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MECHANICS

STATICS

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MECHANICS

STATICS

AN ELEMENTARY TEXT-BOOK

THEORETICAL AND PRACTICAL

BY

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table. On this table are placed all the apparatus for the experiments which are to be performed. Thus for a class of 20 there would be 10 tables and 10 specimens of each of the pieces of apparatus. With some of the more elaborate experiments this plan is not possible. For them the class is taken in groups of five or six, the demonstrator in charge performs the necessary operations and makes the observations, the class work out the results for themselves.

It is with the hope of extending some such system as this in Colleges and Schools that I have undertaken the publication of the present book and others of the Series. My own experience has shewn the advantages of such a plan, and I know that that experience is shared by other teachers. The practical work interests the student. The apparatus required is simple; much of it might be made with a little assistance by the pupils themselves. Any good-sized room will serve as the Laboratory. Gas should be laid on to each table, and there should be a convenient water supply accessible; no other special preparation is necessary.

The plan of the book will, I hope, be sufficiently clear; the subject-matter of the various Sections is indicated by the headings in Clarendon type; the Experiments to be performed by the pupils are shewn thus:

EXPERIMENT 1. *To verify by experiment the parallelogram of forces.*

These are numbered consecutively. Occasionally an account of additional experiments, to be performed with the same apparatus, is added in small type. Besides this the small-type articles contain some numerical examples worked out, and, in many cases, a notice of the principal sources of error in the experiments, with indications of the method of making the necessary corrections. These latter portions may often with advantage be omitted on first reading. Articles or Chapters of a more advanced character, which may also at first be omitted, are marked with an asterisk.

I believe it to be desirable that a student should commence the study of Mechanics with Kinematics and Kinetics, and have therefore arranged the book on this plan. At the same time it will, I hope, be found that the Statics is independent of the other part of the subject, though at the cost of some repetition. It will be possible therefore for a teacher to take it before the Kinematics.

I have to thank many friends for help. Mr Wilberforce and Mr Fitzpatrick have assisted in arranging and devising many of the experiments. Mr Fitzpatrick has also read all the proofs. My pupil, Mr G. G. Schott of Trinity College, collected for me many of the Examples, while Mr Green of Sidney College has most kindly worked through all the Examples and furnished me with the answers.

The illustrations have for the most part been drawn by Mr Hayles from the apparatus used in the class.

R. T. GLAZEBROOK.

CAVENDISH LABORATORY,
January, 1895.

THE issue of a Second Edition has given an opportunity for the correction of various misprints. In other respects the book is unchanged.

November, 1895.

CONTENTS.

If we have to consider the motion due to a force so measured we must remember that the weight of a gramme contains g (981) dynes, where g denotes the acceleration of a falling body in centimetres per second per second, while the weight of a pound contains g (32·2) poundals, g being in this case measured in feet per second per second.

4. Resultant Force. We proceed now to find the resultant of two or more forces impressed on a particle.

DEFINITION. *If two or more forces P, Q... be impressed on a rigid body, the actual acceleration of the body is found by compounding the accelerations communicated by each force separately. If a single force R can be found which, when impressed alone, will communicate to the body this acceleration, this force is called the Resultant of the two or more forces, and these forces are called its components.*

PROPOSITION 1. *When two or more forces are impressed on a particle in the same direction their resultant is the sum (the difference if the forces act in opposite directions) of the forces.*

For let OA, Fig. 1, represent the one force P and AB the second Q, then OA and AB represent also the accelerations of a unit mass on which these forces are impressed. The resultant acceleration is (Dyn. § 29) represented by OB. Thus the resultant force R is represented by OB, and since

Fig. 1.

$$OB = OA + AB,$$
$$R = P + Q.$$

Similarly if the forces be impressed in opposite directions, we have, Fig. 2, $OB = OA - AB$.

Thus $R = P - Q$.

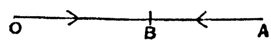

Fig. 2.

If the lines of action of the forces P and Q are not in the same straight line we find the resultant by the parallelogram law.

5. Parallelogram of Forces.

PROPOSITION 2. *If two forces represented in direction and magnitude by two straight lines OA, OB be impressed on a* *particle their resultant is represented by OC the diagonal through O of the parallelogram which has OA and OB for adjacent sides.*

Let OA, OB, Fig. 3, represent two forces P and Q impressed on a particle. Complete the parallelogram $AOBC$ and draw the diagonal OC. Then OC shall represent the resultant force R.

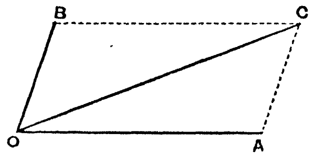

Fig. 3.

For OA, OB represent also the accelerations of a particle of unit mass on which the forces P, Q are each separately impressed; and by the parallelogram of accelerations OC represents the resultant acceleration of such a particle; but the resultant acceleration measures the resultant impressed force.

Hence OC the diagonal of the parallelogram represents R the resultant impressed force.

Thus forces are compounded and resolved according to the parallelogram law and Propositions 9 to 13 of the Kinematics, relating to displacements, apply equally to forces.

We may notice that the resultant force does not depend on the mass of the particle. If in Fig. 3 above, the mass of the particle be m, and the accelerations corresponding to P, Q, and R be p, q, r; then since P is equal to mp, Q to mq, and R to mr, the lines OA, OB represent mp and mq respectively. Thus OC represents mr on the same scale, hence it represents the resultant force R.

The proof of the parallelogram of forces depends therefore on that of the parallelogram of accelerations.

6. Experiments on the Parallelogram of Forces.

The parallelogram of forces can be verified in various ways by direct experiment; we shall describe two such experiments. A student who has difficulty in following the dynamical proof may base his acceptance of the proposition on the direct results of the experiments. A statical proof by Duchayla is often given; there are many reasons however why its use should be avoided and we shall not include it.

In many statical experiments the impressed force is measured by the weight of a body suspended by a string from some part of the apparatus, and it is often necessary to vary this weight. A convenient arrangement is shewn in Fig. 4. An iron rod about 15 cm. in length has a hook at the upper end; to the lower end a flat circular disc is rivetted and the whole is adjusted to have some definite mass such as 1 lb. or, if C.G.S. units are being employed, $\frac{1}{2}$ a kilogramme. The weights take the form of flat circular discs of iron or brass; each disc has a slot cut out as shewn in the figure, the slot reaches to just beyond the centre and is wide enough to admit the vertical rod which supports the scale-pan. Two sizes of "weights," say pounds and $\frac{1}{2}$ pounds or $\frac{1}{2}$ and $\frac{1}{4}$ kilos., will be found convenient; when in use they rest one on the top of the other on the scale-pan forming a pile through the centre of which the supporting rod runs; thus a force equal to the weight of a definite number of half-pounds is easily applied.

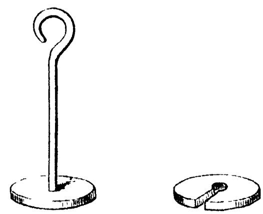

Fig. 4.

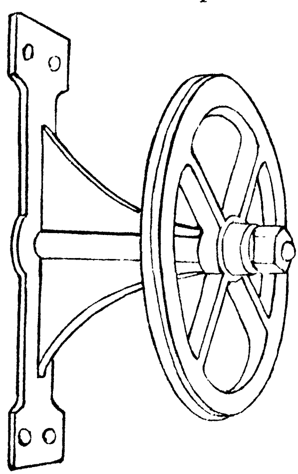

Fig. 5.

In other experiments the force is most easily applied and measured by means of a spring balance.

A useful form of pulley is illustrated in Fig. 5.

EXPERIMENT 1. *To verify by experiment the parallelogram of forces.*

(*a*) The apparatus required for this is illustrated in Fig. 6.[1]

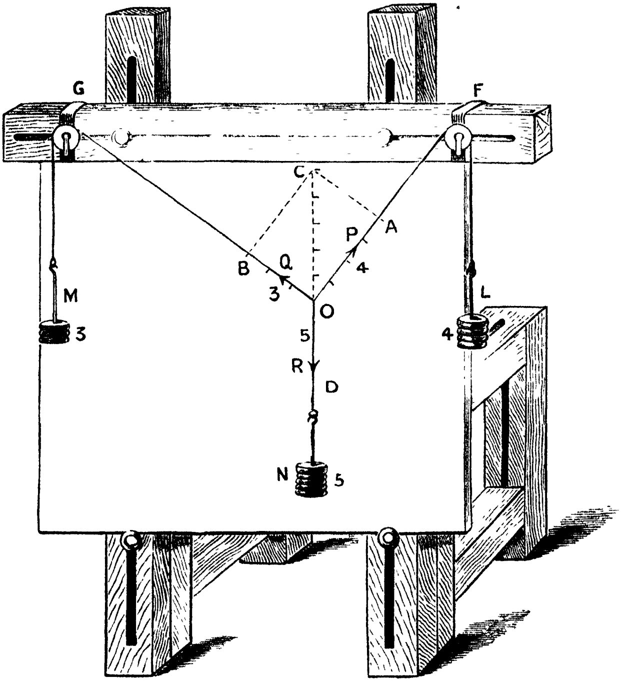

Fig. 6.

Two pulleys *F*, *G* are attached to a horizontal support. A

[1] In the figure the apparatus is shewn supported by a Willis framework. Such a framework consists of a number of bars which can be secured together in various positions by suitable screw bolts, and is very useful for various statical experiments. In a laboratory where a large number of sets of apparatus for the same experiment are required, it is

string AOB passes over these and carries two of the scale-pans just described: these are shewn at L and M. A second string OD knotted at O to the first carries a third pan N.

Some convenient number of weights is put on each scale-pan and the whole system is allowed to come into equilibrium.

Let us suppose the total weights supported at L, M, N respectively including the weight of the scale-pans to be P, Q and R pounds-weight, these weights measure the tensions of the respective strings. Thus forces of P, Q and R lb.-weight act along OA, OB and OD respectively. Since there is equilibrium R is clearly equal and opposite to the resultant of the other two forces P and Q.

Now adopt some convenient length to represent the unit of force, e.g. represent a force of 1 lb.-weight by a length of 10 centimetres.

Draw on the board lines OA, OB, OD, parallel to the strings and measure off along OA and OB lengths OA and OB to represent the forces P and Q; complete the parallelogram $AOBC$ and join OC. Measure the length of OC. It will be found to represent in magnitude the force R on the same scale as OA and OB represent P and Q. Place a straight-edge against OC, it will be found that CO when prolonged is in the same straight line as OD the line of action of R. Thus OC represents R in magnitude but is opposite to it in direction, and since the three forces P, Q, R are in equilibrium, R must be opposite to the resultant of P and Q; hence OC represents the resultant of P and Q represented by OA and OB respectively, and OC is the diagonal of the parallelogram of which OA and OB are adjacent sides. Thus the parallelogram of forces is verified.

By taking along OD a length OD to represent the force R, and constructing a parallelogram with OA and OD as sides, we could shew that Q is represented by the diagonal of this parallelogram.

convenient to have a board fastened to the walls, but projecting some little distance from them, and running round the room at some suitable height. Apparatus such as pulleys, etc. can be secured to this and the weights conveniently suspended from them without coming in contact with the walls. Arrangements should be made for supporting a drawing-board behind the strings which carry the weights, in order to solve questions easily by graphical construction.

In the figure as drawn P, Q, R are forces of 3, 4 and 5 lb. weight. It will be noticed that in this case the angle AOB is a right angle and that since

$$5^2 = 3^2 + 4^2,$$

we have $$R^2 = P^2 + Q^2.$$

(*b*) Knot three strings together at O, Fig. 7, and attach

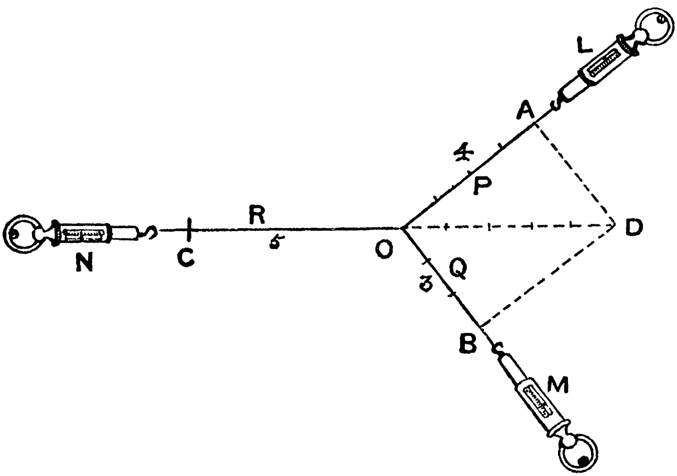

Fig. 7.

their ends to three spring balances L, M, N. Fix the balances to hooks on the edges of a drawing-board in such a way that the strings may be all drawn tight and the balances stretched. The readings of the balances will give us the forces acting along the strings, let them be P, Q, R respectively. Mark off along OL, OM and ON lengths OA, OB, OC to represent the forces P, Q, R; thus if the balances read in lb. we might take a length of 1 inch to represent 1 lb. weight, a force of P lb. weight is then represented by a line P inches in length. Draw lines OA, OB, OC on the paper under the string to represent these forces; then by constructing a parallelogram as before with OA

and OB as sides, we can shew that OC is equal and opposite to the diagonal of this parallelogram, the diagonal represents the resultant of P and Q.

We may use this construction to verify another important formula. Measure with a protractor the angles BOC, COA and AOB opposite respectively to the forces P, Q and R. Look out in a trigonometrical table the sines of these angles and then calculate the values of the fractions

$$\frac{P}{\sin BOC}, \quad \frac{Q}{\sin COA} \text{ and } \frac{R}{\sin AOB},$$

the ratio that is of each force to the sine of the angle between the other two.

It will be found that the ratios are all equal. We thus have the equations

$$\frac{P}{\sin BOC} = \frac{Q}{\sin COA} = \frac{R}{\sin AOB}.$$

Thus in the case shewn in Figure 7 in which

$$P = 3, \quad Q = 4 \text{ and } R = 5,$$

we find $BOC = 143°$, $COA = 127°$ and $AOB = 90°$.

Hence $\sin BOC = \sin 143° = \sin (180° - 143°) = \sin 37°$
$= \cdot 6$ approximately.

Also $\sin COA = \cdot 8$; $\sin AOB = 1$.

Hence
$$\frac{P}{\sin BOC} = \frac{3}{\cdot 6} = 5,$$
$$\frac{Q}{\sin COA} = \frac{4}{\cdot 8} = 5,$$
$$\frac{R}{\sin AOB} = \frac{5}{1} = 5.$$

Thus the relation is verified.

7. Further experiments on the equilibrium of forces.

The spring balance may be employed in various other experiments to measure force and verify the results of theory.

Thus in Fig. 8 a weight is supported by two strings each

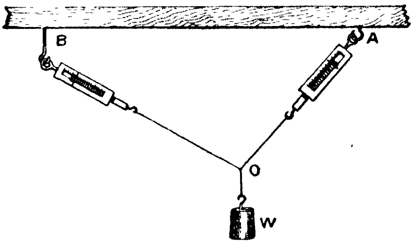

Fig. 8.

of which is attached to a spring balance. The balances are fastened to two points A and B, thus their readings give the tensions in the strings; if a parallelogram be constructed with its sides representing these tensions the diagonal will be vertical and will represent the weight. Moreover we should find in this case that each of the tensions is greater than the weight.

In a similar manner we may support a weight as in Fig. 9 by three or more strings, each of which is attached to a spring balance: the readings of the balances give the tensions, and if,

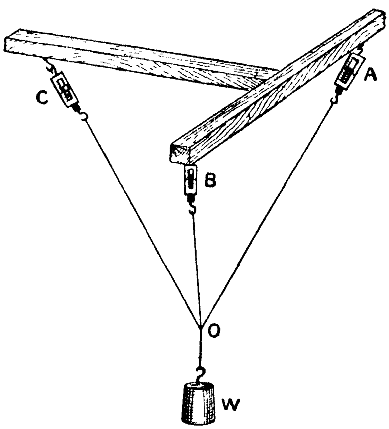

Fig. 9.

starting from any point, we construct a polygon whose sides represent the tensions in direction and magnitude, it will be found that the line joining the starting point to the extremity of the last line so drawn is vertical and represents the weight supported.

8. Composition and Resolution of Forces. Since forces like displacements are combined according to the parallelogram law, the various propositions which have been given for the composition and resolution of displacements apply to forces; for the sake of completeness in this part of the subject we repeat them here. We will first put the parallelogram of forces into a slightly different form.

In order to find the resultant of two forces P, Q acting at a point we draw OA, OB, as in Fig. 3 above, to represent the forces and complete the parallelogram $AOBC$. We have seen that OC is the resultant. Now the position of C can be found somewhat more simply: in the figure AC is equal and parallel to OB, hence AC will represent the force Q in magnitude and direction though not in point of application, for both forces P and Q act at O. We may then clearly find the point corresponding in any given case to C thus.

From O, Fig. 10, the point of action of the forces draw OA to represent the force P, from A the extremity of this line draw AC to represent the force Q in magnitude and direction. Join OC, then OC represents the resultant of P and Q in point of action, magnitude and direction. For by completing the parallelogram by drawing a line from O equal and parallel to AC and another line through C parallel to AO, it is clear that OC is the diagonal of a parallelogram whose two sides meeting at O represent the forces; hence OC represents the resultant of P and Q.

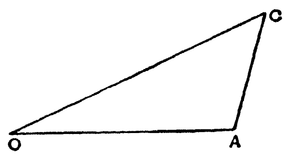

Fig. 10.

This construction can be generalized thus.

Proposition 3. *To find by a graphical construction the resultant of a number of forces impressed on a particle.*

Let OA, OA', OA'', etc. Fig. 11, represent the forces P, P', P'', etc. From A draw AB equal and parallel to OA' to represent Q in magnitude and direction. Then OB is the resultant of P and P'. From B draw BC equal and parallel to OA'' to represent P'' in magnitude and direction: then OC is the resultant of forces represented by OB and BC, and OB represents the resultant of P and P'; hence OC represents the resultant of P, P' and P''. Proceeding in this way we find the resultant of any number of forces acting at a point; for if L is the last point found, then the resultant is OL.

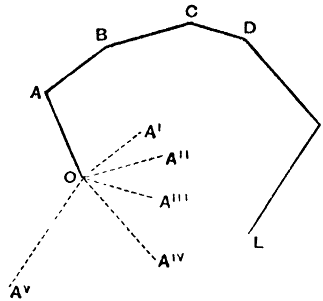

Fig. 11.

Corollary. If L coincide with O the resultant is zero and the forces are in equilibrium. In this case the forces are represented in direction and magnitude by the sides of a closed polygon taken in order and we have the result that:

If a number of forces impressed on a particle be represented in direction and magnitude by the sides of a closed polygon taken in order, the forces are in equilibrium.

This proposition is called the Polygon of forces.

A special case of this is the **Triangle of forces**, of this on account of its importance we give a formal proof.

Proposition 4. *If three forces impressed on a particle be represented in direction and magnitude by the sides of a triangle taken in order, the particle is in equilibrium.*

For let the sides OA, AC, CO, Fig. 12, taken in order, represent in direction and magnitude three forces P, Q, R

acting on a particle at O. By completing the parallelogram $OACB$ we see that OB is equal and parallel to AC and therefore represents Q completely. Hence the resultant of P and Q is represented by OC; this resultant is therefore equal and opposite to R, hence the particle is at rest.

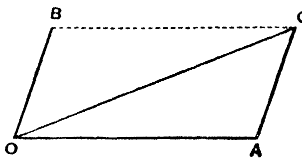

Fig. 12.

It should be noticed that the sides are to be taken in the same direction round the triangle. Thus forces represented by OA, AC, and OC are *not* in equilibrium.

The converse of the above proposition is also true.

PROPOSITION 5. *If three forces impressed on a particle are in equilibrium they can be represented in direction and magnitude by the sides of any triangle drawn so as to have its sides parallel to the forces.*

Let P, Q, R be three forces impressed on a particle at O which are in equilibrium. In Fig. 13 take OA to represent the force P, from A draw AC to represent Q in direction and magnitude and join OC. Then OC represents the resultant of P and Q, and since P, Q and R are in equilibrium, R must be equal and opposite to the resultant of P and Q, thus CO must represent R; hence the forces P, Q, R are represented by OA, AC and CO respectively.

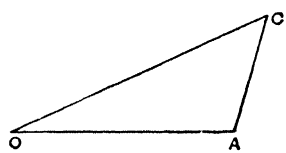

Fig. 13.

Again, any convenient length along OA may be taken to represent P, hence any triangle with its sides parallel to P, Q and R will represent the forces.

The converse of the polygon of forces is not true. All triangles whose sides are parallel to the forces are similar and have their corresponding sides proportional, hence any one of them may be taken to represent the forces: this is not the case for polygons. A number of polygons can be found whose sides represent the forces, all these polygons are similar; but *any* polygon with its sides parallel to the forces will *not* represent them.

The following examples illustrate this graphic method.

Examples. (1) *Find the resultant of forces of 2 to the North, 3 to the East, 3 to the South, and 4 to the West, impressed on a particle.*

Draw a vertical line OA (Fig. 14) upwards, 2 cm. in length, to represent the first force. Draw AB to the right, at right angles to OA, 3 cm. in length; BC downwards, at right angles to AB, 3 cm. in length; CD horizontal to the left, 4 cm. in length; then OD is the required resultant.

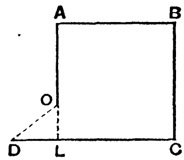

Fig. 14.

Also if OL be perpendicular on CD, it is clear that OL is 1 cm. and LD is also 1 cm. Hence OD is $\sqrt{2}$ cm. Thus the resultant force is $\sqrt{2}$ to the South-West. This example is the same as Example 1 on page 43, (Dynamics).

(2) *Six forces of 1, 9, 2, 7, 3, and 8 lb. weight respectively are impressed on a particle in directions parallel to the sides of a regular hexagon taken in order. Shew that the particle is in equilibrium.*

The adjacent sides of a regular hexagon make angles of 120° with each other. Take a line of 1 cm. to represent a force of 1 lb. wt., and draw AB 1 cm. in length to represent the first force, BC inclined at 120° to it 9 cm. in length to represent the second, CD 2 cm. in length to represent the third, and so on. If the figure be carefully drawn, the end of the sixth line representing the force of 3 lb. wt. will be found to coincide with A. The hexagon is a closed one, and the particle is in equilibrium. The student should construct this figure for himself to scale.

Aliter. The forces of 1 and 7 lb. weight acting in opposite directions are equivalent to a force of 6 lb. wt. acting in the same direction as the 7 lb. wt., the forces of 9 and 3 lb. wt. are equivalent to a force of 6 lb. wt. acting in the direction of the 9 lb. wt., the forces of 2 and 8 lb. wt. are equivalent to a force of 6 lb. acting parallel to the 8 lb.; thus we have three equal forces of 6 lb. acting away from the particle in directions inclined to each other at 120°, and these form a system in equilibrium.

(3) *The ends of two strings are secured to two fixed points L and M, and are knotted together at O; a 5 kilogramme weight is suspended from O; find by a graphical construction the tensions in the strings.*

Fix a drawing-board in a vertical plane behind the strings, and trace on the board their directions. Take some length (say 5 cm.) to represent a force equal to the weight of 1 kilogramme. From O draw OA (Fig. 15) vertically upwards 25 cm. in length, then OA represents the suspended weight. From A draw AB parallel to OM to meet the string OL in B. Let T_1, T_2 be the tensions in OL and OM; W the weight in direction AO.

Then the three forces T_1, T_2 and 5 kilos weight are parallel to the sides of the triangle OBA taken in order. They are therefore represented by its three sides. Measure the lengths of OB and BA in cm.

Then we have

$$\frac{T_1}{OB}=\frac{T_2}{AB}=\frac{W}{AO}=\frac{5}{25}=\frac{1}{5}.$$

Hence

$$T_1=\frac{OB}{5}\text{ kilos wt.},\quad T_2=\frac{AB}{5}\text{ kilos wt.}$$

In the figure drawn it is clear that

$T_1=6$ kilos wt., $T_2=4$ kilos wt.

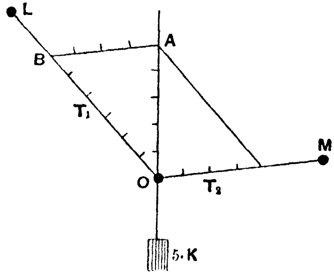

Fig. 15.

(4) *The bob of a pendulum weighing 5 kilos is pulled aside by a horizontal string until its thread is inclined at 30° to the vertical. Find the impressed force in the string and the tension of the pendulum thread.*

Let O (Fig. 16) be the point of suspension, A the pendulum when displaced, OC vertical. Draw AC horizontal meeting OC in C. The impressed forces are 5 kilos weight vertical, parallel therefore to OC, the tension T' of the horizontal string parallel to CA, and the tension T of the pendulum thread parallel to AO. The impressed forces therefore are proportional to the sides of the triangle AOC.

Also since the angle at O is 30° we have

$$AC=\tfrac{1}{2}AO;\quad OC=\tfrac{1}{2}\sqrt{3}\,AO,$$

and

$$\frac{5}{OC}=\frac{T'}{CA}=\frac{T}{AO}.$$

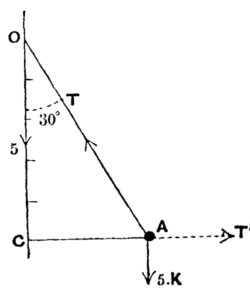

Fig. 16.

$$\therefore\ T=5\frac{AO}{OC}=\frac{10}{\sqrt{3}}\text{ kilos weight,}$$

$$T'=5\frac{AC}{OC}=\frac{5}{\sqrt{3}}\text{ kilos weight.}$$

(5) *Weights W_1, W_2, W_3 are attached to three points A_1, A_2, A_3 in a string the ends of which are secured to two fixed points A, B. The whole hangs in a vertical plane and the form taken by the string is drawn to scale. W_1 is known. Find the weights of W_2 and W_3 and the tensions of the parts of the string.*

Take a vertical line X_1X_2 (Fig. 17) to represent W_1. From X_1 draw

X_1O parallel to AA_1, and from X_2 draw X_2O parallel to A_1A_2. The three

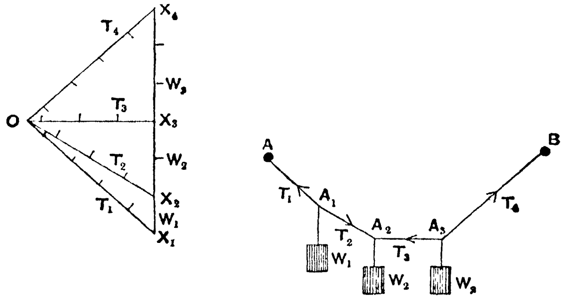

Fig. 17.

forces at A_1 are parallel to the sides of the triangle OX_1X_2, we have therefore

$$\frac{W_1}{X_1X_2} = \frac{T_1}{OX_1} = \frac{T_2}{OX_2}.$$

Thus T_1 and T_2 can be found by measuring OX_1 and OX_2. Draw OX_3 parallel to A_2A_3 meeting X_1X_2 produced in X_3; the three forces T_2, T_3 and W_2 at A_2 are parallel to the sides of the triangle X_2OX_3, they are therefore represented by these sides, thus

$$\frac{T_2}{OX_2} = \frac{T_3}{OX_3} = \frac{W_2}{X_3X_2}.$$

Similarly draw OX_4 parallel to A_3B and produce X_2X_3 to meet OX_4 in X_4. Then T_3, T_4 and W_3 are parallel to X_3O, OX_4 and X_4X_3.

Hence $$\frac{T_3}{X_3O} = \frac{T_4}{OX_4} = \frac{W_3}{X_4X_3}.$$

Thus T_4 and W_3 can be found.

Hence if W_1 be known, the values of W_2 and W_3 together with the tensions in the different parts of the string are given graphically.

In the diagram as drawn it will be found that

$$X_2X_3 = 2X_1X_2, \quad X_3X_4 = 3X_1X_2.$$

Hence if $W_1 = 1$ kilo, then $W_2 = 2$ kilos, $W_3 = 3$ kilos.

Also $T_1 = 4{\cdot}2$ kilos, $T_2 = 3{\cdot}7$ kilos, $T_3 = 3{\cdot}3$ kilos,

and $T_4 = 4{\cdot}6$ kilos.

Hence the tensions and two of the weights are found.

When two or more forces impressed on a particle are given in direction and magnitude it is possible to find expressions for their resultant. We have in the case of two forces to determine the diagonal of a parallelogram two of whose sides are given, while the case of more than two forces involves an extension of the same process.

Thus the propositions given on pp. 33—38 of the Dynamics with regard to displacements apply to forces. For the sake of completeness in this part of the book they are repeated here.

PROPOSITION 6. *To find an expression for the resultant of two forces at right angles.*

Let OA, OB, Fig. 18, represent two forces P, Q respectively at right angles to each other. Complete the rectangle $AOBC$. Let R be the resultant of P and Q, then R is represented by OC.

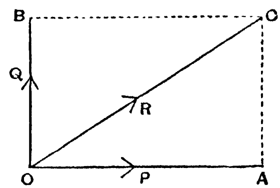

Fig. 18.

Since the angle OAC is a right angle we have

$$OC^2 = OA^2 + AC^2$$
$$= OA^2 + OB^2,$$
$$\therefore\ R^2 = P^2 + Q^2.$$

Hence $$R = \sqrt{\{P^2 + Q^2\}}.$$

PROPOSITION 7. *To find an expression for the resultant of two forces inclined to each other at any angle.*

Let OA, OB represent respectively two forces P, Q inclined to each other at an angle γ.

Complete the parallelogram $AOBC$. OC represents R the resultant of P and Q. Draw CD perpendicular to OA meeting OA produced, Fig. 19 (a), or OA, Fig. 19 (b), in D. Then $AOB = \gamma$; in Fig. 19 (a) the angle γ is less than a right angle; in Fig. 19 (b) it is greater.

Now in Fig. 19 (*a*),

$$OD = OA + AD = OA + AC \cos DAC$$
$$= OA + OB \cos AOB = P + Q \cos \gamma,$$
$$CD = AC \sin DAC = OB \sin \gamma = Q \sin \gamma.$$

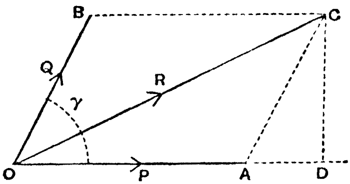

Fig. 19 (*a*).

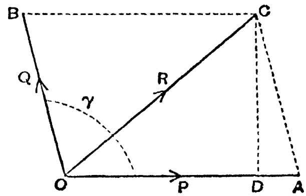

Fig. 19 (*b*).

In Fig. 19 (*b*),

$$OD = OA - AD = OA - AC \cos DAC$$
$$= OA - OB \cos (180 - \gamma) = OA + OB \cos \gamma$$
$$= P + Q \cos \gamma,$$
$$CD = AC \sin DAC = OB \sin (180 - \gamma) = Q \sin \gamma.$$

Hence in either case we have

$$R^2 = OC^2 = OD^2 + DC^2$$
$$= (P + Q \cos \gamma)^2 + Q^2 \sin^2 \gamma$$
$$= P^2 + Q^2 + 2PQ \cos \gamma;$$
$$\therefore \quad R = \sqrt{\{P^2 + Q^2 + 2PQ \cos \gamma\}}.$$

There are many special cases of this last proposition which can be solved by Geometry without reference to Trigonometry. Thus, suppose the angle between the two forces to be 45°.

Hence, constructing Fig. 20 as above, we have

$$AD^2 + CD^2 = AC^2 = Q^2.$$

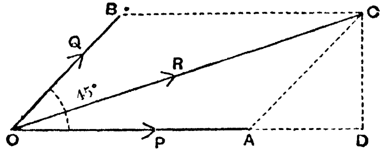

Fig. 20.

Also $$AD = DC;$$

$$\therefore\ AD = DC = \frac{Q}{\sqrt{2}}.$$

And $$OD = OA + AD = P + \frac{Q}{\sqrt{2}}.$$

Hence $$R^2 = OC^2 = OD^2 + DC^2$$

$$= \left(P + \frac{Q}{\sqrt{2}}\right)^2 + \frac{Q^2}{2}$$

$$= P^2 + Q^2 + PQ\sqrt{2}.$$

Or again, if $\gamma = 60°$, we have, fig. 21,

$$AD = \tfrac{1}{2}AC = \tfrac{1}{2}Q,\ \ CD = \frac{Q\sqrt{3}}{2}.$$

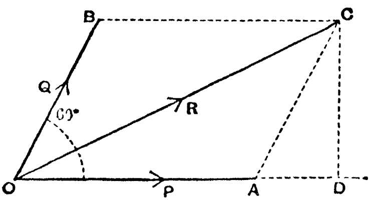

Fig. 21.

$$R^2 = \left(P + \frac{Q}{2}\right)^2 + \frac{3Q^2}{4}$$

$$= P^2 + Q^2 + PQ.$$

These are both given by the general formula by putting $\gamma = 45°$, $\cos\gamma = \sqrt{\tfrac{1}{2}}$ and $\gamma = 60°$, $\cos\gamma = \tfrac{1}{2}$.

If the two forces be equal the resultant bisects the angle between them; for, Fig. 22, if

$$OA = AC,$$

then $$\angle AOC = \angle ACO$$

$$= \angle BOC.$$

Join AB, cutting OC in D, then AB bisects OC at right angles. And

$$R = OC = 2OD = 2OA \cos AOC$$

$$= 2P\cos\tfrac{1}{2}\gamma.$$

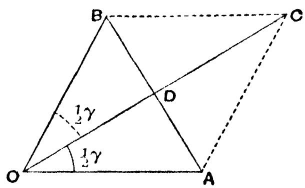

Fig. 22.

9. The Resolution of Forces. Just as we can combine or compound two or more forces and find their resultant, so conversely we can resolve a single force into a number of others, called its components, which are equivalent to it.

PROPOSITION 8. *To find, by a graphical construction, the components of a force in any two directions.*

Let OC, Fig. 23, be the given force, and LM, LN the two given directions. Through O draw OA parallel to LM and through C draw AC parallel to LN. These two forces OA, AC

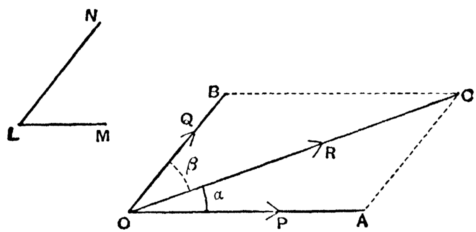

Fig. 23.

acting at O have OC for their resultant, hence OA, AC are components of OC and they are parallel respectively to LM and LN, that is, they are drawn in the given directions.

PROPOSITION 9. *To find an expression for the components of a force in two given directions.*

Let OC, Fig. 23, represent R the given force, and let OA, OB be the components in directions making angles α, β, respectively with OC.

Then
$$AOC = \alpha,$$
$$BOC = ACO = \beta.$$

Hence
$$OAC = 180 - (\alpha + \beta).$$

Now in the triangle OAC the sides are proportional to the sines of the opposite angles.

Hence $$\frac{OC}{\sin OAC} = \frac{OA}{\sin ACO} = \frac{AC}{\sin AOC},$$

$$\therefore \quad \frac{R}{\sin(\alpha+\beta)} = \frac{P}{\sin\beta} = \frac{Q}{\sin\alpha}.$$

Moreover from the figure $\alpha + \beta = \gamma$.

Hence $$P = R\frac{\sin\beta}{\sin\gamma},$$

$$Q = R\frac{\sin\alpha}{\sin\gamma}.$$

PROPOSITION 10. *To find the components of a force in two directions at right angles.*

Let OC, Fig. 24, represent the force R, OA, OB two directions at right angles in which the components are required.

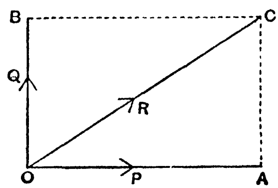

Fig. 24.

Let $AOC = \alpha$.

Draw CA, CB perpendicular on the two directions. Then OA, OB represent the components P, Q.

Also $\frac{OA}{OC} = \cos AOC = \cos\alpha$,

$$\therefore \quad OA = OC\cos\alpha.$$

Hence $$P = R\cos\alpha.$$

Again $\frac{OB}{OC} = \cos BOC = \sin AOC = \sin\alpha$,

$$\therefore \quad OB = OC\sin\alpha.$$

Hence $$Q = R\sin\alpha.$$

If we put $BOC = \beta$ we have clearly

$$OB = OC\cos\beta,$$

$$Q = R\cos\beta.$$

And in this case $\alpha + \beta = 90°$.

Thus, when a force is resolved into two others mutually at right angles, the component in each direction is found by multiplying the original force by the cosine of the angle between it and the direction of the component.

It must be remembered that this result is only true when the two components are at right angles.

Thus, let OA, OB (Fig. 25) be two components of OC at right angles. Draw OB' making an angle γ with OA' and through C draw CA' parallel to OB'. If now OC be resolved into two forces in directions OA and OB' inclined at an angle γ, the component in the direction OA is no longer OA but OA'.

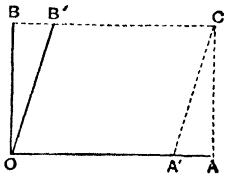

Fig. 25

A force represented by OA is $R \cos \alpha$, where α is the angle between OA and OC, that represented by OA' has not this value.

