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COLLEGE PHYSICS

COLLEGE PHYSICS

MECHANICS, HEAT, AND SOUND

by

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and

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THIRD EDITION



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PREFACE

College Physics is available in one complete volume or as two separate parts: Part 1 covers the subjects of mechanics, heat, and sound; Part 2 includes electricity and magnetism, light, and atomic physics. The total number of topics is small enough so that the complete text may be taught in two semesters.

The text consists exclusively of material suitable for students whose mathematical preparation goes no further than algebra and the elements of trigonometry; no calculus is used. The emphasis is on physical principles; historical background and practical applications have been given a place of secondary importance.

Three systems of units are used: the British gravitational system because it is the one used in engineering work throughout the country; the cgs system because some familiarity with it is essential for any intelligent reading of the literature of physics; and the mks system because of its increasing use in electricity and magnetism, as well as because it seems destined eventually to supplant the cgs system. The symbols and terminology, with few exceptions, are those recommended by the Committee on Letter Symbols and Abbreviations of the American Association of Physics Teachers as listed in the American Standard, ASA-Z10, published in 1947.

The main features of the preceding editions have been retained in the third. The supplementary problems that appeared at the end of the second edition have been incorporated with the problems that follow each chapter. Many figures have been redrawn to achieve greater uniformity and clarity, and many of the optical ray diagrams have been reproduced in white on black. Numerous illustrative problems are worked out in the body of the text; in each of them, all physical quantities are expressed by numerics along with the appropriate units.

The following new material has been incorporated in the third edition: Reynolds number, extension of Bernoulli's equation to include friction, thermoelectricity, Ampere's law, a simple derivation of the expression for the speed of an electromagnetic wave, and the Galilean telescope. Projectile motion and circular motion have been brought into the same chapter, entitled "Motion in a Plane." A new sign convention in geometrical optics allows the same formulas to be used for both mirrors and lenses. The chapter on energy has been completely rewritten from a simple point of view that prepares the student for later, more sophisticated treatments. In the interests of simplicity and clarity, much of the material from earlier editions has been rewritten.

In preparing this revision the authors have referred constantly to the many letters and reviews sent to the publishers by colleagues and friends. We have tried to incorporate most of the suggestions based upon actual teaching experience and have rejected only those which would have required too much space or would have involved too great a departure from normal practice. It is with a feeling of sincere gratitude that we acknowledge the help we have received from the following teachers of physics: C. Adler, R. S. Anderson, B. Bederson and S. Borowitz, C. J Brasefield, A. Capecelatro, H. Conrad, F. S. C., L. H. Fisher, S. Freeman Jr., N. S. Gingrich, S. Goodwin, L. E. Hammitt, C. D. Hodgman, M Iona, R. Katz, S. W. Leifson, M. S. Livingston, R. J. Maurer, L. O. Olsen, R. Pinkston, J. G. Potter, R. Resnick, F. A. Scott, C. P. Slichter, F. W. Thiele, F. Verbrugge, A. E. Werbrouck, L. Wolfenstein, and R. E. Worley.

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M. W. Z.

July 1959

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CHAPTER 1

COMPOSITION AND RESOLUTION OF VECTORS

1–1 The fundamental indefinables of mechanics. Physics has been called the science of measurement. To quote from Lord Kelvin (1824–1907), "I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of *Science*, whatever the matter may be."

A definition of a quantity in physics must provide a set of rules for calculating it in terms of other quantities that can be measured. Thus, when momentum is defined as the product of "mass" and "velocity," the rule for calculating momentum is contained within the definition, and all that is necessary is to know how to measure mass and velocity. The definition of velocity is given in terms of length and time, but there are no simpler or more fundamental quantities in terms of which length and time may be expressed. *Length and time are two of the indefinables of mechanics.* It has been found possible to express all the quantities of mechanics in terms of only three indefinables. The third may be taken to be "mass" or "force" with equal justification. *We shall choose mass as the third indefinable of mechanics.*

In geometry, the fundamental indefinable is the "point." The geometer asks his disciple to build any picture of a point in his mind, provided the picture is consistent with what the geometer *says* about the point. In physics, the situation is not so subtle. Physicists from all over the world have international committees at whose meetings the rules of measurement of the indefinables are adopted. The rule for measuring an indefinable takes the place of a definition.

1–2 Standards and units. The measurement of any indefinable of physics involves the application of a simple set of rules. Instead of referring to these rules in the abstract, let us employ them in connection with the quantity "length." The first step is to choose an arbitrary *standard* of length, in the form of an inanimate, solid, durable material. The international standard of length is a bar of platinum-iridium alloy of X-shaped cross section (see Fig. 1–1) called the *standard meter*, which is kept at the International Bureau of Weights and Measures at Sevres, near Paris.

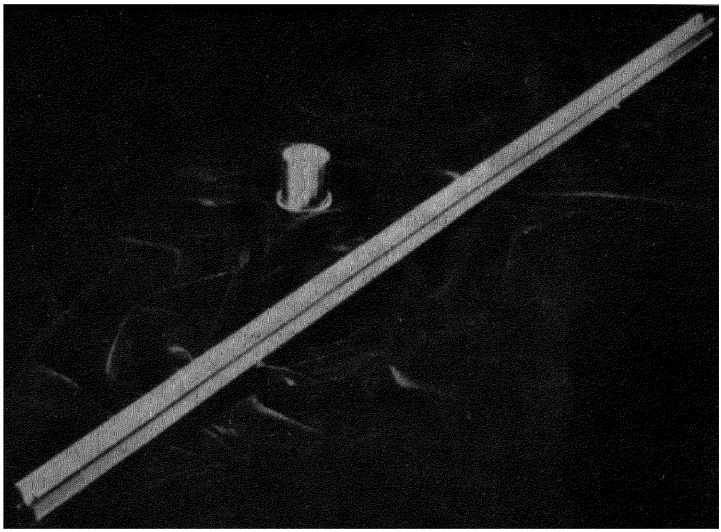


FIG. 1–1. The standard meter and the standard kilogram.

The distance between two lines engraved on gold plugs near the ends of the bar, when the bar is at the temperature of melting ice, is called *one meter*. The meter was originally intended to represent one ten-millionth of the earth's quadrant through Paris, but later, more accurate measurements, have shown that it differs from its intended value by a small amount. This is unimportant. There is no reason why a standard of length should have any connection with the earth's quadrant or with any other length. *A standard is arbitrary, and its virtue lies in the fact that all the scientists of the world accept it.*

The next step is to adopt, also by international agreement, a device or a method by which other lengths may be made equal to the standard, or any multiple or any fraction of the standard. The device for this purpose is a *dividing engine* in which a microscope, mounted on a movable stand, advances along a helical screw when the screw is rotated. With this device two lengths are said to be equal when they both require the same number of rotations of the screw. If one requires one-half the number, it is regarded as half the length, etc. Nowadays, the number of rotations of the screw is replaced by the number of wavelengths of cadmium red light. This requires the addition of an optical interferometer to the dividing engine.

Any multiple or fraction of the standard, and sometimes the standard itself, may be chosen as a convenient *unit* for some specific purpose. Thus,

to measure the dimensions of objects like books or tables, a unit called the *centimeter* is often used, where

$$1 \text{ centimeter} = 10^{-2} \text{ meter}.$$

For atomic dimensions, however, a unit called the *angstrom unit* is found more convenient, where

$$1 \text{ angstrom unit} = 10^{-10} \text{ meter}.$$

In astronomy, a convenient unit of length is the distance light travels in a year, or

$$1 \text{ light year} = 9.45 \times 10^{15} \text{ meters}.$$

The *yard*, originally embodied by a physical standard like the meter, is now defined as

$$1 \text{ yard} = 0.9144 \text{ meter (exactly)},$$

so that the same physical standard serves for both the yard and the meter. The *foot* is defined as one-third of a yard.

The standard of time is our rotating earth and the standard time interval is called the *mean solar day*, the average time interval between successive appearances of the sun overhead. The device for determining whether two time intervals are equal, and also for subdividing the standard time interval into any number of equal subintervals, has been chosen to be the pendulum. The pendulum will be discussed fully in Chapter 11. It is sufficient at this time to know that two time intervals are equal when the same number of pendulum swings occur in each interval. If a pendulum makes 86,400 swings ($60 \times 60 \times 24$) in one mean solar day, the time interval for each swing is said to be one *second*. The second is the most convenient unit of time for most purposes.

The international standard of mass is a cylinder of platinum-iridium (see Fig. 1-1) called the standard *kilogram* (abbreviated kgm). It was originally intended to have a mass equal to that of 1000 cm^3 of pure water at a temperature of 4°C but, as with the standard meter, more precise measurements have shown that this is not exactly true. It is important to emphasize again that the standard kilogram does not have to correspond to a mass of water or of anything else. Whatever it is, it must be accepted by all scientists over the world. The device for determining whether two masses are equal, and also for subdividing the standard kilogram into any number of equal parts or of defining any multiple of the kilogram, is an *equal-arm balance*. The concept of mass as well as the operation of an equal-arm balance will be fully discussed in Chapter 5. It suffices at this time to know that two masses are said to be equal if they balance on an equal-arm balance. If two equal masses that are placed on the left pan

balance the standard kilogram on the right pan, then each of the two masses on the left pan is equal to one-half a kilogram. A convenient unit of mass for many purposes is provided by one-one thousandth of a kilogram, or one gram (1 gm).

The pound mass, like the yard, was formerly embodied by a physical standard of its own, but the *standard avoirdupois pound* is now defined in the U.S. as a body of mass 0.4535924277 kgm.

1–3 Symbols for physical quantities. We shall adopt the convention that an algebraic symbol representing a physical quantity, such as F, p, or v, stands for both a *number* and a *unit*. For example, F might represent a force of 10 lb, p a pressure of 15 lb/ft^2, and v a velocity of 15 ft/sec.

When we write

$$x = v_0 t + \tfrac{1}{2}at^2,$$

if x is in feet then the terms $v_0 t$ and $\frac{1}{2}at^2$ must be in feet also. Suppose t is in seconds. Then the units of v_0 must be ft/sec and those of a must be ft/sec^2. (The factor $\frac{1}{2}$ is a *pure number*, without units.) As a numerical example, let $v_0 = 10$ ft/sec, $a = 4$ ft/sec^2, $t = 10$ sec. Then the preceding equation would be written

$$x = 10\,\frac{\text{ft}}{\text{sec}} \times 10\,\text{sec} + \tfrac{1}{2} \times 4\,\frac{\text{ft}}{\text{sec}^2} \times 100\,\text{sec}^2.$$

The units can now be treated like algebraic symbols. The sec's cancel in the first term and the sec^2's in the second, and

$$x = 100\text{ ft} + 200\text{ ft} = 300\text{ ft}.$$

The beginning student will do well to include the units of all physical quantities, as well as their magnitudes, in all his calculations. This will be done consistently in the numerical examples throughout the book.

1–4 Force. Mechanics is the branch of physics which deals with the motion of material bodies and with the forces that bring about the motion. We shall postpone a discussion of motion until Chapter 4, and start with a study of forces.

When we push or pull on a body, we are said to exert a *force* on it. Forces can also be exerted by inanimate objects; a stretched spring exerts forces on the bodies to which its ends are attached, compressed air exerts a force on the walls of its container, a locomotive exerts a force on the train it is drawing. The force of which we are most aware in our daily lives is the force of gravitational attraction exerted on every body by the earth and called the *weight* of the body. Gravitational forces (and electrical and magnetic forces also) can act through empty space without con-

tact. In this respect they differ from the forces mentioned above, where the body doing the pushing or pulling must make contact with the body being pushed or pulled.

We are not yet in a position to show how a unit of force can be defined in terms of the units of mass, length, and time. This will be done in Chapter 5. For the present, a unit of force can be defined as follows. We select as a standard body the standard pound, defined in the preceding section as a certain fraction (approximately 0.454) of a standard kilogram. The force with which the earth attracts this body, at some specified point on the earth's surface, is then a perfectly definite, reproducible force and is called a force of *one pound* (avoirdupois). A particular point on the earth's surface must be specified, since the attraction of the earth for a given body varies slightly from one point to another. If great precision is not required it suffices to take any point at sea level and 45° latitude. A more precise specification is given in Section 5–3.

In order that an unknown force can be compared with the force unit, and thereby measured, some measurable effect produced by a force must be used. One such effect is to alter the dimensions or shape of a body on which the force is exerted; another is to alter the state of motion of the body. Both of these effects are used in the measurement of forces. In this chapter we shall consider only the former; the latter will be discussed in Chapter 5.

The instrument most commonly used to measure forces is the spring balance, which consists of a coil spring enclosed in a case for protection and carrying at one end a pointer that moves over a scale. A force exerted on the balance changes the length of the spring. The balance can be calibrated as follows. The standard pound is first suspended from the balance at sea level and 45° latitude and the position of the pointer is marked 1 lb. Any number of duplicates of the standard can then be prepared by suspending a body from the balance and adding or removing material until the index again stands at 1 lb. Then when two, three, or more of these are suspended simultaneously from the balance the force stretching it is 2 lb, 3 lb, etc., and the corresponding positions of the pointer can be labeled 2 lb, 3 lb, etc. This procedure makes no assumption about the elastic properties of the spring except that the force exerted on it is always the same when the pointer stands at the same position. The calibrated balance can then be used to measure any unknown force.

1–5 Graphical representation of forces. Vectors. Suppose we are to slide a box along the floor by pulling it with a string or pushing it with a stick, as in Fig. 1–2. That is, we are to slide it by exerting a force on it. The point of view which we now adopt is that the motion of the box is caused not by the *objects* which push or pull on it, but by the *forces* which

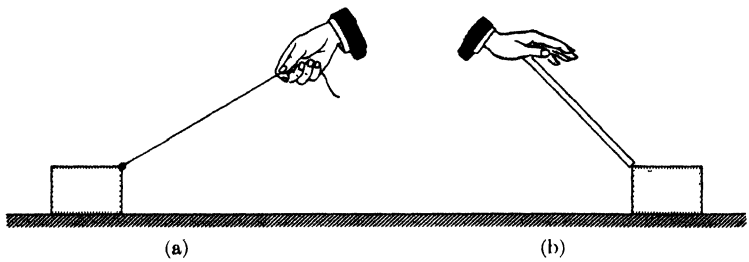


FIGURE 1–2

these exert. For concreteness, assume the magnitude of the push or pull to be 10 lb. It is clear that simply to write "10 lb" on the diagram would not completely describe the force, since it would not indicate the direction in which the force was acting. One might write "10 lb, 30° above horizontal to the right," or "10 lb, 45° below horizontal to the right," but all the above information may be conveyed more briefly if we adopt the convention of representing a force by an arrow. The length of the arrow, to some chosen scale, indicates the size or magnitude of the force, and the direction in which the arrow points indicates the direction of the force. Thus Fig. 1–3 is the force diagram corresponding to Fig. 1–2. (There are other forces acting on the box, but these are not shown in the figure.)

Force is not the only physical quantity which requires the specification of direction as well as magnitude. For example, the velocity of an aircraft is not completely specified by stating that it is 300 miles per hour; we need to know the direction also. The concept of volume, on the other hand, has no direction associated with it.

Quantities like volume, which involve a magnitude only, are called *scalars*. Those like force and velocity, which involve both magnitude and direction, are called *vector quantities*. A vector quantity is represented mathematically by a letter printed in boldface (heavy) type. When this letter appears in an equation, it is understood that both the magnitude and

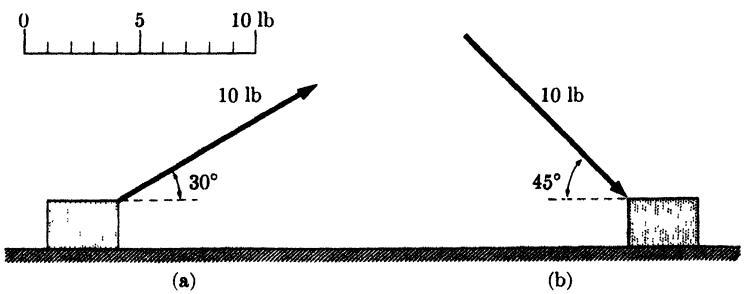


FIGURE 1–3

the direction of the vector quantity are included within the one symbol. The same letter in ordinary (lightface) type represents the magnitude only. Thus, the magnitude of the vector **F** is represented by F. In writing the symbol for a vector quantity on paper or on the blackboard it is often convenient to double one of the vertical lines of the letter, like $\mathbb{F}$. When a vector quantity is represented graphically by an arrow, the arrow is called a vector (or if a more specific statement is needed, a force vector or a velocity vector).

Some vector quantities, of which force is one, are not *completely* specified by their magnitude and direction alone. Thus the effect of a force depends also on its *line of action* and its *point of application.* (The line of action is a line of indefinite length, of which the force vector is a segment.) For example, if one is pushing horizontally against a door, the effectiveness of a force of given magnitude and direction depends on the distance of its line of action from the hinges. If a body is deformable, as all bodies are to a greater or lesser extent, the deformation depends on the point of application of the force. However, since many actual objects are deformed only very slightly by the forces acting on them, we shall assume for the present that all objects considered are perfectly rigid. The point of application of a given force acting on a rigid body may be transferred to any other point on the line of action without altering the effect of the force. Thus *a force applied to a rigid body may be regarded as acting anywhere along its line of action.*

1–6 Components of a vector. When a box is pulled or pushed along the floor by an inclined force as in Fig. 1–2, the effectiveness of the force in moving the box along the floor depends upon the direction in which the force acts. Furthermore, each of the forces in Fig. 1–2 is producing another effect in addition to moving the box ahead. That is, the pull of the cord is in part tending to lift the box off the floor, and the push of the stick is in part forcing the box down against the floor. We are thus led to the idea of the *components* of a force, that is, the effective values of a force in directions other than that of the force itself.

The component of a force in any direction can be found by a simple graphical method. Let a given force be represented by the vector **F** in Fig. 1–4, from O to A. To find the component of **F** in the direction of the line Ob, drop a perpendicular from A to this line, intersecting it at point B. The vector $\mathbf{F}_b$, from O to B, to the same scale as that used for the vector **F**, then represents the component of **F** in the direction Ob, or the effective value of the force in this direction.

Similarly, the vector $\mathbf{F}_c$ from O to C represents the component of **F** in the direction Oc. The component along Od, at right angles to **F**, is zero, and the component along Oa is equal to **F**.

The magnitude of a component of a vector in any direction can be calculated as follows. Referring to Fig. 1–4, we have from the right triangle OAB,

$$\cos\theta_b = \frac{OB}{OA} = \frac{F_b}{F},$$

$$F_b = F\cos\theta_b.$$

If $F = 10$ lb and $\theta_b = 60°$, $\cos\theta_b = 0.500$ and

$$F_b = 10\text{ lb} \times 0.500 = 5.00\text{ lb}.$$

In the same way,

$$\cos\theta_c = \frac{OC}{OA} = \frac{F_c}{F},$$

$$F_c = F\cos\theta_c.$$

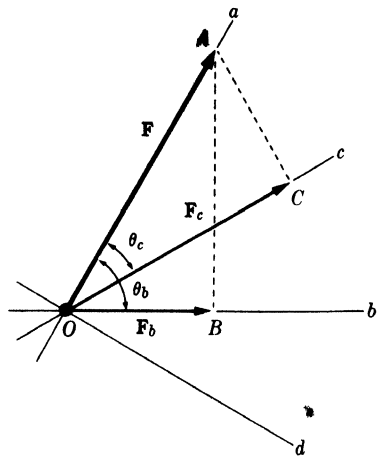


FIG. 1–4. Vectors $\mathbf{F}_b$ and $\mathbf{F}_c$ are the components of $\mathbf{F}$ in the directions Ob and Oc. The component along Od, perpendicular to $\mathbf{F}$, is zero.

If $\theta_c = 30°$, $\cos\theta_c = 0.866$ and $F_c = 10\text{ lb} \times 0.866 = 8.66$ lb.

In general, then, the magnitude of the component of a vector $\mathbf{F}$ in any direction making an angle θ with that of the vector is equal to $F\cos\theta$. If $\theta = 90°$, $\cos\theta = 0$ and the component at right angles to the vector is zero. If $\theta = 0$, $\cos\theta = 1$ and the component is equal to F.

Figure 1–5 shows the same box as in Figs. 1–2(a) and 1–3(a). The vectors $\mathbf{F}_x$ and $\mathbf{F}_y$ are the components of $\mathbf{F}$ in the mutually perpendicular directions Ox and Oy, and are called the *rectangular components* of $\mathbf{F}$ in these two directions. But since a vector has no component at right angles

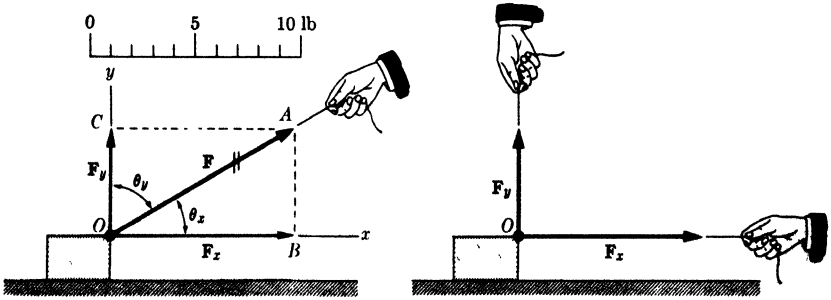


FIG. 1–5. The inclined force $\mathbf{F}$ may be replaced by its rectangular components $\mathbf{F}_x$ and $\mathbf{F}_y$. $F_x = F\cos\theta_x$, $F_y = F\cos\theta_y$.

to its own direction, $\mathbf{F}_x$ has no component along Oy and $\mathbf{F}_y$ has no component along Ox. No further resolution of the force into x- and y-components is therefore possible. Physically, this means that the two forces $\mathbf{F}_x$ and $\mathbf{F}_y$, acting simultaneously as in Fig. 1-5(b), are equivalent in all respects to the original force $\mathbf{F}$. *Any force may be replaced by its rectangular components.*

As a numerical example, let the force $\mathbf{F}$ in Fig. 1-5 have a magnitude of 10 lb, and let $\theta_x = 30°$, $\theta_y = 60°$. Then $F_x = 8.66$ lb, $F_y = 5.00$ lb, and these two forces applied simultaneously as in part (b) are found to produce exactly the same effect as the single force of 10 lb in part (a).

It is often convenient to express both the x- and y-components of a vector in terms of the angle between the vector and the x-axis. We see from Fig. 1-5(a) that

$$\sin \theta_x = \frac{BA}{OA} = \frac{OC}{OA} = \frac{F_y}{F},$$

$$F_y = F \sin \theta_x.$$

Hence with the understanding that θ refers to the angle between a vector $\mathbf{F}$ and the x-axis, we can say, in general, that

$$F_x = F \cos \theta,$$

$$F_y = F \sin \theta.$$

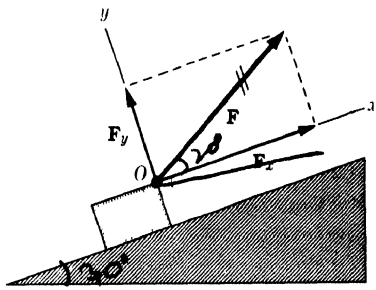


FIG. 1-6. $\mathbf{F}_x$ and $\mathbf{F}_y$ are the rectangular components of $\mathbf{F}$, parallel and perpendicular to the sloping surface of the inclined plane.

The directions in which the rectangular components of a vector may be desired are not necessarily horizontal and vertical. For example, Fig. 1-6 shows a block being dragged up an inclined plane by a force $\mathbf{F}$. The vectors $\mathbf{F}_x$ and $\mathbf{F}_y$ are the components of $\mathbf{F}$ parallel and perpendicular to the sloping surface of the plane.

1-7 Resultant or vector sum. Ordinarily a body is acted on simultaneously by a number of forces having different magnitudes, directions, and points of application. To begin with, we shall consider only sets of forces which lie in the same plane (*coplanar* forces) and which have the same point of application (*concurrent* forces). It is found by experiment that any set of coplanar, concurrent forces can be replaced by a single force whose effect is the same as that of the given forces and which is called their *resultant.*

In Fig. 1-7, a body is acted on by the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, both applied at point O. To find their resultant, construct the parallelogram $OACB$, of which the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ form two adjacent sides. The concurrent

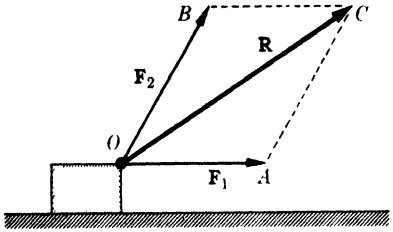


FIG. 1–7. Force **R**, the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$, is obtained by the parallelogram method.

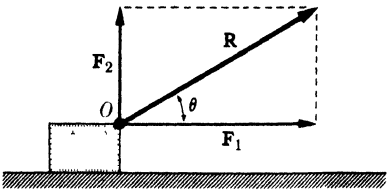


FIG. 1–8. Resultant of two forces at right angles to each other.

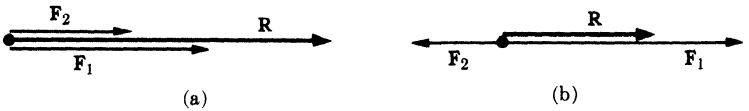


FIG. 1–9. Resultant of (a) two parallel forces, (b) two antiparallel forces.

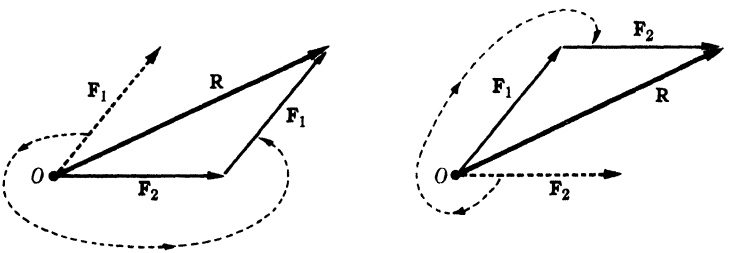


FIG. 1–10. The triangle method for finding the resultant of two forces.

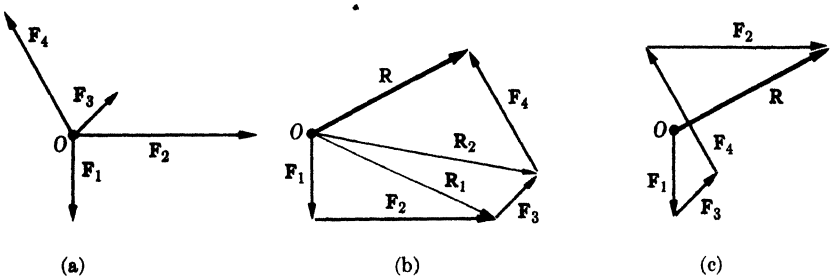


FIG. 1–11. Polygon method.

diagonal of the parallelogram, that is, the vector **R** from O to C, is called the *vector sum* of the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$, and we find by experiment that it represents the resultant force in magnitude and direction.

In the special case in which the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are at right angles to each other, as in Fig. 1-8, the triangle OAC is a right triangle whose sides are the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. The magnitude and direction of the resultant are then given by

$$R = \sqrt{F_1^2 + F_2^2}, \qquad \tan\theta = \frac{F_2}{F_1}.$$

Another special case is that in which two forces having the same line of action are parallel, as in Fig. 1-9(a), or antiparallel, as in Fig. 1-9(b). If they are parallel, the magnitude of the resultant **R** equals the sum of the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$. If they are antiparallel the magnitude of the resultant equals the difference between the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$.

The construction in Fig. 1-7 is called the *parallelogram* method of finding the resultant of two vectors. The *triangle* method, shown in Fig. 1-10, consists of displacing either vector parallel to itself until its tail coincides with the head of the other vector. The resultant **R** is then represented by the closing side of the triangle.

When more than two forces are to be combined, one may first find the resultant of any two, then combine this resultant with a third, and so on. The process is illustrated in Fig. 1-11, which shows the four forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ acting simultaneously at the point O. In Fig. 1-11(b), forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are first combined by the triangle method, giving a resultant $\mathbf{R}_1$; force $\mathbf{R}_1$ is then combined by the same process with $\mathbf{F}_3$, giving a resultant $\mathbf{R}_2$; finally $\mathbf{R}_2$ and $\mathbf{F}_4$ are combined to obtain the resultant **R**. Evidently the vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ need not have been drawn—one need only draw the given vectors in succession, with the tail of each at the head of the one preceding it, and complete the polygon by a vector **R** from the tail of the first to the head of the last vector. The order in which the vectors are drawn makes no difference, as shown in Fig. 1-11(c).

1-8 Resultant by rectangular resolution. While the polygon method is a satisfactory graphical one for finding the resultant of a number of forces, it is awkward for computation because one must, in general, solve a number of oblique triangles. Therefore the usual analytical method of finding the resultant is first to resolve all forces into rectangular components along any convenient pair of axes and then combine these into a single resultant. This makes it possible to work with right triangles only.

Figure 1-12(a) shows three concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, whose resultant we wish to find. Let a pair of rectangular axes be constructed in any arbitrary direction. Simplification results if one axis coincides with

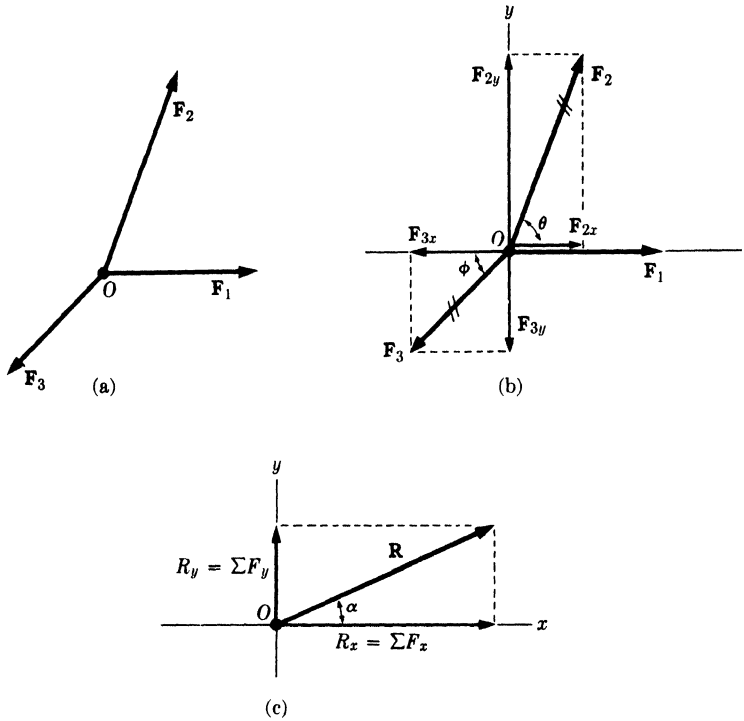


FIG. 1–12. Vector **R**, the resultant of $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, is obtained by the method of rectangular resolution. The rectangular components of **R** have the magnitudes $R_x = \sum F_x$, $R_y = \sum F_y$.

one of the forces, which is always possible. In Fig. 1–12(b), the x-axis coincides with $\mathbf{F}_1$. Let us first resolve each of the given forces into x- and y-components. According to the usual conventions of analytic geometry, x-components toward the right are considered positive and those toward the left, negative. Upward y-components are positive and downward y-components are negative.

Force $\mathbf{F}_1$ lies along the x-axis and need not be resolved. The magnitudes of the components of $\mathbf{F}_2$ are $F_{2x} = F_2 \cos\theta$, $F_{2y} = F_2 \sin\theta$. Both of these are positive, and F_{2x} has been slightly displaced upward to show it more clearly. The components of $\mathbf{F}_3$ are $F_{3x} = F_3 \cos\phi$, $F_{3y} = F_3 \sin\phi$. Both of these are negative.

We now imagine $\mathbf{F}_2$ and $\mathbf{F}_3$ to be removed and replaced by their rectangular components. To indicate this, the vectors $\mathbf{F}_2$ and $\mathbf{F}_3$ are crossed out lightly. All the x-components can now be combined into a single force $\mathbf{R}_x$ whose magnitude equals the algebraic sum of the x-components,

or ΣF_x, and all the y-components can be combined into a single force $\mathbf{R}_y$ of magnitude ΣF_y.

$$R_x = \Sigma F_x, \qquad R_y = \Sigma F_y.$$

Finally, these can be combined as in part (c) of the figure to form the resultant $\mathbf{R}$ whose magnitude, since R_x and R_y are perpendicular to each other, is

$$R = \sqrt{R_x^2 + R_y^2}.$$

The angle α between $\mathbf{R}$ and the x-axis can now be found from any one of its trigonometric functions. For example,

$$\tan \alpha = \frac{R_y}{R_x}.$$

EXAMPLE. In Fig. 1-12, let $F_1 = 120$ lb, $F_2 = 200$ lb, $F_3 = 150$ lb, $\theta = 60°$, $\phi = 45°$. The computations can be arranged systematically as follows:

Force	Angle	x-component	y-component
$F_1 = 120$ lb	0	+120 lb	0
$F_2 = 200$ lb	60°	+100 lb	+173 lb
$F_3 = 150$ lb	45°	−106 lb	−106 lb
		$\Sigma F_x =$ +114 lb	$\Sigma F_y =$ +67 lb

$$R = \sqrt{(114 \text{ lb})^2 + (67 \text{ lb})^2} = 132 \text{ lb},$$

$$\alpha = \tan^{-1} \frac{67 \text{ lb}}{114 \text{ lb}} = \tan^{-1} 0.588 = 30.4°.$$

1-9 Vector difference. It is sometimes necessary to subtract one vector from another. The process of subtracting one *algebraic* quantity from another is equivalent to adding the negative of the quantity to be subtracted. That is,

$$a - b = a + (-b).$$

Similarly, the process of subtracting one *vector* quantity from another is equivalent to adding (vectorially) the negative of the vector to be subtracted, where the negative of a given vector is defined as a vector of the same magnitude but in the opposite direction. That is, if $\mathbf{A}$ and $\mathbf{B}$ are two vectors,

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

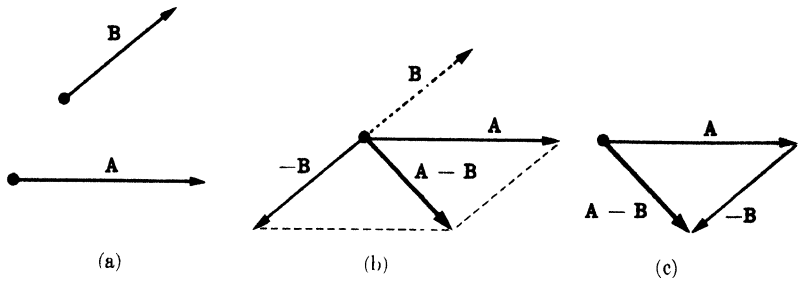


FIG. 1-13. The vector difference **A** − **B** is found in part (b) by the parallelogram method and in part (c) by the triangle method.

Vector subtraction is illustrated in Fig. 1-13. The given vectors are shown in part (a). In part (b), the vector sum of **A** and −**B**, or the vector difference **A** − **B**, is found by the parallelogram method. In part (c), the triangle method has been used.

Vector differences may also be found by the method of rectangular resolution. Both vectors are resolved into x- and y-components. The difference between the x-components is the x-component of the desired vector difference, and the difference between the y-components is the y-component of the vector difference.

Vector subtraction is not often used when dealing with forces, but we shall use it frequently in connection with velocities and accelerations.

Problems

1–1. (a) Find graphically the horizontal and vertical components of a 40-lb force the direction of which is 50° above the horizontal to the right. Let 1/16 in. = 1 lb. (b) Check your results by calculating the components.

1–2. A box is pushed along the floor as in Fig. 1–2 by a force of 40 lb making an angle of 30° with the horizontal. Using a scale of 1 in. = 10 lb, find the horizontal and vertical components of the force by the graphical method. Check your results by calculating the components.

1–3. A block is dragged up an inclined plane of slope angle 20° by a force **F** making an angle of 30° with the plane, as in Fig. 1–6. (a) How large a force **F** is necessary in order that the component F_x parallel to the plane shall be 16 lb? (b) How large will the component F_y then be? Solve graphically, letting 1 in. = 8 lb.

1–4. The three forces shown in Fig. 1–14 act on a body located at the origin. (a) Find the x- and y-components of each of the three forces. (b) Use the method of rectangular resolution to find the resultant of the forces (c) Find the magnitude and direction of a fourth force which must be added to make the resultant force zero. Indicate the fourth force by a diagram.

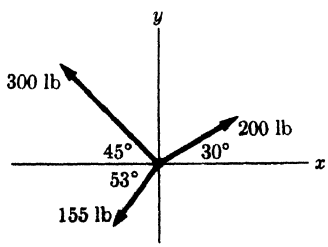


Figure 1–14

1–5. Find graphically the magnitude and direction of the resultant of the three forces in Fig. 1–14. Use the polygon method.

1–6. Two men and a boy want to push a crate in the direction marked x in Fig. 1–15. The two men push with forces $\mathbf{F}_1$ and $\mathbf{F}_2$ whose magnitudes and directions are indicated in the figure. Find the magnitude and direction of the smallest force which the boy should exert.

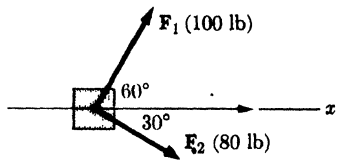


Figure 1–15

1–7. Find graphically the resultant of two 10-lb forces applied at the same point: (a) when the angle between the forces is 30°; (b) when the angle between them is 130°. Use any convenient scale.

1–8. Two men pull horizontally on ropes attached to a post, the angle between the ropes being 45°. If man A exerts a force of 150 lb and man B a force of 100 lb, find the magnitude of the resultant force and the angle it makes with A's pull. Solve: (a) graphically, by the parallelogram method; (b) graphically, by the triangle method (c) analytically, by the method of rectangular resolution. Let 1 in. = 50 lb in (a) and (b).

1–9. Use the method of rectangular resolution to find the resultant of the following set of forces and the angle it makes with the horizontal: 200 lb, along the x-axis toward the right; 300 lb, 60° above the x-axis to the right;

100 lb, 45° above the x-axis to the left; 200 lb, vertically down.

1–10. (a) Find graphically the vector sum $\mathbf{A} + \mathbf{B}$ and the vector difference $\mathbf{A} - \mathbf{B}$ in Fig. 1–16. (b) Use the method of rectangular resolution to find the magnitude and direction of the vector sum and the vector difference.

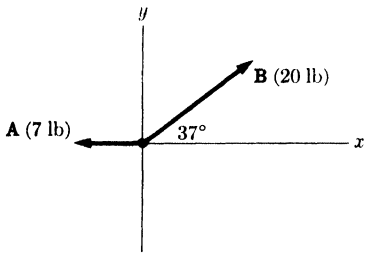


FIGURE 1–16

1–11. Vector **A** is 2 inches long and is 60° above the x-axis in the first quadrant. Vector **B** is 2 inches long and is 60° below the x-axis in the fourth quadrant. Find graphically (a) the vector sum $\mathbf{A} + \mathbf{B}$, and (b) the vector differences $\mathbf{A} - \mathbf{B}$ and $\mathbf{B} - \mathbf{A}$.

1–12. A vector **A** of length 10 units makes an angle of 30° with a vector **B** of length 6 units. Find the magnitude of the vector difference $\mathbf{A} - \mathbf{B}$ and the angle it makes with vector **A**: (a) by the parallelogram method; (b) by the triangle method; (c) by the method of rectangular resolution.

1–13. Two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, act at a point. The magnitude of $\mathbf{F}_1$ is 8 lb and its direction is 60° above the x-axis in the first quadrant. The magnitude of $\mathbf{F}_2$ is 5 lb and its direction is 53° below the x-axis in the fourth quadrant. (a) What are the horizontal and vertical components of the resultant force? (b) What is the magnitude of the resultant? (c) What is the magnitude of the vector difference $\mathbf{F}_1 - \mathbf{F}_2$?

1–14. Two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, act upon a body in such a manner that the resultant force **R** has a magnitude equal to that of $\mathbf{F}_1$ and makes an angle of 90° with $\mathbf{F}_1$. Let $F_1 = R = 10$ lb. Find the magnitude of the second force, and its direction (relative to $\mathbf{F}_1$).

1–15. Find the resultant of the following set of forces by the method of rectangular resolution: 80 lb, vertically down; 100 lb, 53° above horizontal to the right; 60 lb, horizontal to the left. Check by the polygon method.

CHAPTER 2

EQUILIBRIUM

2–1 Introduction. The science of mechanics is based on three natural laws which were clearly stated for the first time by Sir Isaac Newton (1643–1727) and were published in 1686 in his *Philosophiae Naturalis Principia Mathematica* ("The Mathematical Principles of Natural Science"). It should not be inferred, however, that the science of mechanics began with Newton. Many men had preceded him in this field, the most outstanding being Galileo Galilei (1564–1642), who in his studies of accelerated motion had laid much of the groundwork of Newton's three laws.

In this chapter we shall make use of only two of Newton's laws, the first and the third. Newton's second law will be discussed in Chapter 5.

2–2 Equilibrium. Newton's first law. One effect of a force is to alter the dimensions or shape of a body on which the force acts; another is to alter the state of motion of the body.

The motion of a body can be considered as made up of its motion as a whole, or its *translational* motion, together with any *rotational* motion the body may have. In the most general case, a single force acting on a body produces a change in both its translational and rotational motion. However, when several forces act on a body simultaneously, their effects can compensate one another, with the result that there is no change in either the translational or rotational motion. When this is the case, the body is said to be in *equilibrium*. This means that (1) the body as a whole either remains at rest or moves in a straight line with constant speed, and (2) that the body is either not rotating at all or is rotating at a constant rate.

Let us consider some (idealized) experiments from which the laws of equilibrium can be deduced. Figure 2–1 represents a flat, rigid object of arbitrary shape on a level surface having negligible friction. If a single force $\mathbf{F}_1$ acts on the body, as in Fig. 2–1(a), and if the body is originally at rest, it at once starts to move and to rotate clockwise. If originally in motion, the effect of the force is to change the translational motion of the body in magnitude or direction (or both) and to increase or decrease its rate of rotation. In either case, the body does not remain in equilibrium.

Equilibrium can be maintained, however, by the application of a second force $\mathbf{F}_2$ as in Fig. 2–1(b), provided that $\mathbf{F}_2$ is equal in magnitude to $\mathbf{F}_1$,

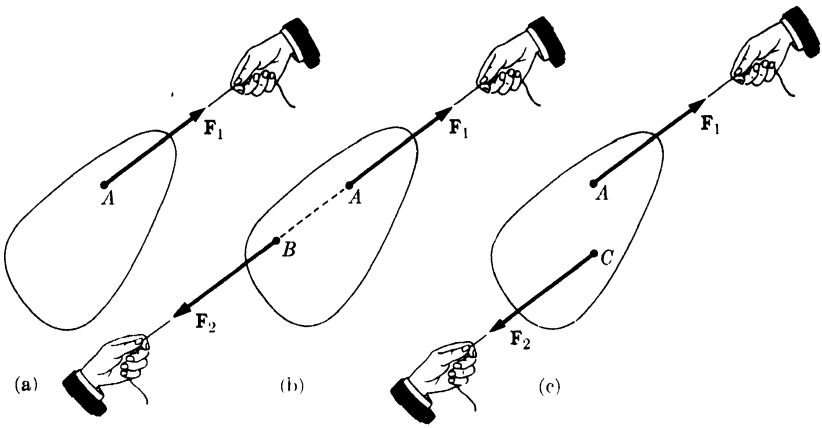


FIG. 2–1. A rigid body acted on by two forces is in equilibrium if the forces are equal in magnitude, opposite in direction, and have the same line of action, as in part (b).

is opposite in direction to $\mathbf{F}_1$, and has the same line of action as $\mathbf{F}_1$. The resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ is then zero. If the lines of action of the two forces are not the same, as in Fig. 2–1(c), the body will be in translational but not in rotational equilibrium.

In Fig. 2–2(a), a body is acted on by three nonparallel coplanar forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$. Any force applied to a rigid body may be regarded as acting anywhere along its line of action. Therefore, let any two of the force vectors, say $\mathbf{F}_1$ and $\mathbf{F}_2$, be transferred to the point of intersection of their lines of action and their resultant $\mathbf{R}$ obtained, as in Fig. 2–2(b). The forces are now reduced to two, $\mathbf{R}$ and $\mathbf{F}_3$, and for equilibrium these must (1) be equal in magnitude, (2) be opposite in direction, and (3) have the same line of action. It follows from the first two conditions that the resultant of the three forces is zero. The third condition can be fulfilled only if the line of action of $\mathbf{F}_3$ passes through the point of intersection of the lines of action of $\mathbf{F}_1$ and $\mathbf{F}_2$. In other words, the three forces must be *concurrent*.

Problems involving more than three forces are usually best treated with the help of the concept of the *moment* of a force, which we shall postpone until the next chapter. This concept will also enable us to find the resultant of parallel forces, a problem we have not yet considered.

The construction in Fig. 2–2 provides a satisfactory graphical method for the solution of problems in equilibrium. For an analytical solution, it is usually simpler to deal with the rectangular components of the forces. We have shown that the magnitudes of the rectangular components of the

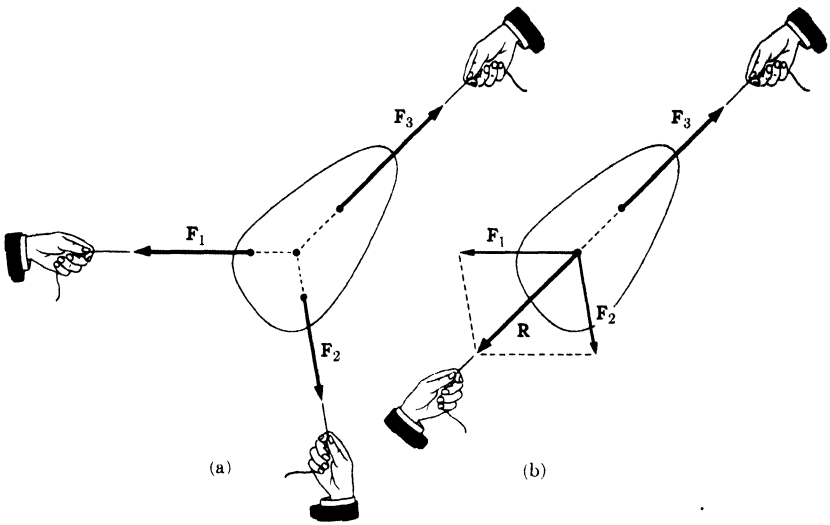


FIG. 2-2. When a body acted on by three nonparallel coplanar forces is in equilibrium, the forces are concurrent and the resultant of any two is equal and opposite to the third.

resultant **R** of any set of coplanar forces are

$$R_x = \Sigma F_x, \qquad R_y = \Sigma F_y.$$

When a body is in equilibrium, the resultant of all of the forces acting on it is zero. Both rectangular components are then zero and hence, for a body in equilibrium,

$$\boxed{\Sigma F_x = 0, \qquad \Sigma F_y = 0.}$$

These equations are called the *first condition of equilibrium.*

The *second condition of equilibrium* is that two forces in equilibrium must have the same line of action, or that three forces in equilibrium must be concurrent.

The first condition of equilibrium ensures that a body shall be in translational equilibrium; the second, that it be in rotational equilibrium. The statement that a body is in complete equilibrium when both conditions are satisfied is the essence of *Newton's first law of motion.* Newton did not state his first law in exactly these words. His original statement (translated from the Latin in which the *Principia* was written) reads: *"Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed on it."*

Although rotational motion was not explicitly mentioned by Newton, it is clear from his work that he fully understood the conditions that the forces must satisfy when the rotation is zero or is constant.

2–3 Stable, unstable, and neutral equilibrium. When a body in equilibrium is displaced slightly, the magnitudes, directions, and lines of action of the forces acting on it may all change. If the forces in the displaced position are such as to return the body to its original position, the equilibrium is *stable*. If the forces act to increase the displacement still further, the equilibrium is *unstable*. If the body is still in equilibrium in the displaced position, the equilibrium is *neutral*. Whether a given equilibrium state is stable, unstable, or neutral can only be determined by considering other states slightly displaced from the first.

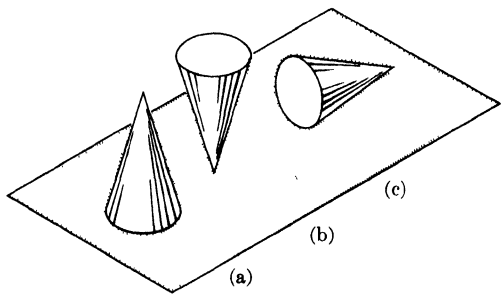


FIG. 2–3. (a) Stable, (b) unstable, and (c) neutral equilibrium.

A right circular cone on a level surface affords an example of the three types of equilibrium. When the cone rests on its base, as in Fig. 2–3(a), the equilibrium is stable. When balanced on its apex, as in part (b), the equilibrium is unstable. When resting on its side, as in part (c), the equilibrium is neutral.

2–4 Newton's third law of motion. Any given force is but one aspect of a mutual interaction between *two* bodies. It is found that *whenever one body exerts a force on another, the second always exerts on the first a force which is equal in magnitude, opposite in direction, and has the same line of action.* A single, isolated force is therefore an impossibility.

The two forces involved in every interaction between two bodies are often called an "action" and a "reaction," but this does not imply any difference in their nature, or that one force is the "cause" and the other its "effect." *Either* force may be considered the "action," and the other the "reaction" to it.

This property of forces was stated by Newton in his *third law of motion*. In his own words, *"To every action there is always opposed an equal reaction:*

or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."

As an example, suppose that a man pulls on one end of a rope attached to a block as in Fig. 2–4. The weight of the block, and the force exerted on it by the surface, are not shown. The block may or may not be in equilibrium. The resulting action-reaction pairs of forces are indicated in the figure. (Actually, the lines of action of all the forces lie along the rope. The force vectors have been offset from this line to show them more clearly.) Vector $\mathbf{F}_1$ represents the force exerted on the rope by the man. Its reaction is the equal and opposite force $\mathbf{F}'_1$ exerted on the man by the rope. Vector $\mathbf{F}_2$ represents the force exerted on the block by the rope. The reaction to it is the equal and opposite force $\mathbf{F}'_2$, exerted on the rope by the block.

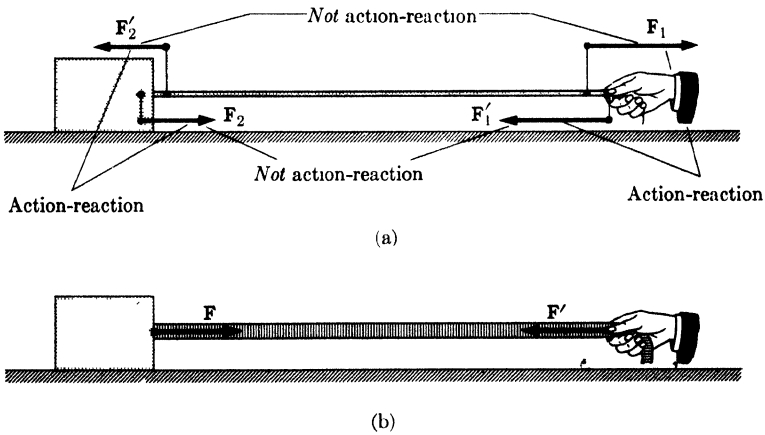


FIG. 2–4. (a) Forces $\mathbf{F}_1$ and $\mathbf{F}'_1$ form one action-reaction pair, and forces $\mathbf{F}_2$ and $\mathbf{F}'_2$ another. $\mathbf{F}_1$ is *always* equal to $\mathbf{F}'_1$, $\mathbf{F}_2$ is *always* equal to $\mathbf{F}'_2$. $\mathbf{F}_1$ and $\mathbf{F}'_2$ are equal only if the rope is in equilibrium, and force $\mathbf{F}'_2$ is *not* the reaction to $\mathbf{F}_1$. (Actually, all forces lie along the rope.) (b) If the rope is in equilibrium it can be considered to transmit a force from the man to the block, and vice versa.

It is very important to realize that the forces $\mathbf{F}_1$ and $\mathbf{F}'_2$, although they are opposite in direction and have the same line of action, do *not* constitute an action-reaction pair. For one thing, both of these forces act on the *same* body (the rope), while an action and its reaction necessarily act on *different* bodies. Furthermore, the forces $\mathbf{F}_1$ and $\mathbf{F}'_2$ are not *necessarily* equal in magnitude. If the block and rope are moving to the right with increasing speed, the rope is not in equilibrium and F_1 is greater than F'_2. Only in the special case when the rope remains at rest or moves with con-

stant speed are the forces $\mathbf{F}_1$ and $\mathbf{F}'_2$ equal in magnitude, but this is an example of Newton's *first* law, not his *third*. Even when the speed of the rope is changing, however, the action-reaction forces $\mathbf{F}_1$ and $\mathbf{F}'_1$ are equal to *each other*, and the action-reaction forces F_2 and F'_2 are equal to *each other*, although then F_1 is not equal to F'_2.

In the special case when the rope is in equilibrium, and when no forces act on it except those at its ends, F'_2 equals F_1 by Newton's *first* law. Since F_2 *always* equals F'_2, by Newton's *third* law, then in this special case F_2 also equals F_1 and the force exerted on the block by the rope is equal to the force exerted on the rope by the man. The rope can therefore be considered to "transmit" to the block, without change, the force exerted on it by the man. This point of view is often useful, but it is important to remember that it applies only under the restricted conditions above.

If we adopt this point of view, the rope itself need not be considered and we have the simpler force diagram of Fig. 2–4(b), where the man is considered to exert a force $\mathbf{F}$ directly on the block. The reaction is the force $\mathbf{F}'$ exerted by the block on the man. The only effect of the rope is to transmit these forces from one body to the other.

A body like the rope in Fig. 2–4, which is subjected to pulls at its ends, is said to be in *tension*. The tension at any point equals the force exerted at that point. Thus in Fig. 2–4(a) the tension at the right-hand end of the rope equals the magnitude of $\mathbf{F}_1$ (or of $\mathbf{F}'_1$) and the tension at the left-hand end equals the magnitude of $\mathbf{F}_2$ (or of $\mathbf{F}'_2$). If the rope is in equilibrium and if no forces act except at its ends, as in Fig. 2–4(b), the tension is the same at both ends. If, for example, in Fig. 2–4(b) the magnitudes of $\mathbf{F}$ and $\mathbf{F}'$ are each 50 lb, the tension in the rope is 50 lb (*not* 100 lb).

2–5 Examples of equilibrium. The conditions of equilibrium express certain relations among the forces acting *on* a body in equilibrium. A carefully drawn diagram, in which each force exerted on a body is represented by a vector, is essential in the solution of problems of this sort. The standard procedure is as follows: First, make a neat sketch of the apparatus or structure. Second, choose some one body which is in equilibrium and show *all* the forces exerted *on* it. This is called "isolating" the chosen body, and the diagram is called a *force diagram* or a *free-body* diagram. On the diagram, which should be sufficiently large to avoid crowding, write the numerical values of all given forces, angles, and dimensions, and assign letters to all unknown quantities. When a structure is composed of several members, a separate force diagram must be constructed for each. Third, construct a set of rectangular axes and indicate on each force diagram the rectangular components of any inclined forces. Cross out lightly those forces which have been resolved. Fourth, obtain

the necessary algebraic or trigonometric equations from the condition of equilibrium,

$$\sum F_x = 0, \qquad \sum F_y = 0.$$

A force which will be encountered in many problems is the *weight* of a body, that is, the force of gravitational attraction exerted on the body by the earth. We shall show in the next chapter that the line of action of this force always passes through a point called the *center of gravity* of the body.

EXAMPLE 1. To begin with a simple example, consider the body in Fig. 2–5, hanging at rest from the ceiling by a vertical cord. Part (b) of the figure is the free-body diagram for the body. The forces on it are its weight $\mathbf{w}_1$ and the upward force $\mathbf{T}_1$ exerted on it by the cord. If we take the x-axis horizontal and the y-axis vertical, there are no x-components of force, and the y-components are the forces $\mathbf{w}_1$ and $\mathbf{T}_1$. Then from the condition that $\sum F_y = 0$, we have

$$\sum F_y = T_1 - w_1 = 0,$$

$$T_1 = w_1. \qquad \text{(1st law)}$$

In order that both forces have the same line of action, the center of gravity of the body must lie vertically below the point of attachment of the cord.

Let us emphasize again that the forces $\mathbf{w}_1$ and $\mathbf{T}_1$ are *not* an action-reaction pair, although they are equal in magnitude, opposite in direction, and have the same line of action. The weight $\mathbf{w}_1$ is a force of attraction exerted on the body by the earth. Its reaction is an equal and opposite force of attraction exerted on the earth by the body. This reaction is one of the set of forces acting *on the earth*, and therefore it does not appear in the free-body diagram of the suspended block.

The reaction to the force $\mathbf{T}_1$ is an equal downward force, $\mathbf{T}_1'$, exerted *on the cord* by the suspended body.

$$T_1 = T_1'. \qquad \text{(3rd law)}$$

The force $\mathbf{T}_1'$ is shown in part (c), which is the free-body diagram of the

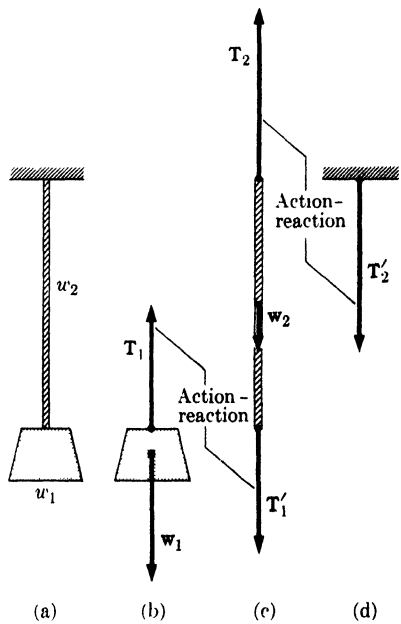


FIG. 2–5. (a) Block hanging at rest from vertical cord. (b) The block is isolated and *all* forces acting *on* it are shown. (c) Forces on the cord. (d) Downward force on the ceiling. Lines connect action-reaction pairs.

cord. The other forces on the cord are its own weight $\mathbf{w}_2$ and the upward force $\mathbf{T}_2$ exerted on its upper end by the ceiling. Since the cord is also in equilibrium,

$$\sum F_y = T_2 - w_2 - T_1' = 0,$$

$$T_2 = w_2 + T_1'. \qquad \text{(1st law)}$$

The reaction to $\mathbf{T}_2$ is the downward force $\mathbf{T}_2'$ in part (d), exerted on the ceiling by the cord.

$$T_2 = T_2'. \qquad \text{(3rd law)}$$

As a numerical example, let the body weigh 20 lb and the cord weigh 1 lb. Then

$$T_1 = w_1 = 20 \text{ lb},$$

$$T_1' = T_1 = 20 \text{ lb},$$

$$T_2 = w_2 + T_1' = 1 \text{ lb} + 20 \text{ lb} = 21 \text{ lb},$$

$$T_2' = T_2 = 21 \text{ lb}.$$

If the weight of the cord were so small as to be negligible, then in effect no forces would act on it except at its ends. The forces $\mathbf{T}_2$ and $\mathbf{T}_2'$ would then each equal 20 lb and, as explained earlier, the cord could be considered to transmit a 20-lb force from one end to the other without change. We could then consider the upward pull of the cord on the block as an "action" and the downward pull on the ceiling as its "reaction." The tension in the cord would then be 20 lb.

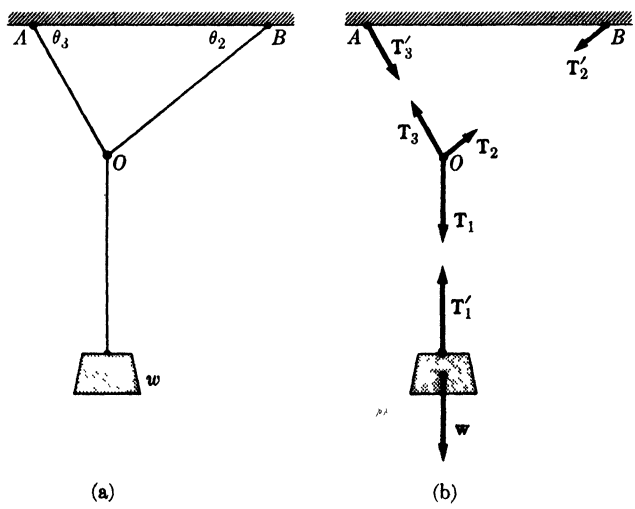


FIG. 2–6. (a) A block hanging in equilibrium. (b) Forces acting on the block, on the knot, and on the ceiling.

EXAMPLE 2. In Fig. 2–6(a), a block of weight w hangs from a cord which is knotted at O to two other cords fastened to the ceiling. We wish to find the tensions in these three cords. The weights of the cords are negligible.

In order to use the conditions of equilibrium to compute an unknown force, we must consider some body which is in equilibrium and on which the desired force acts. The hanging block is one such body and, as shown in the preceding example, the tension in the vertical cord supporting the block is equal to the weight of the block. The inclined cords do not exert forces on the block, but they do act on the knot at O. Hence we consider the *knot* as a small body in equilibrium whose own weight is negligible.

The free-body diagrams for the block and the knot are shown in Fig. 2–6(b), where $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$ represent the forces exerted *on the knot* by the three cords and $\mathbf{T}'_1$, $\mathbf{T}'_2$, and $\mathbf{T}'_3$ are the reactions to these forces.

Consider first the hanging block. Since it is in equilibrium,

$$T'_1 = w. \qquad \text{(1st law)}$$

Since $\mathbf{T}_1$ and $\mathbf{T}'_1$ form an action-reaction pair,

$$T'_1 = T_1. \qquad \text{(3rd law)}$$

Hence

$$T_1 = w.$$

To find the forces $\mathbf{T}_2$ and $\mathbf{T}_3$, we resolve these forces (see Fig. 2–7) into rectangular components. Then, from Newton's *first* law,

$$\sum F_x = T_2 \cos \theta_2 - T_3 \cos \theta_3 = 0,$$

$$\sum F_y = T_2 \sin \theta_2 + T_3 \sin \theta_3 - T_1 = 0.$$

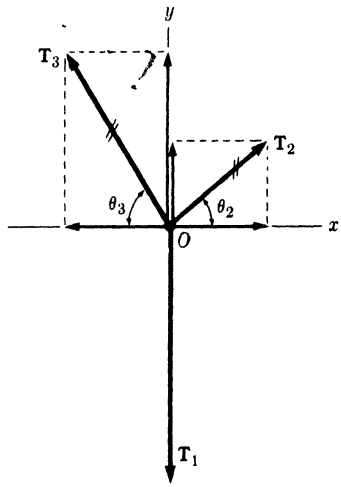


FIG. 2–7. Forces on the knot O in Fig. 2–6, resolved into x- and y-components.

As a numerical example, let $w = 50$ lb, $\theta_2 = 30°$, $\theta_3 = 60°$. Then $T_1 = 50$ lb, and from the two preceding equations,

$$T_2 = 25 \text{ lb},$$

$$T_3 = 43.3 \text{ lb}.$$

Finally, we know from Newton's *third* law that the inclined cords exert on the ceiling the forces $\mathbf{T}'_2$ and $\mathbf{T}'_3$, equal and opposite to $\mathbf{T}_2$ and $\mathbf{T}_3$ respectively.

EXAMPLE 3. Figure 2–8(a) shows a strut AB, pivoted at end A, attached to a wall by a cable, and carrying a load w at end B. The weights of the strut and of the cable are negligible. Suppose the weight w, and the angles θ_1 and θ_2 are known.

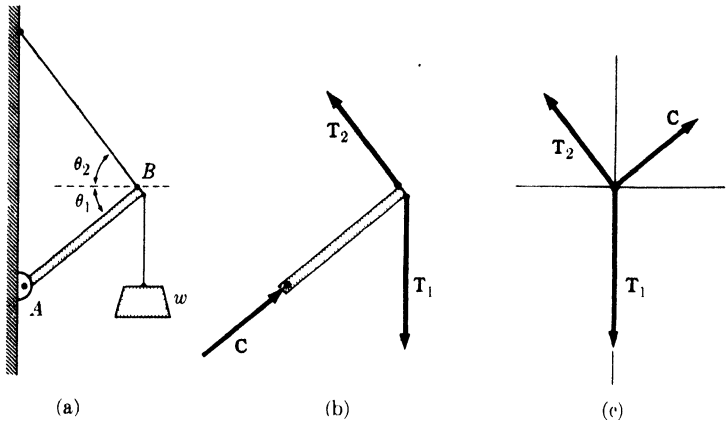


FIGURE 2–8

Figure 2–8(b) shows the forces acting on the strut. $\mathbf{T}_1$ is the force exerted by the vertical cable, $\mathbf{T}_2$ is the force exerted by the inclined cable, and C is the force exerted by the pivot. Force $\mathbf{T}_1$ is known both in magnitude and in direction; force $\mathbf{T}_2$ is known in direction only, and neither the magnitude nor the direction of C is known. However, the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ must intersect at the outer end of the strut, and since the strut is in equilibrium under three forces, the line of action of force C must also pass through the outer end of the strut. In other words, the direction of force C is along the line of the strut.

Hence the resultant of $\mathbf{T}_1$ and $\mathbf{T}_2$ is also along this line and the strut, in effect, is acted on by forces at its ends, directed toward each other along the line of the strut. The effect of these forces is to compress the strut, and it is said to be in *compression.*

If the forces acting on a strut are *not* all applied at its ends, the direction of the resultant force at the ends is *not* along the line of the strut. This is illustrated in the next example.

In Fig. 2–8(c), the force C has been transferred along its line of action to the point of intersection of the three forces. The force diagram is exactly like that of Fig. 2–7, and the problem is solved in the same way.

EXAMPLE 4. In Fig. 2–9 a ladder, which is in equilibrium, leans against a vertical frictionless wall. The forces on the ladder are (1) its weight **w**, (2) the force $\mathbf{F}_1$ exerted on the ladder by the vertical wall and which is perpendicular to the wall if there is no friction, and (3) the force $\mathbf{F}_2$ exerted by the ground on the base of the ladder. The force **w** is known in magnitude and in direction, the force $\mathbf{F}_1$ is known in direction only, and the force $\mathbf{F}_2$ is unknown in both magnitude and direction. As in the preceding example, the ladder is in equilibrium under three forces, which must be concurrent. Since the lines of action of $\mathbf{F}_1$ and **w** are known, their point of intersection (point *O*) can be located. The line of action of $\mathbf{F}_2$ must then pass through this point also. Note that neither the direction of $\mathbf{F}_1$ nor that of $\mathbf{F}_2$ lies along the line of the ladder. In part (b) the forces have been transferred

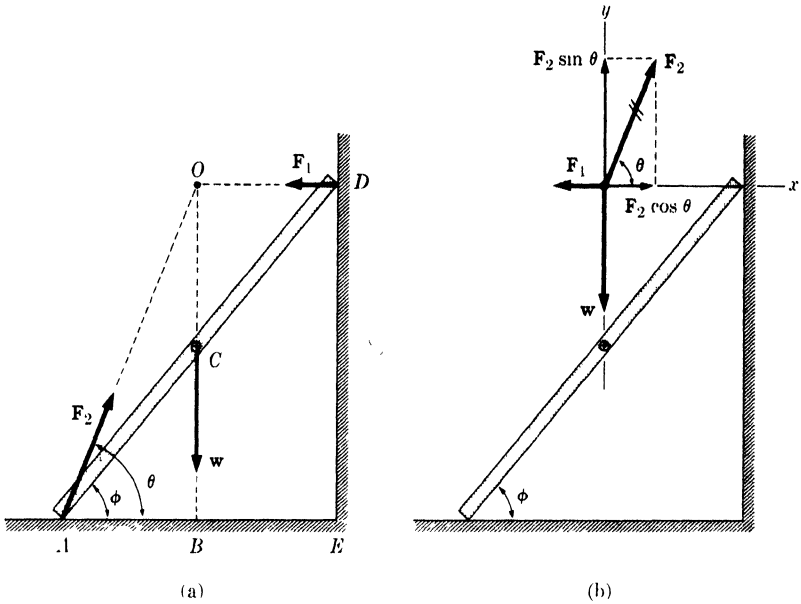


FIG. 2-9. Forces on a ladder leaning against a vertical frictionless wall.

to the point of intersection of their lines of action, and

$$\sum F_x = F_2 \cos\theta - F_1 = 0, \tag{2-1}$$

$$\sum F_y = F_2 \sin\theta - w = 0. \tag{2-2}$$

As a numerical example, suppose the ladder weighs 80 lb, is 20 ft long, has its center of gravity at its center, and makes an angle $\phi = 53°$ with the ground. We wish to find the angle θ, and the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. To calculate θ, we first find the lengths of AB and BO. From the right triangle ABC, we have

$$AB = AC \cos\phi = 10 \text{ ft} \times 0.60 = 6.0 \text{ ft},$$

and from the right triangle AED,

$$DE = AD \sin\phi = 20 \text{ ft} \times 0.80 = 16 \text{ ft}.$$

Then, from the right triangle AOB, since $OB = DE$, we have

$$\tan\theta = \frac{OB}{AB} = \frac{16 \text{ ft}}{6.0 \text{ ft}} = 2.67,$$

$$\theta = 69.5°, \quad \sin\theta = 0.937, \quad \cos\theta = 0.350.$$

$$F_2 = \frac{w}{\sin\theta} = \frac{80 \text{ lb}}{0.937} = 85.5 \text{ lb},$$

and from Eq. (2–1),

$$F_1 = F_2 \cos\theta = 85.5 \text{ lb} \times 0.350 = 30 \text{ lb}.$$

The ladder presses against the wall and the ground with forces which are equal and opposite to $\mathbf{F}_1$ and $\mathbf{F}_2$ respectively.

In the next chapter we shall explain another method of solving this problem, using the concept of the moment of a force.

EXAMPLE 5. In Fig. 2–10, block A of weight w_1 rests on a frictionless inclined plane of slope angle θ. The center of gravity of the block is at its center. A flexible cord is attached to the center of the right face of the block, passes over a frictionless pulley, and is attached to a second block B of weight w_2. The weight of the cord and friction in the pulley are negligible. If w_1 and θ are given, find the weight w_2 for which the system is in equilibrium, that is, for which it remains at rest or moves in either direction at constant speed.

The free-body diagrams for the two blocks are shown at the left and right. The forces on block B are its weight $\mathbf{w}_2$ and the force $\mathbf{T}$ exerted on it by the cord. Since it is in equilibrium,

$$T = w_2. \qquad \text{(1st law)} \tag{2–3}$$

Block A is acted on by its weight $\mathbf{w}_1$, the force $\mathbf{T}$ exerted on it by the cord, and the force $\mathbf{N}$ exerted on it by the plane. We can use the same symbol $\mathbf{T}$ for the force exerted on each block by the cord, because, as explained in Section 2–4, these forces are equivalent to an action-reaction pair and have the same magnitude. The force $\mathbf{N}$, if there is no friction, is perpendicular or *normal* to the surface of the plane. Since the lines of action of $\mathbf{w}_1$ and $\mathbf{T}$ intersect at the center of gravity of the block, the line of action of $\mathbf{N}$ passes through this point also. It is simplest to choose x- and y-axes parallel and perpendicular to the surface of the plane, because then only the weight $\mathbf{w}_1$ needs to be resolved into components. The conditions of equilibrium give

$$\left.\begin{aligned} \textstyle\sum F_x &= T - w_1 \sin\theta = 0, \\ \textstyle\sum F_y &= N - w_1 \cos\theta = 0. \end{aligned}\right\} \quad \text{(1st law)} \qquad \begin{matrix} \text{(2–4)} \\ \text{(2–5)} \end{matrix}$$

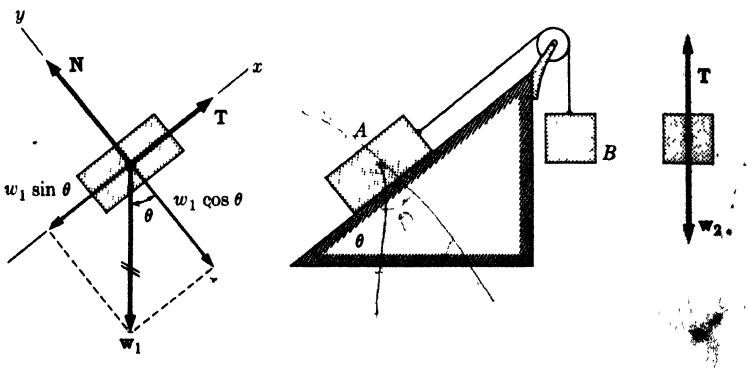


FIG. 2–10. Forces on a block in equilibrium on a frictionless inclined plane.

Thus if $w_1 = 100$ lb and $\theta = 30°$, we have from Eqs. (2–3) and (2–4),

$$w_2 = T = w_1 \sin\theta = 100 \text{ lb} \times 0.500 = 50 \text{ lb},$$

and from Eq. (2–5),

$$N = w_1 \cos\theta = 100 \text{ lb} \times 0.866 = 86.6 \text{ lb}.$$

Note carefully that *in the absence of friction* the same weight w_2 of 50 lb is required whether the system remains at rest or moves with constant speed in *either* direction. This is not the case when friction is present.

2–6 Friction. Whenever the surface of one body slides over that of another, each body exerts a frictional force on the other, parallel to the surfaces. The force *on* each body is opposite to the direction of its motion relative to the other. Thus when a block slides from left to right along the surface of a table, a frictional force to the left acts on the block and an equal force toward the right acts on the table. Frictional forces may also act when there is no relative motion. A horizontal force on a heavy packing case resting on the floor may not be enough to set the case in motion, the applied force being balanced by an equal frictional force exerted on the case by the floor.

The origin of these frictional forces is not fully understood, and the study of them is an important field of research. When one unlubricated metal slides over another, there appears to be an actual momentary welding of the metals together at the "high spots" where they make contact. The observed friction force is the force required to break these tiny welds. The mechanism of the friction force between two blocks of wood, or between two bricks, must be quite different.

Friction forces also act on a body moving through a fluid (liquid or gas). The fluid is said to exhibit *viscosity*. The motion of one body rolling on another is opposed by a force called *rolling friction* and which results from the deformation of the two bodies where they make contact.

In this chapter we shall consider only so-called "dry friction," the force observed when the surface of one unlubricated solid slides over that of another.

In Fig. 2–11(a) a block is at rest on a horizontal surface, in equilibrium under the action of its weight **w** and the upward force **P** exerted on it by the surface.

Suppose now that a cord is attached to the block as in Fig. 2–11(b) and the tension **T** in the cord is gradually increased. Provided the tension is not too great, the block remains at rest. The force **P** exerted on the block by the surface is inclined toward the left as shown, since the three forces **P**, **w**, and **T** must be concurrent. The component of **P** parallel to the surface is called the *force of static friction*, $\mathbf{f}_s$. The other component,

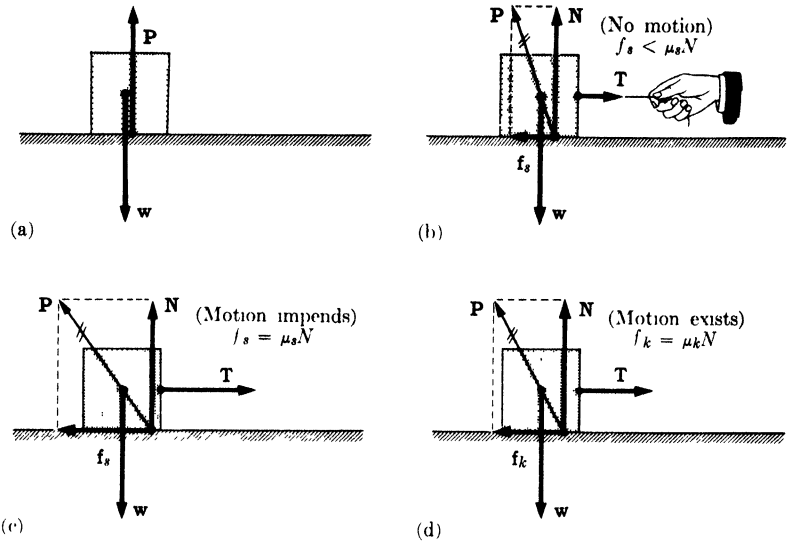


FIG. 2–11. The friction force f is less than or equal to $\mu_s N$ when there is no relative motion, and is equal to $\mu_k N$ when motion exists.

N, is the *normal* force exerted on the block by the surface. From the conditions of equilibrium, the force of static friction $\mathbf{f}_s$ equals the force **T**, and the normal force **N** equals the weight **w**.

As the force **T** is increased further, a limiting value is reached at which the block breaks away from the surface and starts to move. In other words, there is a certain maximum value which the force of static friction $\mathbf{f}_s$ can have. Figure 2–11(c) is the force diagram when **T** is just below its limiting value and motion impends. If the force **T** exceeds this limiting value the block is no longer in equilibrium.

For a given pair of surfaces, the maximum value of f_s is nearly proportional to the normal force N. The actual force of static friction can therefore have any value between zero (when there is no applied force parallel to the surface) and a maximum value proportional to N or equal to $\mu_s N$. The factor μ_s is called the *coefficient of static friction*. Thus,

$$f_s \leqq \mu_s N. \tag{2–6}$$

The equality sign holds only when the applied force **T**, parallel to the surface, has such a magnitude that motion is about to start [Fig. 2–11(c)]. When T is less than this, [Fig. 2–11(b)], the inequality sign holds and the

magnitude of the friction force must be computed from the conditions of equilibrium.

As soon as sliding begins, it is found that the friction force decreases. This new friction force, for a given pair of surfaces, is also nearly proportional to the normal force. The proportionality factor, μ_k, is called the *coefficient of sliding friction* or *kinetic* friction. Thus, when the block is in motion, the force of sliding or kinetic friction is given by

$$f_k = \mu_k N. \tag{2–7}$$

This is illustrated in Fig. 2–11(d).

The coefficients of static and sliding friction depend primarily on the nature of both of the surfaces in contact, being relatively large if the surfaces are rough and small if they are smooth. Typical numerical values are given in Table 2–1. The coefficient of sliding friction varies somewhat

TABLE 2–1

COEFFICIENTS OF FRICTION

Materials	Static, μ_s	Kinetic, μ_k
Steel on steel	0.74	0.57
Aluminum on steel	0.61	0.47
Copper on steel	0.53	0.36
Brass on steel	0.51	0.44
Zinc on cast iron	0.85	0.21
Copper on cast iron	1.05	0.29
Glass on glass	0.94	0.4
Copper on glass	0.68	0.53
Teflon on teflon	0.04	0.04
Teflon on steel	0.04	0.04

with the relative velocity, but for simplicity we shall assume it to be independent of velocity. It is also nearly independent of the contact area. However, since two real surfaces actually touch each other only at a relatively small number of high spots, the true contact area is very different from the over-all area. Equations (2–6) and (2–7) are useful empirical relations, but do not represent fundamental physical laws like Newton's laws.

EXAMPLE 1. In Fig. 2–11, suppose that the block weighs 20 lb, that the tension T can be increased to 8 lb before the block starts to slide, and that a force of 4 lb will keep the block moving at constant speed once it has been set in motion. Find the coefficients of static and kinetic friction.

From Fig. 2–11(c) and the data above, we have

$$\left.\begin{aligned} \textstyle\sum F_y &= N - w = N - 20\text{ lb} = 0, \\ \textstyle\sum F_x &= T - f_s = 8\text{ lb} - f_s = 0, \end{aligned}\right\} \quad \text{(1st law)}$$

$$f_s = \mu_s N \text{ (motion impends).}$$

Hence

$$\mu_s = \frac{f_s}{N} = \frac{8\text{ lb}}{20\text{ lb}} = 0.40.$$

From Fig. 2–11(d), we have

$$\left.\begin{aligned} \textstyle\sum F_y &= N - w = N - 20\text{ lb} = 0, \\ \textstyle\sum F_x &= T - f_k = 4\text{ lb} - f_k = 0, \end{aligned}\right\} \quad \text{(1st law)}$$

$$f_k = \mu_k N \text{ (motion exists).}$$

Hence

$$\mu_k = \frac{f_k}{N} = \frac{4\text{ lb}}{20\text{ lb}} = 0.20.$$

EXAMPLE 2. What is the friction force if the block in Fig. 2–11(b) is at rest on the surface and a horizontal force of 5 lb is exerted on it?

We have

$$\textstyle\sum F_x = T - f_s = 5\text{ lb} - f_s = 0, \quad \text{(1st law)}$$

$$f_s = 5\text{ lb}.$$

Note that in this case

$$f_s < \mu_s N.$$

EXAMPLE 3. What force **T**, at an angle of 30° above the horizontal, is required to drag a 20-lb block to the right at constant speed as in Fig. 2–12, if the coefficient of kinetic friction between block and surface is 0.20?

The forces on the block are shown in the diagram. From the first condition of equilibrium,

$$\textstyle\sum F_x = T\cos 30° - 0.2N = 0,$$

$$\textstyle\sum F_y = T\sin 30° + N - 20\text{ lb} = 0.$$

Simultaneous solution gives

$$T = 4.15\text{ lb},$$

$$N = 17.9\text{ lb}.$$

Notice that in this example the normal force N is not equal to the weight of the block, but is less than the weight by the vertical component of the force T.

EXAMPLE 4. In Fig. 2–13, a block has been placed on an inclined plane and the slope angle θ of the plane has been adjusted until the block slides down the plane at constant speed, once it has been set in motion. Find the angle θ.

The forces on the block are its weight **w** and the normal and frictional components of the force exerted by the plane. Since motion exists, the friction force $f_k = \mu_k N$. Take axes perpendicular and parallel to the surface of the plane. Then

$$\left.\begin{aligned} \Sigma F_x &= \mu_k N - w \sin\theta = 0, \\ \Sigma F_y &= N - w \cos\theta = 0. \end{aligned}\right\} \quad \text{(1st law)}$$

Hence

$$\begin{aligned} \mu_k N &= w \sin\theta, \\ N &= w \cos\theta. \end{aligned}$$

Dividing the former by the latter, we get

$$\mu_k = \tan\theta.$$

It follows that a block, regardless of its weight, slides down an inclined plane with constant speed if the tangent of the slope angle of the plane equals the coefficient of kinetic friction. Measurement of this angle then provides a simple experimental method of determining the coefficient of kinetic friction.

It is left as a problem for the reader to show that if a block is placed on an inclined plane and the slope angle of the plane is slowly increased, the block breaks away from the plane and starts to slide when the tangent of the slope angle becomes equal to the coefficient of static friction, μ_s.

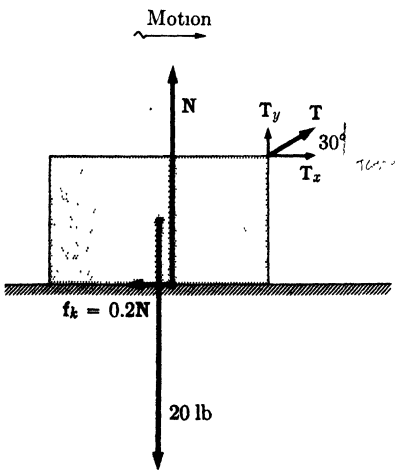


FIG. 2–12. Forces on a block being dragged to the right on a level surface at constant speed.

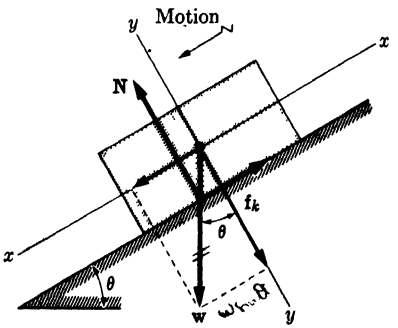


FIG. 2–13. Forces on a block sliding down an inclined plane (with friction) at constant speed.

PROBLEMS

2–1. A block rests on a horizontal surface. (a) What two forces act on it? (b) By what bodies are each of the forces exerted? (c) What are the reactions to these forces? (d) On what body is each reaction exerted, and by what body is each exerted?

2–2. A block is given a push along a table top, and slides off the edge of the table. (a) What force or forces are exerted on it while it is falling from the table to the floor? (b) What is the reaction to each force, that is, on what body and by what body is the reaction exerted? Neglect air resistance.

2–3. Two 10-lb weights are suspended at opposite ends of a rope which passes over a light frictionless pulley. The pulley is attached to a chain which goes to the ceiling. (a) What is the tension in the rope? (b) What is the tension in the chain?

2–4. In Fig. 2–6, let the weight of the hanging block be 50 lb. Find the tensions T_2 and T_3, (a) if $\theta_2 = \theta_3 = 60°$, (b) if $\theta_2 = \theta_3 = 10°$, (c) if $\theta_2 = 60°$, $\theta_3 = 0$, and (d) if $AB = 10$ ft, $AO = 6$ ft, $OB = 8$ ft.

2–5. Find the tension in each cord in Fig. 2–14 if the weight of the suspended body is 200 lb.

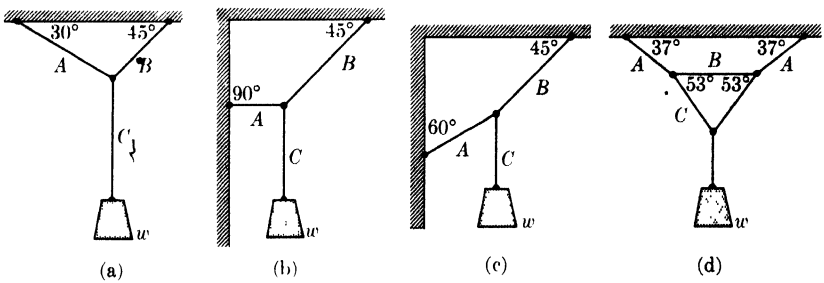


FIGURE 2–14

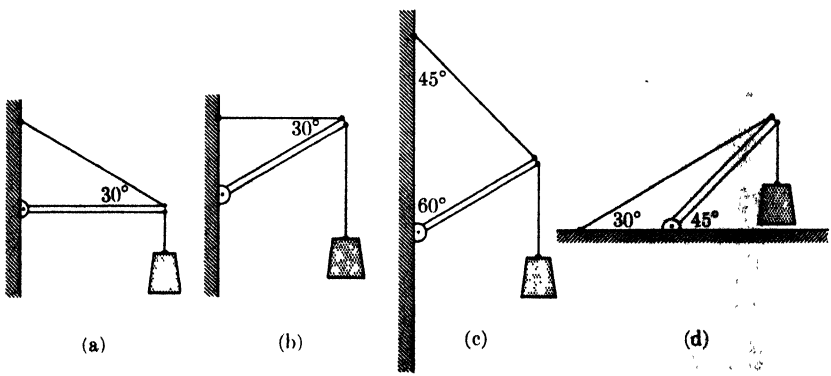


FIGURE 2–15

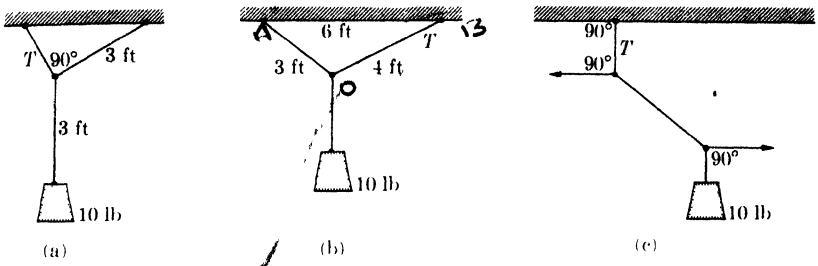


FIGURE 2–16

2–6. Find the tension T in the cable, and the magnitude and direction of the force C exerted on the strut by the pivot in the arrangements in Fig. 2–15. Let the weight of the suspended object in each case be 1000 lb. Neglect the weight of the strut.

2–7. (a) In which of the arrangements in Fig. 2–16 can the tension T be computed, if the only quantities known are those explicitly given? (b) For each case in which insufficient information is given, state one additional quantity, a knowledge of which would permit solution.

2–8. A horizontal boom 8 ft long is hinged to a vertical wall at one end, and a 500-lb body hangs from its outer end. The boom is supported by a guy wire from its outer end to a point on the wall directly above the boom. (a) If the tension in this wire is not to exceed 1000 lb, what is the minimum height above the boom at which it may be fastened to the wall? (b) By how many pounds would the tension be increased if the wire were fastened 1 ft below this point, the boom remaining horizontal? Neglect the weight of the boom.

2–9. One end of a rope 50 ft long is attached to an automobile. The other end is fastened to a tree. A man exerts a force of 100 lb at the mid-point of the rope, pulling it 2 ft to the side. What is the force exerted on the automobile?

2–10. Find the largest weight w which can be supported by the structure in Fig. 2–17 if the maximum tension the upper rope can withstand is 1000 lb and the maximum compression the strut can withstand is 2000 lb. The vertical rope is strong enough to carry any load required. Neglect the weight of the strut.

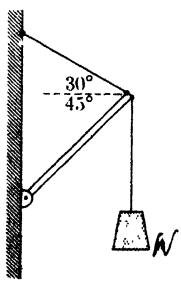


FIGURE 2–17

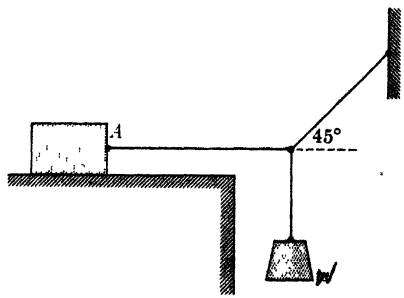


FIGURE 2–18

2–11. (a) Block A in Fig. 2–18 weighs 100 lb. The coefficient of static friction between the block and the surface on which it rests is 0.30. The weight w is 20 lb and the system is in equilibrium. Find the friction force exerted on block A. (b) Find the maximum weight w for which the system will remain in equilibrium.

2–12. A block hangs from a cord 10 ft long. A second cord is tied to the midpoint of the first, and a horizontal pull equal to half the weight of the block is exerted on it, the second cord being always kept horizontal. (a) How far will the block be pulled to one side? (b) How far will it be lifted?

2–13. A flexible chain of weight w hangs between two hooks at the same height, as shown in Fig. 2–19. At each end the chain makes an angle θ with the horizontal. (a) What is the magnitude and direction of the force **F** exerted by the chain on the hook at the left? (b) What is the tension **T** in the chain at its lowest point?



FIGURE 2–19

2–14. A 50-lb block is pulled at constant speed up a frictionless inclined plane by a weight of 10 lb hanging from a cord attached to the block and passing over a frictionless pulley at the top of the plane. (See Fig. 2–10.) Find (a) the slope angle of the plane, (b) the tension in the cord, and (c) the normal force exerted on the block by the plane.

2–15. A block rests upon a rough horizontal surface. A horizontal force **T** is applied to the block and is slowly increased from zero. Draw a graph with T along the x-axis and the friction force f along the y-axis, starting at $T = 0$ and showing the region of no motion, the point where motion impends, and the region where motion exists.

2–16. A block weighing 20 lb rests on a horizontal surface. The coefficient of static friction between block and surface is 0.40 and the coefficient of sliding friction is 0.20. (a) How large is the friction force exerted on the block? (b) How great will the friction force be if a horizontal force of 5 lb is exerted on the block? (c) What is the minimum force which will start the block in motion? (d) What is the minimum force which will keep the block in motion once it has been started? (e) If the horizontal force is 10 lb, how great is the friction force?

2–17. A block is pulled to the right at constant velocity by a 10-lb force acting 30° above the horizontal. The coefficient of sliding friction between block and surface is 0.5. What is the weight of the block? Assume all of the forces on the block to act at its center.

2–18. A block weighing 14 lb is placed on an inclined plane and connected to a 10-lb block by a cord passing over a small frictionless pulley, as in Fig. 2–10. The coefficient of sliding friction between the block and the plane is 1/7. For what two values of θ will the system move with constant velocity? Assume that all the forces on the 14-lb block act at its center. [*Hint:* $\cos\theta = \sqrt{1 - \sin^2\theta}$.]

2–19. A block weighing 100 lb is placed on an inclined plane of slope angle 30° and is connected to a second hanging block of weight w by a cord passing over a small frictionless pulley, as in Fig. 2–10. The coefficient of static friction is 0.40 and the coefficient of sliding friction is 0.30. (a) Find the weight w for which the 100-lb block moves up the plane at constant speed.

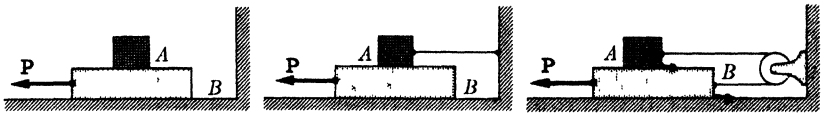


FIGURE 2–20

(b) Find the weight w for which it moves down the plane at constant speed. (c) For what range of values of w will the block remain at rest?

2–20. What force P at an angle ϕ above the horizontal is needed to drag a box of weight w at constant speed along a level floor if the coefficient of sliding friction between box and floor is μ?

2–21. A safe weighing 600 lb is to be lowered at constant speed down skids 8 ft long, from a truck 4 ft high. If the coefficient of sliding friction between safe and skids is 0.30, (a) will the safe need to be pulled down or held back? (b) How great a force parallel to the plane is needed?

2–22. If a force of 86 lb parallel to the surface of a 20° inclined plane will push a 120-lb block up the plane at constant speed, (a) what force parallel to the plane will push it down at constant speed? (b) What is the coefficient of sliding friction?

2–23. Block A in Fig. 2–20 weighs 4 lb and block B weighs 8 lb. The coefficient of sliding friction between all surfaces is 0.25. Find the force **P** necessary to drag block B to the left at constant speed (a) if A rests on B and moves with it, (b) if A is held at rest, and (c) if A and B are connected by a light flexible cord passing around a fixed frictionless pulley.

2–24. Block A, of weight w, slides down an inclined plane S of slope angle 37° at constant velocity while the plank B, also of weight w, rests on top of A. The plank is attached by a cord to the top of the plane (Fig. 2–21). (a) Draw a diagram of all the forces acting on block A. (b) If the coefficient of kinetic friction is the same between the surfaces A and B and between S and A, determine its value.

2–25. Two blocks, A and B, are placed as in Fig. 2–22 and connected by ropes to block C. Both A and B weigh 20 lb and the coefficient of sliding friction between each block and the surface is 0.5. Block C descends with constant velocity. (a) Draw two separate force diagrams showing the forces acting on A and B. (b) Find the tension in the rope connecting blocks A and B. (c) What is the weight of block C?

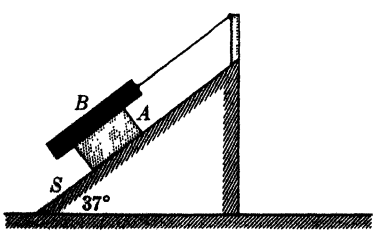


FIGURE 2–21

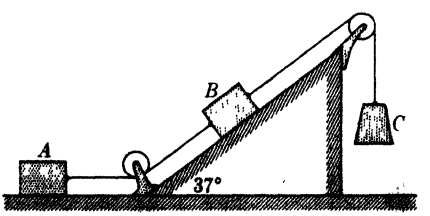


FIGURE 2–22

CHAPTER 3

EQUILIBRIUM. MOMENT OF A FORCE

3–1 Moment of a force. The effect produced on a body by a force o given magnitude and direction depends on the position of the *line o action* of the force. Thus in Fig. 3–1 the force $\mathbf{F}_1$ would produce a counter clockwise rotation (together with a translation toward the right), whil $\mathbf{F}_2$ would produce a clockwise rotation.

The line of action of a force can be specified by giving the perpendicular distance from some reference point to the line of action. In many instances, we shall be studying the motion of a body which is free to rotate about some axis, and which is acted on by a number of coplanar forces all lying in a plane perpendicular to the axis. It is then most convenient to select as the reference point the point at which the axis intersects the plane of the forces. The perpendicular distance from this point to the line of action of a force is called the *force arm* or the *moment arm* of the force about the axis. The product of the magnitude of a force and its force arm is called the *moment* of the force about the axis, or the *torque*.

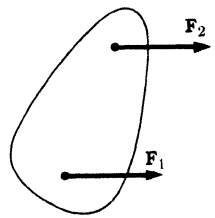


FIGURE 3–1

Thus Fig. 3–2 is a top view of a flat object, pivoted about an axis perpendicular to the plane of the diagram and passing through point O. The body is acted on by the forces $\mathbf{F}_1$ and $\mathbf{F}_2$, lying in the plane of the diagram. The moment arm of $\mathbf{F}_1$ is the perpendicular distance OA, of length l_1, and the moment arm of $\mathbf{F}_2$ is the perpendicular distance OB of length l_2.

The effect of the force $\mathbf{F}_1$ is to produce counterclockwise rotation about the axis, while that of $\mathbf{F}_2$ is to produce clockwise rotation. To distinguish between these directions of rotation, we shall adopt the convention that counterclockwise moments are positive, and that clockwise moments are negative. Hence the moment Γ_1 (Greek gamma) of the force $\mathbf{F}_1$ about the axis through O is

$$\Gamma_1 = +F_1 l_1,$$

and the moment Γ_2 of $\mathbf{F}_2$ is

$$\Gamma_2 = -F_2 l_2.$$

If forces are expressed in pounds and lengths in feet, torques are expressed in pound-feet.

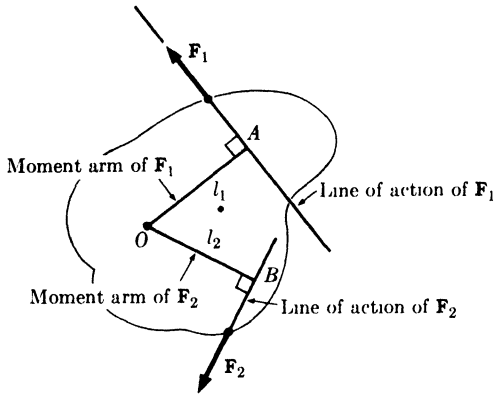


FIG. 3-2. The moment of a force about an axis is the product of the force and its moment arm.

3-2 The second condition of equilibrium. We saw in Section 2-2 that when a body is acted on by any number of coplanar forces, the forces can always be reduced to two, as in Fig. 2-2. If the body is in equilibrium, these forces must (a) be equal in magnitude and opposite in direction, and (b) must have the same line of action.

Requirement (a) is satisfied by the *first condition of equilibrium*,

$$\Sigma F_x = 0, \qquad \Sigma F_y = 0.$$

Requirement (b), which is the second condition of equilibrium, can be simply expressed in terms of the moments of the forces. Figure 3-3 again shows a flat object acted on by two forces $\mathbf{F}_1$ and $\mathbf{F}_2$. If the object is in equilibrium, the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal and both forces have the same line of action. Hence they have the same moment arm OA, of length l, about an axis perpendicular to the plane of the body and passing through any arbitrary point O. Their moments about the axis are therefore equal in magnitude and opposite in sign, and the algebraic sum of their moments is zero. The necessary and sufficient condition, therefore, that two equal and opposite forces have the same line of action is that the algebraic sum of

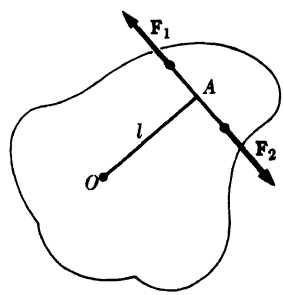


FIG. 3-3. When two forces are in equilibrium, their resultant moment about any axis is zero.

their moments, about any axis, shall be zero. The *second condition of equilibrium* may therefore be expressed analytically as

$$\sum \Gamma = 0 \text{ (about any arbitrary axis).}$$

It is not necessary to first reduce a set of coplanar forces to two forces in order to calculate the sum of their moments. One need only calculate the moment of each force separately, and then add these moments algebraically. Thus if a body is in equilibrium under the action of any number of coplanar forces, the algebraic sum of the torques about any arbitrary axis is zero.

EXAMPLE 1. A rigid rod whose own weight is negligible (Fig. 3–4) is pivoted at point O and carries a body of weight w_1 at end A. Find the weight w_2 of a second body which must be attached at end B if the rod is to be in equilibrium, and find the force exerted on the rod by the pivot at O.

Figure 3–4(b) is the free-body diagram of the rod. The forces T_1 and T_2 are equal respectively to w_1 and w_2. The conditions of equilibrium, taking moments about an axis through O, perpendicular to the diagram, give

$$\sum F_y = P - T_1 - T_2 = 0, \qquad \text{(1st condition)}$$

$$\Gamma_O = T_1 l_1 - T_2 l_2 = 0. \qquad \text{(2nd condition)}$$

Let $l_1 = 3$ ft, $l_2 = 4$ ft, $w_1 = 4$ lb. Then, from the equations above,

$$P = 7 \text{ lb}, \qquad T_2 = w_2 = 3 \text{ lb}.$$

To illustrate that the resultant moment about *any* axis is zero, let us compute moments about an axis through point A.

$$\sum \Gamma_A = P l_1 - T_2(l_1 + l_2) = 7 \text{ lb} \times 3 \text{ ft} - 3 \text{ lb} \times 7 \text{ ft} = 0.$$

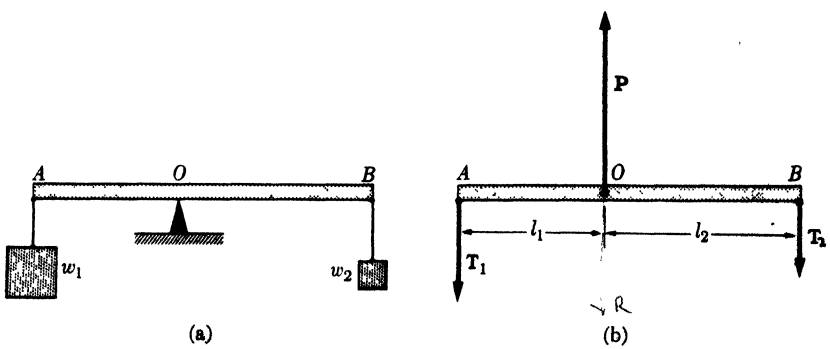


FIG. 3–4. A rod in equilibrium under three parallel forces.

The point about which moments are computed need not lie on the rod. To verify this, let the reader calculate the resultant moment about a point 1 ft to the left of A and 1 ft above it.

EXAMPLE 2. Figure 3–5 illustrates a problem that has already been solved in Example 5 at the end of Section 2–5. A ladder 20 ft long, of weight $w = 80$ lb, with its center of gravity at its center, in equilibrium, leans against a vertical frictionless wall and makes an angle of 53° with the horizontal. We wish to find the magnitudes and directions of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$.

If the wall is frictionless, $\mathbf{F}_1$ is horizontal. The direction of $\mathbf{F}_2$ is unknown (except in special cases, its direction does *not* lie along the ladder). Instead of considering its magnitude and direction as unknowns, it is simpler to resolve the force $\mathbf{F}_2$ into x- and y-components and solve for these. The magnitude and direction of $\mathbf{F}_2$ may then be computed. The first condition of equilibrium therefore provides the equations

$$\left.\begin{aligned} \textstyle\sum F_x &= F_2 \cos\theta - F_1 = 0, \\ \textstyle\sum F_y &= F_2 \sin\theta - 80 \text{ lb} = 0. \end{aligned}\right\} \quad \text{(1st condition)}$$

In writing the second condition, moments may be computed about an axis through any point. The resulting equation is simplest if one selects a point through which two or more forces pass, since these forces then do not appear in

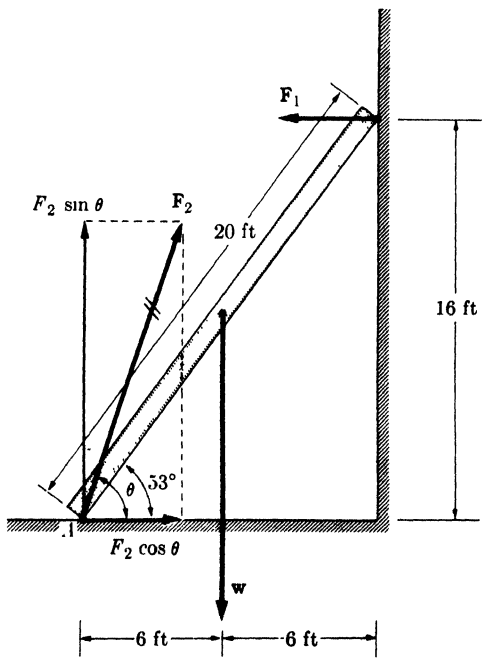


FIG. 3–5. Forces on a ladder in equilibrium, leaning against a frictionless wall.

the equation. Let us therefore take moments about an axis through point A.

$$\sum \Gamma_A = F_1 \times 16 \text{ ft} - 80 \text{ lb} \times 6 \text{ ft} = 0. \qquad \text{(2nd condition)}$$

From the second equation, $F_2 \sin \theta = 80$ lb, and from the third,

$$F_1 = \frac{480 \text{ lb·ft}}{16 \text{ ft}} = 30 \text{ lb}.$$

Then from the first equation,

$$F_2 \cos \theta = 30 \text{ lb}.$$

Hence

$$F_2 = \sqrt{(80 \text{ lb})^2 + (30 \text{ lb})^2} = 85.5 \text{ lb},$$

$$\theta = \tan^{-1} \frac{80 \text{ lb}}{30 \text{ lb}} = 69.5°.$$

This method is simpler than that used in Section 2–5, since it is unnecessary to locate the point of intersection of the concurrent forces, $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{w}$.

EXAMPLE 3. Figure 3–6 illustrates a problem that has already been solved, in part, in Example 3 at the end of Section 2–6. (a) What force $\mathbf{T}$, at an angle of 30° above the horizontal, is required to drag a block of weight $w = 20$ lb to the right at constant speed along a level surface if the coefficient of sliding friction between block and surface is 0.20? (b) Determine the line of action of the normal force $\mathbf{N}$ exerted on the block by the surface. The block is 1 ft high, 2 ft long, and its center of gravity is at its center.

Since motion exists,

$$f_k = \mu_k N.$$

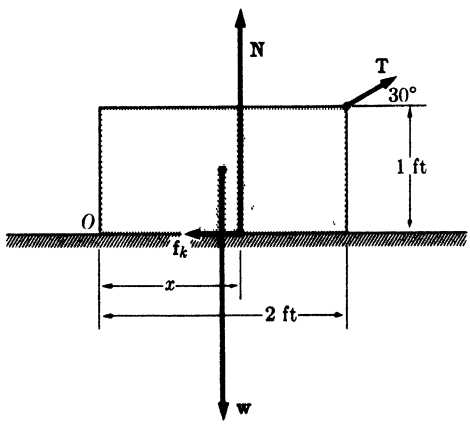


FIGURE 3–6

From the first condition of equilibrium,

$$\left.\begin{aligned} \Sigma F_x &= T\cos 30^\circ - f_k = 0, \\ \Sigma F_y &= T\sin 30^\circ + N - 20\text{ lb} = 0. \end{aligned}\right\} \quad \text{(1st condition)}$$

Let x represent the distance from point O to the line of action of **N**, and take moments about an axis through O. Then, from the second condition of equilibrium,

$$\Sigma\Gamma_O = T\sin 30^\circ \times 2\text{ ft} - T\cos 30^\circ \times 1\text{ ft} + N \times x - 20\text{ lb} \times 1\text{ ft} = 0. \quad \text{(2nd condition)}$$

From the first two equations, we get

$$T = 4.15\text{ lb}, \qquad N = 17.9\text{ lb},$$

and from the third equation,

$$x = 1.08\text{ ft}.$$

The line of action of **N** therefore lies 0.08 ft to the right of the center of gravity.

3-3 Resultant of parallel forces. The direction of the resultant of a set of parallel forces is the same as that of the forces, and its magnitude equals the sum of their magnitudes. The line of action of the resultant can be found from the requirement that the moment of the resultant, about any axis, shall equal the sum of the moments of the given forces.

Consider the parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in Fig. 3-7. Point O is any arbitrary point and the x-axis has been taken at right angles to the directions of the forces. The forces have no x-components, so the magnitude of the resultant is

$$R = \Sigma F_y = F_1 + F_2.$$

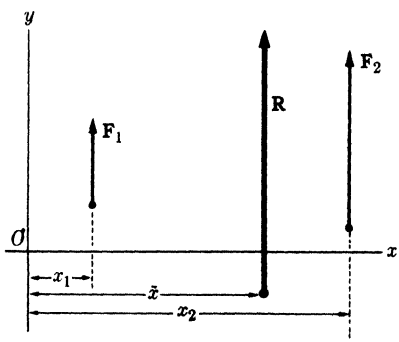


FIG. 3-7. Vector **R** represents the resultant of the parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$, in magnitude, direction, and line of action.

If x_1 and x_2 are the perpendicular distances from O to the lines of action of the forces, their resultant moment about an axis through O is

$$\Sigma\Gamma_O = x_1F_1 + x_2F_2.$$

Let $\bar{x}$ represent the distance from O to the line of action of the resultant. The moment of the resultant about O is then

$$R\bar{x} = (F_1 + F_2)\bar{x},$$

and since this equals the resultant moment, we have

$$(F_1 + F_2)\bar{x} = F_1x_1 + F_2x_2.$$

Therefore

$$\bar{x} = \frac{F_1x_1 + F_2x_2}{F_1 + F_2},$$

and the magnitude, direction, and line of action of the resultant are determined.

The resultant of any number of parallel forces is found in the same way. The magnitude of the resultant is

$$R = \Sigma F,$$

and, if the forces are parallel to the y-axis, the x-coordinate of its line of action is

$$\bar{x} = \frac{\Sigma Fx}{\Sigma F} = \frac{\Sigma Fx}{R}.$$

EXAMPLE. When a body is in equilibrium under the action of three forces, the resultant of any two is equal and opposite to the third and has the same line of action. Show that these conditions are satisfied by the three parallel forces in Fig. 3–4(b).

It was shown in Example 1 at the end of Section 3–2 that if $l_1 = 3$ ft, $l_2 = 4$ ft, and $T_1 = 4$ lb, then $T_2 = 3$ lb and $P = 7$ lb.

Let us first find the resultant of $\mathbf{T}_1$ and $\mathbf{T}_2$. Take the x-axis along the rod with origin at point A. The magnitude of the resultant is

$$R = \Sigma F = -4 \text{ lb} - 3 \text{ lb} = -7 \text{ lb}.$$

The coordinate of its line of action is

$$\bar{x} = \frac{\Sigma Fx}{\Sigma F} = \frac{4 \text{ lb} \times 0 - 3 \text{ lb} \times 7 \text{ ft}}{-7 \text{ lb}} = 3 \text{ ft}.$$

Hence the resultant of $\mathbf{T}_1$ and $\mathbf{T}_2$ is equal and opposite to $\mathbf{P}$ and has the same line of action.

The resultant of **P** and $\mathbf{T}_2$ has a magnitude

$$R = \sum F = 7 \text{ lb} - 3 \text{ lb} = 4 \text{ lb}.$$

The coordinate of its line of action is

$$\bar{x} = \frac{\sum Fx}{\sum F} = \frac{7 \text{ lb} \times 3 \text{ ft} - 3 \text{ lb} \times 7 \text{ ft}}{4 \text{ lb}} = 0,$$

so the resultant of **P** and $\mathbf{T}_2$ is equal and opposite to $\mathbf{T}_1$ and has the same line of action.

3-4 Center of gravity. Every particle of matter in a body is attracted by the earth, and the single force which we call the *weight* of the body is the resultant of all these forces of attraction. The direction of the force on each particle is toward the center of the earth, but the distance to the earth's center is so great that for all practical purposes the forces can be considered parallel to one another. Hence the weight of a body is the resultant of a large number of parallel forces.

Figure 3-8(a) shows a flat object of arbitrary shape in the xy-plane, the y-axis being vertical. Let the body be subdivided into a large number of small particles of weights $\mathbf{w}_1$, $\mathbf{w}_2$, etc., and let the coordinates of these particles be x_1 and y_1, x_2 and y_2, etc. The total weight W of the object is

$$W = w_1 + w_2 + \cdots = \sum w. \tag{3-1}$$

The x-coordinate of the line of action of **W** is

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \cdots}{w_1 + w_2 + \cdots} = \frac{\sum wx}{\sum w} = \frac{\sum wx}{W}. \tag{3-2}$$

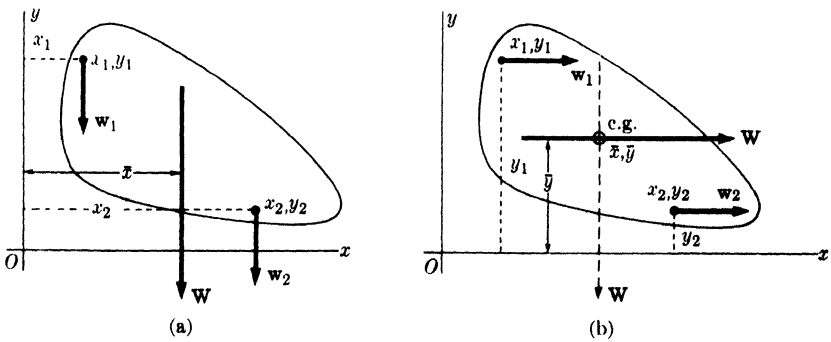


FIG. 3-8. The body's weight **W** is the resultant of a large number of parallel forces. The line of action of **W** always passes through the center of gravity.

Now let the object and the reference axes be rotated 90° clockwise or, which amounts to the same thing, let us consider the gravitational forces to be rotated 90° counterclockwise as in Fig. 3–8(b). The total weight W is unaltered and the y-coordinate of its line of action is

$$\bar{y} = \frac{w_1 y_1 + w_2 y_2 + \cdots}{w_1 + w_2 + \cdots} = \frac{\sum wy}{\sum w} = \frac{\sum wy}{W}. \tag{3–3}$$

The point of intersection of the lines of action of **W** in the two parts of Fig. 3–8 has the coordinates $\bar{x}$ and $\bar{y}$ and is called the *center of gravity* of the object. By considering some arbitrary orientation of the object, one can show that the line of action of **W** *always* passes through the center of gravity.

If the centers of gravity of each of a number of bodies have been determined, the coordinates of the center of gravity of the combination can be computed from Eqs. (3–1) and (3–2), letting w_1, w_2, etc., be the weights of the bodies and x_1 and y_1, and y_2, etc., be the coordinates of the center of gravity of each.

Symmetry considerations are often useful in finding the position of the center of gravity. Thus the center of gravity of a homogeneous sphere, cube, circular disk, or rectangular plate is at its center. That of a cylinder or right circular cone is on the axis of symmetry, and so on.

EXAMPLE. Locate the center of gravity of the machine part in Fig. 3–9, consisting of a disk 2 inches in diameter and one inch long, and a rod 1 inch in diameter and 6 inches long, constructed of a homogeneous material.

By symmetry, the center of gravity lies on the axis and the center of gravity of each part is midway between its ends. The volume of the disk is π in^3 and that of the rod is $3\pi/2$ in^3. Since the weights of the two parts are proportional to their volumes,

$$\frac{w\,(\text{disk})}{w\,(\text{rod})} = \frac{w_1}{w_2} = \frac{\pi}{3\pi/2} = \frac{2}{3}.$$

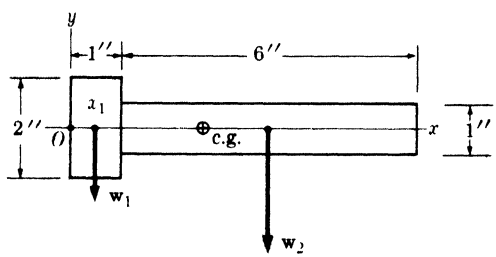


FIGURE 3–9

Take the origin O at the left face of the disk, on the axis. Then $x_1 = 0.5$ inch, $x_2 = 4.0$ inches, and

$$\bar{x} = \frac{w_1 \times 0.5 \text{ in.} + \frac{3}{2}w_1 \times 4.0 \text{ in.}}{w_1 + \frac{3}{2}w_1} = 2.6 \text{ in.}$$

The center of gravity is on the axis, 2.6 inches to the right of O.

The center of gravity of a flat object can be located experimentally as shown in Fig. 3–10. In part (a) the body is suspended from some arbitrary point A. When allowed to come to equilibrium, the center of gravity must lie on a vertical line through A. When the object is suspended from a second point B, as in part (b), the center of gravity lies on a vertical line through B and hence lies at the point of intersection of this line and the first. If the object is now suspended from a third point C, as in part (c), a vertical line through C will be found to pass through the point of intersection of the first two lines.

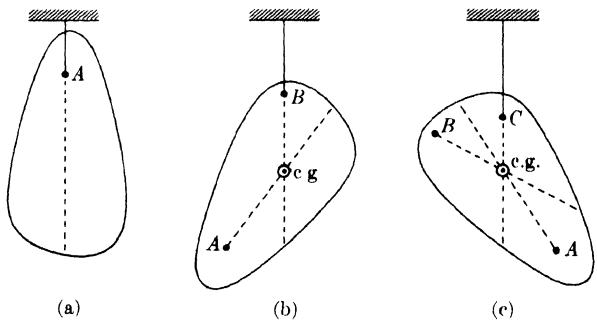


FIG. 3–10. Locating the center of gravity of a flat object.

The center of gravity of a body has another important property. A force **F** whose line of action lies at one side or the other of the center of gravity, as in Fig. 3–11(a), will change both the translational and rotational motion of the body on which it acts. However, if the line of action passes through the center of gravity, as in part (b), only the translational motion is affected and the body remains in rotational equilibrium. Thus when a body is tossed in the air with a whirling motion it continues to rotate at a constant rate, since the line of action of its weight passes through the center of gravity.

It may also be pointed out that when one object rests on or slides over another, the normal and frictional forces are sets of parallel forces distributed over the area in contact. The single vectors which have been used

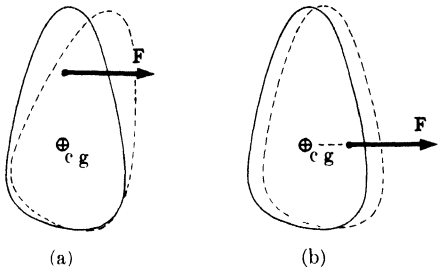


FIG. 3–11. A body is in rotational but not translational equilibrium when acted on by a force whose line of action passes through the center of gravity, as in (b).

to represent these forces are therefore actually the resultants of sets of parallel forces.

3–5 Couples. It often happens that the forces on a body reduce to two forces of equal magnitude and opposite direction, having lines of action which are parallel but do not coincide. Such a pair of forces is called a *couple*. A common example is afforded by the forces on a compass needle in the earth's magnetic field. The north and south poles of the needle are acted on by equal forces, one toward the north and the other toward the south as shown in Fig. 3–12. Except when the needle points in the N-S direction, the two forces do not have the same line of action.

Figure 3–13 shows a couple consisting of two forces, each of magnitude F, separated by a perpendicular distance l. The resultant R of the forces is

$$R = F - F = 0.$$

The fact that the resultant force is zero means that a couple has no effect in producing translation as a whole of the body on which it acts. The only effect of a couple is to produce rotation.

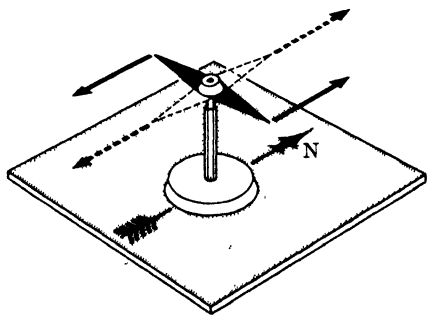


FIG. 3–12. Forces on the poles of a compass needle.

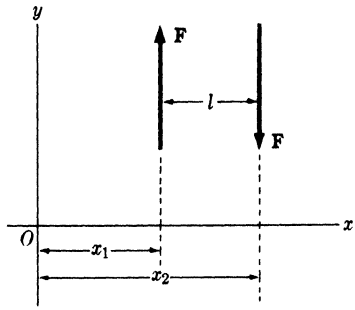


FIG. 3–13. Two equal and opposite forces having different lines of action are called a couple. The moment of the couple is the same about all points, and is equal to lF.

The resultant torque of the couple in Fig. 3–13, about an arbitrary point O, is

$$\begin{aligned} \Gamma_O &= x_1F - x_2F \\ &= x_1F - (x_1 + l)F \\ &= -lF. \end{aligned}$$

Since the distances x_1 and x_2 do not appear in the result, we conclude that the torque of the couple is the same about *all* points in the plane of the forces forming the couple and is equal to the product of the magnitude of either force and the perpendicular distance between their lines of action.

A body acted on by a couple can be kept in equilibrium only by another couple of the same moment and in the opposite direction. As an example, the ladder in Fig. 3–5 can be considered as acted on by two couples, one formed by the forces $F_2 \sin \theta$ and w, the other by the forces $F_2 \cos \theta$ and F_1. The moment of the first is

$$\Gamma_1 = 6 \text{ ft} \times 80 \text{ lb} = 480 \text{ lb·ft}.$$

The moment of the second is

$$\Gamma_2 = 16 \text{ ft} \times 30 \text{ lb} = 480 \text{ lb·ft}.$$

The first moment is clockwise and the second is counterclockwise.

PROBLEMS

3–1. Give an example to show that the following statement is false: Any two forces acting on a body can be combined into a single resultant force that would have the same effect.

3–2. The center of gravity of a log 10 ft long and weighing 100 lb is 4 ft from one end of the log. It is to be carried by two men, one at the heavy end. Where should the other man hold the log if each is to carry half the load?

3–3. A rod AB which is 3 ft long and whose own weight can be neglected rests on a knife-edge 1 ft from end A. A 10-lb weight hangs from end B. (a) What weight must be hung from A to maintain equilibrium? (b) What is then the force exerted by the knife-edge?

3–4. The rod in Fig. 3–14 is pivoted about an axis through O. (a) Find the moment of each force, and the resultant torque, about the axis. (b) Find the magnitude and direction of the force which must be exerted at the right end, perpendicular to the rod, to maintain equilibrium.

3–5. A single force is to be applied to the bar in Fig. 3–15 to maintain it in equilibrium in the position shown. The weight of the bar can be neglected. (a) What are the x- and y-components of the required force? (b) What is the tangent of the angle which the force must make with the bar? (c) What is the magnitude of the required force? (d) Where should the force be applied?

3–6. The strut in Fig. 3–16 weighs 40 lb and its center of gravity is at its center. Find (a) the tension in the cable, and (b) the horizontal and vertical components of the force exerted on the strut at the wall.

3–7. The boom in Fig. 3–17 is uniform and weighs 500 lb. Find the tension in the guy wire, and the horizontal and vertical components of the force exerted on the boom at its lower end.

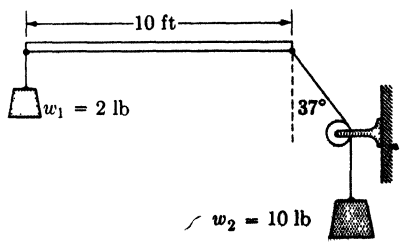


FIGURE 3–15

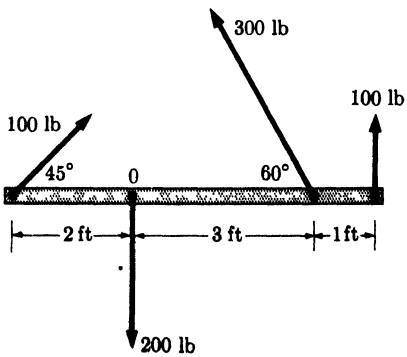


FIGURE 3–14

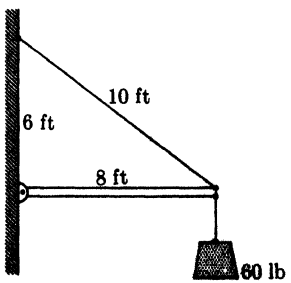


FIGURE 3–16

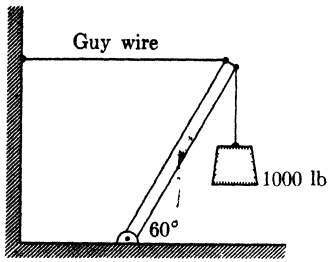


FIGURE 3–17

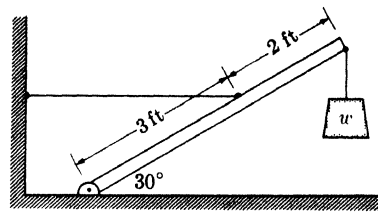


FIGURE 3–19

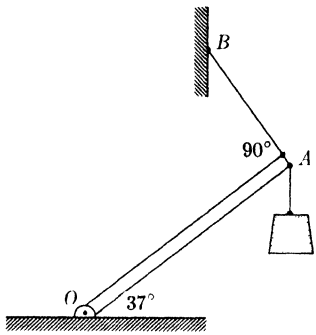


FIGURE 3–18

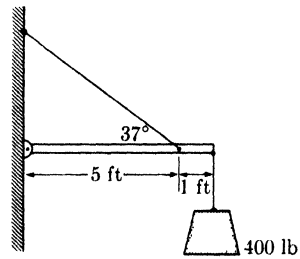


FIGURE 3–20

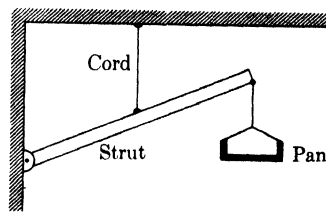


FIGURE 3–21

3–8. The uniform boom OA in Fig. 3–18 is 20 ft long and supports a weight of 3600 lb at end A. (a) If the weight of the boom is negligible, find the tension in the cable AB and the magnitude and direction of the force exerted on the pin at O. (b) If the boom weighs 2800 lb, find the tension in the cable and the magnitude and direction of the force on the pin at O.

3–9. If the weight of the strut in Fig. 3–19 is neglected, (a) what weight w is necessary to produce a tension of 200 lb in the horizontal cable? (b) What will then be the magnitude and direction of the force exerted on the strut at its lower end?

3–10. Find the tension in the cable and the force exerted on the strut by the wall in Fig. 3–20. Neglect the weight of the strut.

3–11. In Fig. 3–21 the vertical cord is attached to the midpoint of the strut, which is weightless. The cord will break if the tension in it exceeds 500 lb. Weights of 100 lb each are placed one by one in the pan. What is the maximum number of such weights that can be safely put into the pan without the cord's breaking?

3–12. End A of the bar AB in Fig. 3–22 rests on a frictionless horizontal surface, while end B is hinged. A hori-

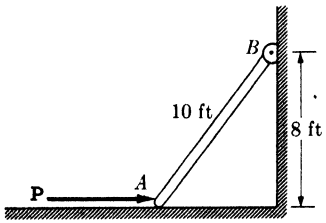


FIGURE 3–22

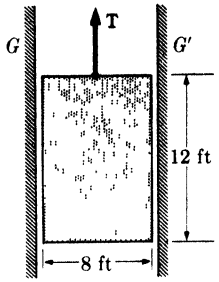


FIGURE 3–23

zontal force **P** of magnitude 12 lb is exerted on end A. Neglect the weight of the bar. What are the horizontal and vertical components of the force exerted by the bar on the hinge at B?

3–13. A ladder 13 ft long leans against a vertical frictionless wall with its lower end 5 ft from the wall. The ladder weighs 80 lb and its center of gravity is at its center. Find the magnitude and direction of the force exerted on the lower end of the ladder.

3–14. A uniform ladder 20 ft long weighing 80 lb, leans against a vertical frictionless wall with its lower end 12 ft from the wall. Compute the magnitude and direction of the resultant force exerted on the lower end of the ladder.

