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A FIRST COURSE IN PROJECTIVE GEOMETRY

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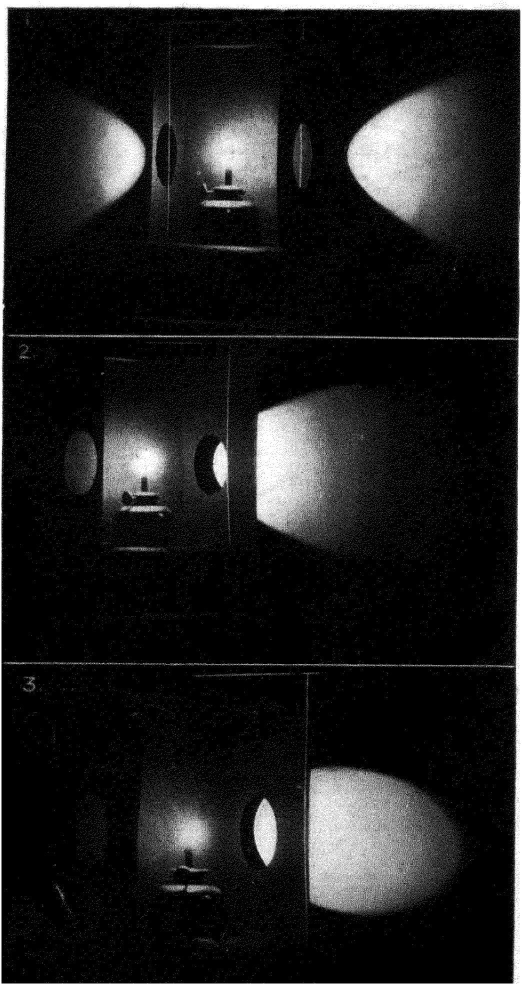

THE CONIC SECTIONS.

1. Hyperbola. 2. Parabola. 3. Ellipse.

[In each of the above photographs the horizontal and vertical threads define a plane through the source of light parallel to the screen on which the light is received. The latter thread, being the common section of this plane with the plane of the circle, represents the position of the vanishing line.

A FIRST COURSE
IN
PROJECTIVE GEOMETRY

BY

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PREFACE

THIS work is intended for the use of students who have read the substance of Euclid, Books I.–XI., and who desire some introduction to the properties of the conic before proceeding to the study of the more advanced works on modern pure geometry.

The subject of Geometrical Conics, through the medium of which such an introduction is usually acquired, often proves repulsive to the average student.

In my opinion this is due to two causes: first, the demands which it makes upon the memory owing to its lack of coherence as commonly treated; and, secondly, to the very slight extension of outlook as regards method which it affords.

In the presentation here adopted, which, I venture to think, is in some respects original, I have endeavoured to overcome as far as possible these defects, while at the same time giving a slight sketch of the method of projection, and of the great principles of homography and duality upon which the further development of pure geometry so greatly depends.

No systematic treatment of imaginary elements or of the theory of involution is attempted, as these subjects are, in my judgment, unsuitable for a first course.

On the other hand, considerable use has been made of concrete illustrations in the form of examples involving practical drawing and calculation, as my teaching experience has convinced me of their value in bringing home to the beginner the extent and variety of application of the principles involved.

With a view to further stimulating interest, a short historical note has been appended to most of the chapters.

In conclusion, I desire to express my hearty thanks to the Senate of the University of London for permission to make use of examples extracted from the degree examination papers of that University; to my friend, Mr. W. H. Salmon, Lecturer in Mathematics at the Northampton Technical Institute, Clerkenwell, for his great kindness in reading the work both in manuscript and proof, and for his helpful criticisms and suggestions; and to Mr. H. H. Smart for his valuable assistance with the photographs reproduced in the frontispiece.

E. HOWARD SMART.

ORPINGTON,
May 8th, 1913.

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CHAPTER I.

INTRODUCTORY.

§1. The Geometric Elements.

Points, straight lines, and planes are called the geometric elements.

§ **2.** The following statements concerning their mutual relations are assumed as fundamental:

(1) Through two points can be drawn one and only one straight line.

(2) Two coplanar straight lines intersect in one point only, or are parallel.

Two non-coplanar straight lines do not intersect.

(3) Three non-collinear points,
or a point and a straight line not passing through it,
or two intersecting straight lines, determine a plane.

(4) A straight line either intersects a plane in one point only or is parallel to it.

(5) Two planes either intersect one another in one straight line only, or are parallel.

(6) Through a given point only one straight line can be drawn parallel to a given straight line.

By the term 'parallel' in the above we mean that the elements concerned do not meet, however far produced.

In the case of two straight lines the term is applied only to such as, lying in one plane, do not meet in any finite point.

§ 3. Elements at Infinity.

An important assumption will now be made by which we can avoid the necessity of the special treatment of cases of parallelism, and aid that process of generalisation which mathematics seeks to develop and extend.

This assumption is that

'*On every straight line of unlimited length there is one, and only one, point at infinity.*'

Consider two straight lines x and p unlimited in length, of which x is fixed, while p turns about a point P on it, in the plane containing P and x. (Fig. 1.)

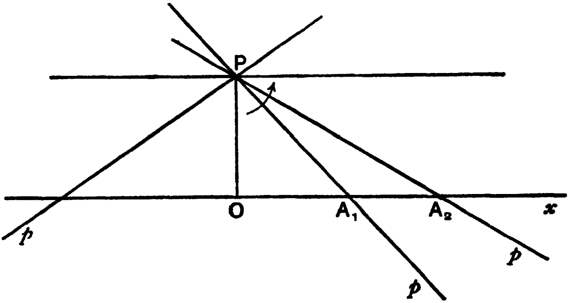


FIG. 1.

If the turning is continued in the same sense, *e.g.* counter-clockwise, the point of intersection of x and p travels along x always in the same direction (in the figure from left to right) till it disappears from view, and, if the turning is continued, reappears, as we should say, at the other end of the line x.

If we travel along x always in the same direction as this point of intersection does, we pass through all the finite points of x and then the infinitely distant point of x which, so to speak, makes the line continuous,* so that after traversing it going still in the same direction, we arrive at the point from which we started.

The above assumption implies that there is one and only one position of the turning line for which its intersection with x is other than finite,† and that in this position the point of intersection is infinitely distant or 'at infinity' on each of the lines.

By § 2 this must therefore be the position in which the two lines are parallel.

Hence we may in future **define two parallel straight lines as straight lines which intersect in one point at infinity**; or otherwise, as **straight lines such that the point at infinity on the one coincides with the point at infinity on the other.**

The proviso 'in one plane' which appears in the last sentence of § 2 may now be omitted. For two straight lines which intersect are necessarily in a plane.

To sum up the results of this article,

(1) On every straight line there is one **and only one** point at infinity (**not** one at each end).

(2) We may consider two parallel straight lines as intersecting 'at infinity'; or such that the point at infinity on the one line coincides with the point at infinity on the other.

(3) We can now say generally that two coplanar straight lines intersect always in one point only (either finite or infinitely distant).

---

* T. F. Holgate, *Monographs of Modern Mathematics.* Ed. J. W. A. Young.

† The assumptions (1) that more than one such line, (2) that no such line exists, form the starting-points of the two great branches of what is called Non-Euclidean Geometry. *Vide* also Henrici, *Congruent Figures*, §§ 111-114.

§ 4. To complete our treatment of parallelism, we **shall** further suppose

(1) That every plane contains one '**line at infinity**' upon which are situated the points at infinity on all lines lying in that plane.

(2) That there is one '**plane at infinity**' which contains all elements (lines and points) at infinity.

Two planes are then said to be parallel when the **line at infinity of the one coincides with the line at infinity of the other.**

A straight line is parallel to a plane when **the point at infinity on the line lies on the line at infinity on the plane.**

We can now say **generally** that

(1) A straight line and a plane have one point and only one in common.

(2) Two planes have one straight line and only one in common.

§ 5. As an illustration of the use of these elements at infinity, consider the following

THEOREM. *If a, b, c be straight lines not all in one plane, of which a and b are each parallel to c, then a is parallel to b.*

For since a is parallel to c, the point at infinity on a coincides with the point at infinity on c. Similarly, the point at infinity on b coincides with the point at infinity on c.

$\therefore$ the point at infinity on a coincides with the point at infinity on b, and $\therefore$ a is parallel to b.

Using the elements at infinity,

Ex. 1. Prove a corresponding theorem for planes.

Ex. 2. Two straight lines a and b are each parallel to a plane. Shew that a is not necessarily parallel to b.

Ex. 3. Two parallel planes are cut by a third plane. Prove that their common sections are parallel.

Ex. 4. Prove that one and only one plane can be drawn through a given point parallel to each of two non-intersecting straight lines.

Ex. 5. Prove that if two intersecting straight lines are parallel respectively to two other intersecting straight lines, the plane of the first pair must be either coincident with, or parallel to, the plane of the second pair.

Ex. 6. If two planes be drawn one through each of two parallel straight lines and perpendicular to a third plane, to which the parallels are not perpendicular, they are parallel.

Ex. 7. If two planes be drawn one through each of two parallel straight lines, their common section is also parallel to the lines.

§ 6. To be given a **direction** in space means that we are given one point at infinity. For by the theorem of § 5 all lines parallel to this direction concur in a single point at infinity.

The importance of this result will be seen later.

§ 7. Notation.

It is of great importance in Modern Geometry to adhere to a definite notation. The following is that usually adopted:

Points are denoted by Roman capitals A, B, C, etc.

Straight lines by small letters a, b, c.

Planes by small Greek letters α, β, γ.

Two letters following one another denote the element determined by those which the letters separately represent.

E.g. AB denotes the straight line determined by the points A, B.

ab denotes the point determined by the straight lines a, b.

$\alpha\beta$ denotes the line of intersection of the planes α, β.

$a\alpha$ denotes the point of intersection of the line a and the plane α; and so on.

§ 8. The Prime Geometric Forms.

A group of like geometric elements constitutes a **Geometric Form.**

These may be classified in three grades according as the number of elements present is singly, doubly or trebly infinite.

In the First Grade.

(1) A set of points lying on a line called a **Range.**

(2) A set of concurrent lines lying in a plane called a **Flat Pencil**, or shortly a **Pencil.**

(3) A set of planes passing through a line called an **Axial Pencil.**

Forms of the Second Grade.

(4) A set of points lying in a plane but not collinear.

(5) A set of lines lying in a plane but not concurrent.

(6) A set of concurrent lines not in one plane.

(7) A set of planes passing through one point.

Forms of the Third Grade.

(8) A set of points, lines and planes in space.

Forms (1) to (5) are those with which we shall be mainly concerned in this book.

The element which is 'incident' to those which constitute a form, *i.e.* that through which they pass, in the case of lines or planes, or on which they lie, in the case of points, is called the **base** of the form; for example, the line on which the points lie, in the case of a Range, or the point of concurrence of the lines (or 'rays' as they are sometimes called), in the case of a Pencil.

Historical Note. The study of Geometry is of great antiquity. Tradition attributes its origin to a utilitarian motive, viz. the periodic resettlement of the boundaries of property on the banks of the river Nile rendered necessary by the annual floods.

Be that as it may, it is certain that at an early date the Greeks began to develop Geometry as an abstract science—the only one which, in their view, was capable of yielding results whose truth was exact. In this way a number of the fundamental propositions of metrical geometry were established, the properties of many figures being supposed to possess a philosophical significance, the details of which cannot be entered upon here. Euclid's *Elements*, the great work of the first professor of Mathematics at the University of Alexandria, was published about 300 B.C., and, though by no means merely a compilation, collected into an organic whole the work of many early investigators, notably Thales* of Miletus (circa 600 B.C.), Pythagoras† (B.C. 580), Hippocrates‡ of Chios (B.C. 450) and Eudoxus§ (B C. 365).

Archimedes of Syracuse (B.C. 287–212) applied this body of geometrical knowledge with striking success to the mensuration of the circle, the parabola and the sphere. Reference will subsequently be made to the work of some of the later geometers of this epoch, which may be said to have come to a close about 350 A.D., when, with the advent of the Dark Ages, ensued a remarkable period of stagnation for all forms of scientific thought and achievement, which lasted for a thousand years

The Greeks seem to have had no idea of the 'principle of continuity,' which asserts that the properties of a geometrical figure are unaffected by continuous change in its parts, provided it is always constructed in the same manner. This conception, which led to the introduction into Geometry of the ideal elements at infinity, whereby the subject gains much in unity of treatment, is due to the astronomer Kepler

---

* Who is said to have discovered Euc. I. 5 and the fact that the angle in a semicircle is a right angle.

† Who discovered the famous proposition Euc. I. 47, commonly known by his name; also the existence of the five regular solids.

‡ Whose name is associated with the famous problem of the 'duplication of the cube.'

§ The investigator of the theory of proportion. The greater part of Euc. Bk. V. is said to be his work.

(1571–1630). Kepler's ideas were developed and extended by Girard Desargues of Lyons (1593–1662), a French engineer who fought under Cardinal Richelieu at the siege of Rochelle. Desargues founded a new school of geometers, and his great work, the *Brouillon Proiect*, may be fairly said to have inaugurated the modern study of Projective Geometry.

Examples on Chapter I.

1. Every straight line not lying in a certain plane, but parallel to a straight line which lies in that plane, is parallel to the plane.

2. If a straight line is parallel to each of two intersecting planes, it is parallel to their common section.

3. An infinite number of straight lines can be drawn to intersect each of three given straight lines in space.

4. Shew how to draw a straight line to intersect two other straight lines in space and to be parallel to a third.

5. Through any two given straight lines in space draw a pair of parallel planes.

6. If three concurrent straight lines are equally inclined to a plane, the intercepts on them between the plane and the point of concurrence are equal.

7. If two pairs of opposite edges of a tetrahedron are perpendicular to one another, the third pair are also perpendicular to one another.

8. Shew how to cut a cube so that the section is a regular hexagon.

9. Through a given point draw a straight line making a given angle with a given plane not passing through the point, and intersecting a given straight line in that plane.

10. The shortest distance between a diagonal and the non-intersecting edge a of a right parallelepiped whose edges are a, b, c is $bc/\sqrt{(b^2+c^2)}$.

11. The shortest distance between two non-parallel non-intersecting edges of a regular octahedron of edge a is $a\sqrt{\frac{2}{3}}$.

12. Shew how to draw through a given point within a given trihedral angle a plane which cuts the faces of the angle in a triangle having the point as centroid. Also if the plane angles which form the trihedral angle be all right angles, shew how to draw the plane so that the point is the orthocentre of the triangle.

CHAPTER II.

PROJECTION. DUALITY.

§ 1. The Fundamental Method of Projection.

Take any geometric figure X in a plane α. (Fig. 2.)

Take any point V not in this plane.

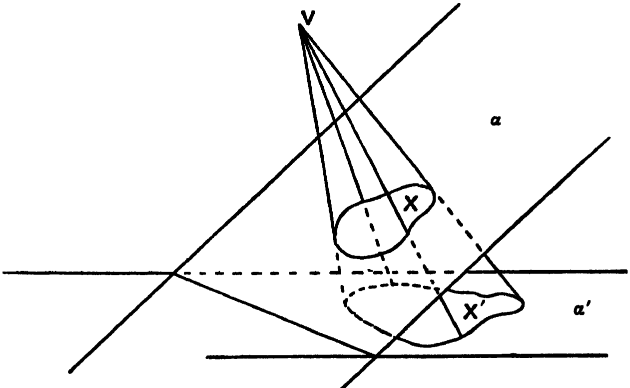


Fig. 2.

Let straight lines or rays be drawn from V to every point f the figure X.

Let another plane α' cut this system of rays.

