frac{N_v}{\gamma N_s L} \tag{1-56}$$

which is the same result determined above from the field picture.

Example 1-10.1. Determine the resistance between faces A and B of the uniformly thick conductor illustrated in Fig. 1-20.

Solution. This is to be done by mapping the field in this conductor and determining the resistance from the field map.

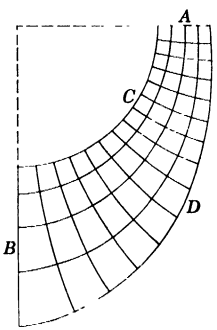


FIG. 1-20. A specified conductor.

$$N_s = 4 \qquad N_v = 14$$

Thus $$R = \frac{1}{\gamma L}\frac{14}{4} \quad \text{ohms}$$

Example 1-10.2. Suppose that Fig. 1-20 is one-quarter of a cable having a circular inner conductor and an elliptical outer conductor, as shown in Fig. 1-21. Determine the sheath, or insulating, resistance of this cable.

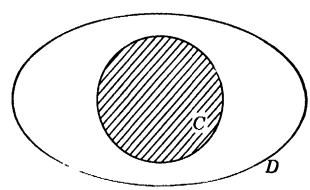


FIG. 1-21. A cable with circular inner conductor and elliptical outer conductor.

Solution. Figure 1-20 now represents the field plot of one-quarter of the cable, but the current is now supposed to flow from surface C to D, rather than between A and B, as in Example 1-10.1. Hence, for the quarter section,

$$N_s = 14 \qquad R = \frac{1}{\gamma L}\frac{4}{14} \quad \text{ohms}$$

$$N_v = 4$$

and, for the total cable,

$$R = \left(\frac{1}{\gamma L}\frac{4}{14}\right)\frac{1}{4} = \frac{1}{14\gamma L} \quad \text{ohms}$$

1-11. The Kirchhoff Laws for the Flow Field. *First Kirchhoff Law.* The first law of Kirchhoff states that a *stationary* electric current behaves like an incompressible fluid; that is, current is neither created nor

destroyed. This means that, if a closed surface is chosen in a multidimensional-flow field, then the total current into the region bounded by the closed surface must be exactly equal to the total current leaving the region. Alternatively, this result may be expressed in the following way: The algebraic sum of the currents entering any finite closed surface in the multidimensional field must be equal to zero. If the flow comprises a group of filamentary conductors connected at a junction, and if the closed surface were chosen to enclose this junction, then the total current at the junction must be zero. It will later be found that the foregoing must be modified if the current is nonstationary or is in a time-varying state.

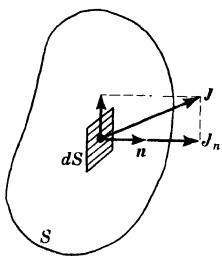


FIG. 1-22. To examine the total current through a closed surface.

This theorem may be formulated mathematically by reference to Fig. 1-22. The current-density vector $\mathbf{J}$ over the surface area dS is considered constant. Hence, the total current passing through the element of area dS is

$$dI = J_n \, dS = \mathbf{J} \cdot \mathbf{n} \, dS$$

The current passing through the total surface is

$$\oint \mathbf{J} \cdot \mathbf{n} \, dS = 0 \tag{1-57}$$

where $\oint$ denotes that the integration is to be carried out over the entire closed surface.

Kirchhoff's Second Law for the Flow Field. The discussion of the current-flow field has been based on the fact that the potential difference between two points in the field is the difference of the potential of these two points relative to some arbitrary datum. This implicitly assumes that no distributed electromotances (electromotive forces, emfs) exist in the field. Then, as discussed in Sec. 1-4, the sum of all potential differences along any arbitrarily chosen closed path in the field is zero. Kirchhoff's second law, which is essentially this statement, may be expressed in the following form: The line integral of the electric field intensity along any arbitrary closed path in a stationary flow field is zero. Mathematically, this result is given by the integral

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{1-58}$$

where $\oint$ indicates that the integration is to be carried out on the closed path. In this case the field is conservative.

Attention is called to the fact that electromotances which are distributed in space may exist; these may be due to electrochemical, electrothermal, or electromotive effects. If such electromotances are repre-

sented by intrinsic electric field intensities which are so defined that the distributed electromotances per unit length are given as the line integrals of the intrinsic electric field intensities, then Kirchhoff's second law may be written as

$$\oint (\mathbf{E}_t - \mathbf{E}_i) \cdot d\mathbf{l} = 0 \tag{1-59}$$

where $\mathbf{E}_t$ denotes the total field, which in general is nonconservative, and where $\mathbf{E}_i$ denotes the intrinsic electric field intensity.

1-12. Joule's Law. It is known that heat is produced in a conductor that carries current. Physically, the heat loss actually occurs because the current carriers are acted upon by the field. Work is done by the field, and some of this work is transferred to the ions of the matter constituting the conductor by collisions of the moving current carriers with the ions. It is desired to examine the law which relates the amount of heat so produced. To do this, refer to Fig. 1-23. The quantity $\mathbf{J} \cdot \mathbf{E}$ is to be examined. From considerations of the units, it is noted that the units of the product are amp/m^2 $\times$ volts/m, which is volt-amp/m^3 or watts/m^3. The quantity $\mathbf{J} \cdot \mathbf{E}$ is therefore written as dp and gives the power converted into heat per unit volume.

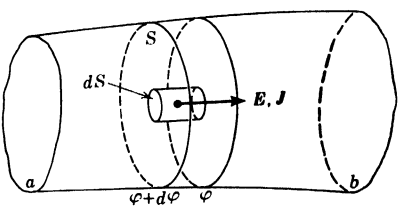


FIG. 1-23. To examine the heating of a conductor.

$$dp = \mathbf{J} \cdot \mathbf{E} \qquad \text{watts/m}^3 \tag{1-60}$$

This is written in the form

$$dp = \frac{dP}{dS\, dl} = \mathbf{J} \cdot \mathbf{E}$$

where dP is the total power converted into heat in the elementary prism $dS\, dl$. Thus, we write

$$dP = \mathbf{J} \cdot \mathbf{E}\, dS\, dl$$

which is regrouped to the expression

$$dP = \mathbf{J}\, dS \cdot \mathbf{E}\, dl$$

The total power is then given by the expression

$$P = -\int_S J\, dS \int_a^b d\phi \qquad \text{watts}$$

By Eq. (1-34), this is

$$P = -I \int_a^b d\phi = -IV_{ba} = IV_{ab}$$

with V_{ab} being the total drop in potential from a to b. By Eq. (1-41), this becomes

$$P = I^2 \int_a^b dR = I^2 R_{ab} \qquad \text{watts} \tag{1-61}$$

This expression gives the rate at which heat is produced in the conductor and is Joule's law.

1-13. Examples of Current Flow-fields. Sufficient theory has been developed to allow the solution of a number of interesting and informative problems. Before this is done, however, it is important to understand that what has actually been accomplished so far in our study is a discussion of one of a large class of conservative scalar field systems. As will later be shown, the entire class of such conservative field systems are summarized and contained in what is known as *Laplace's equation.* It is a consequence of the uniqueness of the solutions of Laplace's equation that a solution to the equation which satisfies the appropriate boundary conditions, whether obtained by intuition, by guess, or by analysis considerations, is a unique solution of the problem.

Formal techniques exist for the solution of Laplace's equation for certain geometric shapes, the boundary conditions being specified over all or part of the geometric boundaries. This technique of solutions is ordinarily referred to as *boundary-value problems.* It will later be shown that Ohm's law in field terms which relate the ordinary potential distribution for points inside a conductor to the current and the statistical property of the substance (the conductivity) is essentially a statement of Laplace's equation. Consequently, the solution of the expression for resistance in Sec. 1-7 is, in fact, a procedure for the solution of Laplace's equation. The method of curvilinear squares discussed in Sec. 1-8 is another technique for the solution of Laplace's equation and is particularly useful when analytic techniques are not possible or are too complicated. Other formal techniques will be pointed out as our theory develops. Some of the problems now to be considered will be solved by applying physical reasoning and intuition and certain of the theory already developed.

Example 1-13.1. Point Source. As a simple example, consider that a sphere, of radius r_0, which is made of a good conducting material (copper) is buried in a material of poor conductivity (the ground). Suppose that the current is conducted to the sphere by an infinitely thin, isolated conductor and that a very large surrounding electrode collects the current and returns it to the source. Find the potential distribution and the ground resistance.

Solution. The geometry of the system is illustrated in Fig. 1-24. Because of the symmetry of the spherical electrode and the fact that it is supposed that the conductor collecting the current is remote from the sphere, thereby not influencing the field in the neighborhood of the sphere, the current lines will be radial, as illustrated. Clearly, under the specified conditions, the total current I conducted to the sphere will be distributed uniformly over the surface of concentric

spheres. The current density at a distance r from the center of the sphere will be

$$J = \frac{I}{4\pi r^2} \tag{1-62}$$

Note that the Kirchhoff current law has been invoked, since the total current to the sphere is assumed to pass through a spherical surface at the distance r. Also, owing to the symmetry of the system, and for the assumed homogeneous earth, it is expected that the equipotential surfaces will be concentric spherical shells. The surface of the metal sphere is also an equipotential, as it was assumed that the material was highly conducting (actually, it is being assumed that the conductivity is infinite).

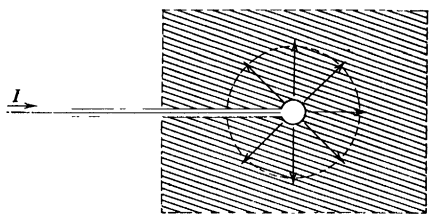


FIG. 1-24. Current flow in the neighborhood of a spherical electrode embedded in the ground.

The electric field intensity in the conducting medium is obtained from Eq. (1-62) by an application of Ohm's law; thus,

$$E = \frac{1}{\gamma} J = \frac{I}{4\pi\gamma r^2} \tag{1-63}$$

The direction of the electric field intensity is the same as that of the current density.

The potential difference between the surface of the metallic sphere and any point P in the region at a distance r from the center of the sphere can be determined by using Eq. (1-34). Thus, the potential drop is

$$V_{OP} = \int_{r_0}^{r} E\, dr = \frac{I}{4\pi\gamma} \int_{r_0}^{r} \frac{dr}{r^2}$$

$$V_{OP} = \frac{I}{4\pi\gamma} \left(\frac{1}{r_0} - \frac{1}{r} \right) \tag{1-64}$$

It is interesting to note that the potential difference between the metallic sphere and an arbitrary point P approaches a limiting value as the distance r increases. The limiting value of this potential drop is

$$V_0 = \frac{I}{4\pi\gamma r_0} \tag{1-65}$$

This value is correct within 1 per cent if the point P is $100r_0$ from the center of the metallic sphere.

Equation (1-64) for the potential difference between the grounding sphere and a point at a distance r therefrom may be combined with Eq. (1-65) to give the potential difference V_{OP} in terms of the total potential difference V_0. The result is

$$V_{OP} = V_0\left(1 - \frac{r_0}{r}\right) \tag{1-66}$$

This function has the form shown in Fig. 1-25.

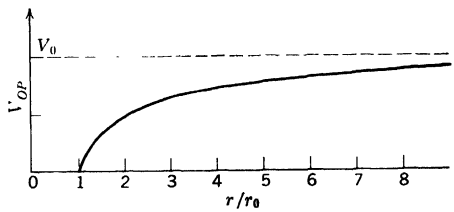


FIG. 1-25. Potential in the neighborhood of a spherical electrode.

The potential difference given by Eq. (1-65) is taken to denote the potential drop across the contact resistance between the metallic sphere and the conducting earth. The contact resistance is then

$$R = \frac{V_0}{I} = \frac{1}{4\pi\gamma r_0} \tag{1-67}$$

For ordinary ground having a conductivity of 10^{-2} mho/m and a radius r_0 of 0.5 m, the ground resistance is

$$R = \frac{1}{4\pi \times 10^{-2} \times 0.5} = 16 \text{ ohms}$$

Note that if a circuit consists of two such spheres, say, several miles apart, embedded in a homogeneous earth, the total resistance in the circuit is equal to twice the resistance of one ground connection, plus the resistance of the metallic circuit.

It is convenient to find an expression for the potential at any point of the field. According to the discussion in Sec. 1-2, this is equal to the potential difference between this point and a reference point. From Eq. (1-64), it appears that the potential of any point at a distance r from the center of the sphere is

$$\phi = \frac{I}{4\pi\gamma r} + k$$

where k is an arbitrary constant. Observe that at large distances the potential ϕ is just equal to k. Therefore, k is the potential of a point at a large distance from the sphere. If, however, this remote point is chosen as the zero, or datum, then the potential at any point in the space is simply

$$\phi = \frac{I}{4\pi\gamma r} \tag{1-68}$$

Observe that ϕ is independent of the radius of the sphere, and is therefore true for the case of a point electrode, for which $r \to 0$.

Example 1-13.2. Hemispherical Ground Connection. As an example of the use of the known solution to one problem in finding the solution of a second problem, consider the problem of finding the resistance of a ground connection consisting of a buried metal hemisphere that has its diametral plane in the surface of the ground.

Solution. As before, the earth is assumed to be homogeneous and isotropic, and the return connection is sufficiently far away so that it has a negligible distorting effect on the current distribution near the hemisphere. A solution to this problem is found by first supposing that the space surrounding the spherical

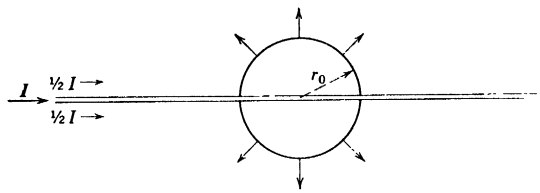


FIG. 1-26. A pair of hemispherical electrodes.

electrode in Example 1-13.1 is divided into two half-spaces by passing a pair of thin, isolated planes through the center of the metallic sphere, as shown in Fig. 1-26. If the symmetry of the system is otherwise not affected, then it follows that each hemisphere will conduct the current of $\frac{1}{2}I$.

Suppose now that the entire half-region above the horizontal plane is removed. This will in no way affect the current distribution or the potential distribution in the lower half-space. Clearly, therefore, if a hemisphere of radius r_0 is buried on the surface of the ground and a current $\frac{1}{2}I$ is conducted to it, the potential distribution is precisely that corresponding to the buried metal sphere of radius r_0 with current I conducted to it. Also, the potential difference across the contact resistance remains the same. Thus, from Eq. (1-65),

$$V_0 = \frac{I}{4\pi\gamma r_0} = \frac{I/2}{2\pi\gamma r_0}$$

from which it follows that the contact resistance is now

$$R = \frac{1}{2\pi\gamma r_0} \qquad (1\text{-}69)$$

which is seen to be twice that of the buried metallic sphere.

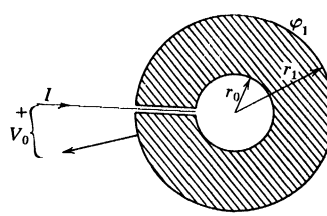


FIG. 1-27. A concentric conducting system.

Example 1-13.3. Concentric Spherical Electrodes. Suppose that the space between a spherical electrode of radius r_0 and a larger spherical shell of radius r_1 is filled with a homogeneous and isotropic medium of conductivity γ. Provision is made for conducting a current I to the inner electrode and away from the outer electrode, as illustrated in Fig. 1-27. It is required to find the resistance of the specified conducting system.

Solution. If V_0 denotes the potential difference between the electrodes, and if the potential of the outer electrode is denoted arbitrarily as ϕ_1, then the poten-

tial of the inner electrode is given by $\phi_1 + V_0$. It follows from Eq. (1-68) that

$$\phi_1 = \frac{I}{4\pi\gamma r_1}$$

$$\phi_1 + V_0 = \frac{I}{4\pi\gamma r_0}$$

It follows from these that

$$V_0 = \frac{I}{4\pi\gamma}\left(\frac{1}{r_0} - \frac{1}{r_1}\right) = \frac{I}{4\pi\gamma}\frac{r_1 - r_0}{r_0 r_1}$$

The resistance between the electrodes is given by the relation

$$R = \frac{1}{\gamma}\frac{r_1 - r_0}{4\pi r_0 r_1} \tag{1-70}$$

If this expression is compared with Eq. (1-42) for the resistance of a cylindrical conductor, it is seen that it may be interpreted as the resistance of a cylindrical conductor of the same material, having a length $r_1 - r_0$, and having a cross section area $A = 4\pi r_0 r_1 = 4\pi(\sqrt{r_0 r_1})^2$.

Example 1-13.4. System of Point Sources. As a somewhat more complicated problem, consider that two point electrodes are deeply embedded in a homogeneous and isotropic medium of conductivity γ and are separated by a distance L. Suppose also that a current I is conducted to one electrode, designated as Q_1, and that the current I is conducted away from the second electrode, designated as Q_2. It is required to find the equipotential distribution in the medium.

Solution. The geometry of the system is illustrated in Fig. 1-28. The potential at the point P is obtained by combining the potentials at P due to each electrode separately. This simple combining of potentials is possible because of the

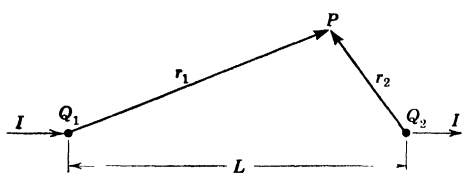


FIG. 1-28. A system of two point electrodes embedded in a conducting medium.

linear relationship that exists between the current and the potential; that is, the superposition theorem applies in such linear systems. The potential at point P is given by

$$\phi_P = \frac{I}{4\pi\gamma}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{1-71}$$

To find the equipotential surfaces, it is recalled that these must satisfy the condition that ϕ = const over the equipotential surface. To find some of the equipotentials, Eq. (1-71) is written in the form

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{4\pi\gamma\phi}{I} = \frac{k}{L} \tag{1-72}$$

where k is a constant. It follows from this that

$$r_1 = \frac{r_2}{1 + k(r_2/L)} \tag{1-73}$$

This expression gives the equation of the radii of the equipotential lines. If k is given the values $k = 0, 1, 2, 3$, etc., equipotential lines having equal potential differences will result. Figure 1-29 gives a sketch of such a potential distribution.

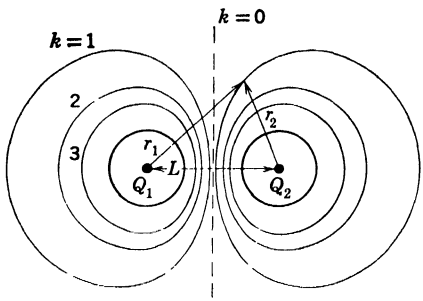


FIG. 1-29. Equipotentials of two point sources of opposite sense.

Example 1-13.5. Ground-current Interference. Suppose that a pair of hemispherical grounding electrodes are buried on the surface of the ground. These are supposed to be the grounding arrangements of a single-wire telegraph circuit or of a trolley wire. A second single-wire telegraph line parallels the first circuit at a distance a away. It is required to evaluate the difference of potential that appears in the second line, due to a current I in the first line. The geometry of the system is that shown in Fig. 1-30.

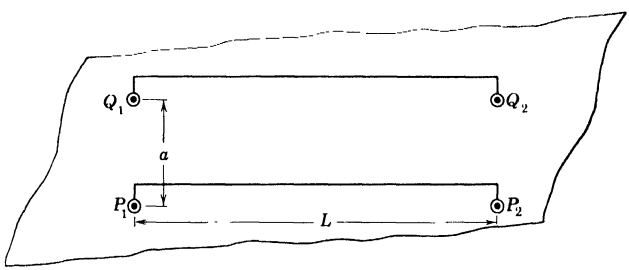


FIG. 1-30. Two parallel single-wire telegraph lines, with grounding electrodes.

Solution. It is first noted that an application of the same reasoning as in Example 1-13.2 permits one to write expressions for the potential at any point on the surface of the earth from Eq. (1-71). The potentials at P_1 and P_2 are given by the expressions shown.

At P_1: $\quad r_1 = a \quad$ and $\quad r_2 = \sqrt{a^2 + L^2}$

$$\phi_{P_1} = \frac{I/2}{2\pi\gamma}\left(\frac{1}{a} - \frac{1}{\sqrt{a^2 + L^2}}\right)$$

At P_2:

$$r_1 = \sqrt{a^2 + L^2} \qquad r_2 = a$$

$$\phi_{P_2} = \frac{I/2}{2\pi\gamma}\left(\frac{1}{\sqrt{a^2 + L^2}} - \frac{1}{a}\right)$$

The difference of potential between P_1 and P_2 is

$$V = \phi_{P_1} - \phi_{P_2} = \frac{I/2}{2\pi\gamma} \times 2 \times \left(\frac{1}{a} - \frac{1}{\sqrt{a^2 + L^2}}\right) \tag{1-74}$$

If the conductors are long compared with the distance between them, that is, if $L \gg a$, then

$$V \doteq \frac{I}{2\pi\gamma}\frac{1}{a} \tag{1-75}$$

Example 1-13.6. Line Source. Consider the line source illustrated in Fig. 1-31. It is required to find an expression for the potential at any point P and the equipotential surfaces.

Solution. To solve the problem, it is supposed that the line source is a continuous and uniform current leak away from the conductor which can be resolved into an array of point sources in a line which are spaced at infinitely small distances. In particular, it is supposed that the line source can be resolved into a large number of very small line elements $d\zeta$, each of which is considered as constituting a point source. If the total current supplied by the line is I, then the current supplied by the element $d\zeta$ is $I\,d\zeta/2L$, where $2L$ is the total length of the line, as illustrated in Fig. 1-31. The potential at the point P due to the current element $I\,d\zeta/2L$ is given by

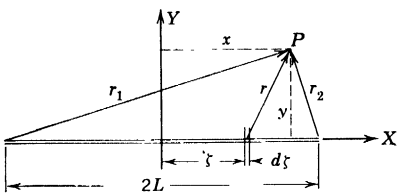


FIG. 1-31. A line source.

$$d\phi = \frac{I\,d\zeta}{2L}\frac{1}{4\pi\gamma r}$$

which, by making use of the geometry of the figure, may be expressed as

$$d\phi = \frac{I}{8\pi\gamma L}\frac{d\zeta}{\sqrt{(x - \zeta)^2 + y^2}} \tag{1-76}$$

where ζ is the distance from the origin to the line element. The total potential of the line source at point P is obtained directly as

$$\phi = \frac{I}{8\pi\gamma L}\int_{-L}^{L}\frac{d\zeta}{\sqrt{(x - \zeta)^2 + y^2}}$$

which integrates to the expression

$$\phi = \frac{I}{8\pi\gamma L}\ln\frac{(x + L) + \sqrt{(x + L)^2 + y^2}}{(x - L) + \sqrt{(x - L)^2 + y^2}} \tag{1-77}$$

The equipotential lines in the XY plane are ellipses if we set $r_1 + r_2 = 2a$, where a is the semimajor axis, and where the foci are at the end points of the line $2L$ (see any book on analytic geometry). In this case,

$$r_1 = a + \frac{xL}{a} = \sqrt{(x + L)^2 + y^2}$$

$$r_2 = a - \frac{xL}{a} = \sqrt{(x - L)^2 + y^2}$$

If these are introduced into Eq. (1-77), the expression for the potential becomes

$$\phi = \frac{I}{8\pi\gamma L} \ln \frac{a + L}{a - L} \tag{1-78}$$

Clearly, for each arbitrary value of a, the potential will be a constant. Owing to symmetry, it is clear that the equipotential surfaces are ellipsoids of rotation,

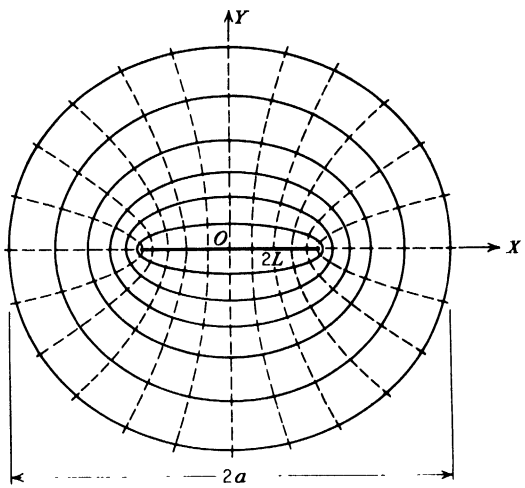


FIG. 1-32. Equipotential and current-flow lines for the line source.

which are obtained by rotating the ellipses about the X axis. Also, the current lines, which must be normal to the ellipsoids, are hyperboloids of two sheets having the same foci as the equipotential ellipsoids. The equipotential ellipses and the current-flow hyperbolas are shown in Fig. 1-32.

Attention is called to the important special case when the minor axis of the ellipse is very small compared with the length, so that $a \doteq L$. The equipotentials are very nearly cylindrical in shape, but with rounded ends. Therefore, the field, owing to such a cylindrical electrode with rounded ends, will be substantially the same as that of a line source. If the diameter of the rod at its center is denoted by d, then the potential at a distance point will be the potential of the equipotential surface. The potential at the surface of the rod will be, since $x = 0$, $y = \frac{1}{2}d$,

$$\phi = \frac{I}{8\pi\gamma L} \ln \frac{L + \sqrt{L^2 + (d/2)^2}}{-L + \sqrt{L^2 + (d/2)^2}} \tag{1-79}$$

Also, if $L \gg d$, the case of a long, thin rod, this expression simplifies to the form

$$\phi = \frac{I}{4\pi\gamma L} \ln \frac{4L}{d} \tag{1-80}$$

Since at large distance $\phi = 0$, then $V_0 = \phi$ gives the total potential difference across the ground resistance.

Example 1-13.7. Grounding Rods.* Suppose that a long grounding rod is half-buried vertically in the ground. It is required to find the resistance of this grounding arrangement.

Solution. The procedure here is the same as in going from Example 1-13.1 to Example 1-13.2, namely, by dividing the region containing the rod into two half-

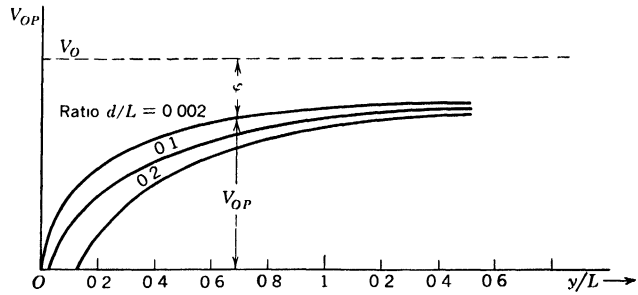


FIG. 1-33. Potential distribution of a cylindrical grounding rod.

spaces. The expression for the potential at any point in the ground surface is, from Eq. (1-77),

$$\phi = \frac{I}{4\pi\gamma L} \ln \frac{L + \sqrt{L^2 + y^2}}{-L + \sqrt{L^2 + y^2}} \tag{1-81}$$

Also, in the manner used in proceeding from Eq. (1-79) to Eq. (1-80), it is found that the total potential difference across the ground resistance for $L \gg d$ is

$$V_0 = \frac{I}{2\pi\gamma L} \ln \frac{4L}{d} \tag{1-82}$$

By combining this expression with Eq. (1-81), there results

$$\phi = \frac{V_0}{2 \ln (4L/d)} \ln \frac{\sqrt{1 + (y/L)^2} + 1}{\sqrt{1 + (y/L)^2} - 1} \tag{1-83}$$

A plot showing the potential-difference distribution for various d/L ratios, as a function of the ratio y/L, is given in Fig. 1-33. Observe that, if $y/L \gg 1$, the condition for large distances from the grounding rod, Eq. (1-81), approximates to

$$\phi \doteq \frac{I}{2\pi\gamma} \frac{1}{y} \tag{1-84}$$

* An extensive discussion is given by J. Zaborsky and J. W. Rittenhouse, Design Charts for Determining Optimum Ground Rod Dimensions, *Trans. AIEE*, pt. III, **72**:810 (1953).

from which $$R = \frac{1}{2\pi\gamma y}$$

which is the same as Eq. (1-69) for the hemispherical grounding electrode. Observe that, for large distances, the potential distribution for this case (see Fig. 1-32) does, in fact, approximate that of a spherical charge distribution.

Example 1-13.8. The Coaxial Cable. Consider a coaxial cable that is long compared with the spacing between the inner and outer electrodes. Find the insulation resistance.

Solution. The potential distribution will be substantially that due to a cylindrical electrode with rounded ends, as discussed in Example 1-13.6 for the corresponding special case, and the equipotential surfaces will be very nearly cylindrical in shape. The current flows uniformly in the radial direction over the whole length of the cylinder through each equipotential surface. However, because of the symmetry of the field, the current density will be

$$J = \frac{I}{2\pi r L} \quad \text{amp/m}^2 \tag{1-85}$$

The current-density vector will be directed outward if the current flow is out of the line conductor.

The electric field intensity is given by

$$E = \frac{1}{\gamma} J = \frac{I}{2\pi\gamma L} \frac{1}{r} \tag{1-86}$$

Therefore, the potential difference between the inner conductor (radius a) and outer conductor (radius b) will be given by

$$V_{ab} = \int_a^b E\, dr = \frac{I}{2\pi\gamma L} \ln \frac{b}{a} \tag{1-87}$$

The insulation resistance thus has the value

$$R = \frac{1}{2\pi\gamma L} \ln \frac{b}{a} \tag{1-88}$$

This expression is in agreement with Eq. (1-44), which was obtained in a somewhat different manner.

1-14. Boundary Conditions: Current through Different Media. In all our prior work, it has been implicitly assumed that the conducting media have been homogeneous and isotropic and of conductivity γ. It often happens that the total conducting medium is made up of more than one type of medium. It is desired to examine the effect on the current-flow field as it passes across the boundary between two media of different conductivities. It will be shown that the current lines will be refracted at the interface between the two media.

Refer to Fig. 1-34, which shows the situation at the interface. As illustrated, it is supposed that a current line makes an angle α_1 with the normal to the interface of the two media. It will enter the second material at an angle α_2 with the normal, which will be different, in general, for different values of conductivities γ_1 and γ_2. Suppose now that the vectors $\mathbf{J}_1$ and $\mathbf{J}_2$ in the regions on both sides of the boundary are resolved into normal components J_n and tangential components J_t. Different conditions apply for J_n and J_t.

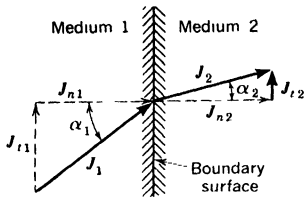


FIG. 1-34. The current-flow lines at the boundary between media of different conductivities.

The Normal Components. Because of the source-free character of the current-flow field, the total current passing through an elementary area dS in the interface must be the same on each side of the interface. This requires that

$$J_{n1}\,dS = J_{n2}\,dS$$

from which it follows that

$$J_{n1} = J_{n2} \tag{1-89}$$

This result states that the normal component of the current density at a boundary is continuous.

Tangential Components. To obtain the boundary conditions on the tangential components, refer to Fig. 1-35, which shows a small rectangular path in the boundary surface or interface between the two different conducting media. Use will now be made of the fact already proved for the current-flow field that the line integral of the electric field intensity around a closed path must be zero. Apply this to the rectangular path shown, which lies half in one medium and half in the other. Choose the sides of the rectangle to be infinitesimal and such that, as $l_1 \to 0$, $l_2/l_1 \to 0$. It is possible to neglect the contribution to the integral from the two short sides l_2 compared with the contribution by the long sides. The line integral thus gives

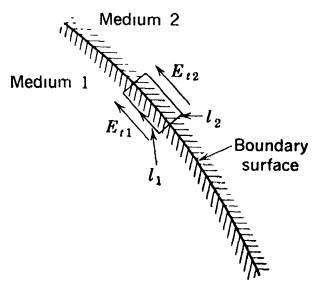


FIG. 1-35. The interface between two conducting media.

$$E_{t1}l_1 - E_{t2}l_1 = 0$$

From this, it follows that

$$E_{t1} = E_{t2} \tag{1-90}$$

which allows the condition to be expressed as

$$\frac{J_{t1}}{J_{t2}} = \frac{\gamma_1}{\gamma_2} \tag{1-91}$$

This shows that the ratio of the tangential components of the current densities at a boundary is equal to the ratio of the conductivities of the two media.

Law of Refraction. Equations (1-89) and (1-91) may be combined into a single expression for the refraction of the current lines. From Eq. (1-89), it follows that

$$J_1 \cos \alpha_1 = J_2 \cos \alpha_2$$

and, from Eq. (1-91),

$$J_1 \sin \alpha_1 = \frac{\gamma_1}{\gamma_2} J_2 \sin \alpha_2$$

Combine these to obtain the expression

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\gamma_1}{\gamma_2} \tag{1-92}$$

It follows from this expression that the angle with the normal of a current line passing from a material of greater conductivity to a material of lesser conductivity will decrease. This is the situation illustrated in Fig. 1-34, whence, in this figure, $\gamma_1 > \gamma_2$.

Suppose that the ratio of conductivities is extremely large, the case when one of the media is a good conductor. It follows from Eq. (1-92) that, for this case, the current lines leave the material with the large conductivity at 90°. Conversely, if the ratio of conductivities is extremely small, the case where one of the media is an insulator, there will be virtually no current flow across the boundary, that is, the normal component of the current density is zero. In this case the current lines must be parallel to the boundary, and the equipotential surfaces must be perpendicular to the boundary.

1-15. Method of Images. A review of the various examples discussed in Sec. 1-13 will show that three classes of problems were considered: (1) electrodes embedded in an infinite medium, (2) half-electrodes embedded in the half-space, and (3) electrode systems with special symmetries. Nothing has yet been said about electrodes which might be buried a small distance below the surface of a half-space. A relatively straightforward technique is available which permits the solution of a number of important problems involving electrodes embedded in a medium of specified boundaries.

Consider that a small spherical electrode is embedded at a distance d below the surface of a half-space, as illustrated in Fig. 1-36. A current I is conducted to the spherical electrode through an isolated conductor, and the current leaves the conducting half-space at a very large distance. It is desired to find the contact resistance of the electrode.

Before discussing the solution of the problem, the boundary conditions that must be fulfilled at the surface of the half-space should be kept clearly in mind. These follow from the above conditions and are given here for convenience:

1. The normal component of the current density is zero, which requires that the current lines be parallel to the boundary.
2. The equipotential surfaces must be perpendicular to the boundary.

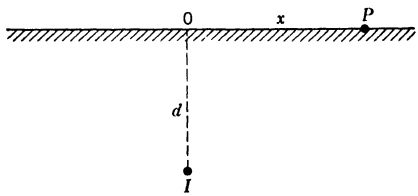


FIG. 1-36. An electrode at a distance below the surface of the ground.

One technique for the solution of this problem has already been discussed. It is the method of field mapping, discussed in Sec. 1-10. This technique allows the specific solution to a given configuration but does not give general analytic results. The method of electrical images will now be discussed, which, if applicable, gives an analytic solution to the problem.

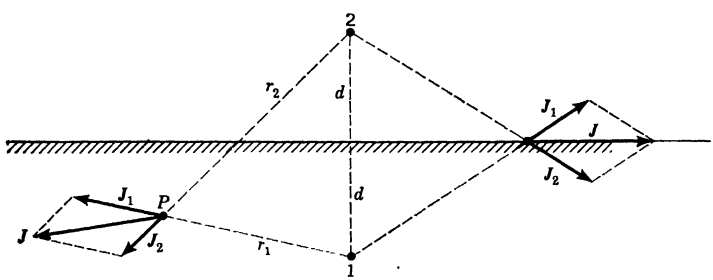


FIG. 1-37. An embedded electrode and its image.

The method of images assumes that the surface of the half-space is a quasi mirror and that the whole half-space below the surface is reflected into the region above the surface. The situation is that shown in Fig. 1-37. Observe that there are now two spherical electrodes, a distance $2d$ apart, each of which receives a current I, and both assumed to be embedded in an unlimited medium of conductivity γ. It is important to establish whether the equivalent problem obtained by an application of the method of images does, in fact, satisfy the boundary conditions. It is important to check this first.

From the fact that the current distribution due to an embedded point electrode is radial (see Example 1-13.1), then, at any point such as P, two components of current density exist, one due to each current. The resultant current density is the vector sum of the two components. Now, if the point P is chosen anywhere along the surface of the half-space, it will be found that the current is everywhere parallel to the boundary, as is required by the boundary condition 1 above. Of course, since the flow lines and the equipotentials are everywhere normal to each other, it follows that the equipotential surfaces will be perpendicular to the surface.

The potential at any point P in the region below the surface is given directly by Eq. (1-71), with due regard for the sense of the currents. The resulting expression is

$$\phi = \frac{I}{4\pi\gamma}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad (1\text{-}93)$$

In fact, by proceeding as in Example 1-13.4, the equipotential distribution will be found to have the form illustrated in Fig. 1-38. Observe from this sketch that the equipotential surfaces are perpendicular to the surface, as required.

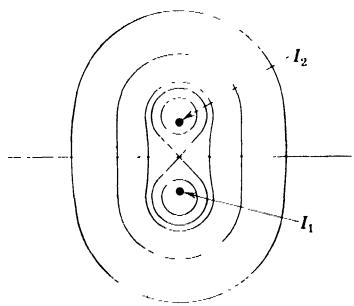


FIG. 1-38. Equipotential distribution of two sources of the same sense.

Attention is called to what is accomplished by the method of images. By the use of this technique, the problem is simplified from that of an electrode embedded below the surface of the half-space to that of two electrodes embedded in an infinite medium. The subsequent results deduced from this new problem are valid only in the region below the surface of the half-space.

The potential difference between the embedded spherical electrode of radius r_0 and a remote point is obtained from Eq. (1-93) by writing $r_1 = r_0$ and $r_2 = 2d$ and is

$$V_0 = \frac{I}{4\pi\gamma}\left(\frac{1}{r_0} + \frac{1}{2d}\right) \qquad (1\text{-}94)$$

The ground resistance is then given by the expression

$$R = \frac{1}{4\pi\gamma r_0}\left(1 + \frac{r_0}{2d}\right) \qquad (1\text{-}95)$$

By comparing this with Eq. (1-67) for the embedded electrode in the total space, it is seen that the contact resistance is higher in the present case. Also, by comparing this with Eq. (1-69), it is seen to be lower than that for the hemispherical conductor in the surface of the half-space. Equa-

tion (1-95) is valid only for $r_0/2d \leq 1$, and the factor $1 + r_0/2d$ will vary from 1 to 2.

PROBLEMS

1-1. Given two vectors in rectangular form

$$\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$$
$$\mathbf{B} = 2\mathbf{i} + 1\mathbf{j}$$

Find the value of $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, $\mathbf{B} - \mathbf{A}$, and sketch the results.

1-2. Given the two vector combinations

$$\mathbf{A} + \mathbf{B} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$
$$\mathbf{A} - \mathbf{B} = 5\mathbf{i} + 1\mathbf{j} + 2\mathbf{k}$$

a. What are the values of **A** and **B**?

b. What is the value of $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})$?

1-3. The potential distribution in a given region of space has the form

$$\phi = k(2x^2 + y^2 - z^2)$$

where k is a positive constant. Find the components of **E**.

1-4. A potential function has the form shown in the neighborhood of the origin.

$$\phi = ax^2 + by$$

where a and b are positive constants.

a. Find the components of E.

b. Where is the minimum of E?

1-5. Prove that Eqs. (1-15) do specify the gradient, defined in Eq. (1-12), in cylindrical and in spherical coordinates.

1-6. Prove that

$$\text{grad } (uv) = u\nabla v + v\nabla u$$

1-7. The accompanying figure represents the equipotential lines of a current-flow field. The equipotential surfaces are cylinders with generatrices perpendicular to the plane of the drawing. All potentials are given in volts measured from a datum not shown. Assume that the conductivity of the material is 0.01 mho/m. Determine the following:

a. The electric field intensity at points C, D, and E. Sketch the electric-field-intensity vector to a reasonable scale at these points. State the scale.

b. Show the potential gradient at the same points.

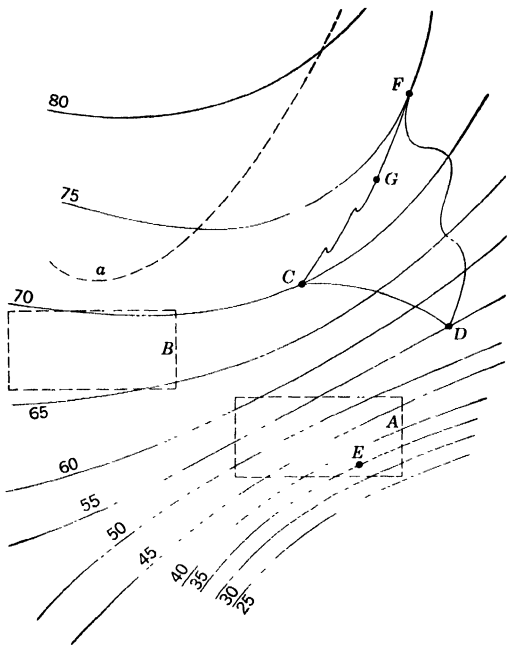
c. Draw the current lines in the area shown in the broken rectangle marked A.

d. Determine and draw the current-density vectors at points C, D, and E.

e. From an inspection of the field plot, determine the approximate ratio of the current density at the center of the equal rectangles A and B.

f. Is it possible that the dashed curve marked a is an equipotential?

g. Determine the value of the line integral along the paths *CD*, *DF*, and *FC*.
h. Determine the sum of the line integrals $V_{CD} + V_{DF} + V_{FC}$.
i. Does the value of the line integral V_{FC} depend on the length of the path?
j. Show at point *G*: the line-element vector, the field-intensity vector, the unit-normal vector.



1-8. The inner diameter of the lead sheath of a lead-sheathed rubber-insulated cable, the inner conductor of which has a cross section of 250,000 cir mils, is 1 in. If the rubber has a resistivity of 1.6×10^9 megohm-cm at 20°C, what is the insulation resistance per mile?

1-9. Given a thick-walled cylinder of copper of inner radius *a*, outer radius *b*, and length *L*. The resistivity of the material is ρ ohms m^3.

a. Deduce an expression for the resistance between the inner and the outer faces of the cylinder.

b. Deduce an expression for the resistance between the two ends of the conductor.

1-10. The region between concentric spherical surfaces is filled by a material of conductivity γ.

a. Compute the resistance between the concentric spherical surfaces.

b. What happens to the resistance, in the limit as the inner radius approaches zero and the shell becomes a solid sphere?

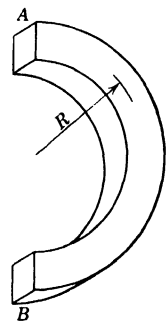
c. For a fixed inner radius, what happens to the value of the resistance as the outer radius approaches infinity?

1-11. Find an approximate expression for the resistance between the plane faces of a right circular cone, the radii of the faces being a and b, and the height h. The conductivity of the material is γ.

1-12. Compare the resistance between faces A and B of the square section of side d shown:

a. By assuming this to be a straight section of length πR

b. By taking into account the effects of the shape



1-13. A linear homogeneous conductor, having a cross section of 10^{-4} m^2 and 10 m long, carries a current of 100 amp. The potential difference across the conductor is 100 volts. Assume a uniform current distribution in the conductor.

a. Determine and show the direction of:

(1) The current-density vector **J**

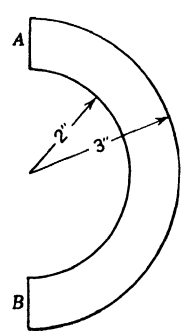
(2) The electric-field-intensity vector **E**

(3) The potential gradient

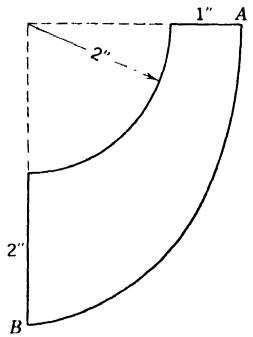
b. Determine the conductivity of the material.

c. Show the equipotential surfaces.

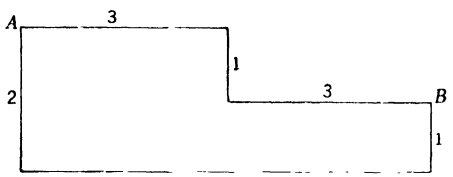
1-14. Find the resistance between faces A and B of the conductor of uniform thickness which has the shape illustrated, by mapping the field and deducing the resistance from this field map.



1-15. Repeat Prob. 1-14 for a conductor having the shape shown.



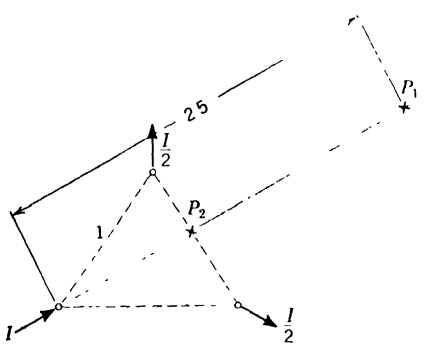
1-16. Consider the conductor illustrated, having a uniform thickness d. The conductivity of the material is γ.



a. Calculate the resistance between faces A and B by considering the conductor to be composed of two simple sections in series.

b. Determine the resistance by considering the conductor to be a single section, with attention to the current-flow lines in the material.

1-17. Three point electrodes which are immersed in a uniform conducting material lie at the vertices of an equilateral tringle, as shown in the diagram.



a. Determine the potential difference between P_1 and P_2.

b. Determine the electric field intensity at P_2.

c. Show how you would find a point of zero potential within the triangle.

1-18. Calculate and plot the ground resistance of spherical grounding arrangements of different radii r_0. Choose the following values of r_0: 0.05, 0.1, 0.5, 1, 10 m. The conductivity of the earth is 10^{-2} mho/m.

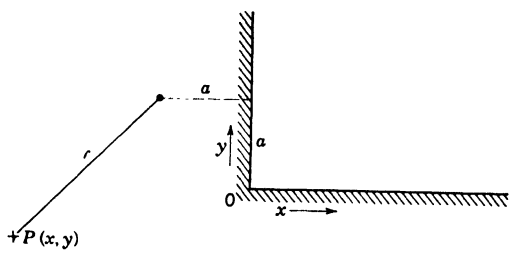
1-19. *a.* Assume that the grounding arrangement of a high-potential transmission line can be replaced by a hemisphere of 1 m radius. If the conductivity of the earth is 10^{-2} mho/m, calculate the contact resistance.

b. Suppose that, owing to a line fault, a current of 100 amp flows from the transmission line to ground. Calculate the potential difference between two points on the ground a distance apart equal to a human step, say, 0.8 m, in the most unfavorable case. HINT: Show that Eq. (1-66) also gives the potential difference between the grounding arrangement and a point on the surface of the ground, a distance r away.

1-20. *a.* Calculate and plot the grounding resistance of grounding rods which have a length of 1 m and which are driven perpendicularly into the ground of conductivity 10^{-2} mho/m. Choose the following values of L/d: 10, 20, 50, 100.

b. Discuss whether it is more desirable to use several short rods or a single, thick, long rod to achieve a low ground resistance.

1-21. A spherical electrode is embedded in a conducting region, as shown in the sketch. Find the potential at any point P in the region within the conducting medium.



CHAPTER 2

STATIC ELECTRIC FIELDS

It was noted in Sec. 1-5 that both positively and negatively charged carriers exist and that it was the steady drift of these carriers that constitutes the current. It is not intended to discuss here the constitution of matter in order to understand the source of such carriers. Rather, it is first intended, in this and in subsequent chapters, to examine certain bulk, or *macroscopic,* phenomena that arise because such charged carriers do exist. Some aspects of the *microscopic* constitution of matter will be considered later in our work.

It will be found that a considerable body of useful knowledge can be developed for a system that consists of stationary charges or stationary-charge distributions. Later considerations will be directed at charges in motion with constant velocity, and it will be shown that the work of Chap. 1 is one aspect of these considerations. The motion of the carriers will be found to introduce certain features which arise from the motion. In fact, it is important to discuss in some detail the fields which arise when the charged carriers are: (1) stationary in space, (2) moving with constant velocity, (3) moving with constant acceleration. This chapter will be confined to charged carriers which are stationary in space.

2-1. The Electric Charge. The basis for the discussion of the static electric field is to be found in Coulomb's law. In 1785, Charles Augustin de Coulomb (1736–1806) reported the results of a series of measurements on the dependence of the force between charged bodies upon the distance between them. To make these measurements, Coulomb used a torsion balance which he had invented. Using small pith balls which could be charged, he placed one at one end of the arm of the torsion balance and fixed the second relative to the balance. The angle through which it was necessary to twist the supporting fiber to which the movable arm was attached, in order to offset the electrical force, furnished a measure of the electrical force. The essential conclusions from Coulomb's experiments are contained in the expression

$$\mathbf{f} \propto \frac{qq_1}{r^2}\mathbf{r}_{01} \tag{2-1}$$

This law states that the force **f** between charged bodies located at point sources is directly proportional to the magnitude of the charges and inversely proportional to the square of the distance between them. Moreover, the force may be one of attraction, if the charges are of opposite sense (positive and negative, respectively). The force may be one of repulsion, if the charges are of the same sense (both positive, or both negative). The unit vector $\mathbf{r}_{01}$ that appears in this expression is in the direction of the line connecting the two charges q and q_1 and denotes specifically that the force acts along the line connecting the two point charges.

If the result is applied to charged bodies which are small compared with the distance between them, Coulomb's law is valid. The limitation to small bodies is, in fact, quite trivial, since any large body may be considered to be made up of a large number of small elements and these small elements may be chosen of any size desired. These small elements, or bodies that are widely separated, are called *point charges*. It should be noted that subsequent experiments have shown that Coulomb's law is valid to distances of 10^{-8} m and less, if the requirement of point charges is met.

2-2. Coulomb's Law. The relation expressed by Eq. (2-1) may be expressed as a mathematical equation by inserting a proportionality factor k. Thus,

$$\mathbf{f} = k \frac{qq_1}{r^2} \mathbf{r}_{01} \tag{2-2}$$

The unit of charge, the unit of force, the unit of distance, and the unit of k may be chosen in a variety of ways, the only requirement being that these form a set which is consistent with Eq. (2-2). Historically, a number of choices have been made, and all the older writings on the subject appear in these. Since 1935, the rationalized mks system of units has been in use by international agreement. In this system, the units are meters (length), kilograms (mass), seconds (time).

In the mks system of units, electrical quantities possess inherent dimensions. Moreover, it has already been noted that our subsequent study will be concerned with stationary charges, with charges in uniform motion, and with charges in accelerated motion. It might appear logical, in this light, that the unit of charge (the coulomb) be adopted as the basic electrical quantity, and some recent textbooks use the mksc system. In 1950, the unit of current (the ampere) was officially designated as the basic electrical quantity, primarily because current can be measured directly with great accuracy and is reproducible in standards laboratories. Since amperes represent coulombs per second, the coulomb is established once the size of the ampere has been established. Thus, the system of units adopted in this text is the mksa system.

The proportionality constant k that appears in Eq. (2-2) is not defined directly but is defined through another constant, ϵ_0, by the equation

$$k = \frac{1}{4\pi\epsilon_0}$$

The constant ϵ_0 is called the *electric permittivity* of free space and is defined to be

$$\epsilon_0 = \frac{10^7 \text{ amp}^2\text{-sec}^4}{4\pi c^2 \text{ kg-m}^3} \cong \frac{10^{-9}}{36\pi} \quad \text{farads/m} \tag{2-3}$$

where c is the velocity of light in free space and is $c \doteq 2.998 \times 10^8$ m/sec. Thus, in free space,

$$k = 9 \times 10^9 \frac{\text{kg-m}^2}{\text{amp}^2\text{-sec}^4} \tag{2-4}$$

Hence, in mksa units, Coulomb's law is written

$$\mathbf{f} = \frac{1}{4\pi\epsilon_0} \frac{qq_1}{r^2} \mathbf{r}_{01} \quad \text{newtons} \tag{2-5}$$

Figure 2-1 illustrates the notation in the vector formulation of Coulomb's law.

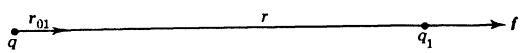


FIG. 2-1. Notation in the vector formulation of Coulomb's law.

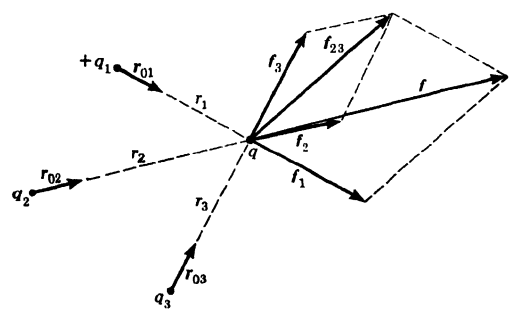


FIG. 2-2. The force on a charge due to an array of charges.

When more than two charges are present, any one of them will experience a force that is the vector sum of the individual forces due to the respective charges, as given by Coulomb's law. This assumption is called the *principle of linear superposition.* It assumes that no saturation effects occur. Suppose, for example, that it is desired to evaluate the force of a group of charges upon a single charge. Let q_1, q_2, q_3, . . . be the group of charges, and let q be the single charge. Let r_i be the distance from the typical charge q_i to the charge q. The situation is illustrated in Fig. 2-2. The force acting upon the charge q will be the vector

sum of the forces due to each of the point charges q_1, q_2, ... Graphically, the result is obtained by the successive applications of the parallelogram law. Analytically, the result is expressed by the relation

$$\mathbf{f} = \frac{1}{4\pi\epsilon_0}\left(\frac{qq_1}{r_1^2}\mathbf{r}_{01} + \frac{qq_2}{r_2^2}\mathbf{r}_{02} + \cdots + \frac{qq_i}{r_i^2}\mathbf{r}_{0i}\right) \tag{2-6}$$

which may be written in the more convenient form

$$\mathbf{f} = \frac{q}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i^2}\mathbf{r}_{0i} \tag{2-7}$$

It must be kept clearly in mind that this is a vector summation, and due account must be taken of the direction of each component of force.

2-3. The Electric Field. Equations (2-5) and (2-7) are expressions, respectively, for the force between two charges and the force on one charge due to an aggregate of fixed charges. Since the charges are not in physical contact, these results may be considered to imply the existence of *action at a distance*. A second and preferred interpretation is to suppose that each charge produces an electric field, and that the force due to one or more charges is exerted on the other charges through the electric field produced by each charge.

In the light of this discussion, a fixed electric charge will be defined as an electric field whose intensity varies inversely with the square of the distance from a point. The point, which is the center of the field, will locate the charge in space. Clearly, on this view, the effects of an electric charge fill all of space. Moreover, Coulomb's law in the simple form of Eq. (2-5) may be considered to show that the interaction of static electric fields produces forces. It may also be interpreted to show that one charge, when situated in the electric field of a second charge, will experience a force. This latter view is to be preferred, and on this view Eq. (2-7) may be interpreted to show that a charge situated in the electric field of an array of charges will experience a force on it. In fact, since the force on the charge q is a vector quantity, and since the charge q is a scalar quantity, the electric field must also be a vector quantity. Clearly, of course, the electric field at a point in space due to an array of fixed charges may be defined as the force per unit charge at the specified point, or

$$\mathbf{E} = \frac{\mathbf{f}}{q} \qquad \text{newtons/coulomb} \tag{2-8}$$

where the electric field intensity $\mathbf{E}$ is a vector, which possesses both magnitude and direction. Therefore,

$$\mathbf{f} = q\mathbf{E} \qquad \text{newtons} \tag{2-9}$$

These expressions are valid only if the presence of the charge q, which may be considered to be a test charge that is introduced in order to explore the field, does not distort or otherwise influence the field. More precisely, therefore, the electric field intensity may be defined as

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{f}}{q} \qquad \text{or} \qquad \mathbf{E} = \frac{d\mathbf{f}}{dq} \tag{2-10}$$

From Eq. (2-9) and Eq. (2-7), it follows that the field due to the specified array of charges $q_1, q_2, \ldots$ is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_i^2} \mathbf{r}_{0i} \tag{2-11}$$

Suppose that one were interested in the electric field intensity due to a charged body, the dimensions of which are not small compared with the distance to the point at which the field is desired. It was noted in Sec. 2-1 that what is required in order to find the field intensity is to consider the body to be made up of a large number of small elements and, in essence, to apply Eq. (2-11) to obtain the desired result. The situation is then as illustrated in Fig. 2-3, where a typical element of the total volume of charge is designated as $\Delta\tau_i$, and where the total charge in $\Delta\tau_i$ is written ΔQ_i. Corresponding to Eq. (2-7), the field at point P is given by the expression

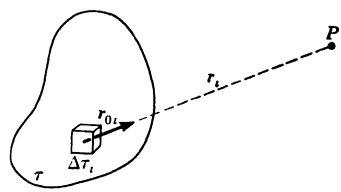


FIG. 2-3. To determine the electric field at a point due to a volume array of charge.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{\Delta\tau_i} \frac{\Delta Q_i}{r_i^2} \mathbf{r}_{0i} \tag{2-12}$$

From purely practical considerations, this expression is of no real value, since, in order to meet the conditions that the dimensions of the elements of charge be small compared with the distance to the point at which the field is desired, the number of elements $\Delta\tau$ would be very large. The number would be so large, in fact, that an impractical mathematical operation is required to carry out the summation in Eq. (2-12). In order to determine the value of $\mathbf{E}$ for a body of reasonable dimensions (these are often referred to as *ponderable bodies*) and at the same time to permit the use of the integral calculus, a quantity is defined that is designated as the *volume density of charge*. By definition,

$$\text{Volume density of charge} = \rho \equiv \lim_{\Delta\tau_i \to 0} \left(\frac{\Delta Q_i}{\Delta\tau_i} \right) = \frac{\sum_i q_i}{d\tau_i} \tag{2-13}$$

where $\sum_i q_i$ represent the sum of all the charges contained in the volume element $d\tau_i$. It is assumed here that it can be possible to subdivide a body into volume cells whose dimensions are very small compared with the dimensions of the body itself and to its distances to other bodies but which nevertheless contain a very large number of individual charges. The subdivisions are chosen small enough to make the density function a continuous function but large enough to contain a sufficient number of charges to wipe out any individual characteristics.

It follows from Eq. (2-13) that

$$\sum_i q_i = \rho \, d\tau \tag{2-14}$$

Furthermore, since the effect of the ponderable body is being considered at a point which is remote from the body, all the q's will have effectively the same r. This follows from the fact that it is possible to take the greatest linear dimensions of the volume element to be sufficiently small to ensure this result. Thus,

$$\sum_i \frac{q_i}{r_i^2} \doteq \frac{1}{r^2} \sum_i q_i = \rho \frac{d\tau}{r^2}$$

from which $$d\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_i^2} \mathbf{r}_{01} = \frac{\mathbf{r}_0}{4\pi\epsilon_0 r^2} \sum_i q_i = \frac{\mathbf{r}_0}{4\pi\epsilon_0 r^2} \rho \, d\tau$$

or, finally,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\rho}{r^2} \mathbf{r}_0 \, d\tau \tag{2-15}$$

where τ denotes the total volume containing the charge.

An entirely parallel development is possible if it is assumed that the charge resides on the surface of a conductor. In this case, a surface density of charge may be defined as

$$\text{Surface density of charge} = \sigma = \frac{\sum_k q_k}{dS} \tag{2-16}$$

from which it follows that

$$\sum_k q_k = \sigma \, dS \tag{2-17}$$

where dS is an element of surface area and $\sum_k q_k$ is the sum of all charges contained in that element. By proceeding in a manner analogous to that which leads to Eq. (2-15), the electric field intensity due to a con-

tinuous surface-charge distribution will have the form

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma}{r^2} \mathbf{r}_0 \, dS \tag{2-18}$$

where S denotes the total surface over which the charge resides.
Of course, if both volume and surface elements contribute to the total electric field, the expression for the electric field intensity has the form

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \left(\int_\tau \frac{\rho}{r^2} \mathbf{r}_0 \, d\tau + \int_S \frac{\sigma}{r^2} \mathbf{r}_0 \, dS \right) \tag{2-19}$$

Equations (2-15), (2-18), and (2-19) can be used to calculate **E** in a relatively limited number of cases. Several examples are given, and it will be found that the integrals can be evaluated in those cases where certain symmetries or relatively simple geometry exist. In general, such vector integrals cannot be evaluated, and hence these equations are not particularly more valuable, when subjected to practical needs, than is Eq. (2-12). Therefore, while these are interesting forms, they cannot be considered as a satisfactory solution to the problem of finding a general method which will permit an evaluation of **E** in a wide variety of cases.

Example 2-3.1. Find the electric field intensity due to a very long, straight, uniformly charged wire. The notation is specified in Fig. 2-4. Let dx represent an element of length of the wire and Λ be the density of charge per unit length. It is required to evaluate **E** at a point a distance d from the wire.

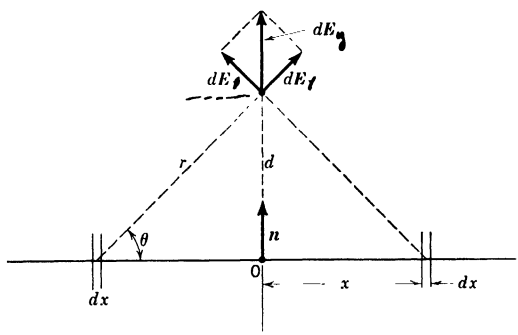


FIG. 2-4. A long, straight, uniformly charged wire.

Solution. Consider two symmetrical elements of the wire, as shown. Owing to the symmetry of the field, the horizontal components of the field intensity will cancel each other, and only the vertical components remain. Because of this, the resulting electric field due to two elements of the charged wire, dx, will be the component dE_y, where

$$dE_y = \frac{1}{4\pi\epsilon_0} \frac{\Lambda}{r^2} dx \times 2 \sin \theta$$

The total field is then given by the relation

$$E_y = \int_0^\infty \frac{1}{2\pi\epsilon_0} \frac{\Lambda\, dx}{r^2} \sin\theta \tag{2-20}$$

Note from the diagram, however, that

$$x = d\cot\theta \qquad \frac{d}{r} = \sin\theta$$

Then,

$$dx = -d\csc^2\theta\, d\theta$$

which, combined with Eq. (2-20), gives

$$E_y = -\frac{\Lambda}{2\pi\epsilon_0}\int_{\pi/2}^{0} \frac{\sin\theta}{d}\, d\theta$$

or, finally

$$E_y = \frac{\Lambda}{2\pi\epsilon_0 d} \tag{2-21}$$

It is seen that the electric field may be written as

$$\mathbf{E} = \frac{\Lambda}{2\pi\epsilon_0 d}\mathbf{n} \tag{2-22}$$

since the field is normal to the wire.

Example 2-3.2. Find the electric field intensity due to an infinite plane sheet of charge. The situation is illustrated in Fig. 2-5.

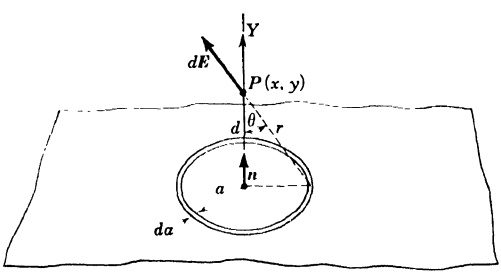


FIG. 2-5. An element of an infinite plane charged sheet.

Solution. Here, as in Example 2-3.1, use is made of the existing symmetry to reduce the vector integral to a scalar integral. Note from the figure that the net vertical contribution to the electric field from a charged surface element dS is

$$dE_y = \frac{1}{4\pi\epsilon_0}\frac{\sigma\, dS}{r^2}\cos\theta$$

The corresponding vertical contribution from the ring element is readily seen to be given by the expression

$$dE_y = \frac{1}{4\pi\epsilon_0}\sigma\frac{2\pi a\, da}{r^2}\cos\theta$$

But, from the diagram,

$$a = r\sin\theta \qquad a = d\tan\theta$$

and so

$$da = d\sec^2\theta\, d\theta$$

whence the contribution to the electric field by the charged ring becomes

$$dE_y = \frac{\sigma}{2\epsilon_0} \sin\theta \, d\theta$$

The total field has the magnitude

$$E_y = \frac{\sigma}{2\epsilon_0} \int_0^{\pi/2} \sin\theta \, d\theta = \frac{\sigma}{2\epsilon_0} \tag{2-23}$$

so that, finally, the electric field intensity is given by

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \mathbf{n} \tag{2-24}$$

which indicates that the **E** vector is directed normal to the charged plane. Attention is called to the fact that this result is independent of the location of the field point P.

Example 2-3.3. As an example of the procedure when symmetry does not exist, it is required to find the electric field intensity at the point P, as shown in Fig. 2-6, due to a uniformly charged wire of length L.

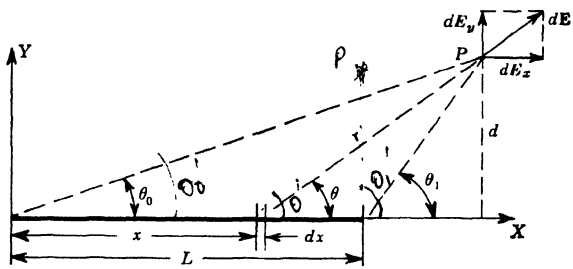


FIG. 2-6. A straight, uniformly charged wire of length L.

Solution. The procedure is somewhat like that given in the foregoing examples, except that, with no obvious symmetry, the vector integral must be evaluated. This is done by dividing the integral into two parts, namely, one which gives the component of the field in the direction of the wire, and one which gives the component of the field normal to the wire. It is thus required to evaluate the two integrals

$$\begin{aligned} dE_x &= \frac{1}{4\pi\epsilon_0} \frac{\Lambda}{r^2} dx \cos\theta \\ dE_y &= \frac{1}{4\pi\epsilon_0} \frac{\Lambda}{r^2} dx \sin\theta \end{aligned} \tag{2-25}$$

Note from the figure that

$$L - x + d \cot\theta_1 = d \cot\theta$$

Then,

$$dx = d \csc^2\theta \, d\theta$$

Therefore,

$$dE_x = \frac{1}{4\pi\epsilon_0} \frac{\Lambda d}{r^2} \cos\theta \csc^2\theta \, d\theta$$

$$dE_x = \frac{1}{4\pi\epsilon_0} \frac{\Lambda}{d} \cos\theta \, d\theta \tag{2-26}$$

and also

$$dE_y = \frac{1}{4\pi\epsilon_0} \frac{\Lambda}{d} \sin\theta \, d\theta$$

The total field components are obtained by integration of these expressions. Thus,

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{\Lambda}{d}\int_{\theta_0}^{\theta_1}\cos\theta\, d\theta$$

which integrates to

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{\Lambda}{d}(\sin\theta_1 - \sin\theta_0) \tag{2-27a}$$

In a similar way, it is found that

$$E_y = -\frac{1}{4\pi\epsilon_0}\frac{\Lambda}{d}(\cos\theta_1 - \cos\theta_0) \tag{2-27b}$$

The total field is given by the vector combination of the E_x and E_y components and is

$$E = \sqrt{E_x^2 + E_y^2} = \frac{1}{4\pi\epsilon_0}\frac{\Lambda}{d}[(\sin\theta_1 - \sin\theta_0)^2 + (\cos\theta_1 - \cos\theta_0)^2]^{1/2}$$

For the case where P is directly above the end of the wire this becomes

$$E = \frac{1}{2\pi\epsilon_0}\frac{\Lambda}{d}\cos\left(\frac{\theta_0}{2} + \frac{\pi}{4}\right) \tag{2-28a}$$

with an angle

$$\theta = \tan^{-1}\frac{\cos\theta_0}{1 - \sin\theta_0} \tag{2-28b}$$

In the limiting case when the charged wire is very long, $\theta_0 \to 0$, Eqs. (2-28) lead to the results

$$E = \frac{1}{2\sqrt{2}\,\pi\epsilon_0}\frac{\Lambda}{d}$$
$$\theta = 45° \tag{2-29}$$

2-4. Lines of Force: Gauss' Law. The foregoing section has presented a discussion of the electric field, and techniques were developed which, in principle at least, permit the evaluation of the magnitude and direction of the electric field at all points in space. Suppose, now, as an aid in visualization of the field, lines are drawn in such a manner that their direction at all points gives the direction of the electric field, and so that their density is numerically equal to the magnitude of the electric field intensity. Such lines, which may be called *lines of electric force*, are an avowedly artificial concept, but they do serve a useful purpose in mapping an electric field, and thereby providing a mental picture of the forces acting on a unit charge situated at different positions in the space of the field. Lines of electric force possess the same utility in electrostatic problems that current-flow lines do in current-flow-field problems.

Electric lines are directed in the same sense as the field. Hence, they must originate on charge centers and end on other charge centers or pass

off to infinity. Examples of such fields are given in Fig. 2-7. If the number of lines per unit area normal to the field is taken to be equal to the field intensity, it appears that a relationship must then exist between the total number of lines passing through a closed surface in space and the charges that produce the electric field. The situation here discussed is precisely analogous to that of the discussion in Sec. 1-5, of the relation between the number of current lines through a closed surface and the strength of the source of current contained in the closed surface. Thus the current density **J** of the current-flow field is somewhat analogous to

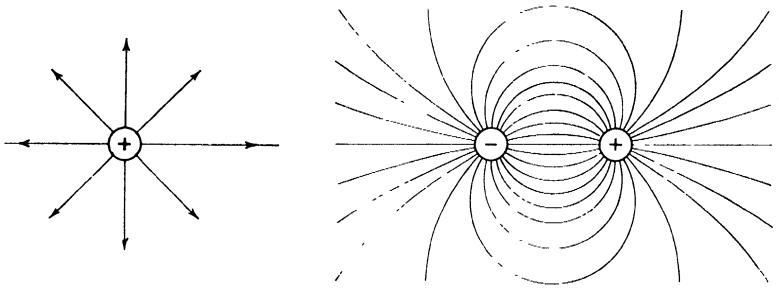


FIG. 2-7. The electric lines of several different charge distributions: (*a*) electric lines due to an isolated charge; (*b*) electric lines due to two equal and opposite charges.

the electric field intensity **E** in the static electric field. The term *flux of electric force* is used for the electric field as the analogue of the current in the current-flow field.

As was surmised above, a relationship does exist between the total number of lines passing through a closed surface in the field and the total charge contained in the surface. This relationship is called *Gauss' law* and states: "The surface integral of the normal component of the flux of electric force over any closed surface in space is directly related to the quantity of electric charge contained in the space bounded by that surface, the relation being Q/ϵ_0."

In mathematical symbols, Gauss' law is written

$$\oint \mathbf{E} \cdot \mathbf{n}\, dS = \frac{Q}{\epsilon_0} \tag{2-30}$$

This theorem may be proved at once from Coulomb's law and from the hypothesis of the validity of the principle of superposition.

PROOF. Let Q represent a point charge situated within a closed surface S. Let P be any point on the surface contained within a small surface element of area dS. Let **E** be the electric field intensity at the point P. The outward-drawn unit normal to the surface element dS is

$\mathbf{n}$; r denotes the distance of the charge Q to the element dS. The geometry is illustrated in Fig. 2-8. The flux passing through the element dS is equal to $\mathbf{E} \cdot \mathbf{n}\, dS$. The total flux passing through the closed surface S is obtained by integrating this expression. Thus,

$$\text{Total flux of electric force} = \oint \mathbf{E} \cdot \mathbf{n}\, dS$$

This expression is combined with Eq. (2-11), which gives the electric field intensity due to a point charge. Thus,

$$\text{Total flux of electric force} = \oint \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \mathbf{r}_0 \cdot \mathbf{n}\, dS$$

Note that the scalar product $\mathbf{r}_0 \cdot \mathbf{n}$ equals the cosine of the angle between $\mathbf{r}_0$ and $\mathbf{n}$. Thus $\mathbf{r}_0 \cdot \mathbf{n}\, dS$ is the projection of dS upon a plane perpendicular to r_0, which is called dA. If $d\Omega$ denotes the solid angle subtended by dS, then since

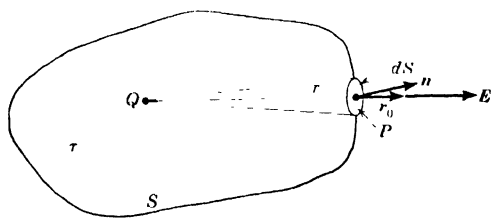


FIG. 2-8. A general surface for proving Gauss' law.

$$dA = r^2\, d\Omega = \mathbf{r} \cdot \mathbf{n}\, dS$$

it follows that

$$\oint \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \mathbf{r}_0 \cdot \mathbf{n}\, dS = \frac{Q}{4\pi\epsilon_0} \int d\Omega$$

This expression is now integrated over the closed surface. Since for the condition shown the surface encloses the charge, then $\int d\Omega = 4\pi$, and

$$\text{Total flux of electric force} = \oint \mathbf{E} \cdot \mathbf{n}\, dS = \frac{Q}{\epsilon_0}$$

If the surface does not enclose the charge, the situation becomes that shown in Fig. 2-9. In this case, $\int d\Omega = 0$, since each cone whose vertex is at the charge will intersect the closed surface twice (or some even number of times, if the surface is provided with a fluted type of contour), and the value of each pair of intersections will have opposite signs, and the elements of the solid angle will cancel each other in the integration.

If, instead of assuming a point charge to be contained within the closed surface S, it is assumed that a volume distribution of charge of density ρ is contained therein, the previous development remains unchanged

except in detail. Now the superposition of the effects of the charged elements making up the total charge, as specified by Eq. (2-14), instead of that due to a single charge, would be used. The final result now has the form

$$\oint \mathbf{E} \cdot \mathbf{n}\, dS = \frac{1}{\epsilon_0} \int_\tau \rho\, d\tau \tag{2-31}$$

A word is in order about the utility of Gauss' law. Two essential properties are worthy of note. One is the general property of the field itself, as specified by Gauss' law. The second is the fact that Gauss' law permits very simple calculations of the field from charge distributions which

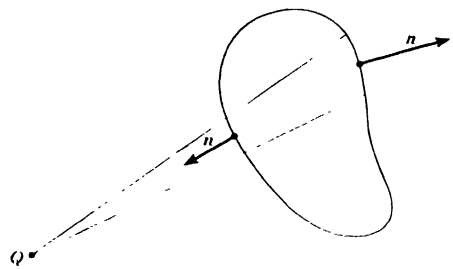


FIG. 2-9. A charge situated outside a closed surface.

may possess certain geometries or symmetry properties. Unfortunately, it is only for these that the integral in Eq. (2-30) can be evaluated. A number of examples of the application of this law will be given.

Example 2-4.1. Calculate the field due to a charged spherical shell.

Solution. A Gaussian surface is constructed concentric with the charged spherical shell. This surface is a sphere, with radius r, where $r > a$, the radius of the charged spherical shell, as shown in Fig. 2-10. From the symmetry of the charge distribution, the field itself must be spherically symmetric, and, in fact, the $\mathbf{E}$ vector will everywhere be perpendicular to the Gaussian surface and of the constant magnitude. Then since $\mathbf{E} \cdot \mathbf{n}$ is everywhere constant over the Gaussian surface, and equal to E, it can be moved from its position under the integral sign. Thus,

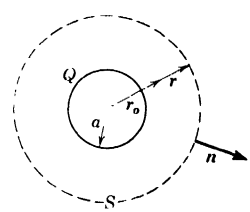


FIG. 2-10. To evaluate the field due to a charged spherical shell.

$$\oint \mathbf{E} \cdot \mathbf{n}\, dS = E \int_S dS = \frac{Q}{\epsilon_0}$$

But since the area of the Gaussian surface is $4\pi r^2$, then

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{r}_0 \tag{2-32}$$

Attention is called to the fact that the field intensity at points outside an isolated charged spherical conductor is the same as that from an equal total charge concentrated in an isolated point. In both instances, the electric field lines are radial.

Note that, if the Gaussian surface had a radius $r < a$, the charge enclosed within the Gaussian surface would be zero. Thus, the electric field intensity inside a charged spherical conductor is zero.

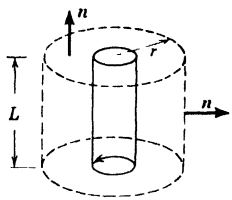


FIG. 2-11. A Gaussian surface around a charged wire.

Example 2-4.2. Find the electric field intensity due to a very long, straight, uniformly charged wire. This is exactly the problem considered in Example 2-3.1, but its solution will be effected by an application of Gauss' law.

Solution. It will be assumed that the charge is uniform along the length of the wire, with a charge per unit length of Λ. The Gaussian surface is a cylinder of radius r and of length L and coaxial with the wire. Moreover, it will be assumed that the length of the wire is sufficient so that, over the Gaussian surface, the electric field lines are radial and independent of the azimuth. Thus, over the cylindrical portion of the Gaussian surface, $\mathbf{E} \cdot \mathbf{n} = E$, a constant. Also, over the plane ends of the Gaussian surface, $\mathbf{E} \cdot \mathbf{n} = 0$. Thus, by Gauss' law,

$$\oint \mathbf{E} \cdot \mathbf{n}\, dS = \int_{\substack{\text{Curved} \\ \text{portion}}} \mathbf{E} \cdot \mathbf{n}\, dS + \int_{\substack{\text{Plane} \\ \text{ends}}} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{Q}{\epsilon_0}$$

$$= E \int_{\substack{\text{Curved} \\ \text{portion}}} dS + 0 = \frac{Q}{\epsilon_0}$$

$$= E \times 2\pi r \times L = \frac{\Lambda L}{\epsilon_0}$$

Therefore, E is radial, of magnitude

$$E = \frac{\Lambda}{2\pi\epsilon_0 r} \tag{2-33}$$

This expression agrees with Eq. (2-22), which was obtained by integration of the vector integral from direct considerations of Coulomb's law.

