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A COURSE OF PLANE GEOMETRY
FOR ADVANCED STUDENTS

PART I.



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TORONTO

A COURSE OF
PLANE GEOMETRY

FOR
ADVANCED STUDENTS

PART I.

BY
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PREFACE

THE present volume is the First Part of a Course in Plane Geometry which, it is hoped, will be found suitable for the higher mathematical divisions of Schools, and for undergraduates attending lectures on Geometry during their first year of residence. It therefore covers the Schedule of Pure Geometry required for the first part of the Cambridge Mathematical Tripos.

Part I. deals with the properties of the straight line, triangle, quadrilateral and quadrangle, and circle.

Part II., which will be published shortly, is concerned with the application of cross-ratio, involution, projection, and reciprocation, to the properties of the conic; together with an account of the meaning, validity, and advantage of the use of imaginary elements, viewed from their analytical source.

The following considerations have guided the development of the present course:

(1) It is only by constant practice in rider work that the many novel ideas in advanced geometry can be appreciated and assimilated. Very numerous exercises have therefore been attached to each set of standard theorems. Some of these are of a simple nature, being partly mere reproductions of the bookwork in different forms; while others will be found hard enough to test the capacities of the best students.

(2) In many places, it is desirable to compare analytical and geometrical methods, as for example in the treatment of inversion, coaxal circles, and polar properties. In such cases, the analysis has been included in the text in preference to relegating it to an appendix, where it is apt to be overlooked.

(3) The connection between Geometry and Statics, supplied by the theory of Vectors, and the graphical interpretation of complex

number which it yields, afford cogent arguments for including some exposition of vector quantities in a regular school course. A chapter has therefore been inserted which introduces the subject on somewhat new lines.

(4) Some familiarity may well be acquired with the Principle of Duality long before the process is studied in detail under reciprocation. Numerous examples present themselves in elementary theorems which pave the way to a better final understanding of the general principle.

(5) Interest is added to the study of Geometry, if some indication is given of its historical development. Accordingly, most chapters have been prefaced by brief historical notes on the subject matter they contain.

The author wishes to acknowledge his indebtedness to Mr. A. E. Broomfield and Mr. G. M. Bell for the valuable assistance received from them.

The Editor of the *Educational Times* has kindly authorised the use of riders which have appeared in the columns of that journal: and the Syndics of the Cambridge University Press have courteously allowed questions to be taken from the Entrance Scholarship and University examination papers.

The historical information is derived mainly from the writings of Professor Cajori, Sir R. Ball, Mr. W. W. Rouse Ball, Professor Casey, and Dr. Taylor.

The author will welcome any corrections or criticisms tending to the improvement of the book.

October, 1908.

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CHAPTER I.

SIMILAR FIGURES.

WHEN constructing a map of a town, or obtaining a numerical solution to a problem involving heights or distances by a drawing to scale, an appeal is made to the theory of similar figures. The picture thrown on a screen by a magic lantern is another example of a figure similar in every respect to the slide from which it is formed.

It is not surprising to find in the early history of geometry some acquaintance with properties of this kind. Thales of Miletus (640–550 B.C.) enunciated the theory of similar triangles, and Euclid (330–275 B.C.) included in his *Elements* a more systematic treatment, which was no doubt largely derived from the work of the three preceding centuries.

Definition.

Polygons which are equiangular and have their corresponding sides proportional are called **similar**.

THEOREM 1.

If two similar polygons are so placed that their corresponding sides are parallel, then the lines joining corresponding vertices are concurrent.

Let $ABCD \dots X$ and $A'B'C'D' \dots X'$ be the polygons

and let
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \dots = k.$$

Produce AA' and BB' to meet at O . Join OC and let it cut BC at C_1 .

By similar triangles, $k = \frac{B'A'}{BA} = \frac{OB'}{OB} = \frac{B'C_1'}{BC}$

but

$$k = \frac{B'C'}{BC},$$

$$\therefore B'C_1' = B'C'$$

$\therefore C_1'$ coincides with C' ,

$\therefore C'$ lies on OC ; and similarly D' lies on OD , etc.,

$\therefore AA', BB', CC', \dots XX'$ intersect at O .

Q. E. D.

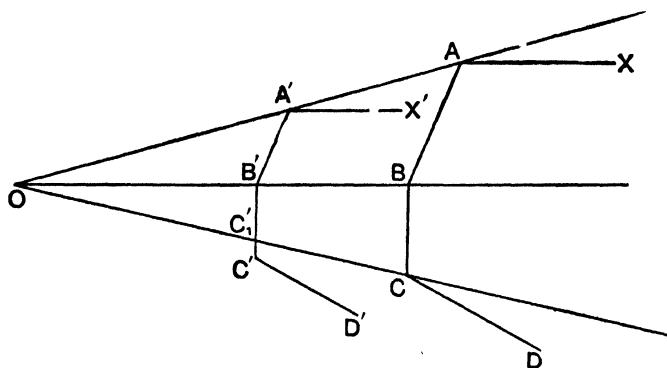


FIG. 1.

In the figure of this theorem., the two polygons have been so drawn that the directions $A \rightarrow B$, $A' \rightarrow B'$ are the same. If however these directions are opposite (Fig. 2) the theorem still holds good, and may be proved in precisely the same way.

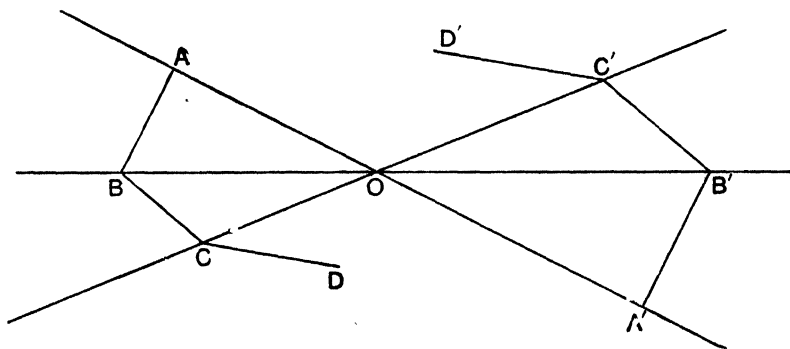


FIG. 2

Corollary.

If P is any point on one of the sides of the polygon $AB \dots X$, and if OP meets the corresponding side of $A'B' \dots X'$ at P' , then

$$\frac{OP'}{OP} = k.$$

1. Prove this Corollary.

THEOREM 2.

If P is a variable point on the side of a given polygon $ABC \dots X$ and if O is any other given point, and if a point P' is taken on OP such that $\frac{OP'}{OP} = \text{a constant } k$, then as P traces out the perimeter of $ABC \dots X$, P' traces out the perimeter of a polygon whose sides are parallel and proportional to the sides of $ABC \dots X$, and the ratio is k .

2. Prove Theorem 2. Notice that there are two cases according as P' is on the same or opposite side of O as P .

Definitions.

✓(a) The polygon traced out by P' in this theorem is said to be similar and similarly situated to the polygon traced out by P ; and O is called their centre of similitude. Or, to use a shorter expression, the polygons are said to be homothetic.

✓(b) If O is a fixed point and if P traces out a given curve S , and if P' is a point on the line OP such that $\frac{OP'}{OP} = k$ (a constant), then P' traces out a curve S' which is said to be homothetic to S ; and P, P' are called corresponding points. If P' traces out S' in the same direction that P traces out S , they are called directly homothetic; but if in opposite directions, they are called inversely homothetic.

The results stated in Ex. 3-5 are of great importance. The proofs are however so simple that they are left to the reader.

✓3. Prove that the curve homothetic to a straight line must be a straight line.

✓4. (1) If P, Q' are two points corresponding to P, Q , prove that $PQ, P'Q'$ are parallel.

(2) If P and P' are corresponding points on two homothetic curves, prove that the tangents at P, P' are parallel.

9 ✓5. If two triangles are homothetic, prove that (1) the centres of their circumscribing circles, (2) the centres of their inscribed circles, are corresponding points.

✓6. If a figure S is gradually transformed into a figure S' and always remains homothetic to it* w.r.t. the same centre of similitude, prove that the path of any point on S , the tangent at which has a given direction, is a straight line.

* w.r.t. will be used as an abbreviation for "with respect to."

7. Prove that the curve homothetic to a circle is another circle and that their centres are corresponding points.

✓ 8. P, Q are two points on the perimeter of a polygon S' , which correspond to the points P, Q on a homothetic polygon S . O is the centre of similitude, prove that the ratio of the areas of S and S' equals the ratio of the areas of the triangles $OPQ, OP'Q'$.

✓ 9. Two plans of the same estate, on different scales, are placed, one wholly on the other, so that a north and south line in one lies on a north and south line in the other, prove that there is one point of contact between the maps at which the same place is represented in each map.

THEOREM 3.

O is any fixed point: P is a variable point on a given circle: P' is a point on OP such that $\frac{OP'}{OP} = k$ (a constant), then the locus of P' is a circle.

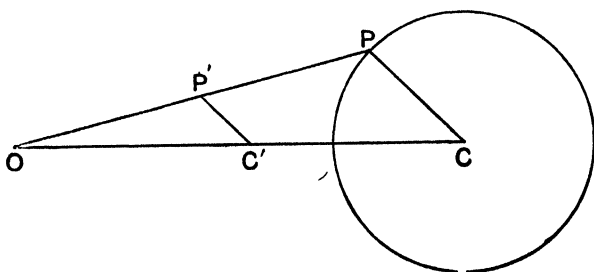


FIG. 3.

Let C be the centre of the given circle.

Join OC ; draw $P'C'$ parallel to PC to meet OC at C' .

By parallels $\frac{OC'}{OC} = \frac{P'C'}{PC} = \frac{OP'}{OP} = k$.

But OC, PC are constants.

$\therefore OC', P'C'$ are constants.

$\therefore C'$ is a fixed point and P' lies on a fixed circle, centre C' .

Q.E.D.

The proof of the theorem remains unaltered when O lies inside the given circle, or when P' lies on PO or OP produced. Notice that the centres of the two circles are corresponding points.

THEOREM 4.

A, B are the centres of two circles of radii a, b ; AB is divided externally at O and internally at O_1 in the ratio of the radii

[i.e. $\frac{OA}{OB} = \frac{AO_1}{O_1B} = \frac{a}{b}$]; then the circles are homothetic w.r.t. both O and O_1 , and corresponding points lie on the extremities of parallel radii.

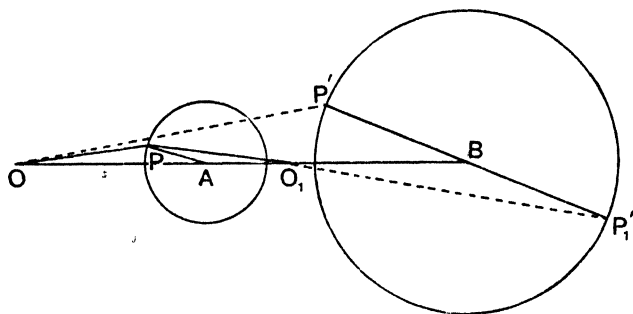


FIG. 4.

AP is any radius of the first circle;

BP' and BP_1' are the radii of the second circle parallel to AP .

Then $\frac{PA}{P'B} = \frac{a}{b} = \frac{OA}{OB}$;

$\therefore \frac{PA}{AO} = \frac{P'B}{BO}$ and angle $PAO = P'BO$ by parallels;

$\therefore$ the triangles $PAO, P'BO$ are similar;

$$\therefore \angle POA = \angle P'OB;$$

$\therefore P'$ lies along OP and $\frac{OP'}{OP} = \frac{P'B}{PA} = \frac{b}{a}$;

$\therefore$ the circles are homothetic w.r.t. O . [Th. 3].

Also P, P' are corresponding points and lie on the extremities of parallel radii.

Precisely the same argument, applied to the triangles O_1AP, O_1BP_1' , proves that the circles are homothetic w.r.t. O_1 and that corresponding points P, P_1' lie on the extremities of parallel radii.

Q.E.D.

Notice that the circles are directly homothetic w.r.t. O and inversely homothetic w.r.t. O_1 ; the centres being corresponding points in each case.

The second part of Theorem 4 may be stated in another way:

If any line through O or O_1 meets the circle, centre A , at P and the circle, centre B , at P' and Q' , the radius AP is parallel to one of the radii BP', BQ' .

10. A is a fixed point on a given circle : a variable chord AP is produced to Q so that $\frac{AP}{AQ}$ is constant, find the locus of Q .

11. Prove that the common tangents to two circles pass through one or other of the two points w.r.t. which the circles are homothetic.

12. The angles of the triangle APQ are given ; A is fixed ; P moves on a given circle, prove that the locus of Q is a circle.

13. Two circles are homothetic w.r.t. O ; a variable line through O cuts the circles in two pairs of corresponding points P, P' ; Q, Q' ; where P, Q are on the same circle. Prove that $OP \cdot OQ' = OQ \cdot OP' = \text{constant}$.

14. A is a fixed point : P is a variable point on a fixed circle centre C ; the line bisecting the angle ACP meets AP at Q ; find the locus of Q .

15. APQ is a variable triangle ; A is fixed, P moves on a fixed line CD , AP meets a fixed line parallel to CD at R ; if $PQ = AR$ and if the angle APQ is constant, find the locus of Q .

It is possible to describe a figure homothetic to a given figure, and drawn to any convenient scale, by means of an instrument, called a **pantograph**.

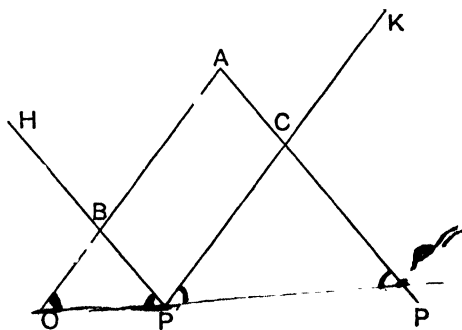


FIG. 5.

In the figure, rods are represented by straight lines. Two equal rods OA, AP are jointed together at A , and the end-point O is fixed. But OA can turn freely about O . Two other rods $P'BH, P'CK$ are jointed together at P' , and are pinned to OA, AP respectively at B, C , so that $\underline{BP' = BO}$ and $BP'CA$ is a parallelogram.

If P is now made to describe any curve S , it can be proved that P' describes a homothetic curve S' .

The proof is left to the reader.

[Show that (1) O, P', P are collinear, (2) $\frac{OP'}{OP} = \frac{OB}{OA} = \text{constant}$.]

In order to obtain any required scale, it is only necessary to adjust the clamp at the position of B for which the ratio $OB':OA$ represents the given scale. If the figure is to be enlarged, it is clear that P' must be made to trace out the given figure.

Certain constructions may be performed by the use of homothetic figures. This is best illustrated by examples.



EXAMPLE I.

In a given triangle ABC , to inscribe a triangle having its sides parallel to those of a given triangle PQR .

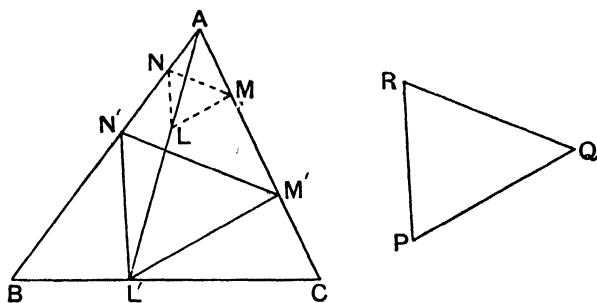


FIG. 6.

Draw any line parallel to QR cutting AC , AB at M , N .

Draw ML , NL parallel to QP , RQ to meet at L .

Join AL and produce it to meet BC at L' .

Through L' draw $L'M'$, $L'N'$ parallel to LM , LN to meet AC , AB at M' , N' .

Then $L'M'N'$ is the required triangle.

✓16. Prove the construction given in Example I.

Another method may be applied to this type of problem. It is not so elegant, but perhaps it brings out more clearly the properties of the centre of similitude. Fundamentally however the two methods are identical.

In the previous construction, through P , Q , R draw lines YZ , ZX , XY parallel to BC , CA , AB to form the triangle XYZ ; then AX , BY , CZ are concurrent, at O say [Theorem 1]. Produce OP , OQ , OR to meet BC , CA , AB at L' , M' , N' ; then $L'M'N'$ is the required triangle.

✓17. Prove this alternative method of construction.

✓18. Apply this method to construct a square having one side along a side of a given triangle, and the other two corners of the square on the remaining sides of the triangle.

EXAMPLE II.

To inscribe a square in a given triangle.

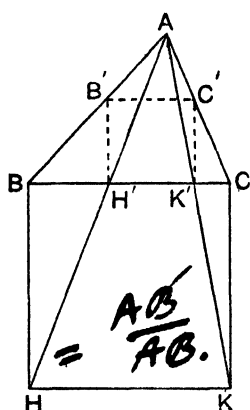


FIG. 7.

ABC is the given triangle.

On BC describe a square $BHKC$, externally to the triangle.

Join AH , AK and let them cut BC at H' , K' .

Draw $H'B'$, $K'C'$ perpendicular to BC to cut AB , AC at B' , C' .

Then $B'H'K'C'$ is the required square.

✓ 19. Prove the construction in Example II.

20. Obtain another solution by describing the square on BC internally to the triangle and using the same construction.

21. If AD is an altitude of the triangle ABC , describe a square on AD , regard B as the centre of similitude, and hence perform the construction of Example II.

THEOREM 5.

A triangle is given in species (*i.e.* its angles are given), one vertex is fixed, another lies on a given circle, then the locus of the third vertex is a circle.

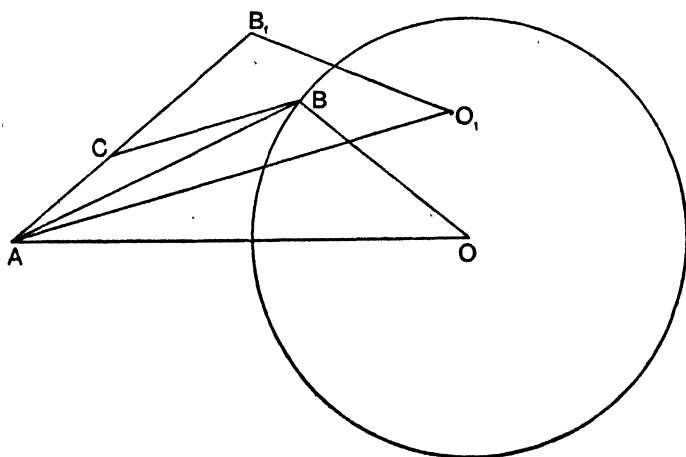


FIG. 8.

ABC is one position of the triangle; A is fixed, B lies on a given circle, centre O .

On AC take a point B_1 such that $AB_1 = AB$.

On AB_1 describe a triangle AO_1B_1 directly similar and $\therefore$ congruent to the triangle AOB .

Since $C_1\hat{A}B_1 = O\hat{A}B$, $O_1\hat{A}O = B_1\hat{A}B = \text{constant}$; also

$$AO_1 = AO = \text{constant};$$

$\therefore O_1$ is a fixed point.

But $O_1B_1 = OB = \text{constant}$;

$\therefore B_1$ describes a circle.

Now $\frac{AC}{AB_1} = \frac{AC}{AB} = \text{constant}$, since triangle ABC is given in species.

$\therefore$ the locus of C is a circle [Th. 3].

Q.E.D.

If the circle, centre O , is rotated rigidly about A through the constant angle BAC , it is clear that B_1 always lies on the new position of this circle. The construction, given above, places this however on a more logical basis.

✓ 22. Two given lines intersect at an inaccessible point H , A is a given point, construct the line AH .

✓ 23. In a given triangle, inscribe an equilateral triangle with one side parallel to a given line.

✓ 24. OB, OC are two given radii of a given circle. Describe a square so that two of its vertices lie on OB, OC and the other two on the arc BC .

✓ 25. $ABCD$ is a quadrilateral such that $BA = BC$ and $DA = DC$; inscribe a square in the quadrilateral.

✓ 26. $ABCD$ is a square: construct a semicircle to touch BA, BC and to have the ends of its base on DA, DC .

✓ 27. Construct a circle to touch two given lines and to pass through a given point.

28. A', B', C' are the mid points of the sides BC, CA, AB of a triangle ABC ; G, H, O are the centroid, orthocentre and circumcentre of the triangle ABC ; prove that

✓ (1) G is the centre of similitude of the triangles $ABC, A'B'C'$;

✓ (2) O is the orthocentre of the triangle $A'B'C'$; and hence

✓ (3) G, H, O are collinear; and find the value of the ratio $\frac{GH}{GO} = \frac{2}{1}$

29. A triangle ABC is given in species; A is fixed, B moves on a fixed straight line; find the locus of C .

30. Two triangles ABC , $A'B'C'$ are directly similar (but corresponding sides are not parallel); BC meets $B'C'$ at D ; the circles DBB' , DCC' meet again at O ; prove that, by a rotation about O , the triangle $A'B'C'$ can be made homothetic to the triangle ABC .]

31. A, B are two fixed points on a circle; P is a variable point on the circle; Q is a point on PA such that $PQ=PB$; prove that the locus of Q is a circle.

32. If S and S' are two curves, each homothetic to a third curve Σ (w.r.t. different centres), prove that S is homothetic to S' .

33. P is a variable point on a fixed line AB ; AC is another fixed line; a circle is drawn to touch AD , AC and the circle on AP as diameter; find the locus of the point of contact of the two circles.

34. A, B are two fixed points on a fixed circle Σ , P is a variable point of Σ ; Q is taken on BP so that $\frac{BQ}{AP}$ is constant, find the locus of Q .

CHAPTER II.

RATIO THEOREMS.

IN the present chapter will be found a few useful theorems on ratios and their application to areas.

Theorem 9 is due to Apollonius, who lectured at Alexandria in the third century B.C. He is said to have been of a conceited and jealous disposition, but little else of a personal nature is known except that his nickname at the university was ϵ , possibly from the name or number of his lecture-room. His great work was a systematic treatise on conic sections.

Theorem 10 is usually ascribed to Ptolemy (*circa* 150 A.D.), but it was probably discovered by Hipparchus nearly three centuries earlier. From this theorem may be deduced all the elementary trigonometrical formulae. This is particularly interesting, because Hipparchus may be regarded as the originator of the science of Trigonometry, which he required for astronomical purposes.

He was the first to systematise astronomical knowledge by compiling a star-catalogue; but his name is immortalised by his discovery of the phenomenon, known as the precession of the equinoxes—a motion so slow that its period is about 25,000 years. Ptolemy stands out as a unique figure in the history of astronomy. From the second to the sixteenth century, *i.e.* till the era of Copernicus, Galileo, Tycho Brahe, Kepler and Newton, the science of astronomy was embodied in Ptolemy's great work, *The Almagest*. His real discoveries, that the earth was spherical and poised in space, and his erroneous beliefs, that the earth was fixed at the centre of a gigantic celestial sphere, rotating daily, to which the stars were attached, and that the paths of the planets were explained by a system of superposed circles, held equal undisputed sway for thirteen or fourteen centuries.

THEOREM 6.

Two triangles ABC , ABD have their vertices C and D on opposite sides of a common base AB . If CD meets AB (or AB produced) at H ; then $\frac{CH}{HD} = \frac{\triangle ACB}{\triangle ADB}$.

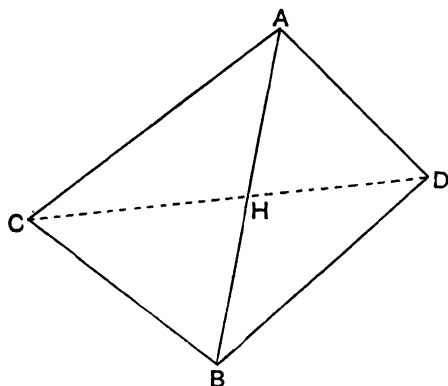


FIG. 9.

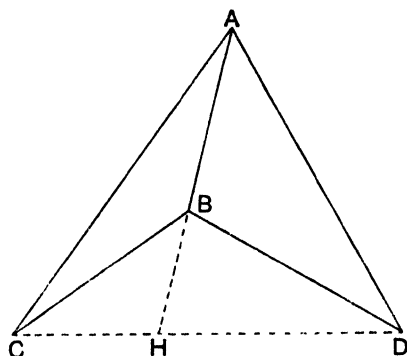


FIG. 10.

$$\begin{aligned} \frac{CH}{HD} &= \frac{\triangle ACH}{\triangle AHD} = \frac{\triangle BCH}{\triangle BHD}; \\ \therefore \frac{CH}{HD} &= \frac{\triangle ACH + \triangle BCH}{\triangle AHD + \triangle BHD} = \frac{\triangle ACH - \triangle BCH}{\triangle ADH - \triangle BDH}; \\ \therefore \text{in either case (Fig. 9 or Fig. 10)} \quad \frac{CH}{HD} &= \frac{\triangle ACB}{\triangle ADB}. \end{aligned}$$

Q.E.D.

1. Prove that Theorem 6 still holds if C and D are on the same side of the common base AB .

THEOREM 7.

If in the triangles ABC , DEF , the angles BAC , EDF are equal; then $\frac{\triangle ABC}{\triangle DEF} = \frac{AB \cdot AC}{DE \cdot DF}$.

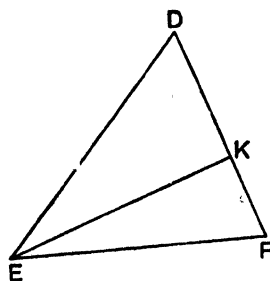
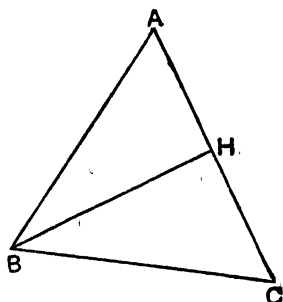


FIG. 11.

Draw BH , EK perpendicular to AC , DF .

Then the triangles ABH , DEK are similar, for

$$\hat{BAH} = \hat{EDK} \text{ and } \hat{BHA} = \hat{EKD}, \text{ right angles.}$$

$$\therefore \frac{BH}{EK} = \frac{AB}{DE};$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BH \cdot AC}{\frac{1}{2}EK \cdot DF} = \frac{AB \cdot AC}{DE \cdot DF}.$$

Q.E.D

Notice that this theorem is an immediate consequence of the trigonometrical formula $\triangle ABC = \frac{1}{2}AB \cdot AC \sin BAC$.

2. O is a point inside the triangle ABC ; AO , BO , CO meet BC , CA , AB at P , Q , R ; prove that

$$(1) \frac{\triangle BOA}{\triangle COA} = \frac{BP}{PC} \text{ and } (2) \frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} = 1.$$

3. With the figure of Ex. 2, prove that

$$(1) \frac{OP}{AP} = \frac{\triangle BOC}{\triangle BAC} \text{ and } (2) \frac{OP}{AP} + \frac{OQ}{BQ} + \frac{OR}{CR} = 1.$$

4. O is a point on the line bisecting the angle BAC of the triangle ABC , prove that $\frac{\triangle BAO}{\triangle CAO} = \frac{BA}{AC}$.

5. Find a point O inside the triangle ABC such that

$$\triangle AOB : \triangle BOC : \triangle COA = p : q : r,$$

where p , q , r are given constants.

6. Points P , Q , R are taken on the sides BC , CA , AB of the triangle ABC so that $BP : PC = CQ : QA = AR : RB = 1 : 2$; prove that $\triangle PQR = \frac{1}{3} \triangle ABC$.

7. Points P , Q are taken on the sides BC , CA of the triangle ABC so that $BP = 2PC$, $CQ = 3QA$; AP meets BQ at O ; calculate the ratio of the areas of the quadrilateral $POQC$ and the triangle ABC . Use Ex. 2.

8. Points P , Q , R are taken on BC , CA , AB so that $\frac{BP}{PC} = \frac{1}{2}$, $\frac{CQ}{QA} = \frac{2}{3}$, $\frac{AR}{RB} = \frac{3}{4}$; the lines AP , BQ , CR form the triangle LMN ; calculate the ratio $\triangle LMN : \triangle ABC$.

9. In the triangles ABC , DEF , if the angles BAC , EDF are supplementary, prove that $\frac{\triangle ABC}{\triangle DEF} = \frac{AB \cdot AC}{DE \cdot DF}$.

10. The diagonals AC , BD of a cyclic quadrilateral $ABCD$ meet at O , prove that $\frac{AB \cdot BC}{AD \cdot DC} = \frac{BO}{OD}$.

11. Two circles touch internally at O : a straight line $ABCD$ cuts the outer at A , D and the inner at B , C ; prove that

$$AB : CD = OA : OB : OC : OD.$$

THEOREM 8.

If two straight lines PRQ , ACB cut three concurrent straight lines OPA , ORC , OQB ; then $\frac{PR}{RQ} : \frac{AC}{CB} = \frac{OP}{OA} : \frac{OQ}{OB}$.

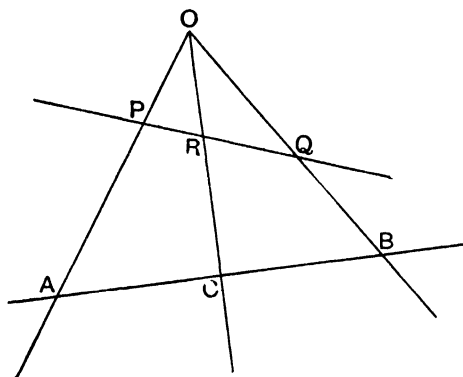


FIG. 12.

$$\begin{aligned}\frac{PR}{RQ} &= \frac{\triangle OPR}{\triangle ORQ} \quad \text{and} \quad \frac{AC}{CB} = \frac{\triangle OAC}{\triangle OCB}; \\ \therefore \frac{PR}{RQ} \times \frac{CB}{AC} &= \frac{\triangle OPR}{\triangle OAC} \times \frac{\triangle OCB}{\triangle ORQ} \\ &= \frac{OP \cdot OR}{OA \cdot OC} \times \frac{OC \cdot OB}{OR \cdot OQ} \\ &= \frac{OP}{OA} \times \frac{OB}{OQ}; \\ \therefore \frac{PR}{RQ} : \frac{AC}{CB} &= \frac{OP}{OA} : \frac{OQ}{OB}.\end{aligned}$$

Q.E.D.

The theorem is still true if OB lies between OA and OC ; or if the line PRQ cuts some or all of the lines OA , OC , OB when produced backwards through O .

12. With the notation of Theorem 8, prove that

$$\frac{PQ}{QR} : \frac{AB}{BC} = \frac{OP}{OA} : \frac{OR}{OC}.$$

13. The medians AA' , CC' of the triangle ABC meet at G , use Theorem 8 to prove that $C'G = \frac{1}{2}GC$.

14. P is any point on AB , the median AA' of the triangle ABC cuts PC at H , prove that $\frac{PH}{HC} = \frac{AP}{AB}$.

15. P, Q, R are points on the sides BC, CA, AB of a triangle such that $\frac{BP}{PC} = \frac{CQ}{QA} = \frac{AR}{RB}$; AP meets QR at D , prove that $\frac{RD}{DQ} = \frac{BP^2}{PC^2}$.

16. A straight line meets the sides BC, CA, AB of a triangle at L, M, N ; prove that $\frac{BL}{CL} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1$.

[Use Theorem 8 to evaluate $\frac{AM}{MC} : \frac{AN}{NB}$.]

17. Two straight lines $ABCD, abcd$ cut four concurrent straight lines OAA, OBB, OCC, ODD ; prove that $\frac{AB}{BC} : \frac{AD}{CD} = \frac{ab}{bc} : \frac{ad}{cd}$.

18. $ABCD$ are four collinear points such that $\frac{AB}{BC} = \frac{AD}{CD}$; O is any point outside AD .

(1) If a straight line cuts OA, OB, OC, OD at a, b, c, d , prove that $\frac{ab}{bc} = \frac{ad}{cd}$.

(2) If the straight abc is parallel to OD , prove that $ab = bc$.

(3) If BOD is a right angle, prove that OB, OD are the bisectors of the angle AOC . [Through B draw PBQ parallel to OD to meet OA, OC at P, Q .]

(4) Prove the Converse of Theorem 9, see page 16.

19. P, Q, R are fixed points on the edges BC, CA, AB of a tetrahedron $ABCD$: any plane meets the edges DA, DB, DC at A', B', C' and the lines DP, DQ, DR at P', Q', R' ; prove that the ratio $\frac{B'P'}{P'C'} \cdot \frac{C'Q'}{Q'A'} \cdot \frac{A'R'}{R'B'}$ is independent of the position of the plane.

20. Points $X_1, X_2, \dots, X_n$ are taken on the sides $A_1A_2, A_2A_3, \dots, A_nA_1$ of a plane polygon $A_1A_2 \dots A_n$. O is any point outside the plane: $OA_1, OA_2, \dots, OA_n, OX_1, OX_2, \dots, OX_n$ meet a second plane at $a_1, a_2, \dots, a_n, x_1, x_2, \dots, x_n$; prove that

$$\frac{a_1x_1}{x_1a_2} \cdot \frac{a_2x_2}{x_2a_3} \dots \frac{a_nx_n}{x_na_1} = \frac{A_1X_1}{X_1A_2} \cdot \frac{A_2X_2}{X_2A_3} \dots \frac{A_nX_n}{X_nA_1}.$$

THEOREM 9.

A, B are two fixed points; P is a moving point such that $\frac{PA}{PB}$ is constant; then P lies on a fixed circle.

Let $\frac{PA}{PB} = \lambda$; produce AP to Q .

Divide AB internally at H and externally at K in the ratio λ .

Since $\frac{AH}{HB} = \frac{AP}{PB} = \frac{AK}{BK}$, PH and PK are the two bisectors of $\hat{APB}$ (internal and external).

$$\therefore \hat{HPK} = \frac{1}{2} [\hat{APB} + \hat{BPQ}] = \text{a right angle};$$

$\therefore P$ always lies on the circle whose diameter is HK .

Q.E.D.

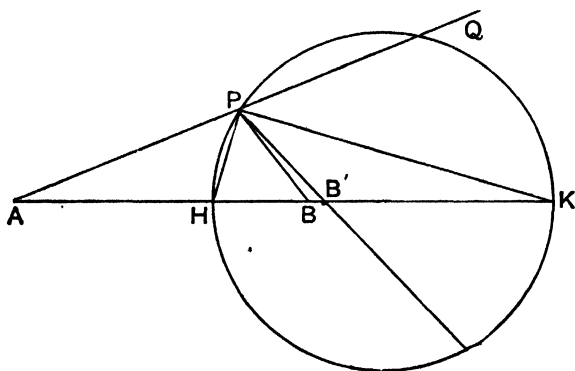


FIG. 13.

The **Converse Theorem** is as follows:

If AB is divided internally at H and externally at K in the same ratio (say λ), and if P is any point on the circle on HK as diameter, then $\frac{PA}{PB} = \lambda$.

Draw PB' , so that $\hat{HPB'} = \hat{APH}$, to meet HK at B' .

$$\text{Since } \frac{AH}{HB} = \frac{AK}{BK}; \therefore \frac{AH}{AK} = \frac{HB}{BK}.$$

Now PH bisects $\hat{APB'}$, but $\hat{HPK}$ is a right angle being in a semicircle;

$\therefore PK$ bisects $\hat{APB'}$ externally;

$$\therefore \frac{AH}{HB'} = \frac{AP}{PB'} = \frac{AK}{B'K};$$

$$\therefore \frac{HB'}{B'K} = \frac{AH}{AK} = \frac{HB}{BK};$$

$$\therefore \frac{HB'}{HB' + B'K} = \frac{HB}{HB + BK} \text{ or } \frac{HB'}{HK} = \frac{HB}{HK};$$

$\therefore HB' = HB$ so that B' coincides with B ;

$\therefore HP$ bisects $\hat{APB}$;

$$\therefore \frac{PA}{PB} = \frac{AH}{HB} = \lambda.$$

Q.E.D.

The circle in Theorem 9 which is the locus of P is called the **circle of Apollonius**.

A simpler proof of the converse of Theorem 9 will be found in the chapter on Harmonic Ranges, page 100. See also Ex. 18 (4).

✓ 21. ABC is a given triangle, find a point P such that

$$PA : PB : PC = 1 : 2 : 4.$$

✓ 22. $ABCD$ are four collinear points. Find a point P at which AB , BC , CD subtend equal angles.

✓ 23. Given the base, vertical angle and the ratio of the other sides of a triangle, construct it.

✓ 24. Given the base, one of the base angles and the ratio of the other sides of a triangle, construct it.

25. AB is a chord of a circle perpendicular to the diameter DE which it meets at H ; the tangent at A meets DE at T , prove that

$$\frac{TE}{EH} = \frac{TD}{HD}.$$

26. A , B are two given points; AP , BQ are parallel chords of a variable circle through A , B . If $\frac{AP}{BQ} = \text{constant}$, prove that the locus of P is a circle.

27. Find the locus of a point at which two given circles subtend equal angles.

28. A , B are two fixed points, P is a variable point, such that $\frac{AP}{PB} = \lambda$. For any particular value of λ , P therefore describes a circle.

Describe the changes in the position and magnitude of this circle as λ increases from 0 to 1 and then increases without limit.

29. With the notation of Ex. 28, if D is the mid point of AB , and if X is the centre of the circle described by P , prove that

$$(1) \text{ its radius} = \frac{\lambda}{\lambda^2 - 1} AB;$$

$$(2) DX = \frac{\lambda^2 + 1}{\lambda^2 - 1} DB;$$

(3) the length of the tangent from D to the circle is independent of the value of λ .

30. A , B are two fixed points; a point P moves so that $\frac{AP}{PB}$ is constant; prove that when P is moving directly towards A , the angle PBA is a right angle.

31. A , B , C are three collinear points; a straight line AP revolves about A ; P is a point on it such that BP , CP are equally inclined to AP , prove that the locus of P is a circle through A .

32. $ABA'B'$ are any four points ; find a point C such that the triangles, CAB , $CA'B'$ are similar, equal angles being denoted by corresponding letters.

33. If two figures are similar find a point O such that if one figure, is rigidly rotated about O , it can be made homothetic to the other figure. [Use Ex. 32.]

34. The internal and external bisectors of the angles of a triangle are drawn and form intercepts on the opposite sides. Three circles are described having these three intercepts as diameters, prove that they have two common points.



THEOREM 10. [Ptolemy's Theorem.]

If $ABCD$ is a cyclic quadrilateral, then

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

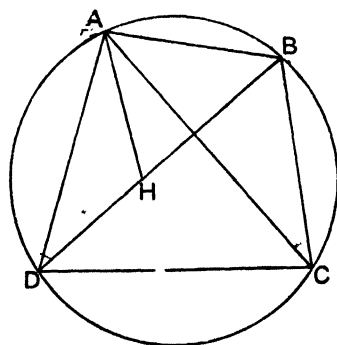


FIG. 14.

Draw AH so that $\angle DAH = \angle BAC$, and let it meet BD at H .

The triangles ADH , ABC are similar,

for
$$\angle DAH = \angle BAC \text{ and } \angle ADH = \angle ACB;$$

$$\therefore \frac{AD}{DH} = \frac{AC}{CB} \text{ or } AD \cdot CB = AC \cdot DH.$$

Also the triangles ADC , ABH are similar,

for
$$\angle DAC = \angle DAH + \angle HAC = \angle BAC + \angle HAC = \angle BAH$$

and
$$\angle DCA = \angle HBA;$$

$$\therefore \frac{DC}{CA} = \frac{HB}{BA} \text{ or } AB \cdot CD = AC \cdot HB;$$

$$\therefore AB \cdot CD + AD \cdot CB = AC \cdot HB + AC \cdot DH$$

$$= AC(DH + HB)$$

$$= AC \cdot DB.$$

Q.E.D.

35. P is a point on the minor arc BC of the circumcircle of an equilateral triangle ABC . prove that $PA = PB + PC$.

36. Show that Pythagoras' theorem on the relations connecting the sides of a right-angled triangle is a special case of Ptolemy's theorem.

37. In the triangle ABC , $AB = AC$; the altitude AD of the triangle meets the circumcircle at P , prove that $AP \cdot BC = 2AB \cdot BP$.

38. C is the mid point of the minor arc AB of a circle; P is a variable point on the major arc, prove that $PA + PB$ is proportional to PC .

39. P is a point on the minor arc AB of the circumcircle of the square $ABCD$, prove that $\frac{PA + PC}{PB + PD} = \frac{PD}{PC}$.

40. P is a point on the minor arc AB of the circumcircle of the regular pentagon $ABCDE$, prove that $\frac{PA + PD}{PB + PE} = \frac{PE}{PC}$.

41. P is a point on the minor arc AB of the circumcircle of the regular hexagon $ABCDEF$, prove that $PE + PD = PA + PB + PC + PF$.

42. $ABCD$ is an isosceles trapezium; AB , CD being the parallel sides, prove that $AC^2 - AD^2 = AB \cdot DC$.

43. If the quadrilateral $ABCD$ is not cyclic, prove that

$$AD \cdot CB + AB \cdot CD > AC \cdot DB.$$

[Draw AH and DH so that angle $DAH = BAC$ and angle $ADH = ACB$.]

44. P is a point on the minor arc AB of the circumcircle of the regular pentagon $ABCDE$, prove that $PA + PB + PD = PC + PE$.

CHAPTER III.

ON LINES AND CIRCLES CONNECTED WITH A TRIANGLE.

It is convenient to discuss the properties of the triangle in two sections. In the first, we shall establish the existence of certain points, lines and circles which are associated with every triangle: and in the second will be found relations of a more metrical character.

Most of the theorems of this chapter were probably known to Euclid. Theorem 25 however is ascribed to Robert Simson (1687-1768) who was a professor at Glasgow University. He undertook a complete revision of Euclid's elements, correcting and supplementing it, and all recent editions of the text are based on his work.

Theorem 18 was first enunciated by an Englishman, named Benjamin Bevan, in 1804: it seems, however, to have been obtained independently by Brianchon.

NOTATION.

It is convenient to denote particular points connected with the triangle ABC by definite letters.

The following notation will be adopted in future, unless otherwise stated:

a, b, c , denote the lengths of BC, CA, AB .

$s = \frac{1}{2}(a + b + c)$.

A', B', C' are the mid points of BC, CA, AB .

D, E, F are the feet of the perpendiculars from A, B, C to BC, CA, AB .

I is the incentre.

r is the radius of the incircle.

I_1, I_2, I_3 are the centres of the circles escribed to BC, CA, AB .

r_1, r_2, r_3 are the radii of these circles.

O is the circumcentre.

R is the radius of the circumcircle.

H is the orthocentre.

G is the centroid.

N is the nine-point centre.

X, Y, Z are the points of contact of the incircle with BC, CA, AB .

X_1, Y_1, Z_1 are the points of contact of the excircle, centre I_1 , with BC, CA, AB , and similarly for X_2, Y_2, Z_2 and X_3, Y_3, Z_3 .

THEOREM 11.

The perpendicular bisectors of the sides of a triangle are concurrent.

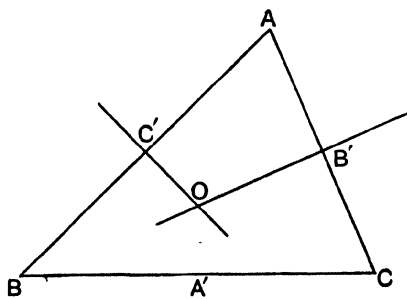


FIG. 15.

Draw the perpendicular bisectors of AB, AC to meet at O .

Every point on the perpendicular bisector of AB is equidistant from A and B ; $\therefore OA = OB$.

Similarly, since O lies on the perpendicular bisector of AC ,

$$OA = OC;$$

$$\therefore OB = OC;$$

$\therefore O$ lies on the perpendicular bisector of BC ;

$\therefore$ the three perpendicular bisectors of the sides meet at O .

Q.E.D.

Corollary.

Since $OA = OB = OC$, a circle centre O , radius OA , passes through B and C and $\therefore$ circumscribes the triangle ABC .

Definition.

The point O is therefore called the **circumcentre** of the triangle ABC .

1. Prove that $\hat{BOA}' = \hat{BAC}$.
2. If $ABCD$ is a parallelogram, prove that the circumcentres of the triangles ABC , ADC are equidistant from AC .
3. L is a point in the base BC of an isosceles triangle ABC ; P , Q are the circumcentres of the triangles ABL , ACL , prove that PL is parallel to AQ .
4. Find a point P such that the feet of the perpendiculars from P to the sides of the triangle ABC are the angular points of an equilateral triangle.
5. Prove that the perpendicular bisector of EF bisects BC .
6. If A , B , C are three collinear points, and if K is any other point, prove that the circumcentres of the triangles KBC , KCA , KAB are concyclic with K .
7. The diagonals AC , BD of a quadrilateral $ABCD$ meet at K , prove that the circumcentres of the triangles KAB , KBC , KCD , KDA are the vertices of a parallelogram.
8. Equilateral triangles are described externally on the sides of a given triangle, prove that their circumcircles meet at a point, and that their circumcentres are the vertices of an equilateral triangle.
9. AB , $A'B'$ are two lines of equal length in a plane; find a point K such that the triangles KAB , $KA'B'$ are congruent.
10. P , Q , R are any points on the sides BC , CA , AB of a triangle, prove that the circumcircles of the triangles AQR , BRP , CPQ are concurrent.

THEOREM 12.

The altitudes of a triangle are concurrent.

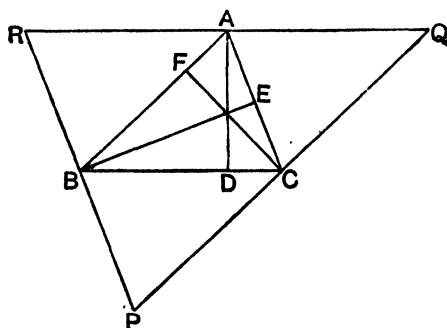


FIG. 16.

Let ABC be the triangle.

Draw AD , BE , CF perpendicular to BC , CA , AB .

Through A , B , C draw lines parallel to BC , CA , AB to form the triangle PQR .

Since $RBCA$ is a parallelogram, $RA = BC$.

Since $ABCQ$ is a parallelogram, $AQ = BC$;

$$\therefore RA = AQ.$$

Similarly, $QC = CP$ and $PB = BR$.

Also, since RQ is parallel to BC , AD is perpendicular to RQ ;

$\therefore AD$ is the perpendicular bisector of RQ .

Similarly, CF , BE are the perpendicular bisectors of QP , PR ;

$\therefore AD$, BE , CF are concurrent. [Th. 11.]

Q.E.D.

Definition.

The point H at which the altitudes concur is called the **orthocentre** of the triangle; and the triangle DEF (see Fig. 16) is called the **pedal triangle** of the triangle ABC .

11. Prove that $\hat{AHE} = \hat{ACB}$

12. Prove that the triangles AEF , ABC are similar.

13. Prove that the triangles AEF , BDF are similar.

14. Prove that $AH \cdot HD = BH \cdot HE = CH \cdot HF$.

15. Prove that $AF \cdot AB = AH \cdot AD = AE \cdot AC$.

16. Prove that $\hat{FDA} = \hat{EDA}$.

17. Prove that $\hat{FDE} = 180^\circ - 2\hat{BAC}$.

18. Prove that $\hat{BHC} + \hat{BAC} = 180^\circ$.

19. If S is the orthocentre of the triangle PQR , prove that P is the orthocentre of the triangle QRS .

20. Given the base BC and the angle $\hat{BAC}$ of the triangle ABC , find the locus of the orthocentre and construct its position when A is very close to B .

21. DP , DQ are the perpendiculars from D to AC , AB ; prove that C , P , Q , B are concyclic and that $\hat{BPD} = \hat{CQD}$.

22. Prove that the circumcentre of the triangle ABC is the orthocentre of the triangle $A'B'C'$.

23. P is a point on the base BC of an isosceles triangle ABC ; K , L are the orthocentres of the triangles APB , APC . Prove that PKL is an isosceles triangle.

24. Prove that OA , OB , OC are perpendicular to the sides of the pedal triangle of the triangle ABC .

Definition.

The line joining a vertex of a triangle to the mid point of the opposite side is called a **median** of the triangle.

THEOREM 13.

The three medians of a triangle are concurrent : and their point of intersection is a point of trisection of each median.

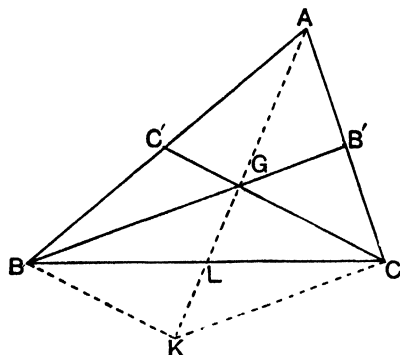


FIG. 17.

Let BB' , CC' be two medians intersecting at G .

Join AG and produce to K , so that $AG = GK$, and let it cut BC at L . Join BK , CK .

Since $AC' = C'B$ and $AG = GK$, $\therefore C'G$ is parallel to BK .

Since $AG = GK$ and $AB' = B'C$, $\therefore GB'$ is parallel to KC ;

$\therefore BGCK$ is a parallelogram ;

$\therefore$ since its diagonals bisect each other,

$BL = LC$ and $GL = LK$;

$\therefore AL$ is a median ;

$\therefore$ the three medians concur at G .

Q.E.D.

Further

$$GL = \frac{1}{2} GK = \frac{1}{2} AG ;$$

$$\therefore GL = \frac{1}{3} AL ;$$

and similarly $GB' = \frac{1}{3} BB'$ and $GC' = \frac{1}{3} CC'$.

Q.E.D.

Alternative Method.

A simpler proof of Theorem 13 may be obtained by using similar triangles.

$B'C'$ is parallel to BC (see Fig. 17) ;

$\therefore$ the triangles BGC , $B'GC'$ are similar.

But $B'C' = \frac{1}{2} BC$;

$$\therefore \frac{C'G}{GC} = \frac{B'G}{GB} = \frac{B'C'}{BC} = \frac{1}{2} ;$$

$\therefore BB'$ cuts CC' at the point of trisection of CC' nearest to C' .

Similarly AA' cuts CC' at this same point.

Q.E.D.

Definition.

The point at which the medians concur is called the **centroid** of the triangle.

25. Construct a triangle given the lengths of the three medians.

26. Construct a triangle given the lengths of two sides and one median. [Two Cases.]

27. If $B'C'$ meets AA' at P , calculate the ratios $\frac{PG}{PA'}$; $\frac{PG}{AA'}$.

28. Prove that a triangle can be drawn with its sides equal and parallel to AA' , BB' , CC' , and that its area = $\frac{3}{4}$ the triangle ABC .

29. In a tetrahedron $ABCD$, the plane angles at each of three corners add up to two right angles, prove that the opposite edges are equal. [Cut down the edges AB , AC , AD and fold it out flat. This appeared in the *Mathematical Gazette*.]

30. If GO meets AD at K , prove that $KG = 2GO$. [Use similar triangles GOA' , GKA .] Hence prove that BE , CF pass through K , so that K is the orthocentre. Hence prove that the orthocentre, centroid and circumcentre are collinear: and that the distance of the orthocentre from A equals $2OA'$.

31. $ALMB$, $APQC$ are the squares described externally on the sides AB , AC of the triangle ABC ; prove that one median of the triangle LAP is an altitude of the triangle ABC .

32. If K is the mid. point of EF , prove that $\hat{KAC} = \hat{A'AB}$.

THEOREM 14.

The internal bisectors of the angles of a triangle are concurrent.

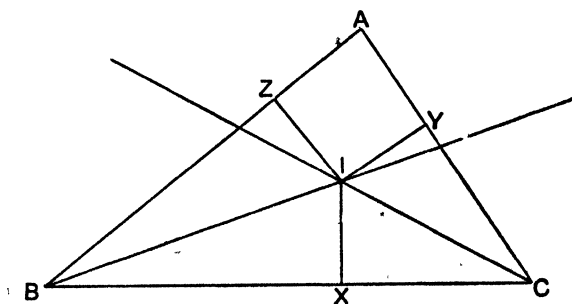


FIG. 18.

Draw the internal bisectors of the angles ABC , ACB to meet at I . From I draw IX , IY , IZ , perpendicular to BC , CA , AB .

Now every point on the line bisecting the angle between BA , BC is equidistant from BA , BC ;

$$\therefore IX = IZ.$$

But I also lies on the bisector of the angle BCA ;

$$\therefore IX = IY;$$

$$\therefore IY = IZ;$$

$\therefore I$ lies on the bisector of the angle BAC ;

$\therefore$ the internal bisectors concur at I . Q.E.D.

Corollary.

Since $IX = IY = IZ$, and since these lines are perpendicular to the sides of the triangle, I is the centre of the circle inscribed in the triangle ABC .

Definition.

(1) The point of concurrence of the internal bisectors of a triangle is called the **incentre** of the triangle: and the circle, centre I , which touches the sides is called the **incircle** of the triangle.

(2) If a circle touches BC and touches AB produced and AC produced it is called an **excircle** of the triangle or a circle escribed to BC : and its centre is called an **excentre** of the triangle.

THEOREM 15.

(1) The vertices of the triangle formed by the external bisectors of the angles of the triangle ABC are the excentres of the triangle.

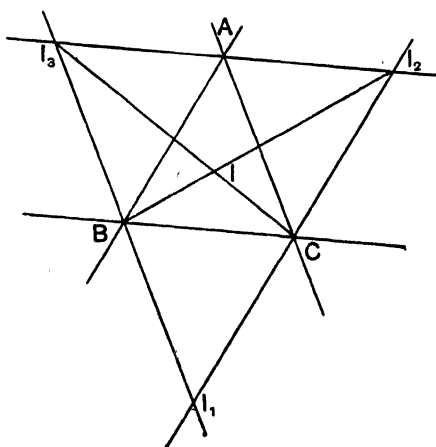


FIG. 19.

(2) The line joining an excentre to the incentre passes through a vertex of the triangle.

(3) The incentre is the orthocentre of the triangle whose vertices are the excentres.

(1) Let I_1, I_2, I_3 be the triangle formed by the external bisectors.

Since BI_1 bisects $\hat{A}BC$ externally, I_1 is equidistant from BC and AB (produced).

Since CI_1 bisects $\hat{A}CB$ externally, I_1 is equidistant from BC and AC (produced);

$\therefore I_1$ is equidistant from the three sides of the triangle ABC ;

$\therefore I_1$ is the centre of the circle escribed to BC ,

and similarly I_2, I_3 are the centres of the circles escribed to CA, AB . Q.E.D.

(2) Since I_1 is equidistant from AB, AC , it follows that AI_1 bisects $\hat{BAC}$; $\therefore I$ lies on AI_1 ,
and similarly I lies on BI_2 and CI_3 . Q.E.D.

(3) Since AI_1 and I_2I_3 are the internal and external bisectors of $\hat{BAC}$, AI_1 is perpendicular to I_2I_3 ; similarly BI_2 is perpendicular to I_3I_1 ;

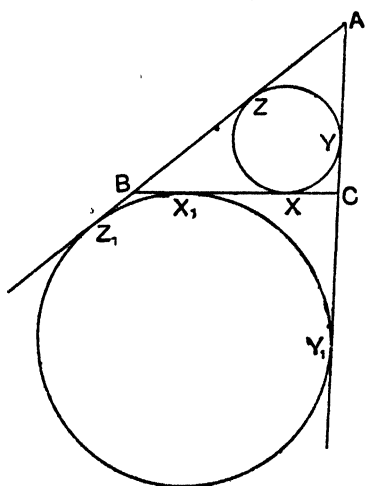
$\therefore I$ is the orthocentre of the triangle $I_1I_2I_3$. Q.E.D.

THEOREM 16.

The incircle of the triangle ABC touches BC, CA, AB at X, Y, Z , and the circle escribed to BC touches BC, CA, AB at X_1, Y_1, Z_1 , then (1) $AZ = AY = s - a$; (2) $AZ_1 = AY_1 = s$; (3) $BX_1 = CX$; where $BC = a, CA = b, AB = c$, and $s = \frac{1}{2}(a + b + c)$.

$$\begin{aligned} (1) \quad 2s &= a + b + c \\ &= a + AY + YC + AZ + ZB \\ &= a + AY + XC + AZ + BX \\ &= a + 2AY + BC \\ &= 2a + 2AY; \end{aligned}$$

$$\therefore AY = s - a.$$



Q.E.D.

FIG. 20.

$$\begin{aligned}
 (2) \quad \angle AZ_1 &= AZ_1 + AY_1 = AB + BZ_1 + AC + CY_1 \\
 &= AB + BX_1 + AC + X_1C \\
 &= c + BC + b \\
 &= a + b + c = 2s;
 \end{aligned}$$

$$\therefore AZ_1 = s.$$

Q.E.D.

$$\begin{aligned}
 (3) \quad BX_1 &= BZ_1 = AZ_1 - AB \\
 &= s - c \\
 &= CX \text{ by part (1).}
 \end{aligned}$$

Q.E.D.

THEOREM 17.

If the area of the triangle ABC equals Δ , then the radius r of the incircle equals $\frac{\Delta}{s}$, and the radius r_1 of the circle escribed to BC equals $\frac{\Delta}{s-a}$.

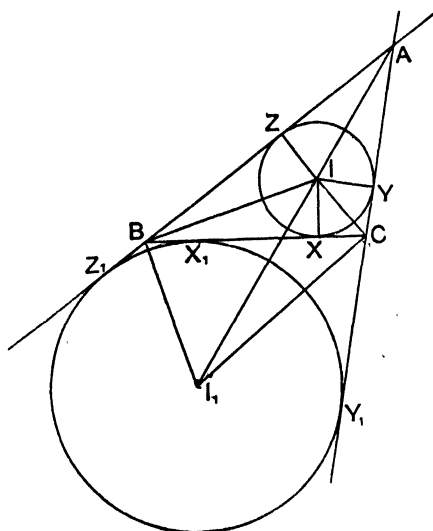


FIG. 21.

$$\begin{aligned}
 (1) \quad \Delta &= \text{triangle } IBC + \text{triangle } ICA + \text{triangle } IAB \\
 &= \frac{1}{2}IX \cdot BC + \frac{1}{2}IY \cdot CA + \frac{1}{2}IZ \cdot AB \\
 &= \frac{1}{2}r \cdot a + \frac{1}{2}r \cdot b + \frac{1}{2}r \cdot c \\
 &= \frac{1}{2}r(a + b + c); \text{ but } a + b + c = 2s \\
 &= r \cdot s;
 \end{aligned}$$

$$\therefore r = \frac{\Delta}{s}.$$

Q.E.D.

$$\begin{aligned}
 (2) \quad \Delta &= \text{triangle } I_1AB + \text{triangle } I_1AC - \text{triangle } I_1BC \\
 &= \frac{1}{2} I_1Z_1 \cdot AB + \frac{1}{2} I_1Y_1 \cdot AC - \frac{1}{2} I_1X_1 \cdot BC \\
 &= \frac{1}{2} r_1 \cdot c + \frac{1}{2} r_1 \cdot b - \frac{1}{2} r_1 \cdot a \\
 &= \frac{1}{2} r_1 (c + b - a) = \frac{1}{2} r_1 (c + b + a - 2a) \\
 &= \frac{1}{2} r_1 (2s - 2a) \\
 &= r_1 (s - a);
 \end{aligned}$$

$$\therefore r_1 = \frac{\Delta}{s - a}.$$

Q.E.D.