Then as each ray can cut a plane in one point only, to each oint of the figure X in plane α corresponds a point in plane α'.

The assemblage of these latter points constitutes a figure X′.

Either of the figures X, X′ is said to be the projection of the other.

§ 2. The point V is called the **Vertex of Projection.**

The rays through V are called the **Projectors.**

The common section of the planes α, α' is called the **Axis of Projection.**

Taking the figure to be projected as X, the plane α', on which the projection X′ is obtained, is called the **Plane of Projection.**

As an illustration, the shadow cast by the figure X on the plane α', taking a point source of light at V, is the projection of X from vertex V on α'.

§ 3. Particular Cases of Projection.

If V is infinitely distant, the rays or projectors are all parallel, and we have what is called **Parallel Projection.**

A further particular case occurs when the parallel rays are all perpendicular to the plane of projection α'. This is called **Orthogonal Projection.**

But unless otherwise specified, we shall by 'projection' understand projection from a finite point as in § 1. This is called **Central or Conical Projection.**

The object of Projective Geometry is to obtain from a given figure by this fundamental method properties of the projected figure. Properties of a figure which hold good also for the projected figure are called **Projective Properties.**

§ 4. We commence by noting some obvious properties of the projection of the point, straight line and the simple coplanar geometric forms.

We have at once:

The projection of a point is a point.

The projection of a straight line is a straight line.

In the latter case V and the straight line to be projected determine a plane β, which contains all the projectors. This plane cuts α' in a straight line, which is the projection required.

It must be further noted that the straight line and its projection cut the axis of projection at the same point, viz. where β cuts it.

It is important to bear this fact in mind when we come to the drawing of projections.

It is also evident that

the distance between two points and
the angle between two straight lines

in general are altered by projection.

THEOREM. *If a triangle be projected on to another plane, the points of intersection of each side with its projection are collinear.*

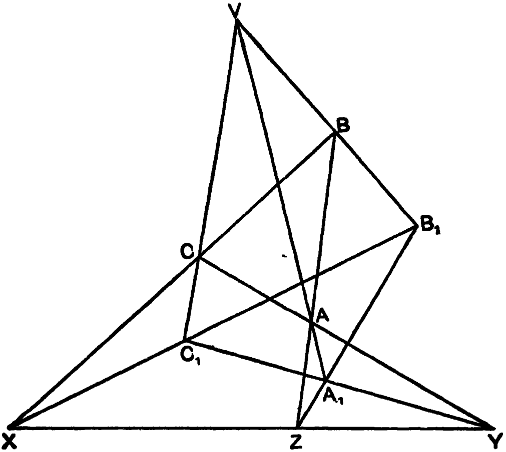


FIG. 2*b*.

Let the triangle ABC be projected on to another plane and let its projection be $A_1B_1C_1$.

Let V be the vertex of projection. (Fig. 2*b*.)

Then BC, B_1C_1 intersect one another, for they lie in a plane through V. But the only common part of the planes ABC, $A_1B_1C_1$ (in which these lines lie) is the common section of these two planes.

$\therefore$ BC, B_1C_1 meet at a point X on this common section. Similarly CA, C_1A_1 meet at Y on it and AB, A_1B_1 at Z. X, Y, Z are therefore collinear.

Also, conversely, if the intersections of corresponding sides of two triangles in different planes are collinear, the joins of corresponding angular points are concurrent.

For as BC, B_1C_1 intersect, they lie in a plane.

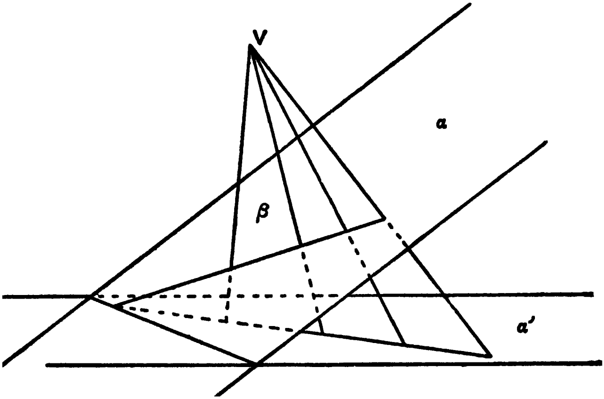
Likewise CA, C_1A_1 lie in a plane, as do also AB, A_1B_1. These three planes have one point V in common, and through this point must pass their common sections, taken in pairs.

But these common sections are AA_1, BB_1, CC_1.

$\therefore$ AA_1, BB_1, CC_1 are concurrent in V.

§ 5. The Projections of the Fundamental Geometric Forms.

A range of points on a line projects into a range of points on the projection of the line.



A pencil of straight lines through a point projects into a pencil of straight lines through the projection of the point.

The first is self-evident (Fig. 3).

For the second we notice that the planes containing V and the several rays of the pencil each pass through the join of V to the vertex of the pencil (Fig. 4).

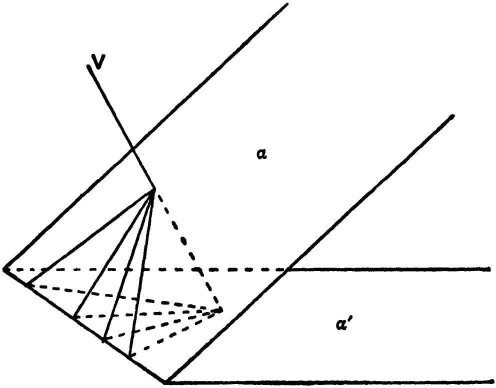


FIG. 4.

Hence they form an axial pencil, the section of which by the plane a' is a flat pencil.

In like manner, forms (4) and (5) of § 8 of the preceding chapter project into similar forms.

To sum up, we can state generally that:

To each point in a given figure corresponds one and only one point in the projected figure.

To each straight line in a given figure corresponds one and only one straight line in the projected figure.

§ 6. Let the projection of a figure X be X′.

Taking another vertex and plane of projection, let the projection of X′ be X″; project X″ in like manner, and let the result be X‴, and so on.

The figures X, X′, X″, X‴, etc., are said to be **projective** with one another.

Two figures are thus said to be projective when one can be obtained from the other by a finite number of projections.

§ 7. Any three Collinear Points are Projective with any other three Collinear Points, the two Lines of Collinearity not necessarily lying in the same Plane.

Let A, B, C, A′, B′, C′ (Fig. 5) be the two sets of points.

Join AA′, and take V any point on AA′.

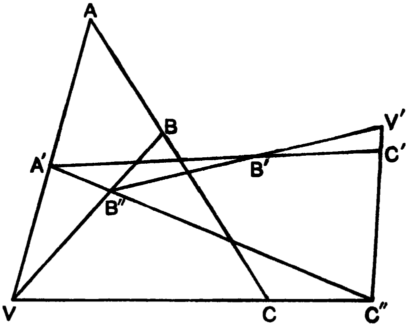


Fig. 5.

Let the plane through V and the line ABC be cut by any plane through A′B′C′ in the line A′B″C″, and let B″, C″ be the points where VB, VC cut the common section of the planes.

Let B″B′, C″C′, which are in the latter plane, cut one another in V′.

Then A, B, C project from V respectively into A′B″C″, and those from V′ into A′, B′, C′ respectively.

We shall return to the subject of projection in Chapter VII. It is necessary first, however, to discuss the properties of the range and pencil in detail.

§ 8. Geometrical Properties—Metrical and Descriptive.

Geometrical properties are of two kinds, (1) those which involve the notions of quantity and measurement, and (2) those in which the relation between the elements is one of position only.

The former, into which enter such considerations as the lengths of lines, the magnitudes of angles, etc., are called **Metrical Properties.**

As examples of such we might take the theorems:

'The angles at the base of an isosceles triangle are equal,' and

'The sum of the squares on the sides of a right-angled triangle is equal to the square on the hypotenuse.'

On the other hand, properties exist into which the idea of quantity does not enter at all, but merely that of relative position.

For example, 'Three intersecting straight lines lie in a plane.'

Such properties are called **Descriptive.**

Euclidean geometry is mainly metrical.

It has been pointed out that magnitudes of lines and angles in general alter by projection. Thus lengths and angles which are equal in a given figure are not necessarily so in the projected figure.

Accordingly metrical properties, **as a rule**, are not projective, while descriptive properties are.

Certain metrical expressions and relations, however, as will be shown in the succeeding chapters, remain unaltered by projection, and the introduction of these vastly extends the scope of projective geometry.

§ 9. The Ideas of Correspondence and Duality.

We have seen that a figure and its projection correspond point for point and line for line: that is to say, for each point on **either** figure corresponds one point (and only one) on the other figure, and the same holds good for lines.

This idea of correspondence enters largely into modern geometry, and as it affects the notation we use, it is necessary to recognise its existence from the outset.

To avoid confusion, it is very desirable to denote corresponding elements of the same kind by the same letter, distinguishing the one from the other by the use of dashes or suffixes:—for example A, A′ for corresponding points, b, b' for lines, and so on.

§ **10.** But a point-to-point and line-to-line relation such as the above is not the only kind of correspondence met with in modern geometry.

Let us confine our attention for the present to geometrical figures in one plane, excluding considerations of magnitude, so that we are concerned with 'descriptive' geometry only.

It is usual at the commencement of the subject to consider geometrical figures as traced out by a point. Our drawing, for instance, is done with a point.

This may thus be taken as the **Primary Element.**

The straight line which is determined by two points is called the **Secondary Element.**

Ordinary rectilineal figures are built up from these elements. A curve is thought of as the **locus** of a point which moves in some known manner.

But suppose that the only available drawing implement at our disposal were a straight edge.*

The straight line would then be the primary element, the point being determined by two intersecting straight lines, and becoming thus the secondary element.

---

* Miss C. A. Scott, *Modern Ideas and Methods in Plane Coordinate Geometry.*

We could still construct rectilineal figures as before. A curve would then be conceived as the **envelope** of (or curve touched by) a moving line (Fig. 6).

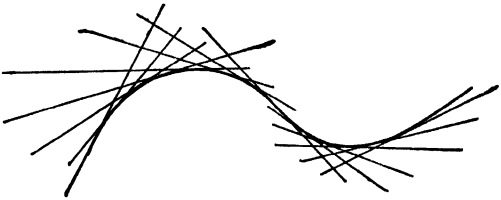


FIG. 6.

We have therefore a **dual** way in which geometrical figures may be obtained according as the generating element is a point or a straight line.

Let us consider a few of the simplest examples of such figures. We arrange them side by side, taking for the left-hand side the point as the primary element, and for the right-hand side the straight line.

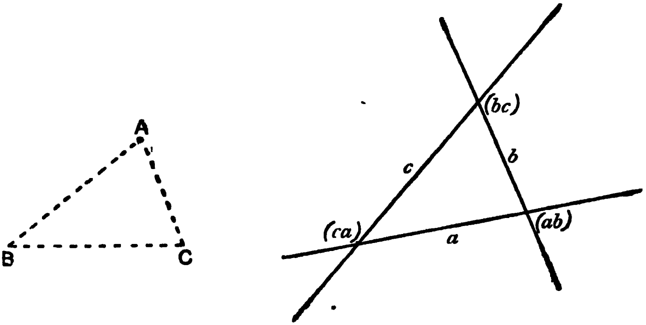


FIG. 7a.

Between such a pair of corresponding figures we have then a point-to-line and line-to-point correspondence.

We have thus as pairs of corresponding figures:

A range of points lying on a straight line.	A pencil of straight lines passing through a point.
The figure obtained by taking 3 non-collinear points and joining them in all possible ways.	The figure obtained by taking 3 non-concurrent straight lines and making them intersect in all possible ways.
(A triangle, but from this aspect we might call it a '3-corner.') Fig. 7*a*.	(A triangle, but from this aspect we might call it a '3-side.') Fig. 7*a*.

Another pair of figures possessing important properties shortly to be investigated may be mentioned. (Fig. 7*b*.)

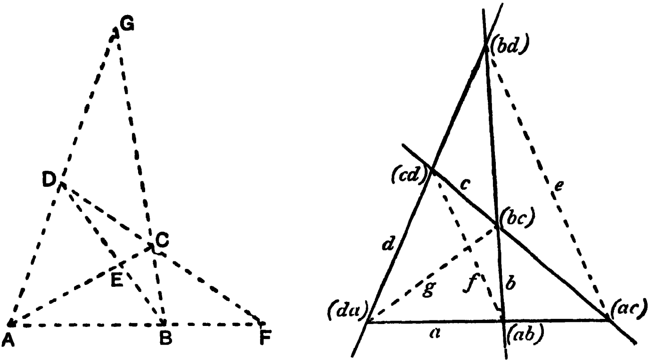


Fig. 7*b*.

The complete quadrangle.	**The complete quadrilateral.**
Take four points A, B, C, D, no 3 of which are collinear.	Take four lines *a*, *b*, *c*, *d*, no 3 of which are concurrent.
Joining these in pairs we have **six** lines called the **Sides**; the joins of any two and of the remaining two are called **Opposite Sides** of the quadrangle; thus AB, CD ; BC, AD ; CA, BD, are opposite sides.	Taking their intersections in pairs we have **six** points called the **Vertices**; the intersections of any two and of the remaining two are called **Opposite Vertices** of the quadrilateral; thus *ab*, *cd* ; *bc*, *ad* ; *ca*, *bd*, are opposite vertices.
The points of intersection E, F, G of the pairs of opposite sides are called the **Diagonal Points.**	The joins *e*, *f*, *g*, of the pairs of opposite vertices are called the **Diagonal Lines.**

Figures of the above kind are called **Correlative.**

It is most important to bear in mind that to one point of the left-hand figure corresponds one and only one straight line of the right-hand figure, and *vice versa.*

This mode of presentation in parallel columns will in general be adopted wherever the principle of duality applies. **Proofs of Theorems**, however, though ranged in dual form, are quite distinct, and the beginner is strongly advised to study them separately.

Later on it will be seen how they may be connected together.

As regards Notation, when point and line correspond the former is denoted by a capital letter and the latter by the corresponding small letter, as in the example just given of the quadrangle and quadrilateral.

§ 11. In the last article we considered the Principle of Duality in its application to plane geometry.

In the geometry of space a duality exists between points and planes.

For example we have the theorems:

(1) Two points determine a straight line.	Two planes determine a straight line.
(2) A point and a straight line not passing through it determine a plane.	A plane and a straight line not lying on it determine a point.
(3) Three points determine a plane provided they are not all on the same straight line.	Three planes determine a point provided they do not all pass through the same straight line.

In this case the range and the axial pencil are correlative forms.

We shall not pursue the subject further than to point out that what we called at the beginning of this chapter the fundamental method of projection really consists of two operations which, like the above theorems, are complementary to one another.

These are (1) the operation of projecting,
(2) the operation of cutting.

Limiting our discussion to conical projection from a vertex V of some figure made up of points and lines, by operation (1) we mean the constructing of the projectors through V and the points of the given figure, or of the planes through V and the lines of the given figure; by operation (2) we mean the section of these projectors or of these planes by another plane.

Historical Note. As was pointed out in Chapter I., to Desargues belongs the honour of applying to geometry the method of projection. His work, however, remained unnoticed for more than a century, being overshadowed by the analytical methods introduced by Descartes (1596–1650), which led on to the discovery of the calculus by Newton (1642–1727) and Leibnitz (1646–1716). It was revised and developed in a remarkable manner by J. V. Poncelet (1788–1867), a general in the French army, who took part in Napoleon's disastrous campaign to Moscow, and, on being captured by the Russians and immured at Saratov, employed his leisure time in writing his masterly treatise, the *Propriétés projectives des figures.*

The principle of duality in a limited form was foreshadowed by Brianchon in 1806, when he discovered the theorem known by his name, which is proved in Chapter XV. In this form it was worked out in considerable detail by Poncelet in the above-mentioned treatise, but the general statement of the principle is due to Gergonne (1826). It has since received considerable extensions at the hands of a brilliant trio of modern geometers, Steiner (1796–1863), Von Staudt (1798–1867) and Chasles (1793–1880).