2-5. The Electric Potential. It will be recalled that the discussion of the electric field intensity in Chap. 1 proceeded as a definition from the gradient of the potential, following considerations of the potential field in current-flow problems. The difference of potential was related to the line integral of the electric field intensity as a mathematical result. Now, having developed considerations of isolated charges in the region of electric fields, it is possible to relate the electric field, the electric potential, and physical work.

Initially, consider an array of fixed charges. It is desired to evaluate the difference of potential, given by the expression

$$\phi_b - \phi_a = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \tag{2-34}$$

between any two points a and b in the field. For the case of one charge of the array, as shown in Fig. 2-12, this becomes

$$\phi_b - \phi_a = -\frac{q}{4\pi\epsilon_0}\int_a^b \frac{\mathbf{r}_0}{r^2} \cdot d\mathbf{l}$$

Observe from the diagram that $\mathbf{r}_0 \cdot d\mathbf{l} = dr$. Then

$$\begin{aligned}\phi_b - \phi_a &= -\frac{q}{4\pi\epsilon_0}\int_a^b \frac{dr}{r^2} \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)\end{aligned} \tag{2-35}$$

For the entire array of charges, by superposition

$$\phi_b - \phi_a = \frac{1}{4\pi\epsilon_0}\sum_i q_i \left(\frac{1}{r_b} - \frac{1}{r_a}\right)_i \tag{2-36}$$

This expression shows that the difference of potential between any two points in a static electric field due to an array of fixed charges is independent of the path between these two points, and depends only on the distances to the end points.

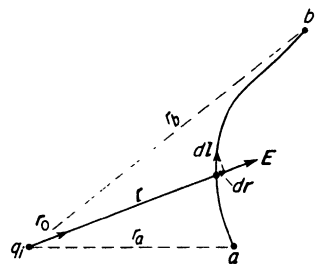


FIG. 2-12. For determining the difference of potential in moving a test charge in an electric field.

Now, since an electric charge experiences a force on it when in an electric field, work must be done in order to move a charge in the field. This work may be positive or negative, depending on the sign of the charge (positive or negative) and the direction in which the motion takes place. Suppose that $\mathbf{f}^*$ denotes the force on a test charge q that is to be moved between points a and b in the field. The force $\mathbf{f}^*$ by the nonelectrical mover is equal and opposite to the reaction of the electric force. Thus

$$\mathbf{f}^* = -\mathbf{f} = -q\mathbf{E}$$

and the work done by $\mathbf{f}^*$ against the electrical force in traversing the path from a to b is given by the expression

$$W = \int_a^b \mathbf{f}^* \cdot d\mathbf{l}$$

which is

$$W = -q\int_a^b \mathbf{E} \cdot d\mathbf{l} \tag{2-37}$$

But by Eq. (2-34) this may be written as

$$W = q(\phi_b - \phi_a)$$

from which

$$(\phi_b - \phi_a) = \frac{W}{q} \qquad \text{joules/coulomb} \tag{2-38}$$

This equation shows that the difference of potential between two points in a static electric field is equal to the work-per-unit charge necessary to carry a test charge from one point to the other.

A discussion was included in Sec. 1-2 to emphasize the significance of the terms "absolute potential" and "potential difference." It was noted that only differences of potential can be observed and measured and that the reference datum for potential can be chosen at will. In electrostatic problems it is customary to choose as the zero reference level the potential at a point infinitely distant from the charges producing the field, where the field is itself zero. On the basis of this reference datum, the *potential at a point is the work that must be done in carrying a unit test charge from infinity to the point in question.* As a note of caution, this choice of reference datum sometimes causes complications, and care must therefore be exercised in considerations of potential.

Example 2-5.1. Find an expression for the potential due to a point charge.

Solution. It has already been shown that the electric field from a point charge is radial and is given by Eq. (2-32). Hence, work will be done by the force only in the radial displacement of the charge q. For this case, Eq. (2-37) becomes

$$\phi(r) = -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{r} = -\int_{\infty}^{r} \mathbf{E} \cdot \mathbf{r}_0 \, dr$$

This is combined with Eq. (2-32) to give

$$\phi(r) = -\frac{1}{4\pi\epsilon_0} Q \int_{\infty}^{r} \frac{dr}{r^2}$$

This integrates to

$$\phi(r) = -\frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_{\infty}^{r}$$

or, finally,

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \tag{2-39}$$

This is the absolute potential at a distance r from a point charge.

Example 2-5.2. Find an expression for the potential due to an array of point charges.

Solution. This is almost a trivial extension of the results of Example 2-5.1, in which Eq. (2-11) for the field due to the specified array of charges is combined with Eq. (2-37). Thus,

$$\phi(r) = -\frac{1}{4\pi\epsilon_0} \sum_i \int_{\infty}^{r} \frac{q_i}{r_i^2} \, dr_i$$

The result is

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_i} \tag{2-40}$$

This shows that the potential at any point due to an array of charges is the arithmetic sum of the potentials due to each charge at the point in question.

Example 2-5.3. Find an expression for the potential due to a continuous distribution of charge.

Solution. The result in this case follows from considerations such as those in Sec. 2-3 for the electric field due to a continuous charge distribution. The result is an extension of Eq. (2-40) and is

$$\phi = \frac{1}{4\pi\epsilon_0} \left[\int_\tau \frac{\rho}{r} d\tau + \int_S \frac{\sigma}{r} dS \right] \tag{2-41}$$

These potential integrals are sometimes referred to as *Poisson integrals*.

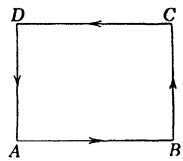


FIG. 2-13. To evaluate the line integral around the path shown.

Example 2-5.4. Evaluate the work done in progressing around the closed path shown in Fig. 2-13 in the field of a specified static charge distribution.

Solution. In order to evaluate the work done in carrying a test charge around the designated path, it is necessary only to apply the basic definition given by Eq. (2-34). Thus, if the potentials at the points A, B, C, D due to the specified charge distribution have the values ϕ_a, ϕ_b, ϕ_c, ϕ_d, respectively, then we may write directly

$$\begin{aligned} W_{AB} &= q(\phi_b - \phi_a) \\ W_{BC} &= q(\phi_c - \phi_b) \\ W_{CD} &= q(\phi_d - \phi_c) \\ W_{DA} &= q(\phi_a - \phi_d) \end{aligned}$$

Therefore, the work done around the closed path, which is the sum of these expressions, is readily seen to be zero. Thus, from this, Eq. (2-38) may be written as

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_B^A \mathbf{E} \cdot d\mathbf{l} + \int_C^B \mathbf{E} \cdot d\mathbf{l} + \int_D^C \mathbf{E} \cdot d\mathbf{l} + \int_A^D \mathbf{E} \cdot d\mathbf{l} = 0 \tag{2-42}$$

This is precisely the situation discussed in Sec. 1-4, and this indicates that the integration around any closed path in the static electric field is zero. It was noted that any field for which the line integral around every closed path is zero is called a conservative field. Hence, the static electric field is conservative.

Attention is called specifically to the fact that the conservative nature of the electric field is implicit in Eq. (2-38). That is, it is only because the line integral is zero in a conservative field that it is permissible to relate the work done in carrying a test charge in the field to the difference of potential.

Example 2-5.5. Find an expression for the potential at all points in the region external to a uniformly charged, long, straight wire.

Solution. According to Eq. (2-33), the electric field is $E = \Lambda/2\pi\epsilon_0 r$, and the potential at any point is given by the expression

$$\phi(r) = \frac{\Lambda}{2\pi\epsilon_0} \int_\infty^r \frac{dr}{r} = \frac{\Lambda}{2\pi\epsilon_0} \ln \frac{r}{\infty} \longrightarrow -\infty \tag{2-43}$$

This is one example of the complications that may sometimes arise owing to the choice of reference datum as being zero at infinity. Physically, this is not a serious complication, as only potential differences can be measured. For example, if the potential is chosen to be zero at some finite value of r, as for a concentric cylindrical cable of inner radius a and outer radius b as shown in Fig. 2-14, and if we set $\phi = \phi_a$ when $r = a$ and $\phi = 0$ when $r = b$, then

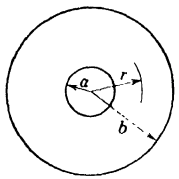


FIG. 2-14. The coaxial cable.

$$\phi = \frac{\Lambda}{2\pi\epsilon_0} \ln \frac{b}{r} \tag{2-44}$$

Note, however, that this expression is valid when $r = a$; then,

$$\phi_a = \frac{\Lambda}{2\pi\epsilon_0} \ln \frac{b}{a}$$

Combine this with Eq. (2-44) to find

$$\phi = \frac{\phi_a}{\ln (b/a)} \ln \frac{b}{r} \tag{2-45}$$

As $b > a$ and the logarithm is never negative, this is the unique solution.

Example 2-5.6. Consider the initially uncharged three-conductor system illustrated in Fig. 2-15, in air, with potentials between wires, as noted. It is required to find the charge density on each of the wires.

Solution. The electric potential between conductors is the resultant of three component terms of the form given in Eq. (2-44). Thus,

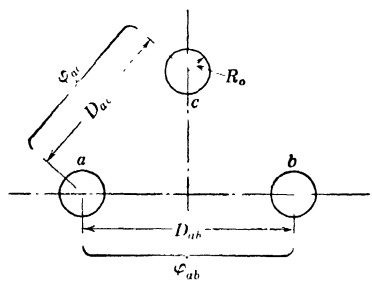


FIG. 2-15. A three-conductor system.

$$\phi_{ba} = \frac{\Lambda_a}{2\pi\epsilon_0} \ln \frac{R_0}{D_{ab}} + \frac{\Lambda_b}{2\pi\epsilon_0} \ln \frac{D_{ba}}{R_0} + \frac{\Lambda_c}{2\pi\epsilon_0} \ln \frac{D_{ca}}{D_{cb}}$$

and

$$\phi_{ca} = \frac{\Lambda_a}{2\pi\epsilon_0} \ln \frac{R_0}{D_{ac}} + \frac{\Lambda_b}{2\pi\epsilon_0} \ln \frac{D_{ba}}{D_{bc}} + \frac{\Lambda_c}{2\pi\epsilon_0} \ln \frac{D_{ca}}{R_0}$$

An expression may be written for the potential between the third pair of wires, but this is not independent since $\phi_{bc} = \phi_{ba} + \phi_{ac}$. A third expression results from the fact that the system possesses no net charge, or

$$\Lambda_a + \Lambda_b + \Lambda_c = 0$$

The three equations are sufficient to solve for the Λ's. These are written in the form

$$\begin{aligned} \Lambda_a \ln \frac{R_0}{D_{ab}} + \Lambda_b \ln \frac{D_{ba}}{R_0} + \Lambda_c \ln \frac{D_{ca}}{D_{cb}} &= 2\pi\epsilon_0\phi_{ba} \\ \Lambda_a \ln \frac{R_0}{D_{ac}} + \Lambda_b \ln \frac{D_{ba}}{D_{bc}} + \Lambda_c \ln \frac{D_{ca}}{R_0} &= 2\pi\epsilon_0\phi_{ca} \\ \Lambda_a + \Lambda_b + \Lambda_c &= 0 \end{aligned} \tag{2-46}$$

The solution of these equations may be carried out most easily with the aid of determinants for each of the Δ's.

2-6. Relationship between Electric Field Intensity and Potential Gradient. Section 2-5 developed an integral relationship between the electric field intensity and the potential. It is now desired to deduce a differential relationship between these two quantities which will permit an expression for the electric field intensity when the formula is known for the potential.

Refer to Fig. 1-11 which is equally applicable in the present case. It follows from this that the element of potential difference is

$$d\phi = -\mathbf{E} \cdot d\mathbf{l} = -E \cos \theta \, dl = -E_l \, dl \tag{2-47}$$

from which may be written

$$E_l = -\frac{d\phi}{dl} \tag{2-48}$$

This expression states that the component of the electric field intensity in any direction l is equal to the negative space rate of change of potential in that direction. The quantity $d\phi/dl$, as discussed in Sec. 1-4, denotes mathematically the directional derivative of the function ϕ in the direction of l.

Suppose that in a given case a cartesian frame of reference is used, whence ϕ would be expressed as a function of x, y, z by the functional relationship

$$\phi = \phi(x,y,z) \tag{2-49}$$

The derivative is given by the expresssion

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \tag{2-50}$$

This expression suggests the scalar product of two vectors. One of these vectors would have as its X, Y, Z components the terms $\partial\phi/\partial x$, $\partial\phi/\partial y$, $\partial\phi/\partial z$. The other vector would have dx, dy, dz as its X, Y, Z components. In order to associate the components with their proper directions along the coordinate axes, the symbols $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ will be used to denote unit vectors in the directions of X, Y, Z, respectively. The first vector is the cartesian expression of what has already been defined as $\nabla\phi$ in Eq. (1-14).

$$\nabla\phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} \tag{2-51}$$

The second vector would be written as

$$d\mathbf{l} = dx \, \mathbf{i} + dy \, \mathbf{j} + dz \, \mathbf{k} \tag{2-52}$$

whence it follows that

$$\nabla\phi \cdot d\mathbf{l} = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz \tag{2-53}$$

Comparing this expression with Eq. (2-50), it is seen that

$$d\phi = \nabla\phi \cdot d\mathbf{l} \tag{2-54}$$

As previously discussed in Chap. 1, the locus of all points at which the potential has the same value forms an equipotential surface. This requires, from Eq. (2-54), that on an equipotential

$$d\phi = 0 = \nabla\phi \cdot d\mathbf{l}$$

Since neither $\nabla\phi$ nor $d\mathbf{l}$ is zero, in general, $\nabla\phi$ must be normal to $d\mathbf{l}$. This means, of course, that the electric field is always normal to an

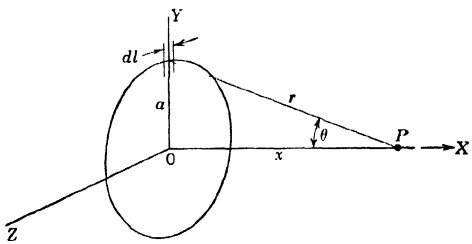


FIG. 2-16 A uniformly charged circular ring.

equipotential surface, a result that has already been discussed at some length in Sec. 1-4. This result is really nothing more than that expressed by Eq. (2-51), but presented in different form.

If $\mathbf{E}$ is written in terms of its cartesian coordinates, then

$$\begin{aligned}\mathbf{E} = \mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z &= -(\mathbf{i}\,\mathbf{i}\cdot\nabla\phi + \mathbf{j}\,\mathbf{j}\cdot\nabla\phi + \mathbf{k}\,\mathbf{k}\cdot\nabla\phi)\\ &= -\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)\end{aligned} \tag{2-55}$$

from which it follows that

$$E_x = -\frac{\partial\phi}{\partial x} \qquad E_y = -\frac{\partial\phi}{\partial y} \qquad E_z = -\frac{\partial\phi}{\partial z} \tag{2-56}$$

Thus the X, Y, Z components of $\mathbf{E}$ may be computed from Eq. (2-56) if ϕ is known as a function of x, y, z.

Example 2-6.1. Find the potential function and the electric field intensity on the axis of a uniformly charged circular ring.[1] The system is shown in Fig. 2-16.

Solution. From the potential integral given by Eq. (2-41) [and this is seen to be just a direct application of Eq. (2-39)],

[1] The more general problem of finding ϕ at any point in space is very difficult.

$$\phi = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi a} \Lambda \frac{dl}{r} = \frac{\Lambda}{4\pi\epsilon_0}\frac{1}{\sqrt{a^2 + x^2}}\int_0^{2\pi a} dl$$

which is

$$\phi = \frac{\Lambda \times 2\pi a}{4\pi\epsilon_0\sqrt{a^2 + x^2}} = \frac{Q}{4\pi\epsilon_0\sqrt{a^2 + x^2}} \tag{2-57}$$

since $2\pi a\Lambda = Q$, the total charge on the ring.

To evaluate the electric field intensity at the point P, use will be made of Eq. (2-55). Since, in this example, y and z do not appear in the expression for ϕ, then

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z} = 0$$

and so

$$\mathbf{E} = -\mathbf{i}\frac{\partial\phi}{\partial x} = -\mathbf{i}\frac{Q}{4\pi\epsilon_0}\frac{\partial}{\partial x}\frac{1}{\sqrt{a^2 + x^2}}$$

$$\mathbf{E} = \mathbf{i}\frac{Q}{4\pi\epsilon_0}\frac{x}{(a^2 + x^2)^{3/2}} = \mathbf{i}\frac{Q}{4\pi\epsilon_0}\frac{\cos\theta}{r^2} \tag{2-58}$$

This equation shows that $\mathbf{E}$ is directed along the positive X axis and has a magnitude $(Q/4\pi\epsilon_0)(\cos\theta/r^2)$.

Example 2-6.2. A small, isolated conducting sphere of radius a is charged with $+Q$ coulombs. Surrounding this sphere and concentric with it is a conducting spherical shell, which possesses no net charge. The inner radius of the shell is b, and the outer radius is c. All nonconducting space is air. It is required to find the absolute potential of each sphere.

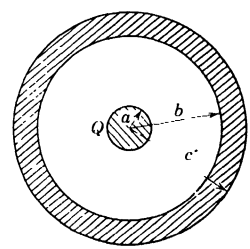


FIG. 2-17. The system under survey.

Solution. Refer to Fig. 2-17, which shows the geometry of the system. To solve this problem, several fundamental concepts already discussed will be employed First, the electric field distribution in space due to the specified system will be found. This is readily accomplished by drawing Gaussian surfaces, in this case spherical shells with their centers at the origin of the figure, but with different radii. Owing to the existing symmetry, the resulting E is radial, and depends only on r:

a. From 0 to a, E is zero, as the inner sphere is a conductor and E is zero within a conductor.

b. From a to b, the charge enclosed is $+Q$, whence E is that due to a spherical charge distribution. The result is given by Eq. (2-32).

c. From b to c, the region is again within a conductor, and E is zero within this region.

d. From c outward, the Gaussian surface contains $+Q$, and Eq. (2-32) is again valid for this region.

A plot showing this E variation is given in Fig. 2-18. Since the absolute potential is that measured with respect to the datum (zero potential at distance infinity) use is made of Eq. (2-38), which is

$$\phi = -\int_\infty^r E_r\, dr$$

The situation will be examined in the various regions specified by the problem:

a. For the region from infinity to c, the potential is

$$\phi = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,dr = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

b. In the region from c to b, since this is a conductor, the potential is a constant and will be equal to that at c.

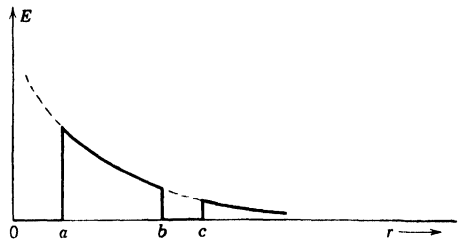


FIG. 2-18. The radial variation of E for the charged system of Fig. 2-17.

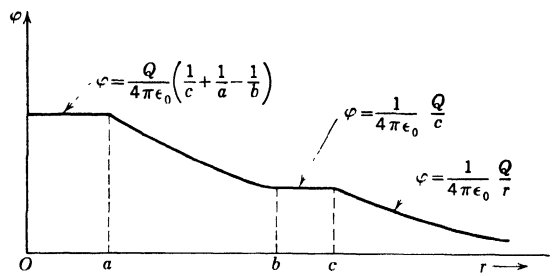


FIG. 2-19. The potential variation for the charged system of Fig. 2-17.

c. The potential at a is that at b, plus the difference of potential between b and a due to the electric field in this region. The potential difference is

$$\phi_{ab} = -\int_{b}^{a} \left(\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\right) dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

so that the total potential at a is

$$\phi_a = \phi_b + \phi_{ab} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{c} + \frac{1}{a} - \frac{1}{b}\right)$$

d. The region from a to 0 is again equipotential and has the value of the potential at a.

A plot showing the potential distribution is given in Fig. 2-19.

2-7. The Divergence of the Field; the Divergence Theorem. It was noted in Sec. 2-4 that Gauss' law, given by Eq. (2-31), relates the flux of the field with the total charge contained within the Gaussian surface. It was also noted that it is ordinarily difficult to carry out the indicated

integrations except in systems with a relatively simple geometry. A question of some interest is whether it is possible to relate the electric field intensity to the density of charge without performing the integrations of Eq. (2-31). Such a relationship is possible.

Consider a very small element of volume $\Delta\tau$. Let it be assumed that the density of charge ρ is a sufficiently slowly varying function so that over the volume $\Delta\tau$ the charge density is substantially constant and of magnitude $\bar{\rho}$. Gauss' law, with the Gaussian surface being that which defines $\Delta\tau$, is then

$$\oint_{\Delta\tau} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{\bar{\rho}}{\epsilon_0} \Delta\tau$$

In the limit as $\Delta\tau \to 0$, the density function $\bar{\rho}$ becomes the actual density ρ, and it follows that

$$\lim_{\Delta\tau \to 0} \frac{\oint_{\Delta\tau} \mathbf{E} \cdot \mathbf{n}\, dS}{\Delta\tau} = \frac{\rho}{\epsilon_0} \tag{2-59}$$

Physically, the expression on the left specifies that the gain in the flux of electric force per unit volume as $\Delta\tau \to 0$ is just equal to (charge density)/ϵ_0.

Attention is now called to the *divergence of a vector*. By definition, the divergence of a vector **A** is its integral over the surface of an infinitesimal volume, divided by that volume as the volume shrinks to zero. That is, by definition

$$\lim_{\Delta\tau \to 0} \frac{\oint_{\Delta\tau} \mathbf{A} \cdot \mathbf{n}\, dS}{\Delta\tau} \equiv \text{div } \mathbf{A} \tag{2-60}$$

The surface integral gives the flux of the vector **A** through the surface. If the divergence of **A** exists at a point in space, then lines of **A** "diverge," or emanate, from that point.

By comparing Eq. (2-59) for the electric field with Eq. (2-60), the mathematical definition of the divergence of a vector point function, it is clear that the electric field may be written in the form

$$\text{div } \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2-61}$$

Thus, lines of the electric field emanate from any point in space at which a charge density exists.

There is considerable advantage in discussing Eqs. (2-31) and (2-61) together. These are rewritten for convenience,

$$\begin{aligned} \oint \mathbf{E} \cdot \mathbf{n}\, dS &= \frac{1}{\epsilon_0} \int_\tau \rho\, d\tau = \frac{Q}{\epsilon_0} \\ \text{div } \mathbf{E} &= \frac{\rho}{\epsilon_0} \end{aligned} \tag{2-62}$$

Gauss' law is an integral statement and gives information concerning the average property of the field taken over a finite region of space. In some cases, where certain symmetries exist, the integral statement may be used to obtain point expressions for the fields from distributed sources. In this case, what has been achieved is to replace the integration over the sources by an integration over the field. The second form expresses the derivative of the field at each point in space. Both these expressions are substantially different statements of the same law, one being an integral statement, and the second being a differential (or point) statement. It will be found that a number of such corresponding integral and differential pairs occur in our study.

A very important result, known as the *divergence theorem*, follows directly by combining both expressions of Eq. (2-62). This is the relationship

$$\oint_\tau \mathbf{E} \cdot \mathbf{n}\, dS = \int_\tau \operatorname{div} \mathbf{E}\, d\tau \tag{2-63}$$

This is a transformation integral which permits the integral of the normal components of a vector over a surface to be expressed in terms of the volume integral of the derivatives of the components of the vector, where the volume is that specified by the given surface.

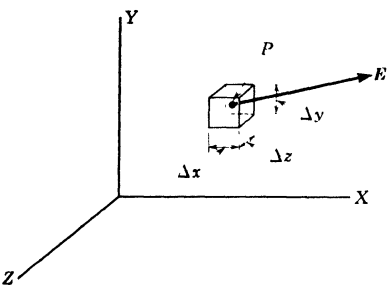


FIG. 2-20. For determining the divergence of a vector in rectangular coordinates.

Example 2-7.1. Obtain an equation for the divergence of the vector **E** in rectangular coordinates from an evaluation of Eq. (2-60).

Solution. Refer to Fig. 2-20, which shows an infinitesimal parallelpiped. The **E** vector is known at every point in the region. At the point P, the mid-point of the cube, the vector has the components E_x, E_y, E_z, with

$$\mathbf{E} = \mathbf{i}E_x(xyz) + \mathbf{j}E_y(xyz) + \mathbf{k}E_z(xyz) \tag{2-64}$$

It is desired to evaluate the surface integral $\oint \mathbf{E} \cdot \mathbf{n}\, dS$ over all faces of the cube.

Consider first the surface integral over the right face. It has the value

$$\int_{\substack{\text{Right}\\\text{face}}} \mathbf{E} \cdot \mathbf{n}\, dS = \int_{\substack{\text{Right}\\\text{face}}} E_x\, dS = E_x\left(x + \frac{\Delta x}{2}, y, z\right) \Delta y\, \Delta z$$

where $E_x(x,y,z)$ is the value at the origin at the center of the figure. The value at the right face is that given by the leading term in the Taylor expansion. For the left face, the outward-drawn normal is opposite to that for the right face, and so a negative sign appears. Thus,

$$\int_{\text{Left face}} \mathbf{E} \cdot \mathbf{n}\, dS = -E_x\left(x - \frac{\Delta x}{2}, y, z\right) \Delta y\, \Delta z$$

The sum of the outward fluxes is then

$$\int_{\substack{\text{Left and}\\ \text{right}\\ \text{faces}}} \mathbf{E} \cdot \mathbf{n}\, dS = E_x\left(x + \frac{\Delta x}{2}, y, z\right) - E_x\left(x - \frac{\Delta x}{2}, y, z\right)$$

$$= \left[\left(E_x + \frac{\partial E}{\partial x}\frac{\Delta x}{2}\right) - \left(E_x - \frac{\partial E}{\partial x}\frac{\Delta x}{2}\right)\right] \Delta y\, \Delta z$$

$$\int_{\substack{\text{Left and right}\\ \text{faces}}} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{\partial E_x}{\partial x} \Delta x\, \Delta y\, \Delta z$$

Similar results are obtained for the other pairs of faces. Thus, the complete integral is the sum of the outward fluxes for the three pairs of faces, namely,

$$\oint_{\text{Cube } \Delta\tau} \mathbf{E} \cdot \mathbf{n}\, dS = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \Delta x\, \Delta y\, \Delta z$$

By Eq. (2-60), therefore,

$$\lim_{\Delta\tau \to 0} \frac{\oint_{\Delta\tau} \mathbf{E} \cdot \mathbf{n}\, dS}{\Delta\tau} = \operatorname{div} \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{2-65}$$

Observe that in rectangular coordinates the divergence is obtained by the vector operation

$$\operatorname{div} \mathbf{E} = \nabla \cdot \mathbf{E} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z) \tag{2-66}$$

By following a similar procedure to that which leads to Eq. (2-65), it will be found that the divergence in cylindrical coordinates is

$$\operatorname{div} \mathbf{E} = \frac{1}{r}\frac{\partial (rE_r)}{\partial r} + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z} \tag{2-67a}$$

and in spherical coordinates is

$$\operatorname{div} \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r \sin \theta}\frac{\partial}{\partial \theta}(E_\theta \sin \theta) + \frac{1}{r \sin \theta}\frac{\partial E_\psi}{\partial \psi} \tag{2-67b}$$

2-8. The Poisson and Laplace Equations. An expression of fundamental importance is obtained by combining Eq. (2-62) with Eq. (2-55). The result is the following:

$$\operatorname{div} \operatorname{grad} \phi = -\frac{\rho}{\epsilon_0} \tag{2-68}$$

which in symbolic notation is

$$\nabla \cdot \nabla\phi = \nabla^2\phi = -\frac{\rho}{\epsilon_0} \tag{2-69}$$

This latter form is known as *Poisson's equation.* If there is no charge present, then $\rho = 0$ and this equation reduces to

$$\nabla^2\phi = 0 \tag{2-70}$$

This is Laplace's equation. The operator ∇^2 is called the *Laplacian operator.*

The gradient in rectangular coordinates is given by Eq. (2-51), and the divergence operation is specified in Eq. (2-66). Thus in rectangular coordinates the Laplacian becomes

$$\nabla^2\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)$$

which is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{2-71}$$

Let us confine our attention initially to Laplace's equation in rectangular coordinates, which is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{2-72}$$

This equation is a partial differential equation in ϕ as a function of x, y, and z. Since this equation is of the second order, its solution will involve in general two arbitrary *functions* of the coordinates x, y, z. The unique form of these functions is determined by the conditions on the boundary of the region. That is, in any problem involving Laplace's equation, some kind of conditions must be specified over the entire surface bounding the region within which Laplace's equation is valid. If the potential is specified at each point on the boundary, the problem is called a *Dirichlet problem.* If the normal derivative of the potential $\partial\phi/\partial n$ is specified at each point on the boundary, the problem is called a *Neumann problem.* Cases may exist in which the potential is specified over a part of the boundary and in which the derivative is specified over the remainder of the boundary, giving a *mixed* D-N problem. More precisely, therefore,[1] if ϕ satisfies Laplace's equation throughout a region R, and if ϕ also satisfies the Dirichlet conditions or the mixed D-N conditions over the complete boundary, then ϕ is the unique solution of the problem. Similarly, if the Neumann conditions are satisfied over the complete boundary, the solution is unique within an additive constant.

[1] For a proof, see O. D. Kellogg, "Foundations of Potential Theory," p. 211, Springer-Verlag OHG, Berlin, 1929.

This means that no matter how a solution is obtained—by intuition, by mathematical reasoning, or by guess—if it satisfies Laplace's equation and fits the boundary conditions over the complete enclosing surface, it is the unique solution. (This point was made in Sec. 1-13.)

Often the boundaries which are considered in engineering practice are circular-cylindrical and spherical ones. The solution of Laplace's equation is considerably simplified if the equation is expressed in terms of the circular-cylindrical or spherical coordinates instead of the cartesian coordinates. From the known form of the divergence and gradient in *circular-cylindrical coordinates*, it may be shown that

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\psi^2} + \frac{\partial^2\phi}{\partial z^2} \tag{2-73}$$

In *spherical coordinates*,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2} \tag{2-74}$$

Before proceeding with specific examples of the solution of Laplace's equation, a word is in order about the type of problems that normally are found in engineering practice, particularly since all our past work in electrostatics has been related to volume, surface, and line distributions of charge. In practice, volume distributions of charge are found in electron tubes, and these are ordinarily dynamic, not static, problems. Volume distributions of charge are also found within matter, and the study of solid-state electronics requires a discussion of such volume distributions of charge.

There are two general types of electrostatic problems.[1] In one class of problem, one or more conductors are embedded in dielectrics. Known charges are placed on the conductors, and it is required to find the distribution of the charge on each, the potentials of the conductors, and the potential at each point in the dielectric. In a variation of this problem, the potentials of the conductors may be specified, with the requirement for finding the charge and its distribution on each and the potential at each point in the dielectric. The second class of problem involves a specified initial field distribution due to external sources (charges). Conducting bodies, dielectric bodies, or both, are introduced into this field. It is usually required to find the charges and potentials of the conductors, the polarization of the dielectrics (polarization will be considered at a later point in the text), and the potential distribution in the dielectric. The solution of all problems of these types proceeds from a

[1] J. A. Stratton, "Electromagnetic Theory," sec. 3.19, McGraw-Hill Book Company, Inc., New York, 1941.

solution of Laplace's equation, subject to the appropriate boundary conditions.

A variety of functions will satisfy Laplace's equation, and it is therefore necessary to select from among all the possible functions which will satisfy the differential equation those which will satisfy the boundary conditions of the problem. It is not intended to examine these general techniques, as such a study is outside the scope of this work.[1] Several simple examples will be discussed to show the manner of finding the solutions. The formal study of the characteristics of these techniques is often referred to as boundary-value problems. Such possible solutions are frequently expressible as Fourier series, Bessel functions, zonal harmonics (Legendre polynomials), etc.

A final word is in order about the progress in our development. The fundamental objective is to find the electric field intensity in the general case. This will become more apparent when the Maxwell field equations have been discussed. It was noted in the discussion of Eq. (2-19) that the integral expression for **E** was of limited usefulness, owing to the few applications for which the solution could be effected. For this reason, attention was turned to the concept of potential in order to examine whether this might be a more general method and one which might permit an indirect, though somewhat more general, method for finding **E** in the general case. It was found that Eq. (2-41) could be developed, but unfortunately the integrals involved in this case are almost as hard to evaluate as the integrals involved in Eq. (2-19). In this section, it has been shown that the general problem has been reduced to that of integrating the second-order partial differential equations (Laplace or Poisson equations). Unfortunately, this is not always easily accomplished, but solutions are possible in a great many cases in complete or approximate form. Consequently, this may be considered as the most general solution to the problem of calculating the electric field intensity at all points in space, due to a specified charge distribution or due to prescribed boundary conditions.

Example 2-8.1. Determine the field between two large parallel plates which are closely spaced.

Solution. In any real capacitor of this type, there is a complicated fringing effect at the edges of the plates. When the plates are large compared with their separation, the fringing effect extends inward only a short distance from the edges (this may be shown to be true from an experimental flux plot, using the techniques of Chap. 1). For many purposes, it is permissible to ignore fringing, and this will be done in the present analysis.

[1] See, for example, Harry W. Reddick and F. H. Miller, "Advanced Mathematics for Engineers," 3d ed., John Wiley & Sons, Inc., New York, 1955; Ivan S. and E. S. Sokolnikoff, "Higher Mathematics for Engineers and Physicists," 2d ed., McGraw-Hill Book Company, Inc., New York, 1941.

The medium between the plates is without charge, and so Laplace's equation applies. Refer to Fig. 2-21. For the geometry involved, Laplace's equation in cartesian form is used. Thus,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

From physical considerations, it is evident that there is no variation of potential in the Y and Z directions. Thus, Laplace's equation reduces, in the present case, to

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} = 0 \qquad (2\text{-}75)$$

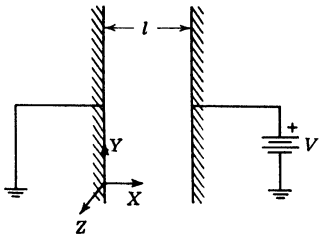


FIG. 2-21. The parallel-plate capacitor.

The general solution of this ordinary differential equation is given by

$$\phi = A + Bx \qquad (2\text{-}76)$$

That this is a solution can be verified by substituting Eq. (2-76) into Eq. (2-75). Note that since Eq. (2-75) is a second-order equation, its general solution must contain two arbitrary constants (A and B). These constants are now evaluated by means of the boundary conditions, which are seen to be, when the origin is taken in the left-hand plate,

Condition 1, when $x = 0$, $\phi = 0$.
Condition 2, when $x = l$, $\phi = +V$.

Substituting these boundary conditions into Eq. (2-76) gives:

By condition 1, $A = 0$.
By condition 2, $B = V/l$.

Thus, the complete, unique solution to this Dirichlet problem is

$$\phi = \frac{V}{l}x \qquad (2\text{-}77)$$

The equipotential surfaces are the planes $x = \text{const}$, and the field is uniform.

To find the **E** vector, use is made of Eq. (2-55). Thus since

$$\mathbf{E} = -\nabla\phi = -\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)$$

and since $\partial\phi/\partial y = 0$, $\partial\phi/\partial z = 0$, then

$$\mathbf{E} = -\mathbf{i}\frac{d\phi}{dx} = -\mathbf{i}\frac{V}{l} \qquad (2\text{-}78)$$

Therefore, the **E** vector points in the negative X direction, with a constant magnitude V/l volts/m.

Example 2-8.2. Find the potential at all points in a concentric cylindrical cable, the sheath being insulated from the core by a dielectric medium.

Solution. The medium between the core and sheath is uncharged, and Laplace's equation applies. Refer to Fig. 2-22 for the geometry involved, Laplace's equation in circular-cylindrical coordinates is used. Thus,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\psi^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

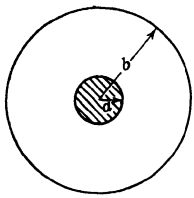


FIG. 2-22. A concentric cylindrical cable.

Since no changes occur along the Z axis and as the variation is symmetrical in ψ, then

$$\frac{\partial^2\phi}{\partial z^2} = 0 \qquad \frac{1}{r^2}\frac{\partial^2\phi}{\partial\psi^2} = 0$$

Hence, Laplace's equation reduces to the form

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = 0$$

which may be written in the form

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0 \tag{2-79}$$

This equation is to be solved and subjected to the boundary conditions:

Condition 1, when $r = b$, $\phi = 0$.
Condition 2, when $r = a$, $\phi = V$.

The general solution of Eq. (2-79) is obtained directly by integrating twice, the result being

$$\phi = A \ln r + B \tag{2-80}$$

Substituting the boundary conditions to evaluate the constants (A and B), there results, by condition 1,

$$0 = A \ln b + B$$

or

$$B = -A \ln b$$

Also, by condition 2,

$$V = A \ln a + B$$

Combine these expressions to find

$$V = A \ln \frac{a}{b}$$

Hence,

$$A = \frac{V}{\ln (a/b)}$$

and

$$B = -\frac{V}{\ln (a/b)} \ln b$$

Substituting the values for A and B, the final solution to the problem becomes

$$\phi = \frac{V}{\ln (a/b)} \ln r - \frac{V}{\ln (a/b)} \ln b$$

or

$$\phi = \frac{V}{\ln (a/b)} \ln \frac{r}{b}$$

This is rewritten in the form

$$\phi = \frac{V}{\ln (b/a)} \ln \frac{b}{r} \tag{2-81}$$

as $b > a$, and the logarithm is never negative. Compare with Eq. (2-45).

To find the **E** vector, use is made of the gradient in circular-cylindrical coordinates, namely,

$$\mathbf{E} = -\nabla\phi = -\left(\mathbf{a}_r \frac{\partial\phi}{\partial r} + \mathbf{a}_\psi \frac{\partial\phi}{r\,\partial\psi} + \mathbf{a}_z \frac{\partial\phi}{\partial z}\right) \tag{2-82}$$

and since symmetry requires that $\partial\phi/r\,\partial\psi = 0$, $\partial\phi/\partial z = 0$, then

$$\mathbf{E} = -\mathbf{a}_r \frac{d\phi}{dr} = +\mathbf{a}_r \frac{V}{\ln (b/a)} \frac{1}{r} \tag{2-83}$$

Therefore, the **E** vector points in the radial direction, with a magnitude $\left(\frac{V}{r} \Big/ \ln \frac{b}{a}\right)$. Unlike the situation in Example 2-8.1, the electric vector is not constant but is a function of the radial distance r from the center of the cable.

Example 2-8.3. Reconsider the parallel-plate capacitor of Example 2-8.1, but now suppose that a uniformly distributed cloud of electrons exists between the plates.

Solution. Since free charges exist in the region between the plates, Poisson's equation

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \tag{2-84}$$

must be solved. To effect the solution, a new variable Φ is introduced, such that the complete solution is of the form

$$\phi = \Phi + f \tag{2-85}$$

where the function f is so chosen that

$$\nabla^2\Phi = 0 \tag{2-86}$$

Thus, the solution of Poisson's equation is reduced to a solution of Laplace's equation. The additional need is to determine the function f, but usually this function can be evaluated by inspection.

In the present example, ρ = const. Also, if Poisson's equation is to be transformed into Eq. (2-86), the f function which is introduced must be such that d^2f/dx^2 will cancel the term $-\rho/\epsilon_0$. Evidently, this can be accomplished by letting

$$f = -\frac{\rho x^2}{2\epsilon_0}$$

Thus, the complete solution is of the form

$$\phi = \Phi - \frac{\rho x^2}{2\epsilon_0}$$

which, in the light of the development in Example 2-8.1, is of the form

$$\phi = A + Bx - \frac{\rho x^2}{2\epsilon_0} \tag{2-87}$$

To evaluate the arbitrary constants, the boundary conditions are introduced. These are:

Condition 1, when $x = 0$, $\phi = 0$.
Condition 2, when $x = l$, $\phi = V$.

The unique solution of the problem is

$$\phi = \left(\frac{V}{l} + \frac{\rho l}{2\epsilon_0}\right) x - \frac{\rho x^2}{2\epsilon_0} \tag{2-88}$$

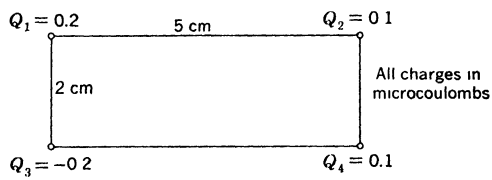
It is noted that Eq. (2-84) could have been solved directly, without introducing the functions Φ and f. In the more complicated cases, however, the auxiliary variable Φ is advantageous.

PROBLEMS

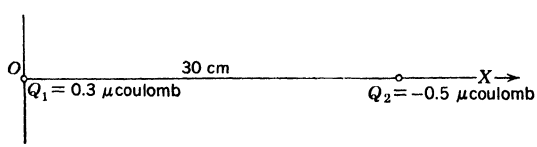
2-1. If it were possible to concentrate charges of 1 coulomb on small spheres which are 1 m apart, what would be the force between these spheres in newtons and in pounds?

2-2. Three concentrated charges of 0.25 μcoulomb are located at the vertices of an equilateral triangle 10 cm on a side. Find the magnitude and direction of the force on one charge due to the other two charges.

2-3. Four concentrated charges are located at the vertices of a plane rectangle, as shown. Determine the magnitude and direction of the resulting force on Q_3.



2-4. Two charges are in the positions shown in the sketch. At what points on the X axis is the electric field intensity equal to zero?

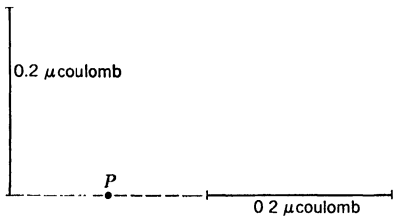


2-5. A fine, straight line of charge of length 8 cm carries a uniform charge of 0.2 μcoulomb/cm. Deduce the magnitude and direction of the electric field intensity at a point that is

a. At a distance of 2 cm from the wire and 2 cm beyond the end of the wire.

b. At a distance of 2 cm beyond the end of the wire, but along the wire.

2-6. Consider the configuration of charge shown, consisting of two uniformly charged wires, each of length 1 m and total charge 0.2 μcoulomb, with their ends separated by 1 m distance. Find the value of E at the point P, the mid-point between the two wires.



2-7. Find the value of the electric field intensity E at a point $2a$ along the axis perpendicular to the plane of a circular wire, of charge Λ coulombs/m, which has a radius a.

2-8. A charged wire of charge $+\Lambda$ coulombs/unit length is in the form of a semicircle of radius R. Determine the expression for E at the center of the circle.

2-9. A charged wire of charge $+\Lambda$ coulombs/unit length is in the form of a semicircle of radius R. Find the direction and magnitude of E at a point a distance a from the plane of the figure and along a line through the center, and normal to the plane of the figure.

2-10. Suppose that a charge of 2.5 μcoulombs is concentrated on a very small spherical conductor.

a. What is the electric field intensity at a point 1 cm from the center of the charge?

b. How many lines of flux of electric force emanate from this charge?

c. What is the density of electric-flux lines at distances 2 cm and 5 cm from the center of the charge?

2-11. Three point charges are located along the X axis, as follows:

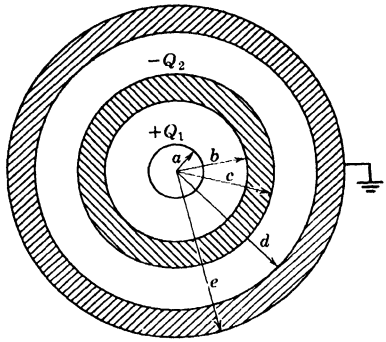
$Q_1 = 0.5$ μcoulomb at the origin
$Q_2 = 1.0$ μcoulomb at $+\frac{1}{2}$ m
$Q_3 = -0.5$ μcoulomb at $-\frac{1}{2}$ m

Find the location of the points of zero potential on the X axis.

2-12. An originally uncharged conducting spherical shell has an outer radius of 10 cm and an inner radius of 6 cm. A charge of 0.2 μcoulomb is uniformly distributed throughout a sphere of radius 2 cm which is concentric with the shell. The space between the inner sphere and the outer shell is air. Determine the absolute potential of each sphere, and plot the variation of potential as a function of distance from the center.

2-13. Repeat Prob. 2-12 if the outer shell is grounded.

2-14. A charge $+Q$ is uniformly distributed on the surface of a sphere of radius a. Concentric with this charged sphere are two conducting shells, as shown in the sketch. The outer shell is grounded. Sketch curves that show as a function of r (*a*) the electric field, (*b*) the potential distribution. Specify the values of all points.



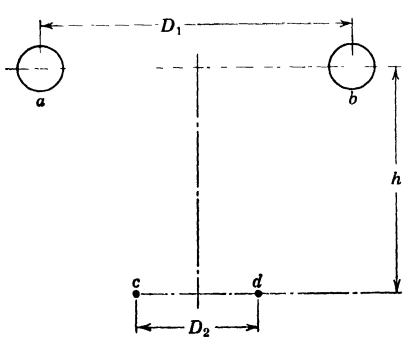
2-15. A system consists of a sphere surrounded by two concentric spherical shells, with dimensions and charges as follows:

	Dimensions, cm	Charges, μcoulombs
Inner sphere	Radius, 2	+1
1st shell..	Inner radius, 8	Grounded
	Outer radius, 10	
2d shell	Inner radius, 16	−2
	Outer radius, 20	

Sketch the potential as a function of distance from the center.

2-16. Find expressions for the electric field intensity and potential per unit length of an infinitely long uniformly charged cylinder of radius R. Sketch the resulting variations of field intensity and potential as a function of r.

2-17. An isolated pair of #0 parallel power lines of radius r_1 which are D_1 apart, have a potential difference V_{ab} and are located a distance h above a pair of telephone wires, as shown. If $r_1 = 0.325$ in., $D_1 = 3$ ft, $V_{ab} = 440$ volts, $h = 2$ ft, $D_2 = \frac{1}{2}$ ft, calculate V_{cd}.



2-18. A three-phase power line consists of wires 1.4 inches in diameter at the vertices of an equilateral triangle 32 ft on a side. At a particular instant, the potential difference $V_{ab} = 220{,}000$ volts, and $V_{bc} = -110{,}000$ volts. Determine the linear charge densities on each conductor.

2-19. At what point in space is the electric field intensity the greatest for the configuration specified in Prob. 2-18? Calculate this value.

2-20. If at a given instant in Problem 2-18 $V_{ab} = 220{,}000\sqrt{2}$ and $V_{bc} = -220{,}000\sqrt{2}$ volts so that lines a and c are at the same potential, determine the linear charge densities on each conductor.

2-21. Prove that:

a. $\text{div}\,(u\mathbf{v}) = \nabla u \cdot \mathbf{v} + u\nabla \cdot \mathbf{v}$.

b. $\nabla \cdot \mathbf{f} = 0$, if $\mathbf{f} = \dfrac{1}{r^2}\mathbf{r}_0$.

2-22. Prove that

$$\int_\tau u\nabla \cdot \mathbf{v}\, d\tau = \int_S u\mathbf{n} \cdot \mathbf{v}\, dS - \int_\tau \nabla u \cdot \mathbf{v}\, d\tau$$

2-23. The potential function in the field of a point charge is, by Eq. (2-39), $\phi = (1/4\pi\epsilon_0)(Q/r)$. Show that this expression satisfies Laplace's equation [Eqs. (2-70) and (2-74)] except at the origin.

2-24. Repeat Prob. 2-23, when ϕ is written in the form

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{x^2 + y^2 + z^2}}$$

CHAPTER 3

CONDUCTORS, DIELECTRICS, CAPACITORS

In the discussion in Chap. 1, certain considerations of conductors led to an expression for the resistance between specified equipotential surfaces. In Chap. 2, certain problems assumed that conductors in an electric field were equipotential surfaces. It is desired to examine the conductor in somewhat greater detail.

3-1. Conductors. By definition, a conductor is a substance containing electrons which are free to move under the action of an electric field through distances which are large compared with atomic dimensions. In the light of the discussion in connection with Ohm's law, Sec. 1-7, the movement of the electrons in the substance occurs even when the electric fields are very small. In fact, a relationship between the current density and the electric field intensity in the conductor, that is, an expression for the conductivity of the metal, will be deduced in Chap. 5.

Suppose that an isolated uncharged conductor is placed in the region containing an electric field, for example, the region between a pair of plates connected by a battery. It is desired to examine the effect on (1) the region inside the conductor and (2) the field outside the conductor. In so far as the region inside the conductor is concerned, since it is isolated, there can be no steady flow of charge. Initially, however, the free electrons redistribute themselves on the surface of the conductor until their field completely neutralizes the impressed field at all points inside the conductor. To see that the redistribution occurs on the surface of the conductor, it is noted from the definition of a conductor that, when no charges are in motion, the electric field inside the conductor must be zero. Consequently, there can be no net charge in any element of volume inside the conductor, since if $\mathbf{E} = 0$, then div $\mathbf{E} = 0$ and, by Eq. (2-62), $\rho = 0$. The entire net charge must reside on the surface of the conductor and must be so distributed as to make $\mathbf{E} = 0$ everywhere inside the conductor. But since

$$E = -\frac{d\phi}{dn} = 0 \tag{3-1}$$

inside the conductor, it is required therefore that

$$\phi = \text{const} \tag{3-2}$$

inside the conductor. Clearly, the surfaces of a conductor must, in the static case, be an equipotential surface, and the entire space within the conductor has the same potential. As noted above, this feature of a conductor has already been used, without proof.

To examine the field outside the isolated conductor, it is noted that, as the redistributed charges on the surface of the conductor are in equilibrium, the electric field must act normally to the surface. That is, if a tangential component of electric field were to exist, it would produce a motion of the charges along the surface. For the case of a long cylindrical conductor in the region between a pair of charged parallel plates, the field distribution becomes that illustrated in Fig. 3-1.

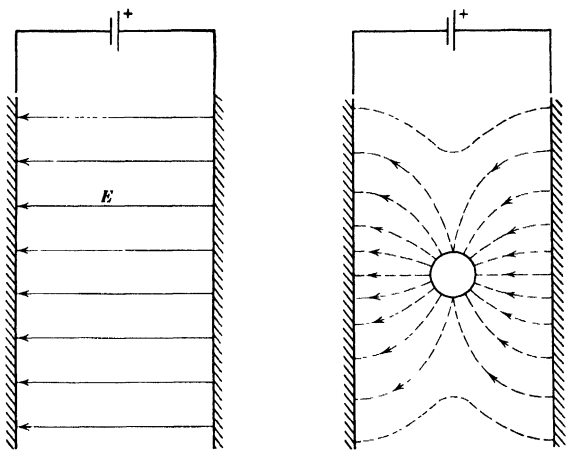


FIG. 3-1. The effect on the static field of a pair of parallel plates due to an isolated cylindrical conductor.

It is desired to express in mathematical terms the fact that the tangential component of the electric field at the surface of the isolated conductor must be zero. This is most conveniently done in terms of the vector, or cross, product of two vectors. As a mathematical digression, the *vector*, or *cross*, product of two vectors, which is quite different from the scalar, or dot, product of two vectors, is defined. Consider two vectors **A** and **B**. By definition, the vector, or cross, product of these vectors is a vector quantity, the magnitude of which is the product of the magnitudes of the two vectors and the sine of the angle between them and the direction of which is normal to the plane of the two vectors and in the direction of advance of a right-handed screw which is turned from one vector to the second. The situation is illustrated in Fig. 3-2. That is, by definition,

$$\mathbf{A} \times \mathbf{B} = AB \sin (\mathbf{A},\mathbf{B})\mathbf{n} \tag{3-3}$$

where **n** is a unit vector, obtained as specified in the definition. It is of some interest to note that the cross product specifies the cross-hatched area shown in Fig. 3-2, the direction of the area being specified by the unit vector (compare with Fig. 1-10). Another fact of importance is noted by considering the cross product $\mathbf{B} \times \mathbf{A}$, which is the cross product of the vectors in reverse order. From the definition, the magnitude is

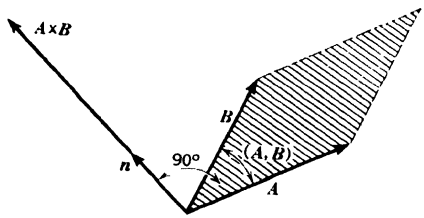


FIG. 3-2. Illustrating the vector product $\mathbf{A} \times \mathbf{B}$.

the same as before, but the direction of the area is just opposite to that above; hence,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{3-4}$$

In mathematical parlance, this states that the vector product of two vectors is *not* commutative. Clearly, the vector product of a vector with itself or with a parallel vector is zero, since the angle between the two vectors is zero and the sine of the angle is therefore zero. Note that the cross between the two vectors is essential to the expression.

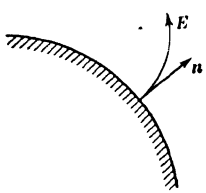


FIG. 3-3. The electric field at the surface of a conductor.

Now return to the essential requirement that the tangential component of the electric field intensity at the surface of a conductor must be zero. Refer to Fig. 3-3. Since **E** must be normal to the conductor, then if **n** denotes a unit outward normal to the surface of the conductor, the condition on **E** is given by

$$\mathbf{E}_t = \mathbf{E} \times \mathbf{n} = 0 \qquad \text{on the surface} \tag{3-5}$$

While Eq. (3-5) specifies that the tangential component of the electric field vanishes, it gives no information concerning the normal component of the electric field. This may be found by constructing a Gaussian surface in the form of a pillbox, which is half in and half out of the surface of the conductor, as illustrated in Fig. 3-4. Owing to the requirement that **E** must be normal to the surface, the contribution to the Gaussian surface integral from the cylindrical sides is zero. Equation (2-30) thus becomes

$$\oint \mathbf{E} \cdot \mathbf{n}\, dS = \int_{\Delta S_1} \mathbf{E} \cdot \mathbf{n}\, dS + \int_{\Delta S_2} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{Q}{\epsilon_0}$$

where a contribution arises only through the flat faces denoted as ΔS_1 and ΔS_2. Note also that the electric field must be zero inside the conductor. Therefore, it follows that

$$\int_{\Delta S_1} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{Q}{\epsilon_0}$$

But $\mathbf{E}$ is a constant over the infinitesimal area ΔS_1. Then

$$\mathbf{E} \cdot \mathbf{n} = \frac{Q}{\Delta S_1\, \epsilon_0} = \frac{\sigma}{\epsilon_0}$$

where the surface-charge density is given by $\sigma = Q/\Delta S_1$. Thus, it follows that

$$\mathbf{E} = \frac{\sigma}{\epsilon_0} \mathbf{n} \tag{3-6}$$

which states in words that, at a point on the surface of a conductor, the field intensity is normal to the surface and is related to the surface-charge

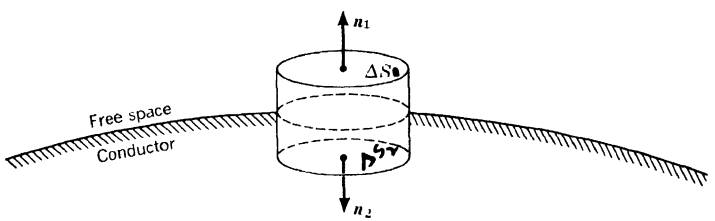


FIG. 3-4. A Gaussian surface in a conductor–free-space interface.

density at that point by Eq. (3-6). A comparison of this result with Eq. (2-24) shows that the electric field intensity changes abruptly from $(\sigma/\epsilon_0)\mathbf{n}$ right at the surface to a value $\sigma/2\epsilon_0$ at points removed from the surface.

3-2. The Method of Electrical Images. In order to apply Eq. (3-6), the value of $\mathbf{E}$ or ϕ must be known. This requires finding a solution of Laplace's equation for the function ϕ which satisfies the prescribed potential boundary conditions. As noted in Sec. 2-8, this involves writing Laplace's equation in the set of orthogonal curvilinear coordinates which is best suited to the description of the surfaces bounding the region in which the value of ϕ is desired. The solution of the resulting equation is expressed in terms of Fourier series, Bessel functions, etc., depending on the system of orthogonal curvilinear coordinates. A detailed study of boundary-value problems will ordinarily provide the competence for carrying out the details of such problems.

There are many problems of importance which do not yield readily to the technique discussed, which is often called *harmonic* analysis. Special

methods have been devised for solving some of these. The method of curvilinear squares discussed in Secs. 1-8ff. is one such method. The method of electrical images discussed in Sec. 1-15 is another such special method. This method, which is due to Lord Kelvin, is of great value in solving for the forms of the lines of force around conductors suspended parallel to the earth.

The essence of the method in its applications to electric field problems is the following: Suppose that a given charge distribution produces a

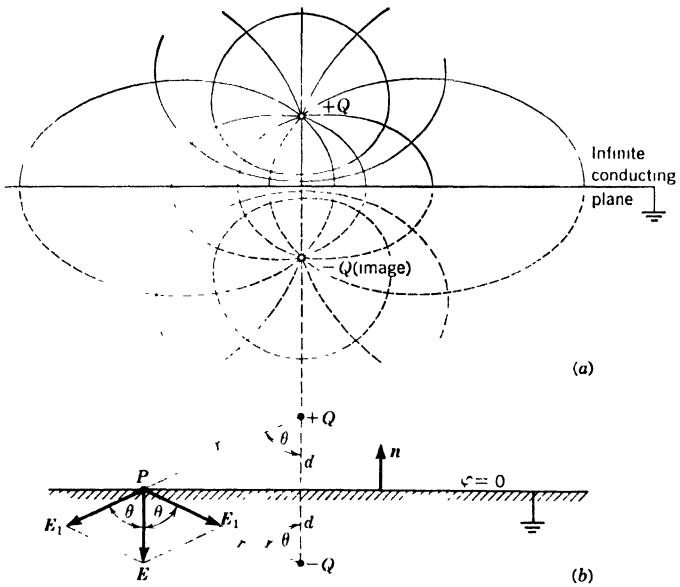


FIG. 3-5. The image of a point charge in an infinite conducting plane: (*a*) Shows the lines of force drawn to show the symmetry of the field at the charge; (*b*) gives the geometry for analysis.

constant potential over a known surface (and this potential is usually taken as zero at the surface). The solution of the problem of finding the field intensity at all points in space due to the charge distribution by means of the method of electrical images involves finding a simple charge distribution reflected in the conductor which, with the distribution of charges outside, will produce the same specified potential surface. From uniqueness considerations of the solution of Laplace's equation, the field will be the same when the conductor is removed and replaced by this new charge distribution. The solution of the problem, in the region normally outside the conductor, is then obtained by considering the field due to the two charge distributions alone.

Example 3-2.1. To illustrate the method, the electrostatic analogue of the problem discussed in Sec. 1-15 is considered. Consider that a point charge $+Q$ is placed at a distance d in front of an infinite plane conductor which is grounded. It is required to find an expression for the charge density induced in the conductor. Refer to Fig. 3-5, which gives some idea of the field distribution.

Solution. In this problem, it is required that the surface of the conductor be an equipotential surface of the system and with $\phi = 0$. Clearly, the choice of an image charge $-Q$ at a distance $-d$ yields a pair of charges which together do, in fact, satisfy the prescribed boundary condition, $\phi = 0$, at all points of the conductor surface. The resulting field of the two specified point charges at a distance $2d$ apart and having equal and opposite charges is the same as that of the original problem.

It follows from Fig. 3-5*b* that the field intensity at the point P is

$$\begin{aligned}\mathbf{E} &= 2\mathbf{E}_1 \cos\theta \\ &= 2\mathbf{E}_1 \frac{d}{r} = 2E_1 \frac{d}{r}(-\mathbf{n})\end{aligned}$$

But since the magnitude of the electric field of the point charge is

$$E_1 = \frac{Q}{4\pi\epsilon_0 r^2}$$

then it follows that

$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 r^2}\frac{d}{r}(-\mathbf{n})$$

which is

$$\mathbf{E} = \frac{Qd}{2\pi\epsilon_0 r^3}(-\mathbf{n})$$

By comparing this expression with Eq. (3-6), it follows that

$$\frac{\sigma}{\epsilon_0} = -\frac{Qd}{\epsilon_0 2\pi r^3}$$

or, finally, the surface-charge density is given by the expression

$$\sigma = -\frac{Qd}{2\pi r^3} \qquad (3\text{-}7)$$

Example 3-2.2. To find the electric field around a spherical conductor maintained at zero potential, when there is a charge $+Q_1$ at a point T near the sphere.

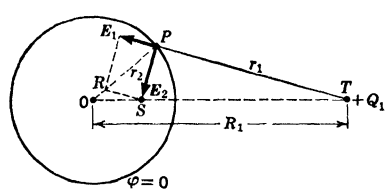


FIG. 3-6. The image of the charge in the sphere.

Solution. Refer to Fig. 3-6, which specifies the system. Consider the field due to the specified charge $+Q_1$ at T and the charge $-Q_2$ at S inside the sphere. From the conditions of the problem, the potential at any point P on the sphere must be zero. This requires that

$$\frac{Q_1}{4\pi\epsilon_0 r_1} - \frac{Q_2}{4\pi\epsilon_0 r_2} = 0$$

or

$$Q_2 = Q_1 \frac{r_2}{r_1} \tag{3-8}$$

This specifies the magnitude of the image charge, which, together with the specified charge, will satisfy the boundary conditions and so specify the field at any point outside the sphere.

Actually, the problem is not yet complete, since the point S or distance r_2 has not yet been uniquely specified. To specify this point, use is made of the fact Eq. (3-8) must be valid for any choice of P on the surface of the sphere. This requires that the ratio r_2/r_1 must be constant. This condition is met if triangle OSP is similar to triangle OTP for all values of P, when

$$\frac{OS}{R} = \frac{R}{OT} \tag{3-9}$$

Then

$$\frac{r_1}{r_2} = \frac{R}{OS} = \text{a constant}$$

Therefore, Eq. (3-8) becomes, in the light of this expression and Eq. (3-9),

$$Q_2 = Q_1 \frac{R}{OT} = Q_1 \frac{R}{R_1} \tag{3-10}$$

It is of some interest to note that, if the potential of the sphere were at the value ϕ instead of 0, this fact can be taken into account by observing that the presence of a charge, say, Q_3, at the center of the sphere would give rise to a spherically symmetrical potential surface. Therefore, the effect of the potential of the sphere can be replaced by the appropriately chosen charge Q_3 which is situated at the center.

This development shows that one pair (the zero potential) of equipotential surfaces due to two point charges Q_1 and $-Q_2$ a distance ST apart are spherical. In the most general case, as illustrated in Fig. 3-5*a*, the equipotential surfaces are not spherical, but vary from the plane to the closed surfaces shown

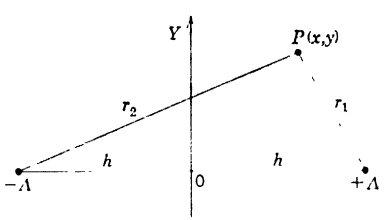


FIG. 3-7. Two parallel, oppositely charged straight lines.

Example 3-2.3. Find the electric field pattern caused by two parallel, straight, oppositely charged infinite lines. From this result, find the image of a charged line in a cylindrical conductor.

Solution. Refer to Fig. 3-7, which shows the geometry of the problem. The origin of a coordinate system is located midway between the lines. It is desired to find the potential at point P. This is readily found by noting that the potential at point P relative to O due to the $+A$ line of charge is

$$V_{PO}^{+} = \frac{2\Lambda}{4\pi\epsilon_0} \ln \frac{h}{r_1}$$

Similarly, the potential at point P due to the $-\Lambda$ line of charge is

$$V_{PO}^{-} = -\frac{2\Lambda}{4\pi\epsilon_0} \ln \frac{h}{r_2}$$

Hence, the total electric potential at point P due to both lines of charge is

$$V = V_{PO}^{+} + V_{PO}^{-} = \frac{2\Lambda}{4\pi\epsilon_0} \ln \frac{r_2}{r_1} \tag{3-11}$$

Note from this that the potential at all points along the Y axis is zero.

In order to determine the locus of an equipotential surface, V is considered constant. This requires that

$$\frac{r_2}{r_1} = \text{constant} \tag{3-12}$$

For convenience, this is called A. Then

$$r_2 = r_1 A \tag{3-13}$$

Now, from the diagram, it is seen that

$$\begin{aligned} r_1^2 &= (x - h)^2 + y^2 \\ r_2^2 &= (x + h)^2 + y^2 \end{aligned} \tag{3-14}$$

and, from Eq. (3-13),

$$r_2^2 = A^2 r_1^2 \tag{3-15}$$

Now, combine Eqs. (3-14) with (3-15) to get

$$(x + h)^2 + y^2 = A^2[(x - h)^2 + y^2]$$

By expansion and rearrangement, it is readily found that

$$x^2 + 2hx \frac{1 + A^2}{1 - A^2} + y^2 = -h^2$$

Now, add to each side of this equation the quantity $h^2(1 + A^2)^2/(1 - A^2)$. This yields the expression

$$x^2 + 2hx \frac{1 + A^2}{1 - A^2} + h^2 \frac{(1 + A^2)^2}{(1 - A^2)^2} + y^2 = h^2 \left[\frac{(1 + A^2)^2}{(1 - A^2)^2} - 1 \right]$$

which may be written as

$$\left(x - h \frac{A^2 + 1}{A^2 - 1} \right)^2 + y^2 = \frac{(2hA)^2}{(A^2 - 1)^2} \tag{3-16}$$

This expression is the equation of a circle of radius $R = 2hA/(A^2 - 1)$, with its

center at the point $[h(A^2 + 1)/(A^2 - 1), 0]$. A plot of these results is shown in Fig. 3-8. Observe the similarity with Fig. 3-6 for the point charge and the sphere.

For convenience, the distance from the origin O to the center of the circle (the equipotential curve) is denoted $D/2$, as shown in the figure. Thus, by writing as noted

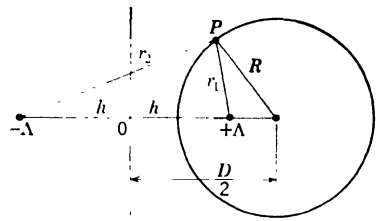


FIG. 3-8. The locus of point P for constant potential.

$$\frac{D}{2} = h\frac{A^2 + 1}{A^2 - 1} \tag{3-17}$$

$$\text{then} \quad R = \frac{2Ah}{A^2 - 1} = \sqrt{\left(\frac{D}{2}\right)^2 - h^2} \tag{3-18}$$

Now, even though the geometry here is that for line charges and that in Fig. 3-6 is for a point charge, for the section shown they are the same. Thus for the locus to be equipotential which requires that r_2/r_1 is a constant in both cases, then by Eq. (3-9)

$$\frac{r_2}{r_1} = A = \frac{2R}{D - 2h} \tag{3-19}$$

Now, combine this with the second form of Eq. (3-18), which is first rewritten as

$$h = \sqrt{\left(\frac{D}{2}\right)^2 - R^2}$$

Then,
$$A^2 = \frac{[D/2 + \sqrt{(D/2)^2 - R^2}]^2}{R^2}$$

from which
$$A = \frac{D}{2R} + \sqrt{\left(\frac{D}{2R}\right)^2 - 1} \tag{3-20}$$

This result is combined with Eq. (3-11) for the potential of the point P (and hence anywhere on the equipotential). The result is

$$V = \frac{2\Lambda}{4\pi\epsilon_0} \ln \frac{r_2}{r_1} = \frac{2\Lambda}{4\pi\epsilon_0} \ln A$$

which may then be written as

$$V = \frac{2\Lambda}{4\pi\epsilon_0} \ln \left[\frac{D}{2R} + \sqrt{\left(\frac{D}{2R}\right)^2 - 1}\right] \tag{3-21}$$

It is interesting to note that this expression may be written in the alternative form involving the inverse hyperbolic cosine, thus:

$$V = \frac{2\Lambda}{4\pi\epsilon_0} \cosh^{-1} \frac{D}{2R} \tag{3-22}$$

The corollary to the foregoing result, which is made evident by reference to Fig. 3-8, is that the image of the negatively charged line in a par-

allel cylinder is the positively charged line, with relative dimensions as illustrated.

Example 3-2.4. A power cable consists of two cylindrical conductors which are symmetrically arranged and enclosed in a lead sheath. The cable is filled with a homogeneous dielectric of permittivity ϵ. A sketch giving some of the pertinent dimensions is contained in Fig. 3-9. If one conductor is at the potential $V+$ relative to the sheath, which is assumed at ground, or zero potential, and if the other conductor is at the potential $V-$ relative to ground, find an expression for the charge per unit length of the cable.

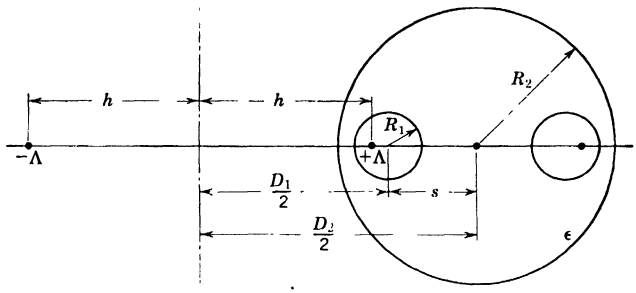


FIG. 3-9 A two-conductor cable within a sheath.

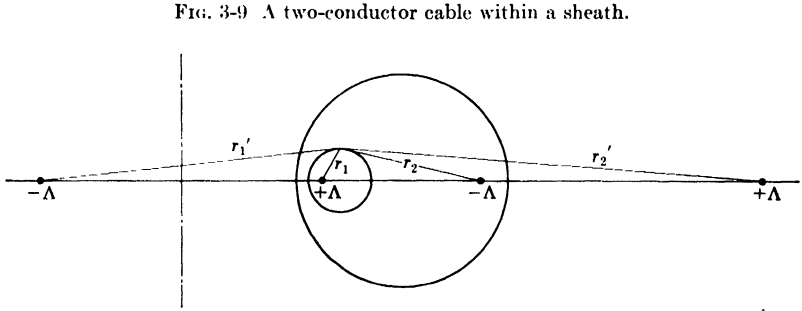


FIG. 3-10. The charge of the equivalent electrical system of the cable of Fig. 3-9.

Solution. Certain of the results of Example 3-2.3 will be used in the solution of the present problem.

First consider the left-hand cylindrical conductor. This is to be replaced by a pair of line charges $+\Lambda$ and $-\Lambda$ which will meet two conditions. Together, these line charges must ensure that the surface of the conductor is an equipotential. Simultaneously, the sheath potential must also be equipotential. A similar pair of line charges replaces the right-hand cylindrical conductor. The sheath potential due to both pairs of line charges will be equipotential, and in fact, will be zero.

In the light of this discussion, the equivalent electrical system of the prescribed cable is that shown in Fig. 3-10.

Now, the potentials of the conductors are known to be $V+$ and $V-$, respectively. Consider the positive conductor, the potential of which is expressed in terms of the system of electrical charges of Fig. 3-10. This is

$$V+ = \frac{\Lambda}{2\pi\epsilon} \ln \frac{r_1' r_2}{r_1 r_2'} \tag{3-23}$$

where

$$\begin{aligned} r_1 &\doteq R_1 \\ r_1' &\doteq 2h \\ r_2 &\doteq 2s \\ r_2' &\doteq 2h + 2s \end{aligned} \tag{3-24}$$

To reduce these quantities, it is noted from Eq. (3-18) that

$$R_1^2 = \left(\frac{D_1}{2}\right)^2 - h^2$$

But, for $R_1 \ll R_2$, this leads to the approximate expression

$$\frac{D_1}{2} \doteq h \tag{3-25}$$

Also, from the fact that

$$R_2^2 = \left(\frac{D_2}{2}\right)^2 - h^2$$

which may be written in the form

$$\left(\frac{D_1}{2} + s\right)^2 = R_2^2 + h^2$$

it is observed, by expanding the left-hand side and noting that $D_1/2 \doteq h$, that

$$\left(\frac{D_1}{2}\right)^2 + D_1 s + s^2 = R_2^2 + h^2$$

which becomes

$$(D_1 + s)s = R_2^2$$

or

$$D_1 + s = \frac{R_2^2}{s} \tag{3-26}$$

The expressions in Eqs. (3-24) become, in the light of Eqs. (3-25) and (3-26),

$$\begin{aligned} r_1 &\doteq R_1 \\ r_1' &\doteq 2h = D_1 = \frac{R_2^2}{s} - s \\ r_2 &\doteq 2s \\ r_2' &\doteq 2h + 2s = \frac{R_2^2}{s} + s \end{aligned}$$

so that Eq. (3-23) attains the form

$$V+ = \frac{\Lambda}{2\pi\epsilon} \ln \frac{R_2^2 - s^2}{R_2^2 + s^2} \frac{2s}{R_1} \tag{3-27}$$

The potential at the negative conductor is

$$V- = \frac{\Lambda}{2\pi\epsilon} \ln \frac{r_1' r_2}{r_2' r_1}$$

where

$$r_2 \doteq R_1$$

$$r_2' \doteq \frac{R_2^2}{s} - s$$

$$r_1 \doteq 2s$$

$$r_1' \doteq \frac{R_2^2}{s} + s$$

Hence, $V-$ may be written as

$$V- = \frac{\Lambda}{2\pi\epsilon} \ln \frac{R_2^2 + s^2}{R_2^2 - s} \frac{R_1}{2s} \tag{3-28}$$

The difference of potential between the two conductors is $(V+) - (V-)$, so that finally

$$\Delta V = (V+) - (V-) = \frac{\Lambda}{2\pi\epsilon} \left[\ln \left(\frac{R_2^2 - s^2}{R_2^2 + s^2} \frac{2s}{R_1} \right) - \ln \left(\frac{R_2^2 + s^2}{R_2^2 - s^2} \frac{R_1}{2s} \right) \right]$$

which is

$$\Delta V = (V+) - (V-) = \frac{2\Lambda}{2\pi\epsilon} \ln \left(\frac{R_2^2 - s^2}{R_2^2 + s^2} \frac{2s}{R_1} \right)$$

Hence, finally

$$\Lambda = \frac{\pi\epsilon[(V+) - (V-)]}{\ln \left(\frac{R_2^2 - s^2}{R_2^2 + s^2} \frac{2s}{R_1} \right)} \text{ coulombs m} \tag{3-29}$$

3-3. Dielectrics. Unlike the conductor, which was defined in Sec. 3-1, a dielectric is defined as a substance containing an array of charges which will remain in equilibrium when the electric field is not zero inside the substance. Thus, a nonconductor, or dielectric, is characterized by the fact that it does not contain free electrons in any appreciable numbers.

Two broad classes of materials exist, depending on the molecular structure; these are the polar and the nonpolar molecules. They differ in their microscopic characteristics. To discuss the differences between the polar and the nonpolar molecules, it is first necessary to discuss what is meant by the term *electric doublet*, or *electric dipole*. An electric dipole is defined as two equal but opposite charges which are a very small distance apart and which are bound together by the action of molecular forces.

A dielectric material is viewed as being composed of aggregates of charges which are normally bound to positions of equilibrium within the matter by molecular or ionic forces. These aggregates of charge may split up into subaggregates of molecules or atoms which behave as units. Two general types of molecules exist, the polar and the nonpolar, as noted.

The nonpolar molecule is normally electrically neutral. This means, since the molecule is known to be composed of electrons and protons, that the centers of positive and negative charges coincide, the resultant molecule having a zero dipole moment (a measure of the effective charges and their displacement from each other). In polar molecules, there is a permanent displacement of the centers of charge, which causes the molecule to act like a dipole.

Suppose that a nonpolar molecule is under the action of an externally applied field. *Polarization* of the molecule, and so of the dielectric, will occur. In this case, the process of polarization consists in the production of electric dipoles within the neutral molecules through the relative displacement of the positive and negative charges which constitute the molecule. That is, such a polarization may be visualized as a strain on

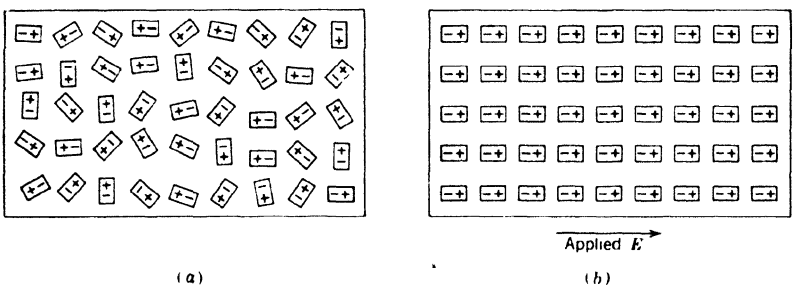


FIG. 3-11. The diagrammatic representing a dielectric: (*a*) the random distribution when no field exists; (*b*) the aligned dipoles in a strong electric field.

the bonds which hold the atoms together. The amount of the shift in the center of the figure of the positive and negative charges which results from the application of the electric field is very small compared with atomic dimensions, but the net result is to produce an effect which is the same as an electric dipole. Such a dipole is said to be an *induced dipole.*

In polar molecules, a permanent displacement of the centers of charge exists, so that the molecules may be looked upon as permanent dipoles. Owing to thermal agitation under normal conditions, the orientation of the dipoles is random so that the dielectric is normally electrically neutral. Under the action of an externally applied electric field, a torque will be exerted on the dipoles, causing a rotation of the axes of the already existing dipoles toward the direction of the impressed field. It is of some interest to note that it is possible in a few materials to cause the alignment produced by the electric field to be retained by the materials. For example, the polarization of molten carnauba wax is retained if the material is allowed to solidify in the field. Such a material is termed an *electret* and is the electric counterpart of the permanent magnet.