33. Prove that (1) $YY_1 = ZZ_1 = a$;

(2) $XX_1 = c \approx b$;

(3) $Z_1Z_2 = a + b$.

34. A polygon, of perimeter l , circumscribes a circle of radius r , prove that its area equals $\frac{1}{2}rl$.

35. From any point inside a regular polygon, perpendiculars are drawn to the sides of the polygon, prove that the sum of their lengths is constant.

36. Prove that the sum of the areas of the triangles I_1AY , I_1AZ is equal to the area of the triangle ABC .

37. If $\hat{BAC} = 60^\circ$, prove that O, H, I, I_1, B, C , lie on a circle.

38. Prove that $\hat{IAO}$ is half the difference of the angles ABC, ACB .

39. M is the foot of the perpendicular from C to AI , prove that the angles BCM, ACM are respectively equal to half the difference and sum of the angles ACB, ABC .

40. The tangents at the ends of a chord AB of a circle meet at C ; prove that the incentre of the triangle ABC lies on the circle.

41. Prove that $\hat{BIC} = 90^\circ + \frac{A}{2}$ where $A = \text{angle } EAC$.

42. Prove that the radius of the circumcircle of the triangle $AYZ = \frac{1}{2}AI$.

43. P, Q are the feet of the perpendiculars from X, Y to YZ and ZX respectively, prove that PQ is parallel to AB .

44. Given the base and vertical angle of a triangle, find the locus of (1) the incentre; (2) the excentres.

45. If $\hat{ACB} = 90^\circ$, prove that $2r = a + b - c$; and that $r_3 = s$.

46. A_1, B_1, C_1 are the excentres of the triangle ABC ; A_2, B_2, C_2 are the excentres of the triangle A_1, B_1, C_1 , and so on; find the value to which $A_n \hat{B}_n \hat{C}_n$ tends, as n becomes very large.

47. The incircle of the triangle ABC touches BC , CA , AB at A_1 , B_1 , C_1 ; the incircle of the triangle $A_1B_1C_1$ touches B_1C_1 , C_1A_1 , A_1B_1 at A_2 , B_2 , C_2 , and so on; prove that $B_n\hat{A}_nC_n = \frac{\pi}{3} + (-2)^{-n} \left[B\hat{A}C - \frac{\pi}{3} \right]$. What limit does $B_n\hat{A}_nC_n$ tend towards, as n becomes very large?

48. In the triangle ABC , the internal and external bisectors of the angles meet the opposite sides at P , p ; Q , q ; R , r respectively. Prove that $\frac{1}{Pp} \pm \frac{1}{Qq} \pm \frac{1}{Rr} = 0$.

49. A variable line XY cuts two fixed lines AB , CD at X , Y ; find the locus of the point of intersection of the lines bisecting the angles AXY , CYX .

50. Prove that H is the incentre of the pedal triangle of ABC .

51. K and L are the feet of the perpendiculars from A to the lines bisecting internally the angles ABC , ACB ; prove that KL is parallel to BC .

52. From a given point, draw a line so as to form with two given lines a triangle of given perimeter.

53. AI_2 meets BC produced at P , prove that $BP \cdot CP = AP^2 + BA \cdot AC$.

54. K is the circumcentre of the triangle $I_1I_2I_3$, prove that KI_1 is perpendicular to BC .

55. Two sides of a triangle, of given perimeter, are given in position, find the envelope of the third side.

THEOREM 18.

$A'B'C'$ are the mid points of the sides of a triangle ABC ;

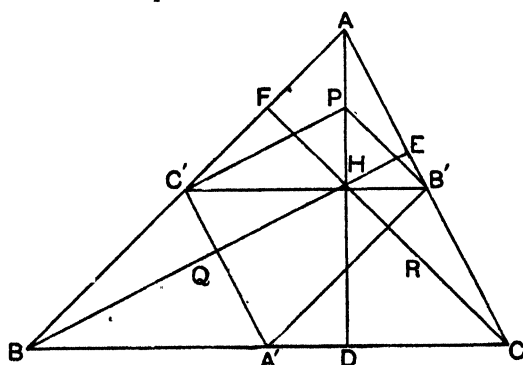


FIG. 22.

AD , BE , CF are the altitudes; H is the orthocentre; P , Q , R are the mid points of HA , HB , HC ; then the nine points

$A'B'C'$, DEF , PQR lie on a circle whose radius is equal to half that of the circumcircle of the triangle ABC .

Since $AP = PH$ and $AC' = C'B$, $\therefore C'P$ is parallel to BH .

Since $BC' = C'A$ and $BA' = A'C$, $\therefore A'C'$ is parallel to AC .

But BH is perpendicular to AC , $\therefore C'P$ is perpendicular to $A'C'$;

$\therefore \angle A'C'P$ is a right angle.

Similarly $\angle A'B'P$ is a right angle.

Further $\angle A'DP$ is a right angle;

$\therefore$ the circle on $A'P$ as diameter passes through B' , C' , D ;

$\therefore D$ and P lie on the circle $A'B'C'$.

Similarly E , Q ; F , R lie on the circle $A'B'C'$;

$\therefore$ the nine points $A'B'C'$, DEF , PQR lie on a circle.

Moreover, each side of the triangle $A'B'C'$ is half the corresponding side of the triangle ABC ; and $\therefore$ the radius of the circle $A'B'C'$ is half that of the radius of the circle ABC . Q.E.D.

Definition.

The circle which passes through these nine points is called the **nine-point circle** of the triangle ABC , and its centre is called the **nine-point centre**.

THEOREM 19.

If PQ is the diameter of the circumcircle of the triangle ABC perpendicular to BC , then AP and AQ are the two bisectors of the angle BAC .

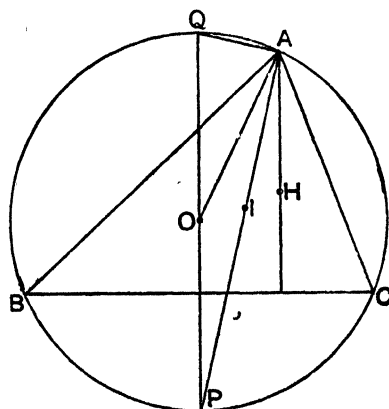


FIG. 23.

Since PQ is a diameter perpendicular to BC , it bisects the arc BPC .

Since $\text{arc } BP = \text{arc } PC,$
 $\hat{BAP} = \hat{PAC};$
 $\therefore AP$ bisects $\hat{BAC}.$

But $\hat{QAP}$ is a right angle, since PQ is a diameter;
 $\therefore AQ$ is perpendicular to AP ;
 $\therefore AQ$ is the other bisector of $\hat{BAC}.$

Q.E.D.

THEOREM 20.

If O, I, H are the circumcentre, incentre and orthocentre of the triangle ABC , then AI bisects the angle OAH .

In Fig. 23, join AH , which is perpendicular to BC , and is $\therefore$ parallel to OP . Also I lies on AP , the bisector of $\hat{BAC}$.

$$OA = OP, \text{ radii, } \therefore \hat{OAP} = \hat{OPA}.$$

But $\hat{OPA} = \hat{PAH}$, alternate angles;
 $\therefore \hat{OAI} = \hat{IAH}.$

Q.E.D.

56. If P is the mid-point of AH , prove that (1) $\hat{A'PC'} = \hat{A'B'C'}$, and (2) $\angle FPC'$ equals the difference of the angles $\angle BAC, \angle ABC$.

57. Prove that the radius of the circumcircle of the pedal triangle equals $\frac{1}{2}R$.

58. Prove that the tangent to the nine-point circle at the mid-point of the base is parallel to the tangent to the incircle from the point at which the bisector of the vertical angle meets the base.

59. Prove that I_1I_2 is bisected by the circumcircle of the triangle ABC .

60. Prove that the triangles ABC, HBC have the same nine-point circle.

61. Prove that the nine-point circles of the triangles $AB'C', BC'A', CA'B'$ each touch the nine-point circle of the triangle $A'B'C'$.

62. If P, Q, R are the mid-points of HA, HB, HC , prove that $A'P, B'Q, C'R$ are equal and concurrent.

63. Prove that the centre N of the nine-point circle is at the mid-point of OH .

64. Given the base BC and vertical angle of a triangle ABC , prove that the locus of N is a circle centre A' .

65. A, B, C are three fixed points. Describe a square with one vertex at A , so that the sides opposite to A pass through B and C .

66. A sphere of radius 6 inches rests on three horizontal wires forming a plane triangle whose sides are 5 inches, 12 inches, 12 inches. Find the height of the top of the sphere above the plane of the wires.

67. A sphere is inscribed in a cube, edge a ; and a plane passing through the extremities of three edges which meet at a corner meets the sphere in a circle, diameter d ; prove that $3d^2 = a^2$.

THEOREM 21.

If $AI I_1$ meets the circumcircle of the triangle ABC at P , then $PI = PB = PC = PI_1$.

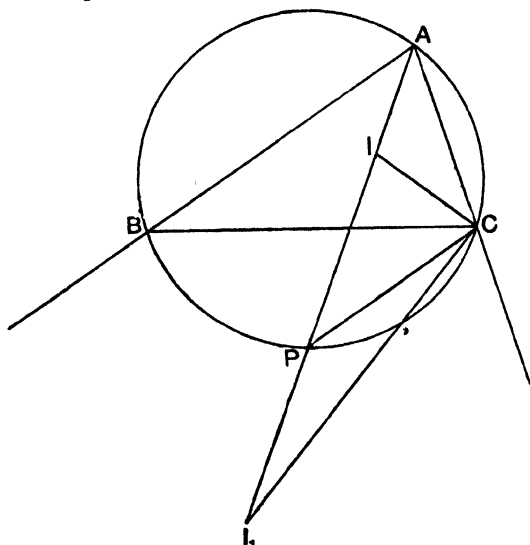


FIG. 24.

$$\begin{aligned} \hat{P}IC &= \hat{I}AC + \hat{ICA} = \frac{1}{2}\hat{B}AC + \frac{1}{2}\hat{B}CA, \\ \text{and } \hat{P}CI &= \hat{P}CB + \hat{BCI} = \hat{P}AB + \hat{BCI} = \frac{1}{2}\hat{B}AC + \frac{1}{2}\hat{B}CA; \\ \therefore \hat{P}IC &= \hat{P}CI; \end{aligned}$$

$$\therefore PI = PC, \text{ and similarly } = PB.$$

Now, $\hat{ICI}_1$ and $\hat{IBI}_1$ are right angles. [Th. 15 (3).]

$\therefore$ the points $BICI_1$ lie on a circle.

But

$$PI = PB = PC;$$

$\therefore P$ is the centre;

$$\therefore PI_1 = PB = PC = PI.$$

Q.E.D.

68. By using Theorem 18 and noting that ABC is the pedal triangle of $I_1I_2I_3$ obtain a shorter proof of Theorem 21.

69. AI, BI, CI are produced to meet the circumcircle of the triangle ABC at P, Q, R ; prove that I is the orthocentre of the triangle PQR .

70. If the circle BIC cuts AC at P , prove that $AP=AB$.

71. Given the incentre, circumcentre and one excentre of a triangle, construct it.

72. Prove that the circumcircle of the triangle formed by any three of the four common tangents to two circles passes through the mid point of the line joining their centres.

73. P, Q, R are the circumcentres of the triangles BIC, CIA, AIB ; find the angles of the triangle PQR in terms of the angles of the triangle ABC , and prove that the two triangles cannot be similar unless ABC is equilateral.

74. A fixed circle X passes through the centre of another fixed circle Y . A variable triangle circumscribes Y and has two of its vertices on X , prove that the locus of the third vertex is a straight line.

75. I_1O is produced to K so that $I_1O = OK$; prove that $IK = 2OA$.

THEOREM 22.

If AD, BE, CF are the altitudes of the triangle ABC , then

- (1) the triangles AEF, ABC are similar,
- (2) AD bisects the angle FDE .

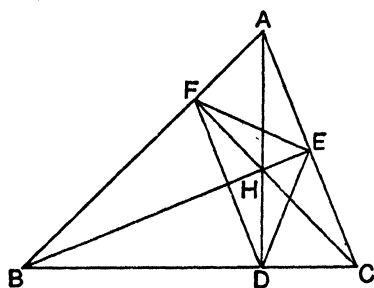


FIG. 25.

(1) $\hat{BFC} = 90^\circ = \hat{BEC}$, $\therefore BFEC$ is a cyclic quadrilateral ;

$$\therefore \hat{AFE} = \hat{BCE};$$

and $\hat{AEF} = \hat{FBC};$

$\therefore$ the triangles AFE, ABC are equiangular and $\therefore$ similar.

Q.E.D.

(2) $\widehat{BFH} = 90^\circ = \widehat{BDH}$, $\therefore BFHD$ is a cyclic quadrilateral ;

$$\begin{aligned}\therefore \widehat{FDH} &= \widehat{FBH} \text{ or } \widehat{ABE} \\ &= 90^\circ - \widehat{BAC}.\end{aligned}$$

Similarly

$$\begin{aligned}\widehat{EDH} &= \widehat{ECH} \text{ or } \widehat{ACF} \\ &= 90^\circ - \widehat{BAC};\end{aligned}$$

$$\therefore \widehat{FDH} = \widehat{EDH};$$

$\therefore AD$ bisects the angle FDE .

Q.E.D.

76. If AP is a diameter of the circumcircle of the triangle ABC , prove that HP bisects BC .

77. If BE , CF meet the circumcircle at Q , R , prove that A is the mid-point of the arc QR .

78. The base BC and the vertical angle of the triangle BPC are fixed. If BE , CF are altitudes of BPC , prove that EF is of constant length.

79. Prove that A is an excentre of the triangle DEF .

80. Deduce from Ex. 70 a corresponding theorem for the pedal triangle.

81. Prove that $FD \cdot DE = BD \cdot DC$.

82. If CO meets the circumcircle at P , prove that $PBHA$ is a parallelogram : hence prove that $AH = 2OA'$.

83. Prove that $\widehat{OAI} = \frac{C+B}{2}$, and hence give an independent proof of Theorem 20.

THEOREM 23.

The altitude AD meets the circumcircle of the triangle ABC at P ; if H is the orthocentre, then $HD = DP$.

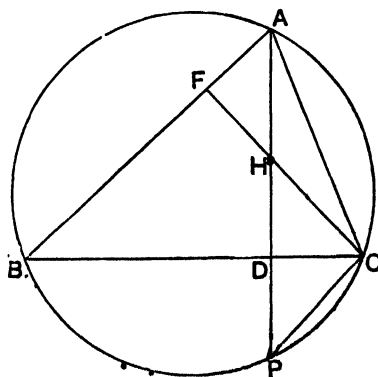


FIG. 26.

Join CP and draw the altitude CHF .

$$\hat{PCD} \text{ or } \hat{PCB} = \hat{PAB} \text{ in the same segment}$$

$$= 90^\circ - \hat{ABC}$$

$$= \hat{FCB}, \text{ since } \hat{BFC} = 90^\circ;$$

$\therefore$ the triangles HDC , PDC are congruent,

for

$$\hat{HDC} = 90^\circ = \hat{PDC},$$

$$\hat{HCD} = \hat{PCD},$$

DC is common;

$$\therefore HD = DP.$$

Q.E.D.

84. Write out the proof of Theorem 23 when $\hat{ACB}$ is obtuse.

85. Prove $AD \cdot HD = BD \cdot DC$.

86. Prove that the circumcircles of the triangles ABC , HBC are equal.

87. Prove that the circumcircles of the triangles HBC , HCA , HAB are equal.

88. If Q , R are the circumcentres of the triangles CHA , AHB , prove that BR and CQ are parallel.

89. P , Q , R are the circumcentres of the triangles HBC , HCA , HAB ; prove that the triangles PQR , ABC are congruent; and that H is the circumcentre of PQR .

90. Prove that the radius of the circle II_1I_2 is equal to the diameter of the circle ABC .

91. AD , BE , CF meet the circumcircle of the triangle ABC at PQR ; prove that H is the incentre of PQR .

92. BE , CF meet the circumcircle of the triangle ABC at Q , R ; prove that A is the centre of the circle HQR .

93. $ABCD$ is a cyclic quadrilateral. H , K are the orthocentres of the triangles ABC , ABD ; prove that A , B , H , K are concyclic.

THEOREM 24.

If AD is an altitude of the triangle ABC , and if R is the radius of the circumcircle, then $AB \cdot AC = 2R \cdot AD$.

Draw the diameter AK of the circumcircle. Join BK . The triangles ADC , ABK are similar,

for $\hat{ADC} = 90^\circ = \hat{ABK}$, since AK is a diameter.

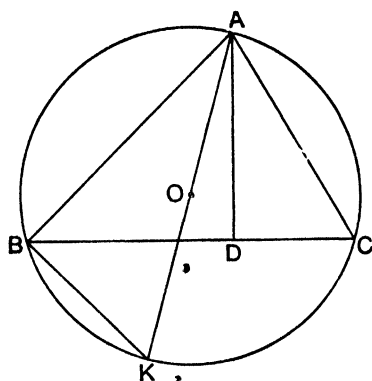


FIG. 27.

$\hat{ACD} = \hat{AKB}$ in the same segment ;

$$\therefore \frac{AB}{AK} = \frac{AD}{AC};$$

$$\therefore AB \cdot AC = AK \cdot AD \\ = 2R \cdot AD.$$

Q.E.D.

94. If $a=40$, $b=68$, $c=84$, prove that $R=42.5$.

95. P is a point on the circle circumscribing a quadrilateral $ABCD$. Perpendiculars PL , PM , PN , PR are let fall to AB , BC , CD , DA , prove that $PL \cdot PN = PM \cdot PR$. [Use Theorem 24.] If PR , PS are the perpendiculars to AC , BD , prove that $PL \cdot PN = PR \cdot PS$.

96. The tangents at the extremities of a chord AB of a circle meet at C . From a point P on the circle, perpendiculars PL , PM , PN are drawn to AB , BC , CA ; prove that $PM \cdot PN = PL^2$.

97. P is any point on the circle circumscribing a hexagon $ABCDEF$; PL_1 , PL_2 , PL_3 , PL_4 , PL_5 , PL_6 are the perpendiculars from P to consecutive sides, prove that $\frac{PL_1}{PL_2} \cdot \frac{PL_3}{PL_4} \cdot \frac{PL_5}{PL_6} = 1$.

98. Generalise Ex. 97 to obtain a theorem for any $2n$ -sided polygon inscribed in a circle.

99. P is a point on the incircle of the triangle ABC : X , Y , Z are the points of contact with BC , CA , AB ; PL , PM , PN , Pl , Pm , Pn are the perpendiculars from P to BC , CA , AB , YZ , ZX , XY ; prove that $\frac{PL}{Pl} \cdot \frac{PM}{Pm} \cdot \frac{PN}{Pn} = 1$.

100. Generalise Ex. 99 to apply to any polygon inscribed in a circle. [Draw tangents at its vertices to form a new polygon.]

101. OP, OQ are two variable chords of a fixed circle. O is a fixed point. If $OP \cdot OQ$ is constant, find the envelope of PQ .

102. P is a point on the circumcircle of an equilateral triangle ABC , and is situated on the minor arc AC ; PL, PM, PN are the perpendiculars to BC, CA, AB ; O is the circumcentre, prove that

$$4AO(PL+PM+PN)=3PB^2-PA^2-PC^2.$$

THEOREM 25. [SIMSON'S LINE.]

If P is any point on the circumcircle of the triangle ABC , and if L, M, N are the feet of the perpendiculars from P to BC, CA, AB , then L, M, N are collinear.

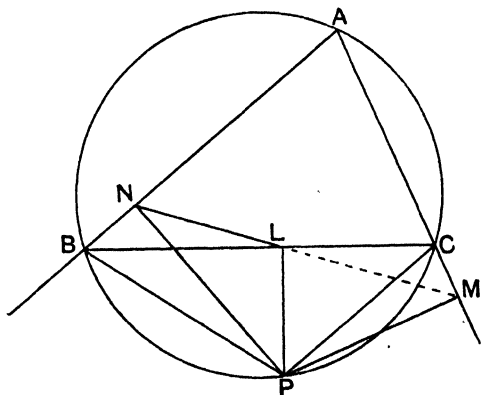


FIG. 28.

Join LN, LM, PB, PC .

Since $\hat{PNB} = 90^\circ = \hat{PLB}$, $PLNB$ is a cyclic quadrilateral.

Since $\hat{PLC} = 90^\circ = \hat{PMC}$, $PLCM$ is a cyclic quadrilateral;

$$\begin{aligned} \therefore \hat{PLM} &= \hat{PCM}, \text{ since } PLCM \text{ are concyclic,} \\ &= \hat{PBA}, \text{ since } PCAB \text{ are concyclic,} \\ &= 180^\circ - \hat{PLN}, \text{ since } PLNB \text{ are concyclic;} \\ \therefore \hat{PLM} + \hat{PLN} &= 180^\circ; \\ \therefore MLN &\text{ is a straight line.} \end{aligned}$$

Q.E.D.

Definition.

The line MLN is called the **Simson Line** or the **pedal line** of P w.r.t. the triangle ABC .

A generalised form of Theorem 25 is given in Ex. 114.

103. If the pedal line of P is parallel to AO , prove that PA is parallel to BC .

104. Find a point P having its pedal line parallel to a given line.

105. Prove that the pedal line of the point at which AD cuts the circumcircle is parallel to the tangent at A to the circumcircle.

106. KL is a diameter of the circumcircle of the triangle ABC , prove that the pedal lines of K and L intersect at right angles on the nine-point circle.

107. The pedal line of a point P meets BC at L and AD at K ; prove that LH is parallel to PK .

108. The perpendiculars from a point P on the circumcircle of the triangle ABC to BC , CA , AB cut the circle again at X , Y , Z ; prove that the triangles ABC , XYZ are congruent.

109. Find the three points, the pedal lines of which are perpendicular to the medians of the given triangle.

110. P , Q , R are the mid-points of the arcs BC , CA , AB of the circumcircle of the triangle ABC ; prove that the triangle formed by the pedal lines of P , Q , R w.r.t the triangle ABC is similar to the triangle formed by joining the points of contact of ABC with its incircle.

111. P is a point on the circle circumscribing a quadrilateral $ABCD$; if AB is parallel to CD , prove that the feet of the perpendiculars from P to AC , AD , BC , BD are concyclic.

112. AD , BE , CF meet the circumcircle of the triangle ABC at P , Q , R ; prove that the pedal line of A w.r.t. the triangle PQR is parallel to BC .

113. P , Q are any two points on the circumcircle of the triangle ABC ; prove that the angle between the pedal lines of P and Q equals $\frac{1}{2}\hat{POQ}$.

114. If in Theorem 25, PL , PM , PN are drawn so that

$$\hat{PNB} = \hat{PLB} = \hat{PMC},$$

prove that L , M , N are collinear.

115. Prove that if P is such a point that the feet of the perpendiculars from P to the sides of a triangle are collinear, then P must lie on the circumcircle of the triangle. (Converse of Theorem 25.)

116. Prove that the circumcircles of the four triangles formed by four straight lines have a common point. [Use Ex. 115.]

117. L , M , N are the feet of the perpendiculars from a point P on the circumcircle of the triangle ABC to the sides BC , CA , AB . Prove that the triangles PLN , PAC are similar, and find the position of P for which $ML \perp LN$.

118. With the notation of Ex. 117, prove that $PL.MN$, $PM.NL$, $PN.LM$ are proportional to BC , CA , AB .

119. With the notation of Ex. 117, prove that of the ratios $\frac{BC}{PL}$, $\frac{CA}{PM}$, $\frac{AB}{PN}$ one of them is equal to the sum of the other two.

120. Prove that the pedal line of P bisects PH .

121. Construct a triangle, given its pedal triangle.

122. With the notation of Fig. 28, if PK is the perpendicular from P to MN , prove that $PM.PN=PA.PK$.

123. Find a point P inside the acute angled triangle ABC such that the circumcircles of the triangles APB , BPC , CPA are equal.

124. P , Q , R are any points on the sides BC , CA , AB of a triangle, prove that the circumcentres of the triangles BPR , CQP , ARQ are the vertices of a triangle similar to ABC .

CHAPTER IV.

ON POINTS CONNECTED WITH A TRIANGLE.

THE theorems of this chapter were enunciated at a much later date than the majority of those already considered. Theorems 26, 27, 29 are due to Euler (1707-1783), a Frenchman by birth, who spent most of his life at Berlin and S. Petersburg. He may be regarded as the first writer on modern analysis: but in addition to his numerous contributions to pure mathematics, there are a number of fundamental theorems in applied mathematics which are first found in his memoirs.

THEOREM 26.

The circumcentre, the centroid, and the orthocentre of a triangle are collinear.

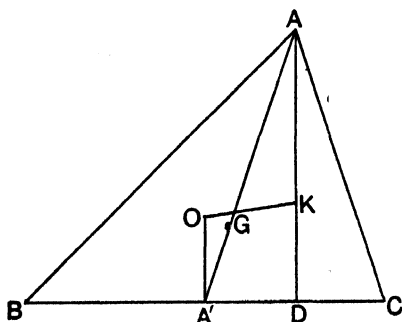


FIG. 29.

Let O , G be the circumcentre and centroid.

Produce OG to meet the altitude AD at K .

The triangles OGA' , KGA are similar, for OA' is parallel to AK , since each is perpendicular to BC ;

$$\therefore \frac{OG}{GK} = \frac{A'G}{GA} = \frac{1}{2}, \text{ since } G \text{ is the centroid;}$$

$$\therefore GK = 2OG.$$

1. The extremities of a straight line PQ of constant length move on two fixed lines AB, AC ; prove that the locus of the orthocentre of the triangle APQ is a circle, centre A .
2. In the triangle ABC , prove that $AH^2 + BC^2 = 4AO^2$.
3. AO meets BC at U ; prove that the line joining D to the mid-point of AU passes through N .
4. If $\hat{BAC} = 45^\circ$, prove that EF bisects OH .
5. Given the nine-point circle and one angular point of a triangle, prove that the locus of the orthocentre is a circle.
6. Given A, N, H , construct the triangle.
7. Given O, A and the lengths of HA, GA , construct the triangle.
8. Given A, B, N , construct the triangle.
9. Prove that the circumcircle and the nine-point circle are homothetic w.r.t. either H or G .
10. Prove that HA' meets the circumcircle at the end P of the diameter through A of the circumcircle.
11. GA', GD meet the circumcircle at P, Q , prove that PQ is parallel to $A'D$ and is equal to $\frac{1}{2}A'D$.
12. P is the mid-point of AH ; GP meets the circumcircle at Q , prove that HQ is parallel to AG .

THEOREM 28.

The internal bisector of the angle BAC meets the circumcircle of the triangle ABC at P . If M, N are the feet of the perpendiculars from P to AB, AC ,

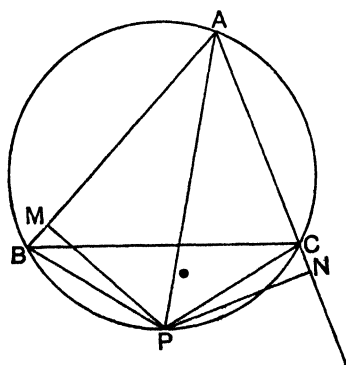


FIG. 31.

then

$$AM = AN = \frac{1}{2}(AB + AC),$$

$$MB = CN = \frac{1}{2}(AB - AC).$$

Join PB , PC .

Since P , B , A , C are concyclic

$$\hat{PBA} + \hat{PCA} = 180^\circ;$$

$\therefore$ one of the angles PBA , PCA is acute and the other is obtuse.

Suppose then (as in Fig. 31) that $\hat{PBA}$ is acute, then M lies between A and B and N lies on AC produced.

[Why is this a necessary part of the proof?]

The triangles AMP , ANP are congruent,

for $\hat{AMP} = 90^\circ = \hat{ANP}$ and $\hat{MAP} = \hat{NAP}$
and AP is common;

$$\therefore AM = AN \text{ and } PM = PN;$$

also the triangles BMP , CNP are congruent,

for $\hat{BMP} = 90^\circ = \hat{CNP}$,

and $PB = PC$

since these chords subtend equal angles at the circumference,

and $PM = PN$;

$$\therefore MB = CN;$$

$$\therefore AB - AM = AN - AC;$$

$$\therefore AB + AC = AM + AN = 2AM;$$

$$\therefore AM = AN = \frac{1}{2}(AB + AC). \quad \text{Q.E.D.}$$

$$\therefore MB = AB - AM = AB - \frac{1}{2}(AB + AC) \\ = \frac{1}{2}(AB - AC);$$

$$\therefore MB = CN = \frac{1}{2}(AB - AC). \quad \text{Q.E.D.}$$

13. Given the base and vertical angle of a triangle, and (1) the difference, or (2) the sum of the other two sides, construct it.

14. In the triangle ABC , $\hat{BAC}$ is fixed in magnitude and position, $AB + AC$ is constant, prove that the locus of A' is a straight line perpendicular to AI .

15. If AI_2 meets the circumcircle at Q , and if QR is drawn perpendicular to AB , prove that AR , RQ are respectively equal to half the difference and the sum of AB , AC .

16. R is the mid-point of the minor arc BC of the circumcircle of the triangle ABC ; the circle on AR as diameter meets AB at S , prove that the tangent from S to the incircle equals $\frac{1}{2}BC$.

17. With the notation of Theorem 28, prove that the tangents from M to the incircle and excircle escribed to BC are equal.

THEOREM 29.

If O and I are the circumcentre and incentre of the triangle ABC , and if R , r are the circumradius and inradius, then $OI^2 = R^2 - 2R \cdot r$.

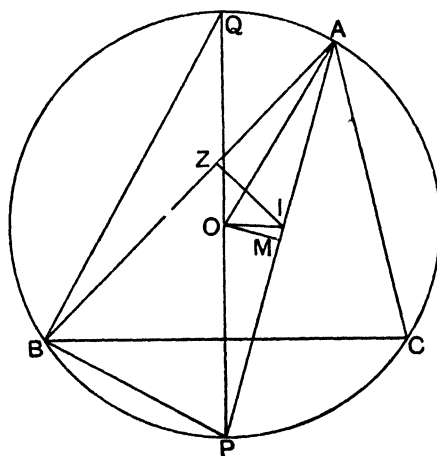


FIG. 32.

Produce AI to meet the circumcircle at P ; produce PO to meet the circumcircle at Q . Draw OM perpendicular to AP ; [it $\therefore$ bisects AP]. Draw IZ perpendicular to AB .

$$\begin{aligned}
 \text{Then} \quad R^2 - OI^2 &= OA^2 - OI^2 \\
 &= OM^2 + MA^2 - (OM^2 + MI^2) \\
 &= MA^2 - MI^2 \\
 &= (MA - MI)(MA + MI) \\
 &= AI \cdot IP, \text{ since } MA = PM, \\
 &= AI \cdot PB, \text{ since } PI = PB.
 \end{aligned}$$

Now the triangles AIZ , QPB are similar,
for $\hat{QBP} = 90^\circ = \hat{AZI}$ and $\hat{BQP} = \hat{ZAI}$ in the same segment;

$$\begin{aligned}
 \therefore \frac{AI}{IZ} &= \frac{PQ}{PB}; \\
 \therefore AI \cdot PB &= PQ \cdot IZ = 2R \cdot r; \\
 \therefore R^2 - OI^2 &= 2R \cdot r; \\
 \therefore OI^2 &= R^2 - 2R \cdot r.
 \end{aligned}$$

Q.E.D.

Corollary.

By the same method, it may be proved that

$$OI_1^2 = R^2 + 2R \cdot r_1.$$

18. Prove the Corollary of Theorem 29.

[You will prove that $OI_1^2 - OP^2 = I_1P \cdot I_1A = PB \cdot I_1A = 2R \cdot r_1$.]

19. A variable triangle circumscribes a fixed circle; if its circumradius is given, find the locus of its circumcentre.

20. Given BC , R , r , construct the triangle.

21. Prove that the ratio of the squares of the tangents from I_1 and I_2 to the circumcircle equals the ratio of the radii r_1 and r_2 .

22. The perpendicular at I to IO meets the incircle at K , K' ; prove that the perimeter of the triangle OKK' equals the diameter of the circumcircle.

23. A circle is drawn to touch OI at I and to touch the circumcircle; prove that it is equal to the incircle.

24. If the circumcircle meets an excircle at P , and if the tangents at P to the two circles are at right angles, prove that the radius of the excircle equals the diameter of the circumcircle.

25. Given the base and vertical angle of a triangle, prove that the incircle is largest when the triangle is isosceles.

A remarkable property of the nine-point circle is due to Feuerbach.

Feuerbach's Theorem. The nine-point circle of a triangle touches the incircle and each of the excircles.

The following sequence of riders indicates one method of proof. Another method will be found on page 149.

26. Let AI meet BC at U ; from U draw UK to touch the incircle at K . Let $A'K$ meet the nine-point circle at R . Draw CM perpendicular to AI and let it meet AI , AB at M , L ; with the usual notation,

(1) prove $A'M = \frac{1}{2}BL = \frac{1}{2}(c - b)$;

(2) prove $A'X = A'M$;

(3) prove $\hat{M}\hat{D}U = \hat{U}\hat{M}A'$ [$MDCA$ is cyclic];

(4) prove $A'U \cdot A'D = A'M^2 = A'X^2$;

(5) prove $K\hat{U}A' = K\hat{I}X = 2U\hat{A}D = A\hat{C}B \sim A\hat{B}C$;

(6) prove that $A'N$ is parallel to OA , and that the tangent to the nine-point circle at A' makes with BC an angle equal to

$$O\hat{A}H = A\hat{C}B - A\hat{B}C;$$

(7) prove that $KRUD$ is a cyclic quadrilateral, and that

$$A'K \cdot A'R = A'X^2;$$

(8) prove that R lies on the incircle;

(9) two circles cut at R , and a line through R meets the circles again at K, A' ; if the tangents at K and A' are parallel, prove that the circles touch at R ;

(10) prove that the nine-point circle touches the incircle. A similar method may be used to prove that it touches each excircle.

27. Find the position of a point X inside the triangle ABC such that the triangles ABX, BCX, CAX are equal in area.

28. Circles are described on AB, AC as diameters, and meet CF, BE at $P, P'; Q, Q'$; prove that $PQP'Q'$ are concyclic.

29. A is a fixed point on the circumference of a circle; PQ is a variable chord of constant length; prove that the locus of the orthocentre of the triangle APQ is a circle.

30. Prove that the line joining the feet of the perpendiculars from H to AI and I_2I_3 bisects BC .

31. M is any point on $A'O$, prove that the lines drawn through A, B, C parallel to MA', MB', MC' are concurrent.

32. Prove that if OA', OB', OC' are placed end to end on the circumference of the nine-point circle, they will just fill half of it.

33. Prove that $\frac{OB' \cdot OC'}{AB \cdot AC} + \frac{OC' \cdot OA'}{BC \cdot BA} + \frac{OA' \cdot OB'}{CA \cdot CB} = \frac{1}{4}$.

34. AI meets BC at P and the circumcircle at P' ; AI_2 meets BC at Q and the circumcircle at Q' ; prove that

$$AP \cdot AP' = QA \cdot AQ' = BA \cdot AC.$$

35. If $\hat{ACB} = 90^\circ$, prove $AZ \cdot ZB = \text{area of triangle } ABC$.

36. If P is the mid-point of HA , prove that $\hat{A'PD} = \hat{ACB} \sim \hat{ABC}$.

37. Prove that the sum of the squares on the six lines joining the four points I, I_1, I_2, I_3 equals four times the sum of the squares on their distances from the circumcentre.

38. Given I, I_1, I_2 , construct the triangle.

39. Given the base and vertical angle of a triangle, prove that the locus of the centroid of the triangle $I_1I_2I_3$ is a circle.

40. Given H, I, G and $\hat{ABC} - \hat{ACB}$, construct the triangle.

41. Given the base, the difference of the base angles, and the magnitude of the nine-point circle, construct the triangle.

42. A triangle of given shape has one vertex fixed and the opposite side passes through a fixed point, find the locus of the orthocentre.

43. Points P, Q, R are taken on BC, CA, AB so that

$$\frac{BP}{PC} = \frac{CQ}{QA} = \frac{AR}{RB} = \frac{m}{n},$$

prove that the area of the triangle $PQR = \frac{m^3 + n^3}{(m+n)^3} \Delta$, where $\Delta = \text{area of the triangle } ABC$.

44. Q and R are the points of trisection of the arcs $B'E$, $C'F$ of the nine-point circle, nearest to B' and C' ; prove that QR is the side of an equilateral triangle inscribed in the nine-point circle.

45. Squares $AEFB$, $ACGH$ are described externally to the triangle ABC , which is right-angled at C ; a point K is taken on AC (produced if necessary) so that $AK=BC$; prove that A is the centroid of the triangle HEK .

46. Points P , Q , R are taken on BC , CA , AB so that

$$\frac{BP}{PC} = \frac{2}{3}, \quad \frac{CQ}{QA} = \frac{3}{4}, \quad \frac{AR}{RB} = \frac{4}{5}.$$

Calculate the ratio of the areas of the triangles PQR , ABC .

47. A tangent LM , parallel to BC , is drawn to the circle escribed to BC ; if AB touches this excircle at Z_1 , and if LM meets AB , AC at P , Q , prove that $\frac{1}{PQ} = \frac{1}{BC} + \frac{1}{AZ_1}$.

48. The base of a triangle passes through a fixed point, and the other sides are bisected at right angles by two fixed lines OP , OQ ; prove that the locus of the vertex is a circle through O .

49. Construct a triangle ABC , given BC , D and the length of a side of the inscribed square which has one of its sides on BC .

50. Construct a triangle given the base and vertical angle and one median. [Two cases.]

51. $ABCD$ is a regular tetrahedron: AE is drawn perpendicular to the face BCD ; prove that $3AE^2 = 2AB^2$.

52. Four equilateral triangles are arranged so that their vertices coincide and their bases form a square. Prove that the opposite edges of the solid angle at the vertex are at right angles.

53. If $ABCD$ is a tetrahedron, prove that

$$\frac{\sin BDC}{\sin BAC} \cdot \frac{\sin DAC}{\sin DBC} \cdot \frac{\sin ABC}{\sin ADC} = 1.$$

54. If $ABCD$ is a tetrahedron, prove that the square of the area of each face is proportional to the product of the sines of the angles of that face and the sines of the angles at the opposite vertex.

55. The sides of the triangle ABC are 6, 6, 8; and a point D is taken such that $DA=DB=DC=9$; prove that the volume of the tetrahedron $ABCD$ is 48.

56. If a cube and an octahedron have a common circumscribing sphere, prove that their surfaces are in the same ratio as their volumes.

It is not proposed to include in this book any formal account of the modern geometry of the triangle which has been developed in great detail in recent years. The following sequence of riders (57-76), however, include a few of the more elementary theorems, and are so arranged that no difficulty will be found in working through them. Those which do not affect the logical connection of the set, regarded as a whole, are marked with an asterisk.

Definition.

If a straight line PQ cuts two fixed straight lines AB , AC at P , Q so that $\hat{APQ} = \hat{ACB}$, then PQ is called an **antiparallel** to BC w.r.t. AB , AC .

57.* With the notation of this definition, prove that $PQCB$ are concyclic.

58.* Prove that the sides of the pedal triangle are antiparallels to the corresponding sides of the original triangle ABC w.r.t. the remaining sides of ABC .

59. If two lines are antiparallel to BC w.r.t. AB , AC , prove that they are parallel to each other.

60. Prove that the antiparallels to BC w.r.t. AB , AC are parallel to the tangent at A to the circumcircle of ABC .

Definition. The line AK which bisects all the antiparallels to BC w.r.t. AB , AC is called a **symmedian line** of the triangle.

A triangle has therefore three symmedians, one through each vertex.

61. If the tangents at A , B , C to the circumcircle of the triangle ABC form the triangle $A_1B_1C_1$ where A_1 , B_1 , C_1 are respectively opposite to A , B , C ; prove that AA_1 , BB_1 , CC_1 are symmedians and that they are concurrent.

[Through A_1 draw an antiparallel QA_1R to BC to meet AB , AC at Q , R ; use Ex. 60 and prove $QA_1 = BA_1 = CA_1 = RA_1$.]

Definition. The point at which the symmedians intersect is called the **symmedian point** of the triangle and will be denoted by K .

62. Prove that $\hat{GAB} = \hat{KAC}$; $\hat{GBC} = \hat{KBA}$; $\hat{GCA} = \hat{KCB}$.

[Reflect the figure about AI so that AB and AC interchange their positions, and note that antiparallels to BC become lines parallel to BC so that AK becomes AG .]

63. P , P' are two points inside the triangle ABC such that $\hat{BAP} = \hat{CAP'}$; perpendiculars PN , $P'N'$; PM , $P'M'$ are drawn to AB , AC respectively. Prove that $PN \cdot P'N' = PM \cdot P'M'$.

[Use similar triangles PAN , $P'AM'$; $P'AN'$, PAM .]

64. Conversely if, with the same construction as in Ex. 63, $PN.P'N' = PM.P'M'$, prove that $\hat{BAP} = \hat{CAP'}$.

65.* Prove, with the same figure as Ex. 63, that MN is perpendicular to AP' .

Definition. AP, AP' are called **isogonal lines** w.r.t. AB, AC if $\hat{BAP} = \hat{CAP'}$; AP, AP' being between AB and AC .

66. If AP, AP' are isogonal lines w.r.t. AB, AC ; and if BP, BP' are isogonal lines w.r.t. BC, BA ; then CP, CP' are isogonal lines w.r.t. CA, CB . [Use Ex. 63, 64.]

Definition. Points P, P' , situated as in Ex. 66 are called **isogonal conjugates**.

67.* Prove that O and H are isogonal conjugates.

68.* If P lies on the circumcircle, prove that its isogonal conjugate is at infinity.

69.* Prove that the incentre and excentres are isogonal self-conjugates (*i.e.* are isogonal to themselves).

70. Prove that G and K are isogonal conjugates.

71. Prove that the perpendiculars from K to BC, CA, AB are respectively proportional to those sides. [Use Ex. 63, 70.]

72. Prove that the sum of the squares on the perpendiculars from a point P to the sides of a triangle is least when P is at K .

[If p, q, r are the lengths of the perpendiculars, use the identity $(p^2 + q^2 + r^2)(a^2 + b^2 + c^2) = (ap + bq + cr)^2 + (aq - bp)^2 + (br - cq)^2 + (cp - aq)^2$, and use Ex. 71.]

73. Prove that it is always possible to find one and only one point Ω such that $\hat{\Omega AB} = \hat{\Omega BC} = \hat{\Omega CA}$.

[Prove that Ω must lie on a circle touching AB at A and passing through C , and also on a circle touching AC at C and passing through B .]

In the same way there exists one and only one point Ω' such that $\hat{\Omega' AC} = \hat{\Omega' CB} = \hat{\Omega' BA}$.

Definition. The points Ω, Ω' obtained in Ex. 73 are called the **Brocard points** of the triangle ABC .

74.* Prove that Ω and Ω' are isogonal conjugates.

75.* If AM is a chord parallel to BC of the circle which touches AB at A and passes through C , prove that BM meets the circle at Ω .

76. Prove that $\hat{B\Omega C} = 180^\circ - \hat{ACB}$. What is the corresponding value for $B\Omega' C$?

CHAPTER V.

CONCURRENCY AND COLLINEARITY.

IN Coordinate Geometry it is of fundamental importance that account should be taken of the direction in which a line is measured. Thus lines measured along the x -axis, Ox , are taken as positive when measured towards the right hand from the point O , and are taken as negative when measured towards the left. This is the usual convention; but it would be equally correct to regard lengths measured to the left as positive and to the right as negative.

Many geometrical theorems can be enunciated in a more general form, if the same connection between sign and direction is observed.

If then A, B are any two points, we say that

$$AB = -BA$$

or

$$AB + BA = 0. \quad (1)$$

This is equivalent to stating that if a point travels from A to B and then from B to A , its final distance from its starting point is zero.

Definition.

Two segments AB, PQ of the same or parallel lines are said to have the same **sense** or opposite senses [or are sometimes called like or unlike] according as the directions $A \rightarrow B$ and $P \rightarrow Q$ are the same or opposite.

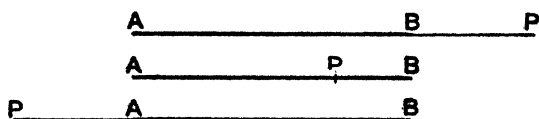


FIG. 33.

1. If ABP are three points in any order on a straight line, then $AB = AP + PB$. ✓

This amounts to stating that if a point travels from A to P and then from P to B , its final distance from A is equal to AB .

2. If $A, P_1, P_2, \dots, P_n, B$ are points in any order on a straight line, then

$$(1) AB = AP_1 + P_1P_2 + \dots + P_nB;$$

$$(2) AP_1 + P_1P_2 + \dots + P_nB + BA = 0.$$

3. If A, B, O are points in any order on a straight line, then

$$AB = OB - OA. \quad \checkmark$$

4. Draw any finite straight line AB and mark roughly on it a point P such that $\frac{AP}{PB}$ equals

$$(1) +1; (2) -\frac{1}{2}; (3) -2; (4) -1.$$

Part 4

5. Describe the changes in the value of the ratio $\frac{AP}{PB}$ as P moves on the unlimited line AB ; and prove that this ratio can never have the same value for two different finite positions of P .

It follows from Ex. 5 that if x is any real number (positive or negative), excluding $x = -1$, there is one and only one position of P on the unlimited line AB for which $\frac{AP}{PB} = x$. And as the distance of P from A or B increases, the ratio $\frac{AP}{PB}$ approaches the value -1 ; and can be made to differ from -1 by a quantity less than any assigned quantity, however small, by increasing sufficiently the distance of P from A .

If $x = -1$, there is no possible finite position for P . To preserve continuity, we shall therefore suppose that there is **one and only one** position of P on the line AB for which $\frac{AP}{PB} = -1$, and we shall call this point, the point at infinity on the line AB , and shall denote it by ∞ . It should be understood that such a point is purely ideal, in other words it is merely a convenient way of expressing in geometrical form an arithmetical idea. A more complete discussion will be found in the Second Part.

A number of relations exist which connect the lengths of various segments of a straight line. For example, if A, B, C, D are any four collinear points, then $AB \cdot CD + BC \cdot AD + CA \cdot BD = 0$. These relations are best proved by taking an origin O on the line and expressing the length of each segment in terms of the distances of its extremities from O .

Thus $AB = OB - OA$, etc.

6. Prove that if $ABCD$ are four collinear points in any order, then $AB \cdot CD + BC \cdot AD + CA \cdot BD = 0$.

✓ 7. C is the mid-point of AB ; P is any other point on the line AB ; prove that

$$(1) PC = \frac{1}{2}(PA + PB),$$

$$(2) AP \cdot BP + CB^2 = CP^2,$$

$$(3) AB^2 + BP^2 = 2AC^2 + 2CP^2, \quad \nearrow - AC^2.$$

8. AB is divided at C so that $AB \cdot BC = AC^2$; prove that

$$AB^2 - AC^2 = AB \cdot AC.$$

9. AB is bisected at C and produced to D ; prove that

$$AC \cdot AD = CB \cdot BD + 2BC^2.$$

Is the result still true if D lies between A and B ? yes

✓ 10. A, B, C, D are four collinear points; P, Q are the mid-points of AB, CD ; prove that $PQ = \frac{1}{2}(AD + BC)$.

11. P is any point on the line ABC ; prove that

$$PA^2 \cdot BC + PB^2 \cdot CA + PC^2 \cdot AB + BC \cdot CA \cdot AB = 0.$$

✓ 12. Prove the result of Ex. 11 when P is any point in the plane. [Draw PO perpendicular to ABC .]

✓ 13. If G is a point on the line $A_1A_2A_3A_4$ such that

$$GA_1 + GA_2 + GA_3 + GA_4 = 0,$$

and if P is any other point on the line, prove that

$$PA_1 + PA_2 + PA_3 + PA_4 = 4PG.$$

Generalise this theorem for n collinear points.

✓ 14. G is a point on the line $A_1A_2A_3A_4$ such that

$$GA_1 + GA_2 + GA_3 + GA_4 = 0,$$

if P is any other point on the line, prove that

$$PA_1^2 + PA_2^2 + PA_3^2 + PA_4^2 - (GA_1^2 + GA_2^2 + GA_3^2 + GA_4^2) = 4PG^2.$$

Generalise this theorem for n collinear points.

✓ 15. Prove the result of Ex. 14 when P is any point in the plane.

Theorem 31 was first discovered by Menelaus who died at the end of the first century A.D. It was, however, forgotten and then discovered afresh by Ceva (1650-1734) and by Carnot (1753-1823). The latter made it the basis of his researches on transversals. Ceva, to whom Theorem 30 is due, was an Italian by birth and an hydraulic engineer by profession. His treatise on geometry includes a number of statical applications. It is interesting to note that he was one of the earliest scientific writers on economics.

THEOREM 30. [CEVA'S THEOREM.]

Three concurrent straight lines are drawn through the vertices A , B , C of a triangle to meet the opposite sides at D , E , F respectively; then $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1$.

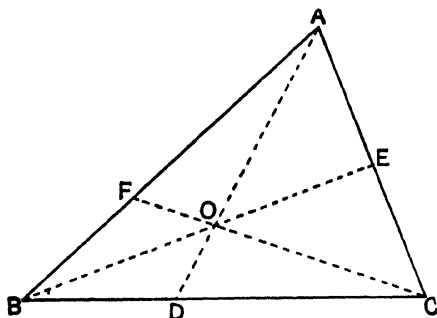


FIG. 34.

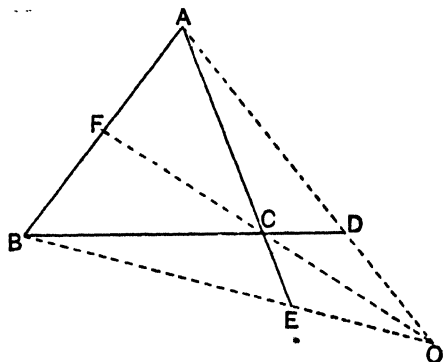


FIG. 35.

Let the three lines concur at O .

If O lies inside the triangle ABC (Fig. 34) each of the ratios $\frac{BD}{DC}$, $\frac{CE}{EA}$, $\frac{AF}{FB}$ is positive; and if O lies outside ABC (Fig. 35) one ratio is positive and the other two are negative;

$\therefore$ in either case the product of the three ratios is positive.

By Theorem 6, $\frac{BD}{DC} = \frac{\triangle BOA}{\triangle COA}$; $\frac{CE}{EA} = \frac{\triangle BOC}{\triangle BOA}$; $\frac{AF}{FB} = \frac{\triangle COA}{\triangle BOC}$;

$$\therefore \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{\triangle BOA}{\triangle COA} \cdot \frac{\triangle BOC}{\triangle BOA} \cdot \frac{\triangle COA}{\triangle BOC} = 1;$$

$$\therefore \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1.$$

Corollary.

If D , E , F are points on BC , CA , AB such that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1,$$

then AD , BE , CF are concurrent.

[This may be easily proved by a 'reductio ad absurdum' method.]

16. Prove the Corollary of Theorem 30.

17. If in Fig. 34, $BD = \frac{1}{2}DC$, $CE = \frac{1}{3}CA$, calculate

(1) $\frac{AF}{FB}$; (2) $\triangle AOB : \triangle BOC : \triangle COA$; (3) $\frac{AO}{AD}$.

18. If points D, E, F are taken on the sides BC, CA, AB of a triangle so that $2BD=DC$, $3CE=2EA$, $AF=3FB$, prove that AD, BE, CF are concurrent. If they meet at O , and if the area of the triangle ABC is 660 sq. cms., calculate the areas of the triangles $BOD, DOC, COE, EOA, AOF, FOB$.

19. In the triangle ABC , BC is produced to D so that $BC=CD$; CA is produced to E so that $CA=AE$; if AD meets BE at O , and if OC meets AB at F , calculate $\frac{BF}{FA}$.

20. Use Ceva's theorem to prove that

- (1) the medians of a triangle are concurrent,
- (2) the internal bisectors of the angles of a triangle are concurrent,
- (3) the altitudes of a triangle are concurrent.

[Note that $\frac{AF}{AE} = \frac{AC}{AB}$, etc., by similar triangles.]

21. If the incircle of a triangle ABC touches BC, CA, AB at D, E, F prove that AD, BE, CF are concurrent. State and prove a similar theorem for an escribed circle.

22. In Fig. 34, prove that (1) $\frac{OD}{AD} = \frac{\triangle BOC}{\triangle BAC}$

$$(2) \frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1.$$

23. In Fig. 34, prove that $\frac{AF}{FB} + \frac{AE}{EC} = \frac{AO}{OD}$.

24. Variable points F, E are taken on the sides AB, AC of the triangle ABC such that $\frac{AF}{FB} + \frac{AE}{EC}$ is constant; if CF, BE intersect at O , prove that the triangle BOC is of constant area, and find the locus of O .

25. In Fig. 34, prove that Ceva's result is equivalent to

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin CBE}{\sin EBA} \cdot \frac{\sin ACF}{\sin FCB} = 1.$$

26. If in Fig. 34 EF is parallel to BC , prove that $BD=DC$.

27. If in Fig. 34 the circle DEF cuts BC, CA, AB again at D', E', F' , prove that AD', BE', CF' are concurrent.

28. If in Fig. 34 lines are drawn through the mid-points of BC, CA, AB parallel to AD, BE, CF , prove that they are concurrent.

29. $ABCD$ is a quadrilateral; AB, CD meet at E ; AC, BD meet at F ; EF cuts AD at H and BC at G ; prove that

$$\frac{EB}{EA} : \frac{EC}{ED} = \frac{DH}{HA} : \frac{CG}{GB}.$$

30. If in Fig. 34 P, Q, R are the mid-points of EF, FD, DE , prove that AP, BQ, CR are concurrent. [Use Theorem 8.]

31. If in Fig. 34 P, Q, R are points on EF, FD, DE such that DP, EQ, FR are concurrent, prove that AP, BQ, CR are concurrent. [Use Theorem 8.]

32. A triangle ABC is inscribed in a triangle XYZ and is circumscribed about a triangle PQR ; the order of the letters round the triangles is $ARBPCQA$ and $XCYAZBX$. If AP, BQ, CR are concurrent, and if AX, BY, CZ are concurrent, prove that PX, QY, RZ are concurrent. [Use Theorem 8.]

33. The sides AB, CD of the quadrilateral $ABCD$ are parallel; CA, DB meet at E ; CB, AD meet at H ; CB, AD meet FEG , a parallel to AB , at G and F respectively; prove that AG, BF, EH are concurrent.

34. AP, BQ, CR are three concurrent lines meeting the opposite sides of the triangle ABC at P, Q, R ; F, G, H, A', B', C' are the mid-points of AP, BQ, CR, BC, CA, AB ; prove that $A'F, B'G, C'H$ are concurrent.

Definition.

Any line cutting a system of straight lines is called a **transversal** of those lines.

THEOREM 31. [MENELAUS' THEOREM.]

If a transversal meets the sides BC, CA, AB of a triangle at D, E, F respectively, then $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$.

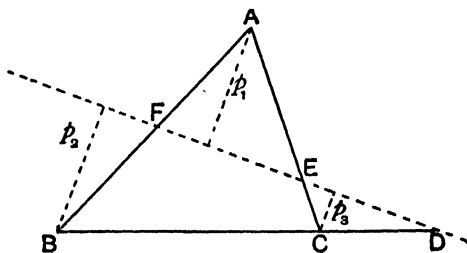


FIG. 36.

The transversal cuts either one side (Fig. 36) or three sides (Fig. 37) of the triangle externally; and therefore either one or three of the ratios $\frac{BD}{DC}, \frac{CE}{EA}, \frac{AF}{FB}$ are negative, and therefore their product is negative in each case.

From A, B, C drop perpendiculars to the transversal, and let p_1, p_2, p_3 be their lengths.

Then by similar triangles

in Fig. 36 $\frac{BD}{DC} = -\frac{p_2}{p_3}, \frac{CE}{EA} = \frac{p_3}{p_1}, \frac{AF}{FB} = \frac{p_1}{p_2}.$

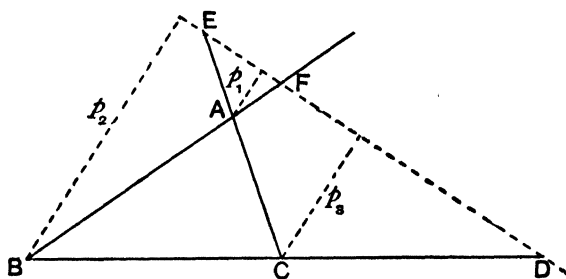


FIG. 37.

[In Fig. 37 each ratio is negative];

$$\therefore \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1.$$

Q.E.D..

Corollary.

If D, E, F are points on the sides BC, CA, AB of a triangle such that $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$, then D, E, F are collinear.

This may be easily proved by a 'reductio ad absurdum' method.

35. Prove the Corollary of Theorem 31.

36. If in Fig. 36 $BD=3CD$ and $CA=3CE$, calculate $\frac{BF}{FA}$.

37. B', C' are the mid-points of the sides AC, AB of the triangle ABC . Q is the mid-point of $B'C'$; BQ meets AC at R . Calculate $\frac{AR}{RC}$.