Examples on Chapter II.

1. Write down the dual (1) *in plano,* (2) in space of a triangle and three concurrent lines through its vertices.

2. Write down the dual in space of a triangle and three collinear points one on each of its sides.

3. Apply the methods of projection and section to prove that the theorem at the end of § 4 holds good when the two triangles are in one plane.

4. Assuming this to be true, write down the reciprocal or dual theorem *in plano.*

5. Write down the dual (*in plano*) of the following theorem, which will be proved later: 'A, B, C; A′, B′, C′ are triads of points on the straight lines p, p'. The points of intersection of BC′, B′C; CA′, C′A; and AB′, A′B are collinear.'

6. Write down the dual *in plano* of the following theorem: Three angles have their apices on a straight line. The six lines obtained by joining the other intersections of the bounding lines of each pair of angles form the sides of a complete quadrangle.

7. What is the reciprocal *in plano* of a hexagon ABCDEF whose diagonals AD, BE, CF are concurrent?

8. Interchanging point and plane after the manner of § 11, obtain the dual in space of the theorem of § 4.

9. Shew that a point-and-plane duality exists between the five regular solids as follows: cube and octahedron; dodecahedron and icosahedron; while the tetrahedron is its own dual.

10. Given a polyhedron having x corners, y edges, and z faces, write down the number of corners, edges, and faces in the dual solid.

11. Euler's theorem states that if S be the number of corners, F the number of faces, E the number of edges in a polyhedron, then $S+F=E+2$. Shew that this theorem is self-reciprocal.

CHAPTER III.

METRICAL RELATIONS. PERSPECTIVE FIGURES.

§ **1.** A **length** in Modern Geometry possesses two properties, **magnitude** and **sign.**

That is to say, AB used quantitatively denotes not merely the **distance** between the points A and B, but that **distance measured from A to B**; in other words, a **step** or **displacement** from the first of these points to the second.

To emphasize this idea let us write for the **moment** $\overline{AB}$ to represent 'the step AB.'

Now $\overline{AB}$ and $\overline{BA}$ are different.

But, inasmuch as AB + $\overline{BA}$ = zero step = 0, we have $\overline{BA} = -\overline{AB}$.

Whatever sign then $\overline{AB}$ has, $\overline{BA}$ will have the opposite sign. Accordingly we may use the signs of operation + and - to distinguish between the **senses** of the two steps or displacements

We shall therefore, in future, treat a length as a quantity of this kind and drop the 'bar' notation.

One advantage of this is that the relation

$$AB = AC + CB$$

holds good **wherever C is taken on the line AB, produced both ways.** For a step from A to B is the sum of steps from A to C and from C to B wherever C is on this line.

We may therefore consider the segments of a length which is divided into two parts, whether the point of division lies between the extremities of the line or outside them. We may speak of a line as 'divided externally' into two segments. One of these segments, and therefore also the ratio of the two, is negative.

§ 2. The following theorem is useful:

If A, B, C, D be four points in any order on a straight line,

$$BC \, . \, AD + CA \, . \, BD + AB \, . \, CD = 0.$$

In proving a relation of this kind it is convenient to select some point O on the line as an origin and express any length XY in terms of distances of its terminal points X, Y from O.

Thus $$XY = XO + OY = OY - OX.$$

Let $$OA = a, \quad OB = b, \quad OC = c, \quad OD = d.$$

Then $$BC \, . \, AD + CA \, . \, BD + AB \, . \, CD$$
$$= (c - b)(d - a) + (a - c)(d - b) + (b - a)(d - c),$$
which is easily seen to vanish identically.

Ex. Prove that if A, B, C, D be collinear points,
$$AD^2 . BC + BD^2 . CA + CD^2 . AB = -AB \, . \, BC \, . \, CA$$

§ 3. Ceva's Theorem.

If AD, BE, CF *be any three concurrent lines passing through the vertices of a triangle and meeting the opposite sides in* D, E, F *respectively, then*

$$AF \, . \, BD \, . \, CE = + FB \, . \, DC \, . \, EA.$$

Conversely. If points D, E, F *be taken on* BC, CA, AB (*or these produced*) *respectively, such that the above relation is satisfied*, AD, BE, CF *will meet in a point.*

NOTE. AF, BD, etc., are segments in the extended meaning of § 1, and are considered positive when measured along their respective sides in the sense ABC.

Let G be the point of concurrence, which must be either inside the triangle, as in Fig. 8*a*, or outside, as in Fig. 8*b*.

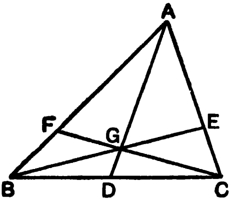


FIG. 8*a*.

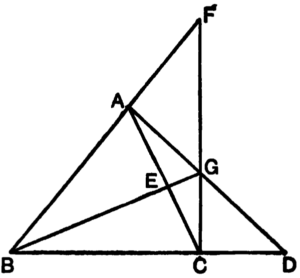


FIG. 8*b*.

Then we have

$$\frac{AF}{FB}=\frac{\triangle ACF}{\triangle BCF}=\frac{\triangle AGF}{\triangle BGF}=\frac{\triangle ACF-\triangle AGF}{\triangle BCF-\triangle BGF}=\frac{\triangle AGC}{\triangle BGC}.$$

Similarly, $\frac{BD}{DC}=\frac{\triangle AGB}{\triangle AGC}$ and $\frac{CE}{EA}=\frac{\triangle BGC}{\triangle AGB}$;

$$\therefore \frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{\triangle AGC}{\triangle BGC}\cdot\frac{\triangle AGB}{\triangle AGC}\cdot\frac{\triangle BGC}{\triangle AGB}=1.$$

The signs of the two products AF . BD . CE and FB . DC . EA are necessarily like. For the above figures represent the only possible cases, and AF and DC in Fig. 8*b*, which are the only negative lengths, appear on opposite sides of the equation.

Conversely. If AF . BD . CE = FB . DC . EA, then AD, BE, CF will meet in a point.

Let BE and CF meet in G.

Join AG, and let it meet BC in D′.

Then, since AD′, BE, CF are concurrent,

$$AF \,.\, BD' \,.\, CE = FB \,.\, D'C \,.\, EA.$$

But

$$AF \,.\, BD \,.\, CE = FB \,.\, DC \,.\, EA;$$

$$\therefore \frac{BD'}{BD}=\frac{D'C}{DC}, \text{ and each of these} = \frac{BD'+D'C}{BD+DC}=1;$$

$\therefore$ D and D′ coincide.

Ex. Apply Ceva's theorem to prove that

(1) The medians of a triangle,

(2) The lines joining the vertices of a triangle to the points of contact of the inscribed circle with the sides,

(3) The joins of the vertices to the points of contact of any one of the escribed circles with the sides,

(4) The perpendiculars from the vertices on the opposite sides, are concurrent.

[In No. 4 note that if AD, BE, CF be the perpendiculars, the triangle AFE is similar to the triangle ABC. Hence AF/EA = b/c, and similarly for FB/BD and DC/CE. The Ceva condition readily follows.]

Mnemonic. This result and that of § 4 are perhaps most easily remembered by writing down the ratios in which the sides are divided by the points D, E, F, starting at a vertex and going round the triangle in the sense ABC. We get thus AF/FB, BD/DC, CE/EA. The product equated to + 1 gives the result of § 3; equated to − 1 the result of § 4.

§ 4. Menelaus' Theorem.

If any transversal cuts the sides BC, CA, AB *of a triangle* ABC *in* D, E, F *respectively,*

$$AF . BD . CE = - FB . DC . EA.$$

Conversely. If this relation connects the distances of points D, E, F *on the sides* BC, CA, AB *respectively from the vertices, these three points are collinear.*

NOTE. The same convention is adopted as in the last proposition.

Any transversal must either cut two sides internally and one externally, or all three externally. For a triangle being a closed figure, if a point travelling along the transversal enters the triangle, it must leave it again. Two cases must therefore be shown in the figure, as in the last proposition.

Also either one or all three of the segments FB, DC, EA will be negative. Hence the minus sign in the equality.

Let p, p', p'' be perpendiculars from A, B, C respectively on this transversal.

Then using the convention that perpendiculars on the same or opposite sides of a line have like or unlike signs, we have in both cases

$$\frac{AF}{FB} = -\frac{p}{p'}, \quad \frac{BD}{DC} = -\frac{p'}{p''}, \quad \frac{CE}{EA} = -\frac{p''}{p}.$$

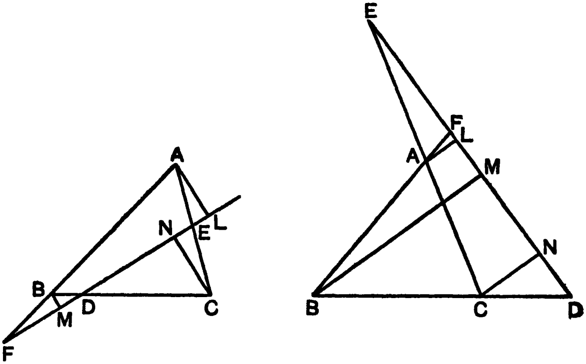


Fig. 9.

It follows then that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1 \text{ in each case}$$

or $$AF \,.\, BD \,.\, CE = -FB \,.\, DC \,.\, EA.$$

Conversely. If D, E, F be taken on the sides so as to fulfil this condition, these points are collinear.

For let EF meet BC in D′.

Then $AF \,.\, BD' \,.\, CE = -FB \,.\, D'C \,.\, EA$ by the preceding.

But $AF \,.\, BD \,.\, CE = -FB \,.\, DC \,.\, EA$ by hypothesis.

$$\therefore \frac{BD'}{BD} = \frac{D'C}{DC} = \frac{BD' + D'C}{BD + DC} = 1 \text{ in each case.}$$

$\therefore$ D and D′ coincide.

Ex. 1. Prove that the sides of the triangle formed by joining the points of contact of the inscribed circle of a triangle with the sides, intersect the sides in three collinear points.

Ex. 2. Prove that the sides of the pedal triangle intersect those of the given triangle in collinear points.

Ex. 3. Examine the case in which the transversal is parallel to a side of the triangle.

§ 5. A Theorem of Desargues.

If the joins of corresponding vertices of two coplanar triangles are concurrent, the intersections of corresponding sides are collinear, and conversely.

[In § 4 of the preceding chapter, this theorem was established for triangles in space. It will now be proved by the method of transversals to hold good when the triangles are coplanar.]

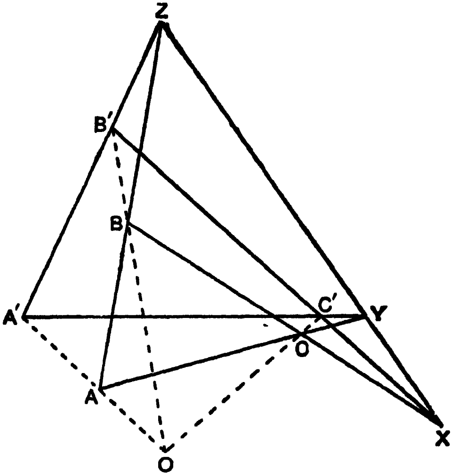


Fig. 10.

Since B′C′X is a transversal cutting the sides of the triangle BCO,

$$\frac{BX}{XC} \cdot \frac{CC'}{C'O} \cdot \frac{OB'}{B'B} = -1.$$

Similarly from the triangle CAO,

$$\frac{CY}{YA}\cdot\frac{AA'}{A'O}\cdot\frac{OC'}{C'C}=-1;$$

and from ABO,

$$\frac{AZ}{ZB}\cdot\frac{BB'}{B'O}\cdot\frac{OA'}{A'A}=-1.$$

From the product of these, it appears that

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=-1.$$

∴ by the converse of Menelaus' theorem X, Y, Z are collinear.

Conversely. If X, Y, Z are collinear, AA′ BB′, CC′ are concurrent.

For consider the triangles BB′Z, CC′Y.

The lines joining corresponding vertices are concurrent.

The intersections of corresponding sides are therefore collinear. ∴ O, A′, A are collinear;

i.e. AA′, BB′, CC′ meet in a point.

§ 6. Perspective.

Two triangles possessing the above property are said to be in **Plane Perspective.**

The point of concurrence of joins of corresponding points is called the **Centre of Perspective.**

The line of collinearity of the intersections of corresponding sides is called the **Axis of Perspective.**

In Chapter II. it was seen that a triangle and its projection in space possess the same properties. Such may be said to be in 'space perspective.' But the term 'perspective' is usually confined to figures in one plane, where, of course, we cannot speak of one figure as the 'projection' of the other.

The terms 'homology,' 'centre and axis of homology' are used by some writers to denote the same ideas. Figures in plane perspective are then called 'homological.'

We define two figures in plane perspective as such that (1) the lines joining corresponding points are concurrent, and (2) the points of intersection of corresponding lines are collinear.

In the case of triangles, § 5 shows that if *either* (1) or (2) holds, the other must of necessity follow; but this is not in general true of all figures.

It may happen, as for example in applying the definition to simple geometric forms composed of points only or lines only, that one half of the definition is irrelevant. See Chap. XIII. § 3.

Historical Note. The property of concurrent lines given in § 3 is due to the Marquis Giovanni Ceva, a philosopher of the seventeenth century.

Menelaus was a geometer and astronomer of the first century A.D. His theorem on transversals is contained in a work called the *Sphaerica*, and is therein extended to triangles on a sphere. The most important applications of it were made by Desargues.

The earliest notions of perspective have been traced by Chasles to Serenus of Antissa, who wrote two books on sections of the cylinder, and come soon after the beginning of the Christian era.

The theorem of § 5 is sometimes attributed to Desargues, but was probably known long before his time. But in any case Desargues' contributions to the theory of perspective were considerable, and his work was carried on after his death by his pupils Bosse, De la Hire and Pascal.

Examples on Chapter III.

1. Prove the theorem of § 5 by projecting one of the triangles in § 4, Chap. II. orthogonally on to the plane of the other.

2. A, B, P are three non-collinear points. In PA a point Q is taken so that PQ $= x$QA; in BQ produced a point R such that BR $= y$RQ; RP produced meets AB produced in S. Express in terms of x and y the ratios AS/AB and RS/RP.

3. Two sides AB, AC of a triangle ABC are divided in the same ratio by a variable line PQ, and PQ is divided in a fixed ratio at R. Prove that the locus of R is a straight line passing through A.

4. Two fixed lines AB, KL are divided in the same ratio by a variable line XY, and XY is divided in a fixed ratio at Z. Prove that the locus of Z is a straight line.

5. ABC is a triangle, D, E, F the middle points of its sides. Points P, Q, R are taken inside the triangle on the perpendiculars from A, B, C respectively to the opposite sides, and such that AP = BQ = CR = radius of the inscribed circle of ABC. Show that the triangles PQR, DEF are in perspective, the centre of perspective being the centre of the inscribed circle of ABC.

6. Prove that the circumcircles of the four triangles formed by four intersecting straight lines meet in a point.

7. Prove that the point in Ex. 6 is fixed if the fourth line be any transversal of the triangle formed by three of the lines, cutting them at X, Y, Z and XY/YZ is a given ratio.