PROPOSITION 11. *To shew that if the components of a force be resolved in any direction, then the sum of these resolved parts is equal to the component of the original force resolved in this same direction.*

Take the case of a force R represented by OC, Fig. 26, which is resolved into two forces P, Q represented by OA and OB respectively.

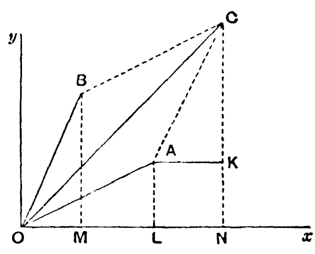

Fig. 26.

Draw Ox, Oy two lines at right angles through O, and draw AL, BM and CN perpendicular to Ox. Then if we suppose all the forces resolved in the directions Ox and Oy it is clear that OL represents the component of P, OM of Q and ON of R.

Draw AK parallel to Ox to meet CN in K.

Then $LN = AK$. And in the triangles BOM and CAK the sides are respectively parallel and OB is equal to AC.

Hence the triangles are equal. Therefore $OM = AK = LN$. Thus LN represents the component of Q in the direction of Ox.

But from the figure

$$ON = OL + LN = OL + OM.$$

Hence

Component of R in the direction $Ox =$

Component of P + Component of Q.

This proposition can readily be extended to the case of any number of forces.

Proposition 12. *To find an expression for the resultant of a number of forces impressed on a particle in given directions lying in one plane.*

Let P_1, P_2 ... be the forces and let them make angles a_1, a_2 ... with a fixed line Ox, Fig. 27, drawn through the point of action. Let Oy be perpendicular to Ox. Let R be the resultant force and θ the angle its direction makes with Ox.

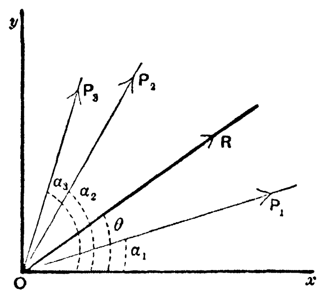

Fig. 27.

Resolve all the forces and the resultant in the two directions Ox and Oy. Then since the resultant is equivalent in its effect to the forces the components of the resultant in each of these two directions are respectively equal to the sum of the components of the forces in these two directions.

The component of the resultant along Ox is $R\cos\theta$, the components of the forces are $P_1 \cos a_1 \ldots$, $P_2 \cos a_2 \ldots$ respectively.

Hence

$$R\cos\theta = P_1\cos a_1 + P_2\cos a_2 + \ldots = \Sigma\{P\cos a\} \ldots(1),$$

where $\Sigma\{P\cos a\}$ means the sum of a number of quantities like $P\cos a$.

Again resolving parallel to Oy

$$R\sin\theta = P_1\sin a_1 + P_2\sin a_2 + \ldots = \Sigma\{P\sin a\} \ldots(2).$$

Thus remembering that $\sin^2\theta + \cos^2\theta = 1$, we have by squaring and adding

$$R^2 = [\Sigma\{P\cos\alpha\}]^2 + [\Sigma\{P\sin\alpha\}]^2$$
$$= \Sigma\{P^2\} + 2\Sigma\{P_1P_2\cos(\alpha_2 - \alpha_1)\},$$

while by dividing (2) by (1) we find

$$\tan\theta = \frac{\Sigma\{P\sin\alpha\}}{\Sigma\{P\cos\alpha\}}.$$

10. Equilibrium of Forces impressed on a particle. We have already found, Prop. 3, the conditions of equilibrium of a set of forces impressed on a particle. If a polygon be drawn whose sides represent the forces in direction and magnitude it will be closed. The same result can be expressed in symbols by the aid of the last proposition thus.

If the particle be in equilibrium the resultant of the forces is zero. Thus the components of the resultant in any two directions must also be zero. Hence the sum of the components of the forces in any two directions at right angles must be zero.

Hence

$$P_1\cos\alpha_1 + P_2\cos\alpha_2 + \ldots = 0 \text{ or } \Sigma(P\cos\alpha) = 0,$$

and $$P_1\sin\alpha_1 + P_2\sin\alpha_2 + \ldots = 0 \text{ or } \Sigma(P\sin\alpha) = 0.$$

This result is of course applicable to the case of three forces, but in this case we know in addition that the forces are represented by the sides of any triangle drawn parallel to their directions; hence for three forces we have the following theorem known as Lami's Theorem.

PROPOSITION 13. *When three forces impressed on a particle are in equilibrium each is proportional to the sine of the angle between the other two.*

Let the three forces be P, Q, R acting in directions OL, OM, ON respectively, Fig. 28.

Let ABC be a triangle whose sides are parallel to the forces, BC being parallel to OL, CA to OM and AB to ON.

Then from the figure

$$CAB = 180 - MON,$$
$$ABC = 180 - NOL,$$
$$BCA = 180 - LOM.$$

Hence $\sin CAB = \sin MON,$

$\sin ABC = \sin NOL,$

$\sin BCA = \sin LOM.$

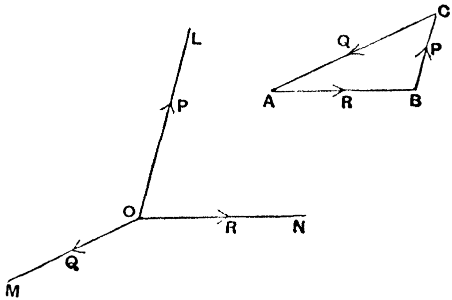

Fig. 28.

But the forces P, Q, R are proportional to the sides BC, CA and AB of the triangle ABC respectively.

Moreover the sides of a triangle are proportional to the sines of the opposite angles. Thus the forces are proportional to the sines of the angles of the triangle which are opposite to them.

Hence $$\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}.$$

Thus $$\frac{P}{\sin MON} = \frac{Q}{\sin NOL} = \frac{R}{\sin LOM}.$$

This theorem has already been verified by experiment (Exp. 1 (*b*).)

We may conclude then that in dealing with questions on the equilibrium of forces impressed on a particle:

(*a*) *The direct method of solution is to resolve the forces in any two convenient directions at right angles and equate to zero each set of components.*

(*b*) *If the forces be only three in number a graphical solution based on the triangle of forces can easily be obtained.*

The following examples will illustrate these various methods.

Examples. (1) *Find the resultant of two forces of* 3 *and* 4 *kilos weight respectively impressed on a particle at right angles.*

Let R be the resultant; then since the forces are at right angles

$$R^2 = 3^2 + 4^2 = 25,$$
$$R = 5 \text{ kilos weight.}$$

This is the result which we verified in Experiment **1.**

(2) *Find the resultant of two forces of* 10 *and* 5 *kilos weight acting at an angle of* 60°.

Let R be the resultant.

Then substituting in the formula

$$R^2 = P^2 + Q^2 + 2PQ \cos \gamma,$$

we have

$$R^2 = 5^2 \{1 + 2^2 + 2 \times 2 \times \tfrac{1}{2}\}$$
$$= 5^2 \times 7,$$
$$R = 5\sqrt{7} \text{ kilos weight.}$$

(3) *A force of* 10 *kilos weight is resolved into two equal forces mutually at right angles; find these forces.*

Since the components are equal, they are equally inclined to the resultant.

Hence the angle between each of them and the resultant is 45°.

Thus if P and Q be their values

$$P = 10 \cos 45° = \frac{10}{\sqrt{2}} \text{ kilos wt.,}$$
$$Q = 10 \sin 45° = \frac{10}{\sqrt{2}} \text{ kilos wt.}$$

(4) *A force of* 15 *kilos weight is resolved into two at right angles, the value of one of these is twice that of the other; find the forces.*

Let them be P and Q kilos weight.

Then
$$P = 2Q,$$
$$P^2 + Q^2 = 15^2,$$
$$Q^2(1 + 4) = 15^2,$$
$$Q^2 = 15 \times 3,$$
$$Q = 3\sqrt{5} \text{ kilos weight.}$$

Hence
$$P = 6\sqrt{5} \text{ kilos weight.}$$

(5) *The resultant of two forces of 3 kilos weight and 5 kilos weight is a force of 7 kilos weight; find the angle between the two.*

Let γ be the angle, then if R be the resultant of two forces P and Q inclined at an angle γ, we know that

$$R^2=P^2+Q^2+2PQ\cos\gamma,$$
$$7^2=5^2+3^2+2\times5\times3\cos\gamma,$$
$$30\cos\gamma=49-25-9=15,$$
$$\cos\gamma=\tfrac{1}{2},$$
$$\gamma=60^\circ.$$

Thus the angle required is 60°.

(6) *Forces equal to the weights of 6, 7, and 8 lb. are impressed on a particle in directions inclined to each other at 120°. Find their resultant.*

Three equal forces at angles of 120° are in equilibrium. The given forces are equivalent to forces respectively of 6 lb., (6+1) lb., and (6+2) lb.

The three forces of 6 lb. are in equilibrium, and may therefore be removed from consideration, and there are left forces of 1 and 2 lb. weight at an angle of 120°. Their resultant may be found graphically, or thus

$$R^2=1^2+2^2+2\times1\times2\times\cos120^\circ$$
$$=5-2=3.$$

Thus $$R=\sqrt{3}\text{ lb. wt.}$$

Or again.

Let R be the resultant force and let it make an angle θ with the direction of the force of 8 lb. weight.

Resolve all the forces parallel and perpendicular to the direction of the 8 lb. force.

$$R\cos\theta=8\cos0+7\cos120+6\cos120$$
$$=8\,.\,1+7\left(-\tfrac{1}{2}\right)+6\left(-\tfrac{1}{2}\right)$$
$$=8-3\tfrac{1}{2}-3=1\tfrac{1}{2}.$$
$$R\sin\theta=8\sin0+7\sin120-6\sin120$$
$$=8.0+7\frac{\sqrt{3}}{2}-6\frac{\sqrt{3}}{2}$$
$$=\frac{\sqrt{3}}{2}.$$
$$\therefore\ R^2=\frac{12}{4}=3.$$

Hence $$R=\sqrt{3}\text{ lb. weight.}$$
$$\tan\theta=\frac{\sqrt{3}}{3}=\frac{1}{\sqrt{3}}.$$
$$\therefore\ \theta=30^\circ.$$

Thus the resultant is a force of $\sqrt{3}$ lb. weight inclined at 30° to the force of 8 lb. weight.

(7) *A body weighing 5 lb. is supported by two strings, the tension in one string is 8 lb. weight and its direction is inclined at 30° to the horizon. Find the direction of the other string and the tension in it.*

(i) *Graphically.*

Draw AB (Fig. 29) vertically down, 5 cm. in length, to represent the weight, draw BC 8 cm. in length and at 30° to the horizon to represent the tension in the first string, and join CA. Then CA represents the direction of the second string, and the number of cm. in CA measures in lb. weight the tension in that string.

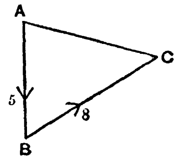

Fig. 29.

(ii) *By resolution of forces.*

Let T be the tension of the string OC, θ the angle it makes with the vertical.

Let OD be the first string and OA the direction of the 5 lb. weight.

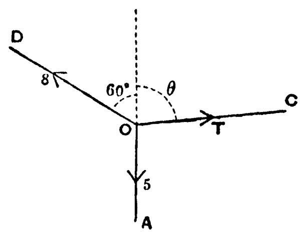

Fig. 30.

Resolving vertically

$$5 = T\cos\theta + 8\cos 60°.$$

Resolving horizontally

$$T\sin\theta = 8\sin 60°.$$

From the first equation

$$T\cos\theta = 1.$$

From the second

$$T\sin\theta = 4\sqrt{3}.$$

Hence

$$T^2 = 49,$$

$$T = 7 \text{ lb. weight},$$

and $$\tan\theta = 4\sqrt{3}.$$

(8) *The resultant R of two forces P, Q impressed on a particle is equal to P and at right angles to it. Find the force Q.*

Let AB (Fig. 31) represent P, BC equal to AB and at right angles to it will represent R. Join CA. Then BC is the resultant of forces represented in magnitude and direction by AB and AC acting at B.

Thus a line through B parallel and equal to AC will represent Q.

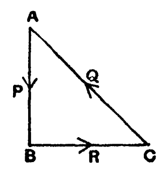

Fig. 31.

But $$AC^2 = AB^2 + BC^2$$

$$= 2AB^2.$$

Therefore $$Q^2 = 2P^2.$$

Hence $$Q = P\sqrt{2}.$$

(9) *Two forces impressed on a particle are represented respectively by* λOA *and* μOB, *A and B being fixed points and* λ *and* μ *constants.*

C is a point in AB such that λAC *is equal to* μBC. *Shew that the resultant of the forces is* $(\lambda + \mu)\, OC$.

By the triangle of forces a force λOA along OA (Fig 32) is equivalent to λOC along OC and λCA acting at O parallel to CA. Again μOB is equivalent to μOC along OC and μCB at O parallel to CB.

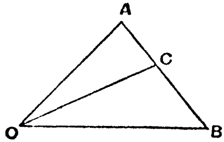

Fig. 32.

Thus the two given forces are equivalent to $(\lambda + \mu)\, OC$ along OC together with λCA and μBC in opposite directions parallel to AB. These last two are equal, they therefore balance and may be removed. Hence the resultant is $(\lambda + \mu)\, OC$.

(10) *Explain the action of the wind in propelling a ship.*

Let AB (Fig. 33) represent the direction of the ship's keel; CD the direction of the sail which we suppose to be flat. Let the pressure of the wind be equivalent to a force P acting on the sail in the direction indicated. Resolve this force into two components, one R at right angles to the sail, the other T along the sail. This last component produces little or no effect, and we may neglect it; it is only the component perpendicular to the sail which we need to consider. This force R acts on the ship through the mast. We may resolve R into two components, the one X parallel to the keel, the other Y at right angles to the keel.

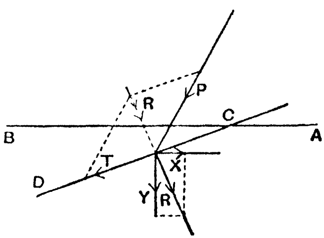

Fig. 33.

The resistance offered by the water to motion in a direction at right angles to the keel is so great that the component Y is almost balanced by it; the ship is built so that the water may offer a small resistance to motion parallel to the keel, and the ship moves in this direction under the impressed force X. The effect of the force in tending to turn the ship round can only be considered later.

EXAMPLES.

TRIANGLE AND PARALLELOGRAM OF FORCES.

1. Forces represented by the weights of 10 lb. and 15 lb. respectively act at a point in northerly and easterly directions. Find the magnitude and direction of their resultant.

2. Find the magnitude of the resultant of two forces $12P$ and $5P$ when they act at a point and in directions at right angles to one another.

3. ABC is a triangle, D, E, and F the middle points of BC, CA, AB respectively. Forces acting at a point are represented in direction and magnitude by the lines AB, AC, BE; shew that their resultant will be similarly represented by $3FD$.

4. The sum of two forces is 36 lb. wt. and the resultant which is at right angles to the smaller of the two is 24 lb. wt. Find the magnitude of the forces.

5. The difference of two forces is 8 lb. wt. and the resultant, which is at right angles to the smaller of the two, is 12 lb. wt. Find the magnitude of the forces.

6. Can three forces which are in the proportion of 7, 10 and 17 keep a point at rest?

7. Forces represented in magnitude and direction by the diagonals of a parallelogram act at one of the corners, what single force will counteract them?

8. Shew by means of a diagram how to find the part of the pressure of the wind which is available for urging on a ship when the wind blows very nearly from the direction in which the ship is going.

9. Explain how the law of composition of forces may be deduced from the second law of motion.

10. Two forces P and Q have a resultant R equal to P. Draw a diagram representing such a system and shew by means of it that the resultant of two forces equal and parallel to P and R respectively would act at right angles to Q.

11. Shew how to find the resultant of two forces acting at the same point and explain how to verify the result by experiment.

12. The wind is blowing from the north-east. Explain with a diagram its action in propelling a ship towards the north.

13. Resolve a force of 15 lb. wt. into two forces each making with it an angle of 30°.

14. Shew how to place three forces which are in the ratio of 3, 4 and 5, so that they may keep a particle at rest.

15. Describe an experiment to prove the parallelogram of forces

16. A pendulum consisting of a bob weighing 1 kilogramme at the end of a string 1 mètre long is drawn aside until the bob is 25 cm. from the vertical through the point of support, and is held in this position by a horizontal string.

Find the forces on the bob (1) when in this position, (2) just after the horizontal string is cut, (3) as the bob swings through its lowest position.

17. Five forces each equal to P act along radii of a circle which are at angular distances $30°$, $60°$, $90°$, $120°$ and $150°$ from a fixed radius; determine the resultant.

18. If a heavy body is supported by two strings one of which is vertical, prove that the other must be vertical also.

19. Shew that a vessel may sail due east against a south-east wind.

20. A straight line is drawn parallel to the base BC of a triangle ABC, cutting AB in a point D such that AD is twice BD. If P be any point on this line, prove that the resultant of forces completely represented by AP, BP, CP is parallel to BC.

21. Find the resultant of two forces represented by the side of an equilateral triangle and the perpendicular on this side from the opposite angle.

22. If the magnitude of one of two forces acting at a point be double that of the other, shew that the angle between its direction and that of their resultant is not greater than thirty degrees.

23. If a uniform heavy bar is supported in a horizontal position by a string slung over a peg and attached to both ends of the bar, prove that the tension of the string will be diminished if its length be increased.

24. A weight hangs at the end of a string attached to a peg. If the weight is held aside by a horizontal force so that the string makes an angle of $30°$ with the vertical, shew that the tension of the string is double the horizontal force.

25. $ABCD$ is a rhombus. Shew, without assuming the Parallelogram of Forces that forces represented by AB, CB, CD and AD are in equilibrium.

26. State how three equal forces must act so as to produce equilibrium and hence find the resultant of two equal forces inclined at $120°$ to each other.

27. A weight is suspended by means of two equal strings attached to points in the same horizontal line. Shew that if the lengths of the strings are increased, their tension is diminished.

28. A weight hangs at the end of a string attached to a peg. If the weight is held aside by a horizontal force, so that the string makes an angle $60°$ with the vertical, shew that the tension of the string is double the weight.

29. Two forces P and Q act at the same point and their directions are inclined to each other at an angle of 45°. Find an expression for the magnitude of their resultant.

Find approximately the magnitude of the resultant if the component forces be respectively equal to weights of 3 lb. wt. and 4 lb. wt.

30. If a heavy uniform bar is supported in a horizontal position by a string slung over a peg and attached to both ends of the bar, prove that the tension of the string will be diminished if its length be increased.

31. A weight is suspended by means of two equal strings attached to points in the same horizontal line. Shew that if the distance between the points is increased, the tension of the strings is increased.

32. Find the resultant of two forces of 5 and 10 lb. weight respectively acting at an angle of 60°.

33. A body is acted upon by two forces, one of 500 dynes due north, and one of 250 dynes north-east; find the resultant force.

34. A mass of 1 lb. is supported by strings of lengths 3 and 4 feet respectively attached to two points in the same horizontal plane 5 feet apart. What is the tension of each string?

35. Prove that when a kite is being flown, the position which the string will take up cannot be at right angles to the body of the kite, but will be less steep.

36. Shew that three forces of 5, 6, and 12 lb. wt. can never be in equilibrium.

37. A body weighing 4 lb. at rest on a smooth table is acted upon by forces of 3 and 4 lb. weight in directions oblique to the table and at right angles to each other. Shew by a diagram how to find the directions of the forces.

38. Three forces keep a particle in equilibrium, one acts towards the east, another towards the north-west, and the third towards the south; if the first be 5, find the other two.

39. $ABCD$ is a parallelogram, and three forces acting at a point are represented in magnitude and direction by AC, BD and $2DA$. Shew that the three forces are in equilibrium.

40. DC and AB are diameters of a circle. Three forces acting at a point are represented in magnitude and direction by AB, DC and $2BD$; shew that they are in equilibrium.

41. Three forces of 5, 12 and 13 lb. wt. are in equilibrium. Shew that two of them are at right angles and find the sines of the angles which the remaining force makes with these two.

42. A weight of 10 lb. is suspended from a fixed point by a string 25 inches in length. The weight is drawn aside until its vertical distance below a horizontal line drawn through the fixed point is 20 inches. Shew that the smallest force which will keep the weight in this position will just support a weight of 6 lb. hanging freely.

43. A force of 7 lb. wt. acts on a particle due north, one of 8 lb. wt. due east, one of 6 lb. wt. N.N.W. Find by a careful drawing the direction and the magnitude of the force which will balance these.

44. If two forces each equal to 1 lb. wt. act at a point and their directions make with each other an angle of 60°, find to the nearest oz. the magnitude of their resultant.

45. Shew that forces of 99 lb. wt. and 5 lb. wt. acting at right angles to each other have a resultant which is 17 times as great as the resultant of 5 lb. wt. and 3 lb. wt. acting at right angles to each other.

46. ABC is an equilateral triangle, AB the perpendicular on BC. Forces each equal to P act along BA, AD and AC respectively, in the directions indicated by the letters. Find the magnitude of their resultant, and shew that it is inclined at an angle 75° to the line AB.

47. Three lines AB, AC and AD in the same plane make the angles BAC, CAD each equal to 30°. Forces, equal to P, act along BA, AC and AD in the directions indicated by the letters. Find the magnitude of their resultant, and shew that it is inclined at an angle 75° to the line AB.

48. $ABCD$ is a rectangle, $AD=15$ inches and $AB=20$ inches; find the magnitude of the resultant of two forces of 16 lb. wt. and 25 lb. wt. acting along AB and AC respectively.

49. To two points A, B, 5 feet apart on a horizontal beam, the ends of a string ACB are attached, AC being 4 feet and BC 3 feet long. From C a weight of 10 lb. is hung. Find the tensions in the strings AC and BC.

50. Four weights of 2, 3, 4 and 5 lb. are hung on a string 5 feet long at points 1 foot apart. The ends of the string are attached to two points 3 feet apart in the same horizontal line, and the form assumed by the string is drawn to scale on a sheet of paper. Shew how to find from the figure the tensions in each part of the string.

CHAPTER II.

PARALLEL FORCES.

11. Rigid Bodies. So far we have dealt only with forces impressed on a particle, or on some body which for our purpose could be treated as a particle. We are to consider now some of the effects of forces impressed on a body, the volume of which cannot be treated as very small. The bodies with which we shall deal are called **Rigid Bodies.** By this it is meant that they do not alter in shape when force is impressed. No body is perfectly rigid, but many substances will resist the application of force and will change in shape by an amount which is practically infinitesimal, when force is applied. Iron, glass and wood have all rigidity, and, though the shape of a body of any of these and other similar materials may vary slightly under the application of a force, the variation will not concern us. Bodies which have rigidity are called **Solids.** Other bodies which we consider in Hydrostatics are **Fluids.** The distinction between Solids and Fluids will best be considered later[1].

12. Superposition of Forces. Now, when a force is impressed on a rigid body, it is found by experiment that any point of the body, which lies in the line of action of the force, may be considered as the point of application of the force.

Thus if a body be in equilibrium under two forces P and Q applied at A and B, Fig. 34, the two forces P and Q must be equal and act in opposite directions along the line AB.

Fig. 34.

Now let a force P act on the body at

[1] See Hydrostatics.

A in direction AB, Fig. 35. At B introduce two equal and opposite forces each equal to P. These forces are in equilibrium and will therefore not disturb the effect of P. Then P at A and P at B directed along BA balance; they therefore produce no effect and may be removed from consideration, and we are then left with P at B in the direction AB producing the same effect as P at A impressed in that same direction.

Fig. 35.

Thus P may be impressed at any point of the body in its line of action without altering its effect.

This is called the principle of the transmissibility of force.

In proving this principle, as well as in some of the Examples solved in Chapter I., we have made use of another principle which is of general application. We have introduced two equal forces in opposite directions and have assumed that this does not affect the equilibrium. The truth of the assumption is obvious.

Thus we may, without altering the conditions of any problem, superpose upon, or remove from, any system of forces any second system which is itself in equilibrium.

13. Parallel Forces. When the lines of action of two or more forces impressed at two or more points in a body meet, we may suppose the forces to be applied at the point of intersection of their lines of action, and find their resultant by the rules established in the last chapter.

In general, however, the lines of action of forces impressed on a body do not all meet in a point; the problem is more complicated. The simplest case of this occurs when the forces are parallel.

DEFINITION. *Two parallel forces are said to be* Like *when they act in the same direction, they are* Unlike *when they act in opposite directions.*

PROPOSITION 13 A. *To find the resultant of two parallel forces impressed at two points of a rigid body.*

(i) *When the forces are like.*

Let the two forces be P, Q acting at the points A, B, Fig. 36,

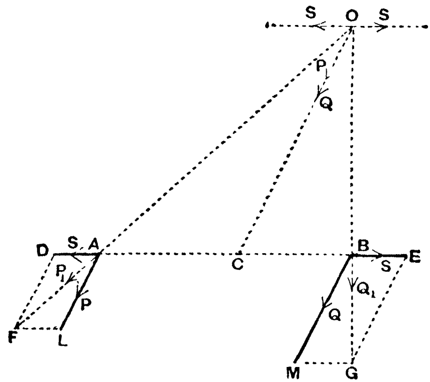

Fig. 36.

in directions AL and BM. Take AL and BM to represent the forces. Join AB, and at A and B introduce two equal and opposite forces S, represented respectively by AD and BE acting in the directions AD and BE. Complete the parallelograms $ADFL$ and $BEGM$.

The resultant of P and S at A is represented by AF and acts along AF, let it be P_1 and replace P and S by their resultant.

The resultant of Q and S at B is expressed by BG, let it be Q_1 acting along BG.

Then the two forces P and Q are equivalent to P_1 and Q_1 impressed at A and B and represented by AF and BG respectively.

The lines of action P_1 and Q_1 when produced will meet. Let them be produced and meet at O. Draw OC parallel to the original direction of the forces P and Q to meet AB in C. Transfer the points of application of P_1 and Q_1 from A and B respectively to O.

At O resolve P_1 into two components parallel to OC and CA respectively, these components must be equal to P and S.

Again resolve Q_1 at O into two components parallel to OC and CB, these two components will be Q and S. Thus we now have, acting at O, $P+Q$ along OC and two equal forces S parallel to CA and CB. These forces are equal and opposite, they may therefore be removed and we are left with a single force R equal to $P+Q$ acting along OC parallel to the original direction of P and Q. Transfer the point of application of R to C. Then the resultant of P and Q is R acting at C parallel to the original direction of P and Q.

We have now to determine the position of the point C.

By the construction, the triangles ADF and ACO are similar.

Hence $$\frac{OC}{CA} = \frac{FD}{DA} = \frac{P}{S},$$

or $$P \cdot AC = S \cdot OC.$$

The triangles BEG and BCO are similar.

Thus $$\frac{OC}{CB} = \frac{GE}{EB} = \frac{Q}{S},$$

or $$Q \cdot BC = S \cdot OC.$$

Thus $$Q \cdot BC = P \cdot AC,$$

or $$\frac{AC}{BC} = \frac{Q}{P}.$$

Again add unity to each side

$$\frac{AC}{BC} + 1 = \frac{Q}{P} + 1,$$

$$\frac{AC + BC}{BC} = \frac{P+Q}{P} = \frac{R}{P},$$

or $$P \cdot AB = R \cdot BC.$$

Similarly $$Q \cdot AB = R \cdot AC.$$

Thus $$\frac{P}{BC} = \frac{Q}{AC} = \frac{R}{AB}.$$

(ii) *When the forces are unlike.*

Let the forces be P and Q acting in the opposite directions AL, BM, Fig. 37, at the points A and B; let P be the greater force.

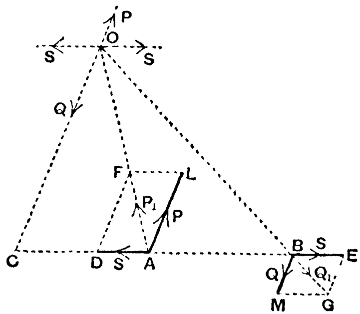

Fig. 37.

Take AL and BM to represent the forces. Join AB, and at A and B introduce two equal and opposite forces S represented respectively by AD and BE impressed in the directions AD and BE.

Complete the parallelograms $ADFL$ and $BMGE$ as before.

Replace P and S, acting at A, by their resultant P_1 along the diagonal AF, replace Q and S at B by their resultant Q_1 along BG.

Since Q is less than P the two diagonals AF and GB must meet. Let them meet at O; draw OC parallel to the original directions of the forces to meet BA produced in C. Remove the points of application of P_1 and Q_1 from A and B respectively to O.

At O resolve P_1 into its two components P and S parallel to AL and AD respectively, and resolve Q_1 into its two components Q and S parallel to BM and BE respectively.

The two components S balance and may be removed, and we have left a force R equal to $P - Q$, in the direction CO produced, which is equivalent to the two forces P and Q at A and B. Transfer the point of application of R to C. Then the resultant required is R equal to $P-Q$ acting at C.

To determine the position of C we have, as before, since the triangles ADF and ACO are similar,

$$\frac{OC}{CA}=\frac{FD}{DA}=\frac{P}{S}.$$

Thus $$P.AC=S.OC.$$

Also, since BEG and BCO are similar,

$$\frac{OC}{CB}=\frac{GE}{EB}=\frac{Q}{S}.$$

Thus $$Q.BC=S.OC.$$

Hence $$P.AC=Q.BC,$$

or $$\frac{P}{BC}=\frac{Q}{CA}=\frac{P-Q}{BC-CA}=\frac{R}{AB}.$$

Thus for like forces we have

$$R=P+Q,$$

for unlike forces

$$R=P-Q,$$

while in either case the position of the point C is given by

$$\frac{P}{BC}=\frac{Q}{CA}=\frac{R}{AB}.$$

14. Couples. We should notice that there is one case of the last proposition (Proposition 13 A ii.) in which the construction fails; if P is equal to Q the lines AF and GB of Fig. 37 will be parallel; we cannot find their point of intersection, it is at an infinite distance away and there is no single force which will replace the two; such a system of two equal unlike parallel forces is called a couple.

We shall consider such a system later.

15. Resultant of Paıallel Forces. When a number of parallel forces are impressed on a body we can find their resultant by taking two and finding the magnitude and line of action of their resultant; then combine with this resultant a third force and so on, in this way the resultant of all the forces can be obtained. It is clearly a force R given by

$$R = P_1 + P_2 + P_3 + \ldots = \Sigma \{P\},$$

where $P_1, P_2 \ldots$ are the individual forces all supposed to act in the same direction; if one or more of the forces acts in the opposite direction its sign, in the expression for R, must be changed.

16. Experiments on Parallel Forces. For experiments on parallel forces a rectangular bar of wood about a metre long with square section, each side of the square being some 2 to 3 cm. in length, is convenient.

One face of the bar should be graduated in centimetres or inches as in Fig. 38.

Three rings of brass wire A, B, C are made to fit the bar

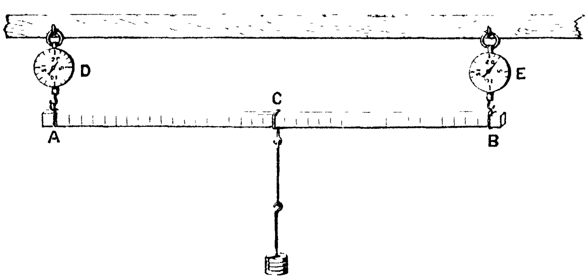

Fig. 38.

and can slide along it. These rings have small hooks attached as shewn in the figure. The bar is suspended in a horizontal position by vertical strings attached to A and B while various weights can be supported from C.

The ends of the strings attached to A and B are secured to two spring balances D and E.

For this experiment Salter's circular balances are the most convenient. In a spiral balance the scale-pan drops considerably when loaded, owing to the stretching of the spring; the drop is much less in the circular form shewn in the figure.

The small motion of the end of the spring is magnified by the pointer.

The spring balances are supported in some convenient way and the bar suspended in a horizontal position.

If the bar be uniform and the rings A, B be equidistant from either end respectively, it will be found that with the central ring C unloaded each balance is equally stretched. This extension is due to the weight of the bar; in making experiments the readings of the balances thus obtained should be subtracted from those observed when the bar is loaded.

EXPERIMENT 2. *To determine by experiment the position and magnitude of the resultant of two parallel forces.*

Suspend the bar as just described from the two balances D and E by the rings A, B. Adjust the length of the strings so that the bar may be horizontal and the rings A, B equidistant from its centre. Take the readings of the spring balances, these should be the same. Suppose them to be ·25 kilo. Suspend a carrier or scale-pan of known weight from C, and place known weights on it. Let the total weight suspended from C be R kilos weight, this is supported by the tensions in the two strings at A and B. Read the spring balances and subtract from each the reading observed before R was suspended. The differences give the values of the forces P, Q which acting at A and B respectively support R.

Note the positions on the scale of the points A, B and C and thus measure the distances AC and BC.

Then it will be found that

$$R = P + Q,$$

and also that

$$P \, . \, AC = Q \times BC.$$

The resultant of P and Q is clearly a force through C equal and opposite to R, thus the formula which gives the magnitude and line of action of the resultant of two parallel forces is verified.

Repeat the experiment; shifting the position of C it will be found that as C is moved P and Q both change, but their sum remains constant while they always satisfy the relation

$$P \cdot AC = Q \cdot BC.$$

Moreover if the value of R be altered by changing the suspended weights it will be found that the values of P and Q are also changed but in such a way that the equation $P + Q = R$ is always true.

Again if O be a point on the bar on the side of A removed from B, and the distances OA, OB and OC be measured, it will be found that

$$R \cdot OC = P \cdot OA + Q \cdot OB.$$

See Section 21.

17. Motion about an axis. We will now consider the equilibrium of a body which can turn round a fixed axis, and on which forces are impressed in a plane perpendicular to the axis. Unless some relation exists between these forces rotation will take place, we wish to determine the relation which must exist if there is to be equilibrium. This condition is easily found; the forces, which we have to consider, are the impressed forces and the reaction at the axis; this reaction is a force which necessarily passes through the axis, and it must balance the resultant of the impressed forces. Hence for equilibrium the resultant of the impressed forces must pass through the axis.

Suppose now that there are only two impressed forces. We can put the conditions into symbols thus.

PROPOSITION 14. *To find the condition of equilibrium of a body which can turn about a fixed axis when acted on by two forces in a plane perpendicular to the axis.*

It is clear from what has been said that the resultant of the forces must pass through the axis.

Let the forces P, Q act in the plane of the paper and let the axis, at right angles to that plane, cut it in C. Let a line ABC through the axis cut the lines of action of the forces in A and B and transfer the points of application of the forces to A and B; then the resultant of two forces P and Q acting at A and B respectively passes through C.

(i) Let the forces be parallel, Fig. 39.

Then their resultant R, which is equal to $P+Q$, passes through C and we have

$$P \, . \, AC = Q \, . \, BC.$$

Draw DCE perpendicular to the forces to meet their lines of action in D and E and let $CD = p$, $CE = q$.

The triangles ACD, BCE are similar.

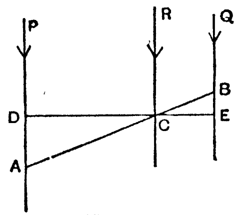

Fig. 39.

Hence $\frac{p}{q} = \frac{CD}{CE} = \frac{CA}{CB} = \frac{Q}{P}$.

Therefore $P \, . \, p = Q \, . \, q$.

Now p and q are the perpendiculars from the axis on the lines of action of P and Q. Thus in this case the condition of equilibrium is that

$P \times$ perpendicular from axis $= Q \times$ perpendicular from axis.

(ii) Let the forces P, Q, Fig. 40, be not parallel but let their lines of action meet at O. Their resultant R acts through O hence R acts along OC. Take OC to represent the resultant and from C draw CF, CG parallel to BO and AO respectively to meet OA and OB in F and G. Draw CD and CE perpendicular to the lines of action of P and Q, and let their lengths be p and q. Then $OFCG$ is a parallelogram and OC represents the resultant of the forces P, Q along OF and OG respectively.

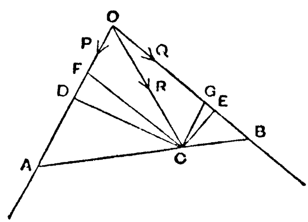

Fig. 40.

Thus OF represents P and OG represents Q.

Now the diagonal of a parallelogram bisects it.

Hence area of triangle OFC = area of triangle OGC,

and area $OFC = \frac{1}{2} OF \, . \, CD$,

also area $OGC = \frac{1}{2} OG \, . \, CE$.

Thus $OF \, . \, CD = OG \, . \, CE.$

Hence $$\frac{p}{q} = \frac{CD}{CE} = \frac{OG}{OF} = \frac{Q}{P}.$$

Therefore $P \, . \, p = Q \, . \, q.$

Thus the condition of equilibrium is the same as before,

$P \times$ perpendicular from axis $= Q \times$ perpendicular from axis.

It may be shewn that, when three or more forces act, the condition of equilibrium is the same in form; the quantities involved are the strength of each force multiplied by the length of the perpendicular from the point C on the line of action of the force.

18. Moment of a Force. The quantity which we have thus been led to consider has been given a name; it is called the moment of the force about the point.