38. If in Fig. 36 BE, CF are the internal bisectors of the angles ABC, ACB , prove that AD is the external bisector of $\hat{BAC}$.

39. Prove that the points at which the external bisectors of the angles of a triangle meet the opposite sides are collinear.

40. H is a point inside the triangle ABC , prove that the external bisectors of the angles AHB, BHC, CHA meet AB, BC, CA respectively at three collinear points.

41. Points P, Q, R are taken on the sides BC, CA, AB of a triangle such that $\frac{BP}{PC} = \frac{CQ}{QA} = \frac{AR}{RB}$; QR and CB produced meet at D , prove that $BD : CD = CP^2 : PB^2$.

42. A circle centre A touches externally each of two given circles centres B, C at D, E ; DE meets BC at F . Another circle centre A' touches the same two circles at D', E' ; prove that $D'E'$ meets BC at F , and that the two original circles are homothetic w.r.t. F .

43. G is the centroid of the triangle ABC ; AG is produced to P so that $GP = AG$. Parallels through P to CA, AB, BC meet BC, CA, AB at L, M, N respectively; prove that L, M, N are collinear.

44. The tangents to the circumcircle of a triangle ABC at the vertices A, B, C meet the opposite sides at L, M, N . Prove that LMN are collinear.

[Show that $\frac{BL}{CL} = \frac{BA^2}{CA^2}$, etc.]

45. If in Fig. 34, EF meets BC at K , prove that $\frac{BD}{DC} = \frac{BK}{CK}$.

46. If in Fig. 35, EF meets BC at K , prove that $\frac{BD}{DC} = \frac{BK}{CK}$.

47. If in Fig. 34, EF meets AD at H and BC at K , prove that

$$\frac{EH}{HF} = \frac{EK}{FK}$$

48. If in Fig. 34, EF meets BC at L , FD meets CA at M , DE meets AB at N ; prove that L, M, N are collinear.

49. Four points a, b, c, d are taken on the sides AB, BC, CD, DA of a quadrilateral $ABCD$ such that $\frac{Aa}{aB} \cdot \frac{Bb}{bC} \cdot \frac{Cc}{cD} \cdot \frac{Dd}{dA} = 1$; prove that ab, cd intersect on AC .

50. If a straight line cuts the sides AB, BC, CD, DA of a quadrilateral at $PQRS$, prove that $\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = +1$.

Generalise this theorem for an n -sided polygon.

51. Points P, P', Q, Q' are taken on the edges AB, CD, AD, BC of a tetrahedron $ABCD$; if the lines PP' and QQ' intersect, prove that $\frac{BP}{PA} \cdot \frac{AQ}{QD} \cdot \frac{DP'}{P'C} \cdot \frac{CQ'}{Q'B} = 1$.

52. Given four lines of lengths a, b, c, d , construct the ratio $\frac{a}{b} \cdot \frac{c}{d}$.

53. The incircle of the triangle ABC touches BC, CA, AB at X, Y, Z ; YZ is produced to meet BC at K , prove that $\frac{BX}{XC} = \frac{BK}{CK}$.

54. A', B', C' are the mid-points of BC, CA, AB ; a transversal cuts $B'C', C'A', A'B'$ at K, L, M ; AK, BL, CM meet BC, CA, AB at K', L', M' ; prove that K', L', M' are collinear.

55. $ABCD$ is a quadrilateral; M, N, P are any points on BC, AD, AB ; PN meets BD at R ; RM meets DC at S ; PM meets AC at Q ; prove that N, Q, S are collinear.

56. A straight line meets the sides BC, CA, AB of a triangle at P, Q, R ; A', B', C' are the mid-points of these sides; AP, BQ, CR meet $B'C', C'A', A'B'$ at P', Q', R' ; prove that P', Q', R' are collinear.

57. A circle cuts each of the sides of a quadrilateral $ABCD$ at two internal points so that the order of the letters round the quadrilateral is $AA_1A_2BB_2B_1CC_1C_2DD_2D_1A$. If D_1C_1, A_1B_1, AC are concurrent, prove the concurrency of the following sets:

(1) D_1A_1, B_1C_1, DB ; (2) D_2C_2, A_2B_2, AC ; (3) D_2A_2, C_2B_2, DB .

58. Three planes AOB, BOC, COA intersect at a point O . Prove that the plane through the bisectors of the angles AOB, BOC cuts the plane COA in a line which bisects the angle COA externally.

59. If one of the transverse common tangents of two circles is perpendicular to one of the direct common tangents, prove that the eight points of contact lie on two straight lines.

60. [Desargue's Theorem on Perspective Triangles.]

$ABC, A'B'C'$ are two triangles such that AA', BB', CC' meet at O ; prove that, if $BC, B'C'$ meet at L ; $CA, C'A'$ at M ; $AB, A'B'$ at N , then LMN are collinear.

[Prove $\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = -1$ by considering the transversals $LB'C', MC'A', NA'B'$ applied to the triangles OBC, OCA, OAB .]

61. [Pascal's Theorem.]

$ABCDEF$ is any hexagon inscribed in a circle; AB, DE meet at L ; BC, EF at M ; CD, FA at N ; prove that LMN are collinear.

[Produce AB, CD, EF to form a triangle XYZ and prove that LMN is a transversal of XYZ by considering BC, DE, FA as transversals of XYZ .]

THEOREM 32.

P, Q, R are points on the sides BC, CA, AB of a triangle, such that the perpendiculars to the sides at these points are concurrent then $BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2$.

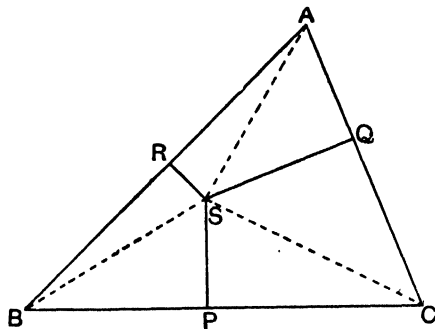


FIG. 38.

Let the perpendiculars at P, Q, R meet at S .

Join SA, SB, SC .

$$\begin{aligned} BP^2 - PC^2 &= (BP^2 + PS^2) - (PC^2 + PS^2) \\ &= BS^2 - SC^2. \end{aligned}$$

Similarly, $CQ^2 - QA^2 = CS^2 - SA^2$ and $AR^2 - RB^2 = AS^2 - SB^2$;

$\therefore$ by addition $BP^2 - PC^2 + CQ^2 - QA^2 + AR^2 - RB^2 = 0$;

$$\therefore BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2.$$

Q.E.D.

Corollary.

If P, Q, R are points on the sides BC, CA, AB of a triangle such that $BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2$, then the perpendiculars at P, Q, R to the sides are concurrent.

This may be easily proved by a 'reductio ad absurdum' method. It is the converse theorem (in the Corollary) which is the more useful in rider work.

62. Prove the Corollary of Theorem 32.

63. Is Theorem 32 still true if S lies outside the triangle ABC ?

64. Use Theorem 32 Corollary to show that the altitudes of a triangle are concurrent.

65. If I_1, I_2, I_3 are the centres of the circles escribed to BC, CA, AB , prove that the perpendiculars from I_1, I_2, I_3 to these sides are concurrent.

66. AD, BE, CF are the altitudes of the triangle ABC , prove that the perpendiculars from A, B, C to EF, FD, DE are concurrent.

67. L, M, N are the feet of the perpendiculars from a point K to the sides BC, CA, AB of a triangle, prove that the perpendiculars from A, B, C to MN, NL, LM are concurrent.

68. Each of three circles touches the other two, prove that the common tangents at the points of contact are concurrent.

69. If each of three circles cuts the other two, prove that the common chords of the circles taken in pairs are concurrent. [Use Theorem 32.]

70. P, Q, R are the feet of the perpendiculars from a point S to the sides BC, CA, AB of a triangle; A', B', C' are the mid-points of these sides, prove that $A'P \cdot BC + B'Q \cdot CA + C'R \cdot AB = 0$.

71. PQR and $P'Q'R'$ are two straight lines; PP', QQ', RR' ; QQ', RR' ; RR', PP' meet at C, A, B respectively, prove that PQ, QR, RP cut AB, BC, CA at three collinear points.

72. $ABCD$ is a parallelogram; AB, AD meet any line through C at H, K ; BK cuts DH at O ; BD cuts AO at P ; prove that $\frac{BP}{PD} = \frac{HC}{CK}$.

73. A', B', C' are the feet of the perpendiculars from the vertices A, B, C of a triangle to any straight line; prove that the perpendiculars from A', B', C' to BC, CA, AB are concurrent.

CHAPTER VI.

VECTOR GEOMETRY.

THE theory of vectors has been constructed entirely during the last century. It is usual to refer to Argand (1768-1825) as the first to attempt to represent graphically imaginary numbers. But this priority is questioned by some authorities. A great development was made by Hamilton (1805-1865) by his famous discovery of the properties of quaternions, published in 1852.

In the following pages, as an introduction to vector addition, the analogous properties which hold for the projections of a set of lines on an arbitrary line are first worked out. The advantage of this method lies in the fact that the law of addition, which is taken as a definition for vectors, is capable of simple proof [Theorem 33] in this special case, and its consequences are therefore more easily understood.

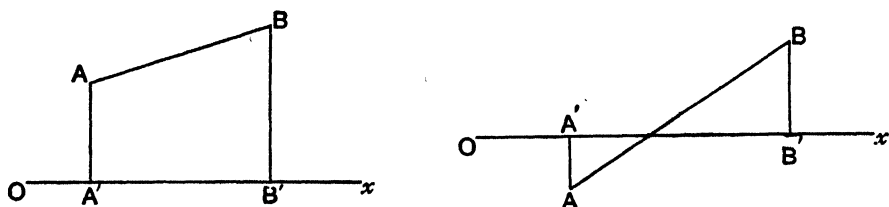


FIG. 39.

Definition.

If perpendiculars AA' , BB' are drawn from the extremities A , B of a given line AB to a line Ox ; $A'B'$ is called the **projection** of AB on Ox and will be denoted by $\overline{A'B'}$. The projection is said to be with reference to Ox . If no special line Ox is mentioned, it is implied that an arbitrary line may be selected for Ox .

THEOREM 33.

If A, B, C are any three points, then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

The proof is left to the reader.

.. 1. If $A_1A_2A_3 \dots A_n$ is any polygon, the sum of the projections of the sides taken in order on any straight line is zero.

$$[i.e. \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \dots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1} = \mathbf{0}.]$$

THEOREM 34.

If D is the mid-point of the base BC of a triangle ABC , then $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$.

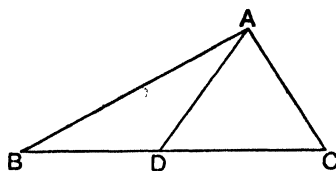


FIG. 40.

$$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} \quad [\text{Th. 33.}]$$

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC};$$

$$\therefore \overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{DC}.$$

But $\overrightarrow{DB} + \overrightarrow{DC} = \mathbf{0}$, for their senses are opposite and their lengths are equal.

$$\therefore \overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}.$$

Q.E.D.

THEOREM 35.

If D is a point on the base BC of a triangle ABC such that $\lambda \cdot BD = \mu \cdot DC$; then $\lambda \cdot \overrightarrow{AB} + \mu \cdot \overrightarrow{AC} = (\lambda + \mu) \overrightarrow{AD}$.

This may be proved in the same way as Theorem 34.

2. Prove Theorem 35.

3. If D is the mid-point of the base BC of a triangle ABC , prove that $\overrightarrow{AB}^2 + \overrightarrow{AC}^2 = 2\overrightarrow{AD}^2 + 2\overrightarrow{DB}^2$.

4. If D is a point on the base BC of a triangle ABC , such that $\lambda \cdot BD = \mu \cdot DC$, prove that

$$\lambda \cdot \overrightarrow{AB}^2 + \mu \cdot \overrightarrow{AC}^2 = (\lambda + \mu) \overrightarrow{AD}^2 + \lambda \cdot \overrightarrow{BD}^2 + \mu \cdot \overrightarrow{DC}^2.$$

5. If $ABCD$ is any quadrilateral, prove that

$$\overrightarrow{AB} \cdot \overrightarrow{CD} + \overrightarrow{BC} \cdot \overrightarrow{AD} + \overrightarrow{CA} \cdot \overrightarrow{BD} = \mathbf{0}.$$

6. If $ABCD$ is any quadrilateral, prove that

$$\overrightarrow{AD}^2 \cdot \overrightarrow{BC} + \overrightarrow{BD}^2 \cdot \overrightarrow{CA} + \overrightarrow{CD}^2 \cdot \overrightarrow{AB} + \overrightarrow{BC} \cdot \overrightarrow{CA} \cdot \overrightarrow{AB} = \mathbf{0}.$$

THEOREM 36.

If two lines AP , AQ are such that their projections on each of two lines at right angles are equal, prove that P coincides with Q .

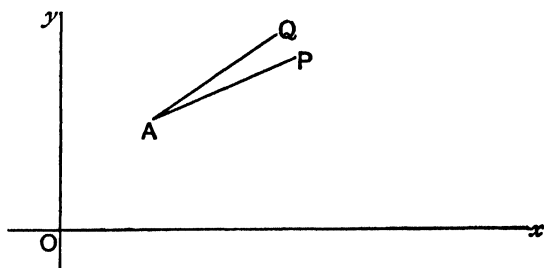


FIG. 41.

Let Ox , Oy be the two given perpendicular lines.

Now $\overline{AP} = \overline{AQ} + \overline{QP}$.

But $\overline{AP} = \overline{AQ}$ referred to Ox and Oy ;

$\therefore \overline{QP} = 0$ referred to Ox and Oy ;

$\therefore QP$ is perpendicular both to Ox and to Oy , or else Q coincides with P ;

$\therefore Q$ must coincide with P .

Q.E.D.

Hence it follows that if $\overline{AP} = \overline{AQ}$ referred to an arbitrary line, P must coincide with Q .

THEOREM 37.

If $\overline{AB} = \overline{CD}$ referred to each of two perpendicular lines, then AB is equal and parallel to CD .

[If possible draw AB' equal and parallel to CD and prove by Theorem 36 that B' coincides with B .]

Hence it follows that if $\overline{AB} = \overline{CD}$ referred to an arbitrary line, then AB and CD are equal and parallel.

7. Prove Theorem 37.

8. If AB is equal and parallel to CD , prove that $\overline{AB} = \overline{CD}$.

9. If AB is parallel to CD and if $\overline{AB} = \overline{CD}$ referred to a given line ox , not parallel to AB , prove that $AB = CD$.

THEOREM 38.

A' is the mid-point of the base BC of the triangle ABC ; G is a point on AA' such that $A'G = \frac{1}{2}GA$. O is a fixed point on an arbitrary line Ox ; then $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$.

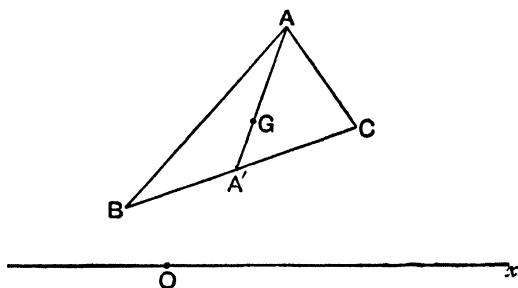


FIG. 424

Since $BA' = A'C$, $\overrightarrow{OB} + \overrightarrow{OC} = 2\overrightarrow{OA'}$. [Th. 34.]

Since $2A'G = GA$, $2\overrightarrow{OA'} + \overrightarrow{OA} = (2+1)\overrightarrow{OG}$; [Th. 35.]

$\therefore$ adding $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3\overrightarrow{OG}$;

$$\therefore \overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}).$$

Q.E.D.

If CA is bisected at B' and G_1 is taken on $B'B$ so that $B'G_1 = \frac{1}{2}G_1B$, in exactly the same way we may prove

$$\overrightarrow{OG_1} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC});$$

$\therefore \overrightarrow{OG} = \overrightarrow{OG_1}$ referred to an arbitrary line;

$\therefore G_1$ coincides with G . [Th. 36.]

And in the same way, if AB is bisected at C' and G_2 is taken on $C'C$ so that $C'G_2 = \frac{1}{2}G_2C$, it follows that G_2 coincides with G .

Hence this method has proved that the medians of a triangle are concurrent, and divide each other at a point of trisection.

Definition.

(1) The point G determined by the equation

$$\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$$

is called the mean centre of the three points A, B, C . It coincides with the centroid or centre of gravity of a triangular plate ABC .

(2) If $A_1A_2 \dots A_n$ are n points, and if G is a point determined by the equation $\overrightarrow{OG} = \frac{1}{n}(\overrightarrow{OA_1} + \overrightarrow{OA_2} + \dots + \overrightarrow{OA_n})$, then G is called the mean centre of the n points $A_1, A_2 \dots A_n$. It coincides with

D.G.

E

the centre of gravity of n equal particles placed at the n points. It is necessary to show that the position of G does not alter if the position of O is changed.

10. Prove that if G is defined by $\overrightarrow{OG} = \frac{1}{n}(\overrightarrow{OA_1} + \overrightarrow{OA_2} + \dots + \overrightarrow{OA_n})$, the position of G is independent of the position of O .

[If O' is any other point, and if $\overrightarrow{O'G'} = \frac{1}{n}(\overrightarrow{O'A_1} + \overrightarrow{O'A_2} + \dots + \overrightarrow{O'A_n})$, show that G' coincides with G .]

11. Prove that the centroid of the triangle ABC coincides with the centroid of the triangle formed by joining the mid-points $A'B'C'$ of its sides. [Express $\frac{1}{3}(\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'})$ in terms of $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$.]

12. Prove that the mean centre of the quadrilateral $ABCD$ is at the mid-point of the line joining the mid-points of AB and CD . Hence, prove that the lines joining the mid-points of opposite sides of a quadrilateral and joining the mid-points of the diagonals are concurrent and bisect each other.

13. If G is the mean centre of ABC , prove that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 0$.

14. If A_1, B_1, C_1, D_1 are the centroids of the triangles BCD, CDA, DAB, ABC , prove that AA_1, BB_1, CC_1, DD_1 are concurrent (at G say), and that $AG = 3GA_1$, etc.

15. $A_1A_2A_3A_4A_5A_6$ are the mid-points of consecutive sides of a hexagon, prove that the triangles $A_1A_3A_5, A_2A_4A_6$ have the same centroid.

16. Points P, Q, R are taken on the sides BC, CA, AB of a triangle so that $\frac{BP}{PC} = \frac{CQ}{QA} = \frac{AR}{RB}$; prove that the centroids of the triangles ABC, PQR are coincident.

17. A_1, B_1, C_1, D_1, E_1 are the mean centres of the quadrilaterals $BCDE, CDEA, DEAB, EABC, ABCD$; prove that the sides of the pentagon $A_1B_1C_1D_1E_1$ are parallel to and equal to one quarter of the corresponding sides of the pentagon $ABCDE$.

18. The sides AB, AC of a triangle are divided at P, Q so that $\frac{AP}{AB} = \frac{CQ}{CA}$; the parallelograms $APRC, ABSQ$ are drawn; prove that PS and QR are parallel to each other and to the median through A of the triangle ABC .

19. (1) O is the centre of a circle of unit radius; $A_1A_2A_3$ are any three points on its circumference; another circle of unit radius is described through A_2 and A_3 ; if a_1 is its centre, prove that

$$\overrightarrow{Oa_1} = \overrightarrow{OA_2} + \overrightarrow{OA_3}.$$

If, further, two other circles of unit radius are described through A_3, A_1 ; A_1, A_2 , and if their centres are a_2 and a_3 , prove that the points $a_1 a_2 a_3$ lie on another circle of unit radius (centre β_4 say).

[β_4 is given by $\overrightarrow{O\beta_4} = \overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3}$.]

(2) If four points $A_1 A_2 A_3 A_4$ lie on a circle of unit radius, and if from every set of three points a circle is constructed as in (1), then the centres $\beta_1 \beta_2 \beta_3 \beta_4$ of these four circles lie on a circle of unit radius, centre γ_5 . [γ_5 is given by $\overrightarrow{O\gamma_5} = \overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \overrightarrow{OA_4}$.]

(3) If five points $A_1 A_2 A_3 A_4 A_5$ lie on a circle of unit radius, and if from every four points a circle of unit radius is constructed as in (2), then the centres $\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5$ of these circles lie on a circle of unit radius, centre δ_6 , and so on indefinitely.

SOLID GEOMETRY.

This method of projection may be readily extended to three-dimensional geometry. Instead of taking two perpendicular lines for reference, it is now necessary to take three mutually perpendicular lines as axes. If AB is a line in space, $\overline{AB}$ will be taken to denote the projection of AB on a specified axis. Where no axis is mentioned, it is implied that an arbitrary line can be selected as the axis of reference.

20. If A, B, C, D are any four points in space, prove that

$$\overline{AB} + \overline{BC} + \overline{CD} = \overline{AD}.$$

21. Prove that $\overline{AB} + \overline{AC} = 2\overline{AD}$ where D is the mid-point of BC .

THEOREM 39.

(1) If $\overline{AB} = \overline{AC}$ referred to an arbitrary line, then B coincides with C .

(2) If $\overline{AB} = \overline{CD}$ referred to an arbitrary line, then AB is equal and parallel to CD .

[The methods of proof employed in Theorems 36, 37 apply here.]

22. Prove Theorem 39.

Definition.

If $A_1 A_2 \dots A_n$ are n points in space, and if G is a point determined by $\overrightarrow{OG} = \frac{1}{n}(\overrightarrow{OA_1} + \overrightarrow{OA_2} + \dots + \overrightarrow{OA_n})$ where O is any fixed point, then G is called the **mean centre** or the **centroid** of the system of points $A_1 A_2 \dots A_n$.

23. Prove that the position of G , as just defined, is independent of the position of O .

24. Prove that the lines joining the mid-points of opposite edges of a tetrahedron are concurrent and bisect each other.

[Prove that the mid-point of each line is at the mean centre of the tetrahedron.]

25. G_1 is the centroid of the base BCD of the tetrahedron $ABCD$; G is a point on AG_1 such that $G_1G = \frac{1}{3}GA$; prove that G is the mean centre of the tetrahedron.

26. G_1, G_2, G_3, G_4 are the centroids of the faces of the tetrahedron $ABCD$ opposite to A, B, C, D , respectively; prove that the lines AG_1, BG_2, CG_3, DG_4 are concurrent.

27. With the notation of Ex. 26, prove that the tetrahedra $G_1G_2G_3G_4, ABCD$ have the same mean centre.

28. With the notation of Ex. 26, prove that G_1G_2 is parallel to and equal to one-third of AB .

29. P, Q, R, S are the mid-points of the edges AB, BC, CD, DA of a tetrahedron $ABCD$, prove that the tetrahedra $PQRS, ABCD$ have the same mean centre.

30. With the notation of Ex. 26, find the ratio in which G_1G_2 divides the line joining the mid-points of AB and CD .

VECTORS.

We have seen that it is possible to represent positive and negative numbers by points on the x -axis referred to a definite point O as origin or zero-point. It is desirable to extend this method of representation.

Certain quantities, such as forces or velocities, etc., are not fully defined by their numerical magnitude; it is necessary in such cases to take account of their directions. These quantities can be represented by a straight line drawn in a definite direction, and of a definite length.

Definition.

Any quantity which can be completely represented by a straight line of appropriate length, drawn in an appropriate direction, is called a **vector**.

It follows from this definition that equal and parallel lines represent the same vector.

Thus a velocity of 20 feet per second in a direction N. 30° E. is a vector, and may be represented by any one of a series of equal and parallel straight lines.

In elementary arithmetic we are concerned with the addition of numbers in which no question of difference of direction—except that of sense in a straight line—occurs. In the case of vectors, however, the idea of addition involves an entirely new conception. For it is necessary to enquire what is meant by *adding* together two quantities which involve, not merely magnitude, but also direction. For example, what is meant by *adding* a velocity of 10 feet per second due North to a velocity of 8 feet per second due East?

The same kind of problem arose when the reader passed from the conception of positive integral indices to fractional or negative indices. The Law of Multiplication, $x^m \times x^n = x^{m+n}$, where m, n are positive integers, no longer admits of proof, just because the symbol x^m has no longer any meaning attaching to it. It is therefore necessary to enlarge the original definition by stating that $x^m \times x^n$ is to be taken as x^{m+n} for all real values of m and n , and using this definition to interpret the meaning of x^m when m is not a positive integer. The present problem must be approached in just the same way.

If A, B, C are any three points on the x -axis, so that OA, OB, OC represent any three real numbers (positive or negative), we know that $AB + BC = AC$. If, however, A, B, C are not collinear so that AB and BC represent two quantities, which involve different directions, we are compelled to enlarge our definition of what is meant by addition under these new circumstances. The original definition is therefore extended as follows:

Definition.

The result of *adding* two vectors AB and BC together is defined to be the vector AC .

This is a definition of what is meant by addition. Notice that it does not violate the theorem that the sum of the lengths of two sides of a triangle is greater than the length of the third; for it does not state that the length of AB + the length of BC = the length of AC ; but that the result of adding (combining) two quantities, whose lengths and directions are represented by AB and BC , is taken to be a quantity whose length and direction is represented by AC .

To avoid confusion between a vector AB and the length AB ,

some difference of notation is desirable. The vector AB will therefore be denoted by $\overrightarrow{AB}$.

The Law of Addition of Vectors is therefore,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

It is now necessary to show that vector quantities obey the fundamental Laws of Algebra, as regards addition.

The meaning of multiplication has yet to be considered.

THEOREM 40.

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{AB}.$$

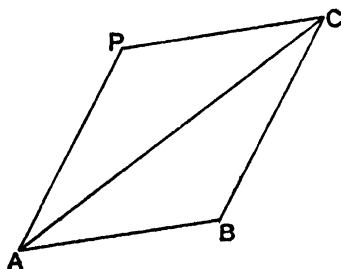


FIG. 43.

Complete the parallelogram $ABCP$.

By the law of addition $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$,

and

$$\overrightarrow{AP} + \overrightarrow{PC} = \overrightarrow{AC};$$

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AP} + \overrightarrow{PC}.$$

But $\overrightarrow{AB} = \overrightarrow{PC}$ and $\overrightarrow{AP} = \overrightarrow{BC}$, since equal and parallel lines represent the same vector;

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{AB}.$$

Q.E.D.

Corollary.

If $\overrightarrow{AB}$, $\overrightarrow{AP}$ are two vectors, their sum is the vector represented by the diagonal through A of the completed parallelogram $PABC$.

THEOREM 41.

$$(\overrightarrow{AB} + \overrightarrow{AC}) + \overrightarrow{AD} = \overrightarrow{AB} + (\overrightarrow{AC} + \overrightarrow{AD}).$$

The proof is left to the reader.

[Complete parallelograms $BACP$, $DAPQ$ so that $\overrightarrow{AQ}$ represents the left hand side; then complete parallelogram $DACR$ and show that $\overrightarrow{AQ} = \overrightarrow{AB} + \overrightarrow{AR}$].

31. Prove Theorem 41.

The process of subtraction can be deduced at once from that of addition.

$$\begin{aligned}\text{For } \overrightarrow{AB} + \overrightarrow{BA} &= \overrightarrow{AA} = \mathbf{0}; \quad \therefore \overrightarrow{AB} = -\overrightarrow{BA}; \\ \therefore \overrightarrow{AB} - \overrightarrow{CB} &= \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.\end{aligned}$$

Starting then with this vector law of addition, it will be found instructive for the reader to work out the theorems and examples (omitting Ex. 2, 6) which have been given under the form of projections of segments of straight lines on an arbitrary line. These can be regarded afresh as vector theorems, for they depended simply on the initial fact $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Thus for example, Theorem 34 may be rewritten as follows: If D is the mid-point of the line joining two points B, C , and if O is the origin, the vector represented by OD equals half the sum of the vectors represented by OB, OC .

The fundamental identity of these two methods may be seen by noting that if two lines have equal projections referred to an arbitrary line, then they represent the same vector quantity for they are equal and parallel. [Th. 37.]

Theorem 35 may be restated as follows:

If D is a point on BC such that $\lambda \cdot BD = \mu \cdot DC$, then

$$\lambda \cdot \overrightarrow{OB} + \mu \cdot \overrightarrow{OC} = (\lambda + \mu) \overrightarrow{OD}.$$

Now, from elementary statics, if $\lambda \cdot BD = \mu \cdot DC$ it follows that D is the centre of gravity of masses λ at B, μ at C . Hence we have the theorem: $m_1 \cdot \overrightarrow{OA_1} + m_2 \cdot \overrightarrow{OA_2} = (m_1 + m_2) \overrightarrow{OG}$, where G is the centre of gravity of masses m_1, m_2 at A_1, A_2 .

This result can be readily extended to give the generalised vector theorem:

$$m_1 \cdot \overrightarrow{OA_1} + m_2 \cdot \overrightarrow{OA_2} + \dots + m_n \cdot \overrightarrow{OA_n} = (m_1 + m_2 + \dots + m_n) \cdot \overrightarrow{OG},$$

where G is the centre of gravity of masses $m_1, m_2, \dots, m_n$ at $A_1, A_2, \dots, A_n$.

In the particular case when $m_1 = m_2 = \dots = m_n$ we have

$$\therefore \overrightarrow{OG} = \frac{1}{n} (\overrightarrow{OA_1} + \overrightarrow{OA_2} + \dots + \overrightarrow{OA_n}),$$

where G is the centre of gravity of n equal masses at $A_1, A_2, \dots, A_n$; this point has been called the mean centre of the system of points $A_1, A_2, \dots, A_n$.

The reader should now work through exercises 10-19.

The extension of vector addition to three dimensional geometry presents no new difficulty, and therefore Exercises 20-30 may be also taken.

VECTOR MULTIPLICATION.

When it is said that a point P on the axis Ox represents a positive or negative number by reference to its distance from O , use is being made of the following idea.

If OA represents a unit of length on Ox , and if OP represents k units of length [k being positive or negative], the ratio $\frac{OP}{OA}$ is by definition equal to k . In speaking of a *length* it is necessary to know the unit (inches, cms., etc.) in terms of which the length is given. But in the case of a *ratio* $\frac{OP}{OA}$, the value does not depend at all on the particular unit chosen, and is in fact a pure number. When considering the meaning of multiplication, it is necessary to work with pure numbers. It is as impossible to multiply 1 inch by 1 inch as it is to multiply 1 chair by 1 chair (unless some new definition is made as to the meaning of multiplication under such circumstances).

For simplicity, however, it is usual to suppose that the product $OP \cdot OQ$ is an abbreviation of $\frac{OP}{OA} \cdot \frac{OQ}{OA}$, where OA represents the unit of length.

Given a unit of length OA on the x -axis, and given any real number k , it is possible to find a point P on Ox such that $\frac{\overline{OP}}{\overline{OA}} = k$; and for any point P on OA , $\frac{\overline{OP}}{\overline{OA}}$ is a real number.

Definition.

If P is a point outside Ox , and if OA is a unit of length on the x -axis, the ratio $\frac{\overline{OP}}{\overline{OA}}$ is called a **complex number**.

If $OP = r$ and $\hat{AOP} = \theta$ (radians), $\frac{\overline{OP}}{\overline{OA}}$ will also be denoted by (r, θ) .

It follows that every positive real number can be denoted by $(r, 0)$ and every negative real number by (r, π) , where r represents any real positive number.

We have now to consider what is meant by multiplying together two complex numbers (r_1, θ_1) , (r_2, θ_2) .

The fundamental definition of multiplication cannot apply here, for as yet (r_1, θ_1) has received no interpretation.

Now in the case of *real* numbers the law of multiplication may be stated in the form

$$(r_1, \theta_1) \times (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2).$$

This may be easily verified.

$$\begin{aligned} \text{E.g.,} \quad -a \times +b &= (a, \pi) \times (b, 0) \\ &= (ab, \pi + 0) = (ab, \pi) \\ &= -ab. \end{aligned}$$

32. Prove that $-a \times -b = +ab$ and $+a \times -b = -ab$ are particular cases of the law enunciated above.

In the case of complex numbers we shall therefore **define multiplication** as the operation of this Law, viz.,

$$(r_1, \theta_1) \times (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2),$$

and then use this Law to discover what a complex number really is.

It is necessary first of all to prove that complex numbers, as now defined, obey the fundamental Laws of Algebra.

It has been already shown that they obey the Laws of Addition, viz.,

$$\begin{aligned} (1) \quad a_1 + a_2 &= a_2 + a_1; \\ (2) \quad (a_1 + a_2) + a_3 &= a_1 + (a_2 + a_3); \end{aligned}$$

where a_1 is written for (r_1, θ_1) , etc.

It remains to prove the following:

$$\begin{aligned} (3) \quad a_1 \times a_2 &= a_2 \times a_1; \\ (4) \quad (a_1 \times a_2) a_3 &= a_1 (a_2 \times a_3); \\ (5) \quad a_1 (a_2 + a_3) &= a_1 a_2 + a_1 a_3. \end{aligned}$$

These may be proved very easily by analysis.

For example, to prove (3);

Let $a_1 = (r_1, \theta_1)$ and $a_2 = (r_2, \theta_2)$.

$$\begin{aligned} \text{Then} \quad a_1 \times a_2 &= (r_1, \theta_1) \times (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2) \\ &= (r_2 r_1, \theta_2 + \theta_1) = (r_2, \theta_2) \times (r_1, \theta_1) = a_2 \times a_1; \end{aligned}$$

for r_1, r_2 and θ_1, θ_2 are real numbers, and therefore satisfy the laws of algebra, so that $r_1 r_2 = r_2 r_1$ and $\theta_1 + \theta_2 = \theta_2 + \theta_1$.

In a similar manner, (4) and (5) may be established.

It is interesting however to regard them from a geometrical standpoint. For this purpose, it is necessary, first of all, to give a geometrical construction for $a_1 \times a_2$.

Let $a_1 = (r_1, \theta_1) = \frac{\overline{OP_1}}{\overline{OA}}$, $a_2 = (r_2, \theta_2) = \frac{\overline{OP_2}}{\overline{OA}}$, where OA represents a unit of length of the x -axis.

On OP_1 construct a triangle OP_1R directly similar to OAP_2 .

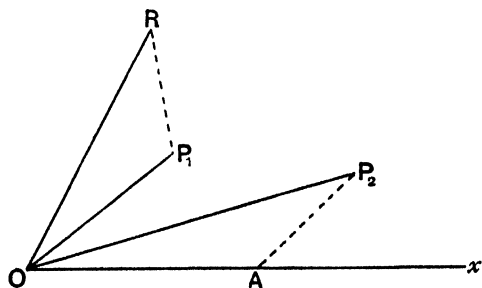


FIG. 44.

Then $\frac{OR}{OP_2} = \frac{OP_1}{OA}$ by similar triangles;

$$\therefore \frac{OR}{OA} = \frac{OR}{OP_2} \cdot \frac{OP_2}{OA} = \frac{OP_1}{OA} \cdot \frac{OP_2}{OA} = r_1 r_2;$$

and $\hat{AOR} = \hat{AOP_1} + \hat{P_1OR} = \hat{AOP_1} + \hat{AOP_2} = \theta_1 + \theta_2;$

$$\therefore \frac{\overline{OR}}{\overline{OA}} = \left(\frac{OR}{OA}, \hat{AOR} \right) = (r_1 r_2, \theta_1 + \theta_2) = (r_1, \theta_1) \times (r_2, \theta_2) = a_1 \times a_2;$$

$\therefore$ this is the required construction.

Q.E.F.

If in Fig. 44 a triangle OP_2R' is drawn directly similar to OAP_1 , then the same reasoning will show that $\frac{\overline{OR'}}{\overline{OA}} = a_2 \times a_1$.

In order therefore to prove that $a_1 \times a_2 = a_2 \times a_1$ it is necessary to show that OR, OR' coincide; this is a simple geometrical rider.

33. If in Fig. 44 the triangles AOP_2, P_1OR are directly similar, prove that the triangles AOP_1, P_2OR are directly similar.

34. By making use of the same construction and method, prove geometrically that $(a_1 a_2) a_3 = a_1 (a_2 a_3)$.

It may be pointed out that, speaking geometrically, to multiply $\frac{\overline{OP_1}}{\overline{OA}}$ by $\frac{\overline{OP_2}}{\overline{OA}}$, the line OP_1 is revolved about O through an angle $\hat{AOP_2}$ and is then stretched in the ratio $\frac{OP_2}{OA}$.

35. Make use of this idea to prove the remaining Law of Algebra, viz., $a_1(a_2 + a_3) = a_1a_2 + a_1a_3$.

36. Draw roughly figures to represent

$$(1) (1, 0) \times \left(2, \frac{\pi}{4}\right);$$

$$(2) (1, \pi) \times \left(2, \frac{\pi}{4}\right);$$

$$(3) \left(1, \frac{\pi}{2}\right) \times \left(1, \frac{\pi}{2}\right);$$

$$(4) (1, 0) \times \left(1, \frac{\pi}{3}\right) \times \left(1, \frac{2\pi}{3}\right);$$

$$(5) \left[\left(1, \frac{\pi}{3}\right)\right]^3 \text{ and } \left[\left(1, \frac{2\pi}{3}\right)\right]^3.$$

INTERPRETATION OF $\frac{\overline{OP}}{\overline{OA}} = (r, \theta)$.

By the law of multiplication $\left(1, \frac{\pi}{2}\right) \times \left(1, \frac{\pi}{2}\right) = (1, \pi)$;

$\therefore \left(1, \frac{\pi}{2}\right)$ is a quantity such that it, when multiplied by itself, becomes -1 , and may therefore be called the square root of -1 , and may be denoted by $\sqrt{-1}$ or i .

37. Prove that $\left(y, \frac{\pi}{2}\right)$ is equal to $y \cdot i$.

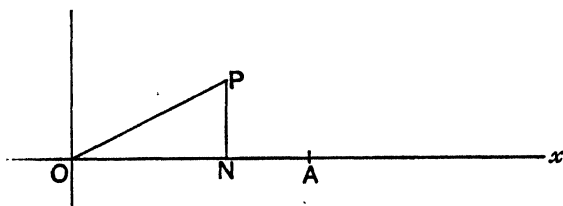


FIG. 45.

From P draw PN perpendicular to Ox .

$$\text{Then } \frac{\overline{OP}}{\overline{OA}} = \frac{\overline{ON} + \overline{NP}}{\overline{OA}} = \frac{\overline{ON}}{\overline{OA}} + \frac{\overline{NP}}{\overline{OA}};$$

$\therefore$ if the lengths of ON , NP are x units and y units,

$$\frac{\overline{ON}}{\overline{OA}} = (x, 0) = x.$$

$$\frac{\overline{NP}}{\overline{OA}} = \left(y, \frac{\pi}{2}\right) = yi;$$

$$\therefore \frac{\overline{OP}}{\overline{OA}} = x + yi.$$

Again, if $\frac{\overline{OP}}{\overline{OA}} = (r, \theta)$, OP contains r units of length ;
 $\therefore ON = r \cos \theta$, $NP = r \sin \theta$;
 $\therefore (r, \theta) = \frac{\overline{OP}}{\overline{OA}} = (r \cos \theta, 0) + \left(r \sin \theta, \frac{\pi}{2} \right)$
 $= r \cos \theta + r \sin \theta \cdot i$
 $= r(\cos \theta + i \sin \theta).$

This then is the important interpretation of the new kind of number (defined above as complex) which is rendered necessary by the Law of Multiplication for vector quantities.

Certain purely geometrical theorems can be established by vector methods. We will prove the theorem of Pythagoras as an example.

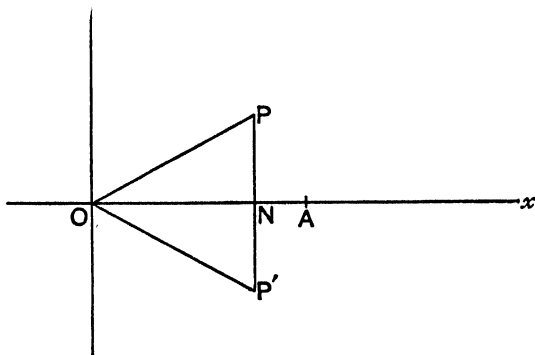


FIG. 46.

Let ONP be a right angled triangle, to prove that
 $ON^2 + NP^2 = OP^2$.

Produce PN to P' so that $PN = NP'$; let $\frac{ON}{OA} = x$, $\frac{NP}{OA} = y$.

Then the triangles ONP , ONP' are congruent.

Let $\frac{\overline{OP}}{\overline{OA}} = (r, \theta)$, $\therefore \frac{\overline{OP'}}{\overline{OA}} = (r, -\theta)$;

$$\therefore \frac{\overline{OP}}{\overline{OA}} \cdot \frac{\overline{OP'}}{\overline{OA}} = (r^2, 0) = \frac{OP^2}{OA^2}.$$

But $\frac{\overline{OP}}{\overline{OA}} = x + iy$, $\frac{\overline{OP'}}{\overline{OA}} = x - iy$;

$$\therefore \frac{\overline{OP}}{\overline{OA}} \cdot \frac{\overline{OP'}}{\overline{OA}} = (x + iy)(x - iy) = x^2 + y^2, \text{ since } i^2 = -1 ;$$

$$\therefore \frac{OP^2}{OA^2} = \frac{ON^2}{OA^2} + \frac{NP^2}{OA^2} ;$$

$$\therefore OP^2 = ON^2 + NP^2.$$

Q.E.D.

DEMOIVRE'S THEOREM.

If n is any positive integer, then

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

$$\begin{aligned} (1, \theta)^n &= (1, \theta) \times (1, \theta) \times \dots \text{ to } n \text{ factors} \\ &= (1, n\theta) \text{ by the Law of multiplication,} \end{aligned}$$

$$\therefore (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

Q.E.D.

This theorem may be extended to the case in which n is any real number. But that would be beyond the scope of this volume.

38. From the identity $(1, \theta) \times (1, \phi) = (1, \theta + \phi)$ deduce the expansions of $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$.

39. From the identity $(1, \theta)^3 = (1, 3\theta)$ deduce the expansions of $\sin 3\theta$ and $\cos 3\theta$.

40. Expand $\sin na$ in terms of $\sin a$, $\cos a$, where n is a positive integer.

41. Verify geometrically that $(1, \theta) + (1, -\theta) = 2 \cos \theta$.

42. Verify geometrically that $(1, 0) + \left(1, \frac{2\pi}{3}\right) + \left(1, \frac{4\pi}{3}\right) = 0$, and that in general $(1, 0) + \left(1, \frac{2\pi}{n}\right) + \left(1, \frac{4\pi}{n}\right) + \dots + \left(1, \frac{2n-2\pi}{n}\right) = 0$, where n is a positive integer.

43. Verify geometrically that $(1, 0) + (1, \theta) = \left(2 \cos \frac{\theta}{2}, \frac{\theta}{2}\right)$.

44. Verify geometrically that

- (1) $i(1+i) = -1+i$;
- (2) $(1+i)(1-i) = 2$;
- (3) $(1-i)^2 = -2i$.

45. Prove by the multiplication Law that $\left(1, \frac{\theta}{3}\right)$ is one of the cube roots of $(1, \theta)$; what are the other two cube roots?

46. Find the three cube roots of $(1, \pi)$; represent them graphically; and compare them with the roots of the equation

$$x^3 + 1 \equiv (x+1)(x^2 - x + 1) = 0.$$

47. Find the three cube roots of i and of $\frac{1-i}{\sqrt{2}}$.

48. Write $\left(4, \frac{\pi}{3}\right)$ in the form $x+iy$; find its square roots by algebraic methods, and compare them with $\left(\sqrt{4}, \frac{\pi}{6}\right)$; $\left(\sqrt{4}, \frac{7\pi}{6}\right)$.

49. AD is a median of the triangle ABC ; interpret trigonometrically the relation $\overline{AD} = \frac{1}{2}(\overline{AB} + \overline{AC})$.

50. Prove that the vector equation $\overline{BC} = \overline{BA} + \overline{AC}$ is equivalent to the two trigonometrical formulae

$$a = b \cos C + c \cos B; \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

51. If $ABCD$ is a parallelogram, prove that $\overline{CA} \cdot \overline{BD} = \overline{AB}^2 - \overline{BC}^2$. Express this result trigonometrically, taking the x -axis along AB .

52. If D is the mid-point of the base BC of the triangle ABC , prove that $\overline{AB}^2 + \overline{AC}^2 = 2\overline{AD}^2 + 2\overline{DB}^2$.

Express this result trigonometrically, taking the x -axis along BC .

53. **Ptolemy's Theorem.** If $ABCD$ is a plane quadrilateral, prove that $\overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD}$. By expressing this result trigonometrically, prove that, if A, B, C, D are concyclic, then

$$\overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD};$$

but that otherwise $\overline{AC} \cdot \overline{BD} < \overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD}$.

54. λ, μ, ν are given real numbers such that $\lambda + \mu + \nu = 0$; if A, B, C are points such that $\lambda \cdot \overline{OA} + \mu \cdot \overline{OB} + \nu \cdot \overline{OC} = 0$, prove that A, B, C are collinear.

55. Use Ex. 54 to establish Menelaus' Theorem, viz. if D, E, F are points on the sides BC, CA, AB of a triangle such that

$$\frac{BD}{DC} = l, \quad \frac{CE}{EA} = m, \quad \frac{AF}{FB} = n,$$

and if $lmn = -1$, then D, E, F are collinear.

CHAPTER VII.

THE MEAN CENTRE.

THIS chapter is concerned with properties which lie on the borderline separating Geometry and Statics. It frequently happens that a statical method affords a simple solution of a geometrical theorem (*e.g.* see Ex. 25); while on the other hand certain types of statical questions are best treated geometrically (*e.g.* Ex. 34). An excellent account of the value of statical methods in geometrical problems is given by Mr. R. F. Davis in Milne's *Companion to Weekly Problem Papers*.

THEOREM 42.

C is a point on the straight line AB such that $\lambda \cdot AC = \mu \cdot CB$; if D, E, F are the feet of the perpendiculars from A, B, C to any straight line Ox , then $(\lambda + \mu)CF = \lambda \cdot AD + \mu \cdot BE$.

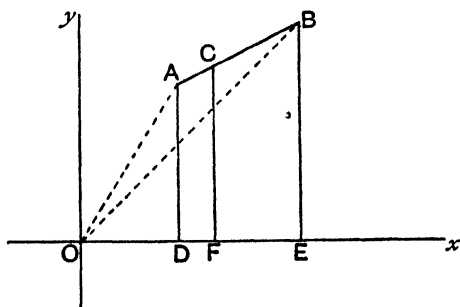


FIG. 47.

Draw Oy perpendicular to Ox .

Apply Theorem 35 to the triangle OAB .

$$\therefore \lambda \cdot \overline{OA} + \mu \cdot \overline{OB} = (\lambda + \mu) \overline{OC},$$

where $\overline{OA}, \overline{OB}, \overline{OC}$ denote the projections of OA, OB, OC on Oy .

$$\begin{aligned}\text{But } \overline{OA} &= DA, \quad \overline{OC} = FC, \quad \overline{OB} = EB; \\ \therefore \lambda \cdot DA + \mu \cdot EB &= (\lambda + \mu)FC, \\ \text{or } (\lambda + \mu)CF &= \lambda \cdot AD + \mu \cdot BE.\end{aligned}$$

Q.E.D.

It should be noticed that C is the centre of two parallel forces λ, μ at A, B respectively: this theorem is therefore equivalent to the statical theorem that the sum of the moments of two parallel forces λ, μ about an axis is equal to the moment of their resultant $\lambda + \mu$ about that axis.

Definitions.

(1) $(x_1y_1), (x_2y_2) \dots (x_ny_n)$ are the coordinates of n coplanar points $A_1A_2 \dots A_n$ referred to two rectangular axes Ox, Oy . If G is a point whose coordinates are

$$\frac{1}{n}(x_1 + x_2 + \dots + x_n), \quad \frac{1}{n}(y_1 + y_2 + \dots + y_n),$$

then G is called the **mean centre** of the system of points.

It is clear that this definition is identical with the definition on page 65.

(2) If G is a point whose coordinates are

$$\frac{m_1x_1 + m_2x_2 + \dots + m_nx_n}{m_1 + m_2 + \dots + m_n}, \quad \frac{m_1y_1 + m_2y_2 + \dots + m_ny_n}{m_1 + m_2 + \dots + m_n},$$

then G is called the **centroid** of the masses $m_1, m_2, \dots, m_n$ at the points $A_1A_2 \dots A_n$.

The identification of G with the statical centre of gravity of the system of masses may be based on the work on page 71. Or again, it may be deduced from the statical theorem relating to moments about an axis.

Another method, which does not, however, differ materially from the first, will be adopted here.

THEOREM 43 (1).

$A_1A_2A_3 \dots A_n$ are n coplanar points. G_1 is taken so as to divide A_1A_2 in the ratio $1:1$; G_2 divides G_1A_3 in the ratio $1:2$; G_3 divides G_2A_4 in the ratio $1:3$; and so on; G_{n-1} divides $G_{n-2}A_n$ in the ratio $1:n-1$. Then G_{n-1} is the mean centre of the points $A_1A_2 \dots A_n$.

Draw two rectangular axes Ox, Oy .

Let $(x_r, y_r), (\xi_r, \eta_r)$ be the coordinates of A_r, G_r respectively.

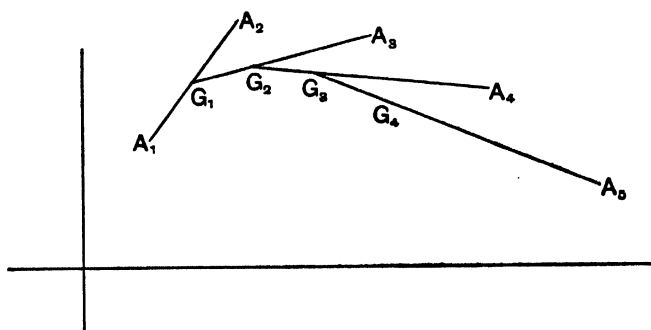


FIG. 48.

By Theorem 42

$$\begin{aligned} 2\eta_1 &= y_1 + y_2, \\ 3\eta_2 &= 2\eta_1 + y_3, \\ 4\eta_3 &= 3\eta_2 + y_4, \\ &\dots\dots\dots \end{aligned}$$

$$n\eta_{n-1} = (n-1)\eta_{n-2} + y_n,$$

adding up $n \cdot \eta_{n-1} = y_1 + y_2 + \dots + y_n;$

$$\therefore \eta_{n-1} = \frac{1}{n}(y_1 + y_2 + \dots + y_n).$$

Similarly, $\xi_{n-1} = \frac{1}{n}(x_1 + x_2 + \dots + x_n);$

$\therefore$ the point $G_{n-1}, (\xi_{n-1}, \eta_{n-1})$ coincides with the mean centre of the n points as just defined.

Q. E. D.

THEOREM 43 (2).

Masses $m_1, m_2, \dots, m_n$ are placed at the n coplanar points $A_1A_2 \dots A_n$; G_1 divides A_1A_2 in the ratio $m_2:m_1$; G_2 divides G_1A_3 in the ratio $m_3:(m_1+m_2)$; G_3 divides G_2A_4 in the ratio $m_4:(m_1+m_2+m_3)$; and so on; G_{n-1} divides $G_{n-2}A_n$ in the ratio $m_n:(m_1+m_2+\dots+m_{n-1})$; then G_{n-1} is the centroid of $m_1, m_2, \dots, m_n$ at $A_1, A_2, \dots, A_n$.

[Use Theorem 42 and the method of Theorem 43 (1).]

The proof is left to the reader.

1. Prove Theorem 43 (2).

D. G.

F

THEOREM 44.

If G is the mean centre of n collinear points $A_1, A_2, \dots, A_n$, then $GA_1 + GA_2 + \dots + GA_n = 0$.

[Take the origin at G and the y -axis along GA_1 and use the definition of the mean centre.]

2. Prove Theorem 44.

3. If G is the mean centre of masses $m_1, m_2, \dots, m_n$ at the n collinear points $A_1, A_2, \dots, A_n$, prove that

$$m_1 \cdot GA_1 + m_2 \cdot GA_2 + \dots + m_n \cdot GA_n = 0.$$

It is convenient to refer at this point to the following **statical theorem**.

If a system of forces, acting at a point O , are represented completely by $m_1 \cdot OA_1, m_2 \cdot OA_2, m_3 \cdot OA_3, \dots, m_n \cdot OA_n$, then they are equivalent to a single force represented completely by

$$(m_1 + m_2 + \dots + m_n) \cdot OG,$$

where G is the centroid of masses $m_1, m_2, \dots, m_n$ at $A_1, A_2, \dots, A_n$.

This theorem is a particular case of the general vector theorem enunciated on page 71, since a force is a special case of a vector quantity. The proof suggested for the vector theorem may be applied equally easily in this case.

THEOREM 45.

If O is the mean centre of masses m_1, m_2, m_3 at A, B, C respectively, then $\frac{m_1}{\triangle OBC} = \frac{m_2}{\triangle OCA} = \frac{m_3}{\triangle OAB}$

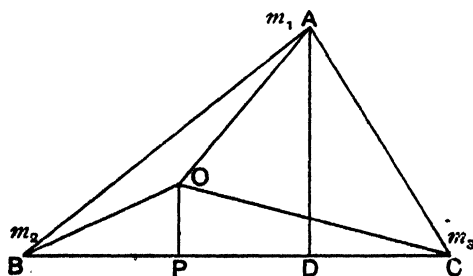


FIG. 49.

Draw OP, AD perpendicular to BC .

By definition, regarding BC as the x -axis,

$$OP = \frac{m_1 \cdot AD + m_2 \cdot 0 + m_3 \cdot 0}{m_1 + m_2 + m_3} = \frac{m_1 \cdot AD}{m_1 + m_2 + m_3};$$

$$\begin{aligned}\therefore \frac{m_1}{OP} &= \frac{m_1 + m_2 + m_3}{AD}; \\ \therefore \frac{m_1}{\frac{1}{2} OP \cdot BC} &= \frac{m_1 + m_2 + m_3}{\frac{1}{2} AD \cdot BC}; \\ \therefore \frac{m_1}{\Delta OBC} &= \frac{m_1 + m_2 + m_3}{\Delta ABC} \\ &= \text{similarly } \frac{m_2}{\Delta OCA} = \frac{m_3}{\Delta OAB}.\end{aligned}$$

Q. E. D.

In the following examples, $\overline{PQ}$ denotes the force represented in magnitude and position by the line PQ .

4. (1) $A_1, A_2 \dots A_n$ are n collinear points; G is their mean centre; if P is any other point on the line $A_1 A_n$, prove that $\sum_{r=1}^n A_r P = n \cdot GP$.

(2) Extend part (1) to apply to the centroid G of masses

$$m_1, m_2, \dots m_n \text{ at } A_1, A_2 \dots A_n.$$

5. G is the mean centre of n collinear points $A_1, A_2, \dots A_n$. H is the mean centre of m points $B_1, B_2, \dots B_m$ on the line $A_1 A_n$; prove that $\sum_{r=1}^n \sum_{s=1}^m A_r B_s = mn \cdot GH$.

6. G is the mean centre of n collinear points $A_1, A_2, \dots A_n$; H is the mean centre of n points $B_1, B_2, \dots B_n$ on $A_1 A_n$; prove that $A_1 B_1 + A_2 B_2 + \dots + A_n B_n = n \cdot GH$.

7. Determine the masses to be placed at **A, B, C**, in order that their centroid may be (1) the incentre, (2) the orthocentre, (3) the circumcentre, (4) the excentre opposite to **A**.

8. Determine the masses to be placed at the excentres so that their centroid may be at the incentre of the triangle.

9. P, Q, R, S are the mid-points of consecutive sides of a quadrilateral $ABCD$; X, Y are the mid-points of the diagonals AC, BD ; prove that the centroid of equal masses placed at A, B, C, D is at the mid-point of each of the lines PR, QS, XY .

10. A, B, C are fixed points: P is a variable point on a fixed line. Find the locus of the centre of four equal and parallel forces at A, B, C, P .

11. A', B', C' are the mid-points of BC, CA, AB ; P is any other point, prove that the forces $\overline{PA}, \overline{PB}, \overline{PC}, \overline{A'P}, \overline{B'P}, \overline{C'P}$ are in equilibrium.

12. If $ABCD$ is a parallelogram, and if P is any other point, prove that the forces $\overline{PA}, \overline{BP}, \overline{PC}, \overline{DP}$ are in equilibrium.

13. O, N are the circumcentre, nine-point centre of the triangle ABC , prove that the forces $\overline{NA}, \overline{NB}, \overline{NC}$ can be balanced by a system of equal forces along OA, OB, OC .

14. H, O are the orthocentre, circumcentre of the triangle ABC , prove that the forces $\overline{HA}, \overline{HB}, \overline{HC}$ are balanced by the forces $2\overline{OA}, 2\overline{OB}, 2\overline{OC}$.

15. Forces P, Q, R, S act along the sides AB, BC, CD, DA of a quadrilateral $ABCD$ and are in equilibrium, prove that

$$\frac{P \cdot R}{AB \cdot CD} = \frac{Q \cdot S}{BC \cdot DA}.$$

16. $ABCD$ is a tetrahedron. P is a variable point such that the resultant of the forces $\overline{PA}, \overline{PB}, \overline{PC}, \overline{PD}$ is of constant magnitude, find the locus of P .

17. P is a variable point on the side CD of the quadrilateral $ABCD$; $\overline{PG}$ is the resultant of the forces $\overline{AP}, \overline{BP}, \overline{PC}, \overline{PD}$; find the locus of G .

18. A', B', C' are the mid-points of BC, CA, AB ; prove that the forces $\overline{AA'}, \overline{BB'}, \overline{CC'}$ are in equilibrium.

19. Find a point P inside (1) the quadrilateral $ABCD$, (2) the tetrahedron $ABCD$, such that the forces $\overline{PA}, \overline{PB}, \overline{PC}, \overline{PD}$ are in equilibrium.

20. **Prove that the straight lines joining the mid-points of opposite edges of a tetrahedron $ABCD$ are concurrent and bisect each other.** [Consider the resultant of the forces $\overline{PA}, \overline{PB}, \overline{PC}, \overline{PD}$, obtained by combining them in pairs in different ways.]