8. Prove that the centres of the circles in Ex. 6 and the point in which they intersect lie on a fifth circle.

9. In a given triangle ABC place a transversal XYZ, so that XY, YZ may have given lengths.

10. Through a point P draw a transversal cutting the sides of a triangle in X, Y, Z, so that XY/YZ is a given ratio.

11. Draw a transversal to cut four given intersecting straight lines in P, Q, R, S, so that PQ : QR : RS is given.

12. Find a point such that the product of its distances from two opposite vertices of a quadrilateral is equal to the product of its distances from two other opposite vertices; and show that each of these is equal to the product of its distances from the third pair of opposite vertices.

13. If three triangles are in perspective, two by two, and have a common axis of perspective, their three centres of perspective are collinear.

14. If three triangles are in perspective, two by two, and have a common centre of perspective, their three axes of perspective are concurrent.

15. ABC is a triangle, V any point in its plane. VP, VQ, VR at right angles respectively to VA, VB, VC cut BC, CA, AB respectively in P, Q, R. Prove that P, Q, R are collinear.

[Draw BM, CN perpendicular to VP.

Then $$\frac{BP}{PC}=\frac{BM}{CN}=-\frac{BV\cos AVB}{CV\cos CVA}.$$

Similarly for CQ/QA and AR/RB.
The converse of Menelaus' theorem gives the required result.]

16. By the use of §5 (or otherwise) prove the theorem enunciated in Ex. 6, Chap. II.

17. Lines AD, BE, CF are drawn making the same angle ω with the sides AB, BC, CA of a triangle respectively. Shew that they will meet in a point Ω if

$$\sin^3\omega=\sin(A-\omega)\sin(B-\omega)\sin(C-\omega),$$

and conversely. Show also that if AD, BE, CF make angles ω with the sides AC, BA, CB respectively, and the same relation holds good, they will meet in a point Ω'.
[Ω and Ω' are called the Brocard points of the triangle.]

18. A line is drawn through the vertex A of a triangle ABC within the triangle, and makes the same angle with the side AC that the median through A makes with AB. If two other similar lines be drawn through B and C, prove that these three lines meet in a point (the symmedian point).

19. Lines drawn through the vertex of an angle and making equal angles (in opposite senses) with the bounding lines of the angle are called 'isogonal conjugates.' Prove that if three lines through the vertices of a triangle are concurrent, their isogonal conjugates are concurrent.

CHAPTER IV.

HARMONIC FORMS AND THEIR ELEMENTARY PROPERTIES.

§ 1. **Def.** If four points A, B, C, D be taken on a straight line such that

$$\frac{AB}{BC} = -\frac{AD}{DC},$$

the range ABCD is said to be **harmonic.**

§ 2. *Given any three collinear points* ABC *to find a fourth point* D, *so that the range* ABCD *may be harmonic.*

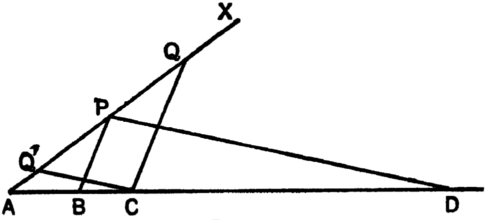


FIG. 11.

Draw a straight line APX making any convenient angle with AC. Take a point P on it. Join BP, and draw CQ parallel to BP. Mark off PQ′ on AP equal to $-$PQ. Join CQ′, and draw PD parallel to CQ′.

Then $$\frac{AB}{BC} = \frac{AP}{PQ} = -\frac{AP}{PQ'} = -\frac{AD}{DC},$$

so that D is the point required.

§ 3. Harmonic Conjugates.

The points A *and* C *divide* BD *in a similar manner to that in which the points* B, D *divide* AC.

For we have $\frac{BC}{CD} = -\frac{BA}{AD}$ from § 1.

Either of the pairs A, C or B, D is said to be harmonically conjugate to the other pair.

§ 4. Properties of the Harmonic Range.

(1) AB, AC, AD *are in Harmonic Progression* (hence the justification for the epithet 'harmonic').

(2) *If* O *be the middle point of the segment joining either pair of harmonic conjugates, the rectangle under the distances of* O *from the other pair is equal to the square on half the length bisected.*

To prove (1) we have, from definition,

$$\frac{AB}{AC-AB}=\frac{AD}{AD-AC}.$$

Invert these ratios and divide each by AC.

$$\frac{AC-AB}{AB.AC}=\frac{AD-AC}{AD.AC}, \text{ whence } \frac{1}{AB}-\frac{1}{AC}=\frac{1}{AC}-\frac{1}{AD},$$

which proves that AB, AC, AD are in H.P.

To prove (2) we have $\frac{AB}{BC}=\frac{AD}{CD}$.

If O be the middle point of AC, we have to prove that $OB.OD=OC^2$.

From the above equality of ratios by componendo-dividendo,

$$\frac{AB+BC}{AB-BC}=\frac{AD+CD}{AD-CD},$$

$$\text{i.e. } \frac{2OC}{2OB}=\frac{2OD}{2OC}; \quad \therefore OB.OD=OC^2.$$

Cor. *Conversely*, if four points A, B, C, D be taken on a line in this order, such that either AB, AC, AD are in harmonical progression, or that (O being the middle point of AC) $OB . OD = OC^2 = OA^2$, the range ABCD is harmonic.

To prove these we have merely to reverse the steps of the preceding argument, and the verification is left as an exercise.

§ **5.** By far the most important property of the harmonic range is contained in the following **fundamental proposition** :

If a pencil be formed by connecting the four points of a harmonic range to any fifth point, the range determined by this pencil on any transversal is also harmonic.

In consequence of this property, such a pencil is called a **Harmonic Pencil.**

Let ABCD be a harmonic range. (Fig. 12.)

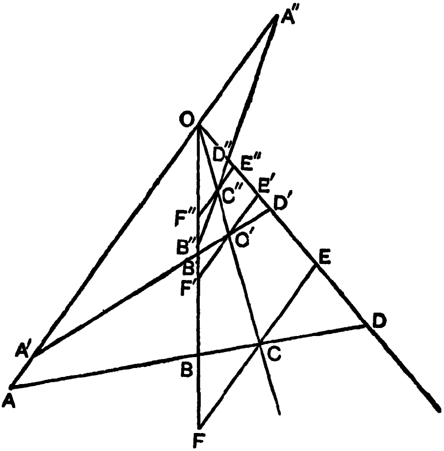


FIG. 12.

Let a pencil be formed by joining these points to O.

Let a straight line ECF be drawn through C parallel to the ray through the conjugate point A, to meet the rays through the other pair of conjugates in E and F.

Then $\frac{AB}{BC} = \frac{OA}{CF}$ and $\frac{AD}{CD} = \frac{OA}{EC}$.

But by definition, $\frac{AB}{BC} = \frac{AD}{CD}$; $\therefore$ $EC = CF$.

If $E'C'F'$ be any parallel to ECF, cutting the same three rays, $E'C' = C'F'$.

But, as before, $\frac{A'B'}{B'C'} = \frac{OA'}{C'F'}$ and $\frac{A'D'}{C'D'} = \frac{OA'}{E'C'}$;

$\therefore$ $\frac{A'B'}{B'C'} = \frac{A'D'}{C'D'}$, and $A'B'C'D'$ is a harmonic range.

If the section of the pencil be taken as $A''B''C''D''$, we have $E''C'' = C''F''$, and the rest of the argument proceeds as before; but we notice that the equal ratios are each negative.

However, the equality still holds, and so the theorem holds for all transversals.

But this proposition may be stated in a yet more general form, viz.

Harmonic Ranges and Pencils project into Harmonic Ranges and Pencils.

Referring to Fig. 3, since a range and its projection lie in one plane, the above theorem applies at once, and the statement is therefore true for ranges.

From Fig. 4 it is seen that a pencil and its projection have a common transversal. Hence, if the former is harmonic, so is the latter.

This establishes the truth of the statement for pencils.

§ 6. Two Points, the Middle Point of the Line joining them, and the Point at Infinity on the same Line form a Harmonic Range.

Let A, C be the points, B the middle point of their join, D_∞ the point at infinity on this line.

Then $\frac{AB}{BC} = 1 = \frac{AD_\infty}{CD_\infty}$; $\therefore$ $ABCD_\infty$ form a harmonic range.

Notice in § 5 that we should expect this, for ECF is a section of the pencil cutting the fourth ray OA at the point at infinity on ECF, and hence E, C, F and this point form a harmonic range.

§ 7. On account of the harmonic property of the pencil, OA, OC are called **Conjugate Rays**, as also OB, OD.

A particular case of a Harmonic Pencil of some importance is that formed by the bounding lines of an angle together with its internal and external bisectors.

Let OB, OD be respectively the internal and external bisectors of the angle AOC. (Fig. 13.)

Then $$\frac{AB}{BC}=\frac{AO}{OC}=\frac{AD}{CD}; \quad \therefore \frac{AB}{BC}=-\frac{AD}{DC}.$$

Hence the range ABCD is harmonic, and by § 5 the pencil connecting it to O is harmonic. Conversely,

If a pair of conjugate rays of a harmonic pencil are at right angles, they are the internal and external bisectors of the angles between the other pair.

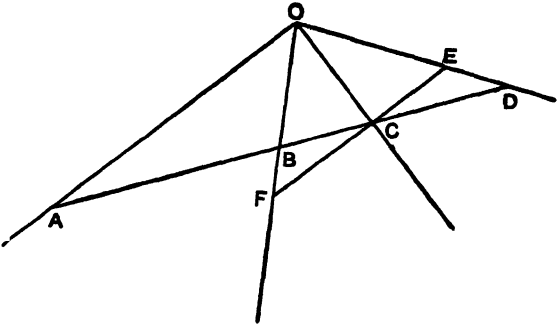


FIG. 13.

For (Fig. 13) if OA, OC be the conjugate pair at right angles, we have as before, ECF being parallel to OA, EC = CF.

But OC is now at right angles to EF.

$\therefore$ the triangles OCE, OCF are congruent, and the angle BOD is bisected by OC.

It must be carefully noted, however, that a harmonic pencil does not in general possess a pair of conjugate rays at right angles.

§ 8. Harmonic Properties of the Complete Quadrangle and the Complete Quadrilateral.

The nature of these figures has already been explained, and the correspondence between them pointed out in Chap. II. § 10.

The properties in question may be exhibited dualistically as follows:

Either pair of the sides of the triangle formed by the diagonal points of the quadrangle forms a harmonic pencil with the pair of opposite sides of the quadrangle which meet at the same diagonal point.	*Either pair of the vertices of the triangle formed by the diagonal lines of the quadrilateral forms a harmonic range with the pair of opposite vertices of the quadrilateral which lie on the same diagonal line.*

We shall prove the theorem for the quadrangle first.

Let E, F, G be the diagonal points (Fig. 14*a*).

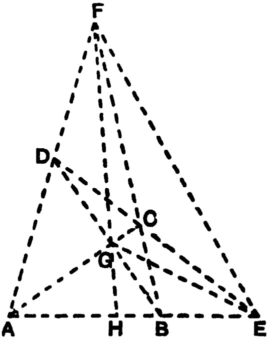


FIG. 14*a*.

Let FG cut AB in H.

Then, applying Ceva's theorem to the triangle FAB, we have

$$FD . AH . BC = + DA . HB . CF.$$

Also considering the transversal DCE cutting the sides of the same triangle, by Menelaus' theorem

$$\mathrm{FD}\,.\,\mathrm{AE}\,.\,\mathrm{BC} = -\mathrm{DA}\,.\,\mathrm{EB}\,.\,\mathrm{CF}.$$

It follows that $$\frac{\mathrm{AH}}{\mathrm{AE}} = -\frac{\mathrm{HB}}{\mathrm{EB}}$$

or $$\frac{\mathrm{AH}}{\mathrm{HB}} = -\frac{\mathrm{AE}}{\mathrm{EB}}.$$

∴ the range AHBE is harmonic.

∴ the pencil obtained by joining F to these points is harmonic.

Similarly the pencil formed by the lines EF, ED, EG, EA is harmonic.

Also the pencil formed by joining G to AHBE is harmonic.

The theorem is therefore proved for the quadrangle.

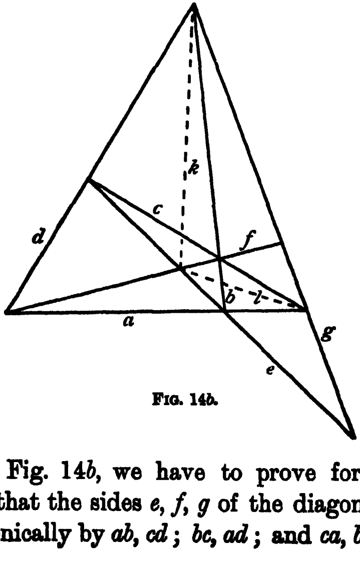


FIG. 14*b*.

Turning to Fig. 14*b*, we have to prove for the **complete quadrilateral** that the sides *e*, *f*, *g* of the diagonal triangle are divided harmonically by *ab*, *cd*; *bc*, *ad*; and *ca*, *bd* respectively.

Let k be the join of bd and ef, l that of ef and ac.

Then it has been proved that d, k, b, g form a harmonic pencil.

∴ the ranges determined by this pencil on f and e are harmonic.

Also k, f, l, e form a harmonic pencil.

∴ the range determined by this pencil on g is harmonic.

And this proves the theorem for the quadrilateral.

The triangle formed by e, f, g is sometimes called the **Harmonic Triangle** of the quadrilateral.

§ 9. (a) To construct the fourth Harmonic of three given Collinear Points by the use of the Ruler only.

This can be done at once by the help of the last theorem.

Through the point whose harmonic conjugate is required draw any ray, and on it take any two points, the joins of which to the other two given points form a complete quadrilateral.

The point of intersection of the remaining two diagonals of this quadrilateral is the fourth harmonic required.

For example, if A, H, B (Fig. 14*a*) are given, and the harmonic conjugate of H is required; take any two points F, G on any ray through H. The joins of F and G to A and B form a complete quadrilateral. whose other diagonals AB and DC intersect in the point E required.

(b) To construct the fourth Harmonic of three Concurrent Rays by the use of the Ruler only.

On the ray whose harmonic conjugate is required take any point, and through it draw any two rays intersecting the other two given rays in the angular points of a quadrangle. The line joining the other two diagonal points of this quadrangle is the fourth harmonic ray required.

This can be illustrated from Fig. 14*b* as above.

Historical Note. The existence of a series of lengths in harmonical progression seems to have been known from the earliest times with which recorded mathematical history deals. It is probable that these magnitudes were obtained at first experimentally (tradition says by the Babylonians) as the lengths of strings, which, when plucked, would give notes in harmony or 'harmonics.' Their accurate determination geometrically was an object of study with the school of philosophers founded by Pythagoras and known as the Pythagoreans.

The harmonic division of a line must have been known to Euclid and Apollonius (the 'great geometer' to whom subsequent reference will be made). It is certainly given in the lemmas of Pappus of Alexandria, who wrote about the beginning of the fourth century A.D. The property of the harmonic pencil by which harmonic ranges can be transferred from one place to another in a plane, or from plane to plane, is first employed by Serenus. The harmonic property of the complete quadrilateral is proved in essence in the lemmas of Pappus, but the result is stated in a clumsy form.

Examples on Chapter IV.

1. The locus of a point whose distances from two fixed points are in a constant ratio is the circle on the line joining a pair of harmonic conjugates of the two fixed points as diameter (circle of Apollonius).

2. If A, B, C, D form a harmonic range, any circle through one pair of conjugates cuts at right angles the circle on the line joining the other pair of conjugates as diameter.