DEFINITION. *The* **Moment** *of a force about a given point is the product of the force and the perpendicular drawn from the point on to the line of action of the force.*

Thus the condition of equilibrium just found may be expressed by the statement that *The moment of P round C is equal to the moment of Q round C.*

Again if we consider the force P only it will turn the body in one direction round the axis, while Q, alone, would turn it in the opposite direction. Under the action of P the body would rotate in a direction opposite to that of the hands of a watch, placed face uppermost on the page, under the action of Q it would rotate in the same direction as the hands of the watch. This fact is generally expressed by the statement that the moments about C of P and Q are opposite in sign.

The moment of P is said to be positive, that of Q is negative.

Thus when the bar is in equilibrium the moments of the forces round C are equal in magnitude and opposite in sign, or in other words, having regard to the difference in sign, we may state that the sum of the moments about C of all the forces is zero.

19. Experiments on Moments. We shall now describe some experiments to verify this result.

EXPERIMENT 3. *To prove that, when a body which can turn about a fixed point is in equilibrium under two forces, the moments of these forces about the point are equal and opposite.*

For this experiment the bar employed in Experiment 2 is again used. It is however suspended by an axis through C, Fig. 41, so that it can rotate in a vertical plane.

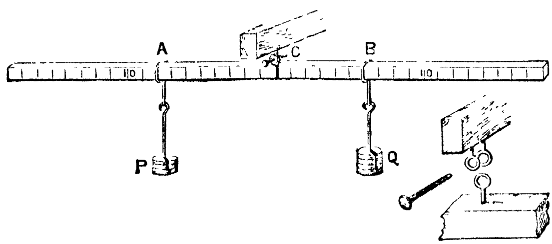

Fig. 41.

This is attained either by boring a hole through the centre of the bar and passing a piece of steel wire through the hole, the ends of the wire pass through two screw-eyes fixed into some convenient support, the wire thus forms the axis; or if more convenient, a screw-eye may be fixed into the bar instead of boring a hole through it. A pin run through these eyes forms the axis. By suspending the bar very close to its centre, the effect of its own weight in tending to turn it will be very small, and may be omitted from consideration. [See §§ 35, 37 *Centre of Gravity.*]

(i) *When the forces are parallel.*

Suspend by means of the wire rings carriers from two points A, B of the bar. Let C be the fixed point or fulcrum. Place weights on the carriers until the bar balances in a horizontal position. Let P be the total weight, including that of the carrier, suspended from A, Q that suspended from B.

Determine the weights P, Q for various positions of the carriers A, B and measure in each case the distances AC and BC. Form a table giving in four columns corresponding values of P, Q, AC and BC, then calculate the products $P.AC$ and $Q.BC$.

It will be found in all cases that they are equal, moreover A and B are on opposite sides of C, hence the moments are opposite in sign. Thus the proposition is verified for two parallel forces.

It can be verified for more than two such forces by placing several carriers on the bar and loading until equilibrium is secured.

In all cases it will be found that the sum of the moments about C, each taken with its proper sign, is zero.

(ii) *When the forces P, Q are not parallel.*
The bar is supported as before and two spring balances are suspended from a point O as shewn in Fig. 42. The hooks of

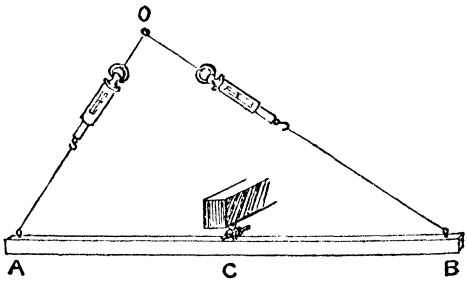

Fig. 42.

the balances are connected by strings to the points A and B and the lengths of the strings are adjusted until the bar is horizontal; the readings of the balances give the tensions P and Q; the perpendicular distances p and q of C from the lines AO and BO are measured, and it will be found as before that

$$P \cdot p = Q \cdot q.$$

In this experiment some small error may be produced by the weights of the balances themselves, but it will not be large if the balances are stretched so that the tension in the strings may be considerable.

Another arrangement of this apparatus is shewn in Fig. 43. The bar is used as before, the strings from A and B pass over pulleys and carry weights. Thus the forces P, Q impressed at

A and *B* are measured by the weights suspended at the ends of the string, the pulleys which should move easily merely serve

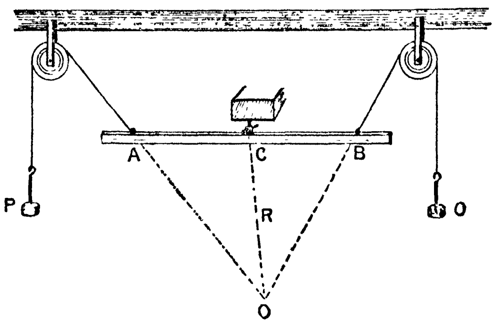

Fig. 43.

to alter the direction in which the forces are impressed. In all these experiments it is necessary that the bar should turn easily on its axis.

20. Principle of the Lever. We have thus verified the law that: When forces in one plane are impressed on a body, which can turn about an axis perpendicular to that plane, and maintain it in equilibrium, the resultant passes through the axis and the sum of the moments of the forces, each taken with its proper sign, about the axis is zero. This law when applied to the case of two forces is often spoken of as the Principle of the Lever.

21. Moments. The moment of a force can be represented geometrically and the representation will be found useful.

Thus let *AB*, Fig. 44, represent a force *P*, and let *C* be a point, about which the moment of *P* is required. Draw *CD* perpendicular to *AB*, Fig. 44 (*a*), or to *AB* produced, Fig. 44 (*b*), and join *AC* and *BC*. Then the moment of *P* about *C* is equal to $P \cdot CD$.

But *P* is represented by *AB*.

Hence $P \,.\, CD$ is represented by $AB \,.\, CD$.

And $AB \,.\, CD = 2$ area triangle ABC.

Thus the moment of a force is twice the area of a

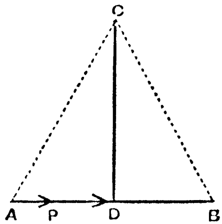

Fig. 44 (*a*).

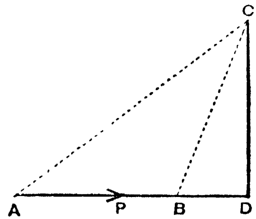

Fig. 44 (*b*).

triangle whose base is the line representing the force completely, and vertex the point round which the moment is being taken.

It should be noticed, that in making use of this proposition, the line AB must represent the force *completely*. It must therefore pass through the point of application of the force. Suppose OA, OB be two lines representing forces P, Q impressed on a particle at O; complete the parallelogram $AOBC$, then for some purposes, e.g. in order to find the resultant, we may treat AC, which is equal and parallel to OB as representing Q, when so doing we bear in mind all the time that it represents it only in magnitude and direction, NOT in point of application, we cannot calculate the moment of Q about some point such as D, by finding the area of the triangle DCA we must find the area of DOB.

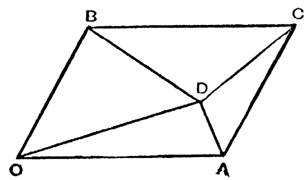

Fig. 45.

We proceed now to some theorems about moments which are of great importance.

PROPOSITION 15. *To prove that the algebraic sum of the moments of two forces about a point in their plane is equal to the moment of their resultant about that point.*

(i) *When the lines of action of the forces meet.*

Let the forces be P, Q impressed on a particle at O in directions OA, OB respectively.

Let H, Fig. 46, be the point about which the moments are

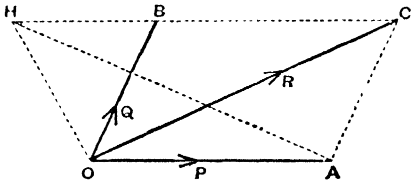

Fig. 46.

required and let OC be the direction of R, the resultant of P and Q.

Draw HC, parallel to OA, to meet OC in C, and OB in B, and from C draw CA parallel to BO to meet OA in A.

Then $AOBC$ is a parallelogram having OC for its diagonal. Take OC to represent the resultant R, then OA and OB must represent P and Q the components of R in the directions OA and OB respectively.

Thus the forces P, Q and R are represented respectively by OA, OB and OC.

Again the moment of P about H is represented by twice the triangle HOA, that of Q by twice the triangle HOB, and that of R by twice the triangle HOC.

Two cases will now arise.

(*a*) If H is outside the angle AOB, Fig. 46, the moments of both forces P and Q about H are the same in sign.

(*b*) If H is within the angle AOB, Fig. 47, the moments of the two forces have opposite signs.

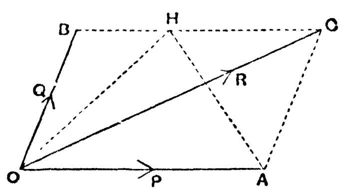

Fig. 47.

In (*a*)

$$\text{Moment of } R = 2\triangle HOC.$$

Now
$$\begin{aligned}\triangle HOC &= \triangle HOB + \triangle COB\\ &= \triangle HOB + \triangle COA\\ &= \triangle HOB + \triangle HOA,\end{aligned}$$

for HC is parallel to OA.

Thus the moment of R = moment of P + moment of Q.

Again (*b*)

$$\text{Moment of } R = 2\triangle HOC.$$

Now
$$\begin{aligned}\triangle HOC &= \triangle COB - \triangle HOB\\ &= \triangle COA - \triangle HOB\\ &= \triangle HOA - \triangle HOB,\end{aligned}$$

for BC is parallel to OA.

Moment of R = moment of P − moment of Q.

In either case *The moment of the resultant of two forces is the algebraic sum of the moments of the forces.*

(ii) *When the lines of action of the forces are parallel.*

Let P, Q be the forces and R their resultant. From H, Fig. 48, the point about which the moments are taken, draw $HACB$ perpendicular to the lines of action of the forces meeting them in A, B and C respectively.

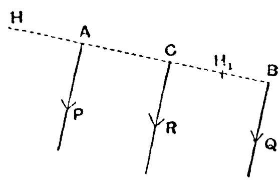

Fig. 48.

In the figure the moments of P and Q about H have the same sign.

Moreover we know that since R is the resultant of P and Q,

$$R = P + Q,$$

and
$$P \,.\, AC = Q \,.\, BC.$$

Now the moment of R about H

$$\begin{aligned}
&= R\,.\,HC = (P+Q)\,HC.\\
&= P\,.\,HC + Q\,.\,HC\\
&= P\,(HA + AC) + Q\,(HB - BC)\\
&= P\,.\,HA + Q\,.\,HB + P\,.\,AC - Q\,.\,BC\\
&= P\,.\,HA + Q\,.\,HB
\end{aligned}$$

Since $$P\,.\,AC = Q\,.\,BC.$$

Thus the moment of R is equal to the moment of P together with the moment of Q.

If the point about which the moments are taken between the lines of action of the forces is H_1 then the two moments have opposite signs.

Also the moment of R

$$\begin{aligned}
&= R\,.\,H_1C = P\,.\,H_1C + Q\,.\,H_1C\\
&= P\,(H_1A - AC) + Q\,(BC - H_1B)\\
&= P\,.\,H_1A - Q\,.\,H_1B - P\,.\,AC + Q\,.\,BC\\
&= P\,.\,H_1A - Q\,.\,H_1B.
\end{aligned}$$

The proofs which have just been given can be extended to the case of three or more forces.

Thus we obtain the result that the sum of the moments of a system of forces in one plane about any point in that plane is equal to the moment of their resultant about that point.

22. Resultant of a number of Parallel Forces. We may notice (1) that this theorem enables us to find the line of action of the resultant of a number of parallel forces in a plane.

For let P_1, P_2 ... be the forces, R their resultant. Let O be any given point in the plane and let OA_1A_2 ... cut the lines of action of the forces at right angles in A_1, A_2,..., and of the resultant in G. Then we wish to determine G.

Now we have

$$R = P_1 + P_2 + P_3 + \ldots.$$

Also taking moments about O

$$R \cdot OG = P_1 \cdot OA_1 + P_2 \cdot OA_2 + \dots$$

Hence $$OG = \frac{P_1 \cdot OA_1 + P_2 \cdot OA_2 + \dots}{P_1 + P_2 + \dots}.$$

Thus (i) The position of G is found; while (ii) The theorem includes that proved in Experiment 3, viz. that the algebraic sum of the moments of a system of forces about a point in the line of action of their resultant is zero. For, if O be in the line of action of the resultant, then OG is zero and the sum of the moments vanishes.

Examples. (1) *Find the resultant of two parallel forces of* 10 *and* 12 *kilos weight acting at two points A and B* 50 *centimetres apart.*

Let the line of action of the resultant R cut AB in C.

Then $$R = 10 + 12 = 22 \text{ kilos weight.}$$

Also taking moments about A,

$$R \cdot AC = 10 \times 0 + 12 \times 50$$
$$= 600.$$

Thus $$AC = \frac{600}{22} = 27 \cdot \dot{2}\dot{7} \text{ cm.}$$

(2) *The line of action of the resultant of two forces of* 10 *and* 15 *kilos weight is* 20 *cm. from that of the smaller force. Find the distance between the lines of action of the two forces.*

Let AB perpendicular to the lines of action of the two forces cut the line of action of the resultant in C. Then $AC = 20$ cm. and it is required to find AB.

Take moments about A

$$(10 + 15)\, AC = 15 \cdot AB,$$

$$AB = \frac{25}{15} \cdot 20 = \frac{100}{3} = 33 \cdot \dot{3} \text{ cm.}$$

(3) *Two men carry a pole* 24 *feet long supporting it at each end; from the middle point of the pole a mass weighing* 3 *cwt. is suspended. The weight of the pole is* 1 *cwt., and may be supposed to act at a distance of* 8 *ft. from one end. Find the weight carried by each man.*

Let AB be the pole, P and Q the upward pressures at A and B, C the middle point and G the point 8 ft. from A, at which the weight may be supposed to act.

Thus P and Q will be the pressures on the men's shoulders at A and B respectively, and their resultant must just balance that of the 3 cwt. and 1 cwt.

Thus $$P+Q=4 \text{ cwt.}$$

Also taking moments about A

$$Q \, . \, AB=1 \, . \, AG+3AC.$$

Thus $$Q=\frac{1\times 8+3\times 12}{24}=\frac{2+9}{6}$$

$$=1\tfrac{5}{6} \text{ cwt. wt.}$$

Hence $$P=2\tfrac{1}{6} \text{ cwt. wt.}$$

(4) *Masses of* 2, 4, 8 *and* 16 *lb. are placed at a series of points in a line and at distances of* 4, 3, 2 *and* 1 *feet from the edge of a table. Find the magnitude and point of application of the resultant force.*

Let the resultant force be R lb. wt. and let it act at a distance of x feet from the edge of the table.

Then $$R=2+4+8+16=30 \text{ lb. wt.}$$

Also taking moments round the edge

$$Rx=2\times 4+4\times 3+8\times 2+16\times 1,$$

$$x=\frac{8+12+16+16}{30}=\frac{52}{30}$$

$$=1\tfrac{11}{15} \text{ feet.}$$

(5) *A series of forces acting at a point are represented in direction and magnitude by the sides of a polygon taken in order. Shew that the sum of their moments about any point in the plane of the polygon is zero.*

The forces are in equilibrium; they have therefore no resultant; the moment therefore of the resultant is zero; hence the sum of the moment of the forces is zero.

(6) *Forces act at the angles A, B, C of a closed polygon, each force being represented* **completely** *by one side of the polygon, and all the forces acting in the same direction round the polygon. Prove that the sum of the moments of the forces about any point in the plane of the polygon is constant.*

Let the point O (Fig. 49) be within the polygon; the forces P, Q, R etc. are completely represented by AB, BC etc.

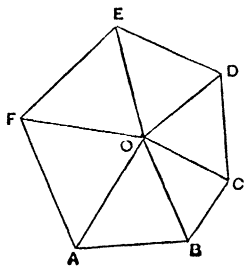

Fig. 49.

Thus moment of $P=2\triangle OAB$,

moment of $Q=2\triangle OBC$, etc.

Hence sum of moments of forces

$$=2\triangle OAB+2\triangle OBC+2\triangle OCD+\ldots$$

$=2$ area polygon, and this is the same for all the positions of O within the polygon.

Now let O be outside the polygon. The moment of P is again represented by $2\triangle OAB$ and so on; but in this case some of the moments are positive, some are negative. If the sum of the areas of the triangles which give negative moments be subtracted from the sum of the areas giving positive moments, it will be found in any case that the difference is the area of the polygon.

Hence in this case also the sum of the moments is twice the area of the polygon.

23. Moment about an axis. So far we have dealt only with forces in one plane. Suppose we have any system of parallel forces and consider a line at right angles to the direction of all these forces. Each of these forces is said to have a moment round this axis; this is found by drawing a line perpendicular both to the direction of the force and to the axis, and finding the product of the length of this line and the force. This common perpendicular is the shortest distance between the direction of the force and the axis; hence the moment of a force about an axis, perpendicular to the direction of the force, is the product of the force and the shortest distance between the axis and the direction of the force. We may put this otherwise thus. Pass a plane perpendicular to the axis through the direction of the force, then the moment of the force round the axis is the same as its moment round the point in which the axis cuts this plane.

If the forces be not all at right angles to the axis each force can be resolved into two components, one at right angles to the axis the other in a plane through the axis. The product of the component at right angles to the axis into the shortest distance from the axis of its line of action is, in this case, called the moment of the force about the axis.

PROPOSITION 16. *To prove that for any system of parallel forces the sum of the moments of the forces about an axis perpendicular to their directions is equal to the moment of the resultant about the same axis.*

Let two parallel forces, P, Q be impressed on two particles A, B, Fig. 50, in directions perpendicular to the plane of the paper. Let Ox be a line in the plane of the paper about which moments are required, and let AL and BM be perpendicular on Ox.

Join AB and divide it in C so that

$$P \,.\, AC = Q \,.\, BC.$$

The resultant $(P+Q)$ of P and Q acts at C.

Draw CN perpendicular to Ox and HCK parallel to Ox, to meet AL and MB (produced) in H and K respectively.

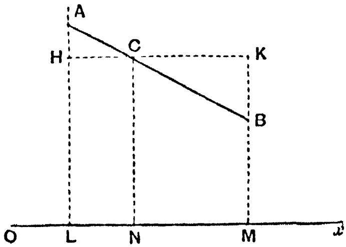

Fig. 50.

Then $$\frac{P}{Q} = \frac{BC}{AC} = \frac{BK}{AH}.$$

Therefore $$P \,.\, AH = Q \,.\, BK.$$

Now the moment of the resultant

$$= R \,.\, CN = (P + Q)\, CN = P \,.\, HL + Q \,.\, KM$$
$$= P\,(AL - AH) + Q\,(BM + BK)$$
$$= P \,.\, AL + Q \,.\, BM - P \,.\, AH + Q \,.\, BK$$
$$= P \,.\, AL + Q \,.\, BM$$
$$= \text{Sum of moments of } P \text{ and } Q.$$

In the same way the proposition can be proved for any number of parallel forces.

The proof may also be extended in a similar way to any system of forces; for our purposes it is sufficient to have established it for a system of parallel forces and to have shown that the sum of the moments of any system of parallel forces, about an axis perpendicular to the direction of the forces, is equal to the moment of their resultant about the same axis.

For examples of the use of this Proposition, see Section 38.

EXAMPLES.

PARALLEL FORCES.

1. Four equal and like parallel forces act at the angular points of a quadrilateral $ABCD$. E is the middle point of AB and F of CD. Prove that the centre of the forces is the middle point of EF. Deduce that the lines joining the middle points of the opposite sides of a quadrilateral bisect one another.

2. A uniform rod $ABCD$ moveable about a fulcrum, and thirty feet in length, has weights P, $3P$, $5P$, $7P$ attached to the rod at A, B, C and D, which are at equal distances apart. If the rod be in equilibrium, find the distance of the fulcrum from A.

3. A uniform bar of length 2 ft. 8 in. and weight $5\frac{1}{4}$ lb. is supported on a smooth peg at one end and by a vertical string distant 4 inches from the other end. Find the tension of the string.

4. Two light rods AB, BC are rigidly connected at B and meet at right angles. Weights W and W' are attached at A and C. If the system can turn about B shew that the tangent of the angle which AB makes with the horizon is

$$\tan^{-1}\frac{W}{W'}\cdot\frac{AB}{BC}.$$

5. A straight uniform heavy rod of length 6 feet has weights of 15 and 22 lb. attached to its ends, and rests in equilibrium when placed across a fulcrum distant $2\frac{1}{2}$ feet from the 22 lb. weight. Find the weight of the rod.

6. A straight uniform rod of length 6 feet and weight 11 lb. is placed across a fulcrum distant $2\frac{1}{2}$ feet from one end to which a weight of 26 lb. is attached. What weight must be attached to the other end so that there may be equilibrium?

7. A uniform bar of length 3 ft. 6 in. and weight 9 lb. is supported on a smooth peg at one end and by a vertical string distant 6 inches from the other end. Find the tension of the string.

8. A straight light rod 2 feet long rests in a horizontal position between two fixed pegs, placed at a distance of 3 inches apart, one of the pegs being at one end of the rod; a weight of 5 lb. is suspended at the other end; find the pressure on each of the pegs.

9. Let P and Q represent two like parallel forces acting at the points A and B respectively of a body. Let C be the point in the straight line AB through which their resultant R acts.

If $R = 14\frac{1}{2}$ lb., $Q = 8$ oz., and $AB = 58$ inches; find AC and BC.

10. A uniform beam weighing 10 cwt. and lying horizontally between supports 50 ft. apart carries additional weights of 3 cwt., 5 cwt. and 8 cwt., at distances of 10 ft., 20 ft., and 35 ft. respectively from one support. Find the proportion of the whole weight born by each support.

11. *ABCD* is a square; *E*, the middle point of *AB*, is joined to *C*; *BD* is joined. Forces of 4 lb. and 6 lb. act in *AB*, *BC* respectively; of 3 lb. and 2 lb. in *AD*, *DC* respectively; a force of $\sqrt{2}$ lb. in *BD*, and one of $5\sqrt{5}$ lb. in *CE*; shew, *by using moments alone*, that the system is in equilibrium.

12. A rod *AB* moveable about a hinge *A*, has a weight of 20 lb. hung on to *B*; *B* is tied by a string to a point *C* vertically above *A* and such that *CB* is 6 times *AC*: find the tension in the string *BC*.

13. A dog-cart loaded with 4 cwt. exerts a pressure on the horse's back of 10 lbs.; find the position of the centre of gravity of the load, the distance between the pad and axle being 6 feet.

14. Two forces act on a body which can move round a fixed point and the body remains at rest. Shew that the moments of the forces round the point are equal.

15. What is meant by the moment of a force about a point?

A man and a boy carry a weight of 55 pounds between them by means of a pole $5\frac{1}{2}$ feet long, weighing 20 pounds. Where must the weight be placed so that the man may bear twice as much of the whole weight as the boy?

16. Find the magnitude and position of the resultant of four forces *P*, 2*P*, 3*P*, and 4*P* acting along the sides of a square taken in order.

17. The sides of a square are 15 inches long, at the ends of one side are two weights of 3 lb. each, at the ends of the opposite side two weights of 5 lb. each; where does the resultant of the four weights act?

18. A thin board in the form of an equilateral triangle, and weighing 1 lb. has one quarter of its base resting on the end of a horizontal table, and is kept from falling over by a string attached to its vertex and to a point on the table in the same vertical plane as the triangle. If the length of the string be double the height of the vertex of the triangle above the base, find its tension.

CHAPTER III.

COUPLES.

24. Theorems about Couples.

DEFINITIONS. (i) *Two equal and opposite parallel forces constitute a* **Couple.**

(ii) *The line drawn at right angles to the directions of the two forces is called the* **Arm** *of the couple.*

(iii) *The product of either force into the perpendicular distance between the lines of action of the two forces—the arm of the couple—is called the* **Moment of the Couple.**

(iv) *A line drawn at right angles to the plane containing the two forces and proportional to the moment of the couple is called the* **Axis** *of the couple.*

(v) *The couple is said to be positive when, to an observer looking along the axis from the point of application of one of the forces, the forces tend to turn the body on which they are impressed in the same direction as the hands of a clock appear to move.*

We are now prepared for some propositions about couples.

PROPOSITION 17. *The algebraic sum of the moments of the forces which constitute a couple about any point in the plane of the couple is constant and is equal to the moment of the couple.*

Let each force of the couple be P and let O, Fig. 51, be any point in the plane of the couple. Draw OAB perpendicular to the lines of action of the two forces.

Then the algebraic sum of the moments of the forces is

$$P \, . \, OB - P \, . \, OA.$$

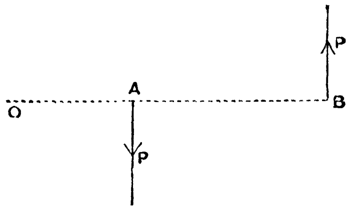

Fig. 51.

Thus the sum of the moments

$$= P\,(OB - OA) = P \, . \, AB$$
$$= \text{moment of couple.}$$

PROPOSITION 18. *Two couples impressed on a rigid body in one plane balance if their moments are equal and opposite.*

Let the forces of the one couple be P, P and its arm p, the forces of the other couple Q, Q and its arm q. Then we have that the moment $P \, . \, p$ is equal to the moment $Q \, . \, q$.

(i) *If the lines of action of P and Q meet.*

Let two of the forces P, Q meet in O, Fig. 52, and the other two P, Q in O'. Draw $O'M$, $O'N$ perpendicular to the directions of P and Q respectively, then $O'M = p$ and $O'N = q$.

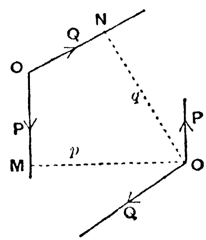

Fig. 52.

Thus we have the result that

$$P \, . \, O'M = Q \, . \, O'N,$$

or the moment of P about O is equal and opposite to that of Q about O. Hence by Section 22, O' is on the line of action of the resultant of P and Q which act at O.

Similarly, by drawing perpendiculars from O on the lines of action of the two forces P, Q which act at O', we can prove that O is on the line of action of the resultant of these two forces. These two resultants are equal in amount and the one acts at O' along $O'O$ the other at O along OO'. Hence they balance each other. Thus the two couples are in equilibrium.

(ii) *If the lines of action of the two forces do not meet but are parallel.*

Let $ACBD$ perpendicular to the common direction meet them as shewn in Fig. 53. We know that $P.AB = Q.CD$.

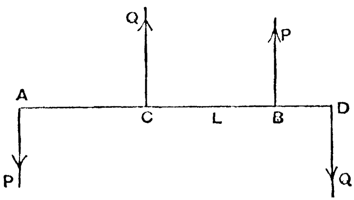

Fig. 53.

Then the resultant of P at B and Q at C is $P+Q$ acting upwards at a point L where

$$\begin{aligned}(P+Q)AL &= Q.AC + P.AB\\ &= Q.AC + Q.CD\\ &= Q(AC+CD) = Q.AD.\end{aligned}$$

Again the resultant of P at A and Q at D is $(P+Q)$ acting downwards at L' where

$$\begin{aligned}(P+Q)AL' &= P.O + Q.AD\\ &= Q.AD.\end{aligned}$$

Thus $$(P+Q)AL = (P+Q)AL',$$

or $$AL = AL'.$$

Hence L and L' coincide, thus we have $P+Q$ acting upwards and $P+Q$ acting downwards at the same point.

Hence the system is in equilibrium.

It follows therefore from this proposition that we may alter the direction of the forces of a couple, keeping the two parallel, without modifying its effect. We may also alter the forces, so long as we alter in the inverse ratio the distance between them and thus keep the moment constant.

Further we may alter the points at which we consider the forces impressed, if only the moment remains unchanged.

Thus the forces of a couple may have any value, and one of them may be supposed to be impressed at any point in the plane of the couple, so long as the moment of the couple remains unchanged in magnitude and direction.

PROPOSITION 19. *Any system of couples impressed on a rigid body in one plane is equivalent to a single resultant couple whose moment is the algebraic sum of the moments of the individual couples.*

Since two couples of equal moment, opposite in direction, balance, any two couples of equal moment, the same in direction, produce equivalent effects.

Thus we may replace any couple by another couple of the same moment with one of its forces passing through any given point.

Suppose now that P_1, P_2... be the forces of the various couples and p_1, p_2... their arms.

Replace the couples by another set of couples having the same forces P_1, P_2... and the same arms but such that all the forces are parallel, and that one force of each couple passes through a fixed point O.

Let OA_1A_2, ... Fig. 54, be perpendicular on the lines of actions of the forces and meet the second force of each couple respectively in A_1, A_2....

Then we have a system of parallel forces P_1, P_2... etc. at O, together with a second system of forces in directions opposite to these, viz. P_1 at A_1, P_2 at A_2 etc.

Moreover

$$OA_1 = p_1,\ OA_2 = p_2 \text{ etc.}$$

Now the first system of forces has a resultant R acting at O. Also $R = P_1 + P_2 + ...$

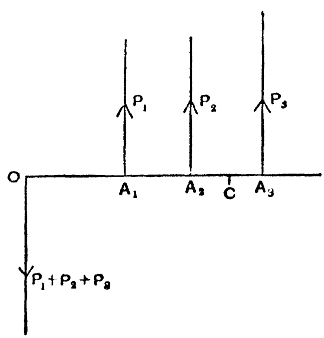

Fig. 54.

The second system has a resultant also equal to R at some point C in $OA_1A_2...$ the direction of this force R being opposite to that of R at O.

The position of C is found by taking moments about O and is given by

$$R \cdot OC = P_1 \cdot OA_1 + P_2 \cdot OA_2 + \dots$$
$$= P_1 \cdot p_1 + P_2 \cdot p_2 + \dots$$
$$= \text{sum of moments of original forces.}$$

Now R at O and R at C, acting in opposite directions form a couple whose moment is $R \cdot OC$, this we have just seen is the sum of the moments of the original couples.

Thus any system of couples in a plane is equivalent to a resultant couple, whose moment is the sum of the moments—each with its proper sign—of the original couples.

PROPOSITION 20. *A force and a couple impressed on a rigid body cannot maintain it in equilibrium.*

Let O, Fig. 55, be any point in the line of action of the force P. Change the arm of the couple, keeping its moment constant, until the forces of the couple are also P. Place the couple so that one of its forces acts at O in a direction opposite to the impressed force P. Then we have two equal and opposite forces P impressed at O and a third force P in a direction parallel to these but at a distance p from them. The two forces at O balance, and we are thus left with an unbalanced force acting at A parallel to the impressed force P but at a distance p from it. Such a force cannot maintain equilibrium.

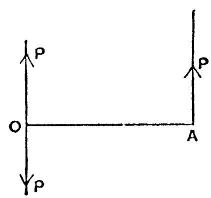

Fig. 55.

PROPOSITION 21. *A force impressed on a rigid body at any point is equivalent to an equal and parallel force, impressed at any other point, together with a couple whose moment is the moment of the force about that point.*

Let a force P be impressed on a body at any point A, Fig. 56. Let O be any other point of the body and draw ON perpendicular to the direction of P. At O introduce two equal and opposite forces P_1, P_2 equal and parallel to P. This will not affect the equilibrium. Then P at A and P_2 at O constitute a couple whose moment is $P \cdot ON$ and there is left a single force P_1 at O, equal and parallel to P.

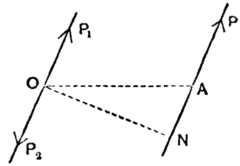

Fig. 56.

Thus the force P at A is equivalent to an equal and parallel force P at O together with a couple of moment $P \cdot ON$ about O.

PROPOSITION 22. *Any system of forces acting at different points of a rigid body in one plane is equivalent to a single resultant force acting at any given point of the plane together with a couple, whose moment is equal to the sum of the moments of the forces about that point.*

For, let there be a number of forces P, Q, R... acting at points A, B, C. Let O be any given point in the plane. Then by the last Proposition each of the given forces is equivalent to an equal and parallel force through O together with a couple whose moment is the moment of the force about O.

The forces P, Q, R... impressed at O have in general a single resultant. The couples have in general a single resultant couple whose moment is equal to the sum of the moments of the couples, and therefore to the sum of the moments of the forces about O.

PROPOSITION 23. *To find the conditions of equilibrium of a system of forces in one plane impressed on a rigid body.*

Such a system is, we have seen, equivalent to a single resultant force and a single resultant couple. Since a single force cannot balance a couple it is necessary and sufficient for equilibrium (i) that the resultant force should be zero, (ii) that the resultant couple should be zero.

In order that the resultant force may be zero the sum of its components in any two directions at right angles must be zero.

Since the moment of the resultant couple about any point is the sum of the moments of the forces about that point, it is necessary in order that the resultant couple may vanish that the sum of the moments of the forces about any point in the plane should vanish.

If the resultant force is zero we are left with a resultant couple; now the moment of a couple is constant, hence in this case the sum of the moments of the forces is constant about any point in the plane; and therefore if the sum of the moments of the forces vanishes about one point, it will vanish about all.

The necessary and sufficient conditions for equilibrium of a system of forces in one plane therefore are

(i) *The sum of the resolved parts of the forces in each of two directions at right angles is zero.*

(ii) *The sum of the moments of the forces about some one point in the plane is zero.*

Conditions equivalent to these hold for the case of forces not acting in one plane.

The following **Examples** illustrate the foregoing propositions.

(1) *Forces of* 4, 5, 6, 7 *lb. weight act at the angular points* A, B, C, D *of a square lamina each side of which is* 1 *ft. parallel to the sides* AB, BC, CD, DA *respectively. Find the resultant force and resultant couple about* A.

For the resultant force. The 4 and 6 lb. wt. are equivalent to a force of 2 lb. wt. parallel to BA, while the 5 and the 7 lb. are equivalent to a force of 2 lb. wt. parallel to DA. Hence the resultant force is $2\sqrt{2}$ lb. wt. in a direction bisecting the angle DAB and acting towards A.

For the resultant couple take moments about A, the moments of the 4 and the 7 lb. weight are zero, for these forces pass through A.

Hence the moment about A is

$$5 \,.\, AB + 6 \,.\, AD,$$

and since AB and AD are each 1 ft. the moment of the couple is 11 ft.-lb.

It may be noticed that the moment of a couple may be measured like work in ft.-lb., or more generally, that it is found by determining the product of a force and a distance. This will be found to be important.

(2) *Six forces impressed on a rigid body are represented completely by the sides of two triangles ABC, DEF; the forces act in opposite directions round the triangles; find the condition that the system may be in equilibrium.*

The resultant force is found by supposing all the forces to be impressed at a point unchanged in magnitude and direction. It is clearly zero, for forces acting at a point represented in direction and magnitude by the sides of the triangle ABC (Fig. 57) are in equilibrium, as are also those

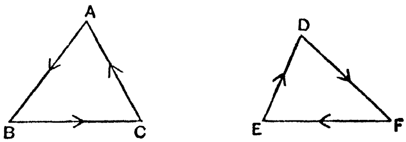

Fig. 57.

represented by the sides of DEF.

Thus the resultant is a couple.

Now the moment of forces represented completely by the sides of a triangle is twice the area of the triangle.

Thus the moment of the resultant couple is twice the difference between the areas of the two triangles. In order therefore that the forces may be in equilibrium, it is necessary that the areas of the two triangles should be equal.

(3) *The three angular points A, B, C of a triangle are each joined to two points P, Q in the plane of the triangle.*

Forces represented completely by PA, PB, and PC, AQ, BQ, and CQ act on the triangle. Shew that the resultant is a force parallel and equal to $3PQ$.

The resultant of PA, AQ (Fig. 58) acting at A is a force at A equal and parallel to PQ, for three forces at A represented by PA, AQ and QP will be in equilibrium.

Similarly the resultant of PB, BQ is PQ at B, while of PC, CQ it is PQ at C.

We have therefore three equal forces each equal to PQ acting at the angular points A, B, C; the resultant of these is a force parallel to PQ and represented by $3PQ$.

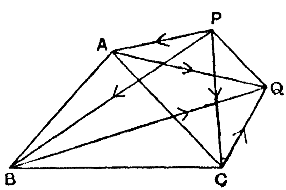

Fig. 58.

CHAPTER IV.

WORK. EQUILIBRIUM.

25. Work done by a Force. A definition of Work has been given (Dynamics, § 105), and it has been seen that work is done when the point of application of a force is moved in the direction of the force. Thus if a point A, Fig. 59, is displaced to A', and $A'A_1$ be drawn at right angles to AB, the line of action of a force F impressed at A, then work is done by the force, and the work done is measured by the product $F \cdot AA_1$.

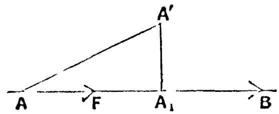

Fig. 59.

This quantity measures the product of either (i) the force and the displacement of the point of application resolved in the direction of the force, or (ii) the product of the displacement and the component of the force resolved in the direction of the displacement.

Moreover we have seen (Dynamics, § 103) that work done is one of the meanings which may be given to the term Action in Newton's statement of the third law of Motion.

26. Projection. In Figure 59, the line AA_1 is often spoken of as the projection of AA' in the direction of the force.

In general, if AB, Fig. 60, be any line and CD a second line, and if further AC and BD be perpendicular from A and B respectively on CD, then CD is the orthogonal projection, or more simply the projection of AB on the second line. If AB represents a force then CD represents the component of the force in the direction CD, it is sometimes spoken of as the projection of the force in that direction.