21. $ABCD$ is a quadrilateral: if the forces $l \cdot \overline{AB}, m \cdot \overline{BC}, n \cdot \overline{CD}, r \cdot \overline{DA}$ are in equilibrium, prove that $ln = mr$ and that

$$\frac{\triangle ABD}{\triangle ABC} = \frac{n-m}{n-r}.$$

22. Four straight lines in a plane intersect at the six points $ABCD, E, F$; O is any other point; construct geometrically the resultant of the forces $\overline{OA}, \overline{OB}, \overline{OC}, \overline{OD}, \overline{OE}, \overline{OF}$.

23. Forces act along the edges BA, BC, DA, DC of a tetrahedron $ABCD$ and are inversely proportional to those edges; prove that if the forces give a single resultant, then $AD \cdot BC = AB \cdot CD$.

24. A', C' are the mid-points of the sides BC, AB of an equilateral triangle ABC ; prove that the magnitude of the resultant of the forces $\overline{AA'}, \overline{AC'}$ equals the magnitude of the resultant of the forces $\overline{AC'}, \overline{AC}$. If H, K are the mid-points of $C'A', C'C$, show that this is equivalent to the theorem that AHK is an isosceles triangle.

25. $ABCD$ is a quadrilateral; AD meets BC at E ; AB meets DC at F ; prove that the resultant of the forces $\overline{AE}, \overline{AD}, \overline{CE}, \overline{CD}$

passes through the mid-points of AC , BD , EF ; hence it follows that the mid-points of these lines are collinear.

26. $A'B'C'$ are the mid-points of the sides BC , CA , AB of a triangle; prove that the centroid of three uniform rods BC , CA , AB is at the incentre of the triangle $A'B'C'$.

27. $ABCD$ is a tetrahedron; P is a point on BC and Q is a point on AD ; if $\frac{BP}{PC}=l$, $\frac{AQ}{QD}=m$, resolve the force $\overline{PQ}$ into components along BA , CA , BD , CD . Is this resolution unique?

28. Three forces P , Q , R act along the edges DA , DB , DC of a tetrahedron $ABCD$; their resultant meets the face ABC at X , find, in terms of the edges of the tetrahedron, the lengths of the perpendiculars from X to BC , CA , AB .

29. $A_1A_2\dots A_n$ and $B_1B_2\dots B_m$ are two polygons of n and m sides; construct the resultant of the forces $\overline{A_sB_t}$, where s, t may have any value from 1 to n and 1 to m respectively.

30. If in Ex. 29 $n=m$, construct the resultant of the forces

$$\overline{A_1B_1}, \overline{A_2B_2}, \dots, \overline{A_nB_n}.$$

31. Find a point P in the plane of the triangle ABC such that the following sets of forces are in equilibrium:

$$(1) a \cdot \overline{PA}, b \cdot \overline{PB}, c \cdot \overline{PC}.$$

$$(2) \sin 2A \cdot \overline{PA}, \sin 2B \cdot \overline{PB}, \sin 2C \cdot \overline{PC}.$$

$$(3) \sec A \cdot \overline{PA}, \sec B \cdot \overline{PB}, \sec C \cdot \overline{PC}.$$

$$(4) a \cdot \overline{AP}, b \cdot \overline{PB}, c \cdot \overline{PC}.$$

32. AD , BE , CF are the altitudes of the triangle ABC , find a point P such that the forces $\sin 2A \cdot \overline{PD}$, $\sin 2B \cdot \overline{PE}$, $\sin 2C \cdot \overline{PF}$ are in equilibrium.

33. With the usual notation, if the forces $\lambda \cdot \overline{II_1}$, $\mu \cdot \overline{II_2}$, $\nu \cdot \overline{II_3}$ are in equilibrium, find the ratios $\lambda : \mu : \nu$.

34. Three forces P , Q , R act along the edges AB , AC , AD of a tetrahedron, if their resultant is parallel to the face BCD , prove that

$$\frac{P}{AB} + \frac{Q}{AC} + \frac{R}{AD} = 0.$$

35. $A_1A_2\dots A_n$ is a plane polygon and H is any point outside the plane: forces $P_1, P_2, \dots, P_n$ act along $HA_1, HA_2, \dots, HA_n$: if their resultant is parallel to the plane, prove that

$$\frac{P_1}{HA_1} + \frac{P_2}{HA_2} + \dots + \frac{P_n}{HA_n} = 0.$$

36. $ABCD$ is a tetrahedron ; G_1, G_2, G_3, G_4 are the centroids of the faces opposite to A, B, C, D ; P is any other point ; prove that the forces $\overline{PA}, 3\overline{PG}_1 ; \overline{PB}, 3\overline{PG}_2 ; \overline{PC}, 3\overline{PG}_3 ; \overline{PD}, 3\overline{PG}_4$; are equivalent : hence show that AG_1, BG_2, CG_3, DG_4 are concurrent (at G , say) ; and that $G_1G = \frac{1}{4}G_1A$.

37. P is any point ; with the usual notation, prove that the resultant of the forces $\frac{1}{r_1} \cdot \overline{PA'}, \frac{1}{r_2} \cdot \overline{PB'}, \frac{1}{r_3} \cdot \overline{PC'}$ passes through the incentre of the triangle ABC .

38. $ABCD$ is a tetrahedron ; I_1, I_2, I_3, I_4 are the incentres of the faces opposite to A, B, C, D ; if AI_1 and BI_2 intersect, prove that

$$(1) AD \cdot BC = AC \cdot BD.$$

$$(2) CI_3 \text{ and } DI_4 \text{ also intersect,}$$

and if $AD \cdot BC = AC \cdot BD = AB \cdot CD$, prove that AI_1, BI_2, CI_3, DI_4 are concurrent.

39. $A_1A_2A_3A_4$ is a tetrahedron ; B_1 is a point on the face opposite to A_1 such that $A_1A_4 \cdot \Delta A_2B_1A_3 = A_1A_2 \cdot \Delta A_3B_1A_4 = A_1A_3 \cdot \Delta A_4B_1A_2$; and similar points $B_2B_3B_4$ are taken on the faces opposite to $A_2A_3A_4$; prove that $A_1B_1, A_2B_2, A_3B_3, A_4B_4$ are concurrent.

40. Tangents are drawn at the angular points of a triangle ABC to its circumcircle so as to form a triangle PQR . If AP, BQ, CR intersect at K , prove that the forces $a^2 \cdot \overline{AK}, b^2 \cdot \overline{BK}, c^2 \cdot \overline{CK}$ are in equilibrium.

41. A_1, A_2, A_3, A_4, A_5 are five points in space ; G_1 is the centroid of the tetrahedron $A_2A_3A_4A_5$ and similarly for G_2, G_3, G_4, G_5 . State what properties connect the lines $A_1G_1, A_2G_2, A_3G_3, A_4G_4, A_5G_5$, and prove them.

42. P, Q, R are points on BC, CA, AB such that $\frac{BP}{PC} = \frac{CQ}{QA} = \frac{AR}{RB}$, prove that the triangles ABC, PQR have the same centroid.

43. Three forces P, Q, R act along the sides BC, CA, AB of a triangle ABC ; if their resultant is along IG , prove that

$$P : Q : R = a(b-c) : b(c-a) : c(a-b).$$

44. Use the system of unit forces along BA, BC, CB, CA to prove that the internal bisectors of the angles of the triangle ABC are concurrent.

45. H is the orthocentre of the triangle ABC ; P, Q, R are the mid-points of AH, BH, CH ; A', B', C' are the mid-points of BC, CA, AB . By placing equal masses at A, B, C, H , prove that the lines $A'P, B'Q, C'R$ are concurrent (at N , say), and if G is the centroid of the triangle ABC , prove that N lies on GH and that $3GN = NH$. Is this theorem true if H is any point in the plane ABC ?

THEOREM 46. [APOLLONIUS' THEOREM:]

(1) If A' is the mid-point of the base BC of a triangle ABC , then $AB^2 + AC^2 = 2AA'^2 + 2A'B^2$.

(2) If A' is a point on the base BC of a triangle ABC such that $m \cdot BA' = n \cdot A'C$, then

$$m \cdot AB^2 + n \cdot AC^2 = (m+n)AA'^2 + m \cdot A'B^2 + n \cdot A'C^2.$$

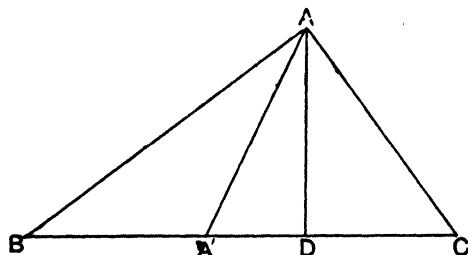


FIG. 50.

Draw AD perpendicular to BC .

$$\begin{aligned} (1) \quad AB^2 &= AD^2 + DB^2 = AD^2 + (BA' + A'D)^2 \\ &= AD^2 + BA'^2 + 2BA' \cdot A'D + A'D^2, \end{aligned}$$

$$\text{and} \quad AC^2 = AD^2 + DC^2 = AD^2 + (A'C - A'D)^2 = AD^2 + (A'C - A'D)^2;$$

$$\therefore AC^2 = AD^2 + A'C^2 - 2A'C \cdot A'D + A'D^2;$$

$$\begin{aligned} \therefore AB^2 + AC^2 &= 2AD^2 + 2BA'^2 + 2A'D^2 \\ &= 2AA'^2 + 2A'B^2. \end{aligned}$$

Q.E.D.

$$\begin{aligned} (2) \quad m \cdot AB^2 &= m \cdot AD^2 + m(BA' + A'D)^2 \\ &= m \cdot BA'^2 + 2mBA' \cdot A'D + m \cdot A'A^2, \end{aligned}$$

$$\text{and} \quad n \cdot AC^2 = n \cdot AD^2 + n(A'C - A'D)^2 = n \cdot A'C^2 - 2nA'C \cdot A'D + n \cdot A'A^2;$$

$$\therefore m \cdot AB^2 + n \cdot AC^2$$

$$= m \cdot BA'^2 + n \cdot A'C^2 + 2A'D(m \cdot BA' - n \cdot A'C) + (m+n)A'A^2;$$

$$\text{but } m \cdot BA' = n \cdot A'C;$$

$$\therefore m \cdot AB^2 + n \cdot AC^2 = (m+n)AA'^2 + m \cdot A'B^2 + n \cdot A'C^2.$$

Q.E.D.

46. Calculate the lengths of the medians of a triangle whose sides are 7 cms., 8 cms., 9 cms.

47. Determine the lengths of the sides of a triangle, if the medians are of lengths x , y , z .

48. AA' is a median of the triangle ABC and is a mean proportional between AB, AC ; prove that $a = \sqrt{2} (b \sim c)$.

49. Prove that three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares of the medians.

50. If $ABCD$ is a parallelogram, prove that

$$2(AB^2 + BC^2) = AC^2 + BD^2.$$

51. P, Q are the mid-points of the diagonals AC, BD of the quadrilateral $ABCD$, prove that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2$.

52. If a, b, c, d are the lengths of the sides of a quadrilateral, and if x, y are the lengths of the diagonals, calculate the length of the line joining the mid-points of a pair of opposite sides.

53. If $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$, prove that $ABCD$ is a parallelogram.

54. If the sum of the squares on each pair of opposite edges of a tetrahedron is the same, prove that the lines joining the mid-points of opposite edges are equal.

55. $ABCD$ is a rectangle: P is any point in space, prove that $PA^2 + PC^2 = PB^2 + PD^2$.

56. The diagonals AC, BD of a cyclic quadrilateral $ABCD$ meet at E ; if $BE = EI$, prove that $2AC^2 = AB^2 + BC^2 + CD^2 + DA^2$.

57. Two straight rods are jointed together at their mid-points: prove that the sum of the squares of the four lines joining their extremities is constant.

58. ABC is an equilateral triangle: AB is produced to D so that $BD = AB$; P is any point such that $DP = DC$; prove that

$$AB^2 + AP^2 = 2BP^2.$$

59. In any tetrahedron, prove that the sum of the squares on any pair of opposite edges together with twice the square on the line joining their mid-points is the same, whatever pair of opposite edges is taken.

60. B is a fixed point, A is the centre of a fixed circle, prove that the locus of the mid-points of chords which subtend a right angle at B is a circle whose centre is the mid-point of AB .

61. B is a fixed point; A is the centre of a fixed circle; PQ is a variable chord parallel to AB , prove that $BP^2 + BQ^2$ is constant.

62. AP, BQ, CR are the tangents from three of the vertices of the rectangle $ABCD$ to a given circle; if $AP^2 + CR^2 = BQ^2$, prove that D lies on the circle.

63. ABC, PQR are two triangles having AB, BC equal to PQ, QR respectively; if the angles ABC, PQR are supplementary, prove that $AC^2 + PR^2 = 2AB^2 + 2BC^2$.

64. Squares are described externally on the sides of a triangle and their adjacent corners are joined; prove that the sum of the squares on the joining lines is equal to three times the sum of the squares on the sides of the triangle.

65. In the triangle ABC , $AB=8$, $AC=6$, $BC=9$; D is a point on BC such that $\frac{BD}{DC}=\frac{4}{5}$. Calculate AD .

66. D , E are points on the sides BC , CA of the triangle ABC , such that $\frac{BD}{DC}=\frac{CE}{EA}=\frac{1}{2}$; if $AD=BE$, prove that $8BC^2=3AB^2+5AC^2$.

67. D is a point on the base BC of the triangle ABC such that $\frac{BD}{DC}=\frac{\lambda}{\mu}$. AD is produced to meet the circumcircle at E , prove that $\mu \cdot AB^2 + \lambda \cdot AC^2 = (\mu + \lambda)AD \cdot AE$.

State what this theorem becomes when (1) AD is a median; (2) AD bisects $\hat{BAC}$.

68. ABC is an equilateral triangle, AB is produced to D so that $BD=\lambda \cdot BA$. P is any point such that $DP=DC$; prove that

$$\lambda \cdot AP^2 + AB^2 = (\lambda + 1)BP^2.$$

69. If AD bisects the angle BAC and meets BC at D , prove that

$$AD^2 = bc \left\{ 1 - \left(\frac{a}{b+c} \right)^2 \right\}. \quad \left[\text{Use } \frac{BD}{DC} = \frac{c}{b} \right]$$

70. A , B are two fixed points. P is a variable point such that $m \cdot AP^2 \pm n \cdot BP^2$ is constant, where m , n are constants, prove that the locus of P is a circle. Does the circle ever degenerate into a straight line?

71. (1) If G is the centroid of the triangle ABC and if P is any other point, prove that $PA^2 + PB^2 + PC^2 = 3PG^2 + \frac{1}{2}BC^2 + \frac{3}{2}AA'^2$.

[Use Apollonius (1) to get $PA^2 + PB^2$ and then use Apollonius (2).]

(2) By taking P at G , prove that

$$PA^2 + PB^2 + PC^2 - (GA^2 + GB^2 + GC^2) = 3PG^2.$$

72. (1) $A_1A_2A_3A_4$ are four fixed points; G_1 is the mid-point of A_1A_2 ; G_2 is a point on G_1A_3 such that $2G_1G_2=G_2A_3$; G is a point on G_2A_4 such that $3G_2G=GA_4$. If P is a variable point, prove that

$$PA_1^2 + PA_2^2 + PA_3^2 + PA_4^2 = 4PG^2 + \text{constant}.$$

(2) Hence prove that $\sum_{r=1}^4 (PA_r^2 - GA_r^2) = 4PG^2$.

[Use the method of Ex. 71 (2).]

(3) State the analogous theorem for n fixed points.

73. ABC is a given triangle; find a point P for which $PA^2 + PB^2 + PC^2$ has its least value.

74. ABC is a given triangle; P is a variable point such that $PA^2 + PB^2 + PC^2$ is constant; prove that the locus of P is a circle.

75. ABC is a given triangle; P is a variable point such that $PA^2 + 2PB^2 + 3PC^2$ is constant; prove that the locus of P is a circle, and construct its centre.

THEOREM 47.

G is the mean centre of the n points $A_1 A_2 \dots A_n$; if P is any other point, then $\sum_{r=1}^n PA_r^2 - \sum_{r=1}^n GA_r^2 = n \cdot PG^2$.

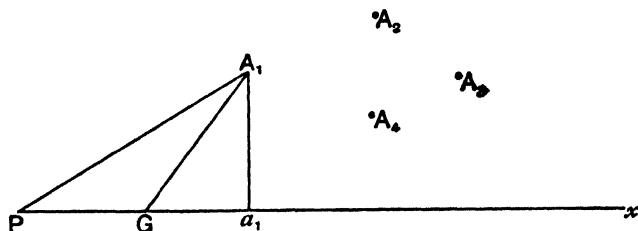


FIG. 51.

Join PG : let $a_1, a_2, a_3 \dots$ be the feet of the perpendiculars from $A_1, A_2, A_3 \dots$ to PG .

Then, regarding G as the origin and PG as the x -axis,

$$\frac{1}{n}(Ga_1 + Ga_2 + \dots + Ga_n) = x\text{-coordinate of } G = 0;$$

$$\therefore \sum_{r=1}^n Ga_r = 0. \quad [\text{Th. 44.}]$$

$$\begin{aligned} PA_1^2 - GA_1^2 &= Pa_1^2 + a_1 A_1^2 - (Ga_1^2 + a_1 A_1^2) \\ &= (PG + Ga_1)^2 - Ga_1^2 \\ &= PG^2 + 2PG \cdot Ga_1; \end{aligned}$$

$$\begin{aligned} \therefore \sum_{r=1}^n PA_r^2 - \sum_{r=1}^n GA_r^2 &= n \cdot PG^2 + 2PG \cdot \sum_{r=1}^n Ga_r \\ &= n \cdot PG^2. \end{aligned} \quad \text{Q.E.D.}$$

THEOREM 48.

If G is the centroid of masses $m_1, m_2, \dots, m_n$ at the n points $A_1, A_2, \dots, A_n$ respectively, and if P is any other point, then

$$\sum_{r=1}^n m_r \cdot PA_r^2 - \sum_{r=1}^n m_r \cdot GA_r^2 = (m_1 + m_2 + \dots + m_n) PG^2.$$

This may be proved in exactly the same way as Theorem 47. Notice that Apollonius' Theorem, parts (1) and (2), are special cases of Theorems 47, 48.

76. Prove Theorem 48.

77. If O , G , N are the circumcentre, centroid and nine-point centre of the triangle ABC , prove that $AN^2 + BN^2 + CN^2 + 9GN^2 = 3OA^2$.

78. If N is the nine-point centre of the triangle ABC and if H is the orthocentre, prove that N is the mean centre of A , B , C , H .

79. If O is the circumcentre and H the orthocentre of the triangle ABC , prove that $AH^2 + BH^2 + CH^2 = 3OA^2 + OH^2$.

80. $ABCD$ is a quadrilateral, find a point P for which

$$PA^2 + PB^2 + PC^2 + PD^2$$

has its least value.

81. G is the mean centre of n points $A_1, A_2 \dots A_n$. Perpendiculars are let fall from them to two parallel lines: P_r, Q_r are the feet of the perpendiculars from A_r ; X, Y are the feet of the perpendiculars from G ; prove that

$$\Sigma A_r P_r^2 - n \cdot GX^2 = \Sigma A_r Q_r^2 - n \cdot GY^2.$$

82. $ABCD$ is a square; P is a variable point such that

$$PA^2 + PB^2 + PC^2 + PD^2 = 3AB^2,$$

find in detail the locus of P .

83. $A_1 A_2 A_3 A_4 A_5$ are five collinear points in order at equal distances apart; P is any other point, prove that

$$PA_1^2 + PA_2^2 + PA_4^2 + PA_5^2 = 4PA_3^2 + 10A_1A_2^2.$$

84. P is a point on the circumcircle of the equilateral triangle ABC , prove that $PA^2 + PB^2 + PC^2 = 2AB^2$.

85. G is the mean centre of n points $A_1, A_2, \dots A_n$; P, Q are any two points equidistant from G , prove that $\Sigma PA_r^2 = \Sigma QA_r^2$.

86. G is the mean centre of n points $A_1, A_2, \dots A_n$; P, Q are any two points, prove that $\Sigma (PA_r^2 - QA_r^2) = n(PG^2 - QG^2)$.

87. A regular pentagon $A_1 A_2 A_3 A_4 A_5$ is inscribed in a circle, centre O , prove that $A_1 A_2^2 + A_1 A_3^2 = 5A_1 O^2$.

88. A regular octagon $A_1 A_2 A_3 \dots A_8$ is inscribed in a circle, centre O , prove that $A_1 A_2^2 + A_2 A_4^2 + A_4 A_7^2 = 6A_1 O^2$.

89. $A_1 A_2 \dots A_n$ is a regular polygon inscribed in a circle, centre O ; P is a point on the circumference, Q is any other point; prove that

$$(1) \Sigma PA_r^2 = 2n \cdot PO^2;$$

$$(2) \Sigma QA_r^2 = n(PO^2 + QO^2);$$

$$(3) \Sigma \Sigma A_r A_s^2 = n^2 \cdot PO^2.$$

90. A regular polygon is inscribed in a circle, prove that the arithmetic mean of the squares of the distances of its corners from any point is equal to the arithmetic mean of the squares of the longest and shortest distances of the point from the circle.

91. $A_1A_2\dots A_n$, $B_1B_2\dots B_n$ are two regular polygons inscribed in two circles, centres C , D respectively; prove that

$$\Sigma \Sigma A_i B_i^2 = n^2 (A_1 C^2 + B_1 D^2 + CD^2).$$

92. Extend Ex. 91 to the case where the regular polygons have m and n sides.

93. $A_1A_2\dots A_n$ is a regular polygon inscribed in a circle, centre O . From any point P , perpendiculars $PN_1, PN_2, \dots, PN_n$ are drawn to the radii $OA_1, OA_2, \dots, OA_n$. Prove that the mean centre of the points $N_1, N_2, \dots, N_n$ is the mid-point of OP . [Draw the circle on OP as diameter.]

94. With the figure of Ex. 93, prove that $\Sigma PN_i^2 = \Sigma ON_i^2$.

95. ABC is a given triangle, find the position of P for which

$$(1) \quad BC \cdot PA^2 + CA \cdot PB^2 + AB \cdot PC^2,$$

$$(2) \quad \lambda \cdot PA^2 + \mu \cdot PB^2 + \nu \cdot PC^2,$$

is a minimum.

96. G is the centroid of masses l, m, n at A, B, C ; GT is the tangent from G to the circumcircle of the triangle ABC . Prove that $l \cdot PA^2 + m \cdot PB^2 + n \cdot PC^2 = (l + m + n)(PG^2 - GT^2)$. [It is supposed that a negative mass may exist.]

97. If G is the centroid of masses m_1, m_2, m_3 at A_1, A_2, A_3 , and if P is a point such that $m_1 \cdot PA_1^2 + m_2 \cdot PA_2^2 + m_3 \cdot PA_3^2 = 0$, prove that the locus of P is a circle cutting the circle $A_1A_2A_3$ at right angles. [I.e. the tangents at a point of intersection are at right angles.]

98. P is a variable point on the nine-point circle of the triangle ABC , prove that $PA^2 \sin A \cos B - C + PB^2 \sin B \cos C - A + PC^2 \sin C \cos A - B$ is constant.

CHAPTER VIII.

HARMONIC RANGES AND PENCILS.

THE fundamental ideas contained in this chapter date back to an early period. The word 'harmonic' was in use at the beginning of the fourth century B.C. But the properties known at that time were probably mainly of a simple metrical character. Theorem 51, which may be regarded as one of the foundation stones of modern descriptive geometry, was probably known to Euclid. Its extension to a pencil formed by any four concurrent lines was incorporated by Pappus in his great work, the *Συναγωγή*, published about the beginning of the fourth century A.D. It is most remarkable that this treatise, which approximates so closely to modern developments, received scarcely any attention before the 17th century.

Definition.

A straight line AB is divided internally at C and externally at D in the same ratio $\left[\text{i.e. } \frac{AC}{CB} = -\frac{AD}{DB} \right]$; then AB is said to be divided harmonically at C and D ; and the points $ACBD$ are said to form a **harmonic range**.

C and D are called harmonic conjugates with respect to A and B .

The range $ACBD$ is denoted by $\{ACBD\}$ or $\{AB; CD\}$.

The reason for the name harmonic will be found in Ex. 3, which is due to Pythagoras.

THEOREM 49.

(1) If C, D divide AB harmonically, then A, B divide CD harmonically.

(2) If O is the mid-point of AB , then $OB^2 = OC \cdot OD$.

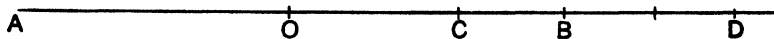


FIG. 52.

(1) Since $\{ACBD\}$ is harmonic,

$$\frac{AC}{CB} = \frac{AD}{BD} \text{ or } AC \cdot BD = AD \cdot CB;$$

$$\therefore \frac{CB}{BD} = \frac{AC}{AD} = -\frac{CA}{AD};$$

$\therefore CD$ is divided internally at B and externally at A in the same ratio. Q.E.D.

$$(2) \text{ Since } \frac{AC}{CB} = \frac{AD}{BD}; \quad \therefore \frac{AC + CB}{AC - CB} = \frac{AD + BD}{AD - BD}.$$

$$\text{But } AC + CB = AB = 2OB,$$

$$AC - CB = (AO + OC) - (OB - OC) = 2OC,$$

$$AD + BD = (AO + OD) + (OD - OB) = 2OD,$$

$$AD - BD = AB = 2OB;$$

$$\therefore \frac{2OB}{2OC} = \frac{2OD}{2OB};$$

$$\therefore OB^2 = OC \cdot OD.$$

Q.E.D.

Corollary.

If O is the mid-point of AB and if C, D are points on AB such that $OB^2 = OC \cdot OD$, then $\{ACBD\}$ is harmonic.

This may be proved by taking each step in (2) in the reverse order.

1. Prove Theorem 49 (2) by putting $OC=c$, $OB=b$, $OD=d$ and substituting in the relation $\frac{AC}{CB} = \frac{AD}{BD}$.

2. Prove the Corollary of Theorem 49 by the method of Ex. 1.

3. If $\{ACBD\}$ is harmonic, prove that AC, AB, AD are in harmonical progression. [Put $AC=c$, $AB=b$, $AD=d$.]

4. If $\{ACBD\}$ is harmonic and if O is the mid-point of AB , prove that

$$(1) DA \cdot DB = DC \cdot DO;$$

$$(2) AB \cdot CD = 2AL \cdot CB = 2AC \cdot BD.$$

5. Two circles intersect at B, C ; a common tangent touches them at P, Q ; any other circle through B, C cuts PQ at L, M ; prove that $\{PQ; LM\}$ is harmonic.

6. O, A, B, C are four collinear points; the lengths of OA, OB, OC are the roots of the equation $x^3 - px^2 + qx + r = 0$; find the relation between p, q, r in order that $\{OABC\}$ may be harmonic.

7. $\{ACBD\}$ is harmonic; O, O' are the mid-points of AB, CD ; the circles on AB, CD as diameters intersect at H , prove that $\angle HOO'$ is a right angle.

8. Two circles intersect at B, C ; a straight line $PQP'Q'$ cuts one circle at P, Q and the other at P', Q' and cuts BC produced at A . On PQ' are taken two points H, K such that $AH^2 = AK^2 = AB \cdot AC$; prove that $\{PQ; HK\}$ and $\{P'Q'; HK\}$ are harmonic.

9. Given two segments $PQ, P'Q'$ of a straight line, which do not overlap, find two points H, K which divide each of the segments harmonically. [Use Ex. 8.]

10. P, Q, R are points on BC, CA, AB such that AP, BQ, CR are concurrent; P', Q', R' are the harmonic conjugates of P, Q, R w.r.t. $B, C; C, A; A, B$: prove that P', Q', R' are collinear.

11. ABC are three given collinear points; construct a point D such that $\{AB; CD\}$ is harmonic.

12. If C is the mid-point of AB , where is the point harmonically conjugate to C w.r.t. A, B ?

13. D is the mid-point of the base BC of the triangle ABC . Any line through D meets AB, AC at F, G and meets a line through A parallel to BC at E ; prove that $\{EGDF\}$ is harmonic.

[Use similar triangles to identify the ratios $\frac{EG}{GD}$ and $\frac{EF}{DF}$.]

14. If $\{ACBD\}$ is harmonic, and if O is any point outside AD , prove that

$$(1) \frac{AC}{CB} = \frac{AO \sin AOC}{OB \sin COB};$$

$$(2) \frac{AC}{CB} : \frac{AD}{BD} = \frac{\sin AOC}{\sin COB} : \frac{\sin AOD}{\sin BOD};$$

(3) if any other straight line meets OA, OB, OC, OD at $A'B'C'D'$, $\{A'C'B'D'\}$ is harmonic.

THEOREM 50.

$\{ACBD\}$ is harmonic: O is any point outside AB . If through B a line PBQ is drawn parallel to OA , meeting OC at P and OD at Q , then $PB=BQ$.

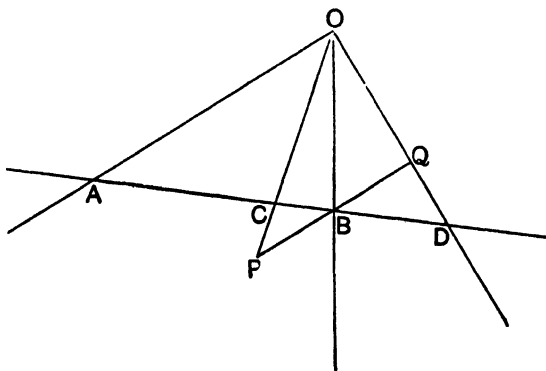


FIG. 53.

Since AO is parallel to PB , $\frac{AC}{CB} = \frac{AO}{PB}$.

Since AO is parallel to BQ , $\frac{AD}{BD} = \frac{AO}{BQ}$.

But $\frac{AC}{CB} = \frac{AD}{BD}$, given;

$$\therefore \frac{AO}{PB} = \frac{AO}{BQ};$$

$$\therefore PB = BQ.$$

Q.E.D.

Corollary.

If OA , OC , OB , OD are four lines such that a parallel to OA through a point B in OB is divided into equal segments (PB , BQ) by OC , OB , OD ; then every line through B is cut harmonically by OA , OC , OB , OD .

This is proved by reversing each step in the above proof.

It is important to notice that if C is the mid-point of AB , then $\{ACB\infty\}$ is harmonic where ∞ denotes a point on AB infinitely distant from A .

15. Prove the Corollary of Theorem 50.

THEOREM 51.

$\{ACBD\}$ is harmonic; O is any point outside AD ; then any transversal is cut in a harmonic range by the four lines OA , OC , OB , OD .

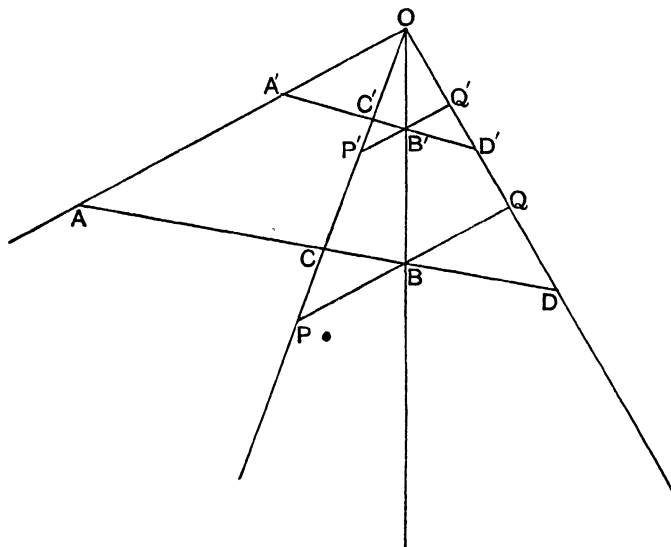


FIG. 54.

Let $A'C'B'D'$ be the transversal.

Through B , B' draw PBQ , $P'B'Q'$ parallel to OA .

Since $\{ACBD\}$ is harmonic, $PB = BQ$;

and by parallels, $\frac{PB}{P'B'} = \frac{OB}{OB'} = \frac{BQ}{B'Q'}$;

$$\therefore P'B' = B'Q';$$

$\therefore \{A'C'B'D'\}$ is harmonic. [Th. 50. Corollary.]

Q.E.D.

This method applies equally well, if the transversal cuts any or all of the lines OA , OC , OB , OD when produced.

16. Draw figures showing the various positions of the range $\{A'C'B'D'\}$ as the transversal moves through the directions AO to CO , to BO , to DO , round to AO again.

Definition. If four concurrent lines OA , OC , OB , OD are such that they are cut harmonically by one (and therefore by every) transversal, then the lines are said to form a **harmonic pencil**; each of the lines OA , OC , OB , OD is called a **ray** of the pencil, and the point O is called the **vertex** of the pencil: OC , OD are called **conjugate ray** w.r.t. OA , OB . The pencil is represented by

$$O\{ACBD\} \text{ or } O\{AB; CD\}.$$

On account of the importance of this theorem, two alternative methods of proof are given below.

THEOREM 51 (SECOND PROOF).

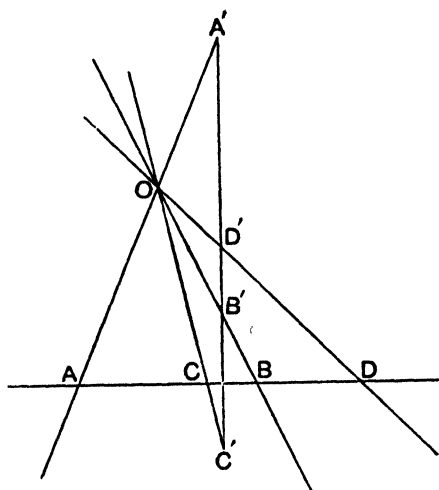


FIG. 55.

Let $A'C'B'D'$ be the transversal.

$$\frac{AC}{CB} = \frac{\triangle AOC}{\triangle COB} = \frac{\frac{1}{2}OA \cdot OC \sin AOC}{\frac{1}{2}OC \cdot OB \sin COB};$$

$$\therefore \frac{AC}{CB} = \frac{OA \sin AOC}{OB \sin COB}.$$

Similarly,

$$\frac{AD}{BD} = \frac{OA \sin AOD}{OB \sin BOD};$$

$$\therefore \frac{AC}{CB} : \frac{AD}{BD} = \frac{\sin AOC \cdot \sin BOD}{\sin COB \cdot \sin AOD},$$

which depends only on the sines of the angles at the vertex ;

$$\therefore \frac{A'C'}{C'B'} : \frac{A'D'}{B'D'} \text{ is equal to the same expression.}$$

[Note that $\sin A'OC' = \sin (180^\circ - AOC) = \sin AOC$];

$$\therefore \frac{AC}{CB} : \frac{AD}{BD} = \frac{A'C'}{C'B'} : \frac{A'D'}{B'D'}.$$

But

$$\frac{AC}{CB} = \frac{AD}{BD} \text{ given ;}$$

$$\therefore \frac{A'C'}{C'B'} = \frac{A'D'}{B'D'};$$

$\therefore \{A'C'B'D'\}$ is harmonic.

Q.E.D.

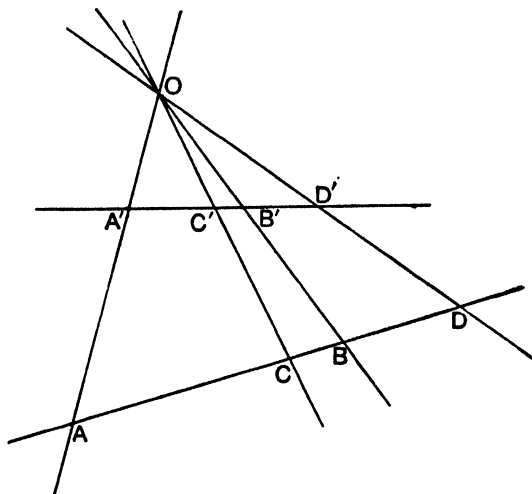
THEOREM 51 (THIRD PROOF).

FIG. 56.

Let $A'C'B'D'$ be the transversal.

Now
$$\frac{A'C'}{C'B'} : \frac{AC}{CB} = \frac{OA'}{OA} : \frac{OB'}{OB}, \quad [\text{Th. 8}]'$$

and
$$\frac{A'D'}{D'B'} : \frac{AD}{DB} = \frac{OA'}{OA} : \frac{OB'}{OB}; \quad [\text{Th. 8}]$$

$$\therefore \frac{A'C'}{C'B'} : \frac{AC}{CB} = \frac{A'D'}{D'B'} : \frac{AD}{DB};$$

$$\therefore \frac{A'C'}{C'B'} : \frac{A'D'}{D'B'} = \frac{AC}{CB} : \frac{AD}{DB}.$$

But
$$\frac{AC}{CB} = -\frac{AD}{DB};$$

$$\therefore \frac{A'C'}{C'B'} = -\frac{A'D'}{D'B'};$$

$\therefore \{A'C'B'D'\}$ is harmonic.

Q.E.D.

The second proof of Theorem 51 puts in evidence the important fact that the harmonic property of a pencil depends on a definite relation between the angles at the vertex: namely, that (with the notation of Fig. 55) $\frac{\sin AOC \cdot \sin BOD}{\sin COB \cdot \sin AOD} = 1$.

17. Use this angular relation to verify that the two bisectors of an angle are harmonic conjugates w.r.t. the arms of the angle.

18. Use this angular relation to prove Theorem 52.

THEOREM 52.

$O\{ACBD\}$ is harmonic; if AO, BO are at right angles, then they are the two bisectors of the angle COD .

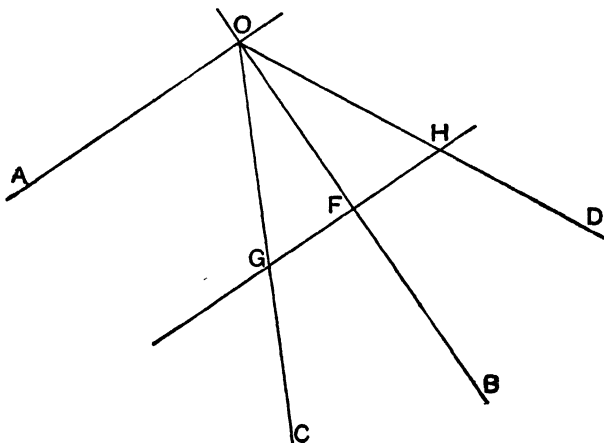


FIG. 57.

Draw any line parallel to OA meeting OB, OC, OD at F, G, H .

Then

$$GF = FH$$

[Th. 50]

also

$$\angle OFG = \angle OFH \text{ right angles, since } \angle AOF = 90^\circ.$$

OF is common;

$\therefore$ the triangles OFG, OFH are congruent;

$$\therefore \angle GOF = \angle FOH;$$

$\therefore OB$ bisects $\angle COD$.

But OA is perpendicular to OB ;

$\therefore OA$ bisects $\angle COD$ externally.

Q.E.D.

Corollary.

If $\{ACBD\}$ is a harmonic range: and if P is any point on the circle on CD as diameter, then $\frac{PA}{PB}$ is constant and equal to $\frac{AC}{CB}$.

This is the converse of Theorem 9. [Apollonius' Circle.]

19. Prove the Corollary of Theorem 52.

20. In the triangle ABC , AI, AI_2 meet BC at P, Q ; prove that $\{BPCQ\}$ is harmonic.

21. In the triangle ABC , prove that (1) $D\{BFAE\}$, (2) $A\{OH; I_2\}$ are harmonic pencils.

22. A, B are two given points, CD is a given straight line, prove that in general there is one and only one position of P on CD such that $\hat{APB}$ is bisected by CD . What is the exceptional case?

23. The lines joining a point on a circle, centre O , to the extremities of a chord PQ meet the diameter perpendicular to PQ at A, B . Prove that $OA \cdot OB = OP^2$.

24. A circle is described touching a straight line AB at C ; if its diameter is the harmonic mean between AC, CB , prove that it touches the circle on AB as diameter.

25. OA, OC, OB are three concurrent lines, construct a line OD such that $O\{ACBD\}$ is harmonic.

26. The incircle of the triangle ABC touches BC, CA, AB at X, Y, Z ; YZ produced meets BC at T , prove that $\{BXCT\}$ is harmonic. [Use Menelaus.]

27. $ABCD$ is a parallelogram; AE is drawn parallel to BD , prove that $A\{EC; BD\}$ is harmonic.

28. A', B', C' are the mid-points of BC, CA, AB , prove that $A'A, A'C$ are harmonic conjugates w.r.t. $A'B', A'C'$.

29. AD is an altitude of the triangle ABC ; A' is the mid-point of BC ; $A'P, A'Q$ are drawn parallel to AB, AC to meet AD at P, Q ; prove that $\{APDQ\}$ is harmonic.

30. With the usual notation for the triangle ABC , if DF meets BE at K , prove that $\{BKHE\}$ is harmonic; also, if EF meets BC at M , prove that $\{BDCM\}$ is harmonic.

31. Through a fixed point K draw a straight line to meet BC, CA, AB at PQR so that $\{KPQR\}$ is harmonic.

32. The tangent at a point C of a circle meets a diameter AB of the circle at T , prove that the other tangent from T to the circle is divided harmonically by CA, CB, T and its point of contact.

33. $A'B'C'$ are the mid-points of BC, CA, AB ; CC' meets $AA', A'B'$ at L, M ; prove that $C'L \cdot MC = C'C \cdot LM$.

34. O is the circumcentre of the triangle ABC ; AK, AL are two lines between AB, AC such that $\hat{BAK} = \hat{LAC}$ and meet the perpendicular from O to BC at K, L ; prove that $OK \cdot OL = OA^2$.

35. $ABCD$ is a quadrilateral; AB, CD meet at F ; AD, BC meet at G ; AC, BD meet at E ; FE meets BC, AD at Q, P ; GE meets AB, CD at H, K ; FG meets AC, BD at R, S . Prove that $\{DKCF\}$ is harmonic. [Apply Menelaus and Ceva to the triangle DGC .] Hence prove that every other set of four collinear points in the figure forms a harmonic range.

36. If $ax^2 + 2hx + b = 0$, $a'x^2 + 2h'x + b' = 0$, give two pairs of points on the x -axis which are harmonic conjugates, prove that $ab' + a'b = 2hh'$.

37. If a variable straight line is cut harmonically by two circles, centres A, B , conjugate points being on the same circle, prove that the locus of the perpendiculars let fall on it from A is a circle whose centre is at the mid-point of AB .

38. $ABCDPQ$ are six points on a circle; if $P\{ABCD\}$ is harmonic, prove that $Q\{ABCD\}$ is harmonic.

Definition. If four points on a circle are such that a harmonic pencil is formed by joining them to one other point on the circle, then by Ex. 38 the same is true for any other point at all on the circle. The four points are then said to form a **harmonic system of points** on the circle.

If four tangents to a circle are cut harmonically by another tangent to the circle, the four tangents are said to form a **harmonic system of tangents** to the circle. It follows from Ex. 40 that four such tangents are cut harmonically by every other tangent to the circle.

39. What does Ex. 38 become when Q is very close to A ?

40. (1) Two tangents TA, TB to a circle, centre C , are met by another tangent at P, Q ; prove that $\hat{PCQ}$ is equal or supplementary to $\frac{1}{2} \hat{ACB}$.

(2) Hence show that if four tangents meet TA at P_1, P_2, P_3, P_4 , and TB at Q_1, Q_2, Q_3, Q_4 , and if $\{P_1P_2P_3P_4\}$ is harmonic, then $\{Q_1Q_2Q_3Q_4\}$ is harmonic.

(3) Hence prove that if four tangents to a circle cut another tangent harmonically, then they cut every other tangent harmonically.

41. PQ, AB are two parallel chords of a circle; C is the mid-point of AB ; PC meets the circle at R ; QT is the tangent at Q ; prove that $Q\{ARBT\}$ is harmonic. [Use Ex. 38.]

42. AB is a diameter of a circle: AP is any chord; the tangents at A, P meet at O ; R is the foot of the perpendicular from P to AB ; prove that BO bisects PR .

[Prove that OP, OB, OR, OA cut AB harmonically.]

43. T is a point on the base BC produced of the triangle ABC ; N is the harmonic conjugate of T w.r.t. BC ; TP is a tangent from T to the circumcircle of the triangle ABC ; PN meets AB, AC at L, M ; and the circle at Q ; prove that $\{PLQM\}$ is harmonic.

44. The tangents at the extremities of a chord AD of a circle meet at P ; PCB is any line through P cutting the circle at C, B and AD at L ; prove that (1) $\frac{BA^2}{CA^2} = \frac{BP}{CP} = \frac{BD^2}{CD^2}$; (2) $\frac{BL}{LC} = \frac{BA^2}{CA^2}$; (3) $\{BLCP\}$ is harmonic.

This proves the important theorem that any straight line drawn from

a point P outside the circle is cut harmonically by the circle and the chord of contact of the tangents from P .

Another method is suggested in Ex. 45.

45. Prove Ex. 44 (3).

[Let E be the mid-point of AD and let CD meet the circle at B' ; produce BE to meet the circle at C' . Join $C'B'$; notice that the figure is symmetrical about EP and that $E\{AP, CC'\}$ is harmonic.]

46. If $A, A'; B, B'; C, C'; D, D'$ are harmonic conjugates w.r.t. the same two points H, K ; prove that $\frac{AB \cdot CD}{AD \cdot CB} = \frac{A'B' \cdot C'D'}{A'D' \cdot C'B'}$.

47. ABC is a straight line: AT is the tangent from A to the circle on BC as diameter: O is its centre; N is the foot of the perpendicular from T to BC ; prove that AO, AT, AN represent the Arithmetic, Geometric and Harmonic means between AB, AC ; and prove that these three quantities are in geometrical progression.

CROSS RATIO.

If a transversal meets four fixed lines, which pass through a point O , at A, C, B, D , it follows from either of the last two methods of proof of Theorem 51 that the value of the ratio $\frac{AC \cdot BD}{AD \cdot BC}$ is equal to $\frac{\sin AOC \cdot \sin BOD}{\sin AOD \cdot \sin BOC}$ and is independent of the position of the transversal. For brevity this ratio is represented by $\{ACBD\}$. In the particular case when $O\{ACBD\}$ is harmonic, this ratio is by definition equal to -1 . Consequently the fact that $O\{ACBD\}$ is harmonic is sometimes denoted by $O\{ACBD\} = -1$.

Definitions.

(1) Any set of collinear points are said to form a range.

(2) Any set of concurrent lines are said to form a pencil: the lines are called rays of the pencil, and the point of concurrency is called the vertex of the pencil.

(3) If a transversal meets a pencil $O\{ACBD\}$ at A, C, B, D , the ratio $\frac{AC \cdot BD}{AD \cdot BC}$ or $\frac{\sin AOC \cdot \sin BOD}{\sin AOD \cdot \sin BOC}$ is called the **cross ratio** of the pencil $O\{ACBD\}$ or the cross ratio of the range $\{ACBD\}$.

(4) If two ranges or pencils are of equal cross ratio, they are said to be equi-cross. To remember the ratio $\{ACBD\}$, place the four points in order on a circle: the numerator is obtained by

starting at A and reading forwards and the denominator by starting at A and reading backwards.

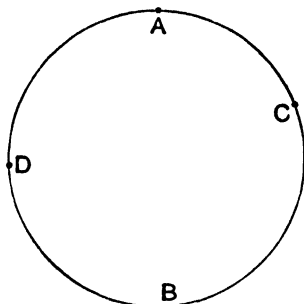


FIG. 58.

48. Prove that the value of the cross ratio $\{ACBD\}$ is unaltered if when any two letters are interchanged, the other two letters are also interchanged.

[E.g. $\{ACBD\} = \{CADB\}$.]

49. If $\{ACBD\} = l$, prove that (1) $\{ABCD\} = 1 - l$; (2) $\{ABDC\} = \frac{l-1}{l}$; (3) $\{ACDB\} = \frac{l}{l-1}$; (4) $\{ADBC\} = \frac{1}{l}$; (5) $\{ADCB\} = \frac{l}{1-l}$.

[The relation in Ex. 6 in chapter V. will be of use.]

50. Prove that there are 24 cross ratios of four collinear points and that they fall into sets of four, each of which is equal to one of the six ratios in Ex. 49.

51. Prove that the six different cross ratios of four collinear points may be written as $l, m, n, \frac{1}{l}, \frac{1}{m}, \frac{1}{n}$ where $l + \frac{1}{m} = m + \frac{1}{n} = n + \frac{1}{l} = -lmn = 1$.

52. Prove that $\frac{AB}{CB} = \{ABC\infty\}$ where ∞ represents a point on AC infinitely distant from A .

53. A variable line meets four fixed planes which have a common line of intersection at $PQRS$, prove that $\{PQRS\}$ is constant.

54. $\{ABCD\}$, $\{AB'C'D'\}$ are two equicross ranges on different lines AD , AD' , prove that BB' , CC' , DD' are concurrent.

55. $O\{ABCD\}$, $O'\{ABCD\}$ are two equicross pencils, if O' lies on OA , prove that B , C , D are collinear.

56. A pencil $O\{ABCD\}$ is cut by a transversal at A , B , C , D ; through B is drawn a line FBG parallel to OD to meet OA , OC at F , G , prove that $\{ABCD\} = \frac{FB}{GB}$. [Use Ex. 52.]

***THEOREM 53.**

If $\{ACBD\} = \{ADBC\}$, then the range $\{ACBD\}$ is harmonic.*

$$\frac{AC \cdot BD}{AD \cdot BC} = \frac{AD \cdot BC}{AC \cdot BD};$$

$$\therefore \frac{AC^2}{CB^2} = \frac{AD^2}{DB^2}; \quad \therefore \frac{AC}{CB} = \pm \frac{AD}{DB}.$$

Here it is necessary to take the negative sign; for if $\frac{AC}{CB} = +\frac{AD}{DB}$, the points C and D coincide.

[The proof is left to the reader, see Ex. 57.]

$$\therefore \frac{AC}{CB} = -\frac{AD}{DB};$$

$\therefore \{ACBD\}$ is harmonic.

Corollary.

If $O\{ACBD\} = O\{ADBC\}$, the pencil $O\{ACBD\}$ is harmonic.

57. If $ABCD$ are collinear, and if $\frac{AC}{CB} = \frac{AD}{DB}$, then C, D coincide.

There are three methods

(α) if $\frac{AC}{CB} = \frac{AD}{DB}$, prove that $\frac{AC}{AB} = \frac{AD}{AB}$.

(β) if $\frac{AC}{CB} = \frac{AD}{DB}$, and if $AC < AD$, then $CB < DB$.

(γ) Draw any other line AX and divide it at P so that

$$\frac{AC}{CB} = \frac{AD}{DB} = \frac{AP}{PX}$$

and show that PC, PD are both parallel to BX .

THEOREM 54.

If $\{ACBD\}$ and $\{A'C'B'D'\}$ are two harmonic ranges, and if AA', CC', BB' are concurrent (at O , say), then DD' also passes through O .

Join OD' and produce it to meet AB at E .

Since $O\{A'C'B'D'\}$ is harmonic, $\{ACBE\}$ is harmonic;

$$\therefore \frac{AC}{CB} = \frac{AE}{BE}.$$

* Theorems similar to Th. 54, Th. 55 hold for equicross ranges and pencil and are of frequent use in rider-work. They are therefore stated below, but their proofs are so similar to those of Th. 54, 55 that they are left as exercises for the reader. Figures should be drawn with the pencils in several different positions, e.g. with O, O' on different sides of AB , or with AD and $A'D'$ intersecting each other, etc., so that familiarity may be obtained in recognising the presence of ranges related to each other in this way.

But $\frac{AC}{CB} = \frac{AD}{BD}$, since $\{ACBD\}$ is harmonic ;

$$\therefore \frac{AE}{BE} = \frac{AD}{BD};$$

$\therefore E$ coincides with D ;

[see Ex. 57]

$\therefore OD'$ passes through D .

Q.E.D.

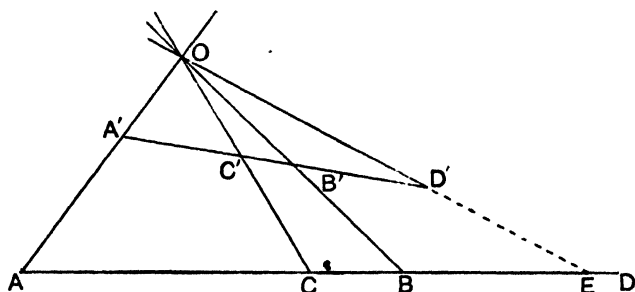


FIG. 59.

Corollary.

If $\{ACBD\}$, $\{AC'B'D'\}$ are harmonic ranges on two lines AD , AD' , then CC' , BB' , DD' are concurrent.

This is a most important theorem.

58. Prove the Corollary of Theorem 54.

THEOREM 55.

If two harmonic pencils $O\{ACBD\}$, $O'\{ACBD\}$ are such that A , C , B are collinear, then D also lies on ACB .

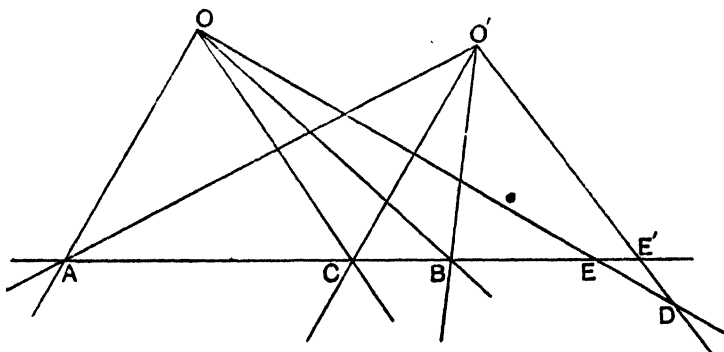


FIG. 60.

Produce ACB to meet OD , $O'D$ at E , E' .

Since $O\{ACBD\}$ is harmonic, $\{ACBE\}$ is harmonic ;

$$\therefore \frac{AC}{CB} = \frac{AE}{BE}.$$

Since $O'\{ACBD\}$ is harmonic, $\{ACBE'\}$ is harmonic ;

$$\therefore \frac{AC}{CB} = \frac{AE'}{BE'};$$

$$\therefore \frac{AE}{BE} = \frac{AE'}{BE'};$$

$\therefore E$ coincides with E' ; (see Ex. 57)

$\therefore ACB$ meets OD , $O'D$ at the same point, and

$\therefore$ at their point of intersection D ;

$\therefore A, C, B, D$ are collinear.

Q.E.D.

Corollary.

If two harmonic pencils $O\{A'CBD\}$, $O'\{ACBD\}$ are such that O' lies on one of the rays of the first pencil, e.g. OA , then C, B, D are collinear.

This is a most important theorem.

59. Prove the Corollary of Theorem 55.

THEOREM 56.

$\{ABCD\}$, $\{A'B'C'D'\}$ are two equicross ranges on different lines AD , AD' , then BB' , CC' , DD' are concurrent.

THEOREM 57.

$\{ABCD\}$, $\{A'B'C'D'\}$ are two equicross ranges so placed that AA' , BB' , CC' intersect at a point O , then DD' passes through O .

THEOREM 58 (1).

$O\{ABCD\}$, $O'\{ABCD\}$ are two equicross pencils; if O' lies on OA , then B, C, L are collinear.

THEOREM 58 (2).

$O\{ABCD\}$, $O'\{ABCD\}$ are two equicross pencils so placed that ABC are collinear, then D also lies on ABC .

It is supposed in the above that OA , $O'A$ are corresponding rays of the two pencils, i.e. $O\{ABCD\} = O'\{ABCD\}$.

60. If $\{APQR\}$, $\{AP'Q'R'\}$ are two harmonic ranges, prove that PR , $P'R$, QQ' are concurrent.

61. In the triangle ABC , the internal and external bisectors of $\hat{BAC}$ meet BC at L , M ; also the internal and external bisectors of $\hat{BCA}$ meet AB at P , Q ; prove that PL , AC , QM are concurrent: and that PM and QL intersect on AC .

62. If $ACBD$ form a harmonic system of points on a circle, prove that the points of intersection of the following pairs of straight lines are collinear: AC , BD ; AB , CD ; the tangents at A , D ; the tangents at B , C .

63. Through a fixed point A is drawn a variable line meeting two fixed lines OC , OD at P_1 , P_2 ; a point Q is taken on AP_1P_2 such that $\frac{1}{AQ} = \frac{1}{AP_1} + \frac{1}{AP_2}$; prove that the locus of Q is a straight line.

64. Through a fixed point A is drawn a variable line meeting n fixed lines at P_1 , P_2 , ... P_n ; on this line is taken a point Q such that

$$\frac{1}{OQ} = \frac{1}{OP_1} + \frac{1}{OP_2} + \dots + \frac{1}{OP_n};$$

prove that the locus of Q is a straight line.

65. With the usual notation for the triangle ABC , if DE , EF meet AB , BC at F' , D' , prove that FD , $F'D'$ intersect on AC . [Use Ex. 21 (1).]

66. With the usual notation for the triangle ABC , if AO , II_1 , I_2I_3 meet BC , DE , DF at P , Q , R , prove that PQR are collinear.

67. If the internal and external bisectors of $\hat{BAC}$ meet BC at L , M , and if C' is the mid-point of AB , apply Theorem 54 Corollary to the harmonic ranges $\{BLCM\}$, $\{BC'A\infty\}$ to invent a property of concurrency of certain lines, where ∞ denotes a point on AB at an infinite distance from A .

68. With the usual notation for the triangle ABC , $C'D$ meets AC at K , EF meets BC at P ; prove that KP is parallel to AB .

69. With the notation of Ex. 68, if $C'P$ meets AC at Q , prove that DQ is parallel to AB .

70. T is a point on the diameter, BA produced, of a circle. TP is the tangent from T to the circle: N is the foot of the perpendicular from P to AB ; the line joining B to the mid-point of TP cuts PN at Q ; prove that AQ is parallel to TP .

* The following exercises involve cross ratios.

71. Prove Theorem 56.

72. Prove Theorem 57.

73. Prove Theorem 58 (1).

74. Prove Theorem 58 (2).

75. If $\{APQR\} = \{AP'Q'R'\}$, prove that PR' , $P'R$, QQ' are concurrent.

76. PQR is a variable triangle; if each side passes through a fixed point, the three fixed points being collinear, and if P , Q move on fixed lines, prove that the locus of R is a fixed line.