It is not intended, in what follows, to develop a theory of the microscopic behavior of dielectric materials. Our purposes are satisfied by considering that a dielectric material is no more complicated than an array of dipoles, permanent or induced, or both. A diagrammatic representation of polarization of a dielectric is given in Fig. 3-11. Figure 3-11*a* shows the dipoles arranged in a random fashion, the space average value of which is supposedly zero. Figure 3-11*b* shows the case of uniform polarization, resulting from the application of a sufficiently strong field.

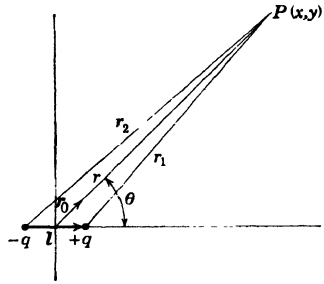


FIG. 3-12. The potential due to an electric dipole.

It is desired to deduce an expression for the potential due to a dipole. Consider Fig. 3-12, which shows such a dipole. The potential at point P due to the two charges is

$$\phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{3-30}$$

In the special but very important case when the distance l is small compared with r_1 and r_2, this expression may be written approximately as

$$\phi = \frac{q}{4\pi\epsilon_0}\left(\frac{r_2 - r_1}{r_1 r_2}\right) \doteq \frac{q}{4\pi\epsilon_0 r^2}(r_2 - r_1) \tag{3-31}$$

But it is noted from the figure that

$$r_2 - r_1 \doteq l\cos\theta$$

then

$$\phi = \frac{ql\cos\theta}{4\pi\epsilon_0 r^2} \tag{3-32}$$

The dipole moment is defined as

$$\mathbf{p} = q\mathbf{l} \tag{3-33}$$

so that finally

$$\phi = \frac{\mathbf{p}\cdot\mathbf{r}_0}{4\pi\epsilon_0 r^2} \tag{3-34}$$

For a volume array of dipoles, the potential follows directly from this expression by superposition. The result may be written as

$$\phi = \int_\tau \frac{\mathbf{P}\cdot\mathbf{r}_0}{4\pi\epsilon_0 r^2}\,d\tau \tag{3-35}$$

where **P**, which is called the *polarization vector*, is the dipole moment per unit volume. This expression may be written in somewhat more convenient form by first noting that it may be expressed relative to the source point as

$$\phi = -\int_\tau \frac{\mathbf{P} \cdot \nabla \left(\frac{1}{r}\right)}{4\pi\epsilon_0} d\tau \tag{3-36}$$

Now use is made of the results of Prob. 2-21 that

$$\text{div } (u\mathbf{v}) = u\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla u$$

Suppose that we write

$$u \equiv \frac{1}{r} \qquad \mathbf{v} \equiv \mathbf{P}$$

Then Eq. (3-36) becomes

$$\phi = \int_\tau \frac{1}{4\pi\epsilon_0} \left[\text{div} \left(\frac{\mathbf{P}}{r}\right) - \frac{1}{r} \nabla \cdot \mathbf{P} \right] d\tau$$

Now applying the divergence theorem given by Eq. (2-63) to the first term on the right, we have, finally,

$$\phi = \frac{1}{4\pi\epsilon_0} \int_S \frac{\mathbf{P} \cdot \mathbf{n}}{r} dS - \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\nabla \cdot \mathbf{P}}{r} d\tau \tag{3-37}$$

This expression shows that where polarization occurs, an equivalent volume-charge density $-\nabla \cdot \mathbf{P}$ is formed throughout the volume, and an equivalent surface-charge density P_n is formed over the surface.

Consider the general case where only free and bound volume charges exist. The total potential at a field point is given by the following expression, which is a combination of Eq. (2-41) and Eq. (3-37)

$$\phi = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\rho - \nabla \cdot \mathbf{P}}{r} d\tau \tag{3-38}*$$

which may be considered as the potential due to an equivalent charge distribution of density $\rho - \nabla \cdot \mathbf{P}$. Correspondingly, Eq. (2-62) in point form for the equivalent charge distribution becomes

$$\nabla \cdot \mathbf{E} = \frac{\rho - \nabla \cdot \mathbf{P}}{\epsilon_0} \tag{3-39}$$

This expression may be rearranged to the form

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho \tag{3-40}$$

It is convenient to define vector **D** by the expression

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{3-41}$$

Here τ is taken larger than the volume of the dielectric, so that this equation includes the contribution from the surface integral.

which is called the *electric displacement.* Clearly, Eq. (3-40) may now be written in the form

$$\nabla \cdot \mathbf{D} = \rho \tag{3-42}$$

The corresponding form of Gauss' law in integral form is

$$\oint \mathbf{D} \cdot \mathbf{n}\, dS = Q \tag{3-42a}$$

These expressions depend on the free-charge density only. They relate the number of lines of electric displacement originating in any closed region of space with the quantity of free electric charge contained within the surface bounding the space. Conversely, of course, if there is no free charge in a given region of space, the lines of electric displacement are continuous, even if one or more dielectric media may exist in the given region of space. Note that since Eqs. (3-42) and (3-42*a*) contain Eqs. (2-62) as special cases, these will be taken as the most general expressions for Gauss' law for the electric field In these forms, they appear in Maxwell's equations.

In a large class of materials, which are the homogeneous and isotropic materials, the polarization vector **P** is directly proportional to the electric field intensity. The proportionality constant is written $\epsilon_0\chi_e$, so that

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} \tag{3-43}$$

where χ_e is known as the *electric susceptibility* of the material. Equation (3-41) may thus be written

$$\mathbf{D} = \epsilon_0(1 + \chi_e)\mathbf{E}$$

which may be written as

$$\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E} = \epsilon\mathbf{E} \tag{3-44}$$

where the quantity

$$\epsilon = \epsilon_0\epsilon_r = \epsilon_0(1 + \chi_e) \tag{3-45}$$

is called the *electric permittivity* of the medium, and where the quantity

$$\epsilon_r = (1 + \chi_e) \tag{3-46}$$

is known by a variety of names, among them being: dielectric constant, relative permittivity, specific inductive capacity. The values of ϵ_r for a few common dielectrics are given in Table 3-1.

Clearly, because of the restrictions on ϵ, Eq. (3-44) is true for the homogeneous isotropic media and is not valid for crystalline materials, which have a definite structure, or for materials under strain. In such nonisotropic media, **D** and **E** do not have a common direction at a point, and ϵ may be a nonlinear function of E and may also be different in different directions.

TABLE 3-1
THE RELATIVE PERMITTIVITY OF SOME COMMON MATERIALS

Material	ϵ_r	*Material*	ϵ_r
Vacuum	1.00000	Liquids:	
Gases at 1 atm pressure:		Castor oil	4.67
Air	1.000590	Transformer oil	2.24
Hydrogen	1.000264	Water, 0°C	88.00
Neon	1.000127	Water, 100°C	48.00
Nitrogen	1.000580	Solids:	
Oxygen	1.000523	Glass	3.8–6.8
Methane	1.00094	Paraffin	2.1–2.5
Sulfur dioxide	1.00993	Polystyrene	2.4–2.8
		Quartz, fused	3.7–4.1
		Mica	4.0–7.0

The properties of the **D** lines are very suggestive of the continuous current-density vector **J** in current-flow fields (see Chap. 1). Moreover, Eq. (3-44) is similar to Ohm's law. Therefore, from this comparative viewpoint, the permittivity of the material plays the same role, mathematically and topographically, as the conductivity. This similarity will

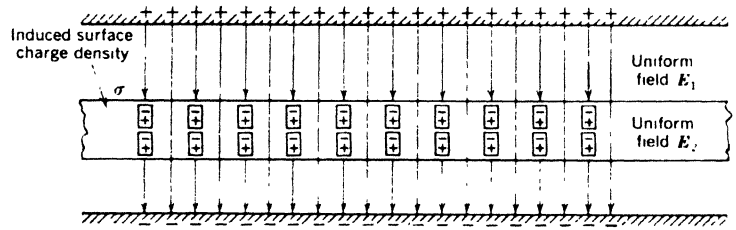


FIG. 3-13 A dielectric slab in a uniform field.

later be exploited to show a relationship of the resistance between indicated surfaces of a specified electrode configuration and that the capacitance between conductors has the same electrode configuration.

The units of **D** may be found directly from Eq. (3-44); thus,

$$\text{Unit of } D = \frac{(\text{coulombs})^2}{\text{newton-m}^2} \times \frac{\text{newtons}}{\text{coulombs}} = \frac{\text{coulombs}}{\text{m}^2}$$

This is further suggestive of the continuous current-density vector **J** in the current-flow fields. That is, if there is associated with free charge a *flux of electric displacement* or a *flux of electric induction*, then **D** is the *density of the flux of electric induction* at any point in the field.

Now refer to Fig. 3-13 which shows a sheet of dielectric material placed in a uniform electric field between a pair of parallel plates. Although dipoles exist throughout the volume of the dielectric, as shown in Fig. 3-11, owing to saturation only the dipoles near the surface are important. This means that $\mathbf{P} = \text{constant}$ throughout the volume, and $\nabla \cdot \mathbf{P} = 0$.

In the free-space region outside the dielectric, the field intensity is denoted as $\mathbf{E}_1$. Because of the induced surface charges, which arise from the alignment of the dipoles of the dielectric, an electric field is produced in the dielectric which opposes the external field. The polarization produces the surface-charge density σ_i on the surface of the polarized dielectric, which is just equal to the normal component of the polarization vector at that point as specified by Eq. (3-37). That is

$$\sigma_i = P_n \tag{3-47}$$

For the uniform dielectric shown, the polarization vector is everywhere constant in magnitude and normal to the slab surfaces. But by the results of Example 2-3.2, the fields due to the oppositely charged layers of surface charge add inside the dielectric and cancel outside the dielectric. Hence the resultant electric field inside the dielectric may be written as

$$\mathbf{E}_2 = \mathbf{E}_1 - \frac{\sigma_i}{\epsilon_0}$$

or

$$\epsilon_0\mathbf{E}_1 = \epsilon_0\mathbf{E}_2 + \mathbf{P} = \epsilon\mathbf{E}_2 \tag{3-48}$$

or simply

$$\mathbf{D}_1 = \mathbf{D}_2 \tag{3-49}$$

This expression shows that the introduction of a dielectric material into an electric field causes a discontinuity in the **E** vector. However, as seen, the **D** vector, which in this case is normal to the surface of the dielectric, is continuous and of constant magnitude. This is a general property of

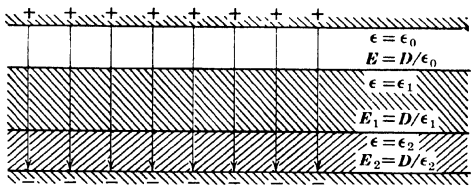


FIG. 3-14. A sketch of the **D** lines in a field containing dielectric media.

the **D** vector for a region of any number of dielectric media. An example of the situation for three dielectrics is shown in Fig. 3-14.

3-4. Boundary Conditions at a Dielectric Surface. For the case of the dielectric in a uniform field considered above, the polarization which reduces the field intensity within the dielectric does not affect the field outside. In the general case for a dielectric in a nonuniform field, or when the dielectric is not an infinite slab, the polarization affects the field both inside and outside the dielectric. Refer to Fig. 3-15, which shows a dielectric cylinder introduced into an originally uniform field. An inspection of the diagram shows the tendency of the **D** lines to run as much as possible through the cylinder. Note, moreover, that the direction of the **D** lines is not normal to the surface of the cylinder since the

surface of the dielectric cylinder is not an equipotential surface, as it would be in the case of a conducting cylinder (compare with Fig. 3-1*b*). It is desired to examine the boundary conditions on the surface between two dielectric media of different relative permittivity.

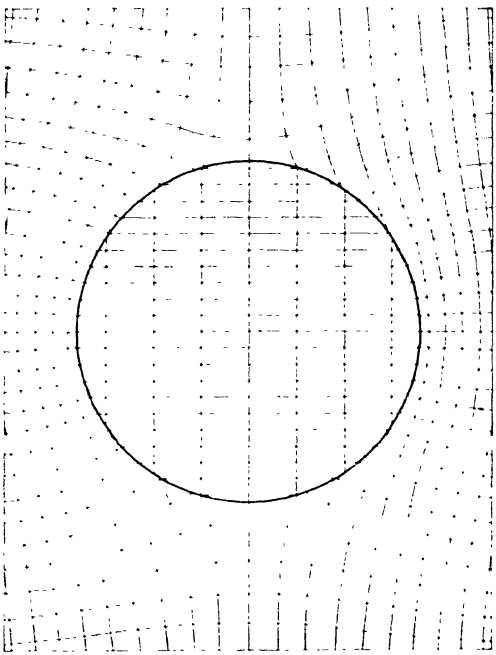


FIG. 3-15. A dielectric cylinder in a uniform field. (*From a drawing by Maxwell.*)

It will now be shown that the following conditions apply on the surface between two dielectric media of different relative permittivity:

1. The tangential component of the electric field intensity **E** is the same on each side of the boundary.

2. The normal component of the density of electric displacement **D** is continuous across the boundary.

In order to prove these two theorems, consider the interface between two dielectric media, as shown in Fig. 3-16.

To prove theorem 1, consider the path specified by the points *a*, *b*, *c*, *d*, which are close to each other and close to the boundary as shown. This path is now assumed centered at the origin of Fig. 3-16*a*. Suppose that the electric field intensity at point *a* is $\mathbf{E}_1$ and that at point *b* is $\mathbf{E}_2$. But the

$$\text{Line integral from } a \text{ to } c = V_{ca} = (E_1 \sin \alpha_1)\ \Delta x$$

Similarly,

$$\text{Line integral from } b \text{ to } d = V_{db} = (E_2 \sin \alpha_2)\ \Delta x$$

Also, in the limit as the distances ab and cd become infinitesimals, the line integrals between these respective points may be neglected. But the line integral around any closed path in a static electric field is zero. It is required, therefore, that

$$E_1 \sin \alpha_1 = E_2 \sin \alpha_2 \tag{3-50}$$

which may be written in the form

$$\mathbf{E}_{t1} = \mathbf{E}_{t2} \tag{3-51}$$

or

$$(\mathbf{E} \times \mathbf{n})_1 = (\mathbf{E} \times \mathbf{n})_2$$

To prove theorem 2, Gauss' theorem will be applied to a small coin-shaped volume enclosing a surface element dS in the boundary between the two dielectric media, as shown in Fig. 3-16c which is centered at the

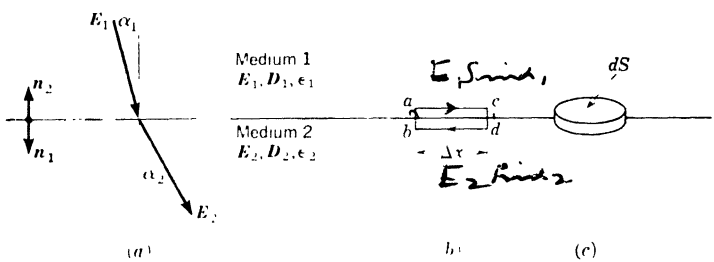


FIG. 3-16 The interface between two dielectric media.

origin. Now, if the volume is shrunk so that the two outer faces dS approach each other, there will be $\mathbf{D}$ lines passing only through the top and bottom surfaces. Moreover, since all charges are bound, Gauss' law requires that

$$\oint \mathbf{D} \cdot \mathbf{n}\, dS = 0$$

from which it follows that

$$\mathbf{D}_{n1}\, dS_1 - \mathbf{D}_{n2}\, dS_2 = 0$$

or since $dS_1 = dS_2$,

$$\mathbf{D}_{n1} = \mathbf{D}_{n2} \tag{3-52}$$

or

$$(\mathbf{D} \cdot \mathbf{n})_1 = (\mathbf{D} \cdot \mathbf{n})_2$$

which proves the theorem.

An expression for the refraction of the lines of electric force is readily obtained. Thus, dividing Eq. (3-50) by Eq. (3-52), it follows that

$$\frac{\mathbf{E}_1 \sin \alpha_1}{\mathbf{D}_1 \cos \alpha_1} = \frac{\mathbf{E}_2 \sin \alpha_2}{\mathbf{D}_2 \cos \alpha_2}$$

Combining this with the known relationship between **D** and **E**, namely, $\mathbf{D}_1 = \epsilon_1\epsilon_0\mathbf{E}_1$ and $\mathbf{D}_2 = \epsilon_2\epsilon_0\mathbf{E}_2$, then

$$\frac{\tan \alpha_1}{\epsilon_1} = \frac{\tan \alpha_2}{\epsilon_2}$$

where

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\epsilon_1}{\epsilon_2} \tag{3-53}$$

A comparison of this expression with Eq. (1-92) for the refraction law of the current-flow lines in the current-flow field between media of different conductivity shows precisely the same relationship.

Refer first to Figs. 3-11 and 3-13. These figures show that, when a dielectric is placed in an electric field, the dipoles are aligned, thereby causing an effective surface polarization charge to appear on each surface of the dielectric. It was, in fact, this surface polarization charge that led to Eq. (3-47). It is clear that the surface polarization charge existing on each dielectric surface will be different, in general, and the **D** lines bend at the boundary although every line of **D** continues across the boundary.

3-5. Dielectric Strength. According to our theory, a dielectric is assumed to be no more complicated than an array of dipoles. This is a simple picture of the situation that permits a consistent theory to be developed. It does suffer because its inherent simplicity does not permit a complete explanation of a wide variety of properties exhibited by many materials. Suffice it to say that a rather complex theory of the solid state has been developed, which is beyond the scope of this text. However, our simple theory does yield reasonable explanations to many problems. It is desired to examine a number of problems which possess considerable engineering importance.

A large general class of problems is concerned with the development of high-voltage cables and bushings. The problem here is to choose a number of dielectric materials and to apportion them so that the optimal breakdown strength results. What is meant by the term *breakdown strength*, and what occurs in a material when breakdown occurs? It is desired to examine these questions.

A gas is normally a good insulator, or dielectric. In a neutral gas molecule, the electrons are bound to the nucleus by electric forces, and in the polyatomic molecules the atoms are bound together by intermolecular forces. When a gas is placed in an electric field, a strain is produced on the elastic forces which permits the polarization of the molecule, but under normal conditions no electrons are freed from the molecule. Should the electric field be made so high that electrons can be torn away from their normal position in the molecule, the electrons will drift to the positive electrode and the positive ion that results when the electron is removed will drift to the negative electrode. Because of this drift of

charged carriers, the gas is no longer an insulator, and dielectric breakdown is said to have occurred.

The same general process occurs in a solid dielectric material, although a number of factors influence the ultimate potential that might be applied before rupture occurs. Among the factors that play a part in ultimate strength of a material are: the material, the impurities present, thickness of the material, temperature, humidity, the presence of flaws, and sometimes the length of time that the potential is applied. Ordinarily, by removing the field from a gaseous or liquid dielectric after breakdown, the dielectric will re-form and heal the break. If chemical changes occur during breakdown, the subsequent breakdown potential may be reduced. But the dielectric does recover its insulating properties. A solid dielectric is usually permanently damaged, as the breakdown will cause a crack or a hole to form in it which does not heal when the field is removed. The subsequent breakdown potential will probably be that of the gas which fills the crack, rather than that of the dielectric material. Since the breakdown of the gas is ordinarily much lower than that of a solid dielectric, permanent degradation of the dielectric strength has occurred.

From what has been said, it is apparent that the magnitude of the electric field at which dielectric breakdown occurs in an insulating material is the dielectric strength of that material. Table 3-2 gives the dielectric strength of a number of important materials.

TABLE 3-2
DIELECTRIC STRENGTH OF SOME COMMON MATERIALS

Material	*Dielectric strength, kv/m*
Air	3,000
Mica	100,000
Polystyrene	20,000
Porcelain	10,000
Rubber	40,000
Transformer oil	12,000

A number of practical examples are considered.

Example 3-5.1. A potential of 50 kv appears between a pair of parallel plates which are 0.02 m apart, the medium between them being air. It has been suggested that a sheet of mica 0.0075 m thick inserted between the plates would give an added margin of safety against breakdown. Examine the merits of the suggestion.

Solution. Refer to Fig. 3-17, which shows the geometry of the problem. Before the addition of the dielectric sheet, the electric field intensity in the region between the plates is, neglecting fringing effects,

$$E = \frac{V}{d} = \frac{50}{0.02} = 2{,}500 \text{ kv/m}$$

Since the dielectric strength of air is 3,000 kv/m, which is higher than the potential gradient in the field, there will be no breakdown.

Now, when the mica plate is inserted, again neglecting fringing, it is necessary to examine the electric field in the air space and in the dielectric. We write directly that

$$V = E_1 d_1 + E_2 d_2$$

But as the electric-displacement vector **D** is constant, then

$$\mathbf{D} = \epsilon \mathbf{E} \text{ everywhere}$$

and

$$\epsilon_1 E_1 = \epsilon_2 E_2$$

Combining this with the known relative permittivity of air and mica, then

$$E_1 = 4E_2$$

or

$$E_2 = \frac{E_1}{4}$$

Combine this expression with that above, to find

$$50 \text{ kv} = 0.0125E_1 + \frac{0.0075}{4} E_1 = 0.0144E_1$$

Solve for E_1, to find

$$E_1 = \frac{50}{0.0144} = 3,440 \text{ kv/m}$$

Since this field strength exceeds the dielectric strength of air, the air will break down. Since the air after breakdown is rendered conducting, then the field intensity in the mica becomes

$$E_2 = \frac{50 \text{ kv}}{0.0075} = 66,700 \text{ kv/m}$$

But the breakdown strength of the mica is greater than this value, and the mica will not rupture

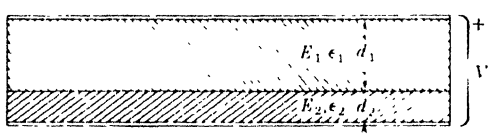


FIG. 3-17. A dielectric sheet between plane electrodes

It must be concluded that the suggestion made was a poor one, as the insertion of the added dielectric actually caused breakdown to occur. While the mica would withstand the resulting field, it would be inadvisable to have continuous sparking exist in the system. What is indicated by this problem is that caution must be exercised in adding dielectric materials to regions of high potential gradients.

Example 3-5.2. It is required to determine the thickness of the layers of dielectric, and their relative permittivities, to be used in a multiple-dielectric coaxial cable, in order to yield a cable design for maximum strength.[1]

[1] H. Waddicor, "Principles of Electric Power Transmission," pp. 213–216, John Wiley & Sons, Inc., New York, 1928.

Solution. The notation is given in Fig. 3-18. To find the electric field intensity at all points in the dielectric, use is made of Gauss' law in the form given by Eq. (3-19). It is noted that free charges exist only on the surface of the inner conductor, whence the total free charge contained in a Gaussian surface, which is a cylinder of unit length and of general radius r, is the same at all points in the region between the inner and the outer conductors. Consequently, the total flux of electric displacement, which is given the symbol ψ, is the same through each dielectric boundary or interface. If the total charge per unit length on the inner conductor is Λ, then the application of Eq. (3-42) gives

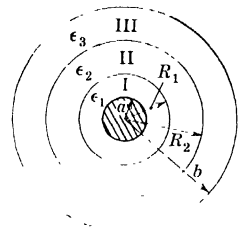


FIG. 3-18. A three-dielectric coaxial cable

$$\oint \mathbf{D} \cdot \mathbf{n}\, dS = \Lambda \tag{3-54}$$

But, from symmetry, D is normal to the surface, whence $\mathbf{D} \cdot \mathbf{n} = D$. Then

$$2\pi r D_r = \Lambda$$

from which

$$D_r = \frac{\Lambda}{2\pi r} \tag{3-55}$$

The corresponding expression for the electric field intensity is then

$$E_r = \frac{\Lambda}{2\pi \epsilon r} \tag{3-56}$$

where the permittivity ϵ is that corresponding to the particular dielectric material.

A plot of the general character of the variation would be somewhat like that

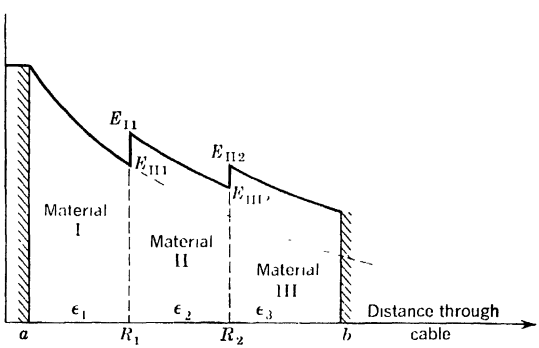


FIG. 3-19. The variation of electric field with distance through the cable.

illustrated in Fig. 3-19. Whether there is a rise or fall in the electric field intensity on the two sides of a dielectric boundary will depend on the respective values of permittivity ϵ. Since the difference of potential may be written as the line integral of the electric field, then the total potential difference between inner and outer conductors is

$$V = \int_a^b \mathbf{E} \cdot d\mathbf{l} = \frac{A}{2\pi}\left(\frac{1}{\epsilon_1}\ln\frac{R_1}{a} + \frac{1}{\epsilon_2}\ln\frac{R_2}{R_1} + \frac{1}{\epsilon_3}\ln\frac{b}{R_2}\right) \tag{3-57}$$

Observe that this expression specifies the area under the curve of Fig. 3-19.

Suppose initially that a single-dielectric cable is considered. For a specified current-handling capacity of the cable, the radius of the inner conductor would be dictated by normal engineering practice for current-handling capacities of conductors. The engineering problem is then that of insulating the inner conductor for a required potential difference between inner and outer conductors. For a single-dielectric material, Eq. (3-57) becomes

$$V = \frac{A}{2\pi\epsilon}\ln\frac{b}{a} \tag{3-58}$$

By eliminating A between Eqs. (3-56) and (3-58), the maximum electric field may be written in the form

$$E_m = \frac{V}{a\ln(b/a)} \tag{3-59}$$

Now, for a specified V and with a dielectric which may be chosen according to its breakdown strength, the ease of handling in manufacture both of the material and of the cable, and the comparative costs (this latter will be affected not only by the dielectric-material costs, but also by the cost of the sheath, since the outer radius will depend on the required thickness of dielectric), Eq. (3-59) may be solved for the required radius of the outer conductor. The result is

$$b = ae^{(V/aE_m)} \tag{3-60}$$

This is independent of the permittivity of the dielectric material. Observe that, for the smallest volume of the cable for a single dielectric, the material chosen should have the highest possible breakdown strength. A factor which will also influence the choice of the material is the capacitance of the cable (this factor is discussed at some length in Sec. 3-6). It will be found that, for low capacitance, the permittivity should be low. Hence, for optimum design, the permittivity should be low, and the dielectric strength should be high.

Refer again to Fig. 3-19 for the case of a single dielectric (which is given by the broken-line curve). Clearly, the dielectric near the inner conductor is stressed to a much greater degree than that near the outer conductor. If the material were stressed almost to its limit at the inner conductor, most of the material is understressed. Thus, the material near the outer conductor increases the volume of dielectric rapidly, but the total addition to the allowable potential increases slowly. The use of multiple dielectrics aids materially in reducing the volume for a specified potential difference.

Consider the situation of Fig. 3-19. Suppose that the three dielectrics have the same breakdown strengths but different permittivities. In this case, the variation is more nearly like that shown in Fig. 3-20. The total area under the composite curve of Fig. 3-20 is considerably greater than that under the broken-line curve, which represents the situation of a cable of a single dielectric of outer radius b and permittivity ϵ_1.

It is of interest to examine the procedure for optimizing the cable design for the situation illustrated in Fig. 3-20. This requires for a total maximum potential that the thickness of the dielectric layers be established for specified per-

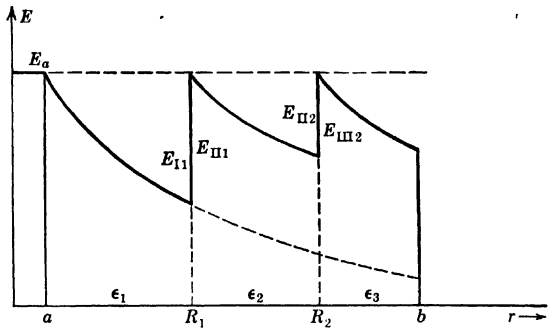


FIG. 3-20. The variation of electric field for three dielectrics with the same breakdown strength.

mittivities. The situation is most easily analyzed by considering the conditions at the several interfaces.

1. Inner conductor, dielectric I interface. The electric displacement is, from Eq. (3-55),

$$D_a = \frac{\Lambda}{2\pi a}$$

2. At the I-II interface. The total flux of electric displacement, designated by the symbol ψ, is unchanged. Thus, $\psi_2 = \psi_a$. Hence,

$$\epsilon_2 R_1 E_2 = \epsilon_0 a E_a$$

For equal breakdown strengths, $E_2 = E_a$. Then,

$$R_1 = \frac{\epsilon_0 a}{\epsilon_2} = \frac{a}{\epsilon_{r2}} \tag{3-61}$$

3. At the II-III interface, $\psi_3 = \psi_a$. Then, as above,

$$R_2 = \frac{a}{\epsilon_{r3}} \tag{3-62}$$

For the radii specified by Eqs. (3-61) and (3-62), the controlling relationship on V is given by (3-57) to be

$$V = E_a \epsilon_0 a \left(\frac{1}{\epsilon_1} \ln \frac{1}{\epsilon_{r2}} + \frac{1}{\epsilon_2} \ln \frac{\epsilon_{r2}}{\epsilon_{r3}} + \frac{1}{\epsilon_3} \ln \frac{b}{a} \epsilon_{r3} \right) \tag{3-63}$$

If the dielectric strengths of the various dielectric materials are not the same, then Fig. 3-20, and the expression deduced therefrom, Eq. (3-63), are no longer valid. If the dielectrics are such that $E_{\text{II-max}}$ is less than $E_{\text{I-max}}$, then the situation is roughly that illustrated in Fig. 3-19 at the I-II interface. The value of R_1 at which the discontinuity in E may occur must be made larger to prevent rupture of material II. One would not choose a dielectric for which $E_{\text{II-max}}$ is greater than $E_{\text{I-max}}$, as the outer materials would then not be stressed to their optimum values. Actually, for optimum results, it would be more desirable to have a graded insula-

tion with a varying permittivity, with the permittivity varying inversely proportional to the radius. Such a material has not yet been developed.

3-6. Capacitors and Capacitance. A simple capacitor is defined as two isolated conductors which are insulated by a nonconductor. One of the conductors carries a charge $+Q$ and the other conductor carries a charge $-Q$.

The capacitance of a capacitor is defined as the ratio of Q to the difference of potential between the two conductors. Accordingly, the capacitance of a capacitor is given by the expression

$$C \equiv \frac{Q}{V_{21}} = \frac{\int_S \sigma \, dS}{\int_1^2 \mathbf{E} \cdot d\mathbf{l}} \tag{3-64}$$

where $\mathbf{E}$ is the electric field intensity due to the charges, σ is the surface-charge density, $d\mathbf{l}$ is an element of length between the conductors, and dS is an element of area of the conductors on which the surface charge resides. From the fact contained in the potential integral, Eq. (2-41), that the potential is proportional to the charge, then the ratio Q/ϕ is a constant that is determined only by the size and shape of the body. That the capacitance is a function of the geometry of the system alone is contained in the form of Eq. (3-64).

In the most common form, the capacitor consists of two large, parallel metal plates which are separated by a thin dielectric. Often, in the interests of saving space, these may be wrapped into helical form. Of course, the dielectric material must be such, in this case, that it will not be damaged by the physical strain imposed upon it in deforming its shape.

A few applications of Eq. (3-64) will now be given.

Example 3-6.1. Find the capacitance of an isolated charged conducting sphere.

Solution. It has been found that the potential of an isolated sphere of charge Q and of radius a is

$$\phi = \frac{Q}{4\pi\epsilon_0 a} \tag{3-65}$$

Since the equal induced charge is at infinity with its associated zero potential, then the difference of potential between the isolated sphere and its induced charge at infinity is also given by Eq. (3-65). The capacitance is then simply

$$C = 4\pi\epsilon_0 a \tag{3-66}$$

Thus, the capacitance of an isolated conducting sphere is directly proportional to the radius of the sphere.

Example 3-6.2. Find the capacitance of a parallel-plate capacitor. Assume that the linear dimensions of the plates are large compared with the separation

(that is, neglect the effect of fringing of the electric field lines at the edges of the plates).

Solution. In Fig. 3-21 is shown a parallel-plate capacitor. We proceed as follows: If the surface-charge density on one plate is σ coulombs/m^2, the boundary condition given by Eq. (3-6) requires that

$$\mathbf{E} = \frac{\sigma}{\epsilon}\mathbf{n}$$

Symmetry requires the field to be uniform, whence the charge density on the second plate must be $-\sigma$. Hence, the total charge on one plate of area A is

$$Q = \sigma A$$

Also, the potential difference between the plates is

$$V = -\int_0^d \mathbf{E} \cdot d\mathbf{l} = Ed = \frac{\sigma}{\epsilon_0} d$$

The capacitance is given by the expression

$$C = \frac{Q}{V} = \frac{\sigma A}{\sigma d/\epsilon_0} = \frac{\epsilon_0 A}{d} \quad \text{farads} \tag{3-67}$$

If fringing the field could not be neglected, the problem would become considerably more complicated, since the charge density would not be a constant over the surface of the conductor. Now one would be faced with the problem of finding a

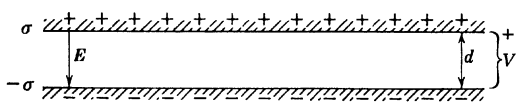


FIG. 3-21. A parallel-plate capacitor.

solution to Laplace's equation in order to find the potential distribution between the plates. From this potential distribution, the electric field intensity at all points at the surface of the plates could be found. From the known electric intensity, the charge distribution is that specified by Eq. (3-6), whence the solution could be found. Of course, any of the known techniques for finding a solution to Laplace's equation could be employed as, for example, the techniques of flux plotting and the method of curvilinear squares, both of which were discussed in some detail in Chap. 1.

Apropos of this discussion, which implies a relationship between capacitance and resistance, the following useful theorem is proved:

THEOREM. The total electrical conductance between any configuration of conductors embedded in a conducting medium of conductivity γ may be obtained at once by replacing ϵ by γ in the expression for the capacitance of this configuration of conductors when they are embedded in a dielectric of permittivity ϵ.

PROOF. The proof of this theorem follows. By definition,

$$C \equiv \frac{Q}{\phi_2 - \phi_1} = \frac{\int_S \sigma \, dS}{\int_1^2 \mathbf{E} \cdot d\mathbf{l}}$$

But at the surface of the conductors, from Eq. (3-6),

$$\epsilon E_n = \sigma$$

Therefore,

$$C = \frac{\epsilon \int_S E_n \, dS}{\int_1^2 \mathbf{E} \cdot d\mathbf{l}}$$

For the case of a current flow, from Sec. 1-6,

$$\Gamma \equiv \frac{1}{R} = \frac{I}{\phi_2 - \phi_1} = \frac{\int_S J_n \, dS}{\int_1^2 \mathbf{E} \cdot d\mathbf{l}} = \gamma \frac{\int_S E_n \, dS}{\int_1^2 \mathbf{E} \cdot d\mathbf{l}}$$

where $\Gamma \equiv 1/R$ is the conductance of the medium. By comparing the above two expressions, it is seen that

$$\text{Conductance} = \text{capacitance} \times \frac{\gamma}{\epsilon}$$

or, in mathematical notation,

$$\Gamma = \frac{\gamma}{\epsilon} C \tag{3-68}$$

In the light of this expression, the techniques of Sec. 1-9 are useful either for determining the resistance between specified faces of a given conductor of conductivity γ or for finding the capacitance between electrodes coincident with the specified faces embedded in a dielectric medium of permittivity ϵ.

Example 3-6.3. Find an expression for the capacitance per unit length of a coaxial cable of inner radius a and outer radius b, separated by a dielectric of permittivity ϵ.

Solution. The relation between charge and potential difference is given by Eq. (3-64), from which it follows that the capacitance becomes

$$C = \frac{2\pi\epsilon}{\log_e (b/a)} \tag{3-69}$$

Example 3-6.4. A capacitor bushing is a physical structure used to insulate a high-potential conductor at its passage through the grounded case of electrical equipment. It consists of alternate layers of dielectric and conducting foil, successive layers becoming progressively shorter in passing from the inner conductor to the mounting bracket on the tank, as illustrated in Fig. 3-22. It is required to design the lengths of the capacitor sections such that the maximum electric potential gradient in each section is the same.

Solution. Neglecting edge effects, the total charge appearing on each conducting layer is the same. Thus, to a first approximation, each layer is a separate

capacitor with equal and opposite charges appearing on the two conducting surfaces. But from Eq. (3-56), writing $\Lambda = Q/L_n$, where Q is the total charge and L_n is the length of the section of mean radius R_n,

$$E_n = \frac{Q}{2\pi\epsilon R_n L_n} \tag{3-70}$$

and it is seen that the field intensity can be held constant throughout the whole structure by holding the quantity $2\pi R_n L_n$, which is the area of the sheet, constant

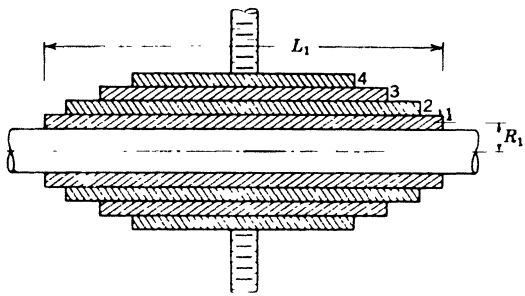


FIG. 3-22 A diagrammatic representation of a capacitor bushing.

as the radius increases. Thus, throughout the bushing it is required that

$$R_1L_1 = R_2L_2 = R_3L_3 = R_4L_4 = \text{const}$$

3-7. Systems of Conductors. The preceding analysis, which dealt with the charge-potential relations of a single conductor, and between two conductors, may be extended and generalized. Consider a system of n conductors which carry charges $Q_1, Q_2, \ldots, Q_n$, respectively, with consequent potentials $\phi_1, \ldots, \phi_n$, as shown in Fig. 3-23. What is desired is a method for evaluating the Q's in terms of the ϕ's and vice versa. Such a relation may be deduced from considerations of the potential integral, Eq. (2-41), which relates the charges and the potentials of the conductors.

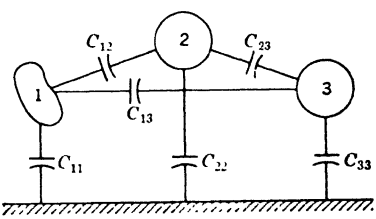


FIG. 3-23. A system of charged conductors.

Suppose that the first conductor is charged to a potential ϕ_1, with all other conductors being grounded. The charge on the first conductor will be $\beta_{11}\phi_1$, that induced on the second conductor will be $\beta_{21}\phi_1$, and that induced on the nth conductor will be $\beta_{n1}\phi_1$. If the second conductor is charged to a potential ϕ_2 and all other conductors are grounded, then the charge on the second conductor will be given by $\beta_{22}\phi_2$ and that induced on the nth conductor will be $\beta_{n2}\phi_2$. If, now, the conductors are charged to potentials $\phi_1, \ldots, \phi_n$, respectively, then the resultant charge on each

conductor is the superposition of the charges resulting from the successive application of each potential ϕ, as noted above. These resultant charges are denoted $Q_1, \ldots, Q_n$ and are related to the potentials by the following set of equations:

$$\begin{aligned} Q_1 &= \beta_{11}\phi_1 + \beta_{12}\phi_2 + \cdots + \beta_{1n}\phi_1 \\ Q_2 &= \beta_{21}\phi_1 + \beta_{22}\phi_2 + \cdots + \beta_{2n}\phi_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ Q_n &= \beta_{n1}\phi_1 + \beta_{n2}\phi_2 + \cdots + \beta_{nn}\phi_n \end{aligned} \tag{3-71}$$

The coefficients β are constants which depend on the shape, relative dimensions, and permittivities of the dielectric space and on the disposition of the conductors. That is, the β's are functions only of the geometry of the configuration. The diagonal terms of the array β_{11}, β_{22}, $\ldots$, β_{nn} are called *coefficients of capacitance*, and the other factors, β_{12}, β_{13}, $\ldots$ (with $\beta_{ij} = \beta_{ji}$), are called *coefficients of induction*. The fact that $\beta_{ij} = \beta_{ji}$ is expressed by saying that reciprocity applies for the β coefficients. As noted above, the β's are such that:

β_{ij} = quantity of electricity on ith conductor which will charge jth conductor to unity potential, when all other conductors are grounded

This consequence follows directly from Eq. (3-71). It should be observed, however, that the total capacitance between pairs of conductors in any given configuration is not directly identified with any one of the coefficients of capacitance or induction.

It is convenient to express Eqs. (3-71) in a form which permits of ready interpretation. Specifically, consider the first of these equations, and suppose that this is written in the following form:

$$Q_1 = (\beta_{11} + \beta_{12} + \cdots + \beta_{1n})\phi_1 - \beta_{12}(\phi_1 - \phi_2) - \cdots - \beta_{1n}(\phi_1 - \phi_n)$$

This may also be expressed as

$$Q_1 = C_{11}\phi_1 + C_{12}(\phi_1 - \phi_2) + \cdots + C_{1n}(\phi_1 - \phi_n)$$

where

$$\begin{aligned} C_{11} &= \beta_{11} + \beta_{12} + \beta_{13} + \cdots + \beta_{1n} \\ C_{12} &= -\beta_{12} \\ C_{13} &= -\beta_{13} \\ &\cdots\cdots\cdots \\ C_{1n} &= -\beta_{1n} \end{aligned} \tag{3-72}$$

Observe that $C_{11}\phi_1$ is the charge on capacitance C_{11} between 1 and ground, between which there is a potential difference ϕ_1. Similarly, $C_{12}(\phi_1 - \phi_2)$ is the charge on capacitance C_{12} between 1 and 2, across which there is a potential difference $\phi_1 - \phi_2$. A similar interpretation of the other terms

may be made. The relationship of this discussion to the system of conductors illustrated in Fig. 3-23 is evident.

It is possible to express the potentials as linear functions of the charges. This is done by solving the foregoing n equations for the potentials. Such a solution has the form

$$\begin{aligned} \phi_1 &= S_{11}Q_1 + S_{12}Q_2 + \cdots + S_{1n}Q_n \\ \phi_2 &= S_{21}Q_1 + S_{22}Q_2 + \cdots + S_{2n}Q_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \phi_n &= S_{n1}Q_1 + \cdots \qquad\qquad + S_{nn}Q_n \end{aligned} \tag{3-73}$$

where the S's are constants which depend only on the geometry of the configuration (since they are expressed in terms of the β's). They are called the *coefficients of potential.*

Example 3-7.1. Consider a system of two conductors separated by a dielectric. Find an expression for the capacitance of this assembly in terms of the β coefficients.

Solution. For a two-conductor system, $Q_2 = -Q_1$, whence

$$\begin{aligned} Q_1 &= \beta_{11}\phi_1 + \beta_{12}\phi_2 \\ -Q_1 &= \beta_{21}\phi_1 + \beta_{22}\phi_2 \end{aligned}$$

These equations are solved simultaneously for the potentials ϕ_1 and ϕ_2, to give

$$\phi_1 = \frac{\beta_{22} + \beta_{12}}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}} Q_1 \qquad \phi_2 = \frac{-(\beta_{11} + \beta_{21})}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}} Q_1$$

The potential difference V_{12} is

$$V_{12} = \phi_1 - \phi_2 = \frac{\beta_{11} + \beta_{12} + \beta_{21} + \beta_{22}}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}} Q_1$$

But since by definition of capacitance $Q_1 = CV_{12}$, then

$$C = \frac{\beta_{11}\beta_{12} - \beta_{12}^2}{\beta_{11} + 2\beta_{12} + \beta_{22}}$$

This shows that C is a complicated function of the coefficients of capacitance and induction.

Example 3-7.2. Calculate the capacitances associated with the three-phase transmission line illustrated in Fig. 3-24.

Solution. The coefficients of potential of Eq. (3-73) may be computed by noting that $S_{11}Q_1$ is the potential on conductor 1 due to charge Q_1 per unit length on 1, all other charges being zero. Similarly, $S_{21}Q_1$ specifies the potential on 2 due to charge Q_1 per unit length on 1, all other charges being zero.

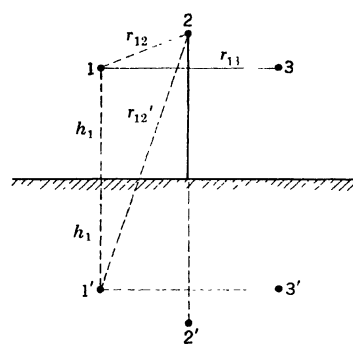


FIG. 3-24. A three-phase transmission line above the ground.

To carry out the calculations of the S coefficients, the effect of the ground is

first replaced by the appropriate choice of images, as illustrated in Fig. 3-24. Considering the charged line and its image, then

$S_{11}Q_1$ = potential on 1 due to $+Q_1$ at the image center of 1 (here taken to be the geometric center of the conductor), together with $-Q_1$ of the image line. This is, by Eq. (3-11),

$$= \frac{Q_1}{2\pi\epsilon_0} \ln \frac{2h_1}{a}$$

where h_1 is the height of the wire above the ground and a is its radius.

Similarly, it is noted that

$S_{21}Q_1$ = potential on 2 due to $+Q_1$ at the image center of 1, together with $-Q_1$ of the image line. This is

$$= -\left(\frac{Q_1}{2\pi\epsilon_0} \ln r_{12} - \frac{Q_1}{2\pi\epsilon_0} \ln r'_{12}\right) = \frac{Q_1}{2\pi\epsilon_0} \ln \frac{r'_{12}}{r_{12}}$$

where r_{12} is the distance between image centers of conductors 1 and 2 and r'_{12} is the distance from the image 1′ to 2.

From the foregoing, it follows that the coefficients of potential are

$$S_{11} = \frac{1}{2\pi\epsilon_0} \ln \frac{2h_1}{a}$$

$$S_{ij} = \frac{1}{2\pi\epsilon_0} \ln \frac{r'_{ij}}{r_{ij}}$$

which may therefore be calculated for specified dimensions. Thus, for the present system,

$$\begin{aligned} \phi_1 &= S_{11}Q_1 + S_{12}Q_2 + S_{13}Q_3 \\ \phi_2 &= S_{21}Q_1 + S_{22}Q_2 + S_{23}Q_3 \\ \phi_3 &= S_{31}Q_1 + S_{32}Q_2 + S_{33}Q_3 \end{aligned}$$

where each of the S coefficients is now known.

The foregoing set of equations is solved for the various values of Q, which become, using determinantal techniques,

$$\begin{aligned} Q_1 &= \beta_{11}\phi_1 + \beta_{12}\phi_2 + \beta_{13}\phi_3 \\ Q_2 &= \beta_{21}\phi_1 + \beta_{22}\phi_2 + \beta_{23}\phi_3 \\ Q_3 &= \beta_{31}\phi_1 + \beta_{32}\phi_2 + \beta_{33}\phi_3 \end{aligned}$$

where $Q_1 = \dfrac{\Delta_1}{\Delta}$ $Q_2 = \dfrac{\Delta_2}{\Delta}$ $Q_3 = \dfrac{\Delta_3}{\Delta}$ $\Delta_1 = \begin{vmatrix} \phi_1 & S_{12} & S_{13} \\ \phi_2 & S_{22} & S_{23} \\ \phi_3 & S_{23} & S_{33} \end{vmatrix}$

$$\Delta = \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} S_{11} & \phi_1 & S_{13} \\ S_{21} & \phi_2 & S_{23} \\ S_{31} & \phi_3 & S_{33} \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} S_{11} & S_{12} & \phi_1 \\ S_{21} & S_{22} & \phi_2 \\ S_{31} & S_{23} & \phi_3 \end{vmatrix}$$

These equations are conveniently written in the form

$$\begin{aligned} Q_1 &= (\beta_{11} + \beta_{12} + \beta_{13})\phi_1 - \beta_{12}(\phi_1 - \phi_2) - \beta_{13}(\phi_1 - \phi_3) \\ Q_2 &= -\beta_{21}(\phi_2 - \phi_1) + (\beta_{22} + \beta_{21} + \beta_{23})\phi_2 - \beta_{23}(\phi_2 - \phi_3) \\ Q_3 &= -\beta_{31}(\phi_3 - \phi_1) - \beta_{32}(\phi_3 - \phi_2) + (\beta_{31} + \beta_{32} + \beta_{31})\phi_1 \end{aligned}$$

from which it follows that the capacitances to ground are

$$C_{11} = \beta_{11} + \beta_{12} + \beta_{13}$$
$$C_{22} = \beta_{21} + \beta_{22} + \beta_{23}$$
$$C_{33} = \beta_{31} + \beta_{32} + \beta_{33}$$

and the capacitances between conductors are

$$C_{12} = -\beta_{12}$$
$$C_{13} = -\beta_{13}$$
$$C_{23} = -\beta_{23}$$

Example 3-7.3. Consider the configuration of conductors illustrated in Fig. 3-25, which shows one conductor shielded from two other conductors by a third conductor. It is required to show that the charge on Q_1 is independent of the potentials on conductors outside conductor 2.

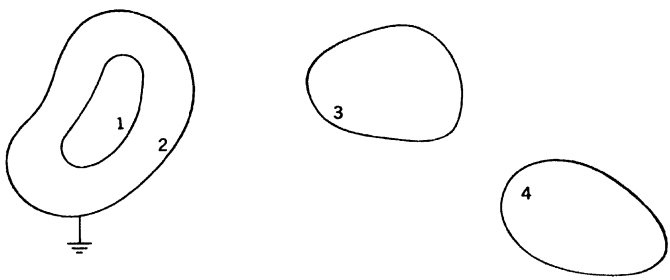


FIG. 3-25. To illustrate electrical shielding.

Solution. Initially, let $Q_1 = 0$. Also, it is assumed that conductor 2 is grounded, as shown, whence $\phi_2 = 0$. Therefore, $\phi_1 = 0$, since conductor 1 is situated within conductor 2. Therefore,

$$0 = 0 + \beta_{13}\phi_3 + \beta_{14}\phi_4$$

But this equation must be satisfied irrespective of the values ϕ_3 and ϕ_4. For this condition to apply, it is required that

$$\beta_{13} = \beta_{14} = 0$$

Hence,

$$Q_1 = \beta_{11}\phi_1 + \beta_{12}\phi_2$$

is independent of the potentials and conductors outside conductor 2.

This problem shows the following: If a charge is placed on conductor 1 with conductor 2 ungrounded, then, by Gauss' law, this charge produces a field outside of 2. Correspondingly, if conductor 2 is grounded, then no field exists outside of 2. If a charge exists outside of conductor 2, then the inner conductor is without charge whether or not the outer conductor is grounded.

Example 3-7.4. Find the partial capacitances of the two-conductor cable illustrated in Fig. 3-26.

Solution. It is observed that, from symmetry,

$$\beta_{11} = \beta_{22}$$
$$\beta_{10} = \beta_{20}$$

Note that, owing to the shielding of conductors 1 and 2 by the sheath 0, any con-

ductors outside the sheath would have no influence on conductors 1 and 2. The general equations become, for this example,

$$Q_0 = \beta_{00}\phi_0 + \beta_{01}\phi_1 + \beta_{01}\phi_2$$
$$Q_1 = \beta_{01}\phi_0 + \beta_{11}\phi_1 + \beta_{12}\phi_2$$
$$Q_2 = \beta_{01}\phi_0 + \beta_{12}\phi_1 + \beta_{11}\phi_2$$

The solution of this problem depends on our ability to find a number of relations among the β's. This is accomplished by considering the effect of making connections among the different conductors, and then finding the capacitances corresponding to each connection.

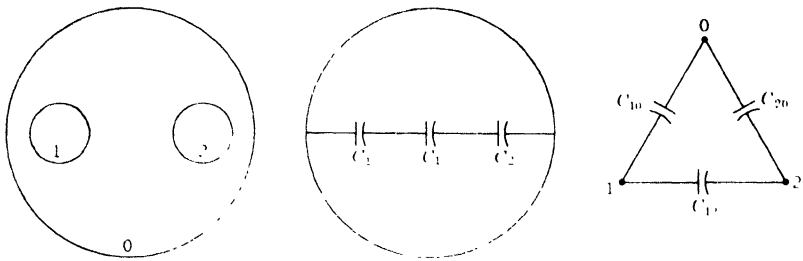


FIG. 3-26. A two-conductor cable and the partial capacitances

Initially, it is noted that since conductors 1 and 2 are screened from the outside by the sheath, then if no charge is introduced,

$$Q_1 = Q_2 = 0$$

whence $\phi_0 = \phi_1 = \phi_2$

Therefore, for conductor 1,

$$0 = \beta_{01} + \beta_{11} + \beta_{12}$$

from which it follows that

$$\beta_{01} = -(\beta_{11} + \beta_{12})$$

which means that, if two coefficients are known, the other is determined.

a. To find the capacitance between conductors 1 and 2, when conductor 0 is grounded (and this is precisely the problem considered in Example 3-2.4), let $Q_1 = Q$; then, $Q_2 = -Q$, $\phi_0 = 0$. Hence,

$$Q = \beta_{01}\phi_0 + \beta_{11}\phi_1 + \beta_{12}\phi_2$$
$$-Q = \beta_{01}\phi_0 + \beta_{12}\phi_1 + \beta_{11}\phi_2$$

Subtracting these expressions yields

$$2Q = (\beta_{11} - \beta_{12})(\phi_1 - \phi_2)$$

Hence, from the definition of capacitance,

$$C_a = \frac{Q}{Q_1 - Q_2} = \tfrac{1}{2}(\beta_{11} - \beta_{12})$$

This is given by Eq. (3-29),

$$C_a = \frac{\pi\epsilon}{\ln\left(\dfrac{R_2^2 - s^2 2s}{R_2^2 + s^2 R}\right)}$$

b. To find the capacitance between the sheath 0 and conductors 1 and 2 together, let $Q_0 = Q$ and $Q_1 = -Q/2$; $Q_2 = -Q/2$; $\phi_0 = 0$; also, $\phi_1 = \phi_2$. Then we have

$$Q = 2\beta_{01}\phi_1$$

and

$$-\frac{Q}{2} = (\beta_{11} + \beta_{12})\phi_1$$

From the second of these expressions,

$$C_b = -2(\beta_{11} + \beta_{12})$$

c. To find the capacitance between the sheath 0 and conductor 1 when conductor 2 is insulated, let $Q_0 = Q$, $Q_1 = -Q$, $Q_2 = 0$. Also, choose $\phi_0 = 0$. Then,

$$\begin{aligned} Q &= \beta_{01}\phi_1 + \beta_{01}\phi_2 \\ -Q &= \beta_{11}\phi_1 + \beta_{12}\phi_2 \\ 0 &= \beta_{12}\phi_1 + \beta_{11}\phi_2 \end{aligned}$$

From the last of this group of equations,

$$\phi_2 = -\frac{\beta_{12}}{\beta_{11}}\phi_1$$

The second of this group becomes, when the relation between ϕ_2 and ϕ_1 is used,

$$-Q = \beta_{11}\phi_1 - \frac{\beta_{12}^2}{\beta_{11}}\phi_1$$

Hence, the capacitance is given by the expression

$$C_c = \frac{Q}{\phi_1} = \frac{\beta_{12}^2}{\beta_{11}} - \beta_{11}$$

Attention is called to the fact that all the desired capacitances are thus expressible in terms of two of the β coefficients. For specific values for any specified multiconductor system of this type, the appropriate two coefficients must be calculated or must be measured.

PROBLEMS

3-1. *a.* Show that in rectangular coordinates

$$\mathbf{A} \times \mathbf{B} = (B_z A_y - B_y A_z)\mathbf{i} + (B_x A_z - B_z A_x)\mathbf{j} + (B_y A_x - B_x A_y)\mathbf{k}$$

b. Show also that this may be conveniently expressed in determinantal form,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{A}_x & \mathbf{A}_y & \mathbf{A}_z \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix}$$

3-2. A scalar triple product is written as $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$. Show by expansion that a cyclic interchange of symbols is possible without altering the value of the resulting quantity. That is, show that

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$$

3-3. Show by direct expansion that the triple vector product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

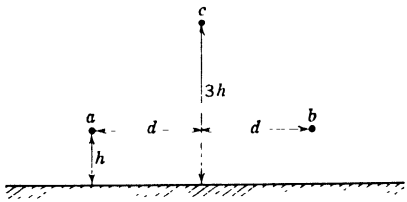
3-4. Show that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \cdot \mathbf{c}\mathbf{b} \cdot \mathbf{d} - \mathbf{a} \cdot \mathbf{d}\mathbf{b} \cdot \mathbf{c}$$

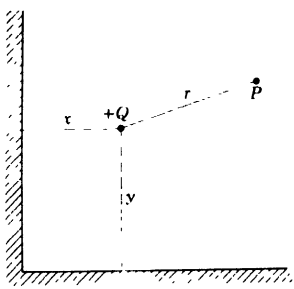
3-5. How much work must be done in moving the charge q under the conditions specified:

a. From position a to position b, as shown in the figure?

b. From position a to position c?



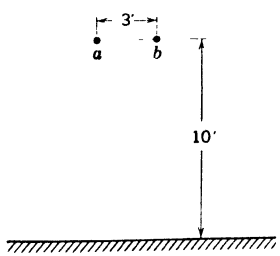
3-6. Find the potential at any point in the region shown, for a charge Q in the position shown. Note the similarity with Prob. 1-21.



3-7. Two #4 wires (dia. = 204.31 mils) are situated as shown with respect to ground. If the difference of potential V_{ab} is 1,600 volts between wires:

a. Determine Λ_a, Λ_b, Λ_g.

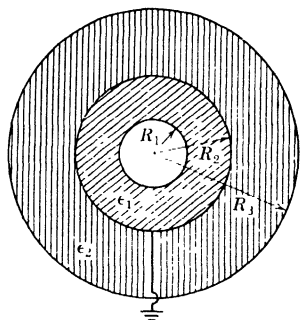
b. What is the potential of a with respect to ground?



3-8. A charge of 2.5 μcoulombs is embedded at the center of a dielectric sphere of $\epsilon_r = 2.5$ and radius 10 cm.

a. Find the electric field intensity at $r = 5$ cm, $r = 12$ cm.
b. Find the electric induction D at these same points.
c. What is the flux of electric force through a shell at $r = 5$?
d. What is the flux of electric induction through a shell at $r = 5$?

3-9. An electrostatic system consists of three coaxial cylinders separated by dielectrics of permittivities ϵ_1 and ϵ_2, as illustrated. A charge $+Q_1$ coulombs/m is placed on the inner conductor, and a charge $-Q_2$ coulombs/m is placed on the outer conductor. The middle conductor is grounded. Sketch a curve showing the potential distribution across the entire section of this multiple coaxial system. Give expressions for the potential variations.



3-10. At the boundary between glass $\epsilon_r = 4$ and air, the lines of electric induction make an angle of 32° with the normal to the interface. If the density of such lines in air is 0.25 μcoulomb m^2:

a. Determine the angle of D in the glass with the normal to the surface.
b. Determine the value of D in the glass.

3-11. A coaxial cable is designed for 7,500 volts. A rubber dielectric having an allowable field strength of 5 kv/mm is used. If the inner conductor is No. 0 solid wire having a radius of 162.4 mils, what is the minimum thickness of dielectric that may be used? $\epsilon_r = 3$.

3-12. A parallel-plate capacitor consists of two metallic plates of area A and separation in air equal to t_1. It is charged initially to a potential difference V, and then the potential source is removed. If the plate separation is reduced to t_2, to what value has the potential difference of the plates changed?

3-13. What is the largest charge density that can be retained by a metal sphere 1 cm in radius in air?

3-14. What is the largest charge density that can be retained on the surface of any conductor in air?

3-15. A parallel-plate capacitor of area A and separation d has a thin coat of varnish of thickness t applied to the inner surface of each plate. Find an expression for the increase in capacitance due to the varnish.

3-16. The permittivity of the dielectric material between the plates of a parallel-plate capacitor varies uniformly from ϵ_1 at one plate to ϵ_2 at the other plate. Show that the capacitance is given by

$$C = \frac{A}{d} \frac{\epsilon_2 - \epsilon_1}{\ln (\epsilon_2 / \epsilon_1)}$$

3-17. A capacitor consists of a stack of 42 sheets of aluminum foil 1 in. by 1 in. 2 mils thick, which are separated from each other by thin sheets of polystyrene, $\epsilon_r = 2.6$, 3 mils thick. Alternate foils are connected together to one terminal, the remaining foils being connected to the second terminal. What is the capacitance? If the dielectric strength of polystyrene is 20,000 kv/m, what is the peak potential rating of the capacitor, allowing a safety factor of 1.5?

3-18. A variable air capacitor consists of a group of 13 parallel semicircular plates 6 cm in diameter. The alternate group of seven plates is connected together into a fixed group, the remaining group of six plates being mounted on a shaft to permit rotation with respect to the fixed set of plates. When the movable plates are fully meshed with the stationary plates, the separation between plates is 1 mm. Neglecting fringing:

a. Deduce the maximum capacitance.

b. Deduce what variation of capacitance with angle of rotation exists for such a capacitor.

3-19. A parallel-plate capacitor, which consists of a stack of parallel plates 3 mm apart, with alternate plates being connected together to one terminal, has an effective area of 25 m^2. Suppose that it is charged to 2,500 volts and then immersed in a liquid dielectric. If the potential difference now becomes 450 volts, what is the relative dielectric constant of the liquid?

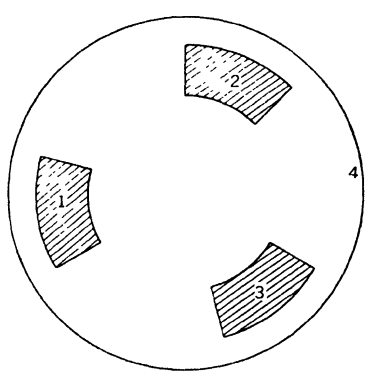
3-20. Calculate the capacitance per 1,000 ft of circuit of two parallel No. 6 wires (radius 81 mils) which are 12 in. apart.

3-21. A three-conductor cable is illustrated in the accompanying diagram. Deduce expressions for the relation among capacitances between conductors and/or sheath:

a. When the sheath is at unit potential, with all conductors insulated

b. Between 1 and 2, when 3 and 4 are insulated

c. Between sheath and 1 and 2, when 3 is insulated



3-22. The capacitance between conductors of a certain standard twin cable is found to be 0.345 μf/mile. If the permittivity of the insulation is 2.4 and its resistivity is 1.5×10^9 megohms/cm^3, what is the insulation resistance between these conductors, in megohms per mile?

3-23. If the capacitor bushing discussed in Example 3-6.4 is to be designed for a stranded center conductor of 500,000 circ mils area (2.06 cm outside diameter), with dielectric sheets 7.5 mm thick and metal sheaths 1.5 mm for a four-

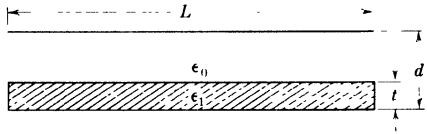
section bushing having a 12-cm shortest section, what must the other lengths be for the same breakdown strength in each section?

3-24. Consider a parallel-plate capacitor, the plates of which are square and the dimensions of which are large compared with their separation. A uniform dielectric is inserted, as shown.

a. Find an expression for the capacitance.

b. Suppose, now, that a conducting foil is placed on the interface between the dielectric and the air. Find an expression for the capacitance in this case.

c. Discuss the results obtained in (*b*) in the light of the discontinuity of the **D** lines.

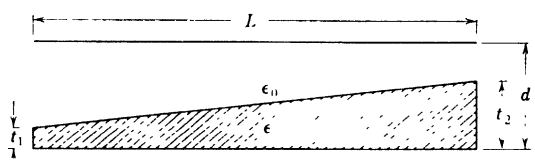


3-25. Consider a parallel-plate capacitor, the plates of which are large compared with their separation, as in Prob. 3-24. A dielectric having a tapered thickness, as illustrated in the accompanying figure, is inserted between the plates.

a. Find an expression for the capacitance.

b. Suppose that a conducting foil is placed on the interface between the dielectric and the air. Find an expression for the capacitance, and compare it with that obtained in (*a*).

c. Discuss the results obtained in (*b*).



Hint: Assume that the taper is small and divide the capacitor into narrow strips in parallel, each of which is assumed to have a dielectric of uniform thickness.

CHAPTER 4

ENERGY AND MECHANICAL FORCES IN THE ELECTRIC FIELD

(Magnetic field).

If an electric field is considered as being developed by moving a group of charges from infinity to their final positions in space, then, owing to the mutual forces on the charges, work must be done in establishing the specified distribution of electric charges. Consequently, an electrostatic potential energy is associated with the charge distribution. If a dielectric medium is introduced into an electric field, work is done in establishing the polarization, thereby changing the electrostatic potential energy of the system. These and related questions are now to be examined.

4-1. The Electrostatic Potential Energy Associated with a Charge Distribution. The total potential energy of the system of charged bodies may be calculated in terms of their charges and potentials by calculating the work done in bringing the charges from infinity, until a specified charge distribution is attained.

It has already been shown that the potential at a point due to a volume distribution of charge is given by the potential integral, Eq (2-41),

$$\phi = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\rho}{r} \, d\tau \tag{4-1}$$

which is taken over all the charged bodies in the field. Of course, if a surface distribution of charge exists, the effects of this must also be taken into account, as this also contributes to the potential at any point. The total expression for the potential at a point is, as before,

$$\phi = \frac{1}{4\pi\epsilon_0} \left(\int_\tau \frac{\rho}{r} \, d\tau + \int_{S'} \frac{\sigma}{r} \, dS \right) \tag{4-2}$$

Of course, if *all* the charges are reduced in the ratio $n/1$, then the potential will be reduced by a factor of n.

We wish to examine the work done in bringing up the charges from infinity, $1/n$ of the total of each element of charge at a time, until the specified charge distribution is established. Suppose that n is a very large number, and suppose initially that all the charges are at an infinite distance from each other. If ϕ is the final potential at any point, then.

during the indicated operation, the potential at any point will change from 0 to ϕ/n. Thus, the work done in bringing up the element of charge $(1/n)\rho\, d\tau$ will lie between 0 and $(\phi/n)(\rho/n\, d\tau)$, whence the work done in bringing up $1/n$th of all the charges will lie between 0 and $(1/n^2)\int_\tau \phi\rho\, d\tau$.

Consider now the second nth part of all the charges. When these are brought up from infinity to their final positions, this will increase the potential at any point from ϕ/n to $2\phi/n$. The work done in this second stage will lie between $(1/n^2)\int_\tau \phi\rho\, d\tau$ and $(2/n^2)\int_\tau \phi\rho\, d\tau$. If this process is continued, the work done in the sth stage will lie between $[(s-1)/n^2]\int_\tau \phi\rho\, d\tau$ and $(s/n^2)\int_\tau \phi\rho\, d\tau$.

The total work done in bringing up the whole of the charges from a state of infinite separation to the final configuration will lie between

$$\frac{1}{n^2}[0 + 1 + 2 + \cdots + (n-1)]\int_\tau \phi\rho\, d\tau \qquad \text{and}$$

$$\frac{1}{n^2}(1 + 2 + 3 + \cdots + n)\int_\tau \phi\rho\, d\tau$$

However, the sums that appear in these expressions are given by

$$\sum_0^{n-1} n = \frac{n(n-1)}{2}$$

and

$$\sum_1^{n} n = \frac{(n+1)n}{2}$$

Thus, the total potential energy of the system lies between

$$\frac{n(n-1)}{2n^2}\int_\tau \phi\rho\, d\tau \qquad \text{and} \qquad \frac{n(n+1)}{2n^2}\int_\tau \phi\rho\, d\tau$$

But when n is increased indefinitely, each of these expressions reduces to the common value $\frac{1}{2}\int_\tau \phi\rho\, d\tau$. Hence, the electrostatic potential energy associated with the volume distribution of charges is given by

$$W_e = \frac{1}{2}\int_\tau \phi\rho\, d\tau \tag{4-3}$$

Of course, if there are surface charges present, the electrostatic potential energy associated with the surface distribution of charges must be added

to Eq. (4-3). Hence, the complete expression for the total potential energy associated with any system of charges is

$$W_e = \frac{1}{2}\int_\tau \phi\rho\, d\tau + \frac{1}{2}\int_{S'} \phi\sigma\, dS \tag{4-4}$$

This expression may be transformed to a more convenient form. Consider, initially, the first term only. This is combined with Poisson's equation, Eq. (2-69), thus,

$$W_e = \frac{1}{2}\int_\tau \phi\rho\, d\tau = -\frac{\epsilon_0}{2}\int_\tau \phi\nabla^2\phi\, d\tau$$

which may be expressed in the form

$$W_e = \frac{\epsilon_0}{2}\int_\tau \phi\nabla\cdot\mathbf{E}\, d\tau \tag{4-5}$$

Attention is now called to the following vector relationship, which may be verified by direct expansion:

$$\nabla\cdot\phi\mathbf{E} = \nabla\phi\cdot\mathbf{E} + \phi\nabla\cdot\mathbf{E} \tag{4-6}$$

Equation (4-5) may now be written as

$$W_e = \frac{\epsilon_0}{2}\int_\tau \nabla\cdot\phi\mathbf{E}\, d\tau - \frac{\epsilon_0}{2}\int_\tau \nabla\phi\cdot\mathbf{E}\, d\tau$$

Now the divergence theorem [see Eq. (2-63)] is applied to the first term on the right of the equation. The result is

$$W_e = \frac{\epsilon_0}{2}\int_S \mathbf{n}\cdot\phi\mathbf{E}\, dS - \frac{\epsilon_0}{2}\int_\tau \nabla\phi\cdot\mathbf{E}\, d\tau$$

where S is the surface that defines the volume τ. The total energy is then given by the expression

$$W_e = \frac{\epsilon_0}{2}\int_\tau \mathbf{E}\cdot\mathbf{E}\, d\tau + \int_S \frac{\epsilon_0}{2}\phi\mathbf{n}\cdot\mathbf{E}\, dS + \int_{S'} \tfrac{1}{2}\phi\sigma\, dS \tag{4-7}$$

Suppose, now, in order to obtain the total energy in all space, that the integration is carried out throughout *all* space. In such an integration, the second integral becomes zero, and the third integral becomes part of the space integral. Specifically, note that the integrand of the second integral is of the order $1/r^3$, and since dS is of the order r^2, then the integral vanishes as $0(1/r)$. The third integral is more complicated to handle, but if it is assumed that the surface-charge distributions are limiting cases of volume-charge distributions, then in the integration over all

space, these surface-charge distributions are contained within the total volume of integration. Thus finally

$$W_e = \frac{\epsilon_0}{2} \int_{\text{All space}} \mathbf{E} \cdot \mathbf{E}\, d\tau \qquad \text{joules} \tag{4-8}$$

for a finite charge distribution. This result may also be written in the following form:

$$W_e = \frac{1}{2} \int_{\text{All space}} \mathbf{E} \cdot \mathbf{D}\, d\tau \qquad \text{joules} \tag{4-9}$$

A word is in order concerning this result, particularly when it is compared with Eq. (4-4). Equation (4-4) seems to imply that the energy is distributed only in the region of charge density, whereas Eq. (4-8) seems to show that the energy resides wherever there is an electric field. In

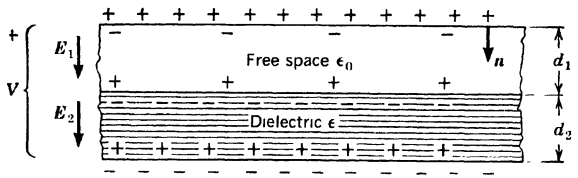


FIG. 4-1. A dielectric slab of permittivity ϵ in a parallel-plate capacitor.

fact, Eq. (4-8) shows that the total energy in an electric field of whatever nature may be regarded as being distributed throughout the entire field with a density at any point equal to

$$w_e = \frac{\epsilon_0}{2} E^2 \qquad \text{joules/unit volume} \tag{4-10}$$

Such a statement involves the assumption that electrostatic energy is something that can be spatially distributed. Clearly, while the foregoing equation can suggest such a hypothesis, it in no way demands it. In fact, we must think of this purely as a definition.

4-2. The Energy Density in Restricted Regions. It is desired to show that the results obtained in the last section are generally valid even in regions containing dielectrics. Consider, therefore, the parallel-plate capacitor with a slab of dielectric between the plates, as illustrated in Fig. 4-1 (fringing is neglected). For an applied potential V between the plates, there will be a surface-charge density σ on the plates. But, by Eq. (3-6), the electric induction between the plates is

$$\mathbf{D} = \sigma \mathbf{n} \tag{4-11}$$

so that the electric field intensity is given by

$$\mathbf{E}_1 = \frac{\mathbf{D}}{\epsilon_0} = \frac{\sigma}{\epsilon_0} \qquad \text{for free space}$$
$$\mathbf{E}_2 = \frac{\mathbf{D}}{\epsilon} = \frac{\sigma}{\epsilon} \qquad \text{for dielectric} \tag{4-12}$$

The potential difference between the plates, which is given by the line integral of the electric intensity between the plates, is

$$V = E_1 d_1 + E_2 d_2$$

or

$$V = \sigma\left(\frac{d_1}{\epsilon_0} + \frac{d_2}{\epsilon}\right) \tag{4-13}$$

It is now desired to calculate the amount of work that was done in establishing the potential difference across the capacitor plates. We consider the process of charging the capacitor from zero charge, when the initial potential difference would be zero, by bringing small amounts of charge across the dielectric space. By following the same reasoning as in Sec. 4-1, the total work done when the capacitor is finally charged will be

$$W_e = \frac{1}{2}\int_S V\sigma \, dS \tag{4-14}$$

where the potential ϕ in Eq. (4-4) has been replaced by V, since the charge is carried through this difference of potential. This is now combined with Eq. (4-13) for the known expression for the potential difference, and there results

$$W_e = \frac{1}{2}\int_S \sigma^2\left(\frac{d_1}{\epsilon_0} + \frac{d_2}{\epsilon}\right) dS \tag{4-15}$$

This expression may be written in the following form, if it is noted that $d(dS) = d\tau$ is the volume of the space over the specified region.

$$W_e = \frac{1}{2}\int_{\text{Free space}} \frac{\sigma^2}{\epsilon_0} d\tau + \frac{1}{2}\int_{\text{Dielectric}} \frac{\sigma^2}{\epsilon} d\tau \tag{4-16}$$

Clearly, the total energy is composed of two terms, that for the free space and that for the dielectric. This is just a special case of Eq. (4-9), which is evident by writing this in the form

$$W_e = \frac{1}{2}\int_{\text{Free space}} \mathbf{E}_1 \cdot \mathbf{D} \, d\tau + \frac{1}{2}\int_{\text{Dielectric}} \mathbf{E}_2 \cdot \mathbf{D} \, d\tau \tag{4-17}$$

which is simply

$$W_e = \frac{1}{2}\int_{\text{All space}} \mathbf{E} \cdot \mathbf{D} \, d\tau \tag{4-18}$$

This development indicates the need for care in applying Eq. (4-9) and shows that the general equation must be properly interpreted in the various regions which may constitute the total space within which the energy is distributed.