3–15. A uniform ladder 35 ft long rests against a frictionless vertical wall with its lower end 21 ft from the wall. The ladder weighs 80 lb. The coefficient of static friction between the foot of the ladder and the ground is 0.4. A man weighing 150 lb starts up the ladder. How far up the ladder can he climb before the ladder starts to slip?

3–16. A freight elevator weighing 2000 lb and having dimensions of 8 × 8 × 12 ft hangs from a cable with a small clearance between vertical frictionless guides, G and G', as shown in Fig. 3–23. A load of 1200 lb is placed in the elevator with its center of gravity 2 ft to the left of the center of the floor. The elevator is then moved upward at constant velocity. (a) Show in a diagram the location and direction of the forces exerted by the guides on the elevator. (b) Compute the magnitude of these forces.

3–17. A roller 20 inches in diameter weighs 72 lb. What horizontal force is necessary to pull the roller over a brick 2 inches high when (a) the force is applied at the center, (b) at the top?

3–18. Find the tension in cord A in Fig. 3–24. The boom is uniform and weighs 400 lb.

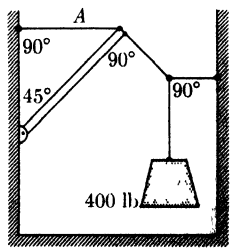


FIGURE 3–24

3–19. A door 7 ft high and 3 ft wide is hung from hinges 6 ft apart and 6 inches from the top and bottom of the door. The door weighs 60 lb, its center of gravity is at its center, and each hinge carries half the weight of the

door. Find the horizontal component of the force exerted on the door at each hinge.

3–20. A gate 8 ft long and 4 ft high weighs 80 lb. Its center of gravity is at its center, and it is hinged at A and B. To relieve the strain on the top hinge a wire CD is connected as shown in Fig. 3–25. The tension in CD is increased until the horizontal force at hinge A is zero. (a) What is the tension in the wire CD? (b) What is the magnitude of the horizontal component of force at hinge B? (c) What is the combined vertical force exerted by hinges A and B?

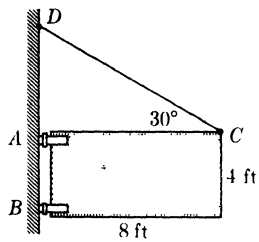


FIGURE 3–25

3–21. A garage door is mounted on an overhead rail as in Fig. 3–26. The wheels at A and B have rusted so that they do not roll, but slide along the track. The coefficient of sliding friction is 0.5. The distance between the wheels is 4 ft, and each is 1 ft in from the vertical sides of the door. The door is symmetrical and weighs 160 lb. It is pushed to the left at constant velocity

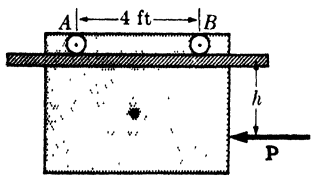


FIGURE 3–26

by a horizontal force **P**. (a) If the distance h is 3 ft, what is the vertical component of the force exerted on each wheel by the track? (b) Find the maximum value h can have without causing one wheel to leave the track.

3–22. A rectangular block 1 ft wide and 2 ft high is dragged to the right along a level surface at constant speed by a horizontal force **P**, as shown in Fig. 3–27. The coefficient of sliding friction is 0.40, the block weighs 50 lb, and its center of gravity is at its center. (a) Find the force **P** required. (b) Find the line of action of the normal force **N** exerted on the block by the surface if the height h is 6 inches. (c) Find the value of h at which the block just starts to tip.

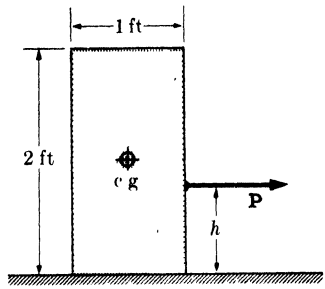


FIGURE 3–27

3–23. A table 8 ft long and 3 ft high, weighing 100 lb, has its center of gravity 6 inches below the center of the table top. The table is pushed at constant speed along a horizontal surface by a horizontal force applied at one end of the table top. The coefficient of sliding friction is 0.40. Compute the upward force and the friction force at each table leg.

3–24. A bench is 6 ft long, 2 ft high, and weighs 50 lb. Its center of gravity is midway between its ends. (a) What force applied at the upper right corner

and 30° above the horizontal will drag the bench at constant speed along a horizontal surface? (b) What is the upward force at each end of the bench? The coefficient of sliding friction is 0.20.

3–25. The chair in Fig. 3–28 is to be dragged to the right at constant speed along a horizontal surface, the coefficient of sliding friction being 0.30. The chair weighs 50 lb. (a) What horizontal force is needed? (b) What is the upward force at each leg if the force dragging the chair is applied at point A? (c) What is the upward force at each leg if the force is applied at point B? (d) What is the maximum height at which the dragging force can be applied without causing the chair to tip?

3–26. Find the magnitude and line of action of the resultant of the four forces in Fig. 3–29.

3–27. Weights of 3, 6, 9, and 12 lb are fastened to the corners of a light wire frame 2 ft square. Find the position of the center of gravity of the weights.

3–28. Find the position of the center of gravity of the T-shaped plate in Fig. 3–30.

3–29. A machine part, shown in cross section in Fig. 3–31, consists of two homogeneous, solid, coaxial cylinders. Where is its center of gravity?

3–30. Find the position of the center of gravity of the L-shaped plate in Fig. 3–32.

3–31. Find the position of the center of gravity of the U-shaped plate in Fig. 3–33.

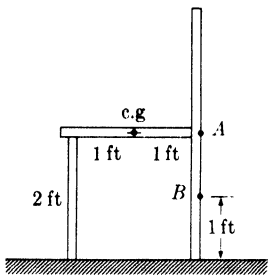


FIGURE 3–28

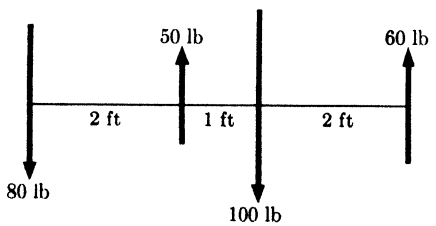


FIGURE 3–29

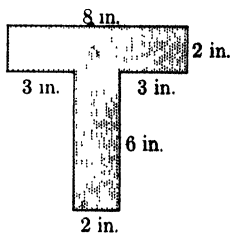


FIGURE 3–30

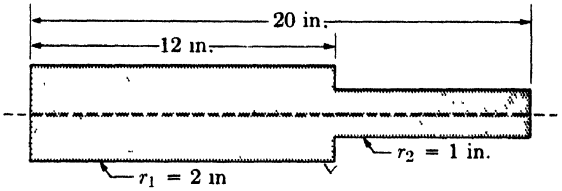


FIGURE 3–31

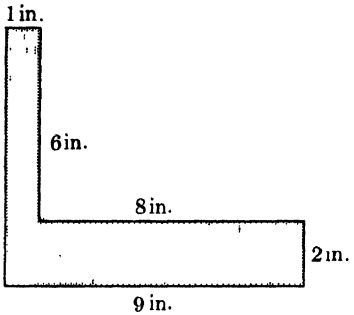


FIGURE 3–32

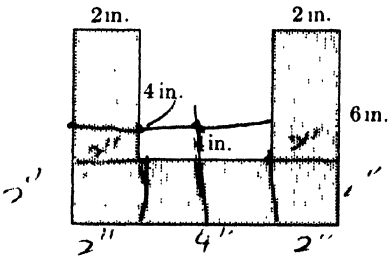


FIGURE 3–33

CHAPTER 4

RECTILINEAR MOTION

4–1 Motion. The preceding chapters have been concerned with forces, and we are now ready to discuss the mathematical methods of describing motion. This branch of mechanics is called *kinematics*.

Motion may be defined as a continuous change of position. In most actual motions, different points in a body move along different paths. The complete motion is known if we know how each point in the body moves, so to begin with we consider only a moving point, or a very small body called a *particle*.

The position of a particle is conveniently specified by its projections onto the three axes of a rectangular coordinate system. As the particle moves along any path in space, its projections move in straight lines along the three axes. The actual motion can be reconstructed from the motions of these three projections, so we shall begin by discussing the motion of a single particle along a straight line, or *rectilinear motion*.

4–2 Average velocity. Consider a particle moving along the x-axis, as in Fig. 4–1(a). The curve in Fig. 4–1(b) is a graph of its coordinate x plotted as a function of time t. At a time t_1 the particle is at point P in Fig. 4–1(a), where its coordinate is x_1, and at a later time t_2 it is at point Q, whose coordinate is x_2. The corresponding points on the coordinate-time graph in part (b) are lettered p and q.

The *displacement* of a particle as it moves from one point of its path to another is defined as the vector $\Delta\mathbf{x}$ drawn from the first point to the second. Thus in Fig. 4–1(a) the vector PQ, of magnitude $x_2 - x_1 = \Delta x$, is the displacement. The *average velocity* of the particle is defined as the ratio of the displacement to the time interval $t_2 - t_1 = \Delta t$. We shall represent average velocity by the symbol $\bar{\mathbf{v}}$ (the bar signifying an average value).

$$\bar{\mathbf{v}} = \frac{\Delta\mathbf{x}}{\Delta t}.$$

Average velocity is a vector, since the ratio of a vector to a scalar is itself a vector. Its direction is the same as that of the displacement vector. The magnitude of the average velocity is

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}. \tag{4–1}$$

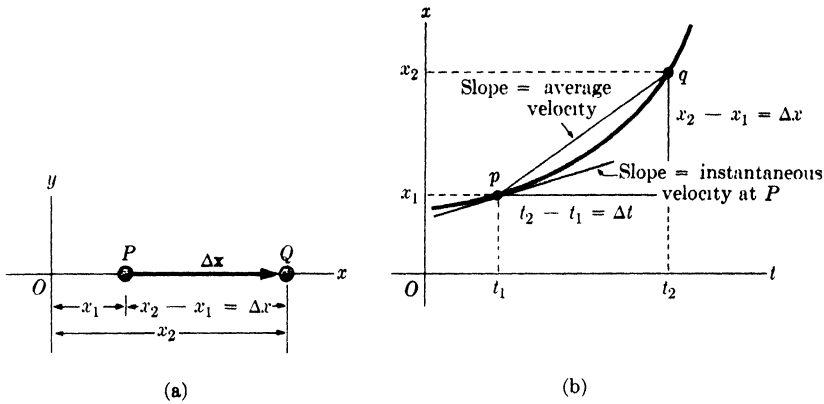


FIG. 4–1. (a) Particle moving on the x-axis. (b) Coordinate-time graph of the motion. The average velocity between t_1 and t_2 equals the slope of the chord pq. The instantaneous velocity at p equals the slope of the tangent at p.

In Fig. 4–1(b), the average velocity is represented by the slope of the chord pq (due allowance being made for the scales to which x and t are plotted), since the slope is the ratio of the "rise," $x_2 - x_1$ or Δx, to the "run," $t_2 - t_1$ or Δt.

Equation (4–1) can be cleared of fractions and written

$$x_2 - x_1 = \bar{v}\,(t_2 - t_1). \tag{4–2}$$

Since our time-measuring device can be started at any instant, we can let $t_1 = 0$ and let t_2 be any arbitrary time t. Then if x_0 is the coordinate when $t = 0$ (x_0 is called the *initial position*), and x is the coordinate at time t, Eq. (4–2) becomes

$$x - x_0 = \bar{v}t. \tag{4–3}$$

If the particle is at the origin when $t = 0$, then $x_0 = 0$ and Eq. (4–3) simplifies further to

$$x = \bar{v}t. \tag{4–4}$$

4–3 Instantaneous velocity. The velocity of a particle at some one instant of time, or at some one point of its path, is called its *instantaneous velocity*. This concept requires careful definition.

Suppose we wish to find the instantaneous velocity of the particle in Fig. 4–1 at the point P. The average velocity between points P and Q is associated with the entire displacement Δx, and with the entire time interval Δt. Imagine the second point Q to be taken closer and closer to the first point P, and let the average velocity be computed over these

shorter and shorter displacements and time intervals. The instantaneous velocity at the first point can then be defined as the *limiting value* of the average velocity when the second point is taken closer and closer to the first. Although the displacement then becomes extremely small, the time interval by which it must be divided becomes small also and the quotient is not necessarily a small quantity.

Then if **v** represents the instantaneous velocity, its magnitude is

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}. \tag{4-5}$$

Instantaneous velocity is also a vector, whose direction is the limiting direction of the displacement vector $\Delta \mathbf{x}$. Since Δt is necessarily positive, it follows that v has the same algebraic sign as Δx. Hence a positive velocity indicates motion toward the right along the x-axis, if we use the usual convention of signs.

As point Q approaches point P in Fig. 4–1(a), point q approaches point p in Fig. 4–1(b). In the limit, the slope of the chord pq equals the slope of the *tangent* to the curve at point p, due allowance being made for the scales to which x and t are plotted. *The instantaneous velocity at any point of a coordinate-time graph therefore equals the slope of the tangent to the graph at that point.* If the tangent slopes upward to the right, its slope is positive, the velocity is positive, and the motion is toward the right. If the tangent slopes downward to the right, the velocity is negative. At a point where the tangent is horizontal, its slope is zero and the velocity is zero.

If we express distance in feet and time in seconds, velocity is expressed in *feet per second* (ft/sec). Other common units of velocity are meters per second (m/sec), centimeters per second (cm/sec), miles per hour (mi/hr), and knots (1 knot = 1 nautical mile per hour).

EXAMPLE. The coordinate of a body moving along the x-axis is given by

$$x = 10 \frac{\text{cm}}{\text{sec}^2} t^2.$$

Compute the average velocity $\bar{v}$ over the time interval from (a) 2 sec to 2.1 sec, (b) from 2 sec to 2 sec + Δt. (c) Compute the instantaneous velocity v at $t = 2$ sec.

(a) At $t_1 = 2$ sec,

$$x_1 = 10 \frac{\text{cm}}{\text{sec}^2} (2 \text{ sec})^2 = 40 \text{ cm}.$$

At $t_2 = 2.1$ sec,

$$x_2 = 10\frac{\text{cm}}{\text{sec}^2}(2.1\text{ sec})^2 = 44.1\text{ cm}.$$

Hence

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{44.1\text{ cm} - 40\text{ cm}}{2.1\text{ sec} - 2\text{ sec}}$$
$$= 41\frac{\text{cm}}{\text{sec}}.$$

(b) At $t_1 = 2$ sec, $x_1 = 40$ cm. At $t_2 = 2\text{ sec} + \Delta t$,

$$x_2 = 10\frac{\text{cm}}{\text{sec}^2}(2\text{ sec} + \Delta t)^2$$
$$= 10\frac{\text{cm}}{\text{sec}^2}(4\text{ sec}^2 + 4\text{ sec}\,\Delta t + \Delta t^2)$$
$$= 40\text{ cm} + 40\frac{\text{cm}}{\text{sec}}\Delta t + 10\frac{\text{cm}}{\text{sec}^2}\Delta t^2.$$

Hence

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{\Delta t[(40\text{ cm/sec}) + (10\text{ cm/sec}^2)\,\Delta t]}{\Delta t}$$
$$= 40\frac{\text{cm}}{\text{sec}} + 10\frac{\text{cm}}{\text{sec}^2}\Delta t.$$

(c) The instantaneous velocity is

$$v = \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0}\left(40\frac{\text{cm}}{\text{sec}} + 10\frac{\text{cm}}{\text{sec}^2}\Delta t\right)$$
$$= 40\frac{\text{cm}}{\text{sec}}.$$

The term *speed* has two different meanings. It is sometimes used to mean the *magnitude* of the instantaneous velocity. In this sense, two automobiles traveling at 50 mi/hr, one north and the other south, are both said to have a speed of 50 mi/hr. In another sense, the *average speed* of a body means the total length of path covered, divided by the elapsed time. Thus if an automobile travels a total distance of 90 miles in 3 hours, its average speed is said to be 30 mi/hr even if the trip starts and ends at the same point. The average *velocity*, in the latter case, would be zero, since the displacement is zero.

4–4 Average and instantaneous acceleration. Except in certain special cases, the velocity of a moving body changes continuously as the motion proceeds. When this is the case the body is said to move with *accelerated motion*, or to have an *acceleration*.

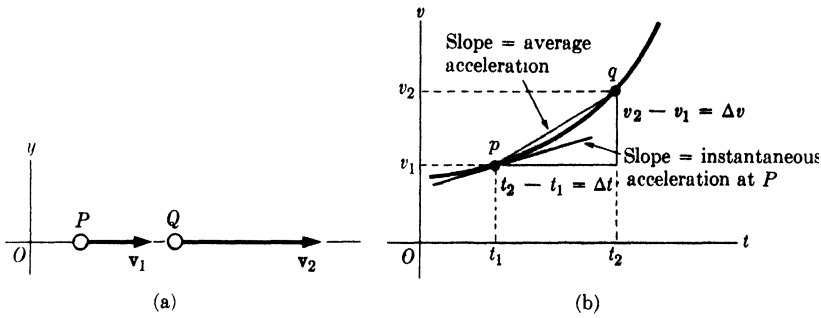


FIG. 4–2. (a) Particle moving on the x-axis. (b) Velocity-time graph of the motion. The average acceleration between t_1 and t_2 equals the slope of the chord pq. The instantaneous acceleration at p equals the slope of the tangent at p.

Figure 4–2(a) shows a particle moving along the x-axis. The vector $\mathbf{v}_1$ represents its instantaneous velocity at point P, and the vector $\mathbf{v}_2$ represents its instantaneous velocity at point Q. Figure 4–2(b) is a graph of the instantaneous velocity v plotted as a function of time, points p and q corresponding to P and Q in part (a).

The *average acceleration* of the particle as it moves from P to Q is defined as the ratio of the change in velocity to the elapsed time.

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} = \frac{\Delta \mathbf{v}}{\Delta t}, \tag{4–6}$$

where t_1 and t_2 are the times corresponding to the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors, the quantity $\mathbf{v}_2 - \mathbf{v}_1$ is a *vector difference* and must be found by the methods explained in Section 1–7. However, since in rectilinear motion both vectors lie in the same straight line, the magnitude of the vector difference in this special case equals the difference between the magnitudes of the vectors. The more general case, in which $\mathbf{v}_1$ and $\mathbf{v}_2$ are not in the same direction, will be considered in Chapter 6.

In Fig. 4–2(b), the magnitude of the average acceleration is represented by the slope of the chord pq.

The *instantaneous acceleration* of a body, that is, its acceleration at some one instant of time or at some one point of its path, is defined in the same way as instantaneous velocity. Let the second point Q in Fig. 4–2(a) be taken closer and closer to the first point P, and let the average acceleration be computed over shorter and shorter intervals of time. The instantaneous acceleration at the first point is defined as the limiting value of the average acceleration when the second point is taken closer and closer to

the first. Then if **a** represents the instantaneous acceleration, its magnitude is

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}.$$

The direction of the instantaneous acceleration is the limiting direction of the vector change in velocity, $\Delta \mathbf{v}$.

Instantaneous acceleration plays an important part in the laws of mechanics. Average acceleration is less frequently used. Hence from now on when the term "acceleration" is used we shall understand it to mean instantaneous acceleration.

The definition of acceleration just given applies to motion along any path, straight or curved. When a particle moves in a curved path the *direction* of its velocity changes and this change in direction also gives rise to an acceleration, as will be explained in Chapter 6.

As point Q approaches point P in Fig. 4–2(a), point q approaches point p in Fig. 4–2(b) and the slope of the chord pq approaches the slope of the tangent to the velocity-time graph at point p. *The instantaneous acceleration at any point of the graph therefore equals the slope of the tangent to the graph at that point.*

If we express velocity in feet per second and time in seconds, acceleration is expressed in feet per second, per second (ft/sec/sec). This is usually written as ft/sec^2, and is read "feet per second squared." Other common units of acceleration are meters per second squared (m/sec^2) and centimeters per second squared (cm/sec^2).

When the absolute value of the magnitude of the velocity of a body is decreasing (in other words, when the body is slowing down), the body is said to be *decelerated* or to have a *deceleration.*

EXAMPLE. Suppose the velocity of the particle in Fig. 4–2 is given by the equation

$$v = m + nt^2,$$

where $m = 10$ cm/sec and $n = 2$ cm/sec^3. (a) Find the change in velocity of the particle in the time interval between $t_1 = 2$ sec and $t_2 = 2 \text{ sec} + \Delta t$.

At time $t_1 = 2$ sec,

$$v_1 = 10 \frac{\text{cm}}{\text{sec}} + 2 \frac{\text{cm}}{\text{sec}^3} \times (2 \text{ sec})^2$$

$$= 18 \frac{\text{cm}}{\text{sec}}.$$

At time $t_2 = 2\ \text{sec} + \Delta t$,

$$v_2 = 10\frac{\text{cm}}{\text{sec}} + 2\frac{\text{cm}}{\text{sec}^3}(2\ \text{sec} + \Delta t)^2$$

$$= 10\frac{\text{cm}}{\text{sec}} + 2\frac{\text{cm}}{\text{sec}^3}(4\ \text{sec}^2 + 4\ \text{sec}\,\Delta t + \Delta t^2)$$

$$= 18\frac{\text{cm}}{\text{sec}} + \Delta t\left(8\frac{\text{cm}}{\text{sec}^2} + 2\frac{\text{cm}}{\text{sec}^3}\Delta t\right).$$

The change in velocity is therefore

$$v_2 - v_1 = \Delta t\left(8\frac{\text{cm}}{\text{sec}^2} + 2\frac{\text{cm}}{\text{sec}^3}\Delta t\right).$$

(b) Find the average acceleration in this time interval.

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta t[(8\ \text{cm/sec}^2) + (2\ \text{cm/sec}^3)\,\Delta t]}{\Delta t}$$

$$= 8\frac{\text{cm}}{\text{sec}^2} + 2\frac{\text{cm}}{\text{sec}^3}\Delta t.$$

(c) Find the instantaneous acceleration at $t = 2$ sec.

$$a = \lim_{\Delta t \to 0}\frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0}\left(8\frac{\text{cm}}{\text{sec}^2} + 2\frac{\text{cm}}{\text{sec}^3}\Delta t\right)$$

$$= 8\frac{\text{cm}}{\text{sec}^2}.$$

4–5 Rectilinear motion with constant acceleration. The simplest kind of accelerated motion is rectilinear motion in which the acceleration is constant, that is, in which the velocity changes at the same rate throughout the motion. The velocity-time graph is then a straight line as in Fig. 4–3, the velocity increasing by equal amounts in equal intervals of time. The slope of a cord between any two points on the line is the same as the slope of a tangent at any point. The instantaneous and average accelerations are therefore equal and, since the slope of a straight line is constant, the acceleration is constant. Its magnitude is

$$a = \frac{v_2 - v_1}{t_2 - t_1}.$$

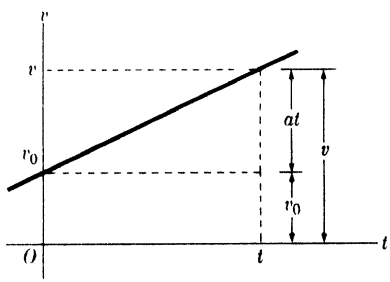


FIG. 4–3. Velocity-time graph for rectilinear motion with constant acceleration.

Now let $t_1 = 0$ and let t_2 be any arbitrary time t. Let v_0 represent the velocity when $t = 0$ (v_0 is called the *initial* velocity), and let

v be the velocity at time t. Then the preceding equation becomes

$$a = \frac{v - v_0}{t - 0},$$

or

$$\boxed{v = v_0 + at.} \tag{4–8}$$

This equation can be interpreted as follows. The acceleration a is the constant rate of change of velocity, or the change per unit time. The term at is the product of the change in velocity per unit time, a, and the duration of the time interval, t. Therefore it equals the total change in velocity. The velocity v at the time t then equals the velocity v_0 at the time $t = 0$, plus the change in velocity at. Graphically, the ordinate v at time t, in Fig. 4–3, can be considered as the sum of two segments: one of length v_0, equal to the initial velocity, the other of length at equal to the change in velocity in time t.

To find the displacement of a particle moving with constant acceleration, we make use of the fact that when the acceleration is constant and the velocity-time graph is a straight line, as in Fig. 4–3, the average velocity in any time interval equals one-half the sum of the velocities at the beginning and the end of the interval. Hence the average velocity between zero and t is

$$\bar{v} = \frac{v_0 + v}{2}. \tag{4–9}$$

This is *not* true, in general, when the acceleration is not constant and the velocity-time graph is curved as in Fig. 4–2.

We have shown that for a particle which is at the origin when $t = 0$, the coordinate x at any time t is

$$x = \bar{v}t, \tag{4–10}$$

where $\bar{v}$ is the average velocity. Hence from the two preceding equations,

$$\boxed{x = \frac{v_0 + v}{2} \times t.} \tag{4–11}$$

Two more very useful equations can be obtained from Eqs. (4–8) and (4–11), first by eliminating v and then by eliminating t. When we substitute in Eq. (4–11) the expression for v in Eq. (4–8), we get

$$x = \frac{v_0 + v_0 + at}{2} \times t,$$

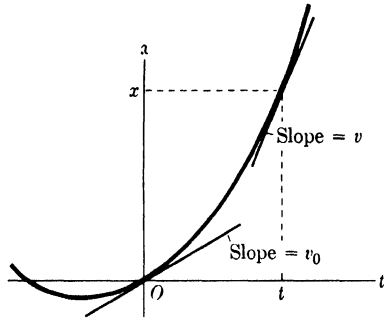


FIGURE 4–4

or

$$\boxed{x = v_0 t + \tfrac{1}{2}at^2.} \tag{4–12}$$

When Eq. (4–8) is solved for t and the result substituted in Eq. (4–11), we have

$$x = \frac{v_0 + v}{2} \times \frac{v - v_0}{a}$$

$$= \frac{v^2 - v_0^2}{2a}$$

or, finally,

$$\boxed{v^2 = v_0^2 + 2ax.} \tag{4–13}$$

Equations (4–8), (4–11), (4–12), and (4–13) are the *equations of motion with constant acceleration*, for the special case where the particle is at the origin when $t = 0$.

The curve in Fig. 4–4 is the coordinate-time graph for motion with constant acceleration. That is, it is a graph of Eq. (4–12). The curve is a *parabola*. The slope of the tangent at $t = 0$ equals the initial velocity v_0, and the slope of the tangent at time t equals the velocity v at that time. It is evident that the slope continually increases, and measurements would show that the *rate* of increase is constant, that is, that the acceleration is constant.

A special case of motion with constant acceleration is that in which the acceleration is zero. The *velocity* is then constant and the equations of motion become simply

$$v = \text{constant},$$

$$x = vt.$$

4–6 Freely falling bodies. The most common example of motion with (nearly) constant acceleration is that of a body falling toward the earth. In the absence of air resistance it is found that all bodies, regardless of their size or weight, fall with the same acceleration at the same point on the earth's surface, and if the distance covered is not too great the acceleration remains constant throughout the fall. The effect of air resistance and the decrease in acceleration with altitude will be neglected. This idealized motion is spoken of as "free fall," although the term includes rising as well as falling.

The acceleration of a freely falling body is called the acceleration due to gravity, or the acceleration of gravity. Its magnitude is denoted by the letter g. At or near the earth's surface its magnitude is approximately 32 ft/sec^2, 9.8 m/sec^2, or 980 cm/sec^2. More precise values, and small variations with latitude and elevation, will be considered later.

NOTE. The quantity "g" is sometimes referred to simply as "gravity," or as "the force of gravity," both of which are incorrect. "Gravity" is a phenomenon, and the "force of gravity" means the force with which the earth attracts a body, otherwise known as the weight of the body. The letter "g" represents the *acceleration* caused by the force resulting from the phenomenon of gravity.

EXAMPLE 1. A body is released from rest and falls freely. Compute its position and velocity after 1, 2, 3, and 4 seconds. Take the origin O at the elevation of the starting point, the y-axis vertical, and the upward direction as positive.

The initial coordinate y_0 and the initial velocity v_0 are both zero. The acceleration is downward, in the negative y-direction, so $a = -g = -32$ ft/sec^2.

From Eqs. (4–12) and (4-8),

$$y = v_0 t + \tfrac{1}{2}at^2 = 0 - \tfrac{1}{2}gt^2 = -16\,\frac{\text{ft}}{\text{sec}^2} \times t^2,$$

$$v = v_0 + at = 0 - gt = -32\,\frac{\text{ft}}{\text{sec}^2} \times t.$$

When $t = 1$ sec,

$$y = -16\,\frac{\text{ft}}{\text{sec}^2} \times 1\ \text{sec}^2 = -16\ \text{ft},$$

$$v = -32\,\frac{\text{ft}}{\text{sec}^2} \times 1\ \text{sec} = -32\,\frac{\text{ft}}{\text{sec}}.$$

The body is therefore 16 ft below the origin (y is negative) and has a downward velocity (v is negative) of magnitude 32 ft/sec.

The position and velocity at 2, 3, and 4 sec are found in the same way. The results are illustrated in Fig. 4–5.

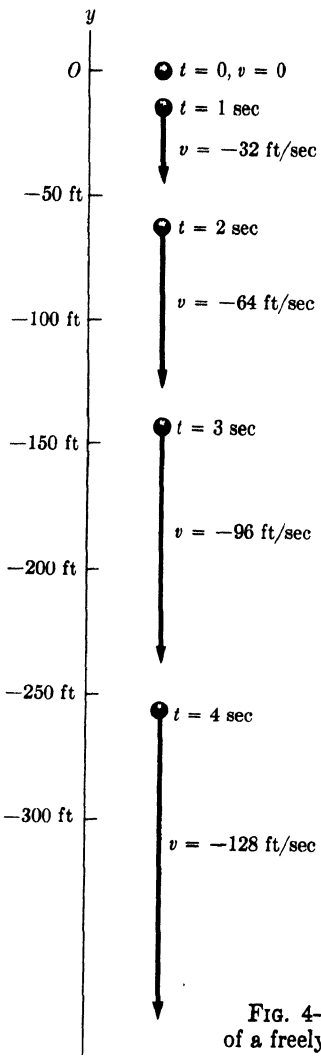


FIG. 4–5. Position and velocity of a freely falling body.

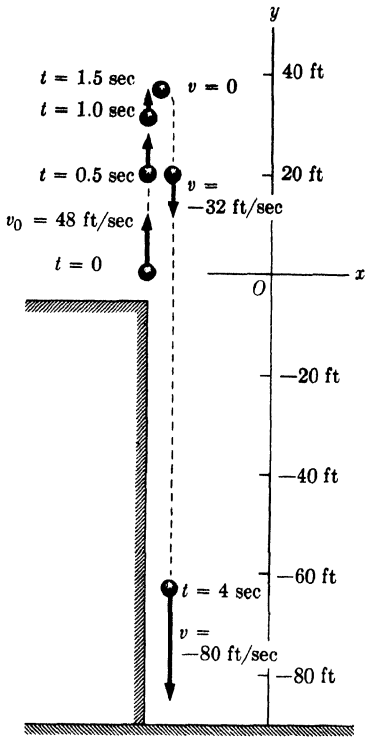


FIG. 4–6. Position and velocity of a body thrown vertically upward.

EXAMPLE 2. A ball is thrown (nearly) vertically upward from the cornice of a tall building, leaving the thrower's hand with a speed of 48 ft/sec and just missing the cornice on the way down. (See Fig. 4–6. The dotted line does not represent the actual path of the body.) Find (a) the position and velocity of the ball, 1 sec and 4 sec after leaving the thrower's hand; (b) the velocity when the ball is 20 ft above its starting point; (c) the maximum height reached and the time at which it is reached. Take the origin at the elevation at which the ball leaves the thrower's hand, the y-axis vertical and positive upward.

The initial position y_0 is zero. The initial velocity v_0 is +48 ft/sec, and the acceleration is −32 ft/sec^2.

The velocity at any time is

$$v = v_0 + at = 48\frac{\text{ft}}{\text{sec}} - 32\frac{\text{ft}}{\text{sec}^2} \times t. \quad (4\text{–}14)$$

The coordinate at any time is

$$y = v_0 t + \tfrac{1}{2}at^2 = 48\frac{\text{ft}}{\text{sec}} \times t - 16\frac{\text{ft}}{\text{sec}^2} \times t^2.$$

The velocity at any coordinate is

$$v^2 = v_0^2 + 2ay = \left(48\frac{\text{ft}}{\text{sec}}\right)^2 - 64\frac{\text{ft}}{\text{sec}^2} \times y. \quad (4\text{–}15)$$

(a) When $t = 1$ sec,

$$y = +32 \text{ ft}, \qquad v = +16\frac{\text{ft}}{\text{sec}}.$$

The ball is 32 ft above the origin (y is positive) and it has an upward velocity (v is positive) of 16 ft/sec.

When $t = 4$ sec,

$$y = -64 \text{ ft}, \qquad v = -80\frac{\text{ft}}{\text{sec}}.$$

The ball has passed its highest point and is 64 ft *below* the origin (y is negative). It has a *downward* velocity (v is negative) of magnitude 80 ft/sec. Note that it is not necessary to find the highest point reached, or the time at which it was reached. The equations of motion give the position and velocity at *any* time, whether the ball is on the way up or the way down.

(b) When the ball is 20 ft above the origin,

$$y = +20 \text{ ft}$$

and

$$v^2 = 1024\frac{\text{ft}^2}{\text{sec}^2}, \qquad v = \pm 32\frac{\text{ft}}{\text{sec}}.$$

The ball passes this point twice, once on the way up and again on the way down. The velocity on the way up is +32 ft/sec, and on the way down it is −32 ft/sec.

(c) At the highest point, $v = 0$. Hence

$$y = +36 \text{ ft}.$$

The time can now be found either from Eq. (4–14), setting $v = 0$, or from Eq. (4–15), setting $y = 36$ ft. From either equation, we get

$$t = 1.5 \text{ sec}.$$

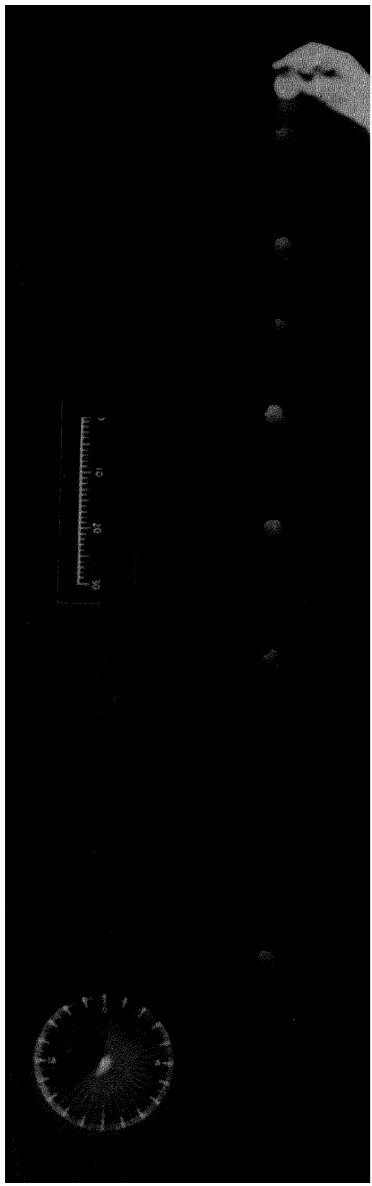


FIG. 4–7. Multiflash photograph (retouched) of freely falling golf ball.

Figure 4–7 is a "multiflash" photograph of a freely falling golf ball. This photograph was taken with the aid of the ultra-high-speed stroboscopic light source developed by Dr. Harold E. Edgerton of the Massachusetts Institute of Technology. By means of this source a series of intense flashes of light can be produced. The interval between successive flashes is controllable at will, and the duration of each flash is so short (a few millionths of a second) that there is no blur in the image of even a rapidly moving body. The camera shutter is left open during the entire motion, and as each flash occurs the position of the ball at that instant is recorded on the photographic film.

The equally spaced light flashes subdivide the motion into equal time intervals Δt. Since the time intervals are all equal, the velocity of the ball between any two flashes is directly proportional to the separation of its corresponding images in the photograph. If the velocity were constant, the images would be equally spaced. The increasing separation of the images during the fall shows that the velocity is continually increasing or the motion is accelerated. By comparing two successive displacements of the ball, the *change* in velocity in the corresponding time interval can be found. Careful measurements, preferably on an enlarged print, show that this change in velocity is the same in each time interval. In other words, the motion is one of *constant* acceleration.

4–7 Velocity components. Relative velocity. Velocity is a vector quantity involving both magnitude and direction. A velocity may therefore be resolved into components, or a number of velocity components combined into a resultant. As an example of the former process, suppose that a ship is steaming 30° E of N at 20 mi/hr in still water. Its velocity may be represented by the arrow in Fig. 4–8, and one finds by the usual method that its velocity component toward the east is 10 mi/hr, while toward the north it is 17.3 mi/hr.

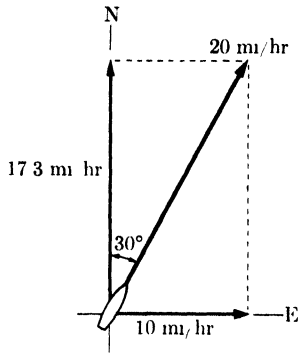


FIG. 4–8. Resolution of a velocity vector into components.

Velocity, like position, can only be specified relative to some reference frame or set of axes; the axes themselves may or may not be in motion. Ordinarily, velocities are specified relative to axes fixed with respect to the earth and considered to be "at rest," although of course they partake of the motion of the earth through space. In what follows, the expression "the velocity of a body" is understood to mean its velocity relative to the earth.

The velocity of one body relative to another when the second is in motion (relative to the earth) *is the vector difference between the velocities of the bodies* (relative to the earth). Specifically, if the bodies are designated by A and B, and their velocities (relative to the earth) by $\mathbf{v}_A$ and $\mathbf{v}_B$, the velocity of A relative to B is

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B \tag{4–16}$$

and the velocity of B relative to A is

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A.$$

Equation (4–16) may be written

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}. \tag{4–17}$$

That is, the velocity of body A (relative to the earth) is the vector sum of the velocity of B (relative to the earth) and the velocity of A relative to B. In general, then, when one body is in motion relative to a second, *the velocity of the first is the vector sum of the velocity of the second and the velocity of the first relative to the second.*

EXAMPLE 1. Automobile A, traveling at 30 mi/hr on a straight level road, is ahead of automobile B traveling in the same direction at 20 mi/hr. What is the velocity of A relative to B and the velocity of B relative to A?

Since both vectors are in the same straight line, the magnitude of their vector difference equals their arithmetic difference. The velocity of A relative to B is

$$v_{AB} = v_A - v_B = 30\frac{\text{mi}}{\text{hr}} - 20\frac{\text{mi}}{\text{hr}} = +10\frac{\text{mi}}{\text{hr}},$$

and the operator of car B sees car A pulling away from him at the rate of 10 mi/hr.

The velocity of B relative to A is

$$v_{BA} = v_B - v_A = 20\frac{\text{mi}}{\text{hr}} - 30\frac{\text{mi}}{\text{hr}} = -10\frac{\text{mi}}{\text{hr}},$$

and the operator of car A (if he looks back) sees car B dropping behind him (v_{BA} is negative) at 10 mi/hr.

EXAMPLE 2. The compass of an airplane indicates that it is headed due north, and its airspeed indicator shows that it is moving through the air at 120 mi/hr. If there is a wind of 50 mi/hr blowing from west to east, what is the velocity of the plane relative to the earth?

The velocity of the air is 50 mi/hr, due east. The velocity of the plane *relative to the air* is 120 mi/hr, due north. The velocity of the plane is the vector sum of these velocities, and from the construction in Fig. 4–9 it is 130 mi/hr, 22.5° E of N. The two velocities of 120 mi/hr and 50 mi/hr can be considered the components of the actual velocity of the plane.

EXAMPLE 3. In what direction should the pilot of the plane set his course in order to travel due north? What will then be his velocity relative to the earth?

The course set by the pilot is the direction in which the plane would actually travel in still air. It is thus in the direction of the velocity of the plane relative to the air. The resultant velocity must be due north. These velocities are related as in Fig. 4–10, from which we find the angle θ to be 24.5° W of N, and the resultant velocity to be 109 mi/hr, due north.

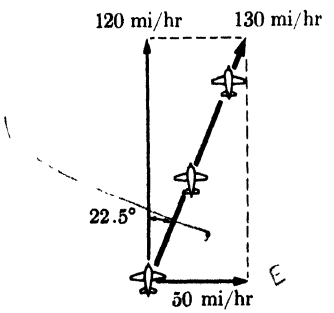


FIGURE 4–9

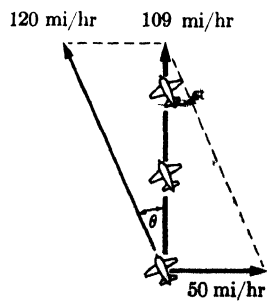


FIGURE 4–10

Problems

4–1. Suppose that a runner on a straight track covers a distance of 1 mile in exactly 4 minutes. What was his average velocity in (a) mi/hr, (b) ft/sec, (c) cm/sec?

4–2. A body moves along a straight line, its distance from the origin at any instant being given by the equation $x = 8t - 3t^2$, where x is in centimeters and t is in seconds. Find the average velocity of the body in the interval from $t = 0$ to $t = 1$ sec, and in the interval from $t = 0$ to $t = 4$ sec.

4–3. Refer to the example at the end of Section 4–3, in which the motion of a body along the x-axis was described by the equation $x = (10 \text{ cm/sec}^2)t^2$. Compute the instantaneous velocity of the body at time $t = 3$ sec. Let Δt first equal 0.1 sec, then 0.01 sec, and finally 0.001 sec. What limiting value do the results seem to be approaching?

4–4. An automobile is provided with a speedometer calibrated to read ft/sec rather than mi/hr. The following series of speedometer readings was obtained during a start.

Time (sec)	0	2	4	6	8	10	12	14	16
Velocity (ft/sec)	0	0	2	5	10	15	20	22	22

(a) Compute the average acceleration during each 2-sec interval. Is the acceleration constant? Is it constant during any part of the time? (b) Make a velocity-time graph of the data above, using scales of 1 in. = 2 sec horizontally, and 1 in. = 5 ft/sec vertically. Draw a smooth curve through the plotted points. What distance is represented by 1 sq in.? What is the displacement in the first 8 sec? What is the acceleration when $t = 8$ sec? When $t = 13$ sec? When $t = 15$ sec?

4–5. The graph in Fig. 4–11 shows the velocity of a body plotted as a function of time.

(a) What is the instantaneous acceleration at $t = 3$ sec?

(b) What is the instantaneous acceleration at $t = 7$ sec?

(c) What is the instantaneous acceleration at $t = 11$ sec?

(d) How far does the body go in the first 5 sec?

(e) How far does the body go in the first 9 sec?

(f) How far does the body go in the first 13 sec?

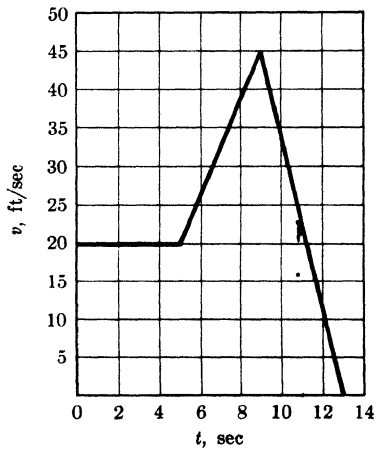


Figure 4–11

4–6. Each of the following changes in velocity takes place in a 10-sec interval. What is the magnitude, the algebraic sign, and the direction of the average acceleration in each interval?

(a) At the beginning of the interval a body is moving toward the right along the x-axis at 5 ft/sec, and at the end of the interval it is moving toward the right at 20 ft/sec.

(b) At the beginning it is moving toward the right at 20 ft/sec, and at the end it is moving toward the right at 5 ft/sec.

(c) At the beginning it is moving toward the left at 5 ft/sec, and at the end it is moving toward the left at 20 ft/sec.

(d) At the beginning it is moving toward the left at 20 ft/sec, and at the end it is moving toward the left at 5 ft/sec.

(e) At the beginning it is moving toward the right at 20 ft/sec, and at the end it is moving toward the left at 20 ft/sec.

(f) At the beginning it is moving toward the left at 20 ft/sec, and at the end it is moving toward the right at 20 ft/sec.

(g) In which of the above instances is the body decelerated?

4–7. The makers of a certain automobile advertise that it will accelerate from 15 to 50 mi/hr in high in 13 sec. Compute (a) the acceleration in ft/sec^2, and (b) the distance the car travels in this time, assuming the acceleration to be constant.

4–8. An airplane taking off from a landing field has a run of 1200 ft. If it starts from rest, moves with constant acceleration, and makes the run in 30 sec, with what velocity in ft/sec did it take off?

4–9. An automobile starts from rest and acquires a velocity of 40 mi/hr in 15 sec. (a) Compute the acceleration in miles per hour per second, and in feet per second per second, assuming it to be constant. (b) If the automobile continues to gain velocity at the same rate, how many more seconds are needed for it to acquire a velocity of 60 mi/hr? (c) Find the distances covered by the automobile in parts (a) and (b).

4–10. A body moving with constant acceleration covers the distance between two points 180 ft apart in 6 sec. Its velocity as it passes the second point is 45 ft/sec. (a) What is its acceleration? (b) What is its velocity at the first point?

4–11. A ball is released from rest and rolls down an inclined plane, requiring 4 sec to cover a distance of 100 cm. (a) What was its acceleration in cm/sec^2? (b) How many centimeters would it have fallen vertically in the same time?

4–12. The "reaction time" of the average automobile driver is about 0.7 sec. (The reaction time is the interval between the perception of a signal to stop and the application of the brakes.) If an automobile can decelerate at 16 ft/sec^2, compute the total distance covered in coming to a stop after a signal is observed: (a) from an initial velocity of 30 mi/hr, (b) from an initial velocity of 60 mi/hr.

4–13. At the instant the traffic lights turn green, an automobile that has been waiting at an intersection starts ahead with a constant acceleration of 6 ft/sec^2. At the same instant a truck, traveling with a constant velocity of 30 ft/sec, overtakes and passes the automobile. (a) How far beyond its starting point will the automobile overtake the truck? (b) How fast will it be traveling?

4–14. The engineer of a passenger train traveling at 100 ft/sec sights a freight train whose caboose is 600 ft ahead on the same track. The freight train is traveling in the same direction as the passenger train with a velocity of 30 ft/sec. The engineer of the passenger train immediately applies the brakes, causing a constant deceleration of 4 ft/sec^2, while the freight train continues with constant speed. (a)

Will there be a collision? (b) If so, where will it take place?

4–15. A sled starts from rest at the top of a hill and slides down with a constant acceleration. The sled is 140 ft from the top of the hill 2 sec after passing a point which is 92 ft from the top. Four sec after passing the 92-ft point it is 198 ft from the top, and 6 sec after passing the point it is 266 ft from the top.

(a) What is the average velocity of the sled during each of the 2-sec intervals after passing the 92-ft point?

(b) What is the acceleration of the sled?

(c) What was the velocity of the sled when it passed the 92-ft point?

(d) How long did it take to go from the top to the 92-ft point?

(e) How far did the sled go during the first sec after passing the 92-ft point?

(f) How long does it take the sled to go from the 92-ft point to the midpoint between the 92-ft and the 140-ft mark?

(g) What is the velocity of the sled as it passes the midpoint in part (f)?

4–16. A subway train starts from rest at a station and accelerates at a rate of 4 ft/sec^2 for 10 sec. It then runs at constant speed for 30 sec, and decelerates at 8 ft/sec^2 until it stops at the next station. Find the *total* distance covered.

4–17. A body starts from rest, moves in a straight line with constant acceleration, and covers a distance of 64 ft in 4 sec. (a) What was the final velocity? (b) How long a time was required to cover half the total distance? (c) What was the distance covered in one-half the total time? (d) What was the velocity when half the total distance had been covered? (e) What was the velocity after one-half the total time?

4–18. The speed of an automobile going north is reduced from 45 to 30 mi/hr in a distance of 264 ft. Find (a) the magnitude and direction of the acceleration, assuming it to be constant, (b) the elapsed time, (c) the distance in which the car can be brought to rest from 30 mi/hr, assuming the acceleration of part (a).

4–19. An automobile and a truck start from rest at the same instant, with the automobile initially at some distance behind the truck. The truck has a constant acceleration of 4 ft/sec^2 and the automobile an acceleration of 6 ft/sec^2. The automobile overtakes the truck after the truck has moved 150 ft. (a) How long does it take the auto to overtake the truck? (b) How far was the auto behind the truck initially? (c) What is the velocity of each when they are abreast?

4–20. (a) With what velocity must a ball be thrown vertically upward in order to rise to a height of 50 ft? (b) How long will it be in the air?

4–21. A ball is thrown vertically downward from the top of a building, leaving the thrower's hand with a velocity of 30 ft/sec.

(a) What will be its velocity after falling for 2 sec?

(b) How far will it fall in 2 sec?

(c) What will be its velocity after falling 30 ft?

(d) If it moved a distance of 3 ft while in the thrower's hand, find its acceleration while in his hand.

(e) If the ball was released at a point 120 ft above the ground, in how many seconds will it strike the ground?

(f) What will be its velocity when it strikes?

•4–22. A balloon, rising vertically with a velocity of 16 ft/sec, releases a sandbag at an instant when the balloon is 64 ft above the ground. (a) Com-

pute the position and velocity of the sandbag at the following times after its release: $\frac{1}{4}$ sec, $\frac{1}{2}$ sec, 1 sec, 2 sec. (b) How many seconds after its release will the bag strike the ground? (c) With what velocity will it strike?

4–23. A stone is dropped from the top of a tall cliff, and 1 sec later a second stone is thrown vertically down with a velocity of 60 ft/sec. How far below the top of the cliff will the second stone overtake the first?

4–24. A ball dropped from the cornice of a building takes 0.25 sec to pass a window 9 ft high. How far is the top of the window below the cornice?

4–25. A ball is thrown nearly vertically upward from a point near the cornice of a tall building. It just misses the cornice on the way down, and passes a point 160 ft below its starting point 5 sec after it leaves the thrower's hand. (a) What was the initial velocity of the ball? (b) How high did it rise above its starting point? (c) What was the magnitude of its velocity as it passed a point 64 ft below the starting point?

4–26. A juggler performs in a room whose ceiling is 9 ft above the level of his hands. He throws a ball vertically upward so that it just reaches the ceiling. (a) With what initial velocity does he throw the ball? (b) What time is required for the ball to reach the ceiling?

He throws a second ball upward with the same initial velocity, at the instant that the first ball is at the ceiling. (c) How long after the second ball is thrown do the two balls pass each other?

(d) When the balls pass each other, how far are they above the juggler's hands?

4–27. An object is thrown vertically upward. It has a speed of 32 ft/sec when it has reached one-half its maximum height. (a) How high does it rise? (b) What is its velocity and acceleration 1 sec after it is thrown? (c) 3 sec after? (d) What is the average velocity during the first half-sec?

4–28. A student determined to test the law of gravity for himself walks off a skyscraper 900 ft high, stopwatch in hand, and starts his free fall (zero initial velocity). Five seconds later, Superman arrives at the scene and dives off the roof to save the student. (a) What must Superman's initial velocity be in order that he catch the student just before the ground is reached? (b) What must be the height of the skyscraper so that even Superman can't save him? (Assume that Superman's acceleration is that of any freely falling body.)

4–29. A ball is thrown vertically upward from the ground and a student gazing out of the window sees it moving upward past him at 16 ft/sec. The window is 32 ft above the ground. (a) How high does the ball go above the ground? (b) How long does it take to go from a height of 32 ft to its highest point? (c) Find its velocity and acceleration $\frac{1}{2}$ sec after it left the ground, and 2 sec after it left the ground.

4–30. A ball is thrown vertically upward from the ground with a velocity of 80 ft/sec. (a) How long will it take to rise to its highest point? (b) How high does the ball rise? (c) How long after projection will the ball have a velocity of 16 ft/sec upward? (d) of 16 ft/sec downward? (e) When is the displacement of the ball zero? (f) When is the magnitude of the ball's velocity equal to half its velocity of projection? (g) When is the magnitude of the ball's displacement equal to half the greatest height to which it

rises? (h) What is the magnitude and direction of the acceleration while the ball is moving upward? while moving downward? when at the highest point?

4–31. A ball rolling on an inclined plane moves with a constant acceleration. One ball is released from rest at the top of an inclined plane 18-m long and reaches the bottom 3 sec later. At the same instant that the first ball is released, a second ball is projected upward along the plane from its bottom with a certain initial velocity. The second ball is to travel part way up the plane, stop, and return to the bottom so that it arrives simultaneously with the first ball. (a) Find the acceleration. (b) What must be the initial velocity of the second ball? (c) How far up the plane will it travel?

4–32. Two piers A and B are located on a river, one mile apart. Two men must make round trips from pier A to pier B and return. One man is to row a boat at a velocity of 4 mi/hr relative to the water, and the other man is to walk on the shore at a velocity of 4 mi/hr. The velocity of the river is 2 mi/hr in the direction from A to B. How long does it take each man to make the round trip?

4–33. A passenger on a ship traveling due east with a speed of 18 knots observes that the stream of smoke from the ship's funnels makes an angle of 20° with the ship's wake. The wind is blowing from south to north. Assume that the smoke acquires a velocity (with respect to the earth) equal to the velocity of the wind, as soon as it leaves the funnels. Find the velocity of the wind.

4–34. An airplane pilot wishes to fly due north. A wind of 60 mi/hr is blowing toward the west. If the flying speed of the plane (its speed in still air) is 180 mi/hr, in what direction should the pilot set his course? What is the speed of the plane over the ground? Illustrate with a vector diagram.

4–35. An airplane pilot sets a compass course due west and maintains an air speed of 120 mi/hr. After flying for one-half hour he finds himself over a town which is 75 mi west and 20 mi south of his starting point. (a) Find the wind velocity, in magnitude and direction. (b) If the wind velocity were 60 mi/hr due south, in what direction should the pilot set his course in order to travel due west? Take the same air speed of 120 mi/hr.

4–36. When a train has a speed of 10 mi/hr eastward, raindrops which are falling vertically with respect to the earth make traces on the windows of the train, which are inclined 30° to the vertical. (a) What is the horizontal component of a drop's velocity with respect to the earth? with respect to the train? (b) What is the velocity of the raindrop with respect to the earth? with respect to the train?

4–37. A river flows due north with a velocity of 3 mi/hr. A man rows a boat across the river, his velocity relative to the water being 4 mi/hr due east. (a) What is his velocity relative to the earth? (b) If the river is 1 mile wide, how far north of his starting point will he reach the opposite bank? (c) How long a time is required to cross the river?

4–38. (a) In what direction should the rowboat in Problem 4–37 be headed in order to reach a point on the opposite bank directly east from the start? (b) What will be the velocity of the boat relative to the earth? (c) How long a time is required to cross the river?

CHAPTER 5

NEWTON'S SECOND LAW. GRAVITATION

5–1 Introduction. In the preceding chapters we have discussed separately the concepts of force and acceleration. We have made use, in problems in equilibrium, of Newton's first law, which states that when the resultant force on a body is zero, the acceleration of the body is also zero. The next logical step is to ask how a body behaves when the resultant force on it is *not* zero. The answer to this question is contained in Newton's second law, which states that when the resultant force is not zero the body moves with accelerated motion, and that the acceleration, with a given force, depends on a property of the body known as its *mass*.

This part of mechanics, which includes both the study of motion and the forces that bring about the motion, is called *dynamics*. In its broadest sense, dynamics includes nearly the whole of mechanics. Statics treats of special cases in which the acceleration is zero, and kinematics deals with motion only.

5–2 Newton's second law. Mass. We know from experience that an object at rest will never start to move of itself; a push or pull must be exerted on it by some other body. It is also a familiar fact that a force is required to slow down or to stop a body which is already in motion, and that a sidewise force must be exerted on a moving body to deviate

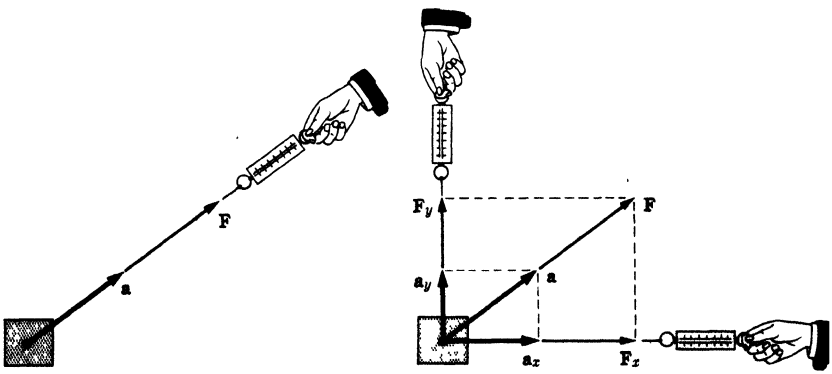


FIG. 5–1. (a) The acceleration **a** is in the same direction as the resultant force **F**. (b) Each component of a force can be considered to produce its own component of acceleration.

it from a straight line. All the processes above (i.e., speeding up, slowing down, or changing direction) involve a change in either the magnitude or the direction of the velocity of the body. In other words, in every case the body is accelerated, and an external force must act on it to produce this acceleration.

Let us consider the following experiments. As shown in Fig. 5–1(a), which is a top view, an arbitrary body is placed on a level frictionless surface and a horizontal force **F** is exerted on it. We shall assume that the body is not rotating and that the line of action of the force passes through the center of gravity of the body. The body acquires no rotational motion and only its translational motion is affected. For concreteness, suppose that the force is measured by a spring balance calibrated in pounds, as described in Section 1–2. With a stopwatch and scale we determine the resulting acceleration a. Let a large number of similar pairs of measurements of force and acceleration be made, first on the same body, using forces that differ in magnitude and direction, and then using different bodies. The results of these experiments show the following:

(a) In every case, the *direction* of the acceleration is the same as that of the force. This is true whether the body is initially at rest or is moving in any direction and with any speed.

(b) For a given body, the ratio of the magnitude of the force to that of the acceleration is always the same, or is a constant.

$$\frac{F}{a} = \text{constant (for a given body)}.$$

The ratio is, in general, different for different bodies.

This constant ratio of force to acceleration can be considered a property of the body called its *mass* m, where

$$m = \frac{F}{a},$$

or

$$F = ma. \tag{5–1}$$

Mass is a scalar quantity.

The mass of a body therefore represents *the force per unit of acceleration.* Since experiment shows that the ratio of force to acceleration is always the same, for a given body, it suffices to make one pair of measurements of force and acceleration to determine the mass. For example, if the acceleration of a certain body is found to be 5 ft/sec^2 when the force is 20 lb, the mass of the body is

$$m = \frac{F}{a} = \frac{20 \text{ lb}}{5 \text{ ft/sec}^2} = 4 \frac{\text{lb}}{\text{ft/sec}^2}.$$

When a large force is needed to speed up a body, slow it down, or deviate it sidewise if it is moving, the mass of the body is large. In everyday language, we would say that the body has a large *inertia*. If only a small force is needed per unit of acceleration, the mass is small and the inertia is small. The mass of a body can therefore be considered to represent in a quantitative way that property of matter which is described qualitatively by the word *inertia*.

The equation

$$\mathbf{F} = m\mathbf{a}$$

is a *vector* equation. That is, the vector $\mathbf{F}$ has the same magnitude and direction as a vector which is m times as great as the vector $\mathbf{a}$. If two vectors are equal their rectangular components are equal also and the vector equation above (for forces and accelerations in the xy-plane) is equivalent to the pair of scalar equations

$$F_x = ma_x, \qquad F_y = ma_y. \tag{5-2}$$

For example, suppose that instead of applying the force $\mathbf{F}$ directly, as in Fig. 5-1(a), we apply instead the force components $\mathbf{F}_x$ and $\mathbf{F}_y$ as in Fig. 5-1(b). We find that the acceleration $\mathbf{a}$ has the same magnitude and direction as in part (a), and that the components a_x and a_y are those given by Eq. (5-2). This means that each component of the force can be considered to produce its own component of acceleration.

It follows that if any number of forces act on a body simultaneously, the forces may be resolved into x- and y-components, the algebraic sums $\sum F_x$ and $\sum F_y$ may be computed, and the components of acceleration are given by

$$\boxed{\sum F_x = ma_x, \qquad \sum F_y = ma_y.} \tag{5-3}$$

This pair of equations is equivalent to the single vector equation

$$\boxed{\sum \mathbf{F} = m\mathbf{a},} \tag{5-4}$$

where $\sum \mathbf{F}$ is the vector sum or *resultant* of *all* the *external* forces acting *on* the body.

Equation (5-4) is the mathematical statement of Newton's *second law of motion*. Newton phrased the law as follows: *"The change of motion is proportional to the motive force impressed; and is made in the direction of the straight line in which that force is impressed."* If for "change of motion" we read "rate of change of velocity," or acceleration, the second

law states that *the acceleration is proportional to the resultant force and is in the same direction as this force.*

We can see from Eq. (5–4) the physical conditions that must be fulfilled for motion with *constant* acceleration; namely, if **a** is constant, then the resultant force $\sum \mathbf{F}$ must be constant also. In other words, motion with constant acceleration is motion under the action of a constant force. If the force varies, the acceleration varies in direct proportion, since the mass m is constant.

It is also evident from Eq. (5–4) that if the resultant force on a body is zero, the acceleration of the body is zero and its velocity is constant. Hence, if the body is in motion, it continues to move with no change in the magnitude or direction of its velocity; if at rest, it remains at rest (its velocity is then constant and equal to zero). But these are evidently the conditions to which Newton's *first* law applies, and we see that the first law is merely a special case of the second when the resultant force and the acceleration are both zero. There are thus only two independent laws of Newton, the second and the third.

5–3 Systems of units. The mass of a body is the ratio of the resultant force on the body to its acceleration. Consider first the *British engineering system* of units, in which forces are expressed in pounds and accelerations in ft/sec^2. In this system, the mass of a body is expressed in lb/(ft/sec^2). It is often convenient to introduce a single term for the combination of units in which a physical quantity is expressed, and the unit above, 1 lb/(ft/sec^2), is called one *slug*. (This term arose from the concept of mass as inertia or sluggishness.) Thus

$$\sum F \text{ (lb)} = m \text{ (slugs)} \times a \text{ (ft/sec}^2\text{)}.$$

For example, if the acceleration of a body is 5 ft/sec^2 when the resultant force on it is 20 lb, its mass is

$$m = \frac{\sum F}{a} = \frac{20 \text{ lb}}{5 \text{ ft/sec}^2} = 4 \frac{\text{lb}}{\text{ft/sec}^2} = 4 \text{ slugs}.$$

The numerical value of the mass of a given body depends on the units in which force and acceleration are expressed. By selecting as a unit of force, that force which acts on any arbitrary body when it has unit acceleration, the mass of that body automatically becomes equal to one unit of mass.

In setting up the *meter-kilogram-second* (mks) system of units, the mass of the *standard kilogram* is chosen to be the unit of mass. The unit of force in this system is then that force which gives a standard kilogram an acceleration of 1 m/sec^2. This force is called *one newton*, and is

TABLE 5–1

SYSTEMS OF UNITS

System of units	Force	Mass	Acceleration
Engineering	pound (lb)	slug	ft/sec^2
mks	newton (n)	kilogram (kgm)	m/sec^2
cgs	dyne	gram (gm)	cm/sec^2

approximately equal to one-quarter of a pound of force (more precisely, 1 newton = 0.22481 lb).

Thus, in the mks system,

$$F \text{ (newtons)} = m \text{ (kgm)} \times a \text{ (m/sec}^2\text{)}.$$

The mass unit in the *centimeter-gram-second* (cgs) system is *one gram*, equal to 1/1000 kilogram. The force unit in this system is the force which gives a body of mass 1 gm an acceleration of 1 cm/sec^2, and is called *one dyne*. In the cgs system, then,

$$\Sigma F \text{ (dynes)} = m \text{ (gm)} \times a \text{ (cm/sec}^2\text{)}.$$

Since 1 kgm = 10^3 gm and 1 m/sec^2 = 10^2 cm/sec^2, it follows that 1 newton = 10^5 dynes.

The units of force, mass, and acceleration in the three systems are summarized in Table 5–1.

It is unnecessary for the National Bureau of Standards to maintain a physical object embodying the engineering mass unit, the slug, or the engineering unit of length, the foot, since both of these are defined in terms of the standard kilogram and the standard meter. One foot, by definition, is exactly 0.3048 meter. One standard pound, by definition, is a body of mass 0.45359237 kgm. The pound of force is the force which gives a standard pound an acceleration equal to the standard acceleration of gravity, 32.1740 ft/sec^2. Finally, the slug is the mass of a body whose acceleration is 1 ft/sec^2 when the force on it is one pound.

5–4 Mass and weight. The *weight* of a body means the gravitational force exerted on it by the earth. When a body is released and allowed to fall freely, the only force acting on it is its weight **w**, and its acceleration is that of any freely falling body, the acceleration of gravity **g**. Its mass is then

$$m = \frac{\Sigma F}{a} = \frac{w}{g}.$$

Since the weight of a body is a force, it must be expressed in units of force. Thus in the engineering system weight is expressed in *pounds;* in the mks system, in *newtons;* and in the cgs system, in *dynes.* The preceding equation gives the relation between the mass and weight of a body in any consistent system of units. In the engineering system a body is usually described by stating its weight w in pounds, and computing its mass m in slugs, if this is desired. Thus the mass of a man whose weight is 160 lb at a point where $g = 32.0$ ft/sec^2 is

$$m = \frac{w}{g} = \frac{160 \text{ lb}}{32.0 \text{ ft/sec}^2} = 5.00 \text{ slugs.}$$

His weight at a second point where $g = 32.2$ ft/sec^2 is

$$w = mg = 5.00 \text{ slugs} \times 32.2 \frac{\text{ft}}{\text{sec}^2} = 161 \text{ lb.}$$

Thus, unlike his mass, which is a constant, his weight varies from one point to another on the earth's surface.

If the mass of a body is 1 slug, its weight at a point where $g = 32.0$ ft/sec^2 is

$$w = mg = 1 \text{ slug} \times 32.0 \frac{\text{ft}}{\text{sec}^2} = 32.0 \text{ lb.}$$

In the mks and cgs systems, a body is usually described by stating its mass m in kilograms or grams, and computing its weight w if this is desired. Thus, at a point where $g = 9.80$ m/sec^2, the weight of a body whose mass is 10 kgm is

$$w = mg = 10 \text{ kgm} \times 9.80 \frac{\text{m}}{\text{sec}^2} = 98.0 \text{ newtons.}$$

Since 10 kgm $= 10^4$ gm, and 9.80 m/sec^2 $= 980$ cm/sec^2, the weight of the same body in cgs units is

$$w = mg = 10^4 \text{ gm} \times 980 \frac{\text{cm}}{\text{sec}^2} = 9.80 \times 10^6 \text{ dynes.}$$

The mass of a body whose weight at a point where $g = 9.80$ m/sec^2 is 1 newton, is

$$m = \frac{w}{g} = \frac{1 \text{ n}}{9.80 \text{ m/sec}^2} = 0.102 \text{ kgm,}$$

and the mass of a body whose weight at the same point is 1 dyne, is

$$m = \frac{w}{g} = \frac{1 \text{ dyne}}{980 \text{ cm/sec}^2} = 1.02 \times 10^{-3} \text{ gm,}$$

or about a milligram.

There are also two other systems of units in which, as in the engineering system, the unit of force rather than of mass is arbitrarily defined. These systems take as their units of force the weight of the standard kilogram and the weight of a gram. The former force is called "one kilogram of force" and the latter "one gram of force." One "kilogram of force" is about 2.2 lb of force or about 9.8 newtons; one "gram of force" is about 0.0022 lb or 980 dynes. The "gram of force" is commonly used as a force unit in elementary physics texts, where the reader has probably met it. The "kilogram force" is used as a force unit in engineering work in those countries which use the metric system exclusively. We shall use neither of these units in this book, and "gram" and "kilogram" will refer to mass only.

5–5 Newton's law of universal gravitation. Throughout our study of mechanics we have been continually encountering forces due to gravitational attraction between the earth and bodies on its surface, forces which are called the weights of the bodies. We now wish to study this phenomenon of gravitation in somewhat more detail.

The law of universal gravitation was discovered by Sir Isaac Newton, and was first announced by him in the year 1686. It may be stated:

Every particle of matter in the universe attracts every other particle with a force which is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

$$F \propto \frac{mm'}{r^2}.$$

The proportion above may be converted to an equation on multiplication by a constant G, which is called the *gravitational constant.*

$$\boxed{F = G\frac{mm'}{r^2}.} \tag{5–5}$$

There seems to be some evidence that Newton was led to deduce this law from speculations concerning the fall of an apple to the earth. However, his first published calculations to justify its correctness had to do with the motion of the moon around the earth.

The numerical value of the constant G depends on the units in which force, mass, and distance are expressed. Its magnitude can be found experimentally by measuring the force of gravitational attraction between two bodies of known masses m and m', at a known separation. For bodies of moderate size the force is extremely small, but it can be measured with an instrument which was invented by the Rev. John Michell, although it was first used for this purpose by Sir Henry Cavendish in 1798. The same type of instrument was also used by Coulomb for studying forces of electrical and magnetic attraction and repulsion.

The Cavendish balance consists of two small spheres of mass m (Fig. 5–2), usually of gold or platinum, mounted at opposite ends of a light horizontal rod which is supported at its center by a fine vertical fiber such as a quartz thread. A small mirror fastened to the fiber reflects a beam of light onto a scale. To use the balance, two large spheres of mass m', usually of lead, are brought up to the positions shown. The forces of gravitational attraction between the large and small spheres result in a couple which twists the fiber and mirror through a small angle, thereby moving the light beam along the scale.

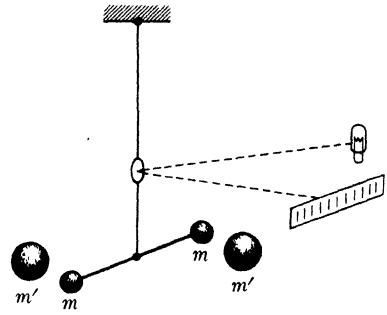


FIG. 5–2. Principle of the Cavendish balance.

By using an extremely fine fiber, the deflection of the light beam may be made sufficiently large so that the gravitational forces can be measured quite accurately. The gravitational constant, measured in this way, is found to be

$$G = 6.670 \times 10^{-8}\ \text{dyne} \cdot \text{cm}^2/\text{gm}^2$$
$$= 6.670 \times 10^{-11}\ \text{newton} \cdot \text{m}^2/\text{kgm}^2.$$

EXAMPLE: Compute the force of gravitational attraction between the large and small spheres of a Cavendish balance, if $m = 1$ gm, $m' = 500$ gm, $r = 5$ cm. (Two spheres attract each other as if the mass of each were concentrated at its center.)

$$F = \frac{(6.67 \times 10^{-8}\ \text{dyne} \cdot \text{cm}^2/\text{gm}^2) \times (1\ \text{gm}) \times (500\ \text{gm})}{(5\ \text{cm})^2} = 1.33 \times 10^{-6}\ \text{dyne},$$

or about one-millionth of a dyne!

5–6 The mass of the earth. Since the constant G in Eq. (5–5) can be found from measurements in the laboratory, the mass of the earth may be computed. From measurements on freely falling bodies, we know that the earth attracts a one-gram mass at its surface with a force of (about) 980 dynes. The distance between the centers of the masses is the radius of the earth, 6380 km or 6.38×10^8 cm. Therefore,

$$980\ \text{dynes} = \frac{(6.67 \times 10^{-8}\ \text{dyne} \cdot \text{cm}^2/\text{gm}^2) \times (1\ \text{gm}) \times m_E}{(6.38 \times 10^8\ \text{cm})^2},$$

where m_E is the mass of the earth. Hence

$$m_E = 5.98 \times 10^{27} \text{ gm}.$$

The volume of the earth is

$$V = \tfrac{4}{3}\pi R^3 = 1.09 \times 10^{27} \text{ cm}^3.$$

The mass divided by the volume is known as the *density*, the value for water being 1 gm/cm^3. The average density of the earth is therefore

$$\frac{m_E}{V} = \frac{5.98 \times 10^{27} \text{ gm}}{1.09 \times 10^{27} \text{ cm}^3} = 5.5 \frac{\text{gm}}{\text{cm}^3}.$$

This is considerably larger than the average density of the material near the earth's surface, so that the interior of the earth must be of much higher density.

5–7 Variations in "*g*." The acceleration of gravity, g, is the acceleration imparted to a body by its own weight. Its weight, however, can be written

$$w = G\frac{mm_E}{R^2},$$

where m is the mass of the body, m_E is the mass of the earth, and R is the distance to the earth's center. Then, since $w = mg$,

$$mg = G\frac{mm_E}{R^2},$$

$$g = \frac{Gm_E}{R^2}. \tag{5–6}$$

Since G and m_E are constants, the acceleration of gravity decreases with increasing distance from the center of the earth, and it is only approximately correct to state that a body falls toward the earth with constant acceleration. Actually, the acceleration continually increases as the body approaches the earth, air resistance being neglected. For most purposes, however, this variation is negligible. The acceleration of gravity varies somewhat from point to point on the earth's surface, partly because of variations in the distance to the earth's center and partly because of local deposits of ore, oil, or other substances whose density is greater or less than the average density of the earth. The greatest variation, however, arises from the rotation of the earth, and will be discussed in Chapter 6.

TABLE 5–2

VARIATIONS OF g WITH LATITUDE AND ELEVATION

Station	North latitude	Elevation, m	g, cm/sec^2	g, ft/sec^2
Canal Zone	9°	0	978.243	32.0944
Jamaica	18°	0	978.591	32.1059
Bermuda	32°	0	979.806	32.1548
Denver	40°	1638	979.609	32.1393
Cambridge	42°	0	980.398	32.1652
Standard station			980.665	32.1740
Greenland	70°	0	982.534	32.2353

From a survey of variations in the value of g, conclusions can be drawn as to the presence of deposits of ore or oil beneath the earth's surface. Hence the precise measurement of g is one of the methods of geophysical prospecting. Some representative values of g are given in Table 5–2.

5–8 Applications of Newton's second law. We now give a number of examples of the application of Newton's second law to specific problems. In all these examples, and in the problems at the end of the chapter, it will be assumed that the acceleration of gravity is 32.0 ft/sec^2 or 9.8 m/sec^2, unless otherwise specified.

EXAMPLE 1. What is the resultant force on a body weighing 48 lb when its acceleration is 6 ft/sec^2?

We must first find the mass of the body. From the relation $m = w/g$,

$$m = \frac{48 \text{ lb}}{32 \text{ ft/sec}^2} = 1.5 \frac{\text{lb}}{\text{ft/sec}^2} = 1.5 \text{ slugs}.$$

Hence

$$\sum F = ma = 1.5 \text{ slugs} \times 6 \frac{\text{ft}}{\text{sec}^2} = 9 \text{ lb}.$$

EXAMPLE 2. What is the resultant force on a body of mass 48 kgm when its acceleration is 6 m/sec^3?

The mass of the body is given, and hence

$$\sum F = ma = 48 \text{ kgm} \times 6 \frac{\text{m}}{\text{sec}^2} = 288 \text{ newtons}.$$

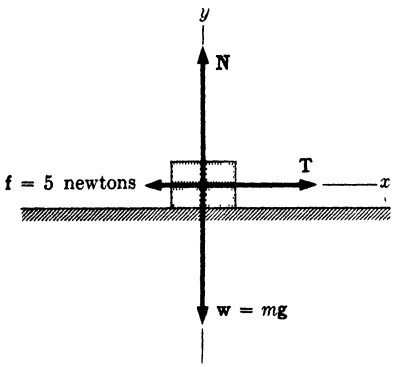


FIGURE 5–3

EXAMPLE 3. A block whose mass is 10 kgm rests on a horizontal surface. What constant horizontal force **T** is required to give it a velocity of 4 m/sec in 2 sec, starting from rest, if the friction force between the block and the surface is constant and is equal to 5 newtons? Assume that all forces act at the center of the block. (See Fig. 5–3.)

The mass of the block is given. Its y-acceleration is zero. Its x-acceleration can be found from the data on the velocity acquired in a given time. Since the forces are constant, the x-acceleration is constant and from the equations of motion with constant acceleration,

$$a_x = \frac{v - v_0}{t} = \frac{4 \text{ m/sec} - 0}{2 \text{ sec}} = 2 \frac{\text{m}}{\text{sec}^2}.$$

Hence from Newton's second law in component form,

$$\sum F_y = ma_y, \qquad N - w = 0,$$

$$N = w = mg = 10 \text{ kgm} \times 9.80 \frac{\text{m}}{\text{sec}^2} = 98 \text{ newtons},$$

$$\sum F_x = ma_x, \qquad T - 5 \text{ n} = 10 \text{ kgm} \times 2 \frac{\text{m}}{\text{sec}^2},$$

$$T = 25 \text{ newtons}.$$

EXAMPLE 4. An elevator and its load weigh a total of 1600 lb. Find the tension **T** in the supporting cable when the elevator, originally moving downward at 20 ft/sec, is brought to rest with constant acceleration in a distance of 50 ft. (See Fig. 5–4.)

The mass of the elevator is

$$m = \frac{w}{g} = \frac{1600 \text{ lb}}{32 \text{ ft/sec}^2} = 50 \text{ slugs}.$$

From the equations of motion with constant acceleration,

$$v^2 = v_0^2 + 2ay, \qquad a = \frac{v^2 - v_0^2}{2y}.$$

The initial velocity v_0 is -20 ft/sec; the velocity v is zero. If we take the origin at the point where the deceleration begins, then $y = -50$ ft. Hence

$$a = \frac{0 - (-20 \text{ ft/sec})^2}{-2 \times 50 \text{ ft}} = 4 \frac{\text{ft}}{\text{sec}^2}.$$

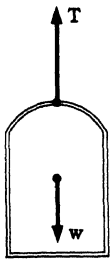


FIG. 5–4. The resultant force is T — w.

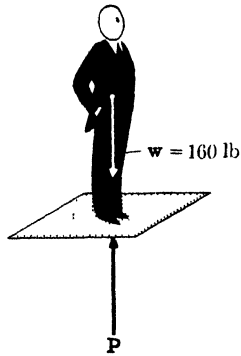


FIG. 5–5. The resultant force is P — w.

The acceleration is therefore positive (upward).

From the free-body diagram (Fig. 5–4) the resultant force is

$$\sum F = T - w = T - 1600 \text{ lb}.$$

Hence

$$\sum F = ma,$$

$$T - 1600 \text{ lb} = 50 \text{ slugs} \times 4 \frac{\text{ft}}{\text{sec}^2} = 200 \text{ lb},$$

$$T = 1800 \text{ lb}.$$

EXAMPLE 5. With what force will the feet of a passenger press downward on the elevator floor when the elevator has the acceleration above, if the passenger weighs 160 lb?

This example illustrates a problem that is frequently encountered, in which it is necessary to find a desired force by first computing the force that is the *reaction* to the one desired, and then using Newton's third law. That is, we first calculate the force with which the elevator floor pushes upward on the passenger; the force desired is the reaction to this.

Figure 5–5 shows the forces acting on the passenger. The resultant force is $P - w$, and since the mass of the passenger is 5 slugs and his acceleration is the same as that of the elevator,

$$\sum F = ma,$$

$$P - 160 \text{ lb} = 5 \text{ slugs} \times 4 \frac{\text{ft}}{\text{sec}^2} = 20 \text{ lb},$$

$$P = 180 \text{ lb}.$$

The passenger exerts an equal and opposite force downward on the elevator floor.

EXAMPLE 6. What is the acceleration of a block on a frictionless plane inclined at an angle θ with the horizontal?

The only forces acting on the block are its weight **w** and the normal force **N** exerted by the plane (Fig. 5–6). Since neither the weight nor the mass of the block is given, a letter must be used to represent one or the other. Let us call the mass m; the weight is then mg. Take axes parallel and perpendicular to the surface of the plane and resolve the weight into x- and y-components. Then

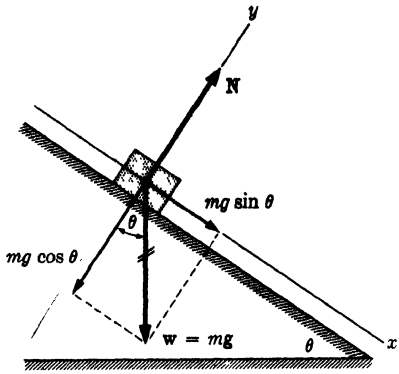


FIG. 5–6. A block on a frictionless inclined plane.

$$\sum F_y = N - mg \cos \theta,$$

$$\sum F_x = mg \sin \theta.$$

But we know that $a_y = 0$, so from the equation $\sum F_y = ma_y$ we find that $N = mg \cos \theta$. From the equation $\sum F_x = ma_x$, we have

$$mg \sin \theta = ma_x,$$

$$a_x = g \sin \theta.$$

The mass does not appear in the final result, which means that any block, regardless of its mass, will slide on a frictionless inclined plane with an acceleration down the plane of $g \sin \theta$. (Note that the *velocity* is not necessarily down the plane.)

EXAMPLE 7. Refer to Fig. 2–4 (Chapter 2). Let the mass of the block be 4 kgm and that of the rope be 0.5 kgm. If the force F_1 is 9 newtons, what are the forces F_1', F_2, and F_2'? The surface on which the block moves is level and frictionless.

We know from Newton's third law that $F_1 = F_1'$ and that $F_2 = F_2'$. Hence F_2 = 9 newtons. The force F_2 could be computed by applying Newton's second law to the block, if its acceleration were known, or the force F_2' could be computed by applying this law to the rope if its acceleration were known. The acceleration is not given, but it can be found by considering the block and rope together as a single system. The vertical forces on this system need not be considered. Since there is no friction, the resultant *external* force acting *on* the system is the force F_1. (The forces F_2 and F_2' are *internal* forces when we consider block and rope as a single system, and the force F_1' does not act on the system, but *on the man.*) Then, from Newton's second law,

$$\sum F = ma,$$

$$9\,\text{n} = (4\,\text{kgm} + 0.5\,\text{kgm}) \times a,$$

$$a = 2\,\frac{\text{m}}{\text{sec}^2}.$$

We can now apply Newton's second law to the block.

$$\sum F = ma,$$

$$F_2 = 4 \text{ kgm} \times 2 \frac{\text{m}}{\text{sec}^2} = 8 \text{ newtons}.$$

Considering the rope alone, the resultant force on it is

$$\sum F = F_1 - F_2' = 9 \text{ n} - F_2',$$

and from the second law,

$$9 \text{ n} - F_2' = 0.5 \text{ kgm} \times 2 \frac{\text{m}}{\text{sec}^2} = 1 \text{ n},$$

$$F_2' = 8 \text{ newtons}.$$

In agreement with Newton's third law, which was tacitly used when the forces F_2 and F_2' were omitted in considering the system as a whole, we find that F_2 and F_2' are equal in magnitude. Notice, however, that the forces F_1 and F_2 are *not* equal and opposite (the rope is not in equilibrium) and that these forces are *not* an action-reaction pair.

EXAMPLE 8. In Fig. 5–7, a block of weight $w_1 = 16$ lb moves on a level frictionless surface, connected by a light flexible cord passing over a small frictionless pulley to a second hanging block of weight $w_2 = 8$ lb. What is the acceleration of the system, and what is the tension in the cord connecting the two blocks?

The diagram shows the forces acting on each block. The forces exerted on the blocks by the cord can be considered an action-reaction pair, so we have used the same symbol T for each. For the block on the surface,

$$\sum F_x = T = 0.5 \text{ slug} \times a,$$

$$\sum F_y = N - w_1 = N - 16 \text{ lb} = 0.$$

For the hanging block,

$$\sum F_y = w_2 - T = 8 \text{ lb} - T$$

$$= 0.25 \text{ slug} \times a.$$

Since a is the same for both blocks, on solving the first and third equations simultaneously, we obtain

$$a = 10.7 \frac{\text{ft}}{\text{sec}^2}, \qquad T = 5.3 \text{ lb}.$$

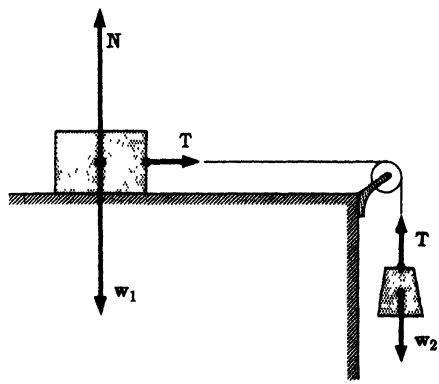


FIGURE 5–7

Note carefully that although the earth pulls on the hanging block with a force of 8 lb, the force exerted on the 16-lb block is only 5.3 lb. It is the connecting cord which pulls on the 16-lb block, and the tension in this cord must be less than 8 lb or the hanging block would not accelerate downward.

5–9 The equal-arm analytical balance. As with many other physical quantities, the mass of a body can be measured in several different ways. One is to use the relation by which the quantity is defined, which in this case is the ratio of the force on the body to its acceleration. A measured force is applied to the body, its acceleration is measured, and the unknown mass is obtained by dividing the force by the acceleration. This method, however, is both cumbersome and inexact because of the difficulty of making precise measurements on an accelerating body.

The second method consists of finding by trial some other body whose mass is (a) equal to that of the given body, and (b) already known. Consider first a method of determining when two masses are equal. It will be recalled that at the same point on the earth's surface all bodies fall freely with the same acceleration g. Since the weight w of a body equals the product of its mass m and the acceleration g, it follows that if, at the same point, the weights of two bodies are equal, their masses are equal also. The *equal-arm balance* is an instrument by means of which one can determine very precisely when the weights of two bodies are equal, and hence when their masses are equal.

The essential feature of the balance, shown schematically in Fig. 5–8, is a light, rigid beam on which are firmly mounted three equally spaced agate knife-edges, parallel to one another and perpendicular to the length of the beam. The central knife-edge at O rests on a polished plane agate plate supported from the floor of the balance case. The scale pans are

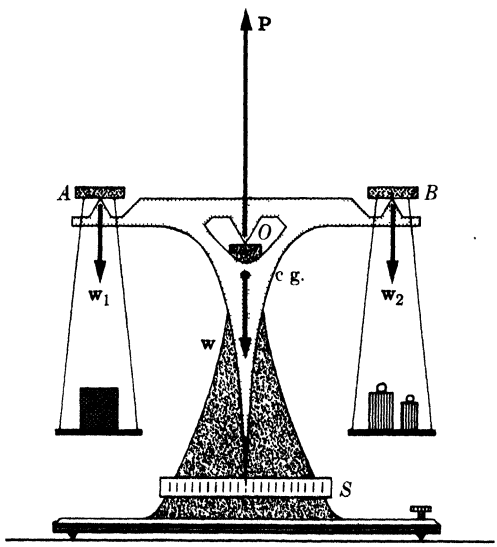


FIG. 5–8. The equal-arm analytical balance.

hung from two similar plates at A and B. A pointer fastened to the beam swings in front of the scale S. The knife-edges and plates act as practically frictionless pivots. Since the scale pans can swing freely about their supporting knife-edges, the centers of gravity of the pans and of any bodies placed on them will always be directly below the knife-edges. The weight of the beam is w, and its center of gravity is directly below the central knife-edge when the beam is horizontal. The weights of the scale pans are equal.

In using the balance, the body whose mass m_1 is desired is placed in the left scale pan, and a combination of known masses m_2 is placed in the right pan. If the combined weights of bodies and scale pans, w_1 and w_2, are exactly equal, the beam remains in stable equilibrium in a horizontal position as in Fig. 5–8, under the action of the four parallel forces $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}$, and the upward force $\mathbf{P}$ exerted at the central knife-edge. Then the weight of the unknown body equals that of the bodies in the right scale pan, and hence their masses m_1 and m_2 are equal also.

There remains the question as to how a set of bodies of known mass (usually referred to as a set of "weights") is obtained in the first place. This is also done with the equal-arm balance. Let us start with a standard kilogram and make two bodies (a) whose masses are *equal*, as determined by the balance, and (b) which when combined will just balance the standard kilogram. Each of these then has a mass of one-half kilogram, or 500 grams. Other submultiples, or multiples, of the standard can be prepared by a similar procedure.

PROBLEMS

(For problem work, use the approximate values of $g = 32$ ft/sec^2 = 9.80 m/sec^2 = 980 cm/sec^2. A force diagram should be constructed for each problem.)

5–1. (a) What resultant force is required to give an automobile weighing 1600 lb an acceleration of 8 ft/sec^2? (b) What resultant force is required to give a block whose mass is 1600 gm an acceleration of 8 cm/sec^2? (c) What resultant force is required to give a block of mass 1600 kgm an acceleration of 8 m/sec^2?

5–2. The Springfield rifle bullet weighs 150 grains (7000 grains = 1 lb), its muzzle velocity is 2700 ft/sec, and the length of the rifle barrel is 30 in. Compute the resultant force accelerating the bullet, assuming it to be constant.

5–3. A body of mass 15 kgm rests on a frictionless horizontal plane and is acted on by a horizontal force of 30 newtons. (a) What acceleration is produced? (b) How far will the body travel in 10 sec? (c) What will be its velocity at the end of 10 sec?

5–4. A body of mass 50 gm is at rest at the origin, $x = 0$, on a horizontal frictionless surface. At time $t = 0$ a force of 10 dynes is applied to the body parallel to the x-axis, and 5 sec later this force is removed. (a) What are the position and velocity of the body at $t = 5$ sec? (b) If the same force (10 dynes) is again applied at $t = 15$ sec, what are the position and velocity of the body at $t = 20$ sec?

5–5. If action and reaction are always equal in magnitude and opposite in direction, why don't they always cancel one another and leave no net force for accelerating a body?

5–6. The mass of a certain object is 10 gm. (a) What would its mass be if taken to the planet Mars? (b) Is the expression $F = ma$ valid on Mars? (c) Newton's second law is sometimes written in the form $F = wa/g$ instead of $F = ma$. Would this expression be valid on Mars? (d) If a Martian scientist hangs a standard pound body on a spring balance calibrated correctly on the earth, would the spring balance read 1 lb? Explain.

5–7. A constant horizontal force of 10 lb acts on a body on a smooth horizontal plane. The body starts from rest and is observed to move 250 ft in 5 sec. (a) What is the mass of the body? (b) If the force ceases to act at the end of 5 sec, how far will the body move in the next 5 sec?

5–8. A .22 rifle bullet, traveling at 36,000 cm/sec, strikes a block of soft wood, which it penetrates to a depth of 10 cm. The mass of the bullet is 1.8 gm. Assume a constant retarding force. (a) How long a time was required for the bullet to stop? (b) What was the decelerating force, in dynes? in lb?

5–9. An electron (mass = 9×10^{-28} gm) leaves the cathode of a radio tube with zero initial velocity and travels in a straight line to the anode, which is 1 cm away. It reaches the anode with a velocity of 6×10^8 cm/sec. If the accelerating force was constant, compute (a) the accelerating force, in dynes, (b) the time to reach the anode, (c) the acceleration. The gravitational force on the electron may be neglected.

5–10. A 5-kgm block is supported by a cord and pulled upward with an acceleration of 2 m/sec^2. (a) What is the tension in the cord? (b) After the

block has been set in motion the tension in the cord is reduced to 49 newtons. What sort of motion will the block perform? (c) If the cord is now slackened completely, the block is observed to move up 2 meters farther before coming to rest. With what velocity was it traveling?

5–11. A body of mass 10 kgm is moving with a constant velocity of 5 m/sec on a horizontal surface. The coefficient of sliding friction between body and surface is 0.20. (a) What horizontal force is required to maintain the motion? (b) If the force is removed, how soon will the body come to rest?

5–12. A hockey puck leaves a player's stick with a velocity of 30 ft/sec and slides 120 ft before coming to rest. Find the coefficient of friction between the puck and the ice.

5–13. A 16-lb block rests on a horizontal surface. The coefficient of sliding friction between block and surface is 0.25, and the coefficient of static friction is 0.30. (a) What is the resultant force on the block when an external horizontal force of 8 lb is exerted on it? (b) If the 8-lb force acts for 4 sec and is then removed, find the total distance moved by the block before coming to rest.

5–14. An elevator weighing 3200 lb rises with an acceleration of 4 ft/sec^2. What is the tension in the supporting cable?

5–15. An 8-lb block is accelerated upward by a cord whose breaking strength is 20 lb. Find the maximum acceleration which can be given the block without breaking the cord.

5–16. A block weighing 10 lb is held up by a string which can be moved up or down. What conclusions can you draw regarding magnitude and direction of the acceleration and velocity of the upper end of the string when the tension in the string is (a) 5 lb, (b) 10 lb, (c) 15 lb?

5–17. A body hangs from a spring balance supported from the roof of an elevator. (a) If the elevator has an upward acceleration of 4 ft/sec^2 and the balance reads 45 lb, what is the true weight of the body? (b) Under what circumstances will the balance read 35 lb? (c) What will the balance read if the elevator cable breaks?

5–18. A transport plane is to take-off from a level landing field with two gliders in tow, one behind the other. Each glider weighs 2400 lb, and the friction force or drag on each may be assumed constant and equal to 400 lb. The tension in the towrope between the transport plane and the first glider is not to exceed 2000 lb. (a) If a velocity of 100 ft/sec is required for the take-off, how long a runway is needed? (b) What is the tension in the towrope between the two gliders while the planes are accelerating for the take-off?

5–19. If the coefficient of friction between tires and road is 0.5, what is the shortest distance in which an automobile can be stopped when traveling at 60 mi/hr?

5–20. An 80-lb packing case is on the floor of a truck. The coefficient of static friction between the case and the truck floor is 0.30, and the coefficient of sliding friction is 0.20. Find the magnitude and direction of the friction force acting on the case (a) when the truck is accelerating at 6 ft/sec^2, (b) when it is decelerating at 10 ft/sec^2.

5–21. A block of mass 2.5 slugs rests on a horizontal surface. The coefficient of static friction between the block and the surface is 0.30, and the coefficient of sliding friction is 0.25. The block is acted upon by a variable horizontal force P. This force is initially zero and

increases with time at the constant rate of 2 lb/sec. (a) When will the block start to move? (b) What is its acceleration 8 sec after it starts to move?

5–22. A 200-gm body starts from rest and slides down a smooth inclined plane. If it travels 120 cm during the third second, what is the angle of inclination of the plane?

5–23. A balloon is descending with a constant acceleration a, less than the acceleration of gravity g. The weight of the balloon, with its basket and contents, is w. What weight, W, of ballast should be released so that the balloon will begin to be accelerated upward with constant acceleration a? Neglect air resistance.

5–24. A 64-lb block is pushed up a 37° inclined plane by a horizontal force of 100 lb. The coefficient of sliding friction is 0.25. Find (a) the acceleration, (b) the velocity of the block after it has moved a distance of 20 ft along the plane, (c) the normal force exerted by the plane. Assume that all forces act at the center of the block.

5–25. A block rests on an inclined plane which makes an angle θ with the horizontal. The coefficient of sliding friction is 0.50, and the coefficient of static friction is 0.75. (a) As the angle θ is increased, find the minimum angle at which the block starts to slip. (b) At this angle, find the acceleration once the block has begun to move. (c) How long a time is required for the block to slip 20 ft along the inclined plane?

5–26. (a) What constant horizontal force is required to drag a 16-lb block along a horizontal surface with an acceleration of 4 ft/sec^2 if the coefficient of sliding friction between block and surface is 0.5? (b) What weight, hanging from a cord attached to the 16-lb block and passing over a small frictionless pulley, will produce this acceleration?

5–27. A block weighing 8 lb resting on a horizontal surface is connected by a cord passing over a light frictionless pulley to a hanging block weighing 8 lb. The coefficient of friction between the block and the horizontal surface is 0.5. Find (a) the tension in the cord, and (b) the acceleration of each block.

5–28. A block having a mass of 2 kgm is projected up a long 30° incline with an initial velocity of 22 m/sec. The coefficient of friction between the block and the plane is 0.3. (a) Find the friction force acting on the block as it moves up the plane. (b) How long does the block move up the plane? (c) How far does the block move up the plane? (d) How long does it take the block to slide down from its position in part (c) to its starting point? (e) With what velocity does it arrive at this point? (f) If the mass of the block had been 5 kgm instead of 2 kgm, would the answers in the preceding parts be changed?

5–29. A 30-lb block on a level frictionless surface is attached by a cord passing over a small frictionless pulley to a hanging block originally at rest 4 ft above the floor. The hanging block strikes the floor in 2 sec. (a) Find the weight of the hanging block. (b) Find the tension in the string while both blocks were in motion.

5–30. Two blocks, each having mass 20 kgm, rest on frictionless surfaces as

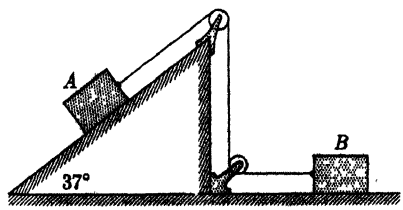
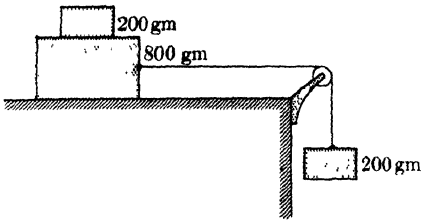


FIGURE 5–9



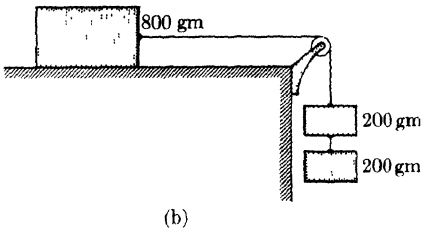


FIGURE 5–10

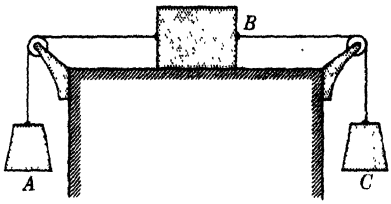


FIGURE 5–11

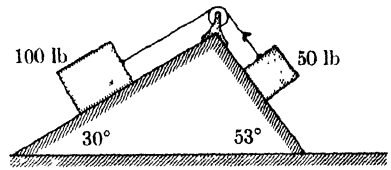


FIGURE 5–12

shown in Fig. 5–9. Assuming the pulleys to be light and frictionless, compute: (a) the time required for block A to move 1 m down the plane, starting from rest, (b) the tension in the cord connecting the blocks.

5–31. A block of mass 200 gm rests on the top of a block of mass 800 gm. The combination is dragged along a level surface at constant velocity by a hanging block of mass 200 gm as in Fig. 5–10(a). (a) The first 200-gm block is removed from the 800-gm block and attached to the hanging block, as in Fig. 5–10(b). What is now the acceleration of the system? (b) What is the tension in the cord attached to the 800-gm block in part (b) of the figure?

5–32. Block A in Fig. 5–11 weighs 3 lb and block B weighs 30 lb. The coefficient of friction between B and the horizontal surface is 0.1. (a) What is the weight of block C if the acceleration of B is 6 ft/sec^2 toward the right? (b) What is the tension in each cord when B has the acceleration stated above?

5–33. Two blocks connected by a cord passing over a small frictionless pulley rest on frictionless planes as shown in Fig. 5–12. (a) Which way will the system move? (b) What is the acceleration of the blocks? (c) What is the tension in the cord?

5–34. Two 100-gm blocks hang at the ends of a light flexible cord passing

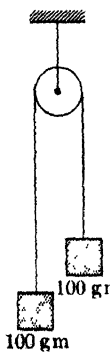


FIGURE 5–13

over a small frictionless pulley as in Fig. 5–13. A 40-gm block is placed on the block on the right, and removed after 2 sec. (a) How far will each block move in the first second after the 40-gm block is removed? (b) What was the tension in the cord before the 40-gm block was removed? after it was removed? (c) What was the tension in the cord supporting the pulley before the 40-gm block was removed? Neglect the weight of the pulley.

5–35. Two 10-lb blocks hang at the ends of a cord as in Fig. 5–13. What weight must be added to one of the blocks to cause it to move down a distance of 4 ft in 2 sec?

5–36. An 8-kgm and a 16-kgm block are suspended at opposite ends of a cord passing over a pulley. Compute (a) the acceleration of the system, (b) the tension in the cord connecting the blocks, and (c) the tension in the cord supporting the pulley. The weight of the pulley may be neglected.

5–37. A force of 10 lb is exerted horizontally against a 60-lb block, which in turn pushes a 40-lb block as in Fig. 5–14. If the blocks are on a frictionless surface, what force does one block exert on the other?

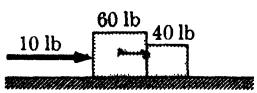


FIGURE 5–14

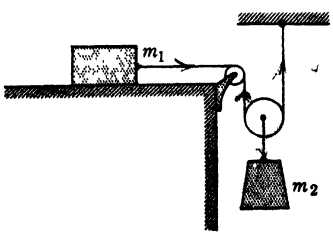


FIGURE 5–15

5–38. In terms of m_1, m_2, and g, find the accelerations of both blocks in Fig. 5–15. Neglect all friction and the masses of the pulleys.

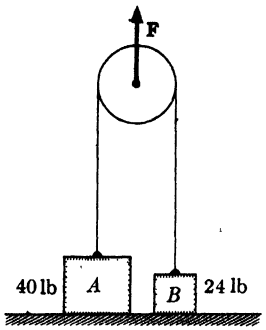


FIGURE 5–16

5–39. The bodies A and B in Fig. 5–16 weigh 40 lb and 24 lb respectively. They are initially at rest on the floor and are connected by a weightless string passing over a weightless and frictionless pulley. An upward force **F** is applied to the pulley. Find the accelerations a_1 of body A and a_2 of body B when F is (a) 24 lb, (b) 40 lb, (c) 72 lb, (d) 90 lb, (e) 120 lb.

5–40. The two blocks in Fig. 5–17 are connected by a heavy uniform rope which weighs 8 lb. An upward force of 48 lb is applied as shown. (a) What is the acceleration of the system? (b) What is the tension at the top of the 8-lb rope? (c) What is the tension at the midpoint of the rope?

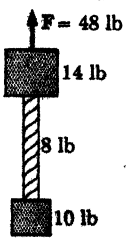


FIGURE 5–17

5–41. Two blocks, weighing 8 and 16 lb respectively, are connected by a string and slide down a 30° inclined plane, as in Fig. 5–18. The coefficient of sliding friction between the 8-lb block and the plane is 0.25, and between the 16-lb block and the plane it is 0.50. (a) Calculate the acceleration of each block. (b) Calculate the tension in the string.

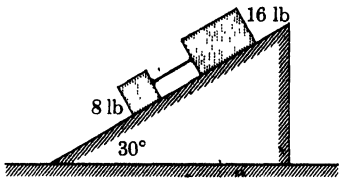


FIGURE 5–18

5–42. Two bodies weighing 10 lb and 6 lb respectively hang 4 ft above the floor from the ends of a cord 12 ft long passing over a frictionless pulley. Both bodies start from rest. Find the maximum height reached by the 6-lb body.

5–43. A man who weighs 160 lb stands on a platform which weighs 80 lb. He pulls a rope which is fastened to the platform and runs over a pulley on the ceiling. With what force does he have to pull in order to give himself and the platform an upward acceleration of 2 ft/sec^2?

5–44. What acceleration must the cart in Fig. 5–19 have in order that the block A will not fall? The coefficient of friction between the block and the cart is μ.

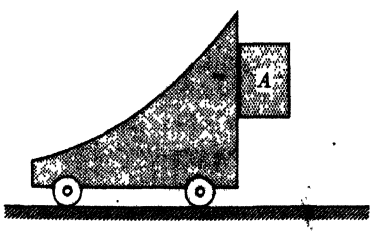


FIGURE 5–19

5–45. In an experiment using the Cavendish balance to measure the gravitational constant G, it is found that a sphere of mass 800 gm attracts another sphere of mass 4 gm with a force of 13×10^{-6} dyne when the distance between the centers of the spheres is 4 cm. The acceleration of gravity at the earth's surface is 980 cm/sec^2, and the radius of the earth is 6400 km. Compute the mass of the earth from these data.

5–46. Two spheres, each of mass 6400 gm, are fixed at points A and B (Fig. 5–20). (a) Find the magnitude and direction of the initial acceleration of a sphere of mass 10 gm if released from rest at point P and acted on only by forces of gravitational attraction of the spheres at A and B.

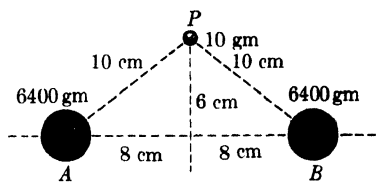


FIGURE 5–20

5–47. The mass of the moon is one eighty-first, and its radius one-fourth, that of the earth. What is the acceleration of gravity on the surface of the moon?

5–48. In round numbers, the distance from the earth to the moon is 250,000 mi, the distance from the earth to the sun is 93 million miles, the mass of the earth is 6×10^{27} gm, and the mass of the sun is 2×10^{33} gm. Approximately, what is the ratio of the gravitational pull of the sun on the moon to that of the earth on the moon?

CHAPTER 6

MOTION IN A PLANE

6–1 Motion of a projectile. Thus far we have considered only motion along a straight line, or *rectilinear motion*. In this chapter we shall discuss two common types of motion along a curved path, namely, the motion of a projectile and of a particle moving in a circle.

The term "projectile" is applied to a baseball or golf ball, a bomb released from an airplane, or a rifle bullet. The path followed by a projectile is called its *trajectory*. The trajectory is affected to a large extent by air resistance, which makes an exact analysis of the motion extremely complex. We shall, however, neglect the important effect of air resistance and assume that the motion takes place in empty space. The only force on the projectile is then its weight mg. Since the force is known, we can use Newton's second law to find the acceleration and then use the principles of kinematics to determine the velocity and position.

Newton's second law, in rectangular component form, is

$$\Sigma F_x = ma_x, \qquad \Sigma F_y = ma_y.$$

In our present problem, if the x-axis is horizontal and the y-axis vertical,

$$\Sigma F_x = 0, \qquad \Sigma F_y = -mg.$$

Therefore

$$a_x = \frac{\Sigma F_x}{m} = 0, \qquad a_y = \frac{\Sigma F_y}{m} = \frac{-mg}{m} = -g.$$

That is, the horizontal component of acceleration is zero, and the vertical component is downward and equal to that of a freely falling body. Since zero acceleration means constant velocity, the motion can be described as a combination of *horizontal motion with constant velocity* and *vertical motion with constant acceleration.*

Consider next the velocity of the projectile. In Fig. 6–1, x- and y-axes have been constructed with the origin at the point where the projectile begins its flight, i.e., just outside the muzzle of the gun or just after leaving the thrower's hand. We shall set $t = 0$ at this point. The velocity at the origin is represented by the vector $\mathbf{v}_0$, called the *initial velocity* or the *muzzle velocity* if the projectile is shot from a gun or rifle. The angle θ_0 is the *angle of departure*. The initial velocity has been resolved into horizontal and vertical components $\mathbf{v}_{0x}$ and $\mathbf{v}_{0y}$, of magnitudes

$$v_{0x} = v_0 \cos \theta_0, \qquad v_{0y} = v_0 \sin \theta_0.$$

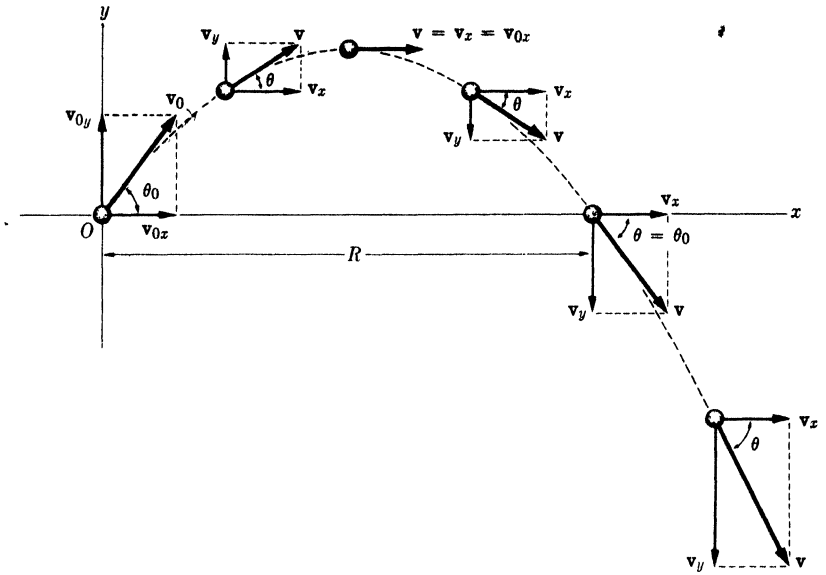


FIG. 6–1. Trajectory of a body projected with an initial velocity $\mathbf{v}_0$ at an angle of departure θ_0. The distance R is the *horizontal range.*

Since the horizontal acceleration a_x is zero, the horizontal component of velocity v_x remains constant throughout the motion, and at any later time t we have

$$v_x = v_{0x} = v_0 \cos \theta_0.$$

The vertical velocity v_y at any time, since the vertical acceleration $a_y = -g$, is

$$v_y = v_{0y} - gt = v_0 \sin \theta_0 - gt.$$

The magnitude of the resultant velocity v at any instant is

$$v = \sqrt{v_x^2 + v_y^2},$$

and the angle θ it makes with the horizontal can be found from

$$\tan \theta = \frac{v_y}{v_x}.$$

The velocity vector $\mathbf{v}$, at every point, is tangent to the trajectory. The x-coordinate at any time, since the x-velocity is constant, is

$$x = (v_0 \cos \theta_0)t, \tag{6–1}$$

The y-coordinate is

$$y = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2. \tag{6-2}$$

The two preceding expressions give the equation of the trajectory in terms of the parameter t. The equation in terms of x and y can be obtained by eliminating t. From Eq. (6-1),

$$t = \frac{x}{v_0 \cos \theta_0},$$

and substitution in Eq. (6-2) gives

$$y = \frac{v_0 \sin \theta_0}{v_0 \cos \theta_0} x - \frac{g}{2v_0 \cos \theta_0} x^2.$$

The quantities v_0, $\sin \theta_0$, $\cos \theta_0$, and g are constants, so the equation has the form

$$y = ax - bx^2,$$

which will be recognized as the equation of a *parabola.*

EXAMPLE 1. A ball is projected horizontally with a velocity v_0 of 8 ft/sec. Find its position and velocity after $\frac{1}{4}$ sec (see Fig. 6-2).

In this case, the departure angle is zero. The initial vertical velocity component is therefore zero. The horizontal velocity component equals the initial velocity and is constant.

The x- and y-coordinates, when $t = \frac{1}{4}$ sec, are

$$x = v_x t = 8 \frac{\text{ft}}{\text{sec}} \times \frac{1}{4} \text{sec} = 2 \text{ ft},$$

$$y = -\frac{1}{2}gt^2 = -\frac{1}{2} \times 32 \frac{\text{ft}}{\text{sec}^2} \times \left(\frac{1}{4} \text{sec}\right)^2 = -1 \text{ ft}.$$

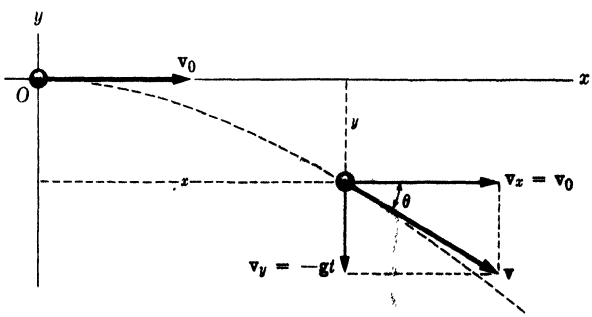


FIGURE 6-2

The components of velocity are

$$v_x = v_0 = 8\,\frac{\text{ft}}{\text{sec}},$$

$$v_y = -gt = -32\,\frac{\text{ft}}{\text{sec}^2} \times \frac{1}{4}\,\text{sec} = -8\,\frac{\text{ft}}{\text{sec}}.$$

The resultant velocity is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(8\ \text{ft/sec})^2 + (-8\ \text{ft/sec})^2} = 8\sqrt{2}\,\frac{\text{ft}}{\text{sec}}.$$

The angle θ is

$$\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{-8\ \text{ft/sec}}{8\ \text{ft/sec}} = -45°.$$

That is, the velocity is 45° *below* the horizontal.

EXAMPLE 2. In Fig. 6–2, let $v_0 = 160$ ft/sec, $\theta_0 = 53°$. Then

$$v_0 \cos\theta_0 = 160\,\frac{\text{ft}}{\text{sec}} \times 0.60 = 96\,\frac{\text{ft}}{\text{sec}},$$

$$v_0 \sin\theta_0 = 160\,\frac{\text{ft}}{\text{sec}} \times 0.80 = 128\,\frac{\text{ft}}{\text{sec}}.$$

(a) Find the position of the projectile, and the magnitude and direction of its velocity, when $t = 2.0$ sec. We have

$$x = 96\,\frac{\text{ft}}{\text{sec}} \times 2.0\,\text{sec} = 192\ \text{ft},$$

$$y = 128\,\frac{\text{ft}}{\text{sec}} \times 2.0\,\text{sec} - \frac{1}{2} \times 32\,\frac{\text{ft}}{\text{sec}^2} \times (2.0\,\text{sec})^2 = 192\ \text{ft},$$

$$v_x = 96\,\frac{\text{ft}}{\text{sec}},$$

$$v_y = 128\,\frac{\text{ft}}{\text{sec}} - 32\,\frac{\text{ft}}{\text{sec}^2} \times 2.0\,\text{sec} = 64\,\frac{\text{ft}}{\text{sec}},$$

$$\vartheta = \tan^{-1}\frac{64\ \text{ft/sec}}{96\ \text{ft/sec}} = \tan^{-1} 0.667 = 33.5°.$$

(b) Find the time at which the projectile reaches the highest point of its flight, and find the elevation of this point.

At the highest point, the vertical velocity v_y is zero. If t_1 is the time at which this point is reached,

$$v_y = 0 = 128\,\frac{\text{ft}}{\text{sec}} - 32\,\frac{\text{ft}}{\text{sec}^2} \times t_1, \qquad t_1 = 4\ \text{sec}$$

The elevation of the point is then

$$y = 128 \frac{\text{ft}}{\text{sec}} \times 4 \text{ sec} - \frac{1}{2} \times 32 \frac{\text{ft}}{\text{sec}^2} \times (4 \text{ sec})^2 = 256 \text{ ft.}$$

(c) Find the *horizontal range* R, that is, the horizontal distance from the starting point to the point at which the projectile returns to its original elevation and at which, therefore, $y = 0$. Let t_2 be the time at which this point is reached. Then

$$y = 0 = 128 \frac{\text{ft}}{\text{sec}} \times t_2 - \frac{1}{2} \times 32 \frac{\text{ft}}{\text{sec}^2} \times t_2^2.$$

This quadratic equation has two roots,

$$t_2 = 0 \qquad \text{and} \qquad t_2 = 8 \text{ sec},$$

corresponding to the two points at which $y = 0$. Evidently the time desired is the second root, $t_2 = 8$ sec, which is just twice the time to reach the highest point. The time of descent therefore equals the time of rise.

The horizontal range is

$$R = v_x t_2 = 96 \frac{\text{ft}}{\text{sec}} \times 8 \text{ sec} = 768 \text{ ft.}$$

The vertical velocity at this point is

$$v_y = 96 \frac{\text{ft}}{\text{sec}} - 32 \frac{\text{ft}}{\text{sec}^2} \times 8 \text{ sec} = -96 \frac{\text{ft}}{\text{sec}}$$

That is, the vertical velocity has the same magnitude as the initial vertical velocity, but the opposite direction. Since v_x is constant, the angle below the horizontal at this point equals the angle of departure.

(d) If unimpeded, the projectile continues to travel beyond its horizontal range. It is left as an exercise to compute the position and velocity at a time 10 sec after the start, corresponding to the last position shown in Fig. 6–2. The results are:

$$x = 960 \text{ ft}, \qquad y = -320 \text{ ft}, \qquad v_x = 96 \frac{\text{ft}}{\text{sec}}, \qquad v_y = -192 \frac{\text{ft}}{\text{sec}}.$$

For a given initial velocity, there is one particular angle of departure for which the horizontal range is a maximum. To find this angle, write the general algebraic expression for R (see the example above) as

$$R = v_x t_2 = v_0 \cos \theta_0 \times \frac{2v_0 \sin \theta_0}{g}$$

$$= \frac{2v_0^2 \sin \theta_0 \cos \theta_0}{g} = \frac{v_0^2 \sin 2\theta_0}{g}.$$

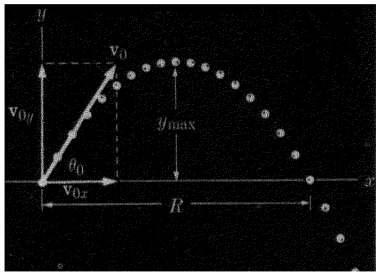


FIG. 6–3. Trajectory of a body projected at an angle with the horizontal.

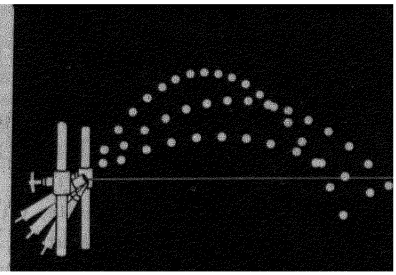


FIG. 6–4. An angle of elevation of 45° gives the maximum horizontal range.

The quantities v_0 and g are constants, so the maximum range results when $\sin 2\theta_0$ is a maximum. This occurs when $2\theta_0 = 90°$, or when $\theta_0 = 45°$.

Figure 6–3 is a multiflash photograph of the trajectory of a ball, to which have been added x- and y-axes and the initial velocity vector. The horizontal distances between consecutive positions are all equal, showing that the horizontal velocity component is constant. The vertical distances first decrease and then increase, showing that the vertical motion is accelerated.

Figure 6–4 is a composite photograph of three trajectories with departure angles of 30°, 45°, and 60°. Note that the horizontal range is the same for the angles of 30° and 60°, and is less than that for the angle of 45°.

6–2 Circular motion. Let us consider next a small body (a particle) revolving in a circle of radius R. Points P and Q in Fig. 6–5 represent two successive positions of the particle. Its displacement as it moves from P to Q is the vector $\Delta\mathbf{s}$. The *average velocity* $\bar{\mathbf{v}}$ is defined, just as in the case of rectilinear motion, as the vector $\Delta\mathbf{s}$ divided by the elapsed time Δt.

$$\bar{\mathbf{v}} = \frac{\Delta\mathbf{s}}{\Delta t}. \qquad (6\text{–}3)$$

The direction of $\bar{\mathbf{v}}$ is the same as that of $\Delta\mathbf{s}$.

The *instantaneous velocity* $\mathbf{v}$, at the point P, is defined as the limit of the average velocity when Q is taken closer

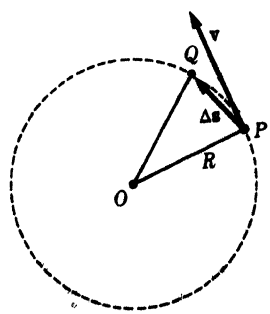


FIG. 6–5. The instantaneous velocity $\mathbf{v}$, at point P, is tangent to the circle. It is defined as the limiting ratio of $\Delta\mathbf{s}$ to Δt when Q approaches P.

and closer to P and when $\Delta\mathbf{s}$ and Δt approach zero.

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{s}}{\Delta t}. \tag{6-4}$$

As point Q approaches point P, the direction of the vector $\Delta\mathbf{s}$ approaches that of the *tangent* to the circle at P, so that *the direction of the instantaneous velocity vector is tangent to the circle.*

At this point we shall discuss only the special case in which the *magnitude* of the velocity is constant. If T is the time of a complete revolution, the magnitude of the velocity is equal to the circumference of the circle, $2\pi R$, divided by T.

$$v = \frac{2\pi R}{T}. \tag{6-5}$$

Although the magnitude of the velocity is constant, its *direction* is not. In Fig. 6-6(a), the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ represent the instantaneous velocities at P and Q. The vectors have the same length, but different directions. Again, just as in rectilinear motion, we define the *average acceleration* $\bar{\mathbf{a}}$ as the *vector change in velocity* between P and Q, divided by the elapsed time. The vector change in velocity means the *vector difference* $\mathbf{v}_2 - \mathbf{v}_1$, which we write as $\Delta\mathbf{v}$.

$$\bar{\mathbf{a}} = \frac{(\mathbf{v}_2 - \mathbf{v}_1)}{\Delta t} = \frac{\Delta \mathbf{v}}{\Delta t}. \tag{6-6}$$

Referring to Section 1-9, recall that the vector difference $\mathbf{v}_2 - \mathbf{v}_1$ can be considered as the vector *sum* of $\mathbf{v}_2$ and $-\mathbf{v}_1$, where $-\mathbf{v}_1$ is a vector of the same length as $\mathbf{v}_1$ but in the opposite direction. Since $\bar{\mathbf{a}}$ is an *average* acceleration over the time interval while the particle moves from P to Q, we construct the vector diagram for obtaining $\Delta\mathbf{v}$ at the midpoint of this interval. This has been done in Fig. 6-6(b), where the vector $\mathbf{v}_2$ has been

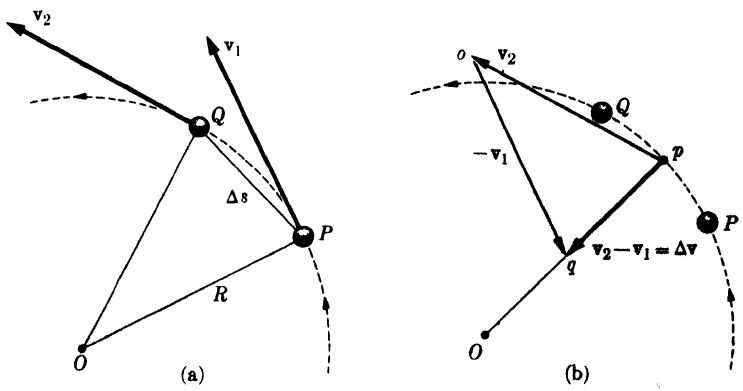


FIG. 6-6. Vector $\Delta\mathbf{v}$ is the vector change in velocity between points P and Q.

transferred to point p, midway between P and Q, and the vector difference $\Delta\mathbf{v}$ has been found by the triangle method. We see that the vector difference $\Delta\mathbf{v}$ points toward the center of the circle, and hence the average acceleration, $\Delta\mathbf{v}/\Delta t$, points in this direction also.

The magnitude of $\Delta\mathbf{v}$ can be expressed as follows. The triangles OPQ and opq in Fig. 6–6(a) and (b) are similar, since both are isosceles triangles and their long sides are mutually perpendicular. Hence if v represents the *magnitude* of either $\mathbf{v}_1$ or $\mathbf{v}_2$, we have

$$\frac{\Delta v}{v} = \frac{\Delta s}{R},$$

or

$$\Delta v = \frac{v}{R}\,\Delta s.$$

The average acceleration $\bar{a}$ is therefore

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v}{R}\,\frac{\Delta s}{\Delta t}. \tag{6–7}$$

The *instantaneous acceleration* $\mathbf{a}$ at the point P is defined in magnitude and direction as the limit of the average acceleration when Q approaches P and $\Delta\mathbf{v}$ and Δt approach zero. As this occurs, the vector $\Delta\mathbf{v}$ in Fig. 6–6(b) becomes shorter (always pointing toward the center, however), while Δt becomes smaller also. In the limit, the *direction* of the acceleration is still toward the center and its magnitude is

$$a = \lim_{\Delta t \to 0} \frac{v}{R}\,\frac{\Delta s}{\Delta t} = \frac{v}{R} \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

But

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = v,$$

so

$$\boxed{a = \frac{v^2}{R}.} \tag{6–8}$$

The magnitude of the instantaneous acceleration is therefore equal to the square of the velocity divided by the radius, and its direction is toward the center of the circle. Because of this it is called a *central*, a *centripetal*, or a *radial* acceleration. (The term "centripetal" means "seeking a center.") The units of radial acceleration are the same as those of an acceleration resulting from a change in the *magnitude* of a velocity. Thus in the engineering system the units or v^2/R are

$$\frac{(\text{ft/sec})^2}{\text{ft}} = \frac{\text{ft}}{\text{sec}^2}.$$

A centrifuge is a device for whirling an object with a high velocity. The consequent large radial acceleration is equivalent to increasing the value of g, and such processes as sedimentation, which would otherwise take place only slowly, can be greatly accelerated in this way. Very high speed centrifuges, called ultracentrifuges, have been operated at velocities as high as 180,000 rev/min, and small experimental units have been driven as fast as 1,300,000 rev/min.

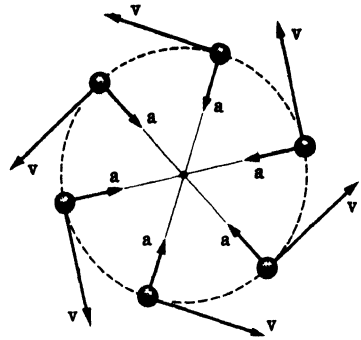


FIG. 6–7. Velocity and acceleration vectors of a particle in uniform circular motion.

Figure 6–7 shows the directions of the velocity and acceleration vectors at a number of points for a particle revolving in a circle with a velocity of constant magnitude.

EXAMPLE. The moon revolves about the earth in a circle (very nearly) of radius $R = 239{,}000$ mi or 12.6×10^8 ft, and requires 27.3 days or 23.4×10^5 sec to make a complete revolution. (a) What is the acceleration of the moon toward the earth?

The velocity of the moon is

$$v = \frac{2\pi R}{T} = \frac{2\pi \times 12.6 \times 10^8 \text{ ft}}{23.4 \times 10^5 \text{ sec}} = 3360 \frac{\text{ft}}{\text{sec}}.$$

Its radial acceleration is therefore

$$a = \frac{v^2}{R} = \frac{(3360 \text{ ft/sec})^2}{12.6 \times 10^8 \text{ ft}} = 0.00896 \frac{\text{ft}}{\text{sec}^2}.$$

(b) If the gravitational force exerted on a body by the earth is inversely proportional to the square of the distance from the earth's center, the acceleration produced by this force should vary in the same way. Therefore, if the acceleration of the moon is caused by the gravitational attraction of the earth, the ratio of the moon's acceleration to that of a falling body at the earth's surface should equal the ratio of the square of the earth's radius (3950 mi or 2.09×10^6 ft) to the square of the radius of the moon's orbit. Is this true?

The ratio of the two accelerations is

$$\frac{8.96 \times 10^{-3} \text{ ft/sec}^2}{32.2 \text{ ft/sec}^2} = 2.78 \times 10^{-4}.$$

The ratio of the squares of the distances is

$$\frac{(2.09 \times 10^6 \text{ ft})^2}{(12.6 \times 10^8 \text{ ft})^2} = 2.75 \times 10^{-4}$$

The agreement is very close, although not exact because we have used average values.

It was the calculation above which Newton made to first justify his hypothesis that gravitation was truly *universal* and that the earth's pull extended out indefinitely into space. The numerical values available in Newton's time were not highly precise. While he did not obtain as close an agreement as that above, he states that he found his results to "answer pretty nearly," and he concluded that his hypothesis was verified.