[Take three positions of the triangle, and prove that the three positions of R are collinear, using Th. 58 (1).]

77. PQR is a variable triangle; PR , QR , PQ pass through three fixed points A , B , H ; P , Q move on two fixed lines CD , CE ; if A , B , C are collinear, prove that the locus of R is a straight line. [Use the method of Ex. 76.]

78. $PQRS$ is a quadrilateral; PQ , PS are given in position, SQ , QR , RS each pass through one of three fixed collinear points, prove that the locus of R is a straight line.

79. PQR is a variable triangle with its vertices on three fixed concurrent lines: PQ , QR pass through fixed points; prove that RP also passes through a fixed point. [Take three positions of PQR and prove that the three positions of PR are concurrent.]

80. $ABCD$ are four fixed points; l is a fixed straight line: any point P is taken on CD ; AP , BP meet l at a , b ; Ca , Db , intersect at Q ; prove that as P traces out CD , Q will trace out a straight line through the point of intersection of AB and l .

81. Ox , Oy are two fixed lines; A , B are two fixed points; P is a variable point on Ox ; PA meets Oy at Q , QB meets Ox at R ; AR meets Oy at S ; prove that PS passes through a fixed point.

82. (Desargues' theorem on perspective triangles). Two triangles ABC , $A'B'C'$ are such that AA' , BB' , CC' intersect at a point O , if BC , $B'C'$ meet at X ; CA , $C'A'$ at Y ; AB , $A'B'$ at Z , prove that XYZ are collinear.

[Show that $A\{XYZO\} = A'\{X'Y'Z'O\}$.]

83. [Desargues' theorem on the quadrangle]. Six lines are drawn joining in pairs the points $ABCD$. Any other straight line meets AB , CD , AD , BC , AC , BD at P , P' , Q , Q' , R , R' , prove that

$$\{PQRR'\} = \{P'Q'R'R\}.$$

[Project the ranges from A , C on to BD .]

84. $ABCD$ is a quadrilateral; AB , CD meet at G ; AD , BC at F ; BD , AC at E . AC is produced to meet FG at P .

Prove that $\{AEC P\} = \{CEAP\}$, and hence that each is harmonic.

[Project the ranges from F , G on to BD .]

85. $\{AB; CD\}$ is a harmonic range; O is the centre of the circle on AB as diameter; a variable tangent to this circle meets lines through C , D perpendicular to CD at H , K ; prove that $\frac{OH}{OK}$ is constant.

CHAPTER IX.

THE QUADRILATERAL AND QUADRANGLE.

THIS chapter is intended to be an introduction to the **Principle of Duality**, which may be most simply illustrated by the relationship of the quadrilateral to the quadrangle. It is essentially a modern idea, to which attention was first drawn in a definite form by Poncelet (1788–1867). His famous treatise on projective geometry was written while he was held a prisoner by the Russians who captured him during the retreat from Moscow. To some extent, however, he was anticipated by Brianchon, another French soldier, who published in 1806 the theorem with which his name is associated and which was obtained by a method of the same kind. Theorem 60 was in substance included by Pappus in his *Συναγωγή*.

Definitions.

- (1) If A and B are two points, the unlimited line joining A and B is called the **join** of A and B .
- (2) If a and b are two lines, the point of intersection of a and b is called the **meet** of a and b .

In the first instance it is usual to regard a curve as consisting of a series of points each of which is so situated as to obey some characteristic law. For example, a circle may be considered as formed by a series of points each of which is at a constant distance from a fixed point. In order to pass from the series of points to the continuous curve, it is necessary to imagine that a number of lines are drawn connecting consecutive points of the series: by taking a large enough number of these points, it is possible to make each connecting line as short as we please:

and the limiting result of this process is said to constitute the curve.

But we may equally well consider a circle as generated by a series of straight lines each of which is at a constant distance from a fixed point. On each line is cut off a segment by the two lines which come, one before and the other after it, in the series. By taking a large enough number of these lines, it is possible to make each segment as short as we please: and in the limit, these segments are said to constitute the curve.

In the first case, the curve is defined as the locus of a series of points: and in the second case as the envelope of a series of lines. Thus the part played by a moving (tracing) point in the first case corresponds to that played by a moving (tangent) line in the second case.

In the first case the point is regarded as the fundamental conception, and the straight line arises as the join of two points: while in the second, the straight line is regarded as the fundamental conception, and the point arises as the meet of two straight lines. If then we have any geometrical system built up on one of these conceptions, it is possible to build up the corresponding geometrical system which is based on the other.

In what follows, it is convenient to take single capital letters to represent points and single small letters to represent lines.

Correspondence will be represented in parallel columns.

AB is the join of two points A, B ; $\{ABCD\}$ denotes the range formed by four collinear points A, B, C, D .	ab is the meet of the two lines a, b ; $\{abcd\}$ denotes the pencil formed by four concurrent lines a, b, c, d .
---	---

$H\{ABCD\}$ denotes the pencil, whose vertex is H and whose rays pass through four points A, B, C, D .	$h\{abcd\}$ denotes the range intercepted on the line h (the base of the range) by the four lines a, b, c, d .
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Hitherto a triangle has been regarded indiscriminately as a figure formed by the joins of three points or the meets of three lines: it is convenient to distinguish by different names these two standpoints.

A figure formed by the joins of three points is called a triangle.	A figure formed by the meets of three lines is called a trilateral.
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This correspondence is exhibited very plainly in theorems which can be stated without reference to the measurement of lines or

angles. It will be found that all such theorems occur in pairs, so that the knowledge of one includes the knowledge of the other. This correspondence is usually spoken of as **The Principle of Duality**.

The following theorems will illustrate the application of this principle :

the joins of n points, no three of which are collinear, taken in pairs, form $\frac{1}{2}n(n-1)$ lines.

$A, B, C; A', B', C'$ are the vertices of two triangles; if the joins of $A, A'; B, B'; C, C'$ are concurrent, then the meets of $BC, B'C'; CA, C'A'; AB, A'B'$ are collinear.

The sides BC, CA, AB of a variable triangle ABC pass through three fixed collinear points P, Q, R respectively; if A, B lie on two fixed lines, then the locus of C is a straight line.

[*I.e.* C lies on a fixed straight line.]

If $\{ABCD\}, \{A'B'C'D'\}$ are two harmonic ranges on different bases x, y ; then the joins of $BB'; CC'; DD'$ are concurrent.

A curve which is met by any straight line at two and only two points is called a **curve of the second degree**.

the meets of n lines, no three of which are concurrent, taken in pairs, form $\frac{1}{2}n(n-1)$ points.

$a, b, c; a', b', c'$ are the sides of two trilaterals; if the meets of $a, a'; b, b'; c, c'$ are collinear, then the joins of $bc, b'c'; ca, c'a'; ab, a'b'$ are concurrent.

The vertices bc, ca, ab of a variable trilateral abc lie on three fixed concurrent lines p, q, r respectively; if a, b pass through two fixed points, then the envelope of c is a point.

[*I.e.* c passes through a fixed point.]

If $\{abcd\}, \{ab'c'd'\}$ are two harmonic pencils with different vertices X, Y ; then the meets of $bb'; cc'; dd'$ are collinear.

A curve to which from any point two and only two tangents can be drawn is called a **curve of the second class**.

Write down without proof the theorems which correspond, in virtue of the principle of duality, to the following: and draw figures for the original and the dual theorem exhibiting this correspondence. [Compare Figs. 61, 62.]

1. Chapter VIII., Ex. 79.

2. $A_1, A_2, \dots, A_n$ are n points in a plane which move on fixed concurrent lines; if $A_1A_2, A_2A_3, \dots, A_{n-1}A_n$, pass through fixed points, then every other join of a pair of the n points passes through a fixed point.

3. From a fixed point D two variable lines DPQ , DRS are drawn to cut two fixed lines AB , AC at P , Q and R , S ; then the locus of the meet of PS , QR is a straight line.

4. $\{ABCD\}$, $\{A'B'C'D'\}$ are two harmonic ranges on different bases x , y ; if the joins of A , A' ; B , B' ; C , C' concur at O , then DD' passes through O .

5. If $\{ABCD\}$, $\{AB'C'D'\}$ are two harmonic ranges on different bases x , y , then the joins of BD' , $B'D$, CC' are concurrent.

6. Two lines FDA , FCB are met by two other lines GBA , GCD ; BD meets AC at E ; then $F\{AB; EG\}$ is a harmonic pencil.

7. One and only one curve of the second degree can be drawn to pass through five given points.

8. $A_1A_2A_3A_4A_5A_6$ are six points on a curve of the second class, then the meets of A_1A_2 , A_4A_5 ; A_2A_3 , A_5A_6 ; A_3A_4 , A_6A_1 are collinear.

9. A variable curve of the second degree passes through four fixed points A , B , C , D ; if two fixed lines AX , BY cut the curve at P , Q ; then PQ passes through a fixed point.

10. The sides of a variable triangle pass through three non-collinear fixed points; if two of the vertices lie on fixed lines, then the locus of the third vertex is a curve of the second degree.

We shall now proceed to work out in detail the dual properties of the figure generated by four straight lines (the quadrilateral) and the figure generated by four points (the quadrangle).

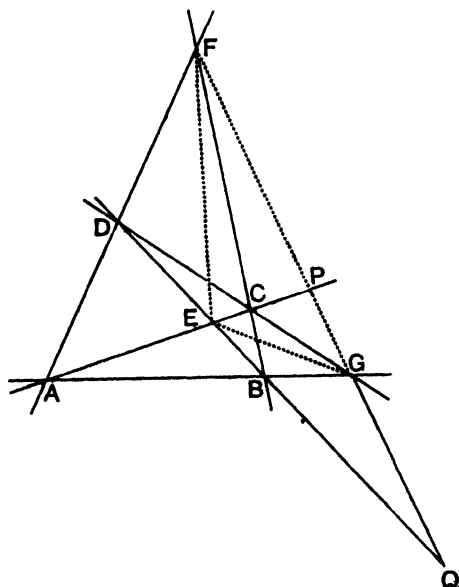


FIG. 61.

A **complete quadrangle** is formed by four points A, B, C, D , called its vertices, no three of which are collinear; their joins form six straight lines, grouped in pairs,

$$AB, CD; BC, AD; CA, BD,$$

called its sides: the sides in each pair being called opposite sides.

The meets of opposite sides form three points

$$G, F, E$$

called diagonal points,

and are the vertices of a triangle called the diagonal point triangle.

If A, B, C, D lie on a circle, the circle is called the circumscribing circle of the quadrangle.

* Now we have

$$\begin{aligned} \{AECF\} &= F\{AECF\} \\ &= \{DEBQ\} \\ &= G\{DEBQ\} \\ &= \{CEAP\}. \end{aligned}$$

$\therefore$ by Theorem 53

$\{AECF\}$ is a harmonic range.

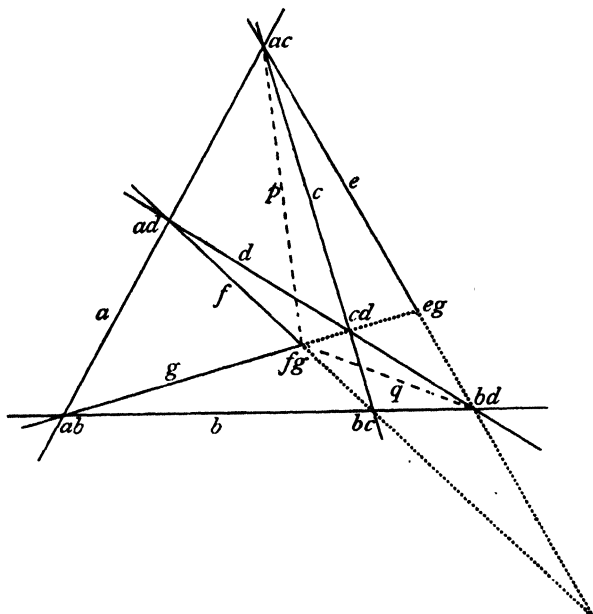


FIG. 62.

A **complete quadrilateral** is formed by four lines a, b, c, d , called its sides, no three of which are concurrent; their meets form six points, grouped in pairs,

$$ab, cd; bc, ad; ca, bd,$$

called its vertices: the vertices in each pair being called opposite vertices.

The joins of opposite vertices form three lines

$$g, f, e$$

called diagonal lines,

and are the sides of a triangle called the diagonal line triangle.

If a, b, c, d envelope a circle, the circle is called the inscribed circle of the quadrilateral.

* Now we have

$$\begin{aligned} \{aecp\} &= f\{aecp\} \\ &= \{debq\} \\ &= g\{debq\} \\ &= \{ceap\}. \end{aligned}$$

$\therefore$ by Theorem 53, Corollary,

$\{aecp\}$ is a harmonic pencil.

The harmonic property of the quadrangle and quadrilateral may be established in a simpler manner.

THEOREM 59.

$ABCD$ is a quadrilateral: AD, BC meet at F ; AB, CD meet at G ; AC, BD meet at E ; FE meets AB at K ; then $\{AKBG\}$ is harmonic.

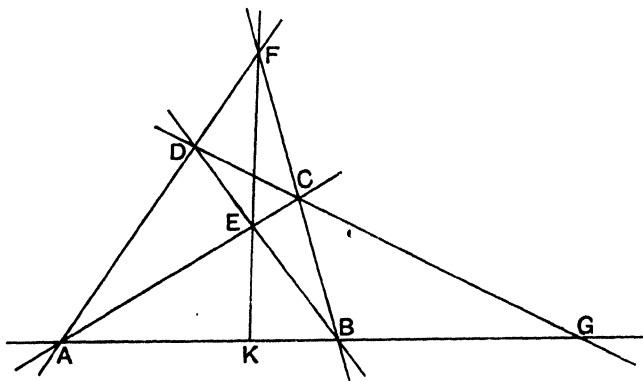


FIG. 63.

From the triangle FAB , since FK, AC, BD concur,

$$\frac{AK}{KB} \cdot \frac{BC}{CF} \cdot \frac{FD}{DA} = 1, \quad \text{by Ceva.}$$

Since DCG is a transversal of the same triangle,

$$\frac{AG}{GB} \cdot \frac{BC}{CF} \cdot \frac{FD}{DA} = -1, \quad \text{by Menelaus;}$$

$$\therefore \frac{AK}{KB} \cdot \frac{BC}{CF} \cdot \frac{FD}{DA} = -\frac{AG}{GB} \cdot \frac{BC}{CF} \cdot \frac{FD}{DA};$$

$$\therefore \frac{AK}{KB} = -\frac{AG}{GB};$$

$\therefore \{AKBG\}$ is harmonic.

Q.E.D.

THEOREM 60.

If in the figure of the complete quadrangle (Fig. 61), FE is produced to meet DC, AB at H, K and GE is produced to meet AD, CB at L, M , then every set of four collinear points forms a harmonic range and every set of four concurrent lines forms a harmonic pencil.

The pencil $F\{AKBG\}$ intercepts on the transversals AP , LG , DQ , DG , the ranges $\{AECP\}$, $\{LEMG\}$, $\{DEBQ\}$, $\{DHCG\}$;
 $\therefore$ by Theorem 59, these four ranges are harmonic;
 $\therefore$ the pencil $G\{AECP\}$ is harmonic.

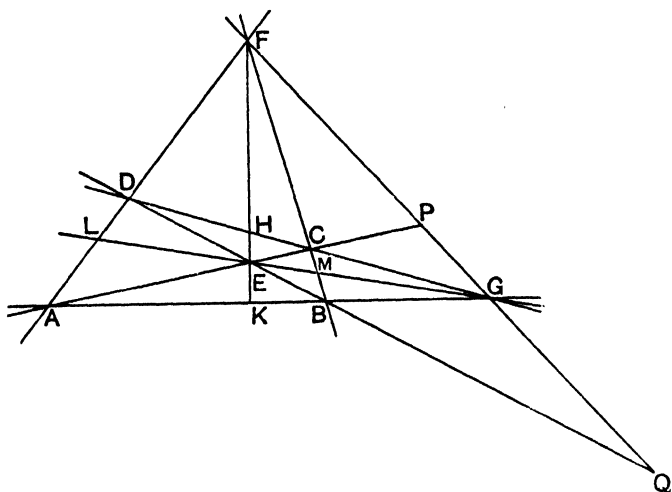


FIG 64.

But it intercepts on the transversals FA , FK , FB the ranges $\{ALDF\}$, $\{KEHF\}$, $\{BMCF\}$ which are therefore harmonic.

Also $E\{BMCF\}$ intercepts on QF the range $\{QGPF\}$ which is therefore harmonic.

Hence each of the nine lines in the figure is cut harmonically. Therefore each of the sets (three in number through E , F , G) of four concurrent lines form harmonic pencils, for each has a transversal which is cut harmonically.

Q.E.D.

11. State the dual theorem of Theorem 60, draw the corresponding figure and work through the proof.

12. If in Fig. 63 AC meets FG at P , use the method of Theorem 59 to prove that $\{AECP\}$ is harmonic.

13. Given three collinear points A , B , C , by using a ruler only, show how to find a point D such that $\{ABCD\}$ is harmonic.

14. Given three concurrent lines PA , PB , PC , by using a ruler only, show how to construct a line PD such that $P\{ABCD\}$ is harmonic.

15. K is the given mid-point of the line joining two given points A , B ; O is a given point outside AB ; draw through O a line parallel to AB , using a ruler only.

16. Given three points ABC on a given circle, determine a point D on the circle such that $ABCD$ may form a harmonic system of points on the circle, using a ruler only.

17. Given a parallelogram, a point A and a straight line l , draw a line through A parallel to l , using a ruler only.

18. The straight lines AD , BC intersect at C ; on BC any point E is taken; AE meets BD at Q ; CQ meets ED at R ; BR cuts AD at P . Prove that the position of P does not vary as E moves on the line BC .

19. With the notation of Theorem 60, Fig. 64, prove that
(1) PD , GE , AQ ; (2) QC , EG , PB
are concurrent.

20. With the notation of Theorem 60, Fig. 64, prove that (1) L , H , P ; (2) K , M , P are collinear.

21. Determine the dual properties which correspond to the properties of Ex. 19, 20, and draw two figures to illustrate the complete correspondence as in Fig. 61, 62.

22. A point C is taken on the line of intersection AB of two planes and points X and X' are taken, one in each plane. A line CHK meets AX , BX at H , K ; a line $CH'K'$ meets AX' , BX' at H' , K' ; AK meets BH at Y ; AK' meets BH' at Y' ; prove that the points $XX'YY'$ lie in one plane.

* THEOREM 61.

The mid-points of the three diagonals of a complete quadrilateral are collinear.

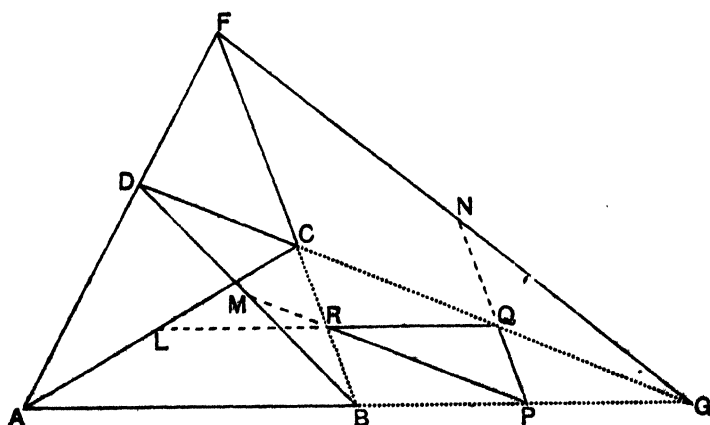


FIG. 65.

* This method of proof was given by Mr. C. E. Hillyer in the *Educational Times*.

AC , BD , FG are the diagonals of the complete quadrilateral $ABCD$.

Let P , Q , R be the mid-points of BG , GC , CB ; and produce PQ , QR , RP to meet FG , AC , BD at N , L , M .

Since P , Q , R are the mid-points of the sides of the triangle BGC , PQ , QR , RP are parallel to CB , BG , GC .

Since then PQN is parallel to BF , and since $GP = PB$, it follows that $GN = NF$, so that N is the mid-point of FG .

Similarly, L , M are the mid-points of AC , BD .

Now by parallels $\frac{RL}{LQ} = \frac{BA}{AG}$; $\frac{QN}{NP} = \frac{CF}{FB}$; $\frac{PM}{MR} = \frac{GD}{DC}$;

$$\therefore \frac{RL}{LQ} \cdot \frac{QN}{NP} \cdot \frac{PM}{MR} = \frac{BA}{AG} \cdot \frac{GD}{DC} \cdot \frac{CF}{FB}$$

= -1 by Menelaus,

since ADF is a transversal of the triangle BCG ;

$\therefore$ by the converse of Menelaus LMN is a transversal of the triangle PQR ;

$\therefore$ the mid-points L , M , N of AC , BD , FG are collinear.

Q.E.D.

Theorem 61 admits of an interesting statical proof.

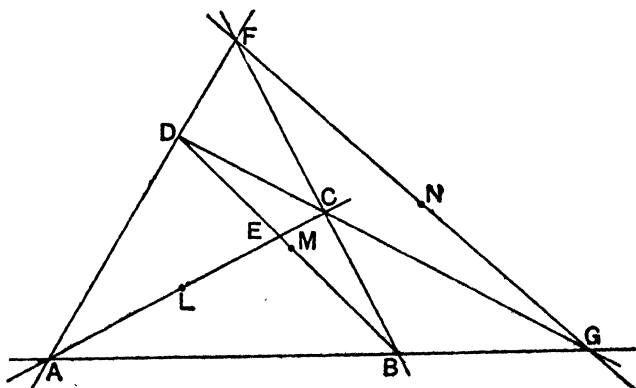


FIG. 66.

L , M , N are the mid-points of AC , BD , FG .

Consider the system of forces $\overline{AD}$, $\overline{AB}$, $\overline{CD}$, $\overline{CB}$.

$\overline{AD}$, $\overline{AB}$ are equivalent to $2\overline{AM}$.

$\overline{CD}$, $\overline{CB}$ are equivalent to $2\overline{CM}$.

$\therefore$ the complete system is equivalent to $2\overline{AM}$, $2\overline{CM}$ or to $4\overline{LM}$.

But the system is also equivalent to

$$\overline{AF}, \overline{FD}; \overline{AG}, \overline{GB}; \overline{CG}, \overline{GD}; \overline{CF}, \overline{FB},$$

or to $\overline{AF}, \overline{AG}; \overline{FD}, \overline{GD}; \overline{FB}, \overline{GB}; \overline{CG}, \overline{CF},$

or to $2\overline{AN}; 2\overline{ND}; 2\overline{NB}; 2\overline{CN};$

$\therefore$ the resultant passes through N ;

$\therefore L, M, N$ are collinear.

Q.E.D.

This method fails if the forces are in equilibrium, in which case $\overline{LM}$ is zero so that the diagonals bisect each other; the quadrilateral is then a parallelogram and FG is at infinity.

If a circle can be inscribed in the quadrilateral its centre must lie on LMN .

For suppose the circle touches AB, BC, CD, DA at P, Q, R, S ; the force-system is then equivalent to $\overline{AP}, \overline{AS}; \overline{PB}, \overline{QB}; \overline{CR}, \overline{CQ}; \overline{RD}, \overline{SD}$; the resultant of each pair bisects the corresponding chord of contact and therefore passes through the centre of the circle; therefore the final resultant passes through the centre of the circle. The reader who is acquainted with the corresponding property of the conic will see that this method yields an immediate proof of the theorem that "The locus of the centres of conics touching four straight lines is a straight line."

Statical methods should be used in Ex. 23-28.

23. Through a point F inside a rectangle $ABCD$ lines are drawn parallel to the sides to meet AB in G, BC in E, CD in K, DA in H . Prove that BH, CF, DG are concurrent.

24. $ABCD$ is a quadrilateral. If $AB=CD$, prove that the line joining the mid-points of BC and AD is equally inclined to the other two sides.

25. On the sides of a quadrilateral four points are taken such that each forms with the extremities of the opposite side a triangle whose area is half that of the quadrilateral. Prove that these points lie on the line joining the mid-points of the diagonals.

26. A parallelogram $ABCD$ is divided into four parallelograms by two lines PKQ, RKS , meeting DC, AB at P, Q and DA, CB at R, S . Prove that the triangle AKC is equal to half the difference of the parallelograms KB, DK .

27. $ABCD$ is a quadrilateral in which AD and BC are parallel; find the magnitude and direction of the resultant of the forces

$$\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}, \overline{BD}, \overline{CA}.$$

28. Prove that the three pairs of bisectors of the opposite angles of a complete quadrilateral intersect at three collinear points.

29. O is any point inside the triangle ABC ; AO, BO, CO meet the opposite sides at D, E, F ; P, Q, R are harmonic conjugates of D, E, F w.r.t. B, C ; C, A ; A, B ; prove that the mid-points of AP, BQ, CR are collinear.

30. If two of the opposite angles of a complete quadrilateral are right angles, prove that the third diagonal is perpendicular to one of the internal diagonals.

31. $ABCD$ is a quadrilateral, UV is the projection of BD on AC ; prove that $AB^2 + CD^2 - BC^2 - AD^2 = 2AC \cdot UV$.

32. With the notation of Fig. 66, if $ABCD$ is inscribed in a circle, prove that $FG^2 = FR^2 + GS^2$ where FR, GS are the tangents from F, G to the circle.

33. $ABCD$ is a quadrilateral having AB parallel to CD ; AC, BD meet at P ; AD, BC meet at Q ; prove that PQ bisects AB and CD .

34. The base BC of a triangle is given, and the vertex A moves on a given straight line. A given line cuts AB, AC at H, K ; find the locus of the meet of CH, BK .

35. If in Fig. 63,

$$(1) FC \cdot FB + GC \cdot GD = FG^2 \text{ or } (2) FA \cdot FD + GA \cdot GB = FG^2,$$

prove that $ABCD$ is a cyclic quadrilateral.

CHAPTER X.

ORTHOGONAL CIRCLES.

Definitions.

(1) If two curves intersect at a point A , and if the tangents at A to the curves make an angle α with each other, the two curves are said to cut each other at A at an angle α .

In particular, if α is a right angle, the curves are said to cut each other orthogonally or at right angles; and the curves are called orthogonal curves.

(2) If CA is a radius of a circle, centre C , and if P, Q are two points on CA such that $CP \cdot CQ = CA^2$, then P, Q are called **inverse points** with respect to the circle.

1. If two circles intersect at A and B , prove that the angles at which they cut each other at A and B are equal.

2. If P, Q are inverse points w.r.t. a circle, and if PQ meets the circle at A, B , prove that $\{AB, PQ\}$ is harmonic.

3. Two circles, centres A, B , intersect orthogonally at C ; prove that CA, CB are the tangents at C to the two circles.

4. Two circles, centres A, B , radii a, b , cut each other orthogonally; prove that $AB^2 = a^2 + b^2$.

5. If a, b are the radii of two circles, centres A, B , and if $AB^2 = a^2 + b^2$, prove that the circles cut orthogonally.

6. P is a point on a circle, centre C ; AB is a diameter; prove that the circles APC, BPC intersect orthogonally.

7. C is a common point of two circles; A is a centre of one; if AC touches the other, prove that the circles are orthogonal.

8. PQ is a common tangent to two circles, P, Q being the points of contact; prove that the circle on PQ as diameter is orthogonal to each of them.

9. $\{ACBD\}$ is a harmonic range; prove that the circles on AB, CD as diameters are orthogonal.

10. Two circles, centres A, B , intersect at C ; prove that the angle at which they cut each other is supplementary to the angle ACB .

11. A circle Σ cuts a circle C_1 at P, Q and a circle C_2 at R, S ; PQ meets RS at H ; prove that it is possible to draw a circle with H as centre to cut S, C_1, C_2 orthogonally, C_1, C_2 being non-intersecting circles.

12. If a straight line cuts a circle orthogonally, prove that it passes through the centre of the circle.

13. $\{ACBD\}$ is a harmonic range; prove that any circle through C, D cuts the circle on AB as diameter orthogonally.

14. Two circles, centres A, B , intersect at C, D ; if $ACBD$ is a cyclic quadrilateral, prove that the circles are orthogonal.

15. P, Q are two points on the major arcs CD of two circles which intersect at C, D ; if $\hat{CPD} + \hat{CQD} = 90^\circ$, prove that the circles cut orthogonally. [Use Ex. 14.]

What is the test if P, Q lie on the minor arcs CD ?

16. Prove the converse of Ex. 15.

17. From a point P outside a circle, centre C , tangents are drawn to the circle; if the chord of contact meets CP at Q , prove that P, Q are inverse points w.r.t. the circle.

18. a, b, c are the given lengths of the sides of the triangle ABC ; circles, centres A, B, C , radii x, y, z , are described so that each is orthogonal to the other two. Calculate x, y, z .

19. If P, Q are inverse points w.r.t. a circle Σ , prove that any circle through P, Q is orthogonal to Σ .

20. If two circles are orthogonal, prove that any diameter of one is cut harmonically by the other circle.

THEOREM 62.

If two circles, centres A and B , intersect orthogonally at C , then (1) CA, CB are tangents to the circles.

$$(2) CA^2 + CB^2 = AB^2.$$

(1) Draw the tangents CP, CQ to the two circles.

Then $\hat{QCP} = 90^\circ$ given;

$\therefore CQ$ is perpendicular to the tangent CP to the circle centre A ;

$\therefore CQ$ passes through A ,

and similarly CP passes through B ;

$\therefore AQC, BPC$ are tangents to the two circles. Q.E.D.

- (2) Since by (1) $\hat{ACB} = 90^\circ$,
 $AB^2 = AC^2 + CB^2$.

Q.E.D.

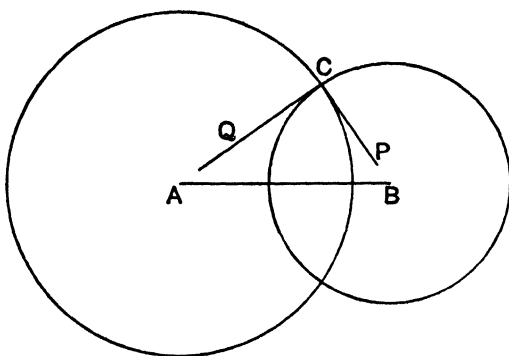


FIG. 67.

Corollary.

Conversely, if two circles, centres A, B , intersect at C , and if *either* $\hat{ACB} = 90^\circ$ *or* $AB^2 = AC^2 + CB^2$ *or* if AC is a tangent to the circle, centre B , then the two circles intersect orthogonally.

This Corollary is most important.

21. Prove the Corollary of Theorem 62.
22. If two circles are orthogonal, prove that the rectangle contained by their common chord and the line joining their centres is equal to twice the rectangle contained by their radii.
23. If two orthogonal circles are equal, calculate the ratio of the area common to both of them to the area of either.
24. The straight line joining the centres O, O' of two orthogonal circles, meets the first in A, B and the second in C, D ; prove that $DA \cdot DB = DC \cdot DO$.
25. Prove that, with the usual notation for the triangle ABC , the three circles whose centres are A, B, C and the squares of whose radii are $AH \cdot AD, BH \cdot BE, CH \cdot CF$ cut orthogonally.
26. A line QPR is drawn through the point of intersection P of two circles, and meets them again at Q, R ; with Q, R as centres two circles are described orthogonal to the given circles, prove that they intersect on the circle on QR as diameter.
27. Prove that the locus of the centres of circles which pass through a given point and cut a given circle orthogonally is a straight line.
28. Prove that the circle, whose diameter is the third diagonal of a quadrilateral inscribed in a circle, is orthogonal to that circle. [Use Ex. 32 in Chapter IX.]

THEOREM 63.

Two circles intersect at C, D ; if the sum of the angles in the segments of the two circles on opposite sides of CD is 90° or 270° , then the circles cut orthogonally.

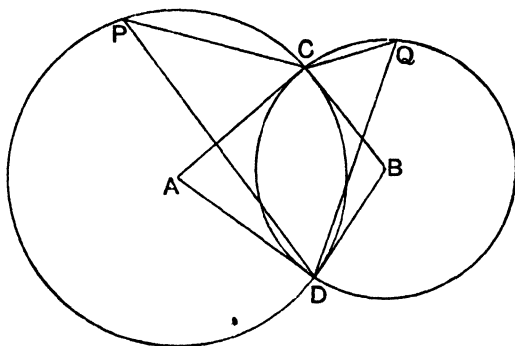


FIG. 68.

A, B are the centres of the circles; P, Q are points on segments of the circles on opposite sides of CD .

In the case of the figure $\hat{CPD} + \hat{CQD} = 90^\circ$.

Now $\hat{CAD} = 2\hat{CPD}$ and $\hat{CBD} = 2\hat{CQD}$;

$$\therefore \hat{CAD} + \hat{CBD} = 2(\hat{CPD} + \hat{CQD}) = 180^\circ;$$

$$\therefore \hat{ACB} + \hat{ADB} = 180^\circ,$$

since the angles of a quadrilateral add up to four right angles.

But $\hat{ACB} = \hat{ADB}$,

$$\therefore \hat{ACB} = 90^\circ;$$

$\therefore$ the circles cut orthogonally.

Q.E.D.

If P, Q had been taken on the minor arcs CD , we should have the case $\hat{CPD} + \hat{CQD} = 270^\circ$.

29. Give the proof of Theorem 63 for the case $\hat{CPD} + \hat{CQD} = 270^\circ$.

30. Use Theorem 63 to prove Ex. 6.

31. H is the orthocentre of the triangle ABC , prove that the circles on AH and BC as diameters are orthogonal.

32. AB is the diameter of a given circle: C is a point off the circle; AC, BC meet the circle again at P, Q ; prove that the circle CPQ is orthogonal to the given circle.

33. TP , TQ are the tangents from a point T to a circle; R is any point such that $TR=TP$. RP , RQ meet the circle again at A , B , prove that AB is a diameter.

34. A point P is taken on the base BC produced of a triangle ABC so that $\hat{APB} = \hat{ACB} - \hat{ABC}$: AD is the internal bisector of the angle BAC and meets BC at D , prove that the circle, centre P , radius PD , is orthogonal to the circumcircle of the triangle ABC .

35. With the usual notation, prove that the circle IBC is orthogonal to the circle on I_2I_3 as diameter.

36. The diagonals of a quadrilateral $ABCD$ meet at O , if the circumcircles of the triangles AOB , COD are orthogonal to each of the circles AOD , BOC , prove that the quadrilateral must be a rectangle.

THEOREM 64.

(1) If $\{ACBD\}$ is harmonic, then the circle on AB as diameter is orthogonal to any circle through C and D .

(2) If C , D are inverse points w.r.t. a circle, any circle through C , D is orthogonal to that circle.

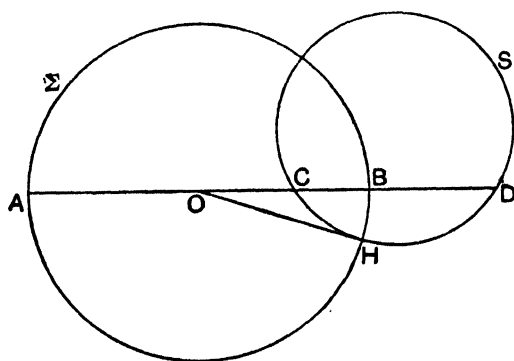


FIG. 69.

(1) Let O be the mid-point of AB ; and let the circle on AB as diameter cut any circle through C , D at H .

Since $\{ACBD\}$ is harmonic, and since $AO = OB$,

$$\begin{aligned}\therefore OC \cdot OD &= OB^2; \\ &= OH^2 \text{ radii};\end{aligned}$$

$\therefore OH$ is a tangent to the circle CHD ;

$\therefore$ the circles cut orthogonally. [Th. 62, Cor.]

(2) If C, D are inverse points

$$\begin{aligned} OC \cdot OD &= OB^2 \text{ by definition} \\ &= OH^2; \end{aligned}$$

$\therefore$ as in (1) the circles cut orthogonally. Q.E.D.

Corollary 1.

If two circles cut orthogonally, then any diameter of one of them is cut harmonically by the other.

Let Σ and S be orthogonal circles (see Fig. 69).

AB , any diameter of Σ , meets S at C, D .

O is the centre of Σ ; Σ cuts S at H .

Since the circles are orthogonal, OH is a tangent;

$$\therefore OC \cdot OD = OH^2 = OB^2 = OA^2 \text{ radii};$$

$\therefore \{ACBD\}$ is harmonic. Q.E.D.

Corollary 2.

If a circle S is orthogonal to a circle Σ and cuts a diameter of Σ at C, D ; then C, D are inverse points w.r.t. Σ .

For from Corollary 1, $OC \cdot OD = OB^2$;

$\therefore C, D$ are inverse points w.r.t. Σ .

37. A fixed diameter AB of a circle S meets a given orthogonal circle Σ at C, D ; P is a variable point on S , prove that $\frac{PC}{PD}$ is constant.

38. Prove that the circumcircle of the triangle formed by the three diagonals of a complete quadrilateral is orthogonal to the three circles whose diameters are the three diagonals.

39. Two circles S_1, S_2 intersect at P, Q ; another circle Σ cuts S_1 and S_2 orthogonally; prove that P, Q are inverse points w.r.t. Σ .

40. Show how to describe a circle to pass through two given points and to be orthogonal to a given circle.

41. Two circles, centres A, B , cut orthogonally at Q, Q' . A line PQR cuts the circles again at P and R ; if L is the mid-point of PR , prove that $\angle ALB = 90^\circ$.

[Use $LQ \cdot LR = LQ \cdot LP$.]

42. Two circles intersect orthogonally at P ; the line joining their centres meets the first circle at A, B ; prove that PA, PB cut the second circle at the ends of a diameter perpendicular to AB .

43. If A, B, C, D are any four points, prove that the circles ABC, ABD cut at the same angle as the circles ACD, BCD .

44. Two circles intersect at O ; POQ , $P'OQ'$ are two lines terminated by the circumferences: prove that PP' , QQ' cut at the same angle as the circles.

45. A , B are two circles which cut a third circle C at equal angles at the points P , Q . The tangent at P to A meets C at R ; the tangent at Q to B meets C at S ; prove that RS is parallel to PQ .

46. If a variable circle is orthogonal to two fixed circles, prove that the locus of its centre is a straight line.

47. From a given point O , three straight lines are drawn cutting another straight line at A , B , C . A circle OBD is drawn orthogonal to the circle OAC and meets ABC again at D ; prove that the angles AOB , COD differ by a right angle.

48. H , K are the inverses of a point P w.r.t. two orthogonal circles, centres A and B . If AK meets BH at L , prove that $PHLK$ is a cyclic quadrilateral.

49. PQ , RS are perpendicular diameters of two orthogonal circles; prove that $PQRS$ form the vertices of a triangle and its orthocentre.

50. S_1 and S_2 are two orthogonal circles, centres O_1 and O_2 ; O_1O_2 meets S_1 at A ; any chord EF of S_2 meets a line through O_1 perpendicular to O_1O_2 at H . Prove that E , F are inverse points w.r.t. the circle, centre H , radius HA .

51. Prove that a line cut harmonically by two circles must be a diameter of one of them.

52. PQ is a variable diameter of a fixed circle; AB is a fixed chord of the circle; prove that AP and BQ intersect on a fixed circle.

53. $\{ACBD\}$ is a harmonic range: a circle S_1 is described on AB as diameter, and S_2 is a circle on CD as diameter in a plane perpendicular to that of S_1 ; P , Q are any two points on S_1 and E , F are any two points on S_2 , prove that $\frac{PE}{PF} = \frac{QE}{QF}$. [Use Ex. 50.]

54. State in general terms the condition that the three circles in Ex. 18 may be real.

CHAPTER XI.

INVERSION.

THE method of Inversion, described in this chapter, is employed to deduce a new theorem or set of theorems from a known property. It appears to be due in the first instance to Quetelet: but it was discovered independently a few years later (1843) by two Dublin University lecturers, Stubbs and Ingram. Since then it has been elaborated by Cayley, Chasles and Roberts. [See Salmon's *Higher Plane Curves*.] And Lord Kelvin has applied it to the solution of certain electrostatical problems.

Many of the properties of a coaxial system of circles can be obtained by using the process of inversion. A discussion of this application is given in Chapter XIII. Since the inverse curve of a conic is in general a curve of degree 3 or 4, inversion may be used to discover properties of higher plane curves from known properties of the conic. The number of useful properties however, which can be thus obtained, is not very considerable; since it will often be found that the new theorem is of so complicated a character, in enunciation, as to be hardly worth possessing.

Definition.

If on the line joining a given point O to a point P , a point P' is taken such that $OP \cdot OP' = k^2$ where k is a given constant, then P' is called the **inverse point** of P with respect to the centre or origin of inversion O ; and k is called the **radius of inversion**.

The circle, centre O , radius k , is called the **circle of inversion**.

It is clear that this definition agrees with the definition of inverse points given in Chapter X.

✓ 1. If P' and Q' are inverse points of P and Q w.r.t. an origin O , prove that $PQP'Q'$ are concyclic.

2. Draw roughly the systems of points given in Figs. 70, 71, 72, and then construct the approximate positions of the inverse points w.r.t. O for any convenient value of k .



FIG. 70.



FIG. 71.



FIG. 72.

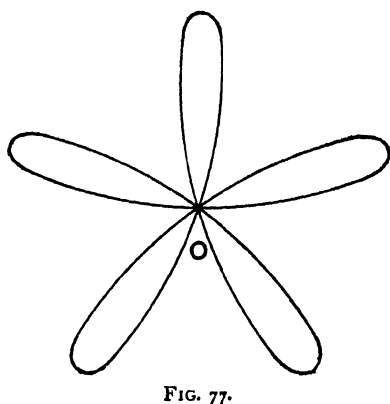
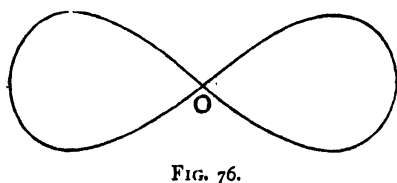
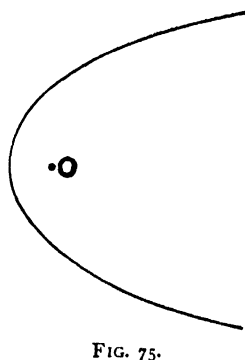
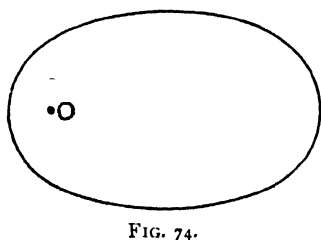
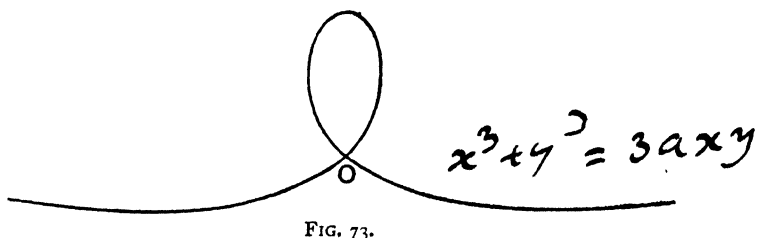
Suppose now that a point P moves along any continuous curve, then it is clear that the point P' , connected with P as in the definition, will trace out another continuous curve which is called the **inverse curve** of the locus of P w.r.t. an origin O and a radius of inversion k . Unless special mention is made of k , it is implied that any constant quantity may be chosen for k . (See Ex. 8.)

3. Draw roughly the curves on the opposite page and then construct the approximate positions of the inverse curves w.r.t. O for any convenient value of k . In Figs. 73, 75, each arm of the curve is supposed to extend to infinity.

✓ 4. If a figure is inverted, which points of the figure will remain unaltered in position?

✓5. A straight line is inverted w.r.t. a point on itself, what is the inverse figure?

6. A given circle is inverted w.r.t. its centre, prove that the inverse figure is a concentric circle : what does the inverse figure become as the given circle shrinks into a point? *str line at inf.*



7. The curve S' is the inverse of the curve S w.r.t. origin O . OA is a tangent to S , prove that it touches S' .

8. If S' and S'' are inverse curves of the curve S w.r.t. the same origin O , but different radii k' , k'' of inversion, prove that S' and S'' are homothetic curves and that O is their centre of similitude.

It is due to this property that the particular value of k to be chosen is *usually* immaterial,

✓9. OT is the tangent from a point O to a circle S ; prove that with origin O and radius of inversion OT , S will coincide with its inverse.

10. If two curves touch each other, prove that their inverse curves w.r.t. any point O also touch each other.

11. If P' is the inverse of P , and if PP' meets the circle of inversion at C, D ; prove that $\{PP'; CD\}$ is harmonic.

12. Regarding a straight line as the limiting form of a circle when the centre is at infinity, if P, P' are inverse points w.r.t. the straight line, prove that it bisects PP' at right angles. [Use Ex. 11.]

13. P', Q', R' are the inverse points of P, Q, R w.r.t. a point O ; prove that $P'\hat{Q}R' = P\hat{Q}R$, if O lies on PR . [Use Ex. 1.]

14. If two curves cut at P , and if their inverse curves cut at P' , prove that the angle of intersection at P is equal to the angle of intersection at P' . [Regard a tangent as the limiting position of a line joining two adjacent points on the curve.]

15. Points P, Q are taken on the diameter AB of a circle so that $\{AB; PQ\}$ is harmonic; prove that P, Q are inverse points w.r.t. the circle.

16. I is the incentre of the triangle ABC ; B', C' are the inverse points of B, C w.r.t. a circle, centre I , radius IA ; prove that I is the orthocentre of the triangle $AB'C'$.

✓17. OA is a diameter of a circle; P is a variable point on the circle. A' and P' are the inverse points of A and P w.r.t. O ; prove that (1) $OA'\hat{P} = 90^\circ$; (2) the locus of P' is a straight line. This proves that the inverse of a circle w.r.t. a point on it is a straight line.

18. From a point O a perpendicular OB is drawn to a given straight line BC ; P is a variable point on BC ; B', P' are the inverse points of B, P w.r.t. O ; prove that (1) $OP'\hat{B} = 90^\circ$; (2) the locus of P' is a circle through O . What therefore is the inverse of a straight line w.r.t. a point outside the line?

Before giving geometrical proofs for the inverse curves of given curves, we shall apply analytical methods for this purpose. By this means it is also possible to include some mention of less familiar curves.

To prove that the inverse of a straight line which does not pass through the origin is a circle through the origin.

Let O be the origin.

Draw OA perpendicular to the given line.

Let P be any point on the given line.

Take Ox along OA . Let $OA = a$.

Produce OP to P' so that $OP \cdot OP' = k^2$.

Let (r_1, θ_1) be coordinates of P .

Then $r_1 \cos \theta_1 = a$.

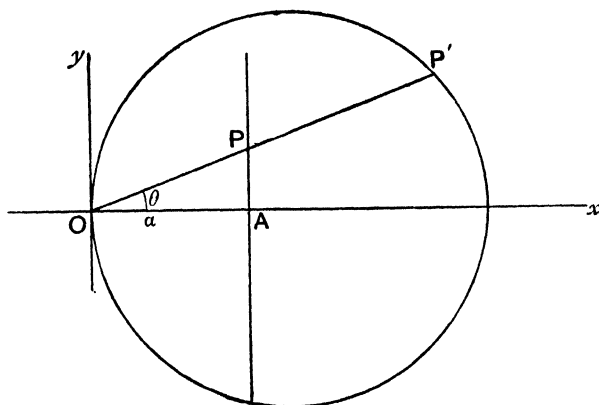


FIG. 78.

If the coordinates of P' are R_1, θ_1

$$r_1 R_1 = k^2;$$

$$\therefore R_1 = \frac{k^2}{r_1} = \frac{k^2}{a} \cos \theta_1;$$

$\therefore$ the equation of the locus of P' is $r = \frac{k^2}{a} \cos \theta$, or $r^2 = \frac{k^2}{a} \cdot r \cos \theta$.

In Cartesians, its equation is $x^2 + y^2 = \frac{k^2}{a} \cdot x$,

which is a circle through O with its centre on OA .

Q.E.D.

19. Prove that the inverse of a circle through the origin is a straight line.

[Take the equation of the circle to be $r = a \cos \theta$.]

20. Prove that the inverse of a circle which does not pass through the origin is another circle.

[The equation of any circle can be written $x^2 + y^2 + 2gx + 2fy + c = 0$, which in polars becomes $r^2 + 2r(g \cos \theta + f \sin \theta) + c = 0$.]

21. Prove that the inverse of a parabola w.r.t. its focus is a cardioid. Draw a rough figure of the cardioid.

[The equation of a parabola referred to its focus is $\frac{l}{r} = 1 + \cos \theta$, and the equation of a cardioid is of the form $r = a(1 + \cos \theta)$. Fig. 75 represents a parabola with O as focus.]

22. Prove that the inverse of any conic w.r.t. its focus is a Limaçon.
 •Draw a rough figure of a Limaçon.

[The equation of any conic referred to its focus is $\frac{l}{r} = 1 + e \cos \theta$; the equation of a Limaçon is of the form $r = a + b \cos \theta$. Fig. 74 represents an ellipse with O as focus.]

23. Find the equation of the inverse of a parabola w.r.t. its vertex and indicate roughly the shape of the curve.

[The Cartesian equation of a parabola with the vertex as origin is $y^2 = px$.] The inverse curve is called the Cissoid of Diocles.

24. Prove that the inverse of a rectangular hyperbola w.r.t. its centre is a Lemniscate whose equation is of the form $r^2 = a^2 \cos 2\theta$. It is represented in Fig. 76. [The Cartesian equation of a rectangular hyperbola with the centre as origin is $x^2 - y^2 = a^2$.]

25. Prove that the inverse of the curve whose equation is $f(r, \theta) = 0$ has an equation of the form $f\left(\frac{k^2}{r}, \theta\right) = 0$.

26. Prove that the inverse of the curve whose equation is $f(x, y) = 0$ has an equation of the form $f\left(\frac{k^2 x}{x^2 + y^2}, \frac{k^2 y}{x^2 + y^2}\right) = 0$.

27. Prove that the degree of the inverse of a curve of degree n cannot be greater than $2n$.

THEOREM 65.

- (1) If a straight line is inverted w.r.t. a point O on it, the inverse is the same straight line.

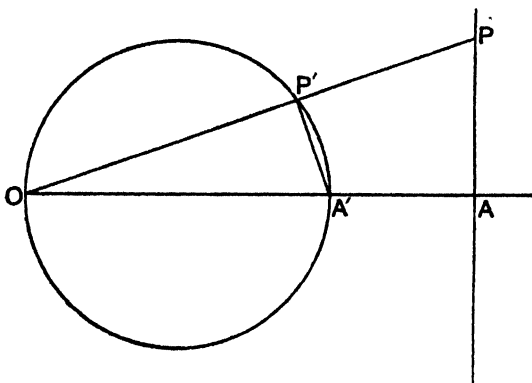


FIG. 79.

- (2) If however O is not on the line, the inverse is a circle through O .

- (3) If a circle is inverted w.r.t. a point O on the circle, the inverse is a straight line.

(1) If Q is any point on the given line OA , the inverse of Q also lies on OA by definition: so that every point on OA inverts into a point on OA .
Q.E.D.

(2) From O draw OA perpendicular to the given line.

Take any point P on the given line.

Let A', P' be the inverses of A, P w.r.t. O .

Since $OA \cdot OA' = OP \cdot OP'$, $A'APP'$ is a cyclic quadrilateral;

$$\therefore \angle P'A' = \angle OAP = 90^\circ;$$

$\therefore P'$ lies on the circle on OA' as diameter;

$\therefore$ the inverse of the straight line is a circle through O , such that its centre lies on the perpendicular from O to the line.

Q.E.D.

(3) Let OA' be the diameter through O of the given circle.

Take any point P' on its circumference.

Let A, P be the inverses of A', P' w.r.t. O .

Then since A', A, P, P' are concyclic, $\angle PAA' = \angle P'A'A = 90^\circ$, since OA' is a diameter;

$\therefore P$ lies on a line through A perpendicular to OA ;

$\therefore$ the locus of P is a straight line;

$\therefore$ the inverse of a circle through O w.r.t. O is a straight line.

Q.E.D.

✓ 28. Prove that a system of parallel lines invert into a system of circles touching each other at a common point.

✓ 29. A parallelogram is inverted w.r.t. any point, what is the inverse figure?

✓ 30. A system of circles have two common points A, B ; when inverted w.r.t. A , what is the inverse figure?

/ 31. What is the inverse of the theorem: a straight line can be drawn to pass through any two points? *O the 3/5.*

✓ 32. Invert w.r.t. O the theorem: from any point outside a circle two lines can be drawn to touch a given circle OAB . [Use Ex. 10.]

✓ 33. Invert w.r.t. A the theorem: ABC is a triangle, DE is drawn parallel to the base BC and meets AB, AC at D, E ; then $\frac{AD}{AB} = \frac{AE}{AC}$.

✓ 34. Invert w.r.t. O the theorem: O is a fixed point; P is a variable point on a fixed line; OP is produced to Q so that $\frac{OP}{OQ}$ is constant, then the locus of Q is a straight line parallel to the fixed line.

THEOREM 66.

If a circle is inverted w.r.t. a point O which does not lie on the circle, then the inverse is another circle.

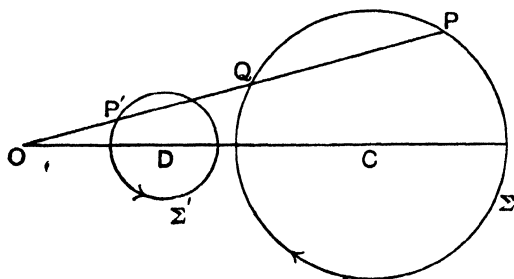


FIG. 80.

Let P be any point on the given circle Σ , and let OP meet Σ again at Q .

Let P' be the inverse of P .

Then $OP \cdot OP' = k^2$, where k = radius of inversion,

also $OP \cdot OQ = t^2$, where t = tangent from O to Σ ;

$$\therefore \frac{OP'}{OQ} = \frac{k^2}{t^2} = \text{constant}; \checkmark$$

$\therefore$ as Q traces out Σ , P' traces out a homothetic circle Σ' .
[Th. 3.] Q.E.D.

It is important to notice that the centre D of Σ' is *not* the inverse of the centre C of Σ . (See Ex. 36.)

Notice also that P' describes Σ' in a sense opposite to the sense in which P describes Σ , if O lies outside Σ .

—What difference is there in the proof when O lies inside Σ ? $\checkmark$

35. Prove Theorem 66 when O lies inside Σ .

36. With the notation of Theorem 66 prove that $\frac{OC}{OD} = \frac{t^2}{k^2}$. And hence show that C, D cannot be inverse points.

37. Prove that a pair of common tangents to a circle and its inverse meet at the centre of inversion.

38. If O lies outside the given circle, radius a , prove with the notation of Theorem 66 that the radius of the inverse circle is $\frac{k^2}{t^2} a$.

39. Two circles intersect at A, B ; O is any point on AB ; prove that it is possible to invert each circle into itself with O as centre of inversion.

40. A circle S passes through the centre of a circle Σ : the external common tangents to S and Σ touch S at P , Q and meet at O . Prove that PQ touches Σ .

[Invert w.r.t. O so that each circle inverts into the other, and consider the original figure and the new figure taken together.]

41. Two non-intersecting circles are inverted w.r.t. any point, what do their four common tangents invert into?

42. Two circles intersect at O ; if the system formed by the two circles and their common tangents is inverted w.r.t. O , what is the inverse system composed of?

43. P , Q , R are three points on a circle, centre A ; PQ , PR are produced to H , K so that $PQ \cdot PH = PR \cdot PK$; prove that PA is perpendicular to HK .

44. A , B , C are three collinear points. Two variable equal circles PAB , PBC meet at P , prove that the locus of P is a straight line. [Invert w.r.t. B .]

45. Prove that the inverse of a plane w.r.t. a point O outside it is a sphere passing through O .

46. Prove that the inverse of a sphere w.r.t. a point on it is a plane.

47. Prove that the inverse of a sphere w.r.t. a point, not lying on it, is another sphere.

48. Prove that the inverse of a circle w.r.t. a point O not in its plane is another circle. [Regard the circle as the section of a sphere through O made by a plane.]

49. If C is the mid-point of AB and if A , C , B are inverted w.r.t. a point O on AB into A' , C' , B' , prove that $\{OC'; A'B'\}$ is harmonic.

50. PQ , RS are two common tangents to two circles PAR , QAS , prove that the circles PAQ , RAS touch each other.

51. If $\{ACBD\}$ is a harmonic range, and if A' , C' , B' , D' are their inverses w.r.t. a point O not on AD , prove that A' , C' , B' , D' form a harmonic system of points on a circle.

52. T is a variable point on a circle Σ ; PQ is the chord of contact of the tangents from T to another given circle, prove that the mid-point of PQ is in general a circle. What is the exceptional case?

53. A , B , C are three collinear points. O is any other point prove that the circumcentres of the triangles OBC , OCA , OAB are concyclic with O .

54. In the triangle ABC , the point A is fixed and the angle BAC is constant; if the area of the triangle ABC is also constant and if B describes a straight line or a circle, prove that the locus of C is in general a circle. Under what conditions is the locus, however, a straight line?

55. Three circles BOC , COA , AOB are such that the centres of BOC , COA lie on OA , OB respectively; prove that the centre of the circle AOB lies on OC , and that the three centres lie on the circum-circle of the triangle ABC .

THEOREM 67.

The angle at which two curves cut is equal to the angle at which their inverse curves cut at the corresponding point of intersection.

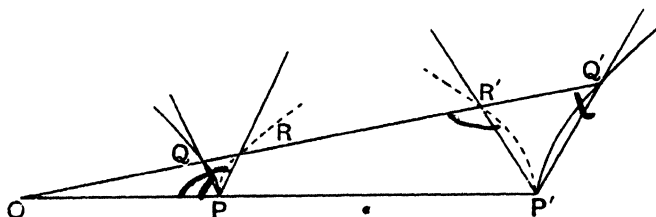


FIG. 81.

O is the centre of inversion.

P , P' are the corresponding points of intersection of the two curves.