For the case of the parallel-plate capacitor, Eq. (4-14) may be written in a convenient form involving the capacitance C. Thus, again assuming that there is no fringing, σ is a constant. For fixed applied potential V, Eq. (4-14) may be written as

$$W_e = \tfrac{1}{2} V \sigma A \tag{4-19}$$

where A is the total area of the plates. But, by definition, the capacitance of a capacitor is

$$C = \frac{Q}{V} = \frac{\sigma A}{V} \tag{4-20}$$

so that, by combining this with the above,

$$W_e = \tfrac{1}{2} C V^2 \tag{4-21}$$

It is of interest to note that the capacitance of the parallel-plate capacitor, which is given by Eq. (4-20), becomes, by Eq. (4-13),

$$C = \frac{A}{d_1/\epsilon_0 + d_2/\epsilon} \tag{4-22}$$

4-3. Pressure in an Electric Field. Consider a simple parallel-plate capacitor, such as that illustrated in Fig. 3-21. It appears reasonable, since one plate is positively charged and the second plate is negatively charged, that there should be a force of attraction between the plates tending to pull them together. An expression for the force can be readily obtained by the application of the principle of virtual work. Suppose, therefore, that the region between the plates is characterized by an electric field intensity $\mathbf{E}$ and a permittivity ϵ. The total energy in the field is given by Eq. (4-18). It is desired, now, to calculate the change in the total energy that would exist were the capacitor plates to be separated by an additional amount dl. The work that would be required to effect the additional separation is equal to the change in energy in the field, and this, in turn, must be related to the product of the force and the distance. That is, the following expression relates the work with the change in energy:

$$dW_e = (\tfrac{1}{2}\mathbf{E} \cdot \mathbf{D}) A \, dl = \mathbf{f} \cdot d\mathbf{l} \tag{4-23}$$

It follows from this that the force acting over the area A is normal to the plates in the direction of $\mathbf{E}$ and of magnitude $\frac{1}{2}(\mathbf{E} \cdot \mathbf{D})A$. But the force per unit area is pressure, and hence the pressure in the electric field is

$$p = \frac{f}{A} = \tfrac{1}{2}\mathbf{E} \cdot \mathbf{D} \qquad \text{newtons/m}^2 \tag{4-24}$$

Attention is called to the fact that, although this pressure exists at any point in an electric field, it is manifest only at two types of surfaces: (1) at a conductor-dielectric boundary, and (2) at the boundary between two dielectrics of different permittivities. These two situations are examined in somewhat greater detail.

1. At a conductor-dielectric boundary, the pressure is in a direction to draw the conductor into the dielectric. The result follows directly from the discussion above that the force between the charged plates tends to draw them together, and hence into the dielectric. The pressure of the conductor on the dielectric is that given by Eq. (4-24), which may be written in the several forms

$$p = {}^1{}_2\mathbf{E} \cdot \mathbf{D} = \frac{1}{2}\frac{D^2}{\epsilon} = {}^1{}_2\epsilon E^2 = \frac{1}{2}\frac{\sigma^2}{\epsilon} \tag{4-25}$$

2. At a dielectric-dielectric boundary, the pressure is in a direction to draw the dielectric of higher permittivity into the dielectric of lower

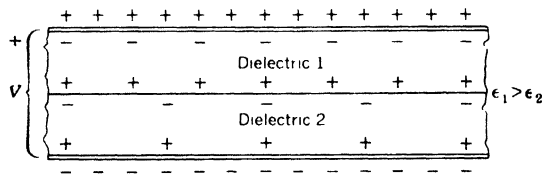


FIG. 4-2 To examine the net pressure at a boundary between two dielectrics

permittivity. In this connection, consider Fig. 4-2, which illustrates the system now under survey. For this case, the expression for the pressure would be determined from Eq. (4-17), where the region previously designated as free space would be called dielectric 1 and the region previously designated as dielectric would be called dielectric 2. The expression for pressure would be obtained as above, and is readily shown to be

$$p = {}^1{}_2\mathbf{E}_2 \cdot \mathbf{D} - {}^1{}_2\mathbf{E}_1 \cdot \mathbf{D}$$

In obtaining this expression, it is assumed that dielectric 2 has increased and dielectric 1 has decreased, by the infinitesimal amount dl, in the development. This expression may be written in the form

$$p = \frac{1}{2}\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right) D^2 \tag{4-26}$$

which is simply the difference of the two pressures.

Note that if $\epsilon_1 > \epsilon_2$ the resulting excess of bound charges adjacent to the boundary between the dielectrics is as illustrated in Fig. 4-2 and tends to cause a force on the boundary toward the conductor of negative charge.

That is, the net pressure seeks to force the dielectric of higher permittivity into the dielectric of lower permittivity.

If the electric field intensities and the displacement flux densities are not normal to the boundary surface, then a somewhat more general consideration of the pressure conditions must be undertaken. The procedure parallels that above, but the results are somewhat more involved. Refer to Fig. 4-3, which illustrates the situation that is assumed to exist. It is desired to find the resulting force components which act on an elementary area dS in the interface and to verify the consistency of the assumptions.

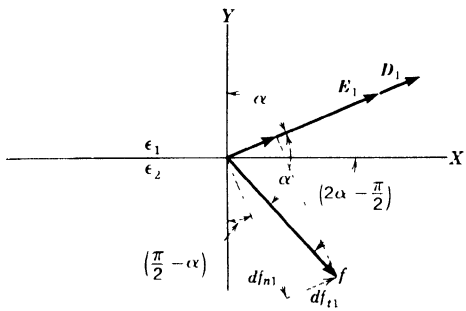


FIG. 4-3. The interface between different dielectrics.

First consider the situation at the interface in dielectric 1. The force is resolved into components along and normal to E_1. These are

$$\begin{aligned} df_{t1} &= p_1\, dS \cos\alpha \\ df_{n1} &= p_1\, dS \sin\alpha \end{aligned} \tag{4-27a}$$

where, of course,

$$p_1 = {}^1{}_2\mathbf{E}_1 \cdot \mathbf{D}_1 = {}^1{}_2 E_1 D_1 \tag{4-27b}$$

Together df_{t1} and df_{n1} completely define the force df, which is seen to be $p_1\, dS$. Therefore, the X and Y components of df become

$$\begin{aligned} df_{x1} &= df \cos\left(2\alpha - \frac{\pi}{2}\right) = df \sin 2\alpha \\ &= 2p_1\, dS \sin\alpha \cos\alpha = E_1 D_1 \sin\alpha \cos\alpha\, dS \end{aligned}$$

and

$$\begin{aligned} df_{y1} &= df \sin\left(2\alpha - \frac{\pi}{2}\right) = -df \cos 2\alpha \\ &= p_1\, dS\, (\sin^2\alpha - \cos^2\alpha) = \frac{E_1 D_1}{2} (\sin^2\alpha - \cos^2\alpha)\, dS \end{aligned}$$

These components are written as

$$\begin{aligned} df_{x1} &= (D_1 \cos\alpha)(E_1 \sin\alpha)\, dS = \epsilon_1 E_{n1} E_{t1}\, dS \\ df_{y1} &= {}^1{}_2\epsilon_1 (E_1^2 \sin^2\alpha - E_1^2 \cos^2\alpha)\, dS = {}^1{}_2\epsilon_1 (E_{t1}^2 - E_{n1}^2)\, dS \end{aligned} \tag{4-28}$$

A corresponding set of expressions are obtained for dielectric 2. But, from the requisite boundary conditions, it follows that

$$\epsilon_2 E_{n2} = \epsilon_1 E_{n1}$$
$$E_{t2} = E_{t1}$$

so that the expressions for dielectric 2 may be written in terms of the field vectors of dielectric 1, thus:

$$df_{x2} = \epsilon_2 E_{n2} E_{t2}\, dS = \epsilon_2 \frac{\epsilon_1}{\epsilon_2} E_{n1} E_{t1}\, dS$$

or
$$df_{x2} = \epsilon_1 E_{n1} E_{t1}\, dS$$

Also
$$df_{z2} = {}^1{}_2\epsilon_2 (E_{t2}^2 - E_{n2}^2)\, dS = {}^1{}_2\epsilon_2 \left(E_{t1}^2 - \frac{\epsilon_1^2}{\epsilon_2^2} E_{n1}^2 \right) dS \tag{4-29}$$

The resultant force is then the difference between the forces on dielectric 1 and dielectric 2. Observe first that the components along the surface of the dielectrics cancel each other; so there is no tendency for the dielectrics to move along the interface. The resultant force tending to force

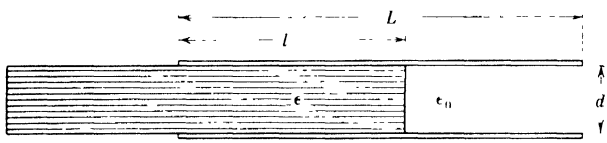


FIG. 4-4. A slab of dielectric partly introduced between capacitor plates.

one dielectric into the other is $df_{y1} - df_{y2}$ over the area dS. The resulting pressure is then

$$p = \frac{df_{y1} - df_{y2}}{dS} = {}^1{}_2(\epsilon_1 - \epsilon_2) E_{t1}^2 + {}^1{}_2 E_{n1}^2 \left(\frac{\epsilon_1^2}{\epsilon_2} - \epsilon_1 \right)$$
$$= {}^1{}_2(\epsilon_1 - \epsilon_2) E_{t1}^2 + {}^1{}_2 \epsilon_1^2 E_{n1}^2 \left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1} \right)$$

or, finally,
$$p = {}^1{}_2(\epsilon_1 - \epsilon_2) E_{t1}^2 + \frac{1}{2} \left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1} \right) D_{n1}^2 \tag{4-30}$$

This expression reduces, in the case where the electric field is normal to the boundary, to Eq. (4-26), as it should.

Example 4-3.1. A parallel-plate capacitor comprising two square plates, L m on a side, separated by a distance d m, is connected to a constant d-c supply. A dielectric slab of width L and thickness d is inserted between the plates, as shown in Fig. 4-4. Neglecting edge effects, find expressions for the following:

a. The capacitance of the capacitor
b. The total energy stored in the capacitor
c. The force tending to draw the dielectric slab into the capacitor

Solution. The solution of this problem proceeds in a straightforward manner. Each part of the problem is examined in order.

a. To find the capacitance of the capacitor, it is noted that the total charge on the capacitor plates is given by the expression

$$Q = \sigma lL + \sigma_0(L - l)L$$

where σ is the charge density in the region of the dielectric slab and where σ_0 is the charge density in the region of space. Noting, however, that

$$\sigma = D = \epsilon E = \frac{\epsilon V}{d}$$

and

$$\sigma_0 = D_0 = \epsilon_0 E_0 = \epsilon_0 \frac{V}{d}$$

then the total charge on the plates becomes

$$Q = \frac{\epsilon V}{d} lL + \epsilon_0 \frac{V}{d} (L - l)L$$

It follows directly from this, since by definition $C = Q/V$, that

$$C = \frac{L}{d} [\epsilon l + \epsilon_0(L - l)] \qquad \text{farads} \tag{4-31}$$

b. To find the total energy in the capacitor, use is made of the expression for energy density. Thus,

$$W_e = \tfrac{1}{2}\epsilon E^2 lLd + \tfrac{1}{2}\epsilon_0 E_0^2 (L - l)Ld$$

But since

$$E = E_0 = \frac{V}{d}$$

then this becomes

$$W_e = \frac{1}{2} \frac{V^2 L}{d} [\epsilon l + \epsilon_0(L - l)] \qquad \text{joules} \tag{4-32}$$

which may be written, by Eq. (4-31), as

$$W_e = \tfrac{1}{2}CV^2$$

c. To find the force tending to draw the dielectric slab into the capacitor, the work done in moving the slab a distance dl will be calculated. This work must equal the change in energy in the field. Thus, from (*b*),

$$d \text{ (work)} = \mathbf{f} \cdot d\mathbf{l} = \frac{1}{2} \frac{V^2}{d} L(\epsilon - \epsilon_0)\, dl$$

Hence, the net force on the dielectric slab is

$$f = \frac{(\epsilon - \epsilon_0)LV^2}{2d} \qquad \text{newtons} \tag{4-33}$$

Example 4-3.2. A parallel-plate capacitor comprises two square plates, L m on a side, separated by a distance d m, as in Fig. 4-4. Initially, the capacitor is

connected to a constant d-c supply, and the capacitor is then disconnected from the source. A slab of dielectric which fully fills the space between the plates is now introduced. Deduce the following:

a. How much work is done in introducing the dielectric into the capacitor?

b. What is the potential across the capacitor?

Solution. The solution is readily found by employing the fundamental results already deduced. First, however, it is noted that removing the capacitor from the source of potential fixes the total charge on the plates without regard for the difference of potential.

a. The work done is found in terms of the change of energy of the system. If the initial energy is denoted

$$W_e = \tfrac{1}{2}CV^2 = \frac{Q^2}{2C}$$

where

$$C = \frac{\epsilon_0 L^2}{d}$$

then, after the dielectric fully fills the space between the plates, the final energy is

$$W_e' = \frac{Q^2}{2C'} = \frac{Q^2}{2C}\frac{\epsilon_0}{\epsilon} = \frac{\epsilon_0}{\epsilon} W_e$$

since

$$C' = \frac{\epsilon}{\epsilon_0} C$$

In the case of any dielectric $\epsilon > \epsilon_0$, then $W_e' < W_e$, or the system has done work in drawing the dielectric into the capacitor. The total work done by the system is

$$\text{Work} = W_e - W_e' = \left(1 - \frac{\epsilon_0}{\epsilon}\right) W_e$$

b. To find the new potential difference between the plates, use is made of the fact that the total charge remains unchanged. Thus, since

$$Q = CV = C'V'$$

then

$$V' = \frac{CV}{C'} = \frac{\epsilon_0}{\epsilon} V$$

Hence, it is observed that the potential between the plates has been reduced by introducing the dielectric

Example 4-3.3. A schematic representation of the construction of an absolute voltmeter is given in Fig. 4-5. Circular plates P_1 and P_2 form a parallel-plate capacitor, and GG is a guard ring which surrounds plate P_1 but is very close thereto. It will be maintained at the same potential as P_1 in order that no appreciable distortion of the field between P_1 and P_2 shall exist. Plate P_1 is suspended from the beam BB of a precision balance that is to measure the force on P_1 with the application of a potential difference between P_1 and P_2. Of course, the entire system must be electrostatically shielded in order that stray fields do not affect the reading of the balance. Also, stops are provided in order to prevent plate P_1 from moving very far from the guard rings. A micrometer screw is provided for changing the plate separation d and for reading the distance d. Under equilibrium, the weight in the scale pan just balances the weight of the movable-plate system.

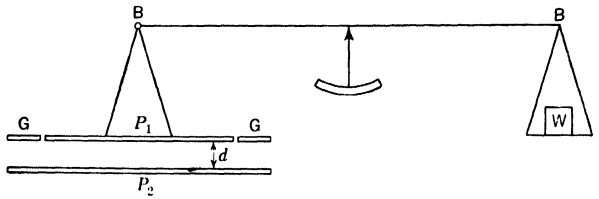


FIG. 4-5. The absolute electrometer.

Solution. Suppose now that a potential difference V is applied between the plates P_1 and P_2. This will cause an unbalance to occur, which is a measure of the force tending to draw the charged plates together. The amount of the force is, from Eq. (4-25),

$$f = pA = \tfrac{1}{2}\epsilon E^2 A = \tfrac{1}{2}\epsilon \frac{V^2}{d^2} A$$

This force is balanced by adding a weight W to the scale pan. From this equation, it follows that the applied potential is given by the expression

$$V = d\sqrt{\frac{2W}{\epsilon A}} \quad \text{volts} \tag{4-34}$$

It should be noted that the word *absolute* appears in the name of this instrument because all factors on the right-hand side of Eq. (4-34) are measured directly and do not depend on a calibrated instrument.

4-4. Forces between Charged Conductors. It was shown in Sec. 4-1 that the total energy of a specified charge distribution may be calculated in terms of the work done in establishing the charge distribution. Consider now the case of an array of conductors $(1, 2, \ldots, k)$ placed in an electric field. Suppose that the final charge on conductor k is Q_k and its final potential is Φ_k. The total potential energy associated with the system of charged conductors is, from Eq. (4-4),

$$W_e = \frac{1}{2}\int_S \phi\sigma \, dS \tag{4-35}$$

since only surface charges exist on the conductors. In this expression the integration is to be taken over the surfaces of the conductors. It may therefore be written as

$$W_e = \sum_k \frac{1}{2}\int_{S_k} \phi\sigma \, dS$$

But the potential over each conductor is a constant. Therefore it follows that

$$W_e = \sum_k \frac{1}{2}\Phi_k \int_{S_k} \sigma \, dS = \sum_{k=1}^{n} \frac{1}{2}\Phi_k Q_k \tag{4-36}$$

Attention is directed to the fact that in this development the assumption is implicit that there is a linear relationship between the charge and the potential on each conductor. This follows directly from the manner of the development leading to Eq. (4-4) for the energy W_e. This is not a serious restriction in the electric-field case, since electric hysteresis and saturation are generally small effects. It will be important to take this nonlinear character of magnetic materials into account when the cognate problem is considered for current circuits.

It is now required to determine the force which the system of charged conductors as a whole exerts on the conductor k or on any particular group of conductors. The desired expression may be obtained by the application of the principle of virtual displacement. This principle, which is simply a statement of the law of conservation of energy, as it applies to the present problem states that, if one of the conductors is given a small displacement consistent with the geometrical constraints on the system, then during this displacement the electrical energy change is such that

Sum of energy increments from all sources	= (mechanical work done on the system by the circuit or group of circuits displaced)	+ (sum of increments in electrical energy storage)	+ (sum of energy losses)

In mathematical form, this is

$$dW_s = dW + dW_e + \text{losses (neglected)} \tag{4-37}$$

But the work done against the electric force in the virtual displacement is expressed by the relation

$$dW = f_r\, dr \tag{4-38}$$

Then, by combining this result with Eq. (4-37), it follows that the force of electrical origin is

$$f_r = \frac{dW_s}{dr} - \frac{dW_e}{dr} \tag{4-39}$$

The value of the force deduced from this expression is the same regardless of the manner of making the virtual displacement, that is, whether the displacement is effected *at constant charge* or *at constant potential*. The two alternatives will be examined.

Constant Charge. In this case, which assumes that the virtual displacement is made at constant charge, it is implied that sources which supplied the energy to establish the field have been removed. It follows directly from Eq. (4-39) that

$$f_r = -\left.\frac{\partial W_e}{\partial r}\right|_{Q_k \text{ const}} \tag{4-40}$$

This expression shows that the force is the rate of decrease in stored electrical energy with displacement at constant charge. This expression may be written in a different form by combining it with Eq. (4-36). The resulting expression is

$$f_r = \sum_{k=1}^{n} \frac{1}{2} Q_k \frac{\partial \Phi_k}{\partial r} \tag{4-41}$$

Constant Potential. When the system is characterized by potentials which are maintained constant during a virtual displacement, charges dQ_k are transferred during the time of the displacement. Under these circumstances each Q_k is a function of the displacement. The electrical energy supplied to the system is

$$dW_s = \sum_{k=1}^{n} \Phi_k \, dQ_k \tag{4-42}$$

If this expression is combined with Eqs. (4-36) and (4-39), the equation for the force becomes

$$f_r = \sum_{k=1}^{n} \Phi_k \frac{\partial Q_k}{\partial r} - \sum_{k=1}^{n} \frac{1}{2} \Phi_k \frac{\partial Q_k}{\partial r}$$

or

$$f_r = \sum_{k=1}^{n} \frac{1}{2} \Phi_k \frac{\partial Q_k}{\partial r} \tag{4-43}$$

This is a very interesting result, since it shows that for a virtual displacement under conditions of constant potential the work done is just equal to the energy stored. By Eq. (4-36) this result may also be written in the form

$$f_r = \frac{\partial W_e}{\partial r}\bigg|_{\Phi_k \text{ const}} \tag{4-44}$$

This expression shows that the force is the rate of increase in stored electrical energy with displacement at constant potential.

Example 4-4.1. The parallel-plate capacitor illustrated in Fig. 4-4 is to be examined from the two viewpoints of Sec. 4-4. In one case, the capacitor will remain connected to the source of potential; in the second case, the capacitor will have been charged to the potential V and then disconnected. The force tending to draw the dielectric into the capacitor is to be found for each case.

Solution. CASE A. POTENTIAL CONSTANT. In this case, it is necessary to evaluate Eq. (4-44) for the present conditions; thus,

$$f_l = + \frac{\partial W_e}{\partial l}\bigg|_{V \text{ const}}$$

where W_e is the energy of the system. But, from Fig. 4-6 for the linear case, we may write

$$W_e = \tfrac{1}{2}CV^2$$

Thus,

$$f_l = \tfrac{1}{2}V^2\frac{\partial C}{\partial l}$$

But, from Eq. (4-31),

$$C = \frac{L}{d}[\epsilon l + \epsilon_0(L - l)]$$

Then,

$$\frac{\partial C}{\partial l} = \frac{L}{d}(\epsilon - \epsilon_0)$$

and the expression for the force then becomes

$$f_l = +\tfrac{1}{2}V^2\frac{L}{d}(\epsilon - \epsilon_0)$$

which is precisely that given in Eq. (4-33).

CASE B. CONSTANT CHARGE. The desired expression for the force is given as Eq. (4-40).

$$f_l = -\left.\frac{\partial W_e}{\partial l}\right|_{Q=\text{const}}$$

where

$$W_e = \frac{Q^2}{2C}$$

It follows, therefore, that

$$f_l = -\frac{Q^2}{2}\left(-\frac{1}{C^2}\frac{\partial C}{\partial l}\right) = +\tfrac{1}{2}V^2\frac{\partial C}{\partial l}$$

This leads to the same expression for the force as above, which it should, since no motion has occurred. Over any finite displacement of the dielectric, the total work done will be different in both cases.

PROBLEMS

4-1. Refer to Prob. 2-14.

a. Determine the total energy stored in the space outside the outer shell.

b. Determine the total energy stored in the space between the inner sphere and the outer spherical shell.

4-2. A concentrated charge of 3×10^{-9} coulomb is located in air at the center of a spherical shell of hard rubber $\epsilon_r = 3$. The inner radius is 6 cm, and the outer radius is 12 cm.

a. How much energy is contained in the hard-rubber shell?

b. How much energy is contained in the region outside the hard-rubber shell?

4-3. *a.* Suppose that the capacitor of Example 4-3.1 with the glass removed is initially charged by connecting it to the 600-volt source and is then disconnected from the source. How much energy is stored in the field of the capacitor?

b. The glass slab is now completely inserted between the capacitor plates.

(1) How much energy is now stored in the field?

(2) What is the potential between the plates?

c. Account for any changes that have occurred in the change from (*a*) to (*b*).

4-4. Suppose that the air space between the two shells of a spherical capacitor is replaced by transformer oil ($\epsilon_r = 2.5$).

a. What are the relative energies that can be stored in the fields in the two cases?

b. If the oil is introduced after the capacitor has been charged to a potential difference V, and the source has been removed, what energy is stored in the field?

c. Account for any differences.

4-5. Suppose that the movable plates of the absolute voltmeter of Example 4-3.3 have an area of 75 cm^2, the spacing between the fixed and the movable plate being 2 mm.

a. What weight must be added to the balance pan if a potential difference of 800 volts is applied?

b. What factors limit the potential difference that may be measured with such an instrument?

4-6. Suppose that the slab in Example 4-3.1 is only t thick, where $t < d$. Deduce an expression for the force acting to pull the slab between the plates.

4-7. *a.* Calculate the pressure acting to draw the capacitor plates into an air dielectric, for the optimal allowable strength of air.

b. If the dielectric is composed of two slabs, half the thickness being air and the second half being mica (see Table 3-2), what is the pressure at the dielectric interface?

CHAPTER 5

ELECTRIC CHARGES IN MOTION

The subject matter of Chap. 1 proceeded from the assumption that both positively and negatively charged carriers exist and that it is the steady drift of these carriers that constitutes the conduction current. Moreover, a conductor was defined (Sec. 3-1) as a substance containing electrons that are free to move through distances that are large compared with atomic dimensions under the action of an electric field. It is the object of this chapter to examine some aspects of the motion of charge centers in solid matter and in free space and to deduce some results of importance to our subsequent development.

5-1. The Electric Current in Metallic Conductors. X-ray and other studies reveal that most metals are crystalline in structure. This means that they consist of a space array of atoms or molecules (strictly speaking, ions) built up by regular repetition in three dimensions of some fundamental structural unit. Depending upon the metal, at least one, sometimes two, and, in a few cases, three electrons per atom are free to move throughout the interior of the metal under the action of applied forces. Thus, a metal is visualized as a region containing a periodic three-dimensional array of heavy, tightly bound ions, permeated with a swarm of electrons that may move about quite freely. However, the space average net charge density is zero in the metal. This is known as the *electron-gas* concept of a metal. The electrons are in incessant motion, owing to the fact that the ions are in a state of agitation due to the thermal agitation of the atoms, and it is assumed that the electrons share the random velocity of thermal agitation of the ions. The average drift of the electrons is zero under the action of the thermal motion alone.

Suppose, now, that an electric field is applied. By Coulomb's law, the effect is to accelerate the electrons uniformly in the direction opposite to the direction of the electric field. However, an indefinite increase in the electronic velocity is prevented by collisions between the electrons and the ions. As a result, the electrons quickly acquire a steady average velocity of drift. This steady drift exists when the average force exerted on the electrons by the ions during the collisions is equal and opposite to the force due to the electric field. It is desired to examine the situation analytically.

Consider an electronic charge q under the action of an external field $\mathbf{E}$. Under the action of the force, the electronic charge will begin to accelerate. Other forces acting on the electrons will be the forces due to the neighboring ions, although the exact nature of these forces is not known at present. However, these forces will counterbalance the action of the electric field, to produce a uniform motion. Let the unknown forces be represented by the symbol $\mathbf{f}$, and let τ_0 be the time between two collisions of electrons with the ions. The average force between collisions may then be written as

$$\bar{\mathbf{f}} = \frac{1}{\tau_0} \int_{\tau_0} \mathbf{f}\, dt$$

Since no acceleration is produced, it is required that

$$\mathbf{E}q + \frac{1}{\tau_0} \int_{\tau_0} \mathbf{f}\, dt = 0 \tag{5-1}$$

It is now convenient to consider the time τ_0 between two collisions to be expressed in terms of the time τ of duration of a collision, and the remainder of the collision time interval, $\tau_0 - \tau$, by writing

$$\frac{1}{\tau_0} \int_{\tau_0} \mathbf{f}\, dt = \frac{1}{\tau_0} \int_{\tau} \mathbf{f}\, dt + \frac{1}{\tau_0} \int_{\tau_0 - \tau} \mathbf{f}\, dt$$

However, it is assumed that

$$\frac{1}{\tau_0} \int_{\tau_0 - \tau} \mathbf{f}\, dt = 0$$

as there is no force brought into play during that interval. Hence,

$$\mathbf{E}q = -\frac{1}{\tau_0} \int_{\tau} \mathbf{f}\, dt \tag{5-2}$$

It is now assumed that:

1. All electrons have the same mean free path λ (average collision distance).
2. All electrons have the same velocity $\mathbf{v}$.
3. The motion after collision is independent of the motion before collision (that is, the motion is totally stopped by the collision).

Then, writing

$\mathbf{v}_2$ = velocity of electron after impact
$\mathbf{v}_1$ = velocity of electron before impact

from considerations of the impulsive force

$$\int_{\tau} \mathbf{f}\, dt = m(\mathbf{v}_2 - \mathbf{v}_1)$$

where m is the mass of the electron. Combine this result with Eq. (5-2), to get

$$\mathbf{E}q = -\frac{1}{\tau_0} m(\mathbf{v}_2 - \mathbf{v}_1)$$

But $\mathbf{v}_2 = 0$ by assumption 3. Therefore,

$$\mathbf{E}q = \frac{1}{\tau_0} m\mathbf{v}_1 = \frac{v_T}{\lambda} m\mathbf{v}_1 \tag{5-3}$$

where v_T is the average velocity of thermal agitation. Of course, the velocity of the electron before impact may vary over wide limits, say, from zero to v_1, and the average velocity is written as

$$\bar{v}_1 = \tfrac{1}{2} v_1 \tag{5-4}$$

This expression is combined with Eq. (5-3), which then yields, for the average value of $\bar{v}_1$,

$$\bar{v}_1 = \frac{1}{2} \frac{q\lambda}{m v_T} E \tag{5-5}$$

Now, consider the motion of a group of charges Σq moving with the average drift velocity $\bar{\mathbf{v}}$. An element of volume $\Delta\tau$ (length Δl, area ΔS)

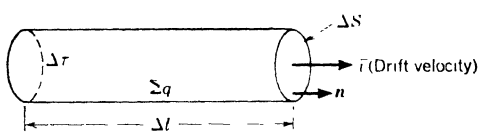


FIG. 5-1 An element of distributed charges.

is now the center of attention. An expression is sought for the number of charges which cross ΔS in the time interval Δt. As the motion of all charges is uniform, all charges inside the volume $\Delta\tau$ will have left the element after a certain interval of time. Hence, if ΔQ denotes the total charge moving out, the rate of transportation of the charges is given by

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\Sigma q}{\Delta l/\bar{v}} = \bar{v}\frac{\Sigma q}{\Delta l} = \bar{v}\frac{\Sigma q}{\Delta\tau}\Delta S = \bar{v}\rho\,\Delta S \tag{5-6}$$

where $\dfrac{\Sigma q}{\Delta\tau} \equiv \rho$

But the transfer of charge per unit time is the current; then it is convenient to write for the current density $\mathbf{J}$ the relation

$$\mathbf{J} \equiv \rho\mathbf{v} \tag{5-7}$$

from which it follows that

$$I = \int_{\Delta S} \mathbf{J}\cdot\mathbf{n}\,dS = \int_{\Delta S} J_n\,dS$$

Now, combine Eqs. (5-7) and (5-5). The result is, since

$$\mathbf{J} = \rho \mathbf{v} = \bar{\mathbf{v}} \frac{\Sigma q}{\Delta \tau} = \bar{\mathbf{v}} N e \tag{5-8}$$

where N is the number of electrons per unit volume and e is the electronic charge,

$$J = \left(\frac{1}{2} \frac{N e^2 \lambda}{m v_T} \right) E \tag{5-9}$$

Observe that this is just in the form of Ohm's law [see Eq. (1-24)], which relates the current density with the electric field intensity. Thus, it follows that the conductivity has the form

$$\gamma = \frac{1}{2} \frac{N e^2 \lambda}{m v_T} \tag{5-10}$$

This expression is temperature-dependent, since both the mean free path λ and the average velocity of thermal agitation v_T are temperature-dependent. Observe that none of the factors that appear in Eq. (5-10) is a function of the electric field, whence γ is independent of E, as discussed in Sec. 1-7.

5-2. Convection Current. The discussion in Sec. 5-1 proceeded from considerations of the motion of electrons in the interstices of the ions under the influence of an applied electric field. Such an electric current in a material medium is usually denoted as *conduction* current. It is also possible for charged masses such as electrons or ions to transport the charge directly from one point to another. In this case, the current is called *convection* current. Thus, the motion of electrons in a conductor under the action of an externally applied electric field is a conduction current, whereas the electrons constituting the beam in a cathode-ray tube constitute a convection current. In steady flow in both cases, charge is transferred from one point to another, or, alternatively, there is a flow, or transport, of charge past a fixed point of observation in the beam even though the charge per unit volume of the beam does not change.

Most of the significant results of current-flow phenomena are quite independent of the character of the current, although certain results are peculiar to one or the other type of current. For example, Eq. (5-7) is true for either, whereas Eq. (5-5) for the relationship between the average velocity of drift of electrons and the applied electric field intensity applies only to the case of conduction current. The appropriate relationship for the velocity of a free charged particle in an electric field must be deduced from the relationship between the difference of potential V between two points in the field and the amount of work necessary to

carry unit positive charge from one of these points to the other. From the results deduced in Sec. 2-5, that

$$W_{ab} = -q \int_a^b \mathbf{E} \cdot d\mathbf{l} = +qV_{ba}$$

and from the fact that for a conservative system the work done may be expressed in terms of the change in kinetic energy of the particle,

$$\begin{aligned} W_{ab} &= \tfrac{1}{2}mv_a^2 - \tfrac{1}{2}mv_b^2 \\ \text{then} \qquad qV_{ba} &= \tfrac{1}{2}mv_a^2 - \tfrac{1}{2}mv_b^2 \end{aligned} \tag{5-11}$$

where v_a and v_b are the values of speed at points a and b, respectively. Thus, for a known charge q, known mass m, and known potential difference through which the particle falls, the change in velocity may be found. For any charged particle and reasonable potential differences, the velocity changes are very high.

Example 5-2.1. An electron in a cathode-ray beam falls through a potential of 1,000 volts. If the electron initially started from rest, what is its velocity?

Solution. In the present case,

$q = -e = 1.6 \times 10^{-19}$ coulomb
$m = 9.1 \times 10^{-31}$ kg
$V = 1{,}000$ volts $= V_{ab}$

Thus, by Eq. (5-11),

$$-1.6 \times 10^{-19} \times 10^3 = 0 - \tfrac{1}{2} \times 9.1 \times 10^{-31} v_b^2$$

or

$$v_b = \left(\frac{2 \times 1.6 \times 10^{-16}}{9.1 \times 10^{-31}} \right)^{1/2}$$

Hence, finally,

$$v_b = 1.88 \times 10^7 \text{ m/sec}$$

This velocity, which is nearly one-tenth the velocity of light in free space, is enormous compared with the usual drift velocity of the electrons in a conductor, which will be of the order of 10^{-4} m/sec.

Another difference to be found between the conduction-current and the convection-current situation is conveniently discussed by reference to Fig. 5-1, which for the conduction current is to represent the boundary of the conductor and for the convection current is to represent a hypothetical boundary of the moving charges. Consider the electric field at a point away from the boundary. For the case of conduction current through a conductor, since the net charge per unit volume of electrons and ions is zero, even though a drift of electrons does exist, the electric field at points outside the conductor is zero. For the case of the convection current, since there are no ions present, there is a net charge per unit volume of beam, and there is an electric field in the region outside the beam. In fact, for a uniform beam current, the electric field at a point in space is

constant even though the electric field associated with each electron is moving.

Clearly, caution must be used in discussing current-flow phenomena to be certain that the limitations of the results, if any, be clearly understood. For example, the entire development of Chap. 1 was deduced from considerations of conduction current.

5-3. Equation of Continuity. The development in Sec. 5-1 has assumed that the conduction charges constituting the current were moving with the average drift velocity. The discussion in Sec. 5-2 involved the matter of convection current. Moreover, the principle of conservation of electric charge has been implicit in the development, as it has been assumed that electricity can neither be created nor destroyed. These considerations are implicit in the Kirchhoff current law (see Sec. 1-11), that "under stationary- or uniform-flow conditions, the rate at which charge enters a given region must be equal to the rate at which it leaves the region," and which was formulated mathematically in the expression, Eq. (1-57),

$$\oint \mathbf{J} \cdot \mathbf{n}\, dS = 0 \tag{5-12}$$

where the integral is evaluated over the entire closed surface specifying the given region. This expression is valid under steady conditions for both conduction and convection current. An application of the divergence theorem to this surface integral leads to the expression

$$\int_\tau \text{div } \mathbf{J}\, d\tau = 0$$

from which, in the manner of Sec. 2-7, one writes the point or differential statement

$$\nabla \cdot \mathbf{J} = 0 \tag{5-13}$$

In differential form, this expresses the fact that in uniform-flow state the number of lines of electric current that originate in any closed region of space is zero, whence the lines of electric current are continuous.

The problem is now to be generalized, in order to provide for the case when there may be a net rate of change of charge within τ, the closed region of space. The problem is no longer a stationary-state one; Eqs. (5-12) and (5-13) must be modified to take into account the fact that the net current into or out of the closed region of space will not vanish. Now, in addition to the term $\oint \mathbf{J} \cdot \mathbf{n}\, dS$, which specifies the difference between the current lines into and out of the closed region of space, it is necessary to consider the rate at which the net charge in τ is decreasing. Again by the principle of conservation of electric charge, it is required that the net change (say, a reduction) in the flow of electric charge

through a given region τ must be equal to the rate at which charge flows out of τ. This result may be expressed mathematically in the form

$$\oint \mathbf{J} \cdot \mathbf{n}\, dS = -\frac{dq}{dt}$$

But the charge q may be expressed in terms of the charge density ρ, since

$$q = \int_\tau \rho\, d\tau$$

This permits the results to be written in the form

$$\oint \mathbf{J} \cdot \mathbf{n}\, dS + \int_\tau \frac{\partial \rho}{\partial t}\, d\tau = 0 \tag{5-14}$$

By the divergence theorem applied to the expression on the left, it follows that

$$\int_\tau \left(\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t}\right) d\tau = 0 \tag{5-15}$$

from which follows the differential equation in $\mathbf{J}$ and ρ, namely,

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{5-16}$$

This is called the *equation of continuity for current.*

5-4. The Displacement Current. Attention is called to the restriction on Eq. (5-16) to conduction currents, or to the convection of electric charge. Should currents exist which are not directly explainable in terms of the flow of electric charge in conductors or in free space, these must be separately examined before one can ascertain their relation to Eq. (5-16). An important example is readily discovered of a current that requires special considerations.

Consider a simple circuit comprising a capacitor and a charging source of fixed potential. Under steady-state conditions with the source connected to the capacitor, there can be no conduction current in the circuit, as the capacitor is an open circuit to conduction. If, however, the currents are changing with time, there will be a conduction current in the conductors as long as the capacitor has not been fully charged. Refer now to Fig. 5-2, which shows a greatly enlarged view of the region near the capacitor. It is desired to examine four separate regions in the neighborhood of the capacitor, as shown: (1) a Gaussian surface wholly in the conductor; (2) a surface partly in the conductor and partly in the dielectric of the capacitor, but extending through the capacitor; (3) a surface partly in the conductor and partly in the dielectric, but terminat-

ing in the dielectric; and (4) a surface wholly in the dielectric. These four surfaces are illustrated in Fig. 5-2.

Suppose at some instant during the charging of the capacitor there is a uniform conduction-current density J in the conductors, and directed perpendicular to the capacitor boundaries. It is desired to examine the current through each of the Gaussian surfaces all of the same ΔS, as illustrated in Fig. 5-2.

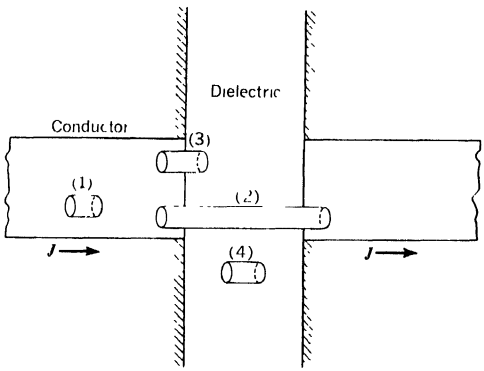


FIG. 5-2. An enlarged portion of a simple circuit in the neighborhood of a capacitor.

1. For surface (1), since the conduction-current density is assumed to be uniform, then by Kirchhoff's first current law, Eq. (5-12),

$$-J\,\Delta S\Big|_{\text{left}} + J\,\Delta S\Big|_{\text{right}} = 0$$

2. For surface (2), irrespective of what may happen in the region between the capacitor plates, if conduction current exists through the left-hand face, and also through the right-hand face, and assuming that the current lines are normal to the Gaussian faces, then

$$-J\,\Delta S\Big|_{\text{left}} + J\,\Delta S\Big|_{\text{right}} = 0$$

3. For surface (3), since a conduction current is assumed to exist through the left-hand face, and since no charged carriers exist in the dielectric to give rise to a conduction current through the right-hand face, it must follow, therefore, that mobile charge is collecting on the surface of the conductor lying inside the Gaussian surface. Under equilibrium conditions, it is necessary that

$$J\,\Delta S\Big|_{\text{left}} = \frac{\partial}{\partial t}(\sigma\,\Delta S) = \Delta S\frac{\partial \sigma}{\partial t} \tag{5-17}$$

However, from Eq. (3-6),

$$D = \sigma$$

from which it follows that

$$-J\,\Delta S\Big|_{\text{left}} + \frac{\partial D}{\partial t}\,\Delta S = 0 \tag{5-18}$$

If now the term $(\partial D/\partial t)\,\Delta S$ is viewed as a current through the right-hand face of the Gaussian surface to balance the conduction current through the left-hand face, the density of this current would be

$$J_d \equiv \frac{\partial D}{\partial t} \tag{5-19}$$

This is adopted as the definition of the *displacement-current density*. Note that, since this current depends on the change of the electric displacement with time, there is no displacement current unless the electric displacement is changing with time. In general, Eq. (5-19) is written

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} \tag{5-20}$$

Note from this that the direction of $\mathbf{J}_d$ is either in the same direction or opposite to $\mathbf{D}$, depending upon whether the time rate of change of $\mathbf{D}$ is increasing or decreasing with time.

4. For surface (4), no conduction current exists through either face. However, with the development of the concept of displacement current, this case would be characterized by the expression

$$-\frac{\partial D}{\partial t}\,\Delta S\Big|_{\text{left}} + \frac{\partial D}{\partial t}\,\Delta S\Big|_{\text{right}} = 0$$

With the development of the displacement current, Eq. (5-16) may now be expressed in a very interesting form. Thus, at any point in space, from Eq. (3-42),

$$\text{div } \mathbf{D} = \rho$$

where ρ is the free charge density. Thus, by differentiating both sides,

$$\text{div } \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \rho}{\partial t}$$

Now, combine this with Eq. (5-16), denoting the conduction-current density as $\mathbf{J}_c$. Then,

$$\text{div } \mathbf{J}_c + \text{div } \frac{\partial \mathbf{D}}{\partial t} = \text{div } \left(\mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}\right) = 0$$

from which it follows that

$$\text{div } \mathbf{J}_t = \text{div } (\mathbf{J}_c + \mathbf{J}_d) = 0 \tag{5-21}$$

This result shows that the divergence of the *total* current density $\mathbf{J}_t$ is zero always, and hence there can be no source or sink of current flow.

Example 5-4.1. Two plane-parallel conductors are separated by two leaky homogeneous media of permittivities ϵ_1 and ϵ_2, of conductivities γ_1 and γ_2, and of thickness d_1 and d_2, as illustrated in Fig. 5-3. It is required to find the electrical conditions for the stationary state.

Solution. From the definition of the line integral of the electric field, it follows that

$$E_1 d_1 + E_2 d_2 = V \tag{5-22}$$

Also, by an application of Gauss' theorem at the interface of the dielectrics assuming that volume densities of charge may be replaced by surface-charge densities

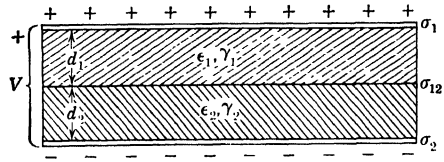


FIG. 5-3. A capacitor with nonhomogeneous media.

and that an accumulation of charge will occur on the line of separation of the media, due to the different conductivities,

$$D_{1n} - D_{2n} = \sigma_{12}$$

where σ_{12} is the surface-charge density, then it follows that

$$\epsilon_1 E_1 - \epsilon_2 E_2 = \sigma_{12} \tag{5-23}$$

Also, since the system is assumed in steady state,

$$J_{1n} - J_{2n} = 0$$

from which

$$\gamma_1 E_1 - \gamma_2 E_2 = 0 \tag{5-24}$$

The three equations (5-22), (5-23), and (5-24) contain three unknowns, σ_{12}, E_1, and E_2. It is therefore easy to deduce the values of these unknowns.

If it is desired to find the capacitance of the capacitor when the steady state is reached, use is made of the fact that by definition

$$C = \frac{\sigma_1 - \sigma_2}{V} \tag{5-25}$$

But the field due to the free charges alone will be given by Eq. (3-6); thus

$$E_1 = \frac{1}{2\epsilon_1}(\sigma_1 - \sigma_{12} + \sigma_2)$$

and

$$E_2 = \frac{1}{2\epsilon_2}(\sigma_1 + \sigma_{12} + \sigma_2) \tag{5-26}$$

Now, since E_1, E_2, and σ_{12} are already known, the capacitance may be determined. If the details are carried through, it will be found that the capacitance is that of a capacitor with perfect dielectric media.

A word of explanation is in order about the phenomenon of *residual charge* that occurs in a leaky homogeneous dielectric (Example 5-4.1).

Suppose that a field is applied. There will be an accumulation of charge in the inhomogeneous spots, due to the fact that the incoming current will not equal the outflowing current in a volume element. This accumulation of charge will create a field which will be opposite to the impressed field. The resulting back field will develop until the effect of the accumulation of charge balances the impressed field in such a way that the incoming current will equal the outflowing current in the element of volume. When this condition is reached, the system will be in its steady state. This is the situation discussed above.

Suppose that such a charged capacitor is now connected through an outside conductor. Just as a certain time was required for the system to reach its steady state, it will require a finite time for the charge inside the dielectric to vanish (to redistribute itself to yield a zero field). A

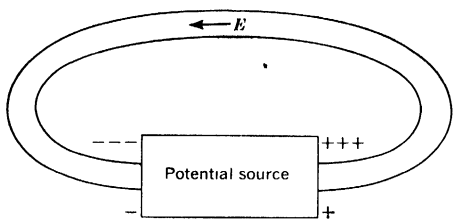


FIG. 5-4. To examine the field inside a conductor connected to the terminals of a potential source.

measure of the time required for the charge to decay to $1/e$th of its original value is of importance, and this *relaxation time* gives a measure of the time for a charged capacitor to decay in potential. As a practical matter, this means that short-circuiting an initially charged capacitor will not remove the charge instantaneously, and the removal of the short circuit after too short a time interval may be hazardous, as the capacitor will not be fully discharged.

5-5. The Steady Flow of Current in Conductors. The character of the redistribution of the electrons within an isolated conductor when it is placed in an electric field has been discussed in Sec. 3-1. It is there shown that the entire net charge must reside on the surface of the conductor and must be so distributed as to make the electric field vanish everywhere inside the conductor. It is now desired to examine the character of the redistribution of the electrons in a conductor which is connected across the terminals of a potential source.

In Fig. 5-4, a conductor is shown connected across the terminals of a potential source, whether this be a chemical battery, an electromagnetic generator, or any other source of electrical power. The function of the potential source is to pile up electrons on one end of the conductor and to remove them from the other end. Due to the presence of the impressed field, there will be a motion of the free electrons inside the conductor.

Let us examine the transient state when the conductor is brought to the immediate neighborhood of the source. The situation now is precisely that discussed in Sec. 3-1, and there will be a motion of free electrons inside the conductor to the surface, where their motion is stopped as they build up a surface layer of charge all around the conductor. This surface layer will produce a field which will oppose the external field, the field of the electrons being just sufficient to neutralize the impressed field at all points inside the conductor.

When the conductor is connected to the terminals of the potential source, an electric field is now caused to exist inside the conductor which is parallel to the surface at all points inside the conductor. Hence, all lines of force originating in the conductor remain in the conductor and constitute the lines of flow of the current.

Note that the net charge in any element of volume inside a conductor must be zero in either the static case or the electron-flow case. That is, the net charge per unit volume when the electrons and the ions of the metal lattice are considered just balances. Otherwise, an unstable component of an electric field will be developed. In the current-flow case, the body electrons are moving through the element of volume, but a uniform flow exists. Hence, all net electric charge in a conductor resides on the surface of the conducting material. It is the function of the generator to pile up electrons on one end of the conductor and to remove them from the other end. The internal field is thus produced by a density gradient of the surface charges.

The surface electrons are not stationary but are also moving with the body electrons, but in such a way that the density remains constant at any point. That is, in the steady state, there will be a motion of the body electrons parallel to the surface and a motion of the surface layer of electrons. Moreover, since the surface-charge density is constant in time at all points of the conductor for steady state, a potential difference exists in the conductor, and it is this potential difference which produces the internal electric field $\mathbf{E}$.

From the requirements of a steady state, by Eq. (5-21),

$$\operatorname{div} \mathbf{J} = 0 \tag{5-27}$$

Also, by Ohm's law,

$$\mathbf{J} = \gamma \mathbf{E} \tag{5-28}$$

Thus, if the medium is homogeneous, then

$$\operatorname{div} \mathbf{J} = \operatorname{div} \gamma \mathbf{E} = \gamma \operatorname{div} \mathbf{E} = 0$$

so that

$$\operatorname{div} \mathbf{E} = 0$$

Also, by Eq. (1-30),

$$\mathbf{E} = -\operatorname{grad} \phi$$

from which it follows that inside the medium

$$\nabla^2\phi = 0 \tag{5-29}$$

That is, Laplace's equation completely specifies the potential distribution within a homogeneous conductor. This means that the solution of the general problem of finding the current distribution in a conducting medium is reduced to finding a value of the function ϕ which will satisfy Eq. (5-29) and which satisfies the following boundary conditions:

1. ϕ shall have specified values over certain surfaces, for example, the surface of the electrodes.

2. $E_n = -\partial\phi/\partial n = 0$ over the surface bounding the conductor.

When ϕ is thus obtained, the current density at any point in the medium is given by

$$J_l = -\gamma \frac{\partial \phi}{\partial l} \tag{5-30}$$

where dl is an element of length in the direction of E_l.

It is here noted that the development of Ohm's law in terms of the resistance parameter R_{ab}, as given in Sec. 1-7, is a procedure by which the solution of Laplace's equation has been reduced to the solution of the expression for the resistance, Eq. (1-41).

PROBLEMS

5-1. A current of 100 amp flows in a No. 0 conductor having a radius of 162.4 mils. What is the average drift velocity of the electrons constituting the current? Assume that there are 3.4×10^{28} electrons/m^3 in copper.

5-2. Suppose that the parallel-plate capacitor of area A illustrated in Fig. 4-1 is being charged at the rate of I amp.

a. What is the displacement-current density at a point between the plates?

b. What is the displacement current?

5-3. At a certain point within a conductor carrying a d-c current, the X and Y components of the current density are

$$J_x = 3az \qquad J_y = 2by$$

where a and b are constants. Find the x dependence of J_z.

5-4. Starting from the equation of continuity, show, for a conducting medium obeying Ohm's law, $J = \gamma E$, and using Gauss' law, that

$$\frac{\partial \rho}{\partial t} + \frac{\gamma}{\epsilon_0}\rho = 0$$

From this, show that, if initially there is charge density at any point inside the conducting medium, the charge density at a later time is given by

$$\rho = \rho_0 e^{[-(\gamma/\epsilon_0)t]}$$

The time constant ϵ_0/γ is known as the relaxation time.

CHAPTER 6

THE MAGNETIC FIELD OF CURRENTS IN FREE SPACE

Heretofore, two classes of phenomena have been under study. One was the properties and characteristics of the current-flow field, both to examine the constitution of the current, and also to deduce from a given flow field a description of the potential distribution within the field. The second dealt with the forces between stationary electric charges, and, from this, the properties of stationary electric fields were examined.

Clearly, since the electric current comprises the transport of electric charges, there must be a moving electric field associated with the moving charges. It is the purpose of this chapter to consider certain properties of such moving electric fields. It is convenient to introduce the subject by considering the forces on charges in motion in these moving electric fields. These forces are called *magnetic forces*, and we shall define a magnetic field in terms of the moving electric field. It should be noted that, even though the chapter is introduced by a discussion of the forces in moving electric fields, it will be confined largely to the subject of the magnetic field itself, a detailed consideration of forces in magnetic fields being reserved for a later chapter.

6-1. Forces in Moving Electric Fields. While the emphasis will later change, it is convenient in what is to follow to recall that an electric field is associated with a charge and is produced by the charge. Hence, the statement that a charge is in motion is equivalent to the statement that an electric field is in motion. The concept of the field associated with the charge is desirable in the present case, as it calls attention to the fact that a space phenomenon rather than a situation at a particular point is under survey.

Consider first two charges, one of which is fixed in space, the second moving relative to it with a *uniform velocity*. Figure 6-1 illustrates the case under discussion, with $\mathbf{u} = 0$. Experiment shows that the electrostatic force experienced by a charge at rest in an electric field is the same whether the electric field is moving or whether it is stationary. Hence, for the situation illustrated in Fig. 6-1, the force on the stationary charge q_1 is given by Coulomb's law and is

$$\mathbf{f} = q_1\mathbf{E} \tag{6-1}$$

whether q_2 is fixed in space or is moving. Clearly, of course, the value of **E** will change as the charge q_2 moves toward and away from q_1. This expression is valid only if the velocity of the moving charge is small compared with the velocity of light.

The results here discussed may be summarized by the statement: The field of moving charges is unaltered by their motion, for small velocities.

Now, consider the situation when both q_1 and q_2 are moving with uniform velocity. The case is now different from before, for an additional force exists. It is desired to examine the nature of this new force. This is conveniently accomplished by first considering an experiment involving the cathode-ray beam within a cathode-ray tube and an external current-carrying wire that is free to be moved relative to the cathode-ray-tube axis. Not only is this a very convenient arrangement which makes possible the direct observation of the effects of any forces, but it likewise avoids the masking effect that would result, due to Coulomb forces, were

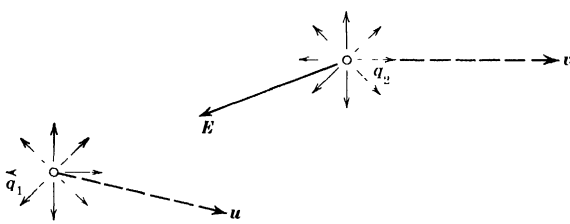


FIG. 6-1. A fixed and a moving charge.

beams of charged particles actually employed. A parallel development is possible from considerations of a pair of current-carrying wires. In fact, the original research work of Ampère was carried out with such conduction currents.

Originally, suppose that the current-carrying wire is below, but parallel to, the tube axis (the X axis) and in the XY plane, as illustrated in Fig. 6-2. With no current through the wire, the cathode-ray beam progresses along the tube axis, and the luminescent spot occurs at the center of the screen. Now, suppose that a steady current is passed through the wire in the direction shown (observe that the actual drift motion of the electrons making up the steady current is opposite to this and is in the direction specified by **v**). The following facts are observed experimentally:

1. A force exists which causes the beam to deflect upward on the screen.
2. The amount of the deflection is proportional to the current.
3. By maintaining the distance of the wire from the tube axis and moving the wire about this axis at a fixed radial distance, it is found that the

deflection of the cathode-ray beam is always away from the current-carrying wire.

4. By rotating the current-carrying wire in a horizontal plane (XZ) about the Y axis, we find for a constant current that the deflection decreases and is zero when the wire is directed along the Z axis. The deflection varies as the sine of the angle between the cathode-ray-beam axis (the X axis) and the direction of the wire.

5. The deflection falls off inversely with the square of the distance between the current and the electron beam, for a constant current.

It is desired to examine the nature of the force. Now we return to the consideration of a single moving charge in place of the cathode-ray

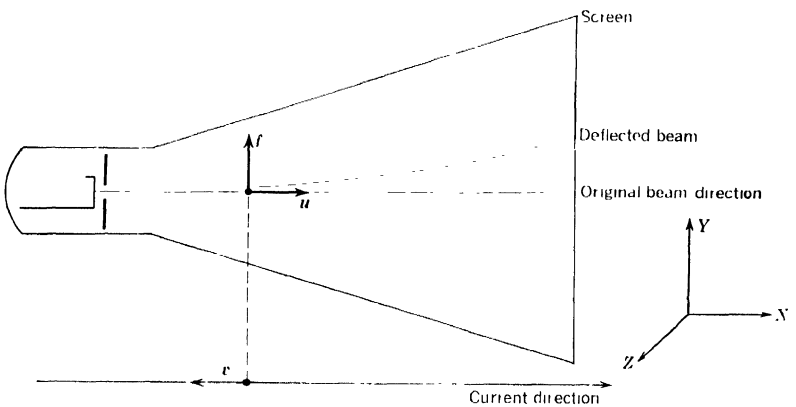


FIG. 6-2. An experiment involving a cathode-ray beam and a current-carrying wire

beam and a single moving charge in place of the current-carrying wire. The following notation is adopted:

q is the exploring charge (for example, electron in cathode-ray beam).

u is the velocity of the exploring charge.

E is the electric field intensity that is associated with q_2 (a moving electron which constitutes a convection current instead of the current-carrying wire) and in the direction from q_2 to q.

v is the velocity of q_2 and so of the electric field.

f is the force on q resulting from the effects of its motion in the moving electric field of q_2.

It is inferred that the experimental observations discussed above remain valid for the present case. Of course, now a Coulomb-type force would also exist. Attention is being confined here to the additional force arising from the motion of both charges. Mathematically, the foregoing observations are contained in the expression

$$f \sim Ev \sin \theta_1 qu \sin \theta_2 \tag{6-2}$$

where θ_1 is the angle between **E** and **v** and θ_2 is the angle between **u** and the normal to the (**E**,**v**) plane.

In the general case, the situation is that presented diagrammatically in Fig. 6-3. The constant of proportionality has been found to be $1/c^2$, where c is the velocity of light. Hence, the expression given in Eq. (6-2) may be written as

$$f = \frac{1}{c^2} qu \sin \theta_2 \, vE \sin \theta_1 \tag{6-3}$$

This expression is conveniently written in vector form. In the light of

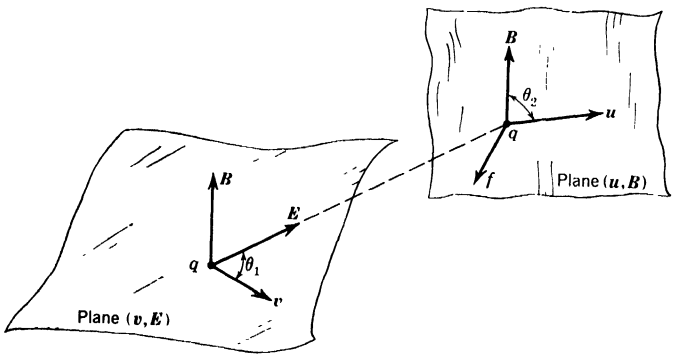


FIG. 6-3. The motional force due to the interaction of two moving electric fields.

the discussion of the vector, or cross, product of two vectors given in Sec. 3-1, it may be written as

$$\mathbf{f} = \frac{1}{c^2} q\mathbf{u} \times (\mathbf{v} \times \mathbf{E}) \tag{6-4}$$

Since, as noted in Sec. 3-1, vector multiplication is not commutative, the order of vectors is important. Moreover, the parentheses in the expression must be retained, as the triple vector product is not associative.

It is convenient in what follows to introduce a new vector **B**, which is called the *magnetic induction, magnetic flux density,* or *magnetic field vector.* This is defined by the expression

$$\mathbf{B} \equiv \frac{1}{c^2} \mathbf{v} \times \mathbf{E} \tag{6-5}$$

The unit of the **B** vector is seen to be

$$\frac{\text{m/sec} \times \text{newtons/coulomb}}{(\text{m/sec})^2} = \frac{\text{volts/m}}{\text{m/sec}} = \frac{\text{volt-sec}}{\text{m}^2} = \frac{\text{webers}}{\text{m}^2} = \text{teslas}^1$$

The name magnetic flux density arises from this viewpoint.

[1] The term for the units of **B** adopted by the International Electrochemical Commission in 1956.

Equation (6-4) now assumes the form

$$\mathbf{f} = q\mathbf{u} \times \mathbf{B} \tag{6-6}$$

This is known as the *Lorentz force*, after Hendrik Lorentz (1892), who first applied it to electric field motion. The following implication of Eq. (6-5) is noted: Every moving electric field is accompanied by a magnetic field. This is a very important result, and this concept is of fundamental significance in radiated field phenomena, which are discussed in Chap. 10.

A word is in order regarding Eq. (6-6). The magnitude of the force specified herein is generally small compared with the Coulomb force, except at the high velocities. The forces involved are sufficient in such devices to produce a noticeable deflection of the electron beam. It is precisely this action which produces the scanning fields in a television

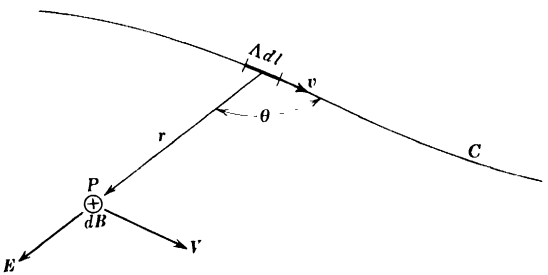


FIG. 6-4. To deduce the magnetic field at a point due to a current.

picture tube, although the magnetic field is usually produced by a current-carrying conductor rather than by a free-charge convection current. In fact, Eq. (6-6) is of basic importance in many devices, including the cyclotron and the mass spectrograph.

6-2. The Magnetic Field of a Current. It is desired to deduce an expression for the magnetic field at all points in space due to a specified system of conductors carrying currents. This will be accomplished by finding the resultant magnetic field at any point in space by the vector addition of the magnetic fields associated with each of the several currents. Initially, therefore, the magnetic field vector due to a single current is to be considered. The more general case is discussed in Sec. 6-6.

Refer to Fig. 6-4, in which C represents a moving charge of density Λ per unit length. The radius vector *from* the current element $\Lambda\, dl$ to the point P at which $\mathbf{B}$ is to be measured is denoted as $\mathbf{r}$. By Eq. (6-5), the magnetic field at the point P due to the current element $\Lambda\, dl$ is

$$d\mathbf{B} = \frac{\Lambda \mathbf{dl} \times \mathbf{r}_0}{4\pi\epsilon_0 c^2 r^2} = \frac{\Lambda\, dl}{4\pi\epsilon_0 r^2 c^2} v \sin\theta$$

since the electric field due to the element of charge $\Lambda\, dl$ is

$$\mathbf{E} = \frac{\Lambda\, dl}{4\pi\epsilon_0 r^2}\mathbf{r}_0$$

Note also that the total current or rate of transport of charge past a given point is given by $I = \Lambda v$. It is convenient to use the following fundamental relation of electromagnetic theory:

$$\epsilon_0\mu_0 = \frac{1}{c^2} \tag{6-7}$$

More will be said about this relationship in Chap. 10. Hence, the total magnetic field at all points in space due to a conduction or convection current of strength I is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_C I\,\frac{d\mathbf{l} \times \mathbf{r}_0}{r^2} \tag{6-8}$$

where μ_0, which is known as the *permeability of free space*, has a value $\mu_0 = 10^{-7} \times 4\pi$ henrys/m.

Observe that the vector $\mathbf{B}$ is directed perpendicular to $d\mathbf{l}$ and $\mathbf{r}$. Hence, each contribution to the integral in Eq. (6-8) is a vector perpendicular to $d\mathbf{l}$ and $\mathbf{r}$, and in the direction of advance of a right-hand screw rotated from $d\mathbf{l}$ to $\mathbf{r}$. The integral is taken around the path of the conducting current. This formula is sometimes referred to as *Ampère's law* and sometimes as the *Biot-Savart law*. It will here be referred to as the *Ampère-Biot-Savart law*.

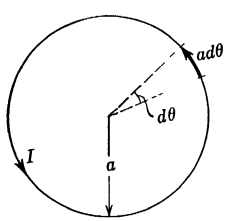


FIG. 6-5. A circular loop carrying a current.

Example 6-2.1. Find an expression for the magnetic induction at the center of a circular current loop.

Solution. Refer to Fig. 6-5, which illustrates the problem. Note from the diagram that

$$d\mathbf{l} = a\, d\theta\, \boldsymbol{\theta}_0$$

where $\boldsymbol{\theta}_0$ is a unit vector in the θ direction. Note also that, since $\boldsymbol{\theta}_0$ and $\mathbf{r}_0$ are perpendicular to each other at all points along the path, then the vector integral given in Eq. (6-8) reduces in this case to

$$\mathbf{B} = (\boldsymbol{\theta}_0 \times \mathbf{r}_0)\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{Ia\, d\theta}{a^2} = (\boldsymbol{\theta}_0 \times \mathbf{r}_0)\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{I\, d\theta}{a}$$

This integral is integrated directly to give

$$\mathbf{B} = \frac{\mu_0 I}{2a}\mathbf{k} \tag{6-9}$$

The unit vector $\mathbf{k}(\equiv \boldsymbol{\theta}_0 \times \mathbf{r}_0)$ is included to call attention to the fact that the magnetic field is directed out of the plane of the paper, which is here taken to be the $+Z$ axis.

Example 6-2.2. Find the magnetic field in the vicinity of a straight current filament of finite length.

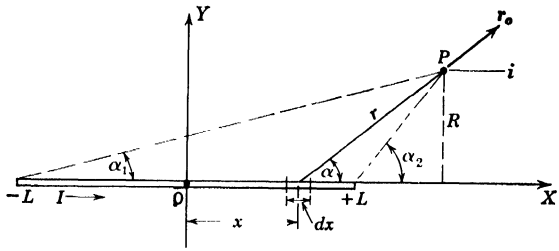


FIG. 6-6 A straight current filament

Solution. Refer to Fig. 6-6 for the geometry and notation to be employed. Consider the contribution to the magnetic field by the current element $L\,d\mathbf{l}$, as illustrated. It follows from Eq. (6-8) that

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\,dx}{[(L - x + R\cot\alpha_2)^2 + R^2]} \mathbf{i} \times \mathbf{r}_0$$

From the figure, it is seen that

$$L - x + R\cot\alpha_2 = R\cot\alpha$$

Then,
$$dx = +R\csc^2\alpha\,d\alpha$$

and also
$$\csc\alpha = \frac{\sqrt{(L - x + R\cot\alpha_2)^2 + R^2}}{R}$$

Combine with the above expression to obtain for $d\mathbf{B}$ the form

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi R}\sin\alpha\,d\alpha\,\boldsymbol{\theta}_0 \tag{6-10}$$

The direction of $\mathbf{B}$ is specified by $(\mathbf{i} \times \mathbf{r}_0) \equiv \boldsymbol{\theta}_0$ and is perpendicular to the plane containing $\mathbf{i}$ and $\mathbf{r}_0$.

This expression is integrated between the limits α_1 and α_2 and gives

$$\mathbf{B} = \frac{\mu_0 I}{4\pi R}(\cos\alpha_1 - \cos\alpha_2)\boldsymbol{\theta}_0 \tag{6-11}$$

Consequently, the magnetic field is circular and by symmetry has its center on the axis.

For the special case when the length of the line becomes very long, thereby approaching the infinite case, the following conditions prevail:

$$L \to \infty \qquad \alpha_1 \to 0 \qquad \alpha_2 \to \pi$$

Subject to these, Eq. (6-11) reduces to

$$\mathbf{B} = \frac{\mu_0 I}{2\pi R}\boldsymbol{\theta}_0 \tag{6-12}$$

This is often called the law of Biot and Savart, rather than the more general expression given as Eq. (6-8). Attention is called to the fact that the magnetic field lines must be closed curves, from the results of Eqs. (6-11) and (6-12). In fact, it will later be shown that the **B** lines are always closed curves and in this respect are quite different from the **E** or **D** lines in electric fields.

A simple and very rapid rule has been devised for specifying the direction of the magnetic field vector due to a current element. This rule, which is known as the *right-hand rule*, is contained in the direction to be associated with the cross product of two vectors. If the wire is gripped by the right hand so that the thumb points in the direction of the current, the fingers point in the direction of the magnetic vector. Correspondingly, if the wire is wound as a coil and the fingers point in the direction of the current in the wires, the thumb points in the direction of the magnetic vector.

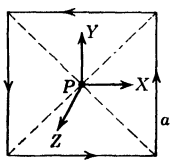


FIG. 6-7. A square current-carrying loop.

Example 6-2.3. Find the magnetic induction at the center of square current loop. The geometry of the problem is illustrated in Fig. 6-7. Note that the situation here is roughly analogous to that in Example 6-2.1, except for the different geometry.

Solution. The solution of this problem is given by a direct application of Eq. (6-11). For the geometry of this problem, the angles must be such that $\alpha_1 = 45°$, $\alpha_2 = 135°$. Observe also that each linear current segment produces an equal contribution to the field. All contributions are in the $+Z$ or the $\mathbf{k}$ direction at the point P, since the $\boldsymbol{\theta}_0$ and the $\mathbf{k}$ directions coincide at this point. The total field at point P is, therefore,

$$\mathbf{B} = 4\left[\frac{\mu_0 I}{4\pi(a/2)}(\cos 45° - \cos 135°)\boldsymbol{\theta}_0\right]$$

which reduces to

$$\mathbf{B} = \frac{2\sqrt{2}\,\mu_0 I}{\pi a}\mathbf{k} \tag{6-13}$$

Observe that the direction of the magnetic vector agrees with that required by an application of the right-hand rule.

Example 6-2.4. Find the magnetic induction at any point on the line through the center, and perpendicular to the plane of a plane circular current loop.

Solution. Refer to Fig. 6-8, which illustrates the problem. This problem is somewhat more complicated than the prior examples, as this requires considerations in three dimensions. The attack on the problem is quite straightforward and employs techniques discussed in solving the examples of Sec. 2-3.

Consider two symmetrically disposed elements of the wire, as shown. Owing to the symmetry of the field, the vertical or X components of the magnetic field vector will cancel, and only the horizontal or Z components remain. Because of this, the resulting magnetic field vector due to the two elements of the current loop dl is

$$dB_z = 2\left(\frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}\right)\cos\beta$$

since the angle between dl and r is 90°, whence sin $(dl,r) = 1$. However, from the figure

$$\cos \beta = \frac{a}{r}$$

and

$$dl = a\, d\theta$$

The total horizontal component of the magnetic field vector is then

$$B_z = \int \frac{\mu_0}{2\pi} \frac{Ia}{r^2} d\theta \frac{a}{r} = \int_0^{\pi} \frac{\mu_0}{2\pi} I \frac{a^2}{r^3} d\theta$$

where the limits on the integral, 0 to π, are appropriate to take into account the

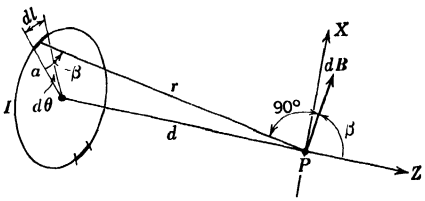


FIG. 6-8. Point P is at a distance d from the plane of a current-carrying loop.

contribution from the entire loop, since each dl of Fig. 6-8 sweeps out one-half the loop. The result of the integration is

$$B_z = \frac{\mu_0 I a^2}{2r^3} \tag{6-14}$$

which becomes

$$B_z = \frac{\mu_0 I a^2}{2(a^2 + d^2)^{3/2}} \tag{6-15}$$

This may, of course, be written in the form

$$\mathbf{B} = \frac{\mu_0 I a^2}{2(a^2 + d^2)^{3/2}} \mathbf{k} \tag{6-16}$$

Example 6-2.5. To find the magnetic field vector at any point on the axis of a solenoid.

Solution. A projection view of the solenoid, a tightly wound coil of cylindrical shape, is shown in Fig. 6-9. To solve this problem, it will be supposed that the diameter of the solenoid is large compared with the diameter of the current-carrying wire. In this way, it is assumed that the solenoid is made up of a stack of current-carrying loops of the type illustrated in Fig. 6-8.

The magnetic field vector at any point on the axis of a current loop is given by Eq. (6-15). Now, considering the elementary current-carrying loop to contain $N\, dz/L$ turns, where N is the total number of turns in the solenoid and L its length, then the expression for the magnetic field vector due to this loop is, by a direct application of Eq. (6-15),

$$dB_z = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \frac{N}{L} dz$$

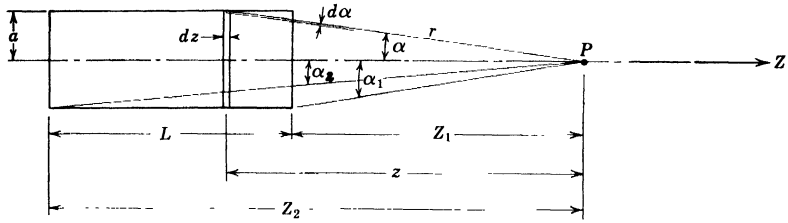


FIG. 6-9. A projection view of a solenoid through a plane containing the diameter.

The total field at point P is then given by the expression

$$B_z = \frac{\mu_0 IN}{2L} \int_{z_1}^{z_2} \frac{a^2}{(a^2 + z^2)^{3/2}}\, dz \tag{6-17}$$

In order to evaluate this expression, it is noted from the figure that

$$z = a \cot \alpha$$

whence

$$a^2 + z^2 = a^2(1 + \cot^2 \alpha) = a^2 \csc^2 \alpha$$

$$dz = -a \csc^2 \alpha\, d\alpha$$

Combining with Eq. (6-17), there results

$$B_z = -\frac{\mu_0 IN}{2L} \int_{\alpha_1}^{\alpha_2} \sin \alpha\, d\alpha$$

This integrates to the expression

$$B_z = -\frac{\mu_0 IN}{2L} (\cos \alpha_1 - \cos \alpha_2) \tag{6-18}$$

as the desired solenoid formula.

Two special cases are of considerable importance.

CASE A. FOR POINT P AT THE END OF A FINITE SOLENOID. In this case, $\alpha_1 = 90°$, whence Eq. (6-18) reduces to

$$B_z = \frac{\mu_0 IN}{2L} \cos \alpha_2$$

But in this case

$$\cos \alpha_2 = \frac{L}{\sqrt{a^2 + L^2}}$$

from which

$$B_z = \frac{\mu_0 IN}{2L} \frac{L}{\sqrt{a^2 + L^2}} \tag{6-19}$$

If the length L is large compared with the dimension a, then this expression reduces approximately to the form

$$B_z \doteq \frac{\mu_0 NI}{2L} \qquad \text{for } L \gg a \tag{6-20}$$

CASE B. FOR POINT P AT THE CENTER OF A FINITE SOLENOID. In this case, $\alpha_2 = \pi + \alpha_1$, whence

$$B_z = \frac{\mu_0 IN}{L} \cos \alpha_2$$

But
$$\cos \alpha_2 = \frac{L/2}{\sqrt{a^2 + (L/2)^2}}$$

Then,
$$B_z = \frac{\mu_0 IN}{L} \frac{1}{\sqrt{1 + (2a/L)^2}} \tag{6-21}$$

For the case of a long solenoid relative to its radius,

$$B_z \doteq \frac{\mu_0 IN}{L} \tag{6-22}$$

A comparison of this expression with Eq. (6-20) reveals the fact that the magnetic field vector at the center of a solenoid is twice its value at the end of the solenoid. This means that half the B lines at the center of the solenoid pass out through the surface of the solenoid before they reach the end of the solenoid.

Example 6-2.6. Find the magnetic field vector at a point outside an infinitely long cylindrical conductor carrying a uniform current density.

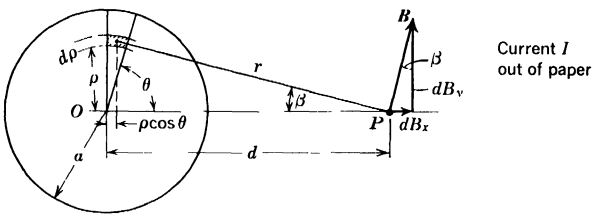


FIG. 6-10. A long cylindrical current-carrying conductor.

Solution. Refer to Fig. 6-10 for the geometry of the problem, with I out of the paper. In this particular problem, it is assumed that the conductor has an appreciable cross section. This requires, in order to find the total magnetic field vector, that the contributions of the type given by Eq. (6-8) for filamentary currents in which the finite current is divided be summed vectorially.

The current density in the conductor, for a total current I, is given by

$$J = \frac{I}{\pi a^2} \tag{6-23}$$

The magnetic field vector due to the current element shown crosshatched in Fig. 6-10, by Eq. (6-12), is

$$dB = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0}{2\pi r} J\rho \, d\rho \, d\theta \tag{6-24}$$

since the total current in the elementary area is

$$dI = J\rho \, d\rho \, d\theta$$

To find the total magnetic field vector at point P requires that the differential effect expressed by Eq. (6-14) must be integrated over the entire surface of the conductor section. Observe that the component dB_x of the figure must yield

zero, because of symmetry. This is evident by choosing a second filamentary current element symmetrical with that shown, but below the line OP, and noting that the contributions to dB_x cancel. It is required only that dB_y be evaluated.

The magnetic-field-vector element is given by the expression

$$dB_y = dB \cos \beta = \frac{\mu_0 J}{2\pi r} \rho \, d\rho \, d\theta \left(\frac{d - \rho \cos \theta}{r} \right)$$

This is integrated over the limits shown, with d being constant.

$$B_y = \frac{\mu_0 J}{2\pi} \int_0^a \rho \, d\rho \left(\int_0^{2\pi} \frac{d - \rho \cos \theta}{d^2 + \rho^2 - 2 d\rho \cos \theta} \, d\theta \right) \tag{6-25}$$

The integral in the parentheses is a standard integral, the solution of which is

$$\int_0^{2\pi} \frac{d - \rho \cos \theta}{d^2 + \rho^2 - 2 d\rho \cos \theta} \, d\theta = \frac{2\pi}{d} \qquad \text{for } d > \rho \tag{6-26}$$

Hence, for $d > \rho$, that is, for points outside the conductor,

$$B_y = \frac{\mu_0 J}{2\pi} \frac{2\pi}{d} \int_0^a \rho \, d\rho$$

which is

$$B_y = \frac{\mu_0 J}{2\pi d} \pi a^2$$

Now combine this with Eq. (6-23) to get

$$B_y = \frac{\mu_0 I}{2\pi d} \tag{6-27}$$

Note that this is the same result that was obtained in Eq. (6-12) for a fine line of current of the same total value, if that fine line of current is supposed concentrated at the center of the cylinder. This result is not valid for points inside the conductor.

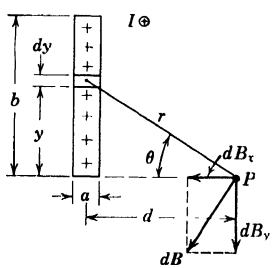


FIG. 6-11. A point adjacent to a long bus bar of current.

Example 6-2.7. The bus bars for distributing large currents are usually rectangular in shape rather than circular, as illustrated in Fig. 6-11, which shows such a bus bar. It is required to find the magnetic field at the point P adjacent to this bus bar and directly opposite one edge of it.