6–3 Centripetal force. Having obtained an expression for the acceleration of a particle revolving in a circle, we can now use Newton's second law to find the resultant force on the particle. Since the magnitude of the acceleration equals v^2/R, and its direction is toward the center, the magnitude of the resultant force on a particle of mass m is

$$\Sigma F = m \frac{v^2}{R}. \tag{6–9}$$

The direction of the resultant force is toward the center also, and it is called a *centripetal force.* (It is unfortunate that it has become common practice to characterize the force by the adjective "centripetal," since this seems to imply that there is some difference in nature between centripetal forces and other forces. This is not the case. Centripetal forces, like other forces, are pushes and pulls exerted by sticks and strings, or arise from the action of gravitational or other causes. The term "centripetal" refers to the *effect* of the force, that is, to the fact that it results in a change in the *direction* of the velocity of the body on which it acts, rather than a change in the *magnitude* of this velocity.)

Anyone who has ever tied an object to a cord and whirled it in a circle will realize the necessity of exerting this inward, centripetal force. If the cord breaks, the direction of the velocity ceases to change (unless other forces are acting) and the object flies off along a tangent to the circle.

EXAMPLE 1. A small body of mass 200 gm revolves in a circle on a horizontal frictionless surface, attached by a cord 20 cm long to a pin set in the surface. If the body makes two complete revolutions per second, find the force **P** exerted on it by the cord. (See Fig. 6–8.)

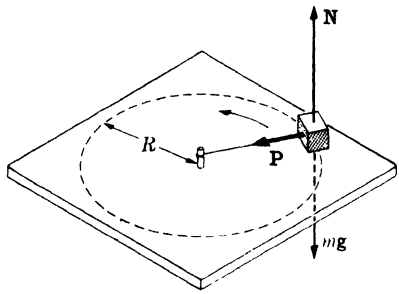


FIGURE 6–8

The circumference of the circle is $2\pi \times 20$ cm $= 40\pi$ cm, so the velocity is 80π cm/sec. The magnitude of the centripetal acceleration is

$$a = \frac{v^2}{R} = \frac{(80\pi \text{ cm/sec})^2}{20 \text{ cm}} = 3150 \frac{\text{cm}}{\text{sec}^2}.$$

Since the body has no vertical acceleration, the forces **N** and $m\mathbf{g}$ are equal and opposite and the force **P** is the resultant force. Therefore

$$\sum F = P = ma = 200 \text{ gm} \times 3150 \frac{\text{cm}}{\text{sec}^2} = 6.3 \times 10^5 \text{ dynes}.$$

EXAMPLE 2. The centripetal force retaining the moon in its orbit is the gravitational attraction exerted on it by the earth. Show that if the gravitational constant G, the radius of the moon's orbit R, and its time of revolution T are known, the mass of the earth can be computed.

Let m_E and m_M be the masses of earth and moon respectively. Then

$$\sum F = m \frac{v^2}{R},$$

$$G \frac{m_E m_M}{R^2} = m_M \frac{(2\pi R/T)^2}{R},$$

and hence

$$m_E = \frac{4\pi^2 R^3}{GT^2}.$$

EXAMPLE 3. Figure 6–9 is a view of the earth looking down on the north pole and showing a (greatly enlarged) view of a body of mass m at the equator, hanging from a spring balance. The forces on the hanging body are the force of gravitational attraction **F** and the upward pull **w** of the spring balance. Since the body is carried along by the earth's rotation, it is not in equilibrium but has a radial acceleration a_R toward the earth's center. The force **F** must therefore be slightly larger than the force **w** in order to provide the necessary centripetal force. That is, the balance reading **w** is somewhat less than the true force of gravitational attraction.

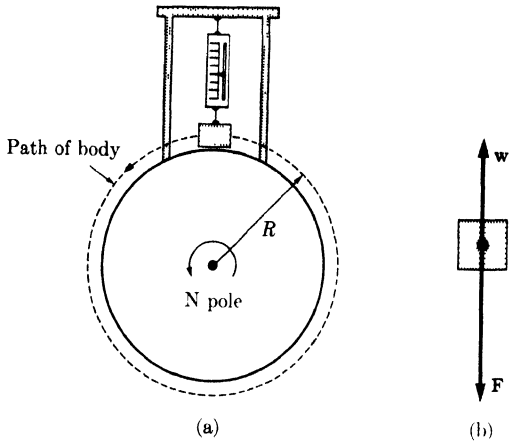


FIG. 6–9. The apparent weight is less than the force of gravitational attraction.

From Newton's law of gravitation,

$$F = G\frac{mm_E}{R^2},$$

where m_E is the mass of the earth.

From Newton's second law,

$$F - w = ma_R.$$

Replacing F by $G(mm_E/R^2)$ and w by mg, where g is the acceleration of gravity at the equator, we get, after canceling m,

$$g \text{ (equator)} = \frac{Gm_E}{R^2} - a_R.$$

At the poles, where v and a_R are zero,

$$g \text{ (poles)} = \frac{Gm_E}{R^2}.$$

This is the expression we previously derived for g in Eq. (5–6), where the rotation of the earth was neglected. We can now see that the effect of the rotation is to reduce the value of g at the equator, relative to that at the poles, by an amount equal to a_R, where for the equator

$$a_R = \frac{v^2}{R} = \frac{(2\pi R/T)^2}{R} = \frac{4\pi^2 R}{T^2}.$$

Approximate numerical values are

$$R = 6.4 \times 10^8 \text{ cm},$$
$$T = 8.64 \times 10^4 \text{ sec}.$$

Hence

$$a_R = 3.37 \frac{\text{cm}}{\text{sec}^2}.$$

Reference to Table 5–2 will show that this effect is sufficient to account for most of the difference between the observed values of g at low and high latitudes.

EXAMPLE 4. Figure 6–10 represents a small body of mass m revolving in a horizontal circle with velocity v of constant magnitude at the end of a cord of length L. As the body swings around its path, the cord sweeps over the surface of a cone. The cord makes an angle θ with the vertical, so the radius of the circle in which the body moves is $R = L \sin \theta$ and the magnitude of the velocity v equals $2\pi L \sin \theta/T$, where T is the time for one complete revolution.

The forces exerted on the body when in the position shown are its weight $m\mathbf{g}$ and the tension $\mathbf{P}$ in the cord. Let $\mathbf{P}$ be resolved into a horizontal component $P \sin \theta$ and a vertical component $P \cos \theta$. The body has no vertical acceleration, so the forces $P \cos \theta$ and mg are equal, and the resultant inward, radial, or centripetal force is the component $P \sin \theta$.

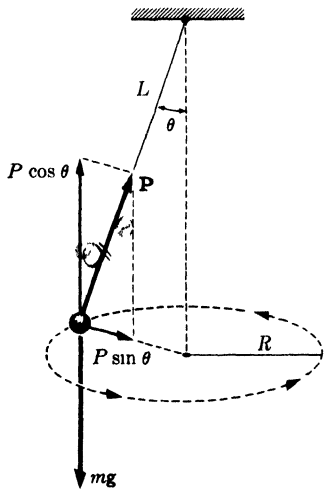


FIG. 6–10. The conical pendulum.

Then

$$P \sin \theta = m \frac{v^2}{R}, \qquad P \cos \theta = mg.$$

When the first of these equations is divided by the second, we get

$$\tan \theta = \frac{v^2}{Rg}. \tag{6–10}$$

Making use of the relations $R = L \sin \theta$ and $v = 2\pi L \sin \theta/T$, we can also write

$$\cos \theta = \frac{gT^2}{4\pi^2 L}, \tag{6–11}$$

or

$$T = 2\pi\sqrt{L \cos \theta/g}. \tag{6–12}$$

Equation (6–12) indicates how the angle θ depends on the time of revolution T and the length L of the cord. For a given length L, $\cos \theta$ *decreases* as the time is made shorter, and the angle θ *increases*. The angle never becomes 90°, however, since this requires that $T = 0$.

Equation (6–12) is similar in form to the expression for the time of swing of a simple pendulum, which will be derived in Chapter 11. Because of this similarity, the present device is called a *conical pendulum*.

Some readers may wish to add to the forces shown in Fig. 6–10 an outward, "centrifugal," force, to "keep the body out there," or to "keep it in equilibrium." ("Centrifugal" means "fleeing a center.") Let us examine this point of view. In the first place, to look for a force to "*keep* the body out there" is an example of

faulty observation, because the body doesn't stay there. A moment later it will be at a different position on its circular path. At the instant shown it is moving toward the reader, and unless a resultant force acts on it, it will, according to Newton's first law, continue to move in this direction. If an outward force *were* acting on it, equal and opposite to the inward component of the force **P**, there would be no resultant inward force to deviate it sidewise from its present direction of motion.

Those who wish to add a force to "keep the body in equilibrium" forget that the term equilibrium refers to a state of rest, or of motion *in a straight line* with constant speed. Here, the body is not moving in a straight line, but in a circle. It is not in equilibrium, but has an acceleration toward the center of the circle and must be acted on by a resultant or *un*balanced force to produce this acceleration.

EXAMPLE 5. Figure 6–11(a) represents an automobile or a railway car rounding a curve of radius R on a level road or track. The forces acting on it are its weight $m\mathbf{g}$, the normal force **N**, and the centripetal force **T**. The force **T** must be provided by friction, in the case of an automobile, or by a force exerted by the rails against the flanges on the wheels of a railway car.

In order not to have to rely on friction, or to reduce wear on the rails and flanges, the road or the track may be banked as shown in Fig. 6–11(b). The normal force **N** then has a vertical component $N \cos \theta$, and a horizontal component $N \sin \theta$ toward the center, which provides the centripetal force. The banking angle θ can be computed as follows. If v is the velocity and R the radius, then

$$N \sin \theta = \frac{mv^2}{R}.$$

Since there is no vertical acceleration,

$$N \cos \theta = mg.$$

Dividing the first equation by the second, we get

$$\tan \theta = \frac{v^2}{Rg}.$$

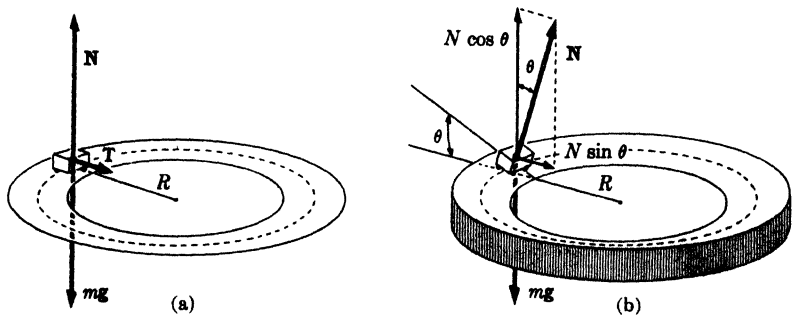


FIG. 6–11. (a) Forces on a vehicle rounding a curve on a level track. (b) Forces when the track is banked.

The tangent of the angle of banking is proportional to the square of the velocity and inversely proportional to the radius. For a given radius no one angle is correct for all velocities. Hence in the design of highways and railroads, curves are banked for the average velocity of the traffic over them.

The same considerations apply to the correct banking angle of a plane when it makes a turn in level flight.

Note that the banking angle is given by the same expression as that for the angle with the vertical made by the cord of a conical pendulum. In fact, the force diagrams in Fig. 6–10 and Fig. 6–11 are identical.

6–4 Motion in a vertical circle. Figure 6–12 represents a small body attached to a cord of length R and whirling in a vertical circle about a fixed point O to which the other end of the cord is attached. The motion, while circular, is not uniform, since the body accelerates on the way down and decelerates on the way up.

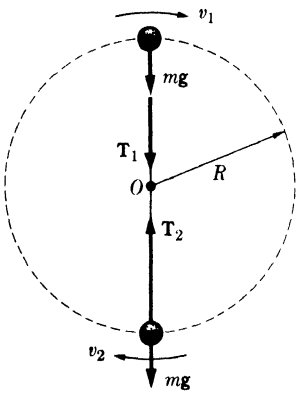


FIG. 6–12. Forces on a body moving in a vertical circle.

Let $\mathbf{v}_1$ represent the velocity of the body as it passes the highest point. The forces acting on it at this point are its weight $m\mathbf{g}$ and the tension $\mathbf{T}_1$ in the cord, both acting downward. The resultant force is $\mathbf{T}_1 + m\mathbf{g}$, and therefore at this point

$$T_1 + mg = \frac{mv_1^2}{R},$$

or

$$T_1 = \frac{mv_1^2}{R} - mg. \qquad (6\text{–}13)$$

Since by definition the centripetal force is the resultant inward force on a body in circular motion, the centripetal force in this instance is provided partly by the body's weight and partly by the tension in the cord.

Similarly, at the lowest point of the circle

$$T_2 - mg = \frac{mv_2^2}{R},$$

or

$$T_2 = \frac{mv_2^2}{R} + mg, \qquad (6\text{–}14)$$

where T_2 and v_2 represent the tension and the velocity at this point, and the upward direction has been taken as positive.

With motion of this sort, it is a familiar fact that there is a certain critical velocity below which the cord becomes slack at the highest point.

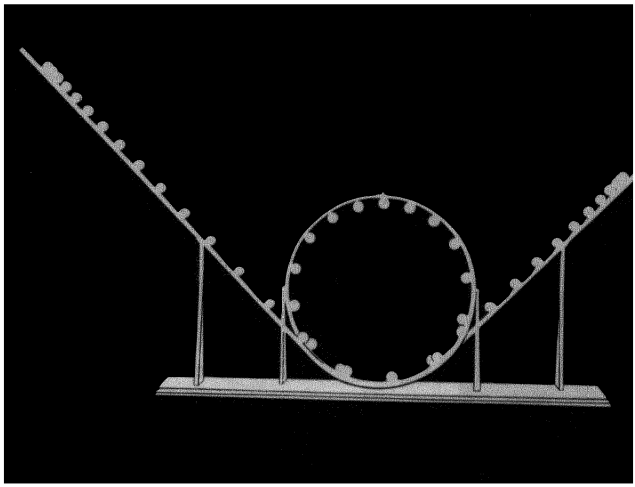


FIGURE 6–13(a)

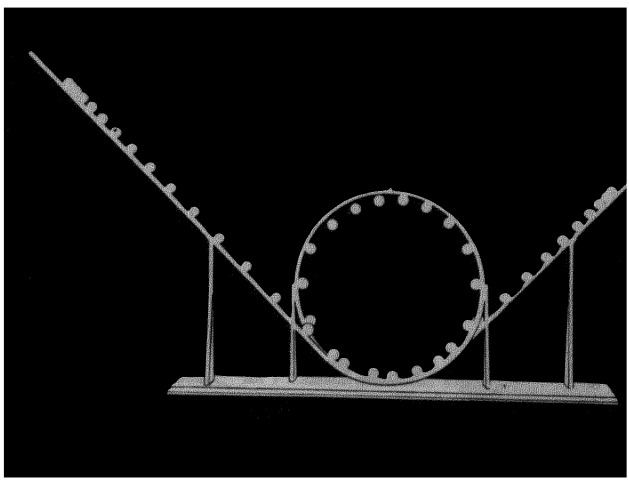


FIGURE 6–13(b)

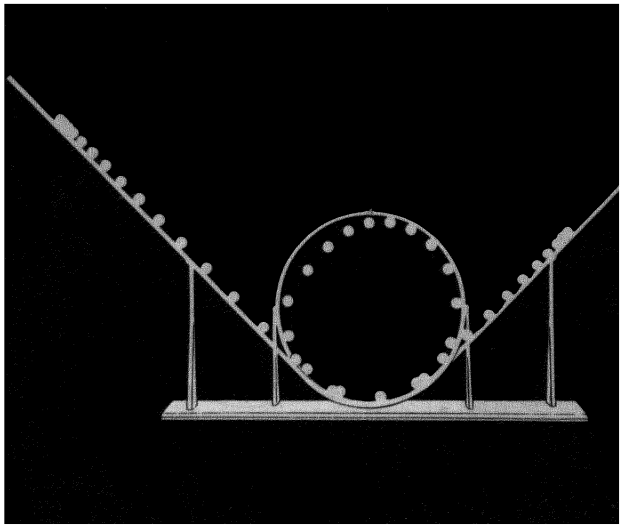


FIGURE 6–13(c)

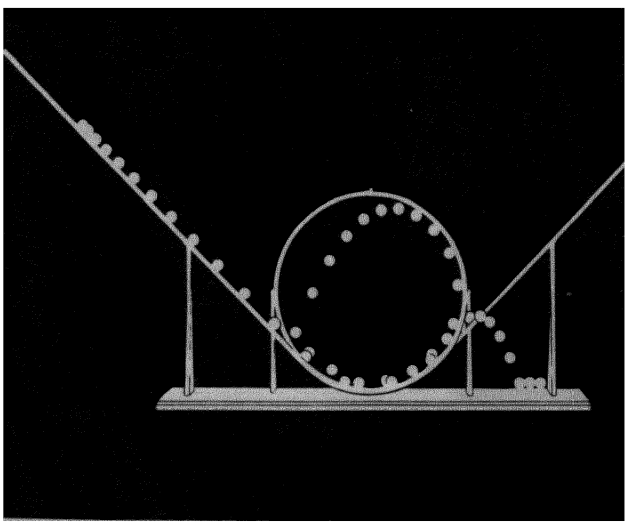


FIGURE 6–13(d)

To find this velocity, set $T_1 = 0$ in Eq. (6–13).

$$0 = \frac{mv_1^2}{R} - mg,$$

$$v_1 = \sqrt{gR}. \tag{6–15}$$

The multiflash photographs of Fig. 6–13 illustrate another case of motion in a vertical circle: a small ball "looping-the-loop" on the inside of a vertical circular track. The inward normal force exerted on the ball by the track takes the place of the tension **T** in Fig. 6–12.

In Fig. 6–13(a) the ball is released from such an elevation that its velocity at the top of the track is greater than the critical velocity, $\sqrt{gR}$. In Fig. 6–13(b) the ball starts from a lower elevation and reaches the top of the circle with a velocity such that its own weight is larger than the requisite centripetal force. In other words, the track would have to pull outward to maintain the circular motion. Since this is impossible, the ball leaves the track and moves for a short distance in a parabola. This parabola soon intersects the circle, however, and the remainder of the trip is completed successfully.

In Fig. 6–13(c) the start is made from a still lower elevation, the ball leaves the track sooner, and the parabolic path is clearly evident. In Fig. 6–13(d), while the ball eventually returns to the track, the collision is so nearly at right angles that it bounces a few times and finally rolls off

6–5 Tangential acceleration. We now take up the more general case c circular motion in which the magnitude of the velocity is not constant. Figure 6–14(a) shows a particle revolving in a circle. In moving from

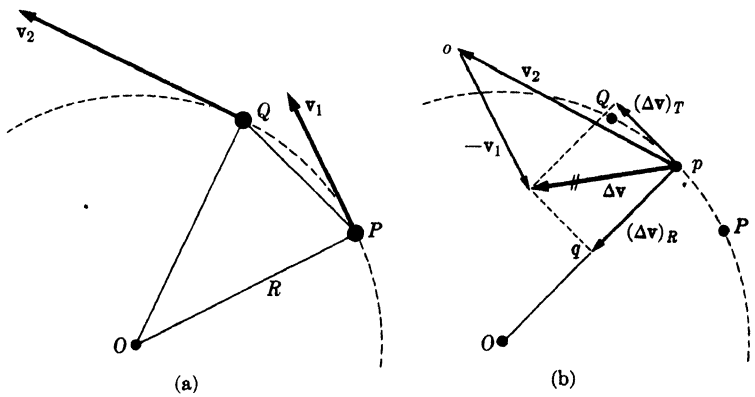


FIG. 6–14. (a) Body moving in nonuniform circular motion. (b) The vector $\Delta\mathbf{v}$ is the change in velocity between points P and Q. It has a tangential component $(\Delta\mathbf{v})_T$ and a radial component $(\Delta\mathbf{v})_R$.

P to Q, its velocity changes from $\mathbf{v}_1$ to $\mathbf{v}_2$, where $\mathbf{v}_2$ has a greater magnitude than $\mathbf{v}_1$ as well as a different direction. In part (b), the vector $\mathbf{v}_2$ has been transferred to point p, midway between P and Q, and the vector change in velocity, $\Delta\mathbf{v}$, has been found as in Fig. 6–6. The ratio $\Delta\mathbf{v}/\Delta t$ is the *average acceleration* between P and Q. Its direction is the same as that of $\Delta\mathbf{v}$, and it does not point toward the center of the circle.

The vector $\Delta\mathbf{v}$ can be resolved into a radial component $(\Delta\mathbf{v})_R$ and a tangential component $(\Delta\mathbf{v})_T$. The ratio $(\Delta\mathbf{v})_R/\Delta t$ is the radial component of the average acceleration, $\bar{\mathbf{a}}_R$, and the ratio $(\Delta\mathbf{v})_T/\Delta t$ is its tangential component, $\bar{\mathbf{a}}_T$.

$$\bar{\mathbf{a}}_R = \frac{(\Delta\mathbf{v})_R}{\Delta t}, \qquad \bar{\mathbf{a}}_T = \frac{(\Delta\mathbf{v})_T}{\Delta t}.$$

The radial and tangential components of the *instantaneous* acceleration are the limits of these ratios as Δt approaches zero or as point Q is taken closer and closer to point P. As Q approaches P, the magnitude of the component $(\Delta\mathbf{v})_T$ becomes more and more nearly equal to the difference between the magnitudes of $\mathbf{v}_1$ and $\mathbf{v}_2$, and the ratio $(\Delta\mathbf{v})_T/\Delta t$ approaches $\mathbf{a}_T$.

$$\mathbf{a}_T = \lim_{\Delta t \to 0} \frac{(\Delta\mathbf{v})_T}{\Delta t}. \tag{6–16}$$

Also, as Q approaches P, the figure opq approaches an isosceles triangle, similar to OPQ. Hence the ratio $(\Delta\mathbf{v})_R/\Delta t$ approaches the same limit as for motion with constant velocity and the magnitude of the instantaneous radial acceleration is

$$a_R = \frac{v^2}{R}. \tag{6–17}$$

The magnitude of the resultant instantaneous acceleration a is

$$a = \sqrt{a_T^2 + a_R^2}. \tag{6–18}$$

EXAMPLE. In Fig. 6–15, a small body of mass $m = 0.10$ kgm swings in a vertical circle at the end of a cord of length $R = 1.0$ m. If its velocity v is 2.0 m/sec when the cord makes an angle $\theta = 30°$ with the vertical, find (a) the radial and tangential components of its acceleration at this instant, (b) the magnitude and direction of its resultant acceleration, and (c) the tension $\mathbf{P}$ in the cord.

(a) The radial component of acceleration is

$$a_R = \frac{v^2}{R} = \frac{(2.0 \text{ m/sec})^2}{1.0 \text{ m}} = 4 \frac{\text{m}}{\text{sec}^2}.$$

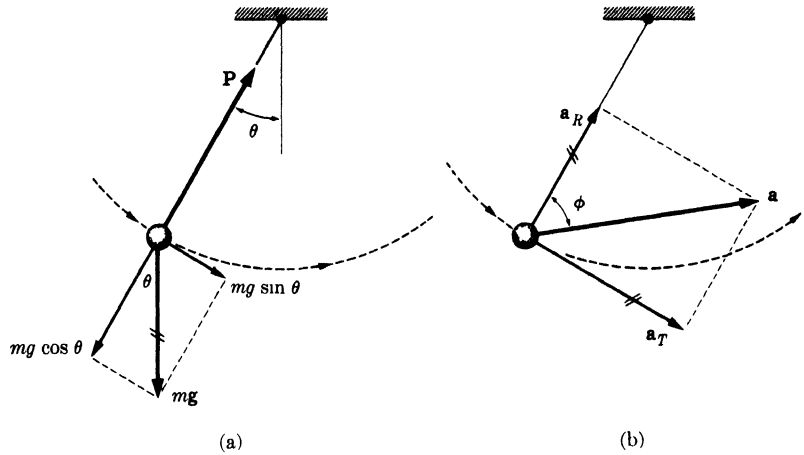


FIG. 6–15. (a) Forces on a body swinging in a vertical circle. (b) The radial and tangential components of acceleration are combined to obtain the resultant acceleration **a**.

The tangential component of acceleration can be found from Newton's second law. Let us take axes in the radial and tangential directions. Then

$$\sum F_T = ma_T, \qquad \sum F_R = ma_R,$$

where $\sum F_T$ and $\sum F_R$ are the resultant tangential and radial forces. We see from the force diagram that

$$\sum F_T = mg \sin \theta,$$

so

$$mg \sin \theta = ma_T,$$

$$a_T = g \sin \theta = 9.8 \frac{\text{m}}{\text{sec}^2} \times 0.50 = 4.9 \frac{\text{m}}{\text{sec}^2}.$$

(b) The resultant acceleration a [see Fig. 6–15(b)] is

$$a = \sqrt{(4.9)^2 + (4.0)^2} \frac{\text{m}}{\text{sec}^2} = 6.3 \frac{\text{m}}{\text{sec}^2}.$$

The angle ϕ is

$$\phi = \tan^{-1} \frac{a_T}{a_R} = \tan^{-1} \frac{4.9}{4.0} = 50.5°.$$

(c) The resultant radial force is

$$\sum F_R = P - mg \cos \theta.$$

Therefore

$$P - mg\cos\theta = ma_R,$$

$$P = m(g\cos\theta + a_R) = 1.25 \text{ newtons}$$

$$= 0.10 \text{ kgm}\left(9.8\,\frac{\text{m}}{\text{sec}^2} \times 0.87 + 4.0\,\frac{\text{m}}{\text{sec}^2}\right).$$

Note that the magnitude of the tangential acceleration is not constant but is proportional to the sine of the angle θ. Hence the equations of motion with constant acceleration cannot be used to find the velocity and displacement along the path. We show in the next chapter how the velocity can be determined quite easily, however, from energy considerations.

Problems

(In all the problems in projectile motion, let $g = 32.0$ ft/sec^2 or 980 cm/sec^2, and neglect air resistance.)

6–1. A ball rolls off the edge of a horizontal tabletop 4 ft high and strikes the floor at a distance of 5 ft horizontally from the edge of the table. What was the velocity of the ball at the instant of leaving the table?

6–2. A block slides off a horizontal tabletop 4 ft high with a velocity of 12 ft/sec. Find (a) the horizontal distance from the table at which the block strikes the floor, and (b) the horizontal and vertical components of its velocity when it reaches the floor.

6–3. A level-flight bomber, flying at 300 ft/sec, releases a bomb at an elevation of 6400 ft. (a) How long before the bomb strikes the earth? (b) How far does it travel horizontally? (c) Find the horizontal and vertical components of its velocity when it strikes.

6–4. A block passes a point 10 ft from the edge of a table with a velocity of 12 ft/sec. It slides off the edge of the table, which is 4 ft high, and strikes the floor 4 ft from the edge of the table. What was the coefficient of sliding friction between block and table?

6–5. A golf ball is driven horizontally from an elevated tee with a velocity of 80 ft/sec. It strikes the fairway 2.5 sec later. (a) How far has it fallen vertically? (b) How far has it traveled horizontally? (c) Find the horizontal and vertical components of its velocity, and the magnitude and direction of its resultant velocity, just before it strikes.

6–6. A .22 rifle bullet is fired in a horizontal direction with a muzzle velocity of 900 ft/sec. In the absence of air resistance, how far will it have dropped in traveling a horizontal distance of (a) 50 yd? (b) 100 yd? (c) 150 yd? (d) How far will it drop in one second?

6–7. A bomb is released from an airplane flying horizontally at an elevation of 1600 ft, with a velocity of 200 mi/hr. (a) How far does the bomb travel horizontally before striking the earth? (b) What will be the magnitude and direction of its velocity just before striking? (c) How long is it in the air?

6–8. A level-flight bombing plane, flying at an altitude of 1024 ft with a velocity of 240 ft/sec, is overtaking a motor torpedo boat traveling at 80 ft/sec in the same direction as the plane. At what distance astern of the boat should a bomb be released in order to hit the boat?

6–9. A bomber is making a horizontal bombing run on a destroyer from an altitude of 25,600 ft. The magnitude of the velocity of the bomber is 300 mi/hr. (a) How much time is available for the destroyer to change its course after the bombs are released? (b) If the bomber is to be shot down before its bombs can reach the ship, what is the maximum angle that the line of sight from ship to bomber can make with the horizontal? Draw a diagram, showing distances approximately to scale.

6–10. A ball is projected with an initial upward velocity component of 80 ft/sec and a horizontal velocity component of 100 ft/sec. (a) Find the position and velocity of the ball after 2 sec; 3 sec; 6 sec. (b) How long a time is required to reach the highest point of the trajectory? (c) How high is this point? (d) How long a time is required for the ball to return to its

original level? (e) How far has it traveled horizontally during this time? Show your results in a neat sketch, large enough to show all features clearly.

6–11. A batted baseball leaves the bat at an angle of 30° above the horizontal, and is caught by an outfielder 400 ft from the plate. (a) What was the initial velocity of the ball? (b) How high did it rise? (c) How long was it in the air?

6–12. A golf ball is driven with a velocity of 200 ft/sec at an angle of 37° above the horizontal. It strikes a green at a horizontal distance of 800 ft from the tee. (a) What was the elevation of the green above the tee? (b) What was the velocity of the ball when it struck the green?

6–13. If a baseball player can throw a ball a maximum distance of 200 ft over the ground, what is the maximum vertical height to which he can throw it? Assume the ball to have the same initial speed in each case.

6–14. A player kicks a football at an angle of 37° with the horizontal and with an initial velocity of 48 ft/sec. A second player standing at a distance of 100 ft from the first in the direction of the kick starts running to meet the ball at the instant it is kicked. How fast must he run in order to catch the ball before it hits the ground?

6–15. A baseball leaves the bat at a height of 4 ft above the ground, traveling at an angle of 45° with the horizontal, and with a velocity such that the horizontal range would be 400 ft. At a distance of 360 ft from home plate is a fence 30 ft high. Will the ball be a home run?

6–16. A ball A is projected from O with an initial velocity $v_0 = 700$ cm/sec in a direction 37° above the horizontal. A ball B 300 cm from O on a line 37° above the horizontal is released from rest at the instant A starts. (a) How far will B have fallen when it is hit by A? (b) In what direction is A moving when it hits B? (See Fig. 6–16.)

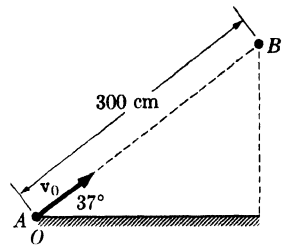


FIGURE 6–16

6–17. A projectile shot at an angle of 60° above the horizontal strikes a building 80 ft away at a point 48 ft above the point of projection. (a) Find the velocity of projection. (b) Find the magnitude and direction of the velocity of the projectile when it strikes the building.

6–18. (a) What must be the velocity of a projectile fired vertically upward to reach an altitude of 20,000 ft? (b) What velocity is required to reach the same height if the gun makes an angle of 45° with the vertical? (c) Compute the time required to reach the highest point in both trajectories. (d) How many feet would a plane traveling at 300 mi/hr move in this time?

6–19. The angle of elevation of an anti-aircraft gun is 70° and the muzzle velocity is 2700 ft/sec. For what time after firing should the fuse be set if the shell is to explode at an altitude of 5000 ft?

6–20. A projectile has an initial velocity of 80 ft/sec at an angle of 53° above the horizontal. Calculate (a) the horizontal distance from the starting point three seconds after it is fired, (b) the vertical distance above the

starting point at the same time, and (c) the horizontal and vertical components of its velocity at this time.

6–21. A trench mortar fires a projectile at an angle of 53° above the horizontal with a muzzle velocity of 200 ft/sec. A tank is advancing directly toward the mortar on level ground at a speed of 10 ft/sec. What should be the distance from mortar to tank at the instant the mortar is fired in order to score a hit?

6–22. The projectile of a trench mortar has a muzzle velocity of 300 ft/sec. (a) Find the two angles of elevation to hit a target at the same level as the mortar and 300 yd distant. (b) Compute the maximum height of each trajectory, and (c) the time of flight of each. Make a neat sketch of the trajectories, approximately to scale.

6–23. A gun fires a projectile with a muzzle velocity of 1200 ft/sec. It is desired to hit a target 1000 yd distant horizontally from the gun, and at an elevation of 980 ft above it. What is the minimum angle of elevation of the gun?

6–24. A bomber, diving at an angle of 53° with the vertical, releases a bomb at an altitude of 2400 ft. The bomb is observed to strike the ground 5 sec after its release. (a) What was the velocity of the bomber, in ft/sec? (b) How far did the bomb travel horizontally during its flight? (c) What were the horizontal and vertical components of its velocity just before striking?

6–25. A 15-lb stone is dropped from a cliff in a high wind. The wind exerts a steady horizontal 10-lb force on the stone as it falls. Is the path of the stone a straight line, a parabola, or some more complicated path? Explain.

6–26. A ball is thrown, as shown in Fig. 6–17, with an initial velocity v_0 at an angle of 37° above the horizontal, from a point 192 ft from the edge of a vertical cliff 160 ft high. The ball just misses the edge of the cliff. (a) Find the initial velocity v_0. (b) Find the distance x beyond the foot of the cliff where the ball strikes the ground.

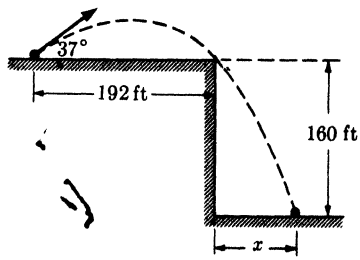


FIGURE 6–17

6–27. A man is riding on a flatcar traveling with a constant velocity of 30 ft/sec (Fig. 6–18). He wishes to throw a ball through a stationary hoop

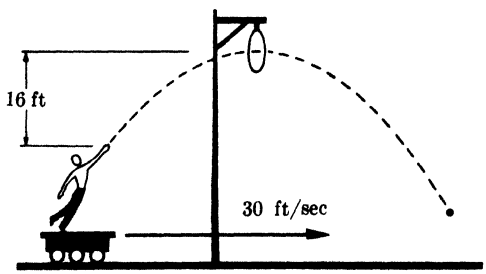


FIGURE 6–18

16 ft above the height of his hands in such a manner that the ball will move horizontally as it passes through the hoop. He throws the ball with a velocity of 40 ft/sec with respect to himself. (a) What must be the vertical component of the initial velocity of the ball? (b) How many seconds after he releases the ball will it pass through the hoop? (c) At what horizontal distance in front of the hoop must he release the ball?

6–28. At time $t = 0$ a body is moving east at 10 cm/sec. At time $t = 2$ sec it is moving 25° north of east at 14 cm/sec. Find graphically the magnitude and direction of its change in velocity and the magnitude and direction of its average acceleration.

6–29. The radius of the earth's orbit around the sun (assumed circular) is 93×10^6 miles, and the earth travels around this orbit in 365 days. (a) What is the magnitude of the orbital velocity of the earth, in mi/hr? (b) What is the radial acceleration of the earth toward the sun, in ft/sec^2?

6–30. A stone of mass 1 kgm is attached to one end of a string 1 m long, of breaking strength 500 newtons, and is whirled in a horizontal circle on a frictionless tabletop. The other end of the string is kept fixed. Find the maximum velocity the stone can attain without breaking the string.

6–31. A phonograph turntable rotates at a constant rate of 78 rev/min. It is found that a small object placed on a record will remain at rest relative to the record if its distance from the center is less than 3 in., but will slip if the distance is any greater. What is the coefficient of static friction between the object and the record?

6–32. (a) At how many revolutions per second must the apparatus of Fig. 6–19 rotate about a vertical axis in order that the cord shall make an angle of 45° with the vertical? (b) What is then the tension in the cord? (Given: $L = 20$ cm, $a = 10$ cm, $m = 200$ gm.)

6–33. The 8-lb block in Fig. 6–20 is attached to a vertical rod by means of two strings. When the system rotates about the axis of the rod the strings are extended as shown in the diagram. (a) How many revolutions per minute must the system make in order that the tension in the upper cord shall be 15 lb? (b) What is then the tension in the lower cord?

6–34. A curve of 600 ft radius on a level road is banked at the correct angle

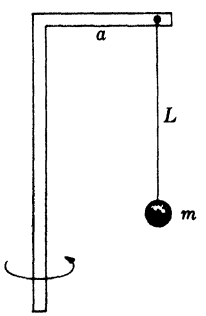


FIGURE 6–19

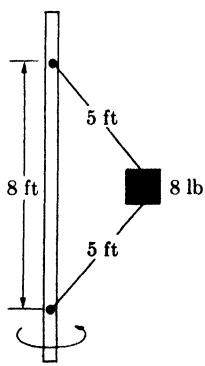


FIGURE 6–20

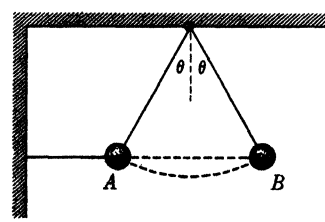


FIGURE 6–21

for a velocity of 30 mi/hr. If an automobile rounds this curve at 60 mi/hr, what is the minimum coefficient of friction between tires and road so that the automobile will not skid? Assume all forces to act at the center of gravity.

6–35. A 2400-lb automobile rounds a level curve of radius 400 ft, on an unbanked road, with a velocity of 40 mi/hr. (a) What is the minimum coefficient of friction between tires and road in order that the automobile shall not skid? (b) At what angle should the roadbed be banked for this velocity?

6–36. An aircraft in level flight is said to make a standard turn when it makes a complete circular turn in two minutes. (a) What is the banking angle in a standard turn if the velocity of the plane is 400 ft/sec? (b) What is the radius of the circle in which it turns?

6–37. The radius of the circular track in Fig. 6–13 is 40 cm and the mass of the ball is 100 gm. (a) Find the critical velocity at the highest point of the track. (b) If the actual velocity at this point is twice the critical velocity, find the force exerted by the ball against the track.

6–38. The pilot of a dive bomber who has been diving at a velocity of 400 mi/hr pulls out of the dive by changing his course to a circle in a vertical plane. (a) What is the minimum radius of the circle in order that the acceleration at the lowest point shall not exceed "7g"? (b) How much does a 180-lb pilot apparently weigh at the lowest point of the pull-out?

6–39. A bicycle and rider, weighing together 160 lb, loop-the-loop in a circular track of radius 8 ft. The velocity at the lowest point is $32\sqrt{2}$ ft/sec. (a) Find the radial acceleration at the highest point. Assume that the bicycle "coasts" without friction. (b) Show in a diagram all the forces acting on the bicycle and rider at the highest point, and compute the force with which the track pushes against the bicycle. (c) With what force does the bicycle press against the track? (d) What is the minimum velocity the bicycle can have at the highest point without leaving the track?

6–40. A cord is tied to a pail of water and the pail is swung in a vertical circle of radius 4 ft. What must be the minimum velocity of the pail at the highest point of the circle if no water is to spill from the pail?

6–41. A ball is held at rest in position A in Fig. 6–21 by two light cords. The horizontal cord is cut, and the ball swings to position B. What is the ratio of the tension in the cord, in position B, to that in position A?

6–42. What would be the length of a day if the rate of rotation of the earth were such that $g = 0$ at the equator?

6–43. Suppose one wishes to establish a space platform or satellite moving in a circle in the earth's equatorial plane and at such a height above the earth's surface that it remains always above the same point. Find the height of the space platform.

6–44. The weight of a man as determined by a spring balance at the equator is 180 lb. By how many ounces does this differ from the true force of gravitational attraction at the same point?

6–45. It is desired to launch a satellite 400 miles above the earth in a circular orbit. If suitable rockets are used to reach this elevation, what horizontal orbital velocity must be imparted to the satellite? The radius of the earth is 3950 miles.

CHAPTER 7

WORK AND ENERGY

7–1 Work. In everyday life, the word *work* is applied to any form of activity that requires the exertion of muscular or mental effort. In physics, however, the term is used in a very restricted sense. Figure 7–1 represents a body moving in a horizontal direction which we shall take as the x-axis. A constant force **F**, at an angle θ with the direction of motion, is exerted on the body by some outside agent. The work W of this force (or the work done by the force) when its point of application undergoes a displacement **s**, is defined as the product of the magnitudes of the displacement and of the component of the force in the direction of the displacement.

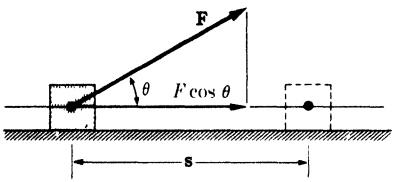


FIG. 7–1. The work done by the force **F** in a displacement **s** is $(F \cos \theta) \cdot s$.

The magnitude of the component of **F** in the direction of **s** is $F \cos \theta$. Then

$$W = F \cos \theta \cdot s. \tag{7–1}$$

Work is a *scalar* quantity, equal to the product of the *magnitudes* of **F** cos θ and **s**.

Work is done only when a force is exerted on a body while the body at the same time moves in such a way that the force has a component along the line of motion of its point of application. If the component of the force is in the *same direction* as the displacement, the work W is *positive*. If it is *opposite* to the displacement, the work is *negative*. If the force is at *right angles* to the displacement, it has no component in the direction of the displacement and the work is *zero*.

Thus when a body is lifted, the work of the lifting force is positive; when a spring is stretched, the work of the stretching force is positive; when a gas is compressed in a cylinder, again the work of the compressing force is positive. On the other hand, the work of the gravitational force on a body being lifted is negative, since the (downward) gravitational force is opposite to the (upward) displacement. When a body slides on a fixed surface, the work of the frictional force exerted *on the body* is negative since this force is always opposite to the displacement of the body. No work is done by the frictional force acting *on the fixed surface* because

there is no motion of this surface. Also, although it would be considered "hard work" to hold a heavy object stationary at arm's length, no work would be done in the technical sense because there is no motion. Even if one were to walk along a level floor while carrying the object, no work would be done since the (vertical) supporting force has no component in the direction of the (horizontal) motion. Similarly, the work of the normal force exerted on a body by a surface on which it moves is zero, as is the work of the centripetal force on a body moving in a circle.

In the engineering system, the unit of force is the pound and the unit of distance is the foot. The unit of work in this system is therefore *one foot·pound* (ft·lb). (The order of terms is interchanged to distinguish this unit from the unit of torque, the pound·foot.) *One foot·pound* may be defined as *the work done when a constant force of one pound is exerted on a body which moves a distance of one foot in the same direction as the force.*

In the mks system, where forces are expressed in newtons and distances in meters, the unit of work is the *newton·meter* (n·m). The reader can supply the definition of a newton·meter from the definition above of a foot·pound. In the cgs system the unit of work is the *dyne·centimeter* (dyne·cm). One dyne·centimeter is called *one erg;* one newton·meter is called *one joule.* There is no corresponding single term for one foot·pound.

Since 1 meter = 100 cm and 1 newton = 10^5 dynes, it follows that 1 newton·meter = 10^7 dynes·centimeters, or

$$1 \text{ joule} = 10^7 \text{ ergs.}$$

Also, from the relations between the newton and pound, and the meter and foot,

$$1 \text{ joule} = 0.7376 \text{ foot·pound},$$

$$1 \text{ foot·pound} = 1.356 \text{ joules}.$$

Figure 7–2 shows *all* the forces on the body of Fig. 7–1. (The inclined force is now represented by **P**.) The earth exerts a downward force $\mathbf{w} = m\mathbf{g}$; the surface exerts an upward force **N** and a friction force **f**. What is the work of each force when the box moves a distance s along the surface toward the right?

The component of **P** in the direction of motion is $P \cos \theta$. The work of the force **P** is therefore

$$W_P = (P \cos \theta) \cdot s.$$

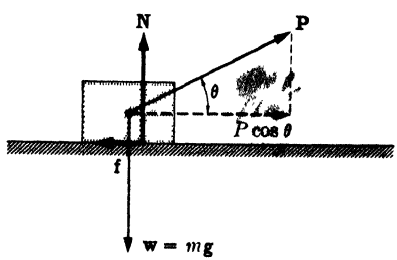


FIG. 7–2. An object on a rough horizontal surface moving to the right under the action of a force **P** inclined at an angle θ.

The forces **w** and **N** are both at right angles to the displacement. Hence

$$W_w = 0, \qquad W_N = 0.$$

The friction force **f** is opposite to the displacement, so the work of the friction force is

$$W_f = -fs.$$

Since work is a scalar quantity, the total work W of all forces on the body is the algebraic (not the vector) sum of the individual works.

$$\begin{aligned} W &= W_P + W_w + W_N + W_f \\ &= (P \cos \theta) \cdot s + 0 + 0 - f \cdot s \\ &= (P \cos \theta - f)s. \end{aligned}$$

But $(P \cos \theta - f)$ is the *resultant* force on the body. Hence *the total work of all forces is equal to the work of the resultant force.*

Suppose that $w = 100$ lb, $P = 50$ lb, $f = 15$ lb, $\theta = 37°$, and $s = 20$ ft. Then

$$W_P = (P \cos \theta) \cdot s = 50 \times 0.8 \times 20 = 800 \text{ ft·lb},$$

$$W_f = -fs = -15 \times 20 = -300 \text{ ft·lb},$$

$$\begin{aligned} W &= W_P + W_w + W_N + W_f \\ &= 800 \text{ ft·lb} + 0 + 0 - 300 \text{ ft·lb} \\ &= 500 \text{ ft·lb}. \end{aligned}$$

As a check, the total work may be expressed as

$$\begin{aligned} W &= (P \cos \theta - f) \cdot s \\ &= (40 \text{ lb} - 15 \text{ lb}) \times 20 \text{ ft} \\ &= 500 \text{ ft·lb}. \end{aligned}$$

7–2 Work done by a varying force. In the preceding section we defined the work of a *constant* force. In many important cases, however, work is done by a force which varies in magnitude or direction during the displacement of the body on which it acts. Thus when a spring is stretched slowly, the stretching force increases steadily as the spring elongates, and when a body is projected vertically upward, the gravitational force exerted on it by the earth decreases inversely with the square of its distance from the earth's center.

The work of a varying force can always be found graphically as follows. In Fig. 7–3(a), the curved line is a graph of the component $F \cos \theta$, for a force which varies in some arbitrary way (in magnitude and direction)

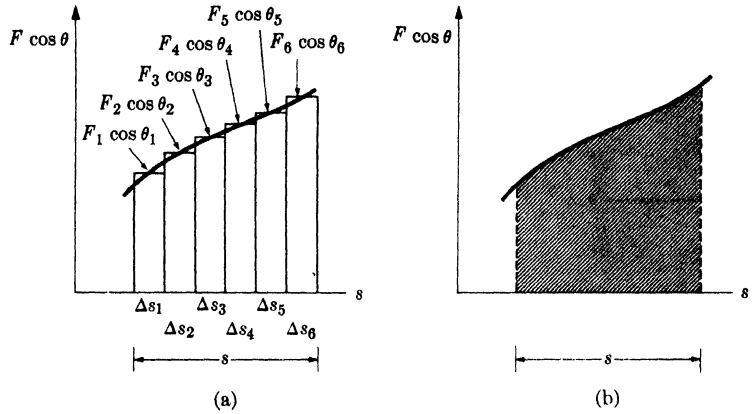


FIG. 7–3. The work of a variable force equals the shaded area in part (b).

with the distance s. Let the distance be subdivided into short segments Δs_1, Δs_2, etc., and approximate the varying force component by one that remains constant at the value $F_1 \cos \theta_1$ over the distance Δs_1, then increases to the constant value $F_2 \cos \theta_2$ over the distance Δs_2, etc., as indicated by the zigzag line. The work that would be done by the constant force $F_1 \cos \theta_1$ in the displacement Δs_1 would be $F_1 \cos \theta_1 \cdot \Delta s_1$; the work done in the displacement Δs_2 would be $F_2 \cos \theta_2 \cdot \Delta s_2$, etc. The work total W would be

$$W = F_1 \cos \theta_1 \cdot \Delta s_1 + F_2 \cos \theta_2 \cdot \Delta s_2 + \cdots .$$

But the products $F_1 \cos \theta_1 \cdot \Delta s_1$, $F_2 \cos \theta_2 \cdot \Delta s_2$, etc., are the *areas* of the various vertical strips, and the total work is evidently equal to the total area of these strips. As the subdivisions are made smaller and smaller, the total area of these strips becomes more and more nearly equal to the shaded area between the smooth curve and the horizontal axis, bounded by vertical lines at the ends of the displacement s as in Fig. 7–3(b).

Hence if a graph is constructed of the component $F \cos \theta$ of any variable force, the work of the force can be determined by computing (or measuring) the area between the curve and the displacement axis.

As an illustration of this method, let us compute the work done when a spring is stretched. To keep a spring stretched at an elongation x above its no-load length, a force **F** must be exerted at one end and a force equal in magnitude but opposite in direction at the other end, as shown in Fig. 7–4. If the elongation is not too great,

$$F = kx, \tag{7-2}$$

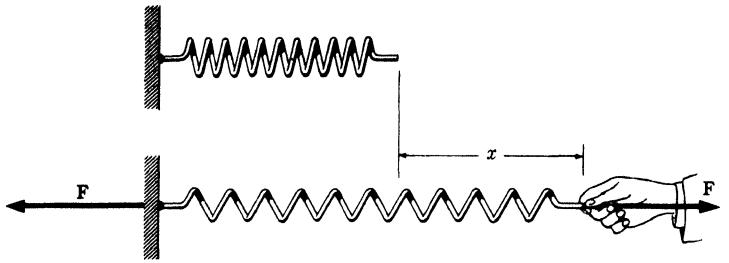


FIG. 7–4. The force to stretch a spring is proportional to its elongation; $F = kx$.

where k is a constant called the *force constant* or the *stiffness* of the spring. This direct proportion between force and elongation, for elongations that are not too great, was discovered by Robert Hooke in 1678 and is known as *Hooke's law*. It will be discussed more fully in a later chapter.

Suppose that forces equal in magnitude and opposite in direction are exerted on the ends of a spring, and that the forces are gradually increased from zero. Let one end of the spring be kept fixed. The work of the force at the fixed end will be zero, but work will be done by the varying force **F** at the moving end. This force is in the same direction as the displacement. In Fig. 7–5, the force F is plotted vertically and the displacement x of the moving end (or the elongation of the spring) is plotted horizontally. The work of the force F when the elongation increases from zero to x is equal to the shaded triangular area. The area of a triangle equals one-half the product of base and altitude. The base of the triangle is x, its altitude is kx, and its area is

$$\text{Area} = \tfrac{1}{2} \cdot x \cdot kx = \tfrac{1}{2}kx^2. \qquad (7\text{–}3)$$

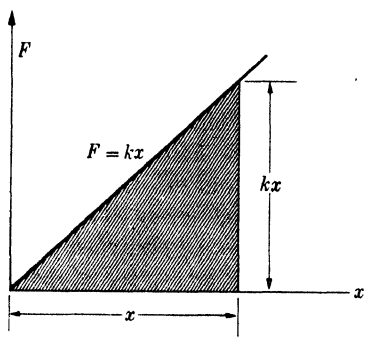


FIG. 7–5. The work done in stretching a spring is equal to the area of the shaded triangle.

The work is therefore equal to one-half the product of the force constant k and the *square* of the elongation x. When the elongation is doubled, the work increases by a factor of 4.

The reader should convince himself that the work done in changing the elongation of a spring from x_1 to x_2 is

$$W = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2.$$

7-3 Work and kinetic energy. Consider again the body illustrated in Fig. 7-2. The resultant force F on the body is $P\cos\theta - f$, and from Newton's second law,

$$F = ma.$$

Suppose the speed of the body increases from v_1 to v_2 while the body undergoes a displacement s. Then

$$v_2^2 = v_1^2 + 2as,$$

$$a = \frac{v_2^2 - v_1^2}{2s}.$$

Hence

$$F = m\,\frac{v_2^2 - v_1^2}{2s},$$

and

$$Fs = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2. \tag{7-4}$$

The product Fs is the work W of the resultant force F. The quantity $\frac{1}{2}mv^2$, one-half the product of the mass of the body and the square of its speed, is called its *kinetic energy*, E_k.

$$\boxed{E_k = \tfrac{1}{2}mv^2.} \tag{7-5}$$

The first term on the right side of Eq. (7-4) is the final kinetic energy of the body, E_{k2}, and the second term is the initial kinetic energy, E_{k1}. The difference between these terms is the *change* in kinetic energy, and we have the important result that *the work of the resultant external force on a body is equal to the change in kinetic energy of the body.* As we have shown, the work of the resultant force is equal to the algebraic sum of the works of the individual forces.

$$\boxed{W = E_{k2} - E_{k1} = \Delta E_k.} \tag{7-6}$$

Kinetic energy, like work, is a scalar quantity. The kinetic energy of a moving body depends only on its speed, or the *magnitude* of its velocity, and not on the direction in which it is moving. The *change* in kinetic energy depends only on the work $W = Fs$ and not on the individual values of F and s. That is, the force F could have been large and the displacement s small, or the reverse might have been true. If the mass m and the speeds v_1 and v_2 are known, the *work* of the resultant force can be found without any knowledge of the force F and the displacement s.

If the work W is *positive*, the final kinetic energy is greater than the initial kinetic energy and the kinetic energy *increases*. If the work is *negative*, the kinetic energy *decreases*. In the special case in which the work is *zero*, the kinetic energy remains *constant*.

Although Eq. (7–4) was derived for the special case of a *constant* resultant force, it is true even when the force varies in any arbitrary way. The work of *any* resultant force on a rigid body equals the change in kinetic energy of the body.

In computing the kinetic energy of a body, consistent units must be used for m and v. In the cgs system, m must be expressed in grams and v in cm/sec. In the mks system, m must be in kilograms and v in m/sec. In the engineering system, m must be in slugs and v in ft/sec. The corresponding units of kinetic energy are 1 gm·cm^2/sec^2, 1 kgm·m^2/sec^2, and 1 slug·ft^2/sec^2. However, the unit of kinetic energy in any system is equal to the unit of work in that system, and kinetic energy is customarily expressed in ergs, joules, or foot·pounds. That is,

$$1 \text{ gm} \frac{\text{cm}^2}{\text{sec}^2} = 1 \frac{\text{dyne}}{\text{cm/sec}^2} \frac{\text{cm}^2}{\text{sec}^2} = 1 \text{ dyne·cm} = 1 \text{ erg}.$$

In the same way,

$$1 \text{ kgm} \frac{\text{m}^2}{\text{sec}^2} = 1 \text{ joule}, \qquad 1 \text{ slug} \frac{\text{ft}^2}{\text{sec}^2} = 1 \text{ ft·lb}.$$

EXAMPLE. Refer again to the body in Fig. 7–2 and the numerical values given at the end of Section 7–1. The total work of the external forces was shown to be 500 ft·lb. Hence the kinetic energy of the body increases by 500 ft·lb. To verify this, suppose the initial speed v_1 is 4 ft/sec. The initial kinetic energy is

$$E_{k1} = \frac{1}{2} m v_1^2 = \frac{1}{2} \frac{100}{32} \text{ slugs} \times 16 \frac{\text{ft}^2}{\text{sec}^2} = 25 \text{ ft·lb}.$$

To find the final kinetic energy we must first find the acceleration.

$$a = \frac{F}{m} = \frac{40 \text{ lb} - 15 \text{ lb}}{(100/32) \text{ slugs}} = 8 \frac{\text{ft}}{\text{sec}^2}.$$

Then

$$v_2^2 = v_1^2 + 2as = 16 \frac{\text{ft}^2}{\text{sec}^2} + 2 \times 8 \frac{\text{ft}}{\text{sec}^2} \times 20 \text{ ft} = 336 \frac{\text{ft}^2}{\text{sec}^2}.$$

and

$$E_{k2} = \frac{1}{2} \frac{100}{32} \text{ slugs} \times 336 \frac{\text{ft}^2}{\text{sec}^2} = 525 \text{ ft·lb}.$$

The increase in kinetic energy is therefore 500 ft·lb.

7–4 Gravitational potential energy. Suppose a body of mass m (and of weight $\mathbf{w} = m\mathbf{g}$) moves vertically as in Fig. 7–6 from a point where its center of gravity is at a height y_1 above an arbitrarily chosen plane (the *reference level*) to a point at a height y_2. The downward gravitational force on the body is its weight $m\mathbf{g}$. Let $\mathbf{P}$ represent the resultant of all other forces acting on the body, and let W' be the work of these forces. The direction of the gravitational force $m\mathbf{g}$ is opposite to the upward displacement, and the work of this force is

$$W_{\text{grav}} = -mg(y_2 - y_1) = -(mgy_2 - mgy_1).$$

(The reader should convince himself that the work of the gravitational force is given by $-mg(y_2 - y_1)$ whether the body moves up or down.)

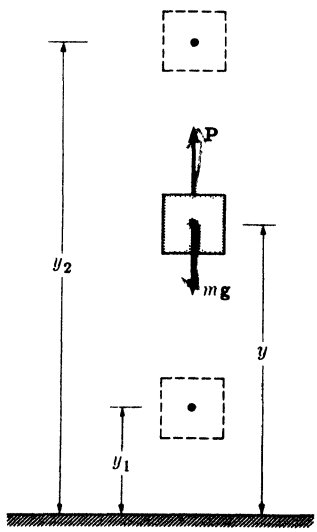


FIG. 7–6. Lifting an object in a gravitational field.

Since the total work equals the change in kinetic energy,

$$W' + W_{\text{grav}} = \Delta E_k,$$

$$W' - (mgy_2 - mgy_1) = (\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2).$$

The quantities $\frac{1}{2}mv_2^2$ and $\frac{1}{2}mv_1^2$ depend only on the final and initial *speeds;* the quantities mgy_2 and mgy_1 depend only on the final and initial *elevations.* Let us therefore rearrange this equation, transferring the quantities mgy_2 and mgy_1 from the "work" side of the equation to the "energy" side.

$$\boxed{W' = (\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2) + (mgy_2 - mgy_1).} \qquad (7\text{–}7)$$

The left side of Eq. (7–7) contains only the work of the force **P**. The terms on the right depend only on the final and initial states of the body (its speed and elevation) and not specifically on the way in which it moved. The quantity *mgy*, the product of the weight *mg* of the body and the height *y* of its center of gravity above the reference level, is called its *gravitational potential energy*, E_p.

$$\boxed{E_p \text{ (gravitational)} = mgy.} \qquad (7\text{–}8)$$

The first expression in parentheses on the right of Eq. (7–7) is the change in kinetic energy of the body, and the second is the change in its gravitational potential energy.

Equation (7–7) can also be written

$$W' = (\tfrac{1}{2}mv_2^2 + mgy_2) - (\tfrac{1}{2}mv_1^2 + mgy_1). \qquad (7\text{–}9)$$

The sum of the kinetic and potential energy of the body is called its *total mechanical energy.* The first expression in parentheses on the right of Eq. (7–9) is the final value of the total mechanical energy, and the second is the initial value. Hence, *the work of all forces acting on the body,* **with the exception of the gravitational force,** *equals the change in the total mechanical energy of the body.*

If the work W' is positive, the mechanical energy increases. If W' is negative, the energy decreases. In the special case in which W' is zero, the mechanical energy remains constant. When a quantity remains constant in a process we say that it is *conserved,* and the preceding statement is a special case of the principle of conservation of mechanical energy.

EXAMPLE 1. A man holds a ball of weight $w = \frac{1}{4}$ lb at rest in his hand. He then throws the ball vertically upward. In this process, his hand moves up

2 ft and the ball leaves his hand with an upward velocity of 48 ft/sec. Discuss the motion of the ball from the work-energy standpoint.

First consider the throwing process. Take the reference level at the initial position of the ball. Then $E_{k1} = 0$, $E_{p1} = 0$. Take point 2 at the point where the ball leaves the thrower's hand. Then

$$E_{p2} = mgy_2 = \tfrac{1}{4}\text{lb} \times 2 \text{ ft} = 0.5 \text{ ft·lb},$$

$$E_{k2} = \tfrac{1}{2}mv_2^2 = \tfrac{1}{2} \times (\tfrac{1}{4}/32) \text{ slug} \times (48 \text{ ft/sec})^2 = 9 \text{ ft·lb}.$$

Let P represent the upward force exerted on the ball by the man in the throwing process. The work W' is then the work of this force, and is equal to the sum of the changes in kinetic and potential energy of the ball.

The kinetic energy of the ball increases by 9 ft·lb and its potential energy by 0.5 ft·lb. The work W' of the upward force P is therefore 9.5 ft·lb.

If the force P is constant, the work of this force is given by

$$W' = P(y_2 - y_1),$$

and the force P is then

$$P = \frac{W'}{y_2 - y_1} = \frac{9.5 \text{ ft·lb}}{2 \text{ ft}} = 4.75 \text{ lb}.$$

However, the *work* of the force P is 9.5 ft·lb whether or not the force is constant.

Now consider the flight of the ball after it leaves the thrower's hand. In the absence of air resistance, the only force on the ball is then its weight mg. Hence the total mechanical energy of the ball remains constant. The calculations will be simplified if we take a new reference level at the point where the ball leaves the thrower's hand. Calling this point 1, we have

$$E_{k1} = 9 \text{ ft·lb}, \qquad E_{p1} = 0,$$

$$E_k + E_p = 9 \text{ ft·lb},$$

and the total mechanical energy at any point of the path equals 9 ft·lb.

Suppose we wish to find the speed of the ball at a height of 20 ft above the reference level. Its potential energy at this elevation is 5 ft·lb. (Why?) Its kinetic energy is therefore 4 ft·lb. To find its speed, we have

$$\tfrac{1}{2}mv^2 = E_k, \qquad v = \pm\sqrt{2E_k/m} = \pm 32 \frac{\text{ft}}{\text{sec}}.$$

The significance of the $\pm$ sign is that the ball passes this point *twice*, once on the way up and again on the way down. Its *potential* energy at this point is the same whether it is moving up or down. Hence its kinetic energy is the same and its *speed* is the same. The algebraic sign of the speed is + when the ball is moving up and — when it is moving down.

Next, let us find the height of the highest point reached. At this point, $v = 0$ and $E_k = 0$. Therefore $E_p = 9$ ft·lb, and the ball rises to a height of 36 ft above the point where it leaves the thrower's hand.

Finally, suppose we were asked to find the speed at a point 40 ft above the reference level. The potential energy at this point would be 10 ft·lb. But the total energy is only 9 ft·lb, so the ball never reaches a height of 40 ft.

Now suppose that a body starts at an elevation y_1 but moves an extremely small distance Δy_1 vertically upward, then is given a very small horizontal displacement, then a small vertical displacement Δy_2, then a small horizontal displacement, and so on until it has been transferred along a zigzag path to any point at a height y_2 above the reference level. The gravitational force $m\mathbf{g}$ does no work in the small horizontal steps since it is at right angles to these steps, and the total work of this force is the sum of the works in the vertical displacements only. Therefore

$$W_{\text{grav}} = -mg(\Delta y_1 + \Delta y_2 + \cdots).$$

But

$$\Delta y_1 + \Delta y_2 + \cdots = y_2 - y_1,$$

so

$$W_{\text{grav}} = -mg(y_2 - y_1).$$

By making the zigzag steps as small as we please, we may cause the body to move from any point at a height y_1 above the reference level to any other point at a height y_2 along a path which is any desired approximation to a smooth curve. Hence the work of the gravitational force will always be $-mg(y_2 - y_1)$ regardless of the nature of the path between the two points. Then if W' represents the work done by all other forces, we have again

$$W' - mg(y_2 - y_1) = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2,$$

or

$$W' = (\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2) + (mgy_2 - mgy_1).$$

Hence when the body moves along *any path whatever*, the work of all forces except the gravitational force is equal to the sum of the changes in kinetic and potential energy. In the special case in which $W' = 0$, there is no change in this sum, and the total mechanical energy remains constant.

EXAMPLE 2. A body slides down a curved track which is one quadrant of a circle of radius R, as in Fig. 7–7. If it starts from rest and there is no friction, find its speed at the bottom of the track.

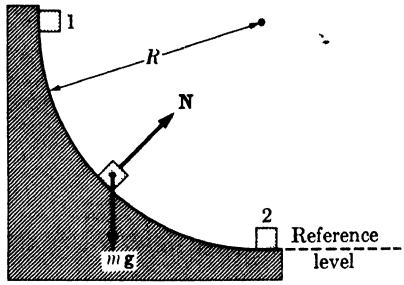


FIG. 7-7. An object sliding down a frictionless curved track.

The equations of motion with constant acceleration cannot be used, since the acceleration decreases during the motion. (The slope angle of the track becomes smaller and smaller as the body descends.) However, if there is no friction, the only force on the body in addition to its weight is the normal force **N** exerted on it by the track. The normal force does no work, so $W' = 0$ and mechanical energy is conserved. Take point 1 at the starting point and point 2 at the bottom of the track. Take the reference level at point 2. Then $y_1 = R$, $y_2 = 0$, and

$$E_{k2} + E_{p2} = E_{k1} + E_{p1},$$

$$\tfrac{1}{2}mv_2^2 + 0 = 0 + mgR,$$

$$v_2 = \pm\sqrt{2gR}.$$

The speed is therefore the same as if the body had fallen *vertically* through a height R. (What is now the significance of the $\pm$ sign?)

As a numerical example, let $R = 1$ m. Then

$$v = \pm\sqrt{2 \times 9.8(\text{m/sec}^2) \times 1\text{ m}} = \pm 4.43\ \frac{\text{m}}{\text{sec}}.$$

EXAMPLE 3. Suppose a body of mass 0.5 kgm slides down a track of radius $R = 1$ m, like that in Fig. 7-7, but its speed at the bottom is only 3 m/sec. How much work was done by the frictional force acting on the body?

In this case, $W' = W_f$, and

$$W_f = (\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2) + (mgy_2 - mgy_1)$$

$$= \left(\frac{1}{2} \times 0.5\text{ kgm} \times 9\ \frac{\text{m}^2}{\text{sec}^2} - 0\right) + \left(0 - 0.5\text{ kgm} \times 9.8\ \frac{\text{m}}{\text{sec}^2} \times 1\text{ m}\right)$$

$$= 2.25\text{ j} - 4.9\text{ j} = -2.65\text{ j}.$$

The frictional work was therefore -2.65 j, and the total mechanical energy *decreased* by 2.65 joules. The mechanical energy of a body is *not* conserved when friction forces act on it.

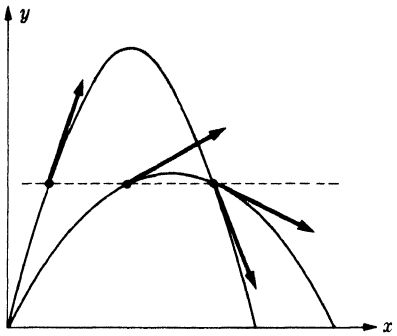


FIG. 7–8. For the same initial velocity, the speed is the same at all points at the same elevation.

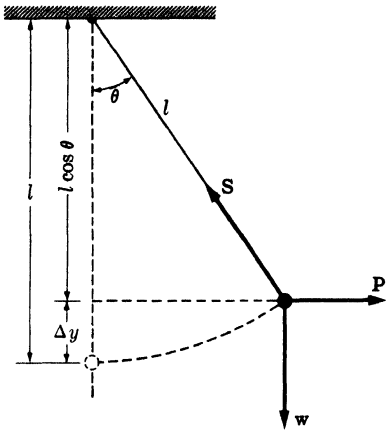


FIG. 7–9. $\Delta y = l(1 - \cos\theta)$.

EXAMPLE 4. In the absence of air resistance, the only force on a projectile is its weight, and the mechanical energy of the projectile remains constant. Figure 7–8 shows two trajectories of a projectile with the same initial velocity (and hence the same total energy) but with different angles of departure. At all points at the same elevation the potential energy is the same; hence the kinetic energy is the same and the speed is the same.

EXAMPLE 5. A small object of weight w hangs from a string of length l, as shown in Fig. 7–9. A *variable* horizontal force **P**, which starts at zero and gradually increases, is used to pull the object very slowly until the string makes an angle θ with the vertical. Calculate the work of the force **P**.

The sum of the works W' of all the forces other than the gravitational force must equal the change of kinetic energy plus the change of gravitational potential energy. Hence

$$W' = W_P + W_S = \Delta E_k + \Delta E_p.$$

Since **S** is perpendicular to the path of its point of application, $W_S = 0$; and since the body was pulled very slowly at all times, the change of kinetic energy is also zero. Hence

$$W_P = \Delta E_p = w\,\Delta y,$$

where Δy is the distance that the object has been raised. From Fig. 7–9, Δy is seen to be $l(1 - \cos\theta)$. Therefore

$$W_P = wl(1 - \cos\theta).$$

7–5 Elastic potential energy. Figure 7–10 shows a body of mass m on a level surface. One end of a spring is attached to the body, the other end being fixed. Take the origin of coordinates at the position of the body

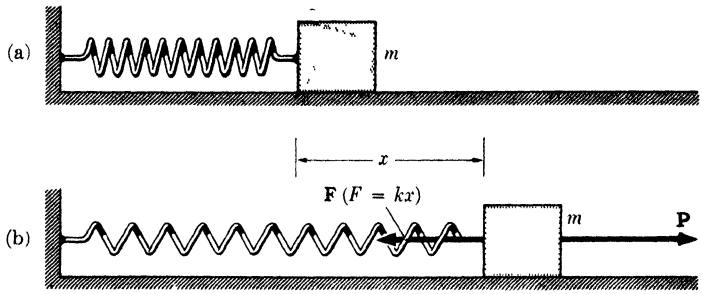


FIG. 7-10. Work done in stretching a spring.

when the spring is unstretched. Then at a position where the coordinate of the body is x, the spring exerts on the body an elastic force **F** of magnitude kx (see Section 7-2).

Let **P** represent the resultant of all other forces acting on the body, and let W' be the work of these forces when the body moves from a coordinate x_1 to a coordinate x_2. We showed in Section 7-2 that when the elongation of a spring is changed from x_1 to x_2, the work of the stretching force exerted *on* the spring is $\frac{1}{2}k(x_2^2 - x_1^2)$. In the present case, we wish to find the work W_{el} of the *reaction* to this force, that is, the force exerted *by* the spring on the body in Fig. 7-10. This will be given by $-\frac{1}{2}k(x_2^2 - x_1^2)$, since the force exerted *by* the spring is equal in magnitude but opposite in direction to that exerted *on* the spring.

$$W_{el} = -\tfrac{1}{2}k(x_2^2 - x_1^2) = -(\tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2).$$

It is easy to show by the same method as in the preceding section that if the body in Fig. 7-10 is moved along *any path whatever* in such a way that the elongation of the spring changes from x_1 to x_2, the work of the elastic force is given by the same expression, $-(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2)$.

Setting the total work equal to the change in kinetic energy of the body, we have

$$W' + W_{el} = \Delta E_k,$$

$$W' - (\tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2) = (\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2).$$

The quantities $\frac{1}{2}kx_2^2$ and $\frac{1}{2}kx_1^2$ depend only on the initial and final positions of the body and not specifically on the way in which it moved. Let us therefore transfer them from the "work" side of the equation to the "energy" side. Then

$$\boxed{W' = (\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2) + (\tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2).} \qquad (7\text{-}10)$$

The quantity $\frac{1}{2}kx^2$, one-half the product of the force constant and the square of the coordinate of the body, is called the *elastic potential energy* of the body, E_p. (The symbol E_p is used for any form of potential energy.)

$$\boxed{E_p \text{ (elastic)} = \tfrac{1}{2}kx^2.} \tag{7-11}$$

Hence the work W' of the force **P** equals the sum of the change in the kinetic energy of the body and the change in its elastic potential energy.

Equation (7–10) can also be written

$$W' = (\tfrac{1}{2}mv_2^2 + \tfrac{1}{2}kx_2^2) - (\tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kx_1^2). \tag{7-12}$$

The sum of the kinetic and potential energies of the body is its total mechanical energy and *the work of all forces acting on the body*, **with the exception of the elastic force**, *equals the change in the total mechanical energy of the body.*

If the work W' is positive, the mechanical energy increases. If W' is negative, it decreases. In the special case in which W' is zero, the mechanical energy remains constant or is *conserved.*

EXAMPLE 1. Let the force constant k of the spring in Fig. 7–10 be 24 newtons/meter, and let the mass of the body be 4 kgm. The body is initially at rest, and the spring is initially unstretched. Suppose that a constant force **P** of 10 newtons is exerted on the body, and that there is no friction. What will be the speed of the body when it has moved 0.50 meters?

The equations of motion with constant acceleration cannot be used, since the resultant force on the body varies as the spring is stretched. However, the speed can be found from energy considerations.

$$W' = \Delta E_k + \Delta E_p,$$

$$10 \text{ n} \times 0.5 \text{ m} = (\tfrac{1}{2} \times 4 \text{ kgm} \times v_2^2 - 0) + \left(\frac{1}{2} \times 24 \frac{\text{n}}{\text{m}} \times 0.25 \text{ m}^2 - 0\right),$$

$$v_2 = 1 \frac{\text{m}}{\text{sec}}.$$

EXAMPLE 2. Suppose the force **P** ceases to act when the body has moved 0.50 m. How much farther will the body move before coming to rest?

The elastic force is now the only force, and mechanical energy is conserved. The kinetic energy is $\frac{1}{2}mv^2 = 2$ joules, and the potential energy is $\frac{1}{2}kx^2 = 3$ joules. The total energy is therefore 5 joules (equal to the work of the force **P**). When the body comes to rest, its kinetic energy is zero and its potential energy is therefore 5 joules. Hence

$$\tfrac{1}{2}kx_{\max}^2 = 5 \text{ j}, \qquad x_{\max} = 0.707 \text{ m}.$$

7–6 Conservative and dissipative forces. When an object is moved from any position above a zero reference level to any other position, the work of the gravitational force is found to be independent of the path and equal to the difference between the final and the initial values of a function called the gravitational potential energy. If the gravitational force alone acts on the object, the total mechanical energy (the sum of the kinetic and gravitational potential energies) is constant or conserved, and therefore the gravitational force is called a *conservative force.* Thus, if the object is ascending, the work of the gravitational force is accomplished at the expense of the kinetic energy. If, however, the object is descending, the work of the gravitational force serves to increase the kinetic energy, or, in other words, this work is *completely recovered.* Complete recoverability is an important aspect of the work of a conservative force.

When an object attached to a spring is moved from one value of the spring extension to any other value, the work of the elastic force is also independent of the path and equal to the difference between the final and initial values of a function called the elastic potential energy. If the elastic force alone acts on the object, the sum of the kinetic and elastic potential energies is conserved, and therefore the elastic force is also a conservative force. If the object moves so as to increase the extension of the spring, the work of the elastic force is accomplished at the expense of the kinetic energy. If, however, the extension is decreasing, the work of the elastic force serves to increase the kinetic energy so that this work is completely recovered also.

To summarize, the work of a conservative force has the following properties:

(1) It is independent of the path.

(2) It is equal to the difference between the final and initial values of an energy function.

(3) It is completely recoverable.

Contrast a conservative force with a friction force exerted on a moving object by a fixed surface. The work of the friction force *depends* on the path; the longer the path, the greater the work. There is *no* function the difference in two values of which equals the work of the friction force. When we slide an object on a rough fixed surface back to its original position, the friction force reverses, and instead of recovering the work done in the first displacement we must again do work on the return trip. In other words, frictional work is *not* completely recoverable. When the friction force acts alone, the total mechanical energy is *not* conserved. The friction force is therefore called a *nonconservative* or a *dissipative* force. The mechanical energy of a body is conserved only when no dissipative forces act on it.

We find that when friction forces act on a moving body, *heat* is always developed. The more general principle of conservation of energy includes this heat as another form of energy, along with kinetic and potential energy, and when it is included the *total* energy of any system remains constant. We shall study this general conservation principle more fully in a later chapter.

EXAMPLE. Example 3 in Section 7–4 illustrates the motion of a body acted on by a dissipative, friction force. The initial mechanical energy of the body is its initial potential energy of 4.9 j. Its final mechanical energy is its final kinetic energy of 2.25 j. The frictional work W_f is −2.65 j. A quantity of heat equivalent to 2.65 j is developed as the body slides down the track. The sum of this heat and the final mechanical energy equals the initial mechanical energy, and the total energy of the system is conserved.

7–7 Internal work and internal potential energy. Thus far in this chapter we have considered only the work of the *external* forces on a *single* body. The work of these forces is called *external work*, W^e. When two (or more) bodies exert forces on each other and we wish to consider them as a single system, the forces between the bodies become *internal* forces. The work of these forces is called *internal work*, W^i.

Suppose that both external and internal forces act on the bodies of a system. The total work W of all forces, external and internal, is the sum of the works W^e and W^i and is equal to the change in the total kinetic energy of the system.

$$W = W^e + W^i = \Delta E_k.$$

EXAMPLE 1. A man stands at rest on frictionless roller skates on a level surface, facing a brick wall. He sets himself in motion (backward) by pushing against the wall. Discuss the problem from the work-energy standpoint.

The *external* forces on the man are his weight, the upward force exerted by the surface, and the horizontal force exerted by the wall. (The latter is the reaction to the force with which the man pushes against the wall.) No work is done by the first two forces because they are at right angles to the motion. No work is done by the third force because there is no motion of its point of application. The external work is therefore zero and the internal work (of the man's muscular forces) equals the change in his kinetic energy.

If any of the *external* forces on the bodies of a system are conservative (gravitational or elastic), the work of these forces can be transferred to the energy side of the work-energy equation and called the change in *external potential energy*. In many instances, the *internal* forces depend

only on the distances between pairs of bodies. The internal work then depends only on the initial and final distances between the bodies, and not on the particular way in which they moved. The work of these internal forces can then also be transferred to the energy side of the work-energy equation and called the change in *internal potential energy of the system.* Internal potential energy is a property of the system as a whole and cannot be assigned to any specific body.

Let W' represent the works of all external and internal forces that have *not* been transferred to the energy side of the work-energy equation and called changes in external and internal potential energy. Let E_p^e and E_p^i represent the external and internal potential energies. The work-energy equation then takes the form

$$W' = \Delta E_k + \Delta E_p^e + \Delta E_p^i, \tag{7-13}$$

where the total mechanical energy now includes the kinetic energy of the system and both its external and internal potential energy. If the work W' is zero, the total mechanical energy is conserved.

EXAMPLE 2. Consider a system consisting of two bodies in "outer space," very far from all other matter. No external forces act on the system, no external work is done when the bodies move, and the system has no external potential energy. A gravitational force of attraction acts between the bodies, which depends only on the distance between them. We can therefore say that the system has an internal potential energy. Suppose the bodies start from rest and accelerate toward each other. The total kinetic energy of the system increases and its internal potential energy decreases. The sum of its kinetic energy and internal potential energy remains constant.

Energy considerations *alone* do not suffice to tell us how much kinetic energy is gained by each body separately. This can be determined, however, from *momentum* considerations, as will be explained in the next chapter.

EXAMPLE 3. In Section 7-5 we computed the elastic potential energy of a body acted on by a force exerted by a spring. This force was an *external* force acting on the body, and the elastic potential energy $\frac{1}{2}kx^2$ should properly be called the *external* elastic potential energy *of the body.*

Let us now consider the spring itself. The spring is a *system,* composed of an enormous number of molecules which exert internal forces on one another. When the spring is stretched, the distances between its molecules change. The intermolecular forces depend only on the distances between pairs of molecules, so that a stretched spring has an internal elastic potential energy. To calculate this, suppose the spring is slowly stretched from its no-load length by equal and opposite forces applied at its ends, as described in Section 7-2. The work of these forces was shown to equal $\frac{1}{2}kx^2$, where x is the elongation of the spring above its no-load length. Let us retain this work on the left side of the work-

energy equation so that it becomes the work W' in Eq. (7–13). There is no change in the kinetic energy of the spring and no change in its external potential energy. If we call the internal potential energy zero when $x = 0$, the *change* in internal potential energy in the stretching process equals the final potential energy E_p'. Then

$$W' = \Delta E_p^i = E_p',$$

$$E_p' = \tfrac{1}{2}kx^2.$$

Hence the *internal elastic potential energy of a stretched spring* is equal to $\frac{1}{2}kx^2$. Note that the internal potential energy can be calculated without any detailed information regarding the intermolecular forces.

EXAMPLE 4. A block of mass m, initially at rest, is dropped from a height h onto a spring whose force constant is k. Find the maximum distance y that the spring will be compressed. See Fig. 7–11.

This is a process for which the principle of the conservation of mechanical energy holds. At the moment of release, the kinetic energy is zero. At the moment when maximum compression occurs, there is also no kinetic energy. Hence, the loss of gravitational potential energy of the block equals the gain of elastic potential energy of the spring. As shown in Fig. 7–11, the total fall of the block is $h + y$, whence

$$mg(h + y) = \tfrac{1}{2}ky^2,$$

or

$$y^2 - \frac{2mg}{k}\,y - \frac{2mgh}{k} = 0.$$

Therefore

$$y = \frac{1}{2}\left[\frac{2mg}{k} \pm \sqrt{(2mg/k)^2 + (8mgh/k)}\right].$$

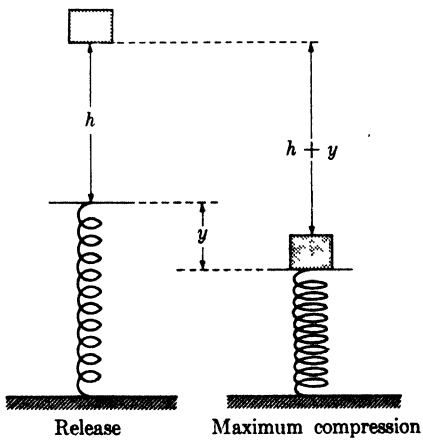


FIG. 7–11. The total fall of the block is $h + y$.

7-8 Power. The time element is not involved in the definition of work. The same amount of work is done in raising a given weight through a given height whether the work is done in one second, or one hour, or one year. In many instances, however, it is necessary to consider the *rate* at which work is done as well as the total amount of work accomplished. The rate at which work is done by a working agent is called the *power* developed by that agent.

If a quantity of work W is done in a time interval t, the average power $\bar{P}$ is defined as

$$\text{Average power} = \frac{\text{work done}}{\text{time interval}}, \qquad \bar{P} = \frac{W}{t}.$$

If the rate of doing work is not uniform, the power at any instant is the ratio of the work done to the time interval, when both are extremely small.

$$\boxed{\text{Instantaneous power } P = \frac{\Delta W}{\Delta t}.} \tag{7-14}$$

In the engineering system, where work is expressed in foot·pounds and time in seconds, the unit of power is 1 foot·pound per second. Since this unit is inconveniently small, a larger unit called the *horsepower* (hp) is in common use. 1 hp = 550 ft·lb/sec = 33,000 ft·lb/min. That is, a 1-hp motor running at full load is doing 33,000 ft·lb of work every minute it runs.

The mks unit of power is 1 joule per second, which is called one *watt*. This is also an inconveniently small unit, and power is more commonly expressed in *kilowatts* (1 kw = 1000 watts = 1000 joules/sec) or *megawatts* (1 megawatt = 1000 kw = 1,000,000 watts).

The cgs power unit is 1 erg per second. No single term is assigned to this unit.

A common misconception is that there is something inherently *electrical* about a watt or a kilowatt. This is not the case. It is true that electrical power is usually expressed in watts or kilowatts, but the power consumption of an incandescent lamp could equally well be expressed in horsepower, or an automobile engine rated in kilowatts.

From the relations between the newton, pound, meter, and foot, it is easy to show that 1 hp = 746 watts = 0.746 kw, or about $\frac{3}{4}$ of a kilowatt. This is a useful figure to remember.

Having defined two units of power, the horsepower and the kilowatt, we may use these in turn to define two new units of work, the *horsepower·hour* and the *kilowatt·hour* (kwh).

One horsepower·hour is the work done in one hour by an agent working at the constant rate of one horsepower.

Since such an agent does 33,000 ft·lb of work each minute, the work done in one hour is $60 \times 33{,}000 = 1{,}980{,}000$ ft·lb.

$$1 \text{ horsepower·hour} = 1.98 \times 10^6 \text{ foot·pounds.}$$

One kilowatt·hour is the work done in one hour by an agent working at the constant rate of one kilowatt.

Since such an agent does 1000 joules of work each second, the work done in one hour is $3600 \times 1000 = 3{,}600{,}000$ joules.

$$1 \text{ kilowatt·hour} = 3.6 \times 10^6 \text{ joules.}$$

Note carefully that the horsepower·hour and the kilowatt·hour are units of *work*, not power.

One aspect of work or energy which may be pointed out here is that although it is an abstract physical quantity, it nevertheless has a monetary value. A pound of force or a foot per second of velocity are not things which are bought and sold as such, but a foot·pound or a kilowatt·hour of energy are quantities offered for sale at a definite market rate. In the form of electrical energy, a kilowatt·hour can be purchased at a price varying from a few tenths of a cent to a few cents, depending on the locality and the quantity purchased. In the form of heat, 778 ft·lb (one Btu) costs about a thousandth of a cent.

7–9 Power and velocity. Suppose a constant force **F** is exerted on a body while the body undergoes a displacement **x** in the direction of the force. The work done is

$$W = Fx,$$

and the average power developed is

$$\bar{P} = \frac{W}{t} = F\frac{x}{t}. \tag{7–15}$$

But x/t is the average velocity, $\bar{v}$. Hence

$$\bar{P} = F\bar{v}.$$

If the time interval is made extremely short, Eq. (7–15) reduces to

$$P = F\frac{\Delta x}{\Delta t}$$

or

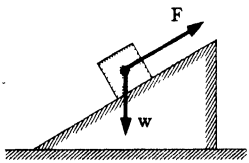
$$\boxed{P = Fv,} \tag{7-16}$$

where P and v are instantaneous values.

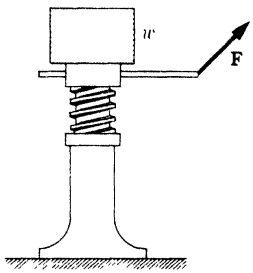
EXAMPLE. A locomotive traveling at 50 ft/sec exerts a draw-bar pull of 20,000 lb. What horsepower does it develop?

$$P = Fv = 20{,}000 \times 50 = 1{,}000{,}000 \frac{\text{ft·lb}}{\text{sec}} = \frac{1{,}000{,}000}{550} = 1820 \text{ hp.}$$

7-10 Simple machines. A machine is a force-multiplying device. Its purpose is not to convert one form of energy into another, but merely to exert a force on an object which is different from (usually larger than) the force which is applied on the machine from the outside. Simple machines



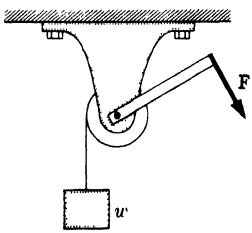
(a) Inclined plane



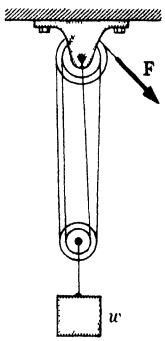
(b) Screw jack



(c) Lever



(d) Crank and axle



(e) Pulley system

FIG. 7-12. Simple machines.

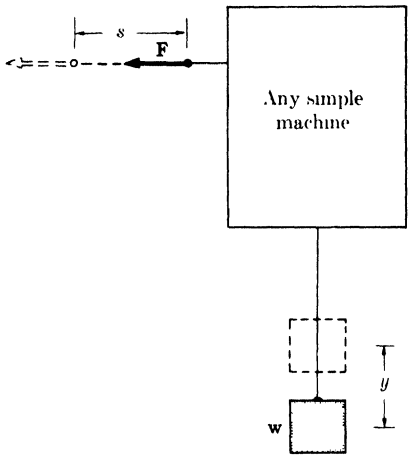


FIG. 7–13. Schematic diagram of a machine. The applied force **F** acts through a distance s, and the body of weight **w** is lifted a height y.

such as an inclined plane, a screw jack, a lever, a crank and axle, or a pully system, shown in Fig. 7–12, play an important role in everyday life. Any one of these machines may be symbolized by the diagram of Fig. 7–13, where a body of weight **w** is imagined to be lifted a height y by the application of a force **F** acting through a distance s.

The force multiplication factor of a machine is expressed by the ratio w/F, known as the *actual mechanical advantage* R_A. Thus

$$R_A = \frac{w}{F}.$$

The work input to the machine (supplied by the external agent responsible for the applied force F) is Fs, whereas the work output of the machine is wy. In general, the work output is less than the work input because of friction. Hence

$$Fs = wy + W_f,$$

where W_f is the work done against friction in the machine. The *ideal mechanical advantage* R_I which the machine would have if there were no friction may be calculated from this equation by setting $W_f = 0$ and solving for w/F, whence

$$R_I = \frac{s}{y}.$$

The efficiency E of the machine is defined as the work output divided by the work input, or

$$E = \frac{wy}{Fs} = \frac{w/F}{s/y},$$

or

$$\boxed{E = \frac{R_A}{R_I}.} \tag{7-17}$$

The ideal mechanical advantage of a machine, i.e., the ratio of s to y, may usually be calculated from simple geometrical considerations. If, therefore, w and F are measured, the efficiency of the machine may be computed from Eq. (7-17).

7-11 Mass and energy. While the mass of a body can ordinarily be considered constant, there is ample experimental evidence that it is actually a function of the velocity of the body, increasing with increasing velocity according to the relation

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where m_0 is the "rest mass" of the body, c is the velocity of light, and v the velocity of the body.

This equation was predicted by Lorentz and Einstein on theoretical grounds based on relativity considerations, and it has been directly verified by experiments on rapidly moving electrons and ions. The increase in mass is not appreciable until the velocity approaches that of light, and therefore it ordinarily escapes detection.

When one calculates the work done in setting a body in motion and takes into account the variation of mass with velocity, the expression for the work, which is equal to the kinetic energy E_k, is

$$\boxed{E_k = mc^2 - m_0c^2 = (m - m_0)c^2.} \tag{7-18}$$

That is, the kinetic energy equals the increase in mass over the rest mass, multiplied by the square of the velocity of light. This is the famous Einstein relation between mass and energy. The kinetic energy is in ergs if m is in grams and c is in cm/sec.

When the velocity is small compared with the velocity of light, the expression above reduces to the familiar form $\frac{1}{2}mv^2$. Let us write it

$$E_k = m_0c^2(1 - \beta^2)^{-1/2} - m_0c^2,$$

where, for brevity, we define a new variable β by the relation

$$\beta = \frac{v}{c}.$$

Now expand $(1 - \beta^2)^{-1/2}$ by the binomial theorem.

$$(1 - \beta^2)^{-1/2} = 1 + \tfrac{1}{2}\beta^2 + \text{terms in } \beta^4, \beta^6, \text{ etc.}$$

Since by hypothesis β is a small quantity, higher powers than the second can be neglected. Then

$$E_k = m_0c^2(1 + \tfrac{1}{2}\beta^2) - m_0c^2 = m_0c^2 \times \tfrac{1}{2}\beta^2 = \tfrac{1}{2}m_0v^2,$$

and at small velocities the kinetic energy equals one-half the product of the rest mass and the square of the velocity.

We shall illustrate the general form of the kinetic energy equation by an example taken from the field of nuclear physics. When the nucleus of a lithium atom is struck by a rapidly moving proton (the nucleus of a hydrogen atom), a momentary union of the two nuclei takes place, after which the compound nucleus breaks up into two alpha particles. (Alpha particles are the nuclei of helium atoms.) The alpha particles recoil in almost opposite directions and move initially with very high velocities. Their combined kinetic energy is much greater than the kinetic energy of the original proton. The source of this kinetic energy is the so-called "binding energy" of the nuclear particles, which is a form of internal potential energy. That is, the internal potential energy of the assemblage of protons and neutrons that makes up the unstable composite nucleus is larger than the potential energy when the same number of particles are combined in the form of two helium nuclei. A crude analogy is that of two masses forced apart by a compressed spring, but tied together by a cord. If the cord is cut, the internal potential energy of the spring is transformed into kinetic energy of the recoiling masses.

The rest mass of a proton is 1.6715×10^{-24} gm. The rest mass of a lithium nucleus is 11.6399×10^{-24} gm, and that of an alpha particle is 6.6404×10^{-24} gm. Although the proton is moving when it collides with the lithium nucleus, its velocity is not great and we may assume its mass equal to its rest mass. Hence the mass of the original system is

$$(1.6715 + 11.6399) \times 10^{-24} \text{ gm} = 13.3114 \times 10^{-24} \text{ gm}.$$

The rest mass of the two alpha particles is

$$2 \times 6.6404 \times 10^{-24} \text{ gm} = 13.2808 \times 10^{-24} \text{ gm}.$$

The alpha particles must therefore be traveling with such velocity that their (combined) masses are increased from 13.2808×10^{-24} gm to 13.3114×10^{-24} gm. Then from Eq. (7-18) their (combined) kinetic

energy is

$$E_k = (13.3114 - 13.2808) \times 10^{-24}\,\text{gm} \times \left(3 \times 10^{10}\,\frac{\text{cm}}{\text{sec}}\right)^2$$
$$= 2.75 \times 10^{-5}\,\text{erg}.$$

Let us compare this with the energy released in a typical chemical reaction. When 2 moles of hydrogen combine with 1 mole of oxygen to form 2 moles of water in the reaction

$$2H_2 + O_2 = 2H_2O,$$

116,000 calories are released. This energy is shared among the H_2O molecules, of which there are $2 \times 6.02 \times 10^{23} = 12 \times 10^{23}$ molecules. (One mole of any substance contains 6.02×10^{23} molecules.) Since 1 calorie = 4.2 joules = 4.2×10^7 ergs, 1.16×10^5 cal = 4.87×10^{12} ergs, and the energy per molecule is

$$\frac{4.87 \times 10^{12}\,\text{ergs}}{12 \times 10^{23}\,\text{molecules}} = 4 \times 10^{-12}\,\text{erg/molecule}.$$

The energy released in the nuclear reaction is therefore about 10 million times as great as that released in the chemical reaction.

The computation of the energy released in the nuclear reaction is verified by observing the distance the alpha particles travel in air at atmospheric pressure before being brought to rest by collisions with other molecules. This distance is found to be 8.31 cm. (One way of making such measurements is with the help of a cloud chamber, illustrated in Fig. 49–5.) A series of independent experiments is then performed in which the range of alpha particles of known energy is measured. These experiments show that in order to travel 8.31 cm, an alpha particle must have an initial kinetic energy of 1.38×10^{-5} erg. The energy of the two alphas together is therefore

$$2 \times 1.38 \times 10^{-5}\,\text{erg} = 2.76 \times 10^{-5}\,\text{erg}.$$

This is in excellent agreement with the energy computed from the excess mass.

PROBLEMS

7–1. The locomotive of a freight train exerts a constant force of 6 tons on the train while drawing it at 50 mi/hr on a level track. How many foot·pounds of work are done in a distance of 1 mi?

7–2. An 80-lb block is pushed a distance of 20 ft along a level floor at constant speed by a force at an angle of 30° below the horizontal. The coefficient of friction between block and floor is 0.25. How many foot·pounds of work are done?

7–3. A horse is towing a canal boat, the towrope making an angle of 10° with the towpath. If the tension in the rope is 100 lb, how many foot·pounds of work are done while moving 100 ft along the towpath?

7–4. A block is pushed 4 ft along a fixed horizontal surface by a horizontal force of 10 lb. The opposing force of friction is 2 lb. (a) How much work is done by the 10-lb force? (b) What is the work of the friction force?

7–5. (a) Compute the kinetic energy of an 1800-lb automobile traveling at 30 mi/hr. (b) How many times as great is the kinetic energy if the velocity is doubled?

7–6. Compute the kinetic energy, in ergs and in joules, of a 2-gm rifle bullet traveling at 500 m/sec.

7–7. An electron strikes the screen of a cathode-ray tube with a velocity of 10^9 cm/sec. Compute its kinetic energy in ergs. The mass of an electron is 9×10^{-28} gm.

7–8. What is the potential energy of a 1600-lb elevator at the top of the Empire State building, 1248 ft above street level? Assume the potential energy at street level to be zero.

7–9. What is the increase in potential energy of a 1-kgm body when lifted from the floor to a table 1 meter high?

7–10. A meter stick whose mass is 300 gm is pivoted at one end as in Fig. 7–14 and displaced through an angle of 60°. What is the increase in its potential energy?

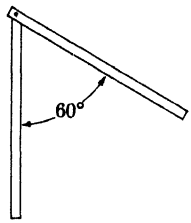


FIGURE 7–14

7–11. The force in pounds required to stretch a certain spring a distance of x ft beyond its unstretched length is given by $F = 10x$. (a) What force will stretch the spring 6 in? 1 ft? 2 ft? (b) How much work is required to stretch the spring 6 in? 1 ft? 2 ft?

7–12. The scale of a certain spring balance reads from zero to 400 lb and is 8 in. long. (a) What is the potential energy of the spring when it is stretched 8 in? 4 in? (b) When a 50-lb weight hangs from the spring?

7–13. A body moves a distance of 10 ft under the action of a force which has the constant value of 5.5 lb for the first 6 ft and then decreases to a value of 2 lb as shown by the graph in Fig. 7–15. (a) How much work is done in the first 6 ft of the motion? (b) How much work is done in the last 4 ft?

7–14. A block weighing 16 lb is pushed 20 ft along a horizontal frictionless surface by a horizontal force of 8 lb. The block starts from rest. (a) How much work is done? What becomes of this work? (b) Check

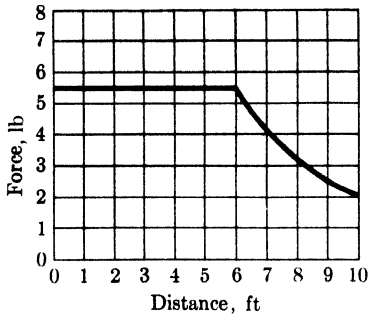


FIGURE 7–15

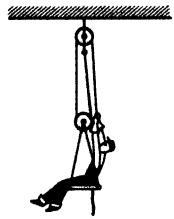


FIGURE 7–16

your answer by computing the acceleration of the block, its final velocity, and its kinetic energy.

7–15. In the preceding problem, suppose the block had an initial velocity of 10 ft/sec, other quantities remaining the same. (a) How much work is done? (b) Check by computing the final velocity and the increase in kinetic energy.

7–16. A 16-lb block is lifted vertically at a constant velocity of 10 ft/sec through a height of 20 ft. (a) How great a force is required? (b) How much work is done? What becomes of this work?

7–17. A 25-lb block is pushed 100 ft up the sloping surface of a plane inclined at an angle of 37° to the horizontal by a constant force F of 32.5 lb acting parallel to the plane. The coefficient of friction between the block and plane is 0.25. (a) What is the work of the force F? (b) Compute the increase in kinetic energy of the block. (c) Compute the increase in potential energy of the block. (d) Compute the work done against friction. What becomes of this work? (e) What can you say about the sum of (b), (c), and (d)?

7–18. A man weighing 150 lb sits on a platform suspended from a movable pulley and raises himself by a rope passing over a fixed pulley (Fig. 7–16). Assuming no friction losses, find (a) the force he must exert, (b) the increase in his energy when he raises himself 2 ft. Answer part (b) by calculating his increase in potential energy, and also by computing the product of the force on the rope and the length of rope passing through his hands.

7–19. A barrel weighing 250 lb is suspended by a rope 30 ft long. (a) What horizontal force is necessary to hold the barrel sideways 5 ft from the vertical? (b) How much work is done in moving it to this position?

7–20. The system in Fig. 7–17 is released from rest with the 24-lb block 8 ft above the floor. Use the principle of conservation of energy to find the

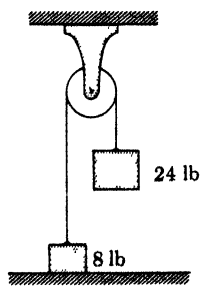


FIGURE 7–17

velocity with which the block strikes the floor. Neglect friction and inertia of the pulley.

7–21. The spring of a spring gun has a force constant of 3 lb per inch. It is compressed 2 inches and a ball weighing 0.02 lb is placed in the barrel against the compressed spring. (a) Compute the maximum velocity with which the ball leaves the gun when released. (b) Determine the maximum velocity if a resisting force of 2.25 lb acts on the ball.

7–22. A block weighing 2 lb is forced against a horizontal spring of negligible mass, compressing the spring an amount $x_1 = 6$ inches. When released, the block moves on a horizontal table top a distance $x_2 = 2$ ft before coming to rest. The spring constant k is 8 lb/ft (Fig. 7–18). What is the coefficient of friction, μ, between the block and the table?

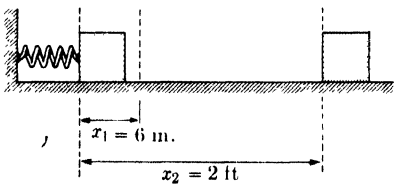


FIGURE 7–18

7–23. A 2-kgm block is dropped from a height of 40 cm onto a spring whose force constant k is 1960 newtons/meter. Find the maximum distance the spring will be compressed.

7–24. A 16-lb projectile is fired from a gun with a muzzle velocity of 800 ft/sec at an angle of departure of 45°. The angle is then increased to 90° and a similar projectile is fired with the same muzzle velocity. (a) Find the maximum heights attained by the projectiles. (b) Show that the total energy at the top of the trajectory is the same in the two cases. (c) Using the energy principle, find the height attained by a similar projectile if fired at an angle of 30°.

7–25. A block weighing 2 lb is released from rest at point A on a track which is one quadrant of a circle of radius 4 ft (Fig. 7–19). It slides down the track and reaches point B with a velocity of 12 ft/sec. From point B it slides on a level surface a distance of 9 ft to point C, where it comes to rest. (a) What was the coefficient of sliding friction on the horizontal surface? (b) How much work was done against friction as the body slid down the circular arc from A to B?



FIGURE 7–19

7–26. A small sphere of mass m is fastened to a weightless string of length 2 ft to form a pendulum. The pendulum is swinging so as to make a maximum angle of 60° with the vertical. (a) What is the velocity of the sphere when it passes through the vertical position? (b) What is the instantaneous acceleration when the pendulum is at its maximum deflection?

7–27. A ball is tied to a cord and set in rotation in a vertical circle. Prove that the tension in the cord at the lowest point exceeds that at the highest point by six times the weight of the ball.

7–28. A small body of mass m slides without friction around the loop-the-loop apparatus shown in Fig. 7–20. It starts from rest at point A at a height $3R$ above the bottom of the loop.

When it reaches point B at the end of a horizontal diameter of the loop, compute (a) its radial acceleration, (b) its tangential acceleration, and (c) its resultant acceleration. Show these accelerations in a diagram, approximately to scale.

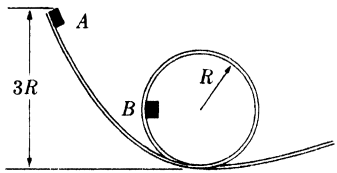


FIGURE 7–20

7–29. A meter stick, pivoted about a horizontal axis through its center, has a body of mass 2 kgm attached to one end and a body of mass 1 kgm attached to the other. The mass of the meter stick can be neglected. The system is released from rest with the stick horizontal. What is the velocity of each body as the stick swings through a vertical position?

7–30. A variable force **P** is maintained tangent to a frictionless cylindrical surface of radius a, as shown in Fig. 7–21. By slowly varying this force, a block of weight w is moved and the spring to which it is attached is stretched from position 1 to position 2. Calculate the work of the force **P**.

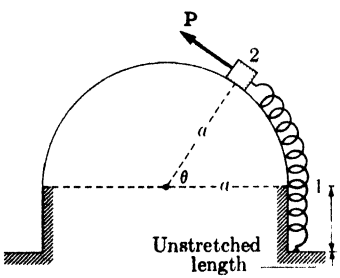


FIGURE 7–21

7–31. What average horsepower is developed by a 180-lb man when climbing in 10 sec a flight of stairs which rises 20 ft vertically? Express this power in watts and kilowatts.

7–32. The hammer of a pile driver weighs 1000 lb and must be lifted a vertical distance of 6 ft in 3 sec. What horsepower engine is required?

7–33. A ski tow is to be operated on a 37° slope 800 ft long. The rope is to move at 8 mi/hr and power must be provided for 80 riders at one time, each weighing, on an average, 150 lb. Estimate the horsepower required to operate the tow.

7–34. (a) If energy costs 5 cents per kwh, how much is one horsepower·hour worth? (b) How many ft·lb can be purchased for one cent?

7–35. Compute the monetary value of the kinetic energy of the projectile of a 14-in. naval gun, at the rate of 2 cents per kwh. The projectile weighs 1400 lb and its muzzle velocity is 2800 ft/sec.

7–36. At 5 cents per kwh, what does it cost to operate a 10-hp motor for 8 hr?

7–37. The engine of an automobile develops 20 hp when the automobile is traveling at 30 mi/hr. (a) What is the resisting force in pounds? (b) If the resisting force is proportional to the velocity, what horsepower will drive the car at 15 mi/hr? At 60 mi/hr?

7–38. The engine of a motorboat delivers 40 hp to the propeller while the boat is making 20 mi/hr. What would be the tension in the towline if the boat were being towed at the same speed?

7–39. A man whose mass is 70 kgm walks up to the third floor of a building. This is a vertical height of 12 meters above the street level. (a) How

many joules of work has he done? (b) By how much has he increased his potential energy? (c) If he climbs the stairs in 20 sec, what was his rate of working in horsepower?

7–40. A pump is required to lift 200 gallons of water per minute from a well 20 ft deep and eject it with a speed of 30 ft/sec. (a) How much work is done per minute in lifting the water? (b) How much in giving it kinetic energy? (c) What horsepower engine is needed?

7–41. A 4800-lb elevator starts from rest and is pulled upward with a constant acceleration of 10 ft/sec^2. (a) Find the tension in the supporting cable. (b) What is the velocity of the elevator after it has risen 45 ft? (c) Find the kinetic energy of the elevator 3 sec after it starts. (d) How much is its potential energy increased in the first 3 sec? (e) What horsepower is required when the elevator is traveling 22 ft/sec?

7–42. An automobile weighing 2000 lb has a speed of 100 ft/sec on a horizontal road when the engine is developing 50 hp. What is its speed, with the same horsepower, if the road rises 1 ft in 20 ft? Assume all friction forces to be constant.

7–43. A string is attached to the upper side of a 350-gm block of wood placed on an inclined plane. When the string is parallel to the plane, a tension of 260,000 dynes in the string will pull the block up the plane at a constant speed. When the tension in the string is 150,000 dynes, the block moves down the plane with constant speed. (a) What is the force of friction between the block and the plane? (b) What is the actual mechanical advantage of the inclined plane? (c) What is the ideal mechanical advantage of the inclined plane? (d) What is the efficiency of the inclined plane?

7–44. An 800-lb weight is to be raised with uniform speed by means of a screw jack. The lever arm of the screw jack is 18 inches long and there are 4 threads to the inch. (a) Find the least force required if the efficiency is 20%. (b) What is the actual mechanical advantage? (c) What is the ideal mechanical advantage?

7–45. A screw jack moves upward $\frac{1}{4}$ in. when turned through one revolution; that is, the pitch is $\frac{1}{4}$ in. What load could be lifted if a force of 20 lb is applied at the end of a bar 2 ft long if friction is neglected?

7–46. A force of 6 lb is required to raise a weight of 30 lb by means of a pulley system. If the weight is raised 1 ft while the applied force acts through a distance of 8 ft, find (a) the ideal mechanical advantage, (b) the actual mechanical advantage, and (c) the efficiency.

7–47. Prove that the ideal mechanical advantage of the inclined plane shown in Fig. 7–12(a) is equal to the length of the plane divided by its height.

7–48. Prove that the ideal mechanical advantage of the lever shown in Fig. 7–12(c) is the distance from the

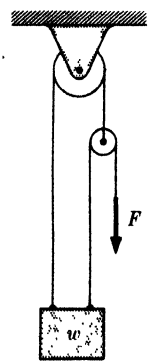


FIGURE 7–22

fulcrum to the point of application of **F** divided by the distance from the fulcrum to the point of application of the load.

7–49. Prove that the ideal mechanical advantage of the crank and axle shown in Fig. 7–12(d) is equal to the length of the crank divided by the radius of the wheel.

7–50. Calculate the ideal mechanical advantage of the device shown in Fig. 7–22.

7–51. An atomic bomb containing 20 kgm of plutonium explodes. The rest mass of the products of the explosion is less than the original rest mass by one ten-thousandth of the original rest mass. (a) How much energy is released in the explosion? (b) If the explosion takes place in one microsecond, what is the average power developed by the bomb? (c) How much water could the released energy lift to a height of one mile?

CHAPTER 8

IMPULSE AND MOMENTUM

8–1 Impulse and momentum. In the preceding chapter it was shown how the concepts of work and energy are developed from Newton's laws of motion. We shall next see how two similar concepts, those of *impulse* and *momentum*, also arise from Newton's laws. The most common use of these concepts is in connection with problems in impact. For the present, we shall consider only problems in which all velocities and all forces are parallel to a fixed direction.

Figure 8–1 represents two bodies that approach one another on a smooth horizontal surface, collide, and then recede from one another. Quantities relating to the first body are unprimed, those relating to the second are primed. The subscript 0 refers to values before the collision, and symbols without subscripts refer to values after the collision.

Since the plane is horizontal and frictionless, the only forces of interest are those which either body exerts on the other during the time the two are in contact. These forces are designated by $\mathbf{F}$ and $\mathbf{F}'$ in Fig. 8–1(b). From Newton's third law, $\mathbf{F}$ and $\mathbf{F}'$ are equal in magnitude and oppositely directed. That is, $F = -F'$. The forces $\mathbf{F}$ and $\mathbf{F}'$ will both vary during the collision. Of course, both are zero before contact. Both are small at the first instant of contact, then both increase to a maximum, and both decrease and become zero when the bodies separate. A force that varies with the time in this manner is called an *impulsive force*, and is plotted on a force-time diagram in Fig. 8–2(a) where the impulsive force is seen to start at time t_0, rise to a large maximum, and end at time t. Suppose the total time interval $t - t_0$ is subdivided into any number of parts Δt_1, Δt_2, . . . , and rectangles are drawn as in Fig. 8–2(b) whose jagged outline approximates the smooth curve. Clearly, the larger the number of subdivisions the more closely will the smooth curve be approximated and the more closely will the sum of the areas of the rectangles approach the area under the smooth curve.

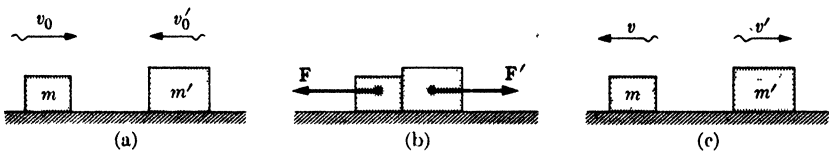


FIGURE 8–1

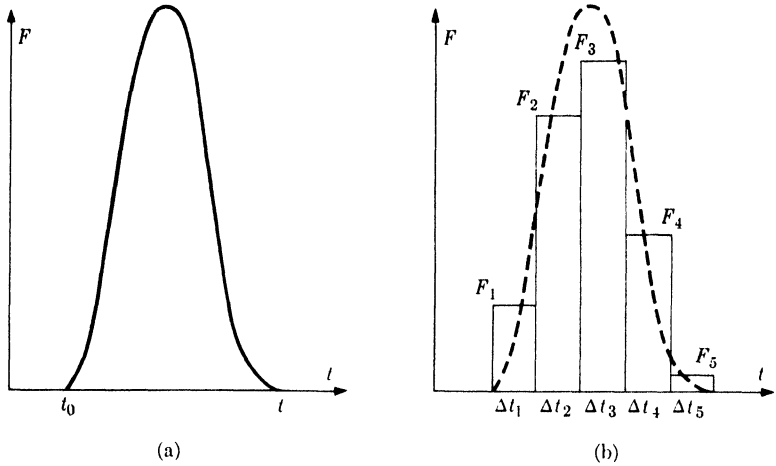


FIGURE 8-2

Let us therefore replace the continuously varying impulsive force by a force which remains constant at the value F_1 during the time interval Δt_1, then jumps instantaneously to the value F_2, maintaining this value for a time Δt_2, and so on. If the velocity of the body of mass m is originally v_0, and changes to the value v_1 at the end of the first time interval, we have, from Newton's second law,

$$F_1 = m \frac{v_1 - v_0}{\Delta t_1}$$

or

$$F_1 \, \Delta t_1 = mv_1 - mv_0;$$

and similarly for the other time intervals,

$$F_2 \, \Delta t_2 = mv_2 - mv_1,$$

$$F_3 \, \Delta t_3 = mv_3 - mv_2,$$

$$F_4 \, \Delta t_4 = mv_4 - mv_3,$$

$$F_5 \, \Delta t_5 = mv - mv_4,$$

where v is the velocity at the end of the last time interval. Adding these equations, we note that all terms in the right member cancel except mv_0 and mv. Hence

$$\sum F \, \Delta t = mv - mv_0. \tag{8-1}$$

If we now imagine the total time interval $t - t_0$ subdivided into a larger number of steps, the left member of Eq. (8-1) becomes the area

under the smooth curve, while the right member remains unaffected. We have, therefore, the *rigorous* result that the area under the $F - t$ curve is equal to the *change* in the value of the product mv.

The area under the $F - t$ curve is called the *impulse* of the force, and is represented by the symbol J.

$$\text{Impulse of a force} = J = \left\{\begin{matrix}\text{Area under} \\ F - t \text{ curve}\end{matrix}\right\}.$$

The product of a body's mass and linear velocity is called its *linear momentum*. In magnitude,

$$\text{Linear momentum} = mv.$$

The result of the calculation of the previous paragraph may therefore be written

$$J = mv - mv_0. \tag{8-2}$$

Equation (8–2) may be stated verbally as follows: *The change in linear momentum of a body equals the impulse of the force exerted on it.*

If in a special case the force is constant, the area is merely $F(t - t_0)$ and

$$\text{Impulse of a constant force} = J = F(t - t_0).$$

The unit of impulse is 1 pound·second in the engineering system, 1 newton·second in the mks system, and 1 dyne·second in the cgs system. The units of momentum in the three systems are 1 slug·ft/sec, 1 kgm·m/sec, and 1 gm·cm/sec. The unit of impulse in any system is equivalent to the corresponding unit of linear momentum, as is easily seen, for example, by recalling from Newton's second law that

$$F = m \times a,$$

$$1 \text{ lb} = 1 \text{ slug} \times 1 \frac{\text{ft}}{\text{sec}^2}.$$

Hence

$$1 \text{ lb·sec} = 1 \frac{\text{slug·ft}}{\text{sec}}.$$

EXAMPLE. A baseball weighing 5.5 ounces and traveling horizontally with a velocity of 80 ft/sec is struck by a bat. Its velocity after leaving the bat is 100 ft/sec, in the direction opposite to its original motion. What was the impulse of the blow?

If the force exerted on the ball were known as a function of time, the impulse could be computed by the method shown in Fig. 8–2. This information is not given, but we can find the impulse from the impulse-momentum theorem. The mass of the ball is 10.7×10^{-3} slug. Hence

$$\begin{aligned} \text{Impulse} &= \text{change in momentum} \\ &= mv - mv_0 = m(v - v_0) \\ &= 10.7 \times 10^{-3}\ \text{slug} \times \left[\left(-100\ \frac{\text{ft}}{\text{sec}}\right) - \left(80\ \frac{\text{ft}}{\text{sec}}\right)\right] \\ &= -1.93\ \text{lb·sec}. \end{aligned}$$

(The positive direction is that of the original velocity.) The *force* of the blow cannot be found from the given data. Any force whose impulse is −1.93 lb·sec will produce the same change in momentum. For instance, if the bat and ball were in contact for 0.001 sec, and if the force were constant during this time, its value would be

$$F = \frac{-1.93\ \text{lb·sec}}{10^{-3}\ \text{sec}} = -1930\ \text{lb}.$$

It is the *impulse* of a blow, rather than the *force* of a blow, which is significant as far as changes in momentum are concerned.

8–2 Conservation of linear momentum. Let us return to a consideration of the colliding bodies in Fig. 8–1. If J' is the impulse of the force F' acting on the right-hand body, then from Eq. (8–2),

$$J' = m'v' - m'v'_0. \tag{8–3}$$

But since at every instant $F' = -F$, it follows that

$$J' = -J.$$

Hence from Eqs. (8–2) and (8–3), we have

$$mv - mv_0 = -(m'v' - m'v'_0),$$

and finally, after rearranging terms,

$$\boxed{mv_0 + m'v'_0 = mv + m'v'.} \tag{8–4}$$

The left side of Eq. (8–4) is the total linear momentum of the system before the collision, the right side is the total momentum after the collision. We have therefore derived the extremely important result that *the total linear momentum of the colliding bodies is unaltered by the collision.* This fact is called the *principle of conservation of linear momentum.* It is one of the most important principles in mechanics.

Note that detailed knowledge of how the forces F and F' vary is unnecessary. The impulses of the forces are necessarily equal in magnitude and opposite in direction and hence they produce equal and opposite changes in momentum. The net change in momentum is therefore zero.

The momentum of a body is a vector quantity, in the same direction as its velocity **v**. If forces and velocities are not all parallel to a given direction but lie in a plane which we shall take as the xy-plane, the preceding calculations can be carried out twice, first for velocities and forces in the x-direction and second for velocities and forces in the y-direction. We then obtain two equations like Eq. (8–4), one for the x-components of momentum and the other for the y-components. That is, both the x-component and y-component of momentum are conserved. It follows that *the linear momentum of the system is conserved in both magnitude and direction.*

A more general statement of the principle of conservation of linear momentum, which does not restrict it to a collision between *two* bodies, is as follows:

The total linear momentum of a system can only be changed by external *forces acting on the system.* The internal forces, being equal and opposite and acting for equal times, produce equal and opposite changes in momentum which cancel one another. Hence, *the total linear momentum of an isolated system is constant in magnitude and direction.*

EXAMPLE 1. The Springfield rifle weighs 9.69 lb and fires a bullet weighing 150 grains (1 lb = 7000 grains) at a muzzle velocity of 2700 ft/sec. Compute the recoil velocity of the rifle if freely suspended.

The linear momentum of rifle and bullet before firing is zero. Hence, after firing, the forward momentum of the bullet is numerically equal to the backward momentum of the rifle. The mass of the rifle is 9.69/32 slugs and that of the bullet is 150/(7000 × 32) slugs. Then

$$\frac{150}{7000 \times 32} \text{ slugs} \times 2700 \frac{\text{ft}}{\text{sec}} = \frac{9.69}{32} \text{ slugs} \times v,$$

$$v = 5.9 \frac{\text{ft}}{\text{sec}}.$$

(The forward momentum of the burnt gases, which is quite appreciable, has been neglected.)

It is important to note that the kinetic energies of the bullet and rifle are *not* equal. The explanation is evident when one considers that a body acquires kinetic energy when work is done on it, the work being the product of the force and the distance moved. While the gases are propelling the bullet forward and

the rifle backward, although the force on each is the same, the distance moved by the bullet is relatively large (the length of the barrel) while the distance moved by the slowly recoiling rifle is much less. Hence the work done on the bullet is much greater than the work done on the rifle, and its kinetic energy is correspondingly greater. Momentum, however, being equal to the product of force and *time*, is the same for both bullet and rifle.

Referring to the example of the Springfield, we find

$$E_{k(\text{bullet})} = \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{150}{7000 \times 32}\text{ slugs}\right)\left(2700\,\frac{\text{ft}}{\text{sec}}\right)^2 = 2440\text{ ft·lb.}$$

$$E_{k(\text{rifle})} = \frac{1}{2}MV^2 = \frac{1}{2}\left(\frac{9.69}{32}\text{ slugs}\right)\left(5.9\,\frac{\text{ft}}{\text{sec}}\right)^2 = 5.25\text{ ft·lb.}$$

EXAMPLE 2. Consider a system consisting of two bodies in "outer space" of mass m and m', very far from all other matter. Suppose the bodies start from rest at a distance r_1 apart and accelerate toward each other until they are at a distance r_2 apart. At this moment, what are the velocities v and v' of the two bodies?

The total initial linear momentum of the system is zero. Since no external force acts on the system, the final momentum must also be zero. Thus

$$mv - m'v' = 0,$$

or

$$v' = \frac{m}{m'}v.$$

An internal force, the gravitational force of attraction, acts between the two bodies, where

$$F = G\frac{mm'}{r^2}.$$

Since this force depends only on the distance between the two bodies, it is possible to ascribe to the system an internal potential energy E_p^i which can be shown to be equal to

$$E_p^i = -G\frac{mm'}{r}.$$

As the bodies approach each other, the total kinetic energy of the system increases and its internal potential energy decreases. The sum of its kinetic energy and internal potential energy remains constant. Hence

$$0 + 0 - G\frac{mm'}{r_1} = \frac{1}{2}mv^2 + \frac{1}{2}m'v'^2 - G\frac{mm'}{r_2},$$

or

$$\frac{1}{2}mv^2 + \frac{1}{2}m'v'^2 = Gmm'\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

Substituting for v' its value mv/m', we get

$$\frac{1}{2}mv^2 + \frac{1}{2}m'\frac{m^2v^2}{m'^2} = Gmm'\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

or

$$\frac{1}{2}mv^2\left(1 + \frac{m}{m'}\right) = Gmm'\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

Therefore

$$v = m'\sqrt{[2G/(m + m')](1/r_2 - 1/r_1)},$$

and

$$v' = m\sqrt{[2G/(m + m')](1/r_2 - 1/r_1)}.$$

8–3 Elastic and inelastic collisions. Although linear momentum is always conserved in a collision, the same is not true of the kinetic energy. If the kinetic energy *does* remain constant, the collision is called *completely elastic.* If the colliding bodies stick together and move as a unit after the collision, it is called *completely inelastic.* These represent the two extremes, and all intermediate cases are possible.

The only completely elastic collisions known are those between atomic and subatomic particles, and even these may not be completely elastic if the energies of the colliding particles are sufficiently great. However, collisions between many objects, such as a pair of billiard balls, are very nearly completely elastic.

If a collision between two bodies is completely elastic, the equations

$$(\tfrac{1}{2}mv_0^2 + \tfrac{1}{2}m'v_0'^2) = (\tfrac{1}{2}mv^2 + \tfrac{1}{2}m'v'^2) \text{ (Conservation of energy)}$$

and

$$(mv_0 + m'v_0') = (mv + m'v') \text{ (Conservation of linear momentum)}$$

must both be satisfied. (The primes and subscripts have the same significance as in Fig. 8–1.) These may be written

$$m(v_0^2 - v^2) = m'(v'^2 - v_0'^2),$$

$$m(v_0 - v) = m'(v' - v_0').$$

When the first is divided by the second, we obtain

$$v_0 + v = v' + v_0'$$

or finally,

$$v_0 - v_0' = -(v - v').$$

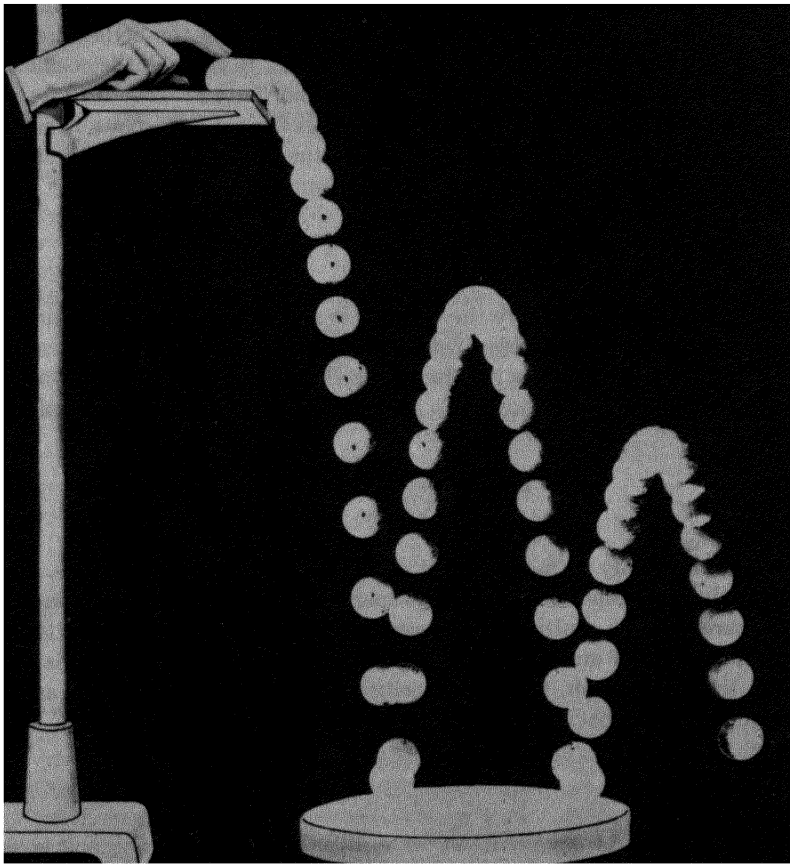


FIGURE 8-3

But $v_0 - v_0'$ is the relative velocity before the collision and $v - v'$ is the relative velocity after the collision. Hence, in a perfectly elastic collision the relative velocity is reversed in direction but unaltered in magnitude.

The degree to which a pair of colliding bodies approach complete elasticity is expressed by their *coefficient of restitution*, e, which is defined as the negative ratio of the relative velocity after collision to the relative velocity before collision.

$$e = -\frac{v - v'}{v_0 - v_0'}. \tag{8-5}$$

From what has just been shown, the coefficient of restitution is unity if the colliding bodies are perfectly elastic, and is zero if the bodies are perfectly inelastic. These are the two extremes and, in general, the coefficient of restitution has some value between zero and unity.

When a ball is dropped onto and rebounds from a fixed plate it has, in effect, collided with the earth. The mass of the earth is so large its velocity is practically unaltered by the collision. Hence in this special case,

$$e = -\frac{v}{v_0}.$$

The relative velocity before the collision is simply the velocity acquired in falling from a height h_0 or $\sqrt{2gh_0}$. If, after colliding, the ball rises to a height h, the relative velocity after the collision is $-\sqrt{2gh}$ (the downward direction is considered positive). Hence the coefficient of restitution is

$$e = -\frac{-\sqrt{2gh}}{\sqrt{2gh_0}} = \sqrt{\frac{h}{h_0}},$$

and a simple way of measuring it is to measure these two heights. The value obtained represents a joint property of the ball and the surface.

Figure 8-3 is a multiflash photograph of a golf ball dropping onto and rebounding from an iron plate. The heights h_0 and h can be measured from the photograph, and the velocities before and after colliding can be found from the spacing of the images before and after impact.

EXAMPLE 1. In Fig. 8-1, let $m = 1$ kgm, $m' = 2$ kgm, $v_0 = 5$ m/sec, $v'_0 = 2$ m/sec. Suppose that, instead of separating after the collision, the two bodies stick together, as would two railroad cars provided with a coupling device, so that $v = v'$. We wish to compute this final velocity. From the principle of conservation of linear momentum,

$$mv_0 + m'v'_0 = (m + m')v,$$

$$v = \frac{mv_0 + m'v'_0}{m + m'} = \frac{1 \text{ kgm} \times 5 \text{ m/sec} + 2 \text{ kgm} \times 2 \text{ m/sec}}{1 \text{ kgm} + 2 \text{ kgm}} = 3 \frac{\text{m}}{\text{sec}}.$$

The system therefore moves to the right after the collision with a velocity of 3 m/sec. This is an example of a completely inelastic collision. The original kinetic energy was

$$E_{k_0} = \frac{1}{2} \times 1 \text{ kgm} \times 25 \frac{\text{m}^2}{\text{sec}^2} + \frac{1}{2} \times 2 \text{ kgm} \times 4 \frac{\text{m}^2}{\text{sec}^2} = 16.5 \text{ joules}.$$

The final kinetic energy was

$$E_k = \frac{1}{2} \times 3 \text{ kgm} \times 9 \frac{\text{m}^2}{\text{sec}^2} = 13.5 \text{ joules}.$$

Therefore, although the momentum did not change in the collision, the kinetic energy decreased from 16.5 to 13.5 joules. The difference, 3 joules, was converted to heat.