3. If ABCD be a harmonic range and P be any other point on the line, prove $(PA+PC)(PB+PD)=2PA.PC+2PB.PD$.

Deduce the relations of § 4.

4. Prove that if four points ABCD in order on a line form a harmonic range, a point O can be found such that the angles AOB, BOC, COD are each half a right angle.

5. ABC, APQ are two straight lines such that BP is parallel to CQ. BP is produced to D, where $PB=PD$; DQ cuts ABC in E. Shew that ABCE is a harmonic range.

6. A variable line is drawn through a fixed point O to cut two fixed coplanar lines in points P and Q. A point R is taken so that

$$\frac{1}{\mathrm{OP}}+\frac{1}{\mathrm{OQ}}=\frac{k}{\mathrm{OR}}, \text{ where } k \text{ is a constant.}$$

Shew that the locus of R is a straight line.

7. Two straight lines OABC, OA′B′C′ are cut by a set of concurrent lines in points A, A′; B, B′; C, C′; etc. Prove that the points of intersection of all cross joins such as AB′, A′B; BC′, B′C; etc., are collinear with O.

8. Two straight lines intersect at a point off the paper. Shew how to draw, by means of the ruler only, a straight line which will pass through their intersection.

9. ABCD is a convex quadrilateral. AB, CD meet in E; BC, AD meet in F; CA, DB meet in G. Complete the parallelograms BFDH, AFCK. Prove that E, H, K are collinear. Hence shew that the middle points of the three diagonals AC, BD, EF of the complete quadrilateral are collinear.

10. Prove the harmonic properties of the complete quadrilateral by drawing in Fig. 14*a* GKL parallel to AD, cutting FC in K and FE in L, and proving that GK = KL.

11. A straight line cuts two given circles in P, P′ and Q, Q′ respectively. If the range PQP′Q′ be harmonic, shew that the middle points of PP′ and QQ′ lie on a fixed circle for all positions of the chord.

12. On the sides BC, CA, AB respectively of a triangle are taken points X, X′; Y, Y′; Z, Z′, such that the ranges BXCX′, CYAY′, AZBZ′ are each harmonic. Prove that if AX, BY, CZ meet in a point, X′, Y′, Z′ are collinear, and conversely.

13. If ABCD, APQR be two harmonic ranges on different lines intersecting at A, BP, CQ, DR will be concurrent.

14. If a, b, c, d be four straight lines in this order forming a harmonic pencil, and if m is the ray which bisects the angle between the rays a and c,

$$\tan^2 \widehat{ma}=\tan \widehat{mb} \tan \widehat{md},$$

where $\widehat{pq}$ denotes the angle between rays p and q.

15. If two complete quadrangles are so situated that five pairs of corresponding sides intersect in points of a certain given straight line, then the sixth pair will also intersect in a point of that straight line.

16. Write down the dual of this last theorem.

CHAPTER V.

INVERSION, SIMILITUDE, COAXAL CIRCLES.

§ 1. Certain methods and results in the plane geometry of the circle and of other figures, of which subsequent use will be made, are dealt with in this chapter. The student who is familiar with them may pass on to the next; they are, however, given here for convenience of reference.

INVERSION.

§ 2. Inverse Points with respect to a Circle.

DEF. If two points P, P′ be taken on the same radius of a circle whose centre is O, such that $OP \, . \, OP' =$ square on the radius of the circle, they are said to be **inverse points** with respect to the circle.

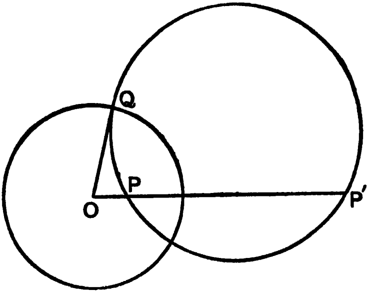


FIG. 15.

§ 3. PROP. **Any circle drawn through a pair of inverse points with respect to a given circle cuts the given circle orthogonally (i.e. at right angles).**

If the circles cut in Q, $OP . OP' = OQ^2$, since P and P′ are inverse points.

∴ OQ is a tangent to the circle PQP′.

∴ the circles cut at right angles.

Conversely. If two circles cut orthogonally, any straight line through the centre of either cuts the other in a pair of inverse points with respect to the first.

§ 4. The Operation of Inversion.

If corresponding to every point on a given curve, its inverse point be taken with respect to a given circle, the locus of these points is called the inverse of the curve with respect to the circle.

More usually, however, we speak of the inverse with respect to the *point* which is the centre of the given circle.

This point is called the **Centre of Inversion.**

The radius of the circle is called the **Radius of Inversion,** so that if P is any point on a given curve, O a fixed point, and if on OP a point P′ is taken, such that $OP . OP' =$ constant (say k^2), the locus of P′ is the inverse of the given curve with respect to O, the radius of inversion being k.

§ 5. PROP. **The inverse of a circle with respect to a point on it is a straight line.**

Let O be the point, k the radius of inversion (Fig. 16).

Take P any point on the given circle. On OP take P′ so that $OP . OP' = k^2$. We want the locus of P′.

Let A be the other end of the diameter through O, and

Also since $OA . OX = OP . OP'$, the quadrilateral XP′PA is cyclic, and since the angle at P is a right angle, so is the angle at X. The perpendicular from any inverse point P′ on OA therefore meets OA at a fixed point; *i.e.* the locus of P′ is a straight line perpendicular to OA and passing through the inverse of A.

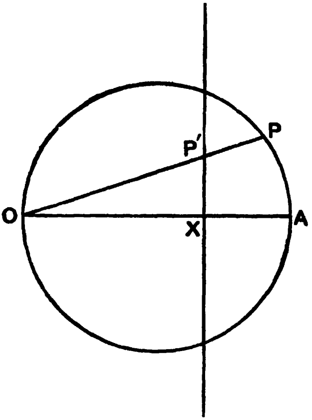


FIG. 16.

§ 6. PROP. **The inverse of a circle with respect to any point is a circle.**

Let O and k be as before.

Let C (Fig. 17) be the centre of the circle to be inverted.

Then if P′ is the inverse of P, $OP . OP' = k^2$.

If OP meet the circle again in Q, $OP . OQ = OT^2$.

Whence $\frac{OP'}{OQ} = \frac{k^2}{OT^2}$, which is constant for all positions of the line OPQ.

Divide OC at X so that $\frac{OX}{OC} =$ this constant ratio.

Then X is a fixed point.

Also, since P′X is parallel to QC,

$$\frac{P'X}{QC} = \text{this same ratio}; \quad \therefore \ P'X \text{ is constant.}$$

∴ the locus of P′ is a circle whose centre is X.

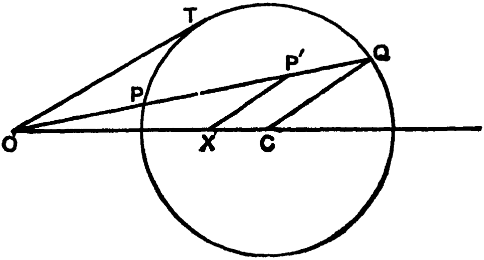


Fig. 17.

It is a good exercise in the application of the principle of continuity to deduce from this the result of § 5.

§ 7. Prop. **If two curves cut one another, their inverses cut at the same angle.**

Consider first a single curve and its inverse.

Take two near positions of the radius vector, and let P, P′, Q, Q′ be pairs of inverse points. (Fig. 18.)

Then, since

$$OP \,.\, OP' = k^2 = OQ \,.\, OQ',$$

the quadrilateral PQQ′P′ is cyclic;

$$\therefore \text{ angle } OPQ = \text{angle } OQ'P'.$$

Now if P, Q are points on the given curve which move up to

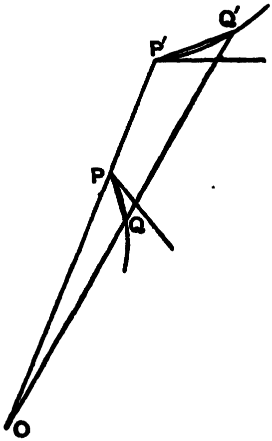


Fig. 18.

ultimate coincidence, P′, Q′ on the inverse curve will do the same, and PQ, P′Q′ become finally the tangents to the given curve and its inverse at the points P, P′ respectively. So that the tangents at P to the original curve and at P′ to the inverse curve make equal angles with the radius vector OPP′ on opposite sides of it.

.It follows that if two curves cut at P, their inverses cut at P′ at the same angle.

SIMILITUDE.

§ 8. Centres of Similitude of two Circles.

The two points which divide the line joining the centres of two given circles internally and externally in the ratio of the radii possess the property that every straight line drawn through them is divided in a constant ratio by the circles.

Take X (Fig. 19) the point of internal division, and draw any secant PXQ. Join P and Q to the centres A and B respectively.

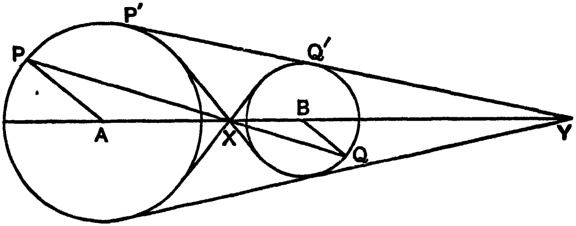


FIG. 19.

Then $\frac{AX}{XB}=\frac{PA}{QB}$, and the angle AXP = angle BXQ; ∴ the triangles, being such that the angles at P and Q are both acute—as in the figure—or both obtuse, are similar;

$$\therefore \frac{PX}{XQ}=\frac{PA}{QB}=\text{constant}.$$

Similarly we could prove the property for any secant through Y

DEF. The points X, Y are called the **centres of similitude** of the two circles.

They are clearly the points where the direct and inverse common tangents cut the line of centres.

For let the direct common tangent P′Q′ cut the line of centres at Y′. Since AP′ is parallel to BQ′, the triangles AP′Y′, BQ′Y′ are similar. Hence AY′/BY′ = AP′/BQ′ = AY/BY.

∴ Y and Y′ must coincide.

Similarly for the inverse common tangents.

§ 9. PROP. **If a variable circle touch two fixed circles, the line joining the points of contact passes through one or other of two fixed points (the centres of similitude of the fixed circles).**

Let A, B (Fig. 20) be the centres of the fixed circles.

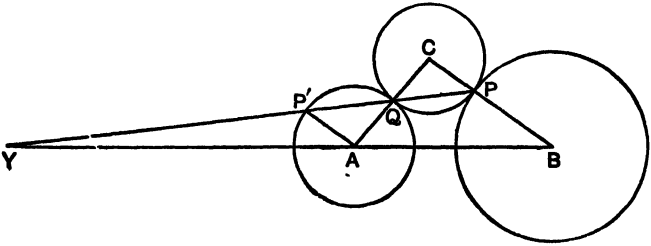


FIG. 20.

Let PQ be the chord of contact of the variable circle whose centre is C. Then CQ and CP pass through A and B respectively.

Let PQ cut the circle, whose centre is A, again in P′.

Then angle AP′Q = angle AQP′ = angle CQP = angle CPQ.

$\therefore$ AP′ is parallel to CPB, and the triangles AP′Y, BPY are similar, so that $\frac{AY}{BY}=\frac{AP'}{BP}$, and Y is the external centre of similitude.

The student should draw figures for the other cases of this proposition in which the variable circle is touched internally by either one or both of the fixed circles. The number of such is increased by including the cases in which one or other of the fixed circles reduces to a point (*i.e.* has zero radius), or has infinite radius (*i.e.* becomes a straight line).

The determination of the centres of similitude in these special cases is a useful exercise.

One such will be considered here.

The centres of similitude of a circle and a straight line are the ends of the diameter of the circle perpendicular to the straight line.

For it follows from the first prop. of this section that the centre of similitude divides the interval between the circumferences in the ratio of the radii (a ratio which is here equal to zero).

Ex. **Prove the theorem of § 9 by the use of Menelaus' theorem.**

§ 10. Similar Curves.

Let O be a fixed point, P any point of a given curve.

On OP take a point P′ so that $\frac{OP'}{OP}=$ a constant ratio k.

Then, as P describes the given curve, P′ will trace out another curve possessing the following properties:

(1) The distances between pairs of corresponding points are always in the constant ratio k.

(2) The tangents at corresponding points are parallel.

Let P and Q be two points on the given curve, P′, Q′ the corresponding points on the derived locus. Then, since

$\frac{OP}{OP'} = \frac{OQ}{OQ'}$ and the angle POQ is common to the triangles POQ, P′OQ′, these triangles are similar, and $\frac{P'Q'}{PQ} = k$. Also P′Q′ is parallel to PQ.

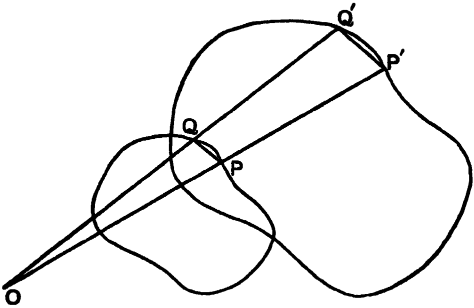


FIG. 21.

Now if P and Q move up to ultimate coincidence, so also will P′ and Q′ on the derived locus, and in the limit the tangent at P to the given curve is parallel to the tangent at P′ to the derived locus.

The derived locus is thus similar and similarly situated to the given locus.

Such figures are called **Homothetic.**

§ **11.** Two figures may be similar without being similarly situated: for in the case last considered either may be turned in the plane about O through any angle.

O is called the **Homothetic Centre.**

Of two homothetic figures one is sometimes said to be derived from the other by 'multiplication.'

Two figures in the same plane, which are such that the one may be derived from the other by multiplication and rotation about the homothetic centre, are said to be **directly similar.**

Referring to the Def. of § 6, Chapter III., we see that **homothetic figures are a particular case of figures in perspective.** For corresponding lines are parallel and the axis of perspective is accordingly at infinity, the centre of perspective being of course the homothetic centre.

§ 12. Symmetry about a Point.

If k the constant ratio is equal to -1, to every point P corresponds a point P′ such that PP′ is bisected at O.

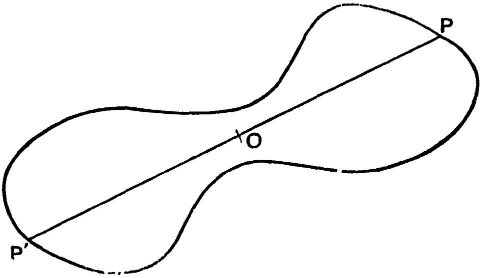


Fig. 22.

Two curves (or two parts of the same curve) which possess this property are said to be symmetrical about O. (Fig. 22.)

§ 13. Symmetry about a Line.

Another particular case of plane perspective may be mentioned here.

If two figures (or parts of the same figure) are such that to each point P of one of them corresponds a point P′ of the other such that PP′ is bisected at right angles by a given straight line, the figures are said to be symmetrical about that line. Either figure (or either half of the same figure as the case may be) is the 'image' of the other in the straight line (Fig. 22*a*).

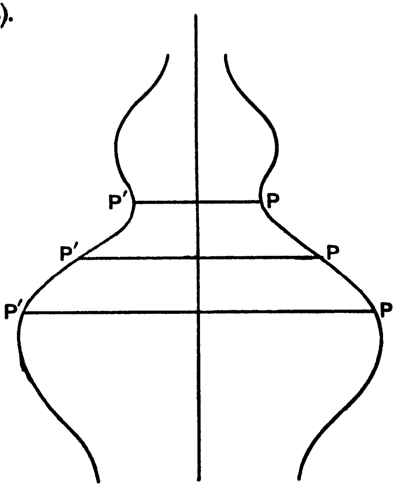


FIG. 22*a*.