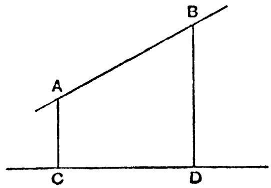

Fig. 60.

In finding, then, the work done by a force, we project the force in the direction of displacement and then multiply together the displacement and the projection of the force. Now let a number of forces P_1, P_2, P_3 represented in direction and magnitude by the lines OA, AB, BC, Fig. 61, etc. be impressed at a point; let Ox be any direction through the point; consider first the case of three forces represented by OA, AB, BC and join OC.

Then OC represents the resultant of the three forces. Draw AL, BM, CN perpendicular to Ox. Then AL is the projection of P_1, LM the projection of P_2 and MN the projection of P_3. Also ON is the projection of R, the resultant.

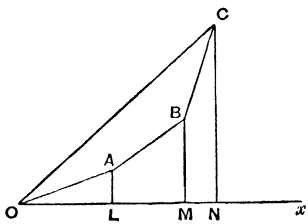

Fig. 61.

And since $ON = OL + LM + MN$, we see that the projection of the resultant is the sum of the projections of the components, or, in other words, the resolved part of the resultant is the sum of the resolved parts of the components. (See Prop. 11.)

27. Work done by a system of forces.

PROPOSITION 24. *If a system of forces be impressed on a particle and the particle receive any displacement the work done by the forces is equal to the work done by the resultant.*

Consider the case of three forces P_1, P_2, P_3 represented by OA, AB, BC, Fig. 62. Let Ox be the direction of the displacement and let a be its amount.

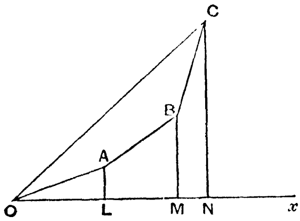

Fig. 62.

Draw AL, BM, CN perpendicular to Ox, then OL, LM, MN represent respectively the components of the forces in the direction Ox and ON represents the component of the resultant.

Hence the work done by the forces

$$= OL \,.\, a + LM \,.\, a + MN \,.\, a$$
$$= (OL + LM + MN)\, a$$
$$= ON \,.\, a$$

= the work done by the resultant.

If one of the forces act in a direction such as AB, Fig. 63, the result is still the same, for the work done by P_2 is in this case negative since the displacement takes place in a direction opposite to the force and M is to the left of L.

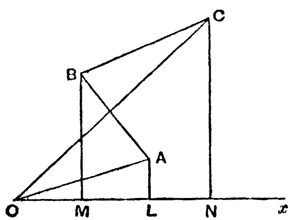

Fig. 63.

Thus the work done by forces

$= (OL - LM + MN)\, a = ON \,.\, a$

= the work done by the resultant.

If the forces form a system in equilibrium, then the point in the diagram, corresponding to C, which forms one extremity of the line representing the last force, coincides with O, the sum of the components of the forces in any direction is zero, and the work done is zero.

Thus, if a system of forces impressed on a particle be in equilibrium, no work is done in any displacement of the particle, provided that the forces are not altered by the displacement.

If we make the displacement very small, we may, even in cases in which the forces do depend on the position of the particle, assume without error in calculating the work that the forces remain unchanged.

Thus the conditions of equilibrium, for a system of forces impressed on a particle, express the fact that the particle can be slightly displaced without expenditure of energy.

Starting from this proposition as an axiom we could establish the various laws already obtained as to the composition and resolution of forces.

***28. Work done on a rigid body.** We proceed now to shew that the conditions of equilibrium for a *body* under a system of forces in one plane also express the fact that if the body be in equilibrium no expenditure of energy is necessary to give it a slight displacement.

PROPOSITION 25. *To find the work done by a force when a body on which it is impressed receives a slight rotation about an axis at right angles to the direction of the force.*

Let AB, Fig. 64, be the direction of the force P impressed in the plane of the paper on a body. Let the body be turned about the point O through a small angle θ so that OA is brought into the position OA'; then AA' is the displacement of A and if the angle be very small we may treat AA' either as a small straight line or as an arc of a circle.

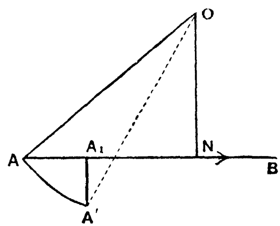

Fig. 64.

Moreover AA' is ultimately, when the angle θ is very small, at right angles to OA.

Draw $A'A_1$ perpendicular to AB the direction of the force, and ON perpendicular from O on the same line AB.

Then since OAA' is a right angle

$$A_1AA' = 90 - OAN = NOA,$$

and the angles at A_1 and N are both right angles, thus the triangles NOA, A_1AA' are similar.

Hence $$\frac{A_1A}{AA'} = \frac{ON}{OA}.$$

Thus $$A_1A = ON \cdot \frac{AA'}{OA}.$$

Now $\frac{AA'}{OA}$ is the circular measure of the angle θ.

Hence $$A_1A = ON \times \theta.$$

Now the work done by the force P is equal to $P \cdot AA_1$.

Thus

Work done $= P \cdot AA_1 = P \cdot ON \times \theta = \theta \times$ moment of P about O.

Hence, when a body is displaced through an angle θ about an axis at right angles to the impressed force, the work done by the force is found by multiplying the moment of the force about the axis by the angular displacement.

Now the circular measure of an angle is merely a number, the ratio of two lines, we see therefore how it is that the moment of a force and work are both measured in the same units, foot-pounds, foot-poundals or centimetre-dynes as the case may be.

PROPOSITION 26. *To find the work done, by a system of forces impressed on a body in one plane, when the body receives a small rotation in that plane about some point.*

If a number of forces are impressed on the body then θ, the angle through which the body is rotated, is the same for them all.

Hence, if O be the point about which rotation takes place, then the work done by each force is found by multiplying its moment about O by the angle θ.

Hence the whole work done

$$= \theta \,\{\text{sum of moments of forces about } O\}.$$

Hence if the sum of the moments about any point is zero no work is done during any small rotation about that point.

***29. Equilibrium of a rigid body.** Now we have seen that one of the conditions of equilibrium of a body under a system of forces in one plane is, that the sum of the moments of the forces about any point in the plane should be zero. If this condition be satisfied no work is done by turning the body through a small angle about any point in the plane.

Again any system of forces is equivalent to a resultant force acting at O and a resultant couple about O. The moment of this couple is the sum of the moments of the forces.

If O remains fixed, work is done only by the resultant couple, none is done by the resultant force, and the work done by the couple is found by multiplying its moment by the circular measure of the angle turned through.

Again if all points on the body are displaced the same amount, no work will be done by the resultant couple For the work done by one force of the couple is equal to that done against the other.

Now any motion of a rigid body in one plane is made up of a translation, in which all points of the body receive equal parallel displacements, and by which some one point of the body is brought to its new position, together with a rotation of the body as a whole about this new position.

For the position of the body will be known if we know the position of two points; suppose then that the two points A, B

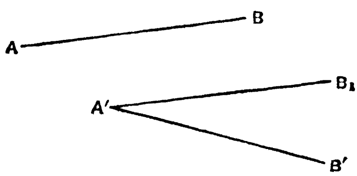

Fig. 65.

(Fig. 65) come to the position A', B', so that $A'B'$ is equal to AB. We can bring AB to the position $A'B'$ in two steps.

First keep it parallel to itself and move A to A'. Then AB will come in the position $A'B_1$ parallel to AB. Secondly turn $A'B_1$ about A' till B_1 comes to B', then AB has been brought to $A'B'$.

If a system of forces be impressed on the body the system is equivalent to a resultant R acting at A and a couple about A; let G be the moment of the couple.

Owing to the first displacement the couple does no work, the whole work done is the product of R into the component of the displacement AA' in the direction of R.

Owing to the second displacement since A' remains fixed the force R does no work. All the work done is done by the couple G and is equal to the product of G and the angle $B_1A'B'$.

Thus, if ϵ denote the angle between the direction of R and AA', and θ the angle $B_1A'B'$, the whole work done is given by

$$R \, . \, AA' \cos \epsilon + G \times \theta.$$

In order then that no work should be done in any small displacement by which the body is brought from one position to another it is necessary and sufficient that both the resultant force and the resultant couple should vanish.

Now the general form of the condition of equilibrium is that both the resultant force and the resultant couple should be zero. Thus in this case the condition of equilibrium expresses the fact that when a body is in equilibrium no expenditure of energy is needed to produce a small displacement.

Work is done by some forces and against others, if there be equilibrium these two amounts of work are equal; in Newton's phrase the action is equal and opposite to the reaction.

30. Virtual Velocities. The principle which we have just proved for the case of forces in one plane is often spoken of as the Principle of Virtual Velocities.

It may be enunciated thus:

Suppose each point of a rigid body to which a system of forces is applied to receive any small displacement consistent with its geometrical relations[1]. *Multiply each force of the*

[1] By this phrase we mean that the displacements of each point must be such as can take place without altering the form or size of the body.

system by the component of the displacement of its point of application estimated in the direction of the force; then form the sum of the products so obtained. If the forces be in equilibrium this sum is zero, and conversely, if the sum be zero for all possible displacements, the system of forces is in equilibrium.

We may state this otherwise thus:

Calculate the total work done in any small possible displacement.

Then (i) if the forces are in equilibrium this work is zero, (ii) if the work is zero *for all such displacements* then the forces are in equilibrium.

The word Virtual is used in describing the Principle because the displacements considered are not actually made; they are hypothetical and the work is calculated *on the supposition* that they are made. The displacements are spoken of as velocities because it is usual to suppose them to take place in the same time. The Principle is also called the Principle of Virtual Work.

A number of problems can most conveniently be solved by the direct application of this Principle. Examples are given in Section 33. For the present we are concerned with shewing that the conditions of equilibrium merely express the fact, *that no expenditure of energy is required in order to displace slightly a body under a system of impressed forces in equilibrium.*

31. Conditions of Equilibrium. It has been shewn that a system of forces impressed in one plane on a rigid body is equivalent to a single force and a couple, and that for equilibrium both the force and the couple must be zero. It has also been pointed out that this condition expresses the fact that no expenditure of energy is needed to give the body a slight displacement.

Now these conditions can be put into various forms each of which may be useful in special cases. We proceed then to illustrate them by some problems and to state various useful theorems.

In solving problems the two following rules will express the conditions of equilibrium in the most straightforward manner.

(i) *Equate to zero the sum of the components of the forces in two convenient directions at right angles.*

(ii) *Equate to zero the sum of the moments of the forces about some convenient point.*

In this way three equations are obtained and the problem, if determinate, can be solved.

Examples. (1) *A uniform ladder of known length l and weight W rests on rough ground against a smooth vertical wall at a known angle α with the horizon. Find the reaction at the points of contact of the wall and the ground, assuming the weight of the ladder to be equivalent to a single force W acting at its middle point.*

Let AB (Fig. 66) be the ladder, BC the wall. Let R be the reaction at the wall. Since the wall is smooth, the direction of R is horizontal. Let P be the reaction at the ground and let it make an angle θ with the horizon.

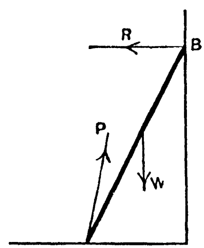

Fig 66.

Since R acts horizontally and W vertically, the horizontal and vertical directions will be two "convenient" directions at right angles in which to resolve.

The vertical components of the forces are W downwards, and $P \sin\theta$ upwards. The horizontal components are R outwards from the wall and $P\cos\theta$ to the wall.

Thus:

Resolving vertically,

$$W = P\sin\theta \quad \text{.............................} \quad (1).$$

Resolving horizontally,

$$R = P\cos\theta \quad \text{.............................} \quad (2).$$

For the third condition we may take moments about *any* point, but since the direction of P passes through A, by taking them about A we eliminate two of our unknowns, P and θ; it will clearly be "convenient" to do this.

Take moments about A,

$$R \,.\, BC = W \,.\, \frac{AC}{2} \quad \text{.............................} \quad (3).$$

But by geometry,

$$BC = AB \sin \alpha = l \sin \alpha,$$

$$AC = AB \cos \alpha = l \cos \alpha.$$

Hence from (3),

$$R = \tfrac{1}{2} W \cot \alpha.$$

From (1) and (2),

$$P^2 = W^2 + R^2$$

$$= W^2 \left(1 + \frac{\cot^2 \alpha}{4}\right)$$

$$P = W \sqrt{(1 + \tfrac{1}{4} \cot^2 \alpha)},$$

and

$$\tan \theta = \frac{W}{R} = 2 \tan \alpha.$$

Hence R, P and θ are determined.

We might have solved the problem by resolving in two other directions or taking moments about some other point, but less conveniently. For a different solution see page 78.

(2) *A uniform rod AB of length l and weight W is hinged at A to a vertical wall; the end B is connected by a horizontal string to the wall, and the rod is inclined to the wall at an angle α; a weight W′ is suspended from B. Determine the tension in the string and the direction and magnitude of the pressure on the hinge.*

Let T be the tension in the string X and Y, in the directions indicated, the horizontal and vertical components of the force at the hinge.

Resolving vertically,

$$Y = W + W'.$$

Resolving horizontally,

$$T = X.$$

Take moments about A,

$$Tl \cos \alpha = W'l \sin \alpha + W \frac{l \sin \alpha}{2},$$

$$T = (W' + \tfrac{1}{2} W) \tan \alpha.$$

Thus the forces are all determined.

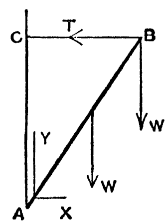

Fig. 67.

32. Problems on Forces in one plane. The conditions of equilibrium can be put into different forms. Thus:

PROPOSITION 27. *To prove that a system of forces is in equilibrium if the sum of the moments of the forces about each of three points not in the same straight line is zero.*

For let the sum of the moments about A be zero, then the system is either in equilibrium or has a resultant force through A.

If the sum of the moments about B is also zero, this resultant must pass through B. Thus it must act in the line AB. But similarly since the sum of the moments about C is also zero the resultant acts along AC; and since A, B, C are not in a straight line this is impossible, hence the resultant is zero, and the system is in equilibrium.

In the case in which the forces acting on the body reduce to three, the following proposition is useful.

PROPOSITION 28. *When three forces in one plane maintain a body in equilibrium their lines of action meet in a point or are parallel*[1].

(i) Let us suppose the directions of two of the forces meet in a point O.

Replace these two by their resultant acting through O. Then the resultant and the third force maintain equilibrium, hence the line of action of the third force passes through O.

(ii) Suppose that the directions of two of the forces are parallel, replace these by their resultant which will also be parallel to the two forces.

Then this resultant must balance the third force. Hence the three forces act in parallel directions.

Corollary. By combining this with the triangle of forces we see that, if three forces whose directions are not parallel maintain a body in equilibrium, they can be represented by the sides of a triangle drawn parallel to their lines of action respectively.

[1] It may be shewn that if three forces maintain a body in equilibrium their lines of action must be in one plane.

In applying this proposition to problems it frequently happens that the lines of action of two of the forces are known. By determining the point of intersection of these two lines of action, the direction of the third force can be found, and then by an application of the triangle of forces the values of the forces can be obtained. We will apply this to the problem of the ladder already solved, and to some other cases.

Examples. (1) *A uniform ladder of known length l and weight W rests on rough ground against a smooth vertical wall at a known angle α with the horizon. Find the reaction at the points of contact of the wall and the ground, assuming the weight of the ladder to be equivalent to a single force W acting at its middle point.*

Let G (Fig 68) be the middle point of the ladder. The weight W acts vertically through G, the pressure of the wall R acts horizontally through B. Let the lines of action of these two forces meet at O, then the pressure of the ground must act along AO. Let OG meet the ground in D, then the three forces W, R, and P are parallel respectively to OD, DA, and AO, and act at O, hence they are proportional to these lines.

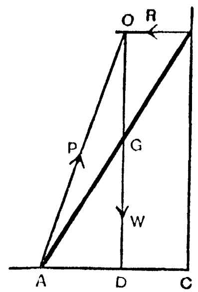

Fig. 68.

Thus $$\frac{W}{OD}=\frac{R}{DA}=\frac{P}{AO}.$$

And $$OD=BC=l\sin\alpha,$$
$$AD=\tfrac{1}{2}AC=\tfrac{1}{2}l\cos\alpha;$$

$$AO=\sqrt{\{AD^2+DO^2\}}=l\left\{\sin^2\alpha+\frac{\cos^2\alpha}{4}\right\}^{\frac{1}{2}}$$
$$=l\sin\alpha\,\{1+\tfrac{1}{4}\cot^2\alpha\}^{\frac{1}{2}}.$$

Hence $$P=W\{1+\tfrac{1}{4}\cot^2\alpha\}^{\frac{1}{2}},$$
$$R=\tfrac{1}{2}W\cot\alpha,$$

while if θ is the angle the direction of P makes with the ground,

$$\tan\theta=\frac{OD}{DA}=2\tan\alpha.$$

(2) *Two equal weightless rods AC, CB are connected by a smooth joint at C, the rod AC is free to turn about a smooth point at A, and the end B of the rod CB is free to move along a smooth groove AB. Forces P, Q in the plane ABC act at the middle points of and perpendicularly to the rods AC, CB respectively, both forces being directed towards the inside of the triangle ACB. Find the position of equilibrium and shew that if $P=2Q$ the triangle is equilateral.*

Since $CA=CB$ (Fig. 69) the triangle CAB is isosceles.

Let the angle $CBA=\theta$.

The forces on the rod CB are the pressure at B normal to AB, let this be R; the force P acting at G, the middle point of CB perpendicular to the rod, and the resistance of the hinge at C, let this be S, and let the lines of action of R and P meet in M.

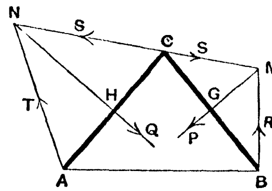

Fig. 69.

Then since there are three forces on the rod, their lines of action meet in a point. Hence S acts along CM.

Moreover $\angle GMB = GMC = \theta$,

for $\angle GBM = 90° - \theta$,

and G is the middle point of BC.

Thus MG, the direction of P, bisects the angle CMB. Hence the two forces R and S are equal, and resolving the forces at M along MG we have

$$P = 2S \cos\theta = 2R \cos\theta.$$

Now the action of BC on CA at the hinge must be equal and opposite to that of CA on BC. Hence if MC be produced to N there is a force S acting on AC at C along CN.

Let H be the middle point of AC and draw HN perpendicular to AC to meet CN in N. The force Q acts along NH.

Thus the lines of action of two of the forces on AC meet in N. The third force acting on AC is T, the pressure at the hinge A. This force must therefore act along AN, and for the rod AC we have forces S, T and Q acting at N.

Moreover $\angle HNA = \angle HNC$,

and therefore $S = T$.

Thus resolving along NH, $Q = 2S \cos HNC$.

Again in the figure

$$\angle ABC = \theta.$$

Therefore, $\angle BCM = \angle CBM = 90 - \theta$,

and $\angle ACB = 180 - 2\theta$.

Hence
$$\angle HCN = 180 - (180 - 2\theta) - (90 - \theta)$$
$$= 3\theta - 90.$$

Thus $\angle HNC = 90 - \angle HCN = 180 - 3\theta$.

Therefore,
$$Q = 2S \cos HNC = 2S \cos(180 - 3\theta)$$
$$= -2S \cos 3\theta.$$

Hence we have

$$P = 2S \cos\theta,\quad Q = -2S \cos 3\theta.$$

Therefore θ is given by the equation

$$\frac{Q}{P} = -\frac{\cos 3\theta}{\cos\theta} = 3 - 4\cos^2\theta$$
$$= 4\sin^2\theta - 1.$$

If the triangle be equilateral

$$\theta = 60^\circ, \quad \sin\theta = \frac{\sqrt{3}}{2}.$$

Hence $$\frac{Q}{P} = 3 - 1 = 2.$$
$$Q = 2P.$$

33. Virtual Velocities. A number of problems may readily be solved by the principle of work.

In applying this principle we have to suppose the system displaced and to calculate the work done in the displacement by each force. We notice (i) that if any part of the system moves over a smooth surface no work is done by the pressure of the surface, for the displacement is at right angles to the direction of the force; (ii) that if any part of the system consists of two rods connected by a smooth joint, no work is done by the mutual forces acting on the rods at the joints, for the displacement is the same for both rods at the joint, but the force which is impressed on the one rod is equal and opposite to that on the other, hence the work done is zero. The following examples illustrate the principle.

Examples. (1) *A weight P is tied to each end of a string which hangs over two smooth pegs AB in the same horizontal line. At a point O in the string midway between A and B a weight W is suspended. Find the angle which the string makes with the vertical when there is equilibrium.*

Let the angle be θ.

Let the weight W be displaced a small vertical distance x so that O (Fig. 70) may come to O', and in consequence let the weights P each rise a distance y bringing them to P'.

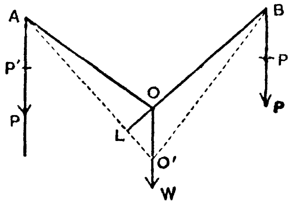

Fig. 70.

Then the work done by W is $W . x$, that done against each of the weights P is $P . y$.

Hence $$W . x = 2P . y.$$

From O draw OL perpendicular to AO'. Then since OO' is very small AL is equal to AO.

Now the length of the string from P to O is the same as that from P' to O'.

Hence

$$PP' + P'A + AO$$
$$= P'A + AL + LO'.$$

Therefore $$LO' = PP' = y.$$

And in the triangle LOO' the angle at L is a right angle, and the angle $LO'O$ is θ.

Hence $$LO' = OO' \cos\theta,$$

or $$y = x \cos\theta.$$

Therefore $$\cos\theta = \frac{y}{x} = \frac{W}{2P}.$$

This result could of course have been more readily obtained by resolving the forces at O in a vertical direction, this at once gives

$$W = 2P\cos\theta.$$

(2) *Apply the principle of work to find the position of equilibrium for the rods in Example 2, p. 78.*

The reactions at A, B, C all disappear from the equation of work for the reasons given above; the only forces which we have to consider are P and Q.

Suppose now that the rod AC is brought into the position AC' by being turned through a small angle about A, and in consequence let CB come to the position $C'B'$.

Let H', G' be the new positions of H and G.

And let HH', which may be treated either as a small arc of a circle about A or a short straight line perpendicular to AC, be equal to x. Then $CC' = 2x$.

Draw CL and $C'M$ perpendicular on AB, and draw $C'N$ parallel to BA to meet CL in N.

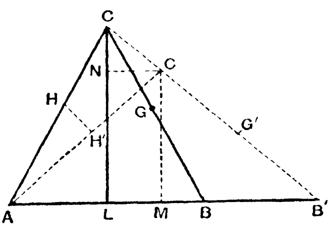

Fig. 71.

Now since G and G' are respectively the middle points of BC and $B'C'$, the component in any direction of the displacement GG' is half the sum of the components in the same direction of the displacements CC' and BB'.

Now $\angle ACC'$ is a right angle and $\angle ACB$ is $180 - 2\theta$. Hence

$$\angle C'CB = 2\theta - 90^\circ,$$

and the projection of CC' in the direction of P is $CC' \sin C'CB$ which is equal to

$$2x \cos 2\theta.$$

Again the projection of BB' in the same direction is, since P acts inward,

$$-BB' \sin ABC.$$

Moreover $\qquad AL = \tfrac{1}{2} AB,$

$$AM = \tfrac{1}{2} AB'.$$

Hence

$$BB' = 2LM = 2C'N$$
$$= 2CC' \sin C'CN$$
$$= 4x \sin \theta.$$

Thus the projection of BB' on the line of action of P is $-4x \sin^2 \theta$.

Hence

Projection of GG' on the line of action of P

$$= \tfrac{1}{2} \{2x \cos 2\theta - 4x \sin^2 \theta\}$$
$$= x \{1 - 4 \sin^2 \theta\}.$$

Thus the work done by Q is Qx, that done by P is

$$Px \{1 - 4 \sin^2 \theta\}.$$

Hence $\qquad Qx + Px \{1 - 4 \sin^2 \theta\} = 0,$

or $\qquad \frac{Q}{P} = 4 \sin^2 \theta - 1,$

as before.

(3) *The opposite angular points A, C, B, D of a rhombus $ABCD$ are connected respectively by two elastic strings. When the whole is in equilibrium the tension in AC is P, that in BD is Q; find the angle of the rhombus.*

Let the diagonals intersect in O (Fig. 72). Let the rhombus be displaced so that it becomes $A'B'C'D'$, so that A moves a distance AA' along OA and B a distance BB' along BO.

Let $\qquad OA = a, \qquad AA' = \alpha,$

$\qquad OB = b, \qquad BB' = \beta.$

Then α and β are small and the principle of work gives

$$-P \cdot AA' + Q \cdot BB' = 0.$$

Hence $\qquad \frac{P}{Q} = \frac{\beta}{\alpha}.$

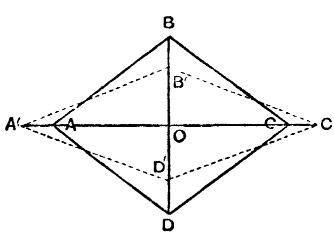

Fig. 72.

Let c be the side of the rhombus, then

$$AB = c = A'B',$$

and

$$a^2+b^2=AB^2=c^2=A'B'^2=(a+\alpha)^2+(b-\beta)^2$$
$$=a^2+2a\alpha+\alpha^2+b^2-2b\beta+\beta^2.$$

And if α and β are very small we may neglect α^2 and β^2.

Hence $$2a\alpha-2b\beta=0,$$

or $$\frac{\beta}{\alpha}=\frac{a}{b}.$$

Hence $$\frac{P}{Q}=\frac{a}{b}=\tan BAC.$$

Thus the angle between the sides of the rhombus is given in terms of the tension.

EXAMPLES.

1. A uniform rod of weight W is supported from a point by two strings. One of these makes an angle of 60°, the other an angle of 30°, with the rod. Find the tensions in the strings.

2. Forces P, Q, and R act along lines OA, OB, and OC which are met by a line through O' in A, B, and C. Shew that if the resultant of the forces passes through O', then

$$P\frac{O'A}{OA}+Q\frac{O'B}{OB}+R\frac{O'C}{OC}=0.$$

3. A rod whose length is 10 feet, and which is thicker at one end than at the other, balances about its centre when 10 lb. is hung from one end and 20 from the other; while if 40 lb. instead of 20 is hung from the second end the fulcrum is at 4 feet from that end. Find the weight of the rod, and the position of its centre of gravity.

4. A heavy uniform plank 9 feet long can turn about a point 3 feet from one end. A man whose weight is double that of the plank stands upon it, with one of his feet midway between the centre of the plank and the point of support, and the other foot 2 feet from the end. Find the pressures which each of the man's feet must exert on the plank in order to preserve equilibrium.

5. A heavy rod equal in length to the radius lies in a smooth hemispherical cup, the centre of gravity of the rod being one-third of its length from one end. Shew that if θ be the angle made by the rod with the vertical $\tan\theta=3\sqrt{3}$.

6. ABC is an equilateral triangle, and forces P, P and $2P$ act along BA, AC and CB respectively. Find the magnitude of the resultant and the point at which it cuts the line AC, produced if necessary.

7. A straight rod has its ends moveable on the arc of a smooth fixed curved wire in a horizontal plane; if a string is fastened to the centre of the rod, find by a geometrical construction the direction in which it can be pulled without moving the rod.

8. Two equal heavy rods AC, BC are jointed together at C, and have their other extremities A and B jointed to fixed pegs in the same vertical line. Prove that the direction of the stress at C is horizontal, and determine, by geometrical construction, the stresses at A and B.

9. If a weight equal to the weight of either rod be attached to the centre of the lower rod, prove that if α is the inclination of each rod to the vertical, and θ the inclination to the vertical of the stress at c,

$$\tan\theta = 3\tan\alpha;$$

and that this stress is to the weight of either rod in the ratio

$$\sqrt{1+9\tan^2\alpha} : 4.$$

10. Two equal heavy uniform beams AB, BC, each of weight W, jointed at B so as to make an angle α with one another, rest in a vertical plane with the ends A, C on a smooth horizontal plane, and AC is joined by an inextensible string. Determine the tension of the string.

11. AB is a uniform rod of weight W attached by two light strings AC, BD to two points C, D in the same horizontal line. Assuming AC to be the shorter string, shew that it is possible to choose a weight P and place it on the rod so as to maintain it in a horizontal position, when P is at a distance from the middle-point of the rod equal to

$$\frac{W+P}{P}\frac{\sin(\theta-\phi)}{\sin(\theta+\phi)}\cdot\frac{AB}{2},$$

where θ and ϕ are the acute angles made with CD by AC and BD respectively.

12. A small bead P is free to move on a given smooth circular wire and is acted on by two forces represented by PA and PB, where A and B are fixed points in the plane of the ring. Find the positions of equilibrium.

13. The sides of a triangular framework are 13, 20, and 21 inches: the longest side rests on a horizontal smooth table and a weight of 63 lb. is suspended from the opposite angle. Find the tension in the side on the table.

14. A gate is hung in the usual manner by two hinges on a gate-post. Indicate the forces acting on the gate when it hangs open and in equilibrium, and shew that it may happen that the reaction of one of the hinges is wholly horizontal. Give a verbal explanation as well as a diagram.

15. A curtain-ring can slide along a horizontal pole, which is at a height of 4 feet above my hand: if a string 10 feet long is attached to the ring, shew by a diagram in what direction I must pull the string so as to move the ring with most effect, and at what point of the string I must take hold.

16. Two small heavy rings of weights W and W' connected by a light string slide on two wires in the same vertical plane making equal angles α with the horizon. If the string makes an angle θ with the horizon shew that

$$(W+W')\tan\theta=(W-W')\cot\alpha.$$

17. A ladder rests against a smooth wall, the ground being also smooth. Compare the horizontal forces which must be applied to the bottom of the ladder to preserve equilibrium, when a weight equal to the weight of the ladder is placed on the ladder at the top and bottom respectively.

CHAPTER V.

CENTRE OF GRAVITY.

34. Centre of Mass. Consider two particles A, B, Fig. 73, of mass m, m' respectively, let them be connected by an inextensible rod whose mass we may neglect, and suppose two parallel forces act on them, the force on each particle being proportional respectively to the mass of that particle. Let the two forces then be ma and $m'a$, a being some constant. These two forces have a resultant which acts at a point C in the line AB, dividing AB so that

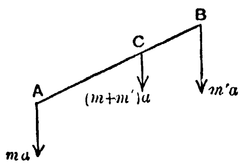

Fig. 73.

$$ma \,.\, AC = m'a \,.\, BC,$$

or

$$\frac{AC}{BC} = \frac{m'}{m}.$$

The point C is called the centre of mass of the two particles. Whatever be the position of the line AB the point C is fixed in that line and the resultant force $ma + m'a$ always acts through it.

Again if we have three masses m_1, m_2, m_3 at A_1, A_2, A_3 Fig. 74, and forces m_1a, m_2a, m_3a are impressed on these, the centre of mass of m_1 and m_2 will be a point C in A_1A_2 such that

$$m_1CA_1 = m_2CA_2.$$

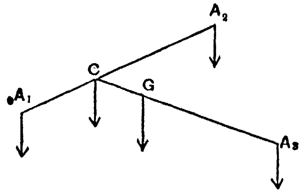

Fig. 74.

We may consider the forces m_1a and m_2a to be replaced by their resultant $(m_1 + m_2)\,a$ impressed at C. Then this force at C and m_3a at A_3 will have a resultant acting always at G a point in A_3C such that

$$(m_1 + m_2)\,CG = m_3GA_3.$$

This resultant will be $(m_1 + m_2 + m_3)\,a$. The point G is the centre of mass of the three particles.

In general, since the resultant of any number of *parallel* forces impressed on a rigid body at various points is a force equal to the sum of the individual forces, whose line of action always passes through a fixed point in the body, we see that if parallel forces m_1a, m_2a.., be impressed respectively on each of the particles m_1, m_2..., of a rigid body these forces will have a resultant equal to $(m_1 + m_2 \ldots)\,a$ whose line of action passes through a fixed point of the body. This point is called the centre of mass of the body.

DEFINITION. *The* **Centre of Mass** *of a body is the point of action of the resultant of a system of parallel forces impressed on each of the particles of the body, each force being proportional to the mass of the particle on which it is impressed. This point is fixed in the body.*

The position of the centre of mass does not depend on the direction of the parallel forces but only on their amounts and on the points at which they are impressed, thus if the body be turned in any way, the forces still remaining parallel, their resultant still acts through the same point in the body.

Again there is only one centre of mass, for if there be two, H_1 and H_2, turn the body if necessary until the line H_1H_2 is at right angles to the forces.

Then the resultant force acts in one line through H_1 and in a second parallel line through H_2, which is impossible. Hence there is only one centre of mass.

35. Centre of Gravity. The weight of a body is the resultant of the weights of the particles of which the body is composed, the weight of each particle is proportional to its mass and, if the body is small compared to the Earth, the lines joining the particles to the centre of the Earth are all parallel, so that the weights of the particles form a system of parallel forces, each being proportional to the mass of the particle on which it is impressed.

The resultant of these forces passes through a fixed point in the body—the centre of mass—or as it is more often called when considered in connexion with the weight of the body the centre of gravity.

DEFINITION. *The weights of the various particles of which a body is composed form a system of parallel forces; these forces have a resultant equal to their sum. This resultant passes through a point which is fixed in the body however it be placed. This point is called the* Centre of gravity *of the body.*

Thus we may treat the weight of a body of finite size as a single vertical force impressed on the body at a definite point; this point is its centre of gravity.

If we allow for the fact that the weights of the various particles are not strictly parallel forces, it does not follow that their resultant passes in all cases through a fixed point in the body. The body has no centre of gravity though it has a centre of mass.

It is clear that if the only impressed force be the weight of the body and the centre of gravity be supported the body will balance in any position in which it may be placed.

We shall describe first an experimental method which can sometimes be applied in order to find the centre of gravity of a body, and then shew how to determine by calculation the position of the centre of gravity for each of certain bodies.

The following proposition will be needed.

PROPOSITION 29. *To prove that, if a body is suspended freely from one point, the centre of gravity is either vertically above or vertically below the point of suspension.*

Two forces only act on the body, viz. its weight and the force exerted at the point of support. These two forces must be equal and their lines of action must coincide.

Now the weight acts vertically through the centre of gravity. Hence the point of support must be in the same vertical as the centre of gravity, and the direction of the force at the point of support must be vertical.

Two cases of this proposition arise.

In the one, Fig. 75, the centre of gravity G is vertically below the point of support O; if the body in this position be slightly displaced it will tend to return to it; the equilibrium is said to be stable.

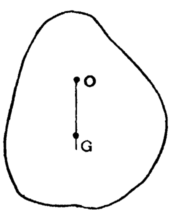

Fig. 75.

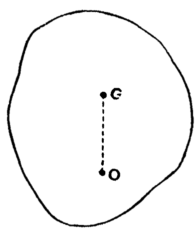

Fig. 76.

In the other case, Fig. 76, the centre of gravity G is vertically above the point of support, the body, if displaced, will not return to its original position; the equilibrium is said to be unstable.

36. Experiments on Centre of Gravity.

EXPERIMENT 4. *To find the centre of gravity of a plane lamina*[1].

Attach a string to any point A, Fig. 77, of the body and suspend it by the string.

Draw, by the aid of a plumb line hanging from the same support as the string, on one face of the lamina a vertical line AG through A The centre of gravity lies in this line, provided the lamina be very thin.

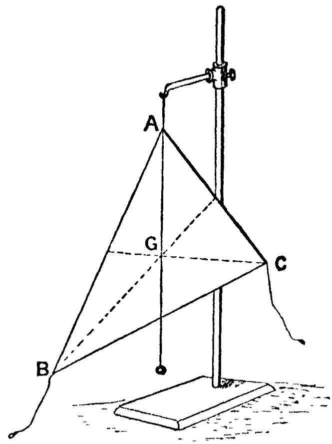

Fig. 77.

Suspend the lamina by a string attached to another point B, and draw on the same face a vertical line BG through B, intersecting AG in G. The centre of gravity lies in this vertical line. Hence the centre of gravity must be G, the point where these two lines intersect. To verify this, suspend the lamina from a third point C, the vertical line through C will also be found to pass through G.

Determine in this way the position of the centre of gravity for a triangular lamina and for a square with a corner cut off; and shew that they agree with their theoretical positions. Sections 37, 39.

EXPERIMENT 5. *To find the centre of gravity of a frame-work.*

On one of the bars of the frame-work a light wire is fixed, this has a ring at one end. Suspend the frame-work from any

[1] A lamina is a thin flat sheet of any material, such as a sheet of paper or cardboard or of very thin metal; for the experiment a thin wooden board will serve. The centre of gravity will be in the interior midway between the surfaces of the board.

point A, Fig. 78, and bend the wire so that a vertical string through A passes through the ring. Suspend the framework from a second point B, and keeping the string from A through the ring determine where the vertical through B cuts this string, this point will be the centre of gravity of the frame-work. Bend the wire until the ring just comes into this position, so that when suspended from A or B the vertical from the point of suspension passes through the ring. It will be found that when suspended from any other point the vertical through that point passes through the ring.