EXAMPLE 2. Suppose the collision in the preceding example had been completely elastic. What would then have been the velocities v and v' after the collision?

From the principle of the conservation of linear momentum,

$$1 \text{ kgm} \times 5 \frac{\text{m}}{\text{sec}} + 2 \text{ kgm} \times 2 \frac{\text{m}}{\text{sec}} = 1 \text{ kgm} \times v + 2 \text{ kgm} \times v'.$$

Since the collision is completely elastic, the coefficient of restitution is 1, or

$$-\frac{v - v'}{(5 \text{ m/sec}) - (2 \text{ m/sec})} = 1.$$

Hence

$$v + 2v' = 9 \frac{\text{m}}{\text{sec}}, \qquad -v + v' = 3 \frac{\text{m}}{\text{sec}}.$$

Solving simultaneously, we obtain

$$v = 1 \frac{\text{m}}{\text{sec}}, \qquad v' = 4 \frac{\text{m}}{\text{sec}}.$$

EXAMPLE 3. The *ballistic pendulum* is a device for measuring the velocity of a bullet. The bullet is allowed to make a completely inelastic collision with a body of much greater mass. The momentum of the system immediately after the collision equals the original momentum of the bullet, but since the velocity is very much smaller it can be determined more easily. Although the ballistic pendulum has now been superseded by other devices, it is still an important laboratory experiment for illustrating the concepts of momentum and energy.

In Fig. 8–4, the pendulum, consisting perhaps of a large wooden block of mass M, hangs vertically by two cords. A bullet of mass m, traveling with a velocity v, strikes the pendulum and remains embedded in it. If the collision time is very small compared with the time of swing of the pendulum, the supporting cords remain practically vertical during this time. Hence no external horizontal forces act on the system during the collision, and the horizontal momentum is conserved. Then if V represents the velocity of bullet and block immediately after the collision,

$$mv = (m + M)V.$$

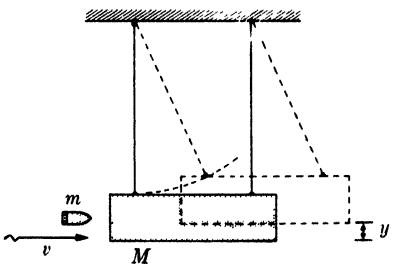


FIG. 8–4. The ballistic pendulum.

The kinetic energy of the system, immediately after the collision, is

$$E_k = \tfrac{1}{2}(m + m')V^2.$$

The pendulum now swings to the right and upward until its kinetic energy is converted to gravitational potential energy. (Small frictional effects can be neglected.) Hence

$$\tfrac{1}{2}(m + m')V^2 = (m + m')gy,$$

and

$$v = \frac{m + m'}{m}\sqrt{2gy}.$$

By measuring m, m', and y, the original velocity v of the bullet can be computed.

It is important to remember that kinetic energy is not conserved *in the collision.* The ratio of the kinetic energy of bullet and pendulum, after the collision, to the original kinetic energy of the bullet, is

$$\frac{\tfrac{1}{2}(m + m')V^2}{\tfrac{1}{2}mv^2} = \frac{m}{m + m'}.$$

Thus if $m' = 1000$ gm and $m = 1$ gm, only about one-tenth of one percent of the original energy remains as kinetic energy; 99.9% is converted to heat.

8–4 Newton's second law. Newton himself did not state his second law in the form in which we have used it. A free translation (Newton's "Principia" was written in Latin) is as follows:

Change of motion is proportional to the applied force, and takes place in the direction of the force. . . . Quantity of motion is proportional to mass and velocity conjointly.

From Newton's definition of "motion," or "quantity of motion," it is evident that he used this term for the concept we now call linear momentum. It is also clear from his writings that the term "change" meant "rate of change" and that the "applied force" referred to the resultant force. Hence in current terminology Newton's statement is:

Rate of change of linear momentum is proportional to the resultant force and is in the direction of this force.

In mathematical language this becomes

$$\frac{\Delta(m\mathbf{v})}{\Delta t} \propto \mathbf{F},$$

or

$$\mathbf{F} = k\frac{\Delta(m\mathbf{v})}{\Delta t}. \tag{8–6}$$

If the mass m is constant, this reduces to

$$\mathbf{F} = km\frac{\Delta \mathbf{v}}{\Delta t} = km\mathbf{a},$$

which is the form we have used, with k made equal to unity by proper choice of units.

We have pointed out in Section 7–11 that while the mass of a body can ordinarily be considered constant, it increases with increasing velocity according to the relation

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}},$$

where m_0 is the "rest mass" of the body, c is the velocity of light, and v the velocity of the body. However, if the mass cannot be considered constant, we cannot set $\mathbf{F} = m\mathbf{a}$, and the original form of Newton's law must be used.

It is a striking example of Newton's genius that, although he could scarcely have foreseen the theory of relativity, he appreciated the fact that momentum is an entity more fundamental than mass.

8–5 The rocket. A rocket motor is merely a combustion chamber in which liquid or solid fuel is burned and which has an opening to direct the gaseous products of combustion in the desired direction. An ascending rocket can be compared to a machine gun pointed vertically downward and firing a steady stream of blank cartridges. The gases expelled from the rocket acquire a downward momentum and the rocket, like a recoiling machine gun, acquires an equal upward momentum.

Let us consider the flight of a rocket directed vertically upward from the earth's surface. Figure 8–5(a) shows the rocket at an instant when the mass of rocket and unburned fuel is m, the upward velocity is v, and the momentum is mv. In a short time interval Δt, a mass Δm of gas is ejected from the rocket. Let v represent the downward velocity of this gas *relative to the rocket.* The velocity v' of the gas relative to the earth is then

$$v' = v - v_r,$$

and its momentum is

$$\Delta m \cdot v' = \Delta m(v - v_r).$$

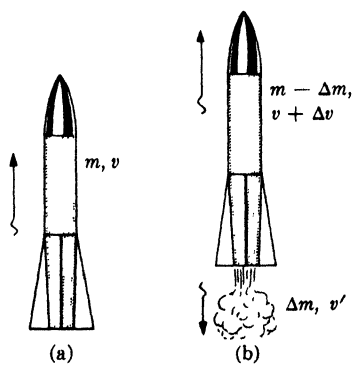


FIGURE 8–5

At the end of the time interval, the mass of rocket and unburned fuel has decreased to $m - \Delta m$, and its velocity has increased to $v + \Delta v$. Its momentum is therefore

$$(m - \Delta m)(v + \Delta v).$$

Figure 8–5(b) represents rocket and ejected gas at this time.

We now make use of Eq. (8–6), with $k = 1$, writing it as

$$F\,\Delta t = \Delta(mv).$$

That is, the product of the resultant external force F on a system, and the time interval Δt during which it acts, is equal to the change in momentum of the system. If air resistance is neglected, the external force F on the rocket is its weight, $-mg$. (We take the upward direction as positive.) The change in momentum, in time Δt, is the difference between the momentum of the system at the end and at the beginning of the time interval. Hence

$$-mg\,\Delta t = [(m - \Delta m)(v + \Delta v) + \Delta m(v - v_r)] - mv.$$

Now expand the right side of this equation and neglect the term $\Delta m\,\Delta v$ (the product of two small quantities). After canceling and rearranging terms, we get

$$m\frac{\Delta v}{\Delta t} = v_r\frac{\Delta m}{\Delta t} - mg.$$

The ratio $\Delta v/\Delta t$ is the acceleration of the rocket, so the left side of this equation (mass times acceleration) equals the resultant force on the rocket. The first term on the right equals the upward thrust on the rocket, and the resultant force equals the difference between this thrust and the weight of the rocket, mg. It will be seen that the upward thrust is proportional both to the relative velocity v_r of the ejected gas and to the mass of gas ejected per unit time, $\Delta m/\Delta t$.

The acceleration is

$$\frac{\Delta v}{\Delta t} = \frac{v_r}{m}\frac{\Delta m}{\Delta t} - g.$$

As the rocket rises, the value of g decreases according to Newton's law of gravitation. (In "outer space," far from all other bodies, g becomes negligibly small.) The values of v_r and $\Delta m/\Delta t$ remain approximately constant while the fuel is being consumed, but the remaining mass m continually decreases. Hence the acceleration *increases* until all the fuel is burned.

EXAMPLE. In the first second of its flight, a rocket ejects 1/60 of its mass with a relative velocity of 6800 ft·sec. What is the acceleration of the rocket?

We have: $\Delta m = m/60$, $\Delta t = 1$ sec.

$$\frac{\Delta v}{\Delta t} = 6800 \frac{\text{ft}}{\text{sec}} \times \frac{1}{60} \times \frac{1}{1 \text{ sec}} - 32 \frac{\text{ft}}{\text{sec}^2}$$

$$= 113 \frac{\text{ft}}{\text{sec}^2} - 32 \frac{\text{ft}}{\text{sec}^2} = 81 \frac{\text{ft}}{\text{sec}^2}.$$

Since the acceleration continually increases, the equations of motion with *constant* acceleration cannot be used to find the final velocity. The methods of calculus lead to the following expression for the velocity v after the fuel is exhausted, assuming that v_r and g are constant:

$$v = 2.3\, v_r \log \frac{m_0}{m} - gt,$$

where m_0 is the initial mass of rocket and fuel, m is the mass of the rocket alone, and t is the time required to burn all the fuel.

EXAMPLE. Suppose the ratio of initial mass to final mass for the rocket above is 4 and that the fuel is consumed in a time $t = 60$ sec. The velocity at the end of this time is then

$$v = 2.3 \times 6800 \frac{\text{ft}}{\text{sec}} \times \log 4 - 32 \frac{\text{ft}}{\text{sec}^2} \times 60 \text{ sec}$$

$$= 9420 \frac{\text{ft}}{\text{sec}} - 1920 \frac{\text{ft}}{\text{sec}} = 7500 \frac{\text{ft}}{\text{sec}}.$$

At the start of the flight, when the velocity of the rocket is zero, the ejected gases are moving downward, relative to the earth, with a velocity equal to the relative velocity v_r. When the velocity of the rocket has increased to v_r, the ejected gases have a velocity zero relative to the earth. When the rocket velocity becomes greater than v_r, the velocity of the ejected gases is in the same direction as that of the rocket. Thus the velocity acquired by the rocket can be greater (and is often much greater) than the relative velocity v_r. In the example above, where the final velocity of the rocket was 7500 ft/sec and the relative velocity was 6800 ft/sec, the last portion of the ejected fuel had an upward velocity of (7500–6800) ft/sec = 700 ft/sec.

PROBLEMS

8–1. (a) What is the momentum of a 10-ton truck whose velocity is 30 mi/hr? At what velocity will a 5-ton truck have (b) the same momentum, (c) the same kinetic energy?

8–2. A baseball weighs 5½ oz. (a) If the velocity of a pitched ball is 80 ft/sec, and after being batted it is 120 ft/sec in the opposite direction, find the change in momentum of the ball and the impulse of the blow. (b) If the ball remains in contact with the bat for 0.002 sec, find the average force of the blow.

8–3. A bullet having a mass of 0.05 kgm, moving with a velocity of 400 m/sec, penetrates a distance of 0.1 m in a wooden block firmly attached to the earth. Assume the decelerating force constant. Compute (a) the deceleration of the bullet, (b) the decelerating force, (c) the time of deceleration, (d) the impulse of the collision. Compare the answer to part (d) with the initial momentum of the bullet.

8–4. An empty freight car weighing 10 tons rolls at 3 ft/sec along a level track and collides with a loaded car weighing 20 tons, standing at rest with brakes released. If the two cars couple together, find (a) their velocity after the collision, and (b) the decrease in kinetic energy as a result of the collision.

8–5. With what velocity should the loaded car in Problem 8–4 be rolling toward the empty one in order that both shall be brought to rest by the collision?

8–6. A bullet weighing 0.02 lb is fired with a muzzle velocity of 2700 ft/sec from a rifle weighing 7.5 lb. (a) Compute the recoil velocity of the rifle, assuming it free to recoil. (b) Find the ratio of the kinetic energy of the bullet to that of the rifle.

8–7. A 75-mm gun fires a projectile weighing 16 lb with a muzzle velocity of 1900 ft/sec. By how many mi/hr is the velocity of a plane mounting such a gun decreased when a projectile is fired directly ahead? The plane weighs 32,000 lb.

8–8. The projectile of a 16-in. seacoast gun weighs 2400 lb, travels a distance of 38 ft in the bore of the gun, and has a muzzle velocity of 2250 ft/sec. The gun weighs 300,000 lb. (a) Compute the initial recoil velocity of the gun, assuming it free to recoil. (b) Find the ratio of the kinetic energy of the projectile to that of the recoiling gun.

8–9. A 4000-lb automobile going eastward on Chestnut Street at 40 mi/hr collides with a truck weighing 4 tons which is going southward across Chestnut Street at 15 mi/hr. If they become coupled on collision, what is the magnitude and direction of their velocity immediately after colliding?

8–10. A body of mass 600 gm is initially at rest. It is struck by a second body of mass 400 gm initially moving with a velocity of 125 cm/sec toward the right along the x-axis. After the collision the 400-gm body has a velocity of 100 cm/sec at an angle of 37° above the x-axis in the first quadrant. Both bodies move on a horizontal frictionless plane. (a) What is the magnitude and direction of the velocity of the 600-gm body after the collision? (b) What is the loss of kinetic energy during the collision?

8–11. (a) Prove that when a moving body makes a perfectly inelastic collision with a second of equal mass,

initially at rest, one-half of the original kinetic energy is "lost." (b) Prove that when a very heavy particle makes a perfectly elastic collision with a very light stationary particle the light one goes off with twice the velocity of the heavy one.

8–12. On a frictionless table, a 3-kgm block moving 4 m/sec to the right collides with an 8-kgm block moving 1.5 m/sec to the left. (a) If the two blocks stick together, what is the final velocity? (b) If the two blocks make a completely elastic head-on collision, what are their final velocities? (c) How much mechanical energy is converted into heat in the collision of part (a)?

8–13. Two blocks of mass 300 gm and 200 gm are moving toward each other along a horizontal frictionless surface with velocities of 50 cm/sec and 100 cm/sec, respectively. (a) If the blocks collide and stick together, find their final velocity. (b) Find the loss of kinetic energy during the collision. (c) Find the final velocity of each block if the collision is completely elastic.

8–14. A 10-gm blocks slides at a velocity of 20 cm/sec on a smooth level surface and makes a head-on collision with a 30-gm block moving in the opposite direction with a velocity of 10 cm/sec. If the collision is perfectly elastic, find the velocity of each block after the collision.

8–15. A block of mass 200 gm, sliding with a velocity of 12 cm/sec on a smooth level surface, makes a perfectly elastic head-on collision with a block of mass m gm, initially at rest. After the collision the velocity of the 200-gm block is 4 cm/sec in the same direction as its initial velocity. Find (a) the mass m, and (b) its velocity after the collision.

8–16. A rifle bullet of mass 10 gm strikes and embeds itself in a block of mass 990 gm which rests on a horizontal frictionless surface and is attached to a coil spring as shown in Fig. 8–6. The impact compresses the spring 10 cm. Calibration of the spring shows that a force of 100,000 dynes is required to compress the spring 1 cm. (a) Find the maximum potential energy of the spring. (b) Find the velocity of the block just after the impact. (c) What was the initial velocity of the bullet?

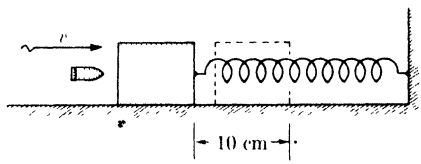


FIGURE 8–6

8–17. A projectile is fired at an angle of departure of 60° and with a muzzle velocity of 1200 ft/sec. At the highest point of its trajectory the projectile explodes into two fragments of equal mass, one of which falls vertically. How far from the point of firing does the other fragment strike if the terrain is level?

8–18. A railroad handcar is moving along straight frictionless tracks. In each of the following cases the car initially has a total weight (car and contents) of 500 lb and is traveling with a velocity of 10 ft/sec. Find the final velocity of the car in each of the three cases. (a) A 50-lb weight is thrown sideways out of the car with a velocity of 8 ft/sec relative to the car. (b) A 50-lb weight is thrown backwards out of the car with a velocity of 10 ft/sec relative to the car. (c) A 50-lb weight is thrown into the car with a velocity of 12 ft/sec relative to the ground and op-

posite in direction to the velocity of the car.

8–19. An open-topped freight car weighing 10 tons is coasting without friction along a level track. It is raining very hard, with the rain falling vertically down. The car is originally empty and moving with a velocity of 2 ft/sec. What is the velocity of the car after it has traveled long enough to collect one ton of rain water?

8–20. A neutron of mass 1.67×10^{-24} gm, moving with a velocity of 2×10^{6} cm/sec, makes a head-on collision with a boron nucleus of mass 17.0×10^{-24} gm, originally at rest. (a) If the collision is completely inelastic, what is the final kinetic energy of the system, expressed as a fraction of the original kinetic energy? (b) If the collision is perfectly elastic, what fraction of its original kinetic energy does the neutron transfer to the boron nucleus?

8–21. A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 9.22×10^{-16} gm·cm/sec, and at right angles to the direction of the electron a neutrino with momentum 5.33×10^{-16} gm·cm/sec. (a) In what direction does the residual nucleus recoil? (b) What is its momentum? (c) If the mass of the residual nucleus is 3.90×10^{-22} gm, what is its kinetic energy?

8–22. A ball is dropped from rest onto a fixed horizontal surface and rebounds to a height which is 64% of the original height. (a) What is the coefficient of restitution? (b) With what vertical velocity must the ball strike the surface to rebound to a height of 25 ft?

8–23. A golf ball is dropped on a hard surface from a height of 1 m and rebounds to a height of 64 cm. (a) What is the height of the second bounce? of the nth bounce? After (about) how many bounces is the height reduced to 1 cm? (b) What is the time of the first bounce, i.e., between the first and second contacts with the surface? What is the time of the nth bounce? (c) What is the coefficient of restitution?

8–24. A 1-lb rubber ball bounces down a flight of stairs, each time rising to the height y of the step above, as shown in Fig. 8–7. (a) Find the coefficient of restitution. (b) If the height of each step, y, is 1 ft, what is the maximum kinetic energy of the ball?

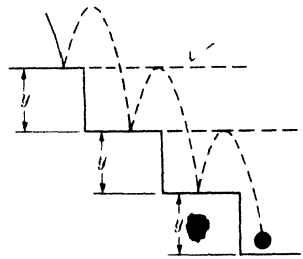


FIGURE 8–7

8–25. A freight elevator is rising with a constant velocity of 5 ft/sec. A stationary observer drops a ball from a point which is 21 ft above the platform at the instant the ball is released. The coefficient of restitution between ball and platform is 0.50. How far above (or below) its starting point does the ball rise on its first bounce?

8–26. A ball is thrown against a vertical wall, striking it at a point 4 ft above the floor with a horizontal velocity of 20 ft/sec. After rebounding from the wall, the ball strikes the floor at a point 8 ft from the wall. (a) What was the coefficient of restitution? (b) If the ball weighed 0.5 lb, how much kinetic energy was "lost" in the impact with the wall?

8–27. When a bullet of mass 10 gm strikes a ballistic pendulum of mass 2 kgm, the center of gravity of the pendulum is observed to rise a vertical distance of 10 cm. The bullet remains embedded in the pendulum. Calculate the velocity of the bullet.

8–28. A bullet weighing 0.01 lb is shot through a 2-lb wooden block suspended on a string 5 ft long. The center of gravity of the block is observed to rise a distance of 0.0192 ft. Find the speed of the bullet as it emerges from the block if the initial speed is 1000 ft/sec.

8–29. A bullet of mass 2 gm, traveling in a horizontal direction with a velocity of 500 m/sec, is fired into a wooden block of mass 1 kgm, initially at rest on a level surface. The bullet passes through the block and emerges with its velocity reduced to 100 m/sec. The block slides a distance of 20 cm along the surface from its initial position. (a) What was the coefficient of sliding friction between block and surface? (b) What was the decrease in kinetic energy of the bullet? (c) What was the kinetic energy of the block at the instant after the bullet passed through it?

8–30. A rifle bullet weighing 0.02 lb is fired with a velocity of 2500 ft/sec into a ballistic pendulum. The pendulum weighs 10 lb and is suspended from a cord 3 ft long. Compute (a) the vertical height through which the pendulum rises, (b) the initial kinetic energy of the bullet, (c) the kinetic energy of bullet and pendulum after the bullet is embedded in the pendulum.

8–31. A 5-gm bullet is fired horizontally into a 3-kgm wooden block resting on a horizontal surface. The coefficient of sliding friction between block and surface is 0.20. The bullet remains embedded in the block, which is observed to slide 25 cm along the surface. What was the velocity of the bullet?

8–32. A bullet of mass 2 gm, traveling at 500 m/sec, is fired into a ballistic pendulum of mass 1 kgm suspended from a cord 1 m long. The bullet penetrates the pendulum and emerges with a velocity of 100 m/sec. Through what vertical height will the pendulum rise?

8–33. A 160-lb man standing on ice throws a 6-oz ball horizontally with a speed of 80 ft/sec. (a) With what speed and in what direction will the man begin to move? (b) If the man throws 4 such balls every 3 sec, what is the average force acting on him?

8–34. Find the average recoil force on a machine gun firing 120 shots per minute. The weight of each bullet is 0.025 lb, and the muzzle velocity is 2700 ft/sec. [*Hint:* average force equals average rate of change of momentum.]

8–35. A rifleman, who together with his rifle weighs 160 lb, stands on roller skates and fires 10 shots horizontally from an automatic rifle. Each bullet weighs 0.0257 lb (180 grains) and has a muzzle velocity of 2500 ft/sec. (a) If the rifleman moves back without friction, what is his velocity at the end of the ten shots? (b) If the shots were fired in 10 sec, what was the average force exerted on him? (c) Compare his kinetic energy with that of the 10 bullets.

8–36. A rocket burns 50 gm of fuel per second, ejecting it as a gas with a velocity of 500,000 cm/sec. (a) What force does this gas exert on the rocket? Give the result in dynes and newtons. (b) Would the rocket operate in free space? (c) If it would operate in free space, how would you steer it? Could you brake it?

CHAPTER 9

ROTATION

9–1 Introduction. The most general type of motion which a body can undergo is a combination of *translation* and *rotation*. Thus far we have considered only the special case of translational motion, along a straight line or along a curve. We next discuss motion of rotation about a fixed axis, that is, motion of rotation without translation. We shall see that many of the equations describing rotation about a fixed axis are exactly analogous to those encountered in rectilinear motion. If the axis is *not* fixed, the problem becomes much more complicated, and we shall not attempt to give a complete discussion of the general case of translation plus rotation.

9–2 Angular velocity. Figure 9–1 represents a rigid body of arbitrary shape rotating about a fixed axis through point O and perpendicular to the plane of the diagram. Line OP is a line fixed with respect to the body and rotating with it. The position of the entire body is evidently completely specified by the angle θ which the line OP makes with some reference line fixed in space, such as Ox. The motion of the body is therefore analogous to the rectilinear motion of a particle, whose position is completely specified by a single coordinate such as x or y. The equations of motion are greatly simplified if the angle θ is expressed in *radians*.

One radian is the angle subtended at the center of a circle by an arc of length equal to the radius of the circle [Fig. 9–2(a)]. Since the radius is contained 2π times ($2\pi = 6.28\ldots$) in the circumference, there are 2π or $6.28\ldots$ radians in

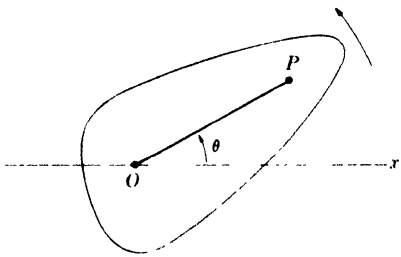


FIG. 9–1. Body rotating about a fixed axis through point O.

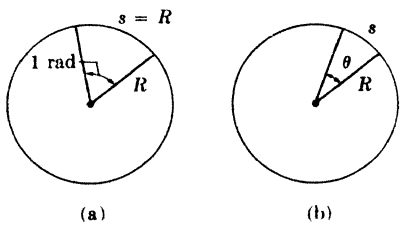


FIG. 9–2. An angle θ in radians is defined as the ratio of the arc s to the radius R.

one complete revolution or 360°. Hence

$$1 \text{ radian} = \frac{360}{2\pi} = 57.3 \ldots \text{ degrees}$$

$$\begin{aligned} 360° &= 2\pi \text{ radians} = 6.28 \ldots \text{ radians} \\ 180° &= \pi \quad " \quad = 3.14 \ldots \quad " \\ 90° &= \pi/2 \quad " \quad = 1.57 \ldots \quad " \\ 60° &= \pi/3 \quad " \quad = 1.05 \ldots \quad " \end{aligned}$$

and so on.

In general [Fig. 9–2(b)], if θ represents any arbitrary angle subtended by an arc of length s on the circumference of a circle of radius R, then θ (in radians) is equal to the length of the arc s divided by the radius R.

$$\theta = \frac{s}{R}, \qquad s = R\theta. \tag{9–1}$$

An angle in radians, being defined as the ratio of a length to a length, is a pure number.

In Fig. 9–3, a reference line OP in a rotating body makes an angle θ_1 with the reference line Ox, at a time t_1. At a later time t_2 the angle has increased to θ_2. The *average angular velocity* of the body, $\bar{\omega}$, in the time interval between t_1 and t_2, is defined as the ratio of the *angular displacement* $\theta_2 - \theta_1$, or $\Delta\theta$, to the elapsed time $t_2 - t_1$ or Δt

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}.$$

The *instantaneous angular velocity* ω is defined as the limit approached by this ratio as Δt approaches zero.

$$\boxed{\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}.} \tag{9–2}$$

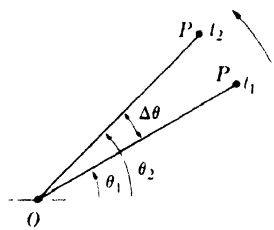


FIG. 9–3. Angular displacement $\Delta\theta$ of a rotating body.

Since the body is rigid, *all* lines in it rotate through the same angle in the same time, and the angular velocity is characteristic of the body as a whole. If the angle θ is in radians, the angular velocity is in *radians per second*. Other units, such as the revolution per minute, are in common use.

9–3 Angular acceleration. When the angular velocity of a body changes, it is said to have an angular acceleration. If ω_1 and ω_2 are the instantaneous angular velocities at times t_1 and t_2, the *average angular acceleration* $\bar{\alpha}$ is

defined as

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t},$$

and the *instantaneous angular acceleration* α is defined as the limit of this ratio when Δt approaches zero:

$$\boxed{\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}.} \tag{9-3}$$

Angular acceleration is expressed in rad/sec^2. Angular velocity and angular acceleration are exactly analogous to linear velocity and acceleration.

9–4 Rotation with constant angular acceleration. When the angular velocity of a body changes by equal amounts in equal intervals of time, the angular acceleration is constant. Under these circumstances the average and instantaneous angular accelerations are equal, whatever the duration of the time interval. One may therefore write

$$\alpha = \frac{\omega - \omega_0}{t - t_0}$$

or

$$\omega = \omega_0 + \alpha(t - t_0),$$

where α is the constant instantaneous angular acceleration.

If $t_0 = 0$,

$$\boxed{\omega = \omega_0 + \alpha t.} \tag{9-4}$$

Equation (9–4) has precisely the same form as Eq. (4–8) for linear motion with constant acceleration and may be interpreted in the same way.

The angular displacement of a rotating body, or the angle turned through by the body, corresponds to the linear displacement of a body moving along a straight line. The expression for the angular displacement can be found with the help of the average angular velocity. If the angular acceleration is constant, the angular velocity increases at a uniform rate and its average value during any time interval equals half the sum of its values at the beginning and end of the interval. That is,

$$\bar{\omega} = \frac{\omega_0 + \omega}{2}.$$

If $\theta_0 = 0$ when $t_0 = 0$, $\theta = \bar{\omega}t$ and hence

$$\boxed{\theta = \frac{\omega_0 + \omega}{2} \cdot t.} \tag{9-5}$$

Equations (9-4) and (9-5) are the fundamental equations applicable to rotation with constant angular acceleration. By combining them we may obtain two more very useful equations. Thus, substituting for ω in Eq. (9-5) the value of ω given by Eq. (9-4), we have

$$\theta = \frac{\omega_0 + \omega_0 + \alpha t}{2} \cdot t$$

or

$$\boxed{\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2.} \tag{9-6}$$

Also, substituting for t in Eq. (9-5) the value of t given by Eq. (9-4), we have

$$\theta = \frac{\omega_0 + \omega}{2} \cdot \frac{\omega - \omega_0}{\alpha} = \frac{\omega^2 - \omega_0^2}{2\alpha},$$

or finally

$$\boxed{\omega^2 = \omega_0^2 + 2\alpha\theta.} \tag{9-7}$$

The following table will serve to emphasize the similarity between the equations for motion with constant linear acceleration and those for motion with constant angular acceleration.

Motion with constant linear acceleration	Motion with constant angular acceleration
a = constant	α = constant
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$x = \frac{v_0 + v}{2} t$	$\theta = \frac{\omega_0 + \omega}{2} t$
$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

EXAMPLE. The angular velocity of a body is 4 rad/sec at time $t = 0$, and its angular acceleration is constant and equal to 2 rad/sec^2. A line OP in the body is horizontal at time $t = 0$. (a) What angle does this line make with the horizontal at time $t = 3$ sec? (b) What is the angular velocity at this time?

$$\text{(a)} \qquad \theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 4\ \frac{\text{rad}}{\text{sec}} \times 3\ \text{sec} + \frac{1}{2} \times 2\ \frac{\text{rad}}{\text{sec}^2} \times (3\ \text{sec})^2$$

$$= 21\ \text{radians} = 3.34\ \text{revolutions}.$$

$$\text{(b)} \qquad \omega = \omega_0 + \alpha t = 4\ \frac{\text{rad}}{\text{sec}} + 2\ \frac{\text{rad}}{\text{sec}} \times 3\ \text{sec} = 10\ \frac{\text{rad}}{\text{sec}}.$$

Alternatively, from Eq. (9–7),

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$= \left(4\ \frac{\text{rad}}{\text{sec}}\right)^2 + 2 \times 2\ \frac{\text{rad}}{\text{sec}^2} \times 21\ \text{rad} = 100\ \frac{\text{rad}^2}{\text{sec}^2},$$

$$\omega = 10\ \frac{\text{rad}}{\text{sec}}.$$

9–5 Relation between angular and linear velocity and acceleration. In Section 6–5 we discussed the linear velocity and acceleration of a *particle* revolving in a circle. When a *rigid body* rotates about a fixed axis, every point in the body moves in a circle whose center is on the axis and which lies in a plane perpendicular to the axis. There are some useful and simple relations between the angular velocity and acceleration of the rotating body and the linear velocity and acceleration of points within it.

Let r be the distance from the axis to some point P in the body, so that the point moves in a circle of radius r, as in Fig. 9–4. When the radius makes an angle θ with the reference axis, the distance s to the point P, measured along the circular path, is

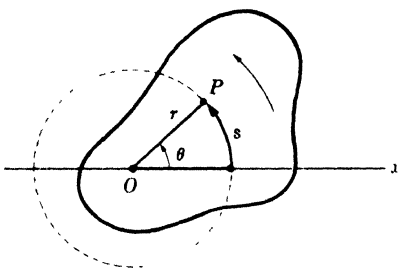


FIG. 9–4. The distance s moved through by point P equals $r\theta$.

$$\boxed{s = r\theta.} \qquad (9\text{–}8)$$

When the angle is very small, say $\Delta\theta$, the length of the arc Δs is also small and the time interval Δt for the body to rotate through the angle $\Delta\theta$ is also small. We have

$$\Delta s = r\,\Delta\theta, \qquad \frac{\Delta s}{\Delta t} = r\,\frac{\Delta\theta}{\Delta t}$$

and, in the limit, as $\Delta t \to 0$,

$$\boxed{v = r\omega.} \quad (9\text{-}9)$$

If the angular velocity about the given axis should change *in magnitude only* (the axis remaining fixed) by $\Delta\omega$, the linear velocity in a direction tangent to a circle of radius r will change by Δv, where

$$\Delta v = r\,\Delta\omega$$

If these changes take place in a short time interval Δt, then

$$\frac{\Delta v}{\Delta t} = r\,\frac{\Delta\omega}{\Delta t}$$

and, in the limit, as $\Delta t \to 0$,

$$\boxed{a_T = r\alpha,} \quad (9\text{-}10)$$

where a_T is the tangential component of the linear acceleration of a point a distance r for the axis.

The *radial* component of acceleration v^2/r of the point P can also be expressed in terms of the angular velocity.

$$\boxed{a_R = \frac{v^2}{r} = \omega^2 r} \quad (9\text{-}11)$$

The tangential and radial components of acceleration of any arbitrary point P in a rotating body are shown in Fig 9–5.

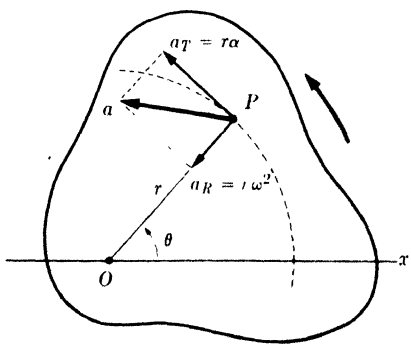


FIG. 9–5. Nonuniform rotation about a fixed axis through point O. The tangential component of acceleration of point P equals $r\alpha$; the radial component equals $r\omega$.[2]

9–6 Kinetic energy of rotation. Moment of inertia. We have shown that the magnitude of the velocity of a particle in a rigid body rotating about a fixed axis is

$$v = r\omega,$$

where r is the distance of the particle from the axis and ω is the angular velocity of the body. The kinetic energy of a particle of mass m is therefore

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mr^2\omega^2.$$

The total kinetic energy of the body is the sum of the kinetic energy of all particles in the body.

$$E_k = \sum \tfrac{1}{2}mr^2\omega^2.$$

Since ω is the same for all particles in a rigid body,

$$E_k = \tfrac{1}{2}[\sum mr^2]\omega^2.$$

To obtain the sum $\sum mr^2$ the body is subdivided (in imagination) into a large number of particles, the mass of each particle is multiplied by the square of its distance from the axis, and these products are summed over all particles. The result is called the *moment of inertia* of the body, about the axis of rotation, and is represented by I.

$$\boxed{I = \sum mr^2.} \qquad (9\text{–}12)$$

Moment of inertia is expressed in *slug·feet*^{2}, *kilogram·meters*^{2}, or *gram·centimeters*^{2}.

The rotational kinetic energy can now be written as

$$\boxed{E_k = \tfrac{1}{2}I\omega^2,} \qquad (9\text{–}13)$$

in which form it is exactly analogous to the translational kinetic energy,

$$E_k = \tfrac{1}{2}mv^2.$$

That is, for rotation about a fixed axis, moment of inertia I is analogous to mass m (or inertia), and angular velocity ω is analogous to linear velocity v.

EXAMPLE. Three small bodies, which can be considered as particles, are connected by light rigid rods as in Fig. 9–6. What is the moment of inertia of the system (a) about an axis through point A, perpendicular to the plane of the diagram, and (b) about an axis coinciding with the rod BC?

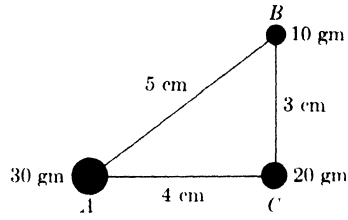


FIGURE 9–6

(a) The particle at point A lies on the axis. Its distance *from* the axis is zero and it contributes nothing to the moment of inertia. Therefore

$$I = \sum mr^2 = 10 \text{ gm} \times (5 \text{ cm})^2 + 20 \text{ gm} \times (4 \text{ cm})^2$$
$$= 570 \text{ gm·cm}^2.$$

(b) The particles at B and C both lie on the axis. The moment of inertia is

$$I = \sum mr^2 = 30 \text{ gm} \times (4 \text{ cm})^2 = 480 \text{ gm·cm}^2.$$

This illustrates the important fact that the moment of inertia of a body, unlike its mass, is not a unique property of the body but depends on the axis about which it is computed.

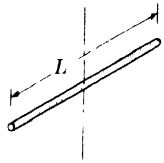
(c) If the body rotates about an axis through A and perpendicular to the plane of the diagram, with an angular velocity $\omega = 4$ rad/sec, what is the rotational kinetic energy?

$$E_k = \tfrac{1}{2} I\omega^2 = \tfrac{1}{2} \times 570 \text{ gm·cm}^2 \times \left(4 \frac{\text{rad}}{\text{sec}}\right)^2$$
$$= 4560 \text{ ergs}.$$

For a body which is not composed of discrete point masses but is a continuous distribution of matter, the summation expressed in the definition of moment of inertia, $I = \sum mr^2$, must be evaluated by the methods of calculus. The moments of inertia of a few simple but important bodies are listed in Fig. 9–7 for convenience.

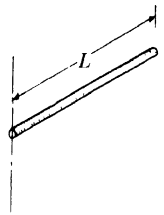
Whatever the shape of a body, it is always possible to find a radial distance from any given axis at which the mass of the body could be concentrated without altering the moment of inertia of the body about that axis. This distance is called the *radius of gyration* of the body about the given axis, and is represented by k.

If the mass m of the body actually were concentrated at this distance, the moment of inertia would be that of a particle of mass m at a distance



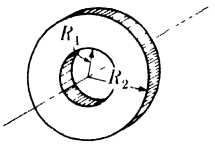
$\frac{1}{12} mL^2$

(a) Slender rod, axis through center



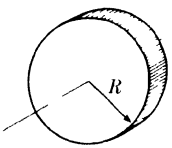
$\frac{1}{3} mL^2$

(b) Slender rod, axis through one end



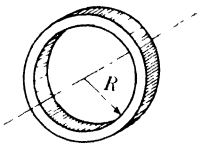
$\frac{1}{2} m(R_1^2 + R_2^2)$

(c) Annular cylinder



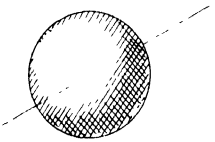
$\frac{1}{2} mR^2$

(d) Solid cylinder



mR^2

(e) Thin-walled hollow cylinder



$\frac{2}{5} mR^2$

(f) Solid sphere

FIG. 9–7. Moments of inertia.

k from an axis, or mk^2. Since this equals the actual moment of inertia, I, then

$$mk^2 = I,$$

$$\boxed{k = \sqrt{I/m}.} \qquad (9\text{–}14)$$

Equation (9–14) may be considered the definition of radius of gyration.

EXAMPLE. What is the radius of gyration of a slender rod of mass m and length L about an axis perpendicular to its length and passing through the center?

The moment of inertia about an axis through the center is $I_0 = \frac{1}{12}mL^2$. Hence

$$k_0 = \sqrt{(mL^2/12)/m} = L/(2\sqrt{3}) = 0.289L.$$

The radius of gyration, like the moment of inertia, depends on the location of the axis.

Note carefully that, in general, the mass of a body can*not* be considered as concentrated at its center of gravity for the purpose of computing its

moment of inertia. For example, when a rod is pivoted about its center, the distance from the axis to the center of gravity is zero, although the radius of gyration is $L/2\sqrt{3}$.

9–7 Work and power in rotational motion. Suppose a force **F** acts as shown in Fig. 9–8 at the rim of a pivoted wheel of radius R while the wheel rotates through a small angle $\Delta\theta$. If this angle is small enough, the force may be regarded as constant during the correspondingly small time interval. By definition, the work of the force **F** is

$$\Delta W = F\,\Delta s.$$

But $\Delta s = R\,\Delta\theta$, so that

$$\Delta W = FR\,\Delta\theta.$$

But FR is the torque, Γ, due to the force **F**, so we have finally

$$\boxed{\Delta W = \Gamma\,\Delta\theta.} \qquad (9\text{–}15)$$

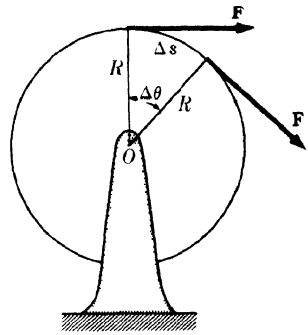


FIGURE 9–8

If the torque is constant while the angle changes by a finite amount from θ_1 to θ_2,

$$W = \Gamma(\theta_2 - \theta_1).$$

That is, the work done by a constant torque equals the product of the torque and the angular displacement.

If Γ is expressed in pound·feet, the work is in foot·pounds. If Γ is in meters·newton, the work is in joules; and if Γ is in centimeters·dyne, the work is in ergs.

When both sides of Eq. (9–15) are divided by the small time interval Δt, we obtain

$$\Delta W/\Delta t = \Gamma\,\Delta\theta/\Delta t\,.$$

But $\Delta W/\Delta t$ is the rate of doing work or the power, and $\Delta\theta/\Delta t$ is the angular velocity. Hence

$$\boxed{P = \Gamma\omega.} \qquad (9\text{–}16)$$

That is, the instantaneous power developed by an agent exerting a torque equals the product of the torque and the instantaneous angular velocity. This is the analog of $P = Fv$ for linear motion.

EXAMPLE. The drive shaft of an automobile rotates at 3600 rpm and transmits 80 hp from the engine to the rear wheels. Compute the torque developed by the engine.

$$\omega = 3600 \times \frac{2\pi}{60} = 120\pi \frac{\text{rad}}{\text{sec}},$$

$$80 \text{ hp} = 44{,}000 \frac{\text{ft·lb}}{\text{sec}},$$

$$\Gamma = \frac{P}{\omega} = \frac{44{,}000 \text{ ft·lb/sec}}{120\pi \text{ rad/sec}} = 117 \text{ lb·ft}.$$

9–8 Torque and angular acceleration. If a number of forces act on a body pivoted about a fixed axis, the torque Γ in Eq. (9–15) must be replaced by the resultant torque $\sum\Gamma$, so in general the work ΔW done in a small angular displacement $\Delta\theta$ is

$$\Delta W = (\sum\Gamma) \cdot \Delta\theta.$$

If the resultant torque remains constant during a finite angular displacement θ, then the work W is given by

$$W = (\sum\Gamma) \cdot \theta.$$

Experiment shows that *a constant resultant torque causes a constant angular acceleration*, just as a constant resultant force produces a constant linear acceleration. When, therefore, a constant resultant torque acts on a body, the body undergoes a constant angular acceleration α, and its angular velocity changes from its original value ω_0 to its final value ω, according to the equation developed earlier in this chapter,

$$\omega^2 = \omega_0^2 + 2\alpha\theta. \tag{9–17}$$

The kinetic energy of rotation changes from $\frac{1}{2}I\omega_0^2$ to $\frac{1}{2}I\omega^2$, and this change is equal to the work done. Thus

$$W = (\sum\Gamma) \cdot \theta = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2,$$

or

$$\sum\Gamma = I\frac{\omega^2 - \omega_0^2}{2\theta}.$$

Since, from Eq. (9–17), $(\omega^2 - \omega_0^2)/2\theta = \alpha$, we write finally

$$\boxed{\sum\Gamma = I\alpha.} \tag{9–18}$$

This is the rotational analog of Newton's second law, $\sum F = ma$, for rotation of a rigid body about a fixed axis.

EXAMPLE. A rope is wrapped around the surface of a flywheel 2 ft in radius, and a 10-lb weight hangs from the rope (Fig. 9–9). The wheel is free to rotate about a horizontal axis through its center. Compute its angular acceleration and the tension in the rope if the moment of inertia of the wheel is 1.5 slug·ft^2.

Isolate the flywheel, as in Fig. 9–9(b). Taking moments about the axis,

$$\sum \Gamma = I\alpha,$$

$$2 \text{ ft} \times T = 1.5 \text{ slug·ft}^2 \times \alpha.$$

Isolating the 10-lb weight, as in Fig. 9–9(c),

$$\sum F = ma,$$

$$10 \text{ lb} - T = \frac{10}{32} \text{ slugs } a.$$

Also, since the linear acceleration of the weight equals the tangential acceleration of the surface of the flywheel,

$$a = R\alpha = 2 \text{ ft} \times \alpha.$$

Simultaneous solution of these equations gives

$$a = 14.5 \frac{\text{ft}}{\text{sec}^2}, \qquad \alpha = 7.27 \frac{\text{rad}}{\text{sec}^2}, \qquad T = 5.45 \text{ lb}.$$

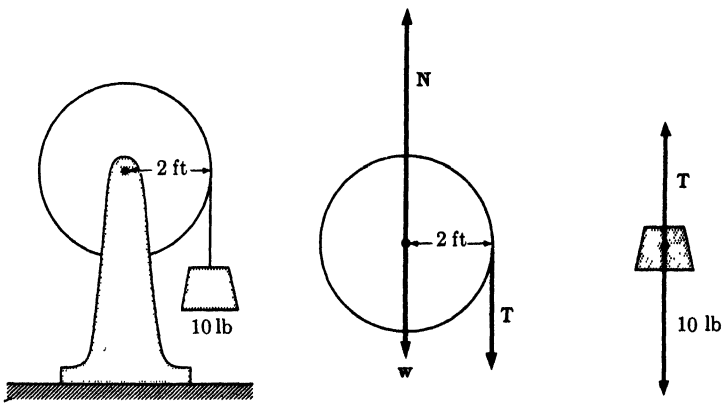


FIGURE 9–9

9–9 Angular momentum and angular impulse. Figure 9–10(a) represents a small body of mass m moving in the plane of the diagram with a speed v and a momentum mv. We define its *angular momentum* about an axis through O perpendicular to the plane of the diagram, as the product of its linear momentum and the perpendicular distance from the axis to its line of motion. That is,

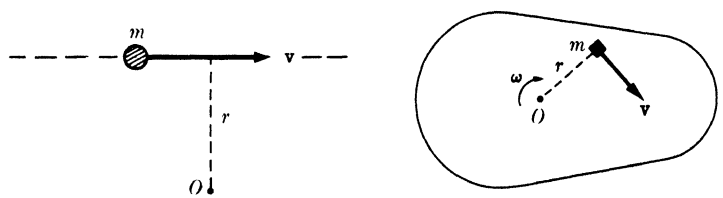
$$\text{Angular momentum of a particle} = mvr.$$

It will be seen that angular momentum is defined in the same way as the moment of a force, and it is often referred to as *moment of momentum*.

Figure 9–10(b) represents a body of finite size rotating in the plane of the diagram about an axis through O. The velocity v of a small element of the body is related to the angular velocity of the body by $v = \omega r$. The angular momentum of the element is therefore $mvr = \omega mr^2$, and the total angular momentum of the body is $\sum\omega mr^2 = \omega\sum mr^2$. But $\sum mr^2$ is the moment of inertia of the body about its axis of rotation. Hence the angular momentum can be written as $I\omega$. In this form it is completely analogous to linear momentum mv. Thus, denoting angular momentum by the symbol L,

$$\boxed{L = I\omega.} \tag{9–19}$$

Figure 9–11(a) represents two disks with moments of inertia I and I', respectively, rotating with angular velocities ω_0 and ω'_0. Suppose at moment t_0 the two disks engage and, after a very short interval of time, say $t - t_0$, the disks have the same final angular velocity ω, as in Fig. 9–11(b). (Subscript zero means before engaging, and the absence of a subscript, after.) During the very short time interval necessary to cause these changes in angular velocity, the larger disk exerts a torque Γ on the smaller, and the smaller exerts a torque Γ' on the larger. From Newton's third law Γ and Γ' are at all times equal in magnitude and oppositely directed. That is, $\Gamma = -\Gamma'$. Both Γ and Γ' vary during the



(a) (b)

FIG. 9–10. Angular momentum.

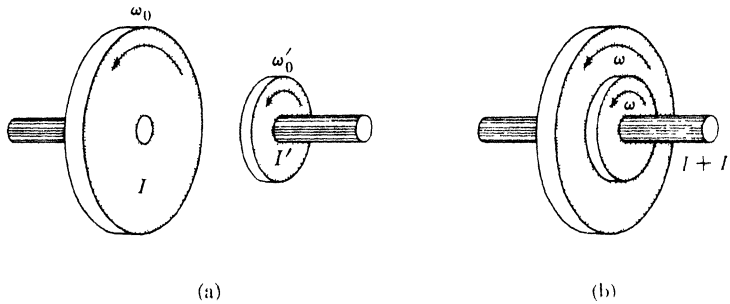


FIG. 9–11. An impulsive torque acts when two rotating disks engage.

contact. Both are zero before contact, both are small at the first instant of contact, then both increase to a maximum, and both decrease and become zero when the disks attain their final angular velocity.

A torque that varies with the time in this manner is called an *impulsive torque* and is plotted on a torque-time diagram in Fig. 9–12(a) and is seen to have the same characteristics as those of the impulsive force shown in Fig. 8–2(a). Suppose the total time interval $t - t_0$ is subdivided into any number of parts Δt_1, Δt_2, . . . , and rectangles are drawn as in Fig. 9–12(b) whose jagged outline approximates the smooth curve. As before, the larger the number of subdivisions, the more closely will the smooth curve be approximated and the more closely will the sum of the areas of the rectangles approach the area under the smooth curve.

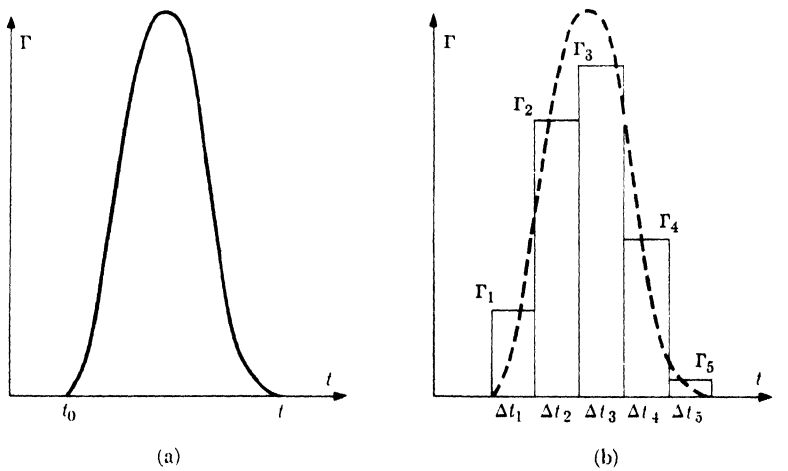


FIGURE 9–12

Let us therefore replace the continuously varying impulsive torque by a torque which remains constant at the value Γ_1 during the time interval Δt_1, then jumps to the value Γ_2, maintaining this value for a time Δt_2, and so on. If the angular velocity of the disk of moment of inertia I changes from its original value ω_0 to the value ω_1 at the end of the first time interval, we have, from Eq. (9–18)

$$\Gamma_1 = I \frac{\omega_1 - \omega_0}{\Delta t_1}$$

or

$$\Gamma_1 \, \Delta t_1 = I\omega_1 - I\omega_0,$$

and similarly for all the other intervals

$$\Gamma_2 \, \Delta t_2 = I\omega_2 - I\omega_1,$$

$$\Gamma_3 \, \Delta t_3 = I\omega_3 - I\omega_2,$$

$$\Gamma_4 \, \Delta t_4 = I\omega_4 - I\omega_3,$$

$$\Gamma_5 \, \Delta t_5 = I\omega_5 - I\omega_4,$$

where ω is the angular velocity at the end of the last time interval. Adding these equations, we get

$$\sum \Gamma \, \Delta t = I\omega - I\omega_0.$$

If we now imagine the total time interval subdivided into a larger and larger number of steps, the left member becomes the area under the smooth curve, while the right member remains unaffected. We have therefore the *rigorous* result that the area under the $\Gamma - t$ curve is equal to the *change* in the value of the product $I\omega$.

The area under the $\Gamma - t$ curve is called the *impulse* of the torque (or the *angular impulse*) and is represented by the symbol J_θ, thus

$$\text{Angular impulse } J_\theta = \left\{ \begin{matrix} \text{Area under} \\ \Gamma - t \text{ curve} \end{matrix} \right\}.$$

We have, therefore,

$$\boxed{\begin{aligned} J_\theta &= I\omega - I\omega_0, \\ J'_\theta &= I'\omega - I'\omega'_0. \end{aligned}} \qquad (9\text{–}20)$$

Equations (9–20) may therefore be stated verbally as follows: *The angular impulse acting on either body is equal to the change in angular momentum of that body about the same axis.*

9–10 Conservation of angular momentum. Since the torque Γ acting on the smaller disk of Fig. 9–11 is at all times equal in magnitude and opposite in direction to the torque Γ' acting on the larger disk, we have

$$J_\theta = -J'_\theta$$

and therefore, from Eqs. (9–20),

$$I\omega - I\omega_0 = -(I'\omega - I'\omega'_0)$$

or

$$\boxed{I\omega_0 + I'\omega'_0 = (I + I')\omega.} \tag{9–21}$$

The left side of Eq. (9–21) is the total angular momentum of the system (consisting of both disks) before engaging, and the right side is the total angular momentum after engaging. We have therefore derived the important result that the total angular momentum of the whole system is unaltered. When both disks are regarded as one system, then the torques Γ and Γ' are *internal* torques and during the engaging process no external torque acts. We therefore have the result that *if the resultant external torque on a system is zero, the angular momentum of the system remains constant;* and hence any interaction between the parts of a system cannot alter its total angular momentum. This is the *principle of conservation of angular momentum*, and it ranks with the principles of conservation of linear momentum and conservation of energy as one of the most fundamental of physical laws.

A circus acrobat, a diver, or a skater performing a pirouette on the toe of one skate, all take advantage of the principle. Suppose an acrobat has just left a swing as in Fig. 9–13, with arms and legs extended and with a small clockwise angular momentum. When he pulls his arms and legs in, his moment of inertia I becomes much smaller. Since his angular momentum $I\omega$ remains constant and I decreases, his angular velocity ω increases.

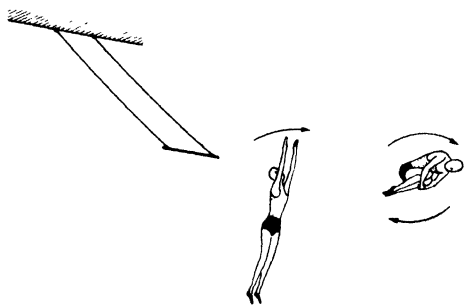


FIG. 9–13. Conservation of angular momentum.

EXAMPLE 1. A 10-lb disk with a radius of gyration of 9 inches and rotating with an angular velocity of 120 rpm engages with another disk of weight 15 lb, radius of gyration 12 inches, originally rotating in the same direction as the first with an angular velocity of 40 rpm. What is the final angular velocity of both disks?

$$I = \frac{w}{g}k^2 = \frac{10}{32}\text{ slugs}\left(\frac{9}{12}\text{ ft}\right)^2 = 0.176\text{ slug}\cdot\text{ft}^2,$$

$$I' = \frac{w'}{g}k'^2 = \frac{15}{32}\text{ slugs }(1\text{ ft})^2 = 0.468\text{ slug}\cdot\text{ft}^2,$$

$$\omega_0 = \frac{120\times 2\pi}{60} = 4\pi\frac{\text{rad}}{\text{sec}},$$

$$\omega_0' = \frac{40\times 2\pi}{60} = \frac{4}{3}\pi\frac{\text{rad}}{\text{sec}}.$$

Since $I\omega_0 + I'\omega_0' = (I + I')\omega$,

$$\omega = \frac{0.176\text{ slug}\cdot\text{ft}^2\times 4\pi\text{ rad/sec} + 0.468\text{ slug}\cdot\text{ft}^2\times\frac{4}{3}\pi\text{ rad/sec}}{(0.176 + 0.468)\text{ slug ft}^2}$$

$$= 2.19\pi\frac{\text{rad}}{\text{sec}} = 65.7\text{ rpm}.$$

EXAMPLE 2. A man stands at the center of a turntable, holding his arms extended horizontally with a 10-lb weight in each hand. He is set rotating about a vertical axis with an angular velocity of one revolution in 2 sec. Find his new angular velocity if he drops his hands to his sides. The moment of inertia of the man may be assumed constant and equal to 4 slugs·ft^2. The original distance of the weights from the axis is 3 ft, and their final distance is 6 inches.

If friction in the turntable is neglected, no external torques act about a vertical axis and the angular momentum about this axis is constant. That is,

$$I\omega = (I\omega)_0 = I_0\omega_0,$$

where I and ω are the final moment of inertia and angular velocity, and I_0 and ω_0 are the initial values of these quantities.

$$I = I_{\text{man}} + I_{\text{weights}},$$

$$I = 4 + 2\left(\frac{10}{32}\right)\left(\frac{1}{2}\right)^2 = 4.16\text{ slugs}\cdot\text{ft}^2,$$

$$I_0 = 4 + 2\left(\frac{10}{32}\right)(3)^2 = 9.63\text{ slugs}\cdot\text{ft}^2,$$

$$\omega_0 = \pi\frac{\text{rad}}{\text{sec}},$$

$$\omega = \omega_0\frac{I_0}{I} = 2.31\pi\frac{\text{rad}}{\text{sec}}.$$

That is, the angular velocity is more than doubled.

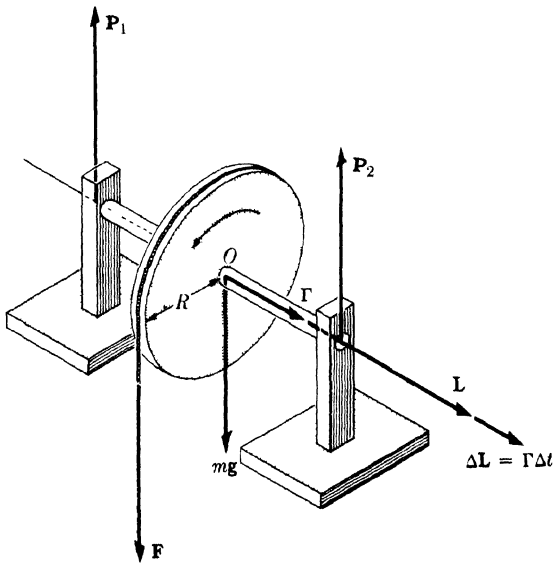


FIG. 9–14. Vector $\Delta\mathbf{L}$ is the change in angular momentum produced in time Δt by the moment $\boldsymbol{\Gamma}$ of the force $\mathbf{F}$. Vectors $\Delta\mathbf{L}$ and $\boldsymbol{\Gamma}$ are in the same direction.

9–11 Vector representation of angular quantities. It is an important point that a quantity associated with an axis, such as angular velocity, angular acceleration, etc., can be represented by a vector along the axis. Thus, torque $\boldsymbol{\Gamma}$ and angular momentum $\mathbf{L}$ are vector quantities. In Fig. 9–14, a disk is mounted on a shaft through its center, the shaft being supported by fixed bearings. If the disk is rotating as indicated, its angular momentum vector $\mathbf{L}$ points toward the right, along the axis.

Suppose a cord is wrapped around the rim of the disk and a force $\mathbf{F}$ is exerted on the cord. The resultant torque on the disk is $\Gamma = FR$, and the torque vector $\boldsymbol{\Gamma}$ also points along the axis. In a time Δt, the torque produces a vector change $\Delta\mathbf{L}$ in the angular momentum, equal to $\boldsymbol{\Gamma}\,\Delta t$ and having the same direction as $\boldsymbol{\Gamma}$. When this change is added vectorially to the original angular momentum $\mathbf{L}$, the resultant is a vector of length $\mathbf{L} + \Delta\mathbf{L}$, in the same direction as $\mathbf{L}$. In other words, the *magnitude* of the angular momentum is increased, its *direction* remaining the same. An increase in the magnitude of the angular momentum simply means that the body rotates more rapidly.

The lengthy argument above appears at first to be nothing more than a difficult way of solving an easy problem in rotation about a fixed axis. However, the vector nature of torque and angular momentum are essential to an understanding of the gyroscope, to be discussed in the next section.

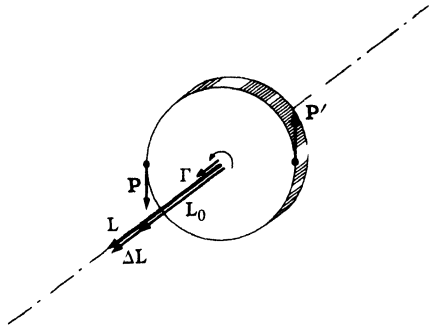


FIG. 9–15. Vector $\Delta\mathbf{L}$ is the change in angular momentum produced by the couple $\mathbf{P}$ - $\mathbf{P}'$.

EXAMPLE. A couple consisting of the two forces $\mathbf{P}$ and $\mathbf{P}'$, each of magnitude 4 lb, is applied for 3 sec to a disk of radius 1.5 ft and moment of inertia 20 slugs·ft^2, pivoted about an axis through its center as in Fig. 9–15. The initial angular velocity of the disk is 5 rad/sec. Show in a vector diagram the torque, the initial angular momentum, and the final angular momentum.

From the preceding discussion, the vectors representing the torque $\boldsymbol{\Gamma}$ due to the couple, and the initial angular momentum $\mathbf{L}_0$, are directed as in Fig. 9–15. The magnitude of the initial angular momentum is

$$L_0 = I\omega_0 = 20 \times 5 = 100 \frac{\text{slugs}\cdot\text{ft}^2}{\text{sec}}.$$

The angular impulse, which was defined as the area under the torque-time curve, is in this problem merely the product of the constant torque Γ and the time interval Δt. Since this must equal the change in angular momentum, we have

$$\Gamma\,\Delta t = I\omega - I\omega_0 = \Delta(I\omega) = \Delta L.$$

But $\Gamma = 4\text{ lb} \times 3\text{ ft} = 12\text{ lb}\cdot\text{ft}$, and $\Delta t = 3$ sec. Hence

$$\Delta L = 12 \times 3 \frac{\text{lb}\cdot\text{ft}}{\text{sec}} = 36 \frac{\text{slugs}\cdot\text{ft}^2}{\text{sec}}.$$

This increase has been represented by the vector $\Delta\mathbf{L}$ in Fig. 9–15.

The final angular momentum, $\mathbf{L}$, is the vector sum of $\mathbf{L}_0$ and $\Delta\mathbf{L}$. Since both are in the same direction, the vector sum is simply the arithmetic sum. That is

$$L = L_0 + \Delta L = 136 \frac{\text{slugs}\cdot\text{ft}^2}{\text{sec}}.$$

9–12 The top and the gyroscope. A symmetrical object rotating about an axis, one point of which is fixed, is called a *top*. If the fixed point is at the center of gravity, the object is called a *gyroscope*. The axis of rotation

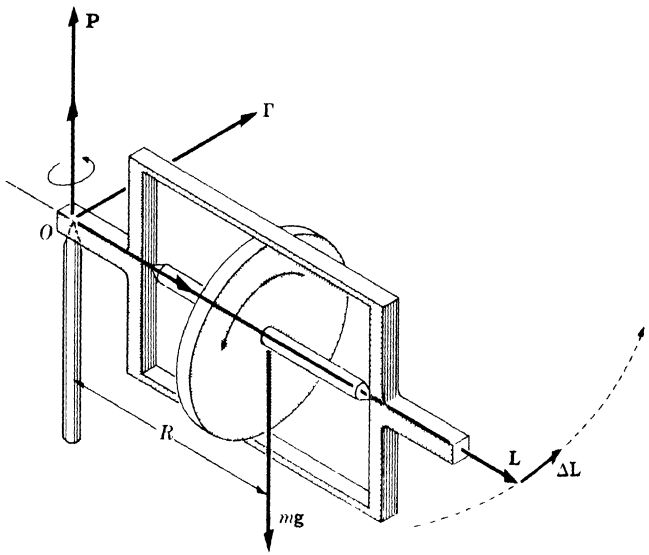


FIG. 9–16. Vector $\Delta\mathbf{L}$ is the change in angular momentum produced in time Δt by the moment $\boldsymbol{\Gamma}$ of the force $m\mathbf{g}$. Vectors $\Delta\mathbf{L}$ and $\boldsymbol{\Gamma}$ are in the same direction.

of a top or gyroscope can itself rotate about the fixed point. When this is the case, the angular momentum vector no longer lies on the axis. However, if the angular velocity *of* the axis is small compared with the angular velocity *about* the axis, the component of angular momentum arising from the former effect is small and we shall neglect it.

Figure 9–16 illustrates the usual mounting of a toy gyroscope, more properly called a top, since the fixed point O is not at the center of gravity. Vector $\mathbf{L}$ is the angular momentum. The upward force $\mathbf{P}$ at the pivot has no moment about this point, and the resultant moment is that due to the weight $m\mathbf{g}$ and given by

$$\Gamma = mgR.$$

Its direction is perpendicular to the axis of rotation as shown. In a time Δt this torque produces a change $\Delta\mathbf{L}$ in the angular momentum, having the same direction as $\boldsymbol{\Gamma}$ and given by

$$\Delta\mathbf{L} = \boldsymbol{\Gamma}\,\Delta t.$$

The angular momentum $\mathbf{L} + \Delta\mathbf{L}$, after a time Δt, is the vector sum of $\mathbf{L}$ and $\Delta\mathbf{L}$. Since $\Delta\mathbf{L}$ is perpendicular to $\mathbf{L}$ and is very small, the new angular momentum vector has the same *magnitude* as the old but a different *direction*. The tip of the angular momentum vector moves as shown,

and as time goes on it swings around a horizontal circle. But since the angular momentum vector lies along the gyroscope axis, the axis turns also, rotating in a horizontal plane about the point P. This motion of the axis of rotation is called *precession*.

The angle $\Delta\phi$, through which the vector $\mathbf{L}$ turns in time Δt, is given by

$$\Delta\phi = \frac{\Delta L}{L} = \frac{\Gamma\,\Delta t}{L}.$$

The *angular velocity of precession* Ω is $\Delta\phi/\Delta t$, and therefore

$$\boxed{\Omega = \frac{\Gamma}{L}.} \tag{9-22}$$

The angular velocity of precession is therefore inversely proportional to the angular momentum. If this is large, the precessional angular velocity will be small.

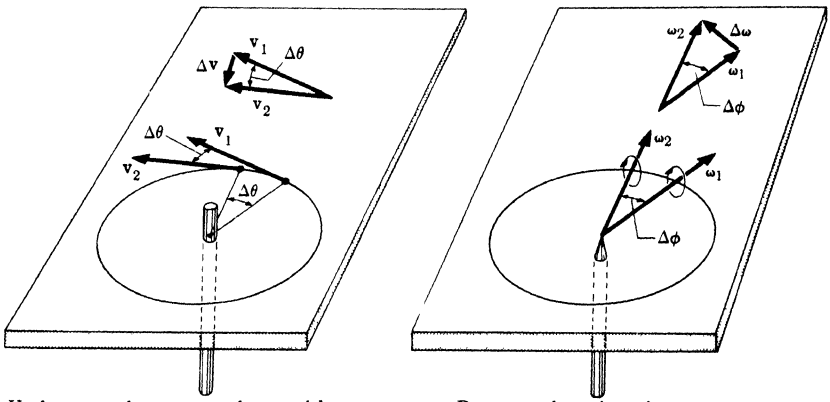
A gyroscope does not necessarily retain a fixed direction in space. We can make the gyroscope turn in any direction we please by applying the appropriate torque.

Why doesn't the gyroscope in Fig. 9–16 fall? The answer is that the upward force $\mathbf{P}$ exerted on it by the pivot is just equal to its weight $m\mathbf{g}$, so that the resultant vertical *force* is zero and the vertical acceleration of the center of gravity is zero. In other words, the vertical component of its linear momentum remains zero, since there is no resultant vertical force. The two forces $\mathbf{P}$ and $m\mathbf{g}$ constitute a *couple* of moment $\Gamma = mgR$, so the resultant moment is *not* zero and the angular momentum changes. If the gyroscope were not rotating, it would have no angular momentum $\mathbf{L}$ to start with. Its angular momentum after a time Δt would be that acquired from the couple acting on it and would be in the same direction as the moment of this couple. In other words, the gyroscope would rotate about an axis through O in the direction of the vector $\boldsymbol{\Gamma}$. But if the gyroscope is originally rotating, the change in its angular momentum produced by the couple adds vectorially to the large angular momentum it already has, and since $\Delta\mathbf{L}$ is horizontal and perpendicular to $\mathbf{L}$, the result is a motion of precession with both the angular momentum vector and the axis remaining horizontal.

To understand why the vertical force $\mathbf{P}$ should equal $m\mathbf{g}$, we must look further into the way in which the precessional motion in Fig. 9–16 originated. If the frame of the gyroscope is initially held at rest, say by supporting the projecting portion of the frame opposite O with one's finger, the upward forces exerted by the finger and by the pivot are each equal to $m\mathbf{g}/2$. If the finger is suddenly removed, the upward force at O,

at the first instant, is still $m\mathbf{g}/2$. The resultant vertical *force* is not zero, and the center of gravity has an initial downward acceleration. At the same time, precessional motion begins, although with a smaller angular velocity than that in the final steady state. The result of this motion is to cause the end of the frame at O to press down on the pivot with a greater force, so that the upward force at O increases and eventually becomes greater than $m\mathbf{g}$. When this happens, the center of gravity starts to accelerate upward. The process repeats itself and the motion consists of a precession together with an up-and-down oscillation of the axis called *nutation*.

To start the gyroscope off with pure precession, it is necessary to give the outer end of the axis a push in the direction in which it would normally precess. This causes the end of the frame at O to bear down on the pivot so that the upward force at O increases. When this force equals $m\mathbf{g}$, the



Uniform circular motion of a particle

Precessional motion of a top

$$v_1 = v_2 = v \qquad \omega_1 = \omega_2 = \omega$$

$$\mathbf{v}_2 = \mathbf{v}_1 + \Delta\mathbf{v} \qquad \boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \Delta\boldsymbol{\omega}$$

$$\Delta v \simeq v\,\Delta\theta \qquad \Delta\omega \simeq \omega\,\Delta\phi$$

$$\lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = v\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t} \qquad \lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} = \omega\lim_{\Delta t\to 0}\frac{\Delta\phi}{\Delta t}$$

$$a = v\omega\left(= \frac{v^2}{R} = R\omega^2\right) \qquad \alpha = \omega\Omega$$

$$F = ma \qquad \Gamma = I\alpha$$

$$F = mv\omega \qquad \Gamma = I\omega\Omega$$

FIG. 9–17. Analogy between uniform circular motion of a particle and precessional motion of a top.

vertical forces are in equilibrium, the outer end can be released, and the precession continues as in Fig. 9–16.

From a purely kinematic point of view, precessional motion of a top is the rotational analog of uniform circular motion of a particle, as pointed out by Benfield in 1958. The analogy is illustrated in Fig. 9–17, where the diagram and calculation for uniform circular motion are shown on he left and the corresponding diagram and calculation for precessional motion of a top are displayed on the right.

Problems

9–1. (a) What angle in radians is subtended by an arc 6 ft in length, on the circumference of a circle whose radius is 4 ft? (b) What angle in radians is subtended by an arc of length 78.54 cm on the circumference of a circle of diameter 100 cm? What is this angle in degrees? (c) The angle between two radii of a circle is 0.60 radian. What length of arc is intercepted on the circumference of a circle of radius 200 cm? of radius 200 ft?

9–2. Compute the angular velocity, in rad/sec, of the crankshaft of an automobile engine rotating at 4800 rev/min.

9–3. (a) A cylinder 6 inches in diameter rotates in a lathe at 750 rev/min. What is the tangential velocity of the surface of the cylinder? (b) The proper tangential velocity for machining cast iron is about 2 ft/sec. At how many rev/min should a piece of stock 2 inches in diameter be rotated in a lathe?

9–4. An electric motor running at 1800 rev/min has on its shaft three pulleys, of diameters 2, 4, and 6 in. respectively. Find the linear velocity of the surface of each pulley in ft/sec. The pulleys may be connected by a belt to a similar set on a countershaft, the 2 in. to the 6 in., the 4 in. to the 4 in., and the 6 in. to the 2 in. Find the three possible angular velocities of the countershaft in rev/min.

9–5. A wheel 2.4 ft in diameter starts from rest and accelerates uniformly to an angular velocity of 100 rad/sec in 20 sec. Find the angular acceleration and the angle turned through.

9–6. The angular velocity of a flywheel decreases uniformly from 1000 rev/min to 400 rev/min in 5 sec. Find the angular acceleration and the number of revolutions made by the wheel in the 5-sec interval. How many more seconds are required for the wheel to come to rest?

9–7. A flywheel requires 3 sec to rotate through 234 radians. Its angular velocity at the end of this time is 108 rad/sec. Find its constant angular acceleration.

9–8. A flywheel whose angular acceleration is constant and equal to 2 rad/sec^2, rotates through an angle of 100 radians in 5 sec. How long had it been in motion at the beginning of the 5-sec interval if it started from rest?

9–9. (a) Distinguish clearly between tangential and radial acceleration. (b) A flywheel rotates with constant angular velocity. Does a point on its rim have a tangential acceleration? a radial acceleration? (c) A flywheel is rotating with constant angular acceleration. Does a point on its rim have a tangential acceleration? a radial acceleration? Are these accelerations constant in magnitude?

9–10. A wheel 30 inches in diameter is rotating about a fixed axis with an initial angular velocity of 2 rev/sec. The acceleration is 3 rev/sec^2. (a) Compute the angular velocity after 6 sec. (b) Through what angle has the wheel turned in this time interval? (c) What is the tangential velocity of a point on the rim of the wheel at $t = 6$ sec? (d) What is the resultant acceleration of a point on the rim of the wheel at $t = 6$ sec?

9–11. A wheel having a diameter of 1 ft starts from rest and accelerates uniformly to an angular velocity of 900 rev/min in 5 sec. (a) Find the position at the end of 1 sec of a point originally at the top of the wheel. (b) Compute and show in a diagram the magnitude

and direction of the acceleration at the end of 1 sec.

9–12. A flywheel of radius 30 cm starts from rest and accelerates with a constant angular acceleration of 0.50 rad/sec^2. Compute the tangential acceleration, the radial acceleration, and the resultant acceleration, of a point on its rim (a) at the start, (b) after it has turned through 120°, (c) after it has turned through 240°.

9–13. A wheel starts from rest and accelerates uniformly to an angular velocity of 900 rmp in 20 sec. At the end of 1 sec, (a) find the angle through which the wheel has rotated, and (b) compute and show in a diagram the magnitude and direction of the tangential and radial components of acceleration of a point 6 in. from the axis.

9–14. A wheel rotates with a constant angular velocity of 10 rad/sec. (a) Compute the radial acceleration of a point 2 ft from the axis, from the relation $a_R = \omega^2 R$. (b) Find the tangential velocity of the point, and compute its radial acceleration from the relation $a_R = v^2/R$.

9–15. Find the required angular velocity of an ultracentrifuge, in rpm, in order that the radial acceleration of a point 1 cm from the axis shall equal 300,000 g (i.e., 300,000 times the acceleration of gravity).

9–16. (a) Prove that when a body starts from rest and rotates about a fixed axis with constant angular acceleration, the radial acceleration of a point in the body is directly proportional to its angular displacement. (b) Through what angle will the body have turned when the resultant acceleration makes an angle of 60° with the radial acceleration?

9–17. Find the moment of inertia of a rod 4 cm in diameter and 2 m long, of mass 8 kgm, (a) about an axis perpendicular to the rod and passing through its center, (b) about an axis perpendicular to the rod and passing through one end, (c) about a longitudinal axis through the center of the rod.

9–18. The inner radius of a hollow cylinder is 3 inches, the outer radius is 4 inches, and the length is 6 inches. What is the radius of gyration of the cylinder about its axis?

9–19. The four bodies shown in Fig. 9–18 have equal masses m. Body A is a solid cylinder of radius R. Body B is a hollow thin cylinder of radius R. Body C is a solid square with length of side = $2R$. Body D is the same size as C, but hollow (i.e., made up of four thin sticks). The bodies have axes of rotation perpendicular to the page and through the center of gravity of each body. (a) Which body has the smallest moment of inertia? (b) Which body has the largest moment of inertia?

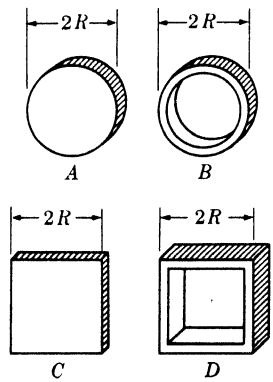


FIGURE 9–18

9–20. Small blocks, each of mass m, are clamped at the ends and at the center of a light rigid rod of length L. Compute the moment of inertia and the radius of gyration of the system about an axis perpendicular to the rod and passing through a point one-

quarter of the length from one end. Neglect the moment of inertia of the rod.

9–21. The radius of the earth is 4000 mi and its mass is 4×10^{21} slugs (approximately). Find (a) its moment of inertia about an axis through its center, and (b) its radius of gyration in miles. Assume the density to be uniform.

9–22. A flywheel consists of a solid disk 1 ft in diameter and 1 inch thick, and two projecting hubs 4 inches in diameter and 3 inches long. If the material of which it is constructed weighs 480 lb/ft^3, find (a) its moment of inertia, and (b) its radius of gyration about the axis of rotation.

9–23. A grinding wheel 6 inches in diameter, weighing 4 lb, is rotating at 3600 rpm. (a) What is its kinetic energy? (b) How far would it have to fall to acquire the same kinetic energy?

9–24. A disk of mass m and radius R is pivoted about a horizontal axis through its center, and a small body of mass m is attached to the rim of the disk. If the disk is released from rest with the small body at the end of a horizontal radius, find the angular velocity when the small body is at the bottom.

9–25. The flywheel of a gasoline engine is required to give up 380 ft·lb of kinetic energy while its angular velocity decreases from 600 rpm to 540 rpm. What moment of inertia is required?

9–26. The flywheel of a punch press has a moment of inertia of 15 slug·ft^2 and it runs at 300 rev/min. The flywheel supplies all the energy needed in a quick punching operation. (a) Find the speed in rev/min to which the flywheel will be reduced by a sudden punching operation requiring 4500 ft·lb of work. (b) What must be the constant power supply to the flywheel in horsepower to bring it back to its initial speed in 5 sec?

9–27. A magazine article described a passenger bus in Zurich, Switzerland, which derived its motive power from the energy stored in a large flywheel. The wheel was brought up to speed periodically, when the bus stopped at a station, by an electric motor which could then be attached to the electric power lines. The flywheel was a solid cylinder of mass 1000 kgm, diameter 180 cm, and its top speed was 3000 rev/min. (a) At this speed, what is the kinetic energy of the flywheel? (b) If the average power required to operate the bus is 25 hp, how long can it operate between stops?

9–28. A grindstone in the form of a solid cylinder has a radius of 2 ft and weighs 96 lb. (a) What torque will bring it from rest to an angular velocity of 300 rev/min in 10 sec? (b) What is its kinetic energy when rotating at 300 rev/min?

9–29. The flywheel of a motor weighs 640 lb and has a radius of gyration of 4 ft. The motor develops a constant torque of 1280 lb·ft, and the flywheel starts from rest. (a) What is the angular acceleration of the flywheel? (b) What will be its angular velocity after making 4 revolutions? (c) How much work is done by the motor during the first 4 revolutions?

9–30. The flywheel of a stationary engine has a moment of inertia of 20 slugs·ft^2. (a) What constant torque is required to bring it up to an angular velocity of 900 rpm in 10 sec, starting from rest? (b) What is its final kinetic energy?

9–31. A grindstone 3 ft in diameter, weighing 96 lb, is rotating at 900 rpm. A tool is pressed normally against the rim with a force of 45 lb, and the grind-

stone comes to rest in 10 sec. Find the coefficient of friction between the tool and the grindstone. Neglect friction in the bearings.

9–32. A constant torque of 20 newtons·meter is exerted on a pivoted wheel for 10 sec, during which time the angular velocity of the wheel increases from zero to 100 rpm. The external torque is then removed and the wheel is brought to rest by friction in its bearings in 100 sec. Compute (a) the moment of inertia of the wheel, (b) the friction torque, (c) the total number of revolutions made by the wheel.

9–33. A cord is wrapped around the rim of a flywheel 2 ft in radius, and a steady pull of 10 lb is exerted on the cord as in Fig. 9–19(a). The wheel is mounted in frictionless bearings on a horizontal shaft through its center. The moment of inertia of the wheel is 2 slugs·ft^2. (a) Compute the angular acceleration of the wheel. (b) Show that the work done in unwinding 20 ft of cord equals the gain in kinetic energy of the wheel. (c) If a 10-lb weight hangs from the cord as in Fig. 9–19(b), compute the angular acceleration of the wheel. Why is this not the same as in part (a)?

9–34. A solid cylinder of mass 15 kgm, 30 cm in diameter, is pivoted about a horizontal axis through its center, and a rope wrapped around the surface of the cylinder carries at its end a block of mass 8 kgm. (a) How far does the block descend in 5 sec, starting from rest? (b) What is the tension in the rope? (c) What is the force exerted on the cylinder by its bearings?

9–35. A bucket of water weighing 64 lb is suspended by a rope wrapped around a windlass in the form of a solid cylinder 1 ft in diameter, also weighing 64 lb. The bucket is released from rest at the top of a well and falls 64 ft to the water. (a) What is the tension in the rope while the bucket is falling? (b) With what velocity does the bucket strike the water? (c) What was the time of fall? Neglect the weight of the rope.

9–36. A 16-lb block rests on a horizontal frictionless surface. A cord attached to the block passes over a pulley, whose diameter is 6 inches, to a hanging block which also weighs 16 lb. The system is released from rest, and the blocks are observed to move 16 ft in 2 sec. (a) What was the moment of inertia of the pulley? (b) What was the tension in each part of the cord?

9–37. Figure 9–20 represents an Atwood's machine. Find the linear accelerations of blocks A and B, the angular acceleration of the wheel C,

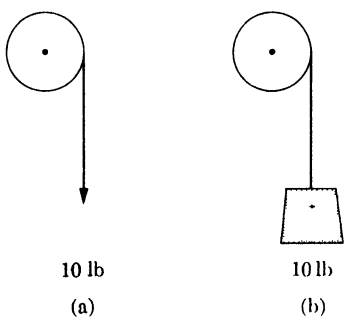


FIGURE 9–19

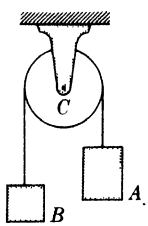


FIGURE 9–20

and the tension in each side of the cord (a) If the surface of the wheel is frictionless, (b) if there is no slipping between the cord and the surface of the wheel. Let the weights of blocks A and B be 8 lb and 4 lb respectively, the moment of inertia of the wheel about its axis be 0.125 slug·ft^2, and the radius of the wheel be 0.5 ft.

9–38. A flywheel 3 ft in diameter is pivoted on a horizontal axis. A rope is wrapped around the outside of the flywheel and a steady pull of 10 lb is exerted on the rope. It is found that 24 ft of rope are unwound in 4 sec. (a) What was the angular acceleration of the flywheel? (b) What was its final angular velocity? (c) What was its final kinetic energy? (d) What is its moment of inertia?

9–39. A light rigid rod 100 cm long has a small block of mass 50 gm attached at one end. The other end is pivoted, and the rod rotates in a vertical circle. At a certain instant the rod makes an angle of 53° with the vertical, and the tangential speed of the block is 400 cm/sec. (a) What are the horizontal and vertical components of the velocity of the block? (b) What is the moment of inertia of the system? (c) What is the radial acceleration of the block? (d) What is the tangential acceleration of the block? (e) What is the tension or compression in the rod?

9–40. (a) Compute the torque developed by an airplane engine whose output is 2000 hp at an angular velocity of 2400 rev/min. (b) If a drum 18 inches in diameter were attached to the motor shaft, and the power output of the motor were used to raise a weight hanging from a rope wrapped around the shaft, how large a weight could be lifted? (c) With what velocity would it rise?

9–41. A block of mass m = 5 kgm slides down a surface inclined 37° to the horizontal as shown in Fig. 9–21, The coefficient of sliding friction is 0.25. A string attached to the block is wrapped around a flywheel on a fixed axis at O. The flywheel has a mass M = 20 kgm, an outer radius R = 0.2 m, and a radius of gyration with respect to the axis k_0 = 0.1 m. (a) What is the acceleration of the block down the plane? (b) What is the tension in the string?

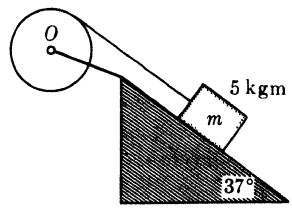


FIGURE 9–21

9–42. A man sits on a piano stool holding a pair of dumbbells at a distance of 3 ft from the axis of rotation of the stool. He is given an angular velocity of 2 rad/sec, after which he pulls the dumbbells in until they are but 1 ft distant from the axis. The moment of inertia of the man about the axis of rotation is 3 slugs·ft^2 and may be considered constant. The dumbbells weigh 16 lb each and may be considered point masses. Neglect friction. (a) What is the initial angular momentum of the system? (b) What is the angular velocity of the system after the dumbbells are pulled in toward the axis? (c) Compute the kinetic energy of the system before and after the dumbbells are pulled in. Account for the difference, if any.

9–43. A block of mass 50 gm is attached to a cord passing through a hole in a horizontal frictionless surface

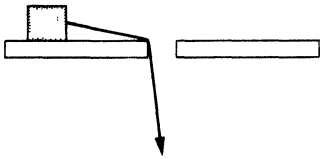


FIGURE 9–22

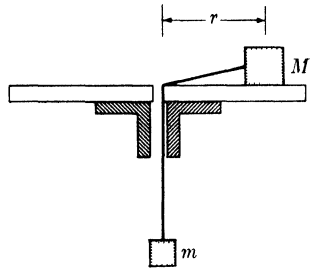


FIGURE 9–23

as in Fig. 9–22. The block is originally revolving at a distance of 20 cm from the hole with an angular velocity of 3 rad/sec. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 10 cm. The block may be considered a point mass. (a) What is the new angular velocity? (b) Find the change in kinetic energy of the block.

9–44. A small block weighing 8 lb is attached to a cord passing through a hole in a horizontal frictionless surface. The block is originally revolving in a circle of radius 2 ft about the hole with a tangential velocity of 12 ft/sec. The cord is then pulled slowly from below, shortening the radius of the circle in which the block revolves. The breaking strength of the cord is 144 lb. What will be the radius of the circle when the cord breaks?

9–45. A block of mass M rests on a turntable which is rotating at constant angular velocity ω. A smooth cord runs from the block through a hole in the center of the table down to a hanging block of mass m. The coefficient of friction between the first block and the turntable is μ. (See Fig. 9–23.) Find the largest and smallest values of the radius r for which the first block will remain at rest relative to the turntable.

9–46. A uniform rod of mass 30 gm and 20 cm long rotates in a horizontal plane about a fixed vertical axis through its center. Two small bodies, each of mass 20 gm, are mounted so that they can slide along the rod. They are initially held by catches at positions 5 cm on each side of the center of the rod, and the system is rotating at 15 rev/min. Without otherwise changing the system, the catches are released and the masses slide outward along the rod and fly off at the ends. (a) What is the angular velocity of the system at the instant when the small masses reach the ends of the rod? (b) What is the angular velocity of the rod after the small masses leave it?

9–47. A turntable rotates about a fixed vertical axis, making one revolution in 10 sec. The moment of inertia of the turntable about this axis is 720 slugs-ft^2. A man weighing 160 lb, initially standing at the center of the turntable, runs out along a radius. What is the angular velocity of the turntable when the man is 6 ft from the center?

9–48. Disks A and B are mounted on a shaft SS and may be connected or disconnected by a clutch C, as in Fig. 9–24. The moment of inertia of disk A is one-half that of disk B. With the

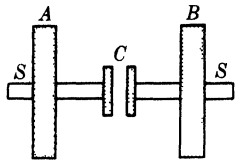


FIGURE 9–24

clutch disconnected, A is brought up to an angular velocity ω_0. The accelerating torque is then removed from A and it is coupled to disk B by the clutch. Bearing friction may be neglected. It is found that 3000 ft·lb of heat are developed in the clutch when the connection is made. What was the original kinetic energy of disk A?

9–49. A man weighing 160 lb stands at the rim of a turntable of radius 10 ft and moment of inertia 2500 slugs·ft^2, mounted on a vertical frictionless shaft at its center. The whole system is initially at rest. The man now walks along the outer edge of the turntable with a velocity of 2 ft/sec, relative to the earth. (a) With what angular velocity and in what direction does the turntable rotate? (b) Through what angle will it have rotated when the man reaches his initial position on the turntable? (c) Through what angle will it have rotated when he reaches his initial position relative to the earth?

9–50. A man weighing 160 lb runs around the edge of a horizontal turntable mounted on a vertical frictionless axis through its center. The velocity of the man, relative to the earth, is 4 ft/sec. The turntable is rotating in the opposite direction with an angular velocity of 0.2 rad/sec. The radius of the turntable is 8 ft and its moment of inertia about the axis of rotation is 320 slugs·ft^2. Find the final angular velocity of the system if the man comes to rest, relative to the turntable.

9–51. Two flywheels, A and B, are mounted on shafts which can be connected or disengaged by a friction clutch C. (Fig. 9–24.) The moment of inertia of wheel A is 4 slugs·ft^2. With the clutch disengaged, wheel A is brought up to an angular velocity of 600 rpm. Wheel B is initially at rest. The clutch is now engaged, accelerating B and decelerating A until both wheels have the same angular velocity. The final angular velocity of the system is 400 rpm. (a) What was the moment of inertia of wheel B? (b) How much energy was lost in the process? Neglect all bearing friction.

9–52. The stabilizing gyroscope of a ship weighs 50 tons, its radius of gyration is 5 ft, and it rotates about a vertical axis with an angular velocity of 900 rev/min. (a) How long a time is required to bring it up to speed, starting from rest, with a constant power input of 100 hp? (b) Find the torque needed to cause the axis to precess in a vertical fore-and-aft plane at the rate of 1 degree/sec.

9–53. The mass of the rotor of a toy gyroscope is 150 gm and its moment of inertia about its axis is 1500 gm·cm^2. The mass of the frame is 30 gm. The gyroscope is supported on a single pivot as in Fig. 9–25, with its center of gravity distant 4 cm horizontally from the pivot, and is precessing in a horizontal plane at the rate of one revolution in 6 sec. (a) Find the upward

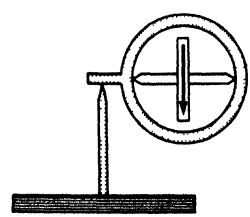


FIGURE 9–25

force exerted by the pivot. (b) Find the angular velocity with which the rotor is spinning about its axis, expressed in rev/min. (c) Copy the diagram, and show by vectors the angular momentum of the rotor and the torque acting on it.

9–54. The moment of inertia of the front wheel of a bicycle is 0.25 slug·ft^2, its radius is 15 inches, and the forward speed of the bicycle is 20 ft/sec. With what angular velocity must the front wheel be turned about a vertical axis to counteract the capsizing torque due to a weight of 120 lb, one inch horizontally to the right or left of the line of contact of wheels and ground? (Bicycle riders: compare with experience and see if your answer seems reasonable.)

CHAPTER 10

ELASTICITY

10–1 Stress. The preceding chapter dealt with the motion of a "rigid" body, a convenient mathematical abstraction since every real substance yields to some extent under the influence of applied forces. Ultimately, the change in shape or volume of a body when outside forces act on it is determined by the forces between its molecules. Although molecular theory is at present not sufficiently advanced to enable one to calculate the elastic properties of, say, a block of copper starting from the properties of a copper atom, the study of the solid state is an active subject in many research laboratories and our knowledge of it is steadily increasing. In this chapter we shall, however, confine ourselves to quantities that are directly measurable, and not attempt any molecular explanation of the observed behavior.

Figure 10–1(a) shows a bar of uniform cross-sectional area A subjected to equal and opposite pulls **F** at its ends. The bar is said to be in *tension.* Consider a section through the bar at right angles to its length, as indicated by the dotted line. Since every portion of the bar is in equilibrium, that portion at the right of the section must be pulling on the portion at the left with a force **F**, and vice versa. If the section is not too near the ends of the bar, these pulls are uniformly distributed over the cross-sectional area A, as indicated by the short arrows in Fig. 10–1(b). We define the *stress* S at the section as the ratio of the force F to the area A.

$$\text{Stress} = \frac{F}{A}. \tag{10–1}$$

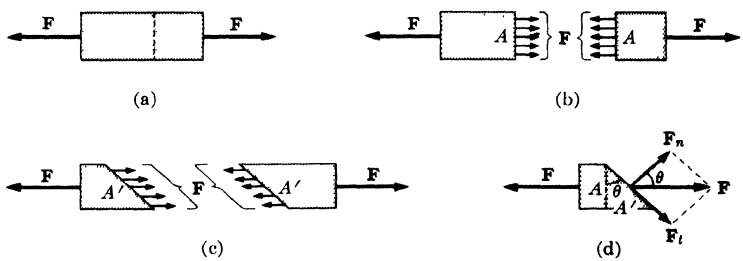


FIG. 10–1. (a) A bar in tension. (b) The stress at a perpendicular section equals F/A. (c) and (d) The stress at an inclined section can be resolved into a *normal* stress, F_n/A', and a *tangential* or *shearing* stress, F_t/A'.

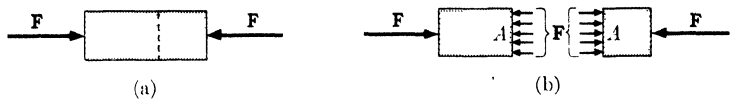


FIG. 10–2. A bar in compression.

The stress is called a *tensile* stress, meaning that each port on *pulls* on the other, and it is also a *normal* stress because the distributed force is perpendicular to the area. Stress is expressed in newtons/m^2, dynes/cm^2, and lb/ft^2. (The lb/in^2 is also commonly used, but it is not a unit of the engineering system.)

Consider next a section through the bar in some arbitrary direction, as in Fig. 10–1(c). The resultant force exerted on the portion at either side of this section, by the portion at the other, is equal and opposite to the force F at the end of the section. Now, however, the force is distributed over a larger area A' and is not at right angles to the area. If we represent the resultant of the distributed forces by a single vector of magnitude F, as in Fig. 10–1(d), this vector can be resolved into a component F_n normal to the area A', and a component F_t tangent to the area. The *normal* stress is defined, as before, as the ratio of the component F_n to the area A'. The ratio of the component F_t to the area A' is called the *tangential* stress or, more commonly, the *shearing* stress at the section.

$$\left.\begin{aligned} \text{Normal stress} &= \frac{F_n}{A'} \\ \text{Tangential (shearing) stress} &= \frac{F_t}{A'} \end{aligned}\right\} \qquad (10\text{–}2)$$

Stress is not a vector quantity since, unlike a force, we cannot assign to it a specific direction. The *force* acting on the portion of the body on a specified side of a section has, of course, a definite direction. Stress is one of a class of physical quantities called *tensors*.

A bar subjected to pushes at its ends, as in Fig. 10–2, is said to be in *compression*. The stress on the dotted section, illustrated in part (b), is also a normal stress but is now a *compressive* stress, since each portion pushes on the other. It should be evident that if we take a section in some arbitrary direction it will be subject to both a tangential (shearing) and a normal stress, the latter now being a compression.

As another example of a body under stress, consider the block of square cross section in Fig. 10–3(a), acted on by two equal and opposite couples produced by the pairs of forces $\mathbf{F}_x$ and $\mathbf{F}_y$ distributed over its surfaces. The block is in equilibrium, and any portion of it is in equilibrium also.

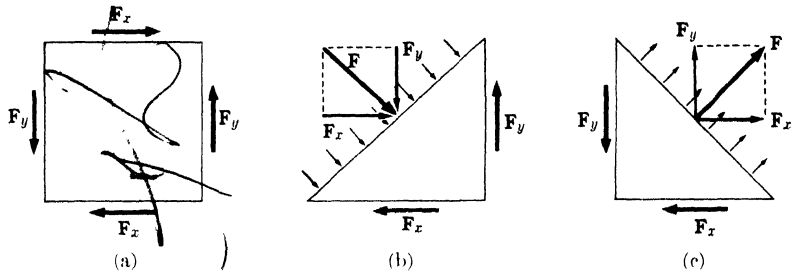


FIG. 10–3. (a) A body in shear. The stress on one diagonal, part (b), is a pure compression; that on the other, part (c), is a pure tension.

Thus the distributed forces over the diagonal face in part (b) must have a resultant **F** whose components are equal to $\mathbf{F}_x$ and $\mathbf{F}_y$. The stress at this section is therefore a pure compression, although the stresses at the right face and the bottom are both shearing stresses. Similarly, we see from Fig. 10–3(c) that the other diagonal is in pure tension.

Consider next a fluid under pressure. The term "fluid" means a substance that can flow, hence the term applies to both liquids and gases. If there is a shearing stress at any point in a fluid, the fluid slips sidewise as long as the stress is maintained. Hence in a fluid at rest, the shearing stress is everywhere zero. (Shearing stresses in moving fluids are discussed in Chapter 14.) Figure 10–4 represents a fluid in a cylinder provided with a piston, on which is exerted a downward force. The triangle is a side view of a wedge-shaped portion of the fluid. If for the moment we neglect the weight of the fluid, the only forces on this portion are those exerted by the rest of the fluid, and since these forces can have no shearing (or tangential) component, they must be normal to the surfaces of the wedge. Let $\mathbf{F}_x$, $\mathbf{F}_y$, and **F** represent the forces against the three faces. Since the fluid is in equilibrium, it follows that

$$F \sin\theta = F_x, \quad F \cos\theta = F_y.$$

Also,

$$A \sin\theta = A_x, \quad A \cos\theta = A_y.$$

Dividing the upper equations by the lower, we find

$$\frac{F}{A} = \frac{F_x}{A_x} = \frac{F_y}{A_y}.$$

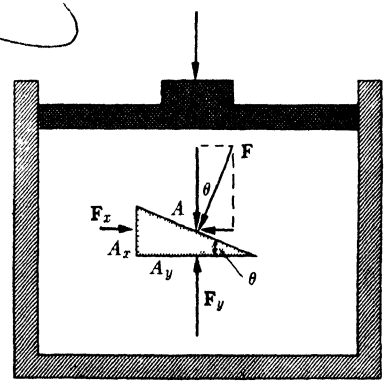


FIG. 10–4. A fluid under hydrostatic pressure. The force on a surface in any direction is normal to the surface.

Hence the force per unit area is the *same*, regardless of the direction of the section, and is always a compression. Any one of the preceding ratios defines the *hydrostatic pressure* p in the fluid.

$$p = \frac{F}{A}, \qquad F = pA. \tag{10–3}$$

Pressure is expressed in newtons/m^2, dynes/cm^2, or lb/ft^2. Like other types of stress, pressure is not a vector quantity and no direction can be assigned to it. The *force* against any area within (or bounding) a fluid at rest and under pressure is normal to the area, regardless of the orientation of the area. This is what is meant by the common statement that "the pressure in a fluid is the same in all directions."

The stress within a solid can also be a hydrostatic pressure, provided the stress at all points of the surface of the solid is of this nature. That is, the force per unit area must be the same at *all* points of the surface, and the force must be normal to the surface and directed inward. This is not the case in Fig. 10–2, where forces are applied at the ends of the bar only, but it is automatically the case if a solid is immersed in a fluid under pressure.

10–2 Strain. The term *strain* refers to the relative change in dimensions or shape of a body which is subjected to stress. Associated with each type of stress described in the preceding section is a corresponding type of strain.

Figure 10–5 shows a bar whose natural length is l_0 and which elongates to a length l when equal and opposite pulls are exerted at its ends. The elongation, of course, does not occur at the ends only; every element of the bar stretches in the same proportion as does the bar as a whole. The *tensile strain* in the bar is defined as the ratio of the increase in length to the original length.

$$\text{Tensile strain} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}. \tag{10–4}$$

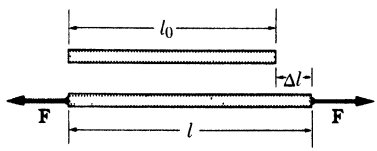


FIG. 10–5. The longitudinal strain is defined as $\Delta l/l_0$.

The *compressive strain* of a bar in compression is defined in the same way, as the ratio of the decrease in length to the original length.

Figure 10–6(a) illustrates the nature of the deformation when shearing stresses act on the faces of a block, as in Fig. 10–3. The dotted outline *abcd* represents the unstressed block, and the full lines $a'b'c'd'$ represent the block under stress. In part (a), the centers of the stressed and unstressed block coincide. In part (b), the edges ad and $a'd'$ coincide. The

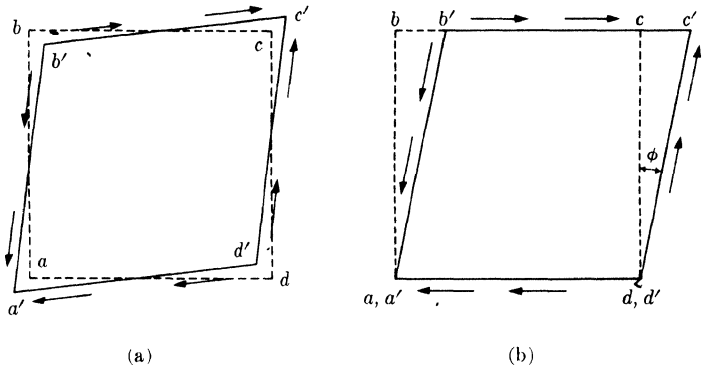


FIG. 10–6. Change in shape of a block in shear. The shearing strain is defined as $\tan \phi \approx \phi$.

lengths of the faces under shear remain very nearly constant, while all dimensions parallel to the diagonal *ac* increase in length, and those parallel to the diagonal *bd* decrease in length. Note that this is to be expected in view of the nature of the corresponding internal stresses (see Fig. 10–3). This type of strain is called a *shearing strain*, and is defined as the tangent of the angle ϕ. Since ϕ is always small, $\tan \phi$ is very nearly equal to ϕ (in radians).

$$\text{Shearing strain} = \tan \phi \approx \phi \text{ (radians)}. \tag{10–5}$$

Like other types of strain, shearing strain is a pure number.

The strain produced by a hydrostatic pressure, called a *volume strain*, is defined as the ratio of the change in volume, ΔV, to the original volume V_0. It also is a pure number.

$$\text{Volume strain} = \frac{\Delta V}{V_0}. \tag{10–6}$$

10–3 Elastic modulus. The ratio of a stress to the corresponding strain is called an elastic modulus and, provided the elastic limit is not exceeded, this ratio is found experimentally to be constant, characteristic of a given material. In other words, the stress is directly proportional to the strain, or is a linear function of the strain (within the elastic limit). This linear relationship between stress and strain is called *Hooke's law*.

Let us first consider longitudinal (i.e., tensile or compressive) stresses and strains. Experiment shows that with a given material, a given longitudinal stress produces a strain of the same magnitude whether the stress is a compression or a tension. Hence the ratio of tensile stress to tensile strain, for a given material, equals the ratio of compressive stress to com-

TABLE 10–1

APPROXIMATE ELASTIC MODULI

Material	Young's modulus, Y		Shear modulus, S		Bulk modulus, B	
	$\frac{10^{12}\text{ dynes}}{\text{cm}^2}$	$\frac{10^6\text{ lb}}{\text{in}^2}$	$\frac{10^{12}\text{ dynes}}{\text{cm}^2}$	$\frac{10^6\text{ lb}}{\text{in}^2}$	$\frac{10^{12}\text{ dynes}}{\text{cm}^2}$	$\frac{10^6\text{ lb}}{\text{in}^2}$
Aluminum	0 70	10	0 24	3 4	0 70	10
Brass	0 91	13	0 36	5 1	0 61	8 5
Copper	1 1	16	0 42	6 0	1 4	20
Glass	0 55	7 8	0 23	3 3	0 37	5 2
Iron	0 91	13	0 70	10	1 0	14
Lead	0 16	2 3	0 056	0 8	0 077	1 1
Nickel	2 1	30	0 77	11	2 6	34
Steel	2 0	29	0 84	12	1 6	23
Tungsten	3 6	51	1 5	21	2 0	29

pressive strain. This ratio is called the *stretch modulus* or *Young's modulus* of the material and will be denoted by Y.

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{\text{compressive stress}}{\text{compressive strain}}$$

or

$$Y = \frac{F_n/A}{\Delta l/l_0}. \tag{10–7}$$

Since a strain is a pure number, the units of Young's modulus are the same as those of stress, namely, force per unit area. Tabulated values are usually in lb/in^2 or dynes/cm^2. Some typical values are listed in Table 10–1.

The ratio of a shearing stress to the corresponding shearing strain is called the *shear modulus* of a material and will be represented by S. It is also called the *modulus of rigidity* or the *torsion modulus*.

$$S = \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{F_t/A}{\phi}. \tag{10–8}$$

(Refer to Fig. 10–6 for the meaning of ϕ.) The shear modulus of a material is also expressed as force per unit area. For most materials it is one-half to one-third as great as Young's modulus.

The modulus relating an increase in hydrostatic pressure to the corresponding fractional decrease in volume is called the *bulk modulus*, and we

shall represent it by B.

$$B = -\frac{p}{\Delta V/V_0}. \tag{10–9}$$

The minus sign is included in the definition of B since an increase of pressure always causes a decrease in volume. That is, if p is positive, ΔV is negative. By including a minus sign in its definition, the bulk modulus itself is a positive quantity.

The reciprocal of the bulk modulus is called the *compressibility*, k. Tables of physical constants often list the compressibility rather than the bulk modulus. From its definition,

$$k = \frac{1}{B} = -\frac{1}{p}\frac{\Delta V}{V_0}. \tag{10–10}$$

The ratio $\Delta V/V_0$ is the fractional change in volume. Hence the compressibility of a substance may be defined as its fractional change in volume per unit increase in pressure.

The units of a bulk modulus, from Eq. (10–9), are the same as those of pressure, and the units of compressibility, from Eq. (10–10), are those of a reciprocal pressure. In tabulating compressibilities, the pressure is often expressed in atmospheres (1 atmosphere = 14.7 lb/in^2). The corresponding units of compressibility are therefore "reciprocal atmospheres," or atm^{-1}. For example, the statement that the compressibility of water (see Table 10–2) is 50×10^{-6} atm^{-1}, or 50×10^{-6} per atmosphere, means that the volume decreases by 50 one-millionths of the original volume for each atmosphere increase in pressure.

Table 10–3 shows the relation between the various types of stress, strain, and elastic moduli.

When a metal rod is subjected to an increasing tensile stress, the strain is found to change as in Fig. 10–7. The first part of the curve, from O to

TABLE 10–2

COMPRESSIBILITY OF LIQUIDS

Liquid	Compressibility, atm^{-1} ($\times 10^{-6}$)
Carbon disulphide	66
Ethyl alcohol	112
Glycerine	22
Mercury	3.8
Water	50

TABLE 10–3

STRESSES AND STRAINS

Type of stress	Stress	Strain	Elastic modulus	Name of modulus
Tension	$\frac{F_n}{A}$	$\frac{\Delta l}{l_0}$	$Y = \frac{F_n/A}{\Delta l/l_0}$	Young's modulus
Compression	$\frac{F_n}{A}$	$\frac{\Delta l}{l_0}$	$Y = \frac{F_n/A}{\Delta l/l_0}$	
Shear	$\frac{F_t}{A}$	$\tan\phi \approx \phi$	$S = \frac{F_t/A}{\phi}$	Shear modulus
Hydrostatic pressure	$p\left(= \frac{F_n}{A}\right)$	$\frac{\Delta V}{V_0}$	$B = -\frac{p}{\Delta V/V_0}$	Bulk modulus

A, is a straight line. That is, in this region there is a linear relationship between stress and strain and the material obeys Hooke's law. If the stress is not carried beyond that corresponding to point A, the specimen returns to its original length when the stress is removed. In other words, the portion of the curve from O to A is the region of perfect elasticity.

If the stress is increased to a value corresponding to point B and then removed, the specimen does not return to its original length but retains a *permanent strain* or a *set*. Point A is called the *elastic limit* or the *proportional limit* of the material. Finally, when the stress is increased sufficiently, the specimen breaks at point C.

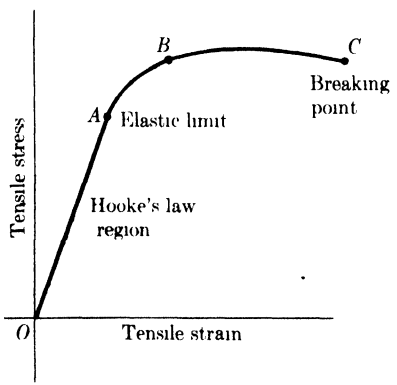


FIG. 10–7. Stress-strain diagram for a metal in tension.

EXAMPLE 1. In an experiment to measure Young's modulus, a load of 1000 lb hanging from a steel wire 8 ft long, of cross section 0.025 in^2, was found to stretch the wire 0.12 inch above its no-load length. What were the stress, the strain, and the value of

Young's modulus for the steel of which the wire was composed?

$$\text{Stress} = \frac{F_n}{A} = \frac{1000}{0.025} = 40{,}000 \frac{\text{lb}}{\text{in}^2}.$$

$$\text{Strain} = \frac{\Delta l}{l_0} = \frac{0.010}{8} = 0.00125.$$

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{40{,}000}{0.00125} = 32 \times 10^6 \frac{\text{lb}}{\text{in}^2}.$$

EXAMPLE 2. Suppose the object in Fig. 10–6 is a brass plate 2 ft square and $\frac{1}{4}$ inch thick. How large a force F must be exerted on each of its edges if the displacement cc' in Fig. 10–6(b) is 0.01 inch? The shear modulus of brass is 5×10^6 lb/in^2.

The shearing stress on each edge is

$$\text{Shearing stress} = \frac{F_t}{A} = \frac{F}{24 \times \frac{1}{4}} = \frac{F}{6} \frac{\text{lb}}{\text{in}^2}.$$

The shearing strain is

$$\text{Shearing strain} = \tan \phi = \frac{0.01}{24} = 4.17 \times 10^{-4}.$$

$$\text{Shear modulus } S = \frac{\text{stress}}{\text{strain}},$$

$$5 \times 10^6 = \frac{F/6}{4.17 \times 10^{-4}},$$

$$F = 12{,}500 \text{ lb}.$$

EXAMPLE 3. The volume of oil contained in a certain hydraulic press is 5 ft^3. Find the decrease in volume of the oil when subjected to a pressure of 2000 lb/in^2. The compressibility of the oil is 20×10^{-6} per atm.

The volume decreases by 20 parts per million for a pressure increase of one atm. Since 2000 lb/in^2 = 136 atm, the volume decrease is $136 \times 20 = 2720$ parts per million. Since the original volume is 5 ft^3, the actual decrease is

$$\frac{2720}{1{,}000{,}000} \times 5 \text{ ft}^3 = 0.0136 \text{ ft}^3 = 23.5 \text{ in}^3.$$

Or, from Eq. (10–10),

$$\Delta V = -kV_0 p = -20 \times 10^{-6} \text{ atm}^{-1} \times 5 \text{ ft}^3 \times 136 \text{ atm}$$
$$= -0.0136 \text{ ft}^3.$$

10–4 The force constant. The various elastic moduli are quantities which describe the elastic properties of a particular *material* and do not directly indicate how much a given rod, cable, or spring constructed of the material will distort under load. If Eq. (10–7) is solved, for F, one obtains

$$F = \frac{YA}{l_0} \Delta l$$

or, if YA/l_0 is replaced by a single constant k, and the elongation Δl is represented by x,

$$F = kx. \tag{10–11}$$

In other words, the elongation of a body in tension above its no-load length is directly proportional to the stretching force. Hooke's law was originally stated in this form, rather than in terms of stress and strain.

When a helical wire spring is stretched, the stress in the wire is practically a pure shear. The elongation of the spring as a whole is directly proportional to the stretching force. That is, an equation of the form $F = kx$ still applies, the constant k depending on the shear modulus of the wire, its radius, the radius of the coils, and the number of coils.

The constant k, or the ratio of the force to the elongation, is called the *force constant* or the *stiffness* of the spring, and is expressed in pounds per foot, newtons per meter, or dynes per centimeter. It is equal numerically to the force required to produce unit elongation.

Problems

10–1. A steel wire 10 ft long and 0.1 square inch in cross section is found to stretch 0.01 ft under a tension of 2500 lb. What is Young's modulus for this steel?

10–2. The elastic limit of a steel elevator cable is 40,000 lb/in^2. Find the maximum upward acceleration which can be given a 2-ton elevator when supported by a cable whose cross section is $\frac{1}{2}$ in^2 if the stress is not to exceed $\frac{1}{4}$ of the elastic limit.

10–3. A copper wire 12 ft long and 0.036 inch in diameter was given the test below. A load of 4.5 lb was originally hung from the wire to keep it taut. The position of the lower end of the wire was read on a scale.

Added load (lb)	Scale reading (in.)
0	3.02
2	3.04
4	3.06
6	3.08
8	3.10
10	3.12
12	3.14
14	3.65

(a) Make a graph of these values, plotting the increase in length horizontally and the added load vertically. (b) Calculate the value of Young's modulus. (c) What was the stress at the elastic limit?

10–4. A steel wire has the following properties:

Length = 10 ft
Cross section = 0.01 in^2
Young's modulus = 30,000,000 lb/in^2
Shear modulus = 10,000,000 lb/in^2
Elastic limit = 60,000 lb/in^2
Breaking stress = 120,000 lb/in^2

The wire is fastened at its upper end and hangs vertically. (a) How great a load can be supported without exceeding the elastic limit? (b) How much will the wire stretch under this load? (c) What is the maximum load that can be supported?

10–5. (a) What is the maximum load that can be supported by an aluminum wire 0.05 inch in diameter without exceeding the elastic limit of 14,000 lb/in^2? (b) If the wire was originally 20 ft long, how much will it elongate under this load?

10–6. A 10-lb weight hangs on a vertical steel wire 2 ft long and 0.001 in^2 in cross section. Hanging from the bottom of this weight is a similar steel wire which supports a 5-lb weight. Compute (a) the longitudinal strain, and (b) the elongation of each wire.

10–7. A 32-lb weight, fastened to the end of a steel wire of unstretched length 2 ft, is whirled in a vertical circle with an angular velocity at the bottom of the circle of 2 rps. The cross section of the wire is 0.01 in^2. Calculate the elongation of the wire when the weight is at the lowest point of its path.

10–8. A copper wire 320 inches long and a steel wire 160 inches long, each of cross section 0.1 in^2, are fastened end-to-end and stretched with a tension of 100 lb. (a) What is the change in length of each wire? (b) What is the elastic potential energy of the system?

10–9. A copper rod of length 3 ft and cross-sectional area 0.5 in^2 is fastened end-to-end to a steel rod of length L and cross-sectional area 0.2 in^2. The compound rod is subjected to equal and opposite pulls of magnitude 6000 lb at its ends. (a) Find the length L of the steel rod if the elongations of the two

rods are equal. (b) What is the stress in each rod? (c) What is the strain in each rod?

10–10. A rod 105 cm long, whose weight is negligible, is supported at its ends by wires A and B of equal length. The cross section of A is 1 mm^2, that of B is 2 mm^2. Young's modulus for wire A is 30×10^6 lb/in^2 and for B it is 20×10^6 lb/in^2. At what point along the bar should a weight w be suspended in order to produce (a) equal stresses in A and B, (b) equal strains in A and B? (Fig. 10–8.)

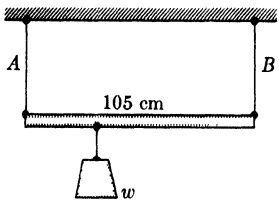


FIGURE 10–8

10–11. A bar of length L, cross-sectional area A, Young's modulus Y, is subjected to a tension F. Represent the stress in the bar by S and the strain by P. Derive the expression for the elastic potential energy, per unit volume, of the bar in terms of S and P.

10–12. The compressibility of sodium is to be measured by observing the displacement of the piston in Fig. 10–4 when a force is applied. The sodium is immersed in an oil which fills the cylinder below the piston. Assume that the piston and walls of the cylinder are perfectly rigid, that there is no friction, and no oil leak. Compute the compressibility of the sodium in terms of the applied force F, the piston displacement x, the piston area A, the initial volume of the oil V_0, the initial volume of the sodium v_0, and the compressibility of the oil k_0.

10–13. Two strips of metal are riveted together at their ends by four rivets, each of diameter 0.25 inch. What is the maximum tension that can be exerted by the riveted strip if the shearing stress on the rivets is not to exceed 10,000 lb/in^2? Assume each rivet to carry one-quarter of the load.

10–14. Find the weight per cubic foot of ocean water at a depth where the pressure is 4700 lb/ft^2. The weight at the surface is 64 lb/ft^3.

10–15. Compute the compressibility of steel, in reciprocal atmospheres, and compare with that of water. Which material is the more readily compressed?

10–16. A steel post 6 inches in diameter and 10 ft long is placed vertically and is required to support a load of 20,000 lb. (a) What is the stress in the post? (b) What is the strain in the post? (c) What is the change in length of the post?

10–17. A bar of cross section A is subjected to equal and opposite tensile forces $\mathbf{F}$ at its ends. Consider a plane through the bar making an angle θ with a plane at right angles to the bar (Fig. 10–9). (a) What is the tensile (normal) stress at this plane, in terms of F, A, and θ? (b) What is the shearing (tangential) stress at the plane, in terms of F, A, and θ? (c) For what value of θ is the tensile stress a maximum? (d) For what value of θ is the shearing stress a maximum?

10–18. Suppose the block in Fig. 10–3 is rectangular instead of square,

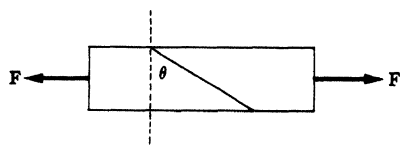


FIGURE 10–9

but is in equilibrium under the action of shearing stresses on those outside faces perpendicular to the plane of the diagram. (Note that then $F_x \neq F_y$.) (a) Show that the shearing stress is the same on all outside faces perpendicular to the plane of the diagram. (b) Show that on all sections perpendicular to the plane of the diagram and making an angle of 45° with an end face, the stress is still a pure tension or compression.

CHAPTER 11

HARMONIC MOTION

11–1 Introduction. The motion of a body when acted upon by a constant force was considered in detail in Chapters 4 and 5. The motion is one of constant acceleration, and it was found useful to derive expressions for the position and velocity of the body at any time, and for its velocity in any position. In the present chapter we are to study the motion of a body when the resultant force on it is not constant, but varies during the motion. Naturally, there are an infinite number of ways in which a force may vary and hence no general expressions can be given for the motion of a body when acted on by a variable force, except that the acceleration at each instant must equal the force at that instant divided by the mass of the body. There is, however, one particular mode of variation which is met with in practice so frequently that it is worth while to develop formulas for this special case. The force referred to is an elastic restoring force, brought into play whenever a body is distorted from its normal shape. When released, the body will be found to vibrate about its equilibrium position.

Examples of this sort of motion are the up-and-down motion which ensues when a weight hanging from a spring is pulled down and released; the vibrations of the strings or air columns of musical instruments; the vibration of a bridge or building under impact loads; and the oscillation of the balance wheel of a watch or of a clock pendulum. Furthermore, many reciprocating motions such as those of the crosshead in a steam engine or the piston of an automobile engine, while not exactly of this type, do approximate it quite closely.

It turns out that the equations of motion involve sines or cosines, and the term *harmonic* is applied to expressions containing these functions. This type of vibratory motion is therefore called *harmonic motion.*

11–2 Elastic restoring forces. It has been shown in Chapter 10 that when a body is caused to change its shape, the distorting force is proportional to the amount of the change, provided the elastic limit is not exceeded. The change may be in the nature of an increase in length, as of a rubber band or a coil spring; or a decrease in length; or a bending as of a flat spring; or a twisting of a rod about its axis; or of many other forms. The term "force" is to be interpreted liberally as the force, or torque, or pressure, or whatever may be producing the distortion. If we restrict the discussion to the case of a push or a pull, where the distortion

is simply the displacement of the point of application of the force, the force and displacement are related by Hooke's law,

$$F = kx,$$

where k is a proportionality constant called the *force constant* and x is the displacement from the equilibrium position.

In this equation, F stands for the magnitude of the force which must be exerted *on* an elastic body to produce the displacement x. The force with which the elastic body pulls back on an object to which it is attached is called the *restoring force* and is equal to $-kx$.

11-3 Definitions. To fix our ideas, suppose that a flat strip of steel such as a hacksaw blade is clamped vertically in a vise and a small body is attached to its upper end, as in Fig. 11-1. We shall assume that the strip is sufficiently long and the displacement sufficiently small so that the motion is essentially along a straight line. The mass of the strip itself is negligible.

Let the top of the strip be pulled to the right a distance A, as in Fig. 11-1, and released. The attached body is then acted on by a restoring force exerted by the steel strip and directed toward the equilibrium position O. It therefore accelerates in the direction of this force, and moves in toward the center with increasing speed. The *rate* of increase (i.e., the acceleration) is not constant, however, since the accelerating force becomes smaller as the body approaches the center.

When the body reaches the center, the restoring force has decreased to zero, but because of the velocity which has been acquired, the body "overshoots" the equilibrium position and continues to move toward the left. As soon as the equilibrium position is passed, the restoring force again comes into play, directed now toward the right. The body therefore decelerates, and at a rate which increases with increasing distance from O.

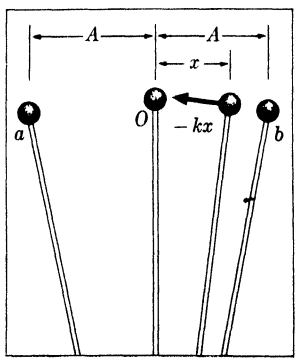


FIG. 11-1. Motion under an elastic restoring force.

It will therefore be brought to rest at some point to the left of O, and repeat its motion in the opposite direction.

Both experiment and theory show that the motion will be confined to a range $\pm A$ on either side of the equilibrium position, each to-and-fro movement taking place in the same length of time. If there were no loss of energy by friction, the motion would continue indefinitely once it had been started. This type of motion, under the influence of an elastic restoring force and in the absence of all friction, is called *simple harmonic motion*, often abbreviated *SHM*.

Any sort of motion which repeats itself in equal intervals of time is called *periodic*, and if the motion is back and forth over the same path, it is also called *oscillatory*.

A *complete vibration* or *oscillation* means one round trip, say from a to b and back to a, or from O to b to O to a and back to O.

The *periodic time*, or simply the *period* of the motion, represented by T, is the time required for one complete vibration.

The *frequency*, f, is the number of complete vibrations per unit time. Evidently the frequency is the reciprocal of the period, or

$$T = \frac{1}{f}.$$

The *coordinate*, x, at any instant, is the distance away from the equilibrium position or center of the path at that instant.

The *amplitude*, A, is the maximum coordinate. The total range of the motion is therefore $2A$.

11–4 Equations of simple harmonic motion. We now wish to find expressions for the coordinate, velocity, and acceleration of a body moving with simple harmonic motion, just as we found those for a body moving with constant acceleration. It must be emphasized that the equations of motion with *constant* acceleration cannot be applied, since the acceleration is continually changing.

Figure 11–2 represents the vibrating body of Fig. 11–1 at some instant when its coordinate is x. The resultant force on it is simply the elastic restoring force, $-kx$, and from Newton's second law,

$$F = -kx = ma,$$

or

$$\boxed{a = -\frac{k}{m}x,} \qquad (11\text{–}1)$$

where m is the mass of the body.

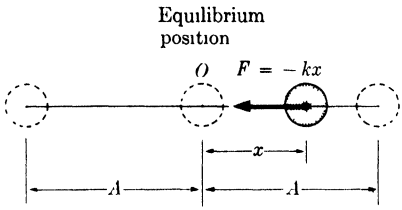


FIGURE 11-2

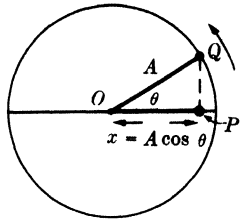


FIG. 11-3. Coordinate of a body in simple harmonic motion.

Since k and m are both constants, the ratio k/m is constant. Hence the acceleration is directly proportional to the displacement and, because of the minus sign, is in the opposite direction to the displacement. That is, when the body is at the right of its equilibrium position, its acceleration is toward the left, and vice versa. Another way of stating this is that the acceleration is always directed toward the center of the path. With the help of calculus, the equation above can be solved at once to find the desired expressions for displacement and velocity. However, we shall make use of a simple geometrical method of deducing these equations.

Consider a type of motion determined as follows. Let Q, Fig. 11-3, be a point revolving in a circle of radius A with a constant angular velocity of ω rad/sec.. Let P be a point on the horizontal diameter of the circle, directly below Q. Point P is called the projection of Q onto the diameter. Point Q is referred to as the *reference point*, and the circle in which it moves as the *reference circle.*

As the reference point revolves, the point P moves back and forth along a horizontal line, keeping always directly below (or above) Q. We shall show that the motion of P is the same as that of a body moving under the influence of an elastic restoring force in the absence of friction.

The displacement of P at any time t is the distance OP or x, and if θ represents the angle which OQ makes with the horizontal diameter,

$$x = A \cos \theta.$$

The angle θ is called the *phase angle*, or simply the *phase* of the motion.

If point Q is at the extreme right end of the diameter at time $t = 0$, the angle θ may be written

$$\theta = \omega t.$$

Hence

$$x = A \cos \omega t.$$

Now ω, the angular velocity of Q in radians per second, is related to f,

the number of complete revolutions of Q per second, by

$$\omega = 2\pi f,$$

since there are 2π radians in one complete revolution. Furthermore, the point P makes one complete vibration for each revolution of Q. Hence f may also be interpreted as the number of vibrations per second or the frequency of vibration of point P. Replacing ω by $2\pi f$, we have

$$\boxed{x = A \cos 2\pi ft.} \tag{11-2}$$

Eq. (11-2) gives the displacement of point P at any time t after the start of the motion, and thus corresponds to

$$x = v_0 t + \tfrac{1}{2}at^2$$

for a body moving with constant acceleration. Note carefully that x, in Eq. (11-2), represents the *distance from the center of the path*, not the distance from the starting point.

The instantaneous velocity of P may be found with the aid of Fig. 11-4. The reference point Q moves with a tangential velocity

$$v_T = \omega A = 2\pi fA.$$

Since point P is always directly below or above the reference point, the velocity of P at each instant must equal the X-component of the velocity of Q. That is, from Fig. 11-4,

$$v = v_T \sin\theta = -2\pi fA \sin\theta$$

or

$$\boxed{v = -2\pi fA \sin 2\pi ft.} \tag{11-3}$$

The minus sign is introduced since the direction of the velocity is toward the left. When Q is below the horizontal diameter, the velocity of P will be toward the right, but since $\sin\theta$ is negative at such points, the minus sign is still needed. Eq. (11-3) gives the velocity of point P at any time, and corresponds to

$$v = v_0 + at$$

for motion with constant acceleration.

Since $\sin\theta = \sqrt{1 - \cos^2\theta}$ and $\cos\theta = x/A$, Eq. (11-3) may be written

$$v = \pm 2\pi fA\sqrt{1 - (x^2/A^2)},$$

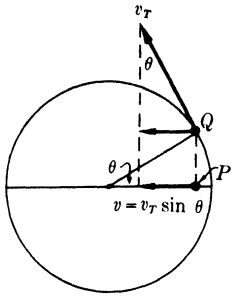


FIG. 11-4. Velocity in simple harmonic motion.