§ 13. (a) Inverse Similarity.

Two similar figures need not possess the property of direct similarity. For (Fig. 23) keeping one of them X′ in its

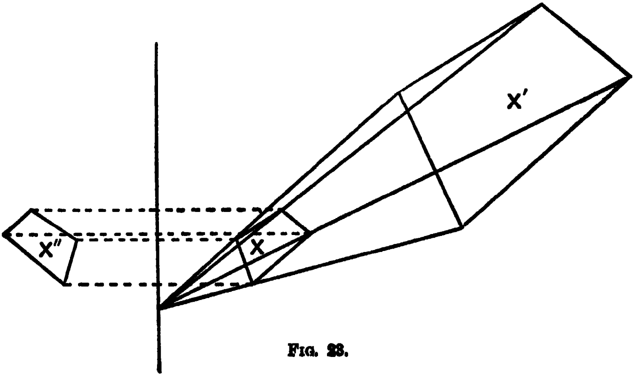


FIG. 23.

original position, we could construct the image of the other X in any straight line passing through the homothetic centre. Calling this image X″ we could not, by any rotation in their own plane, bring X and X″ into coincidence.

X′ and X″ are then called 'inversely similar.'

COAXAL CIRCLES.

§ 14. PROP. **The locus of points from which tangents of equal length can be drawn to two circles is a straight line.**

Let C, C′ (Fig. 24) be the centres of the two circles, T any point such that the tangents TP, TP′ to the circles are equal.

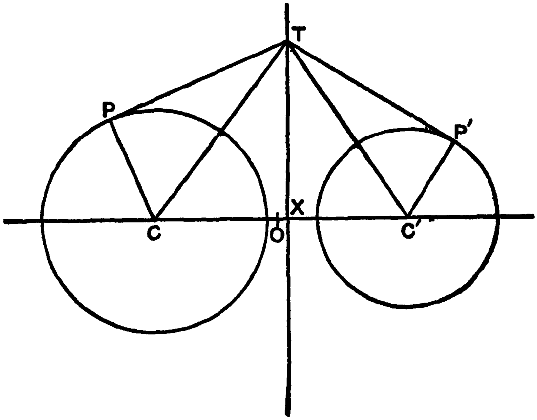


FIG. 24.

Draw TX perpendicular to CC′, and let O be the middle point of CC′.

Then $$CT^2 - C'T^2 = CP^2 + PT^2 - C'P'^2 - P'T^2.$$

But $$PT = P'T;$$

$\therefore$ $CT^2 - C'T^2 = CP^2 - C'P'^2 =$ constant for all positions of T.

But $$CT^2 - C'T^2 = (CX^2 + XT^2) - (C'X^2 + XT^2)$$
$$= CX^2 - C'X^2 = 2CC' . OX.$$

But CC' is constant; ∴ OX is constant.

∴ X is a fixed point.

The locus of T is therefore the straight line through this point at right angles to CC'.

DEF. **This straight line is called the Radical Axis of the two circles.**

If the two circles intersect at Q and Q', the radical axis is their common chord QQ'. For if T be any point on the chord, $TP^2 = TQ . TQ' = TP'^2$.

Ex. Prove that the radical axis is nearer to the larger circle.

§ **15.** PROP. **The radical axes of any three circles taken in pairs are concurrent.**

For let the circles be X, Y and Z, and let the radical axis of X and Y meet that of Y and Z at T.

Then, since T is on the former line, the length of the tangent from T to X = length of tangent from T to Y, and, since T is on the radical axis of Y and Z, the length of the tangent from T to Y = length of tangent from T to Z.

∴ tangent from T to X = tangent from T to Z.

∴ T is on the radical axis of X and Z.

∴ the three axes meet in a point.

§ **16.** Hence we may **construct the radical axis** of two circles which do not intersect.

For if A and B (Fig. 25) be the circles, draw any circle X cutting each of them.

The common chords of X with A and B being the radical axes of X and A and X and B respectively will meet on the radical axis of A and B.

Another point on the axis required is obtained by the intersection of the common chords of A and B with any other circle Y.

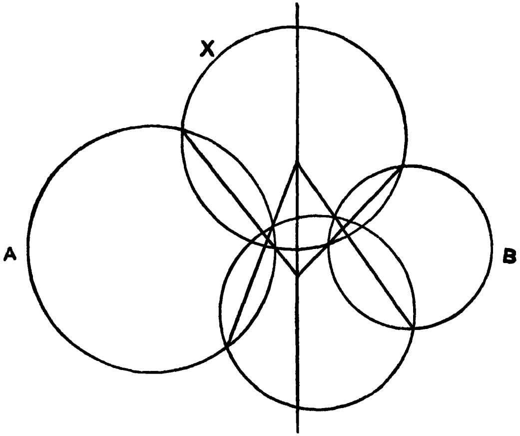


FIG. 25.

Two points being thus known, the radical axis can be drawn.

Since the radical axis is the locus of the intersection of tangents of equal length, it passes through the middle points both of the direct and inverse common tangents. This property affords another construction for its position.

Also any circle whose centre is a point on the radical axis, and whose radius is the length of the tangent from this point to one of the circles, cuts both circles orthogonally. It should be carefully noted, however, that all circles whose centres are on the radical axis do not cut the circles orthogonally.

The student should examine the particular cases which the preceding theorems assume when one or more of the circles has zero radius—*i.e.* shrinks up into a point—and, in particular, should work the following exercises:

(1) Construct the radical axis of a circle and a point outside it.

(2) Describe a circle to cut three circles orthogonally.

(3) Describe a circle to cut two circles orthogonally, and to pass through a given point outside both.

(4) Describe a circle to cut a given circle orthogonally, and to pass through two given points outside it.

§ **17.** Def. Circles which have a common radical axis are said to be **coaxal** or to form a **coaxal system.** Since the radical axis is perpendicular to the line joining the centres of any two of the circles, all the circles must have their centres on the same straight line.

Construction of circles of the system. Let the radical axis meet the line of centres at X (Fig. 26). Then a circle whose

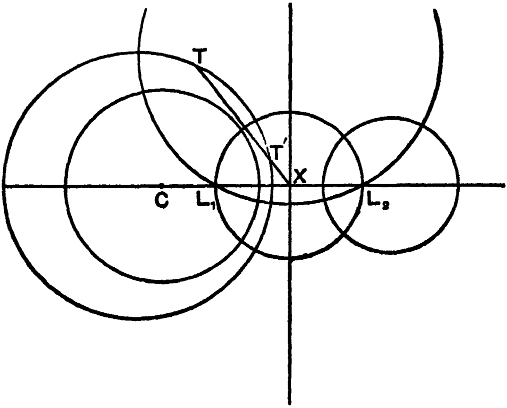


Fig. 26.

centre is X and radius equal to the tangent from X to any one circle of the system cuts all the circles orthogonally.

We can construct as many circles of the system as we please by drawing tangents to this circle from points on the line of centres. The circle whose centre is the point selected, and whose radius is the length of the tangent, is clearly a circle of the system.

§ 18. PROP. **Through any given point one and only one circle can be drawn coaxal with two given circles.**

Let X be the foot of the radical axis of the given circles (Fig. 26), T the point through which the required circle is to be drawn.

Draw the circle, centre X, which cuts the system orthogonally.

The required circle will be cut orthogonally also.

It will therefore cut TX at T′, the inverse of T with respect to the circle whose centre is X.

The required circle passes through T and T′, and has its centre on the given line of centres. It can therefore be constructed uniquely.

§ 19. Limiting Points.

Let L_1 and L_2 be the points where the circle whose centre is X cuts the line of centres.

If these points be selected as the centres of circles of the system, we see by the foregoing construction that each such circle must have zero radius.

They are therefore point circles of the system, being the ultimate positions of such circles when the radius becomes indefinitely small.

Hence the term 'limiting points.'

These points are imaginary when the circles of the system cut in real points; for in this case tangents from the foot X of the radical axis to the circles are imaginary.

The limiting points are inverse points for every circle of the system.

For if C be the centre of any circle of the system, $CL_1 . CL_2$ = sq. on the tangent from C to the circle whose centre is X = sq. of radius of circle whose centre is C.

∴ L_1 and L_2 are inverse points for this circle.

It follows that any circle through L_1 and L_2 cuts all circles of the system orthogonally and has its centre on the radical axis.

Ex. 1. The student should repeat the exercises in § 16, taking some or all of the given points *inside* the circles.

Ex. 2. If each member of one set of circles cuts each member of another set orthogonally, the two sets are each coaxal, one having real and the other imaginary limiting points.

Ex. 3. Prove that a common tangent to any two circles of the system subtends a right angle at either limiting point.

§ 20. PROP. **If T be any point, TP, TP′ tangents from it to two circles whose centres are C, C′,**

$$TP^2 - TP'^2 = 2CC' . TN,$$

where N is the foot of the perpendicular from T on the radical axis of the two circles.

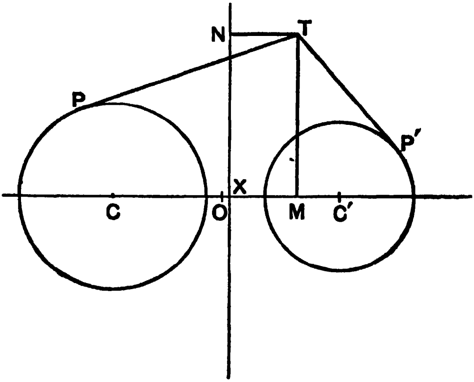


FIG. 27.

Let X be as before, TM perpendicular to CC′, O the middle point of CC′.

Then
$$\begin{aligned}\mathrm{TP}^2-\mathrm{TP}'^2&=\mathrm{CT}^2-\mathrm{CP}^2-(\mathrm{C'T}^2-\mathrm{C'P'}^2)\\&=\mathrm{CT}^2-\mathrm{C'T}^2-(\mathrm{CP}^2-\mathrm{C'P'}^2)\\&=\mathrm{CT}^2-\mathrm{C'T}^2-(\mathrm{CX}^2-\mathrm{C'X}^2)\quad(\S\,14)\\&=2\mathrm{CC'}.\mathrm{OM}-2\mathrm{CC'}.\mathrm{OX}\\&=2\mathrm{CC'}.\mathrm{XM}\\&=2\mathrm{CC'}.\mathrm{TN}.\end{aligned}$$

COR. 1. If T be on a circle, centre C″, coaxal with the other two,
$$\mathrm{TP}^2=2\mathrm{CC''}.\mathrm{TN},$$
$$\mathrm{TP'}^2=2\mathrm{C'C''}.\mathrm{TN}.$$

$\therefore\ \dfrac{\mathrm{TP}^2}{\mathrm{TP'}^2}=\dfrac{\mathrm{CC''}}{\mathrm{C'C''}}=$ constant for all positions of T on the circle.

COR. 2. The locus of a point which moves so that the tangents from it to two given circles are in a constant ratio, is a circle coaxal with the given circles. For, by the preceding, T lies on a circle, centre C″, coaxal with the given circles, where $\dfrac{\mathrm{CC''}}{\mathrm{C'C''}}=$ the square of the given ratio.

We have shown that only one such circle can be drawn.

Historical Note. The method of inversion is said to have been known to geometers from very early times; Chasles even states that it was used by Ptolemy (2nd century A.D.).

Its modern resuscitation is, however, due to Drs. Stubbs and Ingram of Trinity College, Dublin, in 1843-4. As a branch of geometry it has been well expounded by Salmon and Townsend, but perhaps the most interesting modern application of the method is due to the late Lord Kelvin, who employed it with considerable success in Electrostatics in the development of his theory of Electric Images, giving thereby elegant solutions of several difficult problems, such as, for example, the distribution of an electric charge on the surface of a spherical bowl.

Several mechanical devices have been constructed for tracing out automatically the inverse of a given curve. The most notable of such 'inversors' is probably the linkage known as 'Peaucellier's cell,' which is described in No. 5 of the following set of examples.

Examples on Chapter V.

1. The ratio of the distances of any point on a circle from a given pair of inverse points with respect to the circle is constant.

2. Chords of a circle subtend a right angle at a fixed point within a circle. Prove that the locus of their middle points is a circle. Also prove that the locus of the intersection of tangents at their extremities is a circle.

3. Any two circles can be inverted into themselves.

4. If two curves touch one another, so do their inverses.

5. A freely-jointed framework is made up of a rhombus ABCD, to opposite corners B, D of which are connected a pair of equal rods BO, DO. Prove that if O is fixed and A describes the perimeter of any curve, C will describe the inverse curve with respect to O. (Peaucellier's cell.)

6. The circle on the line joining the internal and external centres of similitude of two given circles as diameter (known as the 'circle of similitude') is coaxal with the given circles.

7. The six centres of similitude of three circles taken in pairs lie in threes on four straight lines.

8. Inscribe a square in a regular pentagon.

9. ABC is a triangle, A′, B′, C′ the points of contact of the inscribed circle with the sides BC, CA, AB respectively. Let A″, B″, C″ be the middle points of the arcs of the circumcircle cut off outside the triangle by the sides. Prove that A′A″, B′B″, C′C″ are concurrent.

10. A triangle of given species has one angular point fixed and a second moves along a given curve. Prove that the third describes a similar curve.

11. Find the radical axes of the inscribed circle of a triangle with its escribed circles.

12. If two circles cut two others orthogonally, the line of centres of either pair is the radical axis of the other pair.

13. The circles on the three diagonals of a complete quadrilateral as diameters are coaxal. Hence verify the second part of Ex. 9, Chap. IV.

14. Shew how to invert two circles into two equal circles.

15. A system of concentric circles inverts from any point into a coaxal system.

16. A system of concurrent straight lines inverts into a system of coaxal circles.

17. If a variable circle touches two fixed circles, its radius bears a constant ratio to the perpendicular distance from its centre on the radical axis of the two circles.

CHAPTER VI.

POLES AND POLARS. HARMONIC PROPERTIES OF THE CIRCLE.

§ 1. PROP. **The locus of the intersection of tangents at the ends of chords of a circle which pass through a fixed point in the plane of the circle is a straight line.**

Let O (Fig. 28) be the fixed point, C the centre of the given circle.

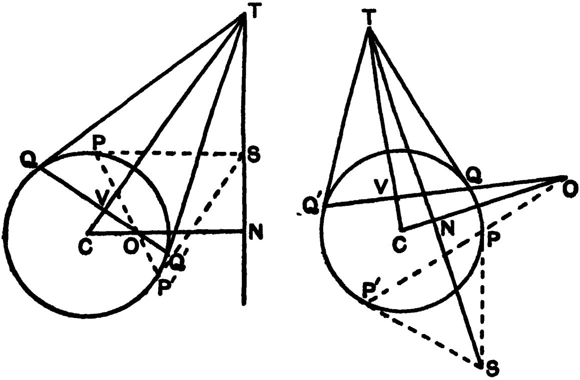


FIG. 28.

CO is a fixed straight line. Let the tangents at the ends of any chord QOQ′ through O meet in T.

Draw TN perpendicular to CO. We shall prove that CN is constant. The locus of T will then be the fixed straight line NT.

Join CT, cutting QQ′ in V.

Then, since QV is perpendicular to CT and CQT is a right angle, $CV . CT = CQ^2$.

Also V, T, N, O being concyclic points, $CV . CT = CN . CO$.

∴ $CN . CO = CQ^2$, but CO and CQ are constant.

∴ CN is constant. Whence the theorem is proved.

§ 2. DEF. **The locus of the intersection of tangents at the ends of any chord drawn through a point is called the Polar of the point with respect to the circle.**

The point itself is called the **Pole.**

Thus, with the figure of § 1, O being the pole, TN is its polar.