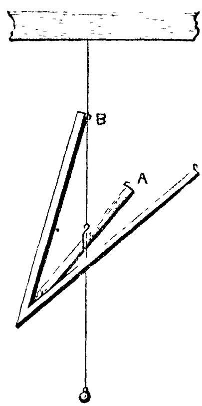

Fig. 78.

The centre of gravity of a solid body cannot be found in this manner, because of the impossibility of reaching the point in the interior of the body which is always vertically below the point of support.

37. Centre of Gravity found by Calculation. The position of the centre of gravity can be found in some cases from consideration of symmetry.

PROPOSITION 30. *To find the centre of gravity of a uniform straight rod.*

The centre of gravity is clearly the middle point of the rod, for let AB, Fig. 79, be the rod. Bisect AB in G. Let P and Q be two equal particles of the rod equidistant from G. Then the resultant of the weights of these two particles passes through G. And the whole rod can be divided into a number of such pairs of equal particles, the centre of gravity of each pair is G, hence the centre of gravity of the rod is G.

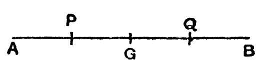

Fig. 79.

PROPOSITION 31. *To find the centre of gravity of a lamina in the form of a parallelogram.*

Let $ABCD$, Fig. 80, be a parallelogram composed of some material of uniform thickness and density.

Bisect the sides AB and CD in E and F and join EF; bisect BC and DA in H and K and join HK intersecting EF in G. Then G shall be the centre of gravity required. Divide the whole parallelogram into a series of narrow strips by lines such as PQ parallel to AB. Let PQ meet EF in R. Then since EF is parallel to BC and bisects AB and CD it also bisects PQ. Thus R is the middle point of PQ. Now we may consider the strip PQ as a uniform straight line; its centre of gravity therefore is at R. The whole parallelogram may be considered as made up of a series of strips such as PQ, the centre of gravity of each of these is on the line EF, thus the centre of gravity of the whole is on the line EF.

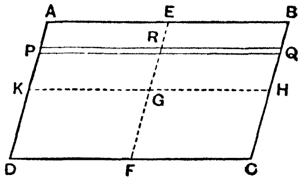

Fig. 80.

In a similar way the parallelogram may be divided up into a series of narrow strips parallel to AD; the centre of gravity of each of these will be at its middle point, that is on the line HK. Hence the centre of gravity of the whole is on the line HK.

Thus the centre of gravity of the whole parallelogram is G, the point of intersection of HK and EF.

PROPOSITION 32. *To find the centre of gravity of a uniform triangular lamina.*

Let ABC, Fig. 81, be a uniform triangular lamina.

Bisect the sides BC, CA, in D and E, and join AD and BE intersecting in G. Then G shall be the centre of gravity required. Divide the triangle into a series of narrow strips, such as PQ, by lines drawn parallel to BC. Let AD meet PQ in R.

Then since PQ and BC are parallel, and are met by the three lines AB, AD and AC we have

$$\frac{PR}{QR} = \frac{BD}{CD} = 1.$$

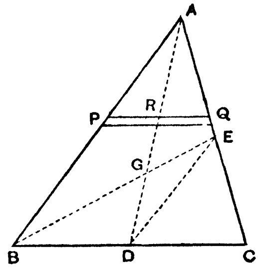

Fig. 81.

Thus R is the middle point of PQ.

Hence R is the centre of gravity of the strip PQ.

Hence the centre of gravity of all strips such as PQ lies on AD.

Thus the centre of gravity of the triangle is in AD.

Similarly, by dividing the triangle into strips parallel to CA, we can prove that the centre of gravity of the triangle is in BE.

Therefore the centre of gravity of the triangle must be G, the point where AD and BE intersect.

Again since D and E are the middle points of CB and CA respectively, DE is parallel to BA and equal to $\frac{1}{2}BA$.

Moreover, since DE and AB are parallel and are met by AD and BE,

$$\angle ADE = \angle DAB,$$

and $$\angle BED = \angle EBA.$$

Hence the triangles GDE and GAB are similar.

Thus $$\frac{DG}{AG} = \frac{DE}{AB} = \frac{1}{2}.$$

Hence $$DG = \tfrac{1}{2}AG = \tfrac{1}{3}AD.$$

Similarly $$EG = \tfrac{1}{3}BE,$$

and if CG be joined it will when produced pass through F the middle point of AB and $FG = \frac{1}{3}FC$.

Thus *the centre of gravity of a triangle is found by joining any angular point to the middle point of the opposite side, and taking a point on this line at a distance from the angle equal to $\frac{2}{3}$ds of the whole length.*

PROPOSITION 33. *To shew that the centre of gravity of a triangle is the same as that of three equal masses at its angular points.*

It is clear that the point G thus found will be the centre of gravity of three equal masses placed at the angular points of the triangle. For consider two equal mases m at B and C, their centre of gravity is D midway between them, and we may replace them by $2m$ at D. Now since $AG = 2GD$ we have

$$m \, . \, AG = 2m \, . \, GD.$$

Thus G, which is the centre of gravity of the triangle, is also the centre of gravity of m at A and $2m$ at D. It is therefore the centre of gravity of the three masses, m at A, B and C.

PROPOSITION 34. *To find the centre of gravity of three uniform rods, BC, CA and AB forming a triangle.*

Let a, b, c be the lengths of the rods respectively, their masses are proportional to a, b and c.

Bisect the rods in D, E, F, Fig. 82, then the points D, E, F are the centres of gravity of the rods and we have to find the centre of gravity of masses a, b, c at the points D, E, F respectively. Let H, a point in FE, be the centre of gravity of masses b at E and c at F, K of masses c at F and a at D, L of masses a at D and b at E.

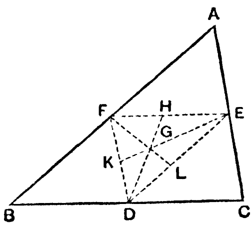

Fig. 82.

Then $\dfrac{FH}{EH} = \dfrac{b}{c} = \dfrac{AC}{AB} = \dfrac{FD}{ED}$.

Therefore DH bisects the angle FDE and we have to find the centre of gravity of $b + c$ at H and a at D. It is clearly a point in DH.

But by considering first c at F and a at D we can shew similarly that the centre of gravity required is a point in EK.

Hence the centre of gravity of the three rods is G, the point in which DH and EK coincide.

This point is clearly the centre of the circle inscribed in the triangle DEF.

38. Formulæ connected with Centre of Gravity. Formulæ can be found which enable us in many cases to obtain by calculation the position of the centre of gravity of a body.

Thus:

PROPOSITION 35. *To find the position of the centre of gravity of a number of particles in a straight line.*

Let A_1A_2, Fig. 83, be the positions of the particles, $m_1m_2\ldots$ their masses. Take any point O in the line and let x_1, $x_2\ldots$ be the distances of the particles from O.

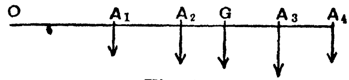

Fig. 83.

Let G be the centre of gravity.

Then G is the point of application of the resultant of a series of parallel forces proportional to m_1, m_2, etc., acting at A_1, A_2, etc., respectively.

And by Proposition 15 the moment of the resultant of these forces about O is equal to the sum of the moments of the forces. Hence, taking moments about O,

$$(m_1 + m_2 + \ldots)\,OG = m_1x_1 + m_2x_2 + \ldots.$$

Hence
$$OG = \frac{m_1x_1 + m_2x_2 + \ldots}{m_1 + m_2 + \ldots}$$
$$= \frac{\Sigma\,(mx)}{\Sigma\,(m)},$$

where as before $\Sigma\,(mx)$ stands for the sum of a series of quantities like mx.

PROPOSITION 36. *To find the centre of gravity of a number of particles in a plane.*

Let m_1, m_2 be the masses of the various particles placed at points A_1, A_2 etc. in a plane.

Let O, Fig. 84, be any fixed point in the plane, and Ox, Oy two lines at right angles meeting in O.

Let A_1M_1, A_2M_2, etc., be perpendicular on Ox, and A_1L_1, A_2L_2, etc., perpendicular on Oy.

Let
$$A_1L_1 = x_1,\ A_2L_2 = x_2, \text{ etc.}$$
$$A_1M_1 = y_1,\ A_2M_2 = y_2, \text{ etc.}$$

Let G be the centre of gravity and let GL, perpendicular on Oy, $= \bar{x}$, and GM, perpendicular on Ox, $= y$.

We require to find the point of application of the resultant of forces proportional to m_1, m_2, etc., impressed on particles at A_1, A_2 etc.

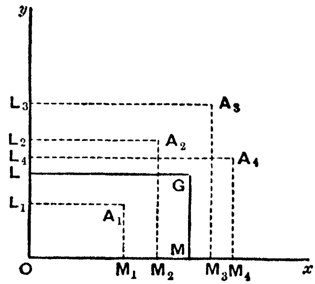

Fig. 84.

The point of application of the resultant is the same whatever be the direction of the forces so long only as they all remain parallel. Let us suppose then that they act perpendicularly to the plane of the paper. They are then at right angles to Ox and Oy and by Prop. 15 the moment of the resultant about Ox and Oy is equal to the sum of the moments of the forces about these lines. Hence, taking moments about Oy,

$$\{m_1 + m_2 + m_3 + \ldots\} GL = m_1 . A_1L_1 + m_2 . A_2L_2 + \ldots.$$

Thus $$\bar{x} = GL = \frac{m_1x_1 + m_2x_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\Sigma(mx)}{\Sigma(m)},$$

while, taking moments about Ox,

$$(m_1 + m_2 + \ldots) GM = m_1 . A_1M_1 + m_2 . A_2M_2 + \ldots$$

$$\bar{y} = GM = \frac{m_1y_1 + m_2y_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\Sigma(my)}{\Sigma(m)}.$$

By these two equations the position of G is determined.

A similar method can be applied to a series of particles in space. In this case we shall have to consider three axes at right angles and obtain the formulæ. We will illustrate the results by some examples.

Examples. (1) *Find the centre of gravity of masses of* 10, 20, 30, 40, *and* 50 *grammes arranged in a straight line at intervals of* 10 *cm. apart.*

Let O be the position of the 10 gramme mass. The distances of the various masses from O are therefore 0, 10, 20, 30 and 40 cm. respectively.

Let G be the centre of gravity,

then, taking moments about O,

$$OG\,(10+20+30+40+50)$$
$$=10.0+20.10+30.20+40.30+50.40,$$
$$OG=\frac{4000}{150}=26\tfrac{2}{3}\text{ cm.}$$

(2) *Masses of* 10, 20, 30 *and* 40 *grammes are placed at the four angles of a square each side of which is* 20 *cm. in length, and a mass of* 50 *grammes at the centre; find the centre of gravity of the whole.*

Let $ABCD$ (Fig. 85) be the square, E the centre, and the masses be placed as shewn in the figure.

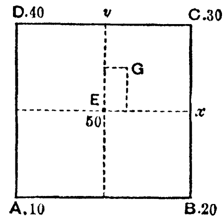

Fig. 85.

(i) *By application of the formula.*

Draw lines Ex, Ey through E parallel to the sides of the square.

Let G be the centre of gravity, x and y its distances from Ey and Ex respectively, x being measured to the right, and y upwards. The distances of the angles of the square from Ex and Ey are each 10 cm., but in taking moments about Ex it must be noted that the moments of the 30 and 40 grammes are of opposite sign to those of the 10 and 20 grammes. And similarly, when taking moments round Ey, the moments of the 10 and 40 grammes are negative, those of the 20 and 30 grammes positive.

Hence, taking moments about Ey,

$$\bar{x}\,(10+20+30+40+50)$$
$$=50.0+40.10+30.10+20.10-10.10$$
$$=500-500=0.$$

Hence $\bar{x}=0$ and G lies in Ey.

This result is obvious from the symmetry.

Taking moments about Ex,

$$\bar{y}\,(10+20+30+40+50)$$
$$=50.0+40.10+30.10-20.10-10.10$$
$$=400.$$

Therefore $$\bar{y}=\frac{400}{150}=2\tfrac{2}{3}\text{ cm.}$$

Thus the centre of gravity is on Ey at a distance of $2\frac{2}{3}$ cm. above E.

(ii) *By geometrical construction.*

Divide DA (Fig. 86) into 5 parts and let H be the first division from D so that

$$\frac{DH}{AH}=\frac{1}{4}.$$

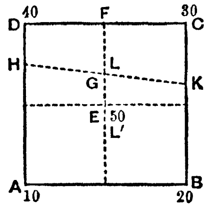

Fig. 86.

H is then the centre of gravity of 40 grammes at D and 10 grammes at A, and it is at a distance of 4 cm. from D.

Similarly, if K be a point on CB at a distance of 8 cm. from C, then K is the centre of gravity of 30 grammes at C and 20 at B.

We have now to find the centre of gravity of 50 grammes at H, 50 grammes at K and 50 grammes at E.

Join HK, cutting in L a line EF through E parallel to the side DA. Let EF meet DC in F.

Then $HL=LK$, and L is the centre of gravity of 50 grammes at each of the points H and K.

Also, since $DH=4$ cm. and $CK=8$ cm., $FL=6$ cm., and therefore $EL=4$ cm.

We have now to find the centre of gravity of 50 grammes at E and 100 grammes at L.

This will be a point G in EL such that

$$EG=2GL=\tfrac{2}{3}EL=\tfrac{2}{3}\,.\,4=2\tfrac{2}{3}\text{ cm.}$$

This is of course the same point as was found by the previous method.

(3) *Where must a mass of* 100 *grammes be placed in order that the centre of gravity of the system and the* 5 *masses described in the previous question may be at the centre of the square?*

The centre of gravity of the four particles at the angles of the square has been shewn to be at L (Fig. 86) in the line EF at a distance of 4 cm. from E. Produce LE to L' making $L'E=EL=4$ cm., and place a mass of 100 grammes at L'. The centre of gravity of 100 grammes at L and 100 grammes at L' is at E half way between them; the 50 grammes is already at E. Hence in order that the centre of gravity of the whole may be at E, the 100 grammes must be placed at L'.

(4) *A uniform wire ABC is bent at B so that the angle ABC is* 60°, *and suspended from the point A. The part AB is* a *cm. long. Find the length of BC in order that when the whole is in equilibrium BC may be horizontal.*

Let $$BC=x\text{ cm.}$$

Let H and K (Fig. 87) be the middle points of AB and BC. Draw AD vertical and HL horizontal.

Then $AB=a$, hence $BD=\frac{a}{2}$,

and $HL=\frac{a}{4}$;

also $BK=\frac{x}{2}$, hence $DK=\frac{x}{2}-\frac{a}{2}$.

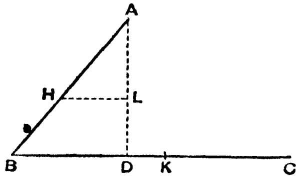

Fig. 87.

Now H is the centre of gravity of AB, and the mass of AB is proportional to its length a. Again K is the centre of gravity of BC, and the mass of BC is proportional to its length x.

Thus, taking moments about A,

$$a\,.\,HL=x\,.\,DK.$$

Hence $$\frac{a^2}{4}=\tfrac{1}{2}x\,(x-a).$$

Thus $$2x^2-2ax-a^2=0,$$

$$x=a\pm a\sqrt{(1+2)}$$

$$=a\,(1+\sqrt{3}),$$

since x must be positive.

(5) *Five pieces of uniform chain are hung at equidistant points on a uniform horizontal rod without weight. The shortest piece of chain is at the same distance from O, the end of the rod, as the interval between any two chains, and the lower ends of the chains lie on a straight line which passes through O; find the centre of gravity of the system.*

Let a be the distance between the chains, b the length of the shortest chain, then the lengths of the other chains are $2b$, $3b$, $4b$, $5b$, and the distances from the chains are a, $2a$,...$5a$.

Let $\bar{x}$ be the distance from O of the centre of gravity measured along the rod. Then, taking moments about O, we have

$$\bar{x}\,(b+2b...5b)=ab+2a\,.\,2b+\,...\,5a\,.\,5b,$$

$$\bar{x}=\frac{a\,(1+2^2+\,...\,5^2)}{1+2\,+...\,5}=\frac{55a}{15}$$

$$=\frac{11a}{3}.$$

The length of the rod is $5a$.

Thus the distance measured parallel to the rod of the centre of gravity of the chains from O is $\frac{11}{16}$ of the length of the rod.

The centre of gravity of each chain is clearly at its middle point, and the middle points of all the chains lie on a straight line through O.

Hence the centre of gravity required is the point in which the vertical from a point $\frac{11}{16}$ of the length of the rod from O cuts the line through O which bisects all the chains.

39. Properties of the Centre of Gravity.

PROPOSITION 37. *A body can be divided into two portions, the centre of gravity of each of which is known, to find the centre of gravity of the whole.*

Let W_1, W_2 be the weights of the two portions, G_1, G_2 their centres of gravity.

Join G_1G_2, Fig. 88, and divide it in G so that

$$W_1 . GG_1 = W_2 . GG_2.$$

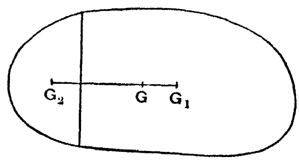

Fig. 88.

Then G is the centre of gravity of weights W_1, W_2 at G_1 and G_2; thus it is the point required.

Moreover by adding $W_1 . GG_2$ to both sides we get

$$W_1(GG_1 + GG_2) = (W_1 + W_2)GG_2.$$

Hence $$GG_2 = \frac{W_1 . G_1G_2}{W_1 + W_2},$$

and $$GG_1 = \frac{W_2 . G_1G_2}{W_1 + W_2}.$$

PROPOSITION 38. *Having given the centre of gravity of a body and of a portion of the body, to find that of the other portion.*

Let W be the weight of the whole body and G, Fig. 88, its centre of gravity, let W_1 be the weight of one portion, G_1 its centre of gravity. Join G_1G and in G_1G produced take a point G_2 such that

$$W . G_2G = W_1 . G_2G_1,$$

then G_2 is the point required.

For subtract from each side of the equation the value $W_1 . G_1G_2$,

then $$(W - W_1)\, G_2G = W_1\,(G_2G_1 - G_2G)$$
$$= W_1 . G_1G_2.$$

Hence G is the centre of gravity of W_1 at G_1 and $(W - W_1)$ at G_2.

But $W - W_1$ is the weight of the second portion of the body and hence G_2 is its centre of gravity.

Examples. (1) *A lamina has the form of a square with an isosceles triangle attached to one side. The side of the square is* a, *the height of the triangle is* h, *find the position of the centre of gravity of the figure, and determine the value of* h *if it lie in the base of the triangle.*

Let $ABCD$ (Fig. 89) be the square, ABE the triangle, G_1 the centre of gravity of the square, G_2 of the triangle. The line G_1G_2 passes through E and bisects AB at right angles.

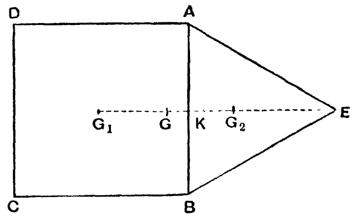

Fig. 89.

Let it cut AB in K, and let G be the centre of gravity of the figure.

The mass of the square is proportional to a^2, that of the triangle to $\frac{1}{2}ah$.

Thus the whole mass is proportional to $a^2 + \frac{1}{2}ah$.

The distance $$KG_1 = \tfrac{1}{2}a,$$
and $$KG_2 = \tfrac{1}{3}h.$$

Hence $$G_1G_2 = \tfrac{1}{2}a + \tfrac{1}{3}h.$$

The moment about any point of the whole weight at G is equal to the sum of the moments about the same point of a^2 at G_1 and $\frac{1}{2}ah$ at G_2.

Take moments about K.

Then $$(a^2 + \tfrac{1}{2}ah)\, KG = a^2\, KG_1 - \tfrac{1}{2}ahKG_2$$
$$= \frac{a^3}{2} - \frac{1}{2}\frac{ah^2}{3}.$$

Thus
$$KG = \frac{3a^2 - h^2}{3(2a + h)},$$
which determines the position of the point required.

If G is in the line AB then $KG = 0$.

Hence $$h^2 = 3a^2,$$
and $$h = a\sqrt{3}.$$

(2) *A portion of a square lamina is removed by a line which passes through the middle points of two adjacent sides; find the centre of gravity of the remainder.*

Let $ABCD$ be the square, G its centre of gravity, E and F the middle points of AB and AD. G_1 the centre of gravity of the part AFE.

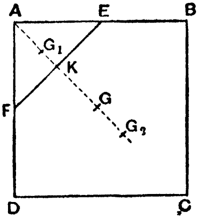

Fig. 90.

Then GG_1 passes through A and bisects FE at right angles. Let K be the point of intersection. Let $GA=a$ so that $2a$ is the diagonal of the square.

Then $$KG=KE=KA=KF=\frac{a}{2}.$$

$$KG_1=\tfrac{1}{3}KA=\frac{a}{6}.$$

Hence $$GG_1=\tfrac{2}{3}a.$$

Also the area of the square is $2a^2$, and the area of the triangle is $\frac{a^2}{4}$.

Thus the area left when the triangle is removed is

$$2a^2-\frac{a^2}{4}, \text{ or } \tfrac{7}{4}a^2.$$

Also its centre of gravity is on G_1G produced, let it be G_2.

Then, taking moments about G,

$$GG_1\cdot\frac{a^2}{4}=GG_2\frac{7a^2}{4}.$$

Hence $$GG_2=\tfrac{1}{7}GG_1=\tfrac{2}{21}a.$$

(3) *From a square, whose side is* a, *an isosceles triangle of altitude* h *is removed, the side of the square being the base of the triangle; find the centre of gravity of the remainder.*

Let AEB (Fig. 91) be the triangle, $ABCD$ the square, G the centre of gravity of the square, G_1 of the triangle.

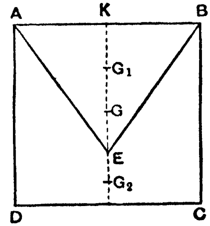

Fig. 91.

EGG_1K is a straight line bisecting AB in K, and the centre of gravity of the figure formed by removing the triangle lies in G_1G produced; let it be G_2.

Area of the square $=a^2$.

Area of the triangle $=\tfrac{1}{2}ah$.

Area of figure formed by removing the triangle

$$=a^2-\tfrac{1}{2}ah=\tfrac{1}{2}a\,(2a-h).$$

$$KG=\tfrac{1}{2}a,\ KG_1=\tfrac{1}{3}h.$$

Hence $GG_1 = \frac{1}{2}a - \frac{1}{3}h = \frac{3a - 2h}{6}$.

Take moments round G.

Then $\frac{1}{2}ah \,.\, GG_1 = \frac{1}{2}a(2a - h)\, GG_2$.

Thus $GG_2 = \frac{h}{2a - h} GG_1 = \frac{h(3a - 2h)}{6(2a - h)}$.

Thus the position of G_2 is found.

The results of these last three examples may be verified by experiment by cutting out in stiff card or sheet metal lamina of the shape indicated, and determining the positions of their centres of gravity by the method indicated in Experiment 4.

40. Equilibrium of a body resting on a horizontal surface. When a heavy body rests on a flat horizontal surface it is in equilibrium under its weight, which acts vertically downwards through its centre of gravity, and the upward pressures at the points of contact with the surface.

These upward pressures have a vertical resultant, and this resultant must balance the weight; it must therefore act vertically upwards in a line which passes through the centre of gravity of the body. If the form and position of the body is such that this is impossible the body cannot be in equilibrium.

Thus suppose the body is a vertical lamina, a sheet of card or metal having the shape shewn in Fig. 92, and that it is in contact with the table at two points A and B. The pressures of the table at A and B are both upward vertical forces. The line of action of their resultant must lie between A and B. Hence for equilibrium the position of the centre of gravity must be such that a vertical drawn through it must fall between A and B. If the centre of gravity be in a position such as G, Fig. 92, the lamina can remain in equilibrium, if it have a position such as G in Fig. 93 equilibrium is impossible.

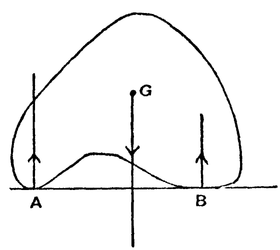

Fig. 92.

In the same way if the body rest on three points, like the legs of a three-legged table, the resultant of the upward pressures balances the weight. If a triangle be formed by joining the points in which the three legs rest on the floor, the resultant upward pressure must act within this triangle, therefore the vertical through the centre of gravity must fall within the triangle.

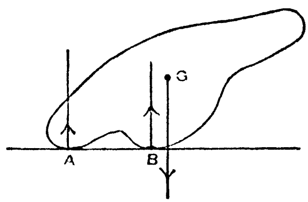

Fig. 93.

Or suppose again that the body rests in contact with the table at more points than three. Imagine a string drawn tightly round the body so as to include all these points of contact, thus forming a closed polygon whose sides are either straight or convex outwards. The pressures at the various points of support all act vertically upwards at points within the area thus defined; the resultant pressure therefore acts vertically upwards at some point within this area, and this resultant pressure, since it balances the weight, must pass through the centre of gravity.

Thus if equilibrium is possible the vertical through the centre of gravity must fall within the area thus defined. This area is sometimes spoken of as *the base of the body*, and the proposition is expressed in the statement that, in order that a body, resting on a plane under gravity, may be in equilibrium, it is necessary that the vertical through the centre of gravity should fall within the convex polygon formed by joining the extreme points of contact of the body and the plane.

This polygon must have no reentrant angles. Thus if $ABCDE$ (Fig. 94) be points of contact having a reentrant angle at D, the boundary of the base is $ABCEA$. A string stretched round the points of support would pass from C to E. The resultant pressure must lie within this polygon, though it might quite well lie without the polygon $ABCDE$.

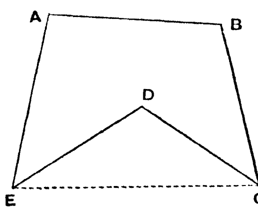

Fig. 94.

Examples. (1) *A right-angled triangle ABC having angles at B and C of 30° and 60° respectively, rests in a vertical plane on a horizontal table, the side AC being vertical and A being the right angle.*

The point C is joined to a point D in AB and the triangle DAC is removed. Find the largest triangle which can thus be removed without disturbing the equilibrium of the rest.

Bisect BC (Fig. 95) in L and join DL. The centre of gravity of the triangle CBD lies in DL. If the angle BDL is acute, a vertical from the centre of gravity must fall within the "base" BD, if it be obtuse the vertical must fall outside the "base," the limiting position of D then will be found by drawing LD vertical, and in this case LD is parallel to CA. Thus since L is the middle point of BC, D is the middle point of BA. Hence the area of the triangle DBC = the area of the triangle DAC $=\frac{1}{2}$ area of the triangle ABC, and half the original triangle may be removed without disturbing the equilibrium.

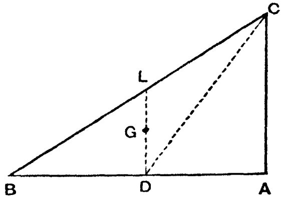

Fig. 95.

(2) *A brick $8 \times 3 \times 4$ inches in size rests with its smallest face on an inclined plane, the 3-inch side being horizontal; the brick is prevented by the friction from slipping down. Find the greatest angle to which the plane can be raised without causing the brick to fall over.*

Let $ABCD$ (Fig. 96) be a section of the brick by a vertical plane, AB being the inclined plane. Let G be the centre of gravity of the brick.

Then so long as the vertical through G falls between A and B, equilibrium is possible. In the limiting position GA is vertical. Thus the angle GAD is equal to the angle of the plane.

Now AG passes through C.

$$\text{Thus } \tan GAD = \tan CAD = \frac{CD}{AD}$$

$$= \tfrac{4}{8} = \tfrac{1}{2}.$$

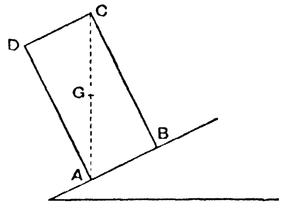

Fig. 96.

Thus the plane can be raised until it makes with the horizon an angle whose tangent is $\frac{1}{2}$.

(3) *A circular table rests on three legs attached to three points in the circumference at equal distances apart. A weight is placed on the table, determine in what position the weight is most likely to upset the table, and find the least value of the weight which when placed in that position will upset the table.*

If the table is upset by placing a weight on it, it will at first turn round an axis passing through the lower ends of two of the legs. A given weight therefore will be most effective in turning the table over when its moment round such an axis is greatest. This will be the case when the weight is as close to the edge of the table as possible, and at a point D midway between two of the legs A and B.

Again let ABC (Fig. 97) represent the top of the table and G its centre of gravity, A, B, C being the points at which the legs are attached, and D the middle point of the arc AB. Join GD cutting the line AB in K. Then if W be the weight of the table, W' that of the weight which is placed on it, the table will not upset so long as the moment of W' about AB is not greater than that of W. Hence in the limiting condition we must have $W'.DK = W.GK$.

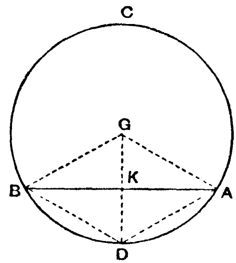

Fig. 97.

But since the angle AGB is 120° and DA is equal to DB, the triangles DGA, DGB are equilateral and the figure $DAGB$ is a rhombus, thus DG is bisected in K.

Hence $DK = KG$.

Thus the table will not upset so long as the weight supported at D is less than that of the table.

41. Stability of Equilibrium. Consider any body supported at one point, such as a lamina, which can turn round a horizontal axis through a point O, Fig. 98. We have seen that the condition for equilibrium is that the centre of gravity should be in the vertical through O. Three cases however may occur; the centre of gravity may be below O, or above O, or it may coincide with O.

Consider the first case and suppose the body to be slightly displaced so that the centre of gravity G is brought to G'. Then the weight of the body W acting through G' has a moment about O which tends to bring the body back to its original position; in this case the equilibrium is said to be *stable*.

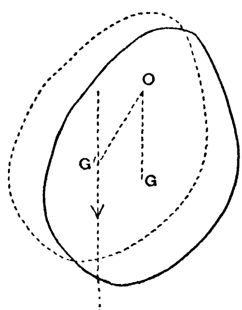

Fig. 98.

If however G be above O as in Fig. 99, and the body be displaced so that G may come to G' the moment of the weight about O tends still further to increase the displacement, the equilibrium is *unstable*.

And thirdly if O and G coincide the body will balance in any position however it may be turned about O, the equilibrium is said to be *neutral*. The above illustration affords an example of what is meant by the terms **stable, unstable** and **neutral**, which are applicable generally to bodies in a position of equilibrium.

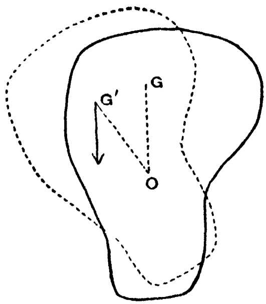

Fig. 99.

DEFINITION. *Consider a body which has been slightly displaced from a position of equilibrium. If the body tends to return to that position, its equilibrium is stable.*

Thus a weight suspended by a string from a fixed point is in stable equilibrium, so is an egg resting with its shortest diameter vertical, or a sphere which has been loaded at one point and rests on the table with the loaded part downwards.

DEFINITION. *A body at rest which, after receiving a small displacement, tends to move further away from its equilibrium position is in unstable equilibrium.*

Thus it is possible to make an egg rest on its point, or to balance a stick on its lower end, but the very slightest disturbance upsets the equilibrium; again a loaded sphere may rest with the load uppermost, but if ever so little displaced it will turn until the load comes to the bottom. These are all cases of unstable equilibrium. A wheel which has a load attached at one point can rest with this load either below or above the axle. In the second position the equilibrium is unstable, if the wheel be disturbed the load will move until it settles itself in the lowest position.

DEFINITION. *A body is in neutral equilibrium when after receiving a small displacement it will rest in its new position.*

A truly balanced wheel or a uniform sphere or cylinder resting on a flat surface are all in neutral equilibrium.

Now in these various cases, in which the weight of the body is the only impressed force in addition to the reaction of the supports; we notice, that for equilibrium the centre of gravity is either as high as possible or as low as possible. The potential energy of the body depends on the position of its centre of gravity; in the first case the potential energy has a maximum value, in the second case it has a minimum value.

In either case, if the body be very slightly displaced, the height of the centre of gravity, and therefore the value of the potential energy, is at first altered by a quantity which is itself very small compared with the change in the position of the body.

Thus let the body be turned through a very small angle θ, so that OG (Fig. 100) becomes OG', and $\angle GOG'=\theta$. Draw $G'K$ perpendicular to OG, then the centre of gravity is raised a distance GK.

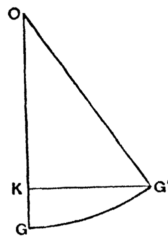

Fig. 100.

Now if $\angle GOG'$ is very small, then $OG'G$ is very nearly a right angle. Hence the triangles KGG' and $G'GO$ are similar.

Thus $$\frac{KG}{GG'}=\frac{GG'}{GO}.$$

Hence $$KG=\frac{GG'^2}{GO}.$$

And if GG' is small GG'^2/GO is very small indeed. Compared with its horizontal displacement, the change in height of the centre of gravity is very small.

This proposition is found always to be true, in an equilibrium position the potential energy has always a maximum or a minimum value, the change in potential energy consequent on a small displacement is very small when compared with the displacement, it depends on the square of the displacement.

Again, in all the cases of unstable equilibrium, the centre of gravity is as high as possible, the potential energy has a maximum value, the change produced by a small displacement is very small but, such as it is, it tends to reduce the potential energy of the body, the energy tends to take the kinetic form, the potential energy tends to decrease.

When the equilibrium is stable the centre of gravity is as low as possible, the potential energy has, in the equilibrium position, a minimum value, the change due to the displacement, though very small, is an increase; to produce it, work must be done against the impressed forces, the body must gain energy; this cannot take place without a supply of energy from without, hence a position of rest in which the potential energy is a minimum is a stable position, for to displace the body work must be done on it from without.

In the unstable position some of the potential energy can be transformed into kinetic, and this is a change which will go on of itself if once started by a small displacement, the position therefore is unstable, the body can do work on external bodies if properly connected with them and when once started will do that work.

If the body in the unstable position be quite free from all impressed force except its weight, and the (frictionless) reaction at the point of support, it acquires kinetic energy in falling as fast as it loses potential energy, it therefore passes through the position of stable equilibrium with an amount of kinetic energy equal to the potential energy it has lost, and this, if there were no friction and no air resistance, would carry it up on the other side to the former unstable position; in practice, however, some of the energy is dissipated as heat and in other ways; the body does not rise to the unstable position but comes instantaneously to rest before reaching it. It then falls back through the stable position and continues to oscillate about this until the kinetic energy it has acquired in the fall is all dissipated, when it comes finally to rest in this position with a minimum of potential energy.

Thus a position of minimum potential energy is one of stable equilibrium because work must be done to displace the body from this position, and to do this work needs a supply of energy from without.

A position of maximum potential energy is one of unstable equilibrium because it is possible for some of the potential energy to be transformed into kinetic energy without an external supply, and this is a change which in nature can take place of itself.

We do not know why there is this tendency for the transformation of energy or by what process it goes on, all that we observe is that motion which can take place without gain of energy to the system, and which merely involves transformation of energy will occur, while motion which involves a gain of energy will not occur unless energy be communicated from without.

EXAMPLES.

1. A cylinder, 50 ft. long, balances on a log put under it at 30 ft. from one end, and it also balances on the log put under its centre, when a weight of 50 lb. is placed at one end and 120 lb. at the other; find the weight of the cylinder.

2. Find by a diagram the centre of gravity of 3 cylindrical rods, of unequal lengths but small uniform thickness, so placed as to form a triangular figure. Where is the centre of gravity, in this case, geometrically situated?

3. A circular table weighing 20 lb. is supported by vertical legs attached to 4 points of the rim forming a square; find from what parts of the rim a hundredweight can be hung without overturning the table.

4. Find the centre of gravity of a lamina formed by a square having a part cut off by means of two cuts reaching from the centre to the two adjacent corners.

5. Twelve equal heavy particles are placed round the circumference of a circle at equal distances from each other. Two of the particles which have three particles between them are now removed; find the centre of gravity of the remaining ten particles.

6. If from a uniform lamina in the form of an equilateral triangle of side a the triangular portion formed by joining the middle points of two of its sides is cut away; find the distance of the centre of gravity of the remaining piece from the centre of gravity of the whole triangle.

7. From a body whose centre of gravity is known a portion whose centre of gravity is known is cut away. Find the centre of gravity of the remaining piece.

If the body is a uniform lamina in the form of a rectangle, and the triangle formed by joining its centre of gravity G to the ends of one of the sides is cut away; find the distance of the centre of gravity of the remaining part from the point G, where a is the length of the adjacent side.

8. A parallelogram $ABCD$ weighs 3 lb. and is divided by its diagonal BD into two parts, one of which, viz. the triangle BCD, is twice as heavy as the other. If a weight of 1 lb. is placed at the corner A of the parallelogram, find the centre of gravity of the system.

9. A regular hexagon is inscribed in a circle, and weights of 1 lb. each are placed at 5 of the angular points of the hexagon, and 3 lb. at the centre of the circle. Find the centre of gravity of the system.

10. If three equal triangles are cut off a triangle by lines respectively parallel to the three sides, shew that the centre of mass of the remaining figure coincides with that of the original triangle.