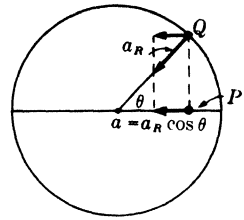


FIG. 11-5. Accceleration in simple harmonic motion.

or

$$\boxed{v = \pm 2\pi f\sqrt{A^2 - x^2}.} \tag{11-4}$$

The symbol $\pm$ is required, since at any given displacement x the point may be moving either toward the right or left. Eq. (11-4) gives the velocity of P at any *displacement*. It thus corresponds to

$$v^2 = v_0^2 + 2ax$$

for constant acceleration.

Finally, the acceleration of point P may be found, making use again of the fact that, since P is always directly below or above Q, its acceleration must equal the X-component of the acceleration of Q. Point Q, since it moves in a circular path with a constant angular velocity ω, has at each instant an acceleration toward the center given by

$$a_R = \omega^2 A = 4\pi^2 f^2 A.$$

From Fig. 11-5, the X-component of this acceleration is

$$a = a_R \cos \theta,$$

or

$$\boxed{a = -4\pi^2 f^2 A \cos 2\pi ft,} \tag{11-5}$$

and since $A \cos 2\pi ft = x$,

$$\boxed{a = -4\pi^2 f^2 x.} \tag{11-6}$$

The minus sign is introduced since the acceleration is toward the left. When Q is at the left of the center, the acceleration of P is toward the right, but since $\cos\theta$ is negative at such points, the minus sign is still required. Eqs. (11–5) and (11–6) give the acceleration of P at any time and at any displacement. There are no corresponding equations in motion with constant acceleration except the extremely simple one

$$a = \text{constant}.$$

Eq. (11–6) states that the acceleration of point P, moving in the manner described, is proportional to the displacement x and is in the direction opposite to x. But this is just the condition which must be fulfilled by a body moving under the influence of an elastic restoring force (Eq. 11–1). Hence it may be concluded that Eqs. (11–2) through (11–6) describe the motion of a body when acted on by such a force, and are the equations which we set out to find.

In order that the motion of point P may coincide in all respects with that of a given vibrating body, the radius A of the reference circle must equal the amplitude A of the actual vibration, and the frequency of revolution of Q must be the same as the frequency of the actual vibration. The proper value of the latter may be found by combining Eq. (11–1),

$$a = -\frac{k}{m}x$$

which gives the acceleration of the vibrating body at any displacement, and Eq. (11–6)

$$a = -4\pi^2 f^2 x,$$

TABLE 11–1

Motion with constant linear acceleration	Simple harmonic motion (in terms of k/m)	Simple harmonic motion (using $2\pi f = \sqrt{k/m}$)
$a = \text{constant}$	$a = -(k/m)x$	$a = -4\pi^2 f^2 x$
	$a = -(k/m)A\cos\sqrt{k/m}\,t$	$a = -4\pi^2 f^2 A\cos 2\pi f t$
$v^2 = v_0^2 + 2ax$	$v = \pm\sqrt{k/m}\,\sqrt{A^2 - x^2}$	$v = \pm 2\pi f\sqrt{A^2 - x^2}$
$v = v_0 + at$	$v = -\sqrt{k/m}\,A\sin\sqrt{k/m}\,t$	$v = -2\pi f A\sin 2\pi f t$
$x = v_0 t + \frac{1}{2}at^2$	$x = A\cos\sqrt{k/m}\,t$	$x = A\cos 2\pi f t$

which gives the acceleration of point P at any displacement. Since these accelerations must be the same,

$$k/m = 4\pi^2 f^2,$$

and therefore

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.} \tag{11-7}$$

Eq. (11-7) may be used to find the vibration frequency of a body of given mass when vibrating under the influence of an elastic restoring force of given force constant.

Since the period is the reciprocal of the frequency, Eq. (11-7) may also be written

$$\boxed{T = 2\pi\sqrt{m/k}.} \tag{11-8}$$

In using Eqs. (11-7) or (11-8), m must be expressed in slugs, kilograms, or grams, and k in lb/ft, newtons/meter, or dynes/cm. The frequency f will then be in vibrations per second, and the period T in seconds per vibration.

A somewhat unexpected conclusion to be drawn from these equations is that the period does not depend on the amplitude, but on the mass and force constant alone.

The equations for simple harmonic motion may be conveniently summarized by comparing them with similar equations for motion with constant linear acceleration, as shown in the first two columns of Table 11-1.

It is helpful in visualizing harmonic motion to represent the coordinate, velocity, and acceleration of the vibrating body graphically. Graphs of these quantities against time are given in Fig. 11-6, which may be considered as a graph of Eqs. (11-2), (11-3), and (11-5). Notice that the velocity is a maximum when the coordinate is zero, that is, at the center; while the velocity is zero when the coordinate is a maximum. The acceleration, on the other

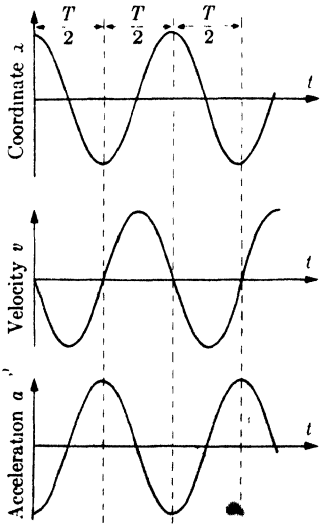


FIG. 11-6. Graphs of coordinate, velocity, and acceleration of a body moving with simple harmonic motion.

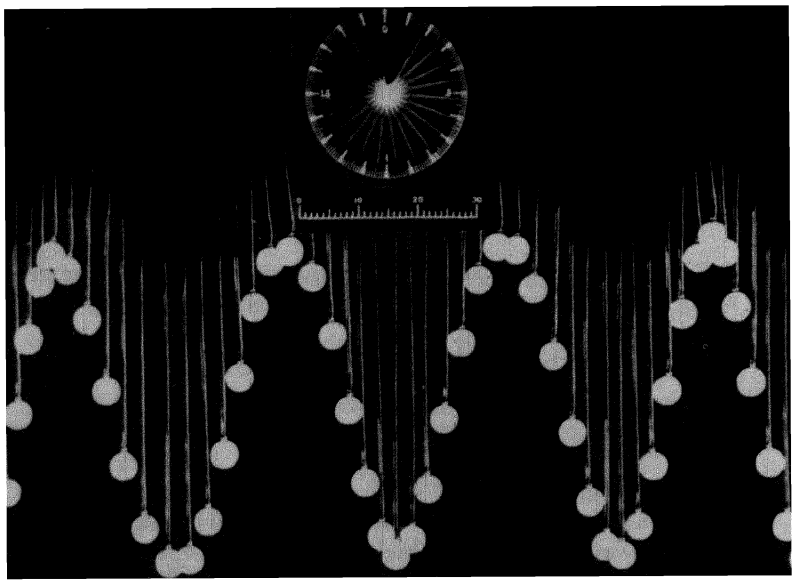


FIG. 11–7. Multiflash photograph of the motion of a ball suspended from a spring.

hand, is zero at the center and a maximum at the ends of the path.

Figure 11–7 is a multiflash photograph of the motion of a mass suspended from a coil spring and set into vertical vibration. The camera was rotated about a vertical axis while the photographs were taken so that each image is displaced laterally from the preceding image. In effect this introduces a horizontal time scale into the motion, and the body traces out its own sinusoidal coordinate-time graph corresponding to the upper diagram in Fig. 11–6. By comparing the *vertical* separation of successive images, it is seen that the velocity is greatest at the center of the path and zero at the ends, while the acceleration is greatest at the ends and zero at the center.

EXAMPLE. A flat steel strip is mounted as in Fig. 11–1. By attaching a spring balance to the end of the strip and pulling it sidewise, it is found that a force of 1 lb will produce a deflection of 6 inches. A 4-lb body is attached to the end of the strip, pulled aside a distance of 8 inches, and released.

(a) Compute the force constant of the spring.

A force of 1 lb produces a displacement of 6 inches or $\frac{1}{2}$ ft. Hence

$$k = \frac{F}{x} = \frac{1}{\frac{1}{2}} = 2\,\frac{\text{lb}}{\text{ft}}.$$

(b) Compute the period of vibration.

$$T = 2\pi\sqrt{m/k} = 2\pi\sqrt{(4/32)/2} = \frac{\pi}{2} \text{ sec.}$$

(c) Compute the maximum velocity attained by the vibrating body.

The maximum velocity occurs at the center, where the coordinate is zero. Since

$$v = \pm 2\pi f\sqrt{A^2 - x^2},$$

then when $x = 0$,

$$v = v_{\max} = \pm 2\pi f A.$$

Also,

$$f = \frac{1}{T} = \frac{2}{\pi} \frac{\text{vib}}{\text{sec}},$$

$$A = 8 \text{ in.} = \tfrac{2}{3} \text{ ft.}$$

$$v_{\max} = \pm 2\pi \times \frac{2}{\pi} \times \frac{2}{3} = \pm \frac{8}{3} \frac{\text{ft}}{\text{sec}}.$$

(d) Compute the maximum acceleration.

The maximum acceleration occurs at the ends of the path where $x = \pm A$. Therefore

$$a_{\max} = \mp 4\pi^2 f^2 A = \mp 4\pi^2 \times \left(\frac{2}{\pi}\right)^2 \times \frac{2}{3} = \mp \frac{32}{3} \frac{\text{ft}}{\text{sec}^2}.$$

(e) Compute the velocity and acceleration when the body has moved halfway in toward the center from its initial position.

At this point,

$$x = \frac{A}{2} = \frac{1}{3} \text{ ft},$$

$$v = -2\pi \times \frac{2}{\pi}\sqrt{(2/3)^2 - (1/3)^2} = -\frac{4}{\sqrt{3}} \frac{\text{ft}}{\text{sec}},$$

$$a = -4\pi^2 \times \left(\frac{2}{\pi}\right)^2 \times \frac{1}{3} = -\frac{16}{3} \frac{\text{ft}}{\text{sec}^2}.$$

(f) How long a time is required for the body to move halfway in to the center from its initial position?

Note that the motion is neither one of constant velocity nor of constant acceleration. The simplest method of handling a problem involving the time required to move from one point to another in harmonic motion is to make use of the reference circle. While the body moves halfway in, the reference point revolves through an angle of 60° (Fig. 11-8). Since the reference point moves with constant angular velocity and makes one complete revolution in $\pi/2$ sec (in this particular example), the time to rotate through 60° is $1/6 \times \pi/2 = \pi/12$ sec.

The time may also be computed directly from the equation

$$x = A \cos 2\pi ft,$$

$$A/2 = A \cos (2\pi \times 2/\pi \times t),$$

$$\cos 4t = \tfrac{1}{2},$$

$$4t = \cos^{-1} \tfrac{1}{2} = \pi/3,$$

$$\therefore\ t = \pi/12 \text{ sec.}$$

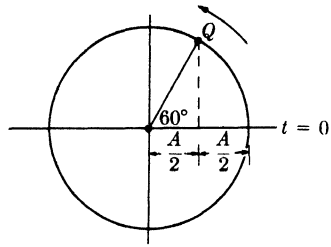


FIGURE 11-8

11-5 The simple pendulum. A simple pendulum consists of a body of small dimensions suspended by an inextensible weightless string. When pulled to one side of its equilibrium position and released, the pendulum bob vibrates about this position with motion which is both periodic and oscillatory. We wish to discover if the motion is simple harmonic.

The necessary condition for simple harmonic motion is that the restoring force F shall be directly proportional to the coordinate x and oppositely directed. The path of the bob, of course, is not a straight line, but the arc of a circle of radius L, where L is the length of the supporting cord. The coordinate refers to distances measured along this arc. (See Fig. 11-9.) Hence if $F = -kx$, the motion will be simple harmonic, or since $x = L\theta$, the requirement may be written $F = -kL\theta$.

Figure 11-9 shows the forces on the bob at an instant when its coordinate is x. Choose axes tangent to the circle and along the radius, and resolve the weight into components. The restoring force F is

$$F = -mg \sin \theta. \quad (11\text{-}9)$$

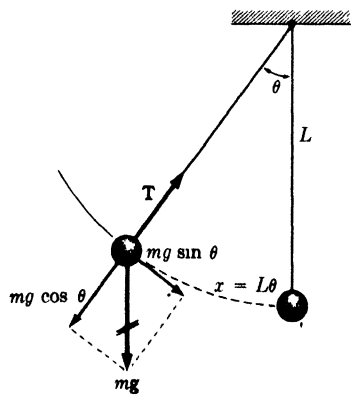


FIG. 11-9. The simple pendulum.

The restoring force is therefore *not* proportional to θ but to sin θ, so the motion is *not* simple harmonic. However, *if the angle* θ *is small*, sin θ is very nearly equal to θ and Eq. (11-9) becomes

$$F = -mg\theta = -mg \frac{x}{L},$$

or

$$F = -\frac{mg}{L} x.$$

The restoring force is then proportional to the coordinate *for small dis-*

placements, and the constant mg/L represents the force constant k. The period of a simple pendulum when its amplitude is small is therefore

$$T = 2\pi\sqrt{m/k} = 2\pi\sqrt{m/(mg/L)},$$

or

$$T = 2\pi\sqrt{L/g}. \tag{11-10}$$

What constitutes a "small" amplitude? It can be shown that the general equation for the time of swing, when the maximum angular displacement is α, is

$$T = 2\pi\sqrt{L/g}\left(1 + \frac{1^2}{2^2}\sin^2\frac{\alpha}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\sin^4\frac{\alpha}{2} + \cdots\right).$$

The time may be computed to any desired degree of precision by taking enough terms in the infinite series. When $\alpha = 15°$ (on either side of the central position), the true period differs from that given by the approximate Eq. (11-10) by less than one-half of one percent.

The utility of the pendulum as a timekeeper is based on the fact that the period is practically independent of the amplitude. Thus, as a clock runs down and the amplitude of the swings becomes slightly smaller, the clock will still keep very nearly correct time.

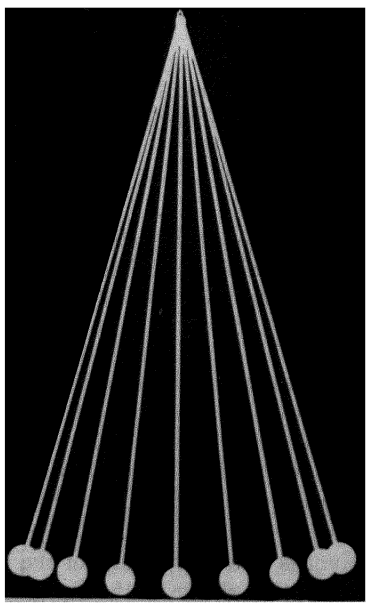


FIG. 11-10. Multiflash photograph of a single swing of a simple pendulum.

The simple pendulum is also a precise and convenient method of measuring the acceleration of gravity g without actually resorting to free fall, since L and T may readily be measured. More complicated pendulums find considerable application in the field of geophysics. Local deposits of ore or oil, if their density differs from that of their surroundings, affect the local value of g, and precise measurements of this quantity over an area which is being prospected often furnish valuable information regarding the nature of underlying deposits.

Figure 11–10 is a multiflash photograph of a single swing of a simple pendulum. The motion is evidently of the simple harmonic type with maximum speed at the center and maximum acceleration at the ends of the swing.

11–6 Angular harmonic motion. Angular harmonic motion results when a body which is pivoted about an axis experiences a restoring torque proportional to the angular displacement from its equilibrium position. This type of vibration is very similar to linear harmonic motion, and the corresponding equations may be written down immediately from the analogies between linear and angular quantities. The oscillatory motion of the balance wheel of a watch is an example of angular harmonic motion.

A restoring torque proportional to angular displacement is expressed by

$$\Gamma = -k'\theta. \tag{11–11}$$

The moment of inertia of the pivoted body corresponds to the mass of a body in linear motion. Hence the period formula for angular harmonic motion is

$$T = 2\pi\sqrt{I/k'}, \tag{11–12}$$

where k' is the constant in Eq. (11–11).

The equations for angular displacement, angular velocity, and angular acceleration can be obtained by comparison with the corresponding equations in Section 11–4.

11–7 The physical pendulum. A so-called "physical" pendulum is any real pendulum, as contrasted with a simple pendulum in which all the mass is assumed to be concentrated at a point. Let Fig. 11–11 represent a body of irregular shape pivoted about a horizontal frictionless axis and displaced from the vertical by an angle θ. The distance from the pivot to the center of gravity is l, the moment of inertia of the pendulum about an axis through the pivot is I, and the mass of the pendulum is m. The restoring torque in the position shown in the figure is

$$\Gamma = -mgl \sin \theta.$$

If θ is small we may replace $\sin\theta$ by θ, and

$$\Gamma = -mgl\theta.$$

Hence the pendulum is acted on, in effect, by an elastic restoring torque with $k' = mgl$. The period of vibration is therefore

$$T = 2\pi\sqrt{I/k'} = 2\pi\sqrt{I/mgl} \tag{11-13}$$

(for small amplitudes).

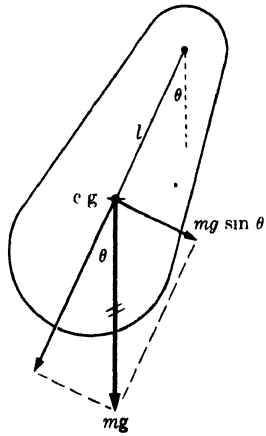


FIG. 11-11. The physical pendulum.

EXAMPLE 1. Let the body in Fig. 11-11 be a meter stick pivoted at one end. Then if L stands for the length of 1 meter,

$$I = \tfrac{1}{3}mL^2, \qquad l = \frac{L}{2}, \qquad g = 9.8\ \frac{\text{m}}{\text{sec}^2},$$

$$T = 2\pi\sqrt{\tfrac{1}{3}mL^2/(mgL/2)} = 2\pi\sqrt{(2/3)(L/g)}$$

$$= 2\pi\sqrt{(2/3)(1/9.8)} = 1.65\ \text{sec}.$$

EXAMPLE 2. Equation (11-13) may be solved for the moment of inertia I, giving

$$I = \frac{T^2 mgl}{4\pi^2}. \tag{11-14}$$

The quantities on the right of the equation are all directly measurable. Hence the moment of inertia of a body of any complex shape may be found by suspending the body as a physical pendulum and measuring its period of vibration. The location of the center of gravity can be found by balancing. Since T, m, g, and l are known, I can be computed. For example, Fig. 11-12 illustrates a connecting rod pivoted about a horizontal knife-edge. The connecting rod weighs 4 lb, and its center of gravity has been found by balancing to be 8 inches below the knife-edge. When set into oscillation, it is found to make 100 complete vibrations in 120 sec, so that $T = 120/100 = 1.2$ sec. Therefore

$$I = \frac{(1.2)^2 \times 4 \times \frac{2}{3}}{4\pi^2} = 0.097\ \text{slug}\cdot\text{ft}^2.$$

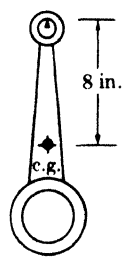


FIGURE 11-12

11–8 Center of oscillation. It is always possible to find an *equivalent* simple pendulum whose period is equal to that of a given physical pendulum. If L is the length of the equivalent simple pendulum,

$$T = 2\pi\sqrt{L/g} = 2\pi\sqrt{I/mgl},$$

or

$$L = \frac{I}{ml}. \tag{11–15}$$

Thus, as far as its period of vibration is concerned, the mass of a physical pendulum may be considered to be concentrated at a point whose distance from the pivot is $L = I/ml$. This point is called the *center of oscillation* of the pendulum.

Figure 11–13 shows a body pivoted about an axis through P and whose center of oscillation is at point C. The center of oscillation and the point of support have the following interesting property, namely, if the pendulum is pivoted about a new axis through point C its period is unchanged and point P becomes the new center of oscillation. The point of support and the center of oscillation are said to be *conjugate* to each other.

The center of oscillation has another important property. Figure 11–14 shows a baseball bat pivoted at O. If a ball strikes the bat at its center of oscillation, no impulsive force is exerted on the pivot and hence no "sting" is felt if the bat is held at that point. Because of this property, the center of oscillation is called the *center of percussion*.

Figure 11–15 is a series of multiflash photographs illustrating the motion of a body suspended by a cord when the body is struck a horizontal blow. The *center of gravity* is marked by a black band. In (a), the body is struck at its center of percussion relative to a pivot at the upper end of the cord,

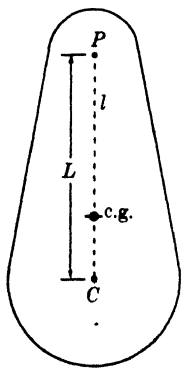


FIG. 11–13. Center of oscillation. The length L equals that of the equivalent simple pendulum.

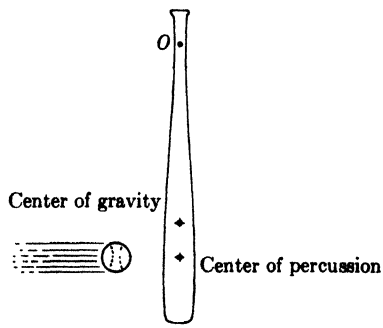


FIG. 11–14. The center of percussion coincides with the center of oscillation.

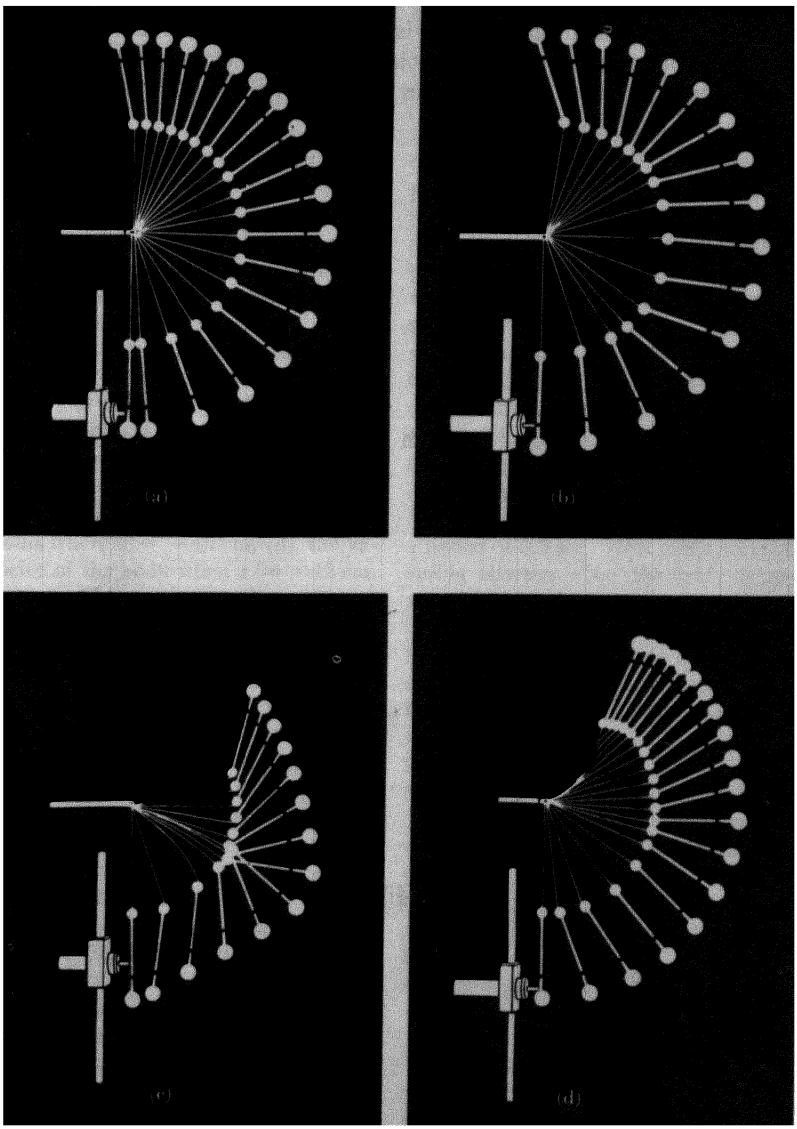


FIG. 11-15. The center of gravity is marked by a black band. A blow is struck (a) at the center of percussion, (b) at the center of gravity, (c) above, and (d) below the center of percussion.

and it starts to swing smoothly about this pivot. In (b), the body is struck at its center of gravity. Note that it does not start to rotate about the pivot, but that its initial motion is one of pure translation. That is, the center of percussion does not coincide with the center of gravity. In (c), the body is struck above, and in (d) below its center of percussion.

Problems

11–1. A body is vibrating with simple harmonic motion of amplitude 15 cm and frequency 4 vib/sec. Compute (a) the maximum values of the acceleration and velocity, (b) the acceleration and velocity when the coordinate is 9 cm, (c) the time required to move from the equilibrium position to a point 12 cm distant from it.

11–2. A body of mass 10 gm moves with simple harmonic motion of amplitude 24 cm and period 4 sec. The coordinate is +24 cm when $t = 0$. Compute (a) the position of the body when $t = 0.5$ sec, (b) the magnitude and direction of the force acting on the body when $t = 0.5$ sec, (c) the minimum time required for the body to move from its initial position to the point where $x = -12$ cm, (d) the velocity of the body when $x = -12$ cm.

11–3. The motion of the piston of an automobile engine is approximately simple harmonic. (a) If the stroke of an engine (twice the amplitude) is 4 inches and the angular velocity is 3600 rev/min, compute the acceleration of the piston at the end of its stroke. (b) If the piston weighs 1 lb; what resultant force must be exerted on it at this point? (c) What is the velocity of the piston, in mi/hr, at the midpoint of its stroke?

11–4. A 4-lb weight hung from a spring is found to stretch the spring 8 inches. (a) What is the force constant of the spring? (b) What would be the period of vibration of the 4-lb weight, if suspended from this spring? (c) What would be the period of an 8-lb weight hanging from the same spring?

11–5. The scale of a spring balance reading from zero to 32 lb is 6 inches long. A body suspended from the balance is observed to oscillate vertically at 1.5 vib/sec. What is the weight of the body?

11–6. A body of mass 100 gm hangs from a long spiral spring. When pulled down 10 cm below its equilibrium position and released, it vibrates with a period of 2 sec. (a) What is its velocity as it passes through the equilibrium position? (b) What is its acceleration when it is 5 cm above the equilibrium position? (c) When it is moving upward, how long a time is required for it to move from a point 5 cm below its equilibrium position to a point 5 cm above it? (d) How much will the spring shorten if the body is removed?

11–7. A body whose mass is 4.9 kgm hangs from a spring and oscillates with a period of 0.5 sec. How much will the spring shorten when the body is removed?

11–8. Four passengers whose combined weight is 600 lb are observed to compress the springs of an automobile by 2 inches when they enter the automobile. If the total load supported by the springs is 1800 lb, find the period of vibration of the loaded automobile.

11–9. A body of mass m is vibrating with simple harmonic motion of amplitude A. (a) At any moment, when the coordinate is x and the velocity is v, what is the total energy? (b) What is the total energy when $x = A$? (c) Derive the equation

$$v = \sqrt{k/m} \cdot \sqrt{A^2 - x^2}.$$

11–10. A block suspended from a spring vibrates with simple harmonic motion. At an instant when the displacement of the block is equal to one-half the amplitude, what fraction of the total energy of the system is kinetic and what fraction is potential?

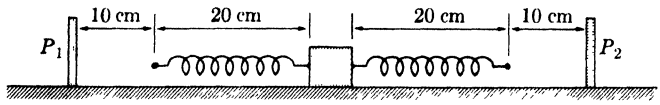


FIGURE 11–16

11–11. Two springs, each of unstretched length 20 cm but having different force constants k_1 and k_2, are attached to opposite ends of a block of mass m on a level frictionless surface. The outer ends of the springs are now attached to the two pins P_1 and P_2, 10 cm from the original positions of the springs. Let $k_1 = 1000$ dynes/cm, $k_2 = 3000$ dynes/cm, $m = 100$ gm. (See Fig. 11–16.) (a) Find the length of each spring when the block is in its new equilibrium position, after the springs have been attached to the pins. (b) Show that the effective force constant of the combination is 4000 dynes/cm. (c) Find the period of vibration of the block if it is slightly displaced from its new equilibrium position and released.

11–12. A body weighing 8 lb is attached to a coil spring and oscillates vertically in simple harmonic motion. The amplitude is 2 ft, and at the highest point of the motion the spring has its natural unstretched length. Calculate the elastic potential energy of the spring, the kinetic energy of the body, its gravitational potential energy relative to the lowest point of the motion, and the sum of these three energies, when the body is (a) at its lowest point, (b) at its equilibrium position, and (c) at its highest point.

11–13. A load of 320 lb suspended from a wire whose unstretched length l_0 is 10 ft, is found to stretch the wire by 0.12 inch. The cross-sectional area of the wire, which can be assumed constant, is 0.016 in^2. (a) If the load is pulled down a small additional distance and released, find the frequency at which it will vibrate. (b) Compute Young's modulus for the wire.

11–14. (a) With what additional force must a vertical spring carrying an 8-lb body in equilibrium be stretched so that, when released, it will perform 48 complete oscillations in 32 seconds with an amplitude of 3 inches? (b) What force is exerted by the spring on the body when it is at the lowest point, the middle, and the highest point of the path? (c) What is the kinetic energy of the system when the body is 1 inch below the middle of the path? its potential energy?

• 11–15. A simple pendulum 8 ft long swings with an amplitude of 1 ft. (a) Compute the velocity of the pendulum at its lowest point. (b) Compute its acceleration at the ends of its path.

• 11–16. Find the length of a simple pendulum whose period is exactly 1 sec at a point where $g = 32.2$ ft/sec^2.

11–17. A pendulum clock which keeps correct time at a point where $g = 980.0$ cm/sec^2 is found to lose 10 sec per day at a higher altitude. Find the value of g at the new location.

11–18. A simple pendulum with a supporting steel wire of cross-sectional area 0.01 cm^2 is observed to have a period of 2 sec when a 10-kgm lead bob is used. The lead bob is replaced by an aluminum bob of the same dimensions having a mass of 2 kgm, and the period is remeasured. (a) What was the

length of the pendulum with the lead bob? (b) By what fraction is the period changed when the aluminum bob is used? Is it an increase or decrease?

11–19. The balance wheel of a watch vibrates with an angular amplitude of π radians and with a period of 0.5 sec. (a) Find its maximum angular velocity. (b) Find its angular velocity when its displacement is one-half its amplitude. (c) Find its angular acceleration when its displacement is 45°.

11–20. A monkey wrench is pivoted at one end and allowed to swing as a physical pendulum. The period is 0.9 sec and the pivot is 6 inches from the center of gravity. (a) What is the radius of gyration of the wrench about an axis through the pivot? (b) If the wrench was initially displaced 0.1 radian from its equilibrium position, what is the angular velocity of the wrench as it passes through the equilibrium position?

11–21. A meter stick is pivoted at one end. At what distance below the pivot should it be struck in order that it start swinging smoothly about the pivot?

CHAPTER 12

HYDROSTATICS

12–1 Introduction. The term "hydrostatics" is applied to the study of fluids at rest, and "hydrodynamics" to fluids in motion. The special branch of hydrodynamics relating to the flow of gases and of air in particular is called "aerodynamics."

A fluid is a substance which can flow. Hence the term includes both liquids and gases. Liquids and gases differ markedly in their compressibilities; a gas is easily compressed, while a liquid is practically incompressible. The small volume changes of a liquid under pressure can usually be neglected in this part of the subject.

The density of a homogeneous material is defined as its mass per unit volume. Densities are therefore expressed in grams per cubic centimeter, kilograms per cubic meter, or slugs per cubic foot. We shall represent density by the greek letter ρ (rho).

$$\rho = \frac{m}{V}, \qquad m = \rho V. \tag{12–1}$$

For example, the weight of 1 cubic foot of water is 62.5 lb; its density is $62.5/32.2 = 1.94$ slugs per cubic foot.

TABLE 12–1

DENSITIES

Material	Density, gm/cm^3	Material	Density, gm/cm^3
Aluminum	2 7	Silver	10 5
Brass	8 6	Steel	7.8
Copper	8 9	Mercury	13 6
Gold	19.3	Ethyl alcohol	0.81
Ice	0.92	Benzene	0.90
Iron	7.8	Glycerin	1.26
Lead	11.3	Water	1.00
Platinum	21.4		

The *specific gravity* of a material is the ratio of its density to that of water and is therefore a pure number. "Specific gravity" is an exceedingly poor term, since it has nothing to do with gravity. "Relative density" would describe the concept more precisely.

12–2 Pressure in a fluid. When the concept of hydrostatic pressure was introduced in Section 10–1, the weight of the fluid was neglected and the pressure was assumed the same at all points. It is a familiar fact, however, that atmospheric pressure decreases with increasing altitude and that the pressure in a lake or in the ocean decreases with increasing distance from the bottom. We therefore generalize the definition of pressure and define the pressure *at any point* as the ratio of the normal force ΔF exerted on a small area ΔA including the point, to the area ΔA.

$$p = \frac{\Delta F}{\Delta A}, \qquad \Delta F = p\,\Delta A. \tag{12–2}$$

If the pressure is the same at all points of a finite plane surface of area A, these equations reduce to Eqs. (10–3),

$$p = \frac{F}{A}, \qquad F = pA.$$

Let us find the general relation between the pressure p at any point in a fluid and the elevation of the point, y. If the fluid is in equilibrium, every volume element is in equilibrium. Consider an element in the form of a thin slab, shown in Fig. 12–1, whose thickness is Δy and whose faces have an area A. If ρ is the density of the fluid, the mass of the element is $\rho A\,\Delta y$ and its weight Δw is $\rho g A\,\Delta y$. The force exerted on the element by the surrounding fluid is everywhere normal to its surface. By symmetry, the resultant horizontal force on its rim is zero. The upward force on

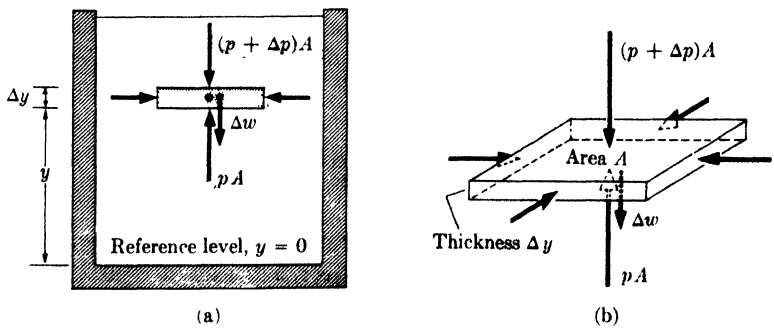


FIG. 12–1. Forces on an element of fluid in equilibrium.

its lower face is pA, and the downward force on its upper face is $(p + \Delta p)A$. Since it is in equilibrium,

$$\sum F_y = 0,$$

$$pA - (p + \Delta p)A - \rho g A\, \Delta y = 0,$$

or

$$\boxed{\Delta p = -\rho g\, \Delta y.} \tag{12-3}$$

Since ρ and g are both positive quantities, it follows that a positive Δy (an increase of elevation) is accompanied by a negative Δp (decrease of pressure). If p_1 and p_2 are the pressures at elevations y_1 and y_2 above some reference level, then

$$p_2 - p_1 = -\rho g(y_2 - y_1).$$

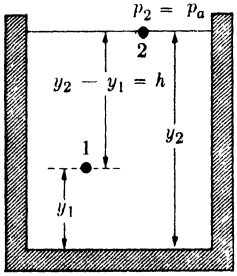


FIGURE 12-2

Let us apply this equation to a liquid in an open vessel, such as that shown in Fig. 12-2. Take point 1 at any level and let p represent the pressure at this point. Take point 2 at the top where the pressure is atmospheric pressure, p_a. Then

$$p_a - p = -\rho g(y_2 - y_1),$$

$$p = p_a + \rho g h. \tag{12-4}$$

Notice that the shape of the containing vessel does not affect the pressure, and that the pressure is the same at all points at the same depth. It also follows from Eq. (12-4) that if the pressure p_a is increased in any way, say by inserting a piston on the top surface and pressing down on it, the pressure p at any depth must increase by exactly the same amount. This fact was stated by the French scientist Blaise Pascal (1623–1662) in 1653 and is called "Pascal's law." It is often stated: "Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel." We can see now that it is not an independent principle but a necessary consequence of the laws of mechanics.

Pascal's law is illustrated by the operation of a hydraulic press, shown in Fig. 12-3. A piston of small cross-sectional area a is used to exert a small force f directly on a liquid such as oil. The pressure $p = f/a$ is transmitted through the connecting pipe to a larger cylinder equipped with a larger piston of area A. Since the pressure is the same in

both cylinders,

$$p = \frac{f}{a} = \frac{F}{A}$$

and

$$F = \frac{A}{a} \times f.$$

It follows that the hydraulic press is a force-multiplying device with an ideal mechanical advantage equal to the ratio of the areas of the two pistons. Barber chairs, dentist chairs, car lifts, and hydraulic brakes are all devices that make use of the principle of the hydraulic press

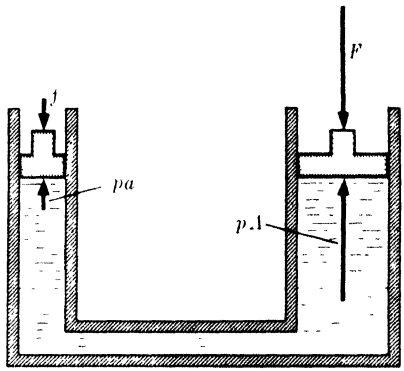


FIG. 12-3. Principle of the hydraulic press.

12-3 The hydrostatic paradox. If a number of vessels of different shapes are interconnected as in Fig. 12-4(a), it will be found that a liquid poured into them will stand at the same level in each. Before the principles of hydrostatics were completely understood, this seemed a very puzzling phenomenon and was called the "hydrostatic paradox." It would appear at first sight, for example, that vessel C should develop a greater pressure at its base than should B, and hence that liquid would be forced from C into B.

Equation (12-4), however, states that the pressure depends only on the depth below the liquid surface and not at all on the shape of the containing vessel. Since the depth of the liquid is the same in each vessel, the pressure at the base of each is the same and hence the system is in equilibrium.

A more detailed explanation may be helpful in understanding the situation. Consider vessel C in Fig. 12-4(b). The forces exerted against

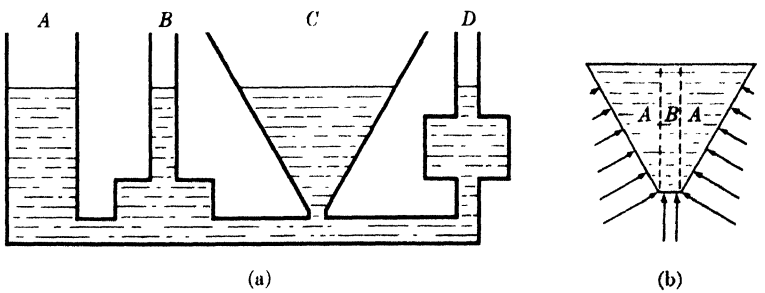


FIG. 12-4. (a) The hydrostatic paradox. The top of the liquid stands at the same level in each vessel. (b) Forces on the liquid in vessel C.

the liquid by the walls are shown by arrows, the force being everywhere perpendicular to the walls of the vessel. The inclined forces at the sloping walls may be resolved into horizontal and vertical components. The weight of the liquid in the sections lettered A is supported by the vertical components of these forces. Hence the pressure at the base of the vessel is due only to the weight of the liquid in the cylindrical column B. Any vessel, regardless of its shape, may be treated the same way.

12–4 Pressure gauges. The simplest type of pressure gauge is the open-tube manometer, illustrated in Fig. 12–5(a). It consists of a U-shaped tube containing a liquid, one end of the tube being at the pressure p which it is desired to measure, while the other end is open to the atmosphere.

The pressure at the bottom of the left column is

$$p + \rho g y_1,$$

while that at the bottom of the right column is

$$p_a + \rho g y_2,$$

where ρ is the density of the manometric liquid. Since these pressures both refer to the same point, it follows that

$$p + \rho g y_1 = p_a + \rho g y_2,$$

and

$$p - p_a = \rho g(y_2 - y_1) = \rho g h.$$

The pressure p is called the *absolute pressure*, whereas the difference $p - p_a$ between this and the atmospheric pressure is called the *gauge pressure*. It is seen that the gauge pressure is proportional to the difference in height of the liquid columns.

The mercury barometer is a long glass tube that has been filled with mercury and then inverted in a dish of mercury, as shown in Fig. 12–5(b). The space above the mercury column contains only mercury vapor whose pressure, at room temperature, is so small that it may be neglected. It is easily seen that

$$p_a = \rho g(y_2 - y_1) = \rho g h.$$

Because mercury manometers and barometers are used so frequently in laboratories, it is customary to express atmospheric pressure and other pressures as so many "inches of mercury" or "centimeters of mercury." An "inch of mercury," however, is *not* a real unit of pressure, since pressure is the ratio of force to area.

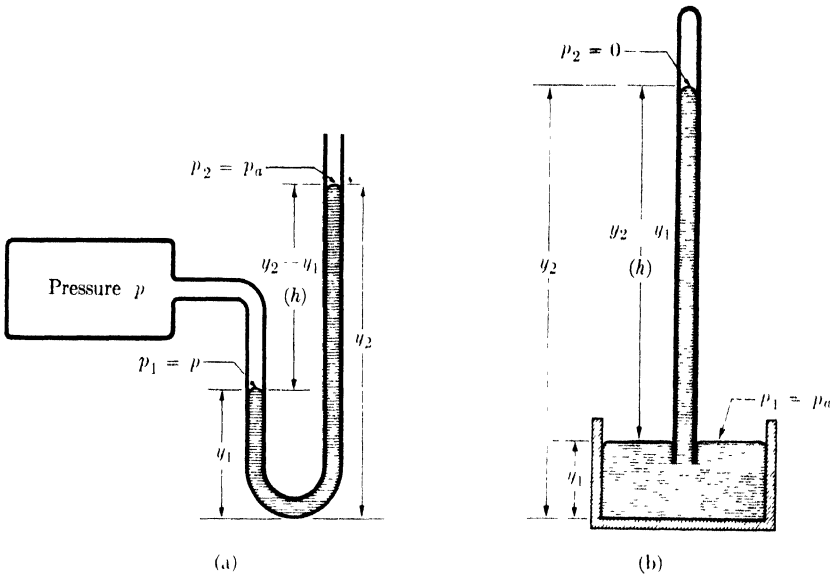


FIG. 12-5. (a) The open-tube manometer. (b) The barometer.

EXAMPLE. Compute the atmospheric pressure on a day when the height of the barometer is 76.0 cm.

The height of the mercury column depends on ρ and g as well as on the atmospheric pressure. Hence both the density of mercury and the local acceleration of gravity must be known. The density varies with the temperature, and g with the latitude and elevation above sea level. All accurate barometers are provided with a thermometer and with a table or chart from which corrections for temperature and elevation can be found. If we assume $g = 980$ cm/sec^2 and $\rho = 13.6$ gm/cm^3,

$$p_a = \rho g h = 13.6 \frac{\text{gm}}{\text{cm}^3} \times 980 \frac{\text{cm}}{\text{sec}^2} \times 76 \text{ cm}$$

$$= 1{,}013{,}000 \frac{\text{dynes}}{\text{cm}^2}.$$

(About a million dynes per square centimeter.) In British engineering units,

$$76 \text{ cm} = 30 \text{ in.} = 2.5 \text{ ft},$$

$$\rho g = 850 \frac{\text{lb}}{\text{ft}^3},$$

$$p_a = 2120 \frac{\text{lb}}{\text{ft}^2} = 14.7 \frac{\text{lb}}{\text{in}^2}.$$

A pressure of 1.013×10^6 dynes/cm^2 = 1.013×10^5 newtons/m^2 = 14.7 lb/in^2, is called *one atmosphere*. A pressure of exactly one million dynes per square centimeter is called one *bar*, and a pressure one one-thousandth as great is one *millibar*. Atmospheric pressures are of the order of 1000 millibars, and are now stated in terms of this unit by the United States Weather Bureau.

The Bourdon-type pressure gauge is more convenient for most purposes than a liquid manometer. It consists of a flattened brass tube closed at one end and bent into a circular form. The closed end of the tube is connected by a gear and pinion to a pointer which moves over a scale. The open end of the tube is connected to the apparatus, the pressure within which is to be measured. When pressure is exerted within the flattened tube, it straightens slightly just as a bent rubber hose straightens when water is admitted. The resulting motion of the closed end of the tube is transmitted to the pointer.

12–5 Archimedes' principle. The irregular outline in Fig. 12–6 represents an imaginary surface bounding an arbitrary portion of a fluid at rest. The short arrows represent the forces exerted by the surrounding fluid against small elements of the boundary surface of equal area ΔA. The force $\Delta\mathbf{F}$ against each element is normal to that element and equal to $p\,\Delta A$, where p depends only on the vertical depth below the free surface and not on the shape or orientation of the boundary surface.

Since the entire fluid is at rest, the x-component of the resultant of these surface forces is zero. The y-component of the resultant, $\mathbf{F}_y$, must equal the weight of the fluid inside the arbitrary surface, $m\mathbf{g}$, and its line of action must pass through the center of gravity of this fluid.

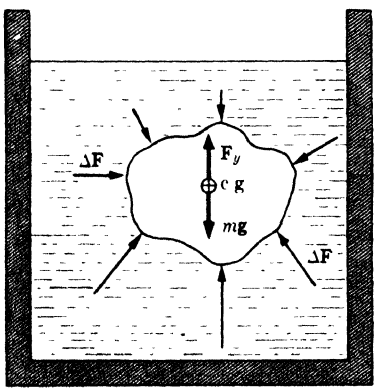


FIG. 12–6. Archimedes' principle. The buoyant force $\mathbf{F}_y$ equals the weight of the displaced fluid.

Now suppose the fluid inside the surface to be removed and replaced by a solid body having exactly the same shape. The pressure at every point will be exactly the same as before, so the force exerted on the body by the surrounding fluid will be unaltered. That is, *the fluid exerts on the body an upward force* $\mathbf{F}_y$, *which is equal to the weight* $m\mathbf{g}$ *of the fluid originally occupying the boundary surface and whose line of action passes through the original center of gravity.*

The submerged body, in general, will *not* be in equilibrium. Its weight may be greater or less than $\mathbf{F}_y$, and

if it is not homogeneous, its center of gravity may not lie on the line of $\mathbf{F}_y$. Therefore, in general, it will be acted on by a resultant force through its own center of gravity and by a couple, and will rise or fall and also rotate.

The fact that a body immersed in a fluid should be "buoyed up" with a force equal to the weight of the displaced fluid was deduced by Archimedes (287–212 B.C.) from reasoning along the same lines as above. It is called *Archimedes' principle* and is, of course, a consequence of Newton's laws and the properties of a fluid. The position of the line of action of the upward force, usually omitted from a statement of the principle, is equal in importance to the magnitude of the force.

The weight of a dirigible floating in air, or of a submarine floating at some depth below the surface of the water, is just equal to the weight of a volume of air, or water, that is equal to the volume of the dirigible or submarine. That is, the average density of the dirigible equals that of air, and the average density of the submarine equals the density of water.

A body whose average density is less than that of a liquid can float *partially* submerged at the free upper surface of the liquid. However, we not only want a ship to float, but to float upright in stable equilibrium without capsizing. This requires that normally the line of action of the buoyant force should pass through the center of gravity of the ship and also, when the ship heels, the couple set up by its weight and the buoyant force should be in such a direction as to right it.

Figure 12-7 represents a section of the hull of a yacht, when on an even keel and when heeled over. In (a), the weight of the yacht **w** and the buoyant force **B** are equal and opposite, and the lines of action of both pass through the center of gravity of the yacht (c.g.). In (b), the center of gravity of the displaced water has shifted to point b and the line of

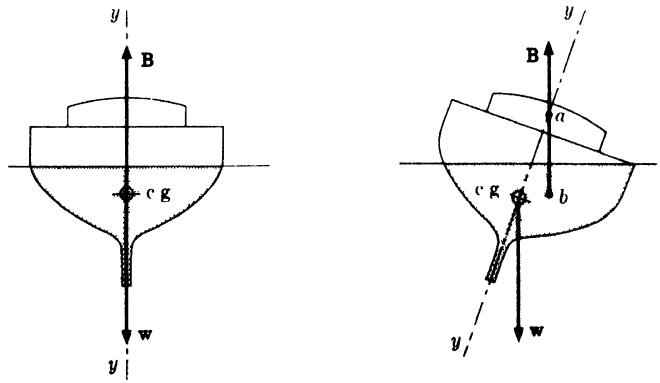


FIG. 12-7. Forces on the hull of a yacht.

action of the buoyant force **B** passes through this point. The line of action of **w** still passes through the center of gravity of the yacht. (Actually, a simple diagram like Fig. 12–7 would apply only if the shape of the hull were the same at all cross sections.) The weight **w** and the buoyant force **B** give rise to a couple in such a direction as to right the yacht. If the line of action of **B** should intersect yy at a point below the center of gravity, the yacht would be unstable and would capsize.

When making "weighings" with a sensitive analytical balance, correction must be made for the buoyant force of the air if the density of the body being "weighed" is very different from that of the standard "weights," which are usually of brass. For example, suppose a block of wood of density 0.4 gm/cm^3 is balanced on an equal-arm balance by brass "weights" of 20 gm, density 8.0 gm/cm^3. The apparent weight of each body is the difference between its true weight and the buoyant force of the air. If ρ_w, ρ_b, and ρ_a are the densities of the wood, brass, and air, and V_w and V_b are the volumes of the wood and brass, the apparent weights, which are equal, are

$$\rho_w V_w g - \rho_a V_w g = \rho_b V_b g - \rho_a V_b g.$$

The true mass of the wood is $\rho_w V_w$, and the true mass of the standard is $\rho_b V_b$. Hence,

$$\begin{aligned} \text{True mass} &= \rho_w V_w = \rho_b V_b + \rho_a(V_w - V_b) \\ &= \text{mass of standard} + \rho_a(V_w - V_b). \end{aligned}$$

In the specific example cited

$$V_w = \frac{20}{0.4} = 50 \text{ cm}^3 \text{ (very nearly)},$$

$$V_b = \frac{20}{8} = 2.5 \text{ cm}^3, \qquad \rho_a = 0.0013 \frac{\text{gm}}{\text{cm}^3}.$$

Hence

$$\rho_a(V_w - V_b) = 0.0013 \times 47.5 = 0.062 \text{ gm}.$$

$$\text{True mass} = 20.062 \text{ gm}.$$

If measurements are being made to one one-thousandth of a gram, it is obvious that the correction of 62 thousandths is of the greatest importance.

EXAMPLE. A tank containing water is placed on a spring scale, which registers a total weight **W**. A stone of weight **w** is hung from a string and lowered into the water without touching the sides or bottom of the tank [Fig. 12–8(a)]. What will be the reading on the spring scale?

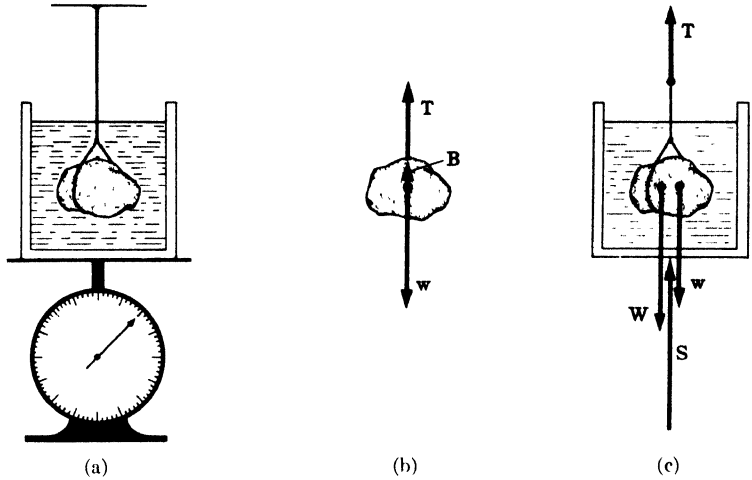


FIGURE 12–8

First, for the stone alone, the forces are as shown in Fig. 12-8(b), where **B** is the buoyant force and **T** is the tension in the string. Since $\sum F_y = 0$,

$$T + B = w.$$

Next, for the tank with the water and stone in it, the forces are as shown in Fig. 12-8(c), where **S** is the force exerted by the spring scale on the isolated system and, by Newton's third law, is equal in magnitude and opposite in direction to the force exerted on the scale. The condition for equilibrium yields the equation

$$T + S = w + W.$$

Subtracting the first equation from the second, we get

$$S = W + B.$$

That is, the reading of the spring scale has been increased by an amount equal to the buoyant force.

PROBLEMS

12–1. The piston of a hydraulic automobile lift is 12 inches in diameter. What pressure, in lb/in^2, is required to lift a car weighing 2400 lb?

12–2. The expansion tank of a household hot-water heating system is open to the atmosphere and is 30 ft above a pressure gauge attached to the furnace. What is the gauge pressure at the furnace, in lb/in^2?

12–3. The submarine Squalus sank at a depth of 240 ft. Compute the absolute pressure at this depth, in lb/in^2 and lb/ft^2. The specific gravity of sea water is 1.025.

12–4. A piece of gold-aluminum alloy weighs 10 lb. When suspended from a spring balance and submerged in water, the balance reads 8 lb. What is the weight of gold in the alloy if the specific gravity of gold is 19.3 and the specific gravity of aluminum is 2.5?

12–5. What is the area of the smallest block of ice 1 ft thick that will just support a man weighing 180 lb? The specific gravity of the ice is 0.917, and it is floating in fresh water.

12–6. A cubical block of wood 10 cm on a side floats at the interface between oil and water as in Fig. 12–9, with its lower surface 2 cm below the interface. The density of the oil is 0.6 gm/cm^3. (a) What is the mass of the block? (b) What is the gauge pressure at the lower face of the block?

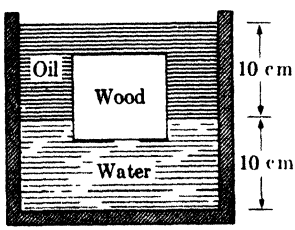


FIGURE 12–9

12–7. The densities of air, helium, and hydrogen (at standard conditions) are, respectively, 0.00129 gm/cm^3, 0.000178 gm/cm^3, and 0.0000899 gm/cm^3. What is the volume in cubic feet displaced by a hydrogen-filled dirigible which has a total "lift" of 10 tons? What would be the "lift" if helium were used instead of hydrogen?

12–8. A piece of wood is 2 ft long, 1 ft wide, and 2 inches thick. Its specific gravity is 0.6. What volume of lead must be fastened underneath to sink the wood in calm water so that its top is just even with the water level?

12–9. A cubical block of wood 10 cm on a side and of density 0.5 gm/cm^3 floats in a jar of water. Oil of density 0.8 gm/cm^3 is poured on the water until the top of the oil layer is 4 cm below the top of the block. (a) How deep is the oil layer? (b) What is the gauge pressure at the lower face of the block?

12–10. A cubical block of steel (density = 7.8 gm/cm^3) floats on mercury (density = 13.6 gm/cm^3). (a) What fraction of the block is above the mercury surface? (b) If water is poured on the mercury surface, how deep must the water layer be so that the water surface just rises to the top of the steel block?

12–11. Block A in Fig. 12–10 hangs by a cord from spring balance D and is submerged in a liquid C contained in beaker B. The weight of the beaker is 2 lb, the weight of the liquid is 3 lb. Balance D reads 5 lb and balance E reads 15 lb. The volume of block A is 0.1 ft^3. (a) What is the weight per unit volume of the liquid? (b) What will each balance read if block A is pulled up out of the liquid?

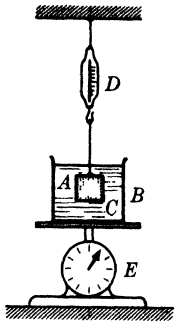


FIGURE 12–10

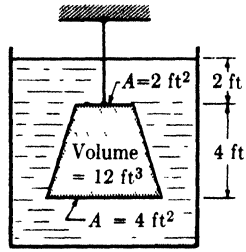


FIGURE 12–11

12–12. A hollow sphere of inner radius 9 cm and outer radius 10 cm floats half submerged in a liquid of specific gravity 0.8. (a) Calculate the density of the material of which the sphere is made. (b) What would be the density of a liquid in which the hollow sphere would just float completely submerged?

12–13. Two spherical bodies having the same diameter are released simultaneously from the same height. If the mass of one is ten times that of the other and if the air resistance on each is the *same*, show that the heavier body will arrive at the ground first.

12–14. When a life preserver having a volume of 0.75 ft^3 is immersed in sea water (specific gravity 1.1) it will just support a 160-lb man (specific gravity 1.2) with 2/10 of his volume above water. What is the weight per unit volume of the material composing the life preserver?

12–15. An object in the shape of a truncated cone weighs 1000 lb in vacuum and is suspended by a rope in an open tank of liquid density 2 slugs/ft^3, as in Fig. 12–11. (a) Find the total downward force exerted by the liquid on the top of the object, of area 2 ft^2. (b) Find the total upward force exerted by the liquid on the bottom of the object, of area 4 ft^2. (c) Find the tension in the cord supporting the object.

12–16. A hollow cylindrical can 20 cm in diameter floats in water with 10 cm of its height above the water line when a 10-kgm iron block hangs from its bottom. If the block is now placed inside the can, how much of the cylinder's height will be above the water line? The density of iron is 7.8 gm/cm^3.

12–17. A block of balsa wood placed in one scale pan of an equal-arm balance is found to be exactly balanced by a 100-gm brass "weight" in the other scale pan. Find the true mass of the balsa wood, if its specific gravity is 0.15.

12–18. A 3200-lb cylindrical can buoy floats vertically in salt water (specific gravity = 1.03). The diameter of the buoy is 3 ft. Calculate (a) the additional distance the buoy will sink when a 150-lb man stands on top, (b) the period of the resulting vertical simple harmonic motion when the man dives off.

12–19. A uniform rod AB, 12 ft long, weighing 24 lb, is supported at end B by a flexible cord and weighted at end A with a 12-lb lead weight. The rod

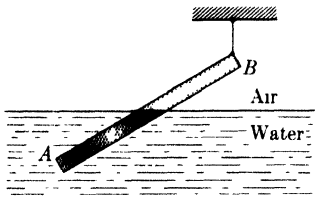


FIGURE 12–12

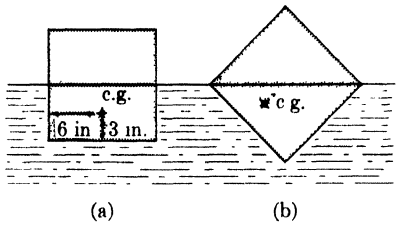


FIGURE 12–13

floats as shown in Fig. 12–12 with one-half its length submerged. The buoyant force on the lead weight can be neglected. (a) Show in a diagram all the forces acting on the rod. (b) Find the tension in the cord. (c) Find the total volume of the rod.

12–20. A cubical block of wood 1 ft on a side is weighted so that its center of gravity is at the point shown in Fig. 12–13(a), and it floats in water with one-half its volume submerged. Compute the righting moment when the block is "heeled" at an angle of 45° as in Fig. 12–13(b).

12–21. A hydrometer consists of a spherical bulb and a cylindrical stem of cross section 0.4 cm^2. The total volume of bulb and stem is 13.2 cm^3. When immersed in water the hydrometer floats with 8 cm of the stem above the water surface. In alcohol, 1 cm of the stem is above the surface. Find the density of the alcohol.

12–22. A 12-lb uniform rod 6 ft long, whose specific gravity is 0.50, is hinged at one end 3 ft below a water surface as in Fig. 12–14. (a) What weight w must be attached to the other end of the rod so that 5 ft of the rod are submerged? (b) Find the magnitude and direction of the force exerted by the hinge on the rod.

12–23. The following is quoted from a letter. How would you reply?

"It is the practice of carpenters hereabouts, when laying out and leveling up the foundations of relatively long buildings, to use a garden hose filled with water, into the ends of the hose being thrust glass tubes 10–12 inches long.

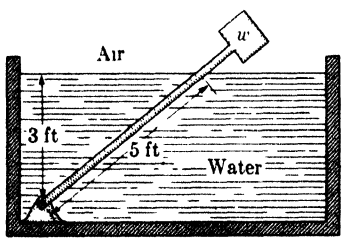


FIGURE 12–14

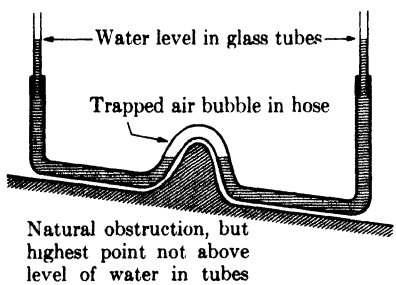


FIGURE 12–15

"The theory is that the water, seeking a common level, will be of the same height in both the tubes and thus effect a level. Now the question rises as to what happens if a bubble of air is left in the hose. Our greybeards contend the air will not affect the reading from one end to the other. Others say that it will cause important inaccuracies.

"Can you give a relatively simple answer to this question, together with an explanation? I include a rough sketch (Fig. 12–15) *of the situation that caused the dispute."*

CHAPTER 13

SURFACE TENSION

13–1 Surface tension. A liquid flowing slowly from the tip of a medicine dropper emerges, not as a continuous stream, but as a succession of drops. A sewing needle, if placed carefully on a water surface, makes a small depression in the surface and rests there without sinking, even though its density may be as much as ten times that of water. When a clean glass tube of small bore is dipped into water, the water rises in the tube, but if the tube is dipped in mercury, the mercury is depressed. All these phenomena, and many others of a similar nature, are associated with the existence of a boundary surface between a liquid and some other substance.

All surface phenomena indicate that the surface of a liquid can be considered to be in a state of stress such that if one considers any line lying in or bounding the surface, the material on either side of the line exerts a pull on the material on the other side. This pull lies in the plane of the surface and is perpendicular to the line. The effect can be demonstrated with the simple apparatus shown in Fig. 13–1. A wire ring a few inches in diameter has attached to it a loop of thread, as shown. When the ring and thread are dipped in a soap solution and removed, a thin film of liquid is formed in which the thread "floats" freely, as shown in part (a). If the film inside the loop of thread is punctured, the thread springs out into a circular shape as in part (b), as if the surfaces of the liquid were pulling radially outward on it, as shown by the arrows. Presumably, the same forces were acting before the film was punctured, but since there was film on *both* sides of the thread the net force exerted by the film on every portion of the thread was zero.

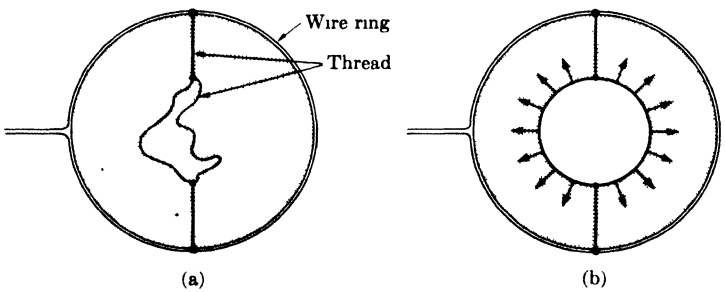


FIG. 13–1. A wire ring with a flexible loop of thread, dipped in a soap solution, (a) before and (b) after puncturing the surface films inside the loop.

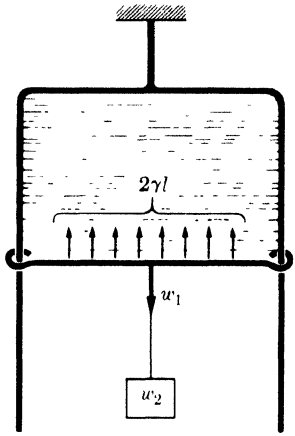


FIG. 13–2. The horizontal slide wire is in equilibrium under the action of the upward surface force $2\gamma l$ and the downward pull $w_1 + w_2$.

Another simple apparatus for demonstrating surface effects is shown in Fig. 13–2. A piece of wire is bent into the shape of a U and a second piece of wire is used as a slider. When the apparatus is dipped in a soap solution and removed, the slider (if its weight w_1 is not too great) is quickly pulled up to the top of the U. It may be held in equilibrium by adding a second weight w_2. Surprisingly, the same total force $F = w_1 + w_2$ will hold the slider at rest in *any* position, regardless of the area of the liquid film, provided the film remains at constant temperature. This is very different from the elastic behavior of a sheet of rubber, for which the force would be greater as the sheet was stretched.

Although a soap film like that in Fig. 13–2 is very thin, its thickness is still enormous compared with the size of a molecule. Hence it can be considered as made up chiefly of bulk liquid, bounded by two surface layers a few molecules thick. When the crossbar in Fig. 13–2 is pulled down and the area of the film is increased, molecules formerly in the main body of the liquid move into the surface layers. That is, these layers are not "stretched" as a rubber sheet would be, but more surface is created by molecules moving from the bulk liquid.

Let l be the length of the wire slider. Since the film has two surfaces, the total length along which the surface force acts is $2l$. The *surface tension* in the film, γ, is defined as *the ratio of the surface force to the length* (perpendicular to the force) *along which the force acts*. Hence in this case,

$$\gamma = F/2l. \qquad (13\text{–}1)$$

Surface tension is expressed in dynes/cm in the cgs system.

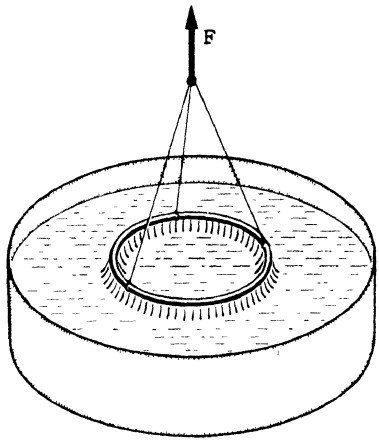


FIG. 13-3. Lifting a circular wire of length l out of a liquid requires an additional force **F** to balance the surface forces $2\gamma l$. This method is commonly used to measure surface tension.

Another less spectacular way of showing a surface force is embodied in the actual apparatus, shown in Fig. 13-3, that is often used to measure surface tension. A circular wire whose circumference is of length l is lifted out from the body of a liquid. The additional force F needed to balance the surface forces $2\gamma l$ due to the two surface films on each side is measured either by the stretch of a delicate spring or by the twist of a torsion wire. The surface tension is then given by

$$\gamma = \frac{F}{2l}.$$

Other methods of measuring surface tension will be apparent in what is to follow. Some typical values are shown in Table 13-1.

The surface tension of a liquid surface in contact with its own vapor or with air is found to depend only on the nature of the liquid and on the temperature. The values for water in Table 13-1 are typical of the general result that surface tension decreases as the temperature increases. Measurements of the surface tension of an extremely thin layer of oil on the surface of water indicate that, in this case, the surface tension depends on the area of the oil film as well as on the temperature.

Another useful viewpoint regarding surface effects is the following. Suppose the wire in Fig. 13-2 is moved down a distance y by applying a downward force $F = w_1 + w_2$. The force F will remain constant provided the temperature of the surface film is kept constant during the motion. The work done is Fy, and the total surface area of the film is

TABLE 13–1

EXPERIMENTAL VALUES OF SURFACE TENSION

Liquid in contact with air	t, °C	Surface tension, dynes/cm
Benzene	20	28 9
Carbon tetrachloride	20	26 8
Ethyl alcohol	20	22.3
Glycerine	20	63.1
Mercury	20	465
Olive oil	20	32 0
Soap solution	20	25 0
Water	0	75 6
Water	20	72 8
Water	60	66 2
Water	100	58 9
Oxygen	−193	15 7
Neon	−247	5 15
Helium	−269	0 12

increased by $2ly$. The *work done per unit area*, in increasing the area, is therefore

$$\frac{\text{Work}}{\text{Increase in area}} = \frac{Fy}{2ly} = \frac{F}{2l}. \tag{13–2}$$

But, from Eq. (13–1), this is equal to the surface tension γ as previously defined, and hence this quantity can be considered either as the *force per unit length* at right angles to the force, or as the *work done per unit area* to increase the area. Using the second definition, the cgs units of γ would be ergs/cm^2, which are equivalent to dynes/cm, since 1 erg = 1 dyne·cm.

13–2 Pressure difference across a surface film. A soap bubble consists of two spherical surface films very close together, with liquid between. If we isolate one-half of the bubble and apply the principles of statics to the equilibrium of this half-bubble, we obtain a simple relation between the surface tension and the difference in pressure of the air inside and that outside the bubble. Consider first a small element ΔA of a surface, shown in Fig. 13–4. Suppose the air pressure on the left of this element is p and that on the right is p_a. The force normal to the element is therefore $(p - p_a)\,\Delta A$. The component of this force in the x-direction is

$$(p - p_a)\,\Delta A \cos\theta.$$

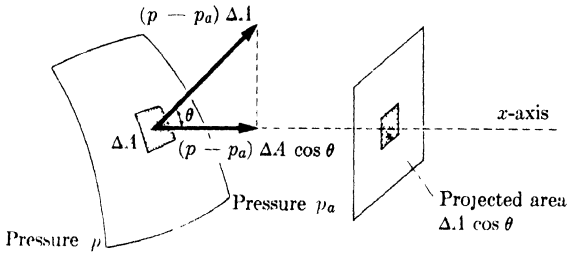


FIG. 13–4. The force in the x-direction is the difference of pressure multiplied by the *projected* area in the x-direction.

But $\Delta A \cos\theta$ is the area projected on a plane perpendicular to the x-axis. The force in the x-direction is therefore the difference of pressure multiplied by the projected area in the x-direction.

Now consider the half-bubble shown in Fig. 13–5. The other half exerts a force to the left equal to twice the surface tension times the perimeter or

$$F \text{ (to the left)} = 2\gamma \times 2\pi R.$$

The force to the right is equal to the pressure difference $p - p_a$ multiplied by the area obtained by projecting the half-bubble on a plane perpendicular to the direction in question. Since this projected area is πR^2,

$$F \text{ (to the right)} = (p - p_a)\pi R^2.$$

Since the half-bubble is in equilibrium,

$$(p - p_a)\pi R^2 = 4\pi R\gamma,$$

or

$$\boxed{p - p_a = \frac{4\gamma}{R}. \qquad \text{(Soap bubble)}} \tag{13–3}$$

It follows from this result that if the surface tension remains constant (this means constant temperature), the pressure difference is larger the smaller the value of R. If two bubbles, therefore, are blown at opposite ends of a pipe, the smaller of the two will force air into the larger. In other words, the smaller one will get still smaller, and the larger will increase.

It may easily be verified that in the case of a liquid drop which has only one surface film, the difference between the pressure of the liquid

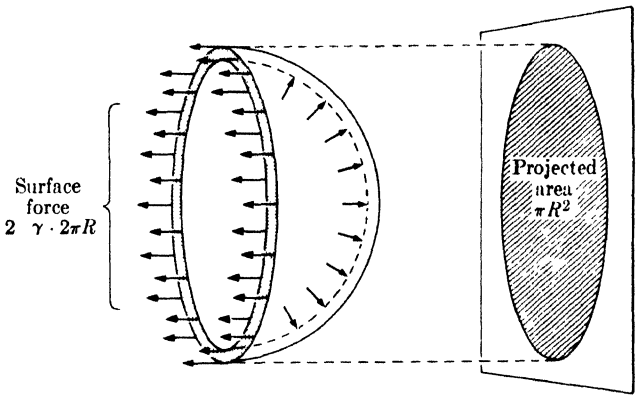


FIG. 13-5. Equilibrium of half of a soap bubble. The force exerted by the other half is $2 \cdot \gamma \cdot 2\pi R$, and the net force exerted by the air inside and outside the bubble is the pressure difference times the projected area, or $(p - p_a)\pi R^2$.

and that of the outside air is given by

$$p - p_a = \frac{2\gamma}{R}. \qquad \text{(Liquid drop)} \tag{13-4}$$

EXAMPLE. Calculate the excess pressure inside a drop of mercury whose temperature is 20°C and whose radius is 4 mm.

$$p - p_a = \frac{2\gamma}{R} = \frac{2 \times 465 \text{ dynes/cm}}{0.4 \text{ cm}} = 2325 \frac{\text{dynes}}{\text{cm}^2}.$$

13-3 Minimal surfaces. Any surface under tension tends to contract until it occupies the minimum area consistent with the boundaries of the surface and with the difference of pressure on opposite sides of the surface. A small volume of heavy engine oil injected into the center of a mixture of alcohol and water whose density is the same as that of the oil will therefore contract until it has the smallest surface area consistent with its volume. The shape of such a surface is spherical.

A more interesting situation is depicted in Fig. 13-6, which shows a series of high-speed photographs of successive stages in the formation of a drop of milk at the end of a vertical tube. The photographs were taken by Dr. Edgerton of M.I.T. It will be seen that the process, if examined in detail, is exceedingly complex. An interesting feature is the small drop that follows the larger one. Both drops execute a few oscillations after their formation (4, 5, and 6) and eventually assume a spherical shape (7),

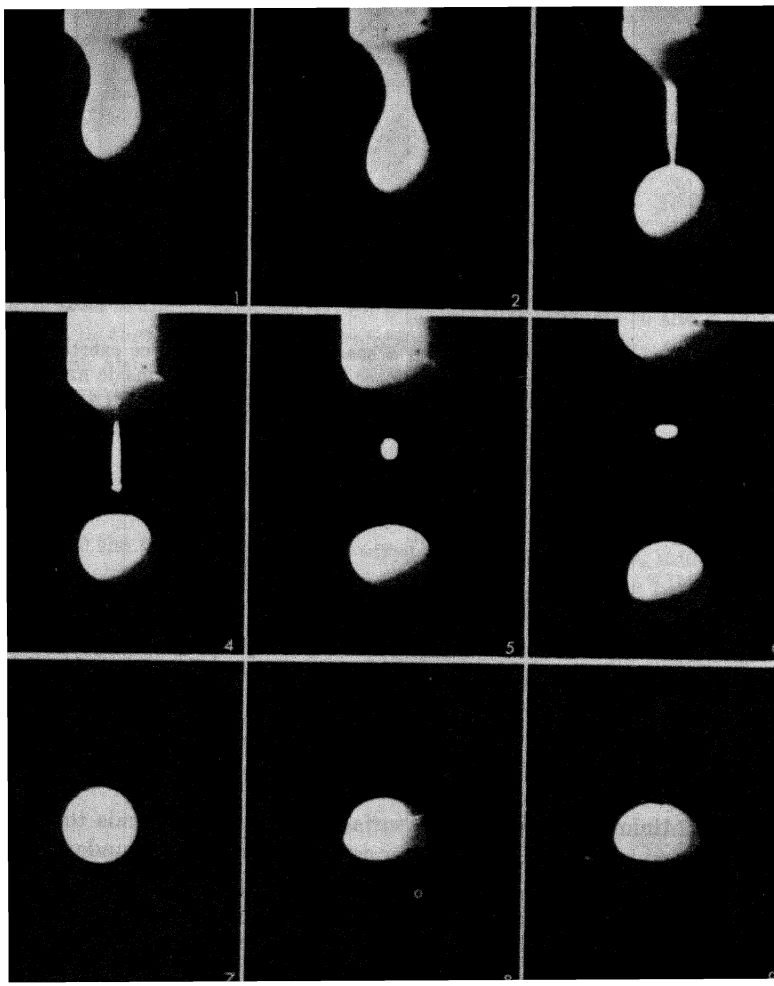


FIG. 13–6. Successive stages in the formation of a drop. (Reproduced from *Flash*, courtesy of Ralph S. Hale & Co.)

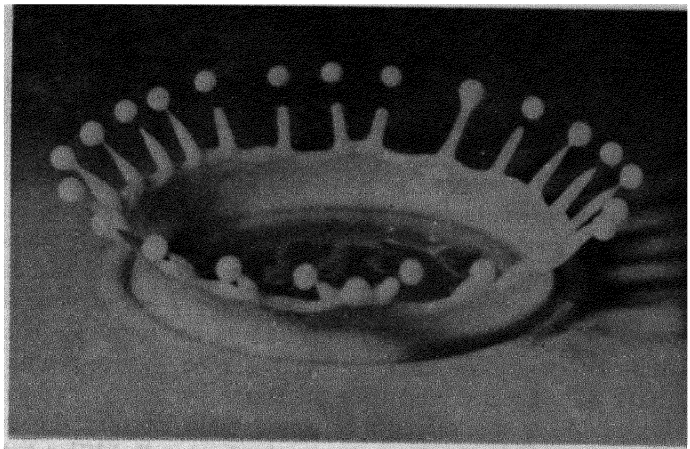


FIGURE 13-7

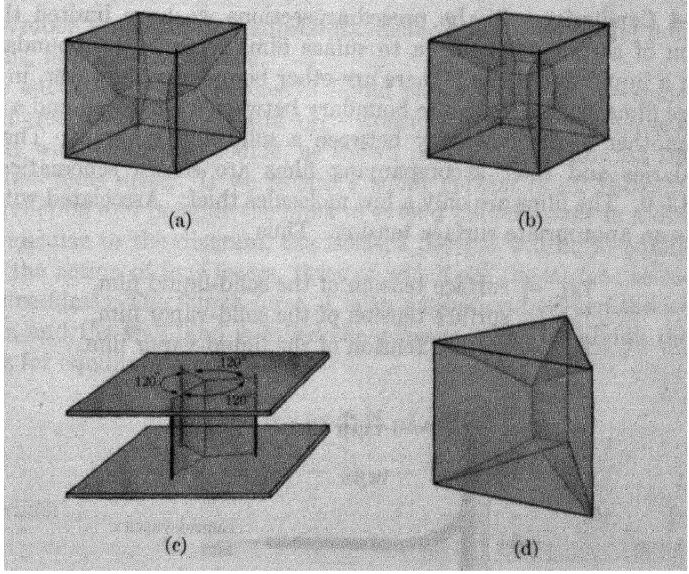


FIG. 13-8. Solution of mathematical problems involving minimal surfaces by means of surface films. (a) Cubical wire framework dipped once. (b) Cubical wire framework dipped twice to entrap an air bubble in the center. (c) Two plastic plates connected by three wires form three plane surface films at angles of 120° to each other. (d) A wire framework in the form of a prism shows that at most three surface films can intersect in a line intersect at a point.

which would be retained but for the effects of air resistance, as shown in 8 and 9. The drop in 9 has fallen 14 ft.

A beautiful photograph of the splash made by a drop of milk falling on a hard surface is reproduced in Fig. 13–7. It also was taken by Dr. Edgerton.

Surface films may be used to solve problems in mathematics whose analytic solution presents great difficulties. If it is required to find the minimal surface (surface of minimum area) bounded by a wire framework bent into an arbitrary shape, the problem may be solved by dipping the wire framework into a soap solution and waiting a few seconds for the film to contract. The results for two frames in the form of a cube and one in the form of a prism are shown in Fig. 13–8. These results could hardly have been guessed, and their prediction by purely mathematical methods would have been attended by considerable difficulty. For further details, the student is referred to a fascinating book called *What is Mathematics?* by Courant and Robbins.

13–4 Capillarity. In the preceding sections we have limited the discussion of surface phenomena to suface films lying in the boundary between a liquid and a gas. There are other boundaries, however, in which surface films exist. One is the boundary between a solid wall and a liquid, and another is the boundary between a solid and a vapor. The three boundaries and their accompanying films are shown schematically in Fig. 13–9. The films are only a few molecules thick. Associated with each film is an appropriate surface tension. Thus

γ_{SL} = surface tension of the solid-liquid film,
γ_{SV} = surface tension of the solid-vapor film,
γ_{LV} = surface tension of the liquid-vapor film.

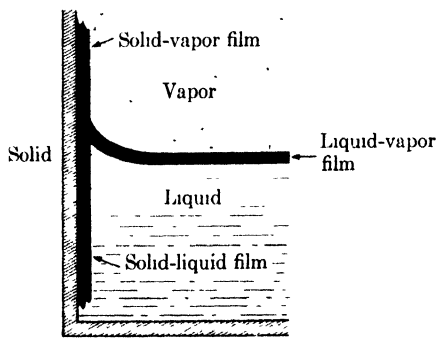


FIG. 13–9. Surface films exist at the solid-vapor boundary and at the solid-liquid boundary as well as at the liquid-vapor boundary.

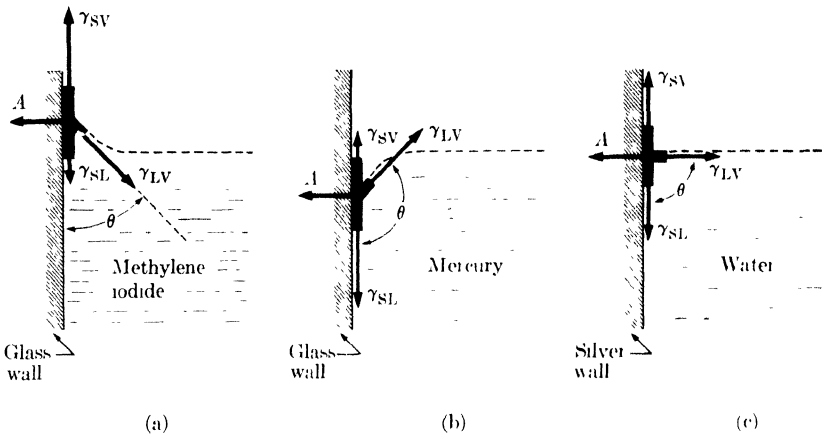


FIG. 13–10. The surface of a liquid near a solid wall is curved if the solid-vapor surface tension γ_{SV} differs from the solid-liquid surface tension γ_{SL}.

The symbol γ without any subscripts, defined and used in the preceding sections, now appears as γ_{LV}.

The curvature of the surface of liquid near a solid wall depends upon the difference between γ_{SV} and γ_{SL}. Consider a portion of a glass wall in contact with methylene iodide, as shown in Fig. 13–10(a). At the wall the three films meet. If we isolate a small portion of all three films at their junction and imagine the films to extend unit distance in a direction perpendicular to the diagram, the isolated portion will be in equilibrium under the action of four forces, three of which are the surface tensions of the three films. The fourth force A is an attraction between the isolated portion and the wall, and is called the *adhesive force*. Applying the conditions for equilibrium, we get

$$\sum F_x = \gamma_{LV} \sin \theta - A = 0,$$

$$\sum F_y = \gamma_{SV} - \gamma_{SL} - \gamma_{LV} \cos \theta = 0,$$

from which

$$A = \gamma_{LV} \sin \theta, \tag{13–5}$$

$$\gamma_{SV} - \gamma_{SL} = \gamma_{LV} \cos \theta. \tag{13–6}$$

The first equation enables us to calculate the adhesive force from measurements of γ_{LV} and the angle θ, known as the *contact angle*. The second equation shows that the contact angle, which is a measure of the curvature of the liquid-vapor surface adjacent to the wall, depends on the difference between γ_{SV} and γ_{SL}. Thus, in Fig. 13–10(a), γ_{SV} is greater than γ_{SL},

TABLE 13-2

CONTACT ANGLES

Liquid	Wall	Contact angle
α-Bromonaphthalene ($C_{10}H_7Br$)	Soda-lime glass	5°
	Lead glass	6°45′
	Pyrex	20°30′
	Fused quartz	21°
Methylene iodide (CH_2I_2)	Soda-lime glass	29°
	Lead glass	30°
	Pyrex	29°
	Fused quartz	33°
Water	Paraffin	107°
Mercury	Soda-lime glass	140°

cos θ is positive, and θ lies between 0° and 90°. The liquid is said to *wet* the glass.

In Fig. 13–10(b) a glass wall is in contact with mercury. The contact angle θ is about 140°, cos θ is negative, and hence γ_{SV} is less than γ_{SL}. When θ lies between 90° and 180°, as it does here, we say that the liquid does *not* wet the glass.

In Fig. 13–10(c) a situation closely approximated by silver in contact with water is shown. In this case, γ_{SV} is very nearly equal to γ_{SL}, cos θ is zero, and θ is 90°.

There are a number of liquids whose contact angles are zero when the liquids are in contact with soda-lime glass, lead glass, pyrex, and fused quartz. Such is the case for water, alcohol, ether, carbon tetrachloride, xylene, glycerine, and acetic acid. Some liquids, however, such as the first two listed in Table 13–2, have contact angles that depend upon the nature of the wall with which they are in contact.

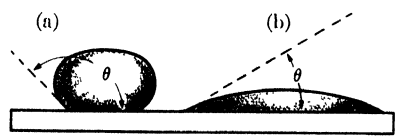


FIG. 13–11. Effect of decreasing contact angle by a wetting agent.

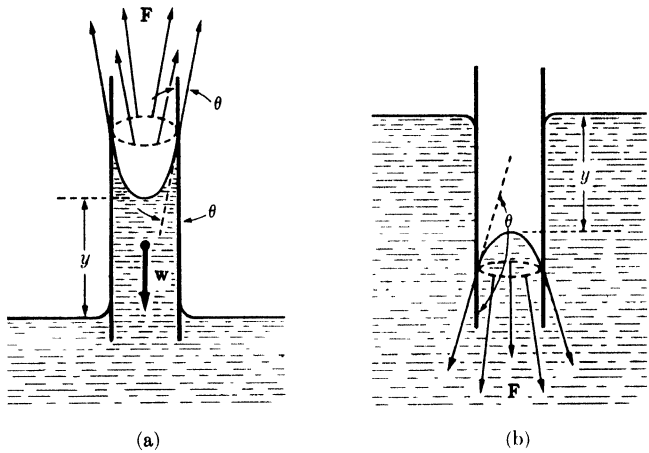


FIG. 13–12. Surface tension forces on a liquid in a capillary tube. The liquid rises if $\theta < 90°$ and is depressed if $\theta > 90°$.

Impurities and adulterants present in or added to a liquid may alter the contact angle considerably. In recent years a number of chemicals have been developed which are very potent as *wetting agents* or *detergents.* These compounds change the contact angle from a large value, greater than 90°, to a value much smaller than 90°. Conversely, waterproofing agents applied to a cloth cause the contact angle of water in contact with the cloth to be larger than 90°. The effect of a detergent on a drop of water resting on a block of paraffin is shown in Fig. 13–11.

The most familiar surface effect is the elevation of a liquid in an open tube of small cross section. The term "capillarity," used to describe effects of this sort, originates from the description of such tubes as "capillary" or "hairlike." In the case of a liquid that wets the tube, the contact angle is less than 90° and the liquid rises until an equilibrium height y is reached, as shown in Fig. 13–12(a). If the tube radius is r, the liquid makes contact with the tube along a line of length $2\pi r$. When we isolate the cylinder of liquid of height y and radius r along with its liquid-vapor film, the total upward force is

$$F = 2\pi r\gamma_{LV} \cos \theta.$$

The downward force is the weight of the cylinder **w**, which is equal to the weight density ρg times the volume $\pi r^2 y$, or

$$w = \rho g \pi r^2 y.$$

The condition for equilibrium is

$$\rho g \pi r^2 y = 2\pi r \gamma_{LV} \cos \theta,$$

or

$$\boxed{y = \frac{2\gamma_{LV} \cos \theta}{\rho g r}.}$$

The same equation holds for the capillary depression, shown in Fig. 13–12(b). Capillarity accounts for the rise of ink in blotting paper, the rise of lighting fluid in the wick of a cigarette lighter, and many other common phenomena.

Problems

13–1. Compare the tension of a soap bubble with that of a rubber balloon in the following respects: (a) Has each a surface tension? (b) Does the surface tension depend on area? (c) Is Hooke's law applicable?

13–2. Water can rise to a height y in a certain capillary. Suppose that this tube is immersed in water so that only a length $y/2$ is above the surface. Will you have a fountain or not? Explain.

13–3. A capillary tube is dipped in water with its lower end 10 cm below the water surface. Water rises in the tube to a height of 4 cm above that of the surrounding liquid, and the angle of contact is zero. What gauge pressure is required to blow a hemispherical bubble at the lower end of the tube?

13–4. A glass tube of inside diameter 1 mm is dipped vertically into a container of mercury, with its lower end 1 cm below the mercury surface. (a) What must be the gauge pressure of air in the tube to blow a hemispherical bubble at its lower end? (b) To what height will mercury rise in the tube if the air pressure in the tube is 3×10^4 dynes/cm^2 below atmospheric? The angle of contact between mercury and glass is 140°.

13–5. On a day when the atmospheric pressure is 950 millibars, (a) what would be the height of the mercury column in a barometric tube of inside diameter 2 mm? (b) What would be the height in the absence of any surface tension effects? (c) What is the minimum diameter a barometric tube may have in order that the correction for capillary depression shall be less than 0.01 cm of mercury?

13–6. (a) Derive the expression for the height of capillary rise in the space between two parallel plates dipping in a liquid. (b) Two glass plates, parallel to each other and separated by 0.5 mm, are dipped in water. To what height will the water rise between them? Assume zero angle of contact.

13–7. A tube of circular cross section and outer radius 0.14 cm is closed at one end. This end is weighted and the tube floats vertically in water, heavy end down. The total mass of the tube and weights is 0.20 gm. If the angle of contact is zero, how far below the water surface is the bottom of the tube?

13–8. Find the gauge pressure, in dynes/cm^2, in a soap bubble 5 cm in diameter. The surface tension is 25 dynes/cm.

13–9. Two large glass plates are clamped together along one edge and separated by spacers a few millimeters thick along the opposite edge to form a wedge-shaped air film. These plates are then placed vertically in a dish of colored liquid. Show that the edge of the liquid forms an equilateral hyperbola.

13–10. A soap bubble may be drawn out into a cylinder by touching to it a ring of the same diameter as the tube from which the bubble is blown, and then "stretching" the bubble between the tube and the ring. By isolating half of the cylindrical surface, show that the gauge pressure within the bubble is given by γ_{LV}/R, where R is the radius of the cylindrical surface.

CHAPTER 14

HYDRODYNAMICS AND VISCOSITY

14–1 Streamline flow. Hydrodynamics is the study of fluids in motion. It is one of the most complex branches of mechanics, as will be realized by considering such common examples of fluid flow as a river in flood or a swirling cloud of cigarette smoke. While it must be true that $F = ma$ at each instant for each drop of water or each smoke particle, imagine attempting to write their equations of motion! However, the problem is not as hopeless as it seems at first sight.

When the proper conditions are fulfilled, the flow of a fluid is of a relatively simple type called *streamline* or *laminar* flow. Figure 14–1 represents a portion of a pipe in which a fluid is flowing from left to right. If the flow is of the streamline type, every particle passing a point such as a follows exactly the same path as the preceding particles which passed the same point. These paths are called *lines of flow* or *streamlines*, and three of them are shown in the figure. If the cross section of the pipe varies from point to point, the velocity of any one particle will vary along its line of flow, but at any fixed point in the pipe the velocity of the particle which happens to be at that point is always the same. The particle which is now at a in the figure will be a moment later at point b, traveling in a different direction with a different speed, and a moment later yet it will be at c, having again changed its velocity. However, if we fix our attention on the point of space marked b, then each successive particle as it passes through b will be traveling in exactly the same direction and with the same speed as is the particle which is at that point now.

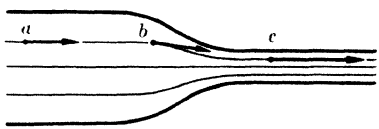


FIG. 14–1. Streamline or laminar flow.

Any real fluid, because of its viscosity, will have a higher velocity at the center of the pipe than at the outside. For the present we shall assume the fluid to be nonviscous and the velocity to be the same at all points of a transverse cross section.

The flow of a fluid is of the streamline type provided the velocity is not too great and the obstructions, constrictions, or bends in the pipe are

not such as to cause the lines of flow to change their direction too abruptly. If these conditions are not fulfilled, the flow is of a much more complicated type called *turbulent.*

14–2 Bernoulli's equation. The fundamental equation of hydrodynamics is Bernoulli's equation, which is a relation between the pressure, velocity, and elevation at points along a line of flow. Figure 14–2 represents a portion of a pipeline in which an incompressible, nonviscous fluid is flowing with streamline flow. The portion of the pipe shown in the figure has a uniform cross section A_1 at the left, followed by a region of diminishing cross section, and then a length of uniform but smaller cross section A_2. Focusing our attention on a portion of the fluid represented by both cross shading and horizontal shading (hereafter called the "system"), let us consider the motion of this system from the position shown in (a) to that in (b).

At all points in the wide part of the pipe, the pressure is p_1 and the velocity v_1. At all points in the narrow part, the pressure is p_2 and the velocity v_2. Since the left end of the system advances a distance l_1 parallel to an external force p_1A_1, it follows that the

$$\text{Work done } \textit{on} \text{ the system} = p_1A_1l_1.$$

The right end advances a distance l_2 while an external force p_2A_2 in the opposite direction is acting. Therefore

$$\text{Work done } \textit{by} \text{ the system} = p_2A_2l_2.$$

To move the system from position (a) to position (b), a net amount of work must be done by some agent (a pump, in this case) equal to

$$\text{Net work done on the system} = p_1A_1l_1 - p_2A_2l_2.$$

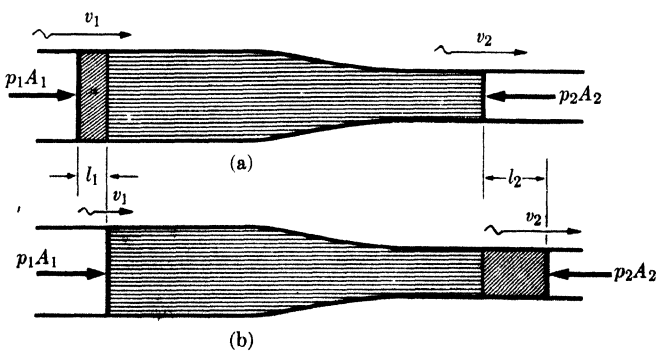


FIGURE 14–2

But $A_1 l_1$ and $A_2 l_2$ are the volumes of the two cross-shaded regions, which must be equal, since the fluid is incompressible. If m is the mass of either cross-shaded region, and ρ is the density of the fluid, then

$$A_1 l_1 = A_2 l_2 = \frac{m}{\rho},$$

and, finally,

$$\text{Net work} = (p_1 - p_2)\frac{m}{\rho}.$$

Since the kinetic energy of the horizontally shaded portion undergoes no change whatever in the transition from (a) to (b), it follows that the total change of kinetic energy of the system is the change in the cross-shaded portions only, or

$$\text{Net change of kinetic energy} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2.$$

In Fig. 14–2, where the pipe is horizontal, there is no change in gravitational potential energy. In general, the cross-shaded portion in (b) will be at a different elevation from that of the cross-shaded portion in (a), and the

$$\text{Net change of gravitational potential energy} = mgy_2 - mgy_1,$$

where y_2 and y_1 are the respective elevations of the cross-shaded portions above some arbitrary reference level.

Strictly speaking, there is always some resistance to the flow of a fluid through a pipe. If the pipe is smooth, large in diameter, and short in length, and if the fluid has a small viscosity and flows slowly, the frictional resistance may be small enough to neglect. For the present let us assume these conditions to be satisfied. We may then equate the net work done on the system to the sum of the increases in its kinetic energy and its gravitational potential energy. This gives

$$(p_1 - p_2)\frac{m}{\rho} = (\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2) + (mgy_2 - mgy_1). \qquad (14\text{–}1)$$

Dividing by mg and rearranging terms, we obtain

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + y_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + y_2, \qquad (14\text{–}2)$$

and since the subscripts 1 and 2 refer to *any* two points along the pipeline, we may write

$$\boxed{\frac{p}{\rho g} + \frac{v^2}{2g} + y = \text{constant.}} \qquad (14\text{–}3)$$

Either Eq. (14–2) or Eq. (14-3) may be considered the special form of *Bernoulli's equation* applicable to streamline flow without resistance.

Note carefully: p is the *absolute* (not gauge) pressure and must be expressed in pounds per square foot, newtons per square meter, or dynes per square centimeter. The density ρ must be expressed in slugs per cubic foot, kilograms per cubic meter, or grams per cubic centimeter. When this is done, every term in Bernoulli's equation has the dimension of length and is called a *head.* The term $p/\rho g$ is called the *pressure head,* $v^2/2g$ the *velocity head,* and y the *elevation head.*

14–3 The equation of continuity. Figure 14–3 represents a pipe in which a fluid is flowing from left to right. Let A_1 be the cross-sectional area at point 1, v_1 the velocity. In time t, those particles of the fluid originally at point 1 will advance a distance $v_1 t$ and a volume of fluid equal to $A_1 v_1 t$ will cross the area A_1. The volume of fluid crossing per unit time is therefore equal to $A_1 v_1$. Similarly, the volume of fluid crossing the area A_2 per unit time is $A_2 v_2$. If the fluid is incompressible, the rates of flow across the two sections must be equal and

$$A_1 v_1 = A_2 v_2$$

or

$$\boxed{Av = \text{constant},} \qquad (14\text{–}4)$$

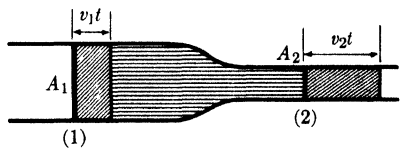


FIG. 14–3. Discharge rate of a pipe.

where A and v are the area and velocity at any point. This is the *equation of continuity* for the steady flow of an incompressible fluid. A consequence of this relation is that the velocity increases if the cross section decreases and vice versa.

14–4 Applications of Bernoulli's equation. (1) The equations of hydrostatics are special cases of Bernoulli's equation, when the velocity is everywhere zero. For example, the variation of pressure with depth in an incompressible liquid may be found by applying Bernoulli's equation to points 1 and 2 in Fig. 14–4. We have

$$p_2 = p_a \text{ (atmospheric)},$$

$$v_1 = v_2 = 0.$$

Let elevations be measured, as usual,

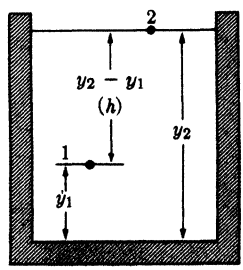


FIGURE 14–4

from the bottom. Then

$$\frac{p_1}{\rho g} + y_1 = \frac{p_a}{\rho g} + y_2,$$

or

$$p_1 = p_a + \rho g(y_2 - y_1) = p_a + \rho g h, \tag{14–5}$$

which is the same as Eq. (12–4).

(2) *Torricelli's theorem.* Figure 14–5 represents a liquid flowing from an orifice in a tank at a depth h below the surface of the liquid in the tank. Take point 1 at the orifice and point 2 at the surface. The pressure at each point is the atmospheric pressure p_a, since both are open to the atmosphere. Take the reference level at the bottom of the tank. If the orifice is small, the level of liquid in the tank will fall only slowly. Hence v_2 is small and we shall assume it zero. Then

$$\frac{p_a}{\rho g} + \frac{v_1^2}{2g} + y_1 = \frac{p_a}{\rho g} + 0 + y_2,$$

or

$$v_1^2 = 2g(y_2 - y_1) = 2gh. \tag{14–6}$$

This is Torricelli's theorem. Note that the velocity of discharge is the same as that which would be acquired by a body falling freely from rest through a height h.

If A is the area of the opening, the volume of fluid flowing out of the tank per unit time is

$$Av = A\sqrt{2gh}. \tag{14–7}$$

Because of the converging of the streamlines as they approach the orifice, the cross section of the stream continues to diminish for a short distance outside the tank. It is the area of smallest cross section, known as the *vena contracta*, which should be used in Eq. (14–7). For a sharp-edged circular opening, the area of the *vena contracta* is about 65% as great as the area of the orifice.

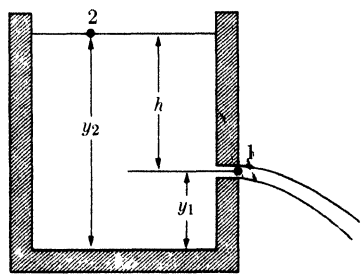


FIG. 14–5. Velocity of efflux = $\sqrt{2gh}$.

(6) *Lift on an aircraft wing.* Figure 14–9 is a photograph of streamline flow around a section in the shape of an aircraft wing or an airfoil, at three different angles of attack. The apparatus consists of two parallel glass plates spaced about 1 mm apart. The wing section, whose thickness equals the separation of the plates, is inserted between them and alternate streams of clear water and ink flow by gravity between the plates and past the section. The photographs have been turned through 90° to give the effect of horizontal air flow past an aircraft wing. Because the fluid is water flowing relatively slowly, the nature of the flow pattern is not identical with that of air moving at high speed past an actual wing.

Consider the first photograph, which corresponds to a plane in level flight. It will be seen that there is relatively little disturbance of the flow below the wing, but because of the shape of the airfoil there is a marked crowding together of the streamlines above it, much as if they were being forced through the throat of a Venturi. Hence the region above the wing is one of increased velocity and reduced pressure, while below the wing the pressure is nearly atmospheric. It is this pressure differential between upper and lower wing surfaces which gives rise to the lift on the wing. The wing is not simply forced up by air blowing against its lower surface.

There is a mistaken impression that the flow around an aircraft wing results in an upward "pull" on the upper surface of the wing. Of course this cannot happen. The air presses against all portions of the wing surface, but the reduction below atmospheric pressure, at the upper surface, usually exceeds the increase above atmospheric pressure at the lower surface.

The second and third photographs show how, as the angle of attack is increased, the streamlines above the wing have to change direction sharply to follow the contour of the wing surface and join smoothly with the streamline flow below the wing. While the slowly moving water in

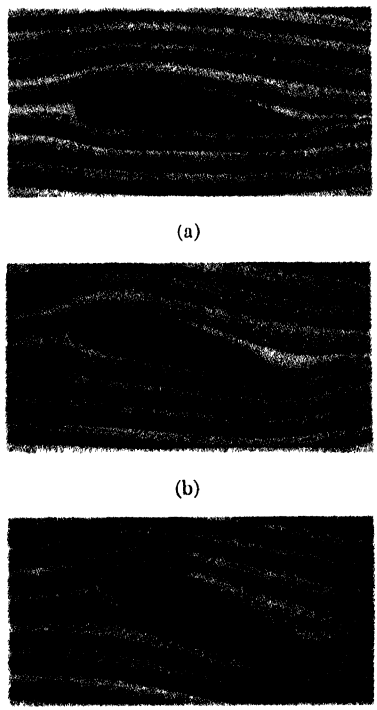


FIG. 14–9. Lines of flow around an airfoil.

Fig. 14–9 does retain its streamline form even at the large angle of attack in the third photograph, it is much more difficult for the air moving rapidly past an aircraft wing to do so. As a consequence, if the angle of attack is too great, the streamline flow in the region above and behind the wing breaks down and a complicated system of whirls and eddies known as *turbulence* is set up. Bernoulli's equation no longer applies, the pressure above the wing rises, and the lift on the wing decreases and the plane stalls.

14–5 Viscosity. Viscosity may be thought of as the internal friction of a fluid. Because of viscosity, a force must be exerted to cause one layer of a fluid to slide past another, or to cause one surface to slide past another if there is a layer of fluid between the surfaces. Both liquids and gases exhibit viscosity, although liquids are much more viscous than gases. In developing the fundamental equations of viscous flow, it will be seen that the problem is very similar to that of the shearing stress and strain in a solid.

Figure 14–10 illustrates one type of apparatus for measuring the viscosity of a liquid. A cylinder is pivoted on nearly frictionless bearings so as to rotate concentrically within a cylindrical vessel. The liquid whose viscosity is to be measured is poured into the annular space between the cylinders. A torque can be applied to the inner cylinder by the weight-pulley system. When the weight is released, the inner cylinder accelerates momentarily but very quickly comes up to a constant angular velocity and continues to

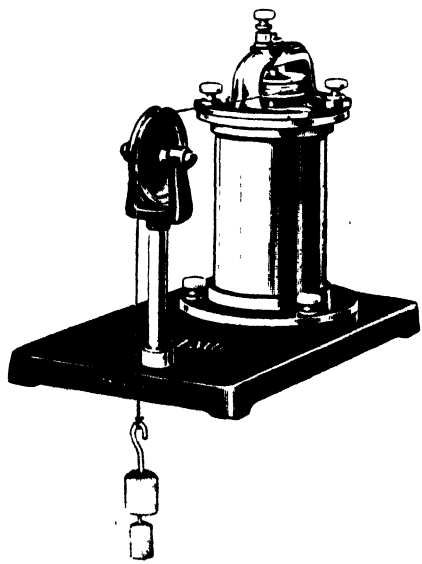


FIG. 14–10. One type of viscosimeter. (Courtesy of Central Scientific Co.)

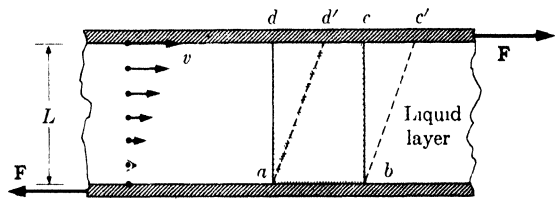


FIG. 14–11. Laminar flow of a viscous liquid.

rotate at that velocity as long as the torque acts. It is obvious that this velocity will be smaller with a liquid such as glycerin in the annular space than it will be if the liquid is water or kerosene. From a knowledge of the torque, the dimensions of the apparatus, and the angular velocity, the viscosity of the liquid may be computed.

To reduce the problem to its essential terms, imagine that the cylinders are of nearly the same size so that the liquid layer between them is thin. A short arc of this layer will then be approximately a straight line. Figure 14–11 shows a portion of the liquid layer between the moving inner wall and the stationary outer wall. The liquid in contact with the moving surface is found to have the same velocity as that surface; the liquid adjacent to the stationary inner wall is at rest. The velocities of intermediate layers of the liquid increase uniformly from one wall to the other as shown by the arrows.

Flow of this type is called *laminar*. (A lamina is a thin sheet.) The layers of liquid slide over one another much as do the leaves of a book when it is placed flat on a table and a horizontal force applied to the top cover. As a consequence of this motion, a portion of the liquid which at some instant has the shape *abcd*, will a moment later take the shape *abc′d′*, and will become more and more distorted as the motion continues. In other words, the liquid is in a state of continually increasing shearing strain.

In order to maintain the motion, it is necessary that a force be continually exerted to the right on the upper, moving plate, and hence indirectly on the upper liquid surface. This force tends to drag the liquid and the lower plate as well to the right. Therefore an equal force must be exerted toward the left on the lower plate to hold it stationary. These forces are lettered **F** in Fig. 14–11. If A is the area of the liquid over which these forces are applied, the ratio F/A is the shearing stress exerted on the liquid.

When a shearing stress is applied to a solid, the effect of the stress is to produce a certain displacement of the solid such as dd'. The shearing strain is defined as the ratio of this displacement to the transverse dimension L, and within the elastic limit the shearing stress is proportional to

TABLE 14-1

TYPICAL VALUES OF VISCOSITY

t, °C	Viscosity of castor oil, poise	Viscosity of water, centipoise	Viscosity of air, μpoise
0	53	1.792	171
20	9.86	1.005	181
40	2.31	0 656	190
60	0.80	0.469	200
80	0.30	0.357	209
100	0.17	0.284	218

the shearing strain. With a fluid, on the other hand, the shearing strain increases without limit so long as the stress is applied, and the stress is found by experiment to be proportional, not to the shearing strain, but to its *rate of change*. The strain in Fig. 14-11 at the instant when the volume of fluid has the shape $abc'd'$ is dd'/ad, or dd'/L. Since L is constant, the rate of change of strain equals $1/L$ times the rate of change of dd'. But the rate of change of dd' is simply the velocity of point d', or the velocity v of the moving wall. Since shearing stress is proportional to rate of change of shearing strain,

$$\frac{F}{A} \propto \frac{v}{L}, \qquad \text{or} \qquad \frac{F}{A} = \eta \frac{v}{L},$$

or

$$\boxed{F = \eta \frac{Av}{L}.} \tag{14-8}$$

The proportionality constant, represented by the Greek letter η (eta), is called the *coefficient of viscosity*, or simply the *viscosity*. It is small for liquids which flow readily, like kerosene, and larger for liquids like molasses or glycerin.

From Eq. (14-8), the unit of viscosity is that of force times distance divided by area times velocity or, in the cgs system, 1 dyne·sec/cm^2. A viscosity of 1 dyne·sec/cm^2 is called a *poise*. Small viscosities are usually expressed in centipoises (1 cp $= 10^{-2}$ poise) or micropoises (1 μp $= 10^{-6}$ poise). Some typical values of viscosity are given in Table 14-1.

The coefficient of viscosity is markedly dependent on temperature, increasing for gases and decreasing for liquids as the temperature is increased.

A common technical method of measuring viscosity makes use of a small container in the bottom of which is an orifice of specified dimensions. A specified volume of liquid is poured into the container and the time required for the liquid to run out through the orifice is measured. The viscosity is then computed by an empirical formula.

Viscosities of lubricating oils are commonly expressed on an arbitrary scale established by the Society of Automotive Engineers. An oil whose SAE number is 10 has a viscosity at 130°F between about 160 and 220 centipoise; the viscosity of SAE 20 is between 230 and 300 centipoise, and that of SAE 30 is between 360 and 430 centipoise.

14–6 Laminar and turbulent flow. In the derivation of Bernoulli's equation and of the equation of continuity, the effect of viscosity was neglected. As a result, the velocities of all particles of fluid at one section of a pipe were equal, and the fluid advanced as a unit along the pipe. In Fig. 14–12(a) the velocity vectors of a number of fluid particles, momentarily at any section of a pipe, are shown when the fluid has no viscosity. The surface determined by the heads of these vectors is a plane, and the fluid flow is said to be characterized by a *plane velocity profile.*

When the fluid is viscous and the velocity is not too great, the flow will be *laminar* and the velocity profile has the shape shown in Fig. 14–12(b). The velocity is a maximum on the axis of the pipe and decreases to zero at the wall. There is therefore a stagnant film of fluid, called the *boundary layer*, at the wall. The laminar motion of the fluid is like that of a number of telescoping tubes sliding relative to one another, with the central tube advancing most rapidly and the outer tube remaining at rest.

When the velocity exceeds a certain critical value, the nature of the flow becomes much more complicated. Random, irregular, local circular currents, called *vortices*, develop throughout the fluid, with a large increase in the resistance to flow. At each point of a cross section, however, the fluid has a forward component of velocity, and the velocity profile of these components has the shape shown in Fig. 14–12(c). Flow of this sort is called *turbulent.*

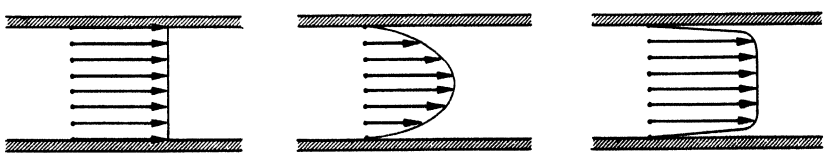


FIG. 14–12. (a) Plane velocity profile. (b) Laminar flow. (c) Turbulent flow.

14–7 Reynolds number. Experiment indicates that there is a combination of four factors which determines whether the flow of viscous fluid through a pipe is laminar or turbulent. This combination is known as the *Reynolds number*, N_R, and is defined as

$$\boxed{N_R = \frac{\rho v D}{\eta},} \tag{14–9}$$

where ρ is the density of the fluid, v the average velocity, η the viscosity, and D the diameter of the pipe. (The actual velocity is not the same over the entire cross section of the pipe, and the average velocity is defined as the uniform velocity which would result in the same discharge rate.)

The Reynolds number is a pure number and therefore its numerical value is the same in any consistent set of units. For example, for water at 20°C flowing in a pipe of diameter 1 cm with an average velocity of 10 cm/sec, the Reynolds number is

$$N_R = \frac{\rho v D}{\eta} = \frac{1 \text{ gm/cm}^3 \times 10 \text{ cm/sec} \times 1 \text{ cm}}{0.01 \text{ dyne·sec/cm}^2},$$

and since 1 dyne = 1 gm·cm/sec^2,

$$N_R = 1000 \frac{\text{gm}}{\text{cm}^3} \times \frac{\text{cm}}{\text{sec}} \times \text{cm} \times \frac{\text{sec}^2\text{·cm}^2}{\text{gm·cm·sec}} = 1000.$$

Had the four quantities been expressed originally in the engineering system of units, the same value of 1000 would have been obtained.

All experiments show that when the Reynolds number lies between 0 and 2000, the flow of viscous fluid is laminar, whereas above about 3000 the flow is turbulent. Between 2000 and 3000 there is a transition region in which the flow is unstable and may change from one type to the other. Thus for water at 20°C flowing in a pipe 1 cm in diameter, the flow is laminar when

$$\frac{\rho v D}{\eta} \lesssim 2000,$$

or when

$$v \lesssim \frac{2000 \times 0.01}{1 \times 1} \frac{\text{cm}}{\text{sec}} \lesssim 20 \frac{\text{cm}}{\text{sec}}.$$

Above about 30 cm/sec the flow is turbulent. If air at the same temperature were flowing at 30 cm/sec in the same pipe, the Reynolds number would be

$$N_R = \frac{0.0013 \times 30 \times 1}{181 \times 10^{-6}} = 216.$$

Since this is much less than 3000, the flow would be laminar and would not become turbulent unless the velocity were as great as 420 cm/sec.

14–8 Flow of a viscous fluid through a pipe. Let us write Bernoulli's equation in the form

$$\frac{p_1 - p_2}{\rho g} = (y_2 - y_1) + \frac{1}{2g}(v_2^2 - v_1^2).$$

This relation was obtained by setting the work done on a moving fluid equal to the change in kinetic energy plus the change in gravitational potential energy. When the effect of viscosity cannot be ignored, some work must also be done against friction, and the pressure difference needed to maintain the flow is greater than that given by the preceding equation. For the special case of a fluid flowing in a pipe of uniform cross section, of length L and diameter D, the velocities v_2 and v_1 are equal. In the absence of friction, the pressure difference between two points would arise only from the difference in elevations of the points. The effect of friction, however, is to introduce an additional pressure head, or *friction head*, given by

$$\frac{\Delta p}{\rho g} = f \frac{L}{D} \frac{v^2}{2g},$$

where v is the average velocity and f, called the *friction factor*, is a function of the Reynolds number N_R. The *total* pressure head between points

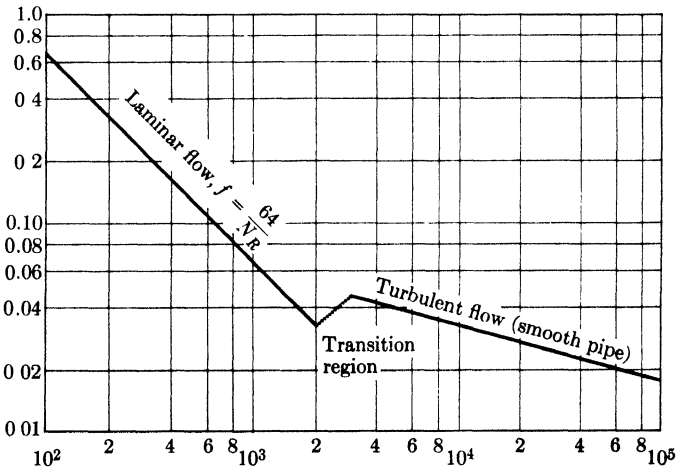


FIG. 14–13. The friction factor f plotted as a function of the Reynolds number

and 2 is thus

$$\frac{p_1 - p_2}{\rho g} = (y_2 - y_1) + f\frac{L}{D}\frac{v^2}{2g}. \qquad (14\text{–}10)$$

When the flow is laminar ($N_R \lesssim 2000$), the friction factor is given by the simple equation

$$f = \frac{64}{N_R}. \qquad \text{(Laminar flow)}$$

When the flow is turbulent ($N_R > 3000$), the friction factor varies as a fractional power of the Reynolds number. Both situations are depicted in the logarithmic graph of Fig. 14–13. (Strictly speaking, the friction factor for turbulent flow is a different function of the Reynolds number for pipes of different degrees of roughness. The curve in Fig. 14–13 applies only to smooth pipes.)

14–9 Stokes' law. When a viscous fluid flows past a sphere with laminar flow, or when a sphere moves through a viscous fluid at rest, a resisting force is exerted on the sphere. (A force, of course, is experienced by a body of any shape, but only for a sphere is the expression for the force readily calculable.) Analysis which is beyond the scope of this book shows that the resisting force R is given by

$$R = 6\pi\eta r v, \qquad (14\text{–}11)$$

where η is the viscosity of the fluid, r the radius of the sphere, and v the relative velocity of sphere and fluid. This relation was first deduced by Sir George Stokes in 1845 and is called *Stokes' law*. We shall consider it briefly in relation to a sphere falling through a viscous fluid. The forces on the sphere are its weight mg, the buoyant force B of the fluid, and the resisting force R (see Fig. 14–14). If ρ is the density of the sphere and ρ_0 the density of the fluid,

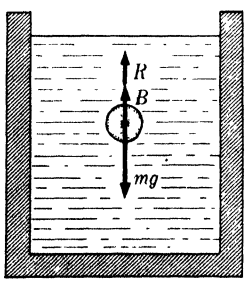


FIGURE 14–14

$$mg = \tfrac{4}{3}\pi r^3\rho g, \quad B = \tfrac{4}{3}\pi r^3\rho_0 g.$$

Since the net downward force on the sphere is equal to the product of its mass times its acceleration,

$$mg - B - R = ma,$$

$$a = g - \frac{B + R}{m}.$$

If the sphere is released from rest ($v = 0$), the viscous force R at the start is zero. The initial acceleration a_0 is therefore

$$a_0 = g - \frac{B}{m} = g\,\frac{\rho - \rho_0}{\rho}.$$

As a result of this acceleration, the sphere acquires a downward velocity and therefore experiences a retarding force given by Stokes' law. As the velocity increases, the retarding force also increases in direct proportion, and eventually a velocity is reached such that the downward force and the retarding force are equal. The sphere then ceases to accelerate and moves with a constant velocity called its *terminal velocity*. This velocity can be found by setting the downward force mg equal to the upward force $B + R$. Thus,

$$\tfrac{4}{3}\pi r^3 \rho g = \tfrac{4}{3}\pi r^3 \rho_0 g + 6\pi\eta r v,$$

or

$$v = \frac{2}{9}\,\frac{r^2 g}{\eta}\,(\rho - \rho_0). \tag{14–12}$$

The relation above holds provided the velocity is not so great that turbulence sets in. When this occurs, the retarding force is much greater than that given by Stokes' law.

EXAMPLE. Find the terminal velocity of a steel ball 2 mm in radius, falling in a tank of glycerin.

$$\rho_{\text{steel}} = \text{(about) } 8\ \frac{\text{gm}}{\text{cm}^3},$$

$$\rho_{\text{glycerin}} = \text{(about) } 1.3\ \frac{\text{gm}}{\text{cm}^3},$$

$$\eta_{\text{glycerin}} = \text{(about) } 8.3 \text{ poise.}$$

$$v = \frac{2}{9}\,\frac{(0.2)^2 \times 980}{8.3}\,(8 - 1.3) = 7\ \frac{\text{cm}}{\text{sec}}.$$

This velocity is attained in a very short distance from the start of the motion. The experiment above is used as one method of measuring viscosity.

PROBLEMS

(In Problems 1–24, neglect friction.)

14–1. A circular hole 1 inch in diameter is cut in the side of a large standpipe, 20 ft below the water level in the standpipe. Find (a) the velocity of efflux, and (b) the volume discharged per unit time. Neglect the contraction of the streamlines after emerging from the hole.

14–2. Water stands at a depth H in a large open tank whose side walls are vertical (Fig. 14–15). A hole is made in one of the walls at a depth h below the water surface. (a) At what distance R from the foot of the wall does the emerging stream of water strike the floor? (b) At what height above the bottom of the tank could a second hole be cut so that the stream emerging from it would have the same range?

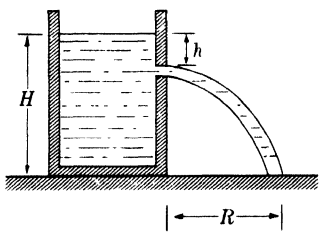


FIGURE 14–15

14–3. Water in an enclosed tank is subjected to a gauge pressure of 4 lb/in^2 applied by compressed air introduced into the top of the tank. There is a small hole in the side of the tank 16 ft below the level of the water. Calculate the velocity with which water escapes from this hole.

14–4. What gauge pressure is required in the city mains in order that a stream from a fire hose connected to the mains may reach a vertical height of 60 ft?

14–5. A tank of large area is filled with water to a depth of one foot. A hole of 1 in^2 cross section in the bottom allows water to drain out in a continuous stream. (a) What is the rate at which water flows out of the tank, in ft^3/sec? (b) At what distance below the bottom of the tank is the cross-sectional area of the stream equal to one-half the area of the hole?

14–6. A sealed tank containing sea water to a height of 5 ft also contains air above the water at a gauge pressure of 580 lb/ft^2. Water flows out from a hole at the bottom. The cross-sectional area of the hole is 1.6 in^2. (a) Calculate the efflux velocity of the water. (b) Calculate the reaction force on the tank exerted by the water in the emergent stream. (Force equals rate of change of momentum.)

14–7. A pipeline 6 inches in diameter, flowing full of water, has a constriction of diameter 3 in. If the velocity in the 6-inch portion is 4 ft/sec, find (a) the velocity in the constriction, and (b) the discharge rate in ft^3/sec.

14–8. A horizontal pipe of 6 in^2 cross section tapers to a cross section of 2 in^2. If sea water of density 2 slugs/ft^3 is flowing with a velocity of 180 ft/min in the large pipe where a pressure gauge reads 10.5 lb/in^2, what is the gauge pressure in the adjoining part of the small pipe? The barometer reads 30 inches of mercury.

14–9. Sea water (weighing 64 lb/ft^3) stands to a height of 4 ft in a tank. The tank contains compressed air at a gauge pressure of 1 lb/in^2. The horizontal outlet pipe has cross-sectional areas of 2.88 in^2 and 1.44 in^2 at the

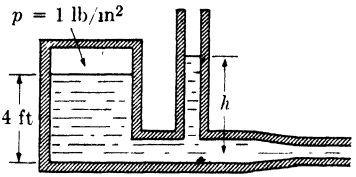


FIGURE 14–16

larger and smaller sections (Fig. 14–16). (a) What is the discharge rate from the outlet? (b) To what height h does water stand in the open end pipe? (c) If now the tank is punctured at the top of the gauge pressure drops to zero, what will be the height h?

14–10. The section of pipe shown in Fig. 14–17 has a cross section of 0.04 ft^2 at the wider portions and 0.01 ft^2 at the constriction. One cubic foot of water is discharged from the pipe in 5 sec. (a) Find the velocities at the wide and the narrow portions. (b) Find the pressure difference between these portions. (c) Find the difference in height between the mercury columns in the U-tube.

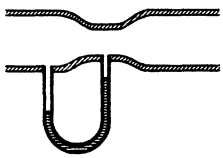


FIGURE 14–17

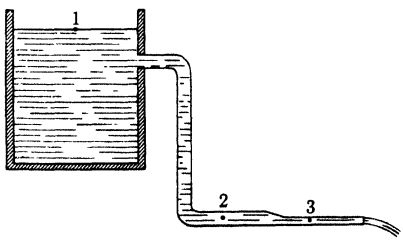


FIGURE 14–18

14–11. Water flows steadily from a reservoir shown in Fig. 14–18. The elevation of point 1 is 40 ft; of points 2 and 3 it is 4 ft. The cross section at point 2 is 0.5 ft^2 and at point 3 it is 0.25 ft^2. The area of the reservoir is very large compared with the cross sections of the pipe. (a) Compute the gauge pressure at point 2. (b) Compute the discharge rate in ft^3/sec.

14–12. Sea water of density 2 slugs/ft^3 flows steadily in a pipeline of constant cross section leading out of an elevated tank. At a point 4.5 ft below the water level in the tank the gauge pressure in the flowing stream is 1 lb/in^2. (a) What is the velocity of the water at this point? (b) If the pipe rises to a point 9 ft above the level of the water in the tank, what are the velocity and the pressure at the latter point?

14–13. Water flows from a reservoir to a turbine 330 ft below. The efficiency of the turbine is 80% and it receives 100 ft^3 of water per minute. Compute the horsepower output of the turbine.

14–14. Sea water weighing 64 lb/ft^3 flows through a horizontal pipe of cross-sectional area 1.44 in^2. At one section the cross-sectional area is 0.72 in^2. The pressure difference between the two sections is 0.048 lb/in^2. How many cubic feet of water will flow out of the pipe in one minute?

14–15. The water level in a tank on the top of a building is 100 ft above the ground. The tank supplies water, through pipes of 0.02 ft^2 cross-sectional area, to the various apartments. Each faucet through which the water emerges has an orifice of 0.01 ft^2 effective area. (a) How long will it take to fill a 1-ft^3 pail in an apartment 75 ft above the ground? (b) What is the gauge pressure in a water pipe (not in

the faucet) on the ground level when the faucet is closed? (c) What is the gauge pressure in a pipe on the ground level when the faucet is open?

14–16. Water in an enclosed tank stands at a level of 16 ft above a short efflux pipe at the base (Fig. 14–19). (a) What pressure (gauge) of compressed air above the tank will result in a velocity of efflux from the pipe of 40 ft/sec? (b) A manometer tube is attached to the efflux pipe close to the tank. To what height will water rise in this tube?

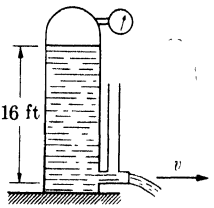


FIGURE 14–19

14–17. At a certain point in a horizontal pipeline the gauge pressure is 6.24 lb/in^2. At another point the gauge pressure is 4.37 lb/in^2. If the areas of the pipe at these two points are 3 in^2 and 1.5 in^2 respectively, compute the number of cubic feet of water which flow across any cross section of the pipe per minute.

14–18. Water flowing in a horizontal pipe discharges at the rate of 0.12 ft^3/sec. At a point in the pipe where the cross section is 0.01 ft^2, the absolute pressure is 18 lb/in^2. What must be the cross section of a constriction in the pipe such that the pressure there is reduced to 15 lb/in^2?

14–19. The pressure difference between the main pipeline and the throat of a Venturi meter is 15 lb/in^2. The areas of the pipe and the constriction are 1 ft^2 and 0.5 ft^2. How many cubic feet per second are flowing through the pipe? The liquid in the pipe is water.

14–20. Assume that air is streaming horizontally past an aircraft wing such that the velocity is 100 ft/sec over the top surface and 80 ft/sec past the bottom surface. If the wing weighs 600 lb and has an area of 40 ft^2, what is the net force on the wing? The density of air is 0.0013 gm/cm^3.

14–21. Modern airplane design calls for a "lift" of about 20 lb/ft^2 of wing area. Assume that air flows past the wing of an aircraft with streamline flow. If the velocity of flow past the lower wing surface is 300 ft/sec, what is the required velocity over the upper surface to give a "lift" of 20 lb/ft^2? The density of air is 0.0013 gm/cm^3.

14–22. At a certain point in a pipeline the velocity is 2 ft/sec and the gauge pressure is 35 lb/in^2. Find the gauge pressure at a second point in the line 50 ft lower than the first, if the cross section at the second point is one-half that at the first. The liquid in the pipe is water.

14–23. The water surface in a large standpipe is at an elevation of 64 ft above the outlet of a pipeline of uniform cross section of 0.10 ft^2. (a) What is the discharge rate, in ft^3/sec? (b) What is the absolute pressure at a point in the line 20 ft above the outlet?