The polar of a point with respect to a circle is therefore a straight line, whether the point is inside or outside the circle.

In the former case the polar **does not cut the circle.**

In the latter case it is the **chord of contact of tangents from O to the circle.** For if TN cut the circle in R, R′ by § 1, $CN . CO = CQ^2 = CR^2 = CR'^2$; ∴ CRO, CR′O are right angles, and OR, OR′ are tangents to the circle.

When O moves up to the circumference from a position outside the circle, the points of contact of the tangents move closer and closer together, so that their chord of contact ultimately becomes, when O is on the circumference, the tangent at O.

The tangent at any point is therefore the polar of its point of contact.

To construct the polar of O. On CO take a point N so that $CO . CN =$ square of radius of the circle. The polar is the straight line through N, perpendicular to CN.

The polar of the centre. If CO is very small, CN must be very great, since the product of CO and CN is constant. The polar of O is then very distant. Ultimately if O coincides with C, the polar of O becomes the **line at infinity** in the plane of the circle. **Hence the polar of the centre is the line at infinity.**

§ 3. PROP. **If the polar of A passes through B, then the polar of B passes through A.**

For, take any point B on the polar of A; draw AN perpendicular to CB. (Fig. 29.)

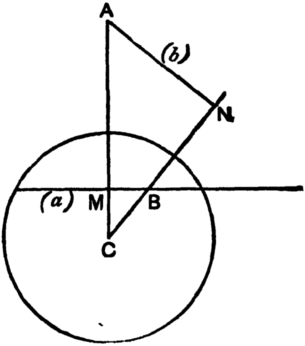


FIG. 29.

Let CA cut the polar of A in M.

Then, since AMBN is cyclic, CM . CA = CB . CN.

But CM . CA = square of radius;

∴ CB . CN = square of radius;

∴ AN is the polar of B.

§ 4. Salmon's Theorem.

The distances of two points from the centre of a circle are proportional to the distances of each from the polar of the other.

Let S be the centre of the circle (Fig. 30), and let KX and QL be the polars of O and P.

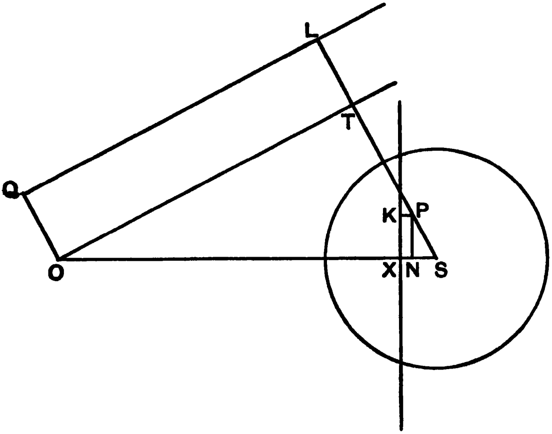


FIG. 30.

Let OT, the parallel through O to QL, cut SPL in T. Let PK, PN be perpendicular to KX and SO respectively.

Then $\quad SX . SO = \text{square of radius} = SP . SL.$

Also, since OTPN is cyclic,

$$SN . SO = SP . ST;$$

$$\therefore \frac{SP}{SO} = \frac{SX}{SL} \text{ and also } \frac{SN}{ST};$$

$$\therefore \frac{SP}{SO} = \frac{SX - SN}{SL - ST} = \frac{NX}{LT} = \frac{PK}{OQ}.$$

§ 5. Conjugate Points and Lines.

DEF. A pair of points such that each lies on the polar of the other with respect to a given circle are called conjugate points with respect to that circle.

DEF. A pair of straight lines such that each contains the pole of the other with respect to a given circle are called conjugate lines with respect to that circle.

PROP. **On any straight line an infinite number of pairs of conjugate points with respect to the circle can be found.**

For taking any point A on the line *a*, the point A′, where the polar of A cuts *a*, is conjugate to A.

A and A′ are therefore a pair of conjugate points on the line *a*; and clearly an infinite number of such pairs can be found.

PROP. **Through any point an infinite number of pairs of conjugate lines with respect to a circle can be drawn.**

For taking any straight line *a* through the point A, the line *a*′, joining A to the pole of *a*, is conjugate to *a*.

a and *a*′ are therefore a pair of conjugate lines through A; and clearly an infinite number of such pairs can be drawn.

§ 6. PROP. **The polars of a pair of conjugate points are a pair of conjugate lines.**

For the polar of each conjugate point passes through the other, and therefore these lines are conjugate, as each contains the pole of the other.

§ 7. Self-Conjugate or Self-Polar Triangles.

PROP. **A pair of conjugate points and the pole of the line joining them form the angular points of a triangle which is self-polar with respect to the circle: that is to say, such that each angular point is the pole of the opposite side with respect to the circle.**

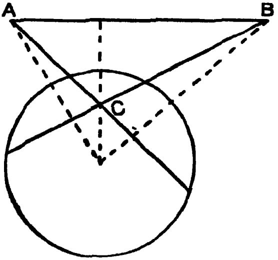


FIG. 31.

For let A and B be the conjugate points and C the pole of AB (Fig. 31).

Since the polar of C passes through A, the polar of A passes through C (§ 3). But it also passes through B, for A and B are conjugate points.

Therefore it is BC. Similarly AC is the polar of B, and the triangle ABC possesses the property stated.

COR. 1. The three perpendiculars from the vertices of a self-polar triangle on the opposite sides each pass through the centre of the circle, and we have therefore the theorem:

The centre of a circle is the orthocentre of every triangle self-conjugate with respect to it.

COR. 2. In like manner it may be proved that a pair of conjugate lines and the polar of their point of intersection form a self-conjugate triangle.

§ 8. The Polars of a Harmonic Range form a Harmonic Pencil (Fig. 32).

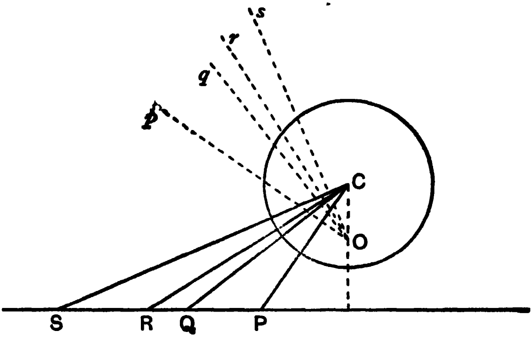


FIG. 32.

The line joining the centre of the circle to any point of the range is perpendicular to the polar of that point.

Also all the polars pass through a common point, viz. the pole of the line on which the range is situated.

Therefore the polars form a pencil, and since the angle between any two straight lines is equal to the angle between perpendiculars to them, the pencil formed by the polars and that formed by the joins of the centre to the points of the range (shown by the full drawn lines in the figure) are superposable. But the latter pencil is harmonic; ∴ so is the other.

Cor. The poles of the rays of a harmonic pencil form a harmonic range.

The theorem of this article is a particular case of a more general one. (See Chap. XIV. § 1.)

§ 9. The fundamental Harmonic Property of Pole and Polar.

Any straight line drawn through a point P to cut a given circle is divided harmonically by P, the circle, and the polar of P.

First Proof. Let KK′, the polar of P, cut the chord PRR′ at Q, and let CP cut KK′ at V (Fig. 33*a*).

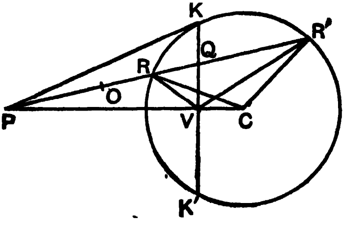


Fig. 33*a*.

Then (§ 2) P and V are inverse points with respect to the circle.

Therefore any circle through them cuts the given circle orthogonally. (Chap. V. § 3.)

Consider the circle on PQ as diameter.

Since it passes through V it cuts the circle KRK′ orthogonally. Let O be the middle point of PQ.

Then it follows that OR . OR′ = sq. on tangent from O to the circle KRK′ = OQ^2.

∴ PRQR′ is a harmonic range. (Chap. IV. § 4 (2) Cor.)

This theorem is of great importance, and a second proof by Euclidean methods is appended.

Second Proof (Casey's). The construction being as before, join CR, CR′ (Fig. 33*a*).

Then PR . PR′ = PK^2 = PV . PC, since PKC is a right angle.

∴ RVCR′ is a cyclic quadrilateral.

∴ angle RVP = angle CR′R = angle CRR′ (since CR = CR′)
= angle CVR′ (in same segment).

∴ angle KVR = angle KVR′.

And since KVP is a right angle, the range PRQR′ is harmonic.

The proof holds equally well when P is inside the circle; the student should draw the figure and verify.

The theorem may be also stated in the following form, which serves to bring out the dual aspect.

'*Any pair of conjugate points with respect to a circle, together with the points where the line joining them cuts the circle, form a harmonic range.*'

Also conversely:

'*If on any line cutting a circle be taken a pair of points harmonically conjugate to those in which the line cuts the circle, these points will be conjugate points with respect to the circle.*'

§ **10.** Prop. **Any pair of conjugate lines (with respect to a circle) through a point, together with the tangents to the circle from that point, form a harmonic pencil.**

For let AB, AC be conjugate lines through A (Fig. 33*b*).

Then the pole of AB lies on AC.

Also, since AB passes through A, the pole of AB lies on BC, the polar of A.

∴ the pole of AB is C, and ABC is, as in the last article, a self-conjugate triangle.

∴ (§ 9) CDBE is a harmonic range.

∴ AC, AD, AB, AE form a harmonic pencil.

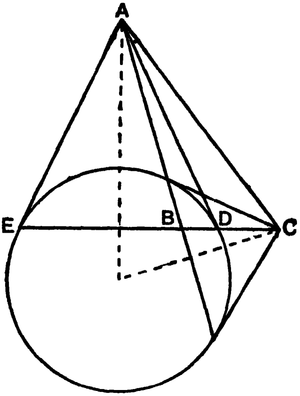


FIG. 33*b*.

Conversely, if four lines through a point form a harmonic pencil, conjugate rays of which are tangents from the point, the other two are conjugate lines with respect to the circle.

This follows easily: the proof is left as an exercise.

HARMONIC PROPERTIES OF THE INSCRIBED QUADRANGLE AND CIRCUMSCRIBED QUADRILATERAL.

§ 11. PROP. **If a quadrangle be inscribed in a circle, the diagonal points determine a triangle self-conjugate with respect to the circle.**

Let ABCD (Fig. 34*a*) be the quadrangle, E, F, G its diagonal points.

Let EG meet FA, FB in H, K.

Then (Chap. IV. § 8) FDHA and FCKB are harmonic ranges; ∴ (§ 9, above) H and K are points on the polar of F.

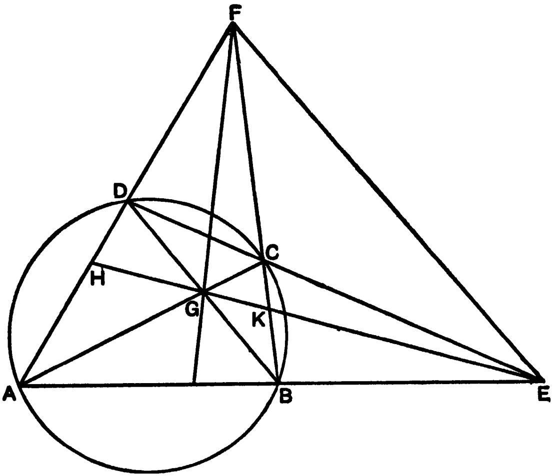


FIG. 34a.

This polar is therefore EG. Similarly FG is the polar of E; ∴ FE is the polar of G, and the triangle EFG is self-conjugate.

§ 12. PROP. **If a quadrilateral be described about a circle, its diagonal triangle is self-conjugate with respect to the circle.**

Let *abcd* (Fig. 34*b*) be the quadrilateral, and *efg* its diagonal triangle.

Let *h*, *k* be the joins of (*eg*) to (*fa*) and (*fb*) respectively.

Then (Chap. IV. § 8) *a*, *f*, *d*, *h* form a harmonic pencil.

So also do *b*, *f*, *c*, *k*.

∴ (§ 9) *h* and *k* are each conjugate lines to *f*.

∴ they meet at the pole of *f*, *i.e.* (*eg*) is the pole of *f*.

Similarly the point (gf) is the pole of e.

∴ the triangle efg is self-conjugate for the circle.

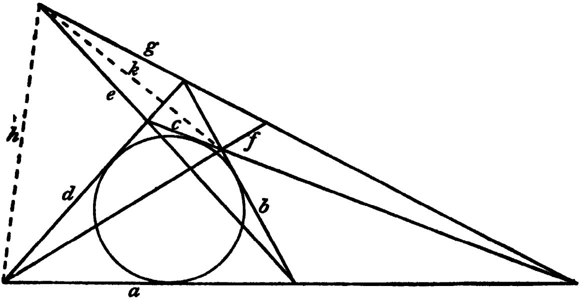


FIG. 34b.

§ 13. PROP. **The angular points of the diagonal triangle of a quadrilateral circumscribed to a circle coincide with the diagonal points of the quadrangle, formed by the points of contact.**

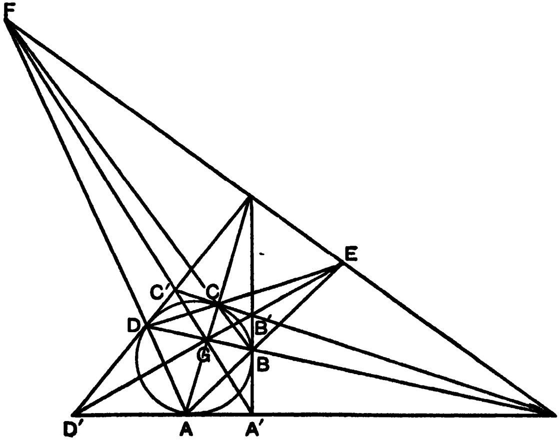


FIG. 35.

For (Fig. 35) the polar of A′ passes through E; ∴ the polar of E passes through A′.

Similarly it passes through C′; ∴ it is A′C′.

But the polar of E is FG, F and G being the other two diagonal points of the inscribed quadrangle.

∴ one side of the diagonal triangle of the quadrilateral coincides with one side of the triangle formed by the diagonal points of the quadrangle.

Similarly for the other two sides. The two triangles are therefore coincident.

§ 14. The Principle of Duality (*continued*).

In Chapter II. § 10, the possibility of a point-to-line and line-to-point correspondence between geometrical figures was suggested.

The theory of pole and polar as developed in the present chapter affords one method of establishing such a correspondence.

For every point A in the plane of a circle there exists a definite unique corresponding straight line *a*—its polar with respect to the circle. For every straight line *a* there exists a definite unique corresponding point A—its pole with respect to the circle.

To the join AB of two points corresponds the point (*ab*) of intersection of their polars.

Starting then with a given figure X composed of **points and lines**, we can **derive** another figure X′ composed of **lines and points**, which are respectively the polars and poles with respect to a given circle of the points and lines of X, *the two figures having a definite relation to one another as regards position.* Again starting with the figure X′ and proceeding in the same way, we should obtain the figure X.

X and X′ are thus called **Reciprocal Figures**, and the method of derivation is called **Reciprocation with respect to a circle.**

As an example, consider the theorem of § 5, Chap. III. 'If the joins of corresponding vertices of two triangles are concurrent, the points of intersection of corresponding sides are collinear.'

Introducing any circle in the plane of the figure, we have the theorem:

'If the points of intersection of corresponding sides of two triangles are collinear, the joins of corresponding vertices are concurrent.'

Reciprocation thus gives us in this case the converse theorem.