11. From a square piece of paper *ABCD* a portion is cut out in the form of an isosceles triangle whose base is *AB* and altitude equal to one third of *AB*. Find the centre of gravity of the remaining portion.

12. A table with a heavy square top *ABCD* rests upon four equal and heavy legs, placed at *A*, *B*, *E* and *F*, where *E* and *F* are the middle points of *BC* and *CD*. Shew that the table will be upset by a weight upon it at *C*, just greater than the weight of the whole table; and find the greatest weight which may be placed at *D* without upsetting the table.

13. A circle of radius r touches internally at a fixed point, a fixed circle of radius R; find the centre of gravity of the area between them, and its ultimate position when r increases and becomes ultimately equal to R.

14. The middle points of two adjacent sides of a uniform rectangular lamina are joined and the lamina is cut in two along the joining line. Find the centre of gravity of the larger portion.

15. From a body, weight W, a piece of weight w is cut and moved a distance x; shew that the centre of gravity of the whole moves a distance $xw \div W$ in the same direction.

ABCD is a trapezium, the angles at *B* and *C* being right angles. Shew that the distance of its centre of gravity from *BC* is

$$\frac{AB^2 + AB \, . \, CD + CD^2}{3(AB + CD)}.$$

16. A triangle is cut off from a uniform square plate by a section along a line joining the middle points of two adjacent sides. Will it be possible to balance the remainder in a vertical position with one of the sides that has been cut in contact with a horizontal plane?

17. *ABC* is a triangle; forces represented by $3AB$ and $4AC$ act along the sides *AB* and *AC*. Prove that their resultant cuts *BC* at a point *G*, such that $BG = \frac{4}{7} BC$.

18. A rod whose length is 10 feet, and which is thicker at one end than at the other, balances about its centre when 10 lb. is hung from one end and 20 from the other; while if 40 lb. instead of 20 is hung from the second end the fulcrum is at 4 feet from that end. Find the weight of the rod and the position of its centre of gravity.

19. A uniform rod of weight W is supported from a point by two strings. One of these makes an angle of 60°, the other an angle of 30° with the rod. Find the tensions in the strings.

20. A thin square board whose weight is 1 lb. has one quarter of one edge resting on the end of a horizontal table, and is kept from falling over by a string attached to an upper corner of the board and to a point on the table in the same vertical plane as the board. If the length of the string be double that of the edge of the board, find its tension.

21. A beam 12 feet long rests on two supports, distant 2 feet from each end. The beam weighs 1 cwt. Find the greatest weight which can be supported from one end without overbalancing the beam. Find also the pressure on each support when this weight is suspended.

22. A circular hole 1 foot in radius is cut out of a circular disc 3 feet in radius. If the centre of the hole be 18 inches from that of the disc, find the centre of gravity of the remainder.

23. Where must a circular hole of 1 ft. radius be punched out of a circular disc of 3 ft. radius, so that the centre of gravity of the remainder may be 2 inches from the centre of the disc?

24. Two isosceles triangles are on the same base but on opposite sides of it, and the altitude of one is 6 inches and of the other 2 inches. Find the distance from the common base of the centre of gravity of the whole figure.

25. A cylinder of wood 12 inches long is 4 inches in diameter for 8 inches of its length, and 3 inches in diameter for the remaining 4 inches. Determine the position of its centre of gravity.

26. ABC is a triangle, whose sides AB, BC, CA are 6, 10 and 8 inches long: at A, B and C respectively, are weights of 7, 8 and 9 lb. Shew that the centre of gravity of the weights coincides with that of the perimeter of the triangle.

27. The middle points of two adjacent sides of a uniform triangular lamina are joined and the lamina is cut in two along the joining line. Find the centre of gravity of the larger portion.

28. How, practically, may the centre of gravity of a heavy beam be found of which one end is heavier than the other? If it be made up of two uniform cylinders whose lengths are as 3 : 5, and weights as 3 : 1, where is the centre of gravity?

29. A uniform bar 4 yards long weighing 12 lb. has three rings each weighing 6 lb. upon it at distances 1 foot, 5 feet and 7 feet from one end. At what point will it balance?

30. One corner of a square is cut off by a straight line passing through the middle points of two adjacent sides. Find the position of the centre of gravity of the remainder.

31. A uniform triangular plate hangs from one angle with the base horizontal; shew that the triangle is isosceles.

CHAPTER VI.

MACHINES.

42. Simple Machines. There are various contrivances by which the amount or the direction of a force impressed on a body can be modified. By impressing a small force at one point of a body we may be able to give rise to a large force acting, it may be, in a different direction at some other point; or *vice versa*, some small force may be produced through the action of a larger force impressed elsewhere.

A contrivance for either of these purposes is called a machine.

DEFINITION. *An apparatus for making a force impressed on a body at a given point and in a given direction available at some other point or in some other direction is called a Machine.*

We should notice at the outset that through the action of a machine the force exerted may be greater than that applied, yet it follows from the principle of energy, that no more work is done by the machine than is done on it. Energy supplied to the machine at one point is transmitted by it to some other point; the amount of such energy, except for frictional loss, remains unchanged.

The words "except for frictional loss" are of course important, for in nature there is very considerable loss in any machine. More work must be done on the machine than it can do.

Now any machine such as a pump, a steam-engine or a crane, consists of a number of simple parts, these we shall find it desirable to classify and deal with separately.

Each of these parts is spoken of as a **Simple Machine.**

We shall suppose further that the machine is in equilibrium, and that a single force impressed at one point just balances another force impressed in general at some other point.

We look upon the first force as exerted by some other body on the machine, it is often called the **Power**, though the name is not a good one, for power means rate of doing work. The second force we consider as a force which the machine exerts on some other body, this is ordinarily described as the **Weight**. In addition to these we have the reactions at the points of support of the machine.

The names Power and Weight come from the fact that the simplest machines were no doubt originally devices to enable men to raise weights. By means of certain contrivances a man is able, though he can exert but little force, to raise a heavy weight; the force he can exert measures the power, the weight he raises is the weight.

The simple machines consist in all cases of bodies which are constrained so as to be capable of motion in some definite manner, two forces applied to such a body balance and the problem is to find the relation between them.

Now we suppose the machines to be frictionless, and the fundamental principle which will apply to all is that if the machine be supposed to receive any very slight possible displacement, the work done by the one force just balances that done against the other.

If then we measure the distance which the point of application of each force is displaced in the direction of the force, and multiply that displacement by the corresponding force, the two products will be numerically equal. Work is done by one force and against the other, the amount of work in each case being the same. If one force be large and the other small, they may balance, but in this case the displacement of the point of application of the first is small, that of the second is large.

Work is measured by the product of two factors, Force and Displacement. In many cases it is convenient to change it from the form of a small force multiplied by a large displacement into the form of a large force multiplied by a small displacement. A machine enables this to be done.

If P, Q are the two forces which balance on a machine, p, q the corresponding displacements of the points at which the forces are impressed measured parallel to the lines of action of P and Q, then

$$P \, . \, p = Q \, . \, q.$$

Hence
$$\frac{p}{q} = \frac{Q}{P},$$

or the displacements are inversely as the corresponding forces.

This result is sometimes expressed by the statement that "what is gained in Power is lost in Speed." If the "Power" P be small and the "Weight" Q large, then p is large and q is small, so that, in order to raise a large "Weight" by the aid of a small "Power," the point at which the "Power" is applied must move through a large distance compared with that traversed by the weight.

In most machines the relation between the "Power" and the "Weight" can be found most simply by making use of this principle; the problems which occur will however be solved in this way and also by the direct application of the conditions of equilibrium of a system of forces. We shall thus obtain verifications of the principle for the simple machines.

DEFINITION. *When two forces, a "Power" and a "Weight," impressed on a machine maintain it in equilibrium, the ratio of the weight to the power is called the* **Mechanical Advantage of the Machine.**

The reason for the name is clear; the object of most machines is to balance a large "Weight" with a small "Power." When this can be done mechanical advantage is gained by the use of the machine.

The Simple Machines may be classified as:

(i) The Lever, including the Wheel and Axle.

(ii) The Pulley.

(iii) The Inclined Plane, including the Wedge.

(iv) The Screw.

43. The Lever. The Lever is a rod or bar which may be either straight or curved and which can move only about a fixed point.

This point is called the fulcrum, and is denoted by C in the figures.

A force P applied at one point A balances another force W applied at a second point B; we wish to find the relation between the two. The lines of action of the two forces and of the reaction at the fulcrum must lie in one plane and meet in a point or be parallel; we will take this plane as the plane of the paper.

The conditions of equilibrium are obtained from the same principle in all cases. The resultant of the forces P and Q impressed at A and B, Fig. 101, must pass through C. Hence the moment of P round C must be equal to that of Q about C. Thus if CL, CM be perpendicular from C on the lines of action of the forces $P \cdot CL = Q \cdot CM$.

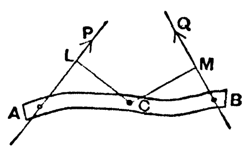

Fig. 101.

We will now consider a little more in detail the various cases which arise, and in the first place we will deal with a straight lever and suppose the forces P and Q to be parallel and at right angles to the length of the lever.

PROPOSITION 39. *To find the mechanical advantage of the straight lever.*

Levers of this kind are usually divided into three classes.

CLASS I. The points A and B at which the Power and the Weight are applied, Fig. 102 (a), are at opposite ends of the lever and the fulcrum C is between them. Levers of this class are a crowbar as ordinarily employed to raise a weight, the beam of a balance, a pair of scissors, or the handle of an ordinary pump.

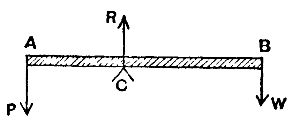

Fig. 102 (a).

For the conditions of equilibrium we have if R be the pressure on the fulcrum and a, b the lengths of the arms CA,

CB respectively,

$$R = P + W,$$

$$P \,.\, CA = W \,.\, CB.$$

Hence $$\frac{W}{P} = \frac{CA}{CB} = \frac{a}{b}.$$

Thus the mechanical advantage may be greater or less than unity according as the Power acts at the end of the longer or the shorter arm.

Class II. The Power and the Weight act on the same side of the fulcrum C but in opposite directions, the Power being applied at a greater distance from the fulcrum than the Weight.

Among levers of this class are an oar and a pair of nut-crackers. In the case of the oar, the portion of the blade in the water is the fulcrum, the power is applied by the oarsman, the pressure of the rowlock corresponds to the weight, the fulcrum is of course not absolutely fixed.

For this class then we have Fig. 102 (b)

$$R = W - P,$$

$$P \,.\, CA = W \,.\, CB.$$

Hence $$\frac{W}{P} = \frac{CA}{CB} = \frac{a}{b}.$$

And since a is greater than b the mechanical advantage is always greater than unity.

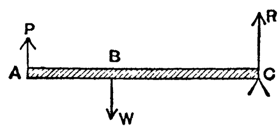

Fig. 102 (b).

Class III. The Power and the Weight act in opposite directions as in Class II., but the Power is nearer the fulcrum than the Weight.

As examples we have some forms of the treadle of a lathe or sewing machine, or a pair of spring shears, the blades of which are held open by a spring and are closed by the pressure of the hand applied at a point between the spring and the blade.

Another important example is the bone of the forearm, the fulcrum is the elbow joint, the power is applied by a muscle attached to the arm not far from the joint, the weight being held in the hand.

In this case we have, Fig. 102 (*c*),

$$R = P - W,$$

$$P \cdot CA = W \cdot CB.$$

Therefore

$$\frac{W}{P} = \frac{CA}{CB} = \frac{a}{b},$$

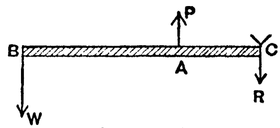

Fig. 102 (*c*).

and since a is less than b the mechanical advantage is less than unity, a small weight is raised by a large power, but the point of application of the power moves over a small distance while the weight is considerably displaced.

44. Bent Levers. If the lever be not straight, or the forces P and W be not parallel, we still find their ratio by taking moments round the fulcrum.

Again let the directions of P and W, Fig. 103, meet at O, then R is the resultant of P and W at O and it passes through C, thus if γ be the angle between OA and OB, R acts along OC while its value is given by

$$R^2 = P^2 + W^2 + 2PW \cos \gamma.$$

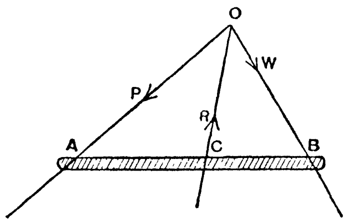

Fig. 103.

We may obtain an equation to find the direction of R thus, supposing ACB to be straight.

Let $OAB = \alpha$, $OBA = \beta$, and let $OCB = \theta$.

Then R acts along OC, P and W along OA and OB respectively.

Resolve the forces perpendicular to AB.

$$R \sin \theta = P \sin \alpha + W \sin \beta \text{ ....(1).}$$

Resolve parallel to AB.

$$R \cos \theta = P \cos \alpha - W \cos \beta \text{ ....(2).}$$

Squaring and adding we have

$$R^2 = P^2 + W^2 - 2PW\cos(\alpha+\beta).$$

Dividing (1) by (2)

$$\tan\theta = \frac{P\sin\alpha + W\sin\beta}{P\cos\alpha - W\cos\beta}.$$

In the above equations we have not taken into account the weight of the lever; this can if necessary be done. Assuming the forces all to be vertical, we have to add the weight of the lever to the pressure on the fulcrum and include the moment of the weight applied at the centre of gravity in the equation of moments.

45. Application of the Principle of Work. We can readily obtain the relation between the power and the weight for the lever by an application of the principle of work.

This has already been done in the general case in Section 22, for it was proved there that when a body can turn about an axis the work done by any force is found by multiplying the moment of the force by the circular measure of the small angle turned through.

Hence if the work done is zero the sum of the moments of the forces is zero and this applied to two forces gives us the principle of the Lever. The ratio of the two forces is equal to the inverse ratio of the arms at which they act. We will however apply the principle of work to the case of a straight lever on which two parallel forces are impressed at right angles to the arms.

Let ACB, Fig. 104, be the lever, ACB being a straight line,

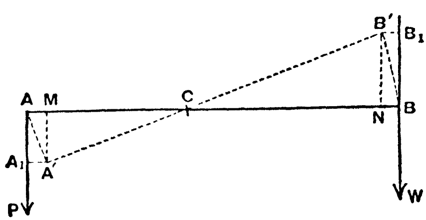

Fig. 104.

let P, W be the forces impressed at the points A and B respectively in directions perpendicular to the lever.

Let $$CA = a, \quad CB = b.$$

Let the lever be turned about C through a small angle θ into the position $A'CB'$. Draw $A'A_1$, $B'B_1$, perpendicular to the directions of P and W respectively, and $A'M$, $B'N$ perpendicular on ACB.

Then $$A_1A = A'M,$$

and $$B_1B = B'N.$$

Work done by $P = P \, . \, AA_1 = P \, . \, A'M.$

Work done against W

$$= W \, . \, BB_1 = W \, . \, B'N.$$

Hence, since these two amounts of work are equal, we have

$$P \, . \, A'M = W \, . \, B'N.$$

Thus $$\frac{P}{W} = \frac{B'N}{A'M} = \frac{A'C}{B'C},$$

for the triangles $A'CM$, $B'CN$ are similar.

Also $$A'C = AC = a,$$

$$B'C = BC = b.$$

Hence $$\frac{W}{P} = \frac{a}{b},$$

which is the result required.

We may also obtain this result by direct experiment. The bar employed has already been described, Section 18, and is shewn in Figure 105.

EXPERIMENT 6. *To find by experiment the relation between the Power and the Weight in a lever and to verify the law that the work done by the Power is equal to that done on the Weight.*

You are given a straight graduated bar ACB, Fig. 105, moveable about C as a fulcrum. This point is very approximately coincident with the centre of gravity of the bar, which will therefore balance about C. The weight of the bar is thus directly supported by pressure at the fulcrum and need not be further considered. Rings A and B slide on the bar and

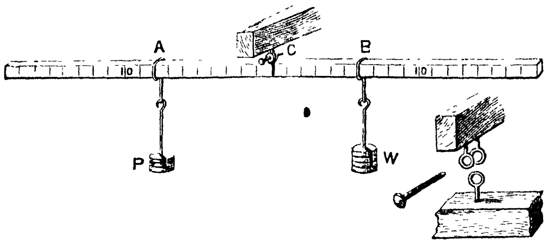

Fig. 105.

from these rings respectively weights which we will call P and W are supported. Place A with its weight P in any convenient position on the bar and adjust either the weight W or the position of the ring B until the bar rests in equilibrium in a horizontal position. Measure the distances AC and BC. Then it will be found that $P \times AC = W \times BC$. Again, measure the heights of A and B above the floor or some other convenient horizontal plane. Then lower the end A and fix the bar in an inclined position $A'CB'$. Measure the heights of A' and B' and hence determine the distances a and b say, through which the Power has been lowered and the Weight raised. It will be found that $P \times a = W \times b$, or the work done by the power is equal to that done on the weight.

Various forms of balances exemplify in their action the principle of the lever. These will be dealt with in a separate section. (See § 59.)

Examples. (1) *Weights of* 10 *and* 15 *lb. are suspended from the ends of a lever* 12 *feet in length; find the point at which they balance.*

Let AB (Fig. 106) be the lever, C the fulcrum, the upward pressure at C is 25 lb.

Let the 10 lb. weight be at A.

Take moments about A.

$$25 . AC = 15 . 12,$$

$$AC = \frac{15 . 12}{25} = \frac{3 . 12}{5} = 7\tfrac{1}{5} \text{ ft.}$$

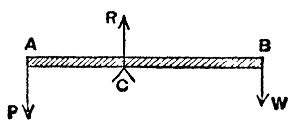

Fig. 106.

(2) *A straight rod is loaded so that its centre of gravity is $\frac{1}{3}$ of its length from one end. When weights of 5 and 10 lb. are supported from the ends, the rod balances about its middle point; find the weight of the rod.*

Let the length of the rod be l feet and let it weigh W lb.

The centre of gravity is $\frac{1}{6}l$ from the centre and the 5 lb. weight clearly hangs on the same side of the centre as the centre of gravity.

Hence, taking moments about the middle point,

$$5 \cdot \frac{l}{2} + W \cdot \frac{l}{6} = 10 \cdot \frac{l}{2}.$$

Thus $$W = 15 \text{ lb.}$$

(3) *Two weights P and Q are suspended from points A and B in a straight rod of weight W. The rod can move about a fulcrum C. If A, C, B and the centre of gravity G be in a straight line and the rod be in equilibrium when inclined at an angle θ to the horizon, shew that it will be in equilibrium in any other position.*

Let the direction of P, W and Q meet a horizontal line through C in M, L and N (Fig. 107) respectively.

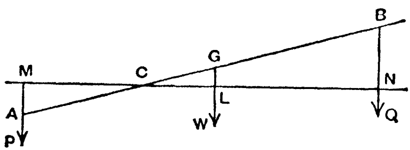

Fig. 107.

Take moments about C.

Then $$P \cdot CM = W \cdot CL + Q \cdot CN.$$

Now $$CL = CG \cos\theta, \quad CM = CA \cos\theta, \quad CN = CB \cos\theta.$$

Hence $$P \cdot CA \cos\theta = W \cdot CG \cos\theta + Q \cdot CB \cos\theta.$$

Thus dividing out by $\cos\theta$ we have

$$P \cdot CA = W \cdot CG + Q \cdot CB.$$

Since this relation does not involve the angle θ it will be true for all values of θ.

It should be noticed however that $\cos\theta$ must not be zero, if it were we should not be justified in dividing out by it. Thus in the initial position the rod must not be vertical; clearly if it were vertical it would be in equilibrium whatever the weights might be.

46. The Wheel and Axle. The apparatus is shewn in Fig. 108. It consists of a wheel or drum of considerable

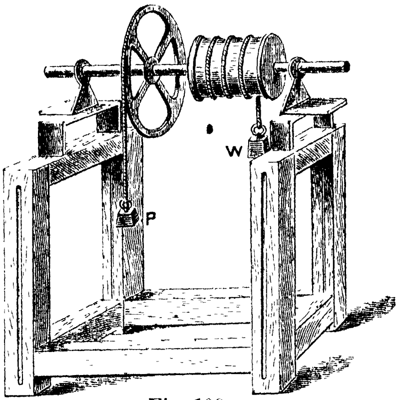

Fig. 108.

diameter round which a rope can be coiled and which can turn about an axis through its centre. A string coiled on this wheel carries the power P. A drum of smaller diameter—the axle—is mounted on the same axis. Round this a rope is coiled in the opposite direction to the first and carries the weight W. Thus when P is lowered the rope round the wheel is uncoiled, that round the axle is coiled up and W is raised.

PROPOSITION 40. *To find the mechanical advantage of the Wheel and Axle.*

Fig. 109 represents a plan of the wheel and axle, perpendicular to the axis round which it can rotate. It is clear that the machine acts like a lever. C the centre is the fulcrum, CA, CB radii of the wheel and axle respectively are the arms.

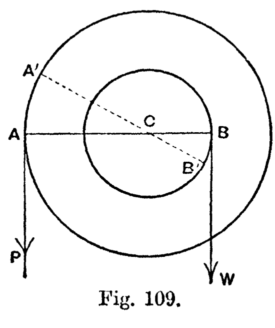

Fig. 109.

Let $CA = a$, $CB = b$.

Then for the conditions of equilibrium we have *either*

(i) By taking moments about C,

$$P \cdot CA = W \cdot CB.$$

Thus
$$\frac{W}{P} = \frac{CA}{CB} = \frac{a}{b},$$

or

(ii) By the principle of work.

Let the apparatus be turned through a small angle θ so that the line $A'CB'$ may become horizontal. Then clearly the power P drops a vertical distance AA', an amount AA' of rope is uncoiled, while W is raised a distance BB'. An amount of rope BB' is coiled up.

Hence
$$W \cdot BB' = P \cdot AA'.$$

Therefore
$$\frac{W}{P} = \frac{AA'}{BB'},$$

and since ACB and $A'CB'$ are straight lines

$$\frac{AA'}{BB'} = \frac{AC}{BC} = \frac{a}{b}.$$

Hence as before

$$\frac{W}{P} = \frac{a}{b}.$$

In the above we have treated the rope as though it were a mathematical line of no thickness. In experiments it may quite well happen that the thickness of the rope is comparable with the radius of the axle, if this is so we may suppose the power and the weight to act respectively at the centre of the rope, and we have then to add to the radii, both of the wheel and of the axle, half the thickness of the rope.

Since the mechanical advantage of the wheel and axle is given by the ratio a/b, we could, by making a large and b small, raise by means of a small power a very large weight, were it not for the fact that if b be too small the axle will not be sufficiently strong to carry the weight. We cannot reduce b beyond a certain limit without endangering the machine. This difficulty is avoided in the differential wheel and axle. See Section 55.

The mechanical advantage of the wheel and axle can be determined by experiment by finding the weight which a given power can support. Friction will however probably prevent any very close agreement between experiment and theory.

47. The Pulley. The Pulley is a small circular disc or wheel with a groove cut in its outer edge round which a string can pass. The wheel can turn on an axis through its centre, the ends of this axis are carried by the block within which the pulley turns.

When the block is fixed as in Fig. 110, the pulley is said to be fixed, in other cases it is moveable. The weight is attached to one end of a string which passes over the groove of the pulley, the power in the case of a fixed pulley can be applied at the other end of the string. If, as we shall suppose, the supports of the pulley are smooth the tension at all points of the string must be the same throughout, and the power will then be equal to the weight.

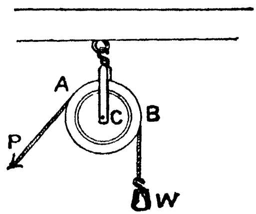

Fig. 110.

For consider the two points A, B, where the string leaves the pulley, and let C be the centre, P the power applied at A, W the weight suspended from B. Then, taking moments about C, we have

$$P \, . \, CA = W \, . \, CB.$$

But $$CA = CB.$$

Hence $$P = W.$$

The fixed pulley is useful only in changing the direction of a force.

48. The single moveable pulley. In this instrument the weight W is suspended from the pulley block; the string passes round the pulley, one end of the string is secured as at C to a fixed support, the power P is applied upwards as at A.

In this case as in others the strings may either be parallel or inclined to each other.

Proposition 41. *To find the relation between the Power and the Weight in a single moveable pulley.*

(i) *When the strings are parallel.*

The forces acting are the tensions of the two parallel strings AD and BC, Fig. 111, and the weight; the weight acts vertically, hence the two strings are vertical. Moreover the tensions are equal and each is equal to P. Thus we have $2P$ upwards balancing W, which acts downwards.

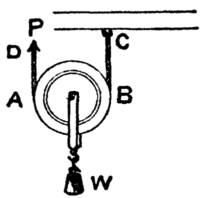

Fig. 111.

Hence $$2P = W.$$

Thus $$\frac{W}{P} = 2,$$

or a given power can raise twice its own weight.

(ii) *When the strings are not parallel.*

Since the tensions in the two strings are equal and are balanced by the weight, their resultant is equal and opposite to the weight, but the resultant of two equal forces bisects the angle between the forces. Thus the two strings are equally inclined to the vertical, let θ, Fig. 112, be the angle between either string and the vertical. The tension in each string is equal to P, hence resolving vertically

$$2P \cos\theta = W.$$

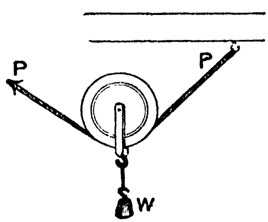

Fig. 112.

Thus $$\frac{W}{P} = 2\cos\theta.$$

If the weight of the pulley be w and it be sufficient to be considered then the downward vertical force is $W + w$, and the last equation becomes

$$\frac{W+w}{P} = 2\cos\theta.$$

If the "Weight" be not vertical but be applied in some other direction by means of a string or rope attached to the pulley block, then its direction still bisects the angle between the strings as in Fig. 113, and a similar equation holds.

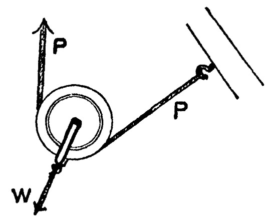

Fig. 113.

PROPOSITION 42. *To apply the Principle of Work to a single moveable pulley with parallel strings.*

Suppose the pulley raised a distance x, so that its centre may move from O to O'.

The simplest way of doing this, Fig. 114, is to suppose both ends of the string to be raised an equal distance x from C and D to C' and D' respectively, the work done, since the tension in each string is P, is $2Px$.

Now suppose that C' is lowered to C again, the pulley being kept fixed, D' will rise an equal distance x to D'', but no work will be done by this, for the work done at one end of the string just balances that done in the other. Thus the end D at which the power P is applied is raised $2x$ and the total work done is as above equal to $2P . x$.

Hence $\quad P . 2x = W . x.$

or $$\frac{W}{P} = 2.$$

If the strings are not parallel we have two equal forces at a point balanced by a third and the problem is the same as that solved in Section 33.

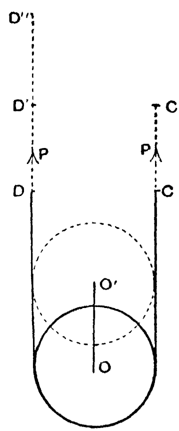

Fig. 114.

49. Systems of Pulleys. Various combinations of Pulleys are in common use. Some of these will be described.

PROPOSITION 43. *To find the mechanical advantage of the first system of pulleys.*

The first system of pulleys consists of a number of pulleys, each of which is suspended by a separate string. One end of each string is attached to a fixed support, the other end of the string after passing round a pulley is fastened as shewn in Fig. 115, to the block of the next pulley. The weight is hung

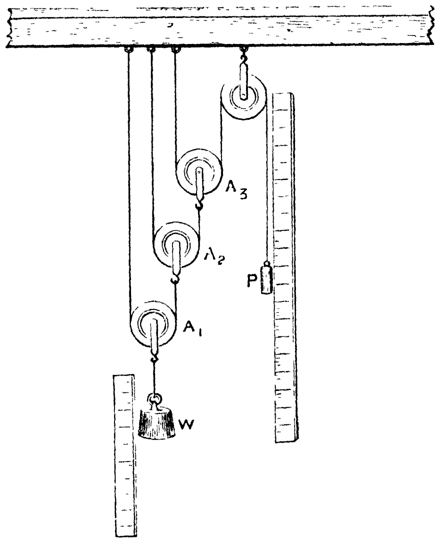

Fig. 115.

from the lowest pulley, the string from the highest moveable pulley passes over a fixed pulley and to it the power is applied.

Thus each pulley after the lowest is acted on downwards by its weight and by the tension of the string which connects it to the pulley next below, and upwards by the two tensions in the parts of the string by which it is supported. Thus the tension in the string round any pulley is half the sum of the weight of that pulley and the tension of the string below it.

Thus let w_1, w_2, w_3... be the weights of the moveable pulleys A_1, A_2,...etc., t_1, t_2, t_3..., the tensions of the strings round A_1, A_2, etc., and suppose there are n moveable pulleys, then

$$t_1 = \frac{1}{2}(W + w_1)$$

$$t_2 = \frac{1}{2}(t_1 + w_2) = \frac{1}{2^2}(W + w_1) + \frac{1}{2}w_2$$

$$t_3 = \frac{1}{2}(t_2 + w_3) = \frac{1}{2^3}(W + w_1) + \frac{1}{2^2}w_2 + \frac{1}{2}w_3,$$

and so on.

Moreover $$t_n = P.$$

Hence

$$P = t_n = \frac{1}{2^n}(W + w_1) + \frac{1}{2^{n-1}}w_2 + \dots \frac{1}{2^{n-2}}w_3 + \dots \frac{1}{2}w_n,$$

or multiplying up by 2^n,

$$2^n P = W + w_1 + 2w_2 + 2^2 w_3 + \dots 2^{n-1} w_n.$$

If the weights of the pulleys be neglected the expressions become simpler though the principle is just the same.

Thus, if there be four moveable pulleys, the tension in the first string is $\frac{W}{2}$, in the next $\frac{W}{2^2}$, and in the fourth $\frac{W}{2^4}$.

Hence $$\frac{W}{2^4} = P.$$

Therefore $$\frac{W}{P} = 2^4 = 16.$$

Thus with this system a given "Power" P could support a "Weight" of $2^n . P$ where n is the number of moveable pulleys.

Proposition 44. *To apply the Principle of Work to the first system of pulleys.*

Let the weight and the first pulley rise a distance x. The end of the string round this pulley rises $2x$; thus the second pulley rises $2x$, the next pulley rises twice this or 2^2x. Thus if there be n pulleys as before the "Power" P moves a distance $2^n x$.

Hence

$$P \,.\, 2^n x = (W + w_1)\, x + w_2 \,.\, 2x + w_3 \,.\, 2^2 x + w_n \,.\, 2^{n-1} x.$$

Therefore

$$2^n P = (W + w_1) + 2w_2 + 2^2 w_3 + \ldots\ 2^{n-1} w_n,$$

which is the same result as previously obtained.

PROPOSITION 45. *To find the mechanical advantage of the second system of pulleys.*

In the second system there are two sheaves of pulleys in separate blocks. The string is attached to one of the blocks, Fig. 116,—in the figure it is the upper—and passes round the pulleys in turn first under one in the lower block, then over one

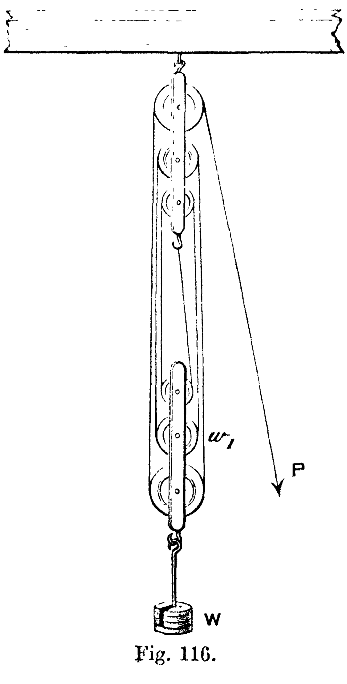

Fig. 116.

in the upper and so on. The pulleys are sometimes arranged with a common axis, sometimes the various pulleys in a block are placed one below the other as in the figure.

In either case the tension of the string is equal to the power; let there be n strings at the lower block. The upward force will be found by multiplying the tension by n, the downward force is the weight supported W, together with the weight of the lower block w_1.

Hence $$nP = W + w_1.$$

We can apply the principle of work thus. If the lower block be raised a height x, a length x of each string will be left slack. Hence the end of the string can move a distance nx.

Thus, $$P \,.\, nx = (W + w_1)\, x,$$

or $$nP = W + w_1.$$

PROPOSITION 46. *To find the mechanical advantage of the third system of pulleys.*

In this system, Fig. 117, one end of each string is attached

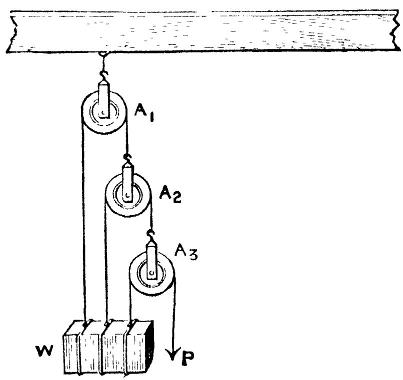

Fig. 117.

to a bar which carries the weight. The uppermost pulley is fixed; a string passes over it and supports the next pulley, another string passes over this and supports the third, and so on, the last string passes over the last moveable pulley and to it the Power is applied.

Now if t_1, t_2 .. t_n be the tension in the strings beginning from that over the topmost pulley, w_1, w_2 ... the weights of the pulleys:

Then
$$t_n = P,$$
$$t_{n-1} = 2t_n + w_n = 2P + w_n,$$
$$t_{n-2} = 2t_{n-1} + w_{n-1} = 2^2P + 2w_n + w_{n-1},$$
$$t_1 = \dots 2^{n-1} P + 2^{n-2}w_n + \dots + w_2.$$

Also
$$W = t_1 + t_2 + \dots t_n$$
$$= P(1 + 2 + 2^2 + \dots + 2^{n-1})$$
$$+ w_n(1 + 2 + 2^2 \dots 2^{n-2})$$
$$+ w_{n-1}(1 + 2 \dots + 2^{n-3})$$
$$+ \dots w_2.$$

Now we know that
$$1 + 2 + 2^2 + \dots + 2^{r-1} = 2^r - 1.$$

Thus
$$W = P(2^n - 1) + w_n(2^{n-1} - 1) + w_{n-1}(2^{n-2} - 1) + \dots + w_2.$$

If the weights of the pulleys be neglected the expression is simplified, for the tension in each string, beginning from the power, is clearly twice that in the string before.

Thus
$$t_1 = P,\ t_2 = 2P\dots$$
$$t_n = 2^{n-1} P.$$
$$W - t_1 + t_2 + \dots t_n$$
$$= P\{1 + 2 + 2^2 + \dots 2^{n-1}\}$$
$$= P\{2^n - 1\}.$$

We may apply the Principle of Work thus. Let the weight and the bar carrying it rise a distance x.

If all the pulleys retained their positions fixed there would be a length x slack in each string.

In consequence of this alone each pulley after the first could be lowered a distance x; the first pulley is fixed, hence the second pulley drops a distance x; in consequence of this drop the third pulley will be lowered through twice this distance or $2x$; to this must be added the direct drop x due to the rise of the weight, which would have taken place even if the second pulley had been fixed.

Thus the actual drop of the third pulley is $2x+x$, or $3x$, and we may write this $(2^2-1)\,x$.

In consequence of this the fourth pulley drops twice as far or $2\,(2^2-1)\,x$, and to this we must add the fall x for the direct rise of the weight.

Thus the drop of the fourth pulley is

$$2\,(2^2-1)\,x+x,$$

or

$$x\,(2^3-2+1),$$

or

$$x\,(2^3-1).$$

The law is now clear, the nth pulley drops $x\,(2^{n-1}-1)$, and the Power

$$x\,(2^n-1).$$

Thus the equation of work gives

$$Wx = Px\,(2^n-1)+w_n x\,(2^{n-1}-1)+\ldots$$
$$+\ldots+w_3 x\,(2^2-1)+w_2 x.$$

Therefore

$$W = P\,(2^n-1)+w_n\,(2^{n-1}-1)+\ldots$$
$$+w_3\,(2^2-1)+w_2.$$

50. Experiments with Pulleys. It is not possible to verify by direct experiment these results. In any system of pulleys the friction is considerable, a smaller power than that given by the equations is sufficient to support a given weight, a larger power than is given is necessary just to raise the weight.

We can however verify by direct measurement the result that when a pulley rises a distance x, and one end of the string round it is held fixed, then the other end rises a distance $2x$.

Thus consider the first system of pulleys. Support a Weight W from the lowest pulley, and let the Power P be another weight supported in a suitable scale-pan. Adjust two vertical scales as shewn in Fig. 115 above, by the side of the power and the weight respectively. Displace the system by raising the weight W a measured distance a on its scale, and observe the distance through which P descends, it will be found to be equal to $2^n P$ where n is the number of moveable pulleys. Similar observations can be made for the other systems of pulleys.