14–24. Two very large open tanks, A and F (Fig. 14–20), both contain the

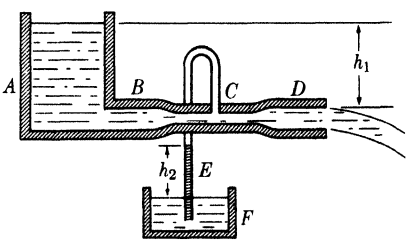


FIGURE 14–20

same liquid. A horizontal pipe BCD having a constriction at C leads out of the bottom of tank A, and a vertical pipe E opens into the constriction at C and dips into the liquid in tank F. Assume streamline flow and no viscosity. If the cross section at C is one-half that at D, and if D is at a distance h_1 below the level of the liquid in A, to what height h_2 will liquid rise in pipe E? Express your answer in terms of h_1. Neglect changes in atmospheric pressure with elevation.

14–25. Water at 20°C flows with a speed of 50 cm/sec through a pipe of diameter 3 mm. (a) What is the Reynolds number? (b) What is the nature of the flow? (c) What is the friction factor? (d) What would be the "friction head" if the pipe were 100 cm long?

14–26. Oil having a viscosity of 300 centipoise and a density of 0.90 gm/cm^3 is to be pumped from one large open tank to another through 1 km of smooth steel pipe 15 cm in diameter. The line discharges into the air at a point 30 m above the level of the oil in the supply tank. (a) What gauge pressure, in atmospheres, must the pump exert in order to maintain a flow of 50 liters/sec? (b) What is the power consumed by the pump?

14–27. Water at 20°C is pumped through a horizontal smooth pipe 15 cm in diameter and discharges into the air. If the pump maintains an absolute pressure of 1.02 atm at a point 300 m from the discharge end of the pipe, (a) what is the velocity of flow? (b) What is the nature of the flow? (c) What is the discharge rate in liters/sec?

14–28. (a) With what terminal velocity will an air bubble 1 mm in diameter rise in a liquid of viscosity 150 cp and density 0.90 gm/cm^3? (b) What is the terminal velocity of the same bubble in water?

14–29. (a) With what velocity is a steel ball 1 mm in radius falling in a tank of glycerin at an instant when its acceleration is one-half that of a freely falling body? (b) What is the terminal velocity of the ball? The densities of steel and of glycerin are 8.5 gm/cm^3 and 1.32 gm/cm^3 respectively.

CHAPTER 15

TEMPERATURE—EXPANSION

15–1 Concept of temperature. To describe the equilibrium states of mechanical systems, as well as to study and predict the motions of rigid bodies and fluids, only three fundamental indefinables were needed: length, mass, and time. Every other physical quantity of importance in mechanics could be expressed in terms of these three indefinables. We come now, however, to a series of phenomena, called *thermal effects* or *heat phenomena*, which involve aspects that are essentially nonmechanical and which require for their description a fourth fundamental indefinable, the *temperature*.

Consider a cake of ice with a volume of about 1 cubic foot at atmospheric pressure. It is in mechanical equilibrium because the vector sum of the forces acting on it is zero. Suppose, in the neighborhood of the ice, there is a small spherical rubber balloon inflated with air. It is also in mechanical equilibrium because the difference between the inside and outside pressure is twice the surface tension divided by the radius. Even though the ice and the air balloon are separately in mechanical equilibrium, it will be found that both systems undergo changes when they are placed together: the volume of the ice decreases (because some of it melts) and both the pressure and volume of the air in the balloon change. After a while these changes cease. The final state of the two systems is called "thermal equilibrium."

The tendency of systems originally in mechanical equilibrium to change their states when placed in contact cannot be explained in terms of length, mass, and time alone. The results of many experiments on systems which change their mechanical states when placed together and finally reach thermal equilibrium lead us to infer that all ordinary objects have a new physical property that determines whether they will be in thermal equilibrium when placed in contact with other objects. This property is called *temperature. The temperatures associated with two objects are said to be equal only when the two objects are in thermal equilibrium.* The temperature of all objects in thermal equilibrium may be represented by a number. The establishment of a temperature scale is merely the adoption of a set of rules for assigning different numbers to different temperatures. Once this is done the necessary and sufficient condition for thermal equilibrium between two systems is that they have the same temperature. Also, when the temperatures are different, we may be sure that the systems are not in thermal equilibrium.

Human beings are endowed with a temperature sense which enables them to tell at least roughly whether two objects will be in thermal equilibrium when placed together. We have learned to describe our sensations when touching other objects by using adjectives such as "cold," "warm," "hot," etc. These words mean roughly the same thing to most people, just as the words "low pitch" and "high pitch" or "red" and "blue" mean about the same thing when used to describe our sensations on hearing sounds or seeing colors. When two objects have been left together for some time and are then touched, their "hotness" or "coldness" feel about the same and the two objects are described as having the same temperature. Temperature is therefore the property of a body that determines our sensation of hotness or coldness when we touch the body.

Our temperature sense is much too unreliable to serve as a means of measurement. An object with at least one easily measurable property that changes appreciably while the object is coming to thermal equilibrium is preferable as a temperature measuring device, or *thermometer*. With the aid of an arbitrarily chosen thermometer it is possible to formulate a set of rules for representing temperatures with numbers, that is, for setting up a *temperature scale*.

15–2 Thermometers. Of all the properties of systems that have been studied with an eye toward possible use in the measurement of temperature, *the pressure of a gas* whose volume is maintained constant has been found to excel in *sensitivity*, *accuracy* of measurement, and *reproducibility*. The constant-volume gas thermometer is illustrated schematically in Fig. 15–1. The materials, construction, and dimensions differ in various laboratories throughout the world and depend on the nature of the gas and the temperature range to be covered.

The gas, usually hydrogen or helium, is contained in bulb C, and the pressure exerted by it can be measured by the open-tube mercury ma-

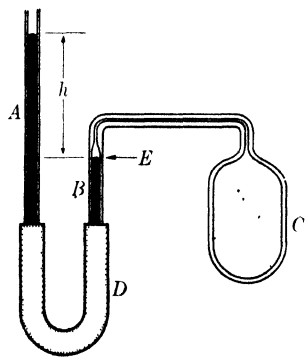


FIG. 15–1. Constant-volume gas thermometer.

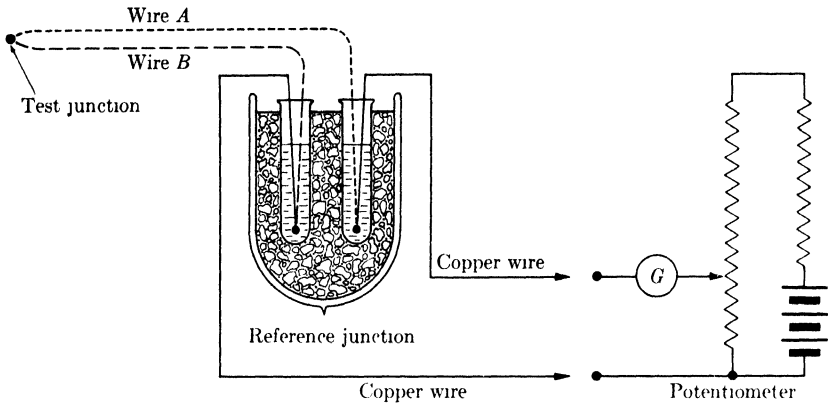


FIG. 15–2. Thermocouple of wires A and B with a reference junction consisting of two junctions with copper, connected to a potentiometer.

nometer. As the temperature of the gas is increased, it expands, forcing the mercury down in tube B and up in tube A. Tubes A and B are connected by the flexible rubber tube D, and by raising A, the mercury level in B may be brought back to the reference mark E. The gas is thus kept at constant volume.

Gas thermometers are absolute instruments and are used mainly in bureaus of standards and in some university research laboratories. They are usually large, bulky, and slow in coming to thermal equilibrium.

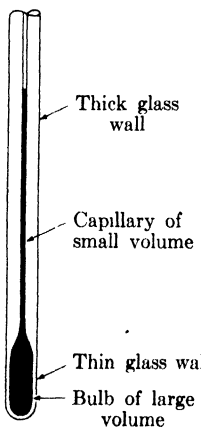


FIG. 15–3. Construction of liquid-in-glass-thermometer.

In most research and engineering laboratories, secondary thermometers that are small and fast must be used. The most commonly used is the *thermocouple*, which consists of a junction of two different metals or alloys. Since the junction is small and has a small mass, it can follow temperature changes rapidly and come to equilibrium quickly. The junction forms part of an electric circuit, as shown in Fig. 15–2. When it is in contact with an object or immersed in a fluid, the temperature measurement is accomplished by measuring a quantity called an emf (ee-em-eff, electromotive force) with an instrument known as a potentiometer. Both the thermocouple and the potentiometer will be studied in later sections of this book.

The best known but least accurate thermometer is a liquid such as mercury or alcohol contained in a bulb which communicates with a very narrow tube or capillary, as shown in Fig. 15–3. The height of the liquid in the capillary is an indication of the temperature. The main fault of the liquid-in-glass thermometer is that its reading changes over the years because of aging of the glass.

In each of the three thermometers mentioned above there is one physical quantity called a *thermometric property* whose change is used to indicate a change of temperature. In the case of a gas at constant volume it is the pressure p; with the thermocouple it is the emf $\mathcal{E}$; and with the liquid-in-glass thermometer it is the height h of the liquid in the capillary.

15–3 The measurement of temperature. Let X stand for any thermometric property such as p, $\mathcal{E}$, or h. Let us define the temperature T common to the thermometer and to all systems in thermal equilibrium with it as a linear function of X. That is,

$$T = aX,$$

where a is an arbitrary constant. It follows that two temperatures on this "linear X scale" are to each other as the ratio of the corresponding X's, or

$$\frac{T_1}{T_2} = \frac{X_1}{X_2}. \tag{15–1}$$

Note that the linear function of X was an *arbitrary choice.* A different algebraic function, or even a logarithmic function, could have been chosen, and the resulting temperature scale would be equally legitimate.

To determine the temperature T of any system, the thermometer must first be brought to thermal equilibrium with *an arbitrarily chosen standard system in an easily reproducible state.* The temperature of the standard system in the chosen state is called a *fixed point.* Before 1954 there were two fixed points: (1) the temperature of pure ice in equilibrium with air-saturated water at one atmosphere pressure (the ice point), and (2) the equilibrium temperature of pure water and pure steam at one atmosphere pressure (the steam point). The temperature interval between these two fixed points was chosen to be 100 degrees.

The use of two fixed points was found unsatisfactory, partly because of the difficulty of achieving equilibrium between pure ice and air-saturated water. When ice melts, it surrounds itself with pure water and thus prevents intimate contact of ice and air-saturated water. Also, the steam point is extremely sensitive to a change in pressure. The temperature scale in use since 1954 is based on one fixed point only. This is the temperature at which ice, liquid water, and water vapor coexist in equilibrium, a state known as the *triple point* of water. To obtain the best agreement between

the old and new scales, the temperature of the triple point is assigned the value 273.16 degrees kelvin, abbreviated 273.16°K. (The reason for the use of Kelvin's name will be made clear later.) Thus, designating the triple point of water by the subscript 3, we have from Eq. (15–1),

$$\frac{T}{T_3} = \frac{X}{X_3},$$

with

$$T_3 = 273.16°\text{K}.$$

Hence,

$$T = 273.16° \frac{X}{X_3}. \tag{15–2}$$

The temperature of the triple point of water is the *standard fixed point* of thermometry. To achieve the triple point, water of the highest purity is distilled into a vessel shown schematically in Fig. 15–4. When all air has been removed, the vessel is sealed off. With the aid of a freezing mixture in the inner well, a layer of ice is formed around the well, as shown in Fig. 15–4(a). When the freezing mixture is replaced by a thermometer bulb, a thin layer of ice is melted nearby, as shown in Fig. 15–4(b). So long as the solid, liquid, and vapor phases coexist in equilibrium, the

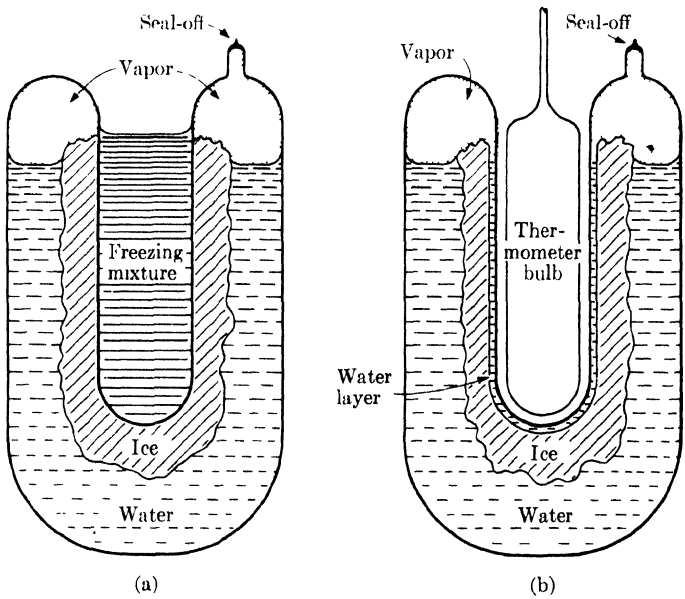


FIG. 15–4. Triple-point cell, (a) with freezing mixture in the central well to freeze a layer of ice; (b) with a thermometer in the well, which melts a thin layer of ice nearby.

system is at the triple point. The actual shape of the apparatus used by the U.S. National Bureau of Standards is shown in Fig. 15–5.

To determine an unknown temperature T using a thermometer with a thermometric property X, two measurements of X have to be made: one at the unknown temperature and the other at the triple point of water. The temperature is then found from Eq. (15–2).

If a series of tests is carried out in which the temperature of a given system is measured with a number of different thermometers, each making use of a different thermometric property, the thermometers will be found to give slightly different results. Even different varieties of the same kind of thermometer yield different results. The smallest variation, however, is found among different gas thermometers. For this reason, and also because of its accuracy and reproducibility, the gas thermometer has been chosen as a world standard. The principles of its operation are as follows. Suppose that an amount of gas is introduced into the bulb of a constant-volume gas thermometer so that the pressure p_3, when the bulb is surrounded by water at its triple point, is equal to 1000 mm Hg. Keeping the volume constant, measure the pressure p when the bulb is at some unknown temperature, such as that of steam condensing at atmospheric pressure, and calculate the quantity

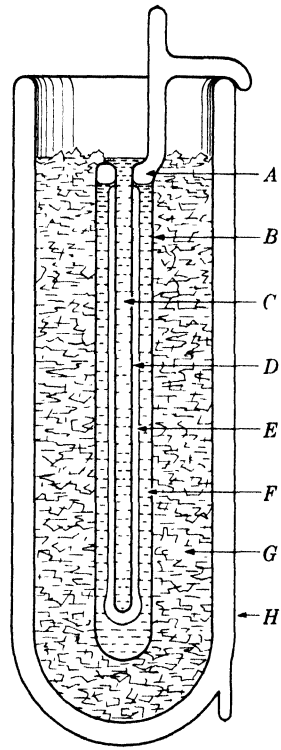


FIG. 15–5. Diagram of the NBS triple-point cell (B, D) in use in an ice bath (G) within a Dewar flask (H). A, water vapor; C, thermometer well; E, ice mantle; F, liquid water.

$$273.16^\circ \frac{p}{1000}.$$

Now remove some of the gas so that the pressure p_3 has a smaller value, say 500 mm Hg. Measure the new pressure p when the bulb is at the temperature of condensing steam, and calculate the quantity

$$273.16^\circ \frac{p}{500}.$$

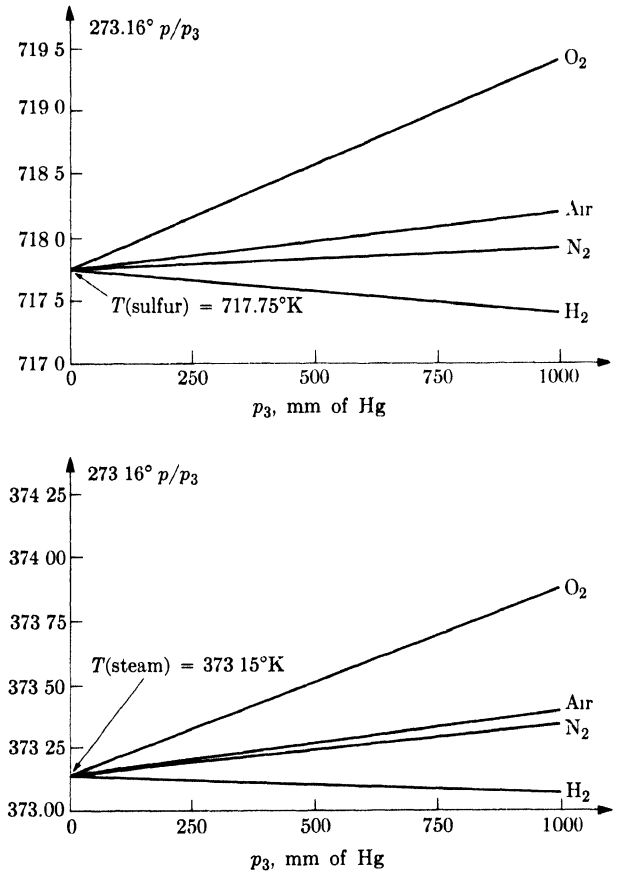


FIG. 15–6. Readings of a constant-volume gas thermometer for the temperature of condensing steam and for that of condensing sulfur, when different gases are used at various values of p_3.

The value of this quantity will be found to be slighly different from that obtained with $p_3 = 1000$ mm Hg.

Continue reducing the amount of gas in the bulb and calculate the quantity 273.16° p/p_3 at each value of p_3. Construct a graph of this quantity against the pressure p_3 and extrapolate the resulting curve to the axis where $p_3 = 0$.

The results of a series of measurements of this sort are plotted in Fig. 15–6, for the temperatures of condensing steam and of condensing sulfur. The graphs show that although the readings of a constant-volume gas

thermometer depend both on the kind of gas used and on the pressure p_3, *all gases indicate the same temperature as* p_3 *is lowered and approaches zero.* We therefore define the gas temperature T by the equation

$$T = [273.16°\text{K}] \lim_{p_3 \to 0} \left(\frac{p}{p_3}\right)_{\text{const vol.}} \tag{15–3}$$

It should be emphasized that this equation, or the set of rules for measuring temperature which it embodies, does not rest on the assumption that "the pressure of a gas at constant volume is directly proportional to the temperature." Such a statement, before a temperature scale is set up, is meaningless.

Although the temperature scale of Eq. (15–3) is independent of the properties of any one particular gas, it still depends on the properties of gases in general. To measure a low temperature, a gas must be used at that low temperature. The lowest temperature that can be measured with a gas thermometer is about 1°K, provided low-pressure helium is used. *The temperature* $T = 0$ *remains as yet undefined.*

The kelvin temperature scale, which is independent of the properties of any particular substance, will be described in Chapter 20. It can be shown that in the temperature region in which a gas thermometer can be used, the gas scale and the kelvin scale are identical. In anticipation of this result, we write °K after a gas temperature. It will also be shown in Chapter 20 how the absolute zero of temperature is defined on the kelvin scale. Until then, the term "absolute zero" will have no meaning. The statement made so often that all molecular activity ceases at the temperature $T = 0$ is entirely erroneous. When it is necessary in statistical mechanics to correlate temperature with molecular activity, it is found that classical statistical mechanics must be modified with the aid of quantum mechanics. When this modification is carried out, the molecules of a substance at absolute zero have a *finite* amount of kinetic energy known as the *zero-point* energy.

15–4 The celsius, rankine, and fahrenheit scales. The celsius temperature scale (formerly called the centigrade scale in the United States and Great Britain) employs a degree of the same magnitude as that of the kelvin scale, but its zero point is shifted so that *the celsius temperature of the triple point of water is* 0.01 *degree celsius,* abbreviated 0.01°C. Thus, if t denotes the celsius temperature,

$$t = T - 273.15°\text{K}. \tag{15–4}$$

The celsius temperature t_S at which steam condenses at 1 atm pressure is

$$t_S = T_S - 273.15°\text{K},$$

and reading T_S from Fig. 15–6,

$$t_S = 373.15° - 273.15°$$

or

$$t_S = 100.00°\text{C}.$$

There are two other scales in common use in engineering and in everyday life in the United States and in Great Britain. The *rankine* temperature T_R (written °R) is proportional to the kelvin temperature according to the relation

$$\boxed{T_R = \tfrac{9}{5}T.} \qquad (15\text{–}5)$$

A degree of the same size is used in the *fahrenheit* scale t_F (written °F), but with the zero point shifted according to the relation

$$\boxed{t_F = T_R - 459.67°\text{R}.} \qquad (15\text{–}6)$$

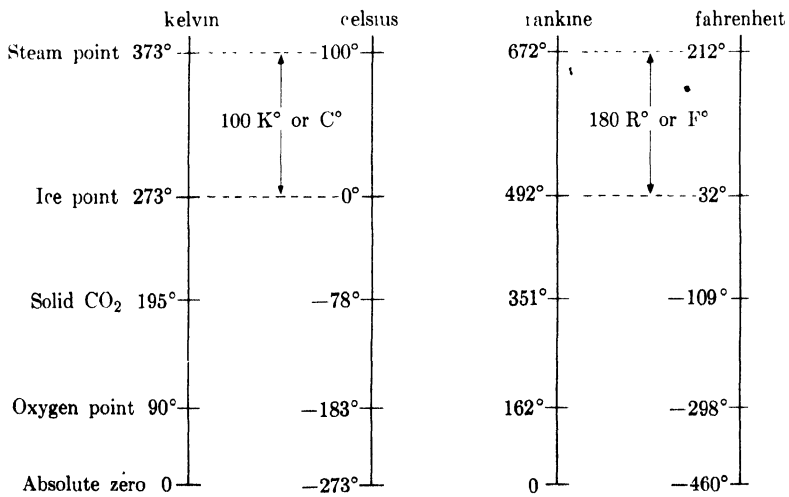


FIG. 15–7. Relation between kelvin ($T = 273.16°\text{K} \times \lim_{p_3 \to 0} (p/p_3)$ for any gas at constant volume), celsius ($t = T - 273.15°\text{K}$), rankine ($T_R = \frac{9}{5}T$), and fahrenheit ($t_F = T_R - 459.67°\text{R}$) temperature scales. Temperatures have been rounded off to the nearest degree.

TABLE 15–1

TEMPERATURES OF FIXED POINTS

Basic fixed points	T, °K	t, °C	T_R, °R	t_F, °F
Standard: Triple point of water	**273.16**	**0.01**	**491.688**	**32.018**
Boiling point of oxygen	90 18	−182 97	162 32	−297 35
{Equil. of ice and air-saturated water} (ice point)	273 15	0 00	491 67	32 00
Boiling point of water (steam point)	373 15	100 00	671 67	212 00
Boiling point of sulfur	717 75	444 60	1291 95	832 28
Melting point of antimony	903 65	630 50	1626 57	1166 90
Melting point of silver	1233 95	960 80	2221 11	1761 44
Melting point of gold	1336 15	1063 00	2405 07	1945 40

Substituting Eqs. (15–4) and (15–5) into Eq. (15–6), we get

$$t_F = \tfrac{9}{5}t + 32°F, \tag{15–7}$$

from which it follows that the fahrenheit temperature of the ice point ($t = 0°C$) is 32°F and of the steam point ($t = 100°C$) is 212°F. The 100 celsius or kelvin degrees between the ice point and the steam point correspond to 180 fahrenheit or rankine degrees, as shown in Fig. 15–7 when the four scales are compared.

The accurate measurement of a boiling point or melting point with the aid of a gas thermometer requires months of painstaking laboratory work and mathematical computation. Fortunately, this has been done for a large number of substances which are obtainable with high purity. Some of these results are shown in Table 15–1. With the aid of these basic fixed points other thermometers may be calibrated. The calibration of the liquid-in-glass thermometer is accomplished by immersing it first in a mixture of ice and water and later in a steam bath and marking the liquid level in both cases. Subdividing the space between these two marks into 100 equal divisions provides a rough celsius scale, and the subdivision into 180 equal divisions, a rough fahrenheit scale.

The calibration of thermocouples and other important types of thermometers will be explained in later portions of the book.

Suppose the temperature of a beaker of water is raised from 20°C to 30°C, through a temperature interval of 10 celsius degrees. It is desirable to distinguish between such a temperature interval and the actual temperature of 10

degrees above the celsius zero. Hence we shall use the phrase "10 degrees celsius," or "10°C," when referring to an *actual temperature*, and "10 celsius degrees," or "10 C°" to mean a temperature *interval*. Thus there is an interval of 10 celsius degrees between 20 degrees celsius and 30 degrees celsius.

15–5 Linear expansion. With a few exceptions, the dimensions of all substances increase as the temperature of the substance is increased. If a given specimen is in the form of a rod or cable, one is usually interested in its change of *length* with changes in temperature. Figure 15–8 represents a rod whose length is L_0 at some reference temperature t_0, and whose length is L at some higher temperature t. The difference $L - L_0 = \Delta L$ is the amount the rod has expanded on heating. It is found experimentally that the increase in length, ΔL, is proportional to the original length L_0, and very nearly proportional to the increase in temperature, $t - t_0$ or Δt. That is,

$$\Delta L \propto L_0 \, \Delta t, \qquad \text{or} \qquad \Delta L = \alpha L_0 \, \Delta t, \tag{15–8}$$

where α is a proportionality constant, different for different materials, and is called the *coefficient of linear expansion.*

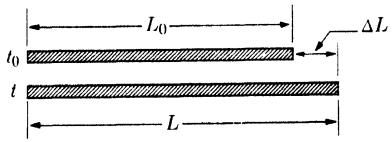


FIG. 15–8. Linear expansion.

Equation (15–8) may be solved for α and written

$$\boxed{\alpha = \frac{\Delta L}{L_0} \frac{1}{\Delta t}.} \tag{15–9}$$

The coefficient of linear expansion of a substance may therefore be described as the fractional change in length per degree rise in temperature. Another useful relation is obtained by replacing ΔL by $L - L_0$ and solving for L.

$$\boxed{L = L_0(1 + \alpha \, \Delta t)} \tag{15–10}$$

Since L_0, L, and ΔL are all expressed in the same unit, the units of α are "reciprocal degrees" (celsius or fahrenheit). Thus the coefficient of

TABLE 15-2

COEFFICIENTS OF LINEAR EXPANSION

Substance	$\alpha\ (C°)^{-1}$
Aluminum	24×10^{-6}
Brass	20×10^{-6}
Copper	14×10^{-6}
Glass	$4\text{-}9 \times 10^{-6}$
Steel	12×10^{-6}
Invar	0.9×10^{-6}
Quartz (fused)	$0\ 4 \times 10^{-6}$
Zinc	26×10^{-6}

linear expansion of copper is written

$$\alpha = 14 \times 10^{-6} \text{ per celsius degree}$$

or

$$\alpha = 14 \times 10^{-6}\ (C°)^{-1}.$$

This means that a copper rod one centimeter long at 0°C, increases in length by 0.000014 cm when heated to 1°C. A rod one foot long at 0°C increases by 0.000014 ft, and so on.

Since the fahrenheit degree is only $\frac{5}{9}$ as large as the celsius degree, coefficients of expansion per fahrenheit degree are $\frac{5}{9}$ as large as their values on the celsius scale.

EXAMPLE. An iron steam pipe is 200 ft long at 0°C. What will be its increase in length when heated to 100°C? $\alpha = 10 \times 10^{-6}$ per celsius degree.

$$L_0 = 200 \text{ ft}, \quad \alpha = 10 \times 10^{-6} \text{ per C°}, \quad t = 100°C, \quad t_0 = 0°C.$$

$$\begin{aligned} \text{Increase in length} = \Delta L &= \alpha L_0 \Delta t \\ &= (10 \times 10^{-6})\ (200)\ (100) \\ &= 0.20 \text{ ft}. \end{aligned}$$

The coefficient of expansion of a substance in the form of a rod is measured by making two fine lines on the rod near its ends, and measuring the displacement of each line with a measuring microscope while the temperature of the rod is changed by a measured amount.

The *bimetallic* element is a device which has come into wide use in recent years, both as a thermometer and as a part of many thermostatic controls.

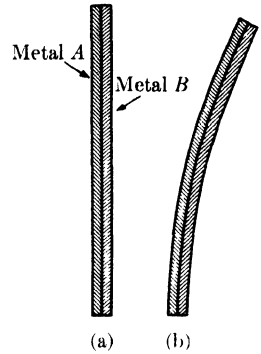


FIG. 15–9 The bimetallic element.

It consists of two thin flat strips of different metals, welded or riveted together as in Fig. 15–9(a). If metal A has a larger coefficient of expansion than metal B, the compound strip, if originally straight, bends into a curve when heated as shown in Fig. 15–9(b). The transverse motion of the end of the strip is very much larger than the increase in length of either metal.

When used as a thermostat, one end of the strip is fixed and the motion of the other end is made to open or close an electrical control circuit. The common oven thermometer consists of a bimetallic strip coiled in a helix. With changes in temperature the helix winds or unwinds, and this motion is transmitted to a pivoted pointer which moves over a calibrated scale. Because of lost motion and friction, such thermometers are not precision instruments.

15–6 Surface and volume expansion. When a plate or sheet of material is heated, both the length and breadth of the plate increase. Consider a rectangular plate whose length and breadth at temperature t_0 are L_0 and b_0 respectively. When heated to a temperature t, these dimensions become

$$L = L_0(1 + \alpha\,\Delta t),$$

and

$$b = b_0(1 + \alpha\,\Delta t).$$

The original area of the plate was

$$A_0 = L_0 b_0,$$

and the area after heating is

$$\begin{aligned} A = Lb &= L_0 b_0(1 + \alpha\,\Delta t)\,(1 + \alpha\,\Delta t). \\ &= A_0[1 + 2\alpha\,\Delta t + (\alpha\,\Delta t)^2]. \end{aligned}$$

TABLE 15–3

COEFFICIENTS OF CUBICAL EXPANSION

Substance	$\beta\ (C°)^{-1}$
Alcohol, ethyl	$0\ 745 \times 10^{-3}$
Carbon disulphide	$1\ 140 \times 10^{-3}$
Glycerin	$0\ 485 \times 10^{-3}$
Mercury	$0\ 182 \times 10^{-3}$
Petroleum	$0\ 899 \times 10^{-3}$

But since α is a small quantity, α^2 will be extremely small and the term $(\alpha\ \Delta t)^2$ may be neglected. Hence

$$A = A_0(1 + 2\alpha\ \Delta t).$$

If we now define a *surface* coefficient of expansion γ so that

$$A = A_0(1 + \gamma\ \Delta t), \tag{15–11}$$

it follows that

$$\boxed{\gamma = 2\alpha,}$$

and the coefficient of surface expansion is twice the coefficient of linear expansion. Although derived for the special case of a rectangular plate, the result holds for a plate of any shape whatever.

If the plate contains a hole, the area *of the hole* expands at the same rate as does the surrounding material. This remains true even if the hole becomes so large that the "plate" is reduced to nothing but a rim around the hole. Thus the area of the "hole" enclosed by a steel wagon tire expands at the same rate as would a disk of this size, if constructed of the same kind of steel as is the rim.

By considering a solid block of material in the form of a rectangular parallelepiped whose dimensions at t_0 are L_0, b_0, and c_0, it is easy to show by the same type of reasoning that

$$V = V_0(1 + 3\alpha\ \Delta t) = V_0(1 + \beta\ \Delta t), \tag{15–12}$$

where V is the volume at the temperature t, V_0 is the volume at t_0, and

$$\boxed{\beta = 3\alpha}$$

is the *volume* coefficient or *cubical* coefficient of expansion. This equation holds regardless of the shape of the body.

It is also true that the volume enclosed by a solid, such as the volume of a tank, a flask, or a thermometer bulb, expands at the same rate as would a solid body of the same material as that of which the walls are composed.

Equation (15–12) may also be used to compute the expansion of a liquid. The linear and surface coefficients of expansion of a liquid are of little significance.

EXAMPLE. The volume of the bulb of a mercury thermometer at 0°C is V_0, and the cross section of the capillary is A_0. The linear coefficient of expansion of the glass is α_G per C°, and the cubical coefficient of expansion of mercury is β_M per C°. If the mercury just fills the bulb at 0°C, what is the length of the mercury column in the capillary at a temperature of t°C?

The volume of the bulb at a temperature t is

$$V = V_0(1 + \beta_G t),$$

where $\beta_G = 3\alpha_G$ is the cubical coefficient of expansion of the glass.

The volume of the mercury at a temperature t is

$$V_M = V_0(1 + \beta_M t).$$

The volume of mercury that has been expelled from the bulb is the difference between these, or

$$V_0(1 + \beta_M t) - V_0(1 + \beta_G t) = V_0 t(\beta_M - \beta_G).$$

This volume is also equal to the length l of the mercury column multiplied by the cross section A of the capillary, where

$$A = A_0(1 + 2\alpha_G t).$$

Table 15–2 shows that the linear coefficient of expansion of glass is of the order of 5×10^{-6} per C°. Hence even if t is as great as 300°C, the term $2\alpha_G t$ is only 0.003. It may therefore be neglected in comparison with unity, which is equivalent to considering the cross section of the capillary constant. Then

$$lA_0 = V_0 t(\beta_M - \beta_G)$$

and

$$l = \frac{V_0}{A_0}(\beta_M - \beta_G)t.$$

The length of the mercury column is therefore proportional to the temperature and to the difference between the cubical coefficients of expansion of mercury and the glass of which the thermometer is constructed.

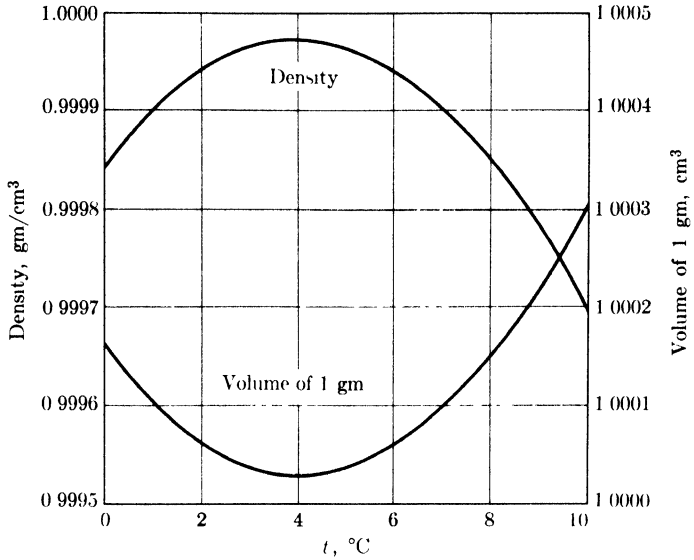


FIG. 15–10. Density of water, and volume of 1 gm, in the temperature range from 0°C to 10°C.

Water, in the temperature range from 0°C to 4°C, *decreases* in volume with increasing temperature, contrary to the behavior of most substances. That is, between 0°C and 4°C, the coefficient of expansion of water is *negative*. Above 4°C, water expands when heated. Since the volume of a given mass of water is smaller at 4°C than at any other temperature, the density of water is a maximum at 4°C. This behavior of water is the reason why lakes and ponds freeze first at their upper surface. Figure 15–10 illustrates the anomalous expansion of water in the temperature range from 0°C to 10°C.

15–7 Thermal stresses. If the ends of a rod are rigidly fixed so as to prevent expansion or contraction and the temperature of the rod is changed, tensile or compressive stresses, called *thermal stresses*, will be set up in the rod. These stresses may become very large, sufficiently so to stress the rod beyond its elastic limit or even beyond its breaking strength. Hence in the design of any structure which is subject to changes in temperature, some provision must, in general, be made for expansion. In a long steam pipe this is accomplished by the insertion of expansion joints or a section of pipe in the form of a U. In bridges, one end may be rigidly fastened to its abutment while the other rests on rollers.

It is a simple matter to compute the thermal stress set up in a rod which is not free to expand or contract. Suppose that a rod at a temperature t has its ends rigidly fastened, and that while they are thus held the temperature is reduced to a lower value, t_0.

The fractional change in length if the rod were free to contract would be

$$\frac{\Delta L}{L_0} = \alpha(t - t_0) = \alpha\,\Delta t. \tag{15–13}$$

Since the rod is not free to contract, the tension must increase by a sufficient amount to produce the same fractional change in length. But from the definition of Young's modulus,

$$Y = \frac{F/A}{\Delta L/L_0},$$

and hence

$$F = AY\frac{\Delta L}{L_0}.$$

Introducing the expression for $\Delta L/L_0$ from Eq. (15–13), we have

$$F = AY\alpha\,\Delta t, \tag{15–14}$$

which gives the tension F in the rod. The *stress* in the rod is

$$\frac{F}{A} = Y\alpha\,\Delta t. \tag{15–15}$$

Problems

15–1. (a) Prove that

$$t_F = \tfrac{9}{5}t + 32°\text{F}.$$

(b) Prove that the celsius and fahrenheit scales have the same reading at $-40°$.

15–2. The pendulum shaft of a clock is of aluminum. What is the fractional change in length of the shaft when it is cooled from 75°F to 45°F?

15–3. The length of Technology Bridge is about 2000 ft. Find the difference between its length on a winter day when the temperature is −20°F, and a summer day when the temperature is 100°F. Use the coefficient of expansion of steel.

15–4. A surveyor's 100-ft steel tape is correct at a temperature of 65°F. The distance between two points, as measured by this tape on a day when the temperature is 95°F, is 86.57 ft. What is the true distance between the points?

15–5. One steel meter bar is correct at 0°C, and another at 25°C. What is the difference between their lengths at 20°C?

15–6. To ensure a tight fit, the aluminum rivets used in airplane construction are made slightly larger than the rivet holes and cooled by "dry ice" (solid CO_2) before being driven. If the diameter of a hole is 0.2500 inch, what should be the diameter of a rivet at 20°C if its diameter is to equal that of the hole when the rivet is cooled to −78°C, the temperature of dry ice?

15–7. A glass flask whose volume is exactly 1000 cm^3 at 0°C is filled level full of mercury at this temperature. When flask and mercury are heated to 100°C, 15.2 cm^3 of mercury overflow. If the cubical coefficient of expansion of mercury is 0.000182 per celsius degree, compute the linear coefficient of expansion of the glass.

15–8. Steel rails 120 ft long are laid with their ends in contact on a day when the temperature is 110°F. What length gap will there be between rails on a day when the temperature is −20°F?

15–9. A steel wire which is 10 ft long at 20°C is found to increase in length by $\frac{3}{4}$ inch when heated to 520°C. Compute its coefficient of linear expansion. (b) Find the stress in the wire if it is stretched taut at 520° and cooled to 20° without being allowed to contract.

15–10. A wire 60 cm long is bent into a circular ring, having a gap of 1.0 cm. The temperature of the wire is increased uniformly by 100°C. At the new temperature the gap is found to be 1.002 cm. What is the linear temperature coefficient of expansion of the wire? Assume no stresses within the wire before or after heating.

15–11. A steel ring of 3.000 inches inside diameter at 20°C is to be heated and slipped over a brass shaft measuring 3.002 inches in diameter at 20°C. (a) To what temperature should the ring be heated? (b) If the ring and shaft together are cooled by some means such as liquid air, at what temperature will the ring just slip off the shaft?

15–12. At a temperature of 20°C, the volume of a certain glass flask, up to a reference mark on the stem of the flask, is exactly 100 cm^3. The flask is filled to this point with a liquid whose cubical coefficient of expansion is 120×10^{-5} per C°, with both flask and liquid at 20°C. The linear coefficient of expansion of the glass is 8×10^{-6} per C°. The cross section of the stem is 1 mm^2 and can be consid-

ered constant. How far will the liquid rise or fall in the stem when the temperature is raised to 40°C?

15–13. A slender steel rod oscillates as a physical pendulum about a horizontal axis through one end. If the rod is 8 ft long at 30°C, compute the fractional change in its period when the temperature is decreased to 0°C.

15–14. A steel rod 1.5 cm^2 in cross section is 70 cm long at 20°C. If it is heated to 520°C and cooled to 20°C without being allowed to contract, compute the stress in the rod.

15–15. The cross section of a steel rod is 1.5 in^2. What is the least force that will prevent it from contracting while cooling from 520°C to 20°C?

15–16. What hydrostatic pressure is necessary to prevent a copper block from expanding when its temperature is increased from 20°C to 30°C?

15–17. A steel bomb is filled with water at 10°C. If the whole is heated to 75°C and no water is allowed to escape, compute the increase in pressure in the bomb. Assume the bomb to be sufficiently strong so that it is not stretched by the increased pressure. The average coefficient of volume expansion of water in this temperature range is $0.39 \times 10^{-3}\ (C°)^{-1}$.

15–18. Steel railroad rails 60 ft long are laid on a winter day when the temperature is 20°F. (a) How much space must be left between rails if they are to just touch on a summer day when the temperature is 110°F? (b) If the rails were originally laid in contact, what would be the stress in them on a summer day when the temperature is 110°F?

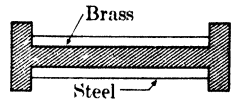


FIGURE 15–11

15–19. A heavy brass bar has projections at its ends, as in Fig. 15–11. Two fine steel wires fastened between the projections are just taut (zero tension) when the whole system is at 0°C. What is the tensile stress in the steel wires when the temperature of the system is raised to 300°C? Make any simplifying assumptions you think are justified, but state what they are.

15–20. A liquid is enclosed in a metal cylinder provided with a piston of the same metal. The system is originally at atmospheric pressure and at a temperature of 80°C. The piston is forced down until the pressure on the liquid is increased by 100 atm, and it is then clamped in this position. Find the new temperature at which the pressure of the liquid is again 1 atmosphere. Assume that the cylinder is sufficiently strong so that its volume is not altered by changes in pressure, but only by changes in temperature. Compressibility of liquid (k) = 50×10^{-6} atm^{-1}. Cubical coefficient of expansion of liquid (β) = $5.3 \times 10^{-4}\ (C°)^{-1}$. Linear coefficient of expansion of metal (α) = $10 \times 10^{-6}\ (C°)^{-1}$.

15–21. A steel rod of length 40 cm and a copper rod of length 36 cm, both of the same diameter, are placed end to end between two rigid supports, with no initial stress in the rods. The temperature of the rods is now raised 50°C. What is the stress in each rod?

CHAPTER 16

QUANTITY OF HEAT

16–1 Heat, a form of energy. Heat was formerly thought to be an invisible weightless fluid called *caloric*, which was produced when a substance burned and which could be transmitted by conduction from one body to another. The abandonment of the caloric theory was a part of the general development of physics during the 18th and 19th centuries. The two men who were probably chiefly responsible for the views we hold today were Count Rumford (1753–1814) (a native of Woburn, Mass.) and Sir James Prescott Joule.

Rumford was engaged in supervising the boring of cannon for the government of Bavaria. To prevent overheating, the bore of the cannon was kept filled with water, and as this boiled away during the boring process the supply had to be continually replenished. It was admitted that caloric had to be supplied to water in order to boil it, and the continual production of caloric was explained by the hypothesis that when matter was more finely subdivided (as in the process of boring) its capacity for retaining caloric grew smaller and the caloric thus released was what caused the water to boil.

Rumford noted, however, that the cooling water continued to boil away even when his boring tools became so dull that they were no longer cutting. That is, even a dull boring tool was apparently an inexhaustible supply of caloric *as long as mechanical work was being done to rotate the tool.*

Now one of the features which justifies our acceptance of many abstract ideas in physics is that they obey a "conservation principle." Here was a process in which *two* quantities failed to be conserved. Mechanical energy was not conserved since work was continually being expended, and caloric was not conserved since it was continually being created. Although Rumford did not express his ideas in just this way, he saw the opportunity to eliminate two cases of nonconservation and at the same time to extend the principle of conservation of energy as it was then understood. He asserted that what had formerly been thought a separate entity, namely caloric, was in reality merely energy in another form. The process was not the continual disappearance of one thing and the appearance of another, but merely the transformation of energy from one form to another. As we would say today, mechanical energy was continually being transformed into heat, the process being one example of conservation of energy.

Rumford made some measurements of the quantities of work done and of cooling water boiled away, but his experiments were not of great pre-

cision. When Joule, in the period from 1843 to 1878, showed that whenever a given quantity of mechanical energy was converted to heat the *same* quantity of heat was always developed, the equivalence of heat and work as two forms of energy was definitely established.

There are, of course, processes for whose explanation the caloric theory is entirely satisfactory. When heat flows from one body to another by conduction, or when substances at different temperatures are mixed in a calorimeter, heat *is* conserved, and for such processes the caloric theory would serve perfectly well.

16–2 Quantity of heat. Heat, like mechanical energy, is an intangible thing and a unit of heat is not something that can be preserved in a standards laboratory. The quantity of heat transferred in a process is measured by some change which accompanies the process, and a unit of heat is defined as the heat necessary to produce some standard, agreed-on change. Three such units are in common use, the calorie (cal), the kilogram·calorie (kcal), and the British thermal unit (Btu).

One calorie is the quantity of heat which must be supplied to one gram of water to raise its temperature through one celsius degree.

One kilogram·calorie is the quantity of heat which must be supplied to one kilogram of water to raise its temperature through one celsius degree.

One Btu is the quantity of heat which must be supplied to one pound of water to raise its temperature through one fahrenheit degree.*

Evidently, 1 kilogram·calorie = 1000 calories.

Since 454 gm = 1 lb, and since 1 F° = $\frac{5}{9}$ C°, the Btu may be defined as the quantity of heat which must be supplied to 454 gm (0.454 kgm) of water to raise its temperature through $\frac{5}{9}$ C°, which is $454 \times \frac{5}{9} = 252$ cal or 0.252 kcal. Hence

$$1 \text{ Btu} = 252 \text{ cal} = 0.252 \text{ kcal}.$$

The heat units here defined vary somewhat with the location of the degree, i.e., whether it is from 0° to 1°, 47° to 48°, etc. It is generally agreed to use the temperature interval from 14.5°C to 15.5°C (the "15° calorie"), and in English units to use the temperature interval from 63°F to 64°F. For most purposes this variation is small enough to be neglected.

It is essential that the distinction between "quantity of heat" and "temperature" shall be clearly understood. The terms are commonly mis-

* In this part of the subject we shall depart from the engineering system of units which we used throughout mechanics, and adopt as a mass unit the *mass of the standard pound.* This unit is also called one pound, and is equal to a mass of 454 grams, or (about) $\frac{1}{32}$ slug. Also, for simplicity, we shall confine metric units chiefly to the cgs system.

used in everyday life. Suppose that two pans, one containing a small and the other a large amount of water, are placed over identical gas burners and heated for the same length of time. It is obvious that at the end of this time the temperature of the small amount of water will have risen higher than that of the large amount. In this instance, equal quantities of heat have been supplied to each pan of water, but the increases in temperature are not equal.

On the other hand, suppose the two pans are both initially at a temperature of 60°F and that both are to be heated to 212°F. It is evident that more heat must be supplied to the pan containing the larger amount of water. The temperature change is the same for both but the quantities of heat supplied are very different.

We shall represent a quantity of heat by the letter Q.

16–3 Internal energy. The temperature of a body may be increased by placing it in contact with a second body at a higher temperature, or by doing mechanical work on the body. For example, the air in a bicycle pump becomes hotter when the piston is pushed down, although it could also be heated by placing it in a furnace.

If one were given a sample of hot air, it would be impossible to tell by any tests whether it had been heated by compression or by heat flow from a hotter body. This raises the question as to whether one is justified in speaking of the "heat in a body," since the present state of the body may have been brought about either by adding heat to it or by doing work on it. We shall show later that the proper term to use is "internal energy," and that the expression "heat energy of a body" is meaningless.

From the atomic point of view, the internal energy of a body is the sum total of the kinetic and potential energies of its molecules, apart from any kinetic or potential energy of the body as a whole. Not enough is known at present about the molecular structure of matter to be able to express internal energies wholly in terms of a molecular model. To a first approximation, the internal energy of a gas at low pressure may be identified with the aggregate kinetic energy of its molecules.

Even though the details of the molecular picture of matter are not fully understood, we do have definite evidence that molecular energies and velocities, whether in a solid, liquid, or gas, increase with increasing temperature. Such statements as "the heat in a body is the energy of motion of its molecules" should, however, be avoided.

16–4 The mechanical equivalent of heat. Energy in mechanical form is usually expressed in ergs, joules, or foot·pounds; energy in the form of heat is expressed in calories or Btu. The relative magnitudes of the "heat units" and the "mechanical units" can be found by an experiment in

which a measured quantity of mechanical energy is completely converted into a measured quantity of heat. The first accurate experiments were performed by Joule, using an apparatus in which falling weights rotated a set of paddles in a container of water. The energy transformed was computed in mechanical units from a knowledge of the weights and their height of fall, and in heat units from a measurement of the mass of water and its rise in temperature. In more recent methods, which are also more precise, electrical energy is converted to heat in a resistance wire immersed in water. The best results to date give:

$$778 \text{ ft·lb} = 1 \text{ Btu},$$
$$4.186 \text{ joules} = 1 \text{ cal},$$
$$4186 \text{ joules} = 1 \text{ kcal}.$$

That is, 778 ft·lb of mechanical energy, when converted to heat, will raise the temperature of 1 lb of water through 1 F°, etc.

These relations are often expressed by the statement that *the mechanical equivalent of heat* is 4.186 joules/cal, or 778 ft·lb/Btu. The phraseology is a carry-over from the early days when the equivalence of mechanical energy and heat was being established.

The precise value of the mechanical equivalent of heat depends on the particular temperature interval used in the definition of the calorie or Btu. To avoid this confusion, an international commission has agreed to *define* 1 kcal as *exactly* 1/860 kilowatt·hour. Then by definition, 1 cal = 4.18605 joules and 1 Btu = 778.26 ft·lb. It follows that 1 Btu = 251.996 cal.

16–5 Heat capacity. Specific heat. Materials differ from one another in the quantity of heat required to produce a given elevation of temperature in a given mass. Suppose that a quantity of heat Q is supplied to a given body, resulting in a temperature rise Δt. The ratio of the heat supplied to the corresponding temperature rise is called the *heat capacity* of the body.

$$\text{Heat capacity} = \frac{Q}{\Delta t}. \tag{16–1}$$

Heat capacities are ordinarily expressed in calories per celsius degree, or Btu per fahrenheit degree. If we set $\Delta t = 1$ degree in Eq. (16–1), it will be seen that the heat capacity of a body is numerically equal to the quantity of heat which must be supplied to it to increase its temperature by one degree.

To obtain a figure that is characteristic of the material of which a body is composed, the *specific heat capacity*, abbreviated *specific heat*, of a

TABLE 16-1

SPECIFIC HEAT CAPACITIES

Substance	Specific heat, cal/gm·C°	Temperature interval
Aluminum	0 217	17–100°C
Brass	0 094	15–100
Copper	0.093	15–100
Glass	0 199	20–100
Ice	0 55	−10–0
Iron	0 113	18–100
Lead	0 031	20–100
Mercury	0 033	0–100
Silver	0 056	15–100

material is defined as the *heat capacity per unit mass* of a body composed of the material. We shall represent specific heat capacity by the letter c.

$$c = \frac{\text{heat capacity}}{\text{mass}} = \frac{Q}{m\,\Delta t}. \tag{16-2}$$

Specific heat capacity is expressed in calories per gram·celsius degree, or Btu per pound·fahrenheit degree.

The specific heat capacity of a material is numerically equal to the quantity of heat which must be supplied to *unit mass* of the material to increase its temperature through 1 degree. The specific heat capacities of a few common materials are listed in Table 16-1. The units are cal/gm·C° or Btu/lb·F°.

It follows from Eq. (16-2) that the heat which must be supplied to a body of mass m, whose specific heat capacity is c, to increase its temperature through an interval Δt, is

$$Q = mc\,\Delta t = mc(t_2 - t_1). \tag{16-3}$$

Strictly speaking, Eq. (16-2) defines the *mean* specific heat capacity over the temperature range Δt. It is found that the quantity of heat required to raise the temperature of a material through a small interval varies with the location of the interval in the temperature scale. At ordi-

nary temperatures, and over temperature intervals which are not too great, however, specific heats may be considered constant. At extremely low temperatures, approaching absolute zero, all specific heats decrease and approach zero.

It should be pointed out that the significance of the word "capacity" in "heat capacity" is not the same as when one speaks of the "capacity" of a bucket. The bucket can hold just so much water and no more, while heat can be added to a body indefinitely with, of course, a corresponding rise in temperature.

For some purposes, particularly in dealing with gases, it is more convenient to express specific heat capacities on the basis of one gram·molecular weight rather than one gram. It was first noted in 1819 by Dulong and Petit that the specific heat capacities of the metals, expressed in this way, were all very nearly equal to 6 cal/gm·molecular wt·C°. This fact is known as the *Dulong and Petit law.* Since the number of molecules is the same in one gram·molecular weight of all substances, this means that the heat capacity of a metallic object depends only on the *number* of molecules it contains, not on the mass of each molecule.

16–6 Calorimetry. The term calorimetry relates to the measurement of quantities of heat. Two types of calorimeter, the *water calorimeter* and the *continuous-flow calorimeter,* will be described.

The water calorimeter in its simple form consists of a thin-walled metal can A (Fig. 16–1) whose capacity is about two liters, and whose outer surface is nickel plated. The can contains a measured quantity of water, and is provided with a cover through which passes thermometer B. Heat losses are further reduced by surrounding the can with the heat-insulating jacket C. If the thermometer is read before and after an unknown quantity of heat Q is introduced into the calorimeter, Q may be found from the measured rise in temperature.

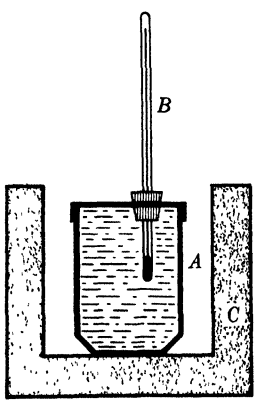


FIG. 16–1. The water calorimeter.

The water calorimeter may be used to measure specific heat as follows: a sample of the material whose specific heat is desired is heated in a furnace or steam bath to a known temperature, say t_s. Let the mass of the sample be m_s and its specific heat c_s.

The water in the calorimeter is thoroughly stirred and its temperature is measured. The sample is then quickly transferred to the calorimeter, the water is again thoroughly stirred, and the new temperature of the water is recorded. Let t_1 and t_2

be the initial and final temperatures of the water, m_w the mass of the water, m_c the mass of the calorimeter can, and c_c its specific heat.

If no heat is lost from the calorimeter during the experiment, the heat given up by the sample in cooling from t_s to t_2 must equal the heat gained by the water and the calorimeter can. Hence

$$m_s c_s(t_s - t_2) = m_w \times 1(t_2 - t_1) + m_c c_c(t_2 - t_1)$$
$$= (m_w + m_c c_c)(t_2 - t_1)$$

and c_s may be found, since the other factors are known.

The effect of the heat capacity of the calorimeter, $m_c c_c$, is evidently equivalent to increasing the mass of the water by an amount $m_c c_c$ and using a calorimeter of zero heat capacity. The product $m_c c_c$ is called the *water equivalent* of the calorimeter.

Actually the calorimeter will gain (or lose) heat from its surroundings during an experiment unless special precautions are taken. One way of minimizing the heat transfer is to start with the calorimeter somewhat cooler than its surroundings and finish with its temperature the same amount higher than the surroundings. Then the heat gained during the first part of the experiment offsets the heat lost in the latter part. Another method (the so-called "adiabatic jacket") is to heat the jacket by an electric heating coil so that its temperature rises at the same rate as does that of the calorimeter. If both temperatures are always equal there will be no gain or loss of heat.

It should be noted that this method of measuring specific heat gives only the *average* specific heat over the temperature range from t_s to t_2.

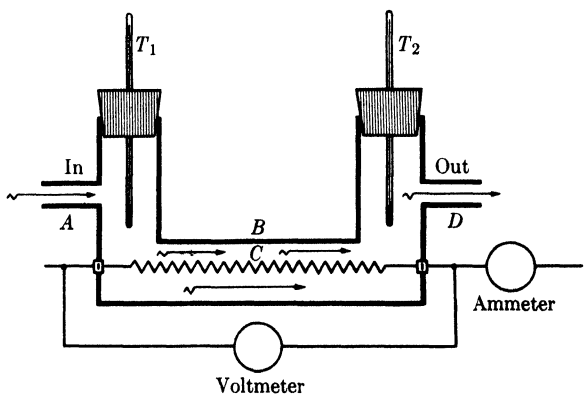


FIG. 16–2. The continuous-flow calorimeter.

Much more elaborate apparatus is required to measure the true specific heat at any desired temperature.

The continuous-flow calorimeter, as used to measure the mechanical equivalent of heat, is illustrated in Fig. 16–2. A continuous stream of water enters the apparatus at A, flows through the tube B around the resistance wire C, and leaves at D. Thermometers T_1 and T_2 read the temperatures t_1 and t_2 at inlet and outlet, and the electrical power expended is measured by the ammeter and voltmeter.

To use the calorimeter, the water is started flowing and the heating current is turned on. Thermometers T_1 and T_2 are read at intervals of, say, one minute, and their temperatures recorded. After sufficient time has elapsed, both thermometer readings become constant. Of course the temperature t_2 at the outlet is higher than the temperature t_1 at the inlet. When this steady state has been reached the apparatus itself is absorbing no heat, since its temperature remains constant. Heat is therefore being

TABLE 16–2

HEATS OF COMBUSTION

Substance	Btu/lb	cal/gm
Solids:		
Charcoal	14,600	8100
Coal	11,000–14,000	6100–7800
Coke	14,300	7950
Paraffin	19,800	11,000
Pitch	15,100	8400
Wood	8300–9200	4600–5100
Liquids:		
Aviation gasoline	20,420	11,300
Diesel fuel oil	19,350	10,700
Domestic fuel oil	19,450	10,800
Ethyl alcohol	14,000	7800
Jet-propulsion fuels	~19,000	~10,500
Kerosene	19,810	11,000
Motor gasoline	20,120	11,200
Gases:	(Btu/ft^3)	(kcal/m^3)
Acetylene	1448	12,870
Coal gas	488	4320
Hydrogen	275	2445
Natural gas	910–1970	8040–17,400
Propane	2330	20,600

carried away by the flowing water at exactly the same rate as it is developed by the heating coil.

If then the mass of water passing through the calorimeter in a certain time is found, usually by catching the water in a beaker placed below the outlet, the quantity of heat developed can be computed from the rise in temperature of this mass of water. The energy input in the same time can be found from the ammeter and voltmeter readings.

16–7 Heat of combustion. The heat of combustion of a substance is the quantity of heat liberated per unit mass, or per unit volume, when the substance is completely burned. Heats of combustion of solid and liquid fuels are usually expressed in Btu/lb or in cal/gm. The heat of combustion of gases is commonly expressed in Btu/ft^3. Some values are given in Table 16–2.

Heats of combustion of solid and liquid fuels are measured with a *bomb calorimeter*. A measured mass of the fuel is inserted in a strong steel bomb which is filled with oxygen under pressure to ensure complete combustion. The bomb is placed in a water calorimeter and the fuel ignited by sending a momentary electric current through a fine heater wire. From the measured temperature rise, the mass of water, and the water equivalent of calorimeter and bomb, the heat of combustion can be computed.

The heat of combustion of gaseous fuels is usually measured with a type of continuous flow calorimeter, illustrated in Fig. 16–3.

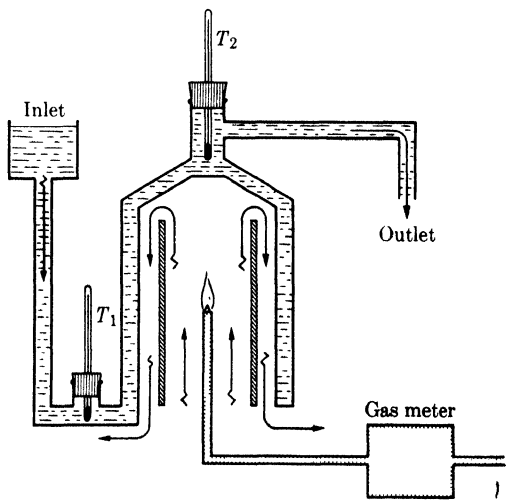


FIG. 16–3. Continuous-flow calorimeter used to measure the heat of combustion of gaseous fuel.

16–8 Change of phase. The term *phase* as used here relates to the fact that matter exists either as a solid, liquid, or gas. Thus the chemical substance H_2O exists in the *solid phase* as ice, in the *liquid phase* as water, and in the *gaseous phase* as steam. Provided they do not decompose at high temperatures, all substances can exist in any of the three phases under the proper conditions of temperature and pressure. Transitions from one phase to another are accompanied by the absorption or liberation of heat and usually by a change in volume.

As an illustration, suppose that ice is taken from a refrigerator where its temperature was, say, −25°C. Let the ice be crushed quickly, placed in a container, and a thermometer inserted in the mass. Imagine the container to be surrounded by a heating coil which supplies heat to the ice at a uniform rate, and suppose that no other heat reaches the ice. The temperature of the ice would be observed to increase steadily as shown by the portion of the graph (Fig. 16–4) from *a* to *b*, or until the temperature has risen to 0°C. As soon as this temperature is reached, some liquid water will be observed in the container. In other words, the ice begins to melt. The melting process is a *change of phase*, from the solid phase to the liquid phase. The thermometer, however, will show no *increase in temperature*, and even though heat is being supplied at the same rate as before, the temperature will remain at 0°C until all the ice is melted (point *c*, Fig. 16–4). (The ice and water mixture must be kept thoroughly stirred, otherwise the temperature of that part of the water closest to the heater will rise above 0°C.)

As soon as the last of the ice has melted, the temperature begins to rise again at a uniform rate (from *c* to *d*, Fig. 16–4) although this rate will be slower than that from *a* to *b* because the specific heat of water is greater than that of ice. When a temperature of 100°C is reached (point *d*),

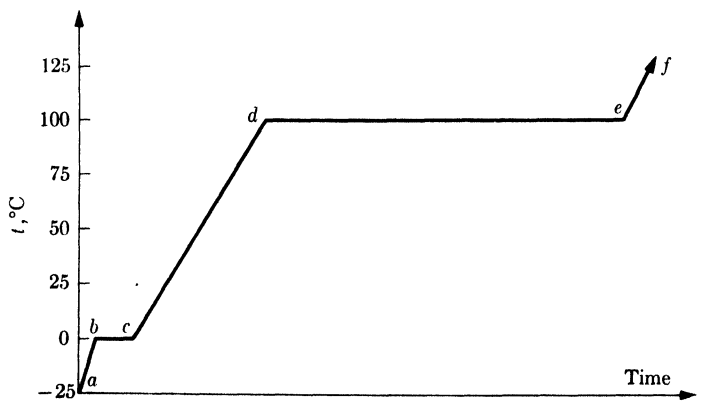


FIG. 16–4. The temperature remains constant during each change of phase.

bubbles of steam (gaseous water or water vapor) start to escape from the liquid surface, or the water begins to boil. The temperature remains constant at 100°C until all the water has boiled away. Another change of phase has therefore taken place, from the liquid phase to the gaseous phase.

If all the water vapor had been trapped and not allowed to diffuse away (a very large container would be needed), the heating process could be continued as from e to f. The gas would now be called "superheated steam."

Although water was chosen as an example in the process just described, the same type of curve as in Fig. 16–4 is obtained for many other substances. Some, of course, decompose before reaching a melting or boiling point, and others, such as glass or pitch, do not change state at a definite temperature but become gradually softer as their temperature is raised. Crystalline substances, such as ice, or a metal, melt at a definite temperature. Glass and pitch are actually supercooled liquids of very high viscosity.

The temperature at which a crystalline solid melts when heat is supplied to it at atmospheric pressure is called its *normal melting point*, and the temperature at which a liquid boils when heat is supplied to it at atmospheric pressure is called its *normal boiling point*. The quantity of heat per unit mass that must be supplied to a material at its melting point to convert it completely to a liquid at the same temperature, is called the *heat of fusion* of the material. The quantity of heat per unit mass that must be supplied to a material at its boiling point to convert it completely to a gas at the same temperature, is called the *heat of vaporization* of the material. Heats of fusion and vaporization are expressed in calories per gram, or Btu per pound. Thus the heat of fusion of ice is about 80 cal/gm or 144 Btu/lb. The heat of vaporization of water (at 100°C) is 539 cal/gm or 970 Btu/lb. Some heats of fusion and vaporization are listed in Table 16–3.

When heat is removed from a gas, its temperature falls and at the same temperature at which it boiled, it returns to the liquid phase, or *condenses*. In so doing it gives up to its surroundings the same quantity of heat which was required to vaporize it. The heat so given up, per unit mass, is called the *heat of condensation* and is equal to the heat of vaporization. Similarly, a liquid returns to the solid phase, or freezes, when cooled to the temperature at which it melted, and gives up heat called *heat of solidification* exactly equal to the heat of fusion. Thus the melting point and the freezing point are at the same temperature, and the boiling point and condensation point are at the same temperature.

Whether a substance, at its melting point, is freezing or melting depends on whether heat is being supplied or removed. That is, if heat is supplied to a beaker containing both ice and water at 0°C, some of the ice will

TABLE 16–3

HEATS OF FUSION AND VAPORIZATION

Substance	Normal melting point		Heat of fusion, cal/gm	Normal boiling point		Heat of vaporization, cal/gm
	°K	°C		°K	°C	
Helium	3 5	−269 65	1 25	4 216	−268 93	5
Hydrogen	13 84	−259 31	14	20 26	−252 89	108
Nitrogen	63 18	−209 97	6 09	77.34	−195 81	48
Oxygen	54 36	−218 79	3 30	90 18	−182 97	51
Ethyl alcohol	159	−114	24 9	351	78	204
Mercury	234	− 39	2 82	630	357	65
Water	273 15	0 00	79 7	373 15	100 00	539
Sulphur	392	119	9 1	717 75	444 60	78
Lead	600 5	327 3	5 86	2023	1750	208
Antimony	903 65	630.50	39 4	1713	1440	
Silver	1233 95	960 80	21 1	2466	2193	558
Gold	1336 15	1063 00	15 4	2933	2660	377

melt; if heat is removed, some of the water will freeze; the temperature in either case remains at 0°C so long as both ice and water are present. If heat is *neither supplied nor removed*, no change at all takes place and the relative amounts of ice and water, and the temperature, all remain constant.

This furnishes, then, another point of view which may be taken regarding the melting point. That is, the melting (or freezing) point of a substance is *that temperature at which both the liquid and solid phases can exist together*. At any higher temperature, the substance can only be a liquid; at any lower temperature, it can only be a solid.

The general term *heat of transformation* is applied both to heats of fusion and heats of vaporization, and both are designated by the letter L. Since L represents the heat absorbed or liberated in the change of phase of unit mass, the heat Q absorbed or liberated in the change of phase of a mass m is

$$Q = mL. \tag{16–4}$$

The household steam-heating system makes use of a boiling-condensing process to transfer heat from the furnace to the radiators. Each pound of water which is turned to steam in the furnace absorbs 970 Btu (the heat of vaporization of water) from the furnace, and gives up 970 Btu when it condenses in the radiators. (This figure is correct if the steam pressure is one atmosphere. It will be slightly smaller at higher pressures.) Thus

the steam-heating system does not need to circulate as much water as a hot-water heating system. If water leaves a hot-water furnace at 140°F and returns at 100°F, dropping 40 F°, about 24 lb of water must circulate to carry the same heat as is carried in the form of heat of vaporization by one pound of steam.

Under the proper conditions of temperature and pressure, a substance can change directly from the solid to the gaseous phase without passing through the liquid phase. The transfer from solid to vapor is called *sublimation*, and the solid is said to *sublime*. "Dry ice" (solid carbon dioxide) sublimes at atmospheric pressure. Liquid carbon dioxide cannot exist at a pressure lower than about 73 lb/in^2.

Heat is absorbed in the process of sublimation, and liberated in the reverse process. The quantity of heat per unit mass is called the *heat of sublimation*.

16–9 Measurement of heats of fusion and vaporization. The method of mixtures is used to measure heats of fusion and of vaporization. For example, the heat of fusion of ice may be found by dropping a weighed sample of ice at 0°C into a calorimeter containing a measured amount of water, and observing the temperature of the water before and after the addition of the ice. Let a mass m_i of ice at 0°C be dropped in a calorimeter containing a mass m_w of water, lowering the temperature of the water from t_1 to t_2. We shall assume that all the ice melts, and neglect the heat capacity of the calorimeter. Then if L represents the heat of fusion of the ice, the ice absorbs a quantity of heat m_iL on melting (this converts it to water at 0°C) and a further quantity of heat m_it_2 on warming up to the final temperature t_2. The water in the calorimeter gives up a quantity of heat $m_w(t_1 - t_2)$. Hence,

$$m_w(t_1 - t_2) = m_i(L + t_2),$$

so that L can be found if the other quantities are known.

The heat of condensation (= heat of vaporization) of steam can be measured in a similar way, by allowing steam from a boiler to condense within a calorimeter. The condensation usually takes place in a coiled tube immersed in the calorimeter, so that the amount of steam condensing may be found by weighing the coil before and after the experiment. We have:

$$\text{Heat given up by condensing steam} = m_sL.$$

$$\left\{\begin{array}{l}\text{Heat given up by condensed steam} \\ \text{(water at 100°C) cooling to } t_2\end{array}\right\} = m_s(100 - t_2).$$

$$\text{Heat absorbed by calorimeter} = m_w(t_2 - t_1).$$

The heat of vaporization may be found by equating heat loss to heat gain.

16–10 Effect of dissolved substances on freezing and boiling points. The freezing point of a liquid is lowered when some other substance is dissolved in the liquid. A common example is the use of an "antifreeze" to lower the freezing point of the water in the cooling system of an automobile engine.

The freezing point of a saturated solution of common salt in water is about −20°C. To understand why a mixture of ice and salt may be used as a freezing mixture, let us make use of the definition of the freezing point as the only temperature at which the liquid and solid phases can exist in equilibrium. When a concentrated salt solution is cooled, it freezes at −20°C, and crystals of ice (pure H_2O) separate from the solution. In other words, ice crystals and a salt solution can exist in equilibrium only at −20°C, just as ice crystals and pure water can exist together only at 0°C.

When ice at 0°C is mixed with a salt solution at 20°C, some of the ice melts, abstracting its heat of fusion from the solution until the temperature falls to 0°C. But ice and salt solution cannot remain in equilibrium at 0°C, so that the ice continues to melt. Heat is now supplied both by the ice and the solution, and both cool down until the equilibrium temperature of −20°C is reached. If no heat is supplied from outside, the mixture remains unchanged at this temperature. If the mixture is brought into contact with a warmer body, say an ice-cream mixture at 20°C, heat flows from the ice-cream mixture to the cold salt solution, melting more of the ice but producing no rise in temperature as long as any ice remains. The flow of heat from the ice-cream mixture lowers its temperature to its freezing point (which will be below 0°C, since it is itself a solution). Further loss of heat to the ice-salt mixture causes the ice cream to freeze.

The boiling point of a liquid is also affected by dissolved substances, but may be either increased or decreased. Thus the boiling point of a water-alcohol solution is *lower* than that of pure water, while the boiling point of a water-salt solution is *higher* than that of pure water.

Both boiling and freezing points are affected by the external pressure, a matter which will be taken up in Chapter 19.

Problems

16–1. How many cubic feet of a coal gas must be burned to raise the temperature of 40 gallons of water from 50°F to 150°F, assuming 25% stack loss? (There are 7.5 gallons in a cubic foot.)

16–2. A certain Diesel engine consumes 20 lb of fuel oil per hour. The heat of combustion of the oil is 20,000 Btu/lb. If the over-all efficiency of the engine is 30%, (a) how many Btu/hr are converted into mechanical work? (b) How many Btu are wasted? (c) What horsepower does the engine develop?

16–3. An automobile weighing 2000 lb is traveling at 10 ft/sec. How many Btu are developed in the brakes when it is brought to rest?

16–4. (a) What quantity of heat, in Btu, is required to raise the temperature of the water in a 40-gallon hot-water tank from 60°F to 140°F? (There are 7.5 gallons in a cubic foot.) (b) If the water is heated by the combustion of gas, how many cubic feet of gas must be burned if stack losses amount to 20%? (c) If the water is heated by an electric heater, how many kwh are required? The efficiency of the electric heater may be assumed 100%.

16–5. (a) A certain house burns 10 tons of coal in a heating season. If stack losses are 15%, how many Btu were actually used to heat the house? (b) In some localities large tanks of water are heated by solar radiation during the summer and the stored energy is used for heating during the winter. Find the required dimensions of the storage tank, assuming it to be a cube, to store a quantity of energy equal to that computed in part (a). Assume that the water is heated to 120°F in the summer and cooled to 80°F in the winter.

16–6. An automobile engine whose output is 40 hp uses 4.5 gallons of gasoline per hour. The heat of combustion of gasoline is 3×10^7 calories per gallon. What is the efficiency of the engine?

16–7. The electric power input to a certain electric motor is 0.50 kw, and the mechanical power output is 0.54 hp. (a) What is the efficiency of the motor? (b) How many Btu are developed in the motor in one hour of operation?

16–8. 400 gm of water are contained in a copper vessel of mass 200 gm. The water is heated by a friction device which dissipates mechanical energy, and it is observed that the temperature of the system rises at the rate of 3 C° per minute. Neglect heat losses to the surroundings. What power in watts is dissipated in the water?

16–9. How long could a 2000 hp motor be operated on the heat energy liberated by one cubic mile of ocean water when the temperature of the water is lowered by 1 C° if all this heat were converted to mechanical energy? Why do we not utilize this tremendous reservoir of energy?

16–10. An aluminum can of mass 500 gm contains 117.5 gm of water at a temperature of 20°C. A 200-gm block of iron at 75°C is dropped into the can. Find the final temperature, assuming no heat loss to the surroundings.

16–11. A casting weighing 100 lb is taken from an annealing furnace where its temperature was 900°F and plunged into a tank containing 800 lb of oil at a temperature of 80°F. The final temperature is 100°F, and the

specific heat of the oil is 0.5 Btu/lb·F°. What was the specific heat of the casting? Neglect the heat capacity of the tank itself and any heat losses.

16–12. Compute from Table 16–1 the heat capacities of one gram-molecular weight of Al, Cu, Pb, Hg, and Ag, and compare with the values predicted by the Dulong and Petit law.

16–13. Compare the heat capacities of equal *volumes* of water, copper, and lead.

16–14. A lead bullet, traveling at 350 m/sec, strikes a target and is brought to rest. What would be the rise in temperature of the bullet if none of the heat developed were lost to the surroundings?

16–15. A copper calorimeter whose mass is 300 gm contains 500 gm of water at a temperature of 15°C. A 560-gm block of copper, at a temperature of 100°C, is dropped into the calorimeter and the temperature is observed to increase to 22.5°C. Neglect heat losses to the surroundings. (a) Find the specific heat of copper. (b) What is the water equivalent of the calorimeter?

16–16. A 50-gm sample of a material, at a temperature of 100°C, is dropped into a calorimeter containing 200 gm of water initially at 20°C. The calorimeter is of copper and its mass is 100 gm. The final temperature of the calorimeter is 22°C. Compute the specific heat of the sample.

16–17. How much heat is required to convert 1 gm of ice at −10°C to steam at 100°C?

16–18. A beaker whose heat capacity is negligible contains 500 gm of water at a temperature of 80°C. How many grams of ice at a temperature of −20°C must be dropped in the water so that the final temperature of the system will be 50°C?

16–19. Ice cubes at 0°C are dropped into a jug of salt water at 0°C. It is observed that the temperature of the mixture drops below 0°C. What happens to the heat that is released by the salt water when its temperature decreases?

16–20. An open vessel contains 500 gm of ice at −20°C. The heat capacity of the container can be neglected. Heat is supplied to the vessel at the constant rate of 1000 cal/min for 100 min. Plot a curve showing the elapsed time as abscissa and the temperature as ordinate.

16–21. A copper calorimeter of mass 100 gm contains 150 gm of water and 8 gm of ice in thermal equilibrium at atmospheric pressure. 100 gm of lead at a temperature of 200°C are dropped into the calorimeter. Find the final temperature if no heat is lost to the surroundings.

16–22. 500 gm of ice at −16°C are dropped into a calorimeter containing 1000 gm of water at 20°C. The calorimeter can is of copper and has a mass of 278 gm. Compute the final temperature of the system, assuming no heat losses.

16–23. A tube leads from a flask in which water is boiling under atmospheric pressure to a calorimeter. The mass of the calorimeter is 150 gm, its heat capacity is 15 cal/C° and it contains originally 340 gm of water at 15°C. Steam is allowed to condense in the calorimeter until its temperature increases to 71°C, after which the total mass of calorimeter and contents is found to be 525 gm. Compute the heat of condensation of steam from these data.

16–24. An aluminum canteen whose mass is 500 gm contains 750 gm of water and 100 gm of ice. The canteen is dropped from an aircraft to the

ground. After landing, the temperature of the canteen is found to be 25°C. Assuming that no energy is given to the ground in the impact, what was the velocity of the canteen just before it landed?

16–25. A calorimeter contains 500 gm of water and 300 gm of ice, all at a temperature of 0°C. A block of metal of mass 1000 gm is taken from a furnace where its temperature was 240°C and is dropped quickly into the calorimeter. As a result, all the ice is just melted. What would the final temperature of the system have been if the mass of the block had been twice as great? Neglect heat loss from the calorimeter, and the heat capacity of the calorimeter.

16–26. An ice cube whose mass is 50 gm is taken from a refrigerator where its temperature was −10°C and is dropped into a glass of water at 0°C. If no heat is gained or lost from outside, how much water will freeze onto the cube?

16–27. A copper calorimeter can, having a heat capacity of 30 cal/deg, contains 50 gm of ice. The system is initially at 0°C. 12 gm of steam at 100°C and 1 atm pressure are run into the calorimeter. What is the final temperature of the calorimeter and its contents?

16–28. A vessel whose walls are thermally insulated contains 2100 gm of water and 200 gm of ice, all at a temperature of 0°C. The outlet of a tube leading from a boiler, in which water is boiling at atmospheric pressure, is inserted in the water. How many grams of steam must condense to raise the temperature of the system to 20°C? Neglect the heat capacity of the container.

16–29. A 2-kgm iron block is taken from a furnace where its temperature was 650°C and placed on a large block of ice at 0°C. Assuming that all the heat given up by the iron is used to melt the ice, how much ice is melted?

16–30. In a household hot-water heating system, water is delivered to the radiators at 140°F and leaves at 100°F. The system is to be replaced by a steam system in which steam at atmospheric pressure condenses in the radiators, the condensed steam leaving the radiators at 180°F. How many pounds of steam will supply the same heat as was supplied by 1 lb of hot water in the first installation?

16–31. A piece of ice falls from rest into a lake at 0°C, and one-half of one percent of the ice melts. Compute the minimum height from which the ice falls.

16–32. What must be the initial velocity of a lead bullet at a temperature of 25°C, so that the heat developed when it is brought to rest shall be just sufficient to melt it?

16–33. Inspect the following data-sheet, which was prepared for an experiment to measure the heat of vaporization of water:

Mass of empty beaker	50 grams
Mass of filled beaker	150 grams
Mass of water	_____
Initial temperature of water	25°C
Final temperature of water	45°C
Change in temperature of water	_____
Mass of water, beaker, and condensed steam	154 grams
Mass of condensed steam	_____
CALCULATED HEAT OF VAPORIZATION OF WATER	_____

Describe the experiment performed,

then fill in the blank spaces and use the data to calculate the heat of vaporization of water. What would you say about the experimental error?

16–34. A "solar house" has storage facilities for 4 million Btu. Compare the space requirements for this storage on the assumption (a) that the heat is stored in water heated from a minimum temperature of 80°F to a maximum at 120°F, and (b) that the heat is stored in Glauber salt ($Na_2SO_4 \cdot 10\ H_2O$) heated in the same temperature range.

Properties of Glauber salt:

Specific heat (solid)	0.46 Btu/lb·F°
Specific heat (liquid)	0.68 Btu/lb·F°
Specific gravity	1.6
Melting point	90°F
Heat of fusion	104 Btu/lb

CHAPTER 17

TRANSFER OF HEAT

17–1 Conduction. If one end of a metal rod is placed in a flame while the other is held in the hand, that part of the rod one is holding will be felt to become hotter and hotter, although it was not itself in direct contact with the flame. Heat is said to reach the cooler end of the rod by *conduction* along or through the material of the rod. The molecules at the hot end of the rod increase the violence of their vibration as the temperature of the hot end increases. Then, as they collide with their more slowly moving neighbors farther out on the rod, some of their energy of motion is shared with these neighbors and they in turn pass it along to those still farther out from the flame. Hence energy of thermal motion is passed along from one molecule to the next, while each individual molecule remains at its original position.

It is well known that metals are good conductors of electricity and also good conductors of heat. The ability of a metal to conduct an electric current is due to the fact that there are within it so-called "free" electrons, that is, electrons that have become detached from their parent molecules. The free electrons also play a part in the conduction of heat, and the reason metals are such good heat conductors is that the free electrons, as well as the molecules, share in the process of handing on thermal energy from the hotter to the cooler portions of the metal.

Conduction of heat can take place in a body only when different parts of the body are at different temperatures, and the direction of heat flow is always from points of higher to points of lower temperature. The phenomenon of heat flow is sometimes made the basis for the definition of temperature equality or inequality. That is, if heat flows from one body to another when the two are in contact, the temperature of the first, by definition, is higher than that of the second. If there is no heat flow, the temperatures are equal.

Figure 17–1 represents a slab of material of cross section A and thickness L. Let the whole of the left face of the slab be kept at a temperature t_2, and the whole of the right face at a lower temperature t_1. The direction of the heat current is then from left to right through the slab.

After the faces of the slab have been kept at the temperatures t_1 and t_2 for a sufficient length of time, the temperature at points within the slab is found to decrease uniformly with distance from the hot to the cold face. At each point, however, the temperature remains constant with time. The slab is said to be in a "steady state." (Nonsteady state

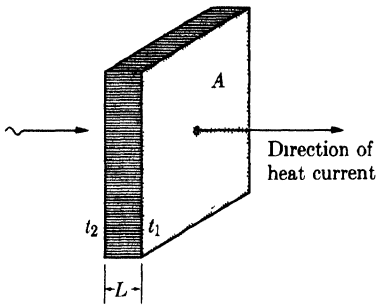


FIG. 17–1. Conduction of heat through a slab.

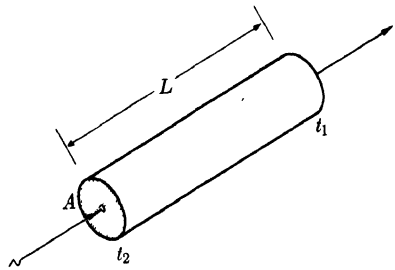


FIG. 17–2. Conduction of heat along a rod.

problems in heat conduction involve mathematical methods beyond the scope of this book.)

It is found by experiment that the rate of flow of heat through the slab in the steady state is proportional to the area A, proportional to the temperature difference $(t_2 - t_1)$, and inversely proportional to the thickness L. Let H represent the quantity of heat flowing through the slab per unit time. Then

$$H \propto \frac{A(t_2 - t_1)}{L}.$$

This proportion may be converted to an equation on multiplication by a constant K whose numerical value depends on the material of the slab. The quantity K is called the *coefficient of thermal conductivity* or simply the thermal conductivity of the material.

$$\boxed{H = \frac{KA(t_2 - t_1)}{L}.} \qquad (17\text{–}1)$$

Equation (17–1) may also be used to compute the rate of heat flow along a rod whose side walls are thermally insulated (see Fig. 17–2, where the letters have the same meaning as in Fig. 17–1).

In some circumstances, either because of nonsteady conditions or because of the geometry of the conductor, the temperature in a body through which heat is flowing does not decrease uniformly along the direction of heat flow as it did in the slab of Fig. 17–1. We can then consider a thin slab of thickness Δx between whose faces the temperature difference is Δt, and Eq. (17–1) becomes

$$H = -KA \frac{\Delta t}{\Delta x}. \qquad (17\text{–}2)$$

TABLE 17-1

THERMAL CONDUCTIVITY

Substance	K, cal/sec·cm·C°
Metals:	
Aluminum	0.49
Brass	0 26
Copper	0.92
Lead	0 083
Mercury	0.020
Silver	0.97
Steel	0.12
Various solids:*	
Firebrick	0.0025
Insulating brick	0 00035
Red brick	0.0015
Concrete	0.002
Cork	0.0001
Felt	0 0001
Glass	0.002
Ice	0.004
Rock wool	0 0001
Wood	0.0003–0.0001
Gases:	
Air	0 000057
Argon	0 000039
Helium	0.00034
Hydrogen	0.00033
Oxygen	0.000056

* Representative values

The minus sign is introduced because if the temperature increases from left to right, the direction of the heat current is from right to left. Equation (17–2) is the general equation of heat conduction. The ratio $\Delta t/\Delta x$ is called the *temperature gradient.* Equation (17–1) evidently relates to a special case in which the temperature gradient is constant and equal to $(t_2 - t_1)/L$.

The cgs unit of rate of heat flow, or heat current, is one calorie per second, and a temperature difference is expressed in celsius degrees. The

unit of thermal conductivity is found from the equation

$$K = \frac{H\,\Delta x}{A\,\Delta t} = \frac{(\text{cal/sec})\cdot\text{cm}}{\text{cm}^2\cdot\text{C}^\circ},$$

or K is expressed in cal/sec·cm·C°. A material for which K is large is therefore a good heat conductor, while if K is small, the material is a poor conductor or a good insulator. There is no such thing as a "perfect heat conductor" ($K = \infty$) or a "perfect heat insulator" ($K = 0$). However, it will be seen from Table 17–1, which lists some representative values of thermal conductivity, that the metals as a group have much greater thermal conductivities than the nonmetals.

17–2 Convection. The term *convection* is applied to the transfer of heat from one place to another by the actual motion of hot material. The hot-air furnace and the hot-water heating system are examples. If the heated material is forced to move by a blower or pump, the process is called *forced convection;* if the material flows due to differences in density, the process is called *natural* or *free* convection. To understand the latter, consider a U-tube as illustrated in Fig. 17–3.

In (a), the water is at the same temperature in both arms of the U and hence stands at the same level in each. In (b), the right side of the U has been heated. The water in this side expands and therefore, being of smaller density, a longer column is needed to balance the pressure produced by the cold water in the left column. The stopcock may now be opened and water will flow from the top of the warmer column into the colder column. This increases the pressure at the bottom of the U produced by the cold column, and decreases the pressure at this point due

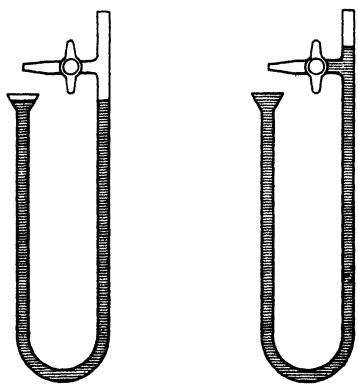


FIG. 17–3. Natural convection is brought about by differences in density.

to the hot column. Hence at the bottom of the U, water is forced from the cold to the hot side. If heat is continually applied to the hot side and removed from the cold side, the circulation continues of itself. The net result is a continual transfer of heat from the hot to the cold side of the column. In the common household hot-water heating system, the "cold" side corresponds to the radiators and the "hot" side to the furnace.

The anomalous expansion of water which was mentioned in Chapter 15 has an important effect on the way in which lakes and ponds freeze in winter. Consider a pond at a temperature of, say, 20°C throughout, and suppose the air temperature at its surface falls to −10°C. The water at the surface becomes cooled to, say, 19°C. It therefore contracts, becomes more dense than the warmer water below it, and sinks in this less dense water, its place being taken by water at 20°C.. The sinking of the cooled water causes a mixing process, which continues until all the water has been cooled to 4°C. Now, however, when the surface water cools to 3°C, it expands, is less dense than the water below it, and hence floats on the surface. Convection and mixing then cease, and the remainder of the water can only lose heat by *conduction*. Since water is an extremely poor heat conductor, cooling takes place very slowly after 4°C is reached, with the result that the pond freezes first at its surface. Then, since the density of ice is even smaller than that of water at 0°C, the ice floats on the water below it, and further freezing can only result from heat flow upward by conduction.

The mathematical theory of heat convection is quite involved. There is no simple equation for convection as in the case of conduction. This arises from the fact that the heat lost or gained by a surface at one temperature in contact with a fluid at another temperature depends on many circumstances, such as

1. Whether the surface is flat or curved.
2. Whether the surface is horizontal or vertical.
3. Whether the fluid in contact with the surface is a gas or a liquid.
4. The density, viscosity, specific heat, and thermal conductivity of the fluid.
5. Whether the velocity of the fluid is small enough to give rise to laminar flow or large enough to cause turbulent flow.
6. Whether evaporation, condensation, or formation of scale takes place.

The procedure adopted in practical calculations is first to define a *convection coefficient* h by means of the equation

$$\boxed{H = hA\,\Delta t,} \qquad (17\text{-}3)$$

TABLE 17-2

COEFFICIENTS OF NATURAL CONVECTION IN AIR AT ATMOSPHERIC PRESSURE

Equipment	Convection coefficient h, cal/sec·cm^2·deg
Horizontal plate, facing upward	$0\ 595 \times 10^{-4}\ (\Delta t)^{1/4}$
Horizontal plate, facing downward	$0\ 314 \times 10^{-4}\ (\Delta t)^{1/4}$
Vertical plate	$0\ 424 \times 10^{-4}\ (\Delta t)^{1/4}$
Horizontal or vertical pipe $\left(\begin{matrix}\text{diameter} \\ D\end{matrix}\right)$	$1\ 00 \times 10^{-4} \left(\frac{\Delta t}{D}\right)^{1/4}$

where H is the heat convection current (the heat gained or lost by convection by a surface per unit of time), A is the area of the surface, and Δt is the temperature difference between the surface and the main body of fluid. The next step is the determination of numerical values of h that are appropriate to a given piece of equipment. Such a determination is accomplished partly by reasoning known as *dimensional analysis* and partly by an elaborate series of experiments. An enormous amount of research in this field has been done in recent years so that, by now, there are in existence fairly complete tables and graphs from which the physicist or engineer may obtain the convection coefficient appropriate to certain standard types of apparatus.

A case of common occurrence is that of natural convection from a wall or a pipe at a constant temperature, surrounded by air at atmospheric pressure and differing in temperature by an amount Δt. The convection coefficients applicable in this situation are given in Table 17-2. As an example of a problem involving natural convection to and from a vertical wall, consider the following.

EXAMPLE. The air in a room is at a temperature of 25°C, and the outside air is at −15°C. How much heat is transferred per unit area of a glass windowpane of thermal conductivity 2.5×10^{-3} cgs unit and of thickness 2 mm?

To assume that the inner surface of the glass is at 25°C and the outer surface is at −15°C is entirely erroneous, as anyone can verify by touching the inner surface of a glass windowpane on a cold day. One must expect a much smaller temperature difference across the windowpane, so that in the steady state, the rates of transfer of heat (1) by convection in the room, (2) by conduction through the glass, and (3) by convection in the outside air, are all equal.

As a first approximation in the solution of this problem, let us assume that the window is at a uniform temperature t. If $t = 5°C$, then the temperature difference between the inside air and the glass is the same as that between the glass and the outside air, or 20 deg. Hence the convection coefficient in both cases is

$$h = 0.424 \times 10^{-4} (20)^{1/4} \frac{\text{cal}}{\text{sec}\cdot\text{cm}^2\cdot\text{deg}}$$

$$= 0.895 \times 10^{-4} \frac{\text{cal}}{\text{sec}\cdot\text{cm}^2\cdot\text{deg}},$$

and, from Eq. (17-4), the heat transferred per unit area is

$$\frac{H}{A} = 0.895 \times 10^{-4} \times 20 = 17.9 \times 10^{-4} \frac{\text{cal}}{\text{sec}\cdot\text{cm}^2}.$$

The glass, however, is not at a uniform temperature; there must be a temperature difference Δt across the glass sufficient to provide heat conduction at the rate of 17.9×10^{-4} cal/sec·cm^2. Using the conduction equation, Eq. (17-1),

$$\Delta t = \frac{L}{K} \times \frac{H}{A} = \frac{0.2}{2.5 \times 10^{-3}} \times 17.9 \times 10^{-4} \text{ deg} = 0.14 \text{ deg}.$$

With sufficient accuracy we may therefore say that the inner surface is at 5.07°C and the outer surface is at 4.93°C.

17-3 Radiation. When one's hand is placed in direct contact with the surface of a hot-water or steam radiator, heat reaches the hand by *conduction* through the radiator walls. If the hand is held above the radiator but not in contact with it, heat reaches the hand by way of the upward-moving *convection* currents of warm air. If the hand is held at one side of the radiator it still becomes warm, even though conduction through the air is negligible and the hand is not in the path of the convection currents. Energy now reaches the hand by *radiation.*

The term radiation refers to the continual emission of energy from the surface of all bodies. This energy is called *radiant energy* and is in the form of electromagnetic waves. These waves travel with the velocity of light and are transmitted through a vacuum as well as through air. (Better, in fact, since they are absorbed by air to some extent.) When they fall on a body which is not transparent to them, such as the surface of one's hand or the walls of the room, they are absorbed and their energy converted to heat.

The radiant energy emitted by a surface, per unit time and per unit area, depends on the nature of the surface and on its temperature. At low temperatures the rate of radiation is small and the radiant energy is chiefly of relatively long wavelength. As the temperature is increased,

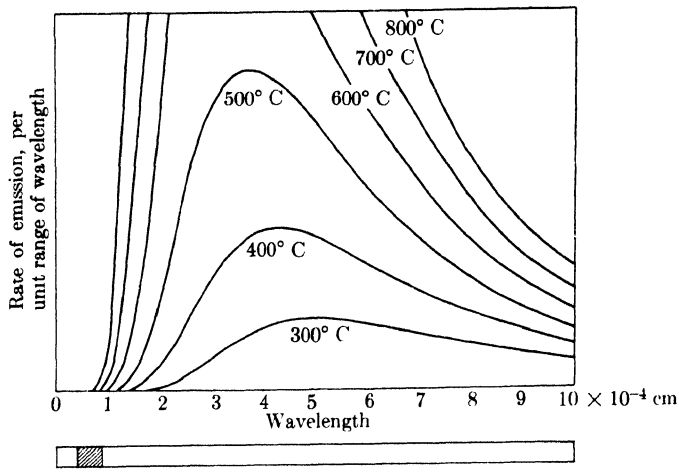


FIG. 17–4. Rate of emission of radiant energy by a blackbody, per unit of wavelength, as a function of wavelength. Shaded area indicates visible spectrum.

the rate of radiation increases very rapidly, in proportion to the 4th power of the absolute temperature. For example, a copper block at a temperature of 100°C (373°K) radiates about 300,000 ergs/sec or 0.03 watt from each square centimeter of its surface. At a temperature of 500°C (773°K) it radiates about 0.54 watt from each square centimeter, and at 1000°C (1273°K) it radiates about 4 watts per square centimeter. This rate is 130 times as great as that at a temperature of 100°C.

At each of these temperatures the radiant energy emitted is a mixture of waves of different wavelengths. At a temperature of 300°C the most intense of these waves has a wavelength of about 5×10^{-4} cm; for wavelengths either greater or less than this value the intensity decreases as shown by the curve in Fig. 17–4. The corresponding distribution of energy at higher temperatures is also shown in the figure. The area between each curve and the horizontal axis represents the total rate of radiation at that temperature. It is evident that this rate increases rapidly with increasing temperature, and also that the wavelength of the most intense wave shifts toward the left, or toward shorter wavelengths with increasing temperature.

At a temperature of 300°C, practically all the radiant energy emitted by a body is carried by waves longer than those corresponding to red light. Such waves are called *infrared*, meaning "beyond the red." At a temperature of 800°C a body emits enough visible radiant energy to be self-luminous and appears "red hot." By far the larger part of the energy emitted, however, is still carried by infrared waves. At 3000°C,

which is about the temperature of an incandescent lamp filament, the radiant energy contains enough of the shorter wavelengths so that the body appears nearly "white hot."

17–4 Stefan's law. Experimental measurements of the rate of emission of radiant energy from the surface of a body were made by John Tyndall (1820–1893) and on the basis of these Josef Stefan (1835–1893), in 1879, concluded that the rate of emission could be expressed by the relation

$$R = e\sigma T^4, \tag{17–4}$$

which is *Stefan's law*. R is the rate of emission of radiant energy per unit area and is expressed in ergs per second per square centimeter, in the cgs system, and in watts per square meter in the mks system. The constant σ has a numerical value of 5.672×10^{-5} in cgs units and 5.672×10^{-8} in mks units. T is the Kelvin temperature of the surface and e is a quantity, sometimes called the *emissivity* or *relative emittance*, which depends on the nature of the surface. Numerical values of e are given in Table 17–3 where it may be seen that e lies between 0 and 1, being small for polished metals and almost 1 for black, rough materials.

TABLE 17–3

EMISSIVITIES OR RELATIVE EMITTANCES OF VARIOUS SURFACES

Material	Temperature range, °C	Emissivity or relative emittance
Polished metals:		
Aluminum	250–600	0.039–0 057
Brass	250–400	0 033–0.037
Copper	100	0 018
Iron	150–1000	0 05–0.37
Filaments:		
Molybdenum	750–2600	0.096–0.29
Platinum	30–1200	0.036–0.19
Tantalum	1300–3000	0.19–0.31
Tungsten	30–3300	0.032–0.35
Other materials:		
Asbestos	40–350	0.93–0.95
Lampblack	20–350	0.95
Rubber	25	0.86

It may be wondered why it is, if the surfaces of all bodies are continually emitting radiant energy, that all bodies do not eventually radiate away all their internal energy and cool down to a temperature of absolute zero [where $R = 0$ by Eq. (17–4)]. The answer is that they would do so if energy were not supplied to them in some way. In the case of a Sunbowl heater element or the filament of an electric lamp, energy is supplied electrically to make up for the energy radiated. As soon as this energy supply is cut off, these bodies do, in fact, cool down very quickly to room temperature. The reason that they do not cool further is that their surroundings (the walls, and other objects in the room) are also radiating, and some of this radiant energy is intercepted, absorbed, and converted into internal energy. The same thing is true of all other objects in the room—each is both emitting and absorbing radiant energy simultaneously. If any object is hotter than its surroundings, its rate of emission will exceed its rate of absorption. There will thus be a net loss of energy and the body will cool down unless heated by some other method. If a body is at a lower temperature than its surroundings, its rate of absorption will be larger than its rate of emission and its temperature will rise. When the body is at the same temperature as its surroundings the two rates become equal, there is no net gain or loss of energy, and no change in temperature.

If a small body of emissivity e at a temperature T is completely surrounded by walls at the same temperature, the rate of *absorption* of radiant energy per unit area by the body is the same as the rate of emission, or

$$R = e\sigma T^4.$$

Hence for such a body at a temperature T_1, surrounded by walls at a temperature T_2, the *net* rate of loss (or gain) of energy per unit area by radiation is

$$R_{\text{net}} = e\sigma T_1^4 - e\sigma T_2^4 = e\sigma(T_1^4 - T_2^4). \tag{17–5}$$

17–5 The ideal radiator. Imagine that the walls of the enclosure in Fig. 17–5 are kept at the temperature T_2 and a number of different bodies having different emissivities are suspended one after another within the enclosure. Regardless of their temperature when they were inserted, it will be found that eventually each comes to the same temperature as that of the walls, T_2, even if the enclosure is evacuated. If the bodies are small compared with the size of the enclosure, radiant energy from the walls strikes the surface of each body at the same rate. Of this energy, a part is reflected and the remainder absorbed. In the absence of any other process, the energy absorbed will raise the temperature of the absorbing body, but since the temperature is observed *not* to change,

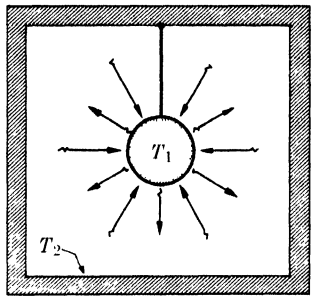


FIG. 17-5. In thermal equilibrium, the rate of emission of radiant energy equals the rate of absorption. Hence a good absorber is a good emitter.

each body must *emit* radiant energy at the same rate as it *absorbs* it. Hence a good absorber is a good emitter, and a poor absorber is a poor emitter. But since each body must either absorb or reflect the radiant energy reaching it, a poor absorber must also be a good reflector. Hence a *good reflector is a poor emitter.*

This is the reason for silvering the walls of vacuum ("Thermos") bottles. A vacuum bottle is constructed with double glass walls, the space between the walls being evacuated so that heat flow by conduction and convection is practically eliminated. To reduce the radiant emission to as low a value as possible, the walls are covered with a coating of silver which is highly reflecting and hence is a very poor emitter.

Since a good absorber is a good emitter, the *best* emitter will be that surface which is the best absorber. But no surface can absorb more than all the radiant energy which strikes it. Any surface which does absorb all the incident energy will be the best emitting surface possible. Such a surface would reflect no radiant energy, and hence would appear black in color (provided its temperature is not so high that it is self-luminous). It is called an *ideally black surface*, and a body having such a surface is called an ideal blackbody, an ideal radiator, or simply a *blackbody.*

No actual surface is ideally black, the closest approach being lampblack, which reflects only about 1%. Blackbody conditions can be closely realized, however, by a small opening in the walls of a closed container. Radiant energy entering the opening is in part absorbed by the interior walls. Of the part reflected, only a very little escapes through the opening, the remainder being eventually absorbed by the walls. Hence the *opening* behaves like an ideal absorber.

Conversely, the radiant energy emitted by the walls or by any body within the enclosure, and escaping through the opening, will, if the walls are of uniform temperature, be of the same nature as that emitted by an ideal radiator. This fact is of importance when using an optical

pyrometer, a device for measuring high temperatures optically. The readings of such an instrument are correct only when it is sighted on a blackbody. If used to measure the temperature of a red-hot ingot of iron in the open, its readings will be too low, since iron is a poorer emitter than a blackbody. If, however, the pyrometer is sighted on the iron while still in the furnace, where it is surrounded by walls at the same temperature, "blackbody conditions" are fulfilled and the reading will be correct. The failure of the iron to emit as effectively as a blackbody is just compensated by the radiant energy which it reflects.

The emissivity e of an ideally black surface is equal to unity. For all real surfaces, it is a fraction, less than one.

Problems

17–1. A slab of a thermal insulator is 100 cm^2 in cross section and 2 cm thick. Its thermal conductivity is 2×10^{-4} cal/sec·cm·C°. If the temperature difference between opposite faces is 100 C°, how many calories flow through the slab in one day?

17–2. One end of a copper bar 18 cm long and 4 cm^2 in cross section is in a steam bath and the other end is in a mixture of melting ice and water. Heat loss across the curved surface can be neglected. (a) What is the heat current in the bar? (b) What is the temperature at a point 4 cm from the cooler end?

17–3. A long rod, insulated to prevent heat losses, has one end immersed in boiling water (at atmospheric pressure) and the other end in a water-ice mixture. The rod consists of 100 cm of copper (one end in steam) and a length, L_2, of steel (one end in ice). Both rods are of cross-sectional area 5 cm^2. The temperature of the copper-iron junction is 60°C, after a steady state has been set up. (a) How many calories per second flow from the steam bath to the ice-water mixture? (b) How long is L_2?

17–4. A rod is initially at a uniform temperature of 0°C throughout. One end is kept at 0°C and the other is brought into contact with a steam bath at 100°C. The surface of the rod is insulated so that heat can flow only lengthwise along the rod. The cross-sectional area of the rod is 2 cm^2, its length is 100 cm, its thermal conductivity is 0.8 cgs units, its density is 10 gm/cm^3, and its specific heat capacity is 0.10 cal/gm·C°. Consider a short cylindrical element of the rod 1 cm in length. (a) If the temperature gradient at one end of this element is 200 C°/cm, how many calories flow across this end per second? (b) If the average temperature of the element is increasing at the rate of 5 C°/sec, what is the temperature gradient at the other end of the element?

17–5. Rods of copper, brass, and steel are welded together to form a Y-shaped figure. The cross-sectional area of each rod is 2 cm^2. The end of the copper rod is maintained at 100°C and the ends of the brass and steel rods at 0°C. Assume there is no heat loss from the surfaces of the rods. The lengths of the rods are: copper,

46 cm; brass, 13 cm; steel, 12 cm. (a) What is the temperature of the junction point? (b) What is the heat current in the copper rod?

17–6. A compound bar 2 meters long is constructed of a solid steel core 1 cm in diameter surrounded by a copper casing whose outside diameter is 2 cm. The outer surface of the bar is thermally insulated and one end is maintained at 100°C, the other at 0°C. (a) Find the total heat current in the bar. (b) What fraction is carried by each material?

17–7. One experimental method of measuring the thermal conductivity of an insulating material is to construct a box of the material and measure the power input to an electric heater, inside the box, which maintains the interior at a measured temperature above that of the outside surface. Suppose that in such an apparatus a power input of 120 watts is required to keep the interior of the box 70 C° above the outside temperature. The total area of the box is 25,000 cm^2 and the wall thickness is 4 cm. Find the thermal conductivity of the material.

17–8. A container of wall area 5000 cm^2 and thickness 2 cm is filled with water in which there is a stirrer. The outer surface of the walls is kept at a constant temperature of 0°C. The thermal conductivity of the walls is 0.000478 cgs units, and the effect of edges and corners can be neglected. The power required to run the stirrer at an angular velocity of 1800 rpm is found to be 100 watts. What will be the final steady-state temperature of the water in the container? Assume that the stirrer keeps the entire mass of water at a uniform temperature.

17–9. A boiler with a steel bottom 1.5 cm thick rests on a hot stove. The area of the bottom of the boiler is 1500 cm^2. The water inside the boiler is at 100°C, and 750 gm are evaporated every 5 minutes. Find the temperature of the lower surface of the boiler, which is in contact with the stove.

17–10. An icebox, having wall area of 2 m^2 and thickness 5 cm, is constructed of insulating material having a thermal conductivity of 10^{-4} cal/sec·cm·C°. The outside temperature is 20°C, and the inside of the box is to be maintained at 5°C by ice. The melted ice leaves the box at a temperature of 15°C. If ice costs one cent per kgm, what will it cost to run the icebox for one hour?

17–11. An electric transformer is in a cylindrical tank 60 cm in diameter and 1 m high, with flat top and bottom. If the tank transfers heat to the air only by natural convection, and the electrical losses are to be dissipated at the rate of 1 kwh, how many degrees will the tank surface rise above room temperature?

17–12. A flat wall is maintained at a constant temperature of 100°C, and the air on both sides is at atmospheric pressure and at 20°C. How much heat is lost by natural convection from 1 sq meter of wall (both sides) in one hour if (a) the wall is vertical, (b) the wall is horizontal?

17–13. A vertical steam pipe of outside diameter 7.5 cm and height 4 meters has its outer surface at the constant temperature of 95°C. The surrounding air is at atmospheric pressure and at 20°C. How much heat is delivered to the air by natural convection in one hour?

17–14. The operating temperature of a tungsten filament in an incandescent lamp is 2450°K and its emissivity is 0.30. Find the surface area of the filament of a 25-watt lamp.

17–15. A blackened solid copper

sphere of radius 2 cm is placed in an evacuated enclosure whose walls are kept at 100°C. At what rate must energy be supplied to the sphere to keep its temperature constant at 127°C?

17–16. A cylindrical metal can 10 cm high and 5 cm in diameter contains liquid helium at 4°K, at which temperature its heat of vaporization is 5 cal/gm. Completely surrounding the helium can are walls maintained at the temperature of liquid nitrogen, 80°K, the intervening space being evacuated. How much helium is lost per hour? Assume the emissivity to be 0.2.

17–17. A solid cylindrical copper rod 10 cm long has one end maintained at a temperature of 20.00°K. The other end is blackened and exposed to thermal radiation from a body at 300°K, no energy being lost or gained elsewhere. When equilibrium is reached, what is the temperature of the blackened end? [*Hint:* Since copper is a very good conductor of heat at low temperature (K = 4 cal/sec·cm·C°), the temperature of the blackened end is only slightly greater than 20°K.]

17–18. An uninsulated steam pipe of diameter 8 cm and absorptivity 0.8 passes vertically through a room in which the air and all solid surfaces are at the average temperature of 27°C. If the surface temperature of the steam pipe is 97°C, compare the rate of heat loss per meter of pipe by radiation with that by natural convection.

CHAPTER 18

THE FIRST LAW OF THERMODYNAMICS

18–1 External work. We have seen that when the temperature of a body is raised, many different effects may take place: gases expand, solids and liquids usually expand but in rare instances they may contract, thermocouples develop an electromotive force, wires undergo a change of resistance, and many other phenomena occur. In such processes there is usually a flow of heat, and also the application of a force which undergoes a displacement, thereby bringing about the performance of work. The study of these phenomena and the energy changes involved in the flow of heat and performance of work constitute a subject known as *thermodynamics.*

Thermodynamics is an intensely practical subject. It concerns itself with a clearly defined *system* (a gas contained in a cylinder, a pound of steam moving through a nozzle, etc.) that is caused to interact directly with its surroundings and, by such interaction, to perform some useful function.

If an external force is exerted on a system by its surroundings, and the point of application of the force is displaced, we say that *external work* is done by the force. Thus when the gas in the cylinder of an automobile engine is compressed, external work is done by the force exerted on it by the moving piston.

Forces acting between parts of the same system are internal forces, and the work of these forces is called *internal work.* [redacted] done by the forces between the molecules or electrons [redacted] is internal work. *Internal work has no place in thermodynamics.*

18–2 Work in changing the volume. Let us consider a system consisting of a fluid contained in a cylinder equipped with a movable piston. Suppose the cylinder has a cross-sectional area A and that the pressure at the piston face is p. The piston therefore exerts an inward force pA on the system. If the piston moves out a small distance Δx, the magnitude of the work ΔW of the force it exerts on the system is

$$\Delta W = pA\,\Delta x.$$

According to the sign conventions used in mechanics, the work would be negative because the (inward) force on the system is opposite to the (outward) displacement of the piston. In thermodynamics it is conven-

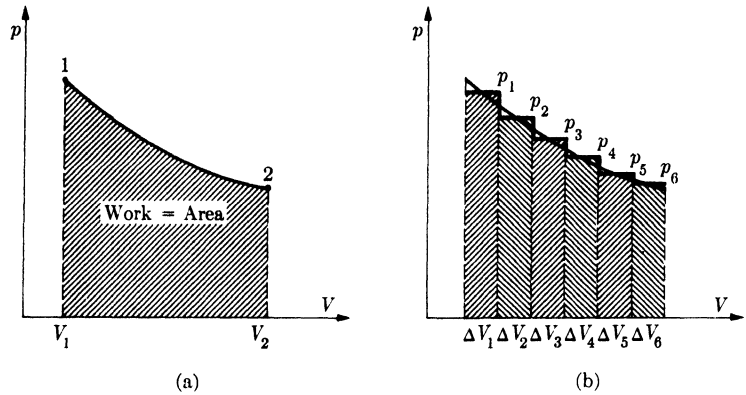


FIG. 18–1. The work done when a substance changes its volume and pressure is the area under the curve on a p-V diagram.

tional to consider work as *positive* when force and displacement are in *opposite* directions. Thus when a system expands against a piston, as in the present example, the work is positive. When a system is compressed, the work is negative.

But $A\,\Delta x = \Delta V$, where ΔV is the change of volume. Therefore

$$\boxed{\Delta W = p\,\Delta V.} \qquad (18\text{–}1)$$

If the pressure remains constant while the volume changes a finite amount, say from V_1 to V_2, then the work W is

$$W = p(V_2 - V_1) \quad \text{(constant pressure only)}. \qquad (18\text{–}2)$$

If, on the other hand, the pressure decreases as the volume increases, then we may imagine that the whole series of changes consists of a small volume change ΔV_1 while the pressure is p_1, then another small volume change ΔV_2 while the pressure is p_2, and so on. The work will then be

$$W = p_1\,\Delta V_1 + p_2\,\Delta V_2 + \cdots.$$

Suppose that the pressure is plotted along the y-axis and the volume along the x-axis. The changes of pressure and volume will then be indicated by a smooth curve such as that shown in Fig. 18–1(a). If the total volume change $V_2 - V_1$ is divided into a number of small steps ΔV_1, ΔV_2, etc., as shown in Fig. 18–1(b), then the smooth curve may be approximated with any desired degree of accuracy by a jagged curve

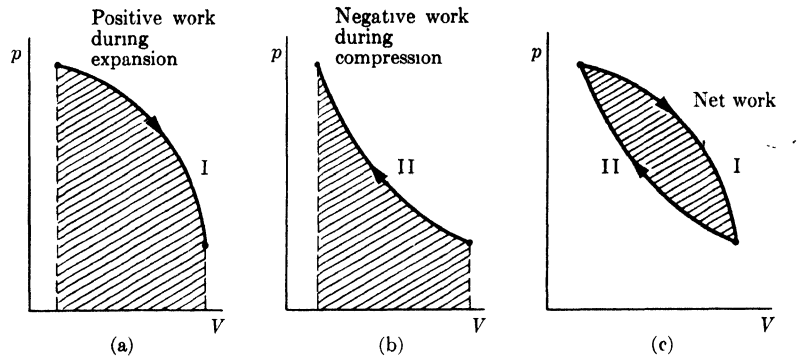


FIG. 18-2. The net work done in a cyclic process equals the area enclosed by the curve representing the process.

where the pressure is p_1 only during the volume change ΔV_1, and is p_2 only during ΔV_2, and so on. The total work, $p_1 \Delta V_1 + p_2 \Delta V_2 + \cdots$, is thus seen to be the sum of the areas of all the rectangles which make up the total area under the jagged curve. If now the number of steps is increased indefinitely, the area under such a jagged curve approaches the area under the smooth curve, and we have the result that

$$W = \begin{Bmatrix}\text{Area under a curve} \\ \text{on a } p\text{-}V \text{ diagram}\end{Bmatrix}. \tag{18-3}$$

Notice that this method of finding the work of a varying pressure is the same as that for finding the work of a varying force, as explained in Section 7-6.

According to the conventions of thermodynamics, the work is positive when a system expands. Thus if a system expands from 1 to 2 in Fig. 18-1(a), the area is regarded as positive. A compression from 2 to 1 would give rise to a negative area.

In Fig. 18-2(a) the pressure and volume changes of a system during expansion are indicated by curve I. The work is indicated by the shaded area under curve I. Similarly, for a compression, the work is represented by the shaded area under curve II in Fig. 18-2(b). In conformity with the sign convention for work, the area under I is positive, and that under II is negative. In Fig. 18-2(c), curves I and II are drawn together so that they constitute a series of processes whereby the system is brought back to its initial state. Such a series of processes, represented by a closed figure, is called a *cycle*. The area within the closed figure is obviously the difference between the areas under curves I and II and therefore

represents the *net* work done in the cycle. It should be noticed that the cycle is traversed in such a direction that the net work is positive. If the direction were reversed, the net work would be negative.

18–3 Work depends on path. On the p-V diagram depicted in Fig. 18–3 an initial state 1 (characterized by pressure p_1 and volume V_1) and a final state 2 (characterized by pressure p_2 and volume V_2) are represented by the two points 1 and 2. There are many ways in which the system may be taken from 1 to 2. For example, the pressure may be kept constant from 1 to 3 (isobaric process) and then the volume kept constant from 3 to 2 (isochoric process), in which case the work done is equal to the area under the line $1 \rightarrow 3$. Another possibility is the path $1 \rightarrow 4 \rightarrow 2$, in which case the work is the area under the line $4 \rightarrow 2$. The jagged line and the continuous curve from 1 to 2 represent other possibilities, in each of which the work done is different. We can see, therefore, that *the work depends not only on the initial and final states but also on the intermediate states, i.e., on the path.*

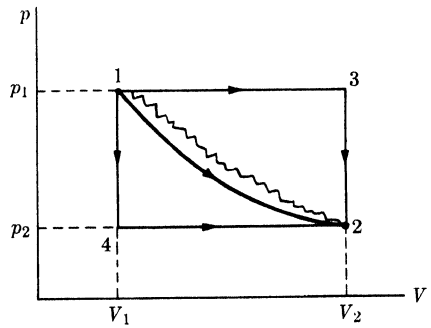


FIG. 18–3. Work depends on the path.

18–4 Work and heat. We have seen how a system may be caused to go from state 1 to state 2 by any number of processes, all of which involve the performance of work. There are other means, however, of changing the state of a system. Consider, for example, the four situations depicted in Fig. 18–4. In (a) the system is a composite one consisting of some water and a paddle wheel, which is caused to rotate and churn the water by means of a falling weight. In (b) both the water and an electric resistance wire embedded in the water constitute the system. An electric current in the resistor is maintained by a generator turned by means of a falling weight. In both cases the state of the system is caused to change and, since the agency for changing the state of the system is a falling weight, *both processes involve the performance of work.*

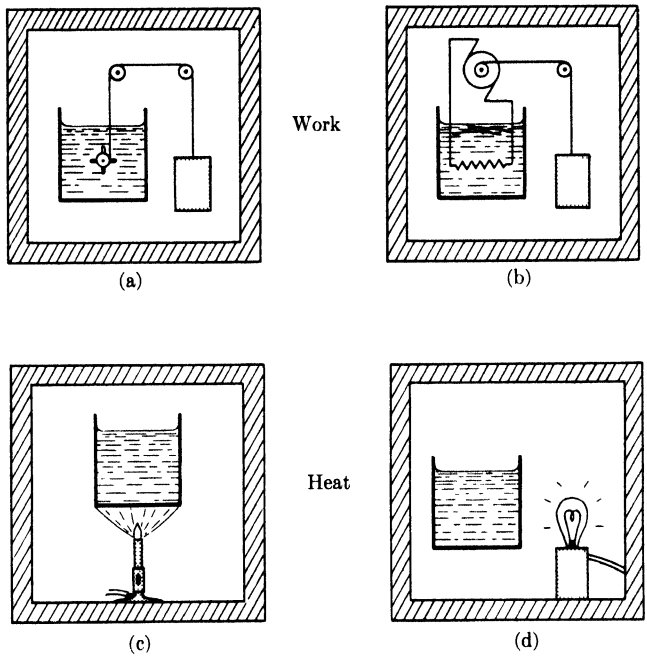


FIG. 18–4. Distinction between work and heat.

In (c) and (d), however, the situation is quite different. The system in both cases is some water in a heat-conducting container. In (c) the system is in contact with the burning gases from a Bunsen burner, i.e., with another body at a higher temperature, whereas in (d) the system is near but not in contact with an electric lamp whose temperature is much higher than that of the water. In both cases the state of the system is caused to change, but in neither case can the agency for the change be described by mechanical means. *In these cases there is a flow of heat.*

It is important to observe that the decision as to whether a particular change of state involves the performance of work or the transfer of heat requires first an unequivocal answer to the questions "What is the system?" and "What are the surroundings?" For example, in Fig. 18–4(b), if the resistor is regarded as the system and the water as the surroundings, then there is a transfer of heat from the resistor by virtue of the temperature difference between the resistor and the water. Also, if a small part of the water is regarded as the system, the rest of the water being the surroundings, then again there is a transfer of heat. Regarding, however, the composite system composed of both the water and the resistor, the surroundings do not contain any object whose temperature

differs from that of the system, and hence no heat is transferred between *this composite system* and its surroundings.

The flow of heat is a *nonmechanical energy transfer* brought about by a temperature difference between two bodies, and we say that heat flows from one place to another. When the flow has ceased, there is no longer any occasion to use the word heat. The performance of work and the flow of heat are methods of supplying energy to a body or extracting energy from a body. We have seen that, in general, the work depends on the path by which the system was brought from the initial to the final state. Exactly the same is true of the heat transferred to or from a body. *Heat also depends on the path.*

It would be just as incorrect to refer to the "heat in a body" as it would be to speak about the "work in a body." For suppose we assigned an arbitrary value to "the heat in a body" in some standard reference state. The "heat in the body" in some other state would then equal the "heat" in the reference state plus the heat added when the body is carried to the second state. But the heat added depends entirely on the path by which we go from one state to the other, and since there are an infinite number of paths which might be followed, there are an infinite number of values which might equally well be assigned to the "heat in the body" in the second state. Since it is not possible to assign any one value to the "heat in the body" we conclude that this concept is meaningless, or at any rate useless.

18–5 The first law of thermodynamics. Suppose a system is caused to change from state 1 to state 2 along a definite path and that the heat absorbed Q and the work W are measured. Expressing both Q and W either in thermal units or in mechanical units we may then calculate the difference $Q - W$. If now we do the same thing over again for many different paths (between the same states 1 and 2), the important result is obtained that $Q - W$ *is the same for all paths connecting* 1 *and* 2. But Q is the energy that has been added to a system by the transfer of heat and W is equal to the energy that has been extracted from the system by the performance of work. The difference $Q - W$, therefore, must represent the internal energy change of the system. It follows that *the internal energy change of a system is independent of the path*, and is therefore equal to the energy of the system in state 2 minus the energy in state 1, or $U_2 - U_1$.

$$U_2 - U_1 = Q - W.$$

If some arbitrary value is assigned to the internal energy in some standard reference state, its value in any other state is uniquely defined, since $Q - W$ is the same for all processes connecting the states.

The simple algebraic statement of these facts in the form

$$\boxed{Q = U_2 - U_1 + W} \tag{18–4}$$

is known as *the first law of thermodynamics.* In applying the law in this form it must be remembered that (1) all quantities must be expressed in the same units, (2) Q is positive when heat goes into the system, (3) W is positive when the force *on* the system is opposite the displacement.

18–6 Adiabatic process. A process that takes place in such a manner that no heat enters or leaves a system is called an *adiabatic process.* This may be accomplished either by surrounding the system with a thick layer of heat insulating material (such as cork, asbestos, firebrick, or any light, porous powder) or by performing the process quickly. The flow of heat is a fairly slow process, so that any process performed quickly enough will be practically adiabatic. Applying the first law to an adiabatic process, we get

$$U_2 - U_1 = -W \quad \text{(Adiabatic process).}$$

Thus the change in the internal energy of a system, in an adiabatic process, is equal in absolute magnitude to the work. If the work W is negative, as when a system is compressed, then $-W$ is positive, U_2 is greater than U_1, and the internal energy of the system increases. If W is positive, as when a system expands, the internal energy of the system decreases. An increase of internal energy is usually accompanied by a rise in temperature and a decrease in internal energy by a temperature drop.

The compression of the mixture of gasoline vapor and air that takes place during the compression stroke of a gasoline engine is an example of an approximately adiabatic process involving a temperature rise. The expansion of the combustion products during the power stroke of the engine is an approximately adiabatic process involving a temperature decrease. Adiabatic processes, therefore, play a very important role in mechanical engineering.

18–7 Isochoric process. If a substance undergoes a process in which the volume remains unchanged, the process is called *isochoric.* The rise of pressure and temperature produced by a flow of heat into a substance contained in a nonexpanding chamber is an example of an isochoric process. If the volume does not change, no work is done and, therefore, from the first law,

$$Q = U_2 - U_1 \quad \text{(Isochoric process),}$$

or all the heat that has been added has served to increase the internal energy. The very sudden increase of temperature and pressure accompanying the explosion of gasoline vapor and air in a gasoline engine may be treated mathematically as though it were an isochoric addition of heat.

18–8 Isobaric process. A process taking place at constant pressure is called an *isobaric process.* When water enters the boiler of a steam engine and is heated to its boiling point, vaporized, and then the steam is superheated, all these processes take place isobarically. Such processes play an important role in mechanical engineering and also in chemistry.

Consider the change of phase of a mass m of liquid to vapor at constant pressure and temperature. If V_L is the volume of liquid and V_V the volume of vapor, the work done in expanding from V_L to V_V at constant pressure p is

$$W = p(V_V - V_L).$$

The heat absorbed by each unit of mass is the heat of vaporization L. Hence

$$Q = mL.$$

From the first law,

$$mL = (U_V - U_L) + p(V_V - V_L). \tag{18–5}$$

EXAMPLE. One gram of water (1 cm^3) becomes 1671 cm^3 of steam when boiled at a pressure of 1 atm. The heat of vaporization at this pressure is 539 cal/gm. Compute the external work and the increase in internal energy.

$$\begin{aligned} \text{External work} &= p(V_V - V_L) \\ &= 1.013 \times 10^6 \text{ dynes/cm}^2\ (1671 - 1)\ \text{cm}^3 \\ &= 1.695 \times 10^9 \text{ ergs} \\ &= 169.5 \text{ joules} \\ &= 41 \text{ calories.} \end{aligned}$$

From Eq. (18–5)

$$U_V - U_L = mL - W = 539 - 41 = 498 \text{ calories.}$$

Hence the external work, or the external part of the heat of vaporization, equals 41 calories, and the increase in internal energy, or the internal part of the heat of vaporization, is 498 calories.

18–9 Free expansion. Imagine a vessel with rigid walls and covered with asbestos. Suppose the vessel is divided into two parts by a thin partition, and that one part contains a gas while the other is evacuated. If the partition is suddenly broken, the gas rushes into a vacuum, under-

going what is known as a *free expansion.* Since the walls of the container are rigid, no external work is done, and since the vessel is heat-insulated, the process is adiabatic. Thus, $Q = 0$ and $W = 0$. Hence, from the first law,

$$U_1 = U_2 \quad \text{(Free expansion)},$$

or the initial and final internal energies are equal.

The magnitude of the temperature change (if any) that takes place as a result of a free expansion is of some theoretical interest and, as a result, many attempts have been made to perform an experiment of this sort. The experimental difficulties, however, are enormous and no one has really succeeded so far. From a practical point of view a free expansion is of no importance whatever.

18–10 Throttling process. A throttling process is one in which a fluid, originally at a constant high pressure, seeps through a porous wall or a narrow opening (needle valve or throttling valve) into a region of constant lower pressure, without a transfer of heat taking place. The experiment is sometimes called the porous plug experiment. Figure 18–5(a) will help to make the process clear. A fluid is discharged from a pump at a high pressure, then passes through a throttling valve into a pipe which leads directly to the intake or low-pressure side of the pump. Every successive element of fluid undergoes the throttling process in a continuous stream.

Consider any element of fluid enclosed between the piston and throttling valve of Fig. 18–5(b). Suppose this piston to move toward the right and another piston on the other side of the valve to move to the right also at such rates that the pressure on the left remains at a constant high value and that on the right at a constant lower value. After all the fluid

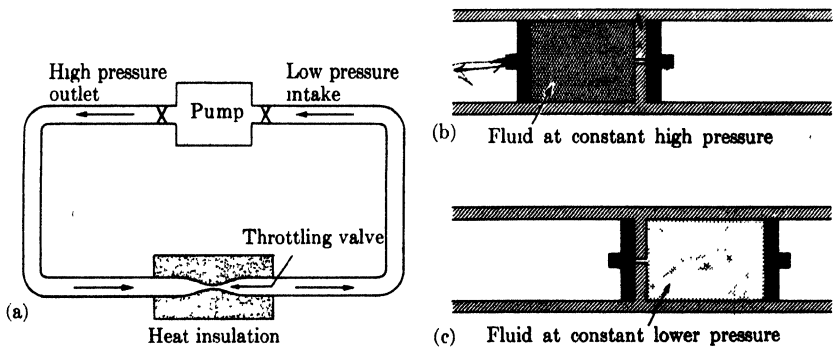


FIG. 18–5. Throttling process.

has been forced through the valve, the final state is that of Fig. 18–5(c).