Let Q , R be points near P on the original curves, such that O , Q , R are collinear, and let Q' , R' be their inverses.

Since $OQ \cdot OQ' = OP \cdot OP' = OR \cdot OR'$

$PQQ'P'$ and $PRR'P'$ are cyclic quadrilaterals;

$$\therefore \angle OPQ = \angle OQ'P' \text{ and } \angle OPR = \angle OR'P';$$

$$\begin{aligned} \therefore \angle QPR &= \angle OPR - \angle OPQ = \angle OR'P' - \angle OQ'P' \\ &= \angle Q'P'R'. \end{aligned}$$

But in the limit when Q , R become very close to P , QP , RP become the tangents at P , and $Q'P'$, $R'P'$ become the tangents at P' to the two curves, which therefore proves what was required.

Q.E.D.

Hence if two circles cut orthogonally at A , it follows that when inverted w.r.t. A , they become two perpendicular straight lines.

If they are inverted w.r.t. a point B on one circle but not on the other, they become a circle and a straight line which is a diameter of the circle.

The following example will illustrate the use of inversion in solving riders. It is convenient to use dashes to denote corresponding points in the inverse figure.

EXAMPLE 1.

Three circles cut one another orthogonally; their common chords are AB, CD, EF ; to prove that the circles ACE, ADF touch at A .

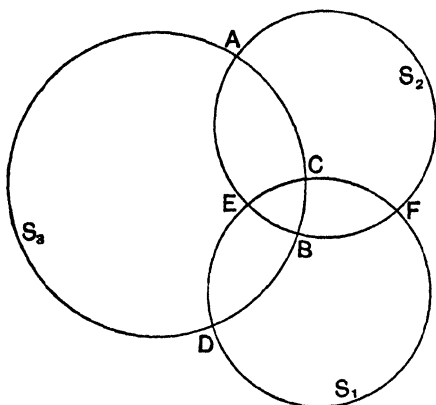


FIG. 82.

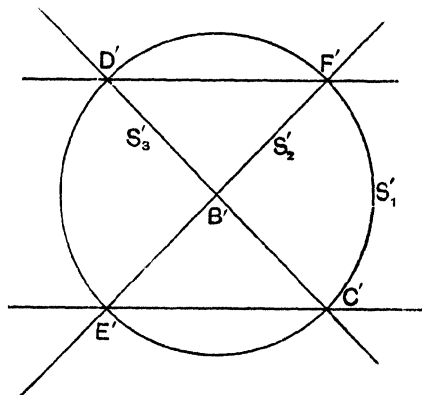


FIG. 83.

Invert w.r.t. A and denote by dashes the corresponding points in the inverse figure.

Let S_1, S_2, S_3 be the three circles $DECF, FBEA, DBCA$.

S_2 and S_3 become straight lines S'_2, S'_3 orthogonal to the circle S'_1 , and are therefore diameters of S'_1 , so that B' is the centre of S'_1 .

Hence $D'B'C', F'B'E'$ are diameters of S'_1 .

Now the circles ACE, ADF become the straight lines $C'E', D'F'$.

But it is obvious that $C'E', D'F'$ are parallel;

$\therefore$ the circles ACE, ADF touch at A . Q.E.D.

Fig. 83 represents the inverse of Fig. 82. It has been drawn out separately to avoid confusion with the original figure. But in reality Figs. 82, 83 are so related that lines joining corresponding points pass through A .

56. $OABCH$ are five points such that the circles OAB, OCH and also the circles OBC, OAH are orthogonal, prove that the circles OCA, OBH are also orthogonal.

57. $\{ACBD\}$ is a harmonic range: O is any point on AD ; A', B', C', D' are the inverse points of A, B, C, D w.r.t. O . By using the fact that the circles on AB, CD as diameters are orthogonal, prove that $\{A'C'B'D'\}$ is harmonic.

58. Four circles, centres $C_1C_2C_3C_4$, pass through two common points A, B . If $A\{C_1C_2C_3C_4\}$ is harmonic, prove that any line through A is cut harmonically by the circles.

59. Prove that if a straight line is cut harmonically by two orthogonal circles it must be a diameter of one of them. [Invert w.r.t. a point of intersection of the line with the circles.]

60. Two circles are inverted w.r.t. any point, what does the line joining their centres invert into?

61. P', Q' are inverse points of P, Q w.r.t. O , prove that

$$OP \cdot P'Q' \cdot PQ = OP' \cdot PQ \cdot PQ'.$$

62. Two lines are drawn from a fixed point to meet a fixed line and to contain a constant angle, prove that the circumcircle of the triangle so formed touches a fixed circle.

63. A variable circle touches a fixed circle and cuts another fixed circle orthogonally, prove that it also touches another fixed circle.

64. A, B are fixed points on a fixed circle Σ , P is a variable point on Σ ; O is any other fixed point. Prove that the circles OAP, OBP cut at a constant angle.

65. Circles are described to touch two given intersecting circles, prove that they cut orthogonally one of the two circles which pass through the points of intersection of the given circles and bisect the angles between them.

66. Invert w.r.t. O the theorem: O is a fixed point on the fixed circle on AB as diameter, P is a variable point, then the angle APB is a right angle.

67. Invert w.r.t. O the theorem: OP, OQ are variable chords drawn through a fixed point O on a fixed circle and are equally inclined to another fixed chord OA , then PQ is fixed in direction.

68. If four points A, B, C, D are inverted w.r.t. any point, prove that $\hat{ABC} + \hat{ADC}$ is unaltered in value.

69. If the circumcircles of the triangles ABC, ABD are orthogonal, prove that the circles CAD, CBD are also orthogonal.

70. Invert w.r.t. any point the theorem: two variable circles touch one another and each of two fixed lines, then their point of contact lies either on one of the fixed lines or on one of the two lines bisecting the angle between the fixed lines.

71. Two circles S_1, S_2 intersect at A, B ; two other circles S_3, S_4 are drawn to touch both S_1 and S_2 externally; prove that a circle can be drawn through A and B to cut S_1 and S_2 orthogonally.

72. Invert w.r.t. A the theorem: if in the quadrilateral $ABCD$, $\hat{BAD} + \hat{BCD} = 180^\circ$, then the points A, B, C, D lie on a circle.

73. Prove that circles which cut one given circle orthogonally and another given circle at a constant angle, also cut a third fixed circle at the same constant angle.

74. A, B are two fixed points on a circle; PQ is a variable diameter; AP meets BQ at R ; prove that the locus of R is a circle through A, B orthogonal to the given circle.

75. From a given point O , straight lines OA, OB, OC are drawn to cut another straight line at A, B, C ; a circle OBD is drawn orthogonal to the circle OAC and meets AC again at D ; prove that the angles AOB, COD differ by a right angle.

76. Two circles intersect orthogonally at P ; O is any point on any circle touching the former circles at Q, R ; prove that the circles OPQ, OPR intersect at an angle of 45° .

77. Explain the fallacy in the following argument:

AB and CD are any two straight lines: O is a point, lying on neither. AB and CD invert w.r.t. O into two circles intersecting at O . But O is the inverse of a point at infinity, therefore the original lines AB and CD intersect at a point at infinity, and are therefore parallel.

78. Invert w.r.t. any point the following theorem: it is possible to draw an infinity of concurrent straight lines cutting a given circle orthogonally.

THEOREM 68.

P' and Q' are the inverse points of P, Q w.r.t. O ; p, p' are the lengths of the perpendiculars from O to PQ and $P'Q'$; then $\frac{P'Q'}{PQ} = \frac{p'}{p} = \frac{k^2}{OP \cdot OQ} = \frac{OP' \cdot OQ'}{k^2}$ where k is the radius of inversion.

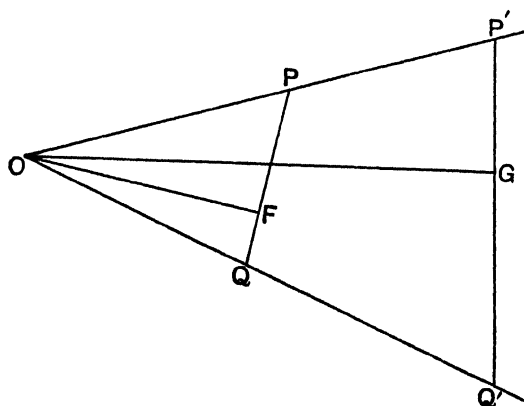


FIG. 84.

Draw OF, OG perpendicular to $PQ, P'Q'$.

Since P, Q, Q', P' are concyclic, $\hat{OPQ} = \hat{OQ'P'}$ and $\hat{OQ'P} = \hat{OP'Q'}$;

$\therefore$ the triangles OPF , $OQ'G$, and also the triangles OPQ , $OQ'P'$, are similar;

$$\therefore \frac{OG}{OF} = \frac{OQ'}{OP} = \frac{P'Q'}{PQ};$$

$$\therefore \frac{P'Q'}{PQ} = \frac{p'}{p}.$$

Also

$$\begin{aligned} \frac{P'Q'}{PQ} &= \frac{OQ'}{OP} = \frac{OQ' \cdot OQ'}{OQ' \cdot OP} = \frac{OQ' \cdot OP'}{OP \cdot OP'} \\ &= \frac{k^2}{OQ' \cdot OP} = \frac{OQ' \cdot OP'}{k^2}; \end{aligned}$$

$$\therefore \frac{P'Q'}{PQ} = \frac{p'}{p} = \frac{k^2}{OP \cdot OQ} = \frac{OP' \cdot OQ'}{k^2}.$$

Q.E.D.

EXAMPLE 2.

[Ptolemy's theorem and its extension.]

In a non-cyclic quadrilateral, the sum of the rectangles contained by two pairs of opposite sides is greater than the rectangle contained by the diagonals.

If, however, the quadrilateral is cyclic, these two expressions are equal.

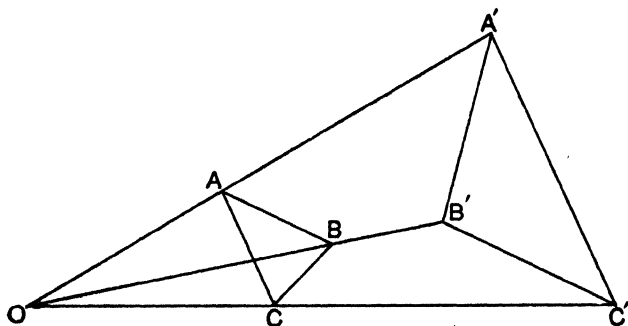


FIG. 85.

Let $OABC$ be the quadrilateral;

A' , B' , C' are the inverse points of A , B , C .

(1) Since $OABC$ is non-cyclic, A' , B' , C' are not collinear.

Now $\frac{AB}{A'B'} = \frac{OA \cdot OB}{k^2}$, where k = radius of inversion; [Th. 68]

$$\therefore \frac{AB}{OA \cdot OB} = \frac{A'B'}{k^2}.$$

Similarly $\frac{BC}{OB \cdot OC} = \frac{B'C'}{k^2}$ and $\frac{AC}{OA \cdot OC} = \frac{A'C'}{k^2}.$

But $A'B' + B'C' > A'C'$, since A', B', C' are not collinear.

$$\begin{aligned}\therefore \frac{AB}{OA \cdot OB} + \frac{BC}{OB \cdot OC} &> \frac{AC}{OA \cdot OC}; \\ \therefore AB \cdot OC + BC \cdot OA &> AC \cdot OB.\end{aligned}$$

Q.E.D.

(2) If $OABC$ is cyclic, $A'B'C'$ are collinear;

$$\therefore A'B' + B'C' = A'C',$$

$$\therefore \text{as before } AB \cdot OC + BC \cdot OA = AC \cdot OB.$$

Q.E.D.

The reader is reminded that a vector proof of this theorem is suggested in Ex. 53 in Chapter VI.

79. Invert w.r.t. O the following theorem:

A variable line is drawn through a fixed point O to meet two fixed lines at P, Q ; on OP is taken a point R such that $\frac{1}{OR} = \frac{1}{OP} + \frac{1}{OQ}$, then the locus of R is a straight line.

80. Invert w.r.t. O the following theorem:

If PAQ, RAS are two chords of a circle, then $PA \cdot AQ = RA \cdot AS$, where O is any point on the circle.

81. Two circles intersect at O, P ; their tangents at O meet the circles again at A, B ; if the circle AOB cuts OP produced at Q , prove that $OQ = 2OP$.

82. A, B, C are three collinear points, O is a point such that $\hat{AOB} = 60^\circ = \hat{BOC}$, prove that $\frac{1}{OB} = \frac{1}{OA} + \frac{1}{OC}$.

83. Three straight lines OP, OQ, OR are of lengths x, y, z ; $\hat{QOR} = \alpha, \hat{ROP} = \beta, \hat{POQ} = \gamma$; if P, Q, R are collinear, prove that $\frac{\sin \alpha}{x} + \frac{\sin \beta}{y} + \frac{\sin \gamma}{z} = 0$. Hence prove by inversion that $PQRO$ are concyclic if $x \sin \alpha + y \sin \beta + z \sin \gamma = 0$.

[This theorem was published in the *Mathematical Gazette*.]

84. O is a point on the minor arc BC of the circumcircle of the triangle ABC ; OP, OQ, OR are the perpendiculars from O , to BC, CA, AB ; prove that $\frac{BC}{OP} = \frac{BA}{OR} + \frac{AC}{OQ}$.

85. State and prove a theorem similar to Ex. 84 which holds for any polygon inscribed in a circle.

86. A line AB of constant length moves with its extremities on two fixed lines ox, oy ; A', B' are inverse points of A, B w.r.t. O . Prove that $A'B'$ touches a fixed circle, centre O .

87. O is a point outside a given circle, centre A ; OPQ is any line cutting the given circle at P, Q and meeting the circle on OA as diameter at R , then $OR = \frac{1}{2}(OP + OQ)$. By inverting w.r.t. O so that the given circle inverts into itself, prove that any line through O is cut harmonically by the given circle, by the chord of contact of the tangents from O to the circle and by O .

88. $OA_1A_2\dots A_n$ are a system of points on a circle, prove that

$$\frac{A_1A_2}{OA_1 \cdot OA_2} + \frac{A_2A_3}{OA_2 \cdot OA_3} + \dots + \frac{A_{n-1}A_n}{OA_{n-1} \cdot OA_n} = \frac{A_1A_n}{OA_1 \cdot OA_n}.$$

89. If A, B, C, D are any four collinear points, prove that the ratio $\frac{AB \cdot CD}{AD \cdot CB}$ is unaltered by inversion w.r.t. any point.

90. Invert w.r.t. any point the theorem that the line joining the centres of two intersecting circles bisects their common chord.

91. Three circles have a common point O ; OA, OB, OC are the common chords and meet the remaining circles at D, E, F , prove that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

92. Three circles OAB, OBC, OCA pass through a common point O ; a line through O meets OAB, OBC, OCA again at F, D, E , prove that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Peaucellier's Cell.

This is a system of linkages for describing mechanically a curve inverse to a given curve. It consists of four equal rods, of length a , jointed together so as to form the rhombus $ABCD$. (See

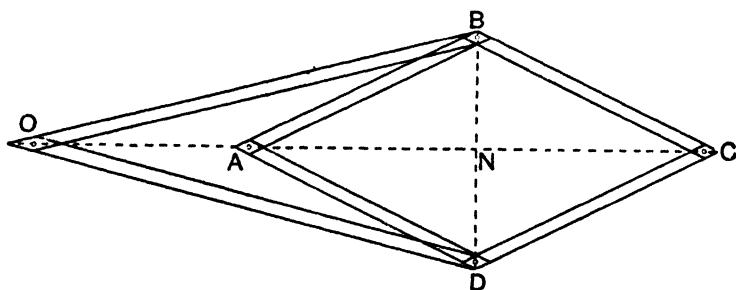


FIG. 86.

Fig. 86.) Two other rods, each of length b ($b > a$), are jointed together at one end O , and the other extremities are jointed to the rhombus at B, D . The point O is kept fixed. If A is made to describe any curve, then a pencil placed at C will trace out the inverse curve.

To prove this, it is necessary to show that (1) O, A, C are collinear, and (2) $OA \cdot OC$ is constant.

This is left as an exercise to the reader.

93. In Fig. 86, prove that OAC are collinear, and that

$$OA \cdot OC = b^2 - a^2.$$

If a rod is fastened at one end to A and at the other to a fixed point H , such that $HA = HO$, then A is compelled to move on a circle through O , and therefore C will describe a straight line.

Now none of the edges of the rods need be straight, the only conditions being that the distances between the points connected by rods should be invariable; hence it follows that this contrivance could be used to construct a straight line without a straight edge of any kind.

A model of this linkage may be easily made by using thin strips of cardboard jointed together with paper fasteners. It will be found that the actual path A can be made to describe, subtends at O an angle of less than two right angles. Another form may be given which is not so restricted in range.

94. Calculate in terms of HA, a, b the angle A_1OA_2 , where A_1, A_2 are extreme positions of A in the linkage which includes the rod HA

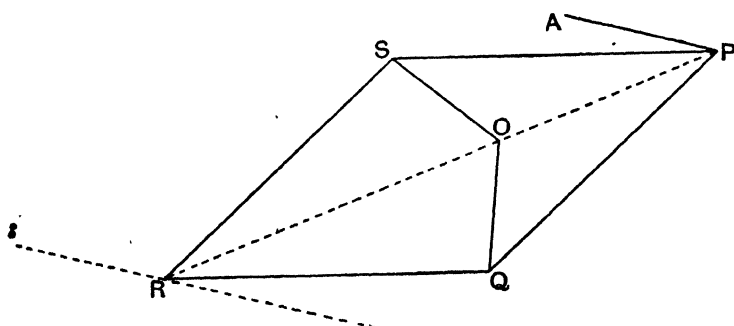


FIG. 87.

In Fig. 87, $PQRS$ is a rhombus formed of four equal rods. OQ, OS are two other equal rods, and the point O is fixed. OQ is less than PQ .

A is a fixed point and AP is a rod of length equal to AO .

Then P is compelled to describe a circle, centre A , which therefore passes through O . It can then be proved that R describes a straight line perpendicular to OA .

It is necessary to prove (1) R, O, P are collinear.

(2) $OR \cdot OP = k^2 = \text{constant}$.

P, R therefore describe inverse curves.

It should be noted that k^2 is negative, so that the radius of inversion is imaginary.

A model can be made without much difficulty and will be found to work well.

95. With the notation of Fig. 87, prove that R, O, P are collinear and that $RO \cdot OP$ is constant.

96. In Fig. 87, $PQ = a$, $OQ = b$, $AP = c$.

(1) Prove that the length of the perpendicular from O to the straight line traced by R is $\frac{a^2 - b^2}{2c}$.

(2) Prove that the length of line which can be traced out by R is

$$2(a+b)\sqrt{1 - \left(\frac{a-b}{2c}\right)^2}.$$

(3) Prove that the length of line traced out by R can, for different values of b, c , never exceed $4a$.

(4) Describe what happens in the limiting case when $a \rightarrow b$. What is the general effect of the radius of inversion becoming zero?

[This question will be made much easier if a model is used.]

THEOREM 69.

If a circle and two inverse points are inverted w.r.t. any point, then the inverse system is a circle and two inverse points.

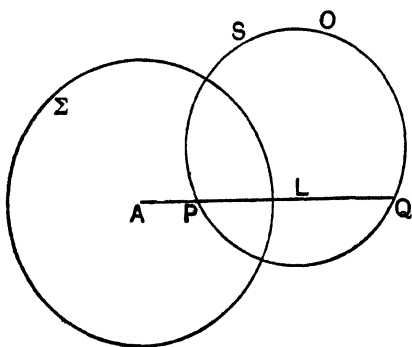


FIG. 88.

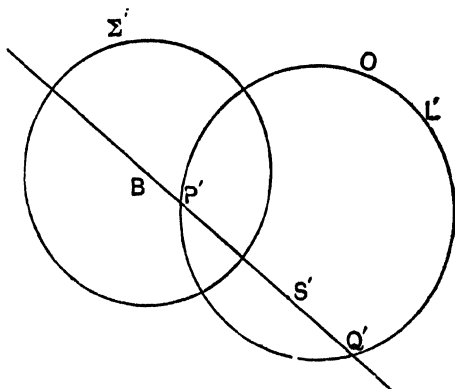


FIG. 89.

Let P, Q be two inverse points w.r.t. circle Σ , centre A .

Any point O is taken as centre of inversion.

Draw a circle through O, P, Q and call it S .

Call the line APQ, L .

When inverted w.r.t. O (see Fig. 89), Σ becomes a circle Σ' and S becomes a straight line S' orthogonal to Σ' (and therefore a diameter of Σ'), because S passes through two inverse points of Σ and is therefore orthogonal to Σ . [Th. 64]

Also the diameter APQ of Σ becomes a circle through O orthogonal to Σ' .

$\therefore$ the inverse points P', Q' of P, Q are the intersections of a diameter of Σ' with a circle orthogonal to Σ' , and are therefore inverse points of Σ' . [Th. 64, Cor. 2.] Q.E.D.

Corollary.

If C_1 and C_2 are inverse curves w.r.t. a circle Σ , and if the system is inverted w.r.t. any point so that C_1, C_2, S become C'_1, C'_2, S' ; then C'_1 and C'_2 are inverse curves w.r.t. the circle S' .

The method of Theorem 69 leads to an important special case.

If the point O lies on Σ , it will be found that Σ inverts into the line bisecting $P'Q'$ at right angles. In other words, Q' is the reflection of P' in the line Σ' .

Hence if C_1 and C_2 are inverse curves w.r.t. a circle Σ , and if the system is inverted w.r.t. a point O on Σ , C_1 and C_2 invert into equal curves, being reflections of one another in the straight line Σ' . [Compare Ex. 12.]

97. Two intersecting circles C_1 and C_2 are orthogonal to a circle S , prove that the points of intersection of C_1 and C_2 are inverse points w.r.t. S .

98. P is the inverse of O w.r.t. a circle Σ , if the system is inverted w.r.t. O , prove that P' is the centre of Σ' .

99. Two given circles intersect at O . By inverting w.r.t. O , prove that the inverse of O w.r.t. any circle touching the given circles lies on one or other of two fixed orthogonal circles.

100. A', B' are inverse points of A, B w.r.t. a given circle Σ ; O is a point on Σ , prove that the circles $OAB, OA'B'$ intersect at a second point on Σ .

101. P is a given point, S and Σ are two given circles; P_1 is the inverse of P w.r.t. S ; P_2 is the inverse of P_1 w.r.t. Σ ; Q_1 is the inverse of P w.r.t. Σ ; Q_2 is the inverse of Q_1 w.r.t. S , prove that P_2 and Q_2 coincide if and only if S and Σ are orthogonal. This may be stated as follows: **the operations of inversion w.r.t. two circles are commutative if and only if the circles are orthogonal.**

102. Q and Q' are inverse points w.r.t. a circle C_1 ; R, R' are the inverse points of Q, Q' w.r.t. a circle C_2 orthogonal to C_1 ; prove that R, R' are inverse points w.r.t. C_1 .

103. Three circles C_1, C_2, C_3 all pass through a point A , and cut each other at angles of 120° . P is any other point; P_1, P_2, P_3 are the inverses of P w.r.t. C_1, C_2, C_3 ; prove that the circles $AP_1P_2, AP_2P_3, AP_3P_1$ also cut each other at angles of 120° .

104. Generalise Ex. 103 so as to obtain a theorem for n circles.

105. A, B, C, D are any four points: three circles are drawn each passing through D , and such that A, B ; B, C ; C, A are respectively inverse points w.r.t. them, prove that the circles have a second common point.

106. C_2 is the inverse of the circle C_1 w.r.t. a circle Σ ; O is any point on Σ ; of the four circles which can be drawn through O to touch C_1 and C_2 , prove that one pair touch each other at O , and that the other pair intersect on Σ .

107. C_2 is the inverse of a circle C_1 w.r.t. a circle Σ ; another circle touches C_1 and C_2 at P, Q ; if both the contacts are internal or both external, prove that P, Q are inverse points w.r.t. Σ .

108. C_1, C'_1 and C_2, C'_2 are pairs of inverse circles w.r.t. a circle Σ ; O is any point on Σ , prove that the inverses of O w.r.t. C_1, C'_1, C_2, C'_2 are concyclic.

THEOREM 70.

(1) O is the point of intersection of the exterior common tangents of two circles C_1, C_2 ; with O as centre of inversion, it is possible to invert either circle into the other.

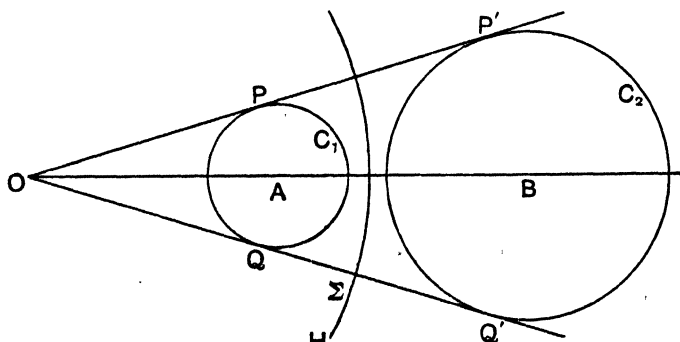


FIG. 90.

(2) If H is any point on the circle of inversion and if the system is inverted w.r.t. H , the circles C_1, C_2 invert into equal circles.

(1) OPP' , OQQ' are the external common tangents of C_1 and C_2 .

$OP \cdot OP' = OQ \cdot OQ'$, for $OP = OQ$ and $OP' = OQ'$.

Take $k^2 = OP \cdot OP'$, where k is the radius of inversion.

Then P , Q invert into P' , Q' ;

$\therefore C_1$ inverts into a circle touching OP' , OQ' at P' , Q' ;

$\therefore C_1$ inverts into C_2 ;

$\therefore C_1$ and C_2 are inverse circles w.r.t. a circle, centre O , radius k .

Q.E.D.

(2) Denote the circle, centre O , radius k , by Σ .

H is any point on Σ .

When inverted w.r.t. H , Σ becomes a straight line Σ' ;

$\therefore C_1, C_2$ invert into circles C'_1, C'_2 which are reflections of C_1, C_2 in the line Σ' ; and are therefore equal circles.

Q.E.D.

Note that Theorem 70 (1), incidentally proves the following theorem:

Through the point of intersection O of the common tangents of two circles any straight line $OPQQ'P'$ is drawn meeting the circles at P, Q and P', Q' , then $OP \cdot OP' = OQ \cdot OQ' = \text{constant}$.

Another proof is given in Theorem 91.

EXAMPLE 3. [FEUERBACH'S THEOREM.]

The nine-point circle of a triangle touches the incircle and each excircle.

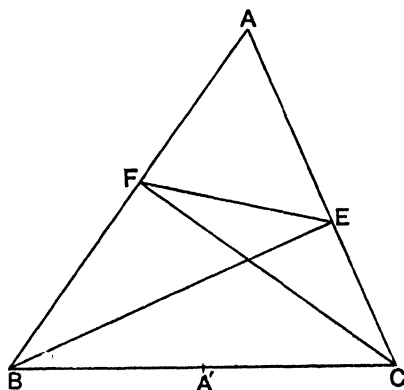


FIG. 91.

The usual notation for the triangle ABC is observed.

AII_1 meets BC at L ; the remaining common tangent $C'LB'$ to the incircle and excircle is the reflection of CLB in AII_1 , so that $\hat{AC'L} = \hat{ACB}$.

Now CI , CI_1 are the bisectors of the angle ACB ;

$\therefore \{AIII_1\}$ is harmonic;

$\therefore \{DXLX_1\}$ is harmonic, by parallels.

But A' is the mid-point of XX_1 , $\therefore A'L \cdot A'D = A'X^2 = A'X_1^2$;

$\therefore$ the inverse of the nine-point circle w.r.t. A' , and radius of inversion $A'X$, is a straight line through L , since the nine-point circle passes through A' , D .

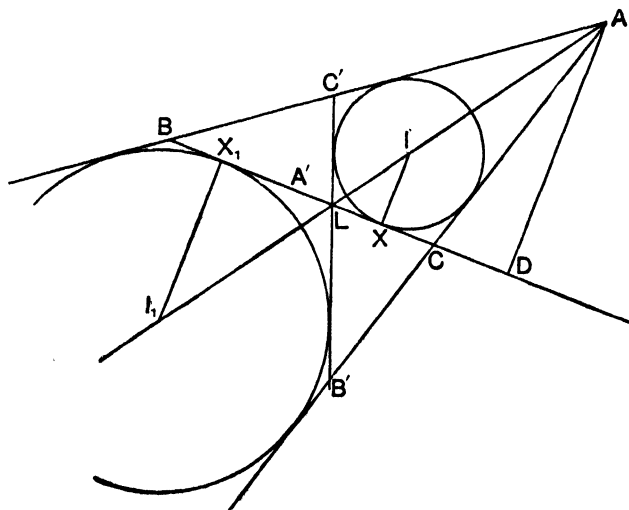


FIG. 92.

Again (see Fig. 91), $A'F = A'E$, for A' is centre of circle $BFEC$;
 $\therefore$ the diameter through A' of the nine-point circle is perpendicular to FE (a chord of the nine-point circle).

But $\hat{AFE} = \hat{ACB} = \hat{AC'L}$;

$\therefore C'L$ is parallel to FE .

$\therefore C'L$ is perpendicular to the diameter through A' of the nine-point circle;

$\therefore C'L$ is the inverse of the nine-point circle w.r.t. A' and with radius of inversion $A'X$.

But the incircle and excircle each invert into themselves in this case;

$\therefore$ since $C'LB'$ touches the incircle and excircle, by considering the inverse figure, it follows that the nine-point circle touches the incircle and excircle.

Q.E.D.

109. Show how to find a centre of inversion such that three given circles can be inverted into themselves.

110. The perpendicular bisector of BC meets the sides AB , AC of the triangle ABC at PQ ; prove that P , Q are inverse points w.r.t. the circumcircle of the triangle ABC .

111. P' is the inverse of P w.r.t. a circle; QPR is any chord; prove that PP' bisects the angle QPR .

112. Invert w.r.t. any point the theorem that the three altitudes of a triangle are concurrent.

113. If the circles inverse to the given circles ACD , BCD w.r.t. a point P are equal, prove that the circle PCD bisects the angle of intersection of the given circles.

114. Deduce from Ex. 113 a method for inverting three circles which cut each other into three equal circles.

115. Use Theorem 70 (2) to invert three given circles into three equal circles.

116. The perpendicular bisectors of the sides AB , AC of a triangle ABC meet AC , AB at Q , R . Circles are drawn with centres Q , R , radii QA , RA ; prove that they intersect on the diameter through A of the circumcircle of the triangle ABC .

117. Four circles A , B , C , D all touch a fifth circle; also A touches B at the point P ; B touches C at Q ; C touches D at R . If the circle PQR cuts A , B , C , D orthogonally, prove that a second circle can be drawn to touch A , B , C , D .

118. Invert w.r.t. O the theorem: AB is a diameter of a circle, centre O ; C is any other point; AC , BC meet the circle again at P , Q ; then the circle CPQ is orthogonal to the given circle.

119. $OABC$, $OA'B'C'$ are two straight lines; if B , B' are the mid-points of AC , $A'C'$, prove that the circles OAA' , OBB' , OCC' have a common chord.

120. A variable circle touches a fixed circle and is such that the tangent to it from a fixed point is of constant length, prove that it also touches another fixed circle.

121. [This requires some knowledge of cross ratios. Th. 56-58.]

Two circles intersect at O , O' ; a variable line OPQ cuts the circles at P , Q and is divided at V so that $\frac{PV}{VQ}$ is constant, prove that the locus of V is a circle through O and O' .

122. G is the centroid of the triangle ABC ; D is the mid-point of BC . The circles CDA , BDG meet at F ; the circles CDG , BDA meet at E ; prove that BC touches the circle DEF ; that DE touches the circle BDG ; and that DF touches the circle CDG .

123. $ABCD$ are any four points. Prove that in an infinite number of ways, two circles can be drawn making an assigned angle with each other and such that A, B are inverse points w.r.t. one circle, and C, D w.r.t. the other.

124. Invert w.r.t. A the theorem: PQ is a variable chord of a fixed circle, centre O , such that OA bisects externally the angle PAQ , where A is a fixed point; then the circle PAQ passes through another fixed point.

125. Invert w.r.t. P the theorem: A, B, C are three collinear points; P is any other point, AFE, BFD, CED are drawn perpendicular to PA, PB, PC , then $PDEF$ are concyclic.

126. A circle touches two other circles at A, B , and meets their common chord at P, Q ; prove that AP, BP, AQ, BQ meet the two circles at the points of contact of their common tangents.

127. A point P is inverted successively w.r.t. four circles each of which is orthogonal to the others; prove that it returns finally to its original position.

[Invert the system w.r.t. a point of intersection of two of the orthogonal circles and note that you obtain two concentric circles, radii r and $r\sqrt{-1}$, and two perpendicular diameters.]

128. OAA', OBB' are two straight lines; $AB', A'B$ intersect on the bisector of the angle between them, prove that

$$\frac{1}{OA} - \frac{1}{OA'} = \frac{1}{OB} - \frac{1}{OB'}.$$

129. BC, CA, AB are arcs of circles orthogonal to the circle ABC . Through A, B, C three circles AD, BE, CF are drawn orthogonal to the circle ABC and to BC, CA, AB respectively. Prove that the circles AD, BE, CF intersect at two common points.

130. Two perpendicular straight lines MX, AD meet at X ; $AX=XD$. Points B, C are taken on AD such that $BX=XC$. If O is any point in AD , and if four circles are drawn through O, M orthogonal to ABM, ACM, BDM, CDM respectively, and cutting them at P, Q, R, S , prove that $PQRS$ are concyclic.

131. The extremities of a line PQ of constant length move on two fixed lines OX, OY ; PK, QK are drawn perpendicular to OP, OQ to meet at K , prove that the locus of K is a circle.

132. From a fixed point O , a tangent OM is drawn to a variable circle through two fixed points C, D ; if OM touches another circle through C, D at N and if P is the harmonic conjugate of O w.r.t. M, N , prove that the locus of P is a circle.

133. Two circles S_1, S_2 touch each other at O . Σ_1, Σ_2 are two other circles also touching each other at O . S_1 meets Σ_1 at D and Σ_2

at A ; S_2 meets Σ_1 at C and Σ_2 at B ; if A, O, C are collinear and if the circle OBD meets AC at H , prove that $\{OCHA\}$ is a harmonic range.

134. A system of circles is drawn to cut a given circle orthogonally at two points of intersection, and to pass through a given point *not* in the plane of the circle. Prove that they intersect in another common point.

Hence show how a circle and a point not in its plane can be inverted into a circle and its centre.

135. Prove that the inverse of a conic w.r.t. any point outside its plane is the section of a cone on a sphere through the vertex of the cone.

136. Casey's Theorem. t is the length of a common tangent of two circles, radii a, b ; t' is the length of the corresponding common tangent of their inverses w.r.t. any point, and a', b' the lengths of their radii; prove that $\frac{t^2}{a \cdot b} = \frac{t'^2}{a' \cdot b'}$.

The following riders (137–167) depend on the analytical results obtained in Ex. 21–24. Where a theorem is to be proved, an elementary knowledge of geometrical conics is required.

These results may be enumerated as follows:

(1) The inverse of a parabola w.r.t. its focus S is a cardioid with its pole at S .

(2) The inverse of a cardioid w.r.t. its pole S is a parabola with its focus at S .

(3) The inverse of a conic w.r.t. a focus S is a Limaçon with its pole at S .

(4) The inverse of a Limaçon w.r.t. its pole S is a conic having S as one focus.

(5) The inverse of a parabola w.r.t. its vertex A is a Cissoid with its cusp at A ; and conversely.

(6) The inverse of a rectangular hyperbola w.r.t. its centre C is a Lemniscate with its double point at C ; and conversely.

137. Invert the following theorem w.r.t. O : if the tangents at the points P, Q, R on a cardioid, whose pole is O , are parallel, then of the quantities $\sqrt{OP}, \sqrt{OQ}, \sqrt{OR}$, the sum of two is equal to the third.

138. Invert w.r.t. S : the circumcircle of the triangle formed by three tangents to a parabola passes through the focus S of the parabola.

139. Invert w.r.t. S : PSP' is a variable chord of a parabola, focus S ; then $\frac{1}{PS} + \frac{1}{SP'}$ is constant.

140. Invert w.r.t. S : PSP' is a variable chord of a conic, focus S ; R is a point on PP' such that $\{PP', SR\}$ is harmonic, then the locus of R is a straight line.

141. Invert w.r.t. S : $LMNPQR$ are six points on a conic, focus S ; then the meets of LM, PQ ; MN, QR ; NP, RL are collinear.

142. Invert w.r.t. C : PCP' is a diameter of a rectangular hyperbola, centre C ; QR is any chord, then the angles $QPR, Q'PR$ are equal or supplementary.

143. A circle is drawn through the pole O of a cardioid to touch the curve at P and meets the axis at T , prove that $OP = OT$.

144. A variable circle is drawn through the pole O of a cardioid to touch the curve; prove that the locus of the extremity of the diameter through O is a circle.

145. If in Ex. 144, P is the point of contact of the variable circle with the curve, prove that its diameter varies as $\sqrt{OP}$.

146. Invert w.r.t. S : (1) if a parabola, focus S , meets a circle, their common chords are equally inclined to the axis of the parabola.

(2) If a conic, focus S , meets a circle, their common chords are equally inclined to the major axis of the conic.

147. A circle is drawn through the pole of a cardioid to touch the curve at a variable point P , prove that the locus of the extremity of the diameter through P is another circle.

148. Invert w.r.t. S : two tangents TP, TQ are drawn from a point T to a parabola, focus S , then

$$\hat{TSP} = \hat{TSQ} = 180^\circ - \hat{QTP}.$$

149. A circle is drawn through the pole O of a cardioid to cut the curve orthogonally at P ; if it cuts the axis at G , prove that $OG = OP$.

150. POQ is a variable chord of a Limaçon whose pole is O ; two circles are drawn through O to touch the curve at P, Q respectively; if they meet again at R , prove that the locus of R is a circle.

151. A variable circle is drawn through the pole O of a Limaçon to touch the curve, prove that the locus of the extremity of the diameter through O is a circle.

152. Invert w.r.t. a focus: the locus of the intersection of perpendicular tangents to a conic is a circle.

153. Obtain a theorem, similar to Ex. 149, for a Limaçon.

154. Invert w.r.t. a focus: PSP', QSQ' are two variable focal chords of a conic perpendicular to each other, then $\frac{1}{PS \cdot SP'} + \frac{1}{QS \cdot SQ'}$ is constant.

155. Invert w.r.t. a focus: a circle of constant radius passes through the centre S of a conic and cuts it at A, B, C, D ; then

(1) $SA \cdot SB \cdot SC \cdot SD$ is constant.

(2) $\frac{1}{SA} + \frac{1}{SB} + \frac{1}{SC} + \frac{1}{SD}$ is constant.

156. Invert w.r.t. a focus: PQ is a variable chord of a conic subtending a constant angle at the focus, then the envelope of PQ is a conic with the same focus.

157. Invert w.r.t. O : P is a variable point on a fixed circle; O is a fixed point; a line PR is drawn so that $\hat{OPR}$ is constant, then PR envelopes a conic, focus O .

158. Invert w.r.t. O : the tangents at the ends of a chord of a cardioid, through the pole O , are at right angles.

159. Invert w.r.t. O : the normals at the ends of a chord through the pole of a Limaçon intersect on a fixed circle.

160. Invert w.r.t. O : two cardioides have the same pole O , and their axes lie in opposite directions along the same line, then they cut orthogonally.

161. Invert w.r.t. C : PP' is a diameter of a rectangular hyperbola, centre C ; a circle is drawn through P, P' and meets the hyperbola at Q, Q' ; then QQ' is a diameter of the circle.

162. Invert w.r.t. C : two rectangular hyperbolas have the same centre C ; also the axes of one are the asymptotes of the other, then they cut orthogonally.

163. Three circles touch a cardioid and pass through its pole, prove that their other three points of intersection are collinear.

164. If two circles touch a cardioid and pass through its pole, prove that their common chord will bisect the angle between the two lines joining the pole to the points of contact of the circles with the cardioid.

165. The cusp of a cissoid is at A , and AX is the tangent to the cusp: a circle is drawn through A to touch the cissoid at P and meets AX at Q ; a line through P perpendicular to AP meets AX at R ; prove that $QA=AR$.

166. Invert w.r.t. A : the locus of the point of intersection of perpendicular tangents to a parabola, vertex A , is a straight line.

167. [This needs the differential Calculus.] If the curve $f(r, \theta)=0$ is transformed by the substitution $r=R^n, \theta=n\theta$, a generalised form of inversion, prove that the angle at which two curves cut is equal to the angle at which the inverse curves cut at the corresponding point.

CHAPTER XII.

POLES AND POLARS.

WHEN considering the Principle of Duality in Chapter IX., nothing was said as to the way in which the correspondence alluded to may be determined. It is essential that this correspondence should be unique. In other words, to any one point A in Fig. 1 there must correspond one and only one line a in Fig. 2, and *vice-versa*. In the following pages some indication will be found of the manner in which this relationship may be effected. It is, however, impossible to realise the generality of the process before the analogous theorems for the conic are enunciated.

The theorems of this chapter cover a wide period of time. The fundamental property, proved in Theorem 73, is given by Apollonius for the case in which the pole lies outside the circle, and he shows further that the same property holds good for the conic. Pappus, in the *Συναγωγή*, completes the theorem by proving it for any position of the pole. The general theory is developed in detail by Desargues, for the conic as well as for the circle, in his treatise on conic sections, published in 1639. Owing to the analytical discoveries of Descartes during the same period, his work received little attention, and was lost until Chasles discovered a copy in 1845. The harmonic properties on pages 168-9 were also in substance contained in this treatise. Dr. Taylor has pointed out that Newton (1647-1727) was also acquainted with the general theory, but whether through Desargues or from independent research, it is impossible to say.

Definition.

P is a point on the diameter AB (or AB produced) of a circle Σ , centre O ; Q is the harmonic conjugate of P w.r.t. A, B ; QR is drawn parallel to the tangent at A to the circle; then QR is called the **polar** of P w.r.t. Σ , and P is called the **pole** of QR w.r.t. Σ .

THEOREM 71.

If the line joining a point P to the centre O of a circle Σ meets the polar of P w.r.t. Σ at Q , then the polar of P is perpendicular to OPQ and

$$OP \cdot OQ = (\text{radius})^2.$$

The proof is very simple and is left to the reader.

The form of the definition given above has been so chosen that it will also apply to the case of a conic.

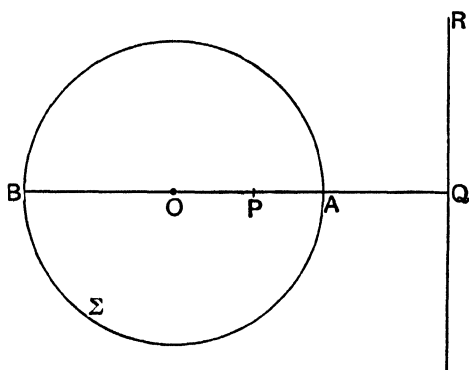


FIG. 93.

1. Prove Theorem 71.
2. If P and P' are inverse points w.r.t. a circle Σ , prove that the polar of P w.r.t. Σ passes through P' .
3. If the point P is outside the circle Σ , prove that the chord of contact of the tangents from P to Σ is the polar of P w.r.t. Σ .
4. What is the polar of a point situated on the circle Σ ?
5. What is the polar of a point situated in a given direction at an infinite distance from the centre of Σ ?
6. What is the polar of the centre of Σ ?
7. What is the pole of a diameter of Σ ?
8. R is a point on the polar QR of a point P w.r.t. a circle, centre O ; PH is drawn perpendicular to OR to meet it at H ; prove that H is the inverse point of R and that HP is the polar of R . Hence prove that if R lies on the polar of P , then P lies on the polar of R .
9. R is any point on the polar of P w.r.t. a circle Σ ; prove that the circle on PR as diameter is orthogonal to Σ . [Note that it passes through Q , Fig. 93.]
10. Any line through a point P meets the polar of P at R , and the circle at H, K , prove that $\{HK; PR\}$ is harmonic. [Use Ex. 9.]
11. PR is a diameter of a circle Σ , orthogonal to another circle S ; O is the centre of S . If OP meets Σ at Q , prove that QR is the polar of P w.r.t. S .

Before giving the geometrical treatment of polar properties, it is instructive to obtain some of the fundamental theorems by analysis.

I. To find the equation of the polar of a point $P(x', y')$ w.r.t. the circle $x^2 + y^2 = a^2$.

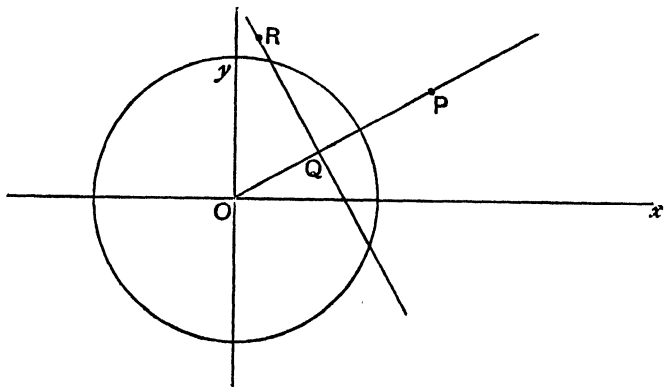


FIG. 94.

The equation of OP is $\frac{x}{x'} = \frac{y}{y'}$;

$\therefore$ any line perpendicular to OP is $xx' + yy' = k$.

The length of the perpendicular from O to this line is $\frac{k}{\sqrt{x'^2 + y'^2}}$.

But if this line is the polar of P , the length of the perpendicular is

$$\frac{a^2}{OP} = \frac{a^2}{\sqrt{x'^2 + y'^2}}; \quad [\text{Th. 71}]$$

$$\therefore k = a^2;$$

$\therefore$ the polar of $P(x', y')$ is $xx' + yy' = a^2$. Q.E.D.

II. If the polar of P passes through R , then the polar of R passes through P . See Fig. 94.

Let the coordinates of P, R be (x_1, y_1) and (x_2, y_2) .

The equation of the circle is $x^2 + y^2 = a^2$.

The polar of P is $xx_1 + yy_1 = a^2$.

Since this passes through R , (x_2, y_2) , $x_2x_1 + y_2y_1 = a^2$ (1)

But the polar of R is $x_2x + y_2y = a^2$;

$\therefore$ by (1) the point P , (x_1, y_1) lies on the polar of R .

Q.E.D.

Definition.

If two points are so situated that the polar of either passes through the other w.r.t. the same circle, then the points are called conjugate points w.r.t. the circle.

It follows from II. that the points (x_1, y_1) and (x_2, y_2) are conjugate points w.r.t. the circle $x^2 + y^2 = a^2$ if $x_1x_2 + y_1y_2 = a^2$.

And the converse is also true.

III. P and Q are conjugate points w.r.t. a circle Σ ; if PH , RK are the tangents from P , R to Σ , then $PR^2 = PH^2 + RK^2$.

With the notation of II.

$$\begin{aligned} PR^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2(x_1x_2 + y_1y_2) \\ &= x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2a^2, \text{ since } P, R \text{ are conjugate points} \\ &= OP^2 - a^2 + OR^2 - a^2 \\ &= PH^2 + RK^2. \end{aligned}$$

IV. **Salmon's Theorem.** P , R are two points: PX , RY are the perpendiculars from P , R to the polars of R , P respectively w.r.t. a circle, centre O : then $\frac{OP}{OR} = \frac{PX}{RY}$.

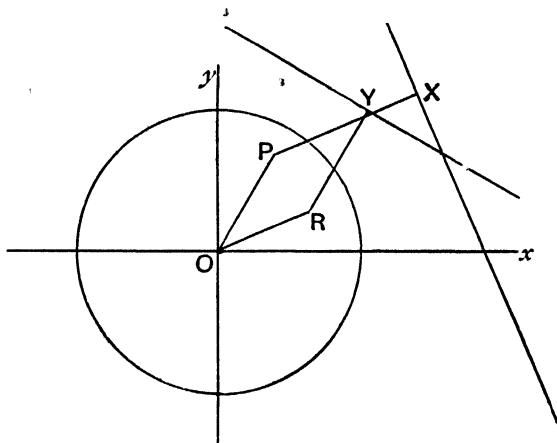


FIG. 95.

Let the coordinates of P , R be (x_1, y_1) , (x_2, y_2) .

Then as before the equation of the polar of R is $xx_2 + yy_2 = a^2$;

$$\therefore PX = \frac{a^2 - x_1x_2 - y_1y_2}{\sqrt{x_2^2 + y_2^2}}.$$

Similarly,

$$RY = \frac{a^2 - x_2x_1 - y_2y_1}{\sqrt{x_1^2 + y_1^2}};$$

$$\therefore \frac{PX}{RY} = \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}} = \frac{OP}{OR}.$$

Q.E.D.

THEOREM 72.

If the polar of P passes through R , then the polar of R passes through P .

O is the centre of the circle.

OP meets the polar of P at Q .

R is a point on the polar QR of P .

Draw PS perpendicular to OR .

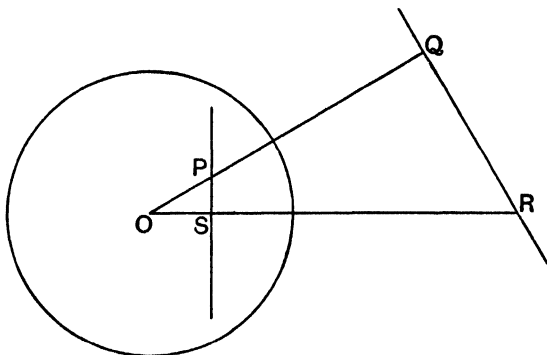


FIG. 96.

Since $\hat{PSR} = 90^\circ = \hat{PQR}$, $PQRS$ is concyclic;

$$\therefore OR \cdot OS = OQ \cdot OP \\ = (\text{radius})^2.$$

[Th. 71]

$\therefore S$ lies on the polar of R .

But PS is perpendicular to OR ;

$\therefore PS$ is the polar of R ;

$\therefore$ the polar of R passes through P .

Q.E.D.

The following Corollaries may be set as exercises.

Corollary 1.

If a line p contains the pole of a line r , then r contains the pole of p .

Let P, R be the poles of p, r .

It is given that R lies on p ;

i.e. the polar of P passes through R ;

$\therefore$ the polar of R passes through P , or, in other words, r passes through P , the pole of p .

Q.E.D.

Corollary 2.

If P is the pole of a line p , and if R is the pole of a line r , then p and r intersect at the pole of PR .

Let p, r meet at K .

The polar of P passes through K , therefore the polar of K passes through P .

Similarly, the polar of K passes through R ;

$\therefore PR$ is the polar of K .

Q.E.D.

Corollary 3.

(1) If any number of points are collinear, then their polars are concurrent.

(2) If any number of lines are concurrent, then their poles are collinear.

For if in 3 (1) P is a variable point on a fixed line l whose pole is L , then the polar of P passes through L , which is a fixed point. And if in 3 (2) p is a variable line through a fixed point L , whose polar is l , then the pole of p lies on l , which is a fixed line.

Corollary 3 is frequently useful, for if it is required to prove three or more particular lines concurrent, it may be easier to prove that their poles are collinear, which would be sufficient; or *vice versa*.

Definitions.

(1) If two points are such that the polar of either w.r.t. a circle Σ passes through the other, then the points are called **conjugate points** w.r.t. Σ .

(2) If two lines are such that the pole of either w.r.t. a circle Σ lies on the other, then the lines are called **conjugate lines** w.r.t. Σ .

12. If P, R are conjugate points w.r.t. a circle Σ , prove that the circle on PR as diameter is orthogonal to Σ .

13. A variable chord PQ of a fixed circle passes through a fixed point H , prove that the tangents at P, Q intersect on a fixed line, viz. the polar of H .

14. T is a variable point on a fixed line; TP, TQ are the tangents from T to a fixed circle, prove that PQ passes through a fixed point.

15. The tangent at A to the circumcircle of the triangle ABC meets BC at T and is produced to U so that $AT= TU$; prove that A and U are conjugate points w.r.t. any circle through B, C .

16. P, R are conjugate points w.r.t. a circle, centre O , prove that the pole of PR is the orthocentre of the triangle OPR .

17. P is a point inside a circle, centre O ; OP meets the polar of P at Q ; any line through P meets the circle at H, K and the polar of P at R ; prove that

(1) OH touches the circle HPQ ;

(2) $\hat{OHP} = \hat{HQP}$ and $\hat{OKP} = \hat{KQP}$;

(3) $\{KPHR\}$ is harmonic.

[This proves that any line through a point P is cut harmonically by P , the polar of P , and the circle.]

18. If P is a point outside a circle, and if $PHRK$ is any line meeting the circle at H, K and the polar of P at R , prove that $\{PR; HK\}$ is harmonic. [Adapt the method of Ex. 17.]

19. Prove that the angle between two lines is equal to the angle which their poles subtend at the centre of the circle.

20. AB is the diameter of a circle C_1 ; if the polar of A w.r.t. a circle C_2 passes through B , prove that C_1 and C_2 are orthogonal.

21. The incircle of the triangle ABC touches BC, CA, AB at X, Y, Z . Prove that the diameter through X of the incircle meets YZ on the line joining A to the mid-point A' of BC .

[Prove that AA' is conjugate to the line through A parallel to BC by Ex. 17 (3), and use Theorem 72, Cor. 3.]

22. P_1, P_2 are two points on the circles S_1, S_2 which have a common centre O . N_1, N_2 are the feet of the perpendiculars from P_1, P_2 to their polars w.r.t. S_2, S_1 respectively. Prove that $\frac{P_1N_1}{P_2N_2} = \frac{OP_2}{OP_1}$.

23. PQ is a diameter of a circle C_1 ; C_2 is a circle orthogonal to C_1 ; prove that the polar of P w.r.t. C_2 passes through Q .

24. I is the incentre of the triangle ABC . The tangent at any point K on the incircle is met by perpendiculars through I to IA, IB, IC at P, Q, R . Construct the positions of the poles of AP, BQ, CR w.r.t. the incircle, and hence prove that these three lines are concurrent.

25. Prove that the locus of a point whose polars w.r.t. three given circles are concurrent is a circle orthogonal to each of the given circles.

26. ABC is a triangle right-angled at B ; BB' is a median. Any circle is drawn to touch BB' at B , prove that A, C are conjugate points w.r.t. it.

27. TP, TQ are the tangents at the extremities of a chord PQ of a circle; the tangent at any other point R on the circle meets PQ at S , prove that TR is the polar of S .

28. PA, QB are the tangents from two points P, Q to a circle, centre C ; if $PQ^2 = PA^2 + QB^2$, prove that Q lies on the polar of P . [Draw QN perpendicular to CP .]

29. ABC is a given triangle. O is the centre of a variable circle through B, C ; find the locus of the point of intersection of AO with the polar of A w.r.t. the circle. [Invert w.r.t. B .]

30. A variable circle passes through a fixed point C , and has two fixed points H, K as conjugate points, prove that the locus of its centre is a straight line perpendicular to the line joining C to the mid-point of HK . [Use Ex. 12.]

31. P, Q are two points subtending a right angle at a point C on a circle; if the tangent at C bisects PQ , prove that P, Q are conjugate points.

32. Prove that the polar of a fixed point w.r.t. a variable circle which touches a fixed line at a fixed point passes through a fixed point.

33. P, R are conjugate points w.r.t. a fixed circle centre O ; if $\hat{POR}$ is constant, and if P moves on a straight line, prove that the locus of R is a circle.

34. Assuming Pascal's theorem that, if a hexagon is inscribed in a circle, then the meets of opposite sides are collinear, deduce Brianchon's theorem that, if a hexagon is circumscribed about a circle the joins of opposite vertices are concurrent. [Use Th. 72, Cor. 2, 3.]

35. Assuming the theorem that, if a circle touches the sides BC, CA, AB of a triangle at X, Y, Z then AX, BY, CZ are concurrent, deduce by using Th. 72, Cor. 3, a theorem for the circumcircle of a triangle. [The incircle of ABC is the circumcircle of XYZ .]

36. PQR is a triangle; P', Q', R' are the poles of QR, RP, PQ w.r.t. a circle; prove that P, Q, R are the poles of $Q'R', R'P', P'Q'$.

Definitions.

(1) If two triangles are such that the vertices of either are the poles of the sides of the other w.r.t. a circle, then the triangles are called **conjugate triangles** w.r.t. the circle.

(2) If a triangle is conjugate to itself (*i.e.* each vertex is the pole of the opposite side w.r.t. a circle), the triangle is called a **self-conjugate triangle** w.r.t. the circle.

37. P, Q are any two conjugate points, R is the pole of PQ ; prove that PQR is a self-conjugate triangle.

38. If PQR is a self-conjugate triangle w.r.t. a circle, prove that the centre of the circle is the orthocentre of the triangle.

39. Given a triangle, prove that there exists one, and only one circle (real or imaginary) w.r.t. which the triangle is self-conjugate, and that the circle is real if, and only if, the triangle is obtuse angled.

40. P is any point on the base BC of an obtuse-angled triangle ABC ; prove that the circle on AP as diameter is orthogonal to the circle w.r.t. which the triangle ABC is self-conjugate.

41. $PQR, P'Q'R'$ are two conjugate triangles of a circle; assuming the theorem that PP', QQ', RR' are concurrent, deduce another theorem by using Th. 72, Cor. 2, 3.

Definition.

The circle (see Ex. 39) for which a triangle is self-conjugate is called the **polar circle** of the triangle.

With the usual notation, Ex. 39 proves that the polar circle of the triangle ABC has its centre at the orthocentre H , and its radius $= \sqrt{HA \cdot HD}$.

THEOREM 73.

If P is the pole of a line QR w.r.t. a circle Σ , then any line through P is cut harmonically by P , QR , and Σ .