In the third system of pulleys the bar to which the weight is attached will not remain horizontal unless the point of attachment of the weight is the point of action of the resultant of the tensions. Thus if K be the point of attachment, A the end of the bar to which the string carrying the weight is attached, and $2a$ the distance between the strings, that is the diameter of each pulley, then taking moments about K

$$W \, . \, AK = t_1 \, . \, 0 + t_2 \, . \, 2a + t_3 \, . \, 4a + \ldots$$

By substituting the values of t_1, t_2, etc., found in Section 49, the position of K can be found.

51. The Inclined Plane. Consider a plane inclined to the horizon at an angle α. A weight could be raised to the top of such a plane—if we neglect friction or take means to make it small—with the application of a smaller force by causing it to slide up the plane, than by lifting it directly. Hence the inclined plane is one of the mechanical powers.

The solution of the problem depends in part on the direction in which the force is impressed.

PROPOSITION 47. *To find the mechanical advantage of an inclined plane when the Power acts parallel to the plane.*

Let BAC, Fig. 118, be an inclined plane, AC being horizontal and BC vertical. Let the angle BAC be equal to α, and consider a body of weight W at rest on the plane, which is supposed to be smooth. Let AB, the length of the plane be equal to l, and let the height BC be h.

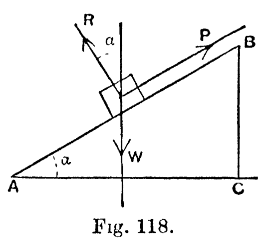

Fig. 118.

The forces acting are the weight W vertical, the power P parallel to the plane, and the resistance R at right angles to the plane. The relation between these quantities can be found in various ways.

(i) *By the resolution of forces.*

The direction along the plane, in which P acts, and a line at right angles to this, along which R acts, will clearly be "convenient" directions in which to resolve the forces.

The angle between the normal to the plane—the direction of R—and the direction of W is clearly α. Hence the component of W, perpendicular to the plane, is $W \cos \alpha$, and along the plane it is $W \sin \alpha$.

Hence resolving along the plane

$$P = W \sin \alpha,$$

and perpendicular to the plane

$$R = W \cos \alpha.$$

Thus P and R are found in terms of W and α.

Moreover, $$\sin \alpha = \frac{BC}{AB} = \frac{h}{l}.$$

Therefore $$\frac{P}{W} = \sin \alpha = \frac{h}{l},$$

or $$P \,.\, l = W \,.\, h.$$

(ii) *By an application of the principle of work.*

The work done by P in moving the body a small distance s along the plane is $P \,.\, s$, if at the same time the body rise a height z, the work done against z is $W \,.\, z$.

No work is done by the resistance R, since the motion is everywhere at right angles to the direction of R.

Hence $$P \,.\, s = W \,.\, z,$$

or $$\frac{W}{P} = \frac{s}{z} = \frac{h}{l},$$

from the figure.

Hence $W \,.\, h = P \,.\, l$, as before.

We can obtain the result directly by considering the work done in moving from A to B.

(iii) *By the triangle of forces.*

Since the forces P, W and R, maintain the body in equilibrium, they must be proportional to the sides of any triangle drawn parallel to them. Let G, Fig. 119, be the particle. Let GK, vertical, meet AC in K, and GL at right angles to the plane meet AC in L. Draw KM parallel to the plane to meet GL in M. Then P, R and W are respectively parallel to KM, MG and GK.

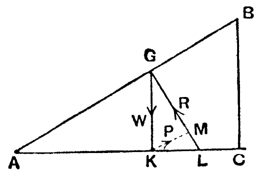

Fig. 119.

Hence $\dfrac{P}{KM} = \dfrac{R}{MG} = \dfrac{W}{GK}$.

Again the triangles CBA, MKG are similar.

Hence
$$\frac{KM}{GK} = \frac{BC}{AB} = \sin \alpha,$$
$$\frac{MG}{GK} = \frac{AC}{AB} = \cos \alpha.$$

Thus
$$\frac{P}{W} = \frac{KM}{GK} = \frac{BC}{AB} = \frac{h}{l} = \sin \alpha,$$
$$\frac{R}{W} = \frac{MG}{GK} = \frac{AC}{AB} = \cos \alpha.$$

PROPOSITION 48. *To find the mechanical advantage of an inclined plane when the power acts horizontally.*

Let AB, Fig. 120, be the plane, BC being vertical and AC horizontal.

Let P be the power acting horizontally, and let α be the angle of the plane. Let $BC = h$ and $AB = l$.

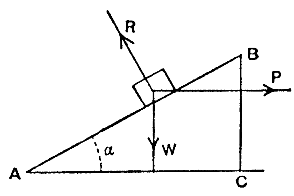

Fig. 120.

Let W be the weight and R the resistance of the plane.

We can find the mechanical advantage in various ways as follows.

(i) *By the resolution of forces.*

Resolve horizontally and vertically. The components of R are $R\cos\alpha$ vertical, and $R\sin\alpha$ horizontal. P is horizontal and W vertical.

Hence, resolving vertically,

$$W = R\cos\alpha.$$

Resolving horizontally

$$P = R\sin\alpha.$$

Thus $$\frac{P}{W} = \frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha.$$

(ii) *By the principle of work.*

In moving the body from A to B, since the displacement in the direction of P is AC, while that opposite to the direction of W is CB, the work done by P is $P\,.\,AC$, and that done against W is $W\,.\,BC$.

Thus $$P\,.\,AC = W\,.\,BC.$$

Hence $$\frac{P}{W} = \frac{BC}{AC} = \tan\alpha.$$

(iii) *By the triangle of forces.*

Let G, Fig. 121, be the body, and let GK vertical meet AC in K, and GL normal to the plane meet it in L.

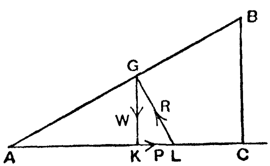

Fig. 121.

Then P, R and W are parallel respectively to KL, LG and GK.

Thus $$\frac{P}{KL} = \frac{R}{LG} = \frac{W}{GK}.$$

But the triangles LKG and BCA are similar.

Hence $$\frac{P}{W} = \frac{KL}{GK} = \frac{BC}{AC} = \tan\alpha,$$

$$\frac{R}{W} = \frac{LG}{GK} = \frac{AB}{AC} = \sec\alpha.$$

If the force be inclined as shewn in Fig. 122 at an angle ϵ to the plane, it is usually simplest to resolve parallel and perpendicular to the plane.

Since three forces which maintain a body in equilibrium are in one plane and R and W lie in a vertical plane through the particle, the direction of P must also be on this vertical plane.

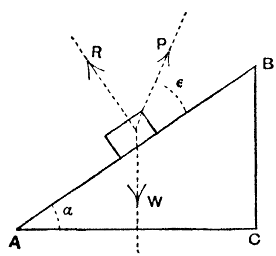

Fig. 122.

Resolving parallel to the plane we have

$$P \cos \epsilon = W \sin \alpha.$$

Resolving perpendicular to the plane we have

$$R + P \sin \epsilon = W \cos \alpha.$$

Some of these results can be verified by experiment.

EXPERIMENT 7. *To prove that on an inclined plane, when the Power acts parallel to the plane, the ratio of the Power to the Weight is equal to that of the height of the plane to its length, and to verify the Principle of Work.*

The plane is a wooden board shewn at AB, Fig. 123, to which a sheet of glass is attached. The board is hinged at A to a second board, which can be clamped to the table. At B, which is at some convenient distance (say 10 inches from A).

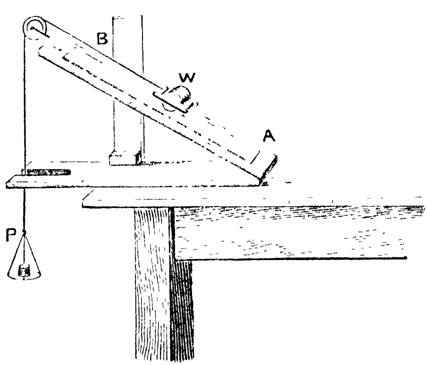

Fig. 123.

there is a thumb-screw. By means of this there can be clamped to the plane a vertical rod with a slot parallel to its length. The rod is graduated in such a way that the height of B above the lowest point A can be read off directly.

Thus the height h can be measured, and the length l is a known constant. A pulley is fixed at the top of the plane. The "Weight" W consists of a heavy brass roller mounted in a frame so as to turn with very little friction, a string attached to the frame passes over the pulley and supports a scale-pan, into which various weights can be placed. The "Power" P is the weight of this scale-pan and weights.

The frame and pulley are arranged so that the string between them when tight is parallel to the plane. Thus the Power P acts on the Weight W parallel to the plane.

Set the plane so that h may have some convenient value, say 5 inches. Observe the value of P required to support W for this value of h. To do this accurately find the value, P_1, which will just drag W up the plane and the value, P_2, which will only just let it roll down. The proper value of P, that is, the value which it would have if there were no friction, is the mean of P_1 and P_2. When the observations are made it will be found that $W . h = P . l$, or that $P : W = h : l$. Again, if α is the angle which the plane makes with the horizon $h/l = \sin \alpha$. Hence $P = W \sin \alpha$. In our case since the length l is 10 inches we have to divide the height by 10 to get $\sin \alpha$, thus if $h = 5$ inches, $\sin \alpha = \cdot 5 = \frac{1}{2}$, and $\alpha = 30°$. Again, suppose the weight is allowed to move down the plane, it will have fallen a vertical height of h inches. Thus the work done by the weight will be $W . h$ units of work; and clearly the power P, since it is attached to W by the string, must have been raised l inches and the work done on it will be Pl units; but by what we have seen $P . l = W . h$, which proves the principle of work for the inclined plane in this case. Take a series of values of P for various values of h and thus shew that in all cases $P . l = W . h$.

52. The Wedge. This is a sort of double inclined plane or prism, Fig. 124, made of iron or steel or some such hard material and used for splitting wood or for other like purposes.

Thus if BAC, Fig. 125, be a wedge of angle α driven into a piece of wood by a weight W applied downwards, and we

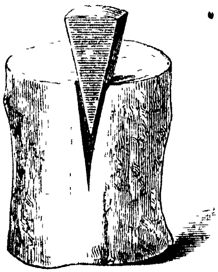

Fig. 124.

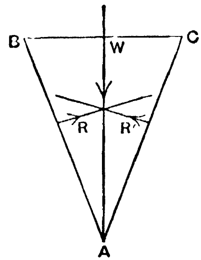

Fig. 125.

suppose R, R', the pressures which the two faces of the wedge exert on the obstacle, to act normally to its faces AB and AC, and further that these two faces are equally inclined to the vertical, at angles therefore of $\frac{\alpha}{2}$, we can find the relation between R, R' and W, thus, supposing the wedge to be smooth.

The vertical components of R and R' balance W, the horizontal components of R and R' are in equilibrium.

Hence resolving vertically

$$W = (R + R') \sin\frac{\alpha}{2},$$

$$R \cos\frac{\alpha}{2} = R' \cos\frac{\alpha}{2}.$$

Hence $$R = R',$$

and $$W = 2R \sin\frac{\alpha}{2}.$$

In reality the friction involved in the use of the wedge is enormous.

We ought to consider two other forces F, F' acting parallel to the faces of the wedge.

If we suppose the machine to be symmetrical with regard to the vertical line bisecting the angle a, we have

$$W = 2R \sin\frac{a}{2} + 2F \cos\frac{a}{2};$$

and unless we know the relation between R and F we cannot carry the solution any further.

53. The Screw. Consider a sheet of paper cut into the form of a right-angled triangle BAC, Fig. 126 (a), and wrap it round a cylinder so that the base AC of the triangle

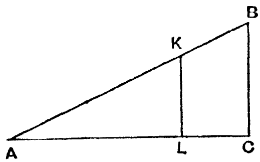

Fig. 126 (a).

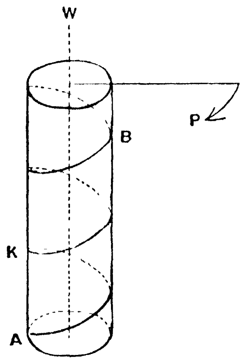

Fig. 126 (b).

may be at right angles to the axis of the cylinder. The hypotenuse AB will form a spiral curve round the cylinder, Fig. 126 (b). Now imagine a projecting thread to be fixed to the outside of this cylinder so as to coincide with the spiral curve thus drawn; the cylinder then becomes a "screw." Suppose now that the paper is wound inside a hollow cylinder of the same radius as the solid cylinder just described and that a hollow groove is cut in the surface of this hollow cylinder, the groove being of such a form that the projecting thread just fits it. The hollow cylinder constitutes a "nut" in which the screw can turn, Fig. 127. Let the nut be held fixed and the end of the screw inserted in it. Turn the screw round its axis by means of a lever; as it is turned its end moves outwards through the nut parallel to the axis. If the axis be vertical, a weight W can be raised by the application of a power P to the end of the lever. In any case if the nut be held fixed, force can be exerted by the end of the screw.

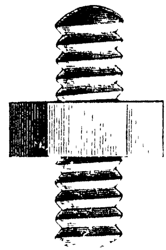

Fig. 127.

The angle α which the thread of the screw makes with a plane at right angles to the axis is called the angle of the screw.

The distance measured parallel to the axis between two consecutive turns of the thread is called the pitch and depends on the angle and on the radius of the cylinder on which the screw is cut.

For let K, Fig. 126 (a), be a point on the paper triangle which when it is rolled on to the cylinder comes directly over the point A. Draw KL perpendicular to the base AC, parallel that is to the axis, when the paper is rolled on the cylinder, L will coincide with A, and KL is the distance between two threads.

Hence $$KL = h.$$

Again AL is clearly the circumference of the cylinder so that if b be its radius we have

$$AL = \text{circumference of a circle of radius } b = 2\pi b.$$

But $$\frac{KL}{AL} = \tan KAL = \tan \alpha.$$

Thus $$KL = AL \tan \alpha,$$

or $$h = 2\pi b \tan \alpha.$$

We notice further that when the screw makes one complete revolution, a point such as K is brought into the position previously occupied by A, the end of the screw advances a distance h parallel to the axis.

It is impossible to obtain a screw in which there is no friction. We must therefore suppose that at each point of the thread there is acting a normal force at right angles to the thread, and a tangential force parallel to it. Suppose these forces to be uniformly distributed, and let R and F be their values for each unit of length of the screw. Let l be the whole length of the screw. R may be resolved into a force $R\cos\alpha$ parallel to the axis, and $R\sin\alpha$ at right angles to it; and F has for its components $F\sin\alpha$ parallel to the axis, $F\cos\alpha$ at right angles to it. The forces at right angles to the axis have moments round the axis, the others have not.

Let us suppose also that we are trying to raise the weight, then the frictional force helps the action of the weight and opposes that of P.

Suppose further that P acts at the end of an arm a in a direction at right angles to the axis.

Then resolving vertically

$$W = Rl\cos\alpha + Fl\sin\alpha.$$

Taking moments about the axis

$$Pa = Rl\sin\alpha \,.\, b + Fl\cos\alpha \,.\, b.$$

If W and P are known these equations will give us R and F; we cannot use them to find the mechanical advantage unless we have some relation between F and R.

If we suppose F is zero, which in practice is never the case, then

$$W = Rl\cos\alpha,$$

$$Pa = Rl\sin\alpha \,.\, b.$$

Hence $$\frac{W}{P} = \frac{a}{b}\cot\alpha.$$

PROPOSITION 49. *To find the mechanical advantage of the Screw.*

We can obtain the result most easily by the Principle of Work. For while W is being raised a distance h, the point of application of P moves once round a circle of radius a, hence its displacement is $2\pi a$; moreover the direction of P is tangential to this circle. Thus the work done is $P \,.\, 2\pi a$.

Hence $$P \,.\, 2\pi a = Wh = W \,.\, 2\pi b\tan\alpha.$$

Thus $$\frac{W}{P} = \frac{2\pi a}{2\pi b\tan\alpha} = \frac{a}{b}\cot\alpha.$$

54. Combinations of Simple Machines. A complex machine is usually made up of a number of Simple Machines. In these the "Weight" of the first becomes the

"Power" of the next and so on. In such a case the mechanical advantage of the whole is found by multiplying together those of all the simple machines.

For let P_1, P_2, P_3, etc., be the powers, W_1, W_2, W_3, etc., the weights, m_1, m_2 ... m_n the mechanical advantages.

Then $$W_1 = P_2, \quad W_2 = P_3 \ldots \quad W_{n-1} = P_n.$$

Hence $$m_1 = \frac{W_1}{P_1} = \frac{P_2}{P_1},$$

$$m_2 = \frac{W_2}{P_2} = \frac{P_3}{P_2},$$

$$m_{n-1} = \frac{W_{n-1}}{P_{n-1}} = \frac{P_n}{P_{n-1}},$$

$$m_n = \frac{W_n}{P_n}.$$

Thus multiplying all together

$$m_1 \,.\, m_2 \,.\, m_3 \ldots m_n$$

$$= \frac{W_n}{P_1} = \text{mechanical advantage of the whole.}$$

Some special forms of machines are described below.

55. The Differential Wheel and Axle. In this apparatus shewn in Fig. 128, the axle consists of two drums or cylinders of different radii, b and c. A rope, the ends of which are coiled in opposite directions round these two drums, passes under a single moveable pulley from which the weight W is supported. As the axle is turned the rope is coiled up on one drum and uncoiled from the other, the motion of the weight depends on the difference of these two effects. The power P is usually applied at the circumference of a wheel of radius a.

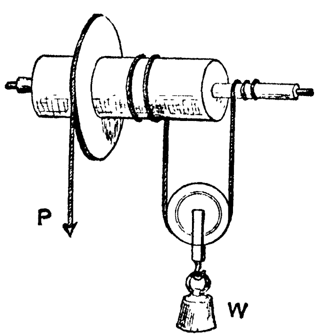

Fig. 128.

In Fig. 129, let $ACOB$ represent the machine as viewed from a point on the axis. Suppose it to be turned through a small angle θ so that $A'C'OB'$ may become horizontal.

The power falls a distance $a\theta$, the end C of the string round the pulley falls a distance $c\theta$, while B rises a distance $b\theta$; hence the loop of string carrying the pulley is shortened by $(b-c)\,\theta$, and therefore the pulley and weight rise $\frac{1}{2}(b-c)\,\theta$.

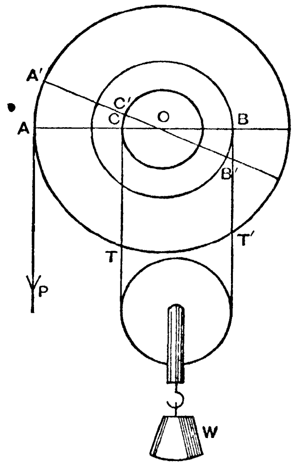

Fig. 129.

Thus the Principle of Work gives

$$P \cdot a\theta = W \cdot \tfrac{1}{2}(b-c)\,\theta,$$

or
$$\frac{W}{P} = \frac{2a}{b-c}.$$

Hence by making b and c nearly equal, W can be made very large compared with P, without unduly reducing the strength of the machine.

For a given motion of the Power the distance traversed by the Weight can be made very small, hence the ratio of the "Weight" to the "Power" can be made very large.

We can solve the problem without using the Principle of Work thus:

Let T be the tension in the string carrying the pulley, then taking moments about O,

$$Pa + Tc = Tb.$$

Hence
$$Pa = T(b-c).$$

But from the equilibrium of the pulley $2T = W$.

Thus
$$P = \frac{T(b-c)}{a} = \frac{W(b-c)}{2a}.$$

56. The Differential Screw. This machine consists of two screws of slightly different pitch h and k. The axes of the two screws coincide, the second screw works inside the cylinder on which the first is cut. Thus if H, Fig. 130, be the outer screw, and K the inner screw, then on giving the outer screw one complete turn its point will move downwards through a distance h, and if the inner screw did not turn in the outer, it too would be displaced this same distance; but the inner screw does turn relatively to the outer, its point is in consequence raised relatively to the outer screw a distance k, thus the point of the inner screw actually descends a distance $(h-k)$. Hence if P be the "Power" impressed at one end of an arm a, we have

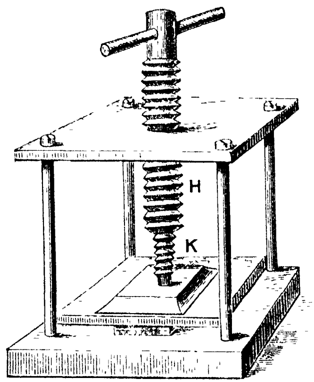

Fig. 130.

$$W(h-k) = P \, . \, 2\pi a.$$

Thus
$$\frac{W}{P} = \frac{2\pi a}{h-k}.$$

57. Cog Wheels. A train of cog wheels is virtually a combination of Wheels and Axles. Consider two such wheels, Fig. 131. Let A and B be their centres and let them be in

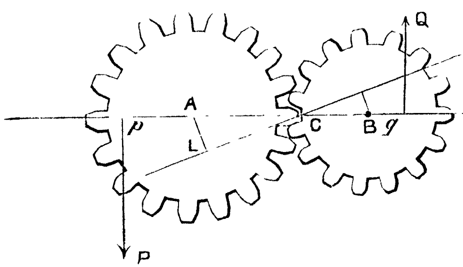

Fig. 131.

contact at C in the line AB. Let R be the force between the wheels at C, and let $AC = a$, $BC = b$. Draw AL, BM perpendicular on the direction of R.

Let the "Power" be a force P acting at an arm p, the "Weight" a force Q acting at an arm q.

Then for the equilibrium of A,

$$P \cdot p = R \cdot AL.$$

For the equilibrium of B,

$$Q \cdot q = R \cdot BM.$$

Also the triangles ACL, BCM are similar.

Thus $$\frac{P \cdot p}{Q \cdot q} = \frac{AL}{BM} = \frac{AC}{BC} = \frac{a}{b}.$$

Hence $$\frac{Q}{P} = \frac{p}{a} \cdot \frac{b}{q}.$$

Now $\frac{p}{a}$ is the mechanical advantage of the wheel A considered as a bent lever; while $\frac{b}{q}$ is that of the other wheel. We have thus found the mechanical advantage of the whole, and see that it is the product of those of the two parts.

58. The Spanish Barton. This forms a useful combination of Pulleys shewn in Fig. 132. A and B are two moveable pulleys which are suspended by a string over a fixed pulley C: the "Weight" W is attached to A. A string passes from a fixed point D under A and over B, and the "Power" P is attached to this.

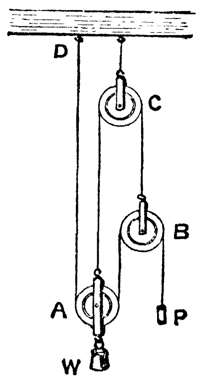

Fig. 132.

Let w_1, w_2 be the weights of the pulleys A and B.

Suppose the weight and the Pulley A raised a distance x. In consequence of this, if the pulley B were fixed, a length $2x$ of the string would be left slack. Thus P would fall a distance $2x$; but since A and B are connected by a string, when A

is raised a distance x, B falls the same distance. In consequence of this the "Power" falls a further distance $2x$: thus on the whole the "Power" falls $4x$. Hence W and A each rise x, P falls $4x$ and B falls x.

Therefore $$(W + w_1)\,x = w_2 x + P\,.\,4x,$$

$$W = w_2 - w_1 + 4P.$$

In practice w_1 and w_2 would usually be equal. Hence the mechanical advantage (W/P) is 4.

We can also find the tension of the strings and solve the problem thus.

Let T be the tension in the string over the fixed pulley, that in the string round the moveable pulleys is P.

Hence for the equilibrium of A

$$W + w_1 = 2P + T,$$

and for that of B,

$$T = 2P + w_2.$$

Therefore $$W + w_1 = 4P + w_2.$$

59. The Balance. The balance, Fig 133, in its simplest form consists of a lever which can turn about a fulcrum: it is used for comparing the masses of two bodies, or rather for

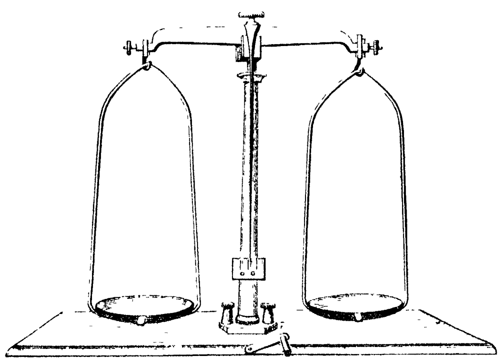

Fig. 133.

determining in terms of the standard mass the mass of any other body. This is done by comparing the weights of the bodies. From the arms of the lever two equal and similar scale-pans are suspended, the standard mass is placed in one of these, the body whose mass is required in the other, and, if the balance be in adjustment, the two masses are equal when the arms are horizontal.

To determine when this is the case with accuracy, a vertical pointer is attached to the beam near the fulcrum, the lower end of this pointer moves over a horizontal scale, being adjusted so as to rest at the centre of the scale when the beam is horizontal. The arms of the beam ought, as we shall see, to be equal and similar if the balance is to be accurate.

The requisites of a good balance are:

(i) Truth. (ii) Sensitiveness. (iii) Stability.

A balance is said to be *True* if the beam be horizontal whenever equal masses are placed in the scale-pans.

A balance is *Sensitive* when the beam deviates appreciably from its horizontal position for a very small difference $P - Q$, in the two masses P and Q.

A balance is *Stable* when the beam if disturbed from its equilibrium position readily comes back to it.

We shall consider how to secure these conditions separately. The points from which the scale-pans are suspended are A and B, Fig. 134, these in a good balance are steel knife-edges,

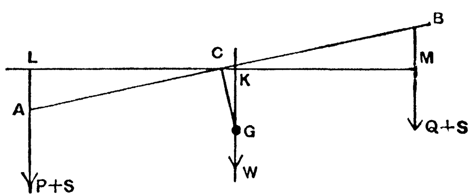

Fig. 134.

secured to the beam in such a way that their edges are at right angles to the length of the beam. The scale-pans are attached to small flat plates of steel, or better of agate, which rest on these knife-edges when the beam is in use and hang with their centres of gravity below the respective knife-edges. The weights thus act vertically through A and B.

The fulcrum C is also a similar knife-edge resting on a steel or agate plate. In good balances there is an arrangement by which the plates are lifted off the knife-edges when the balance is not in use.

It is desirable[1] in a balance that the fulcrum C and the points of support of the scale-pans A and B should be in one straight line, and we shall assume this condition satisfied.

When the balance is loaded, the forces acting are the weight of the beam, let this be W, the weights of the scale-pans S, S' respectively and the weights P, Q of the masses in the scale-pans. The weight W acts at the centre of gravity of the beam; if the beam remains horizontal when the scale-pans are removed, this point must be vertically below the fulcrum, let it be G, then CG is at right angles to ACB.

Let $$CG = h, \ AC = a, \ BC = b.$$

PROPOSITION 50. *To find the condition that a balance may be true.*

The balance is true provided the beam be horizontal whatever equal masses are placed in the scale-pans; if the beam be horizontal the centre of gravity G is vertically below the fulcrum, thus the weight of the beam W has in this case no moment about the fulcrum.

Suppose now the scale-pans are empty, and the beam horizontal; the only forces which have a moment about the fulcrum are the weights S, S' of the scale-pans; these act vertically through A and B respectively. Hence taking moments about the fulcrum,

$$S \cdot a = S' \cdot b.$$

[1] It can be shewn that if this condition be not true the sensitiveness of the balance will vary with the load; the condition cannot be always accurately satisfied, for as the load increases the beam bends and the points A and B are brought down below the fulcrum.

Suppose now that two equal masses P, P are placed one in either scale-pan: if the balance be true the beam will still be horizontal, thus taking moments again

$$(P+S)\,a=(P+S')\,b,$$

or

$$P\,.\,a+S\,.\,a=P\,.\,b+S'\,.\,b.$$

But $$S\,.\,a=S'\,.\,b.$$

Hence subtracting $$P\,.\,a=P\,.\,b.$$

Thus $$a=b,$$

and since $S\,.\,a=S'\,.\,b$, we must have also

$$S=S'.$$

Therefore if a balance is to be true the arms must be equal in length and the scale-pans equal in weight.

Proposition 51. *To find the condition that a balance may be sensitive.*

In finding this condition we assume that the balance is true. The condition required is that, when the weights P and Q differ slightly, the beam should be sensibly inclined to the horizon.

Let the centre of gravity of the beam be at G, a distance h below the fulcrum. Then if the beam is displaced, its weight W will have a moment about the fulcrum tending to restore it; in order that the displacement for a small difference $(P-Q)$ may be considerable the moment of the weight about C must be small; this moment will depend on the value of $W\,.\,h$, and can be made small by making W small or h small or both, thus for sensibility the beam must be light and its centre of gravity near the fulcrum.

But the displacement will also be great if the moment due to the difference of the weights is large.

This moment, since the balance is true, is $(P-Q)\,a$, and for a given difference $P-Q$ can be made large by making a large. Thus the sensitiveness is increased by lengthening the arms. The sensitiveness therefore may be made considerable either by (α) increasing the length of the arms, or (β) reducing the weight of the beam, or (γ) bringing the centre of gravity of the beam near the fulcrum.

In order to secure this last condition there is usually, in good balances, a short vertical wire attached to the beam above the fulcrum. This has a screw thread cut on it and a metal sphere can be screwed up or down on this wire; by raising the sphere the centre of gravity of the beam is raised and the sensitiveness increased.

We may put the results just obtained more briefly thus.

Let the beam be displaced through an angle θ and let a horizontal line through C meet in L, M and K, Fig. 134, the verticals through A, B and C, which are the lines of action of $P+S$, $Q+S$ and W respectively.

Then the angles ACL and CGK are both equal to θ.

And
$$CM = CL = CA \cos\theta = a\cos\theta,$$
$$CK = CG \sin\theta = h \sin\theta.$$

Thus taking moments about C,
$$(P+S)\,CL = (Q+S)\,CM + W\,.\,CK.$$

Hence
$$(P-Q)\,a\cos\theta = Wh\sin\theta,$$

thus
$$\frac{\tan\theta}{P-Q} = \frac{a}{W\,.\,h}.$$

Now $\tan\theta/(P-Q)$ is a measure of the sensitiveness. The balance is sensitive if this fraction be large when $P-Q$ is small. Thus for sensitiveness a/Wh is large.

Hence a must be large or Wh small, or both these conditions must hold.

PROPOSITION 52. *To find the condition that a balance may be stable.*

For this it is necessary that, when the pans are equally loaded, the beam after displacement should return rapidly to its position of equilibrium. Now if the pans be equally loaded and the balance be displaced the moments of the loads about the fulcrum balance, and the only moment tending to restore equilibrium is that of the weight of the beam. This depends on the product $W\,.\,h$; hence for stability this moment must tend to bring the beam back, thus G must be below the fulcrum, and for rapid action the product $W\,.\,h$ must be considerable.

It will not however be sufficient to make W large, for if this only is done the mass to be moved is increased as well as the impressed force and the acceleration is not changed. For great stability then h must be considerable, the centre of gravity must be well below the fulcrum.

It will be noticed that this condition is antagonistic to one of those obtained for sensitiveness; fair sensitiveness and reasonable stability can be secured by making h not very small and giving the balance long arms.

The relative importance of the two conditions depends on the purposes for which the balance is required.

Stability and rapid action would be the main desideratum in a balance employed for weighing coals; a man conducting a chemical or physical research would attach greater importance to sensitiveness, though in this case also rapid action is important. A complete discussion of the question would take us beyond the limits of this book.

60. Use of a Balance. It is important to be able to test a balance for accuracy and to determine any errors which it may possess. Now in using a balance we have always to bring the beam back to the horizontal position. This may be done by placing weights in one or other of the pans until the pointer comes back to zero at the middle of the scale; instead of waiting however till the pointer is actually at the middle of the scale, we may notice the distance it moves on either side of the zero mark, and adjust the weights until these oscillations are equal, we then know that when the beam is at rest the pointer will be at zero[1].

We now proceed to describe some experiments with a balance.

EXPERIMENT 8. *To determine the ratio of the arms of a balance, and to weigh a body correctly in a balance with unequal arms.*

Let the balance come to rest with its pans unloaded, let S and S' be the weights of the scale-pans, a and b the lengths of the arms.

[1] A more exact method of "weighing by oscillation" is given in Glazebrook and Shaw's *Practical Physics*, Section 12, p. 107.

If the pointer does not come to zero, it may be because the pans are unequal in mass, or the arms in length, or it may be that the balance case is not level so that the stem which carries the beam is not vertical. Test for this by a spirit level, assuming the maker has set the stem at right angles to the bottom of the case so that the knife-edges and the plate on which the fulcrum rests will be horizontal when the bottom of the case is.

If the pointer does not come to zero[1], load one of the scale-pans with some shot or bits of lead foil until it does.

Then, since the beam is horizontal, we have if S and S' now denote the weights of the scale-pans and loads used to bring the beam horizontal,

$$Sa = S'b.$$

Let W be the weight of the body whose accurate weight is required.

Place it in the left-hand scale-pan and place weights P in the right-hand scale-pan until the beam is again horizontal.

Then $$(W + S)\,a = (P + S')\,b.$$

Hence $$Wa = Pb.$$

Now interchange W and P: if the beam remain horizontal the arms are equal, if not, let Q be the weights in the left-hand scale-pan which are required to balance W when in the right-hand scale-pan.

Then $$(Q + S)\,a = (W + S')\,b.$$

But $$Sa = S'b.$$

Hence $$Qa = Wb \quad\text{.........................(i)}$$

And we have already seen that

$$Wa = Pb \quad\text{.........................(ii).}$$

[1] It is not important that the pointer should read zero exactly, provided its position with the balance unloaded is noted, and that it is always brought back to this position when a body is being weighed. To secure this it is usually simplest to adjust the balance so that the unloaded reading may be at the middle of the scale or very close to it.

Hence $$QWa^2 = PWb^2,$$

or $$\frac{a^2}{b^2} = \frac{P}{Q}.$$

Therefore $$\frac{a}{b} = \sqrt{\frac{P}{Q}} \quad \text{.................} \quad \text{(iii)}.$$

Also dividing (i) by (ii)

$$\frac{Q}{W} = \frac{W}{P}.$$

Hence $$W^2 = PQ,$$

$$W = \sqrt{PQ}.$$

Thus P and Q are known weights: hence the ratio of the arms and the true value of W is found.

EXPERIMENT 9. *To determine the difference between the weights of the scale-pans.*

Level the balance as in Experiment 8, and note the position of the pointer on the scale. Interchange the scale-pans so that S may now hang from the arm b, S' from the arm a. If the pointer remains in the same position, the pans S and S' are equal in weight. If not, we can find their difference thus.

Let S be suspended from the arm a, and suppose that a small weight w must be added to S to make the beam horizontal.

Then $$(S + w)\,a = S'b.$$

Interchange S and S' and suppose now that w' in the pan S is required for equilibrium

$$S'a = (S + w')\,b.$$

Thus

$$S + w = S'\,\frac{b}{a},$$

$$S' = (S + w')\,\frac{b}{a}.$$

Hence

$$S - S' + w = (S' - S)\,\frac{b}{a} - w'\,\frac{b}{a}.$$

Thus

$$(S'-S)\left(1+\frac{b}{a}\right)=w+w'\,\frac{b}{a}.$$

Therefore

$$S'-S=\frac{w+w'\,\dfrac{b}{a}}{1+\dfrac{b}{a}}.$$

Hence if b/a is known we can find $S'-S$.

If w and w' be both very small and b/a very nearly unity, we shall not alter the value of this quantity much by putting b/a equal to 1, and then

$$S'-S'=\tfrac{1}{2}(w+w').$$

If b be accurately equal to a then clearly w is equal to w', but by proceeding as above and interchanging the pans we take into account the only important part of the effect due to a difference between b and a.

61. The common or Roman Steelyard. This balance consists of a lever AB, Fig. 135, supported on a knife-

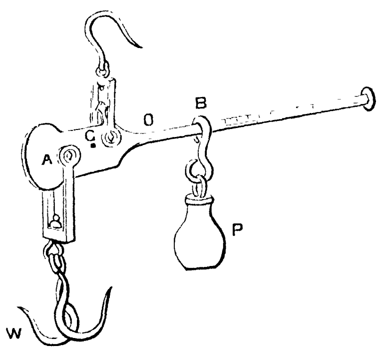

Fig. 135.

edge at C. A hook at A carries the pan in which the object to be weighed is supported. The arm BC has a number of divisions marked on it, and from these a standard weight P of constant magnitude can be suspended. The weight Q is determined by finding the division at which B must be suspended in order that the beam may be horizontal. Practically then the steelyard is a balance such that the length of one arm can be adjusted: we weigh by adjusting this length, not by altering the weight. If the centre of gravity of the steelyard itself coincided with C the fulcrum, the weight W would be directly proportional to the distance from C of the point from which P is suspended, the beam could be graduated by dividing it into equal parts from C, each of these parts being equal to CA. In practice this is not the case: we have always to take into account the weight of the steelyard. This is done in the following way.

Suppose G is the centre of gravity and W the weight of the steelyard.

Let the scale-pan be unloaded and adjust P until the beam is horizontal. Let the position of P so found be O. The point G is usually between C and A, while O is between C and B. Then if a weight P were always kept at O it would just balance W at G. If then a weight Q be put into the scale-pan and another weight equal to P supported at some point along AC, the weight Q will be measured by the distance of this point from C. But the effect of the two weights P is the same as that of a single weight P placed at a distance beyond the second weight equal to CO. The weight Q is then measured by the distance of this weight from O, in other words the steelyard is graduated from O, not from C.