The net work done in this process is the difference between the work done in forcing the right-hand piston out and the work done in forcing the left-hand piston in. Let

p_1 = high pressure (on the left),
V_1 = volume of fluid at the high pressure,
p_2 = lower pressure (on the right),
V_2 = volume of fluid at the low pressure.

Since the low-pressure fluid changes in volume from zero to V_2 at the constant pressure p_2, the work is

$$p_2(V_2 - 0),$$

and since the high-pressure fluid changes in volume from V_1 to zero at the constant high pressure p_1, the work is

$$p_1(0 - V_1).$$

The net work W is therefore

$$W = p_2V_2 - p_1V_1.$$

Since the process is adiabatic, $Q = 0$, and hence from the first law,

$$0 = U_2 - U_1 + (p_2V_2 - p_1V_1)$$

or

$$U_1 + p_1V_1 = U_2 + p_2V_2.$$

This result is of great importance in steam engineering and in refrigeration. The sum $U + pV$, called the *enthalpy*, is tabulated for steam and for many refrigerants. The throttling process plays the main role in the action of a refrigerator, since this is the process that gives rise to the drop in temperature needed for refrigeration. Liquids that are about to evaporate (saturated liquids) always undergo a drop in temperature and partial vaporization as a result of a throttling process. Gases, however, may undergo a temperature rise or drop depending on the initial temperature and pressure and on the final pressure.

Problems

18–1. A pound of water whose volume is 0.0190 ft^3 is converted into steam of volume 1.50 ft^3 at the constant pressure of 309 lb/in^2. How many foot-pounds of work are done?

18–2. A gas contained in a cylinder surrounded by a thick layer of felt is quickly compressed, the temperature rising several degrees. (a) Has there been a transfer of heat? (b) Has work been done?

18–3. A combustion experiment is performed by burning a mixture of fuel and oxygen in a constant-volume "bomb" surrounded by a water bath. During the experiment the temperature of the water is observed to rise. Regarding the mixture of fuel and oxygen as the system, (a) Has heat been transferred? (b) Has work been done? (c) What is the sign of ΔU?

18–4. A liquid is irregularly stirred in a well-insulated container and thereby undergoes a rise in temperature. Regarding the liquid as the system, (a) Has heat been transferred? (b) Has work been done? (c) What is the sign of ΔU?

18–5. A resistor, immersed in running water, carries an electric current. Consider the resistor as the system under consideration. (a) Is there a flow of heat into the resistor? (b) Is there a flow of heat into the water? (c) Is work done? (d) Assuming the state of the resistor to remain unchanged, apply the first law to this process.

18–6. In a certain process, 500 cal of heat are supplied to a system, and at the same time 100 joules of work are done. What is the increase in its internal energy?

18–7. 200 Btu are supplied to a system in a certain process, and at the same time the system expands against a constant external pressure of 100 lb/in^2. The internal energy of the system is the same at the beginning and end of the process. Find the increase in volume of the system.

18–8. A substance undergoes a series of processes which bring it back to its initial state. In this cycle, heat Q_2 is absorbed by the substance and heat Q_1 is rejected. What is the net amount of work done?

18–9. An inventor claims to have developed an engine which takes in 100,000 Btu from its fuel supply, rejects 25,000 Btu in the exhaust, and delivers 25 kwh of mechanical work. Do you advise investing money to put this engine on the market?

18–10. A vessel with rigid walls and covered with asbestos is divided into two parts by an insulating partition. One part contains a gas at temperature T and pressure P. The other part contains a gas at temperature T' and pressure P'. The partition is removed. What conclusion may be drawn by applying the first law of thermodynamics?

18–11. A mixture of hydrogen and oxygen is enclosed in a rigid insulating container and exploded by a spark. The temperature and pressure both increase considerably. Neglecting the small amount of energy provided by the spark itself, what conclusion may be drawn by applying the first law of thermodynamics?

18–12. When water is boiled under a pressure of 2 atm, the heat of vaporization is 946 Btu/lb and the boiling point is 250°F. One lb of steam occupies a volume of 14 ft^3, and 1 lb of water a volume of 0.017 ft^3. (a) Compute the external work, in ft-lb and in

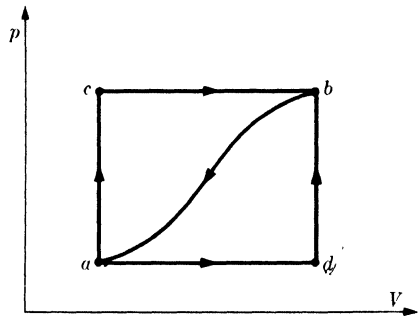


FIGURE 18-6

Btu, when 1 lb of steam is formed at this temperature. (b) Compute the increase in internal energy, in Btu.

18–13. When a system is taken from state *a* to state *b*, in Fig. 18–6, along the path *acb*, 80 Btu of heat flow into the system, and 30 Btu of work are done. (a) How much heat flows into the system along path *adb* if the work is 10 Btu? (b) When the system is returned from *b* to *a* along the curved path, the work is 20 Btu. Does the system absorb or liberate heat and how much? (c) If $U_a = 0$ and $U_d = 40$ Btu, find the heat absorbed in the processes *ad* and *db*.

18–14. A steel cylinder of cross-sectional area 0.1 ft^2 contains 0.4 ft^3 of glycerin. The cylinder is equipped with a tightly fitting piston which supports a load of 6000 lb. The temperature of the system is increased from 60°F to 160°F. Neglect the expansion of the steel cylinder. Find (a) the increase in volume of the glycerin, (b) the mechanical work of the 6000-lb force. (c) the amount of heat added to the glycerin (specific heat of glycerin = 0.58 Btu/lb·F°), (d) the change in internal energy of the glycerin.

CHAPTER 19

THERMAL PROPERTIES OF SOLIDS, LIQUIDS, AND GASES

19–1 The ideal gas law. We know that the atoms of a gas are much more widely separated than are those of a liquid or solid; hence the forces between atoms are of less consequence and the behavior of a gas is governed by simpler laws than those applying to liquids and solids.

Robert Boyle, in 1660, reported on one of the first quantitative experiments relating to gaseous behavior. He found that if the temperature of a fixed mass of gas was held constant while its volume was varied over wide limits, the pressure exerted by the gas varied also, and in such a way that the product of pressure and volume remained approximately constant. This relation is known as *Boyle's law.* If p stands for the *absolute* pressure and V for the volume of any constant mass of gas, then Boyle's law may be written

$$pV = \text{constant} \left\{ \begin{matrix} \text{constant temperature} \\ \text{constant mass} \end{matrix} \right\}. \qquad (19\text{–}1)$$

Many measurements of the pressure and volume of gases carried out since Boyle's time have shown that, at a temperature near that at which the gas condenses into a liquid, the product pV varies considerably. Only at temperatures far above the condensation point is Boyle's law obeyed. In the case of oxygen, nitrogen, air, hydrogen, and helium, Boyle's law is obeyed quite well in the neighborhood of room temperature and atmospheric pressure.

In describing the behavior of gases, it is very useful to use the *gram-mole* as a unit of mass. *A gram-mole of gas is M grams where M is the molecular weight.* Thus, a mole of hydrogen, H_2, is 2 gm, a mole of oxygen, O_2, is 32 gm, etc. Suppose the pressure p, volume V, and the number of moles n of a gas are measured at constant temperature and the product pv (v is the volume per mole, V/n, or *molar volume*) is calculated. The remarkable property of gases that makes them so valuable in thermometry is displayed in Fig. 19–1 where the product pv is plotted against p for four different gases, all at the temperature of boiling sulfur in the top graph, all at the temperature of boiling water in the one beneath, all at the triple point of water in the next lower graph, and all at the temperature of solid CO_2 in the lowest. In each case it is seen that, as the pressure approaches zero, *the product pv approaches the same value for all gases at the same temperature.*

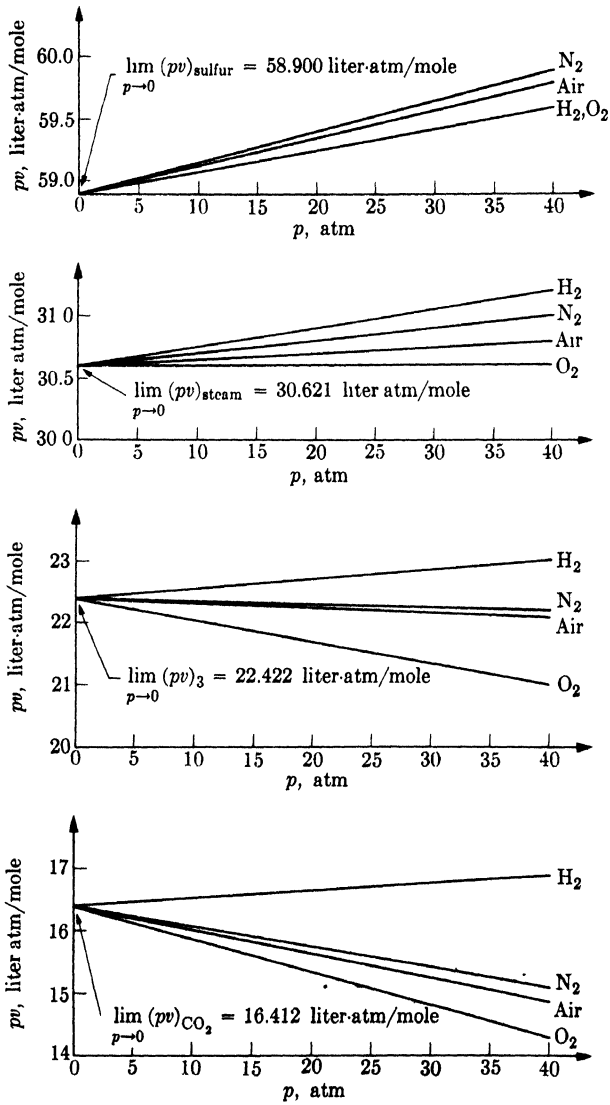


FIG. 19–1. The fundamental property of gases is that $\lim_{p\to 0} (pv)_T$ is independent of the nature of the gas and depends only on T.

According to the rules developed in Chapter 15, the kelvin temperature divided by $T_3 (T_3 = 273.16°\text{K})$ is obtained by taking the limiting value, as the pressure approaches zero, of the ratio p/p_3, *no matter what gas is chosen.* We have, therefore, two fundamental properties that hold *for all gases in the limit of low pressures,* thus:

$$pv = \text{constant} \qquad \text{(constant temperature)}, \tag{19–2}$$

$$\frac{p}{p_3} = \frac{T}{T_3} \qquad \text{(constant volume)}. \tag{19–3}$$

In Fig. 19–2, the relation between p and v at small values of p (Boyle's law) is plotted for two different temperatures: an arbitrary temperature T and the triple point temperature T_3.

At temperature T,

$$pv = k,$$

where k is a constant.

At temperature T_3,

$$(pv)_3 = k_3,$$

where k_3 is another constant.

Dividing the first equation by the second, we get that

$$\frac{pv}{(pv)_3} = \frac{k}{k_3} = \text{constant}, \tag{19–4}$$

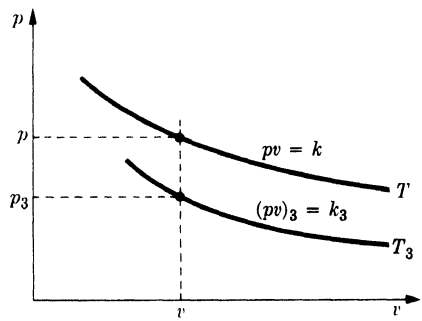


FIG. 19–2. The relation between p and v at small pressures and at two different temperatures.

which must hold for *any two points,* one on the upper curve and the other on the lower. If we apply this result to the two points shown in Fig. 19–2 *on the same constant volume line,* Eq. (19–4) reduces to

$$\frac{p}{p_3} = \frac{k}{k_3} \qquad \text{(constant } v\text{)}.$$

But p/p_3 is none other than the ratio T/T_3 as given in Eq. (19–3). It follows therefore that

$$\frac{k}{k_3} = \frac{T}{T_3},$$

and after substitution back into Eq. (19–4),

$$\frac{pv}{(pv)_3} = \frac{T}{T_3}.$$

Therefore, for all gases in the limit of low pressures,

$$pv = \left[\frac{(pv)_3}{273.16°}\right] \cdot T. \tag{19–5}$$

The expression in brackets is called the *universal gas constant* and is denoted by R. The best value obtained so far for the limiting value of $(pv)_3$ is 22.4216 liters·atm/mole, or 2271.87 joules/mole. Hence

$$R = \frac{22.4216 \text{ liters·atm/mole}}{273.16°\text{K}} = \frac{2271.87 \text{ joules/mole}}{273.16°\text{K}},$$

$$R = \begin{cases} 0.08208 \text{ liter·atm/mole·K°}, \\ 8.317 \times 10^7 \text{ ergs/mole·K°}, \\ 8.317 \text{ joules/mole·K°}, \\ 1.987 \text{ cal/mole·K°}. \end{cases}$$

We may now write Eq. (19–5)

$$pv = RT,$$

and replacing the molar volume v by the ratio of the total volume V to the number of moles n, we get finally

$$\boxed{pV = nRT.} \tag{19–6}$$

Equation (19–6) expresses the behavior of real gases only at *low* pressures where the forces of attraction among the molecules are very weak. If there were no forces whatever among the molecules, the equation would hold exactly. It is found useful to define an *ideal gas* as one that would obey Eq. (19–6) at *all* pressures, and hence this equation is called the *ideal gas law.*

19–2 Properties of an ideal gas. The relation between the pressure and molar volume of an ideal gas at constant temperature is shown by the curves of Fig. 19–2. The curves are equilateral hyperbolas, asymptotic to the p- and v-axes. Each curve corresponds to a different temperature. That is, while pv = a constant at any one temperature, the constant is larger the higher the temperature.

Each of the curves in Fig. 19–2 can be considered to represent a *process* through which the gas is carried, for example, a process in which the gas is compressed from a large to a small volume in a bicycle pump. We shall show later that to keep the temperature constant in such a compression it would be necessary to remove heat from the gas, and therefore the process would have to be carried out slowly in order to keep

the temperature the same throughout the gas. Such a slow compression carries the gas through a series of states, each of which is very nearly an equilibrium state, and it is called a *quasi-static*, or a "nearly static" process. Unless stated otherwise, we shall assume in what follows that all processes are to be carried out in this way, so that at each stage of the process the gas is, for all practical purposes, in an equilibrium state.

Any process in which the temperature remains constant is called *isothermal*, and the curves in Fig. 19–2 are the isothermal curves or, more briefly, the *isotherms* of an ideal gas.

The first accurate statement of the law connecting the volume changes of a gas with changes in its temperature was published by Joseph Louis Gay-Lussac in 1802. Earlier work on the subject had been carried on by many other investigators, among them Jacques A. C. Charles, whose name is often associated with Gay-Lussac's in connection with the relation between volume and temperature of a gas at constant pressure. The proportionality between volume v and kelvin temperature T follows from the ideal gas law, Eq. (19–6). Thus

$$v = \text{constant } T \qquad (\text{constant } p) \tag{19–7}$$

is the mathematical statement of Gay-Lussac's law. A graph of this law (not to scale) is shown in Fig. 19–3 for three different pressures. If an actual gas behaved like an ideal gas at very low temperatures, the straight lines shown in Fig. 19–3 would remain straight and pass through the origin. The absolute zero of temperature could then be defined as the temperature at which the volume of a gas held at constant pressure shrinks to zero! The stubborn fact remains, however, that at some low temperature, a gas maintained at constant pressure becomes a liquid, and Gay-Lussac's law breaks down.

Any process in which the pressure remains constant is called *isobaric*, and the lines in Fig. 19–3 are the isobaric lines or, more briefly, the *isobars* of an ideal gas.

The number of moles n in a sample of gas equals the mass m of the gas divided by its molecular weight. If the latter is represented by M, then $n = m/M$. Hence we may write,

$$pV = m\frac{R}{M}T.$$

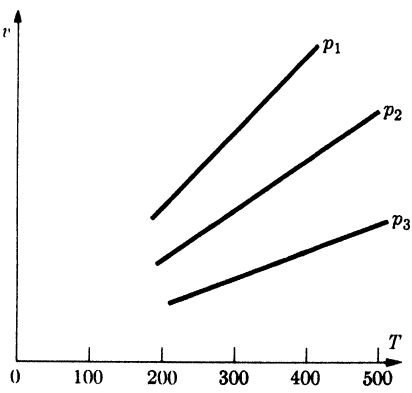


FIG. 19–3. The volume of an ideal gas is proportional to the kelvin temperature at constant pressure (Gay-Lussac's law).

TABLE 19–1

Gas	Density, gm/cm^3 at 1 atm, 0°C
Air	1.2929×10^{-3}
Argon	1.7832×10^{-3}
Carbon dioxide	$1\ 9769 \times 10^{-3}$
Helium	$0\ 1785 \times 10^{-3}$
Hydrogen	$0\ 0899 \times 10^{-3}$
Nitrogen	1.2506×10^{-3}
Oxygen	$1\ 4290 \times 10^{-3}$

Dividing both sides by V and remembering that the definition of the density ρ is m/V, we get

$$\rho = \frac{Mp}{RT}. \tag{19–8}$$

The density of a gas is seen to depend on its pressure and temperature as well as on its molecular weight. Hence in tabulating gas densities the pressure and temperature must be specified. The densities of a few common gases are listed in Table 19–1.

In numerical calculation on gases which may be assumed to be ideal, it is often necessary to compute a final value of p, V, T, or n in terms of given values. For this purpose we note that the quantity pV/nT has the constant value R at all stages of any process through which the gas may go. Hence if subscripts 1 and 2 refer to any two states,

$$\boxed{\frac{p_1V_1}{n_1T_1} = \frac{p_2V_2}{n_2T_2}.} \tag{19–9}$$

In this, as in all forms of the ideal gas equation, the temperature *must* be expressed on either the kelvin scale or on the rankine scale. A great advantage of Eq. (19–9) lies in the fact that no special units of p and V are needed, provided *both* p_1 and p_2 are expressed in the same units, and V_1 and V_2 are expressed in the same units.

EXAMPLE 1. How many kilograms of O_2 are contained in a tank whose volume is 2 ft^3 when the gauge pressure is 2000 lb/in^2 and the temperature is 27°C?

Assume the ideal gas laws to hold. The molecular weight of oxygen is 32 gm/mole.

$$2 \text{ ft}^3 = 2 \times 28.3 = 56.6 \text{ liters.}$$

$$p_{\text{abs}} = 2015 \frac{\text{lb}}{\text{in}^2} = \frac{2015}{14.7} = 137 \text{ atm.}$$

$$T = t + 273 = 300°\text{K.}$$

Hence, from Eq. (19–6),

$$n = \frac{pV}{RT} = \frac{137 \text{ atm} \times 56.6 \text{ liters}}{0.082 \text{ liter·atm/mole·K°} \times 300°\text{K}} = 315 \text{ moles.}$$

$$m = 315 \times 32 = 10{,}100 \text{ gm} = 10.1 \text{ kgm.}$$

EXAMPLE 2. What volume would be occupied by this gas if it were allowed to expand to atmospheric pressure at a temperature of 50°C?

Since we are dealing with a fixed mass of gas, we may write

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2},$$

$$V_2 = V_1 \frac{p_1}{p_2} \frac{T_2}{T_1} = 2 \times \frac{137}{1} \times \frac{323}{300} = 295 \text{ ft}^3.$$

19–3 Molecular theory of matter. The fact that matter is compressible suggests that all substances have a granular or spongy structure with spaces into which the granules can penetrate when the external pressure is increased. The ease with which liquids can flow and gases can diffuse points more to a collection of tiny particles than to a spongelike structure. Thousands of physical and chemical facts support the contention that matter in all phases is composed of tiny particles called *molecules*. The molecules of any one substance are identical. They have the same structure, the same mass, and the same mechanical and electrical properties. Many of the large-scale properties of matter which have been discussed heretofore, such as elasticity, surface tension, condensation, vaporization, etc., can be comprehended with deeper understanding of their significance in terms of the molecular theory. For this purpose we may conceive of a molecule as a rigid sphere, like a small billiard ball, capable of moving, of colliding with other molecules or with a wall, and of exerting attractive or repulsive forces on neighboring molecules. In other parts of physics and chemistry it is important to consider the structure of the molecule, but this is not necessary at this point.

One of the outstanding characteristics of a molecule is the force that exists between it and a neighbor. There is, of course, a force of gravitational attraction between every pair of molecules, but it turns out that

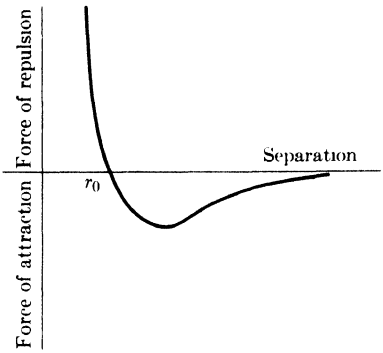


FIG. 19–4. The force between two molecules changes from an attraction, when the separation is large, to a repulsion when the separation is small.

this is negligible in comparison with the forces we are now considering. The forces that hold the molecules of a liquid (or solid) together are, in part at least, of electrical origin and do not follow a simple inverse square law. When the separation of the molecules is large, as in a gas, the force is extremely small and is an attraction. The attractive force increases as a gas is compressed and its molecules brought closer together. But since tremendous pressures are needed to compress a liquid, i.e., to force its molecules closer together than their normal spacing in the liquid state, we conclude that at separations only slightly less than the dimensions of a molecule the force is one of repulsion and is relatively large. The force must then vary with separation in somewhat the fashion shown in Fig. 19–4. At large separations the force is one of attraction but is extremely small. As the molecules are brought closer together the force of attraction becomes larger, passes through a maximum, and then decreases to zero at a separation r_0. When the distance between the molecules is less than r_0, the force is one of repulsion.

A single pair of molecules could remain in equilibrium at a center-to-center spacing equal to r_0 in Fig. 19–4. If they were separated slightly, the force between them would be attractive and they would be drawn together. If they were forced closer together than the distance r_0, the force would be one of repulsion and they would spring apart. If they were either pulled apart or pushed together, and then released, they would oscillate about their equilibrium separation r_0.

If the only property of a molecule were the force of attraction between it and its neighbors, all matter would eventually coalesce into the liquid or solid phase. The existence of gases points to another property which enables molecules to stay apart. This is accomplished by molecular motion. The more vigorous the motion the less chance there is for con-

TABLE 19–2

Phase	Attractive forces among molecules	Molecular kinetic energy	Temperature
Solid	Strong	Small	Low
Liquid	Moderate	Moderate	Medium
Gas	Weak	Large	High

densation into the liquid or solid phase. In solids the molecules execute vibratory motion about more or less fixed centers. This vibratory motion is relatively weak, and the centers remain fixed at regularly spaced positions which comprise a *space lattice*. This gives rise to the extraordinary regularity and symmetry of crystals. In liquids the molecules execute vibratory motion of greater energy about centers which are free to move but which remain at approximately the same distance from one another. The molecules of gases have the greatest kinetic energy. The motion is linear until collision takes place either with another molecule or with a wall. The average distance between molecules of a gas is so great that only small attractive forces exist.

A common substance like water exists in the solid phase at low temperatures. When the temperature is raised beyond a definite value, the liquid phase results, and when the temperature of the liquid is raised further, the water exists in the gaseous phase. That is, from a large-scale or *macroscopic* point of view, the transition from solid to liquid to gas is in the direction of increasing temperature. From a *molecular* point of view this transition is in the direction of increasing molecular kinetic energy. Evidently, *there must be some connection between temperature and molecular kinetic energy.*

The preceding paragraphs may be conveniently summarized by means of Table 19–2.

19–4 Kinetic theory of an ideal gas. If the molecules of a gas are, on the average, far enough apart, the forces of attraction are exceedingly weak. We define an ideal gas as one whose molecules exert no forces on one another, except when they collide. Collisions are assumed to be completely elastic, so that the total kinetic energy of two molecules before collision is the same as that after collision. Suppose that the gas is in a container in the shape of a parallelepiped, and that the total number of molecules is enormously large. The gas is homogeneous; that is, the number of molecules in any small volume element is the same regardless of the position of the element. Moreover, the rectilinear motions

of the molecule are completely chaotic, so that, at any moment, there are as many molecules moving one way as another, although all molecules do not have the same speed.

The pressure exerted by the gas is due to molecular collisions with the walls of the container. These collisions are assumed to be completely elastic, so that a molecule approaching a wall perpendicularly with momentum mv will recede from the wall with momentum $-mv$. The change of momentum per molecule per collision will therefore be $-2mv$. If we calculate the rate at which molecules strike a wall and multiply this by the change of momentum per molecule per collision, we shall have the rate of change of momentum which, by Newton's second law, is the force exerted by the wall on the gas. Dividing by the area A of the wall, the pressure may then be obtained. The pressure calculated in this way may then be compared with pressure measured experimentally, i.e., the pressure expressed by the ideal gas equation. If the two expressions agree, the assumptions of the kinetic theory are confirmed.

In general, a molecule near the center of the container will undergo many collisions with other molecules before it arrives at a wall. For a given density of molecules (i.e., a given number of molecules per unit volume), there exists an average distance that a molecule may traverse before it collides with another molecule. This is called its *mean free path.* Imagine a plane situated near a wall of the containing vessel at a distance much less than the mean free path. Between this plane and the wall all molecules moving in any direction with a component toward the wall will reach the wall without undergoing an appreciable number of collisions with other molecules on the way. In this space, let

v_1 = a velocity component perpendicular to the wall,

N_1 = the number of molecules per unit volume which have the velocity component v_1,

t = the average time it takes such a molecule to reach the wall.

Then the number of molecules striking the wall in time t is equal to

$$\frac{N_1}{2} Av_1t,$$

since only half have velocity components directed *toward* the wall. Each of these molecules undergoes a change of momentum upon collision equal to $-2mv_1$. Therefore, the rate of change of momentum of these molecules equals

$$\frac{(N_1/2)Av_1t \times (-2mv_1)}{t}$$

and, by Newton's second law, this must represent the force exerted by the wall on these molecules. From Newton's third law, however, this is equal in magnitude but opposite in direction to the force exerted *on* the wall. The pressure, therefore, due to the collision of these molecules only is equal to

$$p_1 = \frac{(N_1/2)Av_1t \times (2mv_1)}{At}$$

or

$$p_1 = N_1mv_1^2.$$

Consider now another group of molecules, N_2 per unit volume, that have a different velocity component v_2 perpendicular to the wall. Their collisions with the wall will give rise to an additional pressure

$$p_2 = N_2mv_2^2.$$

The total pressure p due to all the molecules is therefore equal to

$$p = N_1mv_1^2 + N_2mv_2^2 + N_3mv_3^2 + \cdots, \tag{19–10}$$

where all possible values of the velocity component perpendicular to the wall are taken into account.

If we regard the direction perpendicular to the wall as that of the x-axis, we may define an average of the squares of the x-components of velocity by the expression

$$\overline{v_x^2} = \frac{N_1v_1^2 + N_2v_2^2 + N_3v_3^2 + \cdots}{N},$$

in which N represents the total number of molecules per unit volume. Substituting this value of $\overline{v_x^2}$ into the expression for the pressure, Eq. (19–10), we get

$$p = Nm\overline{v_x^2}. \tag{19–11}$$

If v is the magnitude of the velocity of any one molecule, then v may be expressed in terms of its x-, y-, and z-components, as follows:

$$v^2 = v_x^2 + v_y^2 + v_z^2.$$

Averaging over all the molecules, we get

$$\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}.$$

But the properties of the gas are uniform throughout the container. The average of the squares of the velocity components in any one direc-

tion is the same as that for any other direction. Therefore

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$$

and

$$\overline{v^2} = 3\overline{v_x^2}.$$

Substituting this result in Eq. (19–11), we get finally

$$\boxed{p = \tfrac{1}{3}N m\overline{v^2}.} \tag{19–12}$$

If $\mathbf{N}$ is the total number of molecules in the container of volume V, then $N = \mathbf{N}/V$ and

$$p = \frac{1}{3}\frac{\mathbf{N}}{V} m\overline{v^2}$$

or

$$pV = \tfrac{2}{3}\mathbf{N} \times \tfrac{1}{2}m\overline{v^2}. \tag{19–13}$$

The kinetic energy of a single molecule is $\frac{1}{2}mv^2$. The average kinetic energy per molecule is therefore $\frac{1}{2}m\overline{v^2}$. We have already come to the conclusion that molecular kinetic energy is somehow connected with temperature, so let us assume, in the case of an ideal gas, that

$$\boxed{\tfrac{1}{2}m\overline{v^2} = \tfrac{3}{2}kT,} \tag{19–14}$$

where k is a universal constant, known as *Boltzmann's constant.* It follows therefore that

$$pV = \tfrac{2}{3}\mathbf{N} \times \tfrac{3}{2}kT,$$

$$\boxed{pV = \mathbf{N}kT.} \tag{19–15}$$

We have therefore derived a relation among pressure, volume, and temperature (an equation of state) of an ideal gas from the laws of mechanics with the aid of the assumptions that:

(1) An ideal gas consists of identical, "rigid sphere," completely elastic molecules.

(2) The molecules move with uniform linear motion between impacts.

(3) The molecules exert no forces on one another, except during impact.

(4) The average kinetic energy per molecule is proportional to the kelvin temperature.

Comparing Eq. (19–15) with the experimental equation of state of an ideal gas,

$$pV = nRT,$$

we see that they are of the same form. Our assumptions are therefore justified. To make both equations identical, we must have

$$\mathbf{N}k = nR$$

or

$$k = \frac{R}{\mathbf{N}/n}.$$

The ratio of the number of molecules $\mathbf{N}$ to the number of moles n is the number of molecules per mole, or *Avogadro's number* N_0. Thus

$$k = \frac{R}{N_0}.$$

Avogadro's number may be measured in a variety of ways, one of which will be encountered when the conduction of electricity in liquids is studied. The latest value is

$$N_0 = 6.02 \times 10^{23} \frac{\text{molecules}}{\text{mole}}.$$

It follows, then, that Boltzmann's constant is

$$k = \frac{R}{N_0} = \frac{8.31 \times 10^7 \text{ ergs/mole·deg}}{6.02 \times 10^{23} \text{ molecules/mole}}$$

or

$$k = 1.38 \times 10^{-16} \frac{\text{erg}}{\text{molecule·deg}}.$$

Since there are no attractive forces among the molecules of an ideal gas, these molecules have no potential energy. The internal energy U of an ideal gas is therefore entirely kinetic. Since the average kinetic energy per molecule is $\frac{3}{2}kT$, we have

$$U = \tfrac{3}{2}\mathbf{N}kT$$

or

$$\boxed{U = \tfrac{3}{2}nRT,} \qquad (19\text{–}16)$$

which says that *the internal energy of an ideal gas is proportional to the kelvin temperature* and depends on temperature only, independent of the pressure or of the volume. It must be emphasized that Eq. (19–16) is of *very restricted validity*. It holds only for an aggregate of rigid, "billiard-ball" molecules which exert no forces on one another, and which do not rotate, vibrate, or dissociate. Equation (19–16) will therefore be expected to break down at high pressures when the average intermolecular distance is small, and also at high temperatures where molecules vibrate, dissociate,

or undergo electronic rearrangements. We *cannot* infer from Eq. (19–16) that at absolute zero the kinetic energy of the molecules equals zero, because long before absolute zero is approached condensation takes place, and Eq. (19–16) breaks down.

It is shown in advanced textbooks that the energy of a substance in a condensed phase is a very complicated function of the temperature, in no way resembling Eq. (19–16). This function has the property that, as the temperature approaches zero, *the energy does not approach zero*, but approaches a constant value, known as the *zero-point energy*. In the case of helium at low temperature, the zero-point energy is so large that the molecules of liquid cannot get close enough together to form a solid, unless the pressure is raised to about 25 atmospheres. In the absence of this high pressure, helium remains a liquid down to absolute zero.

19–5 Specific heats of an ideal gas. The temperature of a gas may be raised under a variety of conditions. The volume may be kept constant, or the pressure may be kept constant, or both may be allowed to vary in some arbitrary manner. In each of these cases, the amount of heat necessary to cause unit rise of temperature in unit mass is different. In other words, a gas has many different heat capacities. Only two, however, are of practical use, namely, those at constant volume and at constant pressure.

Let us choose as a convenient unit of mass the number of grams equal to the molecular weight, that is, a mole. The corresponding heat capacity is called the *molar heat capacity*, represented by C_v. If we have n moles of an ideal gas with a molar heat capacity at constant volume C_v and we raise its temperature at constant volume from T_1 to T_2, then the heat transferred is $nC_v(T_2 - T_1)$ and the work done is zero. From the first law we get

$$nC_v(T_2 - T_1) = U_2 - U_1 + 0. \tag{19–17}$$

If the same amount of the same gas were heated at constant pressure p until the temperature changed the same amount, then the heat transferred would be $nC_p(T_2 - T_1)$, where C_p is the molar heat capacity at constant pressure. In this process, however, the gas would expand, say, from volume V_1 to V_2 and an amount of work $p(V_2 - V_1)$ would be done. Therefore, using the first law,

$$nC_p(T_2 - T_1) = U_2' - U_1' + p(V_2 - V_1). \tag{19–18}$$

Now U_1 and U_1' are the internal energies, respectively, of an ideal gas at the same temperature T_1 but at different pressures. Since the energy

of an ideal gas is a function of temperature only, $U_1 = U_1'$, and similarly, $U_2 = U_2'$.

Subtracting Eq. (19–17) from Eq. (19–18) we get

$$n(C_p - C_v)(T_2 - T_1) = p(V_2 - V_1). \tag{19–19}$$

But

$$pV_1 = nRT_1,$$

and

$$pV_2 = nRT_2,$$

so that

$$p(V_2 - V_1) = nR(T_2 - T_1).$$

Substituting this result in Eq. (19–19), we get

$$n(C_p - C_v)(T_2 - T_1) = nR(T_2 - T_1),$$

or finally

$$\boxed{C_p - C_v = R.} \tag{19–20}$$

This equation shows that the molar heat capacity at constant pressure of an ideal gas is always larger than that at constant volume, the difference being the universal gas constant R. Of course R must be expressed in the same units as C_p and C_v, usually cal/mole C°. Since $R =$ 8.31 joules/mole C° and 4.19 joules = 1 cal,

$$R = \frac{8.31}{4.19} = 1.99 \frac{\text{cal}}{\text{mole C°}},$$

or very nearly 2 cal/mole C°.

Equation (19–20) is strictly true only for an ideal gas but is very nearly true for real gases at moderate pressures. Measured values of C_p and C_v are given in Table 19–3 for some real gases at low pressures, and the difference is seen to be very nearly 2 cal/mole·C°.

In the last column of Table 19–3 are listed the values of the ratio C_p/C_v denoted by the Greek letter γ (gamma). It is seen that γ is 1.67 for monatomic gases, and is very nearly 1.40 for the so-called permanent diatomic gases. There is no simple regularity for polyatomic gases.

Solids and liquids also expand when heated, if free to do so, and hence perform work. The coefficients of volume expansion of solids and liquids are, however, so much smaller than those of gases that the external work is small. The internal energy of a solid or liquid *does* depend on its volume as well as its temperature, and this must be considered when evaluating the difference between specific heats of solids or liquids. It turns out that here also $C_p > C_v$, but the difference is small and is not expressible as simply as that for a gas. Because of the large stresses set up when

TABLE 19–3

MOLAR HEAT CAPACITIES OF GASES AT LOW PRESSURE

Type of gas	Gas	C_p, cal/mole·C°	C_v, cal/mole·C°	$C_p - C_v$	$\gamma = C_p/C_v$
Monatomic	He	4 97	2.98	1 99	1.67
"	A	4 97	2 98	1 99	1 67
Diatomic	H_2	6 87	4 88	1.99	1.41
"	N_2	6 95	4.96	1 99	1 40
"	O_2	7.03	5 04	1.99	1 40
"	CO	6.97	4 98	1 99	1 40
Polyatomic	CO_2	8.83	6 80	2 03	1.30
"	SO_2	9.65	7 50	2 15	1 29
"	H_2S	8 37	6 2	2 1	1.34

solids or liquids are heated and *not* allowed to expand, most heating processes involving them take place at constant pressure, and hence C_p is the quantity usually measured for a solid or liquid.

19–6 Adiabatic compression or expansion of an ideal gas. When an ideal gas is compressed adiabatically, the work done on the gas serves to increase the internal energy. Since the internal energy is a function of the temperature only, the temperature therefore rises. Conversely, in an adiabatic expansion of an ideal gas, the temperature decreases.

In Fig. 19–5, the heavy curves represent adiabatic curves on a p-V diagram. The dashed curves represent isotherms for comparison. The adiabatic curves, at any point, have a somewhat steeper slope than the isothermal curve passing through the same point. That is, as one follows along an adiabatic from right to left (compression process) the curve continually cuts across isotherms of higher and higher temperatures, in agreement with the fact that the temperature continually increases in an adiabatic compression. It can be shown that the equation of an adiabatic curve for an ideal gas is

$$\boxed{pV^\gamma = \text{constant}} \quad \text{(Adiabatic)}, \tag{19–21}$$

where γ is the ratio of the heat capacities listed in the last column of Table 19–3. Equation (19–21) may be rewritten in the more useful form

$$p_1 V_1^\gamma = p_2 V_2^\gamma \quad \text{(Adiabatic)}, \tag{19–22}$$

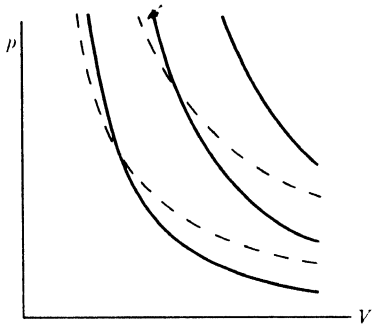


FIG. 19–5. Adiabatic curves (full lines) vs. isothermal curves (dashed lines).

where *any* unit of pressure may be used, provided it holds for *both* p_1 and p_2 and any unit of volume for both V_1 and V_2.

The states traversed by an ideal gas undergoing an adiabatic expansion or compression may be described by a relation between T and V just as well as by one between p and V. If we substitute for p in Eq. (19–21) the value given by the ideal gas equation, we get

$$\frac{nRT}{V} \cdot V^{\gamma} = \text{constant},$$

or

$$\boxed{TV^{\gamma-1} = \text{constant},} \tag{19–23}$$

or

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}. \tag{19–24}$$

Both T_1 and T_2 must be expressed either on the kelvin scale or on the rankine scale. V_1 and V_2 must, of course, be expressed in the same units.

EXAMPLE. The compression ratio of a diesel engine, V_1/V_2, is about 15. If the cylinder contains air at 15 lb/in^2 (absolute) and 60°F (= 520°R) at the start of the compression stroke, compute the pressure and temperature at the end of this stroke. Assume that air behaves like an ideal gas and that the compression is adiabatic. The value of γ for air is 1.40.

From Eq. (19–22),

$$p_2 = p_1 (V_1/V_2)^{\gamma}.$$

Or,

$$\log p_2 = \log p_1 + \gamma \log \frac{V_1}{V_2} = \log 15 + 1.4 \log 15 = 1.176 + 1.646 = 2.822.$$

$$\therefore p_2 = 663 \text{ lb/in}^2.$$

The temperature may now be found either from Eq. (19–24) or by the ideal gas law. Thus, from Eq. (19–24), we have

$$T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1},$$

$$\log T_2 = \log T_1 + (\gamma - 1)\log\frac{V_1}{V_2}$$
$$= \log 520 + (1.4 - 1)\log 15$$
$$= 2.716 + 0.470 = 3.186,$$
$$T_2 = 1535°\text{R} = 1075°\text{F}.$$

Or, from the ideal gas law,

$$T_2 = T_1 \times \frac{p_2V_2}{p_1V_1} = 520 \times \frac{663 \times 1}{15 \times 15} = 1535°\text{R}.$$

19–7 Liquefaction of gases. An ideal gas, if compressed isothermally, remains a gas no matter how great a pressure is applied to it; the volume decreases continually with increasing pressure according to Boyle's law. All real gases, however, become liquids when the pressure is increased sufficiently (provided the temperature is below a value known as the critical temperature of the gas).

The difference between the behavior of an ideal and a real gas in an isothermal compression is illustrated in Fig. 19–6(a) and (b). Imagine the gases to be contained within two similar cylinders, each provided with a piston and a pressure gauge. The volume at each stage of the compression process is proportional to the distance of the piston from the closed end of the cylinder, and the pressure at each stage may be read from the gauge. Several stages of the compression are illustrated in each part of the figure, and corresponding pressures and volumes are plotted in the p-V diagrams. Each gas is initially at the same pressure and volume (point a), and the temperature of each is held constant throughout by the removal of heat.

As the piston in (a) is forced to the left, the reading of the pressure gauge rises steadily, and the relation between p and V is the familiar Boyle's law. As the volume of the real gas is decreased, in Fig. 19–6(b), the pressure rises at first along the curve ab in a manner not very different from that of the ideal gas. When point b is reached, however, a sharp break occurs in the curve, and the volume continues to decrease *without further increase of pressure.* At the same time, drops of liquid begin to appear on the walls of the cylinder. In other words, the process of *liquefaction* or *condensation* begins at point b.

As the volume is further reduced, from b to c and d, the quantity of liquid in the cylinder increases and the quantity of gas decreases. At point e, all the substance has been converted into the liquid phase. Of

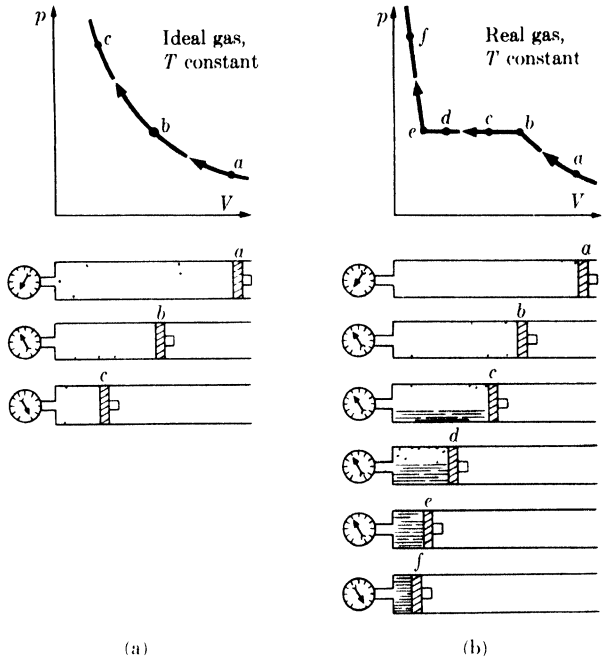


FIG. 19–6. Isothermal compression of (a) an ideal gas, (b) a real gas.

course, during the transition from b to e, it was necessary to remove from the substance its heat of condensation.

Since liquids are nearly incompressible, a very large pressure increase is necessary to reduce the volume below that at point e. That is, the curve rises nearly vertically from e to f, having a very small inclination toward the left.

As an example, if the cylinder in part (b) of the figure had initially contained one gram of steam at 100°C, at a pressure of about $\frac{1}{2}$ atm and a volume of about 3000 cm^3 (point a), condensation would begin when the pressure had increased to 1 atm and the volume decreased to 1670 cm^3 (point b). The pressure would remain constant at 1 atm from b to e, while the volume decreased from 1670 cm^3 to 1 cm^3. 539 cal would have to be removed during the condensation process. To produce a further decrease in volume of 0.001 cm^3, it would be necessary to increase the pressure to about 20 atm (point f).

If the experiment illustrated in Fig. 19–6(b) is repeated, starting at higher and higher temperatures, it is found that greater and greater pressures must be exerted on the gas before condensation begins. Figure

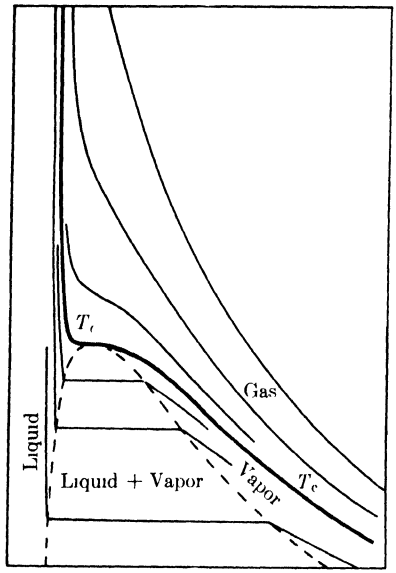


FIG. 19–7. Isotherms of a real gas.

19–7 illustrates the curves obtained. It will be seen that point b moves toward the left and point e toward the right until at the particular temperature lettered T_c in Fig. 19–7 the two points coincide. Above this temperature there is no straight horizontal portion to the curve. In other words, there is no stage in the compression process at which the substance separates into two distinct portions, one of which is a gas while the other is a liquid. The temperature T_c is called the *critical* temperature, and it is now evident why a gas must first be cooled below its critical temperature before it can be liquefied by compression.

It is customary to refer to a gas below its critical temperature as a *vapor*, although this distinction is not rigidly adhered to. Critical temperatures of some common gases are given in Table 19–4.

The dotted line in Fig. 19–7 divides the p-V plane into three regions. At all values of p, V, and T underneath this line, the substance is partly in the liquid and partly in the vapor phase. At the right of the curve it is a vapor or gas, at the left of the curve it is a liquid.

An examination of Fig. 19–7 shows that at any given temperature below the critical temperature there is one pressure and one pressure only at which the substance can exist in either the liquid or the vapor phase, or in both phases simultaneously. This is the pressure corresponding to the horizontal portion of the isothermal curve for that particular temperature. If the pressure is any higher than that at the horizontal

TABLE 19–4

CRITICAL TEMPERATURES, PRESSURES, AND VOLUMES

Substance	Critical temperature (°C)	Critical pressure (atm)	Critical volume (cm^3/gm)
Ammonia	132	112	4.25
Argon	−122	48	1.88
Carbon dioxide	31	73 0	2 17
Helium	−268	2 26	14.4
Hydrogen	−240	12.8	32 3
Oxygen	−119	49.7	2.33
Sulfur dioxide	157	77.7	1 92
Water	374	218	3 14

portion, the substance can only be a liquid. If the pressure is any lower, it can only be a vapor. Precisely at this pressure both liquid and vapor can exist together. The pressure at which a liquid and its vapor can exist in equilibrium, at any given temperature, is called the *vapor pressure* at that temperature. A vapor whose pressure and temperature are those corresponding to the horizontal portion of any of the curves in Fig. 19–7 is called a *saturated vapor*. A saturated vapor may also be defined as one which is in equilibrium with its liquid.

From the straight portions of the curves of Fig. 19–7, one can read off a series of vapor pressures and their corresponding temperatures. If these are plotted as in Fig. 19–8, one obtains the *vapor pressure curve* of the substance. All vapor pressure curves are similar in form to that of Fig. 19–8, rising with a continually increasing slope and ending at the critical point. An abridged table of the vapor pressure of water is given in Table 19–5 in the next section.

Reference to Table 19–4 will show that the critical temperatures of carbon dioxide, ammonia, and sulfur dioxide are higher than "room temperature." Hence these gases can be liquefied at room temperature without precooling, simply by increasing the pressure. Oxygen, nitrogen, or hydrogen, however, must evidently be precooled below room temperature before they can be liquefied. The Linde process for producing liquid air (or liquid oxygen or nitrogen) will be described briefly.

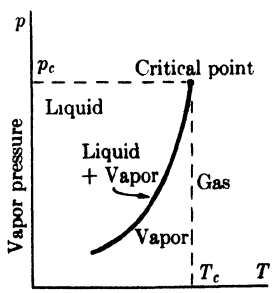


FIG. 19–8. Vapor pressure vs. temperature.

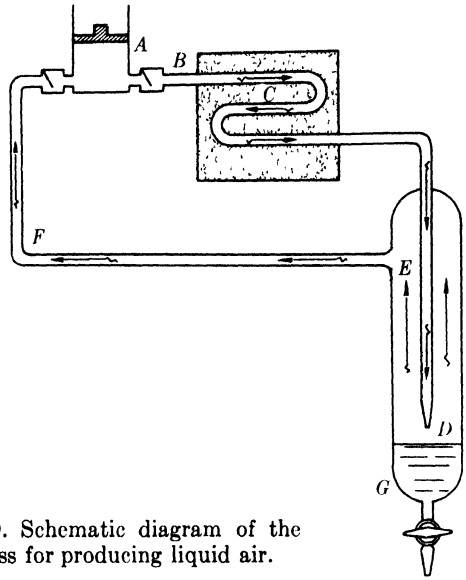


FIG. 19–9. Schematic diagram of the Linde process for producing liquid air.

In Fig. 19–9, compressor A maintains a continuous circulation of air as shown by the arrows. At B, the air leaving the compressor is at high pressure and high temperature. It enters cooling coils C, where it is cooled by air or water, and, still at high pressure, escapes through the small orifice or nozzle D, performing a *throttling* process (see Section 18–10). If air were an ideal gas, no temperature change would result from a throttling process. Real gases, however, undergo marked temperature changes in such a process, and if not too hot to begin with, are cooled in passing through the orifice. The pressure in E and F is kept low by the pump, and the cooled air passes up through E and F and repeats the cycle. The cooled air in E, circulating around the incoming air in D, cools it still further, hence an even lower temperature is reached by the air escaping from D, until eventually the temperature falls sufficiently so that some of the air liquefies as it leaves the nozzle. The liquid air collects at G, where it may be drawn off.

19–8 Effect of pressure on boiling and freezing points. A pan of water exposed to the air of the room will evaporate, whatever its temperature, provided only that there is opportunity for the vapor to diffuse away, or to be removed in some manner from above the water surface. If the temperature of the water is increased to 100°C (at normal atmospheric pressure), the nature of the evaporation process changes completely. Vapor is formed not only at the liquid surface but throughout the entire volume of the liquid, which becomes violently agitated by the bubbles

TABLE 19–5

VAPOR PRESSURE OF WATER (ABSOLUTE)

t, °C	Vapor pressure		t_F, °F
	mm of mercury	lb/in^2	
0	4 58	0 0886	32
5	6 51	0 126	41
10	8 94	0 173	50
15	12 67	0 245	59
20	17 .5	0 339	68
40	55 1	1 07	104
60	149	2 89	140
80	355	6 87	176
100	760	14 7	212
120	1490	28 8	248
140	2710	52 4	284
160	4630	89 6	320
180	7510	145	356
200	11650	225	392
220	17390	336	428

of vapor which rise through it and break at the surface. What distinguishes this violent process of *boiling* from the slow evaporation which goes on at temperatures below the boiling point?

It will be recalled that every liquid has a certain *vapor pressure* which depends on the temperature of the liquid. If, keeping the temperature constant, an attempt is made to increase the pressure *above* the vapor pressure, the vapor immediately condenses. The water in a vessel open to the atmosphere is subjected to atmospheric pressure. Say that the temperature of the water is 80°C. Its vapor pressure, from Table 19–5, is then 355 mm of mercury or 6.87 lb/in^2. Hence if a small bubble of vapor should chance to form within the liquid, where it is subjected to a pressure of 760 mm or 14.7 lb/in^2, it would immediately collapse under the pressure and condense.

Suppose now the temperature of the liquid is increased to 100°C. At this temperature the vapor pressure is 760 mm or 14.7 lb/in^2. Hence bubbles of vapor can form at this temperature, and if the temperature were to increase only slightly above 100°C, the entire mass of water would change to the vapor phase *if its heat of vaporization could be supplied to it.* What actually happens is that the water does so change as fast as

heat is supplied. As long as any liquid water remains, the temperature cannot rise above 100°C and all the heat supplied is used to produce a *change of phase* rather than an *increase in temperature.*

If the external pressure is suddenly increased above 14.7 lb/in^2, boiling immediately ceases, since the pressure is higher than the vapor pressure of water at 100°C. Assuming that heat is still supplied to the water, its temperature increases until the vapor pressure equals the applied pressure, when boiling commences again.

It is evident that under an external pressure less than atmospheric, boiling will take place at a temperature below 100°C. From Table 19–5, if the pressure is reduced to 0.339 lb/in^2, water will boil at room temperature (20°C).

> *The boiling point of a liquid is that temperature at which the vapor pressure of the liquid is equal to the external pressure.*

Freezing points as well as boiling points are affected by external pressure. The freezing point of a substance like water, which expands on solidifying, is *lowered* by an increase in pressure. The reverse is true for substances which contract on solidifying. The change in the freezing point temperature is much smaller than is that of the boiling point; an increase of one atmosphere lowers the freezing point of water by only about 0.007°C.

The lowering of the freezing point of water (or the melting point of ice) can be demonstrated by passing a loop of fine wire over a block of ice and hanging a weight of a few pounds from each end of the loop. Suppose that the main body of the ice is at 0°C and at atmospheric pressure. The temperature of the small amount of ice directly under the wire decreases until it achieves the melting point appropriate to the pressure under the wire. During this increase of pressure and decrease of temperature, a small amount of melting takes place. The water thus formed is squeezed out from under the wire and, coming to the top of the wire where the pressure is atmospheric, it refreezes and liberates heat which passes through the wire and serves to melt the next bit of ice below the wire.

The wire thus sinks farther and farther into the block, eventually cutting its way completely through, but leaving a solid block of ice behind it. This phenomenon is known as *regelation* (refreezing). Since heat is conducted from the top to the bottom of the wire while the wire is cutting through the ice, the greater the thermal conductivity of the wire, the faster will the wire cut through the ice. Even a perfectly conducting wire would not cut through the ice very rapidly, however, because of the very low thermal conductivity of the water film which is always present beneath the wire.

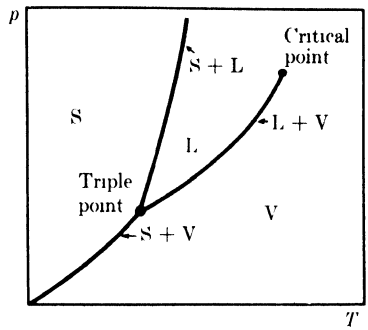


FIG. 19–10. p-T diagram.

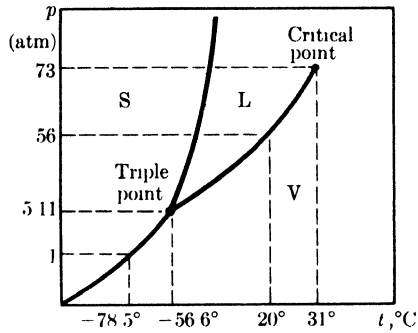


FIG. 19–11. p-t diagram of CO_2. (Not to a uniform scale.)

19–9 The triple point. The vapor pressure curve of Fig. 19–8 can be considered to represent the pressure and temperature at which a change of phase from liquid to vapor will occur, or as the pressure and temperature at which both phases can remain in equilibrium with each other. There exist similar curves representing the pressure and temperature at which solid and liquid can be in equilibrium, and at which solid and vapor can be in equilibrium. These curves are shown in Fig. 19–10, and they intersect at a common point, called the *triple point*, at which all three phases can exist simultaneously.

The three curves in Fig. 19–10 divide the p-T plane into three regions. At any point (i.e., any pair of values of p and T) within one of these regions, the substance can exist in one phase only: solid, liquid, or vapor. Along each line, two phases can exist together, while only at the triple point can all three phases coexist.

For example, the triple-point temperature of CO_2 is -56.6°C, and the triple-point pressure is 5.11 atm. It is evident from Fig. 19–11 (not to a uniform scale) that at atmospheric pressure CO_2 can exist only as a solid or a vapor. Hence solid CO_2 (dry ice) transforms directly to CO_2 vapor when open to the atmosphere, without passing through the liquid state. This direct transition from solid to vapor is called *sublimation*. Liquid CO_2 can exist only at a pressure greater than 5.11 atm. The steel tanks in which CO_2 is commonly stored contain liquid and vapor. The pressure in these tanks is the vapor pressure of CO_2 at the temperature of the tank. If the latter is 20°C, the vapor pressure is about 56 atm or 830 lb/in^2.

While most substances increase in volume in the change of phase from solid to liquid, there are a few for which the reverse is true. Water, of course, is one of the latter. For such a substance an *increase* of pressure

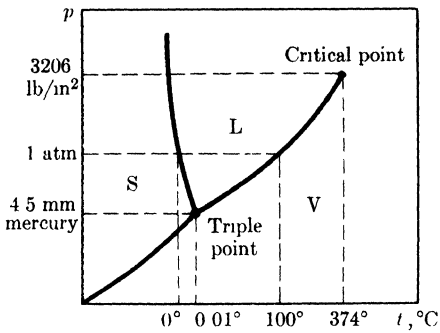


FIG. 19–12. p-t diagram of H_2O. (Not to a uniform scale.)

produces a *lowering* of the freezing point, as shown in Fig. 19–12. This effect was mentioned earlier in connection with regelation.

19–10 Humidity. Atmospheric air is a mixture of gases, consisting of about 80% nitrogen, 18% oxygen, and small amounts of carbon dioxide, water vapor, and other gases. The mass of water vapor per unit volume is called the *absolute humidity*. The total pressure exerted by the atmosphere is the sum of the pressures exerted by its component gases. These pressures are called the *partial pressures* of the components. It is found that the partial pressure of each of the component gases of a gas mixture is very nearly the same as would be the actual pressure of that component alone if it occupied the same volume as does the mixture, a fact known as *Dalton's law*. That is, each of the gases of a gas mixture behaves independently of the others. The partial pressure of water vapor in the atmosphere is ordinarily a few millimeters of mercury.

It should be evident that the partial pressure of water vapor at any given air temperature can never exceed the vapor pressure of water at that particular temperature. Thus at 10°C, from Table 19–5, the partial pressure cannot exceed 8.94 mm, or at 15°C it cannot exceed 12.67 mm. If the concentration of water vapor, or the absolute humidity, is such that the partial pressure equals the vapor pressure, the vapor is said to be *saturated*. If the partial pressure is less than the vapor pressure, the vapor is *unsaturated*. The ratio of the partial pressure to the vapor pressure at the same temperature is called the *relative humidity*, and is usually expressed as a percentage.

$$\text{Relative humidity } (\%) = 100 \times \frac{\text{partial pressure of water vapor}}{\text{vapor pressure at same temperature}}.$$

The relative humidity is 100% if the vapor is saturated and zero if no water vapor at all is present.

EXAMPLE. The partial pressure of water vapor in the atmosphere is 10 mm of mercury and the temperature is 20°C. Find (a) the relative humidity, and (b) the absolute humidity.

(a) From Table 19–5, the vapor pressure at 20°C is 17.5 mm of mercury. Hence

$$\text{Relative humidity} = \frac{10}{17.5} \times 100 = 57\%.$$

(b) Consider any volume V. From Dalton's law, the partial pressure of the water vapor present in this volume is the pressure that the water vapor would exert if it occupied the entire volume alone at 20°C. Using the ideal gas law,

$$pV = nRT,$$

we have

$$\frac{n}{V} = \frac{p}{RT} = \frac{10 \text{ mm}/(760 \text{ mm/atm})}{0.082 \text{ liter·atm/mole·deg} \times 293 \text{ deg}}$$

$$= 0.000548 \frac{\text{mole}}{\text{liter}}$$

Since the molecular weight of water is $M = 18$ gm/mole, and $1 \text{ m}^3 = 10^6 \text{ cm}^3 = 10^3$ liters, we get

$$\frac{m}{V} = \frac{nM}{V} = \frac{0.000548 \times 18}{10^{-3}} = 9.86 \frac{\text{gm}}{\text{m}^3}.$$

Since the water vapor in the atmosphere is saturated when its partial pressure equals the vapor pressure at the air temperature, saturation can be brought about either by increasing the water vapor content or by lowering the temperature. For example, let the partial pressure of water vapor be 10 mm when the air temperature is 20°C, as in the preceding example. Saturation, or 100% relative humidity, could be attained either by introducing enough more water vapor (keeping the temperature constant) to increase the partial pressure to 17.5 mm, *or by lowering the temperature* to 11.4°C, at which, by interpolation from Table 19–5, the vapor pressure is 10 mm.

If the temperature were to be lowered *below* 11.4°C, the vapor pressure would be less than 10 mm. The partial pressure would then be higher than the vapor pressure and enough vapor would condense to reduce the partial pressure to the vapor pressure at the lower temperature. It is this process which brings about the formation of clouds, fog, and rain. The phenomenon is also of frequent occurrence at night when the earth's surface becomes cooled by radiation. The condensed moisture is called *dew*. If the vapor pressure is so low that the temperature must fall below 0°C before saturation exists, the vapor condenses into ice crystals in the form of frost.

The temperature at which the water vapor in a given sample of air becomes saturated is called the *dew point.* Measuring the temperature of the dew point is the most accurate method of determining relative humidity. The usual method is to cool a metal container having a bright, polished surface, and to observe its temperature when the surface becomes clouded with condensed moisture. Suppose the dew point is observed in this way to be 10°C, when the air temperature is 20°C. We then know that the water vapor in the air is saturated at 10°C, hence its partial pressure, from Table 19–5, is 8.94 mm, equal to the vapor pressure at 10°C. The pressure necessary for saturation at 20°C is 17.5 mm. The relative humidity is therefore

$$\frac{8.94}{17.5} \times 100 = 51\%.$$

A simpler but less accurate method of determining relative humidity employs a *wet-and-dry bulb thermometer.* Two thermometers are placed side by side, the bulb of one being kept moist by a wick dipping in water. The lower the relative humidity, the more rapid will be the evaporation from the wet bulb, and the lower will be its temperature below that of the dry bulb. The relative humidity corresponding to any pair of wet-and-dry bulb temperatures is read from tables.

The *hair hygrometer* makes use of the fact that human hair absorbs or gives up moisture from the air in an amount which varies with relative humidity, and changes its length slightly with moisture content. Several strands of hair are wrapped around a small pivoted shaft to which is attached a pointer. The hair is kept taut by a light spring, and changes in its length cause the shaft to rotate and move the pointer over a scale.

19–11 Thermodynamic surfaces. The equation of state of a substance is a relation among the three variables, p, V, and T. If these quantities are laid off on three mutually perpendicular axes, the equation of state defines a *surface* in the p-V-T space. All possible states of the substance are represented by points on this surface, and all processes through which the substance may be carried are represented by lines in the surface. An isothermal process is a line in the surface at all points of which T is constant; in other words, it is the intersection of the surface with a plane perpendicular to the temperature axis. Similarly, processes at constant pressure or volume are intersections of the surface with planes perpendicular to the pressure or volume axes. Such a surface is called a thermodynamic surface, although variables other than p, V, and T are often used.

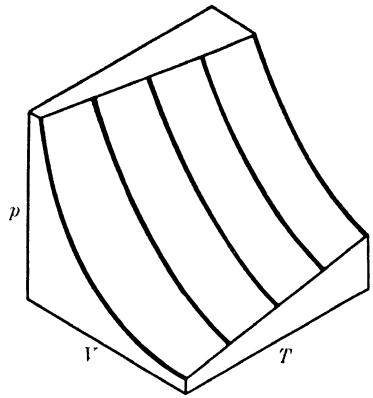


FIG. 19–13. p-V-T surface of an ideal gas.

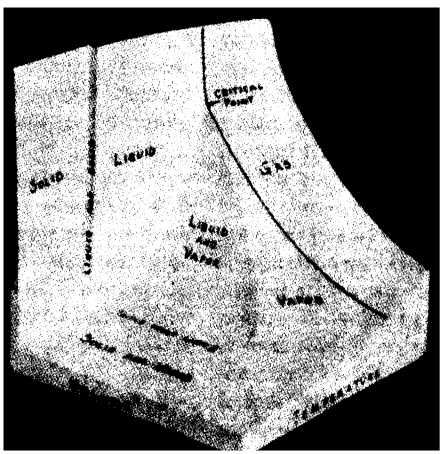


FIG. 19–14. p-V-T surface of a real substance. (Not to a uniform scale.)

Figure 19–13 illustrates the simplest of all p-V-T surfaces, that of an ideal gas. A few isotherms are indicated by the heavy lines. When this surface is viewed in a direction perpendicular to the p-V plane, the isotherms appear as in Fig. 19–2.

Figure 19–14 is a photograph of the p-V-T surface of a real substance, not to a uniform scale. When viewed perpendicular to the p-V plane, the isotherms appear as in Fig. 19–7. The triple "point" is actually not

a point but a line, since, although the pressure and temperature are fixed at that point, the volume is not. That is, the volume depends on the relative masses of the substance which are in each phase. If, for example, the substance is mostly in the vapor phase, with only small amounts of solid and liquid present, the volume will be large. The surface is not constructed to scale because of the large volume changes involved in changes of phase.

Problems

(Assume all gases to be ideal.)

19–1. A tank contains 1.5 ft^3 of nitrogen at an absolute pressure of 20 lb/in^2 and a temperature of 40°F. What will be the pressure if the volume is increased to 15 ft^3 and the temperature raised to 440°F?

19–2. A tank having a capacity of 2 ft^3 is filled with oxygen which has a gauge pressure of 60 lb/in^2 when the temperature is 47°C. At a later time it is found that because of a leak the gauge pressure has dropped to 50 lb/in^2 and the temperature has decreased to 27°C. Find (a) the mass of the oxygen in the tank under the first set of conditions, (b) the amount of oxygen that has leaked out.

19–3. A flask of volume 2 liters, provided with a stopcock, contains oxygen at 300°K and atmospheric pressure. The system is heated to a temperature of 400°K, with the stopcock open to the atmosphere. The stopcock is then closed and the flask cooled to its original temperature. (a) What is the final pressure of the oxygen in the flask? (b) How many grams of oxygen remain in the flask?

19–4. A barrage balloon whose volume is 20,000 ft^3 is to be filled with hydrogen at atmospheric pressure. If the hydrogen is stored in cylinders of volume 2 ft^3 at an absolute pressure of 200 lb/in^2, how many cylinders are required?

19–5. A bubble of air rises from the bottom of a lake, where the pressure is 3.03 atm, to the surface, where the pressure is 1 atm. The temperature at the bottom of the lake is 7°C and the temperature at the surface is 27°C. What is the ratio of the size (i.e., the volume) of the bubble as it reaches the surface to the size of the bubble at the bottom?

19–6. (a) Two gm of nitrogen at 27°C occupy a volume of 2 liters. What is the pressure? (b) If the pressure is doubled and the temperature raised to 127°C, calculate the final volume.

19–7. A liter of helium under a pressure of 2 atm and at a temperature of 27°C is heated until both pressure and volume are doubled. (a) What is the final temperature? (b) How many grams of helium are there?

19–8. A flask contains 1 gm of oxygen at an absolute pressure of 10 atm and at a temperature of 47°C. At a later time it is found that because of a leak the pressure has dropped to $\frac{5}{8}$ of its original value and the temperature has decreased to 27°C. (a) What is the volume of the flask? (b) How many grams of oxygen leaked out between the two observations?

19–9. Give an example of some process in which no heat is added to or removed from a system, but the temperature of the system decreases. If this is not possible, state why.

19–10. Give an example of some process in which heat is added to an object without changing its temperature.

19–11. Ten liters of air at atmospheric pressure are compressed isothermally to a volume of 2 liters. Show the process on a p-V diagram, carefully plotting the points. Approximately how much work is done by the gas during the compression? Find the answer directly from your p-V diagram.

19–12. The submarine Squalus sank at a point where the depth of water was 240 ft. The temperature at the surface is 27°C and at the bottom it is 7°C. The density of sea water may be taken as 2 slugs/ft^2. (a) If a diving bell in a form of a circular cylinder 8 ft high, open at the bottom and closed at the top, is lowered to this depth, to what height will the water rise within it when it reaches the bottom? (b) At what gauge pressure must compressed air be supplied to the bell while on the bottom to expel all the water from it?

19–13. A bicycle pump is full of air at an absolute pressure of 15 lb/in^2. The length of stroke of the pump is 18 inches. At what part of the stroke does air begin to enter a tire in which the gauge pressure is 40 lb/in^2? Assume the compression to be isothermal.

19–14. A vertical cylindrical tank 1 m high has its top end closed by a tightly fitting frictionless piston of negligible weight. The air inside the cylinder is at an absolute pressure of 1 atm. The piston is depressed by pouring mercury on it slowly. How far will the piston descend before mercury spills over the top of the cylinder? The temperature of the air is maintained constant.

19–15. A barometer is made of a tube 90 cm long and of cross section 1.5 cm^2. Mercury stands in this tube to a height of 75 cm. The room temperature is 27°C. A small amount of nitrogen is introduced into the evacuated space above the mercury and the column drops to a height of 70 cm. How many grams of nitrogen were introduced?

19–16. A large tank of water has a hose connected to it, as shown in Fig. 19–15. The tank is sealed at the top and has compressed air between the water surface and the top. When the water height h_2 is 10 ft, the gauge pressure p_1 is 15 lb/in^2. Assume that the air above the water surface expands isothermally and that water weighs 64 lb/ft^3. (a) What is the velocity of flow out of the hose when h_2 = 10 ft? (b) What is the velocity of flow of the hose when h_2 has decreased to 8 ft? Neglect friction.

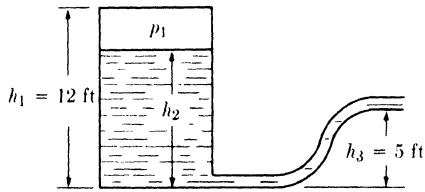


FIGURE 19–15

19–17. The volume of an ideal gas is 4 liters, the pressure 2 atm, and the temperature is 300°K. The gas first expands at constant pressure to twice its original volume, it is then compressed isothermally to its original volume, and finally cooled at constant volume to its original pressure. (a) Show the process in a p-V diagram. (b) Compute the temperature during the isothermal compression. (c) Compute the maximum pressure. (d) Explain how to find the *net* work done by the gas in the process.

19–18. Ten cubic feet of air initially at 140°F expand at a constant gauge pressure of 20 lb/in^2 to a volume of 50 ft^3, and then expand further adiabatically to a final volume of 80 ft^3 and a final gauge pressure of 3 lb/in^2. Sketch the process in a p-V diagram.

19–19. The cylinder of a pump compressing air from atmospheric pressure into a very large tank at 60 lb/in^2 gauge pressure is 10 inches long. (a) At what position in the stroke will air

begin to enter the tank? Assume the compression to be adiabatic. (b) If the air is taken into the pump at 27°C, what is the temperature of the compressed air?

19–20. At the beginning of the compression stroke, the cylinder of a diesel engine contains 48 in^3 of air at atmospheric pressure and a temperature of 27°C. At the end of the stroke, the air has been compressed to a volume of 3 in^3, and the gauge pressure has increased to 600 lb/in^2. Compute the temperature, (a) in degrees kelvin, (b) degrees celsius, (c) degrees fahrenheit.

19–21. Ten liters of air at atmospheric pressure is compressed isothermally to a volume of 2 liters and is then allowed to expand adiabatically to a volume of 10 liters. Show the process in a p-V diagram.

19–22. An ideal gas is contained in a cylinder closed with a movable piston. The initial pressure is 1 atm and the initial volume is 1 liter. The gas is heated at constant pressure until the volume is doubled, then heated at constant volume until the pressure is doubled, and finally expanded adiabatically until the temperature drops to its initial value. Show the process in a p-V diagram.

19–23. An ideal gas at a pressure of 1 atm is heated at constant pressure until its volume is doubled, then heated at constant volume until its pressure is doubled, and finally allowed to expand isothermally until its pressure drops to 1 atm. Show the process in a p-V diagram.

19–24. Two moles of oxygen are initially at a temperature of 27°C and volume 20 liters. The gas is expanded first at constant pressure until the volume has doubled, and then adiabatically until the temperature returns to the original value. (a) What is the total increase in internal energy? (b) What is the final volume?

19–25. Make two plots for a real gas, one showing pressure as a function of volume, and the other showing pressure as a function of temperature. Show on each graph the region in which the substance exists as (a) a gas or vapor, (b) a liquid, (c) a solid. Show also the triple point and the critical point.

19–26. (a) What is the relative humidity on a day when the temperature is 68°F and the dew point is 41°F? (b) What is the partial pressure of water vapor in the atmosphere? (c) What is the absolute humidity, in gm/m^3?

19–27. The temperature in a room is 40°C. A can is gradually cooled by adding cold water. At 10°C the surface of the can clouds over. What is the relative humidity in the room?

19–28. A pan of water is placed in a sealed room of volume 60 m^3 and at a temperature of 27°C. (a) What is the absolute humidity in gm/m^3 after equilibrium has been reached? (b) If the temperature of the room is then increased 1 C°, how many more grams of water will evaporate?

19–29. (a) What is the dew point temperature on a day when the air temperature is 20°C and the relative humidity is 60%? (b) What is the absolute humidity, in gm/m^3?

19–30. The volume of a closed room, kept at a constant temperature of 20°C, is 60 m^3. The relative humidity in the room is 10%. If a pan of water is brought into the room, how many grams will evaporate?

19–31. An air conditioning system is required to increase the relative humidity of 10 ft^3 of air per second from 30% to 65%. The air temperature is

Molecular Data

N_0 = Avogadro's number = 6.02×10^{23} molecules/mole,
Mass of a hydrogen atom = 1.66×10^{-24} gm,
Mass of a nitrogen molecule = $28 \times 1.66 \times 10^{-24}$ gm,
Mass of an oxygen molecule = $32 \times 1.66 \times 10^{-24}$ gm,
Diameter of a typical molecule = 3×10^{-8} cm.

68°F. How many pounds of water are needed per hour?

19–32. Calculate Boltzmann's constant $k = R/N_0$ in ergs per molecule per degree.

19–33. Consider an ideal gas at 0°C and at 1 atm pressure. Imagine each molecule to be, on the average, at the center of a small cube. (a) What is the length of an edge of this small cube? (b) How does this distance compare with the diameter of a molecule?

19–34. A mole of liquid water occupies a volume of 18 cm^3. Imagine each molecule to be, on the average, at the center of a small cube. What is the length of an edge of this small cube? How does this distance compare with the diameter of a molecule?

19–35. The velocity of sound in air at 27°C is about 1100 ft/sec. Compare this with the velocity of a nitrogen molecule at the same temperature.

19–36. (a) Compute the total random translational kinetic energy of the molecules in one mole of oxygen gas at a temperature of 27°C. (b) What is the velocity of a projectile having the same mass as one mole of oxygen if its translational kinetic energy equals the random kinetic energy of the molecule in part (a)?

19–37. At what temperature is the velocity of an oxygen molecule equal to the velocity of a hydrogen molecule at a temperature of 27°C?

CHAPTER 20

THE SECOND LAW OF THERMODYNAMICS

20–1 Introduction to the second law. The dominating feature of an industrial society is its ability to utilize, whether for wise or unwise ends, sources of energy other than the muscles of men or animals. Except for water power, where mechanical energy is directly available, most energy supplies are in the form of fuels such as coal or oil, where the energy is stored as internal energy. The process of combustion releases the internal energy and converts it to heat. In this form the energy may be utilized for heating habitations, for cooking, or for maintaining a furnace at high temperature in order to carry out other chemical or physical processes. But to operate a machine, or to propel a vehicle or a projectile, the heat must be converted to mechanical energy, and one of the problems of the mechanical engineer is to carry out this conversion with the maximum possible efficiency.

There is only one type of process in which internal energy can be converted directly to mechanical energy, and that is when the chemical substances can be combined in an electrolytic cell. All other methods involve the intermediate step of transforming internal energy into heat. The changes may be represented schematically by

$$\text{Internal energy} \rightarrow \text{Heat} \rightarrow \text{Mechanical energy}$$

The process represented by

$$\text{Internal energy} \rightarrow \text{Heat}$$

presents few difficulties. The most common example is, of course, the combustion of coal, oil, or gas. The problem then reduces to

$$\text{Heat} \rightarrow \text{Mechanical energy}$$

It is evident in the first place that this transformation always requires the services of some sort of *engine*, such as a steam engine, gasoline engine, or diesel engine.

At first sight the problem does not seem difficult, since we know that 1 Btu = 778 ft·lb, and it appears that every Btu of thermal energy should provide us with 778 ft·lb of mechanical energy. A pound of coal, for

instance, develops about 13,000 Btu when burned, and might be expected to provide 13,000 × 778 ft·lb of mechanical work. Actual steam engines, however, furnish only from about 5% to about 30% of this value. What becomes of the remaining 70% to 95%?

Stack and friction losses account for only a small part, by far the largest part appearing as heat rejected in the exhaust. No one has ever constructed a heat engine which does not throw away in its exhaust a relatively large fraction of the heat supplied to it, and it is safe to say that no one ever will. The impossibility of constructing an engine which with no other outstanding changes will convert a given amount of heat *completely* into mechanical work is a fundamental law of Nature, known as *the second law of thermodynamics.* The first law, it will be recalled, is a statement of the principle of conservation of energy, and merely imposes the restriction that one can obtain *no more* than 778 ft·lb of mechanical work from every Btu of heat. It does not in itself restrict the fraction of a given amount of heat which an engine can convert into mechanical energy. The second law goes beyond the first, and states that 100% conversion is not possible by any form of engine. Of course, for that fraction of the heat supplied to it which an engine *does* convert to mechanical form, the equivalence expressed by the first law must hold true.

A young French engineer, Sadi Carnot, was the first to approach the problem of the efficiency of a heat engine from a truly fundamental standpoint. Improvements in steam engines, up to the time of Carnot's work in 1824, had either been along the lines of better mechanical design or, if more basic improvements had been made, they had come about by chance or inspiration and had not been guided by any knowledge of basic principles. Carnot's contribution was a "theoretical" one, but it had more influence on the development of our industrial society in the 19th Century than the work of any of the "practical" men who had preceded him in this field.

Briefly, what Carnot did was to disregard the details of operation of a heat engine and focus attention on its truly significant features. These are, first, the engine is supplied with energy, in the form of heat, at a relatively high temperature. Second, the engine performs mechanical work. Third, the engine rejects heat at a lower temperature. In Carnot's time the caloric theory of heat as an indestructible fluid was still generally accepted. It is difficult to ascertain from Carnot's published articles exactly what his beliefs were as to the nature of work and heat, largely because of his failure to give *precise* definitions of the terms used. But although Carnot himself may have intuitively understood the problem, most scientists of his time pictured the flow of heat through an engine, from a higher to a lower temperature, as analogous to the flow of water

through a water wheel or turbine from a higher to a lower elevation. In any time interval, equal amounts of water enter the turbine and are discharged from it, but in the process some mechanical energy is abstracted from the water. Carnot believed that a similar process took place in a heat engine—some mechanical energy was abstracted from the heat but the amount of heat rejected by the engine in any time interval was equal to that delivered to the engine. We know now that this idea is incorrect, and that the heat rejected by the engine is less than the heat supplied to it by the amount that has been converted to mechanical work. In spite of his erroneous concept of the nature of heat, Carnot did in fact obtain the correct expression for the maximum efficiency of any heat engine operating between two given temperatures.

Since it is only heat and work that are of primary concern in a heat engine, we consider for simplicity an engine working in *closed cycles*. That is, the material that expands against a piston is periodically brought back to its initial condition so that in any one cycle the change in internal energy of this material, called the *working substance*, is zero. The condensing type of steam engine actually does operate in this way; the exhaust steam is condensed and forced back into the boiler so that the working substance (in this case, water) is used over and over again. The working substance then merely serves to transfer heat from one body to another and, in virtue of its changes in volume, to convert some of the heat to mechanical work.

The energy transformations in a heat engine are conveniently represented schematically by the *flow diagram* of Fig. 20–1. The engine itself is represented by the circle. The heat Q_2 supplied to the engine is proportional to the cross section of the incoming "pipeline" at the top of the diagram. The cross section of the outgoing pipeline at the bottom is proportional to that portion of the heat, Q_1, which is rejected as heat in the exhaust. The branch line to the right represents that portion of the heat supplied which the engine converts to mechanical work, W. Since the working substance is periodically returned to its initial state and the change in its internal energy in any number of complete cycles is zero, it follows from the first law of thermodynamics that

$$W = Q_2 - Q_1.$$

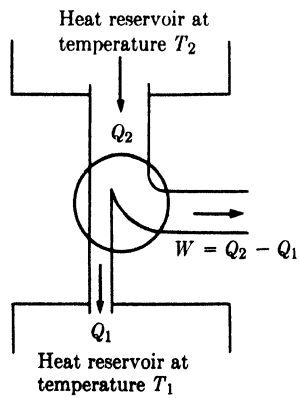


FIG. 20–1. Schematic flow diagram of a heat engine.

That is, the mechanical work done equals the difference between the heat supplied and the heat rejected. (For convenience, W, Q_2, and Q_1 are all considered positive.)

The *thermal efficiency* E of the engine is the ratio of work output to heat input. The output is the mechanical work W. The exhaust heat is not considered a part of the output. The input is the heat Q_2. Hence

$$E = \frac{\text{Work output}}{\text{Heat input}} \rightarrow \frac{W}{Q_2},$$

$$\boxed{E = \frac{Q_2 - Q_1}{Q_2}.} \qquad (20\text{–}1)$$

In terms of the flow diagram, the most efficient engine is the one for which the branch pipeline representing the work obtained is as large as possible, and the exhaust pipeline representing the heat rejected is as small as possible, for a given incoming pipeline or quantity of heat supplied.

We shall now consider, without going into the mechanical details of their construction, the internal combustion engine, the diesel engine, and the steam engine.

20–2 The internal combustion engine. The common internal combustion engine is of the four-cycle type, so called because four processes take place in each cycle. Starting with the piston at the top of its stroke, an explosive mixture of air and gasoline vapor is drawn into the cylinder on the downstroke, the inlet valve being open and the exhaust valve closed. This is the *intake* stroke. At the end of this stroke the inlet valve closes and the piston rises, performing an approximately adiabatic compression of the air-gasoline mixture. This is the *compression* stroke. At or near the top of this stroke a spark ignites the mixture of air and gasoline vapor, and combustion takes place very rapidly. The pressure and temperature increase at nearly constant volume.

The piston is now forced down, the burned gases expanding approximately adiabatically. This is the *power stroke* or *working stroke*. At the end of the power stroke the exhaust valve opens. The pressure in the cylinder drops rapidly to atmospheric and the rising piston on the *exhaust stroke* forces out most of the remaining gas. The exhaust valve now closes, the inlet valve opens, and the cycle is repeated.

For purposes of computation, the internal combustion cycle is approximated by the *air standard* or *Otto* cycle illustrated in Fig. 20–2.

Starting at point a, air at atmospheric pressure is compressed adiabatically in a cylinder to point b, heated at constant volume to point c, allowed to expand adiabatically to point d, and cooled at constant volume to point a, after which the cycle is repeated. Line ab corresponds to the compression stroke, bc to the explosion, cd to the working stroke, and da to the exhaust of an internal combustion engine. V_1 and V_2, in Fig. 20–2, are respectively the maximum and minimum volumes of the air in the cylinder. The ratio V_1/V_2 is called the *compression ratio*, and is about 7 for an internal combustion engine.

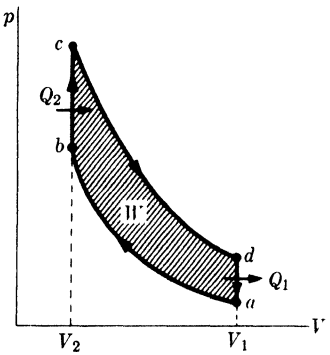


FIG. 20–2. p-V diagram of the Otto cycle.

The work output in Fig. 20–2 is represented by the shaded area enclosed by the figure $abcd$. The heat *input* is the heat supplied at constant volume along the line bc. The exhaust heat is removed along da. No heat is supplied or removed in the adiabatic processes ab and cd.

The heat input and the work output can be computed in terms of the compression ratio, assuming air to behave like an ideal gas. The result is

$$\text{Eff}(\%) = 100\left(1 - \frac{1}{(V_1/V_2)^{\gamma-1}}\right),$$

where γ is the ratio of the specific heats. For a compression ratio of 7 and a value of $\gamma = 1.4$, the efficiency is about 54%. It will be seen that the higher the compression ratio, the higher the efficiency. Friction effects, turbulence, loss of heat to cylinder walls, etc., have been neglected. All these effects reduce the efficiency of an actual engine below the figure given above.

20–3 The diesel engine. In the diesel cycle, air is drawn into the cylinder on the intake stroke and compressed adiabatically on the compression stroke to a sufficiently high temperature so that fuel oil injected at the end of this stroke burns in the cylinder without requiring ignition by a spark. The combustion is not as rapid as in the gasoline engine, and the first part of the power stroke proceeds at essentially constant pressure. The remainder of the power stroke is an adiabatic expansion. This is followed by an exhaust stroke which completes the cycle.

The idealized air-diesel cycle is shown in Fig. 20–3. Starting at point a, air is compressed adiabatically to point b, heated at constant pressure

to point c, expanded adiabatically to point d, and cooled at constant volume to point a.

Since there is no fuel in the cylinder of a diesel engine on the compression stroke, pre-ignition cannot occur and the compression ratio V_1/V_2 may be much higher than that of an internal combustion engine. A value of 15 is typical. The *expansion* ratio V_1/V_3 may be about 5. Using these values, and taking $\gamma = 1.4$, the efficiency of the air-diesel cycle is about 56%. Hence somewhat higher efficiencies are possible than for the Otto cycle. Again, the actual efficiency of a real diesel must be smaller than the value given above.

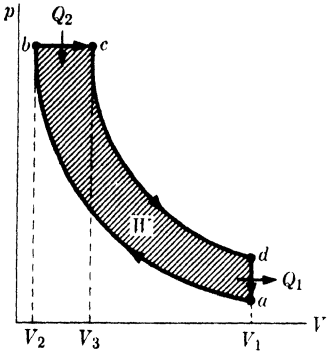


FIG. 20–3. p-V diagram of the diesel cycle.

20–4 The steam engine. The condensing type of steam engine performs the following sequence of operations. Water is converted to steam in the boiler, and the steam thus formed is superheated above the boiler temperature. Superheated steam is admitted to the cylinder, where it expands against a piston; connection is maintained to the boiler for the first part of the working stroke, which thus takes place at constant pressure. The inlet valve is then closed and the steam expands adiabatically for the rest of the working stroke. The adiabatic cooling causes some of the steam to condense. The mixture of water droplets and steam (known as "wet" steam) is forced out of the cylinder on the return stroke and into the condenser, where the remaining steam is condensed into water. This water is forced into the boiler by the feed pump, and the cycle is repeated.

An idealized cycle (called the rankine cycle) which approximates the actual steam cycle is shown in Fig. 20–4. Starting with liquid water at low pressure and temperature (point a), the water is compressed adiabatically to point b at boiler pressure. It is then heated at constant pressure to its boiling point (line bc), converted to steam (line cd), superheated (line de), expanded adiabatically (line ef), and cooled and condensed (along fa) to its initial condition.

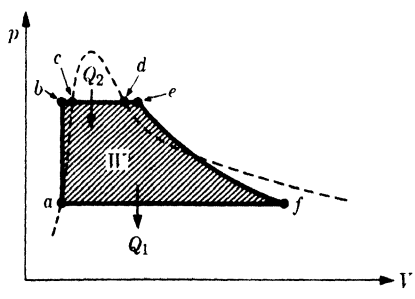


FIG. 20–4. The rankine cycle.

The efficiency of such a cycle may be computed in the same way as was done in the previous examples, by finding the quantities of heat taken in and rejected along the lines *be* and *fa*. Assuming a boiler temperature of 417°F (corresponding to a pressure of 300 lb/in^2), a superheat of 63°F above this temperature (480°F), and a condenser temperature of 102°F, the efficiency of a rankine cycle is about 32%. Efficiencies of actual steam engines are, of course, considerably lower.

20–5 The second law of thermodynamics. Our experience with actual engines therefore leads us to the second law of thermodynamics, which may be stated rigorously as follows:

It is impossible to construct an engine that, operating in a cycle, will produce no effect other than the extraction of heat from a source and the conversion of this heat completely into work.

If the second law were not true, it would be possible to drive a steamship across the ocean by extracting heat from the ocean, or to run a power plant by extracting heat from the surrounding air. It should be noted that neither of these "impossibilities" violates the first law of thermodynamics. After all, both the ocean and the surrounding air contain an enormous store of internal energy which, in principle, may be extracted in the form of a flow of heat. There is nothing in the first law to preclude the possibility of converting this heat completely into work. The second law, therefore, is not a deduction from the first but stands by itself as a separate law of nature, referring to an aspect of nature different from that contemplated by the first law. The first law denies the possibility of creating or destroying energy; the second denies the possibility of utilizing energy in a particular way.

The fact that work may be dissipated completely into heat, whereas heat may not be converted entirely into work, expresses an essential one-sidedness of nature. All natural, spontaneous processes may be studied in the light of the second law, and in all such cases, this peculiar one-sidedness is found. Thus, heat always flows spontaneously from a hotter to a colder body; gases always seep through an opening spontaneously from a region of high pressure to a region of low pressure; gases and liquids left by themselves always tend to mix, not to unmix. Salt dissolves in water but a salt solution does not separate by itself into pure salt and pure water. Rocks weather and crumble; iron rusts; people grow old. These are all examples of *irreversible* processes that take place naturally in only one direction and, by their one-sidedness, express the second law of thermodynamics.

Irreversible, natural processes may be regarded from another point of view. A piece of pure salt and a volume of pure water represent an

orderly arrangement of molecules. The solution of the salt in the water involves an increase in molecular disorder. If all the molecules of gas in a container were in one corner of the container, that would constitute an orderly arrangement. The uniform distribution throughout the container which actually exists is a much more disorderly arrangement. The sand that results from the weathering of rocks over a long period of time represents a greater disorder than the original well-formed rocks. Thus, the one-sidedness of nature may be redescribed by stating that *there is a tendency in nature to proceed toward a state of greater molecular disorder.*

20–6 The refrigerator. A refrigerator may be considered to be a heat engine operated in reverse. That is, a heat engine takes in heat from a *high* temperature source, converts a part of the heat into mechanical work output, and rejects the difference as heat in the exhaust at a *lower* temperature. A refrigerator takes in heat at a *low* temperature, the compressor supplies mechanical work *input*, and the sum is rejected as heat at a *higher* temperature.

The flow diagram of a refrigerator is given in Fig. 20–5. In terms of the processes in a household mechanical refrigerator, Q_1 represents the heat removed from the refrigerator by the cooling coils within it, W the work done by the motor, and Q_2 the heat delivered to the external cooling coils and removed by circulating air or water. It follows from the first law that

$$Q_2 = Q_1 + W.$$

That is, the circulating air or water must absorb both the heat "pumped" out of the refrigerator and the heat equivalent of the work done by the motor.

From an economic point of view, the best refrigeration cycle is one that removes the greatest amount of heat Q_1 from the refrigerator, for

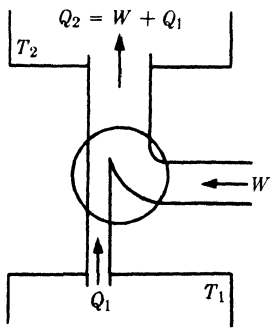


FIG. 20–5. Schematic flow diagram of a refrigerator.

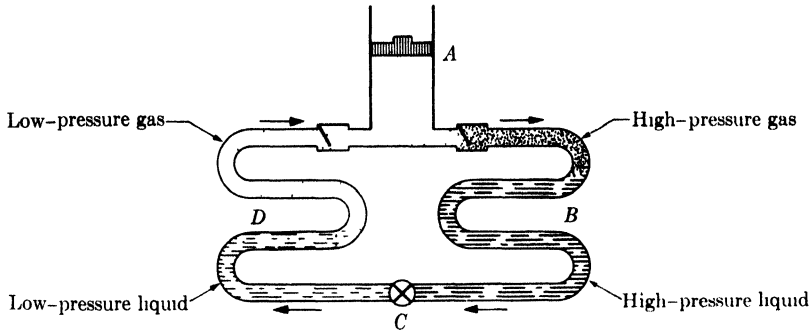


FIG. 20–6. Principle of the mechanical refrigeration cycle.

the least expenditure of mechanical work W. We therefore define the *coefficient of performance* (rather than the efficiency) of a refrigerator as the ratio Q_1/W, and since $W = Q_2 - Q_1$,

$$\text{Coefficient of performance} = \frac{Q_1}{Q_2 - Q_1}. \tag{20–2}$$

The principles of the common refrigeration cycle are illustrated schematically in Fig. 20–6. Compressor A delivers gas (CCl_2F_2, NH_3, etc.) at high temperature and pressure to coils B. Heat is removed from the gas in B by water or air cooling, resulting in condensation of the gas to a liquid, still under high pressure. The liquid passes through the throttling valve or expansion valve C, emerging as a mixture of liquid and vapor at a lower temperature. In coils D, heat is supplied that converts the remaining liquid into vapor which enters compressor A to repeat the cycle. In a domestic refrigerator, coils D are placed in the ice compartment, where they cool the refrigerator directly. In a larger refrigerating plant, these coils are usually immersed in a brine tank and cool the brine, which is then pumped to the refrigerating rooms.

A simplified diagram of the so-called *gas refrigerator* is given in Fig. 20–7. In the generator, a solution of ammonia in water is heated by a small gas flame. Ammonia is driven out of solution and ammonia vapor rises in the liquid lift tube, carrying with it some of the water in the same way that water is raised in the central tube of a coffee percolator. This water collects in the separator, from which point it flows back through the absorber, while the ammonia vapor rises to the condenser. Here the ammonia vapor is liquefied, its heat of condensation being removed by air circulating around the cooling vanes. The liquid ammonia then flows into the evaporator, located in the cooling unit of the refrigerator,

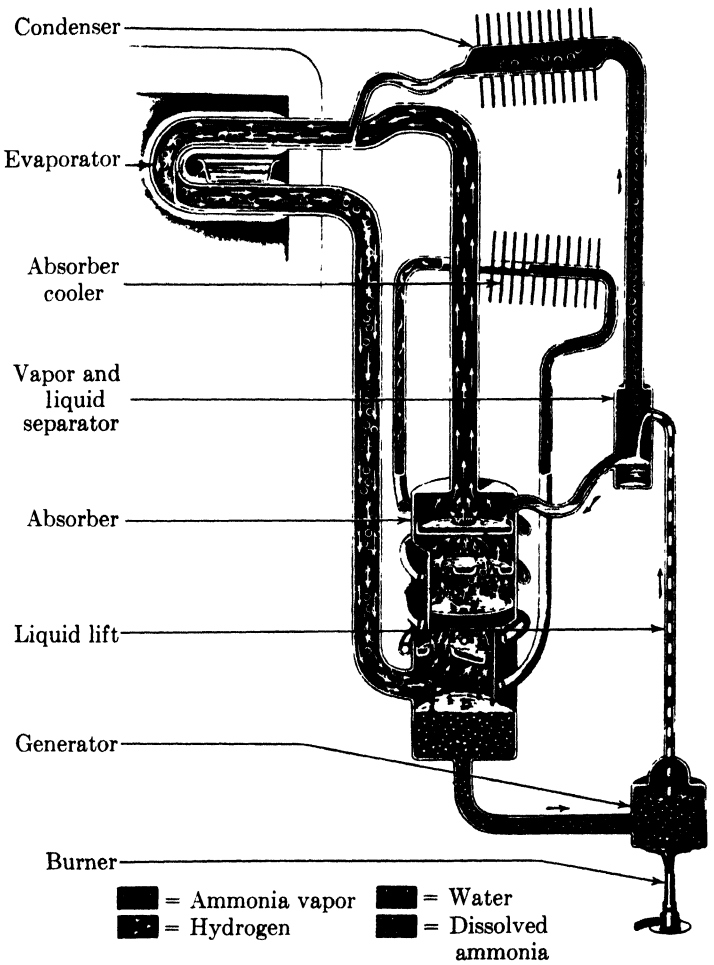


FIG. 20–7. Simplified diagram of the gas refrigerator. (Courtesy of Servel-Electrolux.)

where it evaporates and in so doing absorbs heat from its surroundings. The ammonia vapor continues on to the absorber, where it dissolves in the water returning from the separator. The ammonia-water solution then flows to the generator, completing the cycle.

The absorber and evaporator also contain hydrogen gas which is maintained in circulation by a convection process, brought about by the fact that the mixture of ammonia and hydrogen in the tube at the extreme left is denser than the pure hydrogen in the tube leading from the top

of the absorber. This current of hydrogen, entering at the top of the evaporator, sweeps the ammonia vapor out of the evaporator and aids in rapid evaporation. Since ammonia is much more readily soluble in water than is hydrogen, most of the ammonia is dissolved in the water trickling down through the absorber, while the hydrogen passes upward through the absorber.

It is necessary that heat be removed from the absorber as well as from the condenser because heat is liberated when ammonia vapor dissolves in water. This is accomplished by the auxiliary circuit made up of the cooling coils around the absorber, and the absorber cooler.

If no work were needed to operate a refrigerator, the coefficient of performance (heat extracted divided by work done) would be infinite. Coefficients of performance of actual refrigerators vary from about 2 to about 6. Experience shows that work is always needed to transfer heat from a colder to a hotter body. This negative statement leads to another statement of the second law of thermodynamics, namely:

> *It is impossible to construct a refrigerator that, operating in a cycle, will produce no effect other than the transfer of heat from a cooler to a hotter body.*

At first sight, this and the previous statement of the second law appear to be quite unconnected, but it can be shown that they are in all respects equivalent. Any device that would violate one statement would violate the other.

20–7 The Carnot cycle. Although their efficiencies differ from one another, none of the heat engines which have been described has an efficiency of 100%. The question still remains open as to what is the maximum attainable efficiency, given a supply of heat at one temperature and a reservoir at a lower temperature for cooling the exhaust. An idealized engine which can be shown to have the maximum efficiency under these conditions was invented by Carnot and is called a *Carnot engine.* The *Carnot cycle,* shown in Fig. 20–8, differs from the Otto and diesel cycles in that it is bounded by two *isothermals* and two adiabatics. Thus all the heat input is supplied at a *single* high temperature and all the heat output is rejected at a *single* lower temperature. (Compare with Figs. 20–2 and 20–3, in which the temperature is different at all points of the lines *bc* and *da.*)

This, however, is not the only feature of the Carnot cycle. There are no "one-way" processes in the Carnot cycle, such as explosions or throttling processes. The isothermal and adiabatic processes of the Carnot cycle may be imagined to proceed in either direction. In the direction shown

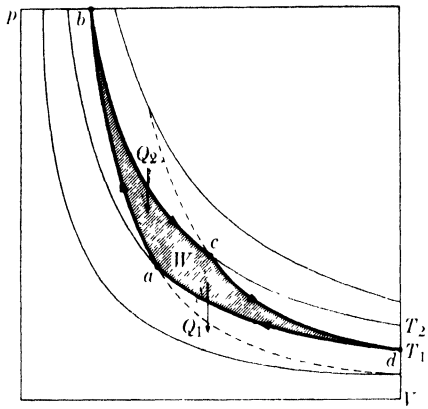


FIG. 20–8. The Carnot cycle.

in Fig. 20–8, heat Q_2 goes in, heat Q_1 goes out, and work W is done by the engine. The arrows in the figure could be reversed, in which case the cycle would be a refrigeration cycle. Then heat Q_2 would go out, heat Q_1 would go in, and work W would have to be done on the refrigerator.

Suppose an engine (not a Carnot engine) were to operate between a source of heat at some temperature and a reservoir of heat at a lower temperature, thereby delivering to the outside an amount of work W. Suppose this work W were used to operate a Carnot refrigerator which extracted heat from the colder reservoir and delivered it to the warmer source. It can be shown that if the first engine were more efficient than the Carnot engine that would result by operating the Carnot refrigerator backward, then the net effect would be a violation of the second law of thermodynamics. Proceeding along these lines, it has been proved that:

No engine operating between two given temperatures can be more efficient than a Carnot engine operating between the same two temperatures,

and also:

All Carnot engines operating between the same two temperatures have the same efficiency, irrespective of the nature of the working substance.

20–8 The kelvin temperature scale. It was shown in the beginning of this chapter that the efficiency of any engine is equal to

$$E = \frac{W}{Q_2} = \frac{Q_2 - Q_1}{Q_2} = 1 - \frac{Q_1}{Q_2}.$$

If, therefore, the efficiencies of all Carnot engines operating between the same two temperatures are the same, irrespective of the working substance, the ratio Q_1/Q_2 must depend only on the two temperatures and on nothing else. Lord Kelvin proposed that this fact be used to define a temperature scale which would be independent of the properties of any particular substance, unlike the gas thermometer scale described in Chapter 15.

The kelvin temperatures of the reservoir and the source between which a Carnot engine operates are defined by the relation

$$\boxed{\frac{Q_1}{Q_2} = \frac{T_1}{T_2},} \tag{20–3}$$

where Q_1/Q_2 is the ratio of the heats rejected and absorbed, and T_1/T_2 is the ratio of the kelvin temperatures of the reservoir and the source.

At first thought it might seem that the ratio of two kelvin temperatures would be impossible to measure, since a Carnot engine is an ideal engine, quite impossible to construct. The situation, however, is not so bad as it seems. The ratio of two kelvin temperatures is the ratio of two heats that are transferred during two isothermal processes bounded by the same two adiabatics. The two adiabatic boundaries may be located experimentally, and the heats transferred during two isothermal "nearly reversible" processes can be measured with considerable precision. As a matter of fact, this is one of the methods used in measuring temperatures below 1°K.

To complete the definition of the kelvin scale we proceed, as in Chapter 15, to assign the arbitrary value of 273.16°K to the temperature of the triple point of water T_3. For a Carnot engine operating between reservoirs at the temperatures T and T_3, we have

$$\frac{Q}{Q_3} = \frac{T}{T_3}$$

or

$$\boxed{T = 273.16°\text{K}\,\frac{Q}{Q_3}.} \tag{20–4}$$

Comparing this with the corresponding expression for the ideal gas temperature, namely,

$$273.16°\text{K} \lim_{p_3 \to 0} \left(\frac{p}{p_3}\right)_{\text{constant volume}},$$

it is seen that, in the kelvin scale, Q plays the role of a "thermometric

property." This does not, however, have the objection attached to a coordinate of an arbitrarily chosen thermometer, inasmuch as the behavior of a Carnot engine is independent of the nature of the working substance.

Suppose we temporarily denote the temperature that is measured with a gas thermometer by the symbol T'. Then, using the ideal gas equation $pV = nRT'$, we may compute the heat rejected, Q_1, and the heat absorbed, Q_2, when an ideal gas undergoes a Carnot cycle. The result of this calculation is that

$$\frac{Q_1}{Q_2} = \frac{T'_1}{T'_2},$$

where T'_1 and T'_2 are the gas temperatures of the reservoir and source respectively. This, however, is exactly the same as the defining equation of the ratio of two kelvin temperatures. Therefore, since the temperature of the triple point of water is the same on both scales, we conclude that the gas temperature and the kelvin temperature are identical.

Now, the efficiency of a Carnot engine rejecting heat Q_1 to a reservoir at kelvin temperature T_1 and absorbing heat Q_2 from a source at kelvin temperature T_2 is, as usual,

$$E = 1 - \frac{Q_1}{Q_2}.$$

But, by definition of the kelvin scale,

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}.$$

Therefore, the efficiency of a Carnot engine is

$$\boxed{E(\text{Carnot}) = 1 - \frac{T_1}{T_2}.} \tag{20–5}$$

Equation (20–5) points the way to the conditions which a real engine, such as a steam engine, must fulfill to approach as closely as possible the maximum attainable efficiency. These conditions are that the intake temperature T_2 must be made as high as possible and the exhaust temperature T_1 as low as possible.

The exhaust temperature cannot be lower than the lowest temperature available for cooling the exhaust. This is usually the temperature of the air, or perhaps of river water if this is available at the plant. The only recourse then is to raise the boiler temperature T_2. Since the vapor pressure of all liquids increases rapidly with increasing temperature, a limit is set by the mechanical strength of the boiler. Another possibility

is to use, instead of water, some liquid with a lower vapor pressure. Successful experiments in this direction have been made with mercury vapor replacing steam. At a boiler temperature of 200°C, at which the pressure in a steam boiler would be 225 lb/in^2, the pressure in a mercury boiler is only 0.35 lb/in^2.

20–9 Absolute zero. It follows from Eq. (20–4) that the heat transferred isothermally between two given adiabatics decreases as the temperature decreases. Conversely, the smaller the value of Q, the lower the corresponding T. The smallest possible value of Q is zero, and the corresponding T is absolute zero. *Thus, if a system undergoes a reversible isothermal process without transfer of heat, the temperature at which this process takes place is called absolute zero.* In other words, at absolute zero, an isotherm and an adiabatic are identical.

It should be noted that the definition of absolute zero holds for all substances and is therefore independent of the peculiar properties of any one arbitrarily chosen substance. Furthermore, the definition is in terms of purely macroscopic concepts. No reference is made to molecules or to molecular energy. Whether absolute zero may be achieved experimentally is a question of some interest and importance. To achieve temperatures below 4.2°K at which ordinary helium (mass number 4) liquefies, it is necessary to lower the vapor pressure by pumping away the vapor as fast as possible. The lowest temperature that has ever been reached in this way is 0.7°K, and this required larger pumps and larger pumping tubes than are usually employed in low-temperature laboratories. With the aid of the light isotope of helium (mass number 3) which liquefies at 3.2°K, vigorous pumping will yield a temperature of about 0.3°K. Still lower temperatures may be achieved magnetically, but it becomes apparent that, the closer one approaches absolute zero, the more difficult it is to go farther. It is generally accepted as a law of Nature that, although absolute zero may be approached as close as we please, it is impossible actually to reach the zero of temperature. This is known as the *"unattainability statement of the third law of thermodynamics."*

PROBLEMS

20–1. The efficiency of an Otto cycle is 50% and $\gamma = 1.50$. What is the compression ratio?

20–2. A Carnot engine whose high-temperature reservoir is at 127°C takes in 100 cal of heat at this temperature in each cycle, and gives up 80 cal to the low temperature reservoir. Find the temperature of the latter reservoir.

20–3. A Carnot engine whose low temperature reservoir is at 7°C has an efficiency of 40%. It is desired to increase the efficiency to 50%. By how many degrees must the temperature of the high-temperature reservoir be increased?

20–4. A Carnot engine is operated between two heat reservoirs at temperatures of 400°K and 300°K. (a) If in each cycle the engine receives 1200 cal of heat from the reservoir at 400°K, how many calories does it reject to the reservoir at 300°K? (b) If the engine is operated in reverse, as a refrigerator, and receives 1200 cal of heat from the reservoir at 300°K, how many calories does it deliver to the reservoir at 400°K? (c) How many calories would be produced if the mechanical work required to operate the refrigerator in part (b) were converted directly to heat?

20–5. What is the efficiency of an engine which operates by taking an ideal monatomic gas through the following cycle? Let $C_v = 3$ cal/mole·C°. (a) Start with n moles at p_0, V_0, T_0. (b) Change to $2p_0$, V_0 at constant volume. (c) Change to $2p_0$, $2V_0$ at constant pressure. (d) Change to p_0, $2V_0$ at constant volume. (e) Change to p_0, V_0 at constant pressure.

20–6. A Carnot refrigerator takes heat from water at 0°C and discards it to the room at a temperature of 27°C. 100 kgm of water at 0°C are to be changed to ice at 0°C. (a) How many calories of heat are discarded to the room? (b) What is the required work in joules?

20–7. A cylinder contains air at a pressure of 2 atm. The volume is 3 liters and the temperature is 300°K. The air is carried through the following processes:

(1) Heated at constant pressure to 500°K.
(2) Cooled at constant volume to 250°K.
(3) Cooled at constant pressure to 150°K.
(4) Heated at constant volume to 300°K.

(a) Show each process on a p-V diagram, giving the numerical values of p and V at the end of each process. (b) Calculate the net work done by the gas.

20–8. For the data in Problem 20–7 find (a) the number of moles of air in the cylinder. (b) For air $C_p = 7$ cal/mole·deg and $C_V = 5$ cal/mole·deg. Find the total heat input to the cylinder in processes (1) and (4). Express the answer in calories. (c) What is the efficiency of this device as a heat engine?

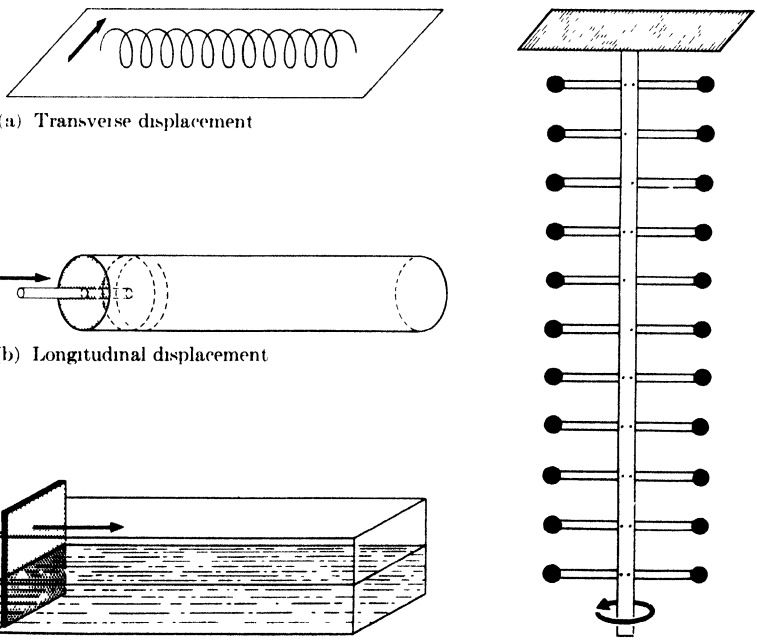
20–9. (a) What is the coefficient of performance of a Carnot refrigerator which removes heat from a reservoir at −10°C and delivers heat to a reservoir at 30°C? (b) How many kwh of energy would have to be supplied to the refrigerator to remove from the low-temperature reservoir an amount of heat equal to that required to melt 100 lb of ice? (c) What would be the cost of this energy, at 5 cents/kwh?

CHAPTER 21

WAVE MOTION

21–1 Propagation of a disturbance in a medium. Imagine a medium consisting of a large number of particles, each connected or coupled to its neighbors by *elastic* material. If one end of the medium is disturbed or displaced in any way, the displacement will not occur immediately at all other parts of the medium. The original displacement will give rise to an elastic force in the material adjacent to it, then the next particle will be displaced, and then the next, and so on. In other words, *the displacement will be propagated along the medium with a definite speed.*

In Fig. 21–1(a) the medium is a spring, or just a wire under tension. If the left end is given a small displacement in a direction perpendicular to the medium, this transverse displacement will occur at successive



(a) Transverse displacement

(b) Longitudinal displacement

(c) Longitudinal and transverse displacement

(d) Torsional displacement

Fig. 21–1. Propagation of disturbances.

intervals of time at each coil of the spring and there will result the propagation of a *transverse pulse* along the spring.

In Fig. 21–1(b) the medium is to be regarded as either a liquid or a gas contained in a tube closed at the right end with a rigid wall and at the left end with a movable piston. If the piston is moved slightly toward the right, a *longitudinal pulse* will be propagated through the medium in the tube.

In Fig. 21–1(c) the medium is a liquid contained in a trough. The horizontal motion of a flat piece of wood at the left end will provide a displacement of the liquid which is both longitudinal and slightly transverse, and this disturbance will travel along the medium.

In Fig. 21–1(d) the medium is a set of "dumbbells" connected to a steel strip. A slight rotation of the lowest dumbbell constitutes a *torsional displacement* which will be propagated up the medium with a finite speed.

It will be shown in the next section that the speed of a pulse produced by a *small* displacement depends only on certain physical properties *of the medium itself* and not on the rapidity of the original displacement.

21–2 Calculation of the speed of a transverse pulse. Consider the string depicted in Fig. 21–2, under a tension T and with linear density (mass per unit length) μ. In Fig. 21–2(a) the string is at rest. At time $t = 0$, the left end of the string is given a transverse speed v. Figure 21–2(b) shows the shape of the string after a time t has elapsed. All points of the string at the left of the point P are moving with speed v, whereas all points at the right of P are still at rest. The boundary between the moving and the stationary portions is traveling to the right with the *speed of propagation* u. The left end of the string has moved up a distance vt, and the boundary point P has advanced a distance ut.

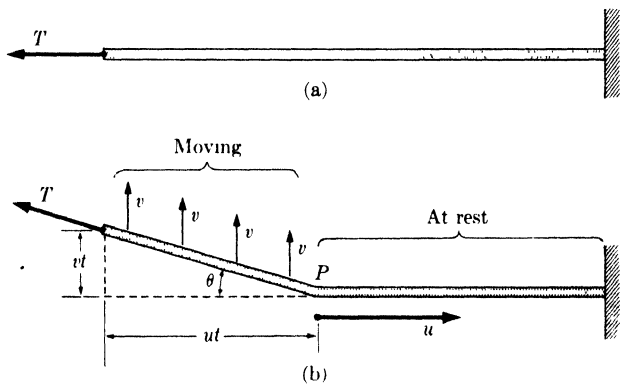


FIG. 21–2. Propagation of a transverse disturbance in a string.

The speed of propagation u can be calculated by setting the transverse impulse (transverse force $\times$ time) equal to the change of transverse momentum of the moving portion (mass $\times$ transverse velocity). The transverse component of the force T is $T \sin \theta$. If the angle θ is very small, $\sin \theta$ is very nearly equal to $\tan \theta$, and hence

$$\sin \theta = \tan \theta = \frac{vt}{ut} = \frac{v}{u}.$$

Hence

$$\text{Transverse impulse} = (T \sin \theta) \cdot t = T \frac{v}{u} t.$$

The mass of the moving portion is the product of the mass per unit length μ and the length ut. Hence

$$\text{Transverse momentum} = \mu utv.$$

Applying the impulse-momentum theorem, we get

$$T \frac{v}{u} t = \mu utv,$$

and therefore

$$\boxed{u = \sqrt{T/\mu}. \quad \text{(Transverse)}} \tag{21-1}$$

Thus it is seen that the velocity of propagation of a transverse pulse in a string depends only on the tension and the mass per unit length.

In numerical applications of Eq. (21-1), attention must be paid to the units employed. With cgs units, T must be expressed in dynes and μ in grams per centimeter. With mks units, T must be in newtons and μ in kilograms per meter. Finally, in engineering units, T must be in pounds and μ in slugs per foot. The corresponding units of velocity will then be, respectively, centimeters per second, meters per second, and feet per second.

EXAMPLE. Calculate the velocity of a transverse pulse in a string under a tension of 20 lb if the string weighs 0.003 lb/ft.

$$T = 20 \text{ lb},$$

$$\mu = \frac{0.003 \text{ lb/ft}}{32 \text{ ft/sec}^2},$$

$$u = \sqrt{\frac{20 \text{ lb} \times 32 \text{ ft/sec}^2}{0.003 \text{ lb/ft}}} = 461 \frac{\text{ft}}{\text{sec}}.$$

21–3 Calculation of the speed of a longitudinal pulse. Figure 21–3 shows a fluid (liquid or gas) of density ρ in a tube of cross-sectional area A and under a pressure p. In Fig. 21–3(a) the fluid is at rest. At time $t = 0$, the piston at the left end of the tube is set in motion toward the right with a speed v. Figure 21–3(b) shows the fluid after a time t has elapsed. All portions of the fluid at the left of point P are moving with speed v, whereas all portions at the right of P are still at rest. The boundary between the moving and the stationary portions travels to the right with the speed of propagation u. The piston has moved a distance vt and the boundary has advanced a distance ut. As for a transverse disturbance in a string, the speed of propagation can be computed from the impulse-momentum theorem.

The quantity of fluid set in motion in time t is that originally occupying a volume of length ut and of cross-sectional area A. The mass of this fluid is therefore ρutA and the longitudinal momentum it has acquired is

$$\text{Longitudinal momentum} = \rho utAv.$$

We next compute the increase of pressure Δp in the moving fluid. The original volume of the moving fluid, Aut, has been decreased by an amount Avt. From the definition of bulk modulus B (see Chapter 10),

$$B = \frac{\text{change in pressure}}{\text{fractional change in volume}} = \frac{\Delta p}{Avt/Aut}.$$

Therefore

$$\Delta p = B\,\frac{v}{u}.$$

The pressure in the moving fluid is therefore $p + \Delta p$, and the force exerted on it by the piston is $(p + \Delta p)A$. The *net* force on the moving

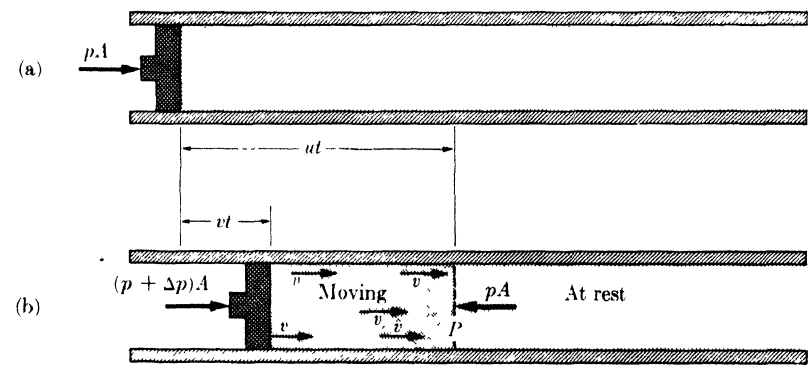


FIG. 21–3. Propagation of a longitudinal disturbance in a fluid confined in a tube.

fluid [see Fig. 21-3(b)] is ΔpA, and the longitudinal impulse is

$$\text{Longitudinal impulse} = \Delta pAt = B\frac{v}{u}At.$$

Applying the impulse-momentum theorem, we get

$$B\frac{v}{u}At = \rho utAv,$$

and therefore

$$\boxed{u = \sqrt{B/\rho}. \qquad \text{(Longitudinal)}} \tag{21-2}$$

The speed of propagation of a longitudinal pulse in a fluid therefore depends only on the bulk modulus and density of the medium.

When a solid bar is struck a blow at one end, the situation is somewhat different from that of a fluid confined in a tube of constant cross section, since the bar will expand slightly sidewise when it is compressed longitudinally. It can be shown by the same type of reasoning as that just given that the velocity of a longitudinal pulse in the bar is given by

$$\boxed{u = \sqrt{Y/\rho}, \qquad \text{(Longitudinal)}} \tag{21-3}$$

where Y is Young's modulus, defined in Chapter 10.

21-4 The motion of a wave. Up to this point we have considered the simplest type of disturbance that could be imparted to a medium, namely, a single transverse or longitudinal displacement. Suppose now that one end of a medium is forced to vibrate periodically, the displacement y (either transverse or longitudinal) varying with the time according to the equation of simple harmonic motion:

$$y = A\cos 2\pi ft.$$

During half a cycle, a displacement in one direction is propagated through the medium, and during the other half, a displacement in the opposite direction is caused to proceed. The resulting continuous train of disturbances traveling with a speed depending on the properties of the medium is called a *wave*.

To fix our ideas, suppose that one end of a stretched string is forced to vibrate periodically in a transverse direction with simple harmonic motion of amplitude A, frequency f, and period $T = 1/f$. For the present we shall assume the string to be long enough so that any effects at the

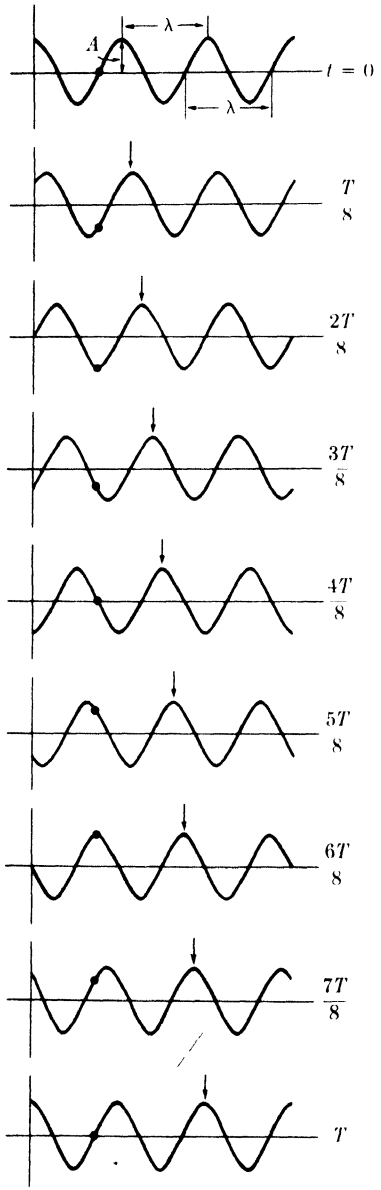


FIG. 21–4. A sinusoidal transverse wave traveling toward the right, shown at intervals of $\frac{1}{8}$ of a period.

far end need not be considered. A *continuous train* of transverse sinusoidal waves then advances along the string. The shape of a portion of the string near the end, at intervals of $\frac{1}{8}$ of a period, is shown in Fig. 21–4 for a total time of one period. The string is assumed to have been vibrating for a sufficiently long time so that the shape of the string is sinusoidal for an indefinite distance from the driven end. It will be seen from the figure that the wave form advances steadily toward the right, as indicated by the short arrow pointing to one particular wave crest, while any one point on the string (see the black dot) oscillates about its equilibrium position with simple harmonic motion. It is important to distinguish between the motion of the *wave form*, which moves with constant velocity u along the string, and the motion of *a particle of the string*, which is simple harmonic and transverse to the string.

The distance between two successive maxima (or between any two successive points in the same phase) is the *wavelength* of the wave and is denoted by λ. Since the wave form, traveling with constant velocity u, advances a distance of one wavelength in a time interval of one period, it follows that $u = \lambda T$, or

$$\boxed{u = f\lambda.} \qquad (21\text{–}4)$$

That is, *the velocity of propagation equals the product of frequency and wavelength.*

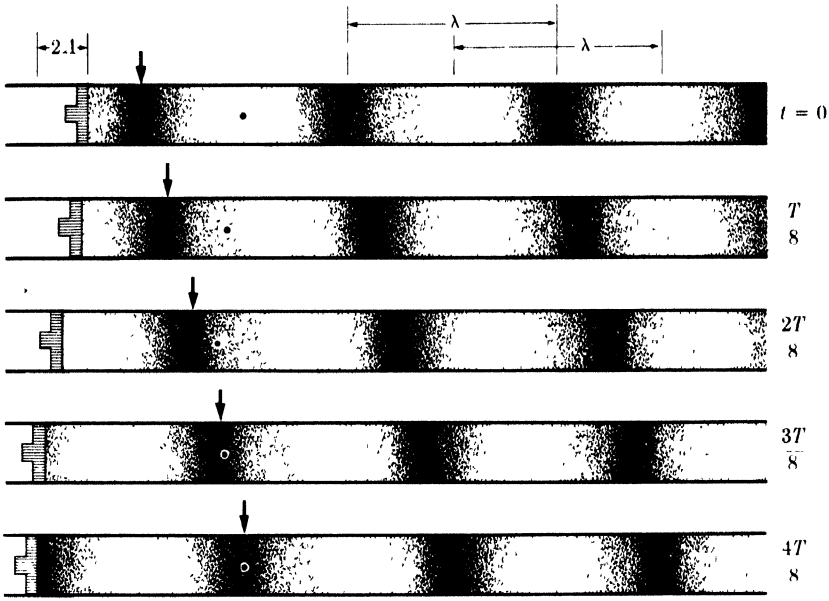


FIG. 21-5. A sinusoidal longitudinal wave traveling toward the right shown at intervals of $\frac{1}{8}$ of a period.

To understand the mechanics of a longitudinal wave, consider the long tube filled with a fluid and provided with a plunger at the left end, as shown in Fig. 21-5. The dots represent particles of the fluid. Suppose the plunger is forced to undergo a simple harmonic vibration parallel to the direction of the tube. During a part of each oscillation, a region whose pressure is above the equilibrium pressure is formed. Such a region is called a *condensation* and is represented by closely spaced dots. Following the production of a condensation, a region is formed in which the pressure is lower than the equilibrium value. This is called a *rarefaction* and is represented by widely spaced dots. The condensations and rarefactions move to the right with constant velocity u. The motion of a single particle of the medium, shown by a heavy black dot, is simple harmonic, parallel to the direction of propagation.

The wavelength is the distance between two successive condensations or two successive rarefactions, and the same fundamental equation, $u = f\lambda$, holds in this, as in all types of waves.

21-5 Adiabatic character of a longitudinal wave. It is a familiar fact that compression of a fluid causes a rise in its temperature unless heat is withdrawn in some way. Conversely, an expansion is accompanied

by a temperature decrease unless heat is added. As a longitudinal wave advances through a fluid, the regions which are compressed at any instant are slightly warmer than those that are expanded. The condition is present, therefore, for the conduction of heat from a condensation to a rarefaction. The quantity of heat conducted per unit time and per unit area depends on the thermal conductivity of the fluid and upon the distance between a condensation and its adjacent rarefaction (half a wavelength). Now for ordinary frequencies, say from 20 to 20,000 vibrations per second, and for even the best known heat conductors, the wavelength is too large and the thermal conductivity too small for an appreciable amount of heat to flow. The compressions and rarefactions are therefore *adiabatic* rather than isothermal.

In the expression for the speed of a longitudinal wave in a fluid, $u = \sqrt{B/\rho}$, the bulk modulus B is defined by the relation

$$B = \frac{\text{change of pressure}}{\text{change of volume per unit volume}}.$$

The change in volume produced by a given change of pressure depends upon whether the compression (or expansion) is adiabatic or isothermal. There are therefore two bulk moduli, the adiabatic bulk modulus B_{ad} and the isothermal bulk modulus. The rigorous expression for the speed of a longitudinal wave should therefore be written

$$\boxed{u = \sqrt{B_{ad}/\rho}.} \qquad (21\text{–}5)$$

In the case of an ideal gas, the relation between pressure p and volume V during an adiabatic process is given by Eq. (19–21),

$$pV^{\gamma} = \text{constant},$$

where γ is the ratio of the heat capacity at constant pressure to the heat capacity at constant volume. It can be shown with the aid of the calculus that

$$B_{ad} = \gamma p. \qquad \text{(Ideal gas)} \qquad (21\text{–}6)$$

Therefore

$$u = \sqrt{\gamma p/\rho}. \qquad \text{(Ideal gas)} \qquad (21\text{–}7)$$

But, for an ideal gas,

$$\frac{p}{\rho} = \frac{RT}{M},$$

where R is the universal gas constant and M the molecular weight. Therefore

$$u = \sqrt{\gamma RT/M}, \qquad \text{(Ideal gas)} \tag{21-8}$$

and since for a given gas, γ, R, and M are constants, we see that the velocity of propagation is proportional to the square root of the absolute temperature.

Let us use Eq. (21-8) to compute the velocity of longitudinal waves in air. The mean molecular weight of air is 28.8, $\gamma = 1.40$, and $R = 8.31 \times 10^7$ ergs/mole·deg. Let $T = 300°\text{K}$. Then

$$u = \sqrt{\frac{1.40 \times 8.31 \times 10^7 \text{ dynes·cm/mole·deg} \times 300 \text{ deg}}{28.8 \text{ gm/mole}}}$$

$$= 34{,}800 \frac{\text{cm}}{\text{sec}} = 348 \frac{\text{m}}{\text{sec}} = 1140 \frac{\text{ft}}{\text{sec}}.$$

This is in good agreement with the measured velocity at this temperature.

Longitudinal waves in air give rise to the sensation of sound. The ear is sensitive to a range of sound frequencies from about 20 to about 20,000 cycles/sec. From the relation $u = f\lambda$, the corresponding wavelength range is from about 56 ft, corresponding to a 20-cycle note, to about 0.056 ft or 5/8 inch, corresponding to 20,000 cycles/sec.