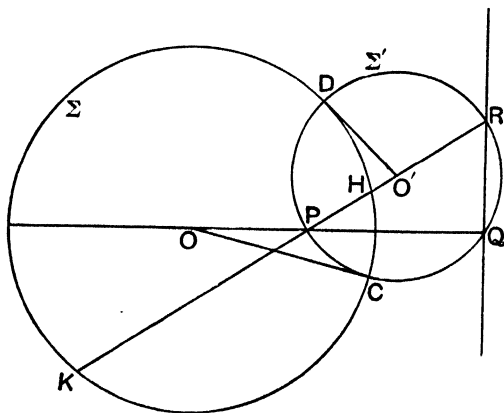


FIG. 97.

Let O be the centre of Σ ; and let OP meet the polar of P at Q .

PR is any line through P cutting Σ at H , K .

On PR as diameter, describe a circle Σ' , and let it cut Σ at C , D .

Since $\angle PQR = 90^\circ$, Σ' passes through Q .

Bisect PR at O' , so that O' is the centre of Σ' .

Now $OP \cdot OQ = OC^2$, since P is the pole of QR ;

$$\therefore \Sigma \text{ is orthogonal to } \Sigma';$$

$$\therefore O'H \cdot O'K = O'D^2 = O'P^2;$$

$$\therefore \{RP; HK\} \text{ is harmonic.}$$

Q.E.D.

Corollary.

If P , R are conjugate points w.r.t. a circle, and if PR meets the circle at H , K , then $\{PR; HK\}$ is harmonic.

Another proof of this fundamental theorem is given on page 187.

42. Prove the Corollary of Theorem 73.

43. Deduce from the Corollary of Theorem 73 a new theorem by using Th. 72, Cor. 3.

THEOREM 74.

p and r are two conjugate lines w.r.t. a circle, and meet each other at a point T outside the circle, then p , r are harmonic conjugates w.r.t. the tangents TX , TY from T to the circle.

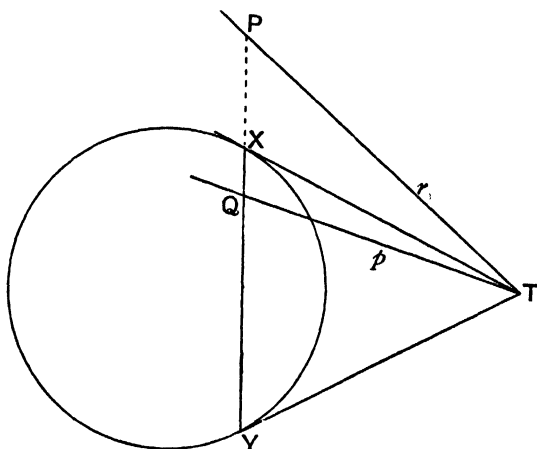


FIG. 98a.

Let P be the pole of p ; and let X , Y be the points of contact of TX , TY .

Then it is given that P lies on r .

Now the polar of P passes through T ;

$\therefore$ the polar of T (i.e. XY) passes through P ;

$\therefore P$ lies on XY .

Then if PXY meets p at Q ,

$\{PQ; XY\}$ is a harmonic range; [Th. 73]

$\therefore T\{PQ; XY\}$ is a harmonic pencil;

$\therefore p$, r are harmonic conjugates w.r.t. TX , TY . Q.E.D.

44. A variable chord PQ of a fixed circle passes through a fixed point H ; K is a point on PQ such that $\{PQ; HK\}$ is harmonic; find the locus of K .

THEOREM 75.

If four points form a harmonic range, then their polars w.r.t. any circle form a harmonic pencil.

Let $\{ABCD\}$ be the harmonic range.

Let P be the pole of AD , O the centre of the circle, radius a .

Let OP meet AD at Q .

Draw PA' , PB' , PC' , PD' perpendicular to OA , OB , OC , OD .

49. AB , CD are chords of a circle, intersecting at O ; the tangents at A , D meet at P ; the tangents at B , C meet at Q , prove that P , O , Q are collinear.

50. A line PAB meets a circle at A , B and the polar of P at Q ; if O is the mid-point of AB , prove that $PQ \cdot PO = PA \cdot PB$.

51. A line PAB meets a circle at A , B and the polar of P at R ; O is the mid-point of AB ; if the polar of P meets the circle at X , Y , prove that PX touches the circle ORX .

52. P is the pole of the chord AB of a circle; O is the mid-point of AB , any line through P meets the circle at C , D ; prove that AB bisects the angle COD .

53. Through the mid-point M of a chord AB of a circle, two straight lines MP , MQ are drawn to the circumference so that AB bisects internally the angle PMQ ; prove that the tangents at P , Q and the chord AB are concurrent.

54. TP , TQ are tangents to a circle; THK is a line cutting the circle at H , K ; PV is a chord of the circle parallel to TH , prove that QV bisects HK . [Prove that $V\{PQ; HK\}$ is harmonic.]

55. C is the mid-point of a chord AB of a circle; PQ is another chord through C ; if the tangents at A , B meet at H , prove that $\hat{PHC} = \hat{QHC}$.

56. Prove that the harmonic mean of the perpendiculars from a given point P inside a circle to the tangents drawn from any point on the polar of P is constant.

57. T is a variable point on a fixed line; TP , TQ are the tangents to a fixed circle; PN , QM are the perpendiculars to the fixed line, prove that $\frac{1}{PN} + \frac{1}{QM}$ is constant.

58. If a given quadrilateral is circumscribed to a given circle, and if a variable tangent to the circle is drawn, prove that the rectangles contained by the perpendiculars on it from each pair of opposite vertices are in a constant ratio. [Use Salmon's Theorem.]

59. PQ , PR are two chords of a circle. The perpendicular bisector of PQ meets PR at H , prove that the line joining H to the pole of QR is parallel to PQ .

60. If two chords AB , CD of a circle are conjugate lines, prove that $ABCD$ form a harmonic system of points on the circle.

61. [Involves cross ratios.] Prove that the cross ratio of four collinear points is equal to the cross ratio of the pencil formed by their polars.

62. [Involves cross ratios.] Two chords APB , CPD of a circle meet at P ; AC and BD meet the minimum chord through P in X and Y ; prove that $PX = PY$.

[Let XY meet the circle in E , F ; use $A\{ECBF\} = D\{ECBF\}$.]

63. [Involves cross ratios.] PQ is a chord of a circle bisecting the chord AB at C ; the tangents at P , Q meet AB at H , K ; prove that $CH = CK$.

THE INSCRIBED QUADRANGLE.

The notation is the same as that on page 114.
The four points A, B, C, D lie on a circle.

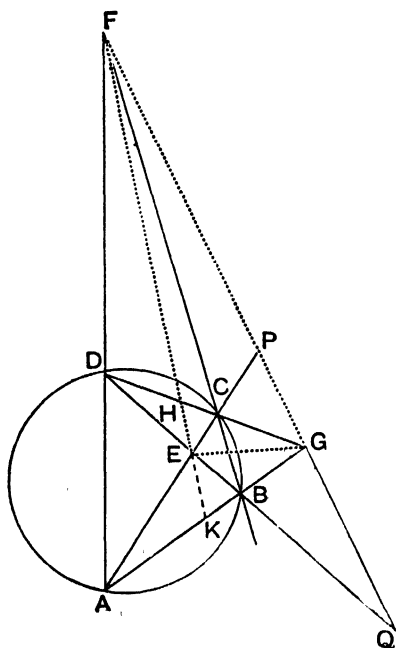
THEOREM 76.

FIG. 99.

To prove that the diagonal point triangle EFG is a self-conjugate triangle w.r.t. the circle.

$\{AEC P\}$ is a harmonic range ;
 $\therefore$ the polar of E passes through P .
[Th. 73]

$\{DER Q\}$ is a harmonic range ;
 $\therefore$ the polar of E passes through Q ;
 $\therefore$ the polar of E is PQ or FG .

Again $\{DHCG\}$ and $\{AKBG\}$ are harmonic ranges ;

$\therefore$ the polar of G passes through H and K ;

$\therefore$ the polar of G is HK or EF .

Similarly, the polar of F is GE ;
 $\therefore$ the triangle EFG is self-conjugate.
Q.E.D.

THEOREM 78.

With the notation of Theorem 76,

To prove that the tangents at A, B, C, D form a quadrilateral having for its diagonal line triangle, the diagonal point triangle of the quadrangle $ABCD$.

BC contains the pole (F) of EG ;

$\therefore EG$ contains the pole of BC ;

$\therefore$ the tangents at B, C meet on EG .

Similarly, the tangents at A, D meet on EG ;
and the tangents at C, D meet on EF ;
and the tangents at A, B meet on EF .

Again, BD contains the pole (E) of FG ,

$\therefore FG$ contains the pole of BD ;

$\therefore$ the tangents at B, D meet on FG .

Similarly, the tangents at A, C meet on FG ;

$\therefore EF, FG, GE$ are diagonal lines of the quadrilateral formed by the tangents at A, B, C, D .

Q.E.D.

[The reader should draw his own figure and draw in the tangents at A, B, C, D .]

THE CIRCUMSCRIBED QUADRILATERAL.

The notation is the same as that on p. 115.

The four lines a, b, c, d touch a circle.

THEOREM 77.

To prove that the diagonal line triangle efg is a self-conjugate triangle w.r.t. the circle.

$\{aecp\}$ is a harmonic pencil ;

$\therefore$ the pole of e lies on p .

[Th. 74]

$\{debq\}$ is a harmonic pencil ;

$\therefore$ the pole of e lies on q ;

$\therefore$ the pole of e is pq or fg .

Again, $\{dhcg\}$ and $\{akbg\}$ are harmonic pencils ;

$\therefore$ the pole of g lies on h and k ;

$\therefore$ the pole of g is hk or ef .

Similarly, the pole of f is ge ;

$\therefore$ the triangle efg is self-conjugate.

Q.E.D.

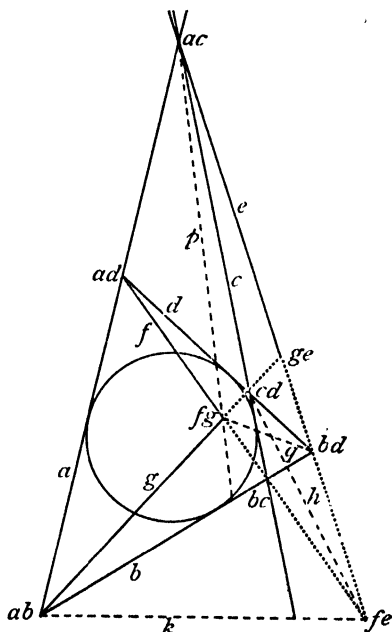


FIG. 100.

THEOREM 79.

With the notation of Theorem 77,

To prove that the points of contact of a, b, c, d form a quadrangle having for its diagonal point triangle, the diagonal line triangle of the quadrilateral $abcd$.

bc lies on the polar (f) of eg ;

$\therefore eg$ lies on the polar of bc ;

$\therefore$ the join of the points of contact of b, c passes through eg .

Similarly, the join of the points of contact of a, d passes through eg ; and the join of the points of contact of c, d passes through ef ; and the join of the points of contact of a, b passes through ef .

Again, bd lies on the polar (e) of fg ;

$\therefore fg$ lies on the polar of bd ;

$\therefore$ the join of the points of contact of b, d passes through fg .

Similarly, the join of the points of contact of a, c passes through fg ; $\therefore ef, fg, ge$ are diagonal points of the quadrangle formed by the points of contact of a, b, c, d .

Q.E.D.

[The reader should draw his own figure and join up the points of contact of a, b, c, d .]

64. PQ is a variable diameter of a fixed circle; A is any fixed point. Find the locus of the orthocentre of the triangle APQ .

65. Two tangents to a fixed circle are given: two others are drawn so as to form with the two fixed tangents a quadrilateral having a pair of opposite sides along the fixed tangents. Prove that the locus of the meet of the internal diagonals of the quadrilateral is a straight line.

66. If a quadrilateral circumscribes a circle, prove that each pair of lines joining the points of contact intersect on some one of the diagonals of the quadrilateral.

67. P is a variable point on the circumcircle of the triangle ABC , AP , BP , CP meet BC , CA , AB at F , G , H ; prove that each side of the triangle FGH passes through a fixed point.

68. PQ is a variable diameter of a fixed circle. A is any fixed point. AP , AQ meet the circle again at R , S ; prove that RS passes through a fixed point.

69. Using a ruler only, construct the polar of a given point w.r.t. a given circle

70. Using a ruler only, construct the pole of a given line w.r.t. a given circle.

71. $ABCD$ is a cyclic quadrilateral inscribed in a circle, centre O ; the diagonals AC , BD meet at G ; TN , $T'N'$ are the perpendiculars from T , T' , the poles of AD , BC , to GO ; prove that $\frac{TN}{T'N'} = \frac{TG}{T'G'}$.

72. If in Fig. 95, PEQ is a chord through E parallel to FG , prove that $\hat{OEP}$ is a right angle, where O is the centre of the circle.

73. PAQ is a chord of a circle; A is a fixed point; B is any other fixed point. BP , BQ meet the circle again at R , S , prove that RS passes through a fixed point.

74. Enunciate the dual theorem corresponding to Ex. 73.

75. The tangents at two points A , B on a circle meet at right angles at T . The tangent at any other point P on the circle cuts AB , AT at Q , R ; prove that TR bisects the angle PTQ .

76. With the notation of Fig. 99, prove that the circle on AC as diameter is orthogonal to the circle PEG .

77. Two sides of a triangle self-conjugate to a given circle cut the circle at four points forming the vertices of a quadrangle $ABCD$, prove that the bisectors of the angles ABC , ADC meet on AC .

78. Any two chords AOB , COD of a fixed circle meet at a given point O ; find the locus of the meet of AC , BD .

79. Enunciate the dual theorem corresponding to Ex. 78.

80. If a circle circumscribes a quadrangle prove that its centre is the orthocentre of the diagonal point triangle.

Enunciate the corresponding theorem for a quadrilateral circumscribing a circle.

81. A quadrangle is inscribed in a circle Σ ; prove that the circles whose diameters are the sides of the diagonal point triangle are orthogonal to Σ .

82. Using a ruler only, construct the tangents to a given circle from a given external point.

83. F, G are conjugate points w.r.t. a circle. A is any point on the circle; FA, GA meet the circle at D, B , GD meets the circle at C . Prove that F, C, B are collinear.

84. Enunciate the dual theorem corresponding to Ex. 83.

85. What is the common self-conjugate triangle of two non-intersecting circles. [Consider the quadrilateral formed by their four common tangents.]

86. The centre A of a circle S lies on a circle Σ ; CD is their common chord: any line through A meets CD at H and Σ at K ; prove that H, K are conjugate points w.r.t. S .

CHAPTER XIII.

COAXAL CIRCLES.

Definition.

THE locus of a point P which moves so that the tangents from P to two circles are of equal length is called the **radical axis** of the two circles.

If A is the centre of a circle, radius a , the length of the tangent from P to the circle is equal to $\sqrt{AP^2 - a^2}$. If P lies inside the circle, the length is therefore imaginary. To secure continuity in the locus of P it is however desirable to include such positions of P (if there are any) in the locus, defined above as the radical axis. The expression $AP^2 - a^2$ is called the **power** of P w.r.t. the circle. Accordingly the definition, given above, may be restated as follows:

Definition.

The locus of a point P which moves so that its powers w.r.t. two circles are equal is called the **radical axis** of the two circles.

1. Through a point P two lines HPK , LPM are drawn to meet two circles, the first at H , K the second at L , M ; if $HP \cdot PK = LP \cdot PM$, prove that P lies on their radical axis.

2. If two circles intersect at A , B , prove that the straight line AB is the radical axis of the two circles.

3. AB , CD are chords of two circles S , Σ ; if $ABCD$ is a cyclic quadrilateral, prove that the meet of AB , CD lies on the radical axis of S , Σ .

4. If two circles touch at A , prove that their radical axis is their common tangent at A .

5. S_1 , S_2 , S_3 are three circles; if the radical axis of S_1 and S_2 meets the radical axis of S_2 and S_3 at H , prove that H lies on the radical axis of S_1 and S_3 .

6. P is a point on the radical axis of two circles, centres A, B , radii a, b ; PN is the perpendicular from P to AB ; prove that $AN^2 - BN^2 = a^2 - b^2$. Hence prove that N is a fixed point and that the radical axis consists of a straight line perpendicular to AB .

7. Σ is a given circle; S_1, S_2, S_3, S_4 , etc. are a system of circles such that the radical axis of any pair is the same as that of any other pair, prove that the radical axis of Σ and S_r passes through a fixed point.

8. H, K are the mid-points of the external common tangents to two circles, prove that HK is their radical axis. [Use Ex. 6.]

9. P, Q, R, S are the mid-points of the common tangents of four non-intersecting circles, prove that P, Q, R, S are collinear. [Use Ex. 6.]

10. Three circles are such that the radical axis of any pair is the same as that of any other pair. If PQ is a real common tangent to two of them, and if it meets the third circle at R, S , prove that $\{PQ; RS\}$ is harmonic.

11. If a circle is orthogonal to each of two circles, prove that its centre lies on their radical axis.

12. A system of circles is such that the radical axis of any pair is the same as that of any other pair. If a circle is drawn orthogonal to two of the circles, prove that it is orthogonal to every circle of the system.

13. Each of a given system of circles is orthogonal to two fixed circles, prove that the radical axis of any two circles of the system is a fixed line.

Many of the properties of coaxal circles admit of a simple analytical treatment. For this purpose the following abbreviations will be useful:

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1,$$

$$S_1(\xi, \eta) \equiv \xi^2 + \eta^2 + 2g_1\xi + 2f_1\eta + c_1,$$

and similarly for S_2, S_3 , etc.

The equation of any circle $S_1 = 0$ can be written in the form

$$(x + g_1)^2 + (y + f_1)^2 = g_1^2 + f_1^2 - c_1.$$

This shows that its centre is $(-g_1, -f_1)$ and its radius

$$= \sqrt{g_1^2 + f_1^2 - c_1}.$$

I. The length of the tangent from (ξ, η) to $S_1 = 0$ is $\sqrt{S_1(\xi, \eta)}$.

TP is the tangent from T to the circle, centre A , whose equation is $S_1 = 0$.

Now, $TP^2 = TA^2 - AP^2$.

But TA^2 equals the square of the line joining (ξ, η) to $(-g_1, -f_1)$;

$$\therefore TA^2 = (\xi + g_1)^2 + (\eta + f_1)^2 ;$$

$$\therefore TP^2 = (\xi + g_1)^2 + (\eta + f_1)^2 - (g_1^2 + f_1^2 - c_1)$$

$$= \xi^2 + \eta^2 + 2g_1\xi + 2f_1\eta + c_1$$

$$= S_1(\xi, \eta) ;$$

$$\therefore TP = \sqrt{S_1(\xi, \eta)}.$$

Q.E.D.

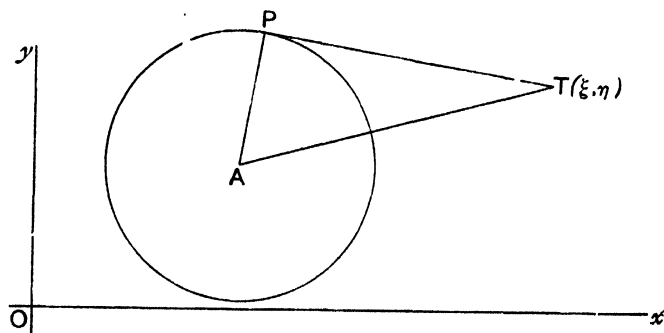


FIG. 101.

II. The radical axis of two circles is a straight line perpendicular to the line joining their centres, and passes through the points of intersection (real or imaginary) of the two circles.

Let $S_1 = 0$ and $S_2 = 0$ be the two circles.

Let (ξ, η) be a point on their radical axis ;

$$\therefore \sqrt{S_1(\xi, \eta)} = \sqrt{S_2(\xi, \eta)} \text{ by definition ;}$$

$$\therefore \text{squaring, } \xi^2 + \eta^2 + 2g_1\xi + 2f_1\eta + c_1 = \xi^2 + \eta^2 + 2g_2\xi + 2f_2\eta + c_2, \quad (1)$$

$$\text{or } 2\xi(g_1 - g_2) + 2\eta(f_1 - f_2) + c_1 - c_2 = 0 ; \dots\dots\dots (2)$$

$$\therefore \text{the locus of } (\xi, \eta) \text{ is a straight line.} \quad \text{Q.E.D.}$$

Now the centres of the circles are $(-g_1, -f_1)$ and $(-g_2, -f_2)$;

$\therefore$ the line joining the centres is $\frac{x+g_1}{g_1-g_2} = \frac{y+f_1}{f_1-f_2}$, which is clearly perpendicular to the radical axis given by

$$x(g_1 - g_2) + y(f_1 - f_2) + \frac{1}{2}(c_1 - c_2) = 0 \text{ from (2).} \quad \text{Q.E.D.}$$

Further, (1) shows that any point whose coordinates satisfy $S_1 = 0$ and $S_2 = 0$ must satisfy the equation of the radical axis.

$\therefore$ the radical axis passes through the common points of $S_1 = 0$ and $S_2 = 0$. Q.E.D.

Definition.

If a system of circles is such that the radical axis of any pair is the same as that of any other pair, the circles are said to form a **coaxal system**.

III. The three radical axes of three circles taken in pairs are concurrent.

Let $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ be the three circles.

Let the radical axis of $S_1 = 0$ and $S_2 = 0$ meet the radical axis of $S_2 = 0$ and $S_3 = 0$ at (ξ, η) ;

$$\therefore S_1(\xi, \eta) = S_2(\xi, \eta),$$

$$\text{and } S_2(\xi, \eta) = S_3(\xi, \eta);$$

$$\therefore S_1(\xi, \eta) = S_3(\xi, \eta);$$

$\therefore (\xi, \eta)$ also lies on the radical axis of $S_1 = 0$ and $S_3 = 0$. Q.E.D.

14. How is this proof affected if two of the radical axes are parallel?

IV. A point P moves so that the lengths of the tangents from P to two given circles are in a constant ratio λ ; then the locus of P is a circle coaxal with the given circles.

Let $S_1 = 0$ and $S_2 = 0$ be the given circles.

Let (ξ, η) be the coordinates of P in any one position.

Then $\frac{S_1(\xi, \eta)}{S_2(\xi, \eta)} = \lambda^2$, or $S_1(\xi, \eta) - \lambda^2 S_2(\xi, \eta) = 0$;

$\therefore$ the locus of P is $S_1 - \lambda^2 S_2 = 0$,

which is a circle, for the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero.

Further, it is coaxal with $S_1 = 0$ and $S_2 = 0$, for the coordinates of any point which satisfy $S_1 = 0$ and $S_2 = 0$ must also satisfy $S_1 - \lambda^2 S_2 = 0$; so that it passes through the points of intersection of $S_1 = 0$ and $S_2 = 0$ and is therefore coaxal with them by II.

Q.E.D.

15. Prove independently, by using the method of I., that the tangents from any point (ξ, η) on the line $S_1 - \lambda^2 S_2 = 0$ to the circles $S_1 = 0$, $S_2 = 0$, $S_1 - \lambda^2 S_2 = 0$ are equal.

V. P is a variable point on a given circle coaxal with two given circles $S_1 = 0$, $S_2 = 0$; then the ratio of the lengths of the tangents from P to these two circles is constant.

The proof is left to the reader.

16. Prove V., using the fact that the equation of any circle coaxal with $S_1 = 0$, and $S_2 = 0$ is of the form $S_1 - k S_2 = 0$, since it must pass through their points of intersection.

VI. To find the general equation of a coaxial system of circles.

Take the line of centres as x -axis, and the radical axis as the y -axis. [All the centres must be collinear, because the line joining every pair is perpendicular to a definite line, viz. the radical axis.]

The equation of any circle whose centre is on the x -axis is

$$x^2 + y^2 + 2g_1x + c_1 = 0.$$

Let $x^2 + y^2 + 2g_2x + c_2 = 0$ represent any other circle of the coaxial system;

$$\therefore \text{ the radical axis is } 2(g_1 - g_2)x + c_1 - c_2 = 0.$$

But the radical axis is $x = 0$;

$$\therefore c_1 - c_2 = 0 \text{ or } c_1 = c_2;$$

$\therefore$ every circle of the coaxial system has an equation of the form $x^2 + y^2 + 2g_r x + c = 0$, where c is the same for every circle; which is the required result.

The equation can be written $(x + g_r)^2 + y^2 = g_r^2 - c$.

Its radius is therefore zero if $g_r^2 - c = 0$ or $g_r = \pm \sqrt{c}$.

In this case, the circle reduces to a point circle

$$(x \pm \sqrt{c})^2 + y^2 = 0.$$

These two points (or point circles), $(+\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$ belong to the coaxial system, and are called the **limiting points** of the system.

17. Prove that the limiting points are real if the radical axis does not meet any circle of the system in real points.

18. Prove that the limiting points are inverse points w.r.t. any circle of the system.

19. Prove that any circle through the points $(+\sqrt{c}, 0)$; $(-\sqrt{c}, 0)$ is orthogonal to any circle of the form $x^2 + y^2 + 2g_r x + c = 0$.

20. Prove that the distance of any point P on the radical axis from either limiting point is equal to the length of the tangent from P to any circle of the coaxial system.

VII. The polars of a given point P , w.r.t. the circles of a coaxial system, all pass through the same point Q ; and PQ is bisected by the radical axis.

Let (ξ, η) be the coordinates of P , and let $x^2 + y^2 + 2g_r x + c = 0$ be any circle of the system.

The polar of (ξ, η) is $x\xi + y\eta + g_r(x + \xi) + c = 0$, or

$$[x\xi + y\eta + c] + g_r[x + \xi] = 0.$$

Now for all values of g , this line passes through the point Q given by

$$\begin{cases} x\xi + y\eta + c = 0, \\ x + \xi = 0. \end{cases} \quad \text{Q. E. D.}$$

The coordinates of Q are therefore $\left(-\xi, \frac{\xi^2 - c}{\eta}\right)$;

$\therefore$ the coordinates of the mid-point of PQ are $\left(0, \frac{\xi^2 - c}{2\eta} + \frac{1}{2}\eta\right)$;

$\therefore$ the mid-point of PQ lies on $x=0$, *i.e.* on the radical axis.

Q. E. D.

21. P is a point on the radical axis of two circles, prove by analysis that the polars of P w.r.t. the circles intersect on their radical axis.

THEOREM 80.

The radical axis of two circles is a straight line perpendicular to the line joining the centres of the circles.

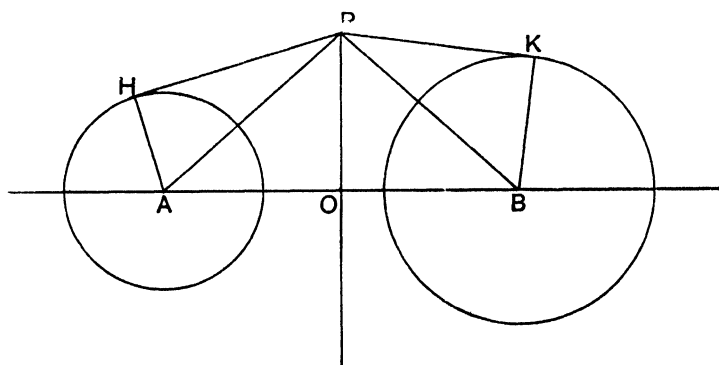


FIG. 102.

A, B are the centres of the two circles.

Let P be a point such that the tangents PH, PK are equal.

Draw PO perpendicular to AB .

$$PH^2 = PK^2;$$

$$\therefore PA^2 - AH^2 = PB^2 - BK^2;$$

$$\therefore PO^2 + OA^2 - AH^2 = PO^2 + OB^2 - BK^2;$$

$$\therefore OA^2 - OB^2 = AH^2 - BK^2;$$

$$\therefore (AO + OB)(AO - OB) \text{ is constant};$$

$$\therefore AB(AO - OB) \text{ is constant};$$

$\therefore$ since $AO + OB$ and $AO - OB$ are each constant, it follows that O is a fixed point;

$\therefore$ the locus of P is a line through O perpendicular to AB .

Q. E. D.

If the circles intersect at real points C, D ; then P lies on the common chord CD . When P lies between C and D , the tangents are imaginary, but the relation $PA^2 - AH^2 = PB^2 - BK^2$ still holds good. The segment CD is therefore also regarded as part of the radical axis. [See the second definition on page 172.]

22. Prove the following construction for the radical axis of two circles, centres A, B .

Draw a circle cutting one circle at L, M and the other at X, Y ; produce LM, XY to meet at P . Draw PO perpendicular to AB ; then PO is the radical axis.

23. Prove that the radical axis of the incircle of the triangle ABC and the circle escribed to BC bisects BC .

24. A, B, C, D are four circles; the radical axis of A and B is perpendicular to that of C and D ; the radical axis of A and C is perpendicular to that of B and D ; prove that the radical axis of A and D is perpendicular to that of B and C .

25. A, B, C are the centres of three circles; if the three radical axes of these circles, taken in pairs, are such that one is equally inclined to the other two, prove that ABC is an isosceles triangle.

THEOREM 81.

S_1, S_2, S_3 are three circles; prove that the radical axes of S_1 and S_2 ; S_2 and S_3 ; S_3 and S_1 are concurrent.

Let L_3, L_1, L_2 be the radical axes of S_1, S_2 ; S_2, S_3 ; S_3, S_1 .

Either L_1, L_2, L_3 are all parallel, or else at least two (say L_1 and L_2) intersect: call their point of intersection O .

Since O lies on L_1 , the tangents from O to S_2 and S_3 are equal.

Since O lies on L_2 , the tangents from O to S_3 and S_1 are equal;

$\therefore$ the tangents from O to S_2 and S_1 are equal;

$\therefore O$ lies on L_3 .

Q.E.D.

Note that the point O may be at infinity. What verbal changes should be made if O lies inside a circle?

Definition.

(1) The point of intersection of the three radical axes of three circles taken in pairs is called the **radical centre** of the three circles.

(2) A system of circles is said to be **coaxal** if the radical axis of any pair is the same as that of any other pair.

Hence if a system of circles pass through two fixed points A, B , they form a coaxal system, having AB as their radical axis.

26. Three circles are such that each touches the other two; if A, B, C are the points of contact, prove that the tangents at A, B, C are concurrent.

27. A variable circle Σ cuts two fixed circles S_1 and S_2 ; prove that the common chords of S_1, Σ and S_2, Σ meet on a fixed line.

28. Three circles are such that each intersects the other two at real points, prove that their common chords are concurrent.

29. PP', QQ', RR' are pairs of points on the sides BC, CA, AB of the triangle ABC . If $PP'QQ'$; $QQ'RR'$; $RR'PP'$ are cyclic quadrilaterals, prove that $PP'QQ'RR'$ lie on the same circle.

30. Straight lines AP, BQ, CR are drawn from the vertices of the triangle ABC to the opposite sides; prove that the radical centre of the circles whose diameters are AP, BQ, CR is the orthocentre of the triangle ABC .

31. A variable circle is drawn through a fixed point to touch a given circle. Find the locus of the point of intersection of the tangent at the point of contact of the two circles with the tangent at the fixed point.

32. Two circles intersect at P, Q and cut a third circle S orthogonally, prove that P, Q are inverse points w.r.t. S .

33. Prove that the orthocentre of a triangle is the radical centre of the three circles described on the sides of the triangle as diameters.

34. $ABCD$ is a cyclic quadrilateral. Two circles PAB, PCD are drawn to touch at P , prove that the locus of P is a circle.

35. Prove that the locus of the centre of a circle which bisects the circumferences of two given circles is a straight line.

36. AD is an altitude of the triangle ABC ; the circles BAD, CAD meet the bisector of the angle BAC at P, Q ; prove that the tangents at P, Q meet on AD .

37. $A'B'C'$ are the mid-points of the sides BC, CA, AB of a triangle; prove that the external bisector of the angle $C'A'B'$ is the radical axis of the incircle of the triangle ABC and the circle described to BC .

38. Show how to construct a circle orthogonal to each of three given circles.

THEOREM 82.

If a system of circles is coaxial, then the centres of the circles are collinear: also, if the line of centres meets the radical axis at O , and if a_r is the radius of a circle of the system whose centre is A_r , then $OA_r^2 - a_r^2$ is a constant for the system.

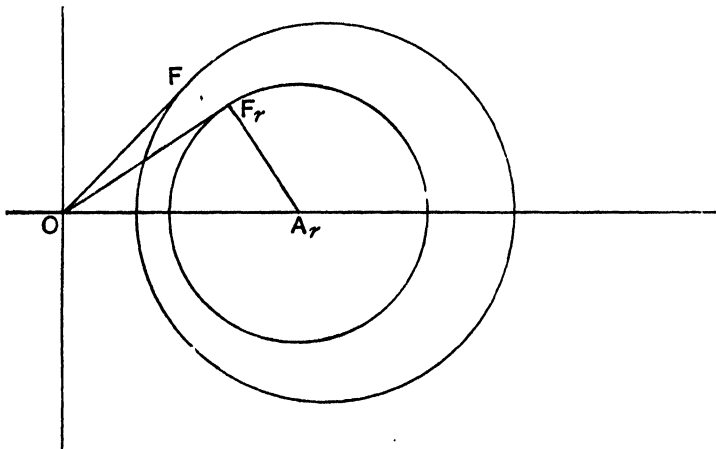


FIG. 103.

(1) If A_s is the centre of any other circle of the system, $A_r A_s$ is perpendicular to the radical axis; [Th. 80]

$\therefore A_s$ lies on the perpendicular from A_r to the radical axis;

$\therefore$ the centre of every circle of the system lies on this line.

Q.E.D.

(2) From O draw OF_r to touch the circle centre A_r , and let OF be a tangent from O to any other circle of the system.

Then $OF_r^2 + F_r A_r^2 = OA_r^2$;

$$\therefore OA_r^2 - a_r^2 = OF_r^2$$

$= OF^2$ since O lies on the radical axis.

$= k^2$, where k is a constant.

Q.E.D.

Corollary.

If the centres of a system of circles are collinear; and if A_r is the centre of one of the circles, radius a_r , and if $OA_r^2 - a_r^2$ is constant, where O is a fixed point on the line of centres, then the circles form a coaxial system.

What modification must be made in this proof; if the circles intersect at real points?

39. Prove the Corollary of Theorem 82.

With the notation of Theorem 82, two points L, L' are taken on the line of centres such that $L'O = OL = k$; then the radii of the two circles of the system, centres L, L' , are given by $OL^2 - (\text{radius})^2 = k^2 = OL^2$, and therefore each radius is zero.

Hence there are two point-circles L and L' which belong to the coaxal system.

Definition.

The two point-circles of a coaxal system are called the **limiting points** of the system.

It should be noted that the positions of L, L' are easily found by observing that the tangents from O to each circle of the system are equal, and that in the case of a point-circle L the tangent is the line OL , so that $L'O^2 = OL^2 = OF_r^2 = OA_r^2 - a^2 = k^2$.

If $OA_r^2 - a^2$ is negative, k is imaginary, and consequently the limiting points are imaginary. This occurs when the radical axis meets one (and therefore every) circle of the coaxal system at real points, for then OA_r is less than the radius a .

In the following examples L, L' will be taken to represent the limiting points of the coaxal system referred to, unless otherwise stated.

40. The radical axis is the perpendicular bisector of the line joining the limiting points.

41. A system of circles are drawn orthogonal to a fixed circle and having their centres on a diameter of that circle, prove that they form a coaxal system with real limiting points.

42. With the notation of Theorem 82, take $k=1$ inch and construct pairs of circles of radii $\frac{1}{2}$ inch, 1 inch, $1\frac{1}{2}$ inches, 2 inches, $2\frac{1}{2}$ inches, 3 inches, to form a coaxal system. Mark in the limiting points. Draw a system of circles passing through the limiting points.

The figure now shows two systems of coaxal circles, one of which has real, and the other imaginary, points of intersection.

43. If L and L' are real, prove that no real circle of the coaxal system can have its centre between L and L' .

44. Prove that the circle on LL' as diameter is orthogonal to every circle of the coaxal system.

45. Prove that L, L' are inverse points w.r.t. any circle of the coaxal system.

46. Prove that any circle through L, L' is orthogonal to any circle of the coaxal system.

47. P is any point on the radical axis of a coaxal system: PH is the tangent from P to a circle of the system, prove that $PH=PL$.

48. PQ is a common tangent to two circles and L is one of their limiting points, prove that $\hat{PLQ}=90^\circ$.

49. Prove that if a circle is orthogonal to each of two circles, centres A and B , it meets AB at the limiting points of the two circles.

50. If two systems of coaxal circles have one limiting point in common, prove that the radical axis of any pair of circles, one from each system, passes through a fixed point.

51. If two systems of coaxal circles have one common circle, prove the property of Ex. 50.

52. Show how to construct a circle of a given coaxal system to touch another given circle: is there more than one solution?

53. If three circles are coaxal, prove that the common tangent of two of them is divided harmonically by the third circle.

54. A system of coaxal circles meet at A, B ; AC is a fixed line drawn through A meeting one of the circles at P , prove that the locus of the foot of the perpendicular from B to the tangent at P is a straight line passing through the foot of the perpendicular from B to AC .

55. A variable circle passes through two fixed points A, B , and cuts a fixed circle at P, Q . Prove that $\frac{AP \cdot AQ}{BP \cdot BQ}$ is constant.

56. If two coaxal systems have one circle in common, prove that they also have one common orthogonal circle.

57. Prove that the radical axes of a fixed circle and each of a system of coaxal circles are concurrent.

58. Prove that the tangent from a limiting point to any circle of the coaxal system is bisected by the radical axis.

59. A variable circle passes through two fixed points A, B and cuts a fixed circle at P, Q ; prove that PQ passes through a fixed point.

60. AB, CD are two parallel chords of two circles S_1, S_2 ; AC and BD meet S_1 at H, K ; prove that the locus of the meet of CD, HK is a straight line.

61. S_1, S_2 are two circles each orthogonal to the two circles Σ_1, Σ_2 ; prove that the radical axis of S_1 and S_2 passes through the centres of Σ_1, Σ_2 .

62. If two circles cut each of a system of circles orthogonally, prove that the system is coaxal.

63. Describe a circle to pass through two given points and to bisect the circumference of a given circle.

64. AD , BE , CF are the altitudes of the triangle ABC ; P is any other point, prove that the circles PAD , PBE , PCF are coaxal.

65. $ABCD$ is a cyclic quadrilateral: S is any circle having A , B as limiting points; Σ is any circle having C , D as limiting points. Prove that the radical axis of S and Σ passes through a fixed point.

66. A system of coaxal circles passes through the points A , B ; another system of coaxal circles has A , B as limiting points; prove that any circle of one system is orthogonal to any circle of the other.

67. A system of circles is drawn so that each bisects the circumferences of two given circles, prove that the system is coaxal.

THEOREM 83.

The limiting points of a coaxal system are inverse points w.r.t. any circle of the system. Conversely, if two points are inverse points w.r.t. each of a system of circles, then the system is coaxal, and has these two points as limiting points.

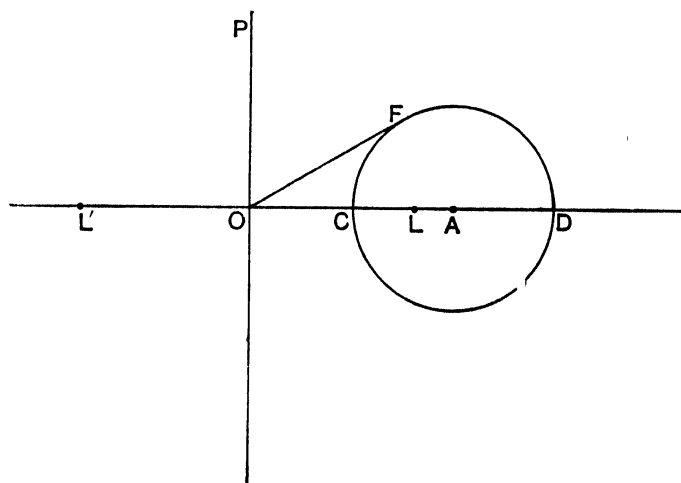


FIG. 104.

(1) Let OP be the radical axis of the system, and L , L' the limiting points.

A is the centre of any one circle of the system: the line of centres meets that circle at C , D and the radical axis at O .

Draw OF to touch the circle.

Then $OL'^2 = OL^2 = OF^2$, since L , L' are point-circles of the system
 $\quad \quad \quad = OC \cdot OD$;

$\therefore \{L'L; CD\}$ is harmonic: but A is the mid-point of CD ;

$\therefore AC^2 = AL \cdot AL'$;

$\therefore L$, L' are inverse points w.r.t. the circle. Q.E.D.

(2) Let L, L' be inverse points w.r.t. each circle of a system.

Then the centres of the circles are collinear, for each must lie on LL' .

Let O be the mid-point of LL' and let LL' meet one of the circles, centre A , radius a , at C, D .

Since L, L' are inverse points $AD^2 = AC^2 = AL \cdot AL'$;

$\therefore \{CD, LL'\}$ is harmonic;

$\therefore OL^2 = OC \cdot OD = OA^2 - a^2$, since O is the mid-point of LL' ;

$\therefore OA^2 - a^2$ is constant and O is a fixed point;

$\therefore$ the system of circles is coaxal, and L, L' are the limiting points. [Th. 82, Cor.]

Q.E.D.

Another method may be used to prove Theorem 83 (2). This is embodied in Ex. 68.

68. L, L' are inverse points w.r.t. each of a system of circles; P is a point on the perpendicular bisector of LL' . Use Theorem 64 (2) to prove that the tangents from P to each of the circles of the system are equal, and hence show that the system is coaxal and has L, L' as limiting points.

69. Show how to draw through a given point a circle orthogonal to two given circles.

70. Draw a circle, coaxal with two given circles, to touch a given straight line. Is there more than one solution? Consider the case where the straight line passes through one or both limiting points.

71. A, B are two fixed points; P is a point such that $\frac{PA}{PB} = \lambda$, then P describes a circle Σ . Prove that, as λ varies, the circles Σ form a coaxal system.

72. L is a limiting point of two circles; the line joining their centres meets one circle at A, B and the other at A', B' . If $\frac{BL}{LA} = \frac{B'L}{A'L}$, prove that the circles are equal.

73. A straight line $ABCD$ meets one circle at A, B and another circle at C, D ; if L is a limiting point, prove that $\hat{ALD} + \hat{BLC} = 180^\circ$.

74. A chord PQ of one circle touches a second circle at R ; if L is a limiting point of the two circles, prove that LR bisects $\hat{PLQ}$.

75. If a circle cuts each of two circles orthogonally, prove that its centre lies on the radical axis, and that it cuts every circle coaxal with the given circles orthogonally. What is meant by the circle cutting the point-circles orthogonally?

THEOREM 84.

If a circle cuts each of two circles orthogonally, it passes through their limiting points, and is orthogonal to every circle coaxal with them.

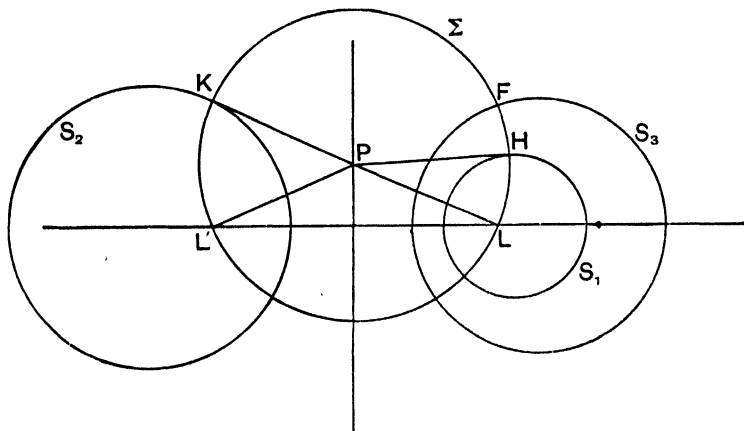


FIG. 105.

Let the circle Σ , centre P , cut the circles S_1 , S_2 orthogonally at H , K .

Then $PH = PK$ radii; but PH , PK are tangents to S_1 , S_2 ;
 $\therefore P$ lies on the radical axis of S_1 and S_2 .

Let L , L' be the limiting points of S_1 and S_2 ; and regard them as point-circles. Since P lies on the radical axis, and since PL , PL' are tangents to the point-circles L , L' ,

$\therefore PL = PL' = PH$; $\therefore L$, L' lie on Σ . Q.E.D.

Let any other circle S_3 of the system cut Σ at F .

Then $PF = PH$, radii.

But P lies on the radical axis, therefore PF is a tangent to S_3 ;
 $\therefore \Sigma$ is orthogonal to S_3 . Q.E.D.

76. Prove that the three pairs of limiting points of three circles taken in pairs are concyclic.

77. AB is a common tangent to two non-intersecting circles, prove that the circle on AB as diameter passes through their limiting points.

78. The points A , B , C , taken in pairs, are the limiting points of three systems of coaxal circles, prove that the radical centre of any three circles, selected one from each system, is a fixed point.

79. Prove that the four circles, whose diameters are the four common tangents of two non-intersecting circles, are coaxal.

80. Prove that any circle through the limiting points of a coaxal system is orthogonal to every circle of the system.

THEOREM 85.

The polars of a given point P w.r.t. each circle of a coaxial system pass through a fixed point R ; and PR is bisected by the radical axis.

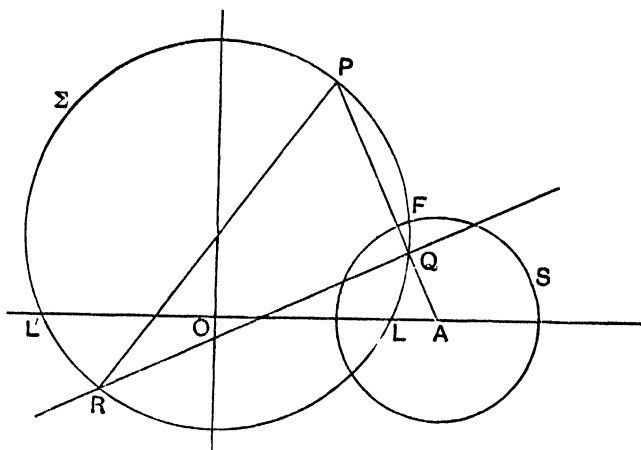


FIG. 106.

P is the given point.

Let L, L' be the limiting points of the coaxial system.

Describe a circle Σ through P, L, L' .

Then Σ is orthogonal to every circle of the coaxial system, for L, L' are inverse points w.r.t. any circle of the system.

Let PR be a diameter of Σ so that R is a fixed point.

Let A be the centre of any circle S of the system.

AP meets Σ at Q ; join QR .

Since S is orthogonal to Σ , AF touches Σ ;

$$\therefore AQ \cdot AP = AF^2;$$

$\therefore Q, P$ are inverse points w.r.t. S .

But $\angle PQR = 90^\circ$ since PR is a diameter of Σ ;

$\therefore RQ$ is the polar of P w.r.t. S ;

$\therefore$ the polar of P w.r.t. S passes through a fixed point R . Q.E.D.

Further, since Σ passes through L, L' , its centre lies on the perpendicular bisector of LL' , i.e. on the radical axis;

$\therefore PR$ meets the radical axis at the centre of Σ ;

$\therefore PR$ is bisected by the radical axis. Q.E.D.

It is well to point out the importance of regarding the limiting points as point-circles of the coaxial system.

Readiness in recognising this fact is frequently of use in solving riders. The fundamental pole and polar property might have been obtained in this way, as the following example will show.

I. P is a given point; a variable line through P meets a given circle Σ , centre O , at H, K ; Q is a point on HK such that $\{HK, PQ\}$ is harmonic; then the locus of Q is a straight line perpendicular to OP .

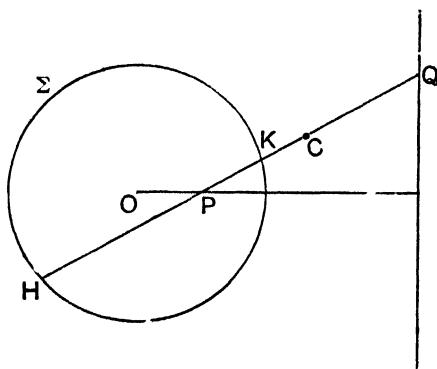


FIG. 107.

Bisect PQ at C .

Since $\{HK, PQ\}$ is harmonic, $CH \cdot CK = CP^2$;

$\therefore C$ is a point on the radical axis of Σ and the point-circle P ;

$\therefore$ the locus of C is a straight line perpendicular to OP .

But P is a fixed point and $PC = CQ$;

$\therefore$ the locus of Q is a straight line perpendicular to OP . Q.E.D.

Another example is given to emphasise the same idea.

II. H, K are the mid-points of the tangents from a point T to a circle Σ ; P is any point on HK ; prove that the tangent PQ from P to Σ is equal to PT .

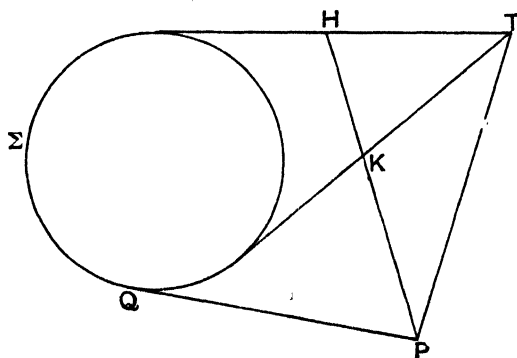


FIG. 108.

Consider the coaxal system formed by Σ and the point-circle T .

Since the tangents from H and also from K are equal, it follows that HK is the radical axis;

$\therefore PT = PQ$, since P lies on the radical axis. Q.E.D.

81. A variable circle passes through a fixed point and cuts a given circle orthogonally; prove that it belongs to a fixed coaxial system. [Use the idea of II.]

82. From a limiting point L a tangent LP is drawn to a circle of the coaxial system and cuts another circle of the system at HK ; prove that $\{LP, HK\}$ is harmonic.

83. L is the polar of a fixed point P w.r.t. any circle of a given coaxial system; by considering P as a point-circle, prove that the line parallel to L and midway between L and P passes through a fixed point, and hence prove Theorem 85.

84. PQ is a variable chord of a fixed circle S ; PQ passes through a fixed point A ; Σ and Σ' are the circles on AP , AQ as diameters. Prove that the common chord of S and Σ meets the common chord of S and Σ' on a fixed straight line.

85. OP , OP' are the tangents from O to a circle; T is any point on the line RR' bisecting OP , OP' ; if the polar of T meets RR' at Q , prove that $\angle TOQ$ is a right angle.

86. QR is the tangent at Q to a circle; P is any other point; PQ is joined, and PR is drawn perpendicular to PQ . If the rectangle $PQRS$ is completed, prove that the polar of P passes through S .

[Prove that the centre of the rectangle lies on the radical axis of the given circle and the point-circle at P .]

87. From two fixed points A , B the tangents AP , AQ , BL , BM are drawn to a variable circle; if P' , Q' , L' , M' are the mid-points of these four lines, prove that the locus of the meet of $P'Q'$ and $L'M'$ is a straight line.

88. The polars of a variable point P w.r.t. two fixed circles meet at Q ; if PQ subtends a right angle at a fixed point A , find the locus of P .

THEOREM 86.

(1) The orthocentres of the four triangles formed by four straight lines are collinear.

(2) The mid-points of the three diagonals of a complete quadrilateral are collinear.

(3) The circles on the three diagonals of a complete quadrilateral are coaxial, and are cut orthogonally by the polar circles of the four triangles formed by the sides of the quadrilateral.

FG is the third diagonal of the quadrilateral $ABCD$.

Let H_1 be the orthocentre of the triangle ABF ; AH_1 , BH_1 , FH_1 meet the opposite sides BF , FA , AB at A_1 , B_1 , F_1 .

Since H_1 is the orthocentre,

$$H_1A \cdot H_1A_1 = H_1B \cdot H_1B_1 = H_1F \cdot H_1F_1 = r_1^2 \text{ (say).}$$

But $\{AA_1, BB_1, FF_1\}$ are pairs of points on the circles whose diameters are AC, BD, FG respectively;

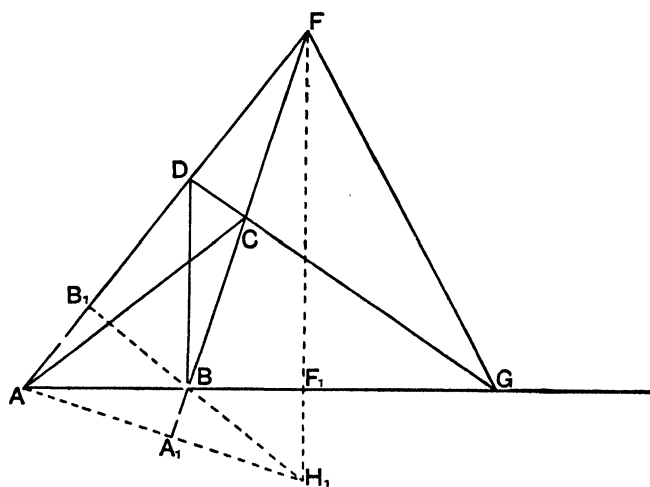


FIG. 109.

$\therefore$ the tangents from H_1 to these three circles are equal. In exactly the same way the tangents from H_2, H_3, H_4 , the orthocentres of the triangles ADG, DCF, CBG , are equal;

$\therefore$ these three circles are coaxal, and H_1, H_2, H_3, H_4 lie on their radical axis;

$\therefore$ the points H_1, H_2, H_3, H_4 are collinear. Q.E.D.

Also the centres of the circles, which are the mid-points of AC, BD, FG are collinear, since the circles are coaxal. Q.E.D.

Finally, the circle, centre H_1 , radius r_1 , is orthogonal to each of these circles. But this is the polar circle of the triangle ABF ;

$\therefore$ the four polar circles are orthogonal to the circles on diameters AC, BD, FG . Q.E.D.

It should be observed that the polar circles are not all real, for the polar circle is real, only if the triangle is obtuse-angled. And the tangents from H_1, H_2, H_3, H_4 to the circles on AC, BD, FG are not all real, but this fact does not affect the proof. [See the second definition, page 172.]

The reader is reminded that the circumcircle of the diagonal line triangle is also orthogonal to these three circles. [See Ex. 89.]

89. If in Fig. 109 AC meets BD at P and FG at R , use the fact that $\{APCQ\}$ is harmonic to prove that the circumcircle of the diagonal line triangle of the complete quadrilateral is orthogonal to the circles on AC , BD , FG as diameters.

90. Find the locus of a point P which moves so that its polars w.r.t. three given circles are concurrent.

91. If each of a system of circles is cut orthogonally by two given circles, prove that the system is coaxial.

92. A variable circle touches a given line at a given point, prove that the polar of any other fixed point passes through a fixed point.

93. A line is drawn cutting two non-intersecting circles; find two points on this line such that each is the point of intersection of the polars of the other w.r.t. the two circles.

94. PQ is a common tangent to two circles; prove that P and Q are conjugate points w.r.t. any coaxial circle.

95. P is a point on the radical axis of two circles, prove that the polars of P w.r.t. the two circles intersect on the radical axis. [Do not use the *result* of Theorem 85.]

96. L, L' are the limiting points of a coaxial system. A is the centre of any circle of the system: AP is a diameter of any circle through A, L ; prove that the polar of P w.r.t. any circle of the system passes through a fixed point.

97. AB is a common tangent of two circles; prove that the polar of A w.r.t. any coaxial circle passes through B .

98. If the polars of a point w.r.t. three circles, whose centres are collinear, are concurrent, prove that the three circles are coaxial.

99. If the polar of a given point w.r.t. a variable circle is fixed, prove that the polar of any other given point passes through another fixed point.

100. A variable circle is drawn through two fixed points. C is a third fixed point. Find the locus of the meet of the diameter through C with the polar of C w.r.t. the circle.

101. A variable circle has two pairs of fixed conjugate points; prove that it belongs to a fixed system of coaxial circles.

102. P is any point on a given circle S_1 ; the tangent at P to S_1 meets the polar of P w.r.t. a second circle S_2 at Q ; prove that the circle on PQ as diameter belongs to a fixed coaxial system.

103. In a trapezium $ABCD$, AB and CD are parallel; AD meets BC at E . Prove that the common chord of the circles on AC and BD as diameters passes through E and is perpendicular to AB .

104. Two circles lying outside each other are met by a variable straight line at A, B and P, Q respectively; prove that there are two

possible positions of a point H such that the bisectors of the angles AHB , PHQ are at right angles.

105. Prove that the circumcentre of the diagonal line triangle of a complete quadrilateral is collinear with the orthocentres of the triangles formed by the sides of the quadrilateral.

106. Prove that the line joining the orthocentres of the triangles formed by the four sides of a quadrilateral is perpendicular to the line joining the mid-points of the diagonals of the quadrilateral.

107. A system of concentric circles, centre A , is inverted w.r.t. any point B ; prove that the inverse system is cut orthogonally by every circle through B and the inverse point A' of A , and hence show that a system of concentric circles inverts into a system of coaxial circles.

108. If a system of coaxial circles has real points of intersection, prove that it can be inverted into a system of concurrent straight lines.

109. If a system of coaxial circles has real limiting points, prove that it can be inverted into concentric circles.

[Invert w.r.t. a limiting point. What does the other limiting point invert into?]

THEOREM 87.

PH , PK are the tangents from a point P to two circles, centres A , B ; PN is the perpendicular to their radical axis; prove that $PK^2 - PH^2 = 2AB \cdot PN$.

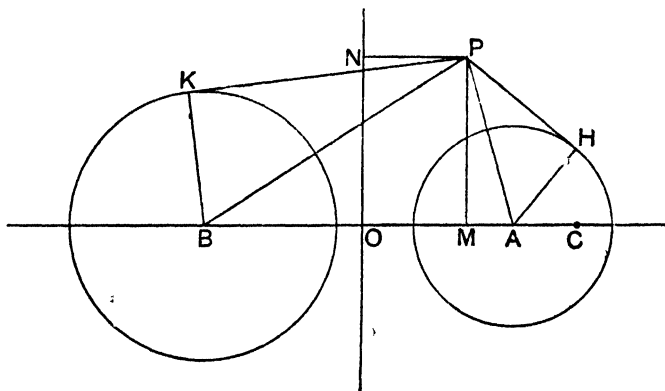


FIG. 110.