Solution. To solve this problem, the bus bar will be considered to be composed of a series of continuous current strips, each of which approximates the filamentary currents of Example 6-2.2. The contribution to the **B** vector due to the current element shown in the diagram for a total current I in the bus bar is given by Eq. (6-9)

$$d\mathbf{B} = \frac{\mu_0 I}{2\pi r} \mathbf{a}_\theta \tag{6-28}$$

For the present problem, this has the form

$$d\mathbf{B} = \frac{\mu_0}{2\pi} \frac{(I/b)\, dy}{(d^2 + y^2)^{1/2}} \mathbf{a}_\theta \tag{6-29}$$

The direction of dB is that shown in Fig. 6-12. It is convenient to evaluate the X and Y components of the resulting **B** vector. Thus, it follows that

$$\begin{aligned} dB_x &= \frac{\mu_0}{2\pi} \frac{Iy\, dy}{b(d^2 + y^2)} \\ dB_y &= \frac{\mu_0}{2\pi} \frac{I}{b} \frac{d}{d^2 + y^2} dy \end{aligned} \tag{6-30}$$

Integrating the first of these expressions gives

$$B_x = \frac{\mu_0 I}{2\pi b} \int_0^b \frac{y\, dy}{d^2 + y^2} = \frac{\mu_0 I}{4\pi b} \ln (d^2 + y^2) \Big|_0^b$$

or

$$B_x = \frac{\mu_0 I}{4\pi b} \ln \frac{d^2 + b^2}{d^2} \tag{6-31}$$

Integrating the second of these expressions yields

$$B_y = \frac{\mu_0 I}{2\pi b} \int_0^b \frac{d\, dy}{d^2 + y^2} = \frac{\mu_0 I}{2\pi b} \tan^{-1} \frac{y}{d} \Big|_0^b$$

or

$$B_y = \frac{\mu_0 I}{2\pi b} \tan^{-1} \frac{b}{d} \tag{6-32}$$

In the general case when the point P may be located at any point opposite the bus bar rather than opposite one edge, the foregoing results may be extended to give the appropriate results. The situation is clarified by referring to Fig. 6-12. Observe that the magnetic field at point P may be considered to be the resultant of two fields; that at the edge, due to a bus bar of length b_1; and that at the point opposite the edge, due to a bus bar of length b_2. The component fields are given in Fig. 6-12. These may be found by first noting that, for a total current I in the bus bar, the current in the b_1 section is

$$I_{b1} = I \frac{b_1}{b_1 + b_2}$$

and the current in the b_2 section is

$$I_{b2} = I \frac{b_2}{b_1 + b_2}$$

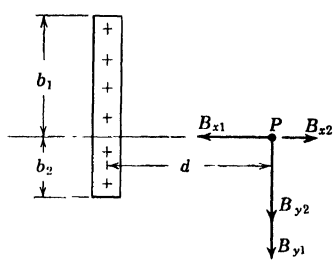


FIG. 6-12. The magnetic field at any point opposite a long bus bar of current.

The components of the field are then deduced from Eqs. (6-31) and (6-32). Note that for the upper portion of the bus bar, the value b_1 is inserted for y, whereas the value $-b_2$ is used for y for the lower portion of the bus bar. The resultant values for the components of B are

$$B_x = B_{x1} + B_{x2} = \frac{\mu_0 I}{4\pi(b_1 + b_2)} \left(\ln \frac{d^2 + b_1^2}{d^2} - \ln \frac{d^2 + b_2^2}{d^2} \right)$$

or

$$B_x = \frac{\mu_0 I}{4\pi(b_1 + b_2)} \ln \frac{d^2 + b_1^2}{d^2 + b_2^2} \tag{6-33}$$

In a similar manner, it is found that

$$B_y = B_{y1} + B_{y2} = \frac{\mu_0 I}{2\pi(b_1 + b_2)} \left(\tan^{-1} \frac{b_1}{d} + \tan^{-1} \frac{b_2}{d} \right) \tag{6-34}$$

Example 6-2.8. Find the magnetic field vector at a point in the plane but outside of a rectangular surface-current element.

Solution. Refer to Fig. 6-13 which illustrates the geometry of the problem. In this problem, the current is assumed to be uniform across the sheet, and in the direction shown. This problem may be approached in two ways. In one way, the current sheet can be divided into narrow current filaments along the X axis.

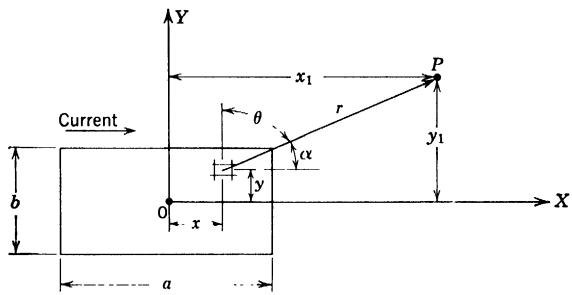


FIG. 6-13. A rectangular surface-current element.

Assume the results of Example 6-2.2, and combine vectorially the effects of the stack of strips that would make up the extension in the Y direction. The way chosen starts back at Eq. (6-8) and takes account of the extension in space by extending the elementary area over the surface and combining the contributions vectorially. These methods are substantially the same, the first method being one step further advanced in the solution.

The magnetic field vector at point P due to the elementary current is directed along the Z axis and is, in accordance with Eq. (6-8),

$$dB_z = \frac{\mu_0}{4\pi} \left(\frac{I}{b}\, dy \right) \frac{dx \sin \alpha}{r^2} \tag{6-35}$$

where the current magnitude is

$$dI = \frac{I}{b}\, dy$$

Note from the figure that

$$r^2 = [(x_1 - x)^2 + (y_1 - y)^2]$$

$$\sin \alpha = \frac{y_1 - y}{r}$$

so that

$$dB_z = \frac{\mu_0 I}{4\pi b} \frac{y_1 - y}{[(x_1 - x)^2 + (y_1 - y)^2]^{3/2}}\, dx\, dy \tag{6-36}$$

The total field is given by the double integral

$$B_z = \frac{\mu_0 I}{4\pi b} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} \frac{y_1 - y}{[(x_1 - x)^2 + (y_1 - y)^2]^{3/2}} dy \tag{6-37}$$

The result of this integration is the very complicated expression shown,

$$B_z = \frac{\mu_0}{4} \frac{I}{b} \left\{ \ln \frac{\left[\left(x_1 - \frac{a}{2}\right) + \sqrt{\left(x_1 - \frac{a}{2}\right)^2 + \left(y_1 + \frac{b}{2}\right)^2}\right]\left[\left(x_1 + \frac{a}{2}\right) + \sqrt{\left(x_1 + \frac{a}{2}\right)^2 + \left(y_1 - \frac{b}{2}\right)^2}\right]}{\left[\left(x_1 + \frac{a}{2}\right) + \sqrt{\left(x_1 + \frac{a}{2}\right)^2 + \left(y_1 + \frac{b}{2}\right)^2}\right]\left[\left(x_1 - \frac{a}{2}\right) + \sqrt{\left(x_1 - \frac{a}{2}\right)^2 + \left(y_1 - \frac{b}{2}\right)^2}\right]} \right\}$$

6-3. Character of B Lines. It was noted in connection with Example 6-2.2 that the B lines produced by currents always close on themselves. This means, of course, that, if an infinitesimal region of space were examined, the total number of B lines entering the specified region must be equal to the total number of B lines leaving the specified region. As has already been found (see Sec. 5-3), a region for which continuity of field lines prevails is characterized mathematically by an expression of the form

$$\text{div } \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

That is, if the divergence of a vector point function vanishes, no sources or sinks exist in the region over which the divergence vanishes. It is of interest to examine div **B**.

It is convenient in the proof to follow to use several results of vector analysis. The proof given is not in the most general form but recognizes the limited familiarity of the student with vector analysis. It is desired to examine the value of the following function, noting that $I\,d\mathbf{l} = \mathbf{I}\,dl$:

$$\text{div } \mathbf{B} = \nabla \cdot \left(\frac{\mu_0}{4\pi} \int \frac{I\,d\mathbf{l} \times \mathbf{r}_0}{r^2}\right) = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\frac{\mathbf{I} \times \mathbf{r}_0}{r^2}\right) dl$$

$$= \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\frac{\mathbf{I} \times \mathbf{r}}{r^3}\right) dl \tag{6-39}$$

which may be written in the form

$$\text{div } \mathbf{B} = -\frac{\mu_0}{4\pi} \int \nabla \cdot \left[\mathbf{I} \times \nabla\left(\frac{1}{r}\right)\right] dl \tag{6-40}$$

But according to the vector-analysis expansion (see Prob. 6-3*b*) it is noted that

$$\nabla \cdot \left[\mathbf{I} \times \nabla\left(\frac{1}{r}\right)\right] = \nabla\left(\frac{1}{r}\right) \cdot \nabla \times \mathbf{I} - \mathbf{I} \cdot \nabla \times \nabla\left(\frac{1}{r}\right) \tag{6-41}$$

To interpret this result it is recalled that **B** is being specified at the field point, whereas the current is that at the source point, specified by $d\mathbf{l}$. If, therefore, the field point is specified by the variables (x,y,z), the source point might be specified by the variables (x',y',z'). But since the divergence operation is being taken with respect to the field-point variables, which are independent of the source-point variables, then

$$\nabla\left(\frac{1}{r}\right) \cdot \nabla \times \mathbf{I} = 0 \tag{6-42}$$

The second term in Eq. (6-41), which is the curl of the gradient of a scalar, is identically zero. Then it follows that *always,*

$$\text{div } \mathbf{B} = 0 \tag{6-43}$$

This means that magnetic field lines are always continuous. If it is recalled that in the discussion of the electric field div $\mathbf{D} = \rho$ was interpreted to show that the D lines originate and terminate on free charges, then the present result may be interpreted to show that free magnetic poles or charges can never exist.

6-4. The Line Integral of the Magnetic Field Vector. In addition to the fundamental Ampère-Biot-Savart relation of Eq. (6-8), a second very useful relation exists between the current and the magnetic field vector. This is the relation between the current I and the line integral of the magnetic field vector. Refer to Fig. 6-14, which shows a straight wire with the current assumed into the paper. The magnetic field lines for this configuration are closed circles, as illustrated, and are specified mathematically by Eq. (6-12). It is desired to examine the value of the line integral $\oint \mathbf{B} \cdot d\mathbf{l}$ when taken around the closed path. Thus, by Eq. (6-12), the integral to be evaluated is the following:

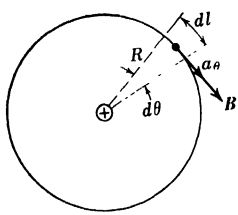


FIG. 6-14. A long, straight wire and the magnetic field line around it.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi R} \int \mathbf{a}_\theta \cdot d\mathbf{l}$$

Note, however, that $\mathbf{a}_\theta$ and $d\mathbf{l}$ are in the same direction, so that the angle between them is 0°. Thus, the integral becomes

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi R} \int dl$$

Moreover, since

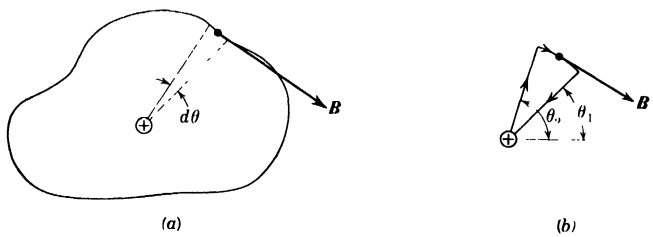
$$dl = R\, d\theta$$

then the integral yields the value

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi R} R \int_0^{2\pi} d\theta = \mu_0 I \tag{6-44}$$

which is independent of R.

Before drawing any conclusions from this result, it is desired to examine the value of the line integral if the closed path of integration around the current was of an irregular shape, as shown in Fig. 6-15. As an aid to the present problem, it is desirable first to determine the value of the line integral along a small section of the total path, which is shown in amplified form in Fig. 6-15*b*. Note that the total path is being divided into a large number of circular arcs.



(*a*) (*b*)

FIG. 6-15. An arbitrary closed path of integration around a current: (*a*) total path; (*b*) an element of the path.

For a single segment of the circle shown, the value of **B** along the arc is that given by Eq. (6-12), whence the line integral along the arc, following the same development as above, is

$$\int_{\text{Arc}} \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} (\theta_2 - \theta_1) \tag{6-45}$$

Along the radial paths, **B** and $d\mathbf{l}$ are perpendicular to each other, whence $\mathbf{B} \cdot d\mathbf{l}$ is zero. Clearly for the element of the path shown, therefore,

$$\int_{\text{Path}} \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} (\theta_2 - \theta_1) \tag{6-46}$$

Since the result is independent of the distance R, then, by taking the summation of the contribution to the total integral by all segments into which the total path has been divided, it is clear that Eq. (6-44) again is the result for the entire path enclosing the current.

If the path is taken either radial or parallel to the current, **B** and $d\mathbf{l}$ are normal to each other and $\mathbf{B} \cdot d\mathbf{l} = 0$. If, therefore, one were to choose an arbitrary path for the integration around the current, the path could again be divided into segments. Now, since any arbitrary $d\mathbf{l}$ can be

analyzed into radial, tangential, and parallel components, it is clear that only the tangential component contributes to the integral, with the final result again being given by Eq. (6-44).

The remaining possibility to be examined is that illustrated in Fig. 6-16, which is the case when the closed path of integration does not enclose the current. Here, too, the path is broken into segments in the manner shown. Based on the considerations that lead to Eq. (6-46)

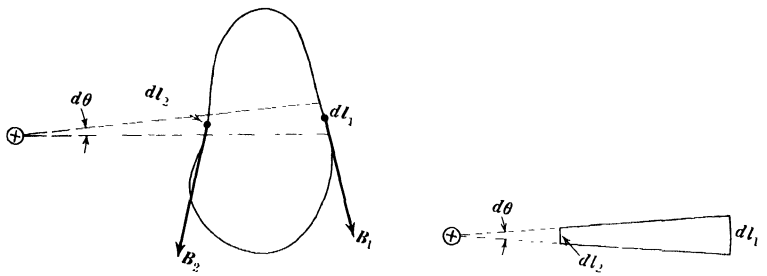


FIG. 6-16. A closed path of integration that does not enclose the current.

for the path of Fig. 6-16*b*, two tangential contributions to the integral exist. Thus,

$$\int_{\text{Total path}} \mathbf{B} \cdot d\mathbf{l} = \int_{\text{Arc 1}} \mathbf{B} \cdot d\mathbf{l}_1 + \int_{\text{Arc 2}} \mathbf{B} \cdot d\mathbf{l}_2$$

But these integrals have equal and opposite values, whence the value of the integral vanishes.

The results of the foregoing development can be summarized in the general result

$$\oint \mathbf{B} \cdot d\mathbf{l} = \begin{cases} \mu_0 I & \text{if path links current} \\ 0 & \text{if path does not link current} \end{cases}$$

If by I is meant the net current which passes through *any surface bounded by the path of integration*, then the result may be written simply as

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \tag{6-47}$$

Attention is called to the fact that this expression has been derived from considerations of a long, straight wire. It is actually true for conductors of other shapes, but the integration is more difficult. This result is known as *Ampère's law*.

By comparing the form of this expression with that given by Eq. (1-22), which was discussed as an example of a conservative field, then evidently it follows that the B field is an example of a nonconservative field. The essential difference between these two types of fields is that,

in the conservative field, the net work in passing around a complete path is zero, whereas, in the nonconservative field, a finite net work must be done in passing around the complete path.

Example 6-4.1. Find the magnetic field vector inside a long, straight wire of cylindrical cross section.

Solution. Refer to Fig. 6-17. By a direct application of Ampère's law, noting that **B** is constant along the circular path and also taking account of the total current contained within the path, then

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \int dl = \mu_0 \frac{I}{\pi a^2} \pi R^2$$

which leads to the expression

$$\mathbf{B} = \frac{\mu_0 R I}{2\pi a^2} \mathbf{a}_\theta \tag{6-48}$$

The approach here has been precisely that which led to Eq. (6-44), although it has been appropriately modified for the special aspects of this problem.

Example 6-4.2. As an interesting variation of Example 6-4.1, it is required to find the magnetic field vector at the center of a cylindrical hole which is displaced from the center of a cylindrical conductor, as illustrated in Fig. 6-18.

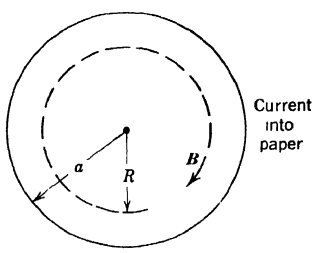


FIG. 6-17. A path inside a cylindrical conductor.

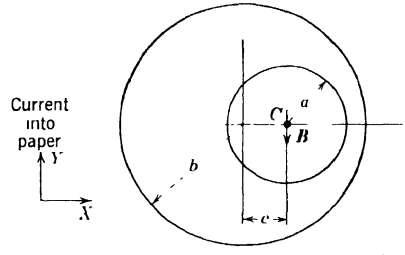


FIG. 6-18. An off-centered hole in a cylindrical conductor.

Solution. The solution of this problem is readily accomplished if it is first noted that any effect of the hole in the conductor can be accounted for by considering the zero current in the hole as being made up of two currents which are in opposite directions. The problem then becomes the superposition of two problems, (*a*) to find the field at point *C* due to a homogeneous conductor carrying a current *I'*, and (*b*) the field at point *C* due to a homogeneous conductor of radius *a* carrying a current of equal current density in the reverse direction to that in (*a*). The resultant magnetic vector is the vector sum of the effects under (*a*) and (*b*). Let I = the actual current in the original conductor, I' = the current in the homogeneous conductor having the same current density. Then

$$\frac{I'}{\pi b^2} = \frac{I}{\pi(b^2 - a^2)}$$

In accordance with the results of Example 6-4.1, the magnetic vector at *C* due to an equivalent solid conductor is

$$\mathbf{B} = \frac{\mu_0 I c}{2\pi(b^2 - a^2)} (-\mathbf{j})$$

The magnetic vector at C due to the equivalent solid conductor of radius a carrying a specified known current opposite that above is zero, since no current is contained within a path of zero radius.

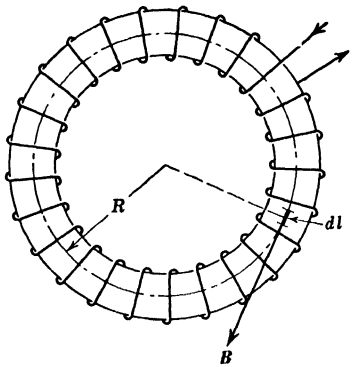


FIG. 6-19. A schematic view of a toroid with N closely spaced turns.

The resultant magnetic vector at point C, which is the summation of the two foregoing factors, is then simply

$$\mathbf{B} = \frac{\mu_0 I r}{2\pi(b^2 - a^2)}(-\mathbf{j}) \qquad (6\text{-}49)$$

Example 6-4.3. Find the magnetic field vector at any point inside a toroid.

Solution. A toroid may be visualized as a doughnut with wire wound uniformly around the doughnut. A schematic view of a toroid is given in Fig. 6-19. From Example 6-2.5, the magnetic field vector is directed along the axis of the solenoid (and the toroid may be considered as a solenoid that has been bent into a circle, with the ends joined). Since the axis of the resulting bent solenoid is circular, the B lines are also circular. Since $\mathbf{B}$ inside the toroid is a constant at a distance R from the center and is directed along $d\mathbf{l}$, a direct application of Ampère's law gives

$$B \oint dl = \mu_0 NI$$

since the total current now linked is NI if there are N turns on the toroid. Thus, the integral leads to

$$B = \frac{\mu_0 NI}{2\pi R} \qquad (6\text{-}50)$$

Observe that B varies inversely with R, so that the value of B is not uniform across a section of the toroid.

Observe that, if the path lies outside of the toroid, the current linked is zero so that, everywhere outside, the magnetic field vector is zero. Note also that, if account is to be taken of the fact that the windings of the toroid must be skewed slightly for the wire to progress along the toroidal core, then the field will not necessarily be zero outside the toroid and, in fact, will be that due to a single circular turn of wire carrying a current I.

Example 6-4.4. Show that the magnetic field vector is constant across a section of a solenoid having a length much greater than the diameter, and for points far from the ends.

Solution. Attention is called to the fact that this example is an extension of Example 6-2.5 and is intended to show that the results given by Eq. (6-21) are valid for points far from the ends and apply for any point across the section of the solenoid, even though it was deduced for a point P on the axis of the solenoid. That is, it will be shown that the magnetic field will be parallel to the axis and uniform across the solenoid.

To show that the result is true, refer to Fig. 6-20, which shows a rectangular path through the surface of the solenoid. It is desired to evaluate the line

integral of the magnetic vector around the rectangular path shown. Over the two radial sides within the solenoid, the path is normal to the field, and so their contribution is zero. Similarly, the contribution is zero from the portions of the paths of d_1 and d_2 which are outside the solenoid, as the magnetic vector is zero in this region. The only contribution comes from that part of the path which is parallel to the axis and inside the solenoid. Thus,

$$\oint \mathbf{B} \cdot d\mathbf{l} = Bd_1 = \mu_0 \frac{N}{L} d_1 I$$

from which it is seen that

$$B = \frac{\mu_0 NI}{L} \tag{6-51}$$

which is precisely Eq. (6-22). Note that, since this result is independent of the position of the path within the solenoid, this expression is completely general within the solenoid.

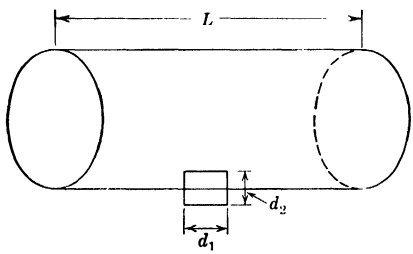


FIG. 6-20. A solenoid which is long compared with its diameter.

6-5. Mathematical Digression—Curl and Stokes' Theorem. In the form given in Eq. (6-40), the Ampère circuital law is a macroscopic law and relates the magnetic vector with a finite current. If it is desired to apply the law to problems of a general nature without a high degree of symmetry, it is necessary to relate the magnetic vector with the current density, both of which are vectors. The situation here is rather like that in Eq. (2-62), which related the divergence of the electric field intensity with the charge density to specify the conditions at a point.

Suppose that Eq. (6-47) is written in the following form, which would apply in the general case,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n}\, dS \tag{6-52}$$

where the area S is that specifying the region through which the current exists. It is desired to examine the situation as the line integral is taken around a very small path in the region of the current. In particular, let the path around which the line integral is taken define the area ΔS. We examine the situation as the path length approaches zero, whence the area also approaches zero. In this case, it follows then that

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{\Delta S} \mathbf{J} \cdot \mathbf{n}\, dS$$

Since the area ΔS is taken very small, it may be assumed that $\mathbf{J}$ is constant over ΔS. Approximately, therefore,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \mathbf{J} \cdot \mathbf{n}\, \Delta S$$

Hence, in the limit, it is permissible to write that

$$\lim_{\Delta S \to 0} \frac{\oint \mathbf{B} \cdot d\mathbf{l}}{\Delta S} = \mu_0 \mathbf{J} \cdot \mathbf{n} = \mu_0 J_n \tag{6-53}$$

By *definition*, the component in the ith direction of the curl $\mathbf{B}$ is

$$\lim_{\Delta S \to 0} \frac{\oint \mathbf{B} \cdot d\mathbf{l}}{\Delta S} \mathbf{i} \equiv (\text{curl } \mathbf{B})_i = (\nabla \times \mathbf{B})_i \tag{6-54}$$

where ΔS is an element of area which is located at the specified point P and which is perpendicular to the ith direction. The integral is taken around the contour of this area, the direction of integration being such that the area is kept to the left in the integration. If $\mathbf{n}$ in Eq. (6-53) is taken in the X direction, then

$$\lim_{\Delta S_x \to 0} \frac{\oint \mathbf{B} \cdot d\mathbf{l}}{\Delta S_x} = (\nabla \times \mathbf{B})_x = \mu_0 \mathbf{J} \cdot \mathbf{i} = \mu_0 J_x$$

Similar expressions result for the Y and Z directions. By adding these components, it follows that Eq. (6-53) may be written

$$\text{curl } \mathbf{B} \equiv \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{6-55}$$

This is the *point statement* of Ampère's law corresponding to the integral statement given by Eq. (6-47). Taken together, the two corresponding forms are therefore

$$\begin{aligned} \oint \mathbf{B} \cdot d\mathbf{l} &= \mu_0 I \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \end{aligned} \tag{6-56}$$

These expressions bear the same general relationship to each other that Eqs. (2-62) do for the electric field.

It is of interest to write down Eq. (6-43) and Eq. (6-55), which together are

$$\begin{aligned} \text{div } \mathbf{B} &= \nabla \cdot \mathbf{B} = 0 \\ \text{curl } \mathbf{B} &= \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \end{aligned} \tag{6-57}$$

Observe, therefore, that both the divergence and the curl of the vector point function $\mathbf{B}$ have known values. According to the Helmholtz

theorem of uniqueness in vector analysis, the specification of the divergence and curl of a vector uniquely defines the vector. Therefore, **B** has been uniquely defined.

A well-known mathematical transformation follows by combining Eq. (6-55) with Eq. (6-52). The result has the form

$$\oint \mathbf{J} \cdot \mathbf{n}\, dS = \int_S \nabla \times \mathbf{B} \cdot \mathbf{n}\, dS = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \qquad (6\text{-}58)$$

This is Stokes' theorem and relates a surface integral with a line integral taken about the contour of the surface. This result is true for any vector.

Expressions for the curl are obtained for various coordinate systems by evaluating Eq. (6-54) for the various coordinate directions. The details are carried out for the rectangular coordinate system. Refer to Fig. 6-21, which shows a small rectangle in the YZ plane with the current

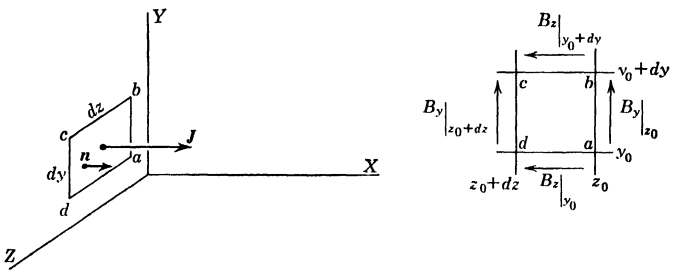


FIG. 6-21. To evaluate the X component of the curl.

density in the X direction. It is required to evaluate the line integral $\oint \mathbf{B} \cdot d\mathbf{l}$ around the indicated contour. Clearly, this may be written as

$$\oint_x \mathbf{B} \cdot d\mathbf{l} = \int_a^b \mathbf{B} \cdot d\mathbf{l} + \int_b^c \mathbf{B} \cdot d\mathbf{l} + \int_c^d \mathbf{B} \cdot d\mathbf{l} + \int_d^a \mathbf{B} \cdot d\mathbf{l}$$

The component values of **B** at the specified positions in space are as illustrated in Fig. 6-21. The respective integrals become as follows, since in general **B** may be written in the form

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

1. Along ab, $d\mathbf{l} = \mathbf{j}\, dy$:

$$\int_a^b \mathbf{B} \cdot d\mathbf{l} = \int_a^b (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \cdot \mathbf{j}\, dy = \int_a^b B_y\, dy = B_y\, dy\Big|_{z_0}$$

2. Along cd:

$$\int_c^d \mathbf{B} \cdot d\mathbf{l} = -B_y\, dy\Big|_{z_0+dz}$$

3. Along bc:

$$\int_b^c \mathbf{B} \cdot d\mathbf{l} = B_z\, dz\Big|_{y_0+dy}$$

4. Along da:

$$\int_d^a \mathbf{B} \cdot d\mathbf{l} = -B_z\, dz\Big|_{y_0}$$

Hence, the total line integral has the value

$$\oint_x \mathbf{B} \cdot d\mathbf{l} = B_y\, dy\Big|_{z_0} - B_y\, dy\Big|_{z_0+dz} + B_z\, dz\Big|_{y_0+dy} - B_z\, dz\Big|_{y_0}$$

Observe, however, that the expressions on the right-hand side may be written in different forms. Thus, it is noted that

$$B_y\, dy\Big|_{z_0} - B_y\, dy\Big|_{z_0+dz} = -\frac{\partial(B_y\, dy)}{\partial z}\, dz = -\frac{\partial B_y}{\partial z}\, dy\, dz$$

and

$$B_z\, dz\Big|_{y_0+dy} - B_z\, dz\Big|_{y_0} = +\frac{\partial(B_z\, dz)}{\partial y}\, dy = \frac{\partial B_z}{\partial y}\, dy\, dz$$

Therefore, it follows that

$$\oint_x \mathbf{B} \cdot d\mathbf{l} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) dy\, dz$$

Thus, by Eq. (6-54), it follows that, since

$$\lim_{dy\, dz \to 0} \frac{\oint_x \mathbf{B} \cdot d\mathbf{l}}{dy\, dz} = (\text{curl } \mathbf{B})_x$$

then the X component of the curl is

$$(\text{curl } \mathbf{B})_x = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)$$

Similarly, it is found that

$$(\text{curl } \mathbf{B})_y = \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)$$

$$(\text{curl } \mathbf{B})_z = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

Hence, finally,

$$\text{curl } \mathbf{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\mathbf{k} \quad (6\text{-}59)$$

A convenient way of remembering how the subscripts are arranged is to think of the determinant shown, noting, however, that it has no real numerical significance. It specifies only the arrangement of the subscripts.

$$\text{curl } \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} \quad (6\text{-}60)$$

Corresponding expressions may be found for the curl in other coordinate systems by the direct application of Eq. (6-54). For the *circular-cylindrical* coordinates,

$$\text{curl } \mathbf{B} = \begin{vmatrix} \frac{\mathbf{a}_r}{r} & \mathbf{a}_\theta & \frac{\mathbf{a}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ B_r & rB_\theta & B_z \end{vmatrix} \tag{6-61}$$

For *spherical coordinates*,

$$\text{curl } \mathbf{B} = \begin{vmatrix} \frac{\mathbf{a}_r}{r \sin \theta} & \frac{\mathbf{a}_\theta}{r \sin \theta} & \frac{\mathbf{a}_\phi}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ B_r & rB_\theta & r \sin \theta B_\phi \end{vmatrix} \tag{6-62}$$

6-6. Self- and Mutual Inductance. In considering Eq. (6-5), it was found that the units of the **B** vector were given as webers per square

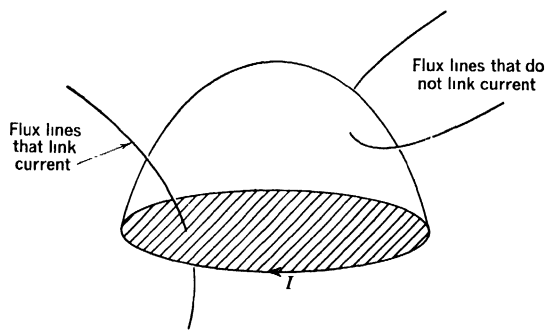


FIG. 6-22. A cap formed across a closed current circuit.

meter, and it was because of this that the **B** vector is also known as the magnetic flux density. Moreover, since the **B** vector is produced by, and is proportional to, the current, the presence of a current implies the presence of a magnetic field.

From the point of view of engineering practice, the following considerations are found to be very useful: Let I represent a closed circuit. Suppose, now, that a cap of any shape is formed across this circuit, such as that illustrated in Fig. 6-22. It was shown in Sec. 6-3 that the B lines are continuous. Consequently, a given B line either links the circuit of Fig. 6-22 or else may pass through the cap without linking the current. Those lines which pass through the cap, but which do not link the current, contribute B lines into and out of the cap.

Suppose that Gauss' law is applied to the closed surface of Fig. 6-22. Since div $\mathbf{B} = 0$ always, then it follows that

$$\oint \mathbf{B} \cdot \mathbf{n}\, dS = 0$$

But this may be written as

$$\int_{\text{Cap}} \mathbf{B} \cdot \mathbf{n}\, dS + \int_{\substack{\text{Cross-hatched} \\ \text{area}}} \mathbf{B} \cdot \mathbf{n}\, dS = 0$$

It follows from this that the total net magnetic flux Φ linking the cap, which is also the total net magnetic flux Φ linking the current, is given by the expression

$$\Phi = \int_{\text{Cap}} \mathbf{B} \cdot \mathbf{n}\, dS = \int_{\text{Cap}} B_n\, dS \qquad \text{webers} \tag{6-63}$$

where B_n is the normal component of $\mathbf{B}$.

In those cases where only a single current circuit exists, the flux lines that link the current are produced by the current. If more than one

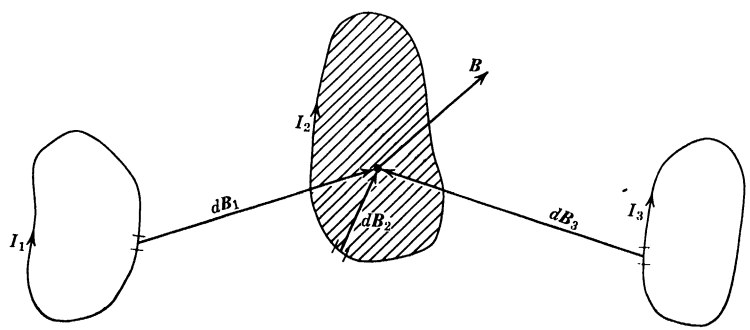


FIG. 6-23. Three interacting current circuits.

current circuit exists, then the situation becomes more complicated, since the flux lines produced by one current may link a number of adjacent other circuits. Specifically, consider the case of three current circuits, as illustrated in Fig. 6-23. It is desired to examine the total magnetic flux λ_2 linking circuit 2. This is accomplished by first drawing a cap with the current circuit 2 as boundary. The flux linkages are given by Eq. (6-63), where the resultant $\mathbf{B}$ at any point on the surface of the cap is the vector sum of the $\mathbf{B}$'s due to each current, or

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3$$

which is

$$\mathbf{B} = \underbrace{\frac{\mu_0}{4\pi} \oint I \frac{d\mathbf{l} \times \mathbf{r}_0}{r^2}}_{\text{Circuit 1}} + \underbrace{\frac{\mu_0}{4\pi} \oint I \frac{d\mathbf{l} \times \mathbf{r}_0}{r^2}}_{\text{Circuit 2}} + \underbrace{\frac{\mu_0}{4\pi} \oint I \frac{d\mathbf{l} \times \mathbf{r}_0}{r^2}}_{\text{Circuit 3}}$$

By combining this expression with Eq. (6-63), there results

$$\Phi = \int_{\text{Cap}} B_{n1}\, dS + \int_{\text{Cap}} B_{n2}\, dS + \int_{\text{Cap}} B_{n3}\, dS$$

Observe that the result of this integration will lead to three terms, a contribution from each of the three components of **B**. Moreover, each contribution is proportional to the value of I in the circuit. The proportionality factor will depend on geometric factors alone, such as spacing between the circuits, their relative orientations, and their physical size and shape. Clearly, for the case discussed, the total magnetic flux linking circuit 2 may be written

$$\lambda_2 = M_{21}I_1 + M_{22}I_2 + M_{23}I_3 \qquad \text{webers} \tag{6-64}$$

Note from this expression that the coefficient M_{21} is the *total flux linking circuit* 2 *when the current in circuit* 1 *is unity and all other currents are zero.* It is known as the *coefficient of mutual induction*, or simply as the *mutual inductance.* Similarly, the coefficient M_{22}, which is called the *self-inductance* of the circuit, is the *total flux linking circuit* 2 *when the current in circuit* 2 *is unity and all other currents are zero.*

In the completely general case of a system of current circuits, the total magnetic flux λ_k linking the kth circuit is a linear function of the currents. Thus,

$$\lambda_k = M_{k1}I_1 + M_{k2}I_2 + \cdots + M_{kp}I_p + \cdots + M_{kn}I_n \tag{6-65}$$

where, by definition,

$$M_{kp} = \frac{1}{I_p}\int_{S_k} \mathbf{B}_p \cdot \mathbf{n}\, dS \qquad \text{and} \qquad \begin{matrix} I_p \neq I_k \\ I_k = 0 \end{matrix} \qquad \text{webers/amp} = \text{henrys} \tag{6-66}$$

where $\mathbf{B}_p$ is the magnetic flux density at points on the surface of the cap S_k bounded by the kth circuit, due to a current, I_p in circuit p.

An explicit relation is possible for the flux linking the kth circuit with the pth circuit. This is done by reference to Fig. 6-24, which shows two circuits, here designated as loops 1 and 2, respectively. In this figure, $d\mathbf{l}_1$ represents an element of the current circuit I_1. It is required to specify the total flux produced by the current element $I_1\, d\mathbf{l}_1$ and which links loop 2. This is given directly by the expression

$$d\Phi_{21} = \frac{\mu_0}{4\pi}\int_{S_2} I_1 \frac{d\mathbf{l}_1 \times \mathbf{r}_0}{r^2} \cdot d\mathbf{S} = -\frac{\mu_0}{4\pi}\int_{S_2} I_1\, d\mathbf{l}_1 \times \nabla\left(\frac{1}{r}\right) \cdot d\mathbf{S} \tag{6-67}$$

But from the results of Prob. 6-3*a*

$$\nabla \times \left(\frac{d\mathbf{l}_1}{r}\right) = \nabla\left(\frac{1}{r}\right) \times d\mathbf{l}_1 + \left(\frac{1}{r}\right)\nabla \times d\mathbf{l}_1 \tag{6-68}$$

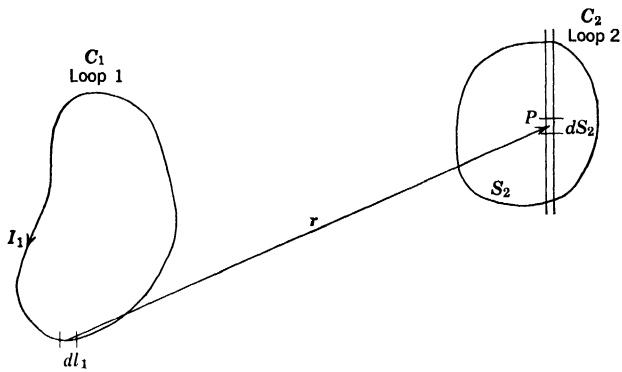


FIG. 6-24. A pair of current-carrying circuits.

Observe that $d\mathbf{l}_1$ is the source point, whereas the curl operation is taken at the field point. Because of this, the second term in the expansion is zero. As a result, Eq. (6-67) may now be written in the form

$$d\Phi_{21} = \frac{\mu_0}{4\pi} \int_{S_2} I_1 \left[\nabla \times \left(\frac{d\mathbf{l}_1}{r} \right) \right] \cdot d\mathbf{S} \tag{6-69}$$

This expression is now transformed through the use of Stokes' theorem, Eq. (6-58). Therefore

$$d\Phi_{21} = \frac{\mu_0}{4\pi} \oint_2 \frac{I_1}{r} d\mathbf{l}_1 \cdot d\mathbf{l}_2 \tag{6-70}$$

The total flux linking circuit 2 due to the entire current I_1 is given by the expression

$$\Phi_{21} = \frac{\mu_0 I_1}{4\pi} \oint_{1,2} \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} \tag{6-71}$$

The total flux linking circuit s due to unit current in circuit k, which is

$$M_{sk} = \Phi_{sk}\Big|_{I_k = 1} = \frac{\mu_0}{4\pi} \oint_{k,s} \oint \frac{d\mathbf{l}_k \cdot d\mathbf{l}_s}{r} \quad \text{henrys} \tag{6-72}$$

is called the *Neumann integral.* An alternate proof of this integral expression is given in Example 9-8.2.

The double integral in Eq. (6-72) is a purely geometrical factor, depending only upon the sizes and shapes of the two circuits, their separation, and their relative orientation. Another point of importance is called to attention, namely, that the same expression would have been obtained had the current I_s been assumed. Then the calculation would have been made of the total flux through the cap with I_k as its boundary. That is,

the integrals in Eq. (6-72) are symmetrical and may be interchanged without influencing the result. It follows at once, therefore, that

$$M_{sk} = M_{ks} \qquad \text{henrys} \tag{6-73}$$

The term M_{kk}, which appears in Eq. (6-64) and which is a measure of the total flux linking the kth circuit when the current in this circuit is unity and all other currents are zero, requires special attention. Ordinarily, the symbol used for this quantity is L, and this is known as the *coefficient of self-induction*, or simply the self-inductance, of the circuit. In this case, Eq. (6-72) becomes

$$L \equiv M_{kk} = \frac{\mu_0}{4\pi} \oint\oint \frac{d\mathbf{l}_{k1} \cdot d\mathbf{l}_{k2}}{r} \qquad \text{henrys} \tag{6-74}$$

with both line integrals taken around the same circuit, as illustrated in Fig. 6-25.

Equation (6-74) is an improper integral because the factor of $1/r$ in the integrand becomes infinite when $d\mathbf{l}_{k1}$ and $d\mathbf{l}_{k2}$ coincide. Without going into all the details of the situation, it is noted that the cross-sectional area of the wire making up the circuit has been neglected. Ordinarily, this is an entirely justifiable procedure in calculating mutual inductances, either by an application of Eq. (6-66) or (6-72) or by techniques to be discussed in Chap. 7. However, the self-inductance of a coil wound with infinitely thin wire would be infinite. For a coil of finite cross section, one might proceed by dividing the finite currents into filamentary circuits. The coefficients M or L would be obtained by integrating for the contributions of the type given by Eqs. (6-72) and (6-74) over the cross section of each filamentary circuit. Thus, in general, the result would be of the form

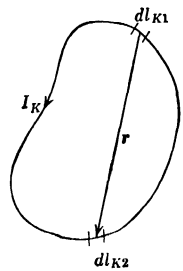


FIG. 6-25. The general circuit for the self-inductance.

$$M_{ks} = \frac{1}{a_k a_s} \int_{a_k} da_k \int_{a_s} da_s \oint\oint_{k,s} \frac{d\mathbf{l}_k \cdot d\mathbf{l}_s}{r} \tag{6-75}$$

where a_k and a_s denote the cross-sectional areas. When this is done for L and the cross-sectional area is multiplied by $1/r$, a finite integral results even though $1/r$ by itself becomes infinite. Example 6-6.1 indicates the character of the mathematical considerations involved in such an evaluation, which leads to a finite result for the self-inductance.

In practice, calculations of both mutual- and self-inductance are usually much easier when based on Eq. (6-66) rather than on integrals like Eqs. (6-72) and (6-74). The essential advantage of these latter forms

is that they show that inductance is a geometric quantity, just as has already been found to be true for resistance and capacitance.

Example 6-6.1. Obtain an expression for the self-inductance per unit length of a coaxial cable of inner radius a and outer radius b, but of negligible thickness. Refer to Fig. 6-26, which shows the geometry.

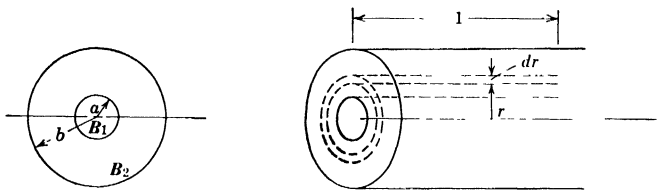


FIG. 6-26. A coaxial cable.

Solution. A direct application of the Ampère circuital law yields for a total current I

a. For region 1, the inner conductor [see Eq. (6-48)],

$$B_1 = \frac{\mu_0 I r}{2\pi a^2} \tag{6-76a}$$

b. For region 2, between the inner and outer conductors [see Eq. (6-12)],

$$B_2 = \frac{\mu_0 I}{2\pi} \frac{1}{r} \tag{6-76b}$$

Consider, now, the partial flux linkages due to the inner conductor. In this connection, refer to Fig. 6-27. Note that, with the differential shell of thickness dr, the differential flux per meter of length is

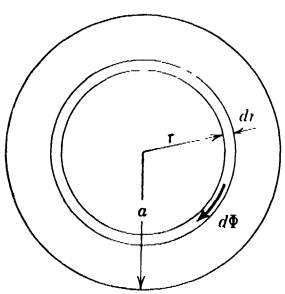


FIG. 6-27. Partial flux linkages inside the cylindrical current.

$$d\Phi_1 = B\, dr = \frac{\mu_0 I}{2\pi a^2} r\, dr$$

Note, however, that the differential flux linkage of this element of flux is the differential flux multiplied by the fraction of the current contained within the path enclosing the flux. Since the ratio of current enclosed to the total current is

$$\frac{I_r}{I} = \frac{r^2}{a^2}$$

then the total flux linkages within the differential flux element are

$$d\lambda_1 = \frac{\mu_0 I}{2\pi a^4} r^3\, dr \tag{6-77}$$

The total flux linkages of the inner conductor per unit length are then

$$\lambda_1 = \int_0^a \frac{\mu_0 I}{2\pi a^4} r^3\, dr = \frac{\mu_0 I}{8\pi} \tag{6-78}$$

In the space between the inner and the outer conductor, the differential flux per meter of length within a differential sheet of thickness dr is

$$d\Phi_2 = B\,dr = \frac{\mu_0 I}{2\pi}\frac{dr}{r} \tag{6-79}$$

But the fraction of current enclosed within this path is unity for all values of r from a to b. Then,

$$\lambda_2 = \Phi_2 = \int_a^b \frac{\mu_0 I}{2\pi}\frac{dr}{r} = \frac{\mu_0 I}{2\pi}\ln\frac{b}{a} \tag{6-80}$$

Thus, the total flux linkages are

$$\lambda = \lambda_1 + \lambda_2 = \frac{\mu_0 I}{8\pi} + \frac{\mu_0 I}{2\pi}\ln\frac{b}{a} \tag{6-81}$$

Since by definition the self-inductance is the flux linkages per unit current, then

$$L = \frac{\lambda}{I} = \frac{\mu_0}{2\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right) \qquad \text{henrys/m} \tag{6-82}$$

Example 6-6.2. Obtain an expression for the self-inductance per unit length of a parallel-wire pair of radius a which are well separated by a distance D, where one wire is a return circuit for the current in the other.

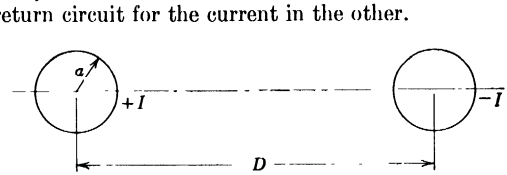


FIG. 6-28. A sectional view of two long parallel wires.

Solution. Refer to Fig. 6-28, which shows the system in a sectional view. The magnetic flux density for the region inside and outside each conductor has the values indicated above [see Eqs. (6-76)]. The flux linkages are found in the manner of Example 6-6.1. Thus, for the region within the conductor, the result is precisely that given by Eq. (6-78) and is

$$\lambda_1 = \frac{\mu_0 I}{8\pi} \tag{6-83}$$

The total flux linkages for the region between the conductors are obtained from the expression

$$\lambda_2 = \Phi_2 = \int_a^{D-a} \frac{\mu_0 I}{2\pi}\frac{dr}{r}$$

which integrates to

$$\lambda_2 = \frac{\mu_0 I}{2\pi}\ln\frac{D-a}{a} \tag{6-84}$$

Attention is called to the fact that partial flux linkages also exist in the region from $D - a$ to $D + a$, as these flux lines partially enclose the return current.

For distances greater than $D + a$, no net current is linked. It will be assumed, however, that this third component of flux linkages is negligible. Hence, the total flux linkages caused by the current I are

$$\lambda = \frac{\mu_0 I}{8\pi} + \frac{\mu_0 I}{2\pi} \ln \frac{D - a}{a} \tag{6-85}$$

It is necessary to consider now the component linkages caused by the current $-I$ in conductor 2. Some thought will show that these are identical with those produced by the current I in conductor 1. Hence, the total flux linkages by the entire loop current per unit length are just twice that given by Eq. (6-85) and are

$$\lambda_{\text{total}} = \frac{2\mu_0 I}{2\pi} \left(\frac{1}{4} + \ln \frac{D - a}{a} \right) \tag{6-86}$$

The self-inductance of the parallel-wire pair per unit length is, then,

$$L = \frac{\lambda_{\text{total}}}{I} = \frac{\mu_0}{\pi} \left(\frac{1}{4} + \ln \frac{D - a}{a} \right) \quad \text{henrys/m} \tag{6-87}$$

Since it is supposed that $D > a$, then this expression reduces to the form

$$L \doteq \frac{\mu_0}{\pi} \left(\frac{1}{4} + \ln \frac{D}{a} \right) \quad \text{henrys/m} \tag{6-88}$$

Example 6-6.3. What is the self-inductance of an infinite solenoid per unit length?

Solution. To solve this problem, use is made of Eq. (6-22) for the magnetic vector inside the solenoid. Since the cross-sectional area is a constant, the total magnetic flux passing through the section is

$$\Phi = BA = \frac{\mu_0 I N A}{L}$$

where A is the area. Since the flux passes through essentially all the solenoid, the flux linkages are approximately

$$\lambda = N\Phi = \frac{\mu_0 I N^2 A}{L} \tag{6-89}$$

The inductance of the very long solenoid is, then, approximately

$$L = \frac{\lambda}{I} = \frac{\mu_0 N^2 A}{L} \tag{6-90}$$

Example 6-6.4. Obtain an expression for the self-inductance of a toroid of circular cross section, with N closely spaced turns.

Solution. The magnetic vector for this problem has been evaluated in Sec. 6-4 and is given by Eq. (6-50), namely,

$$B = \frac{\mu_0 N I}{2\pi r} \tag{6-91}$$

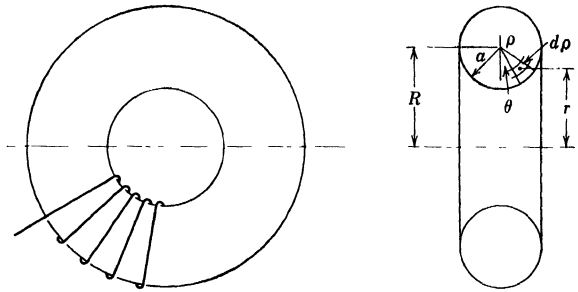


FIG. 6-29. A toroid of circular cross section.

where r is the distance from the axis to the point within the torus. Now, refer to Fig. 6-29, which gives a sectional view of the toroid. Note that, for the specified geometry, Eq. (6-83) may be written as

$$B = \frac{\mu_0 NI}{2\pi(R - \rho \cos \theta)} \tag{6-92}$$

The differential flux in an element of core area $\rho\, d\rho\, d\theta$ is

$$d\Phi = \frac{\mu_0 NI}{2\pi(R - \rho \cos \theta)} \rho\, d\rho\, d\theta \tag{6-93}$$

But this differential flux links N turns of current. Thus, the flux linkages are specified by the equation

$$d\lambda = \frac{\mu_0 N^2 I}{2\pi} \frac{\rho\, d\rho\, d\theta}{R - \rho \cos \theta} \tag{6-94}$$

The total flux linkages are then given by the integral expression

$$\lambda = \frac{\mu_0 N^2 I}{2\pi} \int_0^a d\rho \int_0^{2\pi} \frac{\rho}{R - \rho \cos \theta}\, d\theta \tag{6-95}$$

The integral in θ is a standard-form integral, thereby leading to the expression

$$\text{Integral} = \int_0^a \frac{2\pi}{\sqrt{R^2 - \rho^2}} \rho\, d\rho \tag{6-96}$$

The integral in this expression is also a standard form and leads to

$$\lambda = \mu_0 N^2 I[-\sqrt{R^2 - \rho^2}]_0^a = \mu_0 N^2 I(R - \sqrt{R^2 - a^2})$$

The self-inductance of the toroid is given by the expression

$$L = \mu_0 N^2 (R - \sqrt{R^2 - a^2}) \tag{6-97}$$

Note that, if $R > a$, the second term in this expression may be expanded by the binomial expansion, and the first two terms of this expansion only are retained. In this case, Eq. (6-97) reduces approximately to the expression

$$L = \mu_0 N^2 \frac{1}{2} \frac{a^2}{R} = \frac{\mu_0 N^2 a^2}{2R} \quad \text{henrys} \tag{6-98}$$

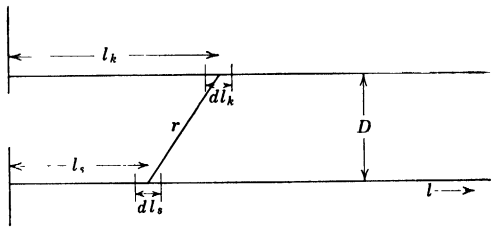


FIG. 6-30. A pair of parallel filamentary circuits.

Example 6-6.5. Derive an expression for the mutual inductance of two straight filamentary circuits of length L and of infinitesimal cross section which are parallel to each other and a distance D apart.

Solution. Refer to Fig. 6-30, which illustrates the system. This problem will be solved by applying the Neumann integral directly. Note, therefore, that in the expression (6-72),

$$M_{sk} = \frac{\mu_0}{4\pi} \oint\oint_{k,s} \frac{d\mathbf{l}_k \, d\mathbf{l}_s \cos(d\mathbf{l}_k, d\mathbf{l}_s)}{r}$$

and that since the circuits are parallel to each other, $\cos(dl_k, dl_s) = 1$. Also,

$$r = \sqrt{(l_k - l_s)^2 + D^2}$$

Thus, $$M = \frac{\mu_0}{4\pi} \int_{0,0}^{L,L} \frac{dl_k \, dl_s}{\sqrt{(l_k - l_s)^2 + D^2}} = \frac{\mu_0}{4\pi} \int_{0,0}^{L,L} \frac{dl_k \, dl_s}{\sqrt{l_k^2 - 2l_s l_k + (D^2 + l_s^2)}}$$

Integrating first in l_k reduces the integral to

$$\begin{aligned} M &= \frac{\mu_0}{4\pi} \int_0^L \left\{ \ln \left[\sqrt{l_k^2 - 2l_s l_k + (D^2 + l_s)^2} - (l_s - l_k) \right] \right\}_0^L dl_s \\ &= \frac{\mu_0}{4\pi} \int_0^L \ln \frac{\sqrt{L^2 - 2l_s L + (D^2 + l_s)^2} - (l_s - L)}{\sqrt{D^2 + l_s^2} - l_s} \, dl_s \\ &= \frac{\mu_0}{4\pi} \int_0^L \ln \frac{\sqrt{(L - l_s)^2 + D^2} + (L - l_s)}{\sqrt{D^2 + l_s^2} - l_s} \, dl_s \end{aligned}$$

Integrating now with respect to l_s gives

$$M = \frac{\mu_0}{4\pi} \left[2l_s \ln \frac{\sqrt{l_s^2 + D^2} + L}{D} - 2(\sqrt{l_s^2 + D^2} - D) \right]_0^L$$

which becomes, by inserting the limits,

$$M = \frac{\mu_0}{2\pi} \left[L \ln \frac{\sqrt{L^2 + D^2} + L}{D} - (\sqrt{L^2 + D^2} - D) \right] \tag{6-99}$$

For the case when $L \gg D$, a somewhat simpler form is possible. That is, since D/L is small, higher powers of D/L will be neglected in the expansions of the

quantities under the square-root signs. Thus, approximately,

$$M = \frac{\mu_0}{2\pi}\left\{L \ln \frac{\sqrt{1 + (D/L)^2} + 1}{D/L} - L\left[\sqrt{1 + \left(\frac{D}{L}\right)^2} - \frac{D}{L}\right]\right\}$$

$$\doteq \frac{\mu_0}{2\pi}\left(L \ln \frac{1 + \frac{1}{2}(D/L)^2 + 1}{D/L} - L\left\{\left[1 + \frac{1}{2}\left(\frac{D}{L}\right)^2 + \cdots\right] - \frac{D}{L}\right\}\right)$$

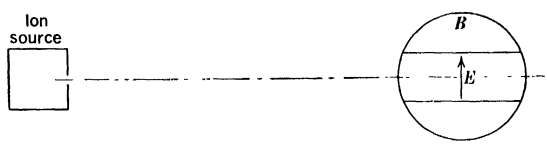
$$\doteq \frac{\mu_0}{2\pi}\left[L \ln \frac{2}{D/L} - L\left(1 - \frac{D}{L}\right)\right]$$

This may then be approximated finally to the form

$$M \doteq \frac{\mu_0 L}{2\pi}\left(\ln \frac{2L}{D} - 1\right) \quad \text{henrys} \tag{6-100}$$

PROBLEMS

6-1. A beam of hydrogen ions travels through a region containing a combined electric and magnetic field distribution. Suppose that each ion has a velocity of 10^5 m/sec. The geometry is illustrated. The **E** field is assumed uniform and has a value of 500 volts/cm in the vertical direction. A magnetic field acts over this same region of space and is of such a magnitude that the ion beam travels along a straight path. Determine the magnitude and direction of the magnetic vector **B**.



6-2. If a rectangular current loop 25 by 50 cm carries a current of 200 amp and is oriented horizontally, what is the value of B (in magnitude and direction) at a point 25 cm directly above the center?

6-3. If u is a scalar and $\mathbf{v}$ is a vector point function:

a. Show that $\nabla \times (u\mathbf{v}) = \nabla u \times \mathbf{v} + u\nabla \times \mathbf{v}$.

b. Show that $\nabla \cdot (\mathbf{a} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{v}$.

c. Show that $\nabla \times \nabla u = 0$.

d. Show that $\nabla \cdot \nabla \times \mathbf{v} = 0$.

e. Prove Eq. (6-61).

f. Prove Eq. (6-62).

6-4. *a.* A conductor is in the form of a regular polygon of n sides inscribed in a circle of radius R. Show that the expression for B at the center for a current I is

$$B = \frac{nI}{2R} \tan \frac{\pi}{n}$$

b. Show that, when n is indefinitely increased, this expression reduces to Eq. (6-9).

6-5. Two long parallel conductors d m apart carry currents in opposite directions. Find the value of B at a point h m above the center line of the conductors and perpendicular to the plane specified by them.

6-6. A spherical surface distribution of uniform charge (the classical picture of an electron) of radius R is rotating about an axis through its center. Find an expression for B at the center of the sphere if the surface-charge density is σ coulombs/m. In what direction is B pointing?

6-7. *a.* Find the expressions for B_x and B_y at any point in the plane of a rectangular current loop.

b. Evaluate the total flux linking the current loop.

c. What is the self-inductance of this rectangular current loop?

6-8. *a.* Find an expression for the **B** vector at any point P in the plane of a circular loop.

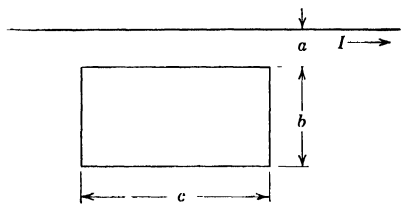
b. Find the total magnetic flux linking the current loop.

c. What is the self-inductance of this circular current loop?

6-9. A rectangular toroid of permeability μ_0, having a height a and width b with an inner radius R and an outer radius $R + b$, is uniformly wound with N turns around the core. If the current in the winding is I amp, find an expression for the self-inductance of the torus.

6-10. Repeat Prob. 6-9 for a toroid of circular section of diameter b.

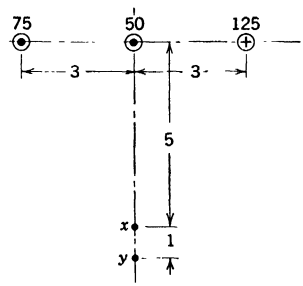
6-11. Given a rectangular loop of wire adjacent to and in the same plane as a long, straight wire carrying a current I, as illustrated. Find an expression for the mutual inductance between these two circuits.



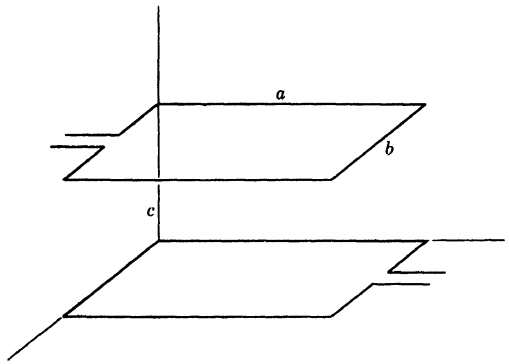
6-12. A single-turn circular loop of wire of radius a is mounted concentric with and at the center of a long solenoid of radius b, where $b > a$. Find the mutual inductance between the two circuits.

6-13. Calculate the mutual inductance between two toroidal windings which are closely wound on an iron core of permeability $\mu_r = 1{,}000$. The mean radius of the torus is 3 in., and the radius of its cross section is ½ in. Each winding has 800 turns.

6-14. A three-wire power line is adjacent to a two-wire communication pair, as illustrated. Find the flux linking the x-y circuit per foot of length.



6-15. Find an expression for the mutual inductance between rectangular coils oriented as shown. What is the numerical value if each side and the spacing between coils is 0.1 m?



6-16. Show the validity of Eq. (6-49) at any point on the axis within the hole of Fig. 6-18.

CHAPTER 7

MAGNETIC EFFECTS OF IRON

In the last chapter, it was assumed that the current was in free space. It is quite important that the situation be examined when the region of the current may contain a material medium, since many of the problems in electrical engineering involve currents either embedded in or surrounding metallic substances. In order to carry out the purposes of this chapter, it is noted that a careful study of the various magnetic phenomena associated with matter indicates that magnetism is an atomic or molecular property and that there is no intrinsic magnetic substance. In the light of this, the hypothesis is adopted that all magnetic properties of matter can be explained in terms of the motions of the electrons and positive charges associated with its constituent atoms and molecules.

7-1. The Atomic Model and Magnetism. According to the Bohr-Rutherford theory of the atom, the atom consists of a nucleus of positive charge that contains nearly all the mass of the atom. Surrounding this central positive core are negatively charged electrons, each of which is moving in some kind of path about the nucleus. Under normal circumstances, the total positive charge of the nucleus is equal to the total charge of all the associated electrons, and the atom as a whole is electrically neutral. The force of attraction between the electron and the proton follows Coulomb's law.

Consider an electron which traverses its path or orbit about the nucleus. The electron moving in its orbit is the equivalent of a convection current traversing a closed path. According to Eq. (6-8), a magnetic field is produced. It is customary in discussing such atomic phenomena to refer also to the magnetic moment of the current loop, which specifies the average magnetic field vector of such a current loop. It may be shown that the magnetic moment of a current loop is $\mathbf{m} = I\mathbf{a}$, where $\mathbf{a}$ is the area of the orbit. This is often referred to as the *orbital angular momentum.*

Another feature of the atomic model, which is based on experimental and theoretical results, is that the electron in the atom also rotates about an axis through the electron. The resulting spinning electron, which is here considered to be a spherical distribution of electric charge, will also

produce a magnetic vector. If the electron spin is considered as the equivalent of a tiny current, there is an associated magnetic moment with the electron spin. This is referred to as the *spin magnetic moment.* Hence, there is associated with an atom the orbital and spin magnetic moments of the electrons. The resultant magnetic moment of the atom is obtained by the proper vector summation of the orbital and spin magnetic moments. These are combined according to quantum-theory rules, which will not be discussed here. It is important to note, however, that the magnetic properties of bulk matter can be described in terms of the orbital and spin motions of the atoms which make up the matter.

It has been established both theoretically and experimentally that all matter may be classified into three groups, according to its fundamentally different behavior under the action of a magnetic field. These are classified under the following headings: (1) diamagnetism, (2) paramagnetism, and (3) ferromagnetism.

1. *Diamagnetic* substances are those which, when placed in a magnetic field, become weakly magnetized *in opposition* to the impressed field. The magnetism which is thus established, in general, varies directly with the impressed field and is independent of the absolute temperature. Examples of diamagnetic substances are Cu, Ag, Zn, Bi, Au, C, Cd, Hg, Pb, and Si.

2. *Paramagnetic* substances are those which, when placed in a magnetic field, become weakly magnetized *in the direction* of the impressed field. The magnetism thus established, in general, is proportional to the impressed field and varies inversely with the absolute temperature. Examples of paramagnetic substances are Pt, Mg, Al, Cr, Mn, Sn, V, and W.

3. *Ferromagnetic* substances are those which, when placed in a magnetic field, become strongly magnetized *in the direction* of the impressed field, and which also exhibit the phenomena of retentivity and hysteresis. The magnetism which is thus established, in general, is proportional to the impressed field and varies inversely with the quantity $T - \theta$, where T is the absolute temperature and where θ is the critical temperature and is characteristic of the material. This variation is known as the *Curie-Weiss law.* Examples of ferromagnetic substances are Fe, Co, and Ni.

It is not intended to discuss the physical properties of matter which account for the three above-mentioned magnetic characteristics, although certain aspects of ferromagnetic materials will receive some consideration at a later point in the chapter. It is sufficient for the present purposes to consider that the atom may be represented by an equivalent current loop.

7-2. B inside a Material Medium. In order to obtain a quantitative measure of the magnetic properties of the materials, it is convenient to consider a sample of matter to be placed in a region of uniform magnetic

induction. This is most easily accomplished by considering the matter to be in the form of a cylinder which may be placed inside an infinite solenoid. More convenient from a practical standpoint would be a specimen of the matter which forms the core of a toroid, the core diameter of which is small compared with the diameter of the toroid. A section of the resulting physical arrangement is illustrated in Fig. 7-1. This figure represents a core of paramagnetic material, since, as noted above, the paramagnetic substance becomes weakly magnetized in the direction of the impressed field. This may be taken to mean that the equivalent atomic, or Amperian, current loops of the matter experience a torque when in an external field which tends to align the magnetic moment of the current loop with the applied field. But, in a uniform magnetic field, the degree of alignment will be the same throughout the core material, thereby yielding a uniformly magnetized core.

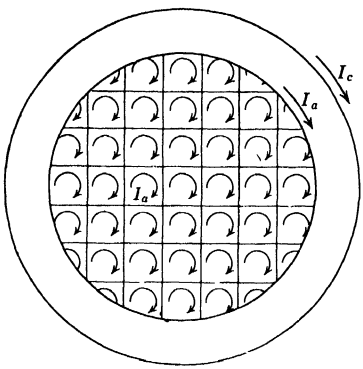


FIG. 7-1. A cylinder of paramagnetic material in a solenoid.

As illustrated, the core is assumed to be divided into regions, supposedly of atomic dimensions, to illustrate the atomic current loops. Note that for an atomic current I_a, owing to the mutual cancellation that occurs throughout the volume, only a surface current I_a remains, and this exists on the periphery of the core, as illustrated. Thus, the atomic current I_a and the external conduction current I_c are effectively parallel current sheets, and the current I_a will therefore strengthen the magnetic field in which such a material is placed.

In order to calculate the **B** vector inside the material medium, it will be assumed that the effect is simply that due to the superposition of the two solenoidal current sheets. Equation (6-47) will be valid in the present case if the current is now taken as the sum of the Amperian current and the conduction current. Thus, the Ampère circuital law is written as

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0(I_c + I_a) \tag{7-1}$$

Attention is called to the fact that the value of **B** computed from this expression for an interior point in a material medium is a space average value, since considerable fluctuations would be expected in an atom and between atoms. The space over which this space average value is taken

is large compared with atomic dimensions but small in terms of the physical dimensions of the specimen.

7-3. The Magnetic Field Intensity H. It is very desirable to distinguish between the cause and effect of the magnetic field on the magnetization of the core material. If there were no core material, then the magnetic vector within the infinite solenoid would be related to the conduction current by the expression

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_c \tag{7-2}$$

As found above, with the core material in the uniform field within the solenoid, the resulting magnetic vector is given by Eq. (7-1). The essential difference is the contribution to the line integral by the resulting effect of the core.

To proceed, Eq. (6-55) is employed to write Eq. (7-1) in point form, namely,

$$\text{curl } \mathbf{B} = \mu_0(\mathbf{J}_c + \mathbf{J}_a) \tag{7-3}$$

As a definition here, but noting that this result may be developed from fundamental considerations relating to the magnetic moment of the atoms per unit volume, the vector **M**, usually called the *magnetization*, is defined by the relation

$$\text{curl } \mathbf{M} \equiv \mathbf{J}_a \tag{7-4}$$

This is combined with Eq. (7-3) to yield

$$\text{curl } \mathbf{B} = \mu_0(\mathbf{J}_c + \text{curl } \mathbf{M})$$

After rearrangement,

$$\text{curl}\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_c \tag{7-5}$$

It was found convenient in electrostatics to introduce the displacement vector **D**, which was defined in such a way as to make **D** depend only on the free charges. Similarly in magnetic fields, it is convenient to define a vector **H**, usually called the *magnetic field intensity*, which depends only on total external currents (in the general case this would consist of the conduction current and the displacement current, if any). This new vector will, therefore, be defined in such a way that the character of the medium in which the current is embedded does not enter into the definition. The magnetic field intensity **H** in a magnetized substance is thus defined by the relation

$$\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad \text{amp/m} \tag{7-6}$$

from which it follows that

$$\text{curl } \mathbf{H} = \mathbf{J}_c \tag{7-7}$$

The corresponding expression in integral form is, therefore,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_c \tag{7-8}$$

This expression is completely general, except that, as noted above, I_c must be interpreted as the total current if both conduction and displacement currents exist. It may be taken as the most general expression of the Ampère circuital law.

A word is in order about the units of, and the name given to, **H**, which is given in Eq. (7-6) in amperes per meter. These units are consistent with Eq. (7-8). Note that owing to the similarity of the units of **H** (amperes per meter) to the units of **E** (volts per meter), both stemming from corresponding line-integral expressions, then, corresponding to the name of **E**, the electric field intensity, the name given to **H** is magnetic field intensity. If, however, **B** is related to the force on a moving charge through Eq. (6-6), and remembering that the force on a fixed charge is related to **E**, it would appear that **B** and not **H** might properly be called the magnetic field intensity. It is less confusing to refer to the two magnetic vectors by the names **B** and **H**.

Another point of importance is the place of **H** in a discussion of magnetics. Recall that, in the discussion of electric field quantities, **E** was taken as the fundamental, or reference, quantity. This was done because the electric potential in a field can be measured relatively easily, and **E** is then directly related to the potential through its negative gradient. Correspondingly, since current is most readily measured and is related to the magnetic field intensity **H** through Eq. (7-7), then **H** is used as the basis of discussion in magnetics rather than the **B** vector, which is the fundamental magnetic vector. That is, it will be found that the magnetic field relations which appear as part of the Maxwell equations are

$$\begin{aligned} \text{div } \mathbf{B} &= 0 \\ \text{curl } \mathbf{H} &= \mathbf{J} \end{aligned} \tag{7-9}$$

These contain, of course, the complete general description of magnetic phenomena.

Equation (7-6) may be written in a somewhat different form for certain classes of material. This follows from the fact that experimentally for most nonferromagnetic materials the magnetization is proportional to and in the same direction as the magnetic field. Thus,

$$\mathbf{M} = \chi_m \mathbf{H} \tag{7-10}$$

where χ_m is called the *magnetic susceptibility* of the material. This relation applies for a large class of linear, homogeneous, and isotropic media (both paramagnetic and diamagnetic). Combine this expression with Eq. (7-6), and

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \chi_m \mathbf{H}$$

which may be rearranged as

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} \tag{7-11}$$

The resulting coefficient of $\mathbf{H}$ is written μ and is called the *magnetic permeability*, or *permeability*. It is

$$\mu = \mu_0(1 + \chi_m) \tag{7-12}$$

The dimensionless quantity

$$\frac{\mu}{\mu_0} \equiv \mu_r = 1 + \chi_m \tag{7-13}$$

is called the *relative permeability*. Finally, the most customary relation between $\mathbf{B}$ and $\mathbf{H}$ results, namely,

$$\mathbf{B} = \mu\mathbf{H} \tag{7-14}$$

Attention is called to the fact that this expression is of limited validity, owing to the manner of the development. It is not valid for certain materials, such as crystals, solids under strain, iron or steel which has been subjected to rolling or hammering, or materials which have a definite structure. In such materials, $\mathbf{B}$ and $\mathbf{H}$ do not have the same direction at a point, and μ may be both a nonlinear function of $\mathbf{B}$ and a function of the coordinates. However, Eq. (7-14) is probably the most widely used relation between $\mathbf{B}$ and $\mathbf{H}$ in magnetic problems. More will be said about its relation to ferromagnetism in Sec. 7-5. Equation (7-6) is generally valid.

7-4. Boundary Conditions at Magnetic Surfaces. The magnetic lines will suffer refraction in passing from one magnetic medium to another, in much the same way that the electric lines were bent in passing through media of different dielectric constant. The expression relating this refraction to the permeabilities of the magnetic media follows from consideration of the continuity of the $\mathbf{B}$ vector (since div $\mathbf{B} = 0$) and the Ampère circuital relation, Eq. (7-8).

Refer to Fig. 7-2, which shows a bundle of B lines in passing through the interface between two magnetic media of permeabilities μ_1 and μ_2.

a. The continuity of B lines is first specified as a necessary condition. Refer to the enlarged view of Fig. 7-2 shown in Fig. 7-2*b*. Since div $\mathbf{B} = 0$,

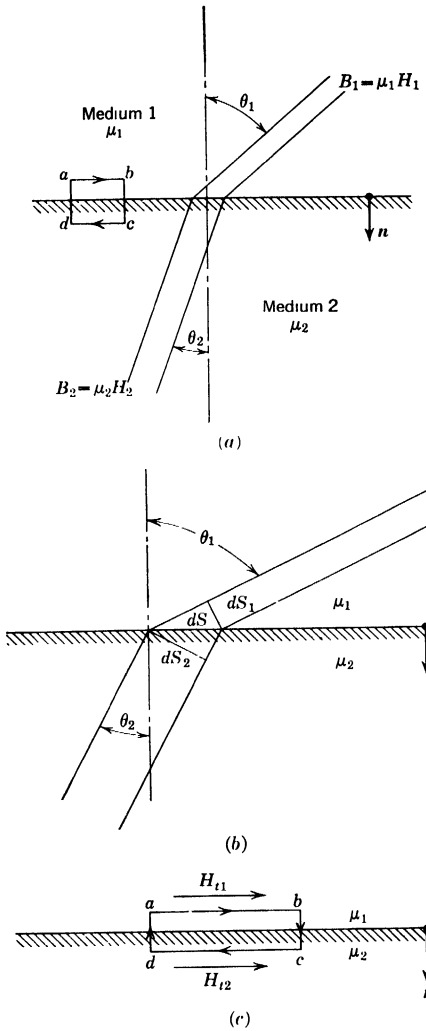


FIG. 7-2. The refraction of magnetic lines.

it is required that the total magnetic flux contained in the bundle of flux lines be constant in passing through the interface. Thus,

$$\Phi = B_1\, dS_1 = B_2\, dS_2$$

where the dS shown is the cross section of the bundle of flux lines in the appropriate medium. Now divide by dS, the corresponding area on the interface, to get

$$B_1 \frac{dS_1}{dS} = B_2 \frac{dS_2}{dS}$$

which from the diagram may be written as

$$B_1 \cos \theta_1 = B_2 \cos \theta_2$$

which may be written as

$$\mathbf{B}_{n1} = \mathbf{B}_{n2}$$

or

$$\mathbf{B}_1 \cdot \mathbf{n} = \mathbf{B}_2 \cdot \mathbf{n} \tag{7-15}$$

This shows that the normal component of the **B** vector is the same on both sides of the boundary.

b. The Ampère circuital law is now applied to the path indicated across the interface. An enlarged view of this portion of Fig. 7-2*c* is shown. No current is assumed to exist in the interface surface, whence, for the path shown,

$$\oint \mathbf{H} \cdot d\mathbf{l} = 0$$

This expression requires the evaluation of the contributions to the integral along each of the four sides, since

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_a^b \mathbf{H} \cdot d\mathbf{l} + \int_b^c \mathbf{H} \cdot d\mathbf{l} + \int_c^d \mathbf{H} \cdot d\mathbf{l} + \int_d^a \mathbf{H} \cdot d\mathbf{l} = 0$$

In the limit as the path is shrunk, to approach the interface,

$$\int_b^c \mathbf{H} \cdot d\mathbf{l} = \int_d^a \mathbf{H} \cdot d\mathbf{l} = 0 \qquad \int_a^b \mathbf{H} \cdot d\mathbf{l} + \int_c^d \mathbf{H} \cdot d\mathbf{l} = 0$$

Thus, it follows that

$$\mathbf{H}_{t1} = \mathbf{H}_{t2} \tag{7-16}$$

which in vector notation is

$$\mathbf{H}_1 \times \mathbf{n} = \mathbf{H}_2 \times \mathbf{n}$$

This shows that the tangential component of the **H** vector is the same on both sides of the boundary.

To find the law of refraction of the B lines on crossing the boundary, it is necessary only to divide Eq. (7-16) by Eq. (7-15). This gives

$$\frac{H_1 \sin \theta_1}{B_1 \cos \theta_1} = \frac{H_2 \sin \theta_2}{B_2 \cos \theta_2}$$

from which it follows that

$$\frac{1}{\mu_1} \tan \theta_1 = \frac{1}{\mu_2} \tan \theta_2$$

which may then be written in the form

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\mu_1}{\mu_2} \tag{7-17}$$

This is the final relationship for the law of refraction.