This may be put more briefly using symbols thus.

Let P suspended at B balance Q at A.

Then $\quad P\,.\,CB = W\,.\,CG + Q\,.\,CA.$

Now $\quad W\,.\,CG = P\,.\,CO$ by experiment.

Hence $\quad P\,.\,CB = P\,.\,CO + Q\,.\,CA.$

Thus
$$Q\,.\,CA = P\,.\,(CB - CO)$$
$$= P\,.\,OB.$$

Hence
$$Q = P\frac{OB}{CA}.$$

Hence if we take points 1, 2... etc., along AC such that their distances from O are AC, $2AC$... respectively, the weight Q is equal to P, $2P$, $3P$, etc., according as P is at 1, 2, 3, etc.

Thus the steelyard is graduated from O.

62. The Danish Steelyard. This consists of a bar AB, Fig. 136, terminating in a ball at B, the weight of which

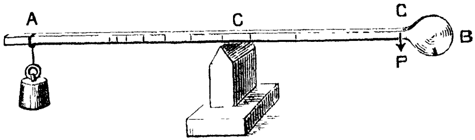

Fig. 136.

constitutes the power, the bar is graduated and the fulcrum is moveable. The body to be weighed is suspended from A, and the fulcrum C is shifted until the steelyard is horizontal. Then if P be the weight of the bar and G its centre of gravity, the moment about C of P acting at G is equal to that of W at A.

Thus to graduate the bar we have

$$W \cdot AC = P \cdot CG = P \cdot (AG - AC).$$

Thus $$(P + W)\, AC = P \cdot AG,$$

or $$AC = \frac{P \cdot AG}{P + W}.$$

Thus by making W equal successively to P, $2P$, $3P$, etc., the successive graduations can be found.

63. The Letter Balance. (Roberval's Balance.) A common form of letter balance is shewn in Fig. 137.

The beam ACB turns about a fulcrum at C. The scale-pans are supported above the beam on knife-edges at A and B.

Two equal vertical rods AD, BE are attached below the scale-pans, and the lower ends D, E of these rods are connected by joints to a horizontal bar DE parallel and equal to the

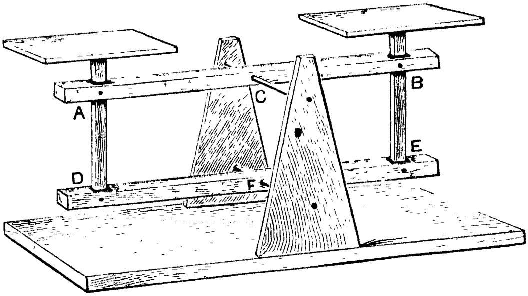

Fig. 137.

beam. This bar can turn about its middle point F, which is vertically below the fulcrum.

By this arrangement, as the balance swings, the rods AD, BE always remain vertical and the scale-pans horizontal.

Moreover it follows readily from the principle of work that if the arms of the beam are equal the weights may be placed anywhere on the scale-pans.

For if the beam be slightly displaced, every point of the one scale-pan rises the same vertical height h, say, while every point of the other falls an equal distance h. Hence if P and Q be the weights at any point of either scale-pan respectively, and there be equilibrium with the beam horizontal, we must have $P = Q$.

In an ordinary balance, if the weights be put on at any point of the scale-pan, the pan swings about its point of support until the centre of gravity of the weights is vertically under this point. The weights therefore always act vertically through the ends of the beam. If the pan be not free so to swing, then in general the arm at which the weights act, the horizontal distance that is between the fulcrum and the vertical line through their centre of gravity, will depend on the position of the weights in the scales and the balance if not actually useless would be very troublesome to use. In the balance just described this difficulty is avoided by securing that each point of the pan rises or falls an equal amount.

EXAMPLES.

1. A balance has unequal arms. A piece of lead placed in the left pan weighs apparently 580 grams; when it is placed in the right pan its weight is apparently 560 grams. Calculate the ratio of the lengths of the arms of the balance.

2. A weightless rod $ABCD$ moveable about a fulcrum and twenty feet in length has weights P, $2P$, $3P$ and $4P$ attached to the rod at A, B, C and D which are at equal distances apart. If the rod be in equilibrium find the distance of the fulcrum from A.

3. In a given Roman steelyard where must the fulcrum be if the smallest weight that can be measured is half the moveable weight, assuming that the beam weighs 4 times the moveable weight?

4. Discuss the effect of an increase in the value of the acceleration of gravity (1) upon the sensitiveness, (2) upon the stability, of a balance.

5. Describe a balance and explain the conditions upon which the sensitiveness of a balance depends.

6. Why is it that the sensitiveness of a balance depends upon the sharpness of its knife-edges?

7. Describe and explain the various precautions which are necessary for the accurate determination of the mass of a body by means of a balance.

8. What is meant by stable, neutral, and unstable equilibrium? Give examples of each of these.

9. A false balance rests with the beam horizontal when unloaded, but the arms are not of equal lengths; a weight W, when hung at the end of the shorter arm b, appears to balance a weight P, and when hung at the end of the longer arm a it appears to balance a third weight Q; shew that

$$W = \sqrt{PQ}.$$

Can you suggest another way of ascertaining correctly the weight of W?

10. How would you compare the "stability" and the "sensibility" of two balances?

11. How may an Inspector with one standard pound test a tradesman's scales and weights?

12. An object is placed in one scale-pan of an ordinary balance and it is balanced by 20 lb. The object is then put into the other scale-pan, and now it takes 21 lb. to balance it. When both scale-pans are empty the scales balance. What is the matter with the balance, and what is the true weight of the object?

13. The arms of a false balance are in the ratio of 20 to 21. What will be the loss to a tradesman who places articles to be weighed at the end of the shorter arm if he is asked for 4 lb. of goods priced at 3*s.* per lb.?

14. The arms of a balance are 2 ft. long. One of the scale-pans is a circular disc, whose diameter is 6 inches, and which is fixed to the end of an arm of the balance by a rod passing through the centre of the pan and rigidly attached to the pan at right angles to its plane. Shew that a 1 lb. mass placed in such a pan may balance any mass between 18 and 14 ounces in the other pan.

15. A weight W is supported on a smooth inclined plane at an angle of 30° to the horizon, by a string attached to a point in the plane; find the tension of the string.

If in the preceding case the pressure on the plane is R, and in the case in which the string is inclined at an angle of 60° to the horizon [the inclination of the plane remaining the same] the pressure is R', prove that

$$2R = 3R'.$$

16. Find the horizontal force that would support a weight W on a smooth plane inclined at an angle of 60° to the horizon.

If on the same inclined plane the weight W is supported by two equal forces one acting horizontally and the other acting along the plane upwards, and the pressure on the plane is R, prove that $R = W$.

17. The height of an inclined plane is 4 feet and it requires a power P acting along the plane to support a weight W. If the height is altered, the length of the plane remaining the same, and $3P$, acting along the plane, is now necessary to support the weight W, find the new height.

18. A weight W is supported on a smooth plane inclined at an angle α to the horizon by means of a force inclined at an angle θ to the plane. Find the magnitude of the force, and the pressure on the plane.

If there is no pressure on the plane, in what direction does the force act?

19. A body of weight W is supported on a smooth plane inclined at an angle α to the horizon by means of a force inclined at an angle $\alpha+\beta$ to the horizon. Find the magnitude of this force.

If the force is vertical find the pressure on the plane.

20. A weight rests on a smooth inclined plane. Shew that the smallest force which will keep it in equilibrium must act along the plane.

If the weight be the weight of a ton, and the inclination of the plane be 45°, what is the power?

21. Two inclined planes of equal heights are so placed that they have a common vertex; a weight lies on each of the planes, and the weights are connected by a string which passes over the common vertex; in this position there is equilibrium; the lengths of the planes are respectively 6 and 3 feet; the weight which rests on the shorter plane is 10 lb. Find the other weight in one of the following ways, neglecting all friction:

(1) by means of the "triangle of forces,"

(2) by means of the "principle of work."

22. Find the ratio of the power to the weight on a smooth inclined plane when the former acts horizontally.

If the weight be the weight of a ton, and the inclination of the plane be 30°, what is the power?

23. A weight W is supported on a smooth plane inclined at an angle 30° to the horizon, by a string inclined at 60° to the horizon; find the tension of the string.

24. Find the horizontal force that would support a weight W on a smooth plane inclined at an angle of 45° to the horizon.

If on the same inclined plane the weight W is supported by two equal forces, one acting horizontally and the other acting along the plane upwards, find the pressure on the plane.

25. Find the ratio of the power to the weight when a body is kept in equilibrium on a smooth plane, inclined at an angle of 30° to the horizon, by a horizontal force.

A given force is applied to support a weight on an inclined plane. Will the greater weight be supported, when the force acts horizontally, and the plane is inclined at an angle of 30° to the horizon; or when the force acts parallel to the plane, and the plane is inclined to the horizon at an angle of 60°?

26. An inclined plane 14 feet long has one end 8 feet and the other end 10 feet above the level of the floor. If 294 ft.-lb. of work are done in dragging a mass of 1 cwt. up the plane, find the friction.

27. Describe the system of pulleys in which each string is attached to a bar from which the weight is suspended, and find an expression for its mechanical advantage, the weights of the pulleys being neglected.

If there are 3 strings attached to the bar, what power will support a weight of 35 lb.?

28. Describe the system of pulleys in which each pulley hangs in the loop of a separate string, and find an expression for its mechanical advantage, the weights of the pulleys being neglected.

If there are four moveable pulleys, what power will support a weight of 50 lb.?

29. Determine the relation of the power to the weight in a system of 4 moveable pulleys, of which the weight may be neglected, one end of each string being fixed to a beam. Find also how far the weight is raised when the power moves through 12 ft.

30. Find the mechanical advantage of the system of weightless pulleys in which each pulley hangs in the loop of a separate string, one end of which is fastened to a fixed beam; all the strings being parallel.

If there are 7 pulleys and the weight is 8 cwt. find the power in lb. wt.

31. Find the ratio of the power to the weight in the system of pulleys in which all the strings are parallel and are attached to the weight.

If there are 6 pulleys and the weight is 9 cwt. find the power in lb. wt.

32. In a certain system of pulleys it is found that the power descends 1 ft., while the weight rises 1 inch. What power will be required to raise a weight of 1 cwt.?

33. Describe the system of pulleys in which each string is attached to a bar from which the weight is suspended, and find an expression for its mechanical advantage, the weights of the pulleys being supposed equal.

If there are three strings attached to the bar, and the weight of each pulley is $1\frac{1}{2}$ lb., what power will support a weight of $42\frac{3}{4}$ lb.?

34. Describe the system of pulleys in which each pulley hangs in the loop of a separate string, and find an expression for its mechanical advantage, the weights of the pulleys being supposed equal.

If there are 4 moveable pulleys, each weighing $\frac{3}{4}$ of a lb., what power will support a weight of $56\frac{3}{4}$ lb.?

35. In the system of three equal pulleys, one fixed and two moveable, in which each string is attached to a bar supporting the weight W, prove that, neglecting the weights of the pulleys and the bar, if P is the power, $7P=W$, and find the point on the bar from which W must be suspended.

If, when there is equilibrium, one-third of the weight falls off, prove that the acceleration of the remainder of the weight will then be one twenty-third of the acceleration due to gravity.

36. Find the ratio of the power to the weight in that system of pulleys in which there is only one string.

In such a system a power P supports a weight W; if P and W are interchanged prove that the weight to be added to P to produce equilibrium is $\dfrac{W^2-P^2}{P}$.

37. Draw carefully a system of pulleys in which each pulley hangs by a separate string and the ratio of the power to the weight is 1 to 32.

38. Find the conditions of equilibrium in the wheel and axle.

Shew that if the axle rest on rough bearings, the least power (acting downwards) that will raise a weight W is

$$\frac{b\,(1 + \sin\lambda)}{a - b\sin\lambda}\,W;$$

where a, b are the radii of the wheel and axle and λ the angle of friction.

39. If in order to raise a weight of 144 lb. through an inch by means of 'a wheel and axle' I must move my hand through a distance of one foot, what power must I exert?

40. A balance is apparently in adjustment when no weights are in the scale-pans. A certain mass is put into the right pan and requires weights of 45·63 grams in the left to maintain equilibrium; on putting the mass into the left pan it is found that 45·81 grams are needed in the right. Find the mass and ratio of the arms.

41. In a wheel and axle the radius of the wheel is 3 feet. The axle is of square section, the side of the square being 6 inches long. Find (i) the greatest, (ii) the least vertical power that must be exerted to slowly lift a weight of 252 lb. in the usual manner.

42. A straight uniform lever AB, 12 feet long, balances about a point in it 5 feet from B, when weights 9 lb. and 13 lb. are suspended at A and B. Find the weight of the lever.

43. A straight uniform lever whose weight is 16 lb. balances about a point one foot from its middle point when weights 6 lb. and 10 lb. are suspended from its ends. Find the length of the lever.

44. Find the power required to support a weight W in a system of 4 pulleys in which each string is attached to the weight and the pulleys are supposed weightless.

If the weights of the pulleys are taken into account and each weighs 1 lb., find what power will support a weight of $78\frac{1}{2}$ lb.

45. Find the power required to support a weight W in a system of 4 pulleys in which each pulley is supported by a separate string, one end of which is fastened to a fixed beam, and the pulleys are supposed weightless.

If the weights of the pulleys are taken into account and each weighs 1 lb, find what power will support a weight of 65 lb.

46. Find the condition of equilibrium in the system of pulleys in which the same string goes round all the pulleys.

If a weight of 6 lb. just supports a weight of 28 lb., and a weight of 8 lb. just supports a weight of 42 lb.; find the number of pulleys, and the weight of the lower block.

47. Find the condition of equilibrium in the system of pulleys in which each string is attached to the weight.

If there are 5 moveable pulleys each weighing half-a-pound, and the weight is 35 lb., what is the power?

CHAPTER VII.

FRICTION.

64. Friction. The term "friction" has been used occasionally in the last chapter: we have seen that in many cases when a body rests on a surface the force between them is not wholly at right angles to the surface, but has a component along the surface. The surface is then said to be rough and the component of the force along the surface is called friction.

It remains now to consider the nature of friction a little more fully.

DEFINITION. *When a body is in contact with a rough surface and the impressed force has a component along the surface a force is called into play tending to balance this component and prevent motion. This force is called* **Friction.**

The **Direction of friction** is opposite to the component of the other forces resolved parallel to the surface, opposite that is to the direction in which motion would take place if there were no friction.

The **Amount of friction** up to a certain limit is always just sufficient to prevent motion, but only a limiting amount of friction can be called into play.

Thus if a body rest on a horizontal table the pressure of the table balances the weight, these forces are both vertical, there is no component in the direction of the surface and no friction is called into play. Apply a small force parallel to the surface, the body does not move, sufficient friction is exerted just to stop the motion; increase the force still further, until the force parallel to the surface reaches a certain limit depending on the normal pressure and on the nature of the surface, the body does not move: when however this limit is exceeded, motion takes place.

Thus consider a ladder resting as in Example (1), p. 75, against a smooth wall on rough ground; when the foot of the ladder is near the wall the friction needed to maintain it in position is small; as the foot is withdrawn from the wall and the slope increases more friction is required and is called into play, until at last there comes a position in which the ladder begins to slip, the friction which needs to be exerted is greater than the ground can exert, the limiting position has been passed.

We may put this in symbols thus:

Let R be the normal pressure of the smooth wall, W the weight of the ladder acting vertically at its centre of gravity, Y the vertical stress at the foot and X the horizontal component of the action—the friction. Then, as in Example (1), p. 75,

Resolving vertically $\qquad Y=W.$

Resolving horizontally $\qquad X=R.$

Taking moments about the lower end

$$Rl\sin a=\tfrac{1}{2}Wl\cos a.$$

Hence $\qquad X=\tfrac{1}{2}W\cot a.$

Now as the angle a decreases this increases: when it becomes greater than the maximum friction which the ground can exert the ladder will slip. Let us call F this maximum friction, then X must be not greater than F.

Hence $\tfrac{1}{2}W\cot a$ is not greater than F, and $\cot a$ must not be greater than $2F/W$.

Many other observations shew us that there is a limit to the maximum amount of friction which can be called into play. The following experiments will help to shew on what the limiting amount depends.

Experiment 10. *To investigate the Laws of Limiting Friction.*

Thus take a well-planed board of hard wood and a small piece of wood, say six inches by nine in size; rub one surface of this small piece of wood as smooth as possible with sandpaper and then cut the wood in two so as to have two pieces, one of about twice the size of the other, with surfaces as nearly alike as possible. Fix a screw eye into one end of each of these pieces of wood and attach a string to each. Fix a pulley at one end of the horizontal board over which the string may pass and support a scale-pan and weights. Secure some pieces of lead to the smaller piece of wood and thus make the weight of the two the same.

Place the larger piece of wood on the board and put a considerable weight, 5 or 6 kilos, on it. Pass the string from the wood over the pulley and suspend the scale-pan as in Fig. 138. Load the scale-pan until the wood just begins to

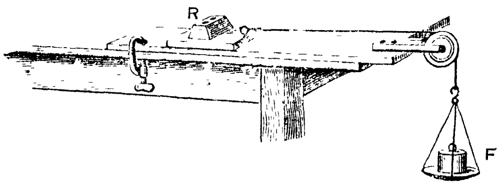

Fig. 138.

slip on the board. The weight required for this can be found fairly closely by gently tapping the board.

Note the total weight suspended; this measures the maximum friction which can be exerted between the board and the wood.

Note also the total weight supported by the board; including in this the weight of the wood itself[1]. This measures the normal stress between the two. Place more weights on the wood so as to increase the force between it and the board; it will be necessary to place more weights in the scale-pan in order to start the motion; the limiting friction is increased. Determine in this way the limiting friction for a number of loads and form a table in which the first column is the Limiting Friction, the second column the Normal Force between the surfaces.

[1] It is convenient to make this up to some fraction of a kilogram such as ½ or ¼ by securing some pieces of lead to its upper side.

Divide the numbers in the first column by the corresponding numbers in the second; it will be found that the quotients are the same.

Thus when the surfaces in contact remain the same the ratio of the limiting friction to the normal force is a constant.

Now repeat the experiment, using the second or smaller piece of wood, it will be found that for the same normal stresses as previously the limiting friction is also practically the same. The areas of the surfaces in contact have been altered but not the nature of those surfaces or the state of their polish; the ratio of the limiting friction to the normal stress is unchanged.

Replace the wooden slide-piece by another of different material; the law, that the ratio of the limiting friction to the normal force is constant, still holds; but the value of this constant ratio differs from that found in the previous experiment.

65. Laws of Limiting Friction. We are thus led to the following laws of limiting friction.

(i) *The ratio of the limiting friction to the normal force between any two given surfaces is a constant.*

(ii) *This constant ratio depends on the material of the surfaces in contact and the state of their polish, but not on their area or shape.*

These two laws of limiting friction together with the definition and statements given in Section 64 are sometimes enunciated together as the Laws of Friction.

These laws, though they express fairly well the result of experiments, are probably not rigorously true; they are however usually employed in considering problems involving friction.

It is customary to give another law which however does not concern us in Statics.

When sliding motion takes place the ratio of the friction to the normal force is still found to be constant for any two given *surfaces and independent of the velocity; this constant however is slightly less than the constant ratio of the limiting friction to the normal force.*

Thus it requires rather greater force to start a body moving on a rough surface against friction than to maintain it in motion with uniform speed when once started.

66. Coefficient of Friction. The constant ratio of the limiting friction to the normal stress for two surfaces of given material in a definite state of polish is called the Coefficient of Friction.

Thus if F is the limiting friction, R the normal force and μ the coefficient of friction, we have

$$\frac{F}{R}=\mu$$

or

$$F=\mu R.$$

The experiments described in Experiment 10 give one method of determining μ. The following method is generally more convenient.

Consider a body lying on a rough horizontal surface under its weight and the reaction of the surface, the body is in equilibrium and there is no friction. Tilt the horizontal surface, the weight will have a component down the surface. Friction is called into play to balance this component. Continue to tilt the surface until the body just begins to slide down; when this is the case the limiting amount of friction has been reached, and this limiting amount of friction is just equal to the component of the weight down the surface when tilted at the angle at which sliding just begins.

The following Experiment will enable us to verify the laws of friction and to find the coefficient of friction by this method.

EXPERIMENT 11. *To prove that on a rough surface the limiting amount of friction is proportional to the normal force, and to find the coefficient of friction.*

The apparatus consists of a mahogany board, Fig. 139, some 12 or 15 inches long and 3 or 4 inches in width. This is hinged at one end to another similar board which can be clamped to the table. At the end of the second board remote

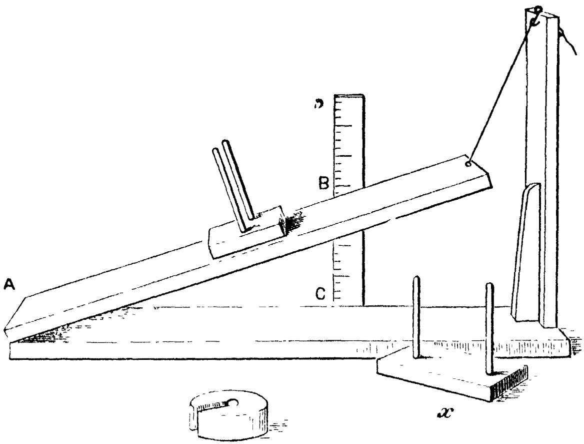

Fig. 139.

from the hinge a vertical support is fixed and the hinged board can be raised and secured in any position by means of a string passing through a small screw eye at the top of this support. A graduated vertical rod is also screwed as shewn at BC to the base board and the height of the plane at B can be easily measured on this rod. The point C is at some convenient distance (say 10 inches) from the hinge A: by dividing the height in inches by 10 we get at once the tangent of the inclination of the plane. Let the angle BAC be α.

One or more small boards of various sizes and materials can be placed on the inclined plane and weights can be placed on these small boards to vary the force between them and the inclined plane. In the apparatus shewn each small board has one or more thin brass rods screwed to it, the weights used in the other experiments can be piled on it so that each weight when the plane is tilted is prevented from slipping by the rod which passes through its centre.

Place one of the small boards with some convenient weight on it on the plane. Raise the plane, gently tapping it from time to time, until the small board begins to slide down. When this takes place note the height BC and thus find the tangent of the inclination of the plane to the horizon.

Now when the board just begins to slide the maximum friction has been reached and this just balances the component of the weight down the board. Thus if R be the normal force, F the friction in any position and W the weight,
we have resolving along the plane,

$$F = W \sin \alpha,$$

and resolving perpendicular to the plane

$$R = W \cos \alpha.$$

Hence

$$\frac{F}{R} = \frac{W \sin \alpha}{W \cos \alpha} = \tan \alpha.$$

Now experiment shews that, so long as the material and state of polish of the surfaces remain the same, the slipping just begins at the same angle. Thus replace the weights by others and repeat the experiment, the slipping takes place at the same angle as before. Replace the small board by another of the same material and polish but of different area; it will just begin to slip at the same angle as previously.

Hence if F now stand for the maximum amount of friction and α for the angle at which slipping takes place we see by the experiment that α is constant so long as the material and polish remain the same.

Hence since the ratio F/R is equal to $\tan \alpha$ we see that F/R is constant under these same conditions.

This ratio is defined to be μ the coefficient of friction.

Hence $$\mu = \frac{F}{R} = \tan \alpha.$$

Thus the coefficient of friction is found by reading BC the height of the plane and dividing it by AC the base.

67. Angle of Friction.

DEFINITION. *The angle α whose tangent gives the coefficient of friction is called the Angle of Friction.*

We can give a more general meaning to this angle thus.

Consider any body resting on a rough surface. Let the friction be F and the normal force R. The resultant of these two will be a force P, which combined with the other forces must maintain equilibrium. Now let P act at an angle θ with the normal force R. Then resolving P along and perpendicular to R, we have

$$P \cos \theta = R,$$
$$P \sin \theta = F.$$

Hence

$$\tan \theta = \frac{F}{R}.$$

Thus in any case the ratio F/R measures the tangent of the angle which the resultant force between the surface and the body makes with the normal to the surface.

Thus, when F reaches its limiting value, θ reaches its greatest value and becomes α the angle of friction. Hence the angle of friction is the angle which the resultant force makes with the normal when the friction has reached its greatest value; or in other words it is the greatest angle which the resultant force can make with the normal.

So long then as the resultant force is inclined to the normal at an angle less than the angle of friction, equilibrium is possible; when this angle becomes greater than the angle of friction motion must take place.

In the case of a body on an inclined plane the resultant force due to the plane must be always vertical, for the only other force is the weight; and the resultant of the normal force and the friction must just balance the weight. Again, the angle which the normal to the plane makes with the vertical is the angle of the plane. Thus the angle between the direction of the resultant force and the normal to the plane is the angle of the plane; so long as the angle of the plane is less than the angle of friction equilibrium is possible; when the angle of the plane is just greater than the angle of friction slipping begins.

The following is a table of approximate values for the coefficient of friction.

Wood upon wood	·5
„ „ „ lubricated	·2
Wood upon polished metal	·6
„ „ „ „ lubricated ...	·12
Metal upon metal	·18
„ „ „ lubricated	·12

EXAMPLES.

FRICTION.

1. Explain what is meant by the coefficient of friction. A cubical block rests on a rough plane, one end of which is gradually raised. Find the greatest value of the coefficient of friction which will just permit the block to slide down the plane before falling over.

2. State the laws of friction; and assuming that the friction is the same when a body is moving as when it is at rest, find the time taken by a body to fall down a rough inclined plane from rest.

3. A brick whose dimensions are $8 \times 4 \times 3$ inches rests on a rough plane in such a way that it cannot slip, and the plane is gradually tilted about a line parallel to one edge of the brick; shew that the angle through which the plane can be raised without upsetting the brick depends on which face of the brick is on the plane; find also the greatest and least angles for which the brick will just not upset.

4. What is meant by the angle of friction? The lower half of an inclined plane is rough, the upper half being smooth. A particle is allowed to slide from the top and is brought to rest by friction just as it reaches the bottom. Find the ratio of the friction to the weight of the particle, assuming it to be independent of the velocity.

5. A block W_1 rests on an inclined plane and is supported partly by friction and partly by the tension of a cord which passes over a pulley at the top of the plane and carries a weight W_2. Shew how to find (graphically or otherwise) the values of W_2 which will (1) just prevent W_1 from slipping down, and (2) just make W_1 begin to slip up, when the coefficient of friction and the inclination of the plane are known.

6. A block of iron weighing 10 lb. rests on a level surface plate. A string attached to the block passes over a pulley so placed above the surface plate that the string makes an angle of 45° with the vertical. After passing over the pulley the string supports a weight. Find the least value of this weight which will make the block slip, the coefficient of friction being $\frac{1}{5}$th.

7. A heavy body is to be drawn up a rough inclined plane. If the force is the least possible prove that its inclination to the plane must equal the angle of friction.

8. Describe an experimental method for finding the coefficient of friction between two substances.

A semicircular disc rests in a vertical plane, with one part of its curved surface touching the ground and another touching a vertical wall. Shew that it will rest in any position in which its straight edge makes an angle with the vertical greater than

$$\cos^{-1}\frac{3\pi}{4}\frac{\mu+\mu^2}{1+\mu^2},$$

where μ is the coefficient of friction between the disc and the ground and between the disc and the wall. [The centre of gravity of a semicircular disc is at a distance from the centre equal to $(4/3\pi)$ times the radius.]

9. A heavy body rests on a rough plane inclined at the angle 30° to the horizontal, the coefficient of friction being $\frac{2}{\sqrt{3}}$.

A horizontal force along the plane is applied to the body, and is gradually increased until the body begins to move; find the direction in which the body begins to move, and the magnitude of the horizontal force.

10. A uniform rod AB of length $2a$ rests with the end B in contact with a rough vertical wall, and is supported by a smooth peg fixed at a distance b from the wall, and the end B is on the point of slipping upwards. Shew that the inclination of the rod to the vertical is then given by the equation $\sin^2\theta\,(\sin\theta-\mu\cos\theta)=\frac{b}{a}$, μ being the coefficient of friction between the rod and the wall.

11. How would you place a brick whose length is double its breadth, and breadth double its thickness, on a rough inclined plane, so as to be least likely to tumble over? Would it be less likely to slide down with one face in contact than another? Give reasons for your answers.

12. A weight W rests in equilibrium on a rough inclined plane, being just on the point of slipping down. On applying a force W parallel to the plane, the weight is just on the point of moving up. Find the angle of the plane and the coefficient of friction.

13. What is the coefficient of friction when a body weighing 50 lb. just rests on a plane inclined at 30° to the horizon? If the plane were horizontal what horizontal force would be required to move the body?

14. Find what horizontal force will be required to support a weight of 3 cwt. upon a smooth inclined plane, whose height is $\frac{3}{5}$ of its length.

15. A force P acting up an inclined plane supports a weight W on it. If R be the reaction of the plane, prove that

$$P : W : R :: \text{height of plane} : \text{length} : \text{base}.$$

16. A mass of 1 cwt. rests on a rough inclined plane of angle 30°. If the coefficient of friction be $1/\sqrt{3}$ find the greatest and least forces which, acting parallel to the plane in both cases, can just maintain the mass in equilibrium.

ANSWERS TO EXAMPLES.

CHAPTER I. (Page 31.)

1. $5\sqrt{13}$ lb.-wt. **2.** $13P$. **4.** 10 lb. and 26 lb.

5. 5 lb. and 13 lb. **13.** $5\sqrt{3}$ lb. each.

14. Place the forces parallel to the sides of a right-angled triangle whose sides are 3, 4 and 5.

16. Weight of bob = 1 kilogramme.
Tension of horizontal string = ·258 kilogramme.
" pendulum " = 1·033 "

17. $P(2+\sqrt{3})$. **21.** $\frac{\sqrt{7}}{2}$ times the side of the triangle.

26. A force equal to the given forces bisecting the angle between them.

29. (i) $\sqrt{P^2+Q^2+\sqrt{2}\,.\,P\,.\,Q}$. (ii) 6·48 lb.-wt.

32. $5\sqrt{7}$ lb.-wt. **33.** 699·5 dynes.

34. $\frac{3}{5}$ lb.-wt. and $\frac{4}{5}$ lb.-wt.

38. $5\sqrt{2}$ to the North-west, 5 to the South.

41. $\sin^{-1}(-\frac{12}{13})$ with the force 5. $\sin^{-1}\frac{5}{13}$ with the force 12.

44. 27·7 oz. **46.** $\sqrt{2}\,.\,P$. **47.** $\sqrt{2}\,.\,P$.

48. 39 lb. **49.** 6 lb. and 8 lb.-wt.

CHAPTER II. (Page 57.)

2. 21 ft. 3 in. **3.** 3 lb.-wt. **5.** 5 lb.

6. 17 lb. **7.** $5\frac{1}{4}$ lb.-wt. **8.** 35 lb. and 40 lb.

9. $AC=2$ in. $BC=56$ in. **10.** $\frac{33}{65}$ and $\frac{32}{65}$.

12. 100 lb. **13.** ·131 ft. from the axle. **15.** $1\frac{1}{2}$ ft. from the man.

16. $2\sqrt{2}\,.\,P$ at an angle of 45° with the force $4P$.

17. In the line joining the middle points of the two sides, and at a distance of $5\frac{5}{8}$ in. from the side which supports the two 5 lb.-wt.

18. $\frac{2}{3}$ lb.-wt.

CHAPTER IV. (Page 83.)

1. $\frac{W}{2}$; $\frac{\sqrt{3}\,.\,W}{2}$. **3.** 50 lb. 4 ft. from the thicker end.

4. Pressure on shorter end $= \frac{6}{7}$ of the man's weight.
" longer " $= \frac{1}{7}$ "

6. P, $\frac{1}{3}AC$ from C. **10.** $\frac{1}{2}W \tan \frac{1}{2}\alpha$.

12. The radius to P bisects the line AB. **13.** 20 lb.

15. The end. **18.** 3 to 1.

CHAPTER V. (Page 110.)

1. 350 lb.

3. From any point on the rim within a distance of $\frac{1}{24}$ of the whole circumference from any of the 4 legs.

4. At a distance of $\frac{1}{6}$ of the side of the square from the centre of the square; the straight line joining the centre of gravity with the centre of the square is parallel to a side of the square.

5. At a point on the diameter drawn from the middle one of the 3 particles mentioned and at a distance of $\cdot 65d$ from that point where d is the length of the diameter.

6. In the perpendicular drawn from the angle which is adjacent to the 2 bisected sides, and at a distance $\frac{a}{6\sqrt{3}}$ below the C. G. of the whole triangle.

7. In the diameter of the rectangle parallel to the side a and at a distance of $\frac{1}{9}a$ from G.

8. At the point G in the straight line AC where $AG = \frac{5}{12}\,.\,AC$.

9. In the diameter drawn from that angular point on which no weight is placed and at a distance of $\frac{9}{16}$ of the diameter from that point.

11. In the straight line drawn parallel to BC from the middle point of AB and at a distance of $\frac{26}{45}$ of the side of the square from this point.

12. Half the weight of the whole table.

13. In the diameter drawn from the point at which the two circles touch one another and at a distance of $\frac{r^2}{R+r}$ from the centre of the larger circle.

14. In the diagonal drawn from the angle enclosed between the two bisected sides and at a distance of $\frac{23}{42}$ of the diagonal from this point.

18. 50 lb. 6 ft. from thinner end. 19. $\frac{\sqrt{3}}{2}W$. $\frac{1}{2}W$.

20. $\frac{2}{1+4\sqrt{3}}$ lb. 2 cwt., 3 cwt., 0.

22. In the straight line joining the two centres, and at a distance of 1 ft. $8\frac{1}{4}$ in. from the centre of the hole.

23. The centre of the circular hole must be 16 in. from that of the disc.

24. $1\frac{1}{3}$ in. 25. $5\frac{13}{41}$ in. from the thicker end.

27. On the line joining the angle removed to the c. of G. of the whole and $\frac{1}{6}$ of this distance from the angle removed.

28. $\frac{5}{16}$ of total length from the heavier end. 29. 5 ft. from the end.

30. In the diagonal drawn from the angle enclosed between the two bisected sides and at a distance of $\frac{23 \cdot \sqrt{2}}{42}$ of the side of the square from this point.

CHAPTER VI. (Page 160.)

1. Left arm $=$ right arm $\times$ 1·018.
2. The fulcrum is at C, that is, $13\frac{1}{3}$ ft. from A.
3. $\frac{8}{9}$ of distance of centre of gravity from the end at which the weight is hung.

12. True weight $=$ 20·494 lb.

13. $7\frac{1}{5}$ pence. 15. $\frac{1}{2}W$. 16. $\sqrt{3} \cdot W$. 17. 12 ft.

18. $\frac{W \cdot \sin a}{\cos \theta}$. $W(\cos a \pm \sin a \tan \theta)$ depending on direction of force. Vertically up.

19. $\frac{W \cdot \sin a}{\cos \beta}$. Nothing. 20. $\frac{1}{\sqrt{2}}$ ton. 21. 20 lb.

22. $\frac{1}{\sqrt{3}}$ ton. 23. $\frac{W}{\sqrt{3}}$. 24. W, W.

25. $1 : \frac{1}{\sqrt{3}}$. When the force acts parallel to the plane.

26. 5 lb.-wt. **27.** 5 lb. **28.** $3\frac{1}{8}$ lb.

29. (1) 1 : 16; (2) 9 in. **30.** 7 lb. **31.** 16 lb. **32.** $9\frac{1}{3}$ lb.

33. $5\frac{1}{4}$ lb. **34.** $4\frac{1}{4}$ lb.

35. From a point distant $\frac{2}{7}$ of the whole length of the bar from the end to which the string attached to the fixed pulley is connected.

38. $\frac{W}{P} = \frac{\text{radius of wheel}}{\text{radius of axle}}$. **39.** 12 lb.

40. (1) 45·72 grams; (2) 1·002 to 1. **41.** $21\sqrt{2}$ lb. 21 lb.

42. 2 lb. **43.** 16 ft. **44.** (1) $\frac{W}{15}$. (2) $4\frac{1}{2}$ lb.

45. (1) $\frac{W}{16}$. (2) 5 lb. **46.** 7 pulleys. 14 lb. **47.** $\frac{13}{128}$ lb.

CHAPTER VII. (Page 174.)

1. $\sqrt{\dfrac{2s}{g \sin \alpha - \mu g \cos \alpha}}$. **3.** $\tan^{-1}\frac{8}{3}$, $\tan^{-1}\frac{3}{8}$. **6.** $2 \sin \alpha$.

9. $\sqrt{2}$ lb.; $\tan^{-1}\frac{1}{\sqrt{3}}$ to the horizon; $\frac{\sqrt{3}}{2}$ times the weight.

11. With the largest face on the plane and the length up the plane. No.

12. 30°; $\frac{1}{\sqrt{3}}$. **13.** $\frac{1}{\sqrt{3}}$; $\frac{50}{\sqrt{3}}$ lb.

14. $2\frac{1}{4}$ cwt. **17.** $\frac{1}{2}$ cwt.; 0.

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