Let the radical axis meet AB at O . Draw PM perpendicular to AB .

$$\begin{aligned} PK^2 - PH^2 &= (PB^2 - BK^2) - (PA^2 - AH^2) \\ &= PB^2 - PA^2 - (BK^2 - AH^2). \end{aligned}$$

Now since O lies on the radical axis, the tangents from O are equal;

$$\therefore OA^2 - AH^2 = OB^2 - BK^2,$$

$$\text{or } BK^2 - AH^2 = OB^2 - OA^2;$$

$$\begin{aligned} \therefore PK^2 - PH^2 &= (PM^2 + MB^2) - (PM^2 + MA^2) - (OB^2 - OA^2) \\ &= MB^2 - MA^2 - (OB^2 - OA^2) \\ &= (BM + MA)(BM - MA) - (BO + OA)(BO - OA) \\ &= BA(BM - MA - BO + OA) \\ &= BA(BM - BO + OA - MA) \\ &= BA \cdot 2OM = 2AB \cdot MO \\ &= 2AB \cdot PN. \end{aligned}$$

Q.E.D.

Notice that if P is to the left of the radical axis, $PK^2 - PH^2$ is negative, but in this case PN and AB are in opposite senses, so that the relation holds good in every case.

Theorem 87 admits of a simple analytical proof.

In Fig. 110, take OA , ON as x -axis and y -axis.

Let (ξ, η) be the coordinates of P .

The equations of the two circles may therefore be written (page 176.)

$$x^2 + y^2 - 2g_1x + c = 0 \text{ and } x^2 + y^2 - 2g_2x + c = 0;$$

$$\begin{aligned} \therefore PK^2 - PH^2 &= (\xi^2 + \eta^2 - 2g_2\xi + c) - (\xi^2 + \eta^2 - 2g_1\xi + c) \\ &= 2\xi(g_1 - g_2). \end{aligned}$$

But $OA = g_1$; $OB = g_2$; $\therefore g_1 - g_2 = OA - OB = OA - BO = BA$ and $\xi = NP$;

$$\therefore PK^2 - PH^2 = 2NP \cdot BA = 2PN \cdot AB. \quad \text{Q.E.D.}$$

Corollary 1.

From a point P on a circle, centre A , a tangent PK is drawn to a circle, centre B ; PN is the perpendicular from P to the radical axis of the two circles; then $PK^2 = 2AB \cdot PN$.

[Take P at H in Theorem 87.]

Corollary 2.

From a variable point P on a fixed circle, centre A , tangents PH , PK are drawn to two other circles, centres B , C ; if the three circles are coaxal, then $\frac{PH}{PK}$ is constant.

The proof is left to the reader (see Ex. 110).

110. Prove Corollary 2 of Theorem 87 by using Corollary 1.

111. Prove Corollary 1 of Theorem 87 without using the *result* of Theorem 87.

THEOREM 88.

A point P moves so that the tangents PH , PK from it to two fixed circles, centres A , B , are in a constant ratio, then the locus of P is a circle coaxal with the given circles.

Take any one position of P and draw through P a circle coaxal with the given circles and let C be its centre.

[See Fig. 110.] PN is the perpendicular from P to the radical axis

By Theorem 87, Corollary 1. $PH^2 = 2CA \cdot PN$,

and $PK^2 = 2CB \cdot PN$;

$$\therefore \frac{CA}{CB} = \frac{PH^2}{PK^2} = \text{constant, by hypothesis};$$

$\therefore C$ is a fixed point;

$\therefore$ every position of P is on the same coaxal circle.

Q.E.D.

112. Prove that the locus of a point, which moves so that the difference of the squares of the tangents from it to two given circles is constant, is a straight line.

Give an analytical *in addition* to a geometrical proof.

113. L is a limiting point of a coaxal system. P is any point on a given circle of the system, prove that PL^2 varies as the distance of P from the radical axis.

114. The external common tangents of two non-intersecting circles meet at O , and the internal common tangents at O' , prove that the circle on OO' as diameter is coaxal with the given circles. [Use Theorem 88, taking P successively at O , O' .]

115. Given three circles S_1 , S_2 , S_3 , find a point P such that the tangents from P to S_1 and S_2 are in a given ratio, and such that the tangent from P to S_3 is of a given length.

116. The square of the tangent from a variable point P to a fixed circle is proportional to the sum of the squares of the tangents from P to two other fixed circles; if the three circles are coaxal, prove that the locus of P is a circle of the same coaxal system.

117. S_1 , S_2 , S_3 , S_4 are four coaxal circles: the product of the tangents from a variable point P to S_1 and S_2 is proportional to the product of the tangents from P to S_3 and S_4 , prove that the locus of P is in general one of two coaxal circles.

118. Generalise Ex. 117 so as to obtain a theorem for $2n$ circles.

119. Find the locus of a point P which moves so that the tangent from P to a fixed circle is (1) equal to the distance of P from a fixed point, (2) in a constant ratio to the distance of P from a fixed point.

120. Two circles touch each other internally at O , and a line $PQRS$ is drawn, meeting one circle at P, S and the other at Q, R . The tangent at P cuts the tangents at Q, R in A, B ; the tangent at S cuts the tangents at Q, R in C, D : prove that $ABCD$ lie on a circle touching the given circles at O .

121. The tangents at A, B, C to the circumcircle of the triangle ABC cut the opposite sides at P, Q, R ; U, V, W are the mid-points of AP, BQ, CR . Prove that UVW is the radical axis of the circumcircle and the polar circle of the triangle ABC .

122. If each of two pairs of opposite vertices of a complete quadrilateral is conjugate w.r.t. a circle, prove that the third pair is also conjugate w.r.t. the circle; and that the circle is one of a coaxial system whose radical axis is the line of collinearity or the mid-points of the diagonals.

123. A straight line PQ cuts two fixed circles so that the chords PP', QQ' are in a constant ratio; the tangents at P and Q meet at T , prove that the locus of T is a circle coaxial with the given circles.

124. A circle Σ touches two given circles and cuts their radical axis at Q, Q' ; prove that the tangents at Q, Q' are parallel to a pair of common tangents of the given circles.

[Use Theorem 57, taking P at the centre of Σ .]

125. From a point P , three tangents PA_1, PB_1, PC_1 are drawn to three coaxial circles, centres A, B, C ; prove that

$$PA_1^2 \cdot BC + PB_1^2 \cdot CA + PC_1^2 \cdot AB = 0.$$

126. A straight line cuts one circle at A, B and a second circle at C, D . The tangent at A meets the tangents at C, D at P, Q , and the tangent at B meets the tangents at C, D at R, S . Prove that $PQRS$ lie on a circle coaxial with the given circles.

THEOREM 89.

A system of coaxial circles can be inverted either into a system of concurrent straight lines or into a system of concentric circles.

Case I. If the circles intersect at two real points A, B , invert w.r.t. A ; then each circle becomes a straight line passing through the inverse of B , and therefore the inverse figure is a system of concurrent straight lines.

Q.E.D.

Case II. If the circles do not intersect at real points, they have two real limiting points L, M (page 181).

Draw any circle Σ through L, M ; and let C be any circle of the coaxal system. Invert w.r.t. L .

Denote by dashes corresponding points and curves in the inverse figure.

Since the centre of C lies on LM , the centre of C' lies also on LMM' .

Since Σ passes through the limiting points, it is orthogonal to C ;

$\therefore \Sigma'$ is a straight line orthogonal to C' ;

$\therefore \Sigma'$ is a diameter of C' .

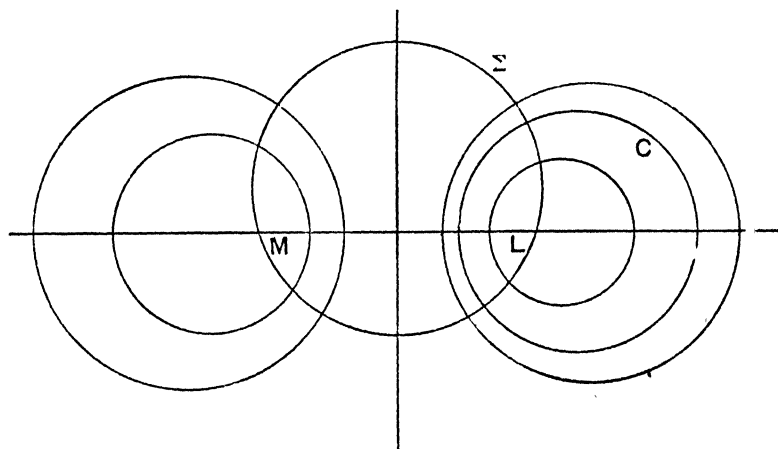


FIG. 111.

² The centre of C' therefore lies on Σ' and LMM' .

But M' is the point of intersection of Σ' with LMM' , each of which is a diameter of C' ;

$\therefore M'$ is the centre of C' ;

$\therefore$ every circle of the system inverts into a circle having its centre at the inverse of the other limiting point. Q.E.D.

The proof of Case I. may run also as follows:

Draw any two circles Σ_1, Σ_2 through L, M . C is orthogonal to Σ_1, Σ_2 . After inversion w.r.t. L , the circles Σ_1, Σ_2 become two straight lines through M' . Since C' is orthogonal to each of these lines, its centre is at M' ;

$\therefore$ the centre of C' is a fixed point.

Q.E.D.

127. Prove that a coaxal system when inverted w.r.t. any point inverts into a coaxal system.

[Consider what the system of circles orthogonal to the given coaxal system invert into, when the points of intersection are imaginary.]

Analysis shows that every two circles have two points of intersection (real or imaginary). If then we take an imaginary point as centre of inversion, it is possible to invert a system of coaxal circles into a system of concurrent straight lines under all conditions. Since we may regard inversion as an analytical process (see pages 133-4) any results which may be obtained by this method will be true. Greater continuity is therefore obtained in the method of proof required to establish many geometrical theorems if this extended form of inversion is admitted.

128. When inverted by the method of Theorem 89, what will the radical axis become?

THEOREM 90.

If a circle S when inverted w.r.t. a circle Σ becomes a circle S' , then the three circles S , Σ , S' are coaxal.

The points common to S and Σ remain unaltered by inversion, and therefore lie on S' ; [See Ex. 129.]

$\therefore$ the three circles have two common points.

Q.E.D.

Corollary.

A system of concentric circles may be regarded as coaxal, having the line at infinity as their radical axis.

[Invert w.r.t. any point, and use the fact that the inverse of a coaxal system is a coaxal system.]

129. Prove Theorem 90 in the case in which S and Σ do not meet at real points without using imaginary points. [Consider what the system of circles orthogonal to S and Σ invert into.]

130. X is a variable circle touching two fixed circles A , B ; the two circles P , Q are drawn, which are orthogonal to A , B , and touch X ; prove that they cut at a constant angle.

131. X is a variable circle touching two fixed circles A , B ; L is a limiting point of A , B ; P is the foot of the perpendicular from L to its polar w.r.t. X , prove that the locus of P is a circle coaxal with A , B .

132. Any circle is drawn through two fixed points. C is a third fixed point, find the locus of the inverse of C w.r.t. the variable circle.

133. T is a point on the radical axis of two circles; the line of centres meets the first circle at A, B and the second at C, D . If TA, TB meet the first circle at P, P' , and if TC, TD meet the second at Q, Q' ; prove that $TQQ'PP'$ are concyclic.

134. S_1 and S_2 are two circles touching externally two circles Σ_1 and Σ_2 , prove that a circle can be drawn coaxal with Σ_1 and Σ_2 to cut S_1, S_2 orthogonally.

135. Two variable circles touch a fixed circle and cut each other at a constant angle, if one of their points of intersection is fixed, find the locus of the other.

136. A variable circle touches each of two fixed circles, prove that it cuts any other fixed circle, coaxal with them, at a constant angle.

137. L is the limiting point of two circles S_1, S_2 ; two circles Σ_1 and Σ_2 are drawn through L to touch S_1 and S_2 respectively and to cut at a given angle, prove that the locus of their other point of intersection, is a circle coaxal with S_1 and S_2 .

138. (1) If a circle C , diameter d , is inverted into a circle C' , diameter d' w.r.t. a point outside C ; and if t is the tangent from O to C' , and if k is the radius of inversion, prove that $\frac{d}{d'} = \frac{k^2}{t^2}$.

(2) Hence if r is the radius of a variable circle, which touches two fixed circles, and if L is one of their limiting points, and if LP is the tangent from L to the variable circle, prove that $\frac{LP^2}{r}$ is constant.

139. Prove that if a variable circle is drawn to touch two fixed circles, the ratio of the tangents drawn from the limiting points to the variable circle is constant.

140. P is a point such that the inverse of two fixed circles is two equal circles, prove that the locus of P is a circle coaxal with the given circles. Hence prove that any three circles can be inverted into three equal circles.

141. If a variable circle P cuts three fixed circles at the same angle, prove by inversion that P belongs to a fixed coaxal system. [Use Ex. 140 (2).]

142. Prove that the circumcircle, the nine-point circle and the polar circle of a triangle are coaxal. [Use Theorem 90.]

143. The line joining the centres of two circles S_1, S_2 meets S_1 at A, B and S_2 at C, D : any circle orthogonal to S_1 and S_2 meets S_1 at P, Q and S_2 at H, K ; prove that AP, BQ, CH, DK are concurrent.

144. A, B are two fixed points on a fixed circle: O is any other fixed point. Two circles are described to touch each other at O and to pass through A and B respectively: if they meet the fixed circle again at

P, Q , prove that the circle OPQ touches a fixed circle coaxal with the given circle and the point circle at O .

145. A straight line $PQRS$ meets one circle at P, Q and another at R, S ; if L is a limiting point, prove that the angles PLS, QLR are equal or supplementary.

146. If three circles are such that two of them do not intersect at real points, show how inversion may be used to describe a circle to touch the three given circles.

147. A system of circles pass through two fixed points A, B ; T is any fixed point on AB : a variable line through T meets any circle of the system at P, Q : O is any other fixed point, prove that the circle OPQ belongs to a fixed coaxal system.

148. Given two circles, one inside the other, prove that a point H can be found such that extreme portions of any straight line cutting both circles subtend equal angles at H .

149. $PT, P'T'$ are the tangents from two points P, P' to a circle; if $PT + P'T' = PP'$, prove that PP' must touch the circle. [Use Theorem 84.]

150. The tangents at A, B, C to the circumcircle of the triangle ABC cut BC, CA, AB at U, V, W ; prove that the circles on AU, BV, CW as diameters are coaxal; and that the line joining the circumcentre to the orthocentre of the triangle ABC is their radical axis.

151. A system of circles have two common points A, B ; any line through A is drawn meeting the circles at $P, Q, R, S \dots$; prove that the ratios $PQ : QR : RS$, etc., are constant.

152. By inverting Ex. 151 w.r.t. A , prove that it is equivalent to a fundamental cross ratio theorem.

153. State and prove the converse of Ex. 151.

154. Two circles S_1, S_2 , centres A, B , touch internally a third circle, centre C , at P, Q . If S_1 and S_2 intersect at H, K , and if H lies on PQ , prove that CK is parallel to AB .

155. a, b, c are the radii of three coaxial circles, centres A, B, C ; prove that $a^2 \cdot BC + b^2 \cdot CA + c^2 \cdot AB + BC \cdot CA \cdot AB = 0$.

156. The tangents at A, A' to a given circle cut a given non-intersecting circle at P, Q and P', Q' ; AA' cuts PP' at X ; O is a limiting point of the two circles; prove that (1) $\frac{PO}{PA} = \frac{P'O}{P'A'}$; and (2) OX bisects $\hat{POP'}$.

157. L, L' are the limiting points of two circles C_1, C_2 which both enclose L ; from any point P on the line through L' perpendicular to

LL' , tangents PHK , PFG are drawn to the inner circle C_1 and meet C_2 at H , K and F , G ; prove that HG and FK intersect at L .

158. A quadrilateral is inscribed in one circle and circumscribed to another circle; prove that the point of intersection of the diagonals is a limiting point of the two circles.

159. AB is a common tangent to two circles HAP , QBK ; O is any fixed point on AB ; any line $OHPQK$ is drawn through O to meet one circle at H , P and the other at Q , K ; AP meets BQ at T ; prove that the locus of T is a coaxal circle.

160. The incircle of the triangle ABC and the circle escribed to BC touch BC at X , X_1 , and CA at Y , Y_1 ; prove that the circles on XX_1 and YY_1 as diameters intersect on the bisector of the angle BAC at the feet of the perpendiculars to it from B and C .

161. Prove that the centroid of three uniform thin rods forming a triangle is the radical centre of the three excircles of the triangle.

162. A , B are limiting points of a coaxal system; C , D are the centres of two circles of the system enclosing A , B respectively. One of the common tangents of these two circles touches the circle, centre D , at E , and cuts the line through B , perpendicular to AB , at F ; prove that AE is parallel to CF .

163. P is any point on a fixed circle S_1 . The tangent at P to S_1 meets a second fixed circle at H , K ; Q is the harmonic conjugate of P w.r.t. H , K ; find the locus of the mid-point of PQ .

164. O is the meet of an exterior common tangent to two circles with their radical axis; L , L' are their limiting points; prove that OL , OL' are parallel to the internal common tangents of the two circles.

165. $P\perp Q$ is a variable chord of a fixed circle of a given coaxal system; L is a limiting point; PH , QK are the perpendiculars from P , Q to the radical axis; prove that $PH \cdot QK$ is constant.

166. A is a fixed point; PQ is a variable chord of a fixed circle Σ such that $\hat{PAQ} = 90^\circ$. Prove that the locus of the mid-point of PQ is a circle coaxal with Σ and the point circle A . [Use Theorem 88.]

167. With the notation of Ex. 166, if the perpendicular from A to PQ meets PQ at D , prove that the locus of D is a circle. [Lachlan.]

168. From a limiting point L of two circles, tangents LP , LQ are drawn to the circles; if LQ meets the other circle at H , K , prove that $LP^2 - LQ^2 = HQ \cdot QK$.

169. By inverting Ex. 166 w.r.t. A , deduce a new theorem.

170. O is a fixed point; POQ , ROS are two variable perpendicular chords of a fixed circle; the tangents at $PRQS$ form a quadrilateral $pqrs$; prove that p , q , r , s lie on the same fixed circle. [Use Ex. 166.]

171. Two circles are drawn in different planes. Prove that there is in general one point on the line of intersection of the planes, from which the tangents to the circles are of equal length; and that if there be more than one such point, then the circles are sections of the same sphere.

172. [Casey's method for proving Poncelet's theorem.]

A, A' are two points on a circle X , and the tangents from them to a circle Y meet at C , and cut X again at B, B' ; O, N are the centres of X, Y ; p, π are the perpendiculars from A and A' to the radical axis of X, Y ; p', π' are the perpendiculars from B, B' to the radical axis, prove that

$$(1) AC + A'C = \sqrt{2 \cdot ON}(\sqrt{p} + \sqrt{\pi});$$

$$(2) AA' : BB' = (\sqrt{p} + \sqrt{\pi}) : (\sqrt{p'} + \sqrt{\pi'}).$$

173. [PONCELET'S THEOREM.] If a variable triangle is inscribed in a fixed circle of a coaxal system, and if two of its sides touch two other fixed circles of the system, prove that the third side touches another fixed circle of the system. [Take two positions of the triangle and use Ex. 172.]

174. Enunciate and prove a theorem similar to Ex. 173 for a polygon.

175. $PQ, P'Q'$ are two chords of a circle; PP', QQ' meet at a point O inside the circle; a circle S is drawn coaxal with the given circle, and the point circle O so as to touch PQ , prove that it will also touch $P'Q'$.

176. LH is the tangent from a limiting point L to any circle, centre A , of the coaxal system, prove that LH^2 varies as LA .

CHAPTER XIV.

CENTRES OF SIMILITUDE.

Definition.

A, B are the centres of two circles, radii a, b ; if two points O, O' are taken on AB such that $\frac{a}{b} = \frac{AO'}{O'B} = \frac{AO}{BO}$, then O and O' are called the external and internal **centres of similitude** of the two circles.

Further, the circle on OO' as diameter is called the **circle of similitude** of the two circles. The reason for this name is contained in the property of Ex. 22.

From Theorem 4, page 5, the following results are evident:

I. The centres of similitude of two circles are points w.r.t. which the circles are homothetic.

II. The common tangents of two circles pass through one or other of the centres of similitude.

III. The lines joining the extremities of parallel diameters of two circles pass through one or other of the centres of similitude.

IV. Any line through a centre of similitude meets the circles in two pairs of corresponding points, which lie at the extremities of parallel radii.

These results should however be worked through independently; they present little difficulty.

1. Work out independently the theorems I.-IV. given above.

Definition.

Two pairs of corresponding points of two circles are said to be **similar**, if their joins pass through the same centre of similitude.

2. Where is the internal centre of similitude of two circles which touch each other? And where is the external centre of similitude of two equal circles.

3. Where are the centres of similitude of a circle and a straight line?
4. Prove that the two centres of similitude are harmonic conjugates w.r.t. the centres of the two circles.
5. Prove that the centroid and orthocentre are the centres of similitude of the nine-point circle and the circumcircle of the triangle.

6. If I, I_1, I_2, I_3 are the incentre and excentres of a triangle, prove that I is a centre of similitude of the circumcircle and the circle $I_1 I_2 I_3$.

7. A straight line drawn through the external centre of similitude O of two circles, centres A, B , meets the first circle at P, Q , and the second at P', Q' . Prove that the lines AP, AQ are parallel to the lines BP', BQ' and that $OP \cdot OQ' = OP' \cdot OQ$, where P', Q are separated by both or neither of the points P, Q' .

8. Prove that the six centres of similitude of three circles, centres A, B, C , taken in pairs, lie in sets of three points, on four straight lines. [Apply Menelaus to the triangle ABC .]

9. $OX, O'X'$ are two of the common tangents to two circles, where O, O' are their centres of similitude. If a, b are the radii, prove that $\frac{OX}{O'V} = \frac{a}{b} = \frac{O'X'}{Y'O'}$. Hence prove that the circle of similitude of the two circles is coaxial with them. [Use Theorem 88.]

10. Prove that the centres A, B of two circles, radii a, b , are inverse points w.r.t. their circle of similitude.

If P is any point on the circle of similitude, prove that $\frac{PA}{PI} = \frac{a}{b}$.

Hence prove that if two circles intersect at P, Q , then their circle of similitude also passes through P, Q .

11. $\Sigma_1, \Sigma_2, \Sigma_3$ are the circles of similitude of the pairs of circles $Z_1, Z_2; Z_2, Z_3; Z_3, Z_1$; A, B, C are the centres and a, b, c the radii of Z_1, Z_2, Z_3 . If Σ_1, Σ_2 intersect at P, Q , prove that

$$\frac{PA}{a} = \frac{PB}{b} = \frac{PC}{c}; \text{ and that } \frac{QA}{a} = \frac{QB}{b} = \frac{QC}{c};$$

and hence prove that the circles $\Sigma_1, \Sigma_2, \Sigma_3$ are coaxial. [Use Ex. 10.]

THEOREM 91.

P, Q are any two points on a circle; P', Q' are similar corresponding points on a second circle: then $PQ, P'Q'$ are parallel.

Further, if PP', QQ' meet at O , and if OP, OQ meet the first circle at X, Y and the second at X', Y' , then

$$OP \cdot OX' = OP' \cdot OX = OQ \cdot OY' = OQ' \cdot OY.$$

A, B are the centres of the two circles.

Since P', P and Q', Q are similar pairs of corresponding points, PP', QQ' meet at a centre of similitude, O .

$$(1) \text{ Now } \frac{OP}{OP'} = \frac{OA}{OB} = \frac{OQ}{OQ'} \text{ by parallels; } \quad [\text{Th. 4}]$$

$\therefore PQ$ is parallel to $P'Q'$. Q.E.D.

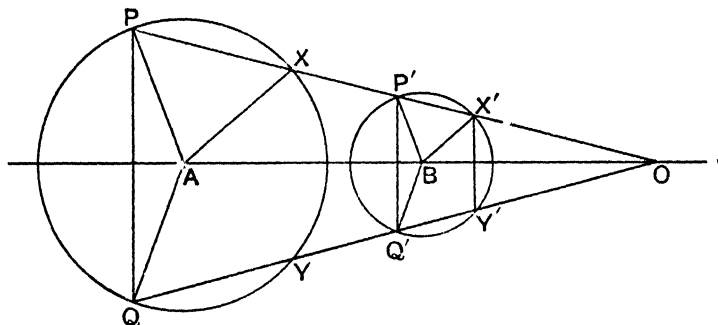


FIG. 112.

$$(2) AX \text{ is parallel to } BX'; \quad [\text{Th. 4}]$$

$$\therefore \frac{OP}{OP'} = \frac{OA}{OB} = \frac{OX}{OX'}$$

$$\therefore OP \cdot OX' = OP' \cdot OX. \quad \text{Q.E.D.}$$

$$(3) \text{ Also } O\hat{X}'Y' = P'\hat{Q}'Y', \text{ since } P'Q'Y'X' \text{ are concyclic}$$

$$= P\hat{Q}Y' \text{ by parallels;}$$

$$\therefore PQY'X' \text{ are concyclic;}$$

$$\therefore OP \cdot OX' = OQ \cdot OY' = OQ' \cdot OY \text{ by a method similar to (2).}$$

Q.E.D.

12. Prove that PQ meets $X'Y'$ on the radical axis of the two circles, with the notation of Theorem 91.

13. Prove that the meets of $PY, Q'X'; QX, P'Y'; XY, P'Q'; PQ, X'Y'$ are collinear, with the notation of Theorem 91.

14. Prove that $\frac{XY}{X'Y'} = \text{ratio of radii}$, with notation of Theorem 91.

15. O is the external centre of similitude of two circles, OTT' is an external common tangent; Q, Q' are the extremities of two parallel radii drawn in the same sense; prove that $QT, Q'T'$ are parallel.

16. Enunciate and prove a theorem similar to Ex. 15 for the internal centre of similitude, O' .

17. Work through the proof of Theorem 91 for an internal centre of similitude O' . Is there any difference, if the circles cut at real points?

THEOREM 92.

The six centres of similitude of three circles, taken in pairs, lie in sets of three points on four straight lines forming a complete quadrilateral, whose diagonal line triangle is the triangle formed by joining the centres of the circles.

Also the three circles of similitude are coaxal.

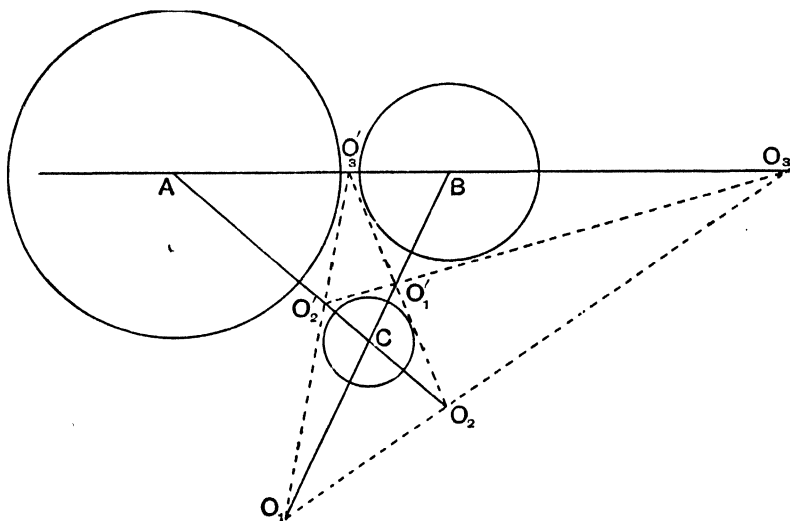


FIG. 113.

O_3, O'_3 are the centres of similitude for the circles, centres A, B ; O_1, O'_1 ; O_2, O'_2 are the centres of similitude for the circles, centres B, C ; C, A .

Let a, b, c be the radii of the circles, centres A, B, C .

$$\begin{aligned} \text{Then} \quad \frac{AO_3}{O_3B} &= -\frac{a}{b}; \quad \frac{BO'_1}{O'_1C} = \frac{b}{c}; \quad \frac{CO'_2}{O'_2A} = \frac{c}{a}; \\ \therefore \frac{AO_3}{O_3B} \cdot \frac{BO'_1}{O'_1C} \cdot \frac{CO'_2}{O'_2A} &= -\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a} = -1; \end{aligned}$$

$\therefore$ by applying Menelaus to the triangle ABC , it follows that O_3, O'_1, O'_2 are collinear.

Similarly, O_3, O_2, O_1 ; O'_3, O'_1, O_2 ; O'_3, O_2, O_1 are sets of collinear points.

Hence clearly ABC is the diagonal line triangle of the quadrilateral formed by the four lines

$$O_3O'_1O'_2; \quad O_3O_2O_1; \quad O'_3O'_1O_2; \quad O'_3O_2O_1. \quad \text{Q.E.D.}$$

Again, the circles of similitude are the circles whose diameters are the three diagonals of this complete quadrilateral and are therefore coaxal. [Th. 86 (3).] Q.E.D.

Corollary

If a circle S touches two circles Σ_1 and Σ_2 at P, Q , then PQ passes through a centre of similitude of Σ_1 and Σ_2 .

For P is a centre of similitude of S and Σ_1 and Q is a centre of similitude of S and Σ_2 .

18. Work out in detail the proof (in Fig. 113) that $O_1O_2O_3$ are collinear.

19. Prove independently the Corollary of Theorem 92.

20. Prove that the centres of the three circles of similitude of three circles, taken in pairs, are collinear.

21. Prove, with the notation of Theorem 92, that the circle ABC is orthogonal to each of the circles of similitude.

THEOREM 93.

The circle of similitude of two circles S_1 and S_2 is coaxal with S_1 and S_2 .

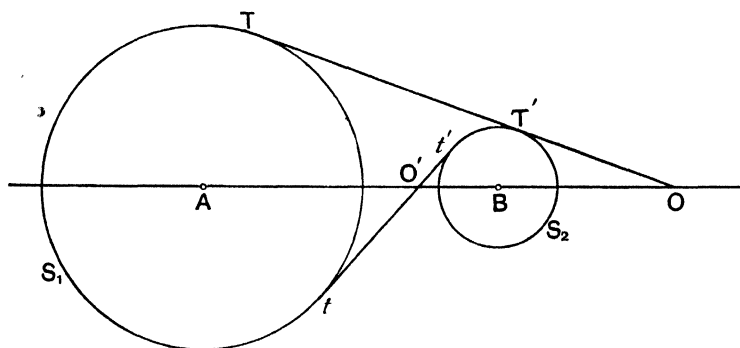


FIG. 114.

Let $OT'T$, $tO't'$ be two common tangents through the centres of similitude O, O' .

Let a, b be the radii of S_1 and S_2 .

Then
$$\frac{OT}{OT'} = \frac{a}{b} = \frac{o't}{o't'}.$$

If then a circle is drawn through O' coaxal with S_1 and S_2 it will also pass through O . [Th. 88]

But its centre must lie on $AO'BO$ since it is coaxial with S_1, S_2 ; therefore the circle through O' coaxial with S_1, S_2 is the circle on $O'O$ as diameter. Q.E.D.

If S_1 and S_2 intersect at real points, the proof suggested in Ex. 10 may be used: and if imaginary points are admitted, from analysis, this proof could be regarded as sufficient in every case.

22. P is any point on the circle of similitude of two circles S_1 and S_2 ; prove that S_2 can be made homothetic to S_1 by a rotation about P.

23. Two circles touch at A ; two variable secants APQ, AHK are drawn at right angles and meet one circle at P, H and the other at Q, K ; prove that PK, QH intersect at a fixed point.

24. Two circles touch externally at S ; a tangent to one of them at P cuts the other at Q, R ; PP' is a diameter of the first circle, prove that SP' bisects $\widehat{QSR}$.

25. Parallel chords of two circles are drawn through the points of contact of a common tangent, prove that the line joining their other extremities passes through a fixed point.

26. Prove that a line through a centre of similitude cuts off from the two circles chords whose lengths are in a constant ratio.

27. If two circles intersect, prove that the tangents at a point of intersection are equidistant from either centre of similitude.

28. Any circle through the centres of two circles is orthogonal to their circle of similitude.

29. Prove that two circles subtend equal angles at any point on their circle of similitude.

30. OPQ is the external common tangent to two circles which intersect at A, B , if O is a centre of similitude, prove that

$$OP \cdot OQ = OA^2.$$

31. The angles of a triangle ABC are given in magnitude. The point A is fixed; B moves on a fixed circle, find the locus of C .

32. O is the external centre of similitude of two circles: one of the internal common tangents touches the circles at P, Q ; OP meets the other circle at L, M , prove that LQ or MQ is a diameter of that circle.

33. H is the orthocentre and O the circumcentre of the triangle ABC ; A' is the mid-point of BC ; HA' meets the circumcircle at P ; prove that $HA' = A'P$ and that A, O, P are collinear.

34. A straight line $ABCD$ cuts two circles at A, B and C, D respectively, cutting off similar segments on the same side of the line;

prove that circles can be drawn to touch the given circles at A, D or at B, C .

35. A straight line $OPQQ'P'$ is drawn through the external centre of similitude of two circles to meet the first at P, Q and the second at Q', P' ; prove that $P'Q.PQ'$ is constant.

36. Through a given point, draw a circle to touch two given circles.

37. Two spheres, radii a and b , ($a > b$) are glued together at a point P . They are placed so as to be both in contact with a horizontal table and are made to roll, without slipping, on the table. Prove that P describes a circle of radius $\frac{4a^{\frac{3}{2}}b^{\frac{3}{2}}}{a^2 - b^2}$.

38. Find when possible two real points P, Q at each of which three given circles subtend equal angles, and prove that the radical centre of the three circles lies on PQ .

39. Two variable circles touch one another and each of them touches two fixed circles: find the locus of the point of contact of the variable circles.

40. S_1, S_2 are two circles touching externally each of two circles Σ_1 and Σ_2 , prove that a circle Z can be described coaxial with Σ_1, Σ_2 and orthogonal to S_1, S_2 .

41. O is a centre of similitude of two circles, P is a variable point; H, K are the poles of OP w.r.t. the two circles; prove that HK passes through a fixed point.

42. O is the external centre of similitude of two circles; prove that the two circles can be inverted into each other w.r.t. O and that their radical axis inverts into the circle of similitude.

43. The incircle of the triangle ABC touches BC at X , and the circle escribed to BC touches BC at X' ; if AX meets the incircle at F, G prove that FX or GX is a diameter of the incircle.

44. Two circles intersect at C, D ; a third circle touches them at P, P' ; prove that $CP:CP' = DP:DP'$.

45. Two circles APQ, AHK cut at A ; FH, QK are their common tangents; prove that the circles APH, AQQ touch each other.

46. Three circles $\Sigma_1, \Sigma_2, \Sigma_3$ are touched externally by a circle Z_1 and internally by a circle Z_2 , prove that the radical centre of $\Sigma_1, \Sigma_2, \Sigma_3$ is a centre of similitude of Z_1 and Z_2 .

47. In the triangle ABC , the circle escribed to BC touches BC at X ; if I is the incentre and if A' is the mid-point of BC , prove that IA' is parallel to AX .

48. If two circles touch each of two other circles, so that four, or two or none of the contacts are external, prove that the radical axis of either pair passes through a centre of similitude of the other pair.

49. The internal bisector of the angle BAC meets BC at D , prove that the circle on AD as diameter is coaxial with the incircle of the triangle ABC and the circle escribed to BC .

50. P is any point on the circumcircle of the triangle ABC ; H is the orthocentre; prove that HP is bisected by the nine-point circle.

51. Given the base BC and the ratio of the sides $\frac{AB}{AC}$ of the triangle ABC , prove that the external centre of similitude of the circles escribed to AB and AC is a fixed point.

52. A, B, C are the centres of three circles, prove that the radical axis of their three circles of similitude passes through the circumcentre of the triangle ABC .

53. O is the circumcentre of the triangle ABC , prove that there are two points P, Q such that $PA:PB:PC = \lambda:\mu:\nu$ and that P, O, Q are collinear, where λ, μ, ν are given constants.

54. Prove that the circle whose diameter is the third diagonal of a cyclic quadrilateral is the circle of similitude of the two circles whose diameters are the other diagonals.

55. The incircle of the triangle ABC touches BC at X ; A', K are the mid-points of BC, AX ; I is the incentre; prove that A', I, K are collinear. [Use Ex. 43.]

56. O is a point of intersection of two circles S_1 and S_2 ; OA, OB are their diameters through O , prove that their circle of similitude is orthogonal to the circle OAB .

MISCELLANEOUS EXAMPLES.

1. AB is a fixed diameter of a given circle, centre O ; P is a variable point on the circumference. BP is produced to Q so that $BP=PQ$. Find the locus of the meet of AP , OQ .

2. The sides of a variable triangle pass through three fixed points. If the triangle is similar to a fixed triangle, prove that the locus of its centroid is a circle.

3. Inscribe in a given triangle a rectangle similar to a given rectangle.

4. A is a fixed point on a fixed circle: AP is a variable chord. Q is a point on AP such that $AP \cdot AQ$ is constant, find the locus of Q .

5. Construct a square $ABCD$ such that $AC-AB$ is equal to a given length.

6. Inscribe in a given triangle a parallelogram similar to a given parallelogram.

7. Construct a quadrilateral, similar to a given quadrilateral, and so that each of its sides passes through a given point.

8. In a given quadrilateral inscribe a quadrilateral, similar to another given quadrilateral.

9. ABC is a given triangle: O is any other point. The figure is rotated about C , in a direction $CA \rightarrow CB$, through an angle ACB ; O' , A' are the new positions of O , A ; $O'A'$ meets OB at H ; and a line through B parallel to $O'A'$ meets CH at K , prove that

$$\frac{CK}{CH} = \frac{CB}{CA}. \quad \text{[Rouché.]}$$

10. Construct a triangle ABC similar to a given triangle, given the point C , and that A , B lie on given lines OA , OB . [Use Ex. 9.]

11. Construct a triangle ABC similar to a given triangle, and such that its vertices lie on three parallel straight lines.

12. B , B' are two points on two lines AB , AB' . O is a point such that, by a rotation about O , the line AB and the point B can be made homothetic to the line AB' and the point B' ; prove that the locus of O is the circle ABB' .

13. AB, AB' are two given lines; B, B' being given points. Find points P, P' on AB, AB' such that $\frac{BP}{B'P'}$ is equal to a given ratio and such that PP' passes through a given point C . [Rouché.]

14. Two sources of light of different intensities are situated at two given points. Find the locus of a point which is equally illuminated by each source.

15. If $ABCD$ is a cyclic quadrilateral, prove that

$$\frac{AC}{BD} = \frac{BA \cdot AD + BC \cdot CD}{AB \cdot BC + AD \cdot DC}$$

16. D, E are points on the sides AB, AC of a triangle; Q is the mid-point of DE ; AQ meets BC at R ; prove that $\frac{BR}{RC} = \frac{AB \cdot AE}{AC \cdot AD}$.

17. C is a point on the diameter AB of a circle; P is any point on the circumference. A line through C perpendicular to AB cut AP, PB at K, H and the circle at L , prove that $CL^2 = CH \cdot CK$.

18. A, B, C are any three points. prove that the locus of a point (in space) which is equally illuminated by each of three sources of light of any intensities situated at A, B, C is a circle.

19. Three points $A'B'C'$ are taken on the sides BC, CA, AB of a triangle: $AB, A'B'$; $BC, B'C'$; $CA, C'A'$ intersect at c, a, b ; prove that

$$\frac{cA}{cB} \cdot \frac{aB}{aC} \cdot \frac{bC}{bA} = \frac{C'B^2}{C'A^2} \cdot \frac{A'C^2}{A'B^2} \cdot \frac{B'A^2}{B'C^2}$$

20. If A', B', C' are three points on the sides BC, CA, AB of a triangle, prove that $\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = \frac{\sin ACC'}{\sin C'CB} \cdot \frac{\sin BAA'}{\sin A'AC} \cdot \frac{\sin CBB'}{\sin B'BA}$.

21. ABC is a triangle right angled at A ; $PQHK$ is a square with its vertices P, Q on BC and H, K on AC, AB respectively; prove that $PQ^2 = BP \cdot QC$.

22. In the triangles ABC, PQR , if

$$\hat{ABC} = \hat{PQR} \text{ and if } \hat{ACB} + \hat{PRQ} = 180^\circ,$$

prove that

$$\frac{AC}{PR} = \frac{AB}{PQ}$$

23. With the usual notation of the triangle ABC , A' meets the circumcircle at P ; prove that the line through A' perpendicular to AP bisects HP .

24. PQ is a diameter of the circumcircle of the triangle ABC . p, q are the pedal lines of P, Q w.r.t. the triangle. The perpendiculars from P, Q to p, q respectively meet at R ; prove that R lies on the circumcircle, and that the pedal line of R is parallel to PQ .

25. ABC is a triangle right angled at B ; the incircle touches AC at Y , prove that $2AY \cdot YC = AB \cdot BC$.

26. With the usual notation of the triangle ABC , if L, M are the feet of the perpendiculars from B, C to AI ; prove that A', D, L, M are concyclic.

27. With the usual notation of the triangle ABC , if the triangles FBD, EDC are equivalent, prove that they are congruent.

28. $ABCD$ is a cyclic quadrilateral; a, b, c, d are the orthocentres of the triangles BCD, CDA, DAB, ABC ; prove that Aa, Bb, Cc, Dd are concurrent.

29. I is the incentre of the triangle ABC ; the circle BIC cuts AB, AC again at M, N ; prove that MN touches the incircle.

30. A variable chord PQ of the circumcircle of the triangle ABC is parallel to BC , prove that the locus of the meet of the pedal lines of P, Q w.r.t. ABC is a straight line.

31. ABC is a triangle right angled at A ; from any point P on AC a perpendicular is drawn to BC cutting AB at Q . If BP cuts QC at R , prove that the locus of R is a circle Σ ; and that the tangent at R to Σ bisects PQ .

32. A chord PQ of the circumcircle of the triangle ABC is parallel to BC , prove that the pedal line of P is perpendicular to AQ .

33. With the notation of Ex. 28, prove that $abcd$ lie on a circle equal to the circle $ABCD$.

34. Two variable chords AB, CD of a circle intersect at C ; if AD is a diameter, prove that $AO \cdot AB + DO \cdot DC$ is constant.

35. ABC is an equilateral triangle inscribed in a circle, centre O . P_2 is any point on the circumference, prove that the pedal line of P bisects OP .

36. With the usual notation of the triangle ABC , prove that

$$AB^2 + BC^2 + CA^2 = 2AH \cdot AD + 2BH \cdot BE + 2CH \cdot CF.$$

37. AD is an altitude of the triangle ABC , right angled at A . r_1, r_2, r are the inradii of the triangles ADC, ADB, ABC ; prove that $r^2 = r_1^2 + r_2^2$.

38. Enunciate and prove a theorem, similar to Ex. 37, connecting the radii of the excircles of the same three triangles.

39. Δ is the area, and R is the circumradius of the triangle ABC , prove that the perimeter of the pedal triangle is equal to $\frac{2\Delta}{R}$.

40. R is the circumradius and σ is the perimeter of a triangle, prove that the area of the triangle formed by its excentres is equal to $R\sigma$.

41. Two circles, centres A, B , touch at C . P is a point outside both circles. If CP bisects $\hat{APB}$, prove that the rectangle contained by the tangents from P to the two circles is equal to PC^2 .

42. ABC is a given triangle; O is any other given point; on OA, OB, OC are taken points a, b, c such that $\frac{Oa}{OA} = \frac{Ob}{OB} = \frac{Oc}{OC} = k$; if A', B', C' are the mid-points of BC, CA, AB , prove that the point P determined by $\vec{OP} = \frac{k}{1+2k}(\vec{OA} + \vec{OB} + \vec{OC})$ lies on aA' ; hence or otherwise prove that aA', bB', cC' are concurrent.

43. Two equilateral triangles $ABC, A'B'C'$ have a common orthocentre. P, P' are any points on the circles $ABC, A'B'C'$ respectively; prove that $PA^2 + PB^2 + PC^2 = P'A^2 + P'B^2 + P'C^2$.

44. $ABCDEF$ is a hexagon; find a line L such that the sum of the perpendiculars from A, B, C to L is in a constant ratio to the sum of the perpendiculars from D, E, F to L , and such that L passes through another given point.

45. PQ is a variable chord of a fixed circle and subtends a right angle at a fixed point O . if the rectangle $PQQR$ is completed, find the locus of R .

46. RS is a diameter of a circle; K is any point on the circumference; the tangent at K meets the tangents at R, S in P, Q ; if PS cuts QR at L , prove that KL is perpendicular to RS .

47. If P, Q, R are points on the sides BC, CA, AB of a triangle, such that the circles on AP, BQ, CR as diameters have a common point, prove that P, Q, R are collinear.

48. $ABCD$ is a quadrilateral: if AB is parallel to CD , prove that $AC^2 + BD^2 = AD^2 + BC^2 + 2AB \cdot CD$.

49. A straight line meets the sides BC, CA, AB , produced of a triangle ABC at P, Q, R ; if p, q, r are the lengths of the tangents from P, Q, R to the circumcircle of ABC , prove that

$$\frac{AR}{r} \cdot \frac{BP}{p} \cdot \frac{CQ}{q} = 1.$$

50. P, Q are points on the sides AB, AC of a triangle ABC such that $\frac{AP}{PB} = \frac{CQ}{QA}$. PQ is divided at D so that $\frac{QD}{DP} = \frac{AP}{PB}$, prove that the area of the triangle APQ is half that of BDC .

51. A straight line meets the sides BC, CA, AB of a triangle at P, Q, R ; three concurrent lines AO, BO, CO meet PR at P', Q', R' ; prove that

$$\frac{QP'}{P'R} \cdot \frac{RQ'}{Q'P} \cdot \frac{PR'}{R'Q} = -1.$$

52. P, Q, R are any points on the sides BC, CA, AB of a triangle; O is any point inside the triangle; AP', BQ', CR' are drawn parallel to OP, OQ, OR to meet BC, CA, AB at P', Q', R' ; prove that

$$\frac{OP}{AP'} + \frac{OQ}{BQ'} + \frac{OR}{CR'} = 1.$$

Is the theorem true for any position of O ?

53. ABC, DEF are any two triangles: if the perpendiculars from A, B, C to EF, FD, DE are concurrent, prove that the perpendiculars from D, E, F to BC, CA, AB are also concurrent.

54. If $O\{AB; CD\}$ is a harmonic pencil, prove that

$$2 \cot AOB = \cot AOC + \cot AOD.$$

55. If $\{AB; CD\}$ is a harmonic range, and if P is any other point on AD , prove that $\frac{2PR}{AB} = \frac{PC}{AC} + \frac{PD}{AD}$.

56. If $\{AB; CD\}$ is a harmonic range, and if H, K are the mid-points of AB, CD , and if P is any other point on AD , prove that $PA \cdot PB + PC \cdot PD = 2PH \cdot PK$.

57. I is the incentre of the triangle ABC ; AI meets BC at Q and the circumcircle at P , prove that $PI^2 = PA \cdot PQ$.

58. With the usual notation of the triangle ABC , if AI cuts BC at Q , prove that $QX \cdot QX' = QA' \cdot QD$.

59. If $O\{AB; CD\}$ is a harmonic pencil, and if P is any other point, prove that $2 \frac{\sin POB}{\sin AOB} = \frac{\sin POC}{\sin AOC} + \frac{\sin POD}{\sin AOD}$.

60. Given two lines p, q which meet off the paper, and a point A , construct, with the use of a ruler only, the join of A to the meet of p, q .

61. Two chords AB, CD of a circle are conjugate lines w.r.t. the circle. P is any other point on the circumference. CD cuts PA, PB at H, K ; prove that H, K are conjugate points w.r.t. the circle.

62. AB is a diameter of a circle: P is any point on the circumference. N is the foot of the perpendicular from P to AB ; T is the pole of AP . Prove that TB bisects PN .

63. A, B are two fixed points. A variable circle is drawn through A, B and cuts a fixed circle at P, Q ; prove that the join of the meets of AP, BQ ; AQ, BP passes through a fixed point.

64. A straight line cuts a circle Σ at A, B and a circle S at C, D ; if A, B are conjugate points w.r.t. S , prove that C, D are conjugate points w.r.t. Σ .

65. The centre A of a circle S lies on a circle Σ : a line through A cuts Σ at K and their radical axis at H ; prove that H, K are inverse points w.r.t. S .

66. Two chords AB , CD of a circle are conjugate lines w.r.t. the circle: any line through A cuts the circle at P and CD at Q ; prove that $B\{CD; PQ\}$ is harmonic.

67. $ABCD$ is a quadrilateral circumscribing a circle, centre O ; A' , B' , C' , D' are the points of contact of AB , BC , CD , DA . If $A'D'$, $B'C'$ meet at P , prove that PO is perpendicular to AC .

68. FG is the third diagonal of a quadrangle $ABCD$, inscribed in a circle Σ ; if K is the mid-point of FG , prove that FG is equal to the sum of the tangents from K to Σ .

69. O is a given point outside a circle. The line bisecting the tangents from O to the circle cuts any chord AB at P , prove that $P\hat{O}B = P\hat{A}O$.

70. A is a fixed point: O is the centre of a fixed circle: PQ is a variable chord of the circle such that OA bisects $P\hat{A}Q$; prove that PQ passes through a fixed point. [Two Cases.]

71. R is the pole of a variable chord PQ of the incircle of the triangle ABC . If $R\{AB, PC\}$ is harmonic, prove that Q is a fixed point.

72. The tangents from the vertices A , B , C of a triangle to any circle meet the opposite sides at P , P' ; Q , Q' ; R , R' ; if P , Q , R are collinear, prove that P' , Q' , R' are also collinear.

73. Two chords AB , CD of a circle are conjugate lines w.r.t. the circle, prove that $AC \cdot BD = BC \cdot AD = \frac{1}{2} AB \cdot CD$.

74. With the notation of Ex. 73, if P is a variable point on AB , and if CP , DP cut the circle again at Q , R , prove that QR passes through a fixed point.

75. ABC is a self-conjugate triangle w.r.t. a circle; P is any point on the circle; BP meets the circle again at Q ; CQ meets the circle again at R ; prove that P , A , R are collinear.

76. With the notation of Ex. 75, prove that BR , AQ , CP are concurrent.

77. L is a limiting point of two circles S_1 , S_2 ; T is a point conjugate to L w.r.t. S_1 ; the tangents TPQ , TRS from T to S_1 cut S_2 at P , Q and R , S ; prove that $Q\hat{L}S = P\hat{L}R$.

78. A , B are the centres of two circles; a variable line through B cuts the circle, centre A , at P , Q ; prove that the circle APQ belongs to a fixed coaxal system.

79. A variable circle subtends given angles at two given points, prove that the locus of its centre is a circle Σ . If the given angles vary, prove that the circles Σ generate a coaxal system.

80. Prove that the six radical axes of the incircle and excircles of a triangle, taken in pairs, form a triangle and its orthocentre.

81. If in Fig. 99, L, M, N are the mid-points of AC, BD, FG , prove that $LM \cdot FG^2 + MN \cdot AC^2 + NL \cdot BD^2 + 4 LM \cdot MN \cdot NL = 0$.

82. AD, BE, CF are the altitudes of the triangle ABC . $EF, BC; FD, CA; DE, AB$ intersect at P, Q, R ; prove that PQR is the radical axis of the circumcircle and nine-point circle of the triangle ABC .

83. A variable circle passes through two fixed points A, B , and cuts two fixed lines AC, BC at P, Q . If AQ, BP and AB, PQ intersect at X, Y , prove that XY passes through a fixed point.

84. "If a circle is inscribed in a quadrilateral $ABCD$, then its centre lies on the line joining the mid-points of the diagonals." What does this theorem become, when AD, DC form one straight line.

85. ABC, PQR are two self-conjugate triangles w.r.t. a circle Σ ; prove that the centre of Σ lies on the radical axis of the circles ABC, PQR .

86. PQ is a variable chord of a given circle: PH, QK are the tangents from P, Q to a second given circle. If $PH^2 + QK^2$ is constant, find the locus of the mid-point of PQ .

87. Give an analytical solution of Ex. 86.

88. $ABCD$ is a quadrilateral: AC, BD intersect at E . Lines drawn through E parallel to AB, BC, CD, DA , meet CD, DA, AB, BC at P, Q, R, S ; prove that P, Q, R, S are collinear, and that PS is parallel to the third diagonal of the quadrilateral.

89. Given three circles S_1, S_2, S_3 and three points A_1, A_2, A_3 , construct a circle Σ such that the radical axes of $\Sigma, S_1; \Sigma, S_2; \Sigma, S_3$ pass through A_1, A_2, A_3 respectively.

90. A_1, A_2, A_3 are points on the radical axes of the circles $S_2, S_3; S_3, S_1; S_1, S_2$. Prove that the meets of $A_1A_2, S_3; A_2A_3, S_1; A_3A_1, S_2$ are concyclic.

91. A point is inverted successively w.r.t. any number of circles of a coaxial system, prove that its final position could be obtained by a single inversion w.r.t. a circle of the system.

92. $ABCD$ is an isosceles trapezium, AB, CD being parallel. In this figure AD, BC, AC, BD represent four rods jointed together at their extremities. O, P, P' are points on DA, DB, CA such that $\frac{DO}{DA} = \frac{DP}{DB} = \frac{CP'}{CA}$. The point O is fixed. Prove that as the system of rods is allowed to move (1) O, P, P' are always collinear; (2) $AB \cdot CD$ is constant; (3) if P describes any curve, then P' describes an inverse curve.

[This instrument was invented by Dr. Hart.]

93. A sphere is inverted w.r.t. any point on its surface; determine the inverses of the parallels of longitude and latitude.

94. (1) C is a fixed point on a fixed chord AB of a circle: a variable chord PQ is divided in a constant ratio at R : if AP , BQ , CR concur at a point O , prove that the locus of O is a circle Σ .

(2) If C moves along the chord AB , prove that the circle Σ generates a coaxal system.

95. a, b, c, d are four points on the sides AB, BC, CD, DA of a quadrilateral inscribed in a circle Σ . k_a denotes the power of a w.r.t. Σ . P_q denotes the inverse of P w.r.t. a circle, centre Q , and the square of whose radius is k_Q . Also P_{qrst} stands for $((P_q)_r)_s$. Prove that

(1) A_{abcd} coincides with A .

(2) If Q is the point d_{cba} , then $(AQ)_{abcd}$ is the same as AQ , and passes through A_{bcd} .

[Note that $(AQ)_{abcd}$ and (AQ) necessarily make the same angle with Σ .]

(3) Hence show how to inscribe in a circle Σ a quadrilateral whose sides pass through four given points a, b, c, d . [Rouché.]

96. Show how to inscribe in a circle Σ a polygon, each of whose sides passes through a given point.

97. S is a variable circle of a given coaxal system. Σ is any other fixed circle. O is a fixed point: a circle S' is drawn through O coaxal with S and Σ , prove that it belongs to a fixed coaxal system.

98. Three given circles have two common points A, B ; a variable line through A cuts the circles at P, Q, R , prove that the tangents at P, Q, R form a triangle whose circumcircle passes through a fixed point.

99. Two chords AB, CD of a circle are conjugate lines w.r.t. the circle; P is any other point on the circumference. CP, DP cut AT at H, K ; prove that HD, KC intersect on the circle.

100. S and Σ are two given circles: the tangent at a variable point P on S meets the polar of P w.r.t. Σ at Q , prove that the circle on PQ as diameter belongs to a fixed coaxal system.

101. Invert w.r.t. P the following theorem: the tangents at the extremities of a chord AB of a circle meet at T ; P is any point on the circle: PL, PM, PN are the perpendiculars to AB, AT, BT , then $PL^2 = PM \cdot PN$.

102. A circle Σ cuts each of three circles S_1, S_2, S_3 orthogonally. If Σ cuts S_1 at P, Q , prove that there are two circles coaxal with S_2, S_3 which touch S_1 , either at P or Q .

103. A, B, C, D form a harmonic system of points on a circle Σ . The figure is inverted w.r.t. any point; prove that the inverses of A, B, C, D form a harmonic system of points on the inverse of Σ .

104. O is a fixed point. POQ , ROS are two variable chords of a fixed circle. Find the locus of the second point of intersection of the circles POR , QOS .

105. OA , OA' are two fixed lines. C is a fixed point. Two lines through C meet OA at P , Q and OA' at P' , Q' ; prove that the circles CPQ , $CP'Q'$ cut at the same angle as the lines OA , OA' .

106. $ABCD$ is a complete quadrangle inscribed in a circle; prove that each pair of diagonal points are the extremities of a diameter of the circle of similitude of two circles whose diameters are a pair of opposite sides.

107. Given the orthocentre and circumcircle of a triangle, find the locus of the mid-points of the sides.

108. Two circles touch at O : PQ is a chord of one; PH , QK are tangents to the other, prove that $\frac{PH}{QK} = \frac{PO}{QO}$.

109. If PQ is a diameter of the circumcircle of the triangle ABC ; prove that the pedal lines of P , Q w.r.t. ABC intersect at right angles on the nine-point circle.

110. S_1 , S_2 , S_3 are three circles. If the centres of S_1 , S_2 lie on the radical axes of S_2 , S_3 and S_3 , S_1 respectively, prove that the centre of S_3 lies on the radical axis of S_1 , S_2 .

111. With the usual notation of the triangle ABC , calculate the values of the different cross ratios formed by the points G , H , O , N .

112. ABC , $A'B'C'$ are two sets of three collinear points. AC , $A'C'$; AC' , $B'C$; $A'C$, AB' ; $A'B$, AB' ; $A'C$, AC' ; $B'C$, $B'C'$ intersect at O , K , H , P , Q , R . Prove that (1) $\{APHB'\} = \{ABCO\} = \{KRCB'\}$
(2) P , Q , R are collinear.

113. C is the mid-point of a chord AB of a circle: two other chords PCQ , RCS are drawn: PS , QR cut AB at L , M , prove that $LC = CM$.

114. $\{ABCD\}$, $\{A'B'C'D'\}$ are two equicross ranges. P , P' are any two points on AA' ; prove that the meets of PB , $P'B'$; PC , $P'C'$; PD , $P'D'$ are collinear.

115. ABC is a given triangle: D is a fixed point on its circumcircle: a variable line through D cuts AB , BC , CA at C' , A' , B' and the circumcircle at P ; prove that $\{PA'B'C'\}$ is of constant cross ratio.

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