7-5. Ferromagnetism. The theory of ferromagnetism is extremely complicated, and only a few selected characteristics of ferromagnetic materials will be discussed. As already noted in Sec. 7-1, ferromagnetic substances are those which, when placed in a magnetic field, become strongly magnetized in the direction of the impressed field; they exhibit the phenomena of remanence and hysteresis. It is desired to discuss in a general way what is meant by these terms, and some characteristics of ferromagnetic matter which will account for this. Note that, from the standpoint of the technologist, ferromagnetic materials are usually of interest and importance and are ordinarily implied when magnetic materials are referred to. The permeability of diamagnetic and paramagnetic materials is so little different from that of free space that free-space conditions are assumed when such materials are used.

As noted in Eq. (7-6), the bulk, or macroscopic, properties of isotropic materials are described in terms of three field vectors, **B**, **H**, and **M**. In the case of ferromagnetism, owing to effects which cannot be described in terms of electric or magnetic interactions alone, the quantities that appear in Eq. (7-6) require special consideration. Without attempting detailed explanations, although the general features of the theory will be given in Sec. 7-6, it is noted that the relation between **B** and **H** for a ferromagnetic material is not the simple linear relation prescribed by Eq. (7-14). Moreover, the value of μ may range over large values, depending on the particular material. Values of μ in the range from 10 to 1,000 are quite common, and values to 10^6 are obtainable.

The relation between B and H for a ferromagnetic material is known as the *magnetization curve*. It is determined for a sample of ferromagnetic material by using an experimental arrangement which permits both the value of H to be calculated and the value of B to be measured. A simple, convenient form for accomplishing the results is to use a specimen of the material as the core of a toroid. The exciting winding is wound uniformly over the surface of this core and permits a calculation of H. A second winding permits measurements to be made (with a ballistic galvanometer) from which B can be calculated. Such an arrangement is often called a *Rowland ring*. The details of measurement are important, if hysteresis effects are to be avoided. When such measurements are correctly made, the results have the form shown in Fig. 7-3. It will be observed that B increases rather rapidly with H, but a region of saturation becomes manifest at the higher values of H. The *knee* of the curve is the rather indefi-

nite region where a noticeable nonlinearity becomes apparent. At this point, the rate of change of B with H decreases. Beyond the knee, saturation is approached, although relatively high values of H may be required for saturation to be reached. At saturation, the Amperian current loops have their magnetic moments aligned parallel to the applied field.

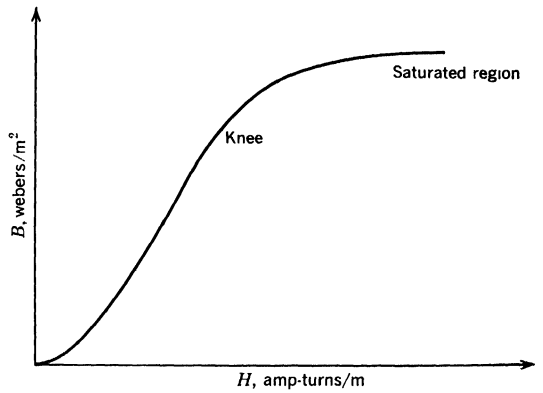


FIG. 7-3. A typical magnetization curve.

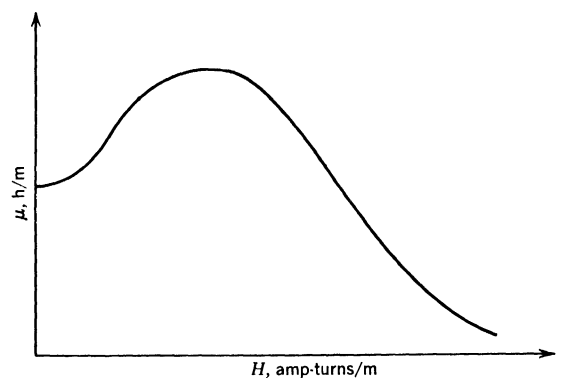


FIG. 7-4. Permeability versus H for the magnetization curve of Fig. 7-3.

The permeability of the material at any point on the magnetization curve is the ratio of B/H at that point. Note that this is the slope of a straight line drawn from the origin to the point. Sometimes the slope of the B/H curve at each point is specified. This gives the ratio dB/dH, which is known as the *incremental permeability*. Clearly, the permeability and the incremental permeability are quite different quantities. The general form of the relationship between the permeability versus H, obtained from a curve such as that in Fig. 7-3, is given in Fig. 7-4.

Suppose, now, that the following cycle is performed: Starting with a sample which is unmagnetized, the magnetization is increased by increasing H from zero to a value H_m. The induction B will follow the magnetization curve, as discussed above, and will reach a value B_m corresponding to H_m. The value of H is now reduced to zero. It will be found that B does not retrace the previous path but decreases along a path, as indicated in Fig. 7-5. The value of B_R at the point $H = 0$ is called the *remanence, retentivity,* or *residual flux density.* This shows that the BH curve is different for decreasing and increasing H, the effect being called *hysteresis.* To reduce the induction B to zero, it is required that a reversed, or negative, value $-H_C$ be applied. This value of H is known as the *coercive force.* By changing H to $-H_m$, the value $-B_m$ results.

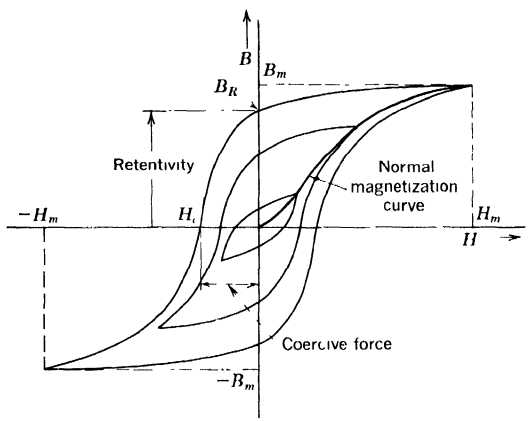


FIG. 7-5. Hysteresis loops and normal magnetization curve.

By returning H to zero, the value of B is found to be $-B_R$. Finally, by completing the cycle from zero to H_m, the results form a closed loop, as illustrated.

It will be shown in Chap. 9 that the energy lost per cycle in carrying a ferromagnetic material through a hysteresis loop is given by the expression

$$\frac{W}{V} = \frac{\text{energy lost/cycle}}{\text{unit volume}} = \oint H\,dB \qquad \text{joules/(cycle)(m}^3\text{)} \qquad (7\text{-}18)$$

But the integral on the right is just the area enclosed in the hysteresis loop. Consequently, materials which possess small hysteresis loops as the material is carried through a magnetic cycle for a specified maximum B_m would possess small hysteresis loss, a characteristic that would normally be desirable in materials for use in a-c apparatus.

An empirical relationship was found by Steinmetz between the energy loss W per cycle of the hysteresis loop and the maximum value of B_m during the cycle. His equation is

$$\frac{W}{\text{Volume}} = \eta B_m^n \tag{7-19}$$

where η is the Steinmetz coefficient for the particular material used. Steinmetz found that n had a value of 1.6 for a wide variety of materials. However, among the new ferromagnetic alloys, n may vary from 1.5 to 2.5; often, in fact, n is not a constant, except for a limited range. If the values of η and n are known for a given material, the Steinmetz equation is very useful for calculation.

The shapes of the hysteresis loops will vary widely, depending upon the material and its past magnetic history. Ferromagnetic materials are often classed as *magnetically hard* or *magnetically soft*. Magnetically hard materials are those which are characterized by a large coercive force and a high remanence. Such materials usually have hysteresis loops of large area. The large value of remanence makes the magnetically hard materials desirable as permanent magnets. Owing to the large hysteresis loops, they are less suitable for use in a-c apparatus. Magnetically soft materials possess properties which are the converse of the magnetically hard materials. They possess a small coercive force, usually a small remanence, and a small hysteresis loop. This makes them desirable for use in a-c apparatus.

Considerable effort has been expended during recent years in developing new magnetic materials with nearly square hysteresis loops. The magnetization curves of such materials are virtually vertical lines through the origin, with a horizontal break at B_m. This means that the material, considered in the light of the hysteresis loop, is fully magnetized either in one direction or in the reverse direction. Such magnetic properties are particularly important for use in magnetic amplifiers, magnetic modulators, memory circuits, and binary counting circuits. One such material is Orthonol (Deltamax), which consists of 50 per cent nickel and 50 per cent iron grain-oriented magnetic material.

7-6. The Domain Theory and the Magnetization Curve. The domain theory of ferromagnetism, a subject that is still the subject of considerable research, furnishes a clear picture of the phenomena that occur in a ferromagnetic material during the process of magnetization. According to this theory, ferromagnetic materials are composed of small regions (with physical dimensions of the order of 10^{-3} to 10^{-4} cm) called *domains*, each of which is magnetized to saturation even though no external field is applied. This saturation effect results from the interaction among the electron-spin magnetic moments and interatomic forces in the crystals

of the ferromagnetic material to cause the alignment of these spins. In the demagnetized state, the magnetization of the domains is randomly oriented, so that the net over-all result is zero magnetization. The effect of an applied field is to cause the assemblage of domains making up the material to align themselves with the direction of the applied field.

Refer to Fig. 7-6, which illustrates the changes in the domain pattern during magnetization as a gradually increasing external field is applied. In very weak fields, two types of behavior can occur: (*a*) the magnetization vector in each domain can rotate slightly in the direction of the

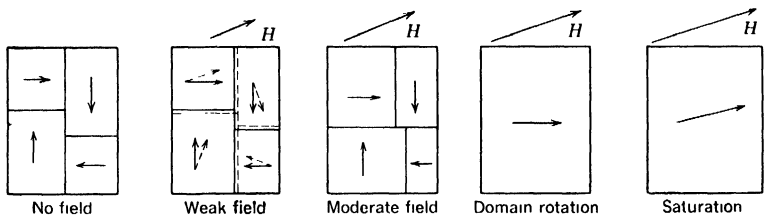


FIG. 7-6. Changes in domain pattern during magnetization.

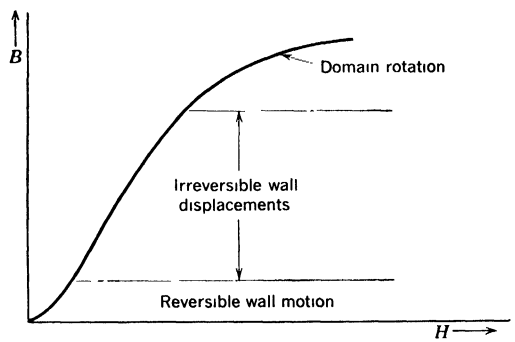


FIG. 7-7. The magnetization curve, with the various regions identified with the domain behavior.

applied field; (*b*) there can be a small motion of the domain walls, with those domains which are favorably oriented with respect to the field direction increasing in size at the expense of those less favorably oriented. If the field is sufficiently small, the processes are reversible with the removal of the field. As the applied field becomes stronger, the boundaries continue to change, but now in an irreversible manner. During this irreversible boundary displacement, the movements often take place in large jumps (this gives rise to the Barkhausen jumps, which produce measurable irregularities in the magnetization curve). With further increases in the applied field, the domain walls tend to disappear, but the magnetization is generally close to the direction of easy magnetization. For

still stronger fields, the magnetization vector rotates into the applied field, and saturation occurs. The several important portions of the magnetization curve identified with domain behavior are included in Fig. 7-7.

7-7. The Magnetic Circuit. A practical problem of considerable importance is the determination of the magnetic flux produced in a closed iron core by the electric current in a winding about the core. A problem of this sort occurs in almost every electromagnetic-field-operated device, including the transformer, the rotating d-c machine, the rotating a-c machine, electromagnetic relays, loudspeakers, and solenoid-operated devices. Owing to the high permeability of the iron core, the magnetic induction within the core will ordinarily greatly exceed that outside. Also, the magnetic flux that passes through the walls of the core will ordinarily be small compared with the flux through the core itself. Under the conditions contemplated, it is assumed that the flux is confined to the core. That is, it is assumed as a first approximation that the air space is a virtual insulator of magnetic flux and that the core is a conductor of magnetic flux. In this light, the magnetic system is being considered as analogous to an electric circuit in the steady state, although a magnetic conductor and magnetic insulator are concepts not nearly so accurate as an electric conductor and insulator.

A second common feature of the magnetic and the electric circuit is that both the flux and the current are without sources or sinks; that is, both are continuous around their respective loops. This means that the flux and the steady-state current through any closed surface are both zero. Mathematically, this latter requires that

$$\oint \mathbf{B} \cdot \mathbf{n}\, dS = 0 \tag{7-20}$$

$$\oint \mathbf{J} \cdot \mathbf{n}\, dS = 0 \tag{7-21}$$

Equation (7-20) requires, if leakage flux is neglected, that the flux through one cross section of the core be the same as that through any other cross section. Correspondingly, if the core is composed of a number of paths, then it is required that the fluxes in the paths which meet in a common point must be zero, or

$$\sum_{\text{Point}} \Phi = 0 \tag{7-22}$$

This is essentially a Kirchhoff flux law and is the exact counterpart of the Kirchhoff current law [see Eq. (1-57)]. Note that the fluxes appearing in this expression are algebraic quantities, and the sign of each must be determined by some consistent method. This requires that a reference direction for flux be prescribed in order to ensure a consistent and correct

result. The magnitude of the flux in any branch of the circuit may be expressed analytically by the expression

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{n}\, dS \tag{7-23}$$

It is now desired to examine the nature of **H** *within the core.* Note that there is no current within the core. Consequently, **J** in Eq. (7-9) is zero, whence the region within the core is characterized by a vector **H** which satisfies the relation

$$\text{curl } \mathbf{H} = 0 \tag{7-24}$$

Now, since it follows mathematically that

$$\text{curl grad } \Psi \equiv 0$$

then **H** in the core may be expressed as the gradient of a scalar potential function Ψ. Thus, in general,

$$\mathbf{H} = -\text{ grad } \Psi = -\nabla\Psi \tag{7-25}$$

But this is precisely the form of the relation that governs the electric field in a conducting medium (see Sec. 1-4) or in free space (see Sec. 2-6). Therefore, it should be noted that the region within the core comprises equipotential surfaces, in this case, surfaces of constant magnetic potential. Also, the difference of magnetic potential may be related to the line integral of the vector **H**, thus,

$$\Psi_b - \Psi_a = -\int_a^b \mathbf{H} \cdot d\mathbf{l} \qquad \text{amp} \tag{7-26}$$

as for the corresponding relation in the electric field [see Eq. (1-34)]. In this line integral, the path *must* always be taken in such a manner that it does not link current. Mathematically, this means that Ψ is a multivalued function in general. In particular, consider a core with a number of turns of current-carrying windings on the surface. The value of the line integral will be different by choosing a path that passes out of the core and which bypasses a different number of turns or which may perform many loops and circuits around current-carrying turns. However, by performing the line integration inside the core, no current is enclosed, and the value of the integral is independent of the path between equipotential surfaces. By carrying the integration around a closed loop, starting and ending at the same point, the value of the line integral is *not* zero. Attention was previously called to the fact that the **B** field is a nonconservative field (see Sec. 6-4), and this is also true of the **H** field. The value of the line integral of **H** taken *once* around the core is

called the *magnetomotive force* (mmf). Thus,

$$\oint \mathbf{H} \cdot d\mathbf{l} = \text{mmf} = F \quad \text{amp} \tag{7-27}$$

If the loop links a current I_c, then, by Eq. (7-8),

$$\text{mmf} = F = I_c$$

Of course, if I_c is carried by a winding of N turns of wire, then

$$F = \oint \mathbf{H} \cdot d\mathbf{l} = I_c N \tag{7-28}$$

Particular attention is called to the misleading character of the name magnetomotive force which is customary for the line integral of **H** around the core. Magnetomotive force is not a force, nor is it work; in fact, the units of mmf are amperes. This means that, while magnetic equipotentials exist in the core, there is no implication that any such thing as a magnetic charge exists in terms of which work in a magnetic field may be specified. For this reason, the term *magnetomotance* is often used. The situation here is not unlike that of a thermal field, with the associated thermal equipotentials. Since thermal charges do not exist, the motion in such thermal fields is not related to work.

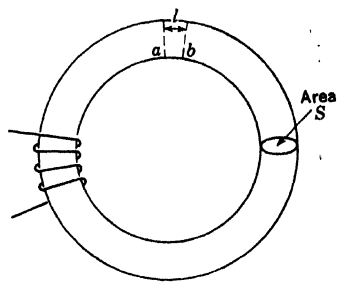


FIG. 7-8. A closed magnetic core.

Consider, now, a closed, uniform iron core, as illustrated in Fig. 7-8, which might be the core of a toroid. In evaluating the line integral, the closed loop is chosen along the axis of this core. In the light of the foregoing discussion, the core may be divided by equipotential surfaces, across each of which there is a definite magnetic potential difference $\Delta\Psi$. Thus, around the complete loop,

$$F = \oint \mathbf{H} \cdot d\mathbf{l} = \sum_{\text{Loop}} \Delta\Psi \tag{7-29}$$

which is the magnetic analogue of the Kirchhoff potential law. But, between the equipotentials, the line integral has the value

$$\Delta\Psi_{ba} = Hl \tag{7-30}$$

Also, the average total flux around the core is given by

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{n}\, dS$$

which is, since B is constant over the area,

$$\Phi = BS \tag{7-31}$$

This may be written in the form

$$\Phi = \mu HS \tag{7-32}$$

Combining Eq. (7-32) with Eq. (7-30), there results

$$\Delta\Psi = \Phi \frac{l}{\mu S} \tag{7-33}$$

By analogy with the resistance of an electric circuit, a quantity known as the *reluctance* R is defined by the relation

$$R \equiv \frac{\Delta\Psi}{\Phi} \qquad \text{amp turns/weber} \tag{7-34}$$

In terms of this new quantity, Eq. (7-29) becomes

$$F = \oint \mathbf{H} \cdot d\mathbf{l} = \Sigma\, \Delta\Psi = \Sigma \Phi R \tag{7-35}$$

For the particular core shape herein discussed, the expression for the reluctance assumes the form

$$R = \frac{l}{\mu S} \tag{7-36}$$

which is identical in form to Eq. (1-42) for the resistance of a rod of uniform cross section.

A feature of general importance relating to the magnetic field inside an iron core follows from the fact that, in general,

$$\text{div } \mathbf{B} = \text{div } \mu\mathbf{H} = 0$$

and

$$\mathbf{H} = -\text{ grad } \Psi$$

By combining these expressions, it is found that the field is described by Laplace's equation,

$$\nabla^2\Psi = 0 \tag{7-37}$$

This means that the discussion in Chap. 1 for the solution of the current-flow field, including flux-plotting methods and the method of curvilinear squares in field plotting, is equally valid for magnetic problems. Furthermore, in special cases of simple geometry, the solution of Laplace's equation is reduced to that of calculating the reluctance of the magnetic circuit.

Example 7-7.1. Suppose that the toroidal core of Fig. 7-8 has a narrow air gap of length l_g. It is required to find the mmf around the complete magnetic circuit.

Solution. It is assumed that the field is confined within the ring and that the field lines are circular in shape. Owing to the requirement that div $\mathbf{B} = 0$, B is constant around the loop. The value of H has the value B/μ in the iron and B/μ_0 in the air gap. The total magnetic potential drop is then made up of two components, that across the gap and that through the iron, or

$$\Delta\Psi = \Delta\Psi_i + \Delta\Psi_g \tag{7-38}$$

Therefore,

$$\Delta\Psi = \int_{\text{Iron}} \mathbf{H} \cdot d\mathbf{l} + \int_{\text{Gap}} \mathbf{H} \cdot d\mathbf{l} \tag{7-39}$$

The total mmf is, therefore,

$$F = \Delta\Psi_i + \Delta\Psi_g = NI \tag{7-40}$$

Observe that the total mmf around the path is independent of the path in the material, although the apportionment of the mmf across different portions of the core does depend on the characteristics of the path.

It is possible to evaluate the integrals in Eq. (7-39) in terms of the reluctance of the path segments. Thus, if the total length of the path in iron is $l - l_g$, then

$$\Delta\Psi_i = \int_{\text{Iron}} \mathbf{H} \cdot d\mathbf{l} = H_i l_i = \frac{B}{\mu} l_i \tag{7-41}$$

Also, for the path of length l_g in air,

$$\Delta\Psi_g = \int_{\text{Gap}} \mathbf{H} \cdot d\mathbf{l} = H_g l_g = \frac{B}{\mu_0} l_g \tag{7-42}$$

The mmf around the total circuit is given by the expression

$$F = B\left(\frac{l_i}{\mu} + \frac{l_g}{\mu_0}\right) \tag{7-43}$$

The total flux through the magnetic circuit is

$$\Phi = BS \tag{7-44}$$

Hence, the reluctance of the circuit is

$$R \equiv \frac{F}{\Phi} = \frac{l_i}{\mu S} + \frac{l_g}{\mu_0 S} = \frac{l - l_g}{\mu S} + \frac{l_g}{\mu_0 S} \tag{7-45}$$

A point of extreme importance is evident by carefully examining the terms in Eq. (7-45). Note in particular that μ_r of iron may be 1,000 or more, while that of air is 1. Clearly, therefore, the reluctance of the short air gap in a magnetic circuit may be considerably higher than the reluctance of the relatively long path in iron.

Owing to the fringing that will occur in the magnetic field in the air gap (and this would correspond to the fringing of the electric field lines at the edge of the plates of a parallel-plate capacitor), the effective area of the portion of the path in the air gap is slightly greater than that in the iron path. An empirical rule that is commonly used in engineering practice for determining the area of the gap is to increase each linear dimension by twice the length of the gap. Under these

conditions, Eq. (7-45) would be replaced by the following approximate expression for a path of circular section

$$R \doteq \frac{l}{\mu(\pi d^2/4)} + \frac{l_g}{\mu_0(\pi/4)(d + 2l_g)^2} \tag{7-46}$$

Example 7-7.2. Consider the magnetic core of square cross section illustrated in Fig. 7-9. It is required to find the air-gap flux density when the total number

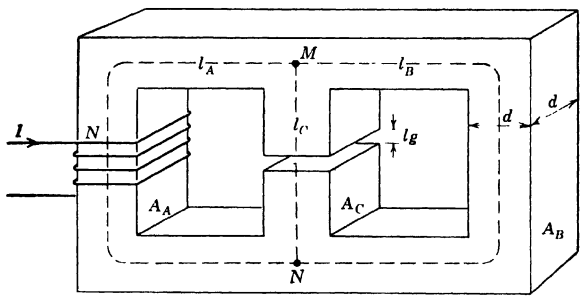


FIG. 7-9. A magnetic circuit of specified dimensions.

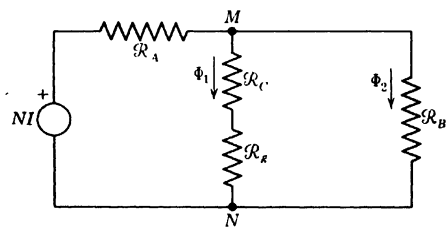


FIG. 7-10. An electric-circuit analogue of Fig. 7-9.

of ampere turns is 200. The magnetic material is silicon steel, which is assumed to have a constant permeability $\mu_r = 4{,}000$.

Solution. The important data are calculated and are tabulated:

$$\mu = \mu_0\mu_r = 4\pi \times 10^{-7} \times 4{,}000 = 1.6\pi \times 10^{-3} \text{ henry/m}$$

$$A_A = A_B = A_C = 3.5 \times 3.5 \text{ in.} = 0.00791 \text{ m}^2 \qquad A_g = 0.00819 \text{ m}^2$$

$$l_A = l_B = 38.5 \text{ in.} = 0.977 \text{ m} \qquad l_c = 13.5 \text{ in.} = 0.343 \text{ m} \qquad l_g = 0.03 \text{ in.} = 0.00762 \text{ m}$$

The general similarity in form between the magnetic circuits and electric circuits has already been noted. It is helpful, therefore, to draw the analogous electric circuit. This is done in Fig. 7-10. The respective resistances (reluctances in the magnetic circuit) are

$$R_A = R_B = \frac{l}{\mu S} = \frac{0.977}{1.6\pi \times 10^{-3} \times 7.91 \times 10^{-3}} = 2.46 \times 10^4$$

$$R_c = \frac{0.343}{1.6\pi \times 10^{-3} \times 7.91 \times 10^{-3}} = 0.862 \times 10^4$$

$$R_g = \frac{7.62 \times 10^{-4}}{4\pi \times 10^{-7} \times 8.19 \times 10^{-3}} = 7.41 \times 10^4$$

The equivalent series circuit is that illustrated in Fig. 7-11, where the parallel combination of the right-hand group is replaced by the equivalent series circuit, where

$$R_{eq} = \frac{1}{1/R_B + 1/(R_c + R_g)} = \frac{1}{1/(2.46 \times 10^4) + 1/(8.27 \times 10^4)}$$
$$= \frac{8.27 \times 2.46 \times 10^8}{(2.46 + 8.27) \times 10^4} = 1.896 \times 10^4$$

The total flux Φ is then

$$\Phi = \frac{NI}{R} = \frac{200}{(2.46 + 1.90) \times 10^4} = 4.59 \times 10^{-3} \text{ weber}$$

The magnetic potential drop between points MN is $\Delta\Psi_{MN}$, where

$$\Delta\Psi_{MN} = \Phi R = 4.59 \times 10^{-3} \times 1.90 \times 10^4 = 87.1 \text{ amp turns}$$

The flux through the center leg of the core is, therefore,

$$\Phi_1 = \frac{\Delta\Psi_{MN}}{R_c + R_g} = \frac{87.1}{8.27 \times 10^4} = 1.05 \times 10^{-3} \text{ weber}$$

Hence, finally,

$$B_g = \frac{1.05 \times 10^{-3}}{8.19 \times 10^{-3}} = 0.129 \text{ weber/m}^2 \text{ (tesla)}$$

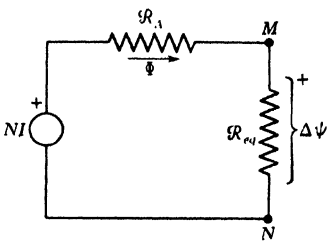


FIG. 7-11. A reduction of the circuit of Fig. 7-10.

Example 7-7.3. Again consider the magnetic core that is illustrated in Fig. 7-9. It is required to find the total number of ampere turns in the exciting winding to produce an air-gap flux of 0.20 weber/m^2. The magnetic material is silicon steel, which is assumed to have a constant permeability $\mu_r = 4{,}000$.

Solution. Use is again made of the electric-circuit analog of Fig. 7-10. Now, it is noted that

$$\Phi_1 = BA_g = 0.20 \times 8.19 \times 10^{-3} = 1.64 \times 10^{-3} \text{ weber}$$

Therefore, the magnetic potential difference $\Delta\Psi_{MN}$ is

$$\Delta\Psi_{MN} = \Phi(R_c + R_g) = 1.64 \times 10^{-3} \times 8.27 \times 10^4 = 135.7 \text{ amp turns}$$

The flux Φ_2 is given by

$$\Phi_2 = \frac{\Delta\Psi_{MN}}{R_B} = \frac{135.7}{2.46 \times 10^4} = 5.51 \times 10^{-3} \text{ weber}$$

Hence, the total flux Φ is

$$\Phi = \Phi_1 + \Phi_2 = 7.15 \times 10^{-3} \text{ weber}$$

The total magnetic potential around the core is

$$\Sigma \Delta\Psi = 7.15 \times 10^{-3} \times 2.46 \times 10^{4} + 135.7 = 175.9 + 135.7 = 311.6 \text{ amp turns}$$

which is the mmf. Thus,

$$F = NI = 312 \text{ amp turns}$$

A direct check is possible in this particular case by comparison with the results of Example 7-7.2, which showed that an mmf of 200 amp turns produced an air-gap flux density of 0.129 weber/m^2.

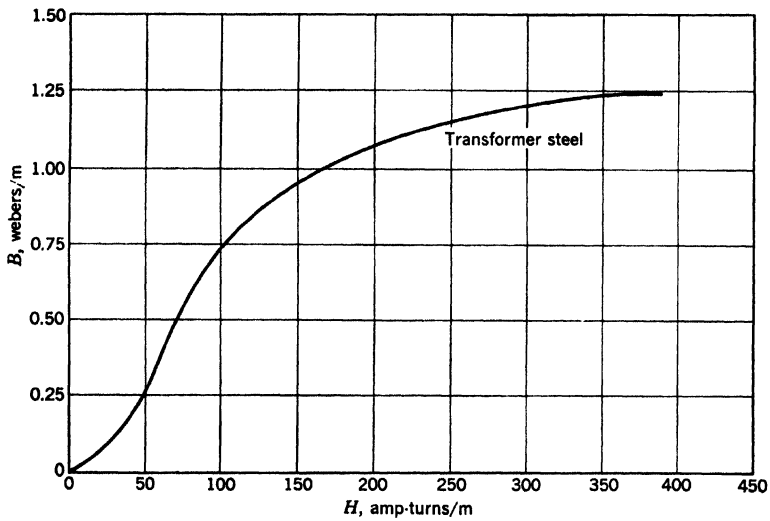


FIG. 7-12. Magnetization curve of transformer steel.

Example 7-7.4. It is required to repeat Example 7-7.3, but now, instead of assuming that the permeability of the steel is constant, the magnetization curve for the material of the core is provided. This is given in Fig. 7-12.

Solution. The difference of magnetic potential between the points MN is

$$\Delta\Psi_{MN} = (Hl)_g + (Hl)_c$$

but

$$(Hl)_g = \left(\frac{Bl}{\mu_0}\right)_g = \frac{0.20 \times 7.62 \times 10^{-4}}{4\pi \times 10^{-7}} = 121.2 \text{ amp turns}$$

It is noted that, in the iron portion of the center leg, the value of B is

$$B_C = 0.20 \times \frac{8.19 \times 10^{-3}}{7.91 \times 10^{-3}} = 0.207 \text{ weber/m}^2$$

The corresponding value of H is, from the curve, 43 amp turns/m. Thus,

$$(Hl)_C = 43 \times 0.343 = 14.8 \text{ amp turns}$$

Therefore,

$$\Delta\Psi_{MN} = 121.2 + 14.8 = 136 \text{ amp turns}$$

Now, consider path B. The magnetic potential difference is still $\Delta\Psi_{MN}$, so that

$$\Delta\Psi_{MN} = (Hl)_B$$

Therefore,
$$H_B = \frac{\Delta\Psi_{MN}}{l_B} = \frac{136}{0.977} = 139 \text{ amp turns/m}$$

From Fig. 7-12, the corresponding value of B is

$$B_B = 0.92 \text{ weber/m}^2$$

For path A,

$$B_A = \frac{\Phi_A}{A_A} = \frac{B_C A_C + B_B A_B}{A_A} = B_C + B_B$$

or
$$B_A = 0.207 + 0.92 = 1.13 \text{ webers/m}^2$$

The corresponding value of H is, from Fig. 7-12,

$$H_A = 238 \text{ amp turns/m}$$

and the magnetic potential drop along path A is

$$\Delta\Psi_A = 238 \times 0.977 = 233 \text{ amp turns}$$

The magnetic potential around the complete circuit is the mmf, and

$$F = \sum_{\text{Circuit}} \Delta\Psi = 233 + 136 = 369 \text{ amp turns}$$

Example 7-7.5. Find the magnetic vector **B** at any point in the region outside an iron block of infinite permeability due to a current in a straight wire parallel to the surface of the block.

Solution. The situation here requires the solution of Laplace's equation, but this is accomplished in precisely the same way as that corresponding to Fig. 1-36 for an electrode which is embedded in a conducting medium at a distance d below the surface, and like that in Fig. 3-5 for an electric charge which is placed at a distance d from a plane conductor. In the present case, a line source rather than a point or spherical source of excitation exists, but this image source has the same direction as the current.

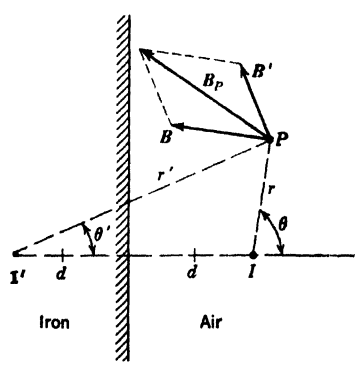


FIG. 7-13. To find the field in air due to a current in the neighborhood of an iron surface.

From the fact specified in Eq. (7-15) that the normal component of the **B** vector must be the same on both sides of the boundary between the iron and the air, and also that $\mathbf{H} = 0$ inside an infinitely permeable medium, the effect of the iron can be replaced by an image current which has the same direction as that in the air. The situation becomes that illustrated in Fig. 7-13. The resulting **B**

is obtained by considering the contribution to the field by the two currents. The result is

$$\mathbf{B}_p = \mathbf{B} + \mathbf{B}'$$

where, by Eq. (6-12),

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{a}_\theta$$

$$\mathbf{B}' = \frac{\mu_0 I}{2\pi r'}\mathbf{a}'_\theta$$

The detail of finding the expressions for the magnitude and angle of $\mathbf{B}_p$ is left as a problem for the student (see Prob. 7-6).

7-8. Permanent Magnets. A study of the characteristics of permanent magnets requires consideration of the hysteresis loop to a much

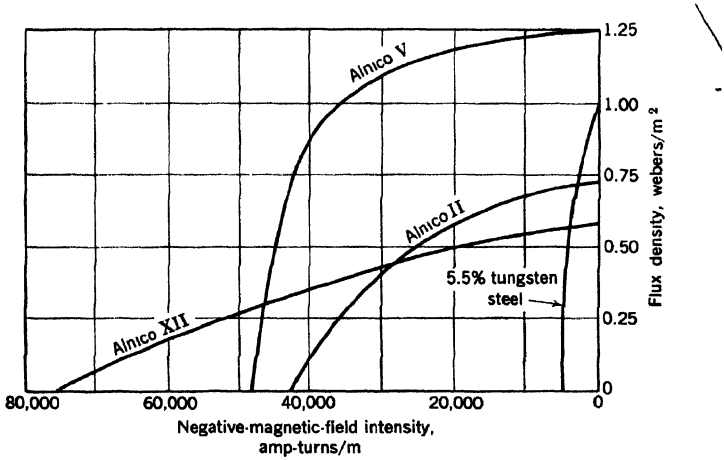


FIG. 7-14. Demagnetization curves of several permanent-magnet materials.

greater extent than is the case for the magnetically soft materials. Actually, it is the section of the major hysteresis loop in the second quadrant of the *BH* diagram that is of particular interest. This portion of the curve is referred to as the *demagnetization curve* and is, of course, a characteristic of the material. Several representative curves are shown in Fig. 7-14. It is usually desirable that permanent magnets have a high remanence, but the coercive force must be large also, so that the magnet is not easily demagnetized. That is, the material of the permanent magnet is of the so-called magnetically hard type.

In most applications, it is desired that magnetic stability prevail, in order that the magnetic characteristics be maintained under the normal operating conditions. Manufacturers and instrument makers often artificially age or mature their magnets by subjecting them to a tempera-

ture cycle. Of course, a natural aging process over several years is equally effective. In addition, magnetic stability is obtained generally by partially demagnetizing the material. Refer to Fig. 7-15, which shows the effect of applying a negative external magnetic field ΔH to a solid core, say, for example, a toroidal core. This causes the material to move to point a on the demagnetization curve. When the negative-magnetizing field is removed, the flux density moves to point b. Successive applications and removal of the magnetizing field ΔH are accompanied by the operating point moving from point b to a and back along the *minor hysteresis loop* shown. Attention is called to the slope of the line joining a and b, which is substantially the same as the slope of the original curve at B_R, regardless of where point a is located on the original curve.

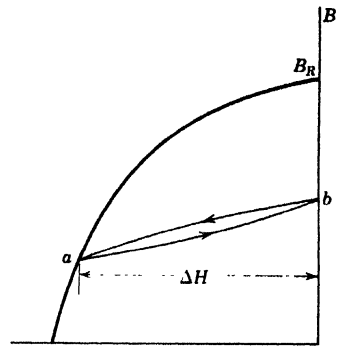


FIG. 7-15. Minor hysteresis loop.

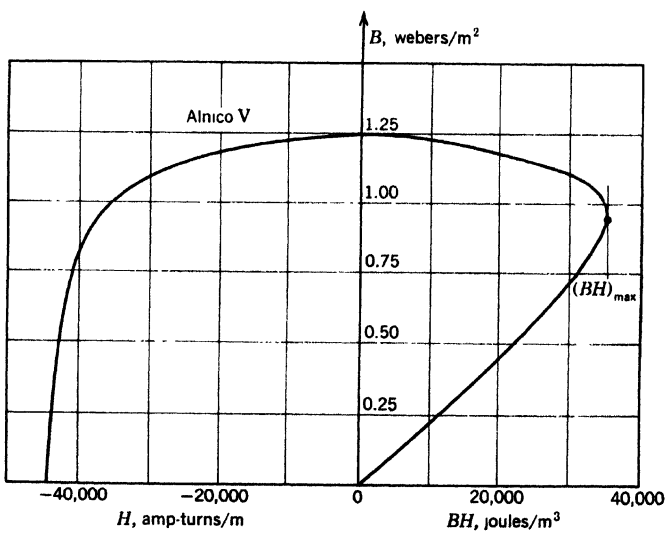


FIG. 7-16. Demagnetization and BH product curves for Alnico V.

A comparison of Alnico V and Alnico XII is interesting. As seen from Fig. 7-14 an altered ΔH in the former is accompanied by a much greater change in flux density than in the latter. That is, even though the remanence of the Alnico XII is lower than that of Alnico V, there is only a

small sacrifice in flux density in passing to a minor hysteresis loop. While this is a very desirable feature of Alnico XII, a more useful criterion for judging the quality of permanent-magnet material is to be found in terms of the amount of magnetic material needed for a given flux. This will be found to be related to the maximum BH product, $(BH)_{max}$, as the operating point a is adjusted to different values along the demagnetization curve. Note that $(BH)_{max}$ indicates the maximum energy density (joules per cubic meter) in the magnet at this operating value a. This product is called the *energy product*. The character of the variation with operating point is illustrated in the curve of Fig. 7-16. Representative materials and their significant constants are given in Table 7-1.

TABLE 7-1
PERMANENT-MAGNET MATERIALS

Material	Remanence, webers/m²	Coercive force, amp/m	$(BH)_{max}$, joules/m³
Chrome steel	1.0	4,000	1,600
Alnico II	0.7	44,800	13,600
Alnico V	1.25	44,000	36,000
Platinum cobalt	0.4	200,000	30,400

7-9. The Magnetic Circuit—Permanent Magnets. It is now desired to examine certain features of a magnetic circuit, which includes permanent magnets in whole or in part. As a means for studying the problem, suppose that the toroidal core of Example 7-7.1 is a magnetically hard material. If there is no air gap and if the ring is magnetized to some saturation value, then, with the removal of the magnetizing current, the flux density in the magnetic core material is equal to the remanence. If, however, a narrow air gap exists in the magnetic circuit, the flux density has a smaller value, say, the point P, which lies somewhere on the demagnetization curve as shown in Fig. 7-17. To find the point P, it is necessary only to calculate the magnetic potential drop across the air gap. From Eqs. (7-39) to (7-42), and noting that ordinarily, when permanent-magnet material is used, no external magnetizing source is applied, whence $NI = 0$, then

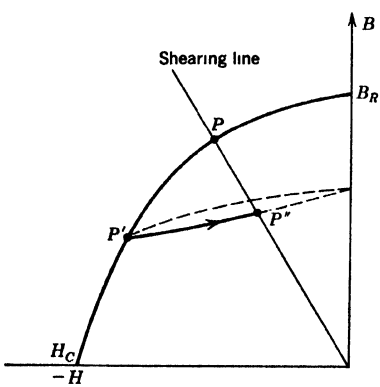


FIG. 7-17. Air-gap shearing line.

$$F = NI = 0 = H_i l_i + H_g l_g \tag{7-47}$$

or

$$H_i l_i = -H_g l_g \tag{7-48}$$

Thus, H_i and H_g are in opposite directions. If leakage is neglected, then B is uniform around the circuit. Also, if fringing in the air gap is neglected, then Eq. (7-48) may be written in the form

$$\frac{\mu_0 H_g}{H_i} = \frac{B}{H_i} = -\mu_0 \frac{l_i}{l_g} \tag{7-49}$$

The ratio, B/H_i, which specifies the permeability of the iron, gives the slope of a line called the *shearing line*, as shown in Fig. 7-17. The intersection of this line with the demagnetization curve specifies the point P.

As already discussed, it ordinarily would be desired that the permanent magnet be stabilized. This requires that a moderate demagnetizing field be applied to the iron, moving the operating point of the iron to P' on the demagnetizing curve. On removing the field, the iron moves to point P'' on the shearing line along the minor hysteresis loop. Just so long as any differences of H no greater than the values between points P' and P'' are applied to the ring, the iron will always return to P''.

Attention is called to the fact that, if, for example, a U magnet is magnetized with a soft-iron keeper in place, the removal of the keeper will cause a given shearing line to be specified. If, now, soft-iron pole pieces are attached, as is usually the case with electrical instruments or with tachometer generators, a new shearing line will be specified.

PROBLEMS

7-1. A sample of medium silicon steel has a hysteresis loss of 120 joules/(m³)(cycle) when the maximum flux density is 0.75 weber/m². Find the loss when the maximum flux density is changed to 1.15 webers/m².

7-2. Consider a steel ring of mean radius R and cross-sectional area A which is uniformly wound with a coil of N turns. A flux density of B webers/m² is produced in the ring by an exciting current of I amp.

a. Show that the inductance of the coil may be written in the form $L = N^2/R$, where R is the reluctance of the magnetic circuit.

b. Show that this expression reduces to the form given in Eq. (6-98).

7-3. A steel ring of mean diameter 50 cm and cross-sectional area 20 cm² is uniformly wound with a coil of 500 turns, and a flux density of 1 weber/m² is produced in the ring by an exciting current of 4 amp.

a. Calculate the coil inductance under these operating conditions.

b. Calculate the exciting current necessary to maintain this flux density when a gap of 0.5 cm width is cut in the ring, and find the new coil inductance. Neglect fringing and leakage.

7-4. The structure of a filter choke that is supposed to have a self-inductance of *approximately* 0.5 henry is made of transformer steel and has the dimensions shown in the accompanying figure. The iron core has a square cross section.

When the current in the winding is 200 ma, the flux density in the air gap is 0.95 weber/m².

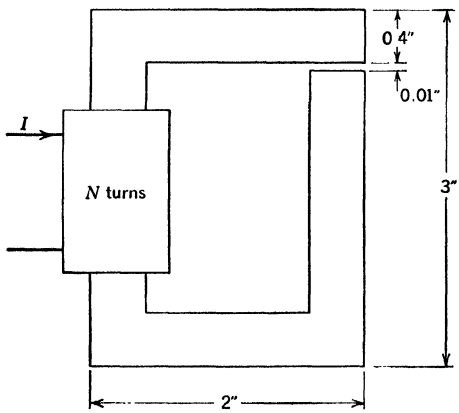
a. How many turns are required in the winding?

b. What is the actual inductance of the arrangement?

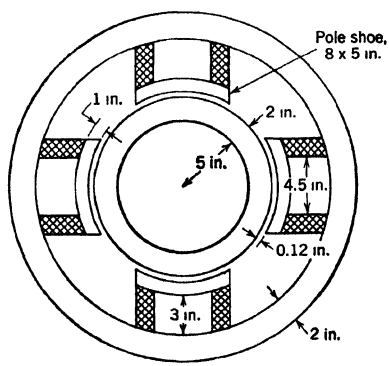
The *BH* characteristic of the core material is specified by the relation

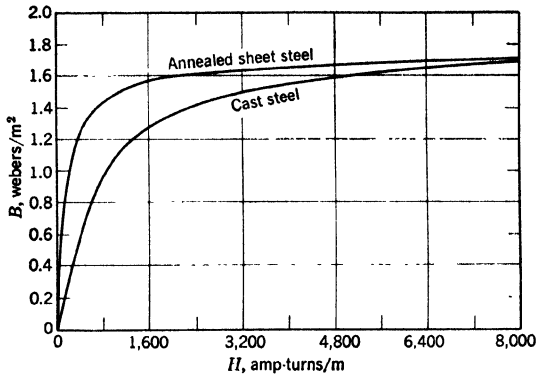
$$B \text{ (webers/m}^2) = 9.3 \times 10^{-3}H - 1.8 \times 10^{-5}H^2 \qquad \text{for } 0 < H < 200$$

with H in ampere turns per meter.



7-5. The accompanying sketch shows the magnetic circuit of a four-pole dynamo. The field cores are made of cast steel and are 4.5 by 4.5 in. The armature is of annealed sheet steel and has a net axial length of 5.3 in. over the iron. The yoke is of cast steel, the cross section being 2 by 6 in. The other necessary dimensions are given on the figure. Using the magnetization curves given, determine the field ampere turns necessary to produce an average air-gap flux density of 0.82 weber/m². Assume that the leakage factor is 1.3 (that is, that the flux in the field cores and yoke is greater than that in the armature by the factor 1.3).





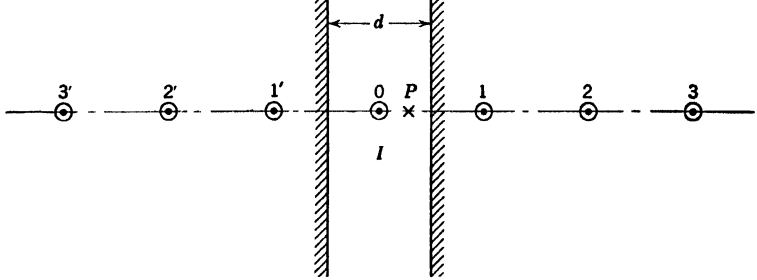
7-6. Carry out the details of Example 7-7.5, to find the magnitude and angle of the resultant $\mathbf{B}_P$.

7-7. To find the field at point P in the air space or the slot between the parallel surfaces of iron of infinite permeability due to a straight parallel current at the center of the slot requires an infinite row of equally spaced images, as shown.*

a. Find an expression for B in the air gap at points remote from the current.

b. Find an expression for B at point P.

c. Sketch the general character of the field throughout the air gap.



7-8. Consider an infinite solenoid of diameter D carrying a current I, thereby yielding a uniform magnetic field distribution across its area. Suppose, now, that a short iron cylinder of diameter $d = 0.1D$ and permeability $\mu_r = 100$ is inserted in the coil with its axis parallel to the axis of the coil. Sketch the expected form of both the B field and the H field distributions.

7-9. Consider a wire embedded in the plane surface between two permeable media, one of which is characterized by a permeability μ_1, the second of which is characterized by a permeability μ_2. Find an expression for the magnetic field intensity H at a distance R from the wire.

See B. Hague, "Electromagnetic Problems in Electrical Engineering," Oxford University Press, London, 1929.

CHAPTER 8

ELECTROMAGNETIC INDUCTION

This chapter may be considered as part 4 of our study of fields and field phenomena. Part 1 was a study of the current-flow field; part 2 was a study of the phenomena associated with static electric charges, and so with static electric fields; part 3 considered the phenomena associated with electric charges in uniform motion, which were referred to as magnetic phenomena. The present chapter will consider the phenomena associated with accelerated charges, or, correspondingly, with time-changing currents or time-changing magnetic fields. Extensive discussion of the phenomenon of electromagnetic induction, which arises in this case, will be undertaken.

8-1. Motional Electromotance—The Flux-cutting Rule. The discussion in Sec. 6-1 led to the result that, when a charge q moves in a magnetic field, it usually experiences a force. The relation between the force $\mathbf{f}$, the velocity of the charge $\mathbf{v}$, and the magnetic induction $\mathbf{B}$ is given by Eq. (6-6), which is here rewritten for convenience.

$$\mathbf{f}_m = q\mathbf{v} \times \mathbf{B} \qquad \text{newtons} \tag{8-1}$$

Note from this expression that the force on the moving charge is a maximum when $\mathbf{v}$ and $\mathbf{B}$ are perpendicular; also, that the force on the moving charge is zero when $\mathbf{v}$ is along $\mathbf{B}$.

Consider, now, the situation illustrated graphically in Fig. 8-1, in which an electric conductor is physically moved across a magnetic field. Owing to this motion, the free electrons in the conductor will experience a force on them in the direction along the conductor and will therefore begin to move in the direction of the force. The direction of the electron flow is that shown by the arrow and implies the existence of a potential difference between the ends of the wire, with the polarity as marked. That is, the situation is the same as if an electric field had been set up in the conductor, which is specified by the relation

$$\mathbf{E}_m = \frac{\mathbf{f}_m}{q} = \mathbf{v} \times \mathbf{B} \qquad \text{volts/m} \tag{8-2}$$

The following implication of Eq. (8-2) is noted: Every moving magnetic field is accompanied by an electric field. Also, attention is again directed to Eq. (6-5), which indicated that every moving electric field is accom-

panied by a magnetic field. There is, therefore, a complete parallelism in the two cases.

Suppose that an external circuit is connected to the moving conductor of the figure, but with the external circuit stationary. Such a system might consist of a pair of fixed rails on which the moving conductor slides and to which the fixed circuit is connected. Since the connecting circuit is at rest, no magnetic force exists on the free electrons of this circuit. However, since a closed circuit now exists, the electrons will flow around the total path in the direction of the force in the moving conductor. If account is taken of the current direction and of the fact that a current-carrying conductor is moving in a magnetic field, it is found that a mechanical force exists, and this is denoted **f** in Fig. 8-1. The work

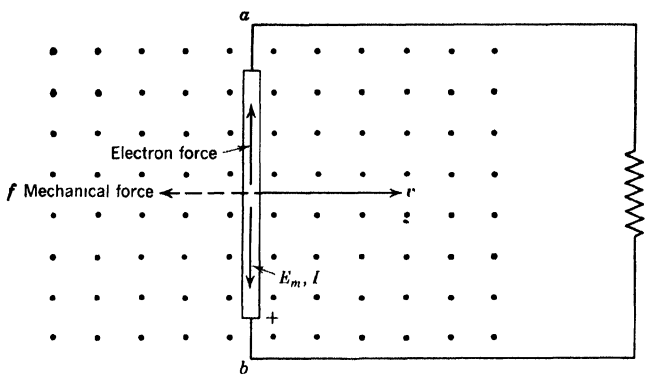


FIG. 8-1. A wire moving with velocity perpendicular to magnetic flux lines. (**B** out of paper.)

done against this force is just the work that must be done to move the electrons through the circuit. The external work is just that to move the conductor through the field. Observe, therefore, that the system or device being discussed is one in which mechanical energy is converted into electrical energy. Any such device is called a source of electromotive force or electromotance, which is often written simply as emf.

In general, any device in which some form of energy is converted into electrical energy is called a source of emf. Some examples of sources of emf are batteries, in which chemical energy is converted into electrical energy, and generators, in which mechanical energy is converted into electrical energy. Only sources of emf involving magnetic fields are surveyed in this chapter. A direct definition follows: *The magnitude of the emf of a source is the energy converted in the source into electrical energy when a unit charge flows around the circuit containing the source.* Observe that energy converted irreversibly into heat is excluded in this definition.

Since there is nothing in the foregoing definition which would exclude the reverse process, namely, that in which electrical energy is converted into some other form of energy, then devices of this type might also be defined as sources of emf. It is preferred, however, to refer to these latter devices as *energy converters*. It is convenient to include a subsidiary definition that applies for the source of emf: *The emf in volts of a source which is furnishing energy is the work in joules possible by the source in transferring a charge of* 1 *coulomb completely around the circuit and back to the starting point.* In particular, it should be noted that, even though the word force appears in the term electromotive force, the concept is that of work. It is for this reason that the word electromotance is preferred to electromotive force.

Attention is called to the fact that a motional electromotance is induced in the moving wire in the field, whether or not a closed external circuit is provided. If only the moving wire is provided, the electrons will accumulate at the top end of the wire (the end marked a in Fig. 8-1) until the electric field set up by the negative charge will just balance the electric field set up by the magnetic force. Thereafter, the charge distribution will remain static, just so long as the wire moves with constant velocity.

It is desired to compute the emf induced in a conductor moving in a magnetic field. From Eq. (8-2) and the direct definition of the emf e of a source, the instantaneous emf of the source is

$$e = \int \mathbf{E}_m \cdot d\mathbf{l} \qquad \text{volts} \tag{8-3a}$$

which may be written as

$$e = \int \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} \qquad \text{volts} \tag{8-3b}$$

This is the general expression for the motional emf. For the special case illustrated in Fig. 8-1 with the closed external circuit outside the magnetic field, $\mathbf{v}$ is zero for all parts of the circuit except for the portion of the circuit illustrated. Therefore, the total integral of Eqs. (8-3) become, since l is the length of the moving conductor,

$$e = \int_0^l \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$$

But, in this case, $\mathbf{v} \times \mathbf{B}$ is constant along l, whence the integration may be effected, with the result

$$e = vB \int_0^l dl = Blv \qquad \text{volts} \tag{8-4}$$

Of course, if l is not perpendicular to $\mathbf{B}$, the appropriate component dictated by the vector relation must be considered.

It is important to note carefully the sense of the electromotance e in Eqs. (8-3) and subsequent expressions. The direction of $\mathbf{E}_m$ is the direction of $\mathbf{v} \times \mathbf{B}$. Thus, the sense of e is the direction of $\mathbf{v} \times \mathbf{B}$ and is illustrated in Fig. 8-1. Observe, therefore, that the sense of the emf e is opposite to that of the sense of the potential difference V in the external circuit. That is, the induced electric field intensity is in the direction of increasing potential, whereas the electric field intensity in the external circuit is in the direction of decreasing potential. While this situation seems to create a fundamental impasse, this is actually not the case if one keeps clearly in mind the fact that $\mathbf{E}_m$ of Eq. (8-2) is *not* an *electrostatic* electric field intensity and is *not* due to a distribution of electric charges. $\mathbf{E}_m$ of Eq. (8-2) is due to the motion of the conductor in the magnetic field, and an external device is doing work to move the conductor. Moreover, $\mathbf{E}_m$ does not exist in the absence of moving charges. Clearly, the motional $\mathbf{E}_m$ is quite a different kind of field vector from the $\mathbf{E}$ of an electrostatic field distribution.

8-2. Time-changing Magnetic Fields and Induced EMF. The experiments of Faraday in 1831 showed that an electric current is induced in a closed circuit when a magnet in the neighborhood of the circuit is moved. Later observation showed that the induced current could be produced in a variety of ways. This would include (1) moving the circuit instead of the magnet, (2) changing the area of the circuit while in a magnetic field, and (3) changing the intensity of the magnetic field (for example, by changing the current in a nearby coil which was producing the field).

A more precise statement relating all the foregoing possibilities was first formulated by Neumann in 1845. The Neumann law states: "An emf is set up in a circuit when the magnetic flux linking the circuit is changed in any manner; the magnitude of the emf is proportional to the time rate of change of flux-linkages with the circuit."

In 1833, Lenz stated the law for determining the sense of the induced emf. The Lenz law states: "The direction of the induced emf is such that any current that it produces tends to oppose the change of flux that produces it."

These two laws can be combined into a single equation,

$$e = -\frac{d\lambda}{dt} = -N\frac{d\Phi}{dt} \quad \text{volts} \tag{8-5}$$

where e is in volts and λ is in weber turns. The minus sign takes account of the sense of the induced emf. However, appropriate interpretation will be necessary, corresponding to the precise manner in which the emf is produced. The induced emf e opposes the change in flux and does not necessarily oppose the flux itself.

It is now desired to show that the motional electromotance may also be computed from Eq. (8-5). This will be done for the situation depicted

in Fig. 8-1. Refer to Eq. (8-4), which applies in this case. In this expression, v is replaced by dx/dt, where dx denotes the distance moved in a time dt. Thus, Eq. (8-4) becomes

$$e = Blv = Bl\frac{dx}{dt}$$

But it is noted that

$$l\,dx = dS$$

where dS is the area swept out by the conductor in time dt. Thus, it follows that

$$e = B\frac{dS}{dt}$$

Now denote by $-d\Phi$ the decrease in the number of flux lines linking the circuit when the conductor has moved the distance dx. Hence,

$$-d\Phi = B\,dS$$

from which the foregoing relation becomes

$$e = -\frac{d\Phi}{dt} \tag{8-6}$$

which is Faraday's law for the single turn of wire represented by the circuit of Fig. 8-1. Clearly, therefore, the motional electromotance may be deduced from Faraday's law if the proper interpretation is put on $d\Phi/dt$. If the circuit is stationary and unchanging, the emf may be computed only from Eq. (8-5).

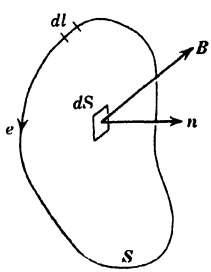


FIG. 8-2. A decreasing magnetic flux through a circuit.

8-3. The Electric Field Intensity and the Changing Magnetic Field. Refer to Fig. 8-2, which shows a closed conducting circuit through which a magnetic flux of uniform density **B** exists. The conducting circuit is the contour of a surface, an element dS of which has a direction specified by the unit normal **n**. If the magnetic flux linking the circuit is decreasing, then the induced emf has the sense shown by the arrow for e. Suppose that $\mathbf{E}_m$ represents the electric field intensity that is produced in the conducting circuit due to the changing magnetic flux. As in Fig. 8-1, this is in the same sense as e, and, by Eq. (8-3),

$$e = \oint \mathbf{E}_m \cdot d\mathbf{l} \tag{8-7}$$

with the integration taken around the path in the sense of e. By combining this expression with Faraday's law, it follows that

$$\oint \mathbf{E}_m \cdot d\mathbf{l} = -\frac{d\lambda}{dt} \tag{8-8}$$

This equation states that every time-changing magnetic field has an electric field associated with it. But since total flux through the surface is

$$\lambda = \int_S \mathbf{B} \cdot \mathbf{n}\, dS$$

then it follows that

$$\frac{d\lambda}{dt} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n}\, dS$$

If the surface and its contour are stationary and only the induction is changing, the d/dt outside the integral may be replaced by $\partial/\partial t$ inside the integral. This expression becomes

$$\frac{d\lambda}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS \tag{8-9}$$

It therefore follows, by combining this expression with Eq. (8-8), that

$$\oint_c \mathbf{E}_m \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS \tag{8-10}$$

This expression shows that the line integral of the electric field intensity due to a changing magnetic flux around any closed fixed path is equal to the negative time rate of change of the magnetic flux through a cap bounded by the closed path. Observe that there is no requirement that the closed path follow a conductor. In fact, the closed path may exist in a region where no conductor exists. An example of this type of path is to be found in the betatron.

It is noted also, as in the case of the motional electromotance, that an emf is induced in the circuit even if the circuit is open-circuited. Suppose that an infinite-impedance voltmeter is connected across the open-circuited terminals. Owing to the field that is induced in the circuit, the electrons will accumulate at one end of the circuit until the electric field set up by the negative charge will just balance the electric field set up by the changing magnetic field. Thereafter, the charge distribution will remain static, the electric field in the conductor remaining zero. The line integral around the complete circuit is necessarily equal to $-d\lambda/dt$, but the entire contribution to the integral is obtained from the nonconducting space between the terminals. Thus, the potential difference between the ends of the circuit is equal to the induced emf in the circuit.

Example 8-3.1. The magnetic circuit illustrated in Fig. 8-3 is energized by a current in the coil of N_1 turns. The material of the magnetic circuit is laminated steel having a constant permeability μ, which is large compared with μ_0 of the air

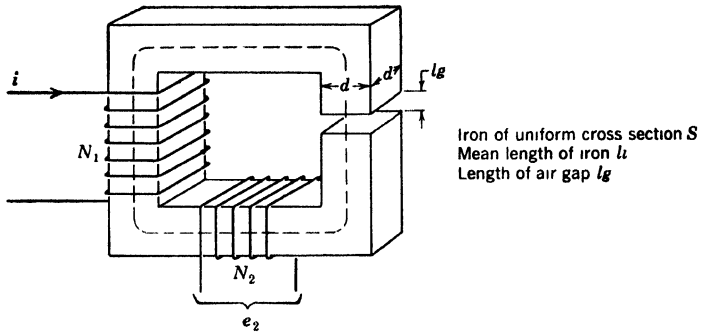


FIG. 8-3. A transformer with an air gap.

gap. The length of the air gap is small compared with the length of the iron circuit. It is required to determine the emf induced in coil N_2.

Solution. The reluctance of the iron path is given by the expression

$$R_i = \frac{l_i}{\mu S}$$

Also, the reluctance of the path through the air gap is

$$R_g = \frac{l_g}{\mu_0(d + 2l_g)^2}$$

Hence, the total reluctance of the magnetic circuit is

$$R = R_i + R_g$$

The mmf around the path indicated by the dotted line is

$$F = N_1 i_1$$

The flux in the circuit is then

$$\lambda = \frac{F}{R} = \frac{N_1 i_1}{R}$$

For the case when the flux is changing with time,

$$\frac{d\lambda}{dt} = \frac{N_1}{R}\frac{di_1}{dt}$$

so that, by Faraday's law, the emf induced in the winding N_2 is

$$e = -N_2\frac{d\lambda}{dt} = -\frac{N_1 N_2}{R}\frac{di_1}{dt} \tag{8-11}$$

Example 8-3.2.[1] An armature conductor of a single-phase a-c motor is an example of a winding that moves in a time-varying magnetic field. It is required to find an expression for the emf induced in the winding.

[1] L. V. Bewley, "Flux Linkages and Electromagnetic Induction," The Macmillan Company, New York, 1952; Flux Linkages and Electromagnetic Induction in Closed Circuits, *Trans. AIEE*, **48**:327 (1929).

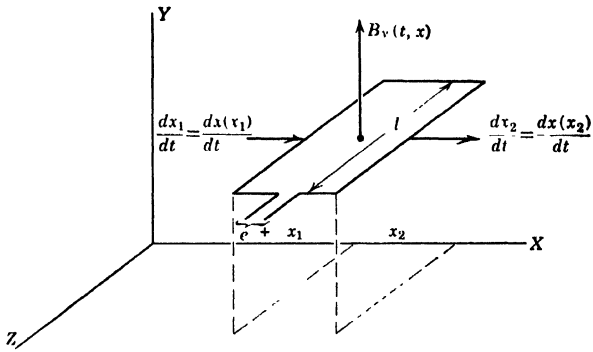


FIG. 8-4. A moving coil in a time-changing magnetic field.

Solution. The situation here under survey is illustrated in Fig. 8-4. As indicated, it is assumed that the magnetic field is a function of the coordinate x and time t, with no motion in the Y and Z direction. Also, the coil sides are assumed to be moving with velocities dx_1/dt and dx_2/dt. An application of Faraday's law for the induced emf leads to the expression

$$e = -N\frac{d\Phi}{dt} = -Nl\frac{d}{dt}\int_{x_1}^{x_2} B_y(x,t)\,dx \tag{8-12}$$

But this expression may be differentiated under the integral sign and written in the form

$$e = -Nl\left(\int_{x_1}^{x_2}\frac{\partial B_y}{\partial t}\,dx + B_y\frac{dx}{dt}\Bigg|_{x_1}^{x_2}\right) \tag{8-13}$$

which is written as

$$e = -N\int_{x_1}^{x_2}\frac{\partial B_y}{\partial t}\,l\,dx - Nl\left(B_{y2}\frac{dx_2}{dt} - B_{y1}\frac{dx_1}{dt}\right) \tag{8-14}$$

But, for a rigid coil, the coil sides have the same velocity, so that this expression becomes

$$e = -N\frac{\partial\Phi}{\partial t} - Nl(B_{y2} - B_{y1})\frac{dx}{dt} \tag{8-15}$$

The first term of this expression is the time rate of change of the flux linkage with the coil, with the coil fixed in space at the position indicated and with the magnetic field assumed fixed in space with respect to the coil. The second and third terms are the components of the motional emf, the velocity of the coil sides being dx/dt, but the magnetic induction being assumed fixed with respect to the coil. The first term in this equation is the transformer potential; the second term is the motional emf and corresponds to the $\mathbf{v} \times \mathbf{B}$ term in Eqs. (8-3).

8-4. Eddy-current Loss. From the foregoing section, the presence of a closed circuit in a changing magnetic field results in current flow, due to the emf that is induced therein. Thus a block of metal in a changing

field, such as that illustrated in Fig. 8-5, has currents in it; and these circulating currents are called *eddy currents.*

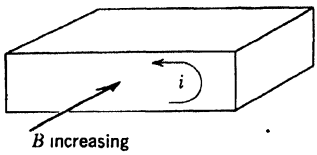


FIG. 8-5. Eddy currents induced in a block of metal in a time-changing magnetic field

It is not difficult to calculate the eddy current in such a metal plate if the following assumptions are made:

1. The flux is a sinusoidal function of the time.

2. The width of the plate is large compared with its thickness.

3. The eddy currents do not modify the distribution of magnetic flux. This effect will be explicitly examined in Chap. 10.

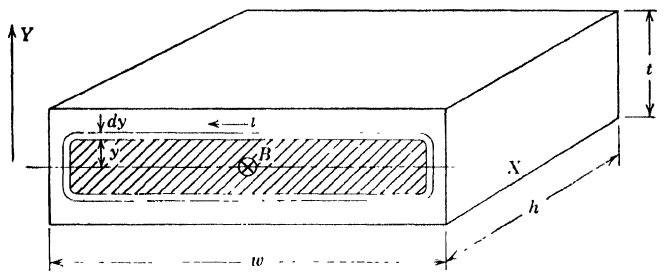


FIG. 8-6. The cross section of a metal lamination.

Consider, therefore, a metal plate, as illustrated in Fig. 8-6, with the flux density assumed to be uniform over the plate but varying sinusoidally with the time according to the relation

$$B = B_m \sin \omega t$$

The emf induced around the loop illustrated is given by the expression

$$e = -\frac{d\Phi}{dt} = -2ywB_m\omega \cos \omega t$$

where the area of the loop shown is approximately $2yw$. But the resistance of the loop through which the eddy current flows is approximately

$$R \doteq \frac{2\rho w}{h\,dy}$$

where h is the length of the lamination in the flux direction and dy is the thickness. Therefore, the average power loss in the path associated with the eddy-current flow is

$$dP = \frac{1}{T}\int_0^T \frac{e^2}{R}\,dt = \frac{1}{2}\frac{(2ywB_m)^2}{(2\rho w)/(h\,dy)} = \frac{(\omega w B_m)^2}{\rho w}\,hy^2\,dy$$

To find the power loss in the whole block, this expression is integrated thus,

$$P = \int_0^{t/2} \frac{\omega^2 B_m^2 hw}{\rho} y^2 \, dy$$

which is

$$P = \frac{4\pi^2 twB_m^2 f^2}{\rho} \frac{t^3}{24} \quad \text{watts}$$

But htw is the volume of the lamination, so that

$$P = \frac{\pi^2 B_m^2 f^2 t^2}{6\rho} \quad \text{watts/m}^3 \tag{8-16}$$

It should be noticed from this expression that the loss is inversely proportional to the resistivity of the material, which indicates the desirability of using high-resistivity materials. In the case of machinery and transformers, there is considerable advantage in using high-resistivity steels, which accounts for the widespread use of silicon steels. Observe also that the power loss varies as the square of the width of the lamination. This accounts for the laminated construction of machine and transformer cores. For example, for a constant volume of iron, by reducing the thickness of the iron from t to insulated laminations t/n thick, there is an n-fold decrease in the eddy-current loss.

It would be expected, in view of the approximations made in the above derivation, that the expression might be in error. Experiments show that Eq. (8-16) gives very closely the correct type of functional relationship between P and B_m, f, and h, but the constant term may be in error by as much as a factor 2, depending upon the character of the machine construction.

8-5. General Notes on Interpreting the Laws of Induction. It was noted in connection with Eq. (8-5) that Faraday's law (as stated by Neumann) applied to the general problem of electromagnetic induction. Also, from the examples discussed in this chapter, it appeared that no fundamental difficulty arose in the application of this law. There are a number of instances when the interpretation of the law becomes far from obvious. It is desired to examine a number of such situations, which require considerable caution in their interpretation. Among such special situations are the following: (1) the problem arising from the definition of flux linkage, (2) problems which involve moving magnetic fields, and (3) problems involving the relative motion between the line of the circuit and the material in which it lies. A discussion is included of each of these special situations.

Faraday's Law and Flux Linkages. The interpretation of Eq. (8-5) carries with it the requirement of a clear definition of the term flux

linkages. While the implication is that the flux linkages denote the sum of the fluxes through all the turns, this is often an inadequate concept. Consider, for example, an air-core solenoid with widely spaced turns, as illustrated in Fig. 8-7. It is not possible to specify the flux linking any particular turn since the turn does not define a closed loop. If the solenoid closed on itself, one might be able to specify the number of turns, but in general this might not be possible.

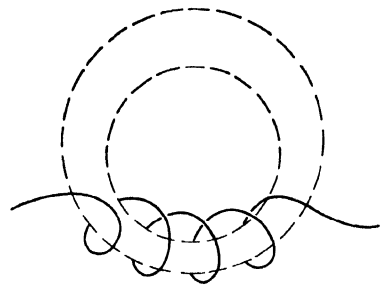


FIG. 8-7. An air-core solenoid with widely spaced turns.

The ambiguity discussed above is avoided by recalling that the flux linkage is a generalization of the idea of the flux across a cap of any shape that is formed with the current circuit as its boundary. Now the difficulty becomes that of drawing or imagining a cap which shall have its edge on the current circuit. As an indication of the situation, consider the two-turn circuit illustrated in Fig. 8-8. Included in Fig. 8-8 are a

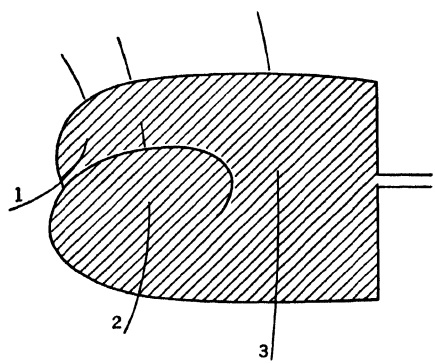


FIG. 8-8. A two-turn circuit with indicated tubes of flux.

number of flux tubes which pass through the cap which is drawn with the current circuit as its boundary. Observe that the flux tubes (1) and (3) link only once with the circuit, whereas flux tube (2) links the circuit twice. Clearly, care must be exercised in determining the effective flux through the cap, although Eq. (6-63) for the total flux through the cap is generally valid. Often, in fact, it may be very difficult to evaluate this total flux, but this is possible in principle.

An experiment by L. V. Bewley[1] shows the need for care in the defini-

[1] L. V. Bewley, "Flux Linkages and Electromagnetic Induction," The Macmillan Company, New York, 1952.

tion of flux linkages. The essentials of the experiment are illustrated in Fig. 8-9. In the diagram, C is an insulating cylinder which is located in an axial magnetic field. The cylinder C is arranged to rotate so that additional turns are added from the wire spool on the drum D. One end of the wire is connected to the slip ring R, to which one voltmeter lead is attached. The second voltmeter lead is attached to a sliding contact, as shown. The essential problem is to examine whether an emf is generated by winding turns on a coil in a magnetic field.

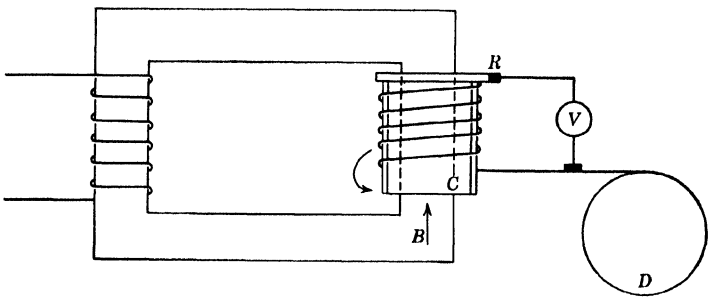


FIG. 8-9. The elements of the Bewley experiment.

It appears that the flux linkage in the coil is increasing, because the number of turns is increasing. This could be interpreted to mean that, since

$$e = -\frac{d}{dt}(N\Phi)$$

where N is the number of turns in the coil and Φ is the flux passing through the coil, upon expansion

$$e = -N\frac{d\Phi}{dt} - \Phi\frac{dN}{dt} \tag{8-17}$$

Obviously, upon winding more turns on the coil through which the flux Φ was passing, it might appear that the second term was satisfied and that an emf might be observed. Actually, no emf is observed. The explanation is found in the exact definition of flux linkage.

In order to study the present situation, the circuit of Fig. 8-9 is represented by two circuits. These are illustrated in Fig. 8-10. Circuit (a) comprises a lengthening spiral, which is joined by a central axis to a radius AM, which is fixed in direction, and a radius BN, which rotates. Circuit (b) comprises a shaft $A'B'$, which is connected to the external circuit through the fixed radius $A'M'$, and rotating radius $B'N'$, which appears as a spoke, or strut, of the contact ring.

Observe that, each time a turn is added, the linkage in circuit (a) is, in fact, increased by the flux through the extra turn. At the same time,

however, an equal flux has been swept through by the moving radius $B'N'$ of circuit (b). The emf produced by each circuit (a) and (b) is equal and opposite, so that the resultant emf is zero. The apparent flux-linkage theory takes account only of circuit (a) and hence does not lead to the correct result.

The Moving Magnetic Field. In the discussion in Sec. 8-1, it was supposed that the conductor was moving through a fixed magnetic field. The converse problem of a magnetic field moving past a fixed conductor was also included in this discussion by implication, since the equations

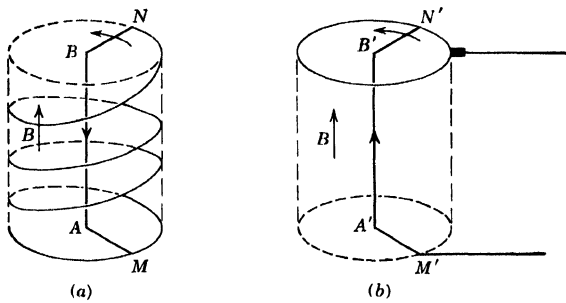


FIG. 8-10. The two contributing circuits to the experiment of Fig. 8-9.

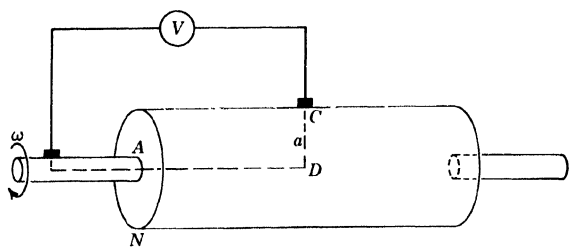


FIG. 8-11. A cylindrical magnet rotating about its axis.

were equally applicable to both cases. The concept of a moving field is a very important one, and one that exists in the alternator and in polyphase induction machinery.

Sometimes a physical picture aids one in field problems, and such concepts as *tubes of electric flux* and *tubes of magnetic flux* are helpful. One concept of a magnetic field which is frequently used is that which likens magnetic-flux lines to rubber bands which are fixed with respect to the magnetic field. It is important not to associate too realistic or too arbitrary a concept to such models or picture aids. For example, consider a cylindrical magnet that rotates along its axis, brushes being provided as illustrated in Fig. 8-11. The question is whether or not an emf is induced, with a subsequent indication on the voltmeter.

It is actually misleading to believe that because the magnet is rotating the lines of magnetic flux, which might be imagined as fixed with respect to the magnet, are rotating with it. That is, the magnetic field at a point on the surface of the magnet remains unchanged even though the magnet itself is rotating. Another way of expressing the situation is to note that the north and south poles of the magnet are magnetic equipotentials, with appropriately located magnetic-equipotential surfaces at intermediate points. There is no change in magnetic potential as the magnet rotates, whence there is no change in flux with time.

To answer the question relative to the reading of the voltmeter, attention is directed to a volume element of matter making up the magnet. Since this volume element of matter is in a moving magnetic field, a redistribution of charge will occur and an electric field will result, exactly in the manner leading to Eq. (8-2). The resulting electric field between the axis and any point in the cylindrical magnet will be $\mathbf{E}_m = -\mathbf{v} \times \mathbf{B}$, and the emf between the axis and the outer surface is

$$e = \int_0^a \mathbf{B} \times \mathbf{v} \cdot d\mathbf{l}$$

which is

$$e = \int_0^a B r\omega \, dr$$

which yields the expression

$$e = \frac{\omega B a^2}{2} \tag{8-18}$$

A convenient way of looking at the situation is to consider any path DC of the magnet to be fixed with respect to the magnet. Thus as the magnet rotates, the path DC also rotates. The situation now is equivalent to that of the single-spoke wheel shown in Fig. 8-12. The emf that is induced in the present case is specified by the flux-cutting rule, and is precisely that given in Eq. (8-18).

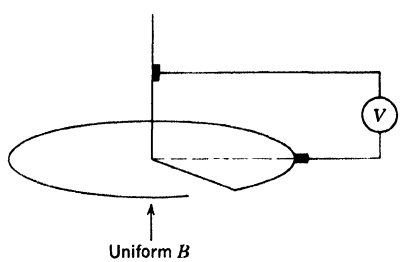


FIG. 8-12. The single-spoke-wheel equivalent of the rotating magnet of Fig. 8-11.