21-6 Mathematical representation of a wave. Suppose that a sinusoidal wave (either transverse or longitudinal) is advancing toward the right in a medium which extends indefinitely in the x-direction. At any given time t, the displacement y of a particle depends on its position, or its coordinate x.

For example, in Fig. 21-4, where we have arbitrarily set $t = 0$ at an instant when the shape of the string is a cosine curve, the transverse displacement y at this instant is given by

$$y = A \cos \frac{2\pi}{\lambda} x \qquad (t = 0). \tag{21-9}$$

It can be seen from this equation, and from the top graph of Fig. 21-4, that $y = A$ when $x = 0$, $x = \lambda$, $x = 2\lambda$, etc. When this equation is plotted, it gives rise to a graph of the *wave form*, as shown in Fig. 21-6.

Every particle of the medium, on the other hand, oscillates with simple harmonic motion. For example, the displacement of the particle at the left end of the string in Fig. 21-4 is given by

$$y = A \cos \frac{2\pi}{T} t \qquad (x = 0). \tag{21-10}$$

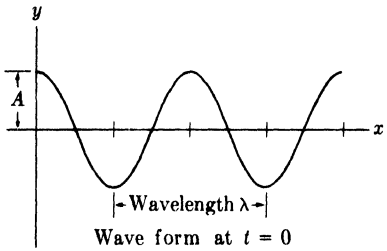


FIGURE 21-6

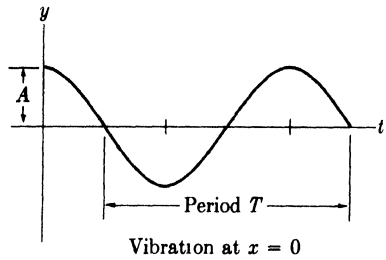


FIGURE 21-7

When this equation is plotted, we obtain a graph of the *vibration*, shown in Fig. 21-7.

We see, therefore, that the displacement y of a particle depends both on the position of the particle and on the time. In other words, y *is a function of two independent variables*, x and t. This function must have the property that when t is constant y will vary with the cosine of $2\pi x/\lambda$, as in Eq. (21-9), and when x is constant y will vary with the cosine of $2\pi t/T$, as in Eq. (21-10). Such a function is

$$y = A \cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right). \tag{21-11}$$

This expression represents a sinusoidal wave (transverse or longitudinal) traveling in the *positive* x-direction. For the negative x-direction one merely replaces the minus sign in the parentheses by a plus sign.

A graph of Eq. (21-11) would be a surface for the same reason that a graph of the relation between p, V, and T for water is a thermodynamic surface. It is customary in plotting Eq. (21-11) to make two plane graphs, either

1. (a) the wave form at one moment and (b) the wave form at a known time later, or
2. (a) the wave form at a moment and (b) the vibration at a point.

Problems

21–1. A steel wire 2 m long has a mass of 20 gm and is stretched with a tension of 1000 newtons. What is the velocity of propagation of a transverse wave in the wire?

21–2. One end of a horizontal string is attached to a prong of an electrically driven tuning fork whose frequency of vibration is 240 vib/sec. The other end passes over a pulley and supports a weight of 6 lb. The linear weight density of the string is 0.0133 lb/ft. (a) What is the speed of a transverse wave in the string? (b) What is the wavelength?

21–3. A metal wire has the following properties: coefficient of linear expansion = $1.5 \times 10^{-5}(C°)^{-1}$, Young's modulus = 2.0×10^{12} dynes/cm^2, density = 9.0 gm/cm^3. At each end are rigid supports. If the tension is zero at 20°C, what will be the speed of a transverse wave at 8°C.

21–4. A metal wire of density 20. slugs/ft^3 with a Young's modulus equal to 15×10^6 lb/in^2 is stretched between rigid supports. At one temperature the speed of a transverse wave is found to be 657 ft/sec. When the temperature is raised 100 F°, the speed decreases to 536 ft/sec. What is the coefficient of linear expansion?

21–5. A steel pipe 200 ft long is struck at one end. A person at the other end hears two sounds as a result of two longitudinal waves, one in the pipe and the other in the air. What is the time interval between the two sounds? Take Young's modulus of steel to be 30×10^6 lb/in^2.

21–6. What must be the stress (F/A) in a stretched wire of a material whose Young's modulus is Y in order that the velocity of longitudinal waves shall equal 10 times the velocity of transverse waves?

21–7. The velocity of longitudinal waves in water is approximately 1450 m/sec at 20°C. Compute the adiabatic compressibility $(1/B_{ad})$ of water and compare with the isothermal compressibility listed in Table 10–2.

21–8. Provided the amplitude is sufficiently great, the human ear can respond to longitudinal waves over a range of frequencies from about 20 vib/sec to about 20,000 vib/sec. Compute the wavelengths corresponding to these frequencies (a) for waves in air, (b) for waves in water. (See Problem 21–7.)

21–9. At a temperature of 27°C, what is the velocity of longitudinal waves in (a) argon, (b) hydrogen? Compare with the velocity in air at the same temperature.

21–10. What is the difference between the velocities of longitudinal waves in air at −3°C and at 57°C?

21–11. The sound waves from a loudspeaker spread out nearly uniformly in all directions when their wavelength is large compared with the diameter of the speaker. When the wavelength is small compared with the diameter of the speaker, much of the sound energy is concentrated in the forward direction. For a speaker of diameter 10 inches, compute the frequency for which the wavelength of the sound waves, in air, is (a) 10 times the diameter of the speaker, (b) equal to the diameter of the speaker, (c) 1/10 the diameter of the speaker.

21–12. Show that Eq. (21–11) may be written

$$y = A \cos \frac{2\pi}{\lambda} (x - ut).$$

21–13. A traveling transverse wave on a string is represented by the equation in Problem 21–12. Let $A = 1$ in., $\lambda = 2$ in., and $u = \frac{1}{4}$ in./sec. (a) At time $t = 0$, compute the transverse displacement y at $\frac{1}{4}$-inch intervals of x(i.e., at $x = 0$, $x = \frac{1}{4}$ in., $x = \frac{1}{2}$ in., etc.) from $x = 0$ to $x = 4$ inches. Show the results in a graph. This is the shape of the string at time $t = 0$. (b) Repeat the calculations, for the same values of x, at times $t = 1$ sec, $t = 2$ sec, $t = 3$ sec, and $t = 4$ sec. Show on the same graph the shape of the string at these instants. In what direction is the wave traveling?

21–14. The equation of a transverse traveling wave on a string is

$$y = 2 \cos [\pi(0.5x - 200t)],$$

where x and y are in cm and t is in sec. (a) Find the amplitude, wavelength, frequency, period, and velocity of propagation. (b) Sketch the shape of the string at the following values of t: 0, 0.0025, and 0.005 sec. (c) If the mass per unit length of the string is 5 gm/cm, find the tension.

FIGURES 21–8 and 21–9

To use these diagrams, cut a slit about $4\frac{1}{4}''$ long and about $\frac{1}{16}''$ wide in a card. Place the card over the diagram and move it vertically with constant velocity. The portions of the curves that appear in the slit will correspond to the oscillations of the particles in a longitudinal traveling wave in Fig. 21–8, and to a longitudinal stationary wave in Fig. 21–9.

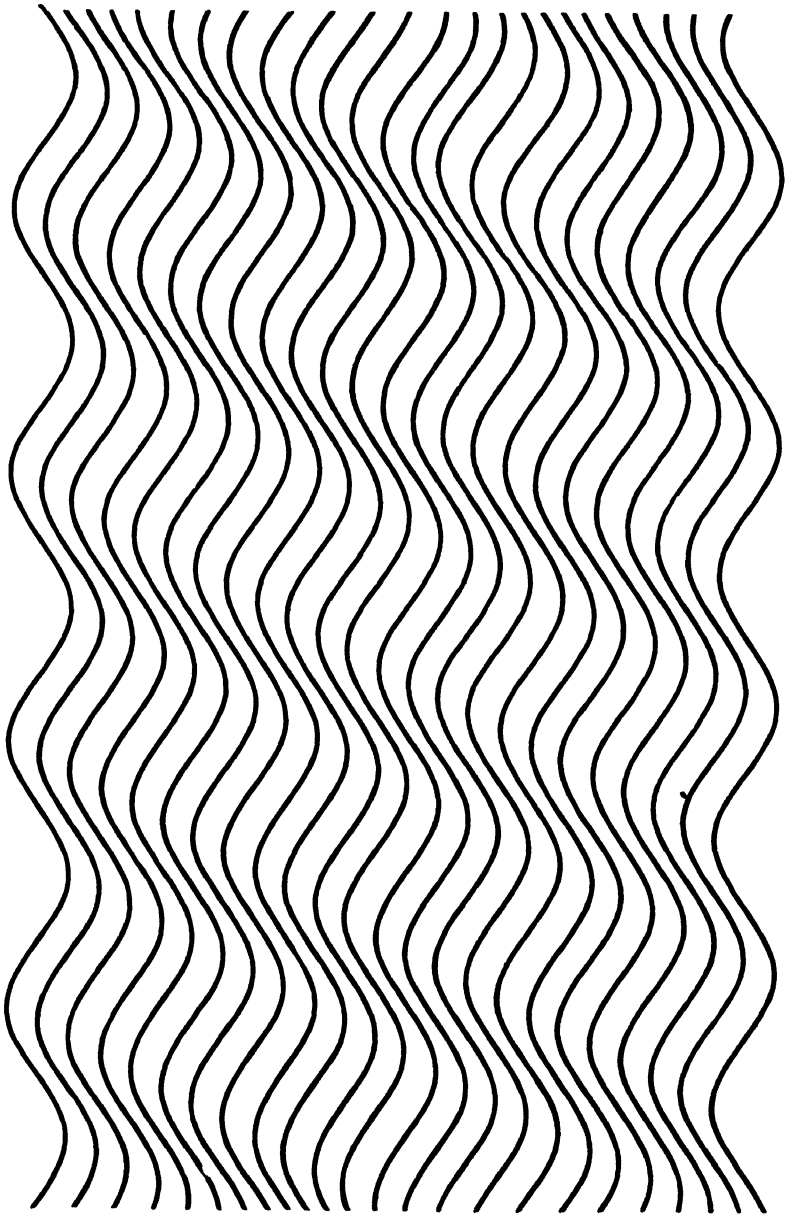


FIGURE 21-8

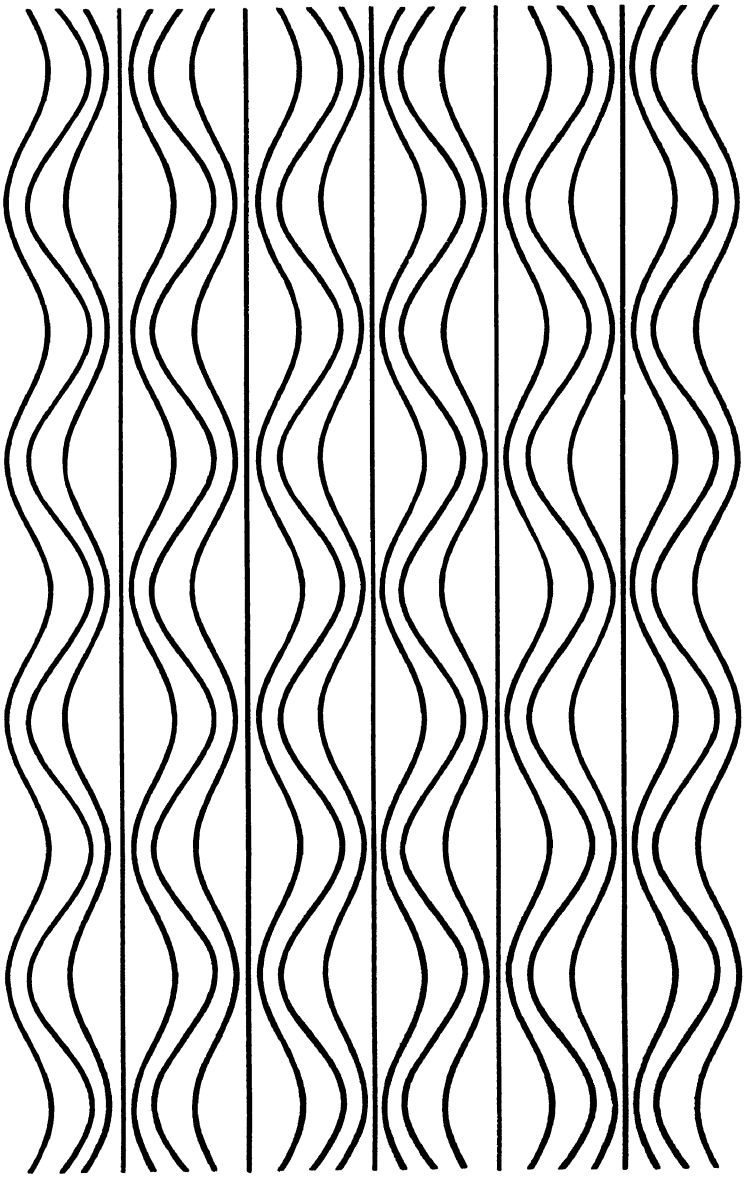


FIGURE 21-9

CHAPTER 22

VIBRATING BODIES

22–1 Boundary conditions for a string. Let us now consider what will happen when a wave pulse or wave train, advancing along a stretched string, arrives at the end of the string. If fastened to a rigid support, the end must evidently remain at rest. The arriving pulse exerts a force on the support, and the reaction to this force "kicks back" on the string and sets up a *reflected* pulse traveling in the reverse direction. At the opposite extreme from a rigidly fixed end would be one which was perfectly free—a case of no great importance here (it may be realized by a string hanging vertically) but which is of interest since its analogue does occur in other types of waves. At a free end the arriving pulse causes the string to "overshoot" and a reflected wave is also set up. The conditions which must be satisfied at the ends of the string are called *boundary conditions.*

The multiflash photograph of Fig. 22–1 shows the reflection of a pulse at a fixed end of a string. (The camera was tipped vertically while the photographs were taken so that successive images lie one under the other. The "string" is a rubber tube and it sags somewhat.) It will be seen that the pulse is reflected with its displacement and its velocity both reversed.

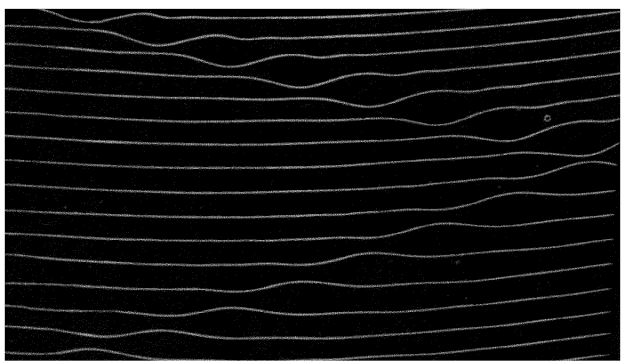


Fig. 22–1. A pulse starts in the upper left corner and is reflected from the fixed end of the string at the right.

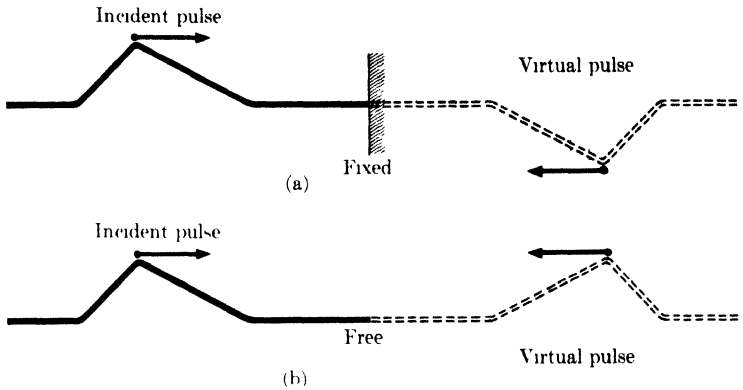


FIG. 22-2. Description of reflection of a pulse (a) at a fixed end of a string, and (b) at a free end, in terms of an imaginary "virtual" pulse.

When reflection takes place at a free end, the direction of the velocity is reversed but the direction of the displacement is unchanged.

It is helpful to think of the process of reflection in the following way. Imagine the string to be extended indefinitely beyond its actual terminus. The actual pulse can be considered to continue on into the imaginary portion as though the support were not there, while at the same time a "virtual" pulse, which has been traveling in the imaginary portion, moves out into the real string and forms the reflected pulse. The nature of the reflected pulse depends on whether the end is fixed or free. The two cases are shown in Fig. 22-2.

The displacement at a point where the actual and virtual pulses cross each other is the algebraic sum of the displacements in the individual pulses. Figures 22-3 and 22-4 show the shape of the end of the string for both types of reflected pulses. It will be seen that Fig. 22-3 corresponds to a free end and Fig. 22-4 to a fixed end. In the latter case, the incident and reflected pulses combine in such a way that the displacement of the end of the string is always zero.

22-2 Stationary waves in a string. When a continuous train of waves arrives at a fixed end of a string, a continuous train of reflected waves appears to originate at the end and travel in the opposite direction. Provided the elastic limit of the string is not exceeded and the displacements are sufficiently small, the actual displacement of any point of the string is the algebraic sum of the displacements of the individual waves, a fact which is called the *principle of superposition.* This principle is extremely important in all types of wave motion and applies not only to

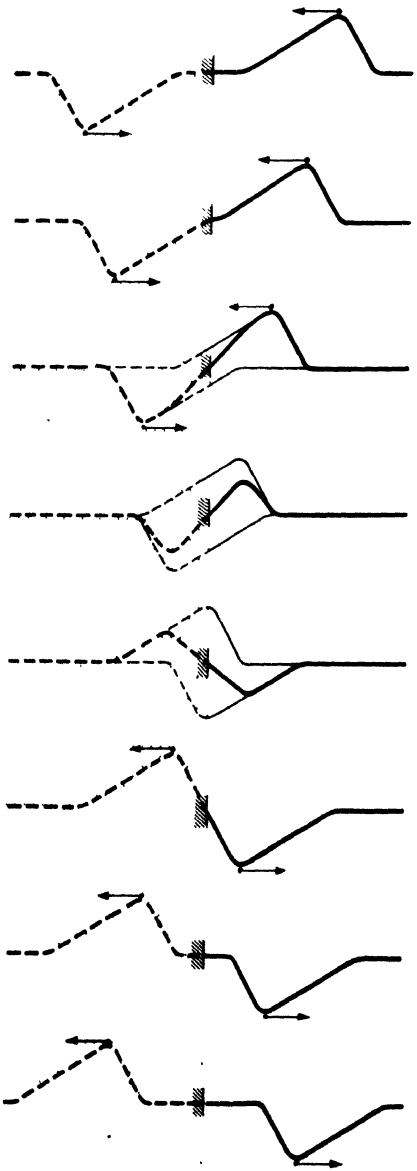


FIG. 22-3. Reflection at a free end.

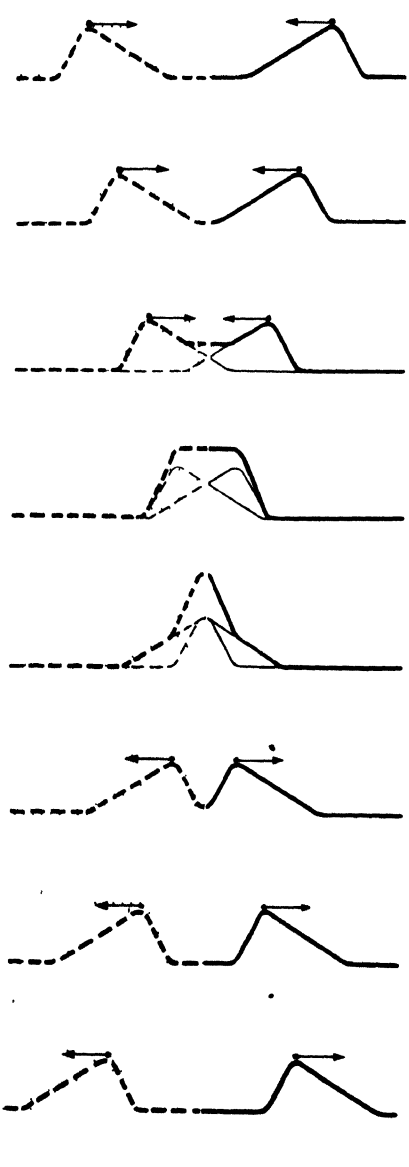


FIG. 22-4. Reflection at a fixed end.

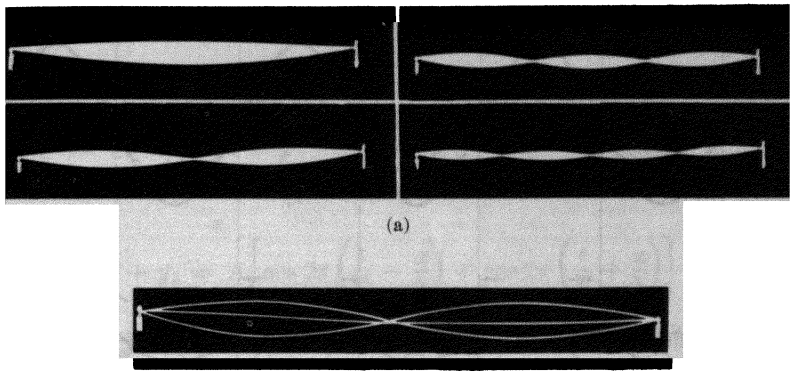


FIG. 22–5. (a) Standing waves in a stretched string (time exposure). (b) Multiflash photograph of a standing wave, with nodes at the center and at the ends.

waves in a string but to sound waves in air, to light waves, and, in fact, to wave motion of any sort. The general term *interference* is applied to the effect produced by two (or more) sets of wave trains which are simultaneously passing through a given region.

The appearance of the string in these circumstances gives no evidence that two waves are traversing it in opposite directions. If the frequency is sufficiently great so that the eye cannot follow the motion, the string appears subdivided into a number of segments as in the time exposure photograph of Fig. 22–5(a). A multiflash photograph of the same string, in Fig. 22–5(b), indicates a few of the instantaneous shapes of the string. At any instant (except those when the string is straight) its shape is a sine curve, but whereas in a traveling wave the amplitude remains constant while the wave progresses, here the wave form remains fixed in position (longitudinally) while the amplitude fluctuates. Certain points known as the *nodes* remain always at rest. Midway between these points, at the *loops* or *antinodes*, the fluctuations are a maximum. The vibration as a whole is called a *stationary* wave.

To understand the formation of a stationary wave, consider the four separate graphs of wave form at four instants $\frac{1}{8}$ of a period apart, shown in Fig. 22–6. The system of short dashed curves represents a wave traveling to the right. The system of long dashed curves represents a wave of the same velocity, same wavelength, and same amplitude traveling to the left. The heavy curves represent the resultant wave form, obtained by applying the principle of superposition, that is, by adding displacements. At those places on the string marked N, the resultant

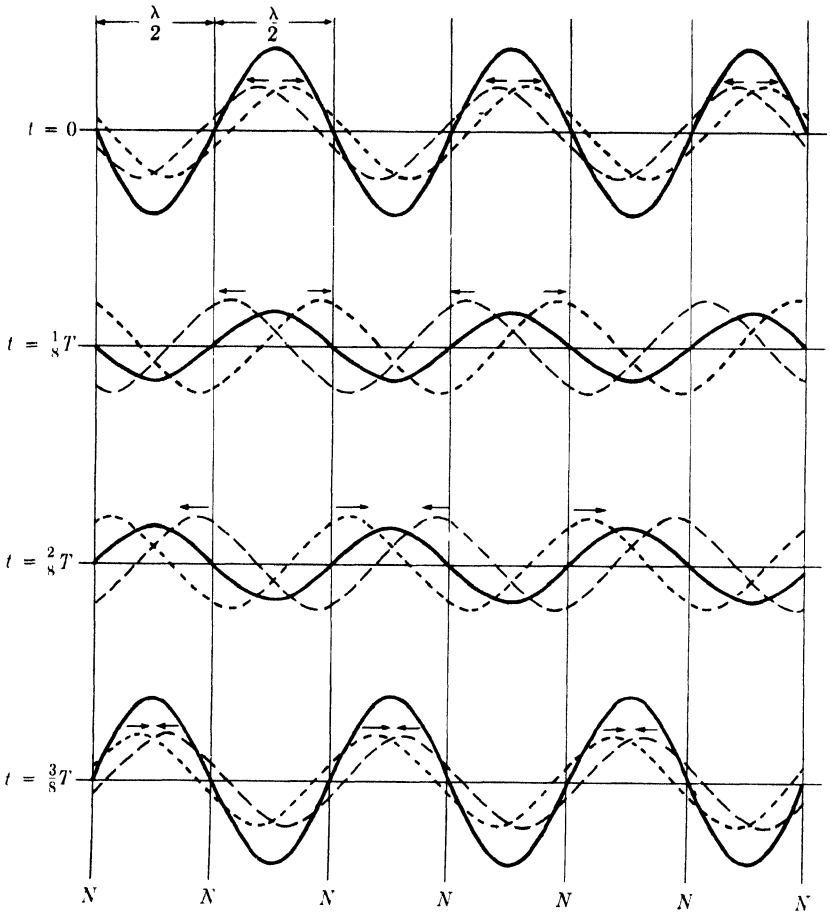


FIG. 22-6. The formation of a stationary wave.

displacements are always zero. These are the nodes. Midway between the nodes, the vibrations have the largest amplitude. These are the antinodes. It is evident from the figure that

$$\left\{\begin{array}{c}\text{Distance between adjacent nodes}\\ \text{or}\\ \text{Distance between adjacent antinodes}\end{array}\right\} = \frac{\lambda}{2}.$$

The equation of a standing wave may be obtained by adding the displacements of two waves of equal amplitude, period, and wavelength,

but traveling in opposite directions. Thus if

$$y_1 = A \cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad \text{(positive } x\text{-direction)},$$

$$y_2 = A \cos 2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right) \qquad \text{(negative } x\text{-direction)},$$

then

$$y_1 + y_2 = A\left[\cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) + \cos 2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right)\right].$$

Introducing the expressions for the cosine of the sum and difference of two angles and combining terms, we obtain

$$y_1 + y_2 = [2A \cos 2\pi ft] \cos \frac{2\pi x}{\lambda}. \tag{22–1}$$

The shape of the string at each instant is, therefore, a cosine curve whose amplitude (the expression in brackets) varies with time.

22–3 Vibration of a string fixed at both ends. Thus far we have been discussing a long string fixed at one end and have considered the stationary waves set up near that end by interference between the incident and reflected waves. Let us next consider the more usual case, that of a string fixed at both ends. A continuous train of sine or cosine waves is reflected and re-reflected, and since the string is fixed at both ends, both ends must be nodes. Since the nodes are one-half a wavelength apart, the length of the string may be $\lambda/2$, $2(\lambda/2)$, $3(\lambda/2)$, or, in general, any integral number of half-wavelengths. Or, to put it differently, if one considers a particular string of length L, stationary waves may be set up in the string by vibrations of a number of different frequencies, namely, those which give rise to waves of wavelengths $2L/1$, $2L/2$, $2L/3$, etc.

From the relation $f = u/\lambda$, and since u is the same for all frequencies, the possible frequencies are

$$\frac{u}{2L}, \qquad 2\,\frac{u}{2L}, \qquad 3\,\frac{u}{2L}, \cdots$$

The lowest frequency, $u/2L$, is called the *fundamental* frequency f_1 and the others are the *overtones*. The frequencies of the latter are, therefore, $2f_1$, $3f_1$, $4f_1$, and so on. Overtones whose frequencies are integral multiples of the fundamental are said to form a *harmonic series*. The fundamental is the *first harmonic*. The frequency $2f_1$ is the *first overtone* or the *second harmonic*, the frequency $3f_1$ is the *second overtone* or the *third harmonic*, and so on.

We can now see an important difference between a spring-mass system and a vibrating string. The former has but one natural frequency, while the vibrating string has an infinite number of natural frequencies, the fundamental and all the overtones. If a body suspended from a spring is pulled down and released, only one frequency of vibration will ensue. If a string is initially distorted so that its shape is the same as *any one* of the possible harmonics, it will vibrate, when released, at the frequency of that particular harmonic. But when a piano string is struck, not only the fundamental, but many of the overtones are present in the resulting vibration. The fundamental frequency of the vibrating string is $f_1 = u/2L$, where $u = \sqrt{T/\mu}$. It follows that

$$f_1 = \frac{1}{2L}\sqrt{T/\mu}. \tag{22-2}$$

Stringed instruments afford many examples of the implications of this equation. For example, all such instruments are "tuned" by varying the tension T, an increase of tension increasing the frequency or pitch, and vice versa. The inverse dependence of frequency on length L is illustrated by the long strings of the bass section of the piano or the bass viol compared with the shorter strings of the piano treble or the violin. One reason for winding the bass strings of a piano with wire is to increase the mass per unit length μ, so as to obtain the desired low frequency without resorting to a string which is inconveniently long.

22–4 Demonstration of the harmonic series in a vibrating string. We have seen that a string is capable of vibrating at a number of different frequencies. That it may vibrate with many different frequencies *at the same time* may be demonstrated very graphically with the aid of the apparatus depicted in Fig. 22–7. A metal string is stretched between two metal posts which are in turn connected to a "step-up" transformer. The secondary of the transformer is then connected to those plates of a cathode-ray oscilloscope which impart vertical motion to the electron beam. If the string is made to oscillate in a magnetic field, an alternating current will be set up whose frequency is exactly the same as that of the string. With proper adjustment of the oscilloscope, this alternating current may be caused to give rise to figures on the screen such as those shown.

Suppose that one small magnet is placed over the center of the string [Fig. 22–7(a)] and the string is plucked near the center so that this part of the string vibrates perpendicular to the magnetic lines of force. The figure on the oscilloscope shows the fundamental frequency.

Placing one magnet $\frac{1}{4}$ of the way along the string and another magnet *with its polarity reversed* at the $\frac{3}{4}$ mark [Fig. 22–7(b)], and plucking the

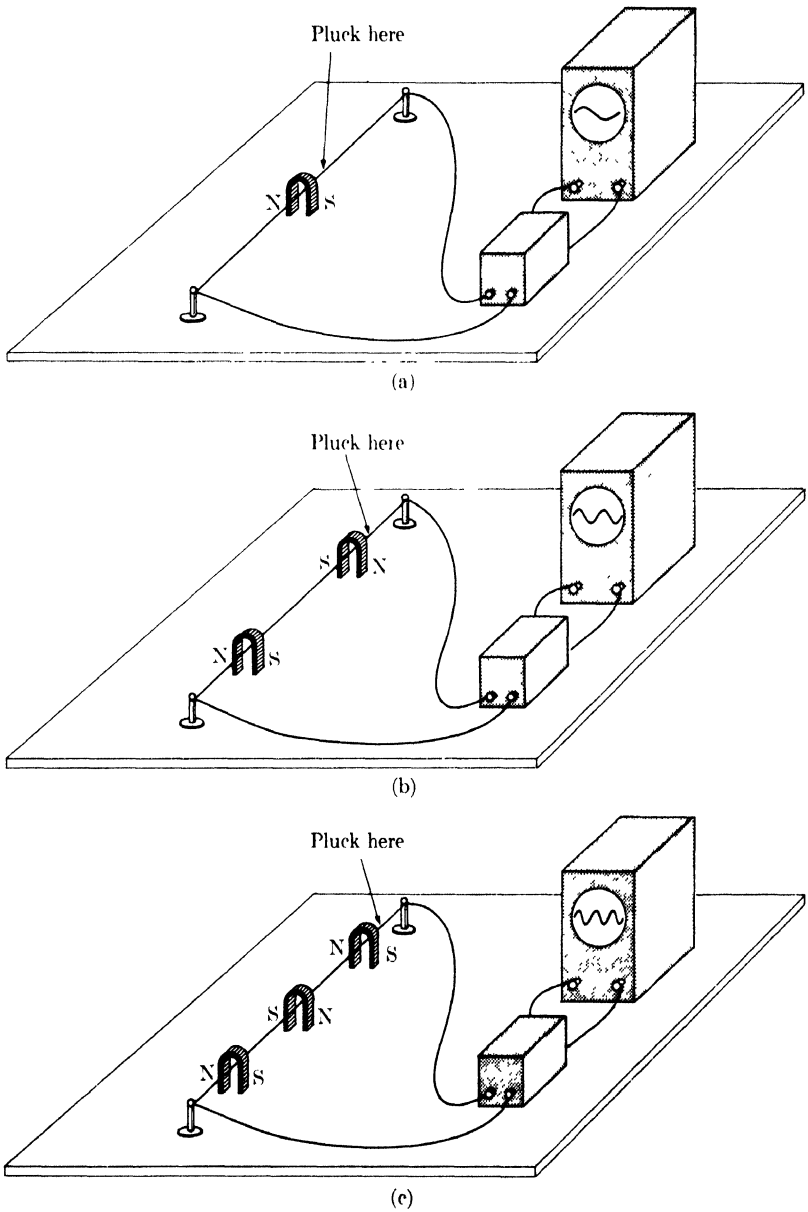


FIG. 22–7. Demonstration of the harmonics present in the vibration of a plucked string.

string near the $\frac{1}{4}$ point, the second harmonic may be obtained. If one of the magnets is quickly reversed while the string is sounding, so as to set the two magnetic fields in the same direction, the fundamental will occur again, showing that the fundamental and second harmonic exist at the same time.

We now place three magnets on the string at the $\frac{1}{6}$, $\frac{1}{2}$, $\frac{5}{6}$ points, with the polarity shown in Fig. 22–7(c). Plucking the string near the $\frac{1}{6}$ point, the third harmonic is obtained. While the string is vibrating, if the middle magnet is reversed so as to make all the magnets point in the same direction, the fundamental will appear. Thus the fundamental and third harmonic exist at the same time.

Proceeding in this manner, we may pick up higher harmonics and demonstrate that a string can vibrate with all these frequencies at the same time.

22–5 Resonance. In general, whenever a body capable of oscillating is acted on by a periodic series of impulses having a frequency equal to one of the natural frequencies of oscillation of the body, the body is set into vibration with a relatively large amplitude. This phenomenon is called *resonance*, and the body is said to *resonate* with the applied impulses.

A common example of mechanical resonance is provided by pushing a swing. The swing is a pendulum with a single natural frequency depending on its length. If a series of regularly spaced pushes is given to the swing, with a frequency equal to that of the swing, the motion may be made quite large. If the frequency of the pushes differs from the natural frequency of the swing, or if the pushes occur at irregular intervals, the swing will hardly execute a vibration at all.

Unlike a simple pendulum, which has only one natural frequency, a stretched string (and other systems to be discussed later in this chapter) has a large number of natural frequencies. Suppose that one end of a stretched string is fixed while the other is moved back and forth in a transverse direction. The amplitude at the driven end is fixed by the driving mechanism. Stationary waves will be set up in the string, whatever the value of the frequency f. If the frequency is not equal to one of the natural frequencies of the string, the amplitude at the antinodes will be fairly small. However, if the frequency is equal to *any one* of the natural frequencies, the string is in resonance and the amplitude at the antinodes will be very much larger than that at the driven end. In other words, although the driven end is not a node, it lies much closer to a node than to an antinode when the string is in resonance. In Fig. 22–5(a), the right end of the string was fixed and the left end was forced to oscillate vertically with small amplitude. Stationary waves of relatively

large amplitude resulted when the frequency of oscillation of the left end was equal to the fundamental frequency or to any of the first three overtones.

A bridge or, for that matter, any structure, is capable of vibrating with certain natural frequencies. If the regular footsteps of a column of soldiers were to have a frequency equal to one of the natural frequencies of a bridge which the soldiers are crossing, a vibration of dangerously large amplitude might result. Therefore, in crossing a bridge, a column of soldiers is ordered to break step.

Tuning a radio is an example of electrical resonance. By turning a dial, the natural frequency of an alternating current in the receiving circuit is made equal to the frequency of the waves broadcast by the desired station. Optical resonance may also take place between atoms in a gas at low pressure and light waves from a lamp containing the same atoms. Thus light from a sodium lamp may cause the sodium atoms in a glass bulb to glow with characteristic yellow sodium light.

The phenomenon of resonance may be demonstrated with the aid of the longitudinal waves set up in air by a vibrating plate or tuning fork. If two identical tuning forks are placed some distance apart and one is struck, the other will be heard when the first is suddenly damped. Should a small piece of wax or modeling clay be put on one of the forks, the frequency of that fork will be altered enough to destroy the resonance.

22–6 Interference of longitudinal waves. The phenomenon of interference between two longitudinal waves in air may be demonstrated with the aid of the apparatus depicted in Fig. 22–8. A wave emitted by an electrically driven diaphragm S is sent into a metal tube, where it divides into two waves, one following the constant path SAR, the other the path SBR, which may be varied by sliding the tube B to the right. Suppose the frequency of the source is 1100 vibrations per second. Then

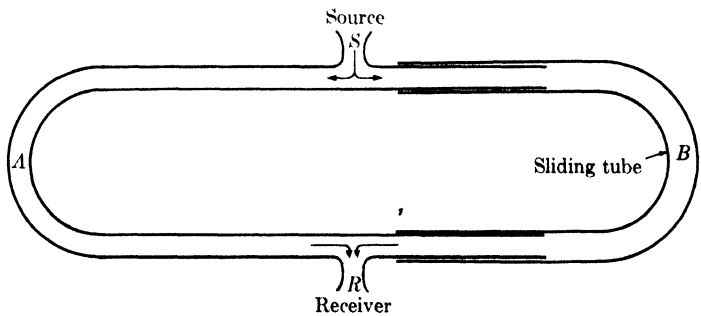


FIG. 22–8. Apparatus for demonstrating interference of longitudinal waves.

the wavelength $\lambda = u/f = 1$ ft. If both paths are of equal length, the two waves will arrive at R at the same time and the vibrations set up by both waves will be in phase. The resulting vibration will have an amplitude equal to the sum of the two individual amplitudes and the phenomenon of *reinforcement* may be detected either with the ear at R or with the aid of a microphone, amplifier, and loudspeaker.

Now suppose the tube B is moved out a distance of three inches, thereby making the path SBR 6 inches longer than the path SAR. The right-hand wave will have traveled a distance $\lambda/2$ greater than the left-hand wave, and the vibration set up at R by the right-hand wave will therefore be in opposite phase to that set up by the left-hand wave. The consequent interference is shown by the marked reduction in sound at R.

If the tube B is now pulled out another 3 inches, so that the *path difference*, SBR minus SAR, is one foot (one wavelength), the two vibrations at R will again reinforce each other. Thus

$$\left\{\begin{matrix}\text{Reinforcement takes place}\\ \text{when the path difference}\end{matrix}\right\} = 0,\ \lambda,\ 2\lambda,\ \text{etc.}$$

$$\left\{\begin{matrix}\text{Interference takes place}\\ \text{when the path difference}\end{matrix}\right\} = \frac{\lambda}{2},\ \frac{3\lambda}{2},\ \frac{5\lambda}{2},\ \text{etc.}$$

An acoustical interferometer of this sort is of value only in demonstrating the phenomenon of interference. Optical interferometers, however, whose principles of operation are the same, have many practical uses in physical optics.

22–7 Stationary longitudinal waves. Longitudinal waves traveling along a tube of finite length are reflected at the ends of the tube in much the same way that transverse waves in a string are reflected at its ends. Interference between the waves traveling in opposite directions gives rise to stationary waves.

If reflection takes place at a closed end, the displacement of the particles at that end must necessarily be always zero. Hence a closed end is a *node*. If the end of the tube is open, the nature of the reflection is more complex and depends on whether the tube is wide or narrow compared with the wavelength. If the tube is narrow compared with the wavelength, which is the case in most musical instruments, the reflection is such as to make the open end an *antinode*. Therefore the longitudinal waves in a column of fluid are reflected at the closed and open ends of a tube in the same way that transverse waves in a string are reflected at fixed and free ends respectively.

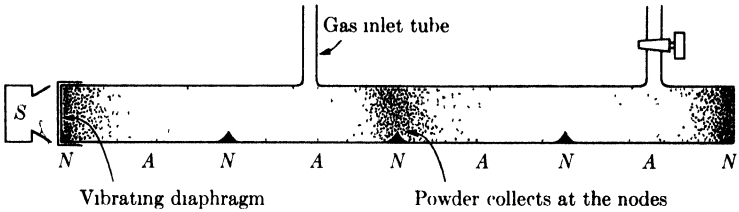


FIG. 22–9. Kundt's tube for determining the velocity of sound in a gas. The dots represent the density of the gas molecules at an instant when the pressure at the displacement nodes is a maximum or a minimum.

The reflections at the openings where the instrument is blown are found to be such that an antinode is located at or near the opening. The effective length of the air column of a wind instrument is thus less definite than the length of a string fixed at its ends.

Stationary longitudinal waves in a column of gas may be demonstrated conveniently with the aid of the apparatus shown in Fig. 22–9, known as Kundt's tube. A glass tube a few feet long is closed at one end with glass and at the other with a flexible diaphragm. The gas to be studied is admitted to the tube at a known temperature and at atmospheric pressure. A powerful source of longitudinal waves S, whose frequency may be varied, causes vibration of the flexible diaphragm. A small amount of light powder or cork dust is sprinkled uniformly along the tube.

When a frequency is found at which the air column is in resonance, the amplitude of the stationary waves becomes large enough for the gas particles to sweep the cork dust along the tube, at all points where the gas is in motion. The powder therefore collects at the displacement nodes, where the gas remains at rest. Sometimes a wire running along the axis of the tube is maintained at a dull red heat by an electric current, and the nodes show themselves as hot points compared with the antinodes.

With careful manipulation and with a good variable frequency source, a fair determination of the velocity of the wave may be obtained with Kundt's tube. Since, in a stationary wave, the distance between two adjacent nodes is one-half a wavelength, the wavelength λ is obtained by measuring the distance between alternate clumps of powder. Knowing the frequency f, the velocity u is then

$$u = f\lambda.$$

A constant frequency source may be used if the vibrating element is a piston which may be moved along the tube until resonance is obtained.

At a displacement node, the pressure variations above and below the average are a maximum, whereas at an antinode, there are no pressure

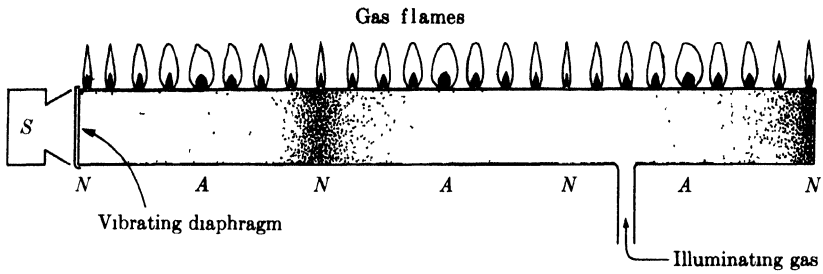


FIG. 22–10. The variations in gas pressure are greatest at the displacement nodes. The dots represent the density of the gas molecules at an instant when the pressure at the displacement nodes is a maximum or a minimum.

variations. This may be understood easily when it is realized that two small masses of gas on opposite sides of a node are vibrating in *opposite phase*. Thus, when they approach each other, the pressure at the node is a maximum, and when they recede from each other, the pressure at the node is a minimum. Two small masses of gas, however, on opposite sides of an antinode vibrate *in phase*, and hence give rise to no pressure variations at the antinode. This may be demonstrated in the case of illuminating gas with the aid of the apparatus shown in Fig. 22–10. At the nodes, where the harmonic variations of gas pressure are greatest, the flames are blue and stiff. At the antinodes, where the variations of gas pressure are small, the flames are yellow and yielding.

22–8 Vibrations of organ pipes. If one end of a pipe is open and a stream of air is directed against an edge, vibrations are set up and the tube resonates at its natural frequencies. As in the case of a plucked string, the fundamental and overtones exist at the same time. In the

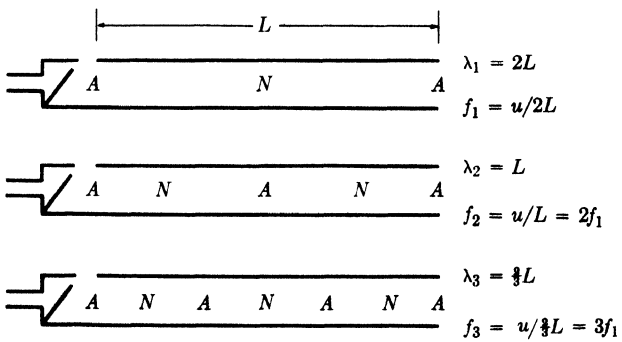


FIG. 22–11. Modes of vibration of an open organ pipe.

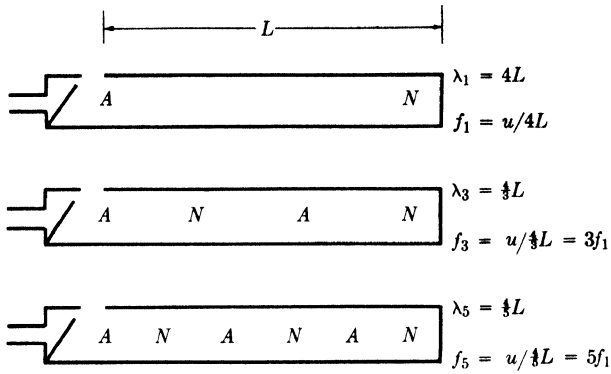


FIG. 22–12. Modes of vibration of a closed organ pipe.

case of an open pipe, the fundamental frequency f_1 corresponds to an antinode at each end and a node in the middle, as shown at the top of Fig. 22–11. Succeeding diagrams of Fig. 22–11 show two of the overtones, which are seen to be the second and third harmonics. *In an open pipe the fundamental frequency is $u/2L$ and all harmonics are present.*

The properties of a closed pipe are shown in the diagrams of Fig. 22–12. The fundamental frequency is seen to be $u/4L$, which is one-half that of an open pipe of the same length. In the language of music, the pitch of a closed pipe is one octave lower than that of an open pipe of equal length. From the remaining diagrams of Fig. 22–12, it may be seen that the second, fourth, etc., harmonics are missing. Hence, *in a closed pipe, the fundamental frequency is $u/4L$ and only the odd harmonics are present.*

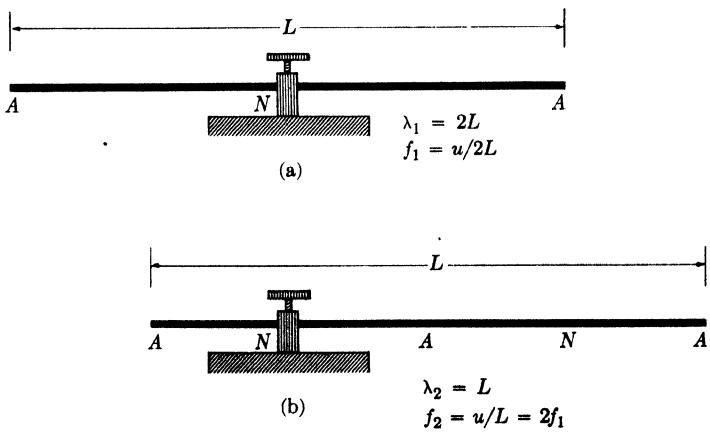


FIG. 22–13. Modes of vibration of a rod.

22–9 Vibrations of rods and plates. A rod may be set in longitudinal vibration by clamping it at some point and stroking it with a chamois skin that has been sprinkled with rosin. In Fig. 22–13(a) the rod is clamped in the middle, and consequently when it is stroked near the end, a stationary wave is set up with a node in the middle and antinodes at each end, exactly the same as the fundamental mode of an open organ pipe. The fundamental frequency of the rod is then $u/2L$, where u is the velocity of a longitudinal wave in the rod. Since the velocity of a longitudinal wave in a solid is much greater than that in air, a rod has a higher fundamental frequency than an open organ pipe of the same length.

By clamping the rod at a point $\frac{1}{4}$ of its length from one end, as shown in Fig. 22–13(b), the second harmonic may be produced.

If a stretched flexible membrane, such as a drumhead, is struck a blow, a two-dimensional pulse travels outward from the struck point and is reflected and re-reflected at the boundary of the membrane. If some point of the membrane is forced to vibrate periodically, continuous trains of waves travel along the membrane. Just as with the stretched string, stationary waves can be set up in the membrane and each of these waves has a certain natural frequency. The lowest frequency is the fundamental and the others are overtones. In general, when the membrane is vibrating, a number of overtones are present.

The nodes of a vibrating membrane are lines (nodal lines) rather than points. The boundary of the membrane is evidently one such line. Some of the other possible nodal lines of a circular membrane are shown in Fig. 22–14, with the modes of vibration arranged in order of increasing frequency. The natural frequency of each mode is given in terms of the

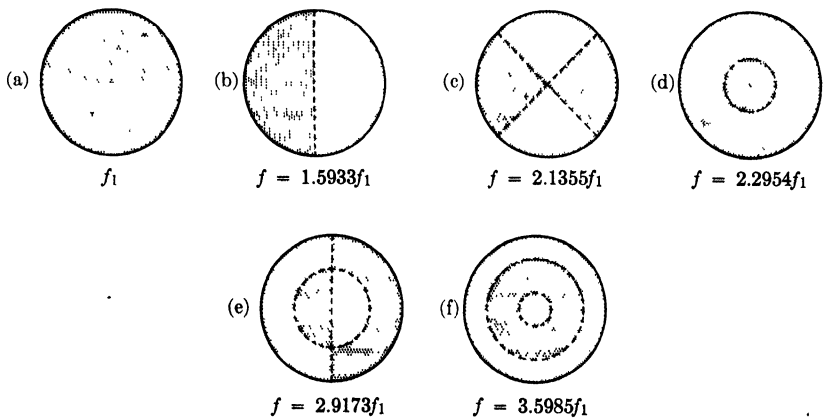


FIG. 22–14. Possible modes of vibration of a membrane, showing nodal lines. The frequency of each is given in terms of the fundamental frequency, f_1.

fundamental f_1. It will be noted that the frequencies of the overtones are *not* integral multiples of f_1. That is, they are not harmonics.

The restoring force in a vibrating flexible membrane arises from the tension with which it is stretched. A metal plate, if sufficiently thick, will vibrate in a similar way, the restoring force being produced by bending stresses in the plate. The study of vibrations of membranes and plates is of importance in connection with the design of loud-speaker diaphragms and the diaphragms of telephone receivers and microphones.

PROBLEMS

22–1. Two transverse cosine waves, each of amplitude 1 inch and wavelength 2 inches, travel in opposite directions in a string with a speed of $\frac{1}{4}$ inch/sec. Construct graphs of the shape of the string at the following times: $t = 0$, $t = 2$ sec, $t = 4$ sec.

22–2. The equation of a transverse wave in a stretched string is

$$y = 4 \cos 2\pi \left(\frac{t}{0.02} - \frac{x}{400} \right),$$

where y and x are in centimeters and t is in seconds. (a) Is the wave a traveling wave or a standing wave? (b) What is the amplitude of the wave? (c) What is its wavelength? (d) What is its velocity of propagation? (e) What is its frequency?

22–3. A steel piano wire 50 cm long, of mass 5 gm, is stretched with a tension of 400 newtons. (a) What is the frequency of its fundamental mode of vibration? (b) What is the number of the highest overtone that could be heard by a person who can hear frequencies up to 10,000 cycles/sec?

22–4. A steel wire of length $L = 100$ cm and density $\rho = 8$ gm/cm^3 is stretched tightly between two rigid supports. Vibrating in its fundamental mode, the frequency is $f = 200$ cycles per sec. (a) What is the velocity of transverse waves on this wire? (b) What is the longitudinal stress in the wire (in dynes/cm^2)? (c) If the maximum acceleration at the mid-point of the wire is 80,000 cm/sec^2, what is the amplitude of vibration at the mid-point?

22–5. A stretched string is observed to vibrate with a frequency of 30 cycles per second in its fundamental mode when the supports are 60 cm apart. The amplitude at the antinode is 3 cm. The string has a mass of 30 gm. (a) What is the velocity of propagation of a transverse wave in the string? (b) Compute the tension in the string.

22–6. Suppose the piano wire in Problem 22–3 is set vibrating at twice its fundamental frequency. (a) What is the wavelength of transverse waves in the wire? (b) What is the wavelength, in air, of the sound waves emitted by the wire?

22–7. A standing wave of frequency 1100 vibrations per second in a column of methane at 20°C produces nodes that are 20 cm apart. What is the ratio of the heat capacity at constant pressure to that at constant volume?

22–8. An aluminum weight is hung from a steel wire. The fundamental frequency for transverse stationary

waves on the wire is 300 cycles/sec. The weight is then immersed in water so that one-half of its volume is submerged. What is the new fundamental frequency?

22–9. Stationary waves are set up in a Kundt's tube by the longitudinal vibration of an iron rod one meter long, clamped at the center. If the frequency of the iron rod is 2480 vibrations per second and the powder heaps within the tube are 6.9 cm apart, (a) what is the velocity of the waves in the iron rod, and (b) in the gas?

22–10. The speed of a longitudinal wave in a mixture of helium and neon at 300°K was found to be 758 m/sec. What is the composition of the mixture?

22–11. The atomic weight of iodine is 127. A stationary wave in iodine vapor at 400°K produces nodes that are 6.77 cm apart when the frequency is 1000 vib/sec. Is iodine vapor monatomic or diatomic?

22–12. A copper rod 1 m long, clamped at the $\frac{1}{4}$ point, is set in longitudinal vibration and is used to produce stationary waves in a Kundt's tube containing air at 300°K. Heaps of cork dust within the tube are found to be 4.95 cm apart. What is the velocity of longitudinal waves in copper?

22–13. Find the fundamental frequency and the first four overtones of a 6-inch pipe (a) if the pipe is open at both ends, (b) if the pipe is closed at one end. (c) How many overtones may be heard by a person having normal hearing for each of the above cases?

22–14. A long tube contains air at a pressure of 1 atm and temperature 77°C. The tube is open at one end and closed at the other by a movable piston. A tuning fork near the open end is vibrating with a frequency of 500 cycles/sec. Resonance is produced when the piston is at distances 18.0, 55.5, and 93.0 cm from the open end. (a) From these measurements, what is the velocity of sound in air at 77°C? (b) From the above result, what is the ratio of the specific heats γ for air?

22–15. An organ pipe A of length 2 ft, closed at one end, is vibrating in the first overtone. Another organ pipe B of length 1.35 ft, open at both ends, is vibrating in its fundamental mode. Take the velocity of sound in air as 1120 ft/sec. Neglect end corrections. (a) What is the frequency of the tone from A? (b) What is the frequency of the tone from B?

22–16. A plate cut from a quartz crystal is often used to control the frequency of an oscillating electrical circuit. Longitudinal standing waves are set up in the plate with displacement antinodes at opposite faces. The fundamental frequency of vibration is given by the equation

$$f_1 = \frac{2.87 \times 10^5}{s},$$

where f_1 is in cycles/sec and s is the thickness of the plate in cm. (a) Compute Young's modulus for the quartz plate. (b) Compute the thickness of plate required for a frequency of 1200 kilocycles/sec. (1 kilocycle = 1000 cycles.) The density of quartz is 2.66 gm/cm^3.

CHAPTER 23

ACOUSTICAL PHENOMENA

23–1 Pressure variations in a sound wave. We shall limit ourselves in this chapter to the consideration of longitudinal waves only, and in particular to those which, when striking the ear, give rise to the sensation of sound. Such waves, within the frequency range from 20 to 20,000 vibrations per second, are called, for simplicity, *sound waves.*

The reception of a sound wave by the ear gives rise to a vibration of the air particles at the eardrum with a definite frequency and a definite amplitude. This vibration may also be described in terms of the variation of air pressure at the same point. The air pressure rises above atmospheric pressure and then sinks below atmospheric pressure with simple harmonic motion of the same frequency as that of an air particle. The maximum amount by which the pressure differs from atmospheric pressure is called the *pressure amplitude.* It can be proved that the pressure amplitude is proportional to the displacement amplitude.

Measurements of sound waves show that the maximum pressure variations in the loudest sounds which the ear can tolerate are of the order of magnitude of 280 dynes/cm^2 (above and below atmospheric pressure of about 1,000,000 dynes/cm^2). The corresponding maximum displacement for a frequency of 1000 vibrations per second is about a thousandth of a centimeter. The displacement amplitudes, even in the loudest sounds, are therefore extremely small.

The maximum pressure variations in the *faintest* sound of frequency 1000 vibrations per second are only about 2×10^{-4} dyne/cm^2. The corresponding displacement amplitude is about 10^{-9} cm. By way of comparison, the wavelength of yellow light is 6×10^{-5} cm, and the diameter of a molecule about 10^{-8} cm. It will be appreciated that the ear is an extremely sensitive organ.

23–2 Intensity. From a purely geometrical point of view, that which is propagated by a traveling wave is the *wave form.* From a physical viewpoint, however, something else is propagated by a wave, namely, *energy.* The most outstanding example, of course, is the energy supply of the earth, which reaches us from the sun via electromagnetic waves. The *intensity* I of a traveling wave is defined as *the time average rate at which energy is transported by the wave per unit area* across a surface perpendicular to the direction of propagation. More briefly, the intensity is the average power transported per unit area.

We have seen that the power developed by a force equals the product of force times velocity. Hence the power per unit area in a sound wave equals the product of the excess pressure (force per unit area) times the *particle* velocity. Averaging over one cycle, it can be proved that

$$I = \frac{P^2}{2\rho u}, \tag{23-1}$$

where P is the pressure amplitude, ρ is the average density of the air, and u is the velocity of the sound wave. It will be noted that the *intensity* is proportional to the *square of the amplitude*, a result which is true for any sort of wave motion.

The intensity of a sound wave of pressure amplitude $P = 280$ dynes/cm^2 (roughly, the loudest tolerable sound) is

$$I = \frac{(280 \text{ dynes/cm}^2)^2}{2 \times 1.22 \times 10^{-3} \text{ gm/cm}^3 \times 3.46 \times 10^4 \text{ cm/sec}}$$

$$= 940 \text{ ergs/sec·cm}^2 = 94 \times 10^{-6} \text{ watt/cm}^2.*$$

The pressure amplitude of the faintest sound wave which can be heard is about 0.0003 dyne/cm^2 and the corresponding intensity is about 10^{-16} watt/cm^2.

The total power carried across a surface by a sound wave equals the product of the intensity at the surface times the surface area, if the intensity over the surface is uniform. The average power developed as sound waves by a person speaking in an ordinary conversational tone is about 10^{-5} watt, while a loud shout corresponds to about 3×10^{-2} watt. Since the population of the city of New York is about six million persons, the acoustical power developed if all were to speak at the same time would be about 60 watts, or enough to operate a moderate-sized electric light. On the other hand, the power required to fill a large auditorium with loud sound is considerable. Suppose the intensity over the surface of a hemisphere 20 meters in radius is 10^{-4} watt/cm^2. The area of the surface is about 25×10^6 cm^2. Hence the acoustic power output of a speaker at the center of the sphere would have to be

$$10^{-4} \times 25 \times 10^6 = 2500 \text{ watts}$$

or 2.5 kilowatts. The electrical power input to the speaker would need to be considerably larger, since the efficiency of such devices is not very high.

* The "watt/cm^2" is a hybrid unit, neither cgs nor mks. We shall retain it to conform with general usage in acoustics.

TABLE 23–1

NOISE LEVELS DUE TO VARIOUS SOURCES

(Representative values)

Source or description of noise	Noise level, db
Threshold of pain	120
Riveter	95
Elevated train	90
Busy street traffic	70
Ordinary conversation	65
Quiet automobile	50
Quiet radio in home	40
Average whisper	20
Rustle of leaves	10
Threshold of hearing	0

23–3 Intensity level and loudness. Because of the large range of intensities over which the ear is sensitive, a logarithmic rather than an arithmetic intensity scale is convenient. Accordingly, the *intensity level* β of a sound wave is defined by the equation

$$\beta = 10 \log \frac{I}{I_0}, \tag{23–2}$$

where I_0 is an arbitrary reference intensity which is taken as 10^{-16} watt/cm^2, corresponding roughly to the faintest sound which can be heard. Intensity levels are expressed in *decibels*, abbreviated db.*

If the intensity of a sound wave equals I_0 or 10^{-16} watt/cm^2, its intensity level is zero. The maximum intensity which the ear can tolerate, about 10^{-4} watt/cm^2, corresponds to an intensity level of 120 db. Table 23–1 gives the intensity levels in db of a number of familiar noises. It is taken from a survey made by the N. Y. City Noise Abatement Commission.

The intensity of a sound wave is a purely objective or physical attribute of a wave, and can be measured by acoustical apparatus without

* Originally, a scale of intensity levels in *bels* was defined by the relation

$$\text{Intensity level} = \log I/I_0.$$

This unit proved rather large and hence the decibel, one-tenth of a bel, has come into general use. The unit is named in honor of Alexander Graham Bell.

making use of the hearing sense of a human observer. However, if we listen to a sound wave whose intensity is gradually increased, the sensation which we describe as *loudness* increases also. The term loudness is reserved to refer to this sensation, and since it is a sensation or a subjective attribute of a sound wave, loudness cannot be measured by physical apparatus. Nevertheless, it is possible to establish a numerical scale of loudness sensation. It is found that while an increase of intensity results in an increase of the loudness sensation, loudness is by no means proportional to intensity. That is, a sound of intensity 10^{-6} watt/cm^2 is not one hundred times as loud as one of intensity 10^{-8} watt/cm^2. Rather, the loudness sensation is more nearly, although not directly, proportional to the logarithm of the intensity or to the *intensity level.* In other words, the loudness sensation produced by a sound wave of intensity level 60 db exceeds the loudness sensation produced by the same type of wave of intensity level 40 db, by approximately the same amount as the sensation produced by the 40-db wave exceeds that produced by a wave of intensity level 20 db.

The preceding discussion holds only for the comparison of the physiological effects produced by the *same type of wave* at various intensities. It may break down completely when comparing waves of different types. For example, it is possible to create one wave of one frequency only and another wave of many frequencies, so that both waves have exactly the same intensity level. The two waves, however, may give rise to entirely different sensations of loudness.

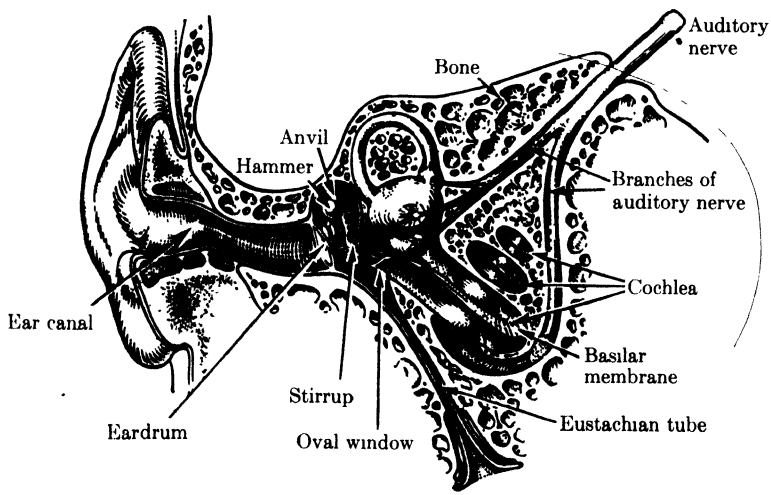


FIG. 23–1. Diagrammatic section of the right ear.

23–4 The ear and hearing. Figure 23–1 is a semidiagrammatic section of the right ear. The scale of the inner ear has been exaggerated to show details. Sound waves traveling down the ear canal strike the eardrum. A linkage of three small bones, the hammer, anvil, and stirrup, transmits the vibrations to the oval window. The oval window in turn transmits them to the inner ear, which is filled with fluid. The terminals of the auditory nerve, of which there are about 30,000 in each ear, are distributed along the basilar membrane which divides the spiral channel or cochlea into two canals. The 30,000 nerve terminals actually occupy an area only about 30 millimeters long and $\frac{1}{3}$ of a millimeter wide, a remarkable feat of engineering.

A great deal of work has been done in recent years, notably by Dr. Harvey Fletcher of the Bell Telephone Laboratories, on the processes by which the sound waves set up in the cochlea are picked up by the nerve endings. To represent the process graphically, the cochlea is drawn as a conventionalized spiral (Fig. 23–2). Each division along the spiral refers to a so-called "patch" of 1% of the nerve endings (about 300 terminals).

The width of the blackened strip in Fig. 23–3 shows the extent to which the corresponding nerve patches are stimulated by a 200-cycle tone at an intensity level of 90 db. Diagrams of this sort are called *auditory patterns*. Although the response of the ear to a pure tone is not localized at any one point, notes of lower frequency stimulate chiefly those patches near the inner portion of the spiral, and vice versa. When listening to a street noise or a symphony orchestra, all portions of the cochlea will be stimulated to a greater or lesser extent.

The range of frequencies and intensities to which the ear is sensitive is conveniently represented by a diagram like that of Fig. 23–4, which is a graph of the *auditory area* of a person of good hearing. The height

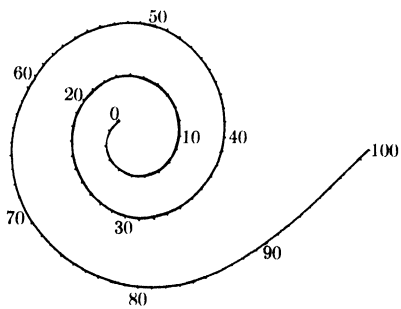


FIG. 23–2. Conventionalized diagram of the cochlea. (Courtesy of Dr. Harvey Fletcher.)

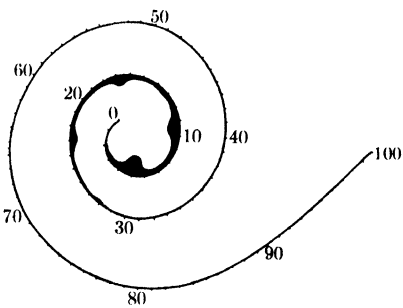


FIG. 23–3. Auditory pattern of a 200-cycle tone at an intensity level of 90 db. (Courtesy Dr. Harvey Fletcher.)

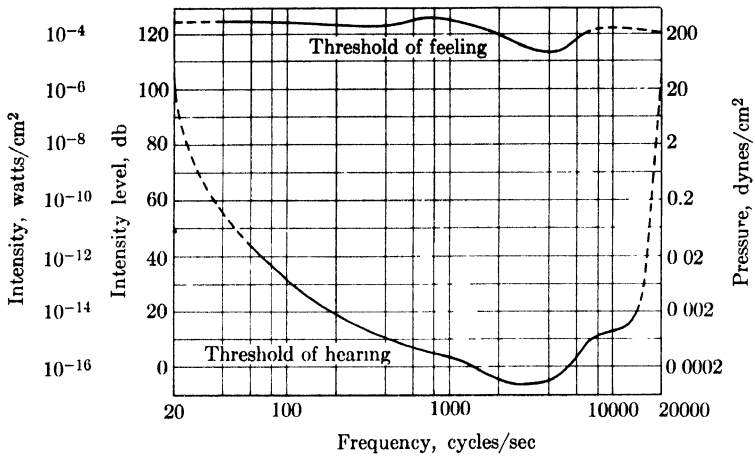


FIG. 23-4. Auditory area between threshold of hearing and threshold of feeling. (Courtesy of Dr. Harvey Fletcher.)

of the lower curve at any frequency represents the intensity level of the faintest pure tone of that frequency which can be heard. It will be seen from the diagram that the ear is most sensitive to frequencies between 2000 and 3000 cycles/sec where the *threshold of hearing*, as it is called, is about −5 db. The height of the upper curve at any frequency corresponds to the intensity level of the loudest pure tone of that frequency which can be tolerated. At intensities above this curve, which is called the *threshold of feeling*, the sensation changes from one of hearing to discomfort or even pain. The height of the upper curve is approximately constant at a level of about 120 db for all frequencies. Every pure tone which can be heard may be represented by a point lying somewhere in the area between these two curves.

Only about 1% of the population has a threshold of hearing as low as the bottom curve in Fig. 23-4; 50% of the population can hear pure tones of a frequency of 2500 cycles when the intensity level is about 8 db, and 90% when the level is 20 db.

It was stated earlier that the frequency range of the ear is from about 20 to about 20,000 cycles/sec and the intensity range from about 10^{-16} to 10^{-4} watt/cm^2, or zero to 120 db. We see now that these statements must be qualified. That is, for a loud tone of intensity level 80 db, the frequency range is from 20 to 20,000 cycles/sec, but at a level of 20 db it is only from about 200 to about 15,000 cycles/sec. At a frequency of 1000 cycles/sec the range of intensity level is from about 3 db to about 120 db, whereas at 100 cycles/sec it is only from 30 db to 120 db.

Sounds which are not pure tones are not associated with any one fre-

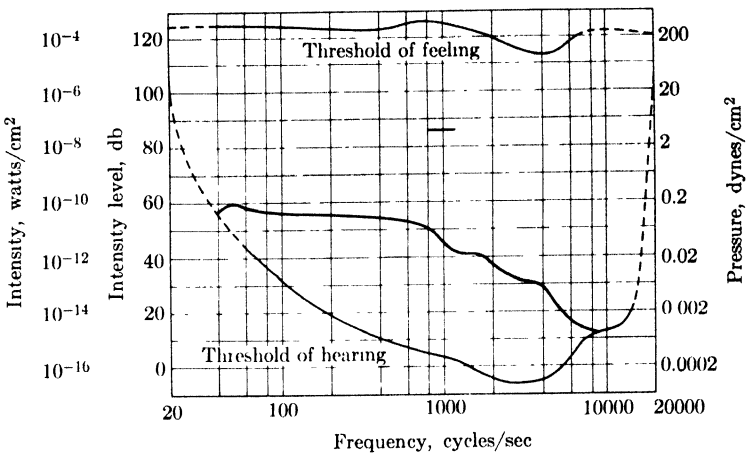


FIG. 23-5. Spectrogram of street noise. (Courtesy of Dr. Harvey Fletcher.)

quency and hence cannot be represented by a single point on the diagram. A sound such as that from a musical instrument, consisting of a mixture of a relatively few frequencies (the fundamental and overtones), can be represented by a set of points, each point giving the intensity and frequency of one particular overtone. A sound such as a street noise, while it cannot be considered as made up of a fundamental and overtones, can nevertheless be represented on the diagram in the following way. The sound is picked up by a microphone and sent through an electrical network which selects a narrow range of frequencies and measures the average intensity within this range. By repeating the process at a large number of frequencies throughout the audible range, a series of points is obtained which can be plotted. A continuous curve drawn through them is called the *spectrogram* of the sound. A typical spectrogram of street noise is shown in Fig. 23-5.

The term "spectrogram" is borrowed from optics. The process just described is entirely analogous to the optical one of dispersing a beam of light waves into a spectrum by means of a prism and measuring the intensity at a number of points throughout the spectrum. The light emitted by a gas in an electrical discharge is a mixture of waves of a number of definite frequencies and corresponds to the sound emitted by a musical instrument. Most light beams, however, are a mixture of all frequencies and are therefore the optical analog of noise.

The total intensity level of a noise can be found from its spectrogram by an integration process. There are also instruments known as noise meters which measure the level directly. The level of the street noise in Fig. 23-5 is about 85 db and is shown by the short heavy line.

23–5 Quality and pitch. A string that has been plucked or a plate that has been struck, if allowed to vibrate freely, will vibrate with many frequencies at the same time. It is a rare occurrence for a body to vibrate with only one frequency. A carefully made tuning fork struck lightly on a rubber block may vibrate with only one frequency, but in the case of musical instruments, the fundamental and many harmonics are usually present at the same time. The impulses that are sent from the ear to the brain give rise to one net effect which is characteristic of the instrument. Suppose, for example, the sound spectrum of a tone consisted of a fundamental of 200 vibrations per second and harmonics 2, 3, 4, and 5, all of different intensity, whereas the sound spectrum of another tone consisted of exactly the same frequencies but with a different intensity distribution. The two tones would sound different; they are said to differ in *quality*.

Adjectives used to describe the quality of musical tones are purely subjective in character, such as reedy, golden, round, mellow, tinny, etc. *The quality of a sound is determined by the number of overtones present*

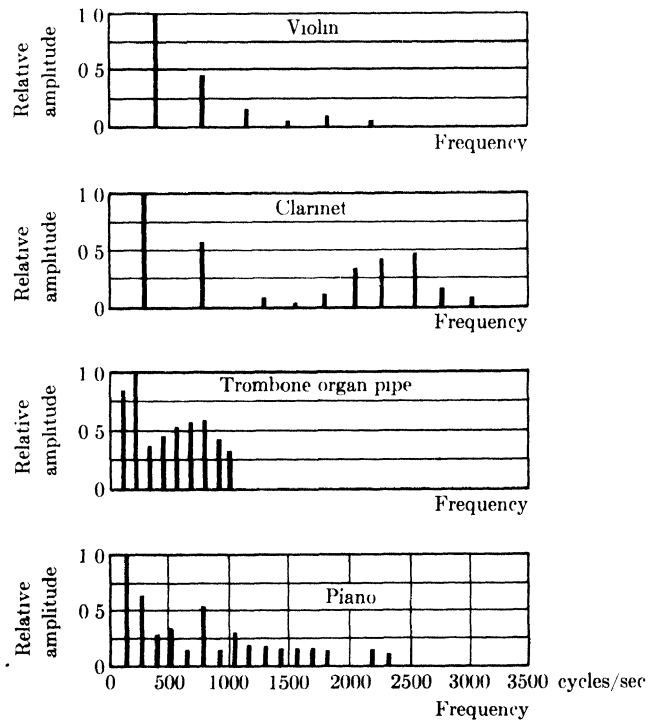


FIG. 23–6. Sound spectra of some musical instruments. (Courtesy of Dr. Harvey Fletcher.)

and their respective intensities. The sound spectra of various musical instruments are shown in Fig. 23–6.

Another subjective attribute of a musical sound that is described with the aid of the adjectives high, medium, low, etc., is called the *pitch.* It is a simple matter to explain the pitch of a sound produced by a wave of only one frequency. If we take a set of tuning forks differing in size and cause each one to sound its fundamental only, then it becomes apparent that the greater the frequency, the higher the pitch. The situation, however, is very much more complicated when the sound is produced by a wave of many frequencies. If the fundamental is considerably more intense than any of the overtones, then the pitch is determined by the frequency of the fundamental. But this is not always the case. It is possible to construct a tone consisting of many frequencies whose pitch corresponds to a frequency that actually does not exist in the tone itself.

23–6 Beats. Stationary waves in an air column have been cited as one example of interference. They arise when two wave trains of the same amplitude and frequency are traveling through the same region in opposite directions. We now wish to consider another type of interference which results when two wave trains of equal amplitude but slightly different frequency travel through the same region. Such a condition exists when two tuning forks of slightly different frequency are sounded simultaneously or when two piano wires struck by the same key are slightly "out of tune."

Let us consider some one point of space through which the waves are simultaneously passing. The displacements due to the two waves separately are plotted as a function of the time on graph (a) in Fig. 23–7. If the total extent of the time axis represents one second, the graphs correspond to frequencies of 16 vibrations per second and of 18 vibrations per second. Applying the principle of superposition to find the resultant vibration, we get graph (b), where it is seen that the amplitude varies

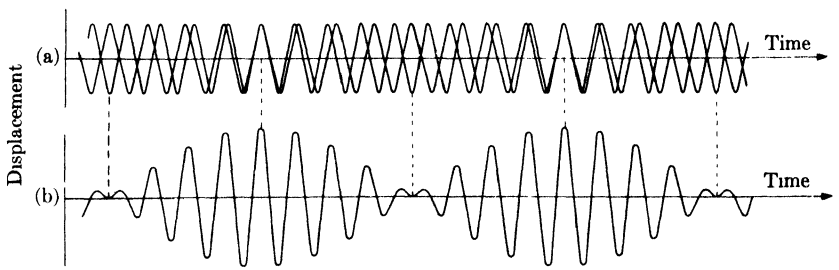


FIG. 23–7. Beats are fluctuations in amplitude produced by two sound waves of slightly different frequency.

with the time. These variations of amplitude give rise to variations of loudness which are called *beats*. Two strings may be tuned to the same frequency by tightening one of them while sounding both until the beats disappear.

The production of beats may be treated mathematically as follows. The displacements due to the two waves passing simultaneously through some one point of space may be written

$$y_1 = A \cos 2\pi f_1 t, \qquad y_2 = A \cos 2\pi f_2 t.$$

(The amplitudes are assumed equal.)

By the principle of superposition, the resultant displacement is

$$y = y_1 + y_2 = A\,[\cos 2\pi f_1 t + \cos 2\pi f_2 t],$$

and, since

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2},$$

this may be written

$$y = \left[2A \cos 2\pi \left(\frac{f_1 - f_2}{2}\right) t\right] \cos 2\pi \frac{f_1 + f_2}{2}\, t.$$

The resulting vibration can then be considered to be of frequency $(f_1 + f_2)/2$, or the average frequency of the two tones, and of amplitude given by the expression in brackets. The amplitude therefore varies with time at a frequency $(f_1 - f_2)/2$. If f_1 and f_2 are nearly equal, this term is small and the amplitude fluctuates very slowly. When the amplitude is large, the sound is loud, and vice versa. A beat, or a maximum of amplitude, will occur when $\cos 2\pi t\,(f_1 - f_2)/2$ equals 1 or -1. Since each of these values occurs once in each cycle, the number of beats per second is twice the frequency $(f_1 - f_2)/2$, or *the number of beats per second equals the difference of the frequencies.*

23–7 Combination tones. Beats between two tones can be detected by the ear up to a beat frequency of 6 or 7 per second. At higher frequencies, individual beats can no longer be distinguished and the sensation merges into one of *consonance* or *dissonance* depending on the frequency ratio of the tones. A beat frequency, even though it lies within the frequency range of the ear, is not interpreted by the ear as a tone of that frequency. Nevertheless, a tone can be heard of frequency equal to the frequency difference between two others sounded simultaneously. Such a tone is called a *difference* tone. Although not as easy to recognize, a frequency called a *summation* tone and equal to the sum of the frequencies of the two tones can also be heard. The general term applied to both difference tones and summation tones is *combination* tones.

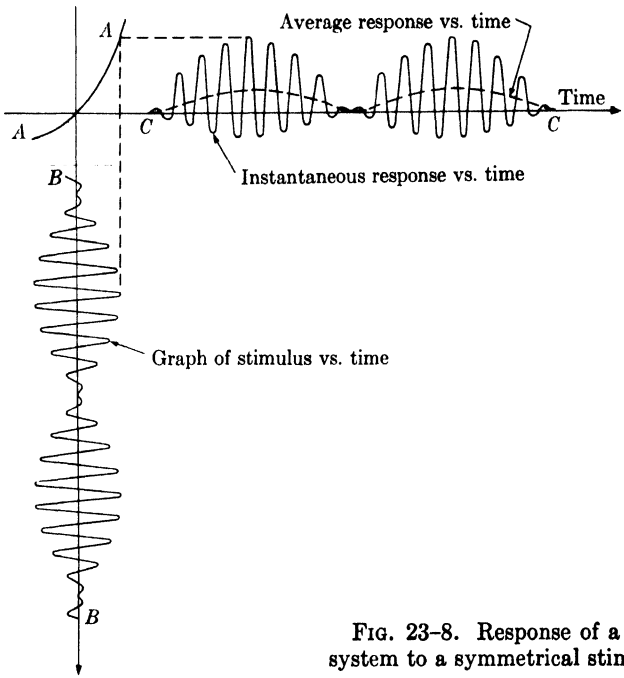


FIG. 23–8. Response of a nonlinear system to a symmetrical stimulus.

Combination tones are similar to beats but are due to the nature of the hearing mechanism. The ear is one of a number of devices whose response is said to be *nonlinear*. A vacuum tube operated on the curved portion of its characteristic is another. Figure 23–8 illustrates the situation schematically. The response to a given stimulus is represented by the curved line AA. (In a vacuum tube the "response" would be the plate current and the "stimulus" the grid potential.) The curve BB extending vertically downward represents the displacement of the air at a point through which two waves of different frequency are passing simultaneously. The response to this stimulus is obtained by projecting upward from every point of this curve to the curve AA and then projecting across. It will be seen that because of the curvature (or nonlinearity) of AA, the response curve CC is not symmetrical but has an "average" upward sweep indicated by the dotted line. Hence, if the graph represents the motion of the eardrum, it is seen that the latter will vibrate at a frequency equal to the beat frequency or the difference in frequencies of the two tones. This vibration is superposed on the much higher frequency of the individual waves in the tone. Note carefully that if the response curve AA were *linear*, the curve CC would be symmetrical and no such motion of the eardrum would ensue.

23–8 The Doppler effect. When a source of sound, or a listener, or both, are in motion relative to the air, the pitch of the sound, as heard by the listener, is in general not the same as when source and listener are at rest. The most common example is the sudden drop in pitch of the sound from an automobile horn as one meets and passes a car proceeding in the opposite direction. This phenomenon is called the *Doppler effect.*

We shall consider only the special case in which the velocities of listener and source, v_L and v_S, lie along the line joining them. Since these velocities may be in the same or opposite directions, and the listener may be either ahead of or behind the source, a convention of signs is required. We shall take the positive directions of v_L and v_S as that *from* the position of the listener *to* the position of the source. The velocity of propagation of sound waves, u, will always be considered positive.

In Fig. 23–9, a listener L is at the left of a source S. The positive direction is then from left to right, and both v_L and v_S are positive in the diagram. The sound source is at point a at time $t = 0$ and at point b at time t. The outer circle represents the wave surface emitted at time $t = 0$. This surface (in free space) is a sphere with center at a, and is traveling radially outward at all points with velocity u. (The fact that the wave originated at a *moving* source does not affect its velocity after leaving the source. The wave velocity u is a property of the *medium* only; the waves forget about the source as soon as they leave it.) The

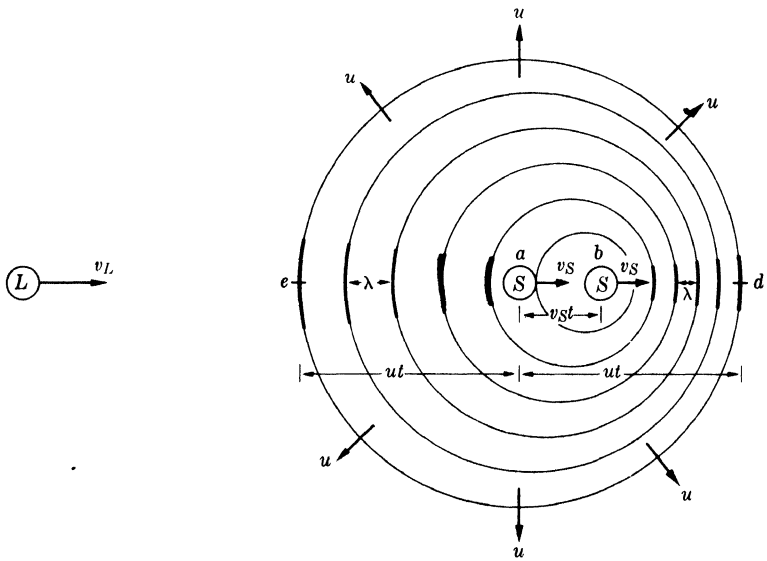


FIG. 23–9. Wave surfaces emitted by a moving source.

radius of this sphere (the distance *ea* or *ad*) is therefore ut. The distance *ab* equals $v_S t$, so

$$eb = (u + v_S)t, \qquad bd = (u - v_S)t.$$

In the time interval between $t = 0$ and $t = t$, the number of waves emitted by the source is $f_S t$, where f_S is the frequency of the source. In front of the source these waves are crowded into the distance *bd*, while behind the source they are spread out over the distance *eb*. The wavelength in front of the source is therefore

$$\lambda = \frac{(u - v_S)t}{f_S t} = \frac{u - v_S}{f_S},$$

while the wavelength behind the source is

$$\lambda = \frac{(u + v_S)t}{f_S t} = \frac{u + v_S}{f_S}.$$

The waves approaching the moving listener L have a velocity of propagation relative to him, given by $u + v_L$. The frequency at which he encounters these waves is

$$f = \frac{u + v_L}{\lambda} = \frac{u + v_L}{(u + v_S)/f_S},$$

or

$$\frac{f}{u + v_L} = \frac{f_S}{u + v_S}, \tag{23–3}$$

which expresses the frequency f as heard by the listener in terms of the frequency f_S of the source. It is unnecessary to derive equations for other special cases if consistent use is made of the sign convention given above. See the following examples.

If the medium in which the waves are traveling has a velocity v_M parallel to the line joining listener and source, then

$$\frac{f}{u + v_L - v_M} = \frac{f_S}{u + v_S - v_M}.$$

EXAMPLE. Let $f_S = 1000$ cycles/sec, $u = 1000$ ft/sec. The wavelength of the waves emitted by a stationary source is then $u/f_S = 1.00$ ft.

(a) What are the wavelengths ahead of and behind the moving source in Fig. 23–9 if its velocity is 100 ft/sec?

In front of the source,

$$\lambda = \frac{u - v_S}{f_S} = \frac{1000 - 100}{1000} = 0.90 \text{ ft}.$$

Behind the source,

$$\lambda = \frac{u + v_S}{f_S} = \frac{1000 + 100}{1000} = 1.10 \text{ ft.}$$

(b) If the listener L in Fig. 23–9 is at rest and the source is moving away from him at 100 ft/sec, what is the frequency as heard by the listener?

Since $v_L = 0$ and $v_S = 100$ ft/sec,

$$f = f_S \frac{u}{u + v_S} = 1000 \frac{1000}{1000 + 100} = 909 \frac{\text{cycles}}{\text{sec}}.$$

(c) If the source in Fig. 23–9 is at rest and the listener is moving toward the left at 100 ft/sec, what is the frequency as heard by the listener?

The positive direction (from listener to source) is still from left to right, so

$$v_L = -100 \frac{\text{ft}}{\text{sec}}, \qquad v_S = 0,$$

$$f = f_S \frac{u + v_L}{u} = 1000 \frac{1000 - 100}{1000} = 900 \frac{\text{cycles}}{\text{sec}}.$$

Thus while the frequency as heard by the listener is less than the frequency f_S both when the source moves away from the listener and when the listener moves away from the source, the decrease in frequency is not the same for the same velocity of recession.

The Doppler effect is not, of course, confined to sound waves. Light waves are emitted as a result of electronic rearrangements in atoms, and presumably the frequency of these light sources is the same whether a given atom is on the earth or in a star. The wavelength of the light from certain stars is found to be slightly longer, and from other stars slightly shorter, than that from the same atoms in a source on the earth. The inference is that the stars are moving toward or away from the earth, and their velocity can be computed from the observed wavelength difference. Even in a gas discharge tube in the laboratory, the atoms are not at rest, but are flying about with relatively large velocities. Hence the light from such a source, resulting from a given atomic process, is a mixture of waves with a slight "spread" of wavelengths. Another interesting example is the reflection of radar waves from a moving object such as a plane or an automobile. The wavelength of the reflected waves is increased if the object is moving toward the source, decreased if it is moving away from the source. The same phenomenon occurs when underwater sound waves (sonar) are reflected by a moving submarine.

TABLE 23–2

FREQUENCY RELATIONS IN THE MAJOR DIATONIC SCALE AND IN THE EQUALLY TEMPERED SCALE

	Do	Re	Mi	Fa	Sol	La	Ti	Do′	Re′
Frequency relations	Middle C	D	E	F	G	A	B	C′	D′
1 Octave, key of C	1							2	
2 Major triad, key of C	4		5		6			(8)	
3 Major triad, key of F				4		5		6	
4 Major triad, key of G		(3)			4		5		6
5 Diatonic scale, key of C	264	297	330	352	396	440	495	528	594
6		9/8 whole	10/9 whole	16/15 half	9/8 whole	10/9 whole	9/8 whole	16/15 half	9/8 whole
7 Equally tempered scale suitable for all keys	261.6	293.7	329.6	349 2	392.0	440	493.9	523.3	587.4
8		$\sqrt[6]{2}$	$\sqrt[6]{2}$	$\sqrt[12]{2}$	$\sqrt[6]{2}$	$\sqrt[6]{2}$	$\sqrt[6]{2}$	$\sqrt[12]{2}$	$\sqrt[6]{2}$

23–9 Musical intervals and scales. If certain musical tones are produced in succession, an untrained listener, as well as an accomplished musician, recognizes a relationship among them. Such relations are described in musical language by words such as octave, major third, minor third, etc. The listener recognizes something basic in these intervals, and experimental measurement of the fundamental frequencies of the separate tones discloses that they bear simple whole-number ratios to one another. Thus, the fundamental frequencies of middle C of the

piano and its *octave* above, C′, are to each other as 1 to 2. Another basic set of tones is obtained by playing C, E, G. These constitute what is known as a *major triad*, and the frequencies are found to be proportional to 4, 5, and 6.

Starting at middle C of the piano, and playing only the white notes toward the right, it is possible to find, within only nine notes, three major triads. These are shown in Table 23–2, and enable one to calculate the frequencies of all the notes, once one of them has been chosen arbitrarily. By international agreement, the frequency of the A above middle C is chosen to be 440 vibrations per second. The frequencies of all the notes, shown in row 5 of Table 23–2, are those that correspond to the *major diatonic scale* of the key of C. The frequency ratios of adjacent notes are seen to be either 9/8, 10/9, or 16/15. The interval between two tones whose frequencies bear the ratio 9/8 or 10/9 are called *whole tones*, whereas the interval between two notes of frequency ratio 16/15 is a *half tone*.

If a major diatonic scale were constructed starting at D instead of at C, four new notes would be needed for a perfect diatonic scale. If all possible musical keys were to be provided for, 72 notes would be needed for each octave. To avoid this tremendous complication, what is known as the *equally tempered scale* has been devised (used by J. S. Bach and probably earlier musicians). In this scheme there are 12 half-tone intervals in every range of an octave, adjacent notes a half tone apart bearing the constant ratio of the twelfth root of 2, i.e., 1.05946. Simple as this scheme is, it results in no one scale being exactly diatonic. Since the ratios of the diatonic scale were originally selected to suit the preferences of the ear (being made up of three sets of major triads, each of which constitutes a harmonious combination), this means that music which makes use of the equally tempered scale is not quite so pleasant to the ear. The difference, however, is apparently slight to any but the most critical.

Problems

23–1. (a) If the pressure amplitude in a sound wave is tripled, by how many times is the intensity of the wave increased? (b) By how many times must the pressure amplitude of a sound wave be increased in order to increase the intensity by a factor of 16 times?

23–2. (a) Two sound waves, one in air and one in water, are equal in intensity. What is the ratio of the pressure amplitude of the wave in water to that of the wave in air? (b) If the pressure amplitudes of the waves are equal, what is the ratio of their intensities?

23–3. (a) Relative to the arbitrary reference intensity of 10^{-16} watt/cm^2, what is the intensity level in db of a sound wave whose intensity is 10^{-10} watt/cm^2? (b) What is the intensity level of a sound wave in air whose pressure amplitude is 2 dynes/cm^2?

23–4. (a) Show that if β_1 and β_2 are the intensity levels in db of sounds of intensities I_1 and I_2 respectively, the difference in intensity levels of the sounds is

$$\beta_2 - \beta_1 = 10 \log \frac{I_2}{I_1}.$$

(b) Show that if P_1 and P_2 are the pressure amplitudes of two sound waves, the difference in intensity levels of the waves is

$$\beta_2 - \beta_1 = 20 \log \frac{P_2}{P_1}.$$

(c) Show that if the reference level of intensity is $I_0 = 10^{-16}$ watt/cm^2, the intensity level of a sound of intensity I is

$$\beta = 160 + 10 \log I.$$

23–5. Two loud-speakers, A and B, radiate sound uniformly in all directions. The output of acoustic power from A is 8×10^{-4} watt, and from B it is 13.5×10^{-4} watt. Both loud-speakers are vibrating in phase at a frequency of 173 cycles/sec. (a) Determine the difference in phase of the two signals at a point C along the line joining A and B, 3 m from B and 4 m from A. (b) Determine the intensity at C from speaker A if speaker B is turned off, and the intensity at C from speaker B if speaker A is turned off. (c) With both speakers on, what is the intensity and intensity level at C?

23–6. The intensity due to a number of independent sound sources is the sum of the individual intensities. How many db greater was the intensity level when all five of the quintuplets cried simultaneously than when a single one cried? How many more crying babies would be required to produce a further increase in the intensity level of the same number of db?

23–7. A window whose area is 1 m^2 opens on a street where the street noises result in an intensity level, at the window, of 60 db. How much "acoustic power" enters the window via the sound waves?

23–8. (a) What are the upper and lower limits of intensity level of a person whose auditory area is represented by the graph of Fig. 23–4? (b) What are the highest and lowest frequencies he can hear when the intensity level is 40 db?

23–9. Two whistles, A and B, each have a frequency of 500 cycles/sec. A is stationary and B is moving toward the right (away from A) at a velocity of 200 ft/sec. An observer is between the two whistles, moving toward the

right with a velocity of 100 ft/sec. Take the velocity of sound in air as 1100 ft/sec. (a) What is the frequency from A as heard by the observer? (b) What is the frequency from B as heard by the observer? (c) What is the beat frequency heard by the observer?

23–10. A railroad train is traveling at 100 ft/sec in still air. The frequency of the note emitted by the locomotive whistle is 500 cycles/sec. What is the wavelength of the sound waves (a) in front of the locomotive, and (b) behind it? What would be the frequency of the sound heard by a stationary listener (c) in front of, (d) behind the locomotive? What frequency would be heard by a passenger on a second train traveling at 50 ft/sec and (e) approaching the first, (f) receding from the first? (g) How is each of the preceding answers altered if a wind of velocity 30 ft/sec is blowing in the same direction as that in which the first locomotive is traveling?

23–11. Compare the frequencies heard by a listener under the following circumstances: (a) A listener in still air moves with a velocity of 100 ft/sec directly toward a stationary source emitting sound waves of a frequency 1000 cycles/sec. (b) The listener remains at rest while the source moves directly toward him at 100 ft/sec.

23–12. A man stands at rest in front of a large smooth wall. Directly in front of him, between him and the wall, he holds a vibrating tuning fork of frequency f_0 cycles/sec. He now moves the fork toward the wall with a velocity v. How many beats per sec will he hear between the sound waves reaching him directly from the fork, and those reaching him after being reflected from the wall? Represent the velocity of sound in air by V. For a numerical example, let $f_0 = 400$ cycles/sec and $v = 4$ ft/sec.

23–13. A source of sound waves S, emitting waves of frequency 1000 cycles/sec, is traveling toward the right in still air with a velocity of 100 ft/sec. At the right of the source is a large smooth reflecting surface moving toward the left with a velocity of 400 ft/sec. (a) How far does an emitted wave travel in 0.01 sec? (b) What is the wavelength of the emitted waves in front of (i.e., at the right of) the source? (c) How many waves strike the reflecting surface in 0.01 sec? (d) What is the velocity of the reflected waves? (e) What is the wavelength of the reflected waves?

ANSWERS TO ODD-NUMBERED PROBLEMS

CHAPTER 1

1–1. 25.7 lb, 30.6 lb
1–3. (a) 18.5 lb (b) 9.2 lb
1–7. (a) 19.3 lb in a direction midway between the 10-lb forces
(b) 8.46 lb in a direction midway between the 10-lb forces
1–9. 308 lb, 25° above the x-axis
1–11. (a) 2 in. (b) 3.46 in.
1–13. (a) 7 lb, 2.9 lb
(b) 7.6 lb
(c) 11 lb
1–15. $R = 0$

CHAPTER 2

2–3. (a) 10 lb (b) 20 lb
2–5. (a) 150 lb(A), 180 lb(B), 200 lb(C)
(b) 200 lb(A), 280 lb(B), 200 lb(C)
(c) 550 lb(A), 670 lb(B), 200 lb(C)
(d) 167 lb(A), 58 lb(B), 125 lb(C)
2–7. (a) Parts (b) and (c) can be solved
(b) In part (a) another side or angle is needed
2–9. 630 lb
2–11. (a) 20 lb (b) 30 lb
2–13. (a) $F = w/2 \sin \theta$
(b) $T = w/2 \tan \theta$
2–17. 22 lb (normal force does *not* equal weight)
2–19. (a) 76 lb (b) 24 lb
(c) from 15.4 to 84.6 lb
2–21. (a) Pulled up
(b) 145 lb
2–23. (a) 3 lb (b) 4 lb (c) $\frac{5}{2}$ lb
2–25. (b) 10 lb (c) 30 lb

CHAPTER 3

3–3. (a) 20 lb (b) 30 lb
3–5. (a) −6 lb, 10 lb (b) $\frac{5}{3}$
(c) 12 lb (d) 2 ft from right end of bar
3–7. 722 lb, 722 lb, 1500 lb
3–9. (a) 69.4 lb (b) 212 lb, 19° above the horizontal
3–11. 2 weights
3–13. 81.5 lb, 78° above the horizontal
3–15. 19.3 ft
3–17. (a) 54 lb (b) 24 lb
3–19. 15 lb each
3–21. (a) 20 lb on A, 140 lb on B
(b) 4 ft
3–23. 32.5 lb and 13 lb on each front leg, 17.5 lb and 7 lb on each rear leg
3–25. (a) 15 lb (b) 5 lb on each front leg, 20 lb on each rear leg
(c) 8.75 lb on each front leg, 16.25 lb on each rear leg
(d) 3.33 ft above floor
3–27. On the perpendicular bisector of the line joining the 9 and 12 lb weights, at a distance of 0.6 ft from this line.
3–29. 7.4 inches from the large end
3–31. 4 inches to the right and 2.5 inches up from the lower left-hand corner.

CHAPTER 4

4–1. (a) 15 mi/hr
(b) 22 ft/sec
(c) 672 cm/sec

4–3. 61 cm/sec, 60.1 cm/sec, 60.01 cm/sec, 60 cm/sec

4–5. (a) 0 (b) 6.3 ft/sec^2
(c) -11 ft/sec^2 (d) 100 ft
(e) 230 ft (f) 320 ft

4–7. 4 ft/sec^2, 620 ft

4–9. (a) 2.67 mi/hr·sec, 3.92 ft/sec^2
(b) 7.5 sec (c) 441 ft, 550 ft between 15 sec and 22.5 sec

4–11. (a) 12.5 cm/sec^2 (b) 7840 cm

4–13. (a) 300 ft (b) 60 ft/sec

4–15. (a) 24 ft/sec, 29 ft/sec, 34 ft/sec
(b) 2.5 ft/sec^2
(c) 21.5 ft/sec (d) 8.6 sec
(e) 23 ft (f) 1 sec
(g) 24 ft/sec

4–17. (a) 32 ft/sec (b) 2.83 sec
(c) 16 ft (d) 22.6 ft/sec
(e) 16 ft/sec

4–19. (a) 8.7 sec (b) 75 ft
(c) v_A = 52 ft/sec, v_T = 35 ft/sec

4–21. (a) 94 ft/sec (b) 124 ft
(c) 53 ft/sec (d) 150 ft/sec^2
(e) 1.96 sec (f) 93 ft/sec

4–23. 39.4 ft

4–25. (a) 48 ft/sec (b) 36 ft
(c) 80 ft/sec

4–27. (a) 32 ft (b) 13 ft/sec, -32 ft/sec^2
(c) -51 ft/sec, -32 ft/sec^2
(d) 37 ft/sec

4–29. (a) 36 ft (b) 0.5 sec
(c) 32 ft/sec, -32 ft/sec^2, -16 ft/sec, -32 ft/sec^2

4–31. (a) 4 m/sec^2 (b) 6 m/sec
(c) 4.5 m

4–33. 6.55 knots

4–35. (a) 50 mi/hr, 53° S of W
(b) 60° S of W

4–37. (a) 5 mi/hr, 53° E of N
(b) 0.75 mi (c) 15 min

CHAPTER 5

5–1. (a) 400 lb (b) 12,800 dynes
(c) 12,800 n

5–3. (a) 2 m/sec^2 (b) 100 m
(c) 20 m/sec

5–7. (a) 0.5 slug (b) 500 ft

5–9. (a) 1.62×10^{-10} dyne
(b) 3.33×10^{-9} sec
(c) 1.8×10^{17} cm/sec^2

5–11. (a) 19.6 n (b) 2.55 sec

5–13. (a) 4 lb (b) 128 ft

5–15. 48 ft/sec^2

5–17. (a) 40 lb (b) 4 ft/sec^2 down
(c) zero

5–19. 240 ft

5–21. (a) 12 sec after force is applied
(b) 8 ft/sec^2

5–23. $W = 2wa/(g + a)$

5–25. (a) 37° (b) 6.4 ft/sec^2
(c) 2.5 sec

5–27. (a) 6 lb, (b) 8 ft/sec^2

5–29. (a) 2 lb (b) 1.9 lb

5–31. (a) 196 cm/sec^2
(b) 314,000 dynes

5–33. (a) to left (b) 2.13 ft/sec^2
(c) 43.3 lb

5–35. 1.33 lb

5–37. 4 lb

5–39.

	a_1	a_2
(a)	0	0
(b)	0	0
(c)	0	16 ft/sec^2
(d)	4 ft/sec^2	28 ft/sec^2
(e)	16 ft/sec^2	48 ft/sec^2

5–41. (a) 4.4 ft/sec^2
(b) 1.2 lb

5–43. 127.5 lb

5–45. 6.2×10^{27} gm

5–47. 6.32 ft/sec^2

CHAPTER 6

6–1. 10 ft/sec
6–3. (a) 20 sec (b) 6000 ft
(c) v_x = 300 ft/sec,
v_y = 640 ft/sec
6–5. (a) 100 ft (b) 200 ft
(c) v_x = 80 ft/sec,
v_y = 80 ft/sec
v = 113 ft/sec at 45° below horizontal
6–7. (a) 0.56 mi (b) 296 mi/hr at 47° below horizontal
(c) 10 sec
6–9. (a) 40 sec (b) 55°
6–11. (a) 121 ft/sec (b) 57.2 ft
(c) 3.8 sec
6–13. 100 ft
6–15. Yes. Ball clears fence by 10 ft
6–17. (a) 67 ft/sec (b) 39 ft/sec at 31° below horizontal
6–19. 1.98 sec, 157 sec
6–21. 1300 ft
6–23. 22°
6–27. (a) 32 ft/sec (b) 1 sec
(c) 54 ft
6–29. (a) 66,700 mi/hr
(b) 0.0193 ft/sec^2
6–31. 0.5
6–33. (a) 38.2 rev/min
(b) 5 lb
6–35. (a) 0.27 (b) 15°
6–37. (a) 198 cm/sec
(b) 29.5×10^4 dynes
6–39. (a) 128 ft/sec^2 (b) 480 lb
(c) 480 lb (d) 16 ft/sec
6–41. $\cos^2 \theta$
6–43. 36,000 km above earth
6–45. 16,900 mi/hr

CHAPTER 7

7–1. 63.4×10^6 ft·lb
7–3. 9850 ft·lb
7–5. (a) 55,000 ft·lb (b) 4 times
7–7. 4.5×10^{-10} erg
7–9. 9.8 joules
7–11. (a) 5 lb, 10 lb, 20 lb
(b) 1.3 ft·lb, 5 ft·lb, 20 ft·lb
7–13. (a) 33 ft·lb
(b) about 15 ft·lb
7–15. (a) 160 ft·lb (b) 160 ft·lb
7–17. (a) 3300 ft·lb (b) 1300 ft·lb
(c) 1500 ft·lb (d) 500 ft·lb
Goes into heat.
(e) b + c + d = a
7–19. (a) 42 lb (b) 105 ft·lb
7–21. (a) 40 ft/sec (b) 20 ft/sec
7–23. 10 cm
7–25. (a) 0.25 (b) 3.5 ft·lb
7–29. 1.81 m/sec
7–31. 0.655 hp, 489 watts, 0.489 kw
7–33. 154 hp
7–35. $1.29
7–37. (a) 250 lb (b) 5 hp (c) 80 hp
7–39. (a) 8200 joules (b) 8200 joules
(c) 0.55 hp
7–41. (a) 6300 lb (b) 30 ft/sec
(c) 68,000 ft·lb (d) 220,000 ft·lb
(e) 250 hp
7–43. (a) 55,000 dynes (b) 1.3
(c) 1.7 (d) 79%
7–45. 12,000 lb
7–51. (a) 18×10^{20} ergs
(b) 18×10^{26} ergs/sec
(c) 1.2×10^{13} gm (about 3 billion gallons)

CHAPTER 8

8–1. (a) 28,000 slug·ft/sec
(b) 60 mi/hr (c) 43 mi/hr
8–3. (a) 8×10^5 m/sec^2
(b) 4×10^4 n (c) 5×10^{-4} sec
(d) 20 n·sec
8–5. 1.5 ft/sec
8–7. 0.65 mi/hr
8–9. 17 mi/hr 53° E of S
8–13. (a) 10 cm/sec (b) 0.14 joule
(c) −70 cm/sec, 80 cm/sec

8–15. (a) 100 gm (b) 16 cm/sec
8–17. 58,700 ft
8–19. 1.82 ft/sec
8–21. (a) 149° from direction of electron
(b) 10.6×10^{-16} gm·cm/sec
(c) 14.5×10^{-10} erg
8–23. (a) 41 cm, $100(0.64)^n$, 10
(b) 0.71 sec, $0.9 \times (0.8)^n$ sec
(c) 0.8
8–25. 7.2 ft below starting point
8–27. 280 m/sec
8–29. (a) 0.16 (b) 240 joules
(c) 0.32 joule
8–31. 596 m/sec
8–33. (a) 0.19 ft/sec (b) 1.3 lb
8–35. (a) 4 ft/sec (b) 2 lb
(c) 25,000 ft·lb for bullets, 40 ft·lb for man

CHAPTER 9

9–1. (a) 1.5 rad (b) 1.6 rad, 90°
(c) 120 cm, 120 ft
9–3. (a) 20 ft/sec (b) 230 rev/min
9–5. 5 rad/sec^2, 1000 rad
9–7. 20 rad/sec^2
9–11. (a) At the bottom of the wheel
(b) 360 ft/sec at 3° from the vertical
9–13. (a) 135°
(b) 2.36 ft/sec^2, 11.03 ft/sec^2
9–15. 164,000 rev/min
9–17. (a) 2.67 kgm·m^2
(b) 10.7 kgm·m^2
(c) 1.6×10^{-3} kgm·m^2
9–19. (a) A smallest (b) D largest
9–21. (a) 2.56×10^{28} slug·mi^2
(b) 2530 mi
9–23. (a) 278 ft·lb (b) 70 ft
9–25. 1.1 slug·ft^2
9–27. (a) 2×10^7 joules (b) 18 min
9–29. (a) 4 rad/sec^2 (b) 14 rad/sec^2
(c) 32,000 ft·lb
9–31. 0.47
9–33. (a) 10 rad/sec^2 (b) 200 ft·lb
(c) 6.15 rad/sec^2
9–35. (a) 21.3 lb (b) 52.3 ft/sec
(c) 2.45 sec
9–37. (a) 10.7 ft/sec^2, 0, 5.33 lb
(b) 4.57 ft/sec^2, 9.14 rad/sec^2, 6.86 lb, 4.57 lb
9–39. (a) 240 cm/sec, 320 cm/sec
(b) 5×10^5 gm·cm^2
(c) 1600 cm/sec^2
(d) 780 cm/sec^2
(e) 51,000 dynes, tension
9–41. (a) 2m/sec^2 (b) 9.8 newtons
9–43. (a) 12 rad/sec (b) 0.027 joule
9–45. $(mg \pm \mu Mg)/M\omega^2$
9–47. 0.08 rev/sec
9–49. (a) −0.04 rad/sec
(b) 60 deg (c) 72 deg
9–51. (a) 2 slug·ft^2
(b) 2620 ft·lb
9–53. (a) 180,000 dynes
(b) 4300 rev/min

CHAPTER 10

10–1. 25×10^6 lb/in^2
10–3. (b) 14×10^6 lb/in^2
(c) 0.016×10^6 lb/in^2
10–5. (a) 27.5 lb (b) 0.028 ft
10–7. 0.0253 in.
10–9. (a) 1.8 ft (b) 30,000 lb/in^2 in steel, 12,000 lb/in^2 in copper (c) 0.001 in steel, 0.0006 in copper
10–13. 2000 lb
10–15. Steel, 0.64×10^{-6} atm^{-1}
Water, 50×10^{-6} atm^{-1}
Water is 78 times more compressible
10–17. (a) $(E \cos^2 \theta)/A$
(b) $(F \sin 2\theta)/2A$
(c) 0° (d) 45°

CHAPTER 11

11–1. (a) 9470 cm/sec^2, 377 cm/sec
(b) 5680 cm/sec^2, 301 cm/sec
(c) 0.0368 sec
11–3. (a) $2400\pi^2$ ft/sec^2
(b) 740 lb (c) 43 mi/hr
11–5. 23 lb
11–7. 6.20 cm
11–9. (a) $\frac{1}{2}mv^2 + \frac{1}{2}kx^2$
(b) $\frac{1}{2}kA^2$
11–11. (a) $L_1 = 35$ cm, $L_2 = 25$ cm
(c) 1 sec
11–13. (a) 9 vib/sec
(b) 20×10^6 lb/in^2
11–15. (a) 2 ft/sec
(b) 4 ft/sec^2
11–17. 979.78 cm/sec^2
11–19. (a) 40 rad/sec (b) 34 rad/sec
(c) 120 rad/sec^2
11–21. 67 cm

CHAPTER 12

12–1. 21 lb/in^2
12–3. 120 lb/in^2, 18,000 lb/ft^2
12–5. 34.8 ft^2
12–7. 270,000 ft^3, 9.3 tons lift, using helium
12–9. (a) 5 cm (b) 4900 dynes/cm^2
12–11. (a) 100 lb/ft^3 (b) E will read 5 lb, D will read 15 lb
12–15. (a) 4500 lb (b) 10,000 lb
(c) 230 lb
12–17. 100.87 gm
12–19. (b) 4 lb (c) 1 ft^3
12–21. 0.781 gm/cm^3

CHAPTER 13

13–3. 13,720 dynes/cm^2
13–5. (a) 70.7 cm of Hg
(b) 71.2 cm of Hg
(c) 11 cm
13–7. 4.3 cm

CHAPTER 14

14–1. (a) 36 ft/sec (b) 0.2 ft^3/sec
14–3. 39.6 ft/sec
14–5. (a) 0.056 ft^3/sec (b) 3 ft
14–7. (a) 16 ft/sec (b) 0.79 ft^3/sec
14–9. (a) 0.2 ft^3/sec (b) 4.7 ft
(c) 3 ft
14–11. (a) 12 lb/in^2 (b) 12 ft^3/sec
14–13. 50 hp
14–15. (a) 2.5 sec (b) 44 lb/in^2
(c) 33 lb/in^2
14–17. 12 ft^3/min
14–19. 27 ft^3/sec
14–21. 325 ft/sec
14–23. (a) 6.4 ft^3/sec
(b) 6.02 lb/in^2
14–25. (a) 1500 (b) laminar flow
(c) 0.043 (d) 18.2 cm of water
14–27. (a) 30.4 cm/sec (b) turbulent flow (c) 53.8 liters/sec
14–29. (a) 0.77 cm/sec
(b) 1.89 cm/sec

CHAPTER 15

15–3. 1.6 ft
15–5. 0.03 cm
15–7. 1.0×10^{-5} (C°)$^{-1}$
15–9. (a) 13×10^{-6} (C°)$^{-1}$
(b) 19×10^4 lb/in^2
15–11. (a) 76°C (b) −63.3°C

15–13. Period decreases, 1.8×10^{-4}
15–15. 270,000 lb
15–17. 460 atm
15–19. 72,000 lb/in^2, assuming length of brass is constant
15–21. $9:35 \times 10^8$ dynes/cm^2

CHAPTER 16

16–1. 74 ft^3
16–3. 4.0 Btu
16–5. Assuming a heat of combustion of 11,000 Btu/lb
(a) 18.7×10^7 Btu
(b) 42.2 ft
16–7. (a) 80.5% (b) 330 Btu
16–9. 370 years
16–11. 0.1 Btu/lb·F°
16–13. 1.00, 0.827, 0.35
16–15. (a) 0.092 cal/gm·C°
(b) 27.6 gm
16–17. 725 cal
16–19. It melts the ice
16–21. 0°C with 0.2 gm of ice left
16–23. 539 cal/gm
16–25. 24°C
16–27. 40°C
16–29. 1.84 kgm
16–31. 17,100 cm
16–33. 445 cal/gm

CHAPTER 17

17–1. 86,400 cal/day
17–3. (a) 1.8 cal/sec (b) 20 cm
17–5. (a) 40°C (b) 2.4 cal/sec
17–7. 1.15×10^{-3} cal/sec·cm·C°
17–9. 110°C
17–11. 84.2 C°
17–13. 4.54×10^5 cal
17–15. 1.78 watts
17–17. 20.03°K

CHAPTER 18

18–1. 6.6×10^4 ft·lb
18–3. (a) Yes (b) No
(c) Negative
18–5. (a) No (b) Yes (c) Yes
(d) Work done on the resistor = heat transferred to the water
18–7. 10.8 ft^3
18–11. $U_1 = U_2$
18–13. (a) 60 Btu (b) liberate 70 Btu
(c) 50 Btu; −10 Btu

CHAPTER 19

19–1. 3.6 lb/in^2
19–3. (a) 0.75 atm (b) 2 gm
19–5. 3.25
19–7. (a) 927°C (b) 0.325 gm
19–9. Adiabatic expansion
19–11. 16 liter·atm
19–13. When the piston has descended 13.12 in.
19–15. 0.0023 gm
19–17. (b) 600°K (c) 4 atm
19–19. (a) When piston is 3.13 inches from bottom (b) 477°K
19–27. 16%
19–29. (a) 12°C (b) 10 gm/cm^3
19–31. 1.3 lb/hr
19–33. (a) About 3×10^{-7} cm
(b) About 10 times as great
19–35. 1700 ft/sec
19–37. 4800°K

CHAPTER 20

20–1. 4
20–3. 93 C°
20–5. 15%
20–7. (a) 2 atm, 5 liters;
1 atm, 5 liters;
1 atm, 3 liters;
2 atm, 3 liters;
(b) 2 liter·atm
20–9. (a) 6.6 (b) 0.64 kwh
(c) 3¢

CHAPTER 21

21–1. 316 m/sec
21–3. 6320 cm/sec
21–5. 0.164 sec
21–7. 48×10^{-6} atm^{-1}
21–9. 322 m/sec, 1320 m/sec, 347 m/sec
21–11. (a) 132 cy/sec (b) 1320 cy/sec
(c) 13,200 cy/sec

CHAPTER 22

22–3. (a) 200 cy/sec
(b) 49th overtone
22–5. (a) 3600 cm/sec
(b) 6.5×10^{6} dynes
22–7. 1.28
22–9. (a) 5000 m/sec
(b) 340 m/sec
22–11. Diatomic
22–13. (a) 1140, 2280, 3420, 4560, 5700 cy/sec
(b) 570, 1710, 2850, 3990, 5130 cy/sec (c) 16, 17
22–15. (a) 420 cy/sec (b) 415 cy/sec

CHAPTER 23

23–1. (a) 9 times (b) 4 times
23–3. (a) 60 db (b) 77 db
23–5. (a) π rad (b) Due to A: 4×10^{-10} watt/cm^2. Due to B: 12×10^{-10} watt/cm^2 (c) 2.1×10^{-10} watt/cm^2, 63.2 db
23–7. 10^{-6} watt
23–9. (a) 454 cy/sec (b) 462 cy/sec
(c) 8 cy/sec
23–11. (a) 1090 cy/sec
(b) 1100 cy/sec
23–13. (a) 11 ft (b) 1 ft
(c) Almost 15 waves
(d) 1100 ft/sec (e) 0.49 ft

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
0		0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
N	0	1	2	3	4	5	6	7	8	9

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	8689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039
N	0	1	2	3	4	5	6	7	8	9

NATURAL TRIGONOMETRIC FUNCTIONS

Angle					Angle				
Degree	Radian	Sine	Cosine	Tangent	Degree	Radian	Sine	Cosine	Tangent
0°	.000	0.000	1.000	0.000					
1°	.017	.018	1.000	.018	46°	0.803	0.719	0.695	1.036
2°	.035	.035	0.999	.035	47°	.820	.731	.682	1.072
3°	.052	.052	.999	.052	48°	.838	.743	.669	1.111
4°	.070	.070	.998	.070	49°	.855	.755	.656	1.150
5°	.087	.087	.996	.088	50°	.873	.766	.643	1.192
6°	.105	.105	.995	.105	51°	.890	.777	.629	1.235
7°	.122	.122	.993	.123	52°	.908	.788	.616	1.280
8°	.140	.139	.990	.141	53°	.925	.799	.602	1.327
9°	.157	.156	.988	.158	54°	.942	.809	.588	1.376
10°	.175	.174	.985	.176	55°	.960	.819	.574	1.428
11°	.192	.191	.982	.194	56°	.977	.829	.559	1.483
12°	.209	.208	.978	.213	57°	.995	.839	.545	1.540
13°	.227	.225	.974	.231	58°	1.012	.848	.530	1.600
14°	.244	.242	.970	.249	59°	1.030	.857	.515	1.664
15°	.262	.259	.966	.268	60°	1.047	.866	.500	1.732
16°	.279	.276	.961	.287	61°	1.065	.875	.485	1.804
17°	.297	.292	.956	.306	62°	1.082	.883	.470	1.881
18°	.314	.309	.951	.325	63°	1.100	.891	.454	1.963
19°	.332	.326	.946	.344	64°	1.117	.899	.438	2.050
20°	.349	.342	.940	.364	65°	1.134	.906	.423	2.145
21°	.367	.358	.934	.384	66°	1.152	.914	.407	2.246
22°	.384	.375	.927	.404	67°	1.169	.921	.391	2.356
23°	.401	.391	.921	.425	68°	1.187	.927	.375	2.475
24°	.419	.407	.914	.445	69°	1.204	.934	.358	2.605
25°	.436	.423	.906	.466	70°	1.222	.940	.342	2.747
26°	.454	.438	.899	.488	71°	1.239	.946	.326	2.904
27°	.471	.454	.891	.510	72°	1.257	.951	.309	3.078
28°	.489	.470	.883	.532	73°	1.274	.956	.292	3.271
29°	.506	.485	.875	.554	74°	1.292	.961	.276	3.487
30°	.524	.500	.866	.577	75°	1.309	.966	.259	3.732
31°	.541	.515	.857	.601	76°	1.326	.970	.242	4.011
32°	.559	.530	.848	.625	77°	1.344	.974	.225	4.331
33°	.576	.545	.839	.649	78°	1.361	.978	.208	4.705
34°	.593	.559	.829	.675	79°	1.379	.982	.191	5.145
35°	.611	.574	.819	.700	80°	1.396	.985	.174	5.671
36°	.628	.588	.809	.727	81°	1.414	.988	.156	6.314
37°	.646	.602	.799	.754	82°	1.431	.990	.139	7.115
38°	.663	.616	.788	.781	83°	1.449	.993	.122	8.144
39°	.681	.629	.777	.810	84°	1.466	.995	.105	9.514
40°	.698	.643	.766	.839	85°	1.484	.996	.087	11.43
41°	.716	.658	.755	.869	86°	1.501	.998	.070	14.30
42°	.733	.669	.743	.900	87°	1.518	.999	.052	19.08
43°	.751	.682	.731	.933	88°	1.536	.999	.035	28.64
44°	.768	.695	.719	.966	89°	1.553	1.000	.018	57.29
45°	.785	.707	.707	1.000	90°	1.571	1.000	.000	∞

PERIODIC TABLE OF THE ELEMENTS

Atomic weights are based on the most recent values adopted by the International Union of Chemistry (For artificially produced elements, the approximate atomic weight of the most stable isotope is given in brackets)

Group→		I	II	III	IV	V	VI	VII	VIII			O
Period	Series											
1	1	1 H 1 0080										2 He 4 003
2	2	3 Li 6 940	4 Be 9 013	5 B 10 82	6 C 12 011	7 N 14 008	8 O 16 0000	9 F 19 00				10 Ne 20 183
3	3	11 Na 22 991	12 Mg 24 32	13 Al 26 98	14 Si 28 09	15 P 30 975	16 S 32 066	17 Cl 35 457				18 A 39 944
4	4	19 K 39 100	20 Ca 40 08	21 Sc 44 96	22 Ti 47 90	23 V 50 95	24 Cr 52 01	25 Mn 54 94	26 Fe 55 85	27 Co 58 94	28 Ni 58 71	
	5	29 Cu 63 54	30 Zn 65 38	31 Ga 69 72	32 Ge 72 60	33 As 74 91	34 Se 78 96	35 Br 79 916				36 Kr 83 80
5	6	37 Rb 85 48	38 Sr 87 63	39 Y 88 92	40 Zr 91 22	41 Nb 92 91	42 Mo 95 95	43 Tc [99]	44 Ru 101 1	45 Rh 102 91	46 Pd 106 4	
	7	47 Ag 107 880	48 Cd 112 41	49 In 114 82	50 Sn 118 70	51 Sb 121 76	52 Te 127 61	53 I 126 91				54 Xe 131 30
6	8	55 Cs 132 91	56 Ba 137 36	57–71 Lanthanide series*	72 Hf 178 50	73 Ta 180 95	74 W 183 86	75 Re 186 22	76 Os 190 2	77 Ir 192 2	78 Pt 195 09	
	9	79 Au 197 0	80 Hg 200 61	81 Tl 204 39	82 Pb 207 21	83 Bi 209 00	84 Po 210	85 At [210]				86 Rn 222
7	10	87 Fr [223]	88 Ra 226 05	89– Actinide series**								

*Lanthanide series	57 La 138 92	58 Ce 140 13	59 Pr 140 92	60 Nd 144 27	61 Pm [147]	62 Sm 150 35	63 Eu 152 0	64 Gd 157 26	65 Tb 158 93	66 Dy 162 51	67 Ho 164 94	68 Er 167 27	69 Tm 168 94	70 Yb 173 04	71 Lu 174 99
**Actinide series	89 Ac 227	90 Th 232 05	91 Pa 231	92 U 238 07	93 Np [237]	94 Pu [242]	95 Am [243]	96 Cm [245]	97 Bk [249]	98 Cf [249]	99 E [253]	100 Fm [255]	101 Md [256]	102 No	103

FUNDAMENTAL CONSTANTS

Largely based on values in the *American Institute of Physics Handbook* (1957). The probable error for each value has been omitted here; it should properly be considered part of the datum.

Name of Quantity	Symbol	Value
Velocity of light in vacuum	c	2.9979×10^8 m/sec
Charge of electron	q_e	-1.602×10^{-19} coul = -4.803×10^{-10} statcoul
Rest mass of electron	m_e	9.108×10^{-31} kgm
Ratio of charge to mass of electron	q_e/m_e	1.759×10^{11} coul/kgm = 5.273×10^7 statcoul/gm
Planck's constant	h	6.625×10^{-34} j·sec
Boltzmann's constant	k	1.380×10^{-23} j/°K
Avogadro's number (chemical scale)	N_0	6.023×10^{23} molecules/mole
Universal gas constant (chemical scale)	R	8.314 j/mole·°K
Mechanical equivalent of heat	J	4.185×10^3 j/kcal
Standard atmospheric pressure	1 atm	1.013×10^5 new/m^2
Volume of ideal gas at 0°C and 1 atm (chemical scale)		22.415 liter/mole
Absolute zero of temperature	0°K	−273.16°C
Acceleration due to gravity (sea level, at equator)		9.78049 m/sec^2
Universal gravitational constant	G	6.673×10^{-11} new·m^2/kgm^2
Mass of earth	m_E	5.975×10^{24} kgm
Mean radius of earth		6.371×10^6 m = 3959 mi
Equatorial radius of earth		6.378×10^6 m = 3963 mi
Mean distance from earth to sun	1 AU	1.49×10^{11} m = 9.29×10^7 mi
Eccentricity of earth's orbit		0.0167
Mean distance from earth to moon		3.84×10^8 m ≐ 60 earth radii
Diameter of sun		1.39×10^9 m = 8.64×10^5 mi
Mass of sun	m_S	1.99×10^{30} kgm = 333,000 × mass of earth
Coulomb's law constant	C	8.98×10^9 new·m^2/coul^2
Faraday's constant (1 faraday)	F	96,500 coul/mole
Mass of neutral hydrogen atom	m_{H^1}	1.008142 amu
Mass of proton	m_p	1.007593 amu
Mass of neutron	m_n	1.008982 amu

FUNDAMENTAL CONSTANTS—*Continued*

Name of Quantity	Symbol	Value
Mass of electron	m_e	5.488×10^{-4} amu
Ratio of mass of proton to mass of electron	m_p/m_e	1836.12
Rydberg constant for nucleus of infinite mass	R_∞	109,737 cm^{-1}
Rydberg constant for hydrogen	R_H	109,678 cm^{-1}
Wien displacement law constant		0.2898 cm·°K

Numerical constants: $\pi = 3.142$; $e = 2.718$; $\sqrt{2} = 1.414$; $\sqrt{3} = 1.732$

GREEK ALPHABET

A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ϵ	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

CONVERSION FACTORS

LENGTH:

1 m = 100 cm = 1000 mm
1 km = 1000 m = 0.6214 mile
1 m = 39.37 in.; 1 cm = 0.3937 in.
1 ft = 30.48 cm; 1 in. = 2.540 cm
1 mi = 5,280 ft = 1.609 km
1 A = 10^{-8} cm; 1μ (micron) = 10^{-4} cm

AREA:

1 cm^2 = 0.155 $in.^2$; 1 m^2 = 10^4 cm^2 = 10.76 ft^2
1 $in.^2$ = 6.452 cm^2; 1 ft^2 = 144 $in.^2$ = 0.0929 m^2

VOLUME:

1 liter = 1000 cm^2 = 10^{-3} m^3 = 0.0351 ft^3 = 61 $in.^3$
1 ft^3 = 0.0283 m^3 = 28.32 liters; 1 $in.^3$ = 16.39 cm^3 = 7.5 gal

VELOCITY:

1 cm/sec = 0.03281 ft/sec; 1 ft/sec = 30.48 cm/sec
1 mile/min = 60 mi/hr = 88 ft/sec

ACCELERATION:

1 cm/sec^2 = 0.03281 ft/sec^2 = 0.01 m/sec^2
30.48 cm/sec^2 = 1 ft/sec^2 = 0.3048 m/sec^2
100 cm/sec^2 = 3.281 ft/sec^2 = 1 m/sec^2

FORCE:

1 dyne = 2.247 $\times$ 10^{-6} lb = 10^{-5} newton
1.383 $\times$ 10^4 dynes = 0.0311 lb = 0.1383 newton
4.45 $\times$ 10^5 dynes = 1 lb = 4.45 newtons
10^5 dynes = 0.2247 lb = 1 newton

MASS:

1 gm = 6.85 $\times$ 10^{-5} slug = 10^{-3} kgm
453.6 gm = 0.0311 slug = 0.4536 kgm
1.459 $\times$ 10^4 gm = 1 slug = 14.59 kgm
10^3 gm = 0.0685 slug = 1 kgm

PRESSURE:

1 atm = 14.7 $lb/in.^2$ = 1.013 $\times$ 10^6 $dynes/cm^2$

ENERGY:

1 joule = 10^7 ergs = 0.239 cal; 1 cal = 4.18 joule
1 ev = 10^{-6} Mev = 1.60 $\times$ 10^{-12} erg = 1.07 $\times$ 10^{-9} amu
1 amu = 1.66 $\times$ 10^{-24} gm = 1.49 $\times$ 10^{-3} erg = 931 Mev

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