The Moving Line of the Circuit and the Material. A class of problem which bears some relation to that in Fig. 8-11 is that in which part of the circuit in which the emf is generated traverses the material of the circuit, which itself is in motion relative to the circuit. The Faraday disk is representative of the problems now under survey; it consists of a thin metallic disk that is situated with its plane normal to a uniform magnetic

field. The system is illustrated in Fig. 8-13. The disk is supposed to rotate at a constant angular velocity of ω radians/sec. The essential question is whether or not an emf is developed between the shaft and the periphery of the disk. Clearly, this situation is the same as the above except for the details of the system. Now the magnetic field is external to the moving material, rather than combined with it.

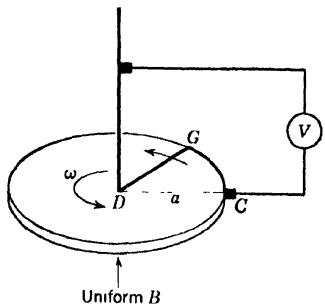


FIG. 8-13. The Faraday disk generator.

Suppose that the circuit is closed by the dotted line DC, as shown in the figure. It would appear that the flux linkage with the circuit is permanently zero, and the induced emf should also be zero. Actually, however, an emf is developed in this device and is expressed analytically by Eq. (8-18). The essential element which must not be ignored in the application of the flux-linkage rule for the evaluation of the induced emf $e = -d\lambda/dt$ is that the flux linkage is evaluated for a circuit that is chosen so that at no point are the particles of the material moving across it. On this basis, the emf is calculated by means of the circuit shown by the solid line of Fig. 8-13, with the radial line DG rotating with angular velocity ω.

8-6. The Faraday Law in Point Form. It was noted in Sec. 8-2 that Eq. (8-8) expresses the total induced emf in a closed circuit due both to motional and time changing fields. That is, the expression

$$e = \oint \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS = -\frac{d}{dt} \int \mathbf{B} \cdot \mathbf{n}\, dS \qquad (8\text{-}19)$$

specifies the total emf, provided, as noted in Sec. 8-5, that there is no relative motion between the line of the circuit and the material in which it lies. Correspondingly, as shown in Eq. (8-10), the emf induced throughout the circuit is given by

$$e = \oint \mathbf{E}_m \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot \mathbf{n}\, dS \qquad (8\text{-}20)$$

Note, however, that, since any field $\mathbf{E}_c$ due to charges is conservative, its line integral is zero. Then it is clear that, if

$$\mathbf{E} = \mathbf{E}_c + \mathbf{E}_m$$

then

$$\oint \mathbf{E} \cdot d\mathbf{l} = \oint \mathbf{E}_m \cdot d\mathbf{l}$$

so that

$$e = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot \mathbf{n}\, dS \qquad (8\text{-}21)$$

That is, the electric field in Eq. (8-21) may be the total field and not just that part which is induced by a changing flux.

It is convenient to transform the line integral appearing in the expression above into a surface integral through the use of Stokes' theorem [see Eq. (6-58)]. Therefore, it follows that

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{E})\, dS$$

For the case where the surface and its contour are stationary and only the induction is changing (and, as already mentioned, this case is often referred to as transformer induction), as already noted in Eq. (8-9), the d/dt outside the integral may be replaced by $\partial/\partial t$ inside the integral. Then,

$$\int_S \mathbf{n} \cdot (\nabla \times \mathbf{E})\, dS = -\int_S \mathbf{n} \cdot \frac{\partial \mathbf{B}}{\partial t}\, dS \tag{8-22}$$

Now, since this expression must be true for any surface, it follows that the two vectors in its integrands can be equated so that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{8-23}$$

This relation, which expresses Faraday's law in differential form, will appear as one of Maxwell's equations.

8-7. Magnetic Vector Potential. It is recalled that in discussing electrostatic problems it was convenient to relate the electric field intensity with the gradient of a scalar potential. This proved to be a very desirable procedure, because techniques exist for the evaluation of the scalar potential which are far more powerful than exist for the direct evaluation of the electric field itself. In discussing magnetic phenomena, it is found convenient to introduce a new magnetic vector, since with this newly defined quantity the solution of certain problems becomes somewhat more direct.

Refer to Eq. (8-21), which relates the emf induced in a circuit through which a changing flux exists, namely,

$$e = \int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \tag{8-24}$$

where C is the contour through which the magnetic flux Φ is contained, as illustrated in Fig. 8-14. We now define a vector $\mathbf{A}$ at any point P, whose magnitude is given by

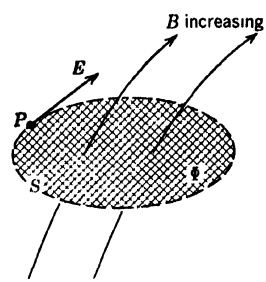


FIG. 8-14. A closed contour around a magnetic flux.

$$A = -\int E\, dt \qquad \text{webers/m} \tag{8-25}$$

and whose direction is the direction of **E**. It follows, since **A** is a function of both the space coordinates and the time, that, upon differentiation,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \tag{8-26}$$

By combining this expression with Eq. (8-24), then

$$-\int \frac{\partial}{\partial t}(\mathbf{A} \cdot d\mathbf{l}) = -\frac{d\Phi}{dt}$$

or

$$\int_C \mathbf{A} \cdot d\mathbf{l} = \Phi \tag{8-27}$$

Observe that this is just a line integral; specifically, it shows that the line integral of **A** around any closed circuit is equal to the magnetic flux linked with that circuit. Observe that Eq. (8-26) yields only that component of the electric field that is due to the time changing magnetic field. It does not include any effects due to static charges or to other emf producing fields.

Owing to the relation between **E** and **A**, it is evident that a relation must exist between the vector potential **A** and the magnetic flux density **B**. This is readily deduced from Eq. (8-23) by first noting, upon taking the curl of both sides of Eq. (8-26), that

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{A} = -\frac{\partial \mathbf{B}}{\partial t}$$

from which it follows that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{8-28}$$

This is the fundamental connection between **B** and **A**. It is noted that this expression is consistent with the requirement specified in Eq. (6-43), that div **B** be zero always, since

$$\nabla \cdot \mathbf{B} = \nabla \cdot \nabla \times \mathbf{A} \equiv 0 \tag{8-29}$$

Consider the situation of the magnetic field in free space due to a d-c or slowly varying current. The mathematical relationship expressing the field in terms of the current is given by Eq. (7-9), which is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Therefore,

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} \tag{8-30}$$

Before we can continue with a discussion, it is necessary to examine the meaning of the function on the left, the curl of the curl of a vector, as this is our first introduction to such a sequence of operations. As a direct approach, suppose that the operations specified are carried out in cartesian form. The expansions specified by

$$\nabla \times \nabla \times \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)$$

will be found to be the following:

$$\nabla \times \nabla \times \mathbf{A} = \left[\mathbf{i}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\,\partial y} + \frac{\partial^2 A_z}{\partial x\,\partial z}\right) + \mathbf{j}\left(\frac{\partial^2 A_x}{\partial x\,\partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y\,\partial z}\right) + \mathbf{k}\left(\frac{\partial^2 A_x}{\partial x\,\partial z} + \frac{\partial^2 A_y}{\partial y\,\partial z} + \frac{\partial^2 A_z}{\partial z^2}\right)\right] - \left[\mathbf{i}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right) + \mathbf{j}\left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right) + \mathbf{k}\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\right]$$

The first group on the right is specified by the mathematical group

$$\nabla(\nabla \cdot \mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

The second group on the right is suggestive of the Laplacian operator and is found to be

$$\nabla^2 \mathbf{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)$$

Therefore, in the notation of vector analysis,

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{8-31}$$

Therefore, Eq. (8-30) may now be written in the form

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \tag{8-32}$$

Another very important aspect of vectors (Helmholtz theorem of uniqueness) permits this expression to be reduced. This theorem notes that the specification of the curl of a vector does not uniquely specify the vector itself. This follows from the fact that one might require the divergence of the vector to be any prescribed quantity and yet not affect its curl. That is, the curl of a vector has the same value regardless of the value of its divergence. Because of this fact, it is most desirable, of course, to choose the vector potential in the most convenient manner for our purposes. In this case, it is found most desirable to choose **A** so as to require its divergence to be zero. Under these circumstances, Eq. (8-32) now reduces to the form

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{8-33}$$

Mathematically, this expression is similar to Poisson's equation [see (Eq. 2-69)], except that vectors are involved instead of scalars. It is noted that the effect of choosing **A** such that its divergence is zero limits the

validity of Eq. (8-33) to the static, or d-c, case. Actually, it is found to be valid when **J** does not change too rapidly, up to the moderate radio frequencies.

Since Eq. (8-33) is a vector equation, it is equivalent to three scalar equations, as shown.

$$\begin{aligned} \nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z \end{aligned} \tag{8-34}$$

Observe that each of these relations is in the form of Poisson's equation and that the many mathematical techniques for the solution of Poisson's equation may be used for the solution of these equations. This is really a very important observation, since it permits us to write down one form of the solution of this expression as a consequence of our prior work. Attention is drawn to Sec. 2-8, which discusses Poisson's equation for the electric case in the form

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

This equation is satisfied by the potential integral

$$\phi = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\rho}{r} d\tau$$

Hence, it follows that the solutions to Eqs. (8-34) may be written directly as

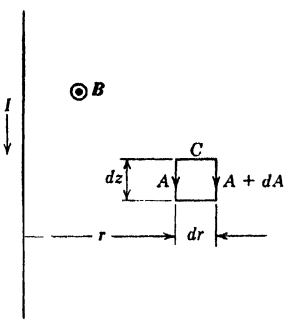


FIG. 8-15. To find the **A** vector of a filamentary current.

$$A_x = \frac{\mu_0}{4\pi} \int \frac{J_x}{r} d\tau$$

$$A_y = \frac{\mu_0}{4\pi} \int \frac{J_y}{r} d\tau$$

$$A_z = \frac{\mu_0}{4\pi} \int \frac{J_z}{r} d\tau$$

which may be recombined to the vector-potential integral

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_\tau \frac{\mathbf{J}}{r} d\tau \tag{8-35}$$

Thus, Eq. (8-35) is one form of the solution of the vector Poisson equation, Eq. (8-33).

Example 8-7.1. Find an expression for the vector potential **A** for a long, straight current filament.

Solution. Refer to Fig. 8-15, which shows the geometry of the system under survey. To solve the present problem, it is necessary to evaluate the line inte-

gral specified by Eq. (8-27). The contour C is that illustrated, whence the solution requires a knowledge of the magnetic flux linking with C. This may be written in terms of the known value of B for a straight filamentary current, which is given by Eq. (6-12). From Eq. (8-35) if $\mathbf{J} = J_z$, then $\mathbf{A} = A_z$, and A contains no radial component,

$$\int_C \mathbf{A} \cdot d\mathbf{l} = A\,dz - (A + dA)\,dz$$
$$= -dA\,dz \tag{8-36}$$

Also, the flux enclosed by the contour C is

$$\Phi = B\,dr\,dz \tag{8-37}$$

Therefore, it follows that

$$-dA\,dz = B\,dr\,dz$$

from which

$$\frac{dA}{dr} = -B = \frac{\mu_0 I}{2\pi r} \tag{8-38}$$

By performing an integration of this expression, there results

$$A = -\frac{\mu_0 I}{2\pi} \ln r + F \tag{8-39}$$

where F is an arbitrary constant.

Actually, this problem is relatively trivial, because it is usually desired to determine the expression for $\mathbf{A}$ directly from the current and then, by an application of Eq. (8-28), one proceeds to the expression for $\mathbf{B}$. The problem does serve to emphasize that $\mathbf{A}$ is parallel to I for the filamentary case. Since a more complicated current-carrying conductor can usually be reduced to the summation of filamentary strips, the vector potential for the more general case will also be parallel to the current. This result will permit a simplification of the general results discussed below.

8-8. Applications of the Vector Potential. Before undertaking the solution of particular problems by means of the vector potential, it is desired to show that several of the results deduced by other methods may be deduced from the known form for $\mathbf{A}$ given by Eq. (8-35).

Example 8-8.1. Show that the magnetic field of a current is expressible in the integral form given in Eq. (6-8).

Solution. In the present case, an examination is made of the expression

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{r}\,d\tau = \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{\mathbf{J}}{r}\right) d\tau \tag{8-40}$$

A direct expansion of the integrand may be effected. There results

$$\nabla \times \frac{\mathbf{J}}{r} = \frac{1}{r} \nabla \times \mathbf{J} + \frac{\mathbf{J} \times \mathbf{r}_0}{r^2}$$

The term $1/r \nabla \times \mathbf{J}$ vanishes because the curl operation is to be taken relative to the field point, a function of (x,y,z), whereas $\mathbf{J}$ is a function of the source point variables (x',y',z'). Thus

$$\nabla \times \frac{\mathbf{J}}{r} = \frac{\mathbf{J} \times \mathbf{r}_0}{r^2} \tag{8-41}$$

For a filament of constant area a, we may write $d\tau = a\, dl$,

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \mathbf{r}_0}{r^2} a\, dl = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \mathbf{r}_0}{r^2} dl$$

which may then be written in the form

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int I \frac{d\mathbf{l} \times \mathbf{r}_0}{r^2} \tag{8-42}$$

which is precisely that given by Eq. (6-8).

Example 8-8.2. Show from considerations involving the vector potential **A** that the Neumann expression for the mutual inductance of two circuits [Eq. (6-72)] is valid.

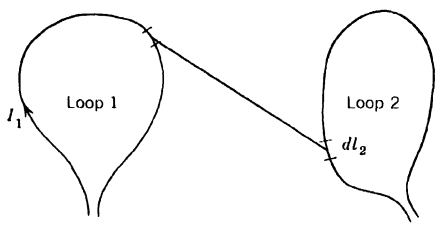


FIG. 8-16. A pair of mutually coupled circuits.

Solution. Refer to Fig. 8-16, which shows two adjacent electric circuits. If the current in circuit 1 is changing with time, there will be a changing magnetic field through circuit 2. As a result, an emf will be induced in circuit 2, which is expressed by Eq. (8-22),

$$e = -\int_{S_2} \mathbf{n} \cdot \frac{\partial \mathbf{B}}{\partial t}\, dS = -\int_{S_2} \mathbf{n} \cdot \frac{\partial}{\partial t} (\nabla \times \mathbf{A})\, dS$$

By interchanging the space (curl) and time derivatives, and by an application of Stokes' theorem, this expression may be written as a line integral as shown:

$$e = -\int_{S_2} \mathbf{n} \cdot \nabla \times \left(\frac{\partial \mathbf{A}}{\partial t}\right) dS = -\oint_2 \frac{\partial}{\partial t} \mathbf{A} \cdot d\mathbf{l} \tag{8-43}$$

where the line integral is taken along circuit 2. The known form for the vector potential is to be combined with this expression. This is first written as

$$\mathbf{A}_1 = \frac{\mu_0}{4\pi} \int_{\tau_1} \frac{\mathbf{J}_1}{r}\, d\tau = \frac{\mu_0}{4\pi} \oint \frac{\mathbf{I}_1}{r}\, dl$$

where the line integral is taken around circuit 1. The result by combining the foregoing two forms is

$$e = -\frac{\mu_0}{4\pi} \left(\oint_2 \frac{\partial}{\partial t} \oint_1 \frac{I_1}{r}\, d\mathbf{l}_1 \cdot d\mathbf{l}_2 \right) \tag{8-44}$$

In those cases where the current is constant throughout the circuit (lumped circuit constants, that is, no distributed parameters), the current may be taken out from under the integral sign. Note also that only the current changes with time, whence we may write the above expression in the form

$$e = -\frac{\mu_0}{4\pi}\frac{dI_1}{dt}\oint_{2,1}\oint\frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r}$$

But by definition the coefficient of $-dI/dt$ is the mutual inductance of the two circuits, so that finally

$$M = \frac{\mu_0}{4\pi}\oint_{2,1}\oint\frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} \tag{8-45}$$

which is precisely the form given in Eq. (6-72). Reference to Sec. 6-6 will show how much more simply this result has been deduced by means of the vector potential.

Example 8-8.3. Find the vector potential of a long wire of circular cross section of radius b carrying a current of density J. From this value of $\mathbf{A}$, deduce the expressions for $\mathbf{B}$.

Solution. Two solutions are required, one for regions outside the current-carrying conductor and the second for the region inside the conductor.

A. REGION OUTSIDE CONDUCTOR. Choose the axis of the conductor as the Z axis. The Poisson equation for $\mathbf{A}$ [Eq. (8-33)] is written in cylindrical form since the currents are axial, and A can only have a Z component. Thus,

$$\nabla^2 A_z = \frac{d^2A_z}{dr^2} + \frac{1}{r}\frac{dA_z}{dr} = 0 \tag{8-46}$$

The solution of this expression, according to the standard methods of differential equations, is the expression

$$A_z = C \ln r + F \tag{8-47}$$

which is precisely that given by Eq. (8-39), which was obtained in a different manner. The appropriate expression for $\mathbf{B}$ is obtained from the general curl relationship connecting $\mathbf{B}$ and $\mathbf{A}$ (see Appendix I). It is found that the terms involving A_z are

$$\mathbf{B} = \text{curl } \mathbf{A} = \mathbf{a}_r\frac{\partial A_z}{r\partial\phi} - \mathbf{a}_\phi\frac{\partial A_z}{\partial r} \tag{8-48}$$

Combine this with Eq. (8-47) to find

$$\mathbf{B} = -\mathbf{a}_\phi\frac{C}{r} \tag{8-49}$$

The unknown constant C may be deduced from the Ampère law taken around a circular path.

$$\oint \mathbf{H} \cdot d\mathbf{l} = I$$

Then,

$$-\frac{C}{r} \times 2\pi r = \mu_0 I$$

from which it follows that

$$C = -\frac{\mu_0 I}{2\pi}$$

or, finally,
$$\mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r} \qquad r \geq b \tag{8-50}$$

B. REGION INSIDE CONDUCTOR. In the present case, the equation for solution is the following,

$$\nabla^2 A_z = -\mu_0 J_z = \text{const} \tag{8-51}$$

which is written in the form

$$\frac{d^2 A_z}{dr^2} + \frac{1}{r}\frac{dA_z}{dr} + \mu_0 J_z = 0 \tag{8-52}$$

To find a solution to this equation, use is made of the technique discussed in connection with Example 2-8.3. Without indicating the details, which are precisely parallel to those used for the former example, the complete solution is given by the expression

$$A_z = C \ln r + F - \frac{\mu_0 J_z r^2}{4} \tag{8-53}$$

To find the expression for **B**, we proceed in precisely the same way as we did in going from Eq. (8-47) to Eq. (8-49). The appropriate form for **B** is again given by Eq. (8-48), which yields, by combining this with Eq. (8-53), the expression

$$\mathbf{B} = -\mathbf{a}_\phi \left[\frac{C}{r} + \frac{\mu_0 J_z r}{2}\right] \tag{8-54}$$

Since this expression indicates a singularity at $r \to 0$, which is physically impossible, then it is necessary to take $C = 0$. It follows from this that

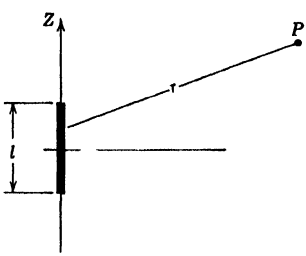


FIG. 8-17. A short length of wire carrying a current.

$$\mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 J_z r}{2} = \mathbf{a}_\phi \frac{\mu_0 I r}{2\pi b^2} \qquad r \leq b \tag{8-55}$$

This solution agrees with that specified above for case *a* at the boundary where $r = a$, where

$$\mathbf{B} = \mathbf{a}_\phi \frac{I\mu_0}{2\pi a} \tag{8-56}$$

As a general comment, it is worth noting that the results obtained in this example were deduced more simply by a direct application of Eq. (6-8) without the use of the vector potential. In more complicated cases, the vector-potential method is often of real help in deducing the results.

Example 8-8.4. Find the vector potential of a short length of wire carrying an a-c current at distances remote from the wire.

Solution. Refer to Fig. 8-17 for the geometry of the system. The practical details of achieving such a short length of current without additional wires are of no immediate concern.

It is assumed that the current is in the z direction and that the current distribution along the length of the wire is constant. The required solution is then given by the expression

$$A_z = \frac{\mu_0}{4\pi} \int \frac{J_z}{r} d\tau$$

appropriately interpreted. Thus, writing

$$d\tau = a\, dz$$

where a denotes the area of the conductor, and noting that

$$J_z a = i$$

then the expression for A_z becomes

$$A_z = \frac{\mu_0}{4\pi} \int_{-l/2}^{l/2} \frac{I_m \sin \omega t}{r} dz \tag{8-57}$$

Now, for remote distances, $r \gg l$, so that the distance r to any point on the line is considered to be a constant. Thus, approximately,

$$A_z \doteq \frac{\mu_0}{4\pi} \frac{I_m \sin \omega t}{r} \int_{-l/2}^{l/2} dz = \frac{l\mu_0 I_m \sin \omega t}{4\pi r}$$

$$= \frac{il\mu_0}{4\pi r} \tag{8-58}$$

In the case where the distance r is not large compared with l, the problem becomes more complicated, and a technique somewhat parallel to that used in the solution of Example 2-3.3 would be necessary.

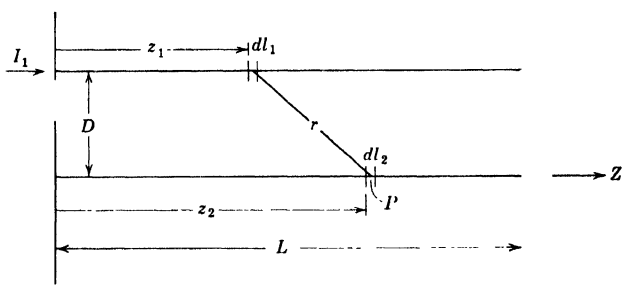


FIG. 8-18. A pair of parallel filamentary currents.

Example 8-8.5. Repeat Example 6-6.5 for the mutual inductance between two parallel wires of length L, by using the magnetic vector potential.

Solution. Refer to Fig. 8-18 for a sketch of the system. The vector potential at point P due to the current element $I_1\, dl_1$ is given by

$$A_z = \frac{\mu_0 I_1}{4\pi} \int_0^L \frac{dl_1}{r}$$

Note from the figure that

$$dl_1 = dz_1$$

and

$$r = [D^2 + (z_2 - z_1)^2]^{1/2}$$

Then, the expression for A_z becomes

$$A_z = \frac{I_1\mu_0}{4\pi}\left\{\int_0^{z_2}\frac{dz_1}{[D^2+(z_2-z_1)^2]^{1/2}}+\int_{z_2}^{L}\frac{dz_1}{[D^2+(z_1-z_2)^2]^{1/2}}\right\}$$

which integrates to

$$A_z = \frac{\mu_0 I_1}{4\pi}\{2\ln(z_2+\sqrt{D^2+z_2^2}) + 2\ln[(L-z_2)+\sqrt{D^2+(L-z_2)^2}] - 2\ln D\}$$

The electric field strength at points P is given by the expression

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$$

from which

$$E_z = -\frac{\mu_0}{2\pi}\frac{dI_1}{dt}\{\ln(z_2+\sqrt{D^2+z_2^2}) + \ln[(L-z_2)+\sqrt{D^2+(L-z_2)^2}] - \ln D\}$$

Hence, the total emf induced in wire 2 is

$$V_2 = \int_0^L E_z\,dz = -\frac{\mu_0}{4\pi}\frac{dI_1}{dt}\{\ln(z_2+\sqrt{D^2+z_2^2}) + \ln[(L-z_2)+\sqrt{D^2+(L-z_2)^2}] - \ln D\}$$

or

$$V_2 = \frac{\mu_0 L}{2\pi}\frac{dI_1}{dt}\left\{\ln\left[\frac{L}{D}+\sqrt{1+\left(\frac{L}{D}\right)^2}\right]-\sqrt{1+\left(\frac{D}{L}\right)^2}+\frac{L}{D}\right\}$$

But, by definition leading to Eq. (8-45),

$$V_2 = M\frac{dI_1}{dt}$$

Then, the expression for mutual inductance is

$$M = \frac{\mu_0 L}{2\pi}\left\{\ln\left[\frac{L}{D}+\sqrt{1+\left(\frac{L}{D}\right)^2}\right]-\sqrt{1+\left(\frac{D}{L}\right)^2}+\frac{D}{L}\right\} \qquad (8\text{-}59)$$

This expression is seen to be in complete agreement with Eq. (6-99) previously deduced.

8-9. Field Theory and Circuit Theory. It is now recalled that the common circuit parameters were introduced from field considerations. Resistance was introduced in Chap. 1 in connection with current-flow fields, capacitance was introduced in Chap. 3 in the discussion of electric fields, and inductance was introduced in Chap. 6 in the discussion of magnetic fields. It is now desired to review these matters and to examine some of the subtleties that exist. To do so requires first that we deduce a general expression for the electric field in a region of space which is small compared with the wavelength of the excitation that produces the fields.

With this objective in mind, consider Eq. (8-23), which expresses Faraday's law in differential form,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This is combined with Eq. (8-28), which relates the magnetic field vector and the magnetic vector potential,

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Thus,
$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\nabla \times \left(\frac{\partial \mathbf{A}}{\partial t}\right)$$

where it is noted that the space and time derivatives have been interchanged. This expression is now rewritten in the form

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{8-60}$$

Now, as a general mathematical result, the curl of the gradient of a scalar is identically zero. That is,

$$\nabla \times (\nabla \phi) \equiv 0 \tag{8-61}$$

This means that Eq. (8-60) may be derived from the gradient of a scalar, or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = -\nabla \times (\nabla \phi) = 0$$

It follows from this, therefore, that

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{8-62}$$

Clearly, this form is reasonable, as it reduces to Eq. (2-55) in the static case. In this expression, the terms are

$-\nabla \phi = \mathbf{E}_c$ due to the static charge distribution

$-\dfrac{\partial \mathbf{A}}{\partial t} = \mathbf{E}_m$ due to the time-changing magnetic field

Suppose, now, that we examine a conductor which is part of a circuit. According to Ohm's law, Eq. (1-35),

$$\mathbf{E}_t = \frac{\mathbf{J}}{\gamma} \tag{8-63}$$

where $\mathbf{E}_t$ is the total field at any point of the conductor without regard for the source of the field. In general, $\mathbf{E}_t$ will be made up of two parts, the field specified by Eq. (8-62) plus any additional field due to charges or currents not included in $\mathbf{E}$. Such additional fields may exist and

may be due to primary sources, arising from chemical, thermal, photoelectric, or piezoelectric action. These are impressed fields, and are denoted by the symbol $\mathbf{E}_i$ (this effect was introduced in Sec. 1-11). Thus, in the general case, the electric field is given by the expression

$$\mathbf{E}_t = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{E}_i$$

or

$$\mathbf{E}_i = \frac{\mathbf{J}}{\gamma} + \text{grad } \phi + \frac{\partial \mathbf{A}}{\partial t} \tag{8-64}$$

The line integral of this field relation is taken around the stationary circuit.

$$\oint \mathbf{E}_i \cdot d\mathbf{l} = \oint \frac{\mathbf{J}}{\gamma} \cdot d\mathbf{l} + \oint \nabla\Phi \cdot d\mathbf{l} + \oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} \tag{8-65}$$

The several terms of this expression are to be interpreted separately.

Consider the term $\oint (\mathbf{J}/\gamma) \cdot d\mathbf{l}$. As shown in Sec. 1-7, this expression yields the total resistive drop around the circuit.

The term $\oint \mathbf{E}_i \cdot d\mathbf{l}$, if it is assumed that $\mathbf{E}_i$ exists only in a limited region of the circuit, may be written as

$$\oint \mathbf{E}_i \cdot d\mathbf{l} = \int_1^2 \mathbf{E}_i \cdot d\mathbf{l} = V_i$$

where V_i denotes the emf applied by this generator between terminals 1 and 2 defining the region over which $\mathbf{E}_i$ acts.

The term $\oint \nabla\phi \cdot d\mathbf{l} = 0$, since the line integral around a closed path due to a conservative field vanishes.

The last term in Eq. (8-65) may be written, in accordance with the results of Example 8-8.2, as

$$\oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} = L \frac{di}{dt}$$

Thus, for the case where the closed circuit consists of a generator, a resistor, and an inductor, the general field relations expressed by Eq. (8-65) reduce to the circuit equation

$$V_i = Ri + L \frac{di}{dt} \tag{8-66}$$

Suppose, now, that the circuit is considered resistanceless, but a capacitor is included in the circuit. The impressed electric field $\mathbf{E}_i$ is again expressible as the sum of a number of contributing components, of the same form as Eq. (8-64). Thus,

$$\mathbf{E}_i = \text{grad } \phi + \frac{\partial \mathbf{A}}{\partial t} \tag{8-67}$$

The line integral of this field relation around the stationary circuit that is closed except for the gap across the capacitor is to be evaluated.

$$\oint \mathbf{E}_i \cdot d\mathbf{l} = \oint \text{grad } \phi \cdot d\mathbf{l} + \oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} \tag{8-68}$$

The first and third terms are interpreted precisely as above. The term $\oint \nabla\phi \cdot d\mathbf{l}$ around the complete path gives zero. However, this should be split into the path around the closed circuit except for the gap across the capacitor, plus the path across the capacitor. But $\int \nabla\phi \cdot d\mathbf{l}$ across the capacitor is $-V_C$, so that $\int \nabla\phi \cdot d\mathbf{l}$ around the balance of the circuit is $+V_C$. Thus Eq. (8-68) yields the expression

$$V_i = V_C + L\frac{di}{dt} \tag{8-69}$$

PROBLEMS

8-1. A large electromagnet which is made of laminated silicon steel core is operated at 60 cps. The volume of steel is 20 ft^3. The magnetic flux density is 1.3 webers/m^2. The laminations are of 29-gauge sheet steel (0.014 in.), and the electrical resistivity of this steel is 60×10^{-8} ohm-m.

a. What is the average power loss due to eddy currents in the core, assuming that the correction factor in Eq. (8-16) is 1.5?

b. If the density of this silicon steel is 468 lb/ft^3, what is the eddy-current power loss per pound for this electromagnet?

8-2. A certain transformer core is made of sheet-steel laminations 0.020 in. thick. It is found that the eddy-current power loss is 380 watts when the transformer is operated on a 60-cps line with a maximum flux density of 0.93 weber/m^2. Calculate the eddy-current loss of a transformer, of the same design and volume of core material, which is made of laminations 0.014 in. thick and which is operated from the 60-cps line with a maximum flux density of 1.3 webers/m^2.

8-3. A magnetic field that varies harmonically with time at a frequency of 10^6 cps induces a peak emf of 5 volts in a single-turn square loop having an area of 8 m^2. The plane of the loop is at an angle of 45° with respect to the direction of the field. Find the peak value of B_m.

8-4. A rigid, straight wire of length L rotates about an axis that is perpendicular to it and fixed at one end, and in a plane normal to a uniform magnetic field of magnitude B. Contact is made to the wire at the axle and also at the moving end by a rail in the form of a circular arc. When the angular speed is ω, what is the induced emf in the wire?

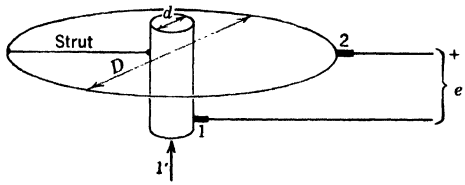
8-5. A copper disk of 15 cm diameter is located at the center of a long solenoid of length 1 m and having a diameter of 16 cm. The solenoid is wound with 1,200 turns. The disk is arranged to rotate about an axis coincident with the axis of the solenoid at 1,800 rpm. Brush takeoffs are provided at the center and at the edge of the disk. If the current in the solenoid is 2 amp, what is the emf between the brushes?

8-6. A rigid, straight wire of length L rotates about, but is insulated from, a straight, vertical wire carrying a current I. It rotates in a plane perpendicular

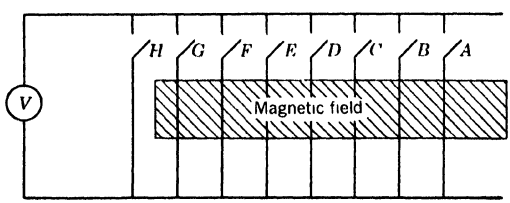
to the current-carrying conductor. Contact is made to the moving wire at the axle and also at the moving end by rails in the form of circular arcs. When the angular speed is ω, what is the induced emf in the wire?

8-7. *a.* A cylindrical bar magnet is rigidly attached to a copper ring by a single copper strut. Brushes are provided for connecting electrically to the magnet and to the copper ring, as illustrated. If the assembly rotates about the axis of the magnet with constant angular velocity ω, what is the value of the emf between the brushes?

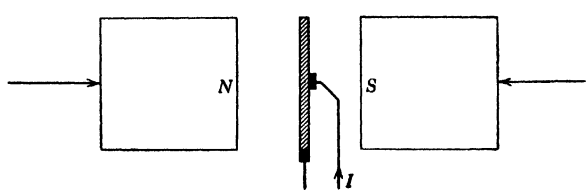
b. If the brush 1 is moved to position 1′ at the center of the magnet, what is the emf now?



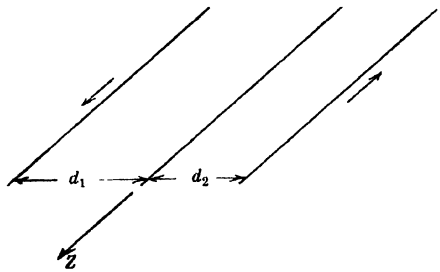
8-8. Refer to the accompanying diagram, which shows a circuit consisting of a number of leads with switches. The switch system is assumed to be such that as one switch closes the prior switch opens. That is, assume that A is closed initially. When B is closed an instant later, A then opens. This process continues, thereby causing the area of the resulting loop to change with time. Suppose that a magnetic field is now introduced which produces magnetic flux perpendicular to the plane of the paper in the position indicated. For the spatially limited field distribution specified, as the switch system operates, the amount of flux linking the circuit decreases, with the total flux linkages falling to zero when the final switch H closes. Is there any reading of the voltmeter during the operation of the switch system? Explain. (L. V. Bewley, "Flux Linkages and Electromagnetic Induction," p. 14, The Macmillan Company, New York, 1952.)



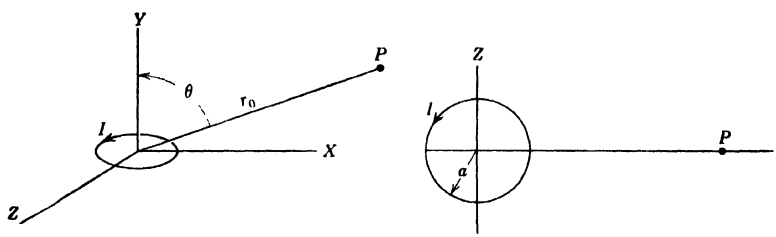
8-9. It is asked whether the Faraday disk will operate as a motor under the following conditions: With the magnet system mounted on bearings and free to rotate, if I is applied to the disk, will the magnets rotate?



8-10. Find the potential induced per meter of length in the center wire of a three-wire parallel group, as shown in the figure. The currents in the outer wires are sinusoidal and of the form $i = I_m \sin \omega t$, but in opposite directions. The center conductor carries no current. The wires may be considered to be infinitely long.



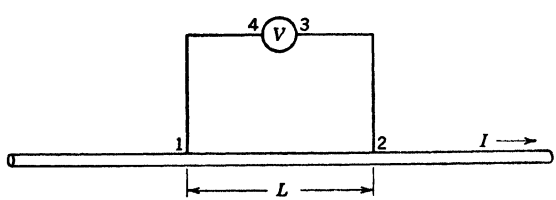
8-11. Consider a circular loop of radius a that carries a steady current I. Find an expression for **B** at point P, at a point above the X axis when $r_0 \gg a$.



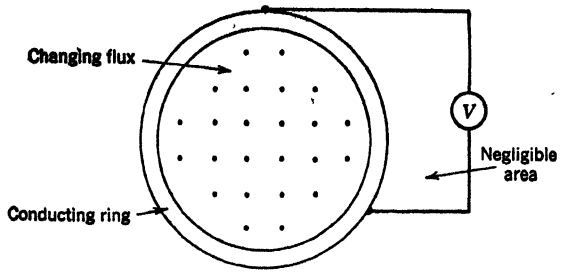
8-12. Carry out the details comparable with those which lead to Eqs. (8-66) and (8-69) for the case where a circuit contains a dielectric and a conductor.

8-13. *a.* A long, straight wire carries an a-c current. To measure the current, it is planned to connect a sensitive a-c voltmeter across a length L of the wire as shown. The voltmeter leads have not been twisted, so that the leads and voltmeter make up a circuit of reasonable area. What is the reading of the voltmeter under the conditions shown?

b. Assume that the resistance of the wire, which is 2 mm in diameter, does not change with frequency and that the current therein is sinusoidal of angular frequency ω. Deduce an explicit expression for the ratio of induced emf to IR drop. If the voltmeter loop area is 1 by 1 m, with a frequency of 1 Mc, what is the value of this ratio?



8-14. A conducting ring is linked by a changing magnetic flux Φ, which is confined within the ring as shown. Will there be a reading of the voltmeter, and, if so, what will the reading be? The contacts subtend 120° at the center.



8-15. A circular disk of radius a and thickness d is placed in a time-changing magnetic field which is perpendicular to and of the same magnitude at all points over the plane of the disk. The conductivity of the material of the disk is γ. Find an expression for the induced field intensity and the current density at every point in the disk. What is the total current?

CHAPTER 9

ENERGY AND MECHANICAL FORCES IN THE MAGNETIC FIELD

It was noted in the introduction to Chap. 6 that the magnetic field was to be introduced through considerations of the force on a charge in motion in a moving electric field but that a detailed discussion of such forces was to be reserved for a later chapter. It is desired to carry out this examination in this chapter. It is also the object of this chapter to evaluate the magnetic energy in a general system of electric currents in free space and in the presence of magnetic materials. General relations are deduced which express the force in terms of changes in the energy in the field.

9-1. The Force on a Current Element in a Magnetic Field. It is possible to describe the mechanism of the force on a conductor carrying a current in a magnetic field. Such a description is to be found by considering the force exerted on the mobile, negatively charged carriers which constitute the current. Refer to Fig. 9-1, which shows a current-carrying conductor which is in the region of a magnetic field. The mobile carriers of the current in the conductor, which are normally uniformly distributed across the area of the conductor, are subject to the force given by Eq. (6-6). As a result, they will move in the direction of this force, which is perpendicular to the plane of $\mathbf{u}$ and $\mathbf{B}$. This causes an accumulation of charge on the surface of the conductor, which continues until the force per unit charge due to the electric field $\mathbf{E}$ thereby established is precisely equal and opposite to the force specified by Eq. (6-6) at all points inside the conductor. Thus, when a steady state is reached inside of the conductor, the electric field that has been established must be

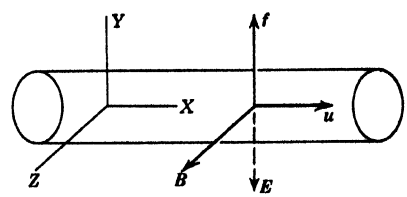


FIG. 9-1. A current-carrying conductor in a magnetic field.

$$\mathbf{E} = \mathbf{u} \times \mathbf{B} \qquad \text{volts/m} \tag{9-1}$$

Consider the stationary charges (atoms) which make up the conductor (Sec. 5-1). These positively charged atoms are in the electric field $\mathbf{E}$.

It is the aggregate force exerted by this field on the stationary charges which constitutes the force now under consideration. In order to calculate this force, it is assumed that the volume density of *positively charged* atoms of the conducting material is equal to that of the negatively charged carriers. (If, for example, each atom provided two free electrons to the current, then each atom would have two positive charges, and the resultant force would be twice that considered, but the force per unit charge would remain unchanged.) Let $d\mathbf{f}$ be the force on the volume element $d\tau$ of length dl and area dS of the conducting material. Then,

$$d\mathbf{f} = \rho \, d\tau \, \mathbf{E}$$

This expression is combined with Eq. (9-1), to give

$$d\mathbf{f} = \rho \, d\tau \, \mathbf{u} \times \mathbf{B} \tag{9-2}$$

But the current-density vector at every point in the conductor, from Eq. (5-7), is

$$\mathbf{J} = \rho \mathbf{u} \qquad \frac{Cou}{m^3} \cdot \frac{m}{s} = \frac{Cou}{m^2} = J$$

Therefore, the force per unit volume in a conductor is

$$\frac{d\mathbf{f}}{d\tau} = \mathbf{J} \times \mathbf{B} \tag{9-3}$$

and the total force on a current-carrying conductor is the vector sum

$$\mathbf{f} = \int_\tau \mathbf{J} \times \mathbf{B} \, d\tau \tag{9-4}$$

where $d\tau$ is the differential element of volume and the integral is to be taken over the volume of the conductor, τ.

For conductors carrying uniform current densities and having a uniform cross-sectional area A, Eq. (9-3) may be written in the form

$$d\mathbf{f} = \mathbf{J}(A \, dl) \times \mathbf{B} \tag{9-5}$$

Now, writing I for JA and including the vector direction of J in dl, which is written $d\mathbf{l}$, the expression for the force on the element dl of a conductor of uniform cross section and uniform current density is

$$d\mathbf{f} = \mathbf{I} \, d\mathbf{l} \times \mathbf{B} \tag{9-6}$$

This expression is often referred to as the *Ampère force law.* The total force on a conductor of uniform cross section and uniform current density is

$$\mathbf{f} = \oint \mathbf{I} \, d\mathbf{l} \times \mathbf{B} \tag{9-7}$$

In this expression, the line integral is evaluated around the entire closed

path of the current. For a section of straight conductor of length L in a uniform magnetic field, the force is

$$\mathbf{f} = I\mathbf{L} \times \mathbf{B} \qquad \text{newtons} \tag{9-8}$$

The direction specified by this vector cross product is often referred to as the *Fleming left-hand rule.* According to this rule, the thumb and the first two fingers are arranged to be mutually perpendicular. If the second finger is directed along the conductor L and the index finger is directed along B, then the thumb points in the direction of f. Actually, it appears easier to determine the direction of the force from the known vector relationships.

Example 9-1.1. Find an expression for the force between two long, parallel, straight lines of current.

Solution. Suppose that the two currents are designated as I_1 and I_2, as in Fig. 9-2. One current lies in the magnetic field of the other. Hence, the force

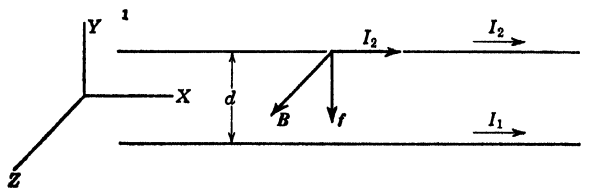


FIG. 9-2. Two parallel, straight current filaments.

per unit length of either conductor is obtained by evaluating the **B** vector due to one current at the second conductor and applying Eq. (9-8). For example, the **B** vector due to I_1 at the conductor I_2 a distance d away, from Eq. (6-12), is

$$\mathbf{B} = \frac{\mu_0 I_1}{2\pi d}(\mathbf{i} \times \mathbf{j}) = \frac{\mu_0 I_1}{2\pi d}\mathbf{k} \tag{9-9}$$

and is directed perpendicular to I_2. The force per unit length on I_2 is

$$\frac{f}{l} = (I_2\mathbf{i}) \times \left(\frac{\mu_0 I_1}{2\pi d}\mathbf{k}\right) = \frac{\mu_0 I_1 I_2}{2\pi d}(-\mathbf{j}) \tag{9-10}$$

The direction of the force is one of attraction between the wires if the currents are in the same direction in the wires. It is a force of repulsion if the currents are in opposite directions in the wires. Note that interchanging the order of evaluating B from I_2 and finding the force per unit length on I_1 leads to the same result.

Example 9-1.2. Find an expression for the approximate force between two closely spaced, coaxial, circular lines of current.

Solution. Refer to Fig. 9-3 for a sketch of the system. It will be assumed that the two circular lines of current are close together, so that the **B** vector at any point of one circular line of current due to the other is approximately that which would exist at a point near a linear current carrying the same current.

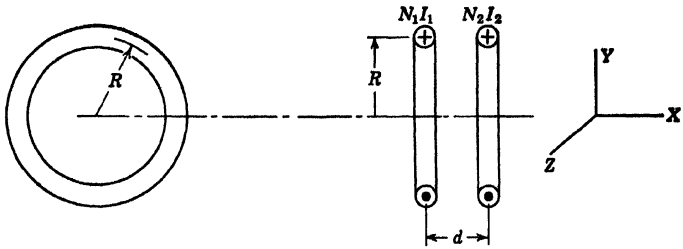


FIG. 9-3. Two closely spaced, coaxial, circular lines of current.

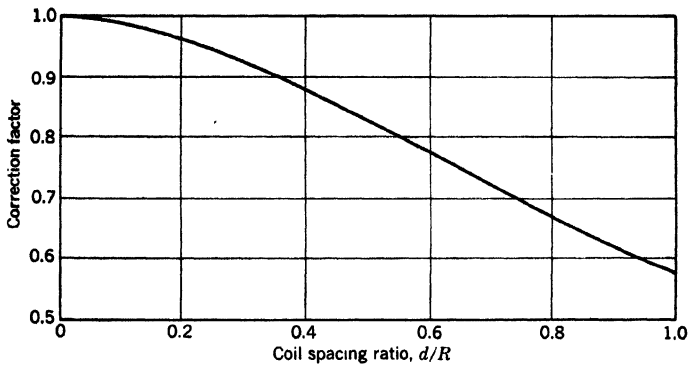


FIG. 9-4. The force-function correction factor for two closely spaced, coaxial, circular lines of current of equal radii.

This approximation avoids the introduction of elliptic integrals, which are required in the general case.

Owing to the approximation that has been made, the **B** vector at the coil designated with the number 2 due to the current in the coil marked as 1 is that given in Eq. (9-9), namely,

$$\mathbf{B} = \frac{\mu_0 N_1 I_1}{2\pi d}(-\mathbf{a}_r) \tag{9-11}$$

and is directed perpendicular to I_2. The force per unit length on I_2 is

$$\frac{f}{l} = \frac{\mu_0 N_1 I_1}{2\pi d} N_2 I_2(-\mathrm{i}) \tag{9-12}$$

whence the total force between the two coils is

$$\mathbf{f} = \frac{R\mu_0 N_1 N_2 I_1 I_2}{d}(-\mathrm{i}) \tag{9-13}$$

The direction of the force is one of attraction if the currents are in the same direction in the wires. The force is one of repulsion if the currents are in opposite directions in the coils.

Owing to the approximation that has been made, the results are not exact, the error increasing with increased spacing between the coils. Figure 9-4 gives the

correction factor by which Eq. (9-13) must be multiplied for increased spacing. This result stems from the work of Maxwell, Lord Rayleigh, and others.

Example 9-1.3. The bus bars for distributing large currents are usually rectangular in shape rather than circular, and the dimensions of these bus bars may be comparable with their spacing. Consider Fig. 9-5, which shows two thin, long, parallel bus bars carrying current in opposite directions. Note that these currents might be different in the general case. It is required to find the force between these two bus bars per unit length.

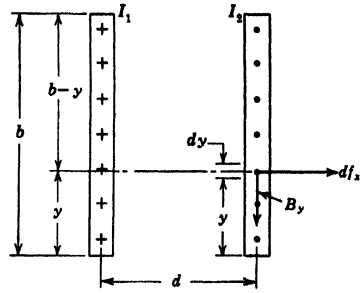


FIG. 9-5. Two thin, long, parallel bus bars.

Solution. Reference is made to Example 6-2.7, which examines the magnetic field at a point adjacent to a bus bar such as that given on the left of Fig. 9-5. Two components of the **B** vector exist, B_x and B_y, which are given, respectively, by Eqs. (6-33) and (6-34). Note, however, that the interaction of I_2 with B_x according to Eq. (9-8) tends to stretch the bar vertically but does not contribute to the force between them. There is a force due to the B_y component of the field. This field component is given by Eq. (6-34), with the appropriate change of symbols. Thus, at the element of current in the right-hand bus,

$$B_y = \frac{\mu_0 I_1}{2\pi b}\left(\tan^{-1}\frac{b-y}{d} + \tan^{-1}\frac{y}{d}\right) \tag{9-14}$$

The differential force per unit length due to the differential current $(I_2/b)\,dy$ is

$$df_x = \frac{\mu_0 I_1 I_2}{2\pi b^2}\left(\tan^{-1}\frac{b-y}{d} + \tan^{-1}\frac{y}{d}\right)dy \tag{9-15}$$

The total force is then given by the expression

$$f_x = \frac{\mu_0 I_1 I_2}{2\pi b^2}\int_0^b \left(\tan^{-1}\frac{b-y}{d} + \tan^{-1}\frac{y}{d}\right)dy$$

This integrates to the form

$$f_x = \frac{\mu_0 I_1 I_2}{2\pi b^2}\left(2b\tan^{-1}\frac{b}{d} - d\ln\frac{d^2+b^2}{d^2}\right) \tag{9-16}$$

Note that in this example the width of the bus has been considered small compared with the spacing. The integrations are somewhat more involved if this condition is not met.

Example 9-1.4. A switch is inserted in a line in the manner illustrated in Fig. 9-6. The current in this system is such that the force always tends to open the switch. It is required to find an expression for the force.

Solution. To evaluate the force, use is made of Eq. (6-11) for the **B** vector at a point opposite the end of a long wire. The appropriate expression is the following for the lower current:

$$B = \frac{\mu_0 I}{4\pi y} \tag{9-17}$$

The corresponding expression for the upper current is

$$B = \frac{\mu_0 I}{4\pi(d - y)} \tag{9-18}$$

Hence, the total **B** vector is given by the expression

$$B = \frac{\mu_0 I}{4\pi}\left(\frac{1}{y} + \frac{1}{d - y}\right) \tag{9-19}$$

An expression for the force on the distance dy, which by an application of the direction rule is found to be in the direction shown in Fig. 9-6, is given by the expression

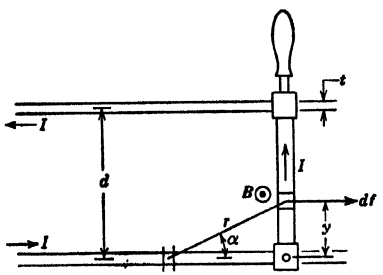


FIG. 9-6. To determine the force in a switch.

$$df = \frac{\mu_0 I^2}{4\pi}\left(\frac{1}{y} + \frac{1}{d - y}\right) dy \tag{9-20}$$

The total force is then obtained by integrating this expression. The result is

$$f = \int_{t/2}^{d-t/2} \frac{\mu_0 I^2}{4\pi}\left(\frac{1}{y} + \frac{1}{d - y}\right) dy \tag{9-21}$$

which integrates to the form

$$f = \frac{\mu_0}{2\pi} I^2 \ln \frac{2d - t}{t} \tag{9-22}$$

where t is the width of the switch contacts, as illustrated.

9-2. Potential Energy in a System of Rigid Currents. First, consider a single circuit carrying a constant current I placed in a magnetic field of fixed external sources. Owing to the mechanical force of magnetic origin that acts on the system, it will be supposed that the circuit is displaced, rotated, or deformed by an infinitesimal amount. This virtual displacement is denoted by $\delta\mathbf{r}$. The work done by the circuit against the force in the virtual displacement must be supplied by the current source and is given by the expression

$$d(\delta W) = \delta\mathbf{r} \cdot d\mathbf{f} \tag{9-23}$$

But, by Eq. (9-6),

$$d\mathbf{f} = I\, d\mathbf{l} \times \mathbf{B} \tag{9-24}$$

Thus,

$$d(\delta W) = I\, \delta\mathbf{r} \cdot d\mathbf{l} \times \mathbf{B} \tag{9-25}$$

The combination of the dot and cross products of the three vectors in Eq. (9-25) is called the *scalar triple product.* It is a property of the scalar

triple product, as may be shown by carrying out the specified vector operations, that this quantity is unchanged by the cyclic interchange of the vectors. Therefore, Eq. (9-25) may be written as

$$d(\delta W) = I\mathbf{B} \cdot \delta\mathbf{r} \times d\mathbf{l}$$

Recall from Sec. 3-1 that the cross product of two vectors specifies the area (one vector times the normal component of the second vector), which is in the direction of a unit normal perpendicular to the area. Thus, $\mathbf{B} \cdot (\delta\mathbf{r} \times d\mathbf{l})$ is an element of magnetic flux linkages swept out by $d\mathbf{l}$ in its motion. For the entire circuit, the increment of flux linkages due to the displacement is

$$\oint \mathbf{B} \cdot \delta\mathbf{r} \times d\mathbf{l} = \delta\lambda$$

Therefore, from Eq. (9-25), the total work done by the system during the displacement is

$$\delta W = I \oint \mathbf{B} \cdot \delta\mathbf{r} \times d\mathbf{l} \tag{9-26}$$

which may then be written in the simple, though very important, form

$$\delta W = I\,\delta\lambda \tag{9-27}$$

If it is now assumed that the external sources and the current I are maintained constant in the virtual displacement, then the work done by the mechanical forces is compensated by a decrease in a potential energy function U. Thus

$$\delta U = -\delta W = -I\delta\lambda \tag{9-28}$$

This implies that the potential energy of the rigid current in the magnetic field is

$$U = -I\lambda \tag{9-29}$$

If $d\mathbf{r}$ is a real rather than a virtual displacement, then work must be done to keep the current constant. Now, because of the change in flux linkages, an emf will be induced

$$V = -\frac{d\lambda}{dt}$$

where dt is the time required to effect the displacement. But this induced emf must be counterbalanced by an equal and opposite applied emf V_i. The work done by the applied voltage on the circuit during this time interval is

$$V_i I\,dt = +I\,d\lambda \tag{9-30}$$

This expression may be interpreted to show that the work done by the mechanical forces in a small displacement of a linear circuit is exactly

compensated by the energy by the source to maintain the current constant. The total work on the circuit is zero.

Let us now evaluate the energy that is being supplied during the process of building up the currents in the system of n rigid currents. This may be done by considering the energy that is supplied to all currents I_k as they are increased from zero to their final values, or, equivalently, as the flux-linking current I_k is increased from zero to its final value Λ_k. It will be supposed that the wires are of infinitesimal thickness so that Λ is exactly defined for each circuit.

To find the desired expression, it is supposed that an external emf V_{ki} which is generated by chemical or mechanical means is connected to the typical circuit k which has a resistance R_k associated with it. If at any instant the current in the circuit is I_k, and if magnetic flux threading the circuit is λ_k, then variations in currents are accompanied by variations in the flux linkages. If V_k is the emf induced by a variation in λ_k, then

$$V_k + V_{ki} = R_k I_k \tag{9-31}$$

or

$$V_{ki} = R_k I_k + \frac{d\lambda_k}{dt} \tag{9-32}$$

The power expended by the impressed source V_{ki} is $V_{ki} I_k$, and the work done on the kth circuit in the interval dt is

$$dW_k = R_k I_k^2 \, dt + I_k \, d\lambda_k \tag{9-33}$$

Of this work the amount $R_k I_k^2 \, dt$ is dissipated as heat, while $I_k \, d\lambda_k$ is stored as magnetic energy. Thus a variation in the magnetic energy of the n current filaments is related to the increments in fluxes by

$$W = \sum_{k=1}^{n} \int_0^{\Lambda_k} I_k \, d\lambda_k \tag{9-34}$$

For a linear system, this expression may be conveniently expressed in terms of the self- and mutual inductances of the circuits. To find this relation, it is first recalled from Eq. (6-64) that the flux linking the kth circuit is a linear function of currents in all the circuits, namely,

$$\lambda_k = \sum_{s=1}^{n} M_{ks} I_s \tag{9-35}$$

This may be written as

$$\lambda_k = L_k I_k + \sum_{s=1}^{n}{}' M_{ks} I_s \qquad \text{with } s \neq k \tag{9-36}$$

where, as noted, the summation denoted by Σ' does not include the term

for which $s = k$, as this has been separately shown. It follows from Eq. (9-36) that

$$\delta\lambda_k = L_k\,\delta I_k + \sum_{s=1}^{n}{}' M_{ks}\,\delta I_s$$

Thus, for the general system, Eq. (9-34) becomes

$$\delta W = \sum_{k=1}^{n} I_k \left(L_k\,\delta I_k + \sum_{s=1}^{n}{}' M_{ks}\,\delta I_s\right) \tag{9-37}$$

To make plausible the manner in which this expression may be reduced, consider the specific case of two current circuits. Equation (9-37) in this case becomes

$$\delta W = L_1 I_1\,\delta I_1 + M_{12} I_1\,\delta I_2 + L_2 I_2\,\delta I_2 + M_{21} I_2\,\delta I_1$$

which may be written in the form

$$\delta W = \delta(\tfrac{1}{2}L_1 I_1^2) + \delta(\tfrac{1}{2}L_2 I_2^2) + \delta(M_{12} I_1 I_2)$$

This shows that we may write, for the total magnetic energy stored in the system of two rigid currents,

$$W_m = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2 \tag{9-38}$$

For the general case, Eq. (9-38), the result becomes

$$W_m = \frac{1}{2}\sum_{k=1}^{n} L_k I_k^2 + \frac{1}{2}\sum_{k=1}^{n}\sum_{s=1}^{n}{}' M_{ks} I_k I_s$$

The factor $\frac{1}{2}$ in the second term arises to compensate for the fact that $M_{ks} = M_{sk}$ will appear twice in the double summation. This becomes

$$W_m = \frac{1}{2}\sum_{k=1}^{n}\left(L_k I_k^2 + \sum_{s=1}^{n}{}' M_{ks} I_k I_s\right) \tag{9-39}$$

or

$$W_m = \frac{1}{2}\sum_{k=1}^{n} I_k\left(L_k I_k + \sum_{s=1}^{n}{}' M_{ks} I_s\right)$$

By Eq. (9-36), this becomes finally

$$W_m = \frac{1}{2}\sum_{k=1}^{n} I_k \lambda_k \tag{9-40}$$

But this is just the result specified by Eq. (9-34). Hence, finally, for a linear system,

$$W_m = \sum_{k=1}^{n} \int_0^{\Lambda_k} i_k \, d\lambda_k = \frac{1}{2} \sum_{k=1}^{n} I_k \lambda_k \tag{9-41}$$

This expression specifies the total energy supplied during the whole process of building up the currents in the system. It is, therefore, the total amount of energy that is stored in the magnetic field of the system of currents. Note that in the development it is implicitly assumed that the system is magnetically isolated from its surroundings, since externally produced magnetic fields would produce fluxes even when all the currents of the system are zero.

It is possible to extend Eq. (9-34) from that for a finite number of current filaments to a continuous distribution of current. Now instead of beginning with Eq. (9-31), we start by writing the field equivalent form

$$\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\gamma} \tag{9-42}$$

where $\mathbf{E}_i$ is the nonelectrostatic field produced by the battery. This expression assumes that the battery feeds energy both into heat losses and also into a magnetic field. Now take the scalar product of Eq. (9-42) with $\mathbf{J}$. Then

$$\mathbf{E}_i \cdot \mathbf{J} = \frac{J^2}{\gamma} - \mathbf{E} \cdot \mathbf{J} \tag{9-43}$$

This expression has the same interpretation as Eq. (9-33). It specifies that the time rate at which the battery does work is just that to supply the heat loss in the current-carrying medium (J^2/γ), and the rate at which energy is fed into the fields produced by the currents ($\mathbf{E} \cdot \mathbf{J}$).

It is now assumed that the fields are slowly varying (this is called a quasi-static field) so that displacement terms need not be taken into account. By writing

$$\mathbf{J} = \nabla \times \mathbf{H}$$

in the right-hand term of Eq. (9-43) only, the resulting expression being integrated over all of space, then

$$\int_\tau \mathbf{E}_i \cdot \mathbf{J} \, d\tau = \int_\tau \frac{J^2}{\gamma} \, d\tau - \int_\tau \mathbf{E} \cdot (\nabla \times \mathbf{H}) \, d\tau \tag{9-44}$$

But by a vector-analysis expansion

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$$

The last term in the expression thus becomes

$$\int_\tau \mathbf{E} \cdot (\nabla \times \mathbf{H}) \, d\tau = \int_\tau \mathbf{H} \cdot (\nabla \times \mathbf{E}) \, d\tau - \int_\tau \nabla \cdot (\mathbf{E} \times \mathbf{H}) \, d\tau$$

Combine the first term on the right with Eq. (8-23), and apply Stokes' theorem to the second term on the right. The result is

$$\int_\tau \mathbf{E} \cdot (\nabla \times \mathbf{H})\, d\tau = -\int_\tau \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\, d\tau - \int_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S}$$

If the integration is taken over all of space, the integral on the right for the static or quasi-static fields vanishes as $0(1/r^2)$, since E varies as $1/r^2$ and H varies as $1/r^2$. Hence Eq. (9-44) becomes

$$\int_\tau \mathbf{E}_i \cdot \mathbf{J}\, d\tau = \int_\tau \frac{J^2}{\gamma}\, d\tau + \int_{\text{Space}} \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\, d\tau \tag{9-45}$$

The second term on the right specifies the rate at which energy is fed into the field. We may write from this that the variation in the magnetic field energy is

$$\delta W_m = \int_{\text{Space}} \mathbf{H} \cdot \delta \mathbf{B}\, d\tau \tag{9-46}$$

For the case of a medium that magnetizes linearly, $\mathbf{B} = \mu\mathbf{H}$, and this expression may be integrated as follows:

$$W_m = \int_0^B \delta W_m = \int_0^B \int_{\text{Space}} \mathbf{H} \cdot \delta \mathbf{B}\, d\tau$$

$$= \int_{\text{Space}} \int_0^H \frac{\mu\, \delta \mathbf{H}^2}{2}\, d\tau = \tfrac{1}{2} \int_{\text{Space}} \mu H^2\, d\tau$$

$$W_m = \tfrac{1}{2} \int_{\text{Space}} \mathbf{H} \cdot \mathbf{B}\, d\tau \tag{9-47}$$

This gives the energy density in terms of a volume integral over the fields. It appears from this expression that the total energy in the magnetic field of any system of electric currents may be regarded as distributed throughout the entire field with a density at any point equal to

$$\frac{\Delta W_m}{\Delta \tau} = \frac{\mu H^2}{2} \qquad \text{joules}/m^3 \tag{9-48}$$

Here, as noted in connection with Eq. (4-10) for the cognate electric field problem, this statement involves the assumption that magnetic energy is, in its nature, something that can be spatially distributed. Clearly, of course, while the foregoing expression can suggest such a hypothesis, it in no way demands it.

For the case of nonlinear materials, special considerations are required in evaluating Eq. (9-46). These will be considered in some detail in Sec. 9-8.

It is readily possible to show that Eq. (9-47) reduces to Eq. (9-41) for the energy expressed in terms of a volume integral over the current sources of the field. To do this, write the expression as

$$W_m = \tfrac{1}{2} \int_{\text{Space}} \mathbf{H} \cdot (\nabla \times \mathbf{A})\, d\tau$$

This expression is expanded in the same manner as the last term in Eq. (9-44). The result is

$$W_m = \tfrac{1}{2} \int_{\text{Space}} \mathbf{A} \cdot (\nabla \times \mathbf{H})\, d\tau + \tfrac{1}{2} \int_S (\mathbf{H} \times \mathbf{A}) \cdot d\mathbf{S}$$

But the right-hand term in this expression vanishes as $0(1/r)$, so that finally

$$W_m = \tfrac{1}{2} \int_{\text{Space}} \mathbf{A} \cdot \mathbf{J}\, d\tau \tag{9-49}$$

This expression is similar in form to that for W_e for the static-electric field, given in Eq. (4-3).

Now since the currents are confined to the filaments, we may write $I = J\, dS$, and Eq. (9-49) may be written as

$$W_m = \tfrac{1}{2} \sum_k I_k \int \mathbf{A} \cdot d\mathbf{l} = \tfrac{1}{2} \sum_k I_k \int_S \mathbf{B} \cdot d\mathbf{S}$$

which is then the expression

$$W_m = \tfrac{1}{2} \sum_k I_k \lambda_k \tag{9-50}$$

which is the result desired.

It is noted that Eqs. (9-46), (9-47), and (9-49) depend only on the final fields and not on the nature of the process in establishing the fields (although they were derived by a particular virtual process). They may be taken to represent the general expressions for the energy of the magnetic field.

Example 9-2.1. Derive an expression for the self-inductance per unit length of a coaxial cable of inner radius a and outer radius b, but of negligible thickness. Use is to be made of energy considerations.

Solution. Refer to Example 6-6.1 for the diagram of the system and the analysis in terms of flux linkages. Now note that the total magnetic energy stored in the field may be expressed by the relation

$$W_m = \tfrac{1}{2} L I^2 = \frac{\mu_0}{2} \int_\tau H^2\, d\tau \tag{9-51}$$

where the volume in which the energy is stored comprises two parts, the region inside the inner conductor and the space between the inner and outer conductor. The magnetic field intensity is zero outside the cable, whence no energy is stored outside the cable.

As noted in Eq. (6-76a), the magnetic field intensity in the region within the inner conductor is given by the expression

$$H_1 = \frac{Ir}{2\pi a^2}$$

The magnetic field intensity for the region between the inner and outer conductors has the form

$$H_2 = \frac{I}{2\pi}\frac{1}{r}$$

The general energy relation, Eq. (9-51), now becomes per unit length

$$W_m = \tfrac{1}{2}LI^2 = \frac{\mu_0}{2}\left[\int_0^a \left(\frac{I}{2\pi a^2}r\right)^2 \times 2\pi r\,dr + \int_a^b \left(\frac{I}{2\pi r}\right)^2 \times 2\pi r\,dr\right]$$

Consider, now, the definite integral

$$\int_0^a \left(\frac{Ir}{2\pi a^2}\right)^2 \times 2\pi r\,dr = \frac{I^2r^4}{8\pi a^4}\bigg|_0^a = \frac{I^2}{8\pi}$$

Also,

$$\int_a^b \left(\frac{I}{2\pi r}\right)^2 \times 2\pi r\,dr = \frac{I^2}{2\pi}\ln r\bigg|_a^b = \frac{I^2}{2\pi}\ln\frac{b}{a}$$

Therefore, it follows that

$$L = \frac{2W_m}{I^2} = \frac{\mu_0}{I^2}\left(\frac{I^2}{8\pi} + \frac{I^2}{2\pi}\ln\frac{b}{a}\right)$$

which may be written in the form

$$L = \frac{\mu_0}{2\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right) \qquad \text{henrys/m}$$

This expression is identical with that given by Eq. (6-82), as it must be.

9-3. Forces in Terms of Energy Changes. The energy expressions discussed in the foregoing section may be used to find expressions for the forces between the currents and certain geometrical parameters of the circuit. Consider therefore that the kth circuit is subject to a virtual displacement. The rate at which the electrical force f_k acting on the kth circuit is doing work is $f_k v_k$. But the system energy balance requires that we write

$$f_k v_k + \frac{dW_m}{dt} + \sum_k I_k^2 R_k - \sum_k V_{ki} I_k = 0 \qquad (9\text{-}52)$$

where the terms in this equation have the following meanings:

1—the rate of mechanical work by the electrical forces
2—the rate of change of magnetic energy
3—rate of heat losses
4—rate at which the battery does work.

Consider the special *constant-current* process in which the external emfs are so controlled that the currents within the system remain constant during the virtual displacements. Now we have that

$$f_k v_k + \frac{1}{2}\sum_k I_k \frac{d\lambda_k}{dt} - \sum_k I_k \frac{d\lambda_k}{dt} = 0 \tag{9-53}$$

This expression shows the very interesting fact that in order to maintain a constant current in the circuits as the geometry changes, the batteries must do exactly twice the amount of work that is done by the external sources, in addition to supplying the heat losses. Physically, this means that in this process when the field does work on the kth conductor, the battery must supply not only this work but at the same time supply a like additional amount of energy to the stored energy. This situation has been referred to as the 50-50 rule.

The converse of the foregoing relates to the case when work is done on the circuits by changing their form against the action of electromagnetic forces or by pulling them apart. In this case, the energy returned to the sources (essentially by charging the batteries) is twice the amount of the mechanical work that is done, the second half being restored to the sources by a reduction in the stored magnetic energy. Note that the 50-50 rule is not valid when iron exists in the neighborhood of the currents which may be subject to saturation.

It is now observed that Eq. (9-53) may be written as

$$f_k v_k = \frac{1}{2}\sum I_k \frac{d\lambda_k}{dt}$$

This is

$$f_k v_k = \frac{\partial W_m}{\partial t} = \frac{\partial W_m}{\partial x_k}\bigg|_{\substack{v_k \\ I \text{ constant} \\ x_i \text{ constant},\ i \neq k}}$$

Hence, finally

$$f_k = \frac{\partial W_m}{\partial x_k}\bigg|_{I \text{ constant}} \tag{9-54}$$

Observe also from Eq. (9-52) that the back emf terms can be ignored in the force calculations if the flux linkages are held constant. It follows directly, as above, that

$$f_k = -\frac{\partial W_m}{\partial x_k}\bigg|_{\Lambda \text{ constant}} \tag{9-55}$$

9-4. Forces and Torques between Circuits in Terms of Changes of Mutual Inductance. An application of Eq. (9-54) which is of considerable practical importance involves two circuits which have self-inductances L_1 and L_2, respectively, and the mutual inductance M. If the currents in the circuits are I_1 and I_2, then the energy stored in the magnetic field is that specified in Eq. (9-38). Suppose, now, that circuit 1 is given a virtual displacement δr. If f_1 is the magnetic force on this circuit

in the direction of δr, the work done by the circuit against the force in displacing the circuit is, by Eq. (9-54),

$$f_{1r} = \tfrac{1}{2}I_1^2 \frac{\partial L_1}{\partial r} + \tfrac{1}{2}I_2^2 \frac{\partial L_2}{\partial r} + I_1 I_2 \frac{\partial M}{\partial r} \tag{9-56}$$

This expression gives the component of the total magnetic force in any direction r on circuit 1 due to circuit 2 in terms of the currents and the space rate of change of inductances in that direction.

An entirely similar procedure may be employed to find an expression for the component of torque on circuit 1 due to circuit 2 tending to give the circuit an angular displacement θ. The result is

$$T_{1\theta} = \tfrac{1}{2}I_1^2 \frac{\partial L_1}{\partial \theta} + \tfrac{1}{2}I_2^2 \frac{\partial L_2}{\partial \theta} + I_1 I_2 \frac{\partial M}{\partial \theta} \tag{9-57}$$

In many cases, the self-inductances are constant, whence the foregoing results are dependent only on the changes in the mutual inductances. The resulting expressions become

$$\begin{aligned} f_{1r} &= I_1 I_2 \frac{\partial M}{\partial r} \\ T_{1\theta} &= I_1 I_2 \frac{\partial M}{\partial \theta} \end{aligned} \tag{9-58}$$

From the fact that the flux linkages in the circuits are

$$\begin{aligned} \lambda_1 &= L_1 I_1 + M I_2 \\ \lambda_2 &= L_2 I_2 + M I_1 \end{aligned} \tag{9-59}$$

then, Eqs. (9-58) may be expressed in the form

$$\begin{aligned} f_{1r} &= I_1 \frac{\partial \lambda_1}{\partial r} \\ T_{1\theta} &= I_1 \frac{\partial \lambda_1}{\partial \theta} \end{aligned} \tag{9-60}$$

These latter expressions may be given a broad interpretation. They show that a force or a torque is produced in a circuit of constant self-inductance regardless of whether the flux linkages are produced by currents in two circuits or by a group of other circuits or magnets. This more general situation will receive further consideration in Sec. 9-8 when magnetic media exist in the field.

Example 9-4.1. Calculate the force between coaxial circular coils, as illustrated in Fig. 9-7, for R_2 much less than R_1.

Solution. Unlike the assumptions made in solving Example 9-1.2, it will here be assumed, since R_2 is much less than R_1, that the magnetic flux density set up by the larger coil will be uniform over the area of the smaller coil. This permits

the use of Eq. (6-16) for the magnetic field vector on the axis of a circular coil, namely,

$$B_z = \frac{\mu_0}{2} \frac{N_1 I_1 R_1^2}{(R_1^2 + d^2)^{3/2}} \tag{9-61}$$

Then, by the definition of mutual inductance (see Sec. 6-6), it follows that

$$M = \frac{\mu_0}{2} N_1 N_2 \frac{\pi R_2^2 R_1^2}{(R_1^2 + d^2)^{3/2}} \tag{9-62}$$

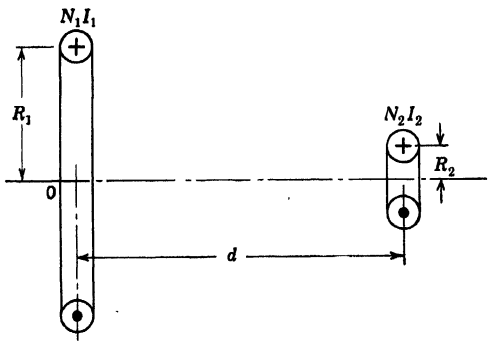


FIG. 9-7. Two widely spaced coils of different radii.

Therefore, according to Eq. (9-58), the force between the coils is given by

$$f_x = I_1 I_2 \frac{\partial M}{\partial x} = I_1 I_2 \frac{\partial M}{\partial d} \tag{9-63}$$

which is

$$f_x = -\tfrac{3}{2}\mu_0 \pi \frac{N_1 N_2 I_1 I_2 R_2^2 R_1^2 d}{(R_1^2 + d^2)^{5/2}} = -\frac{3}{2}\frac{\mu_0}{\pi} \frac{(\pi R_1^2 N_1 I_1)(\pi R_2^2 N_2 I_2) d}{(R_1^2 + d^2)^{5/2}} \tag{9-64}$$

Observe that, if R_1 and R_2 are both very small compared with d, this expression reduces to

$$f_x = -\frac{3}{2}\frac{\mu_0}{\pi} \frac{(\pi R_2^2 N_2 I_2)(\pi R_1^2 N_1 I_1)}{d^4} \tag{9-65}$$

The minus sign that appears in Eqs. (9-64) and (9-65) indicates that the force is one of attraction between the coils.

A comparison of Eq. (9-64) with (9-13) shows quite different force laws between the coils. This difference arises because of the different coil dimensions and the different spacing between the coils.

9-5. Energy Storage in a Region Containing Iron. In the derivation of Eq. (9-40) for the magnetic energy in a system of rigid currents, it was assumed that the system was magnetically isolated. That is, it was assumed that the fluxes were linear functions of the currents, which is not true when iron is present. It is desired to examine the situation

when iron exists in the field. For convenience, the simple case of an ideal toroidal coil wound on an iron core, as illustrated in Fig. 6-19, is examined. The iron cross section is taken to be small.

Suppose initially that the iron core is demagnetized and it is to be magnetized by passing a current through the solenoid winding. For a core material that is isotropic and homogeneous, the magnetizing force **H** at any point inside of the iron is parallel to the axis. When the current in the coil is I, then, for a core of small cross section, the magnetizing force is almost uniform. It is given by Eq. (6-50) appropriately modified, or

$$H = \frac{NI}{2\pi R} \tag{9-66}$$

Correspondingly, the magnetic field vector is almost uniform in the core and is directed, of course, normal to the loops of the winding. The total flux linkage is

$$\lambda = \pi a^2 NB \tag{9-67}$$

where a is the radius of the iron making up the core of the torus. Therefore, the total energy which is transferred from the electric circuit to the magnetic field, which is entirely within the iron, is, from Eq. (9-34),

$$W_m = \int_0^{\lambda} I \, d\lambda \tag{9-68}$$

which becomes, by combining with the foregoing expressions,

$$W_m = (2\pi R)(\pi a^2) \int_0^B H \, dB \tag{9-69}$$

But the volume of the torus is just

$$\text{Volume} = (2\pi R)(\pi a^2)$$

Therefore, the energy transferred is

$$W_m = \int_0^B H \, dB \times \text{volume of iron} \tag{9-70}$$

for the case when the iron starts from a completely demagnetized state.

It is natural to assume in the light of the foregoing that the energy stored in any small volume $\Delta\tau$ of iron is

$$\Delta W_m = \Delta\tau \int_0^B H \, dB \tag{9-71}$$

Attention is called to the fact that, even though it has been deduced for an iron ring, this expression is true in general. Its significance is made evident by an examination of Fig. 9-8. That is, if the iron starts in the demagnetized state and the flux density is carried to the point B_1, then

the shaded area represents the value of the integral contained in Eq. (9-71), which is the energy input to the magnetic field per unit volume. If it is assumed that the relation $B = \mu H$ is valid over the entire range of variation of B, although it is known that μ may vary considerably for ferromagnetic substances which are subjected to wide variations in H, then Eq. (9-71) may be written as

$$\Delta W_m = \Delta\tau \int_0^B \frac{B}{\mu}\, dB$$

which attains the alternate form

$$\frac{\Delta W_m}{\Delta\tau} = \frac{B^2}{2\mu} = \frac{\mu H^2}{2} = \frac{HB}{2} \qquad \text{joules/unit volume} \qquad (9\text{-}72)$$

Observe that this is precisely in the form given by Eq. (9-48) for the energy storage per unit volume of the magnetic field for a system of electric currents. This means, of course, that an expression of the form given in Eq. (9-72) is valid for any composite medium when due account is taken of the different characteristics of the various media. The situation here is precisely like that which was found to exist for a composite dielectric medium, with the appropriate expression for the energy storage in the electric field of such a system being expressed by Eqs. (4-17) and (4-18).

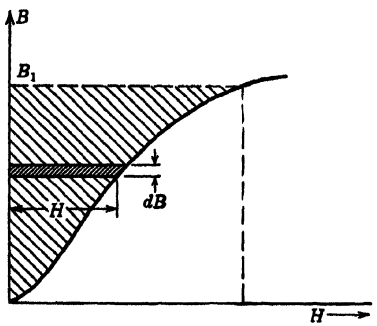


FIG. 9-8. The energy density stored in the magnetic field.

9-6. Hysteresis Loss. Suppose that the foregoing considerations are now applied to the case when the iron is taken completely through a magnetic cycle, as illustrated by the hysteresis loop of Fig. 7-5. Refer to Fig. 9-9 for the prescribed notation. As an extension of Eq. (9-71), the energy input to the magnetic field during the complete cycle is written in the form

$$\begin{aligned}\frac{W}{\text{Volume}} &= \oint H\, dB \\ &= \int_{B_m}^{B_R} H\, dB + \int_{B_R}^{-B_m} H\, dB + \int_{-B_m}^{-B_R} H\, dB + \int_{-B_R}^{B_m} H\, dB \\ &= \int_{(1)} H\, dB + \int_{(2)} H\, dB + \int_{(3)} H\, dB + \int_{(4)} H\, dB \qquad (9\text{-}73)\end{aligned}$$

Each of these integrals is examined separately. Observe that for the

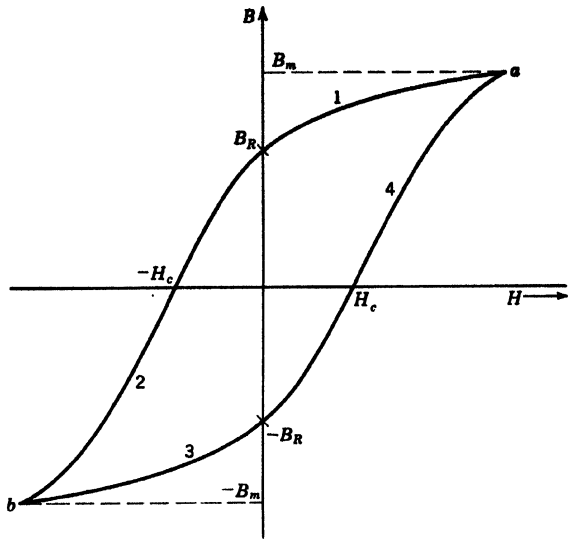


FIG. 9-9. Calculation of the energy loss due to hysteresis.

portion of the cycle from B_m to B_R, which is designated as path (1), dB is negative. Hence,

$$\frac{\Delta W_1}{\Delta \tau} = -\text{area } (aB_mB_R)$$

Also, along path (4) from $-B_R$ to B_m the energy input is

$$\frac{\Delta W_4}{\Delta \tau} = +\text{area } (-B_RaB_m)$$

Adding these expressions shows that the reversal of B from $-B_R$ around the complete cycle to $-B_R$ has been accompanied by a net input of energy

$$\frac{\Delta W_1 + \Delta W_4}{\Delta \tau} = \text{area } (-B_RaB_m) - \text{area } (aB_mB_R) = \text{area } (-B_RaB_R) \tag{9-74}$$

In contrast to this, it is observed from Eq. (9-71) that, if the direction of **B** is reversed in a medium which does not possess hysteresis, the net input energy is zero. The extra energy required by the material that possesses hysteresis is not recoverable and denotes a loss of energy.

Observe that a second reversal to carry the cycle along paths (2) and (3) from B_R to $-B_R$ requires another insertion of energy which is equal to that given in Eq. (9-74). Therefore, it is seen that the total energy input per cycle per unit volume is

$$\frac{\text{Energy input/cycle}}{\text{Volume}} = \text{area of hysteresis loop}$$

or

$$\frac{\Delta W}{\Delta \tau} = \oint H \, dB \tag{9-75}$$

This energy loss is converted into heat and represents a loss that is known as the hysteresis loss. The Steinmetz equation, which relates the energy loss per cycle of the hysteresis loop per unit volume and the maximum value B_m during the cycle, has already been discussed and is given in Eq. (7-19).

9-7. Attraction between Magnetized Iron Surfaces. It is possible to obtain an expression for the force that exists on a piece of magnetic material which is inserted into a magnetic field. This would be accomplished in a manner somewhat like that of Sec. 4-3, where the cognate electric

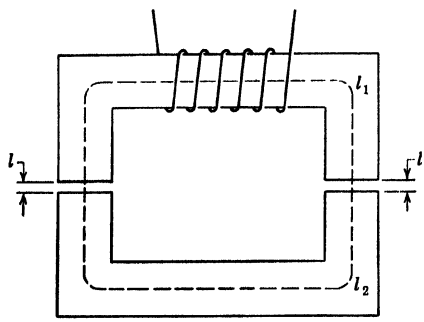


FIG. 9-10. A typical electromagnetic circuit.

field problems are discussed. Actually, however, our concern is not with the general problem; our interest is confined to the force between parallel iron surfaces which are separated by a small air gap. Examples of the distributions here contemplated can be found in the electromagnetic relay, in d-c instruments of the D'Arsonval type, and in the magnetic circuit of a rotating machine. An expression for the force will be obtained by the application of the principle of virtual work.

Consider the magnetic circuit illustrated in Fig. 9-10. The total energy stored in the magnetic field of this magnetic circuit per unit volume is expressed analytically by Eq. (9-72). Therefore, since

$$\Delta W_m = \Delta \tau \, \tfrac{1}{2} BH \qquad \text{joules/unit volume}$$

then the total magnetic energy in the field, which is

$$W_m = \frac{1}{2} \int_\tau BH \, d\tau \tag{9-76}$$

may be written as shown, where the volume element is written as $d\tau = dS\,dl$,

$$W_m = \frac{1}{2}\int_S B\,dS \int_L H\,dl$$

or, finally, $$W_m = \tfrac{1}{2}\Phi F = \frac{1}{2}\frac{F^2}{R} = \tfrac{1}{2}\Phi^2 R \quad \text{joules} \tag{9-77}$$

by Eq. (7-35).

To find the force of attraction between the two sections shown, it is necessary to calculate the change in the total energy that would result were the two sections of the magnetic circuit separated by an additional amount dl. The work that would be required to effect the additional separation, since for constant flux the battery does no work, is found to be equal to the change in energy in the field, and therefore this must be related to the product of the force and the distance. That is, noting that the reluctance of the path after the change is $R + (2\,dl/\mu_0 A)$, and that the reluctance of the path before the change is R, then the resulting change in energy, which is the work done, is

$$dW_m = \tfrac{1}{2}\Phi^2\left[R + \frac{2\,dl}{\mu_0 A} - R\right] = \tfrac{1}{2}\Phi^2\frac{2\,dl}{\mu_0 A}$$

Hence, it follows that

$$f\,dl = \tfrac{1}{2}\Phi^2\frac{2\,dl}{\mu_0 A} = 2\left(\frac{1}{2}\frac{B^2 A}{\mu_0}\right)dl$$

from which the force of attraction is given by the relation

$$f = 2\left(\frac{1}{2}\frac{B^2 A}{\mu_0}\right) \quad \text{newtons} \tag{9-78}$$

Observe that this expression has assumed that the total magnetic flux Φ, or, alternatively, the flux density B, remains constant during the virtual change.

The expression given by Eq. (9-78) may be generalized to show that the local value of the tractive pressure on any magnetized surface, even when the magnitude of B varies from point to point, is

$$p = \frac{f}{A} = \tfrac{1}{2}BH = \frac{1}{2}\frac{B^2}{\mu} \tag{9-79}$$

This expression corresponds exactly to the pressure at the conductor-dielectric interface, which is given in Eq. (4-25).

A force exists at the boundary surface between two surfaces of different permeabilities in a magnetic field in much the same way that a force exists at the boundary between two dielectric materials in an electric field (see Sec. 4-3). In fact, the force, as in the electric case, is always

perpendicular to the boundary surface and is given by an expression having precisely the same mathematical form, as Eq. (4-30). Thus,

$$p = \tfrac{1}{2}(\mu_2 - \mu_1)H_t^2 + \frac{1}{2}\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)B_n^2 \tag{9-80}$$

For a surface between iron ($\mu_2 = \mu_r\mu_0$) and air ($\mu_1 = \mu_0$), this expression may be written as

$$p = \frac{1}{2}\frac{\mu_r - 1}{\mu_r\mu_0}(B_n^2 + \mu_r B_t^2) \tag{9-81}$$

where B_t and B_n are respectively the tangential and normal components of the flux density in the air region.

These latter expressions permit the tractive pressure to be calculated when the field distribution is known. Moreover, in contradistinction to Eq. (9-78), they permit the evaluation of the tractive pressure when μ_r is almost equal to unity. In fact, Eq. (9-81) shows that $p = 0$ for $\mu_r = 1$, the case of a magnetically neutral medium.

It is of considerable interest to note that the tractive force which has been found to exist at the boundary surface between two surfaces of different permeability—in Eq. (9-78) at the air-iron boundary surface—and for a single gap is

$$f = \frac{1}{2}\frac{B^2A}{\mu_0} \qquad \text{newtons}$$

and may be written in several alternative forms. These forms are readily seen to be

$$f = \tfrac{1}{2}\Phi^2\left(\frac{\partial R}{\partial l}\right)_{\Phi=\text{const}}$$

$$f = \tfrac{1}{2}\Phi\left(\frac{\partial F}{\partial l}\right)_{\Phi=\text{const}}$$

$$f = \left(\frac{\partial W_m}{\partial l}\right)_{\Phi=\text{const}}$$

This general problem will receive detailed consideration in the next section.

If there is an actual physical motion resulting from the force of attraction, due account must be taken of the fact that the force may change during the motion. For example, if it is desired to evaluate the work done by the field in closing the gaps in Fig. 9-10, the total change in energy in the field between the initial and final states of the system must be calculated. Specifically for the system illustrated, it is noted that the reluctance of the magnetic path with the gap open is

$$R_1 = R_i + 2R_a$$

where R_i denotes the reluctance of the iron path and R_a is the reluctance of the air gap. With the gap closed, the reluctance of the magnetic path is

$$R_2 = R_i$$

If it is assumed that the current in the coil is constant, which means that the mmf remains constant, then the total change in energy that results from closing the gap is

$$\begin{aligned} \Delta W_m &= \frac{1}{2}\frac{F^2}{R_2} - \frac{1}{2}\frac{F^2}{R_1} \\ \Delta W_m &= \tfrac{1}{2}F^2\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \end{aligned} \tag{9-82}$$

9-8. Forces and Torques on Circuits with Associated Iron.[1] It is desired, now, to extend the considerations of Sec. 9-3 to the more general case of a current-carrying circuit which is situated in a field which may contain iron. Suppose that the circuit is given a virtual displacement δr. If f_r is the magnetic force on this circuit in the direction of δr, there will be a change in flux linkages in the virtual displacement by an amount $I\,\delta\lambda$. Also, a change will occur in the energy stored in the field. The energy supplied in the virtual displacement thus is partly stored and partly converted into external work. By the principle of conservation of energy, we may write

$$I\,\delta\lambda = f_r\,\delta r + \delta W$$

In the limit as δr becomes vanishingly small, this expression becomes

$$f_r = I\frac{\partial\lambda}{\partial r} - \frac{\partial W}{\partial r} \tag{9-83}$$

An entirely similar procedure may be employed to find an expression for the component of torque in giving the circuit a virtual angular displacement $\delta\theta$. The result is

$$T_\theta = I\frac{\partial\lambda}{\partial\theta} - \frac{\partial W}{\partial\theta} \tag{9-84}$$

Equation (9-83) may be generalized by combining this expression with Eq. (9-34), then

$$f_r = \sum_1^n I_k\frac{d\Lambda_k}{dr} - \sum_1^n \frac{d}{dr}\int_0^{\Lambda} i_k\,d\lambda_k \tag{9-85}$$

The value of the force deduced from this expression is the same, regardless of the manner of making the virtual displacement, that is, whether it is

[1] R. E. Doherty and R. H. Park, Mechanical Force between Electric Circuits, *Trans. AIEE*, **45**:240 (1926).

effected *at constant flux linkages* or *at constant current.* The expression may often be written in simpler form, depending on the method of effecting the virtual displacement. The two alternatives will be examined separately.

Constant Linkages. Suppose that the virtual displacement is made at constant flux linkage. In this case, the first term on the right-hand side of Eq. (9-85) vanishes. Note that the limits in the integrals of the remaining term are not functions of r, and partial differentiation under the integral sign is permitted. Thus, Eq. (9-85) becomes, in the present case,

$$f_r = -\sum_1^n \int_0^{\Lambda} \frac{\partial i_k}{\partial r}\, d\lambda_k \tag{9-86}$$

which may be written by Eq. (9-85) in the form

$$f_r = -\left.\frac{\partial W_m}{\partial r}\right|_{\Lambda_k \text{ (const)}} \tag{9-87}$$

This shows that the force is the rate of decrease in stored magnetic energy with displacement at constant linkages, and is identical with Eq. (9-55) for the linear case.

Constant Current. Under the present conditions, it is found convenient to express Eq. (9-85) in slightly different form. This is accomplished by carrying out a partial integration on the right-hand member of the expression. Thus, by writing

$$\sum \frac{d}{dr} \int_0^{\Lambda} i_k\, d\lambda_k = \sum_1^n \frac{d}{dr}\left(I_k\Lambda_k - \int_0^{I_k} \lambda_k\, di_k\right)$$

then Eq. (9-85) becomes

$$f_r = \sum_1^n I_k \frac{d\Lambda_k}{dr} - \sum_1^n \frac{d}{dr}\left(I_k\Lambda_k - \int_0^{I_k} \lambda_k\, di_k\right) \tag{9-88}$$

But at constant current independent of r, the first term on the right cancels the second term, with the remainder

$$f_r = \sum_{k=1}^n \int_0^{I_k} \frac{\partial \Lambda}{\partial r}\, di \qquad \text{joules/m} \tag{9-89}$$

A form somewhat like that of Eq. (9-86) is possible if one defines a quantity which may be called the *magnetic coenergy*, as distinguished from the magnetic energy. This is most conveniently done by reference to

Fig. 9-11, which shows a typical graphical plot of λ versus i for a single circuit. Note that saturation is placed in evidence, but hysteresis is neglected. In this graph, the crosshatched area directly denotes the magnetic energy, according to Eq. (9-34). The area under the curve, in the manner noted, is the magnetic coenergy. In terms of the coenergy, the force equation above becomes

$$f_r = + \left. \frac{\partial W'_m}{\partial r} \right|_{I \text{ const}} \tag{9-90}$$

Note, therefore, that the force is the rate of increase of coenergy with displacement at constant current. This expression should be compared with Eq. (9-54). Clearly, when the medium is linear, the results are identical.

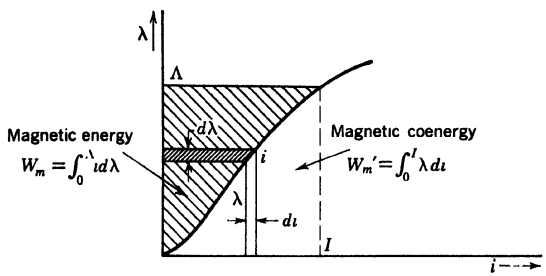


FIG. 9-11. To define the magnetic coenergy.

A particularly important example of these expressions is that in which some of the iron moves with the circuit. This is the case of a coil that has an attached iron core and is exactly the situation of an armature coil in a rotating machine. Equations (9-83) and (9-84) are still valid, but they now express the forces and torques on the entire coil and core assembly without giving any idea of the distribution of the force and torque between the coil and the iron. However, it is possible to deduce some ideas from these equations. Mechanical systems ordinarily move in a manner to reduce the stored energy in the field. Consequently, the system tends to move in a direction to increase the flux linkage with the coil. In air-cored coils, as by Eq. (9-60), only the second tendency prevails, whereas, in fields containing iron but no coils, only the first tendency prevails. Iron-clad coils involve both effects, and the general problem of forces on such coils becomes rather difficult.

Consider specifically the armature of a d-c motor, which is represented schematically in Fig. 9-12 (in actual form in Fig. 9-12*a* and in ideal form in Fig. 9-12*b*). For convenience, it will be assumed that the flux distribution is uniform under each pole, with a sudden change occurring at the mid-point between the poles. Figure 9-12*a*, shows the normal sym-

metrical configurations of conductors in the slots in the armature, with AB representing a particular coil on the armature surface, and with $A'B'$ representing the same coil at a later instant of time as the armature rotates in the frame of the machine. For the disposition illustrated in Fig. 9-12*a*, there will be no change in the stored energy in the coil except possibly as it moves past the pole tips. The average torque associated

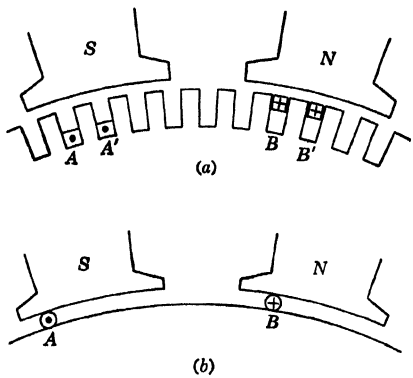


FIG. 9-12. An armature coil in (*a*) an actual machine and (*b*) an ideal machine.

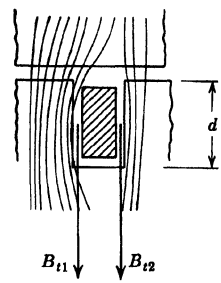


FIG. 9-13. To compute the force in the armature of a motor.

with this coil during the interval for the coil to move from its position AB to the new position $A'B'$ is obtained from Eq. (9-81) and is

$$T = I \frac{\partial \lambda}{\partial \theta} \tag{9-91}$$

where I is the current in the conductor and $\partial\lambda/\partial\theta$ is the mean rate at which flux linkage with the coil changes with respect to the angular movement of the rotor. But since the flux linkage is expressed by the relation

$$\lambda = B_0 l R \theta \tag{9-92}$$

where l is the length of the conductor lying at right angles to the field, B_0 is the flux density in the teeth, R is the armature radius, and θ is the angle subtended at the center of the rotor by the coil, then it follows that

$$T_\theta = B_0 l I R \tag{9-93}$$

Now, consider the corresponding coil AB on an ideal motor having a smooth rotor, with the conductors carried on its outside surface. The flux density at a point in the air gap of this ideal machine is the same as that in the actual machine, except that the variations due to the teeth

are not present in the ideal machine. This means that, while there may be differences in the flux-linkage distribution between the actual and the ideal machine, the average rate at which flux linkage with the coil changes, $\partial\lambda/\partial\theta$, is the same in both machines, whence both machines produce the same torque. This is a most remarkable result, for the torque in the ideal machine is due entirely to forces on the armature conductors, whereas the torque in the actual machine is produced almost entirely by forces on the iron since the conductors lie in a very weak field (see Prob. 9-16 for the case of a conductor embedded in iron).

The foregoing conclusions follow also directly from an application of Eq. (9-81). Refer to Fig. 9-13, which shows the field distribution in the teeth adjacent to the conductor. Observe that the flux lines are tangential to the tooth walls. Therefore, the driving force appears as two forces: one is a force on the conductor, and the second is a force on the tooth walls. Specifying, as before, that B_0 is the flux density in the teeth, then the magnetic flux density in the slot is

$$B = \frac{B_0}{\mu_r}$$

If the conductor in the slot has the length l, then the force on the conductor is, for a current I,

$$f_1 = \frac{B_0}{\mu_r} Il \tag{9-94}$$

Observe that this force varies inversely with the permeability of the iron and becomes smaller with increasing permeability of the iron.

To find the force on the tooth walls, it is necessary to find an expression for the effective magnetic flux density in the tooth walls. Two cases are of importance: (1) a very narrow or closed slot, the depth a of which is large compared with its width; and (2) the opposite situation of a broad slot. For the case of the deep, narrow slot, owing to the current I through the conductor in the slot, the magnetic flux density will be different on the two sides of the tooth wall. An application of the line integral around the slot boundary gives, approximately,

$$\oint H\,dl = (H_{t1} - H_{t2})a = I \tag{9-95}$$

Therefore, the tangential components of the magnetic induction on the two walls of the tooth are

$$\begin{aligned} B_{t1} &= \frac{B_0}{\mu_r} + \frac{\mu_0 I}{2a} \\ B_{t2} &= \frac{B_0}{\mu_r} - \frac{\mu_0 I}{2a} \end{aligned} \tag{9-96}$$

As a result, opposing forces act on the tooth-wall surfaces. The resulting force is obtained by applying Eq. (9-79) to each wall. This force difference is

$$f_2 = \frac{\mu_r - 1}{2\mu_r\mu_0}\,\mu_r\,\frac{2B_0}{\mu_r}\,\frac{\mu_0 I}{a}\,al = \frac{\mu_r - 1}{\mu_r}\,B_0 Il \tag{9-97}$$

Note, by comparing this expression with Eq. (9-92), that the force on the iron of the armature is $\mu_r - 1$ times as large as that on the conductor. The sum of both forces is

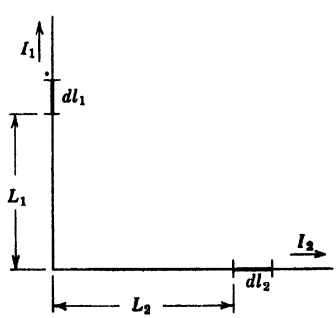
$$f = f_1 + f_2 = B_0 Il \tag{9-98}$$

This is the same value as that of a conductor in a field with a magnetic induction B_0, as already noted above.

For the case of the broad slot, Eq. (9-95) is no longer valid, since the contribution to the line integral by the upper and lower paths around the slot is no longer negligible. However, the component of the flux density produced by the current in the conductor divides into two paths, one part traversing the tooth wall, air gap, and pole face, and the second part traversing the tooth wall and the slot. This second component is negligible compared with the first, owing to the length of the air path through the slot. In any case, this component remains relatively constant. The total flux density on the tooth wall is thus the resultant of the contribution by the main poles and the component due to the current in the coil. The total flux linkage is then essentially that of the ideal machine, whence the reasoning that led to Eq. (9-91) is again invoked to show the validity of the results.

PROBLEMS

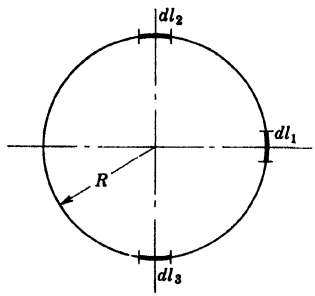
9-1. Two elements of current are oriented in air, as illustrated. Find the direction and magnitude of the force on each element of current.



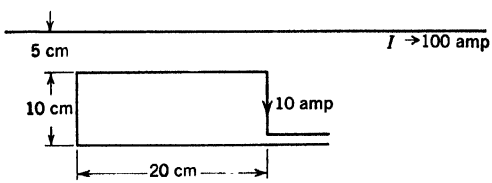
9-2. Consider the three elements of a circular current, as shown. Compare the following:

a. The differential force on dl_2 due to dl_1 and dl_3

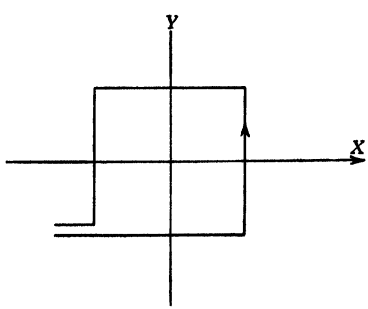
b. The differential force on dl_1 by dl_2 and dl_3



9-3. Given a rectangular loop of wire carrying a current of 10 amp adjacent to and in the same plane as a long, straight wire carrying a current of 100 amp, find the magnitude and direction of the force on the loop. The geometry is illustrated in the accompanying diagram.

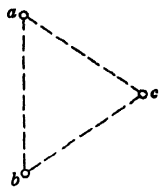


9-4. A current of 100 amp flows in a long wire along the X axis, as shown. A rigid, square frame loop in the XY plane of one turn carries a current of 10 amp. The loop is 1 m square. Find the magnitude and direction of the force on the loop.

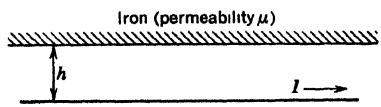


9-5. The three conductors of a transmission line are supported at point 1 m apart. They form the vertices of an equilateral triangle, as shown. At a cer-

tain instant, the currents in conductors a and b are each 120 amp, with the return current of 240 amp in conductor c. Find the mechanical force on conductor c.



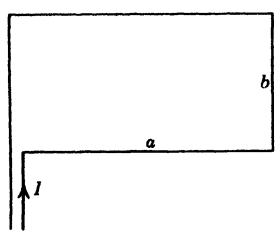
9-6. Calculate the force on a current-carrying wire parallel to an iron block of permeability μ and a distance h away, as illustrated.



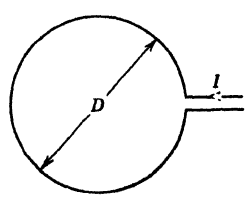
9-7. Suppose that the parallel bus bars of Fig. 9-5 carry currents of 1,000 amp each, with $b = 8$ cm and $d = 12$ cm. Find the approximate force per meter acting on each bus bar.

9-8. Suppose that the switch of Fig. 9-6 represents a disconnect switch. During a short circuit and before the circuit breaker opens, a current of 5,000 amp flows. What is the force tending to open the switch if $d = 8$ in. and $l = 1$ in.?

9-9. Find an expression for the force tending to open a rectangular current circuit.

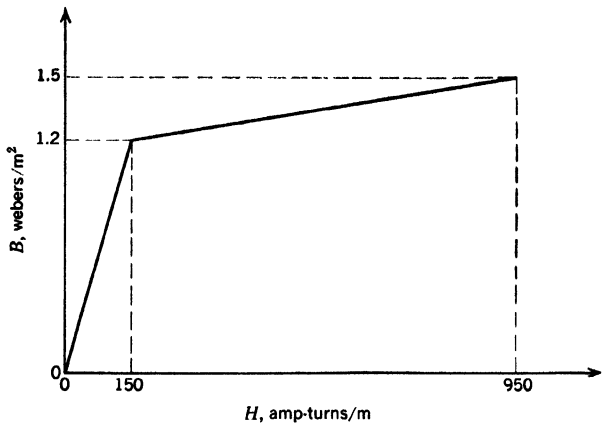


9-10. Find an expression for the force tending to open a circular current circuit.

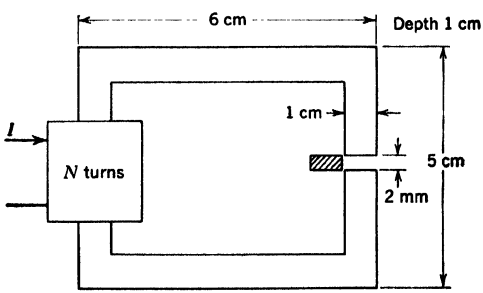


9-11. The BH characteristic of a certain transformer steel is specified by the equation in Prob. 7-4. Determine the energy stored in the iron per unit volume when $B = 1.2$ weber/m^2.

9-12. Consider the reactor containing a short air gap, as illustrated in Prob. 7-4, with a depth of 0.75 in. An idealized BH characteristic of the magnetic material is shown in the diagram. Find the magnetic energy stored in the magnetic field when the flux density in the material is 1.5 webers/m^2. Neglect fringing in the air gap; also neglect leakage.

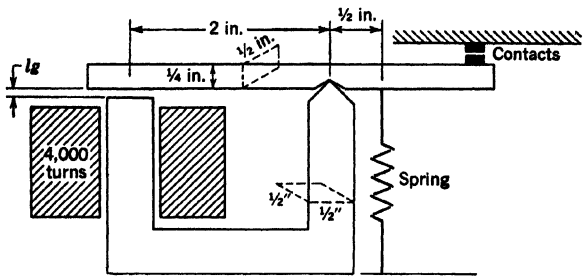


9-13. A magnetic circuit has the dimensions indicated in the figure. The magnetizing coil has 150 turns and carries a current of 1 amp. A slab of iron which is just sufficient to fill the air gap is available. If the permeability of the iron is assumed to be constant, with $\mu_r = 4{,}000$, how much work is done in pulling the slab into the air gap?

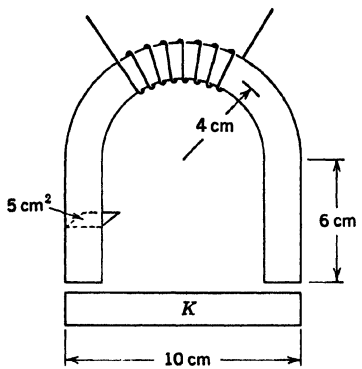


9-14. The general construction of a relay is illustrated in the accompanying figure. The spring exerts a pull of 4 oz, the relay being *nominally open* when there is no current in the coil. When sufficient current exists in the exciting coil of 4,000 turns, the relay closes. The air gap l_g reduces from $\frac{3}{32}$ in. to $\frac{1}{32}$ in. during the closing of the relay. The material of the relay is transformer iron having the magnetizing curve of Fig. 7-12. The mean length of the iron path is 7.5 in.

a. How much current is required to close the relay?
b. After the relay is closed, how much current is required to keep it closed?



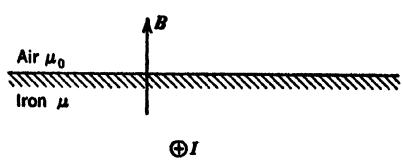
9-15. An electromagnet is illustrated in the figure. This unit is designed to support a weight of 20 lb (which includes the weight of the keeper K). The



cross section of the magnetic portions of the circuit is 5 cm², with other dimensions as shown. The air gaps are each 0.15 mm long. It is required to find the number of turns in the magnetizing coil to support this weight when the exciting current is 2 amp. The permeability of the material varies with B as specified in the tabulation.

B, webers/m²	0.36	0.44	0.48	0.6	0.72
μ_r	3,300	3,000	2,900	2,600	2,300

9-16. A conductor carrying a current I is embedded in a semi-infinite plane mass of iron of permeability μ. A magnetic field B is normal to the surface, as shown.

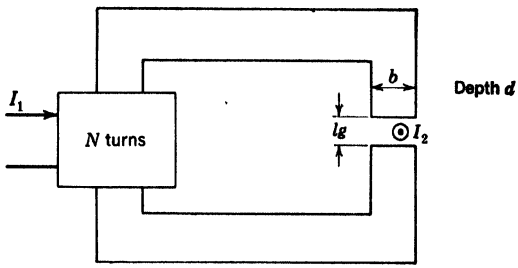


a. Using the method of electrical images, find the force on the conductor-iron system.

b. Suppose that the iron is removed so that the conductor is in air. Find the force on the conductor.

c. What can be said about the total force on the system in each case?

9-17. A conductor carrying a current is placed in the air-gap field of an electromagnet, as illustrated in the accompanying figure. Deduce an expression for the force on the conductor. Assume that the permeability of the iron is infinite, that there is no fringing of the magnetic flux in the air gap, and that the return path of I_2 is remote from the field. The conductor I_2 is a distance x from the edge of the core.



CHAPTER 10

MAXWELL'S EQUATIONS; ELECTROMAGNETIC WAVES

The set of equations which have become known as the *Maxwell equations* are those which have been developed in the foregoing chapters and will therefore have associated with them the names of other investigators. These equations describe the sources and the field vectors in the broad fields of electrostatics, magnetostatics, and electromagnetic induction. There might therefore be a tendency to underestimate the real contribution of Maxwell. It was the work of Maxwell which introduced the interesting and important phenomena of electromagnetic wave motion and radiation. Moreover, he developed the theory which enables an examination of the space and time variations of changing fields in the same way that the equations of electrostatics and magnetostatics permit an examination of stationary fields.

10-1. The Maxwell Equations. The following set of four differential equations, which express point statements concerning the field quantities, express all the phenomena which have been studied. These, with an indication of where they appear in the text, are:

$\nabla \cdot \mathbf{D} = \rho$	(3-42)	Gauss	(10-1*a*)
$\nabla \cdot \mathbf{B} = 0$	(6-43)		(10-1*b*)
$\nabla \times \mathbf{H} = \mathbf{J}_c$	(7-9)	Ampère	(10-1*c*)
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	(8-23)	Faraday's law in differential form	(10-1*d*)

To these are often added several auxiliary equations, including

$$\nabla \cdot \mathbf{J}_c + \frac{\partial \rho}{\partial t} = 0 \qquad \text{continuity equation} \qquad (10\text{-}2)$$

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H} \qquad \mathbf{J} = \gamma \mathbf{E} \qquad \text{constitutive equations} \qquad (10\text{-}3)$$

Equations (10-1) are known as Maxwell's equations.

A feature of Ampère's law, Eq. (10-1*c*), that is worthy of special mention follows from the incompleteness of the magnetic-circuit law for the time-varying case and was, in fact, one of the important contributions of Maxwell. The initial problem is made evident by reference to Fig. 10-1,

which represents a portion of a circuit containing a capacitor, in which a current is assumed to flow. To evaluate the Ampère circuital law, $\oint \mathbf{H} \cdot d\mathbf{l}$, requires that the line integral taken around the contour of any circuit L be related to the total current through a cap with the contour

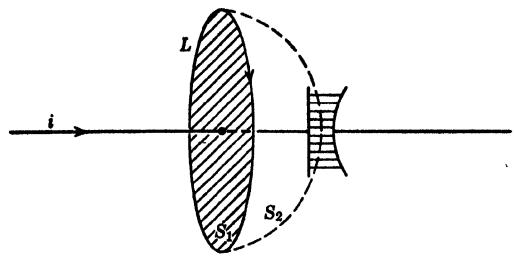


FIG. 10-1. To demonstrate the incompleteness of Ampère's law.

as its edge. For the cap specified by the surface S_1, the total current is i, whence in this case, if the law is still correct,

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = i$$

Now, consider the cap specified by the surface S_2, which is so chosen that it passes between the plates of the capacitor. Clearly, in this case, assuming a perfect dielectric, no conduction current passes through the surface S_2, and it appears that

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = 0$$

unless the circuital law itself undergoes some change when it goes from the static case to the time-varying case. This is found to be contrary to reason, since if a unique solution exists it cannot possibly be a function of the choice of the cap with L as perimeter. The introduction by Maxwell of the postulate of the displacement current avoided this inconsistency and permitted the generalization of the Ampère law to the form

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} \tag{10-4}$$

As known from our past studies, for each of the equations in Eqs. (10-1) which specifies derivatives of the field quantities at a point in space, there is a corresponding statement which applies for a finite region of space. Although the point statement was obtained from considerations of the integral statement, it is not inconsistent to state that the integral statement may be obtained from the point statement. This is done by performing an integration over a surface or a volume and then

transforming the integrals by an application of the divergence theorem or Stokes' theorem. The integral forms corresponding to the differential equations in Eqs. (10-1) as modified by Eq. (10-4) are the following:

$$\begin{aligned} \oint \mathbf{D} \cdot \mathbf{n}\, dS &= \int_\tau \rho\, d\tau \\ \oint \mathbf{B} \cdot \mathbf{n}\, dS &= 0 \\ \oint \mathbf{H} \cdot d\mathbf{l} &= \int_S \left(\mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{n}\, dS \\ \oint \mathbf{E} \cdot d\mathbf{l} &= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS \end{aligned} \tag{10-5}$$

Before extending the Maxwell equations to the general problems of electromagnetic waves, some comments concerning the several equations of Eqs. (10-1) are in order. Consider the third equation of the group, which expresses Ampère's law in the extended form given in Eq. (10-4), containing the displacement current as well as the conduction current. The divergence of this expression is examined, thus,

Max well's eq. from Ampere's law:—

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J}_c + \nabla \cdot \left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

But as may be readily demonstrated by performing the indicated operations, div (curl A) $\equiv 0$ for any vector; then,

$$\nabla \cdot \left(\frac{\partial \mathbf{D}}{\partial t}\right) + \nabla \cdot \mathbf{J}_c = 0 \tag{10-6}$$

By combining this expression with the continuity equation, Eq. (10-2), there results

$$\nabla \cdot \left(\frac{\partial \mathbf{D}}{\partial t}\right) - \frac{\partial \rho}{\partial t} = 0$$

An interchange of the order of partial differentiation yields the expression

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{D} - \rho) = 0$$

which is always satisfied if, in general,

$$\nabla \cdot \mathbf{D} = \rho \tag{10-7}$$

This expression is known to hold for static fields, as discussed in Chap. 3. It now appears as a general expression implied by Eqs. (10-1c) and (10-2), although Eq. (10-6) merely requires that the quantity $\nabla \cdot \mathbf{D} - \rho$ be independent of time and, in general, a constant. However, a situation

is possible in which both ρ and $\nabla \cdot \mathbf{D}$ are zero in a region of space so that $\nabla \cdot \mathbf{D} - \rho = 0$ in the region. Equation (10-6) then specifies that this relationship must be independent of time, whence Eq. (10-7) follows.

There is no direct proof that the Maxwell generalization of Ampère's law is valid, but there is very convincing indirect proof of its validity. In fact, it is only through the existence of the displacement current that Maxwell showed there should be electromagnetic fields capable of being propagated through space as waves. The velocity and other characteristics of the waves were predictable from the field equations. Subsequently, Hertz, working in Germany in the 1890s, showed the experimental existence of electromagnetic waves, which possessed all the properties predicted by Maxwell. It is now known, of course, that radio and light waves are Maxwell electromagnetic waves.

Another interesting feature of the postulate of displacement current is the symmetry that is added to the laws of electromagnetism. The generalized Ampère law indicates that an electric displacement current (a time-changing electric field) produces a magnetic field. Correspondingly, Faraday's law, Eq. (10-1*d*), specifies that a time-changing magnetic field produces an electric field. In fact, by comparing Eqs. (10-1*c*) and (10-1*d*), it is clear that $-\partial \mathbf{B}/\partial t$ behaves like a magnetic displacement-current density. However, magnetic conduction currents do not exist in nature, because there are no free magnetic poles.

It is of some interest to consider finally Faraday's law, specified in Eq. (10-1*d*). Proceeding as above, by taking the divergence of both sides of this equation, and noting again that div (curl $\mathbf{A}$) $\equiv 0$ for any vector, then it is required that Maxwell's eq. from Farad

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot \left(\frac{\partial \mathbf{B}}{\partial t}\right) \equiv 0$$

The order of the partial derivative is changed without affecting the results and yields

$$\frac{\partial}{\partial t}(\nabla \cdot B) = 0 \tag{10-8}$$

This expression states that the quantity $\nabla \cdot \mathbf{B}$ is a constant, in general, independent of the time. But the $\mathbf{B}$ function may actually be a function of time. Moreover, a nonzero value of the divergence of $\mathbf{B}$ has never been found in nature, and it is postulated therefore that the constant is zero. Equation (10-8) requires that $\nabla \cdot \mathbf{B}$ remain zero and, in fact, that it has always been zero. The deduction made from Eq. (10-8) is that everywhere and always

$$\nabla \cdot \mathbf{B} = 0 \tag{10-9}$$

This property was shown to exist for the steady-state case [see Eq. (6-43)].

From the present viewpoint, if Eq. (10-1*d*) is postulated, then Eq. (10-1*b*) follows as a deduction. However, Eq. (10-1*b*) is so important that it is usually listed as one of the Maxwell equations.

10-2. The Wave Equations. Maxwell's equations supply us with a statement of the space and time variations of **E** and **H** at all points in an electromagnetic field. Consequently, if a set of initial values of **E** and **H** at all points in the electromagnetic field are given, their values at all future times may be found. Also, if values of **E** and **H** are specified as a function of time over any surface (for example, the surface of a radio antenna, or the surface of a transformer core), it is possible to find their values at all other points in the field.

To determine the wave equations satisfied by the field quantities, the following limitations will be imposed on the system:

1. The medium obeys Ohm's law, $\mathbf{J}_c = \gamma\mathbf{E}$.
2. The free charge density in the field is zero, that is, $\rho = 0$.
3. The medium is assumed to be both magnetically and electrically homogeneous and isotropic.

Subject to these limitations, the corresponding Maxwell equations become

$$\nabla \cdot \mathbf{D} = 0 \tag{10-10a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{10-10b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10-10c}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} \tag{10-10d}$$

where

$$\mathbf{J}_c = \gamma\mathbf{E}$$
$$\mathbf{D} = \epsilon\mathbf{E}$$
$$\mathbf{B} = \mu\mathbf{H}$$

To find the wave equations, a number of mathematical manipulations are necessary. First, the curl of Eq. (10-10*c*) is taken. This yields

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) \tag{10-11}$$

But it has been shown [see Eq. (8-31)] as a mathematical expansion that the operation

$$\text{curl curl} = \text{grad div} - \nabla \cdot \nabla \tag{10-12}$$

so that Eq. (10-11) may be written as

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - (\nabla \cdot \nabla)\mathbf{E} = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) \tag{10-13}$$

By interchanging the order of the partial derivative and combining the result with Eq. (10-10*d*), there results

$$\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = \mu \frac{\partial}{\partial t}\left(\gamma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}\right)$$

Also, by including (10-10*a*), which requires that

$$\nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{E} = 0$$

then

$$(\nabla \cdot \nabla)\mathbf{E} = \mu\left(\gamma \frac{\partial \mathbf{E}}{\partial t} + \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}\right)$$

By rearranging, there results finally

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu\gamma \frac{\partial \mathbf{E}}{\partial t} = 0 \tag{10-14}$$

free space

This is the three-dimensional wave equation for the absorbing medium. The absorption, which represents a power loss in the medium accompanying the propagation, arises from the term involving the first time derivative. For a medium with no absorption, $\gamma = 0$, whence the *free-space* wave equation for the medium without absorption is

$$\nabla^2 \mathbf{E} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{10-15}$$

A somewhat similar procedure may be followed for deriving the wave equation for the other three field quantities, **D**, **B**, and **H**, and all are found to satisfy exactly the same type of wave equation. This may be indicated symbolically by the expression

$$\left(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2} - \mu\gamma \frac{\partial}{\partial t}\right)\begin{bmatrix}\mathbf{E}\\ \mathbf{D}\\ \mathbf{B}\\ \mathbf{H}\end{bmatrix} = 0 \tag{10-16}$$

The solution of these equations subject to the appropriate initial and boundary conditions will provide answers to problems such as those mentioned above, namely:

1. Given a set of initial values of **E** and **H** at all points in an electromagnetic field, to find their values at all future times.
2. Given **E** and **H** as functions of time over any surface, to find their values at all other points in the field.

Several representative problems will be discussed below.

10-3. Plane Electromagnetic Waves in Free Space. Equations of the form given by Eq. (10-15) apply for homogeneous, isotropic, nonconducting media and are applicable, therefore, for radio-wave propagation. In order to examine some of the specific properties of such wave propagation, we examine whether a function of the form $\mathbf{E} = E_y(x,t)$ is a possible solution to Maxwell's equations. This denotes an electric field which is

directed along the Y axis and which is being propagated in the X direction. Such a field is said to be a plane linearly polarized wave in the Y direction. For these conditions, Eq. (10-15) reduces to

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2} \tag{10-17}$$

This equation is of a type that is well known in mathematical physics and is called the *equation of wave propagation.* Solutions of this equation are of the form

$$E_y = f(x - vt)$$

and

$$E_y = F(x + vt)$$

or

$$E_y = f(x - vt) + F(x + vt) \tag{10-18}$$

provided that

$$v \equiv \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{10-19}$$

In fact, for free-space propagation, the velocity of propagation v is given by

$$v = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = (2.9978 \pm 0.0001) \times 10^8 \text{ m/sec} \tag{10-20}$$

which is the velocity of light in free space. In the general solution, Eq. (10-18), the functions $f(x - vt)$ and $F(x + vt)$ denote any function of $x - vt$ and $x + vt$, respectively, and might represent, for example, the functions $(x - vt)$, $\sin(x - vt)$, etc.

To show that the designated functions do satisfy the wave equation, consider the function $f(x - vt)$. Write, for convenience,

$$\eta = x - vt$$

Then,

$$\frac{\partial E_y}{\partial x} = \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial f}{\partial \eta}$$

and

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = \frac{\partial^2 f}{\partial \eta^2} \tag{10-21}$$

In a similar way, by treating x as a constant,

$$\frac{\partial E_y}{\partial t} = \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial t} = -v\frac{\partial f}{\partial \eta}$$

$$\frac{\partial^2 E_y}{\partial t^2} = \frac{\partial}{\partial \eta}\left(-v\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial t} = v^2\frac{\partial^2 f}{\partial \eta^2} \tag{10-22}$$

Similar expressions result for the function $F(x + vt)$. By combining Eqs. (10-21) and (10-22) with Eq. (10-17), it follows that

$$\frac{\partial^2 f}{\partial \eta^2} = \mu_0 \epsilon_0 v^2 \frac{\partial^2 f}{\partial \eta^2} \tag{10-23}$$

whence, as already noted, for $f(x - vt)$ to be a solution, it is required that

$$v^2 = \frac{1}{\mu_0 \epsilon_0}$$

It is readily shown that Eq. (10-18) will represent a wave that travels unchanged in form with a velocity $1/\sqrt{\mu_0 \epsilon_0}$ parallel to the X axis. It is evident that $E_y = f(x - vt)$ will remain constant so long as $x - vt$ remains constant. Note, moreover, that if attention is fixed on a point which is supposed to travel with a velocity v in the direction of X positive, then $x - vt$ will remain constant with respect to this moving point. Refer to Fig. 10-2, which represents the function E at different points in space. In this diagram, x_1 denotes the position of the moving

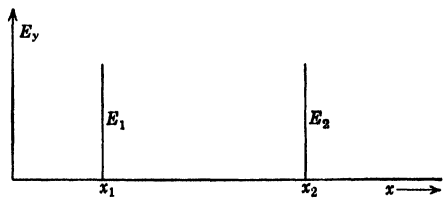


FIG. 10-2. The function E_y at different points in space.

point at time t_1; x_2 is the position of the point at time t_2. Writing E_1 for the function at x_1 and t_1,

$$E_1 = f(x_1 - vt_1)$$

with a corresponding form for point x_2, t_2, then

$$E_2 = f(x_2 - vt_2)$$

Write this in the form

$$E_2 = f[x_1 + (x_2 - x_1) - vt_2]$$

But since $x_2 - x_1 = v(t_2 - t_1)$, then

$$E_2 = f[x_1 + v(t_2 - t_1) - vt_2]$$

or
$$E_2 = f(x_1 - vt_1)$$

Therefore,
$$E_2 = E_1$$

In the same manner, it may be shown that $F(x + vt)$ represents a wave traveling in the negative-X direction. Clearly, therefore, Maxwell's equations describe mathematically the property of the electromagnetic field in which the disturbance established at any point in the field is propagated from that point with a velocity $v = 1/\sqrt{\mu_0 \epsilon_0}$.

Another point of some interest in wave propagation arises if the value of H corresponding to E is calculated. This may be done directly through Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

An examination of the expression for the curl in rectangular form shows that it reduces in the present case to

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

Upon performing the integration in time, and choosing the function of integration to be zero, we find for B_z

$$B_z = \frac{1}{v} f(x - vt) \tag{10-24}$$

Note that the pattern of the variations of B is exactly the same as that of the variations of E, except that E_y and B_z are directed perpendicularly to each other. In general, **E** is perpendicular and right-handed to **B** about the direction of propagation. Now, from this equation, the value for H is

$$H_z = \frac{1}{\mu_0 v} f(x - vt)$$

so that

$$E_y = \mu_0 v H_z$$

which may be written, by Eq. (10-18), as

$$E_y = \sqrt{\frac{\mu_0}{\epsilon_0}}\, H_z \tag{10-25}$$

It is common to write this result in the form

$$E_y = Z_0 H_z$$

where

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.6 \text{ ohms} \tag{10-26}$$

The units of Z_0 are seen to be volts per ampere or ohms, whence the term *impedance* is applied to Z_0, which is called the *wave impedance of free space.*

10-4. Sinusoidal Electromagnetic Waves. In many of the important field problems, it is known that the time dependence of the field is harmonic (sinusoidal, having a single frequency). This is approximately the situation with radio-wave phenomena and with light or optical phenomena. Also, in the more general case of complex periodic waves, the representation of the waves in a Fourier series essentially reduces these problems to one involving sinusoidal waves. It is worth treating this specific type of variation in some detail, as it leads to some interesting

conclusions. Suppose, therefore, that it is assumed the electric field is given by the function

$$\mathbf{E} = \mathbf{E}_0 e^{-j\omega t} \tag{10-27}$$

This expression assumes, in effect, that the complete solution of the differential equation for **E**, given by Eq. (10-14), is expressible as a product solution consisting of a sinusoidal time function and a space function, as yet unknown. By substituting this expression into Eq. (10-14), there results

$$\nabla^2 \mathbf{E}_0 e^{-j\omega t} + \omega^2 \mu\epsilon \mathbf{E}_0 e^{-j\omega t} + j\omega\mu\gamma \mathbf{E}_0 e^{-j\omega t} = 0$$

from which the space variation of $\mathbf{E}_0$ is that prescribed by the following differential equation:

$$\nabla^2 \mathbf{E}_0 + \omega^2 \mu\epsilon \left(1 + j\frac{\gamma}{\omega\epsilon}\right) \mathbf{E}_0 = 0 \tag{10-28}$$

This is the *wave equation* for harmonic time dependence, but for the most general sort of medium, as discussed in Sec. 10-2.

In free space and in most dielectric materials, $\gamma \to 0$, whence $\gamma/\omega\epsilon$ is negligibly small. In this case, Eq. (10-28) simplifies to the form

$$\nabla^2 \mathbf{E}_0 + \omega^2 \mu\epsilon \mathbf{E}_0 = 0 \tag{10-29}$$

This is known as the *Helmholtz equation* and is the time-independent wave equation, since the time does not appear in the expression.

In metals and in highly conducting media (and this applies even in sea water up to relatively high frequencies), the converse approximation is valid; that is, in these media, $\gamma/\omega\epsilon \gg 1$. The resulting expression has the form

$$\nabla^2 \mathbf{E}_0 + j\omega\mu\gamma \mathbf{E}_0 = 0 \tag{10-30}$$

and is known as the *diffusion equation*. A specific example of the applicability of the diffusion equation will be found in the next section.

10-5. Field Penetration. A class of problems of considerable importance is that in which the frequency is sufficiently low to permit neglecting the displacement current in comparison with the conduction current. For example, see Prob. 10-2. The diffusion equation controls the action in this case. An example of considerable practical importance will be considered.

Power transformers are constructed with large steel cores upon which are placed the windings. The purpose of the core is, as already discussed in Chap. 7, to provide a low-reluctance path for the magnetic flux. The operation of the transformer depends on a time-varying flux within the core to induce an emf in the windings, in accordance with the Faraday law. But, as found in discussing the eddy-current problem in Chap. 9, not only does the changing flux produce an emf in the windings, but it also

produces an emf within the core itself. The emf produced within the core material would produce large circulating currents, were it not for the fact that high-resistivity core material is used, and also that the core is broken into laminations, each of which is electrically insulated from the other. The laminated structure serves to reduce the eddy-current losses to a reasonable value and to maintain the reluctance predicted from d-c magnetization considerations.

The over-all eddy-current problem is that of evaluating the effect of the eddy currents in one lamination; one phase of this investigation is the determination of the magnetic flux distribution within the lamination. Clearly, two contributions to the mmf in the core exist, that produced

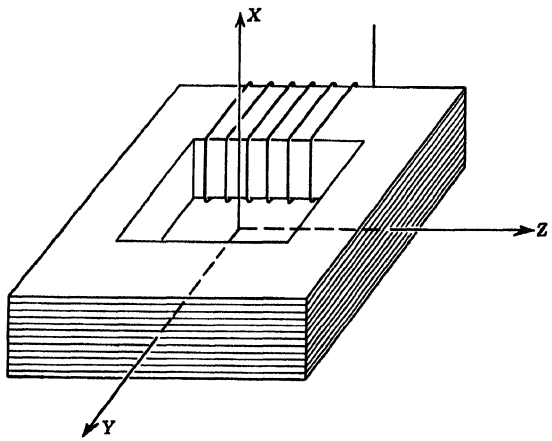


FIG. 10-3. A transformer core for study.

by the currents in the external windings, and that produced by the eddy currents within the core. It is expected, therefore, that the flux distribution in the core would not be uniform, and it is this problem to which attention is being addressed.

Refer to Fig. 10-3, which is to represent the transformer core. It is assumed that a magnetic field of strength $H = H_0 \cos \omega t$ is produced parallel to the laminations by the winding, which is chosen to be along the Z axis.

For the range of frequencies over which the displacement-current term may be neglected in comparison with the conduction-current term, Eq. (10-16) reduces to the form

$$\nabla^2 \mathbf{H} - \mu\gamma \frac{\partial \mathbf{H}}{\partial t} = 0 \tag{10-31}$$

But attention is being directed to a lamina which is oriented as in Fig.

10-4. For the situation illustrated in Fig. 10-4, the controlling differential equation reduces to the one-dimensional expression

$$\frac{\partial^2 H_z}{\partial x^2} - \mu\gamma \frac{\partial H_z}{\partial t} = 0 \qquad (10\text{-}32)$$

For the assumed case that H varies sinusoidally with respect to the time, and with H_0 uniform over the lamination surface, this equation reduces to

$$\frac{\partial^2 H_z}{\partial x^2} = j\omega\mu\gamma H_z \qquad (10\text{-}33)$$

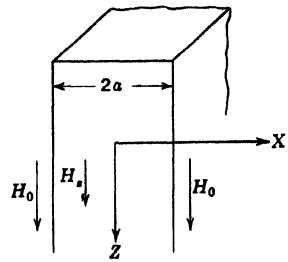


FIG. 10-4. A lamina of the power transformer of Fig. 10-3.

A solution to Eq. (10-33) which satisfies the equation and the boundary conditions is the following,[1]

$$H_z = H_0 \frac{\cosh 2\alpha x + \cos 2\alpha x}{\cosh 2\alpha a + \cos 2\alpha a} \cos(\omega t + \theta)$$

where

$$\tan\theta = \frac{\sinh\alpha(a-x)\sin\alpha(a+x) + \sinh\alpha(a+x)\sin\alpha(a-x)}{\cosh\alpha(a-x)\cos\alpha(a+x) + \cosh\alpha(a+x)\cos\alpha(a-x)} \qquad (10\text{-}34)$$

and where x is the distance from the center of the lamina, $2a$ is the thickness of the lamina, H_0 is the amplitude of the applied field, ω is the angular frequency of the applied field, $\alpha = \sqrt{\omega\mu\gamma/2}$, and θ is the phase difference between the applied field and the field inside the lamina.

The value of the field across the lamina is desired. Observe first that:

1. At the boundary, $x = \pm a$, and

$$H_z = H_0$$

which is the required boundary condition.

2. At the center of the lamina, $x = 0$, and

$$H_{zc} = \frac{H_0\sqrt{2}}{(\cosh 2\alpha a + \cos 2\alpha a)^{1/2}} \qquad (10\text{-}35)$$

$$\tan\theta_c = \tanh\alpha a \tan\alpha a$$

An indication of the character of the variation specified by Eq. (10-34) for material having the characteristics specified in the table below is given in Fig. 10-5.

[1] A. Russell, "The Theory of Alternating Currents," vol. 1, p. 491, Cambridge University Press, New York, 1914.

Data: $\mu_r = 2{,}500$, $\omega = 200\pi$, $\gamma = 10^{-4}$ mho/cm^3, $\alpha = 10\pi$.

Thickness $2a$	H_0/H center	θ_c
0.5	1.1	45°
1.0	2.4	90°
2.0	11.6	180°

The graphs of Fig. 10-5 indicate an appreciable variation of H with thickness and indicate the desirability of having thin lamina in order to provide a relatively uniform field intensity throughout the lamina.

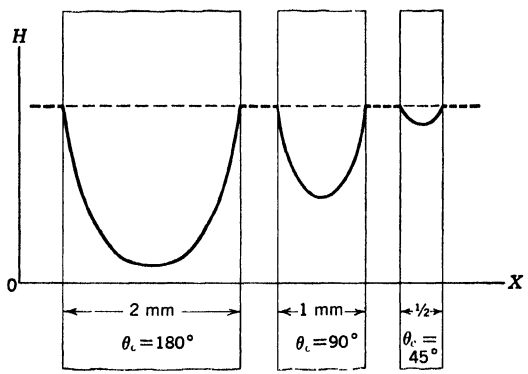


FIG. 10-5. Graphs of the amplitude of H at various depths in iron sheets of different thickness.

10-6. Poynting's Theorem. It was found in Chap. 4 that the energy associated with a given configuration of charge could be expressed in two forms, given, respectively, by Eqs. (4-9) and (4-21), namely,

$$W_e = \frac{1}{2} \int_{\text{Space}} \mathbf{E} \cdot \mathbf{D}\, d\tau = \tfrac{1}{2}CV^2 \quad \text{joules} \tag{10-36}$$

Similarly, the energy associated with a system of rigid current circuits was given, respectively, by the forms in Eqs. (9-47) and (9-39), namely,

$$W_m = \frac{1}{2} \int_{\text{Space}} \mathbf{B} \cdot \mathbf{H}\, d\tau = \tfrac{1}{2}LI^2 \quad \text{joules} \tag{10-37}$$

It is expected, therefore, that the total energy stored in a volume τ of any electromagnetic field, in which all field vectors exist simultaneously, may be taken as

$$W = W_e + W_m = \frac{1}{2} \int_{\text{Space}} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H})\, d\tau \tag{10-38}$$

Moreover, the time rate of decrease of stored energy gives the available instantaneous power and may be written as

$$P(t) = -\frac{\partial}{\partial t}\left[\frac{1}{2}\int_{\tau} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H})\, d\tau\right] \tag{10-39}$$

But, by the conservation of energy, this power may be changed to thermal form, or it may leave through the surface that encloses τ. This condition is written

$$P(t) = \int_{\tau} \mathbf{J} \cdot \mathbf{E}\, d\tau + \oint \mathbf{S} \cdot \mathbf{n}\, dS \tag{10-40}$$

where the first term is, by Eq. (1-60), the Joule heat that is developed in the volume element and the second term denotes a vector whose magnitude expresses the power flow per unit area through the surface and whose direction is the direction of maximum flow. It is desired to find an explicit form for the power-flow density vector **S**.

To find the form for **S**, it is noted that Eq. (10-39),

$$P(t) = -\frac{\partial}{\partial t}\left[\frac{1}{2}\int_{\tau} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H})\, d\tau\right]$$

may be written as

$$P(t) = -\int_{\tau} \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\right) d\tau \tag{10-41}$$

Combine this expression with Maxwell's equations to find

$$P(t) = -\int_{\tau} (\mathbf{E} \cdot \nabla \times \mathbf{H} - \mathbf{E} \cdot \mathbf{J} - \mathbf{H} \cdot \nabla \times \mathbf{E})\, d\tau$$

which is

$$P(t) = \int_{\tau} \mathbf{E} \cdot \mathbf{J}\, d\tau + \int_{\tau} (\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H})\, d\tau \tag{10-42}$$

Use is now made of the vector identity

$$\text{div } (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{10-43}$$

which permits writing $P(t)$ in the form

$$P(t) = \int_{\tau} \mathbf{E} \cdot \mathbf{J}\, d\tau + \int_{\tau} \nabla \cdot (\mathbf{E} \times \mathbf{H})\, d\tau \tag{10-44}$$

This is in the general form expected and was written originally in Eq. (10-40). For equivalence, it is required that the following expressions be related.

$$\oint \mathbf{S} \cdot \mathbf{n}\, dS = \int_{\tau} \nabla \cdot (\mathbf{E} \times \mathbf{H})\, d\tau \tag{10-45}$$

By applying the divergence theorem to the integral on the right-hand side, then

$$\int_\tau \nabla \cdot (\mathbf{E} \times \mathbf{H})\, d\tau = \oint \mathbf{n} \cdot (\mathbf{E} \times \mathbf{H})\, dS$$

which therefore permits identifying the vector **S** with

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{10-46}$$

The vector **S** is called the *Poynting vector.*

The Poynting vector **S** specifies the instantaneous direction and magnitude of energy-flow density at any point in the medium and from Eq. (10-45) may be interpreted as the flux of power per unit area. From Eq. (10-46), it is seen that **S** is always perpendicular to the plane determined by **E** and **H**. For example, if **E** and **H** are perpendicular to each other, which is the situation with plane, linearly polarized waves in space, then **S** is in the direction of propagation of the wave, which is perpendicular to the plane of **E** and **H**.

Note carefully that our development interprets only $\nabla \cdot \mathbf{S}$ or $\oint \mathbf{n} \cdot \mathbf{S}\, dS$ and not the vector **S** itself. In fact, all that is contained in the foregoing development is the conclusion that the flux of **S** into any closed volume is equal to the rate of storage of energy, plus the rate of dissipation of energy within that volume. It has not been proved that there is an energy flow of S watts/m^2 at every point. Clearly, if the vector **S** has no flux into any closed volume, there is no storage or dissipation of energy taking place in the system. For example, a static system which is characterized by E lines and H lines which cross one another may contain a nonzero **S** vector, but $\nabla \cdot \mathbf{S} = 0$ and no flow of energy is taking place. In nonstatic problems where energy may be transferred, the vector **S** does relate to energy flow.

Example 10-6.1. Evaluate the power that is being conveyed along a coaxial cable by a d-c current I, which flows outward along the inner conductor and returns along the sheath. Neglect the resistance of the conductors.

Solution. The electric field of the cable is radial and is given by Eq. (3-59),

$$\mathbf{E} = E_r = \frac{V}{r \ln (b/a)} \tag{10-47}$$

Correspondingly, the magnetic field intensity is circumferential and is given by Eq. (6-12),

$$\mathbf{H} = H_\theta = \frac{I}{2\pi r} \tag{10-48}$$

Observe that the angle between **E** and **H** is 90°. Therefore,

$$\mathbf{S} = S_z = \frac{VI}{2\pi r^2 \ln (b/a)} \tag{10-49}$$

and the direction of S is that of the current in the positive conductor.

The power flow, which in this case is confined to the region between the conductors, is for any plane perpendicular to the axis of the cable,

$$P = \oint \mathbf{S} \cdot \mathbf{n}\, dS = \int_a^b S_z\, 2\pi r\, dr$$

$$P = \int_0^b \frac{VI}{2\pi r^2 \ln (b/a)}\, 2\pi r\, dr = VI \tag{10-50}$$

But this is the entire power transmitted by the cable. Therefore, the Poynting theorem locates the entire flow of energy in the field space between the conductors.

It is noted that, if the conductor resistance is not negligible, the potential difference V is no longer constant. Now, an axial component of $\mathbf{E}$ is necessary to keep

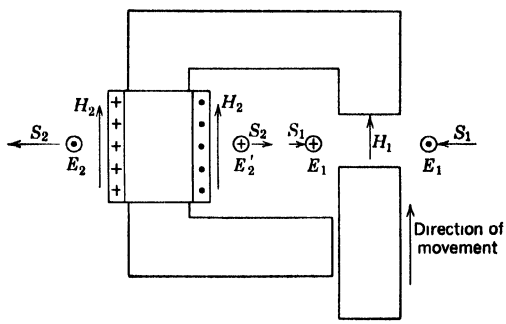


FIG. 10-6. A traction magnet, the armature of which is moving in the closing direction.

the current flowing in the conductors. This results in an inward radial component $-S_r$ which is exactly that required to supply the ohmic energy loss (see Prob. 10-5).

Example 10-6.2. A traction magnet, as illustrated in Fig. 10-6, has a constant current in the excitation coil. The armature is allowed to move in the closing direction. Discuss the operation of the device from considerations of the Poynting vector.

Solution. Before seeking a solution to this problem, several aspects of the fields should be discussed. It is assumed for convenience that the winding approximates a continuous distribution. From the magnetic circuit law, it follows that

$$NI = \left[\frac{L_i}{\mu_0 \mu_i A} + \frac{L_g}{\mu_0 A}\right] \Phi$$

where L_i and L_g denote the iron and air-gap path lengths respectively, and Φ denotes the total flux in the magnetic path of area A (that is, fringing at the air gap is being neglected). It follows from this expression that

$$H_i = \frac{\Phi}{\mu_0 \mu_i A} = \frac{NI}{(L_i + \mu_i L_g)}$$

which, for large μ_i shows that H_i in the iron core is small. Clearly, for an infinite permeability medium, the value of H in the medium is zero. By choosing

a closed path through the current sheet and into the iron core, and evaluating the line integral around this closed path, there will be found a field $\mathbf{H}_2$ which exists outside and adjacent to the coil, as shown in Fig. 10-6. If the value of H_i in the iron is small, then the field H_2 will have a magnitude

$$H_2 = nI$$

where n is the number of turns per unit length along the core. This expression may be written as

$$H_2 = J$$

where J is the linear-current density.

Consider the situation which exists when the armature is stationary. Owing to the constant current in the excitation coil and fixed magnetic circuit, no E fields will exist anywhere, since in the present case $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t = 0$. In this case the Poynting vector is zero everywhere, and no energy is transformed or converted, although static magnetic energy is stored in the magnetic field of the system. As shown in Sec. 9-7, a force will be exerted on the armature.

Suppose now that the armature of the magnet moves in the closing direction. It will be assumed that the only effect of this is a decreasing reluctance of the magnetic path. (This neglects any effects due to the changing magnetization of the iron itself.) As a result of the decreasing reluctance of the magnetic path, the flux will increase. This means that a time-changing B field exists both in the air gap and in the core. Because of these time-changing B fields, E fields will be generated. These are designated as $\mathbf{E}_1$ and $\mathbf{E}_2$ in the figure, and in both cases, these are closed lines. The field $\mathbf{E}_1$ in the neighborhood of the air gap will be out of the paper to the right and into the paper to the left of the armature. In the neighborhood of the excitation winding, the field $\mathbf{E}_2$ will be into the paper on the right and out of the paper on the left.

Because of the $\mathbf{E}$ and $\mathbf{H}$ fields in the neighborhood of the excitation winding, there will be a flow of energy, which is specified by $\mathbf{S}_2$ in the figure. Observe that this energy flow is outward from the coil. Note also that there is an energy flow $\mathbf{S}_1$ into the air gap. Moreover, if the tangential component H_i along the surface of the iron is small, there will be very little energy flow from the coil into the core. Thus it is seen that energy is flowing from the coil and into the air gap. Actually, part of this energy is converted into mechanical energy, and part is added to the storage in the field (and principally in the air gap).

10-7. Poynting Theorem and the Slepian Vector. As noted above, the development of the Poynting theorem interprets only div $\mathbf{S}$ or $\oint \mathbf{n} \cdot \mathbf{S}\, dS$ and not the vector itself. Thus, the interpretation of

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

as a vector density of power flow must be taken as a postulate, since other vectors can be formed which would serve as well. That is, if $\mathbf{N}$ denotes a vector which is defined at every point such that its flux out of any volume is zero, that is, such that div $\mathbf{N}$ is everywhere zero, then the vector $\mathbf{S} + \mathbf{N}$, for which

$$\text{div } (\mathbf{S} + \mathbf{N}) = \text{div } \mathbf{S}$$

would be an acceptable form, as well as **S**, to represent the flow of energy. Thus, depending on the choice of **N**, an unlimited number of possible Poynting vectors exist.

An interesting choice of **N** was discussed by Slepian,[1] his choice being

$$\mathbf{N} = \text{curl } \phi\mathbf{H}$$

where ϕ is the electric potential of the charges in the field. Thus, we now examine the vector

$$\mathbf{S}' = \mathbf{S} + \text{curl } \phi\mathbf{H}$$

or

$$\mathbf{S}' = \mathbf{E} \times \mathbf{H} + \nabla \times (\phi\mathbf{H}) \tag{10-51}$$

Use is now made of the fact that

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned} \tag{10-52}$$

from which it follows that

$$\mathbf{S} = \left[\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\right) \times \mathbf{H}\right] + \nabla \times (\phi\mathbf{H})$$

But the vector quantity

$$\nabla \times (\phi\mathbf{H}) = \phi\nabla \times \mathbf{H} + \nabla\phi \times \mathbf{H}$$

so that

$$\begin{aligned} \mathbf{S}' &= \left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\right) \times \mathbf{H} + \phi\nabla \times \mathbf{H} + \nabla\phi \times \mathbf{H} \\ &= \phi\nabla \times \mathbf{H} - \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{H} \end{aligned}$$

or, finally,

$$\mathbf{S}' = \phi\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) - \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{H} \tag{10-53}$$

Thus, using the Slepian vector, the expression for power flow is

$$P = \oint (\mathbf{S}' \cdot \mathbf{n})\, dS = \oint \left[\phi\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) - \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{H}\right] \cdot \mathbf{n}\, dS$$

which may be written in the form

$$P = \int_\tau \left[\nabla\phi \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\right] d\tau - \oint \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{H} \cdot \mathbf{n}\, dS \tag{10-54}$$

Example 10-7.1. A two-wire line feeds a resistive load. Find an expression for the power according to Eq. (10-54).

[1] J. Slepian, Energy Flow in Electric Systems—the VI Energy-flow Postulate, *Trans. AIEE*, **61**:835 (1942).

Solution. Each of the terms in Eq. (10-54) will be examined separately. The first term, which is $\int \nabla\phi \cdot \mathbf{J}\, d\tau$, is written in the form

$$\int_\tau \nabla\phi \cdot \mathbf{J}\, d\tau = I \int \nabla\phi \cdot d\mathbf{l} = IV$$

since over the volume over which the current exists $d\tau = a\, dl$.

The second term may be transformed into the form

$$\int \nabla\phi \cdot \frac{\partial \mathbf{D}}{\partial t}\, d\tau = -\int \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}\, d\tau = -\frac{1}{2}\frac{\partial}{\partial t}\int \mathbf{E} \cdot \mathbf{D}\, d\tau$$

which is a measure of the capacitive power into or out of the electric field surrounding the conductors.

The third term $\oint (\partial \mathbf{A}/\partial t) \times \mathbf{H} \cdot \mathbf{n}\, dS$ is of the form given in Eq. (10-45) and accounts for the inductive power into or out of the magnetic field around the conductor.

It is noted, therefore, that, even though the product of VI in circuit theory and $\mathbf{E} \times \mathbf{H}$ in field theory measures power flow, the two are not equal in general, owing to the inductive and capacitive power that is transferred into or out of the magnetic and electric fields around the conductor. Often the difference is very small, and in the static problem the difference is zero. The importance of the terms will actually depend on the relative magnitudes of the power being measured and the energy being stored.

PROBLEMS

10-1. The electric vector of a certain radio wave is vertical and has a magnitude of 30 μv/m. The frequency of the wave is 10^6 cps. Calculate the maximum value in microamperes of the total displacement current through a horizontal area of 1.5 m^2.

10-2. For what frequency does each of the following materials have unity for the ratio of displacement to conduction currents, $\epsilon\omega/\gamma$?

Copper, conductivity 5.8×10^7 mho/m, permittivity $\epsilon = \epsilon_0$

Ordinary water, conductivity 4.0×10^{-3} mho/m, permittivity $\epsilon_r = 81$

Marble, resistivity 10^8 ohm-m, permittivity $\epsilon_r = 8$

10-3. A radio wave for which the electric vector is nearly vertical travels with approximately the speed of light.

a. Calculate the magnetic field intensity at a point where the electric field intensity is 10 μv/m.

b. Calculate the maximum magnetic flux linking a square-loop antenna, 3 ft on a side.

c. In what direction must the loop be oriented in order to include this flux?

d. Draw a diagram showing E, H, v, and the loop thus oriented.

10-4. A loop antenna consists of 10 turns, each having an area of 0.75 m^2. A radio wave having a frequency of 1 Mc induces a sinusoidally varying emf of 100 μv in this antenna. Calculate the magnetic field intensity H of the radio wave.

10-5. A straight, isolated copper wire of diameter D, conductivity γ, carries a steady current I. Integrate Poynting's vector over the entire boundary of a piece of wire of length L, and compare with the Joule heat developed in this wire.

10-6. A 10-Mc plane wave traveling in free space has an amplitude of $E_0 = 50\ \mu$v/m.

a. What is the average energy density of the wave?

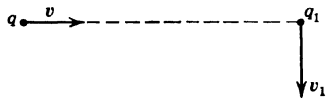
b. What is the peak energy density?

c. What is the average Poynting vector?

d. What is the peak Poynting vector?

e. How much energy is contained in a cube 10 km on a side?

10-7. Consider two charges moving in paths at right angles to each other at the instant when they are disposed, as shown.



a. It might be expected from Newton's third law that the force at each charge due to the influence of the other should be equal. Show that this is not true for the situation given.

b. What explanation can you put forward for the apparent violation of this basic principle of physics? [See J. M. Keller, Newton's Third Law and Electrodynamics, *Am. J. Phys.*, **10**:302 (1942); L. Page and N. I. Adams, Jr., Action and Reaction between Moving Charges, *Am. J. Phys.*, **13**:141 (1945).]

APPENDIX

SUMMARY OF VECTOR FORMULAS

1. $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$

2. $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}(B_z A_y - B_y A_z) + \mathbf{j}(B_x A_z - B_z A_x) + \mathbf{k}(B_y A_x - B_x A_y)$

3. $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

4. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$

5. $\text{grad} = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}$

6. $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

7. $\nabla(UV) = U(\nabla V) + V(\nabla U)$

8. $\nabla \cdot (U\mathbf{A}) = \mathbf{A} \cdot (\nabla U) + U(\nabla \cdot \mathbf{A})$

9. $\nabla \times (U\mathbf{A}) = \nabla U \times \mathbf{A} + U\nabla \times \mathbf{A}$

10. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$

11. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

12. $\nabla \times \nabla U \equiv 0$

13. $\nabla \cdot \nabla \times \mathbf{A} = 0$

If $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ $\mathbf{r}_0 = \mathbf{r}/r$:

14. $\nabla \cdot \mathbf{r} = 3$

15. $\nabla \times \mathbf{r} = 0$

16. $\nabla r = \mathbf{r}_0$

17. $\nabla \dfrac{1}{r} = -\dfrac{\mathbf{r}_0}{r^2}$

If τ is a volume bounded by surface S and $d\mathbf{a}$ is normal to S and directed outward:

18. $\displaystyle\int_\tau \nabla u \, d\tau = \int_S u \, d\mathbf{a}$

19. $\displaystyle\int_\tau \nabla \cdot A \, d\tau = \int_S \mathbf{A} \cdot d\mathbf{a}$

20. $\displaystyle\int_\tau \nabla \times \mathbf{A} \, d\tau = \int_S d\mathbf{a} \times \mathbf{A}$

If S is a two-sided surface bounded by contour C, with $d\mathbf{a}$ normal to S:

21. $\displaystyle\int_S d\mathbf{a} \times \nabla \mathbf{u} = \oint_c u \, d\mathbf{l}$

22. $\displaystyle\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint_c \mathbf{A} \cdot d\mathbf{l}$

Theorem of uniqueness: "A vector field is uniquely determined if the divergence and curl are specified and if the normal component of the vector is known over a closed surface, or if the field vanishes (at least as rapidly as $1/r^2$) at infinity."

Differential operators

Gradient

$$\text{Rect. grad } \phi = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}$$

$$\text{Cyl. grad } \phi = \mathbf{a}_r\frac{\partial \phi}{\partial r} + \mathbf{a}_\psi\frac{1}{r}\frac{\partial \phi}{\partial \psi} + \mathbf{a}_z\frac{\partial \phi}{\partial z}$$

$$\text{Sph. grad } \phi = \mathbf{a}_r\frac{\partial \phi}{\partial r} + \mathbf{a}_\psi\frac{1}{r}\frac{\partial \phi}{\partial \psi} + \mathbf{a}_\theta\frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \theta}$$

Divergence

$$\text{Rect. div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\text{Cyl. div } \mathbf{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\psi}{\partial \psi} + \frac{\partial E_z}{\partial z}$$

$$\text{Sph. div } \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(E_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\psi}{\partial \psi}$$

Curl

$$\text{Rect. curl } \mathbf{E} = \mathbf{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\text{Cyl. curl } \mathbf{E} = \mathbf{a}_r\left(\frac{1}{r}\frac{\partial E_z}{\partial \psi} - \frac{\partial E_\psi}{\partial z}\right) + \mathbf{a}_\psi\left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right) + \mathbf{a}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\psi) - \frac{1}{r}\frac{\partial E_r}{\partial \psi}\right]$$

$$\text{Sph. curl } \mathbf{E} = \mathbf{a}_r\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(E_\psi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial E_\theta}{\partial \psi}\right] + \mathbf{a}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial E_r}{\partial \psi} - \frac{1}{r}\frac{\partial}{\partial r}(rE_\psi)\right] + \mathbf{a}_\psi\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta}\right]$$

Laplacian

$$\text{Rect. } \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\text{Cyl. } \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \psi^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\text{Sph. } \nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial \psi^2}$$

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