§ 7 of this chapter furnishes another example of the same kind.

Since the reciprocal of a harmonic range is a harmonic pencil, and *vice versa*, we see that the two theorems of § 8, Chapter IV., are reciprocal to one another, and likewise the construction of § 9 of the same chapter.

The proofs there given were quite distinct from one another, but we now see that, having proved one theorem, the other can be at once deduced from it by reciprocation.

Further, in cases in which the argument is purely 'descriptive' (see § 8, Chap. II.), the proofs can be correlated step by step, and it is a useful exercise to set out the reasoning in parallel columns in this way.

As regards theorems relating to the circle, we can reciprocate with respect to the circle itself. It has been shown in § 2 that the tangent is the polar of its point of contact, and accordingly to a system of points on the circle corresponds a system of tangents to the circle in the reciprocal figure. Also the circle itself regarded as a locus of points is *its own reciprocal* regarded as an envelope of tangents (Fig. 36).

§ 6 shows that conjugate points reciprocate into conjugate lines and *vice versa*.

With this explanation it follows easily that the theorems of § 9 and § 10 are reciprocal, as also those of § 11 and § 12, while § 13 practically carries out the process of reciprocation in the case of the last two.

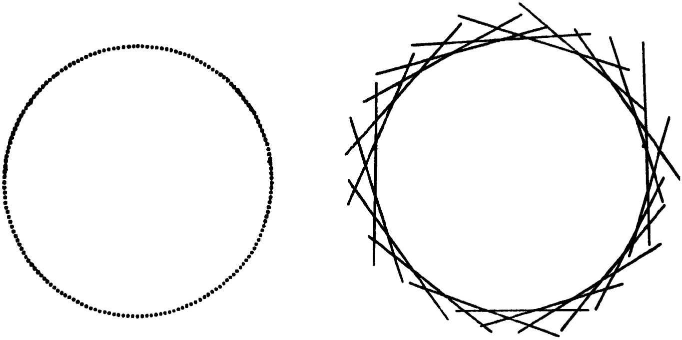


FIG. 36.

§ **15**. To sum up, we can translate a theorem into its reciprocal by interchanging terms as follows:

point	straight line
join of two points	intersection of two st. lines
lie on	pass through
collinear	concurrent
range	pencil
harmonic range	harmonic pencil
inscribed (figure)	circumscribed (figure)
circle (locus)	circle (envelope)

It must be borne in mind

(1) that we are at present reciprocating theorems about rectilineal figures only with respect to a circle chosen arbitrarily and circle theorems *with respect to the circle itself*;

(2) that for the present we are reciprocating *descriptive theorems only*, *i.e.* those in which lengths of lines and magnitudes of angles do not appear.

The extension of the method to the case in which the circle is replaced by any conic will be dealt with in Chapter VIII., and to the cases in which we reciprocate circle and conic theorems with respect to another circle or conic in Chapter XVIII.

Examples on Chapter VI.

1. POQ is any chord of a circle. If O′ be the inverse of O with respect to the circle, prove that the angle PO′Q is bisected by OO′.

2. Any tangent meets two tangents and their chord of contact in three points, which with the point of contact form a harmonic range.

3. Diameters of a circle which are conjugate lines are perpendicular. If one of two conjugate lines passes through the centre, the other is at right angles to it.

4. The circle on the third diagonal of a cyclic quadrilateral cuts the circumcircle of the quadrilateral orthogonally.

5. The square on the third diagonal of a cyclic quadrilateral is equal to the sum of the squares on the tangents from its extremities to the circle.

6. Construct a circle with respect to which a given triangle is self-conjugate. [This is called the Polar Circle of the triangle.]

7. Prove that the polar circle of a triangle cuts the circles described on the sides as diameters orthogonally.

8. A is a point in a given line AB and C is a given circle. A variable circle D touches AB at A and cuts C in P and Q; X is the point where D meets the harmonic conjugate of AB with respect to AP and AQ.

Prove (1) that PQ and the tangent at X cut AB at the same point; (2) that this is a fixed point; and (3) deduce that the locus of X for different positions of the variable circle is a circle which cuts C orthogonally.

9. One point (or line) exists which is conjugate to each of two given points (or lines) with respect to a circle.

10. Given one vertex of a quadrangle, a pair of diagonal points and the circle inscribed in the triangle formed by the diagonal points, find the other angular points of the quadrangle.

11. All circles which have a common pair of inverse points are coaxal.

12. The limiting points of a coaxal system are a common pair of conjugate points for all circles of the system.

13. The polars of a point with respect to a coaxal system of circles are concurrent.

14. The circumcircle of the diagonal triangle of a complete quadrilateral cuts the circles on the three diagonals as diameters orthogonally.

15. X, Y are a pair of conjugate points with respect to a given circle, centre C. CX, CY cut the circle on XY as diameter in X′, Y′ respectively. Prove that YX′, Y′X intersect at the pole of XY with respect to the circle whose centre is C.

16. The polar circles of the four triangles formed by four intersecting straight lines are each cut orthogonally by the circles whose diameters are the three diagonals of the complete quadrilateral formed by the four lines.

17. Hence prove the result of Ex. 13, Chap. V.

18. Use the result of Ex. 13, Chap. V., to deduce that the three middle points of diagonals of the complete quadrilateral are collinear; also that the orthocentres of the four triangles are collinear, the two lines of collinearity being perpendicular.

19. If two triangles are reciprocal with respect to a circle, they are in perspective.

20. Prove that the centre of the circle being O and its radius r, the ratio of the area of the triangle A′B′C′, reciprocal to ABC with respect to the circle, to the area of ABC is

$$\frac{1}{4}r^4\frac{ABC}{AOB\,.\,BOC\,.\,COA}.$$

CHAPTER VII.

PROJECTION.

§ **1.** In the preceding six chapters we have been collecting and arranging methods and results in Plane Geometry with a view to their application in the theory of projection. We now resume the treatment of the latter from the point at which it was left in Chapter II.

§ 2. The Vanishing Lines.

We have seen that the projection of a straight line is the common section of a plane through V, the vertex of projection, and that line with the plane of projection.

If these two planes are parallel, however, the projection becomes the line at infinity on the plane of projection.

Let π be the plane of the figure and π' the plane of projection.

Draw through V planes parallel to π and π', meeting respectively π' and π in the lines l', l.

Then l is the straight line in the plane π which is 'projected to infinity' on π', and is on this account called the **Vanishing Line of the plane π.**

Similarly l' is the vanishing line of π'.

Fig. 37 shows the position of the vertex of projection, the vanishing lines, and a point P and its projection P′.

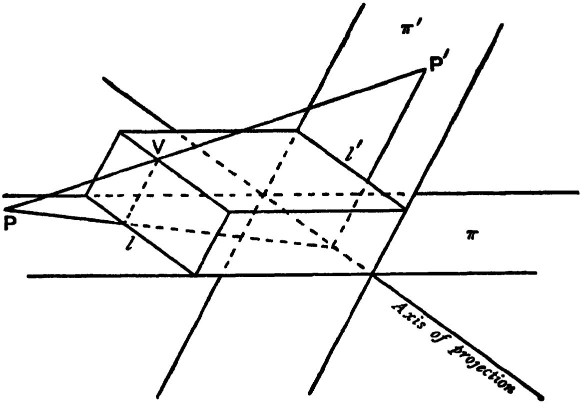


Fig. 37.

§ 3. The Projection of an Angle.

Let the bounding lines of the angle QPR in the plane π cut the vanishing line of that plane in Q and R, and let the planes VPQ, VPR cut π′ in P′Q′, P′R′. Then Q′P′R′ is the projected angle. Since the plane VQR is parallel to π′, VQ and P′Q′, being sections of these planes by the plane VPQ, are parallel.

Similarly VR and P′R′ are parallel.

∴ angle QVR = angle Q′P′R′. It follows that:

The projected angle is equal to the angle subtended at the vertex of projection by the intercept made by the given angle on the vanishing line of its plane.

A very important particular case occurs when the apex of the given angle is on the vanishing line.

The projection of the apex is then a point at infinity, and accordingly the bounding lines of the angle project into lines which have a common point at infinity. Hence **straight lines which intersect on the vanishing line of their plane project into parallel straight lines.**

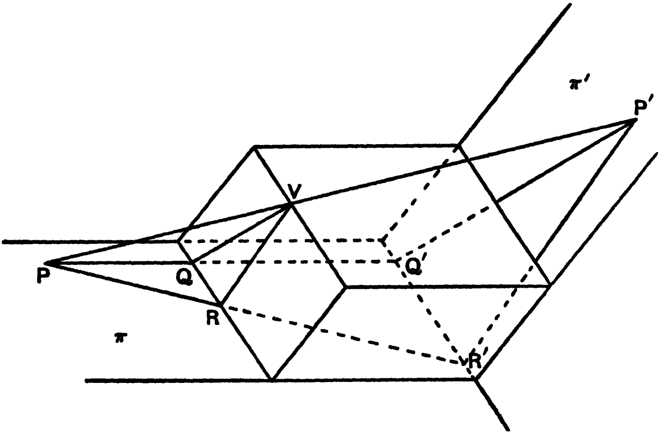


FIG. 38.

Let P be the apex of the angle as before; the planes through V and either bounding line of the given angle will be cut by the plane π′ in straight lines, which are each parallel to VP.

It is a good exercise to draw the figure.

§ 4. PROBLEM. **To project any two angles into angles of given size, and simultaneously a given straight line to infinity.**

Let the bounding lines of the given angles HKL, QPR intersect the line which is to be projected to infinity at H, L and Q, R respectively (Fig. 39). It is required to find a suitable vertex and plane of projection so that these angles may be projected into angles of given size, and that simultaneously HR may be projected to infinity.

On HL, QR describe segments of circles containing angles equal to those into which HKL and QPR are respectively to be projected. Let these intersect at V.

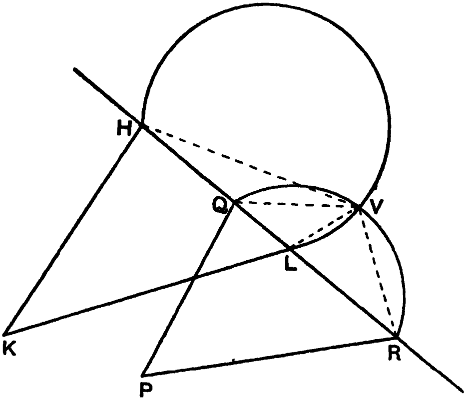


FIG. 39.

Rotate the segments about the line HQLR through any angle, so that the plane of the segments is now inclined to the plane containing HKL, QPR.

If the angles be projected from V on to any plane parallel to VHR, they will be projected into angles equal to HVL, QVR (§ 3); and these are of the required magnitude.

§ 5. PROBLEM. **To project a triangle into a triangle of given size, and simultaneously a given straight line to infinity.**

By the last article we can project the given straight line to infinity, and any two angles of the given triangle into the corresponding pair of the triangle of given size.

Since the sum of the angles of a triangle is two right angles, the third angle of the one will project into the third angle of the other.

The projected triangle will now be of the required *shape*, but not necessarily of the required *size*.

But by moving the plane of projection parallel to itself, we can adjust the size as desired without altering the shape.

Ex. Project a given triangle into an equilateral triangle, the line joining the middle points of a pair of sides being projected to infinity.

§ 6. Problem. **To project any quadrilateral into a square.**

Let FGEH (Fig. 40) be the third diagonal of the quadrilateral ABCD, and let it be intersected by the other two at G and H.

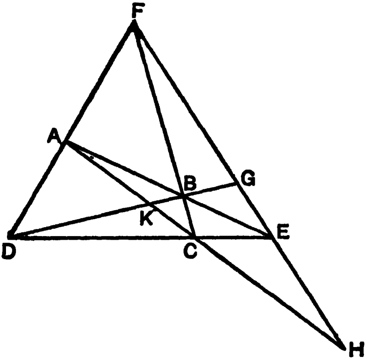


Fig. 40.

Then if FGEH be projected to infinity, ABCD is projected into a parallelogram, as the points of intersection of opposite sides project to infinity.

If also FBE be projected into a right angle, the projected figure will be a rectangle. Lastly, if the angle GKH be projected into a right angle, the diagonals of the projected figure will be at right angles, and the figure will be a square.

By § 4 these projections can be *simultaneously* made.

§ 7. Proof by Projection of the Harmonic Property of the Complete Quadrilateral.

Projecting the third diagonal FGH (Fig. 40) to infinity, the figure becomes a parallelogram (Fig. 41). But the diagonals of a parallelogram bisect one another; ∴ the ranges A′K′C′ ∞ and D′K′B′ ∞′ are harmonic (Chap. IV. § 6), ∞ and ∞′ denoting the points at infinity on A′C′ and B′D′ respectively.

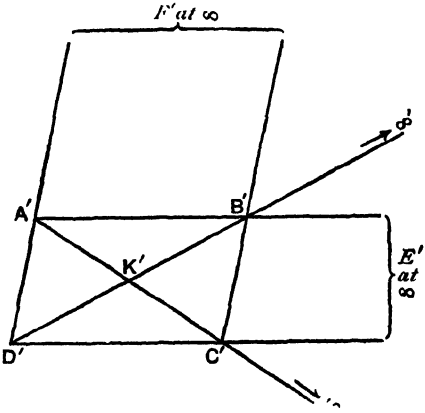


FIG. 41

∴ the ranges which project into these must be harmonic also; ∴ DKBG and AKCH are harmonic (Fig. 40).

∴ the range FGEH, which is a section of the pencil D(AKCH), is harmonic.

§ 8. The Drawing of Projections.

It is desirable that at this stage the student should be able to draw for himself the actual forms of the projected figures. We proceed to explain how this may be done.

It will be found helpful to construct a model in stiff paper of Fig. 42 as an assistance in following the steps of the reasoning. This model takes the form of a hollow box open at the ends, two adjacent faces of which are the planes π, π'.

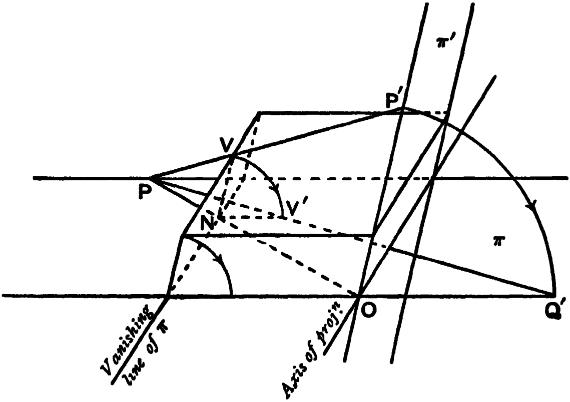


FIG. 42.

Now let π' be turned about its common section with π until it coincides with π. This process is called 'rabatting' the plane of projection into the plane of the figure.

As this is done, the opposite side of the box—the plane through V and the vanishing line of π—will turn about this vanishing line till it also coincides with the plane π, and the whole box is folded flat.

V moves on the arc of a circle whose centre is the foot N of the perpendicular from V on the vanishing line, and whose plane is perpendicular to this line.

Let V′ be its position on arriving at the plane π.

It will be proved that **after rabatment**:

(1) A point and its rabatted projection are collinear with V′, the rabatted vertex.

(2) Corresponding lines intersect on the axis of projection.

Let P be any point on the plane π. The plane PNV will cut π' in P′O, parallel to VN.

P′O is therefore perpendicular to the axis of projection.

P′ moves by rabatment to Q′, where Q′O = P′O, and is perpendicular also to the axis of projection.

But originally P, V, P′ were collinear.

$$\therefore \frac{PN}{PO} = \frac{VN}{P'O} = \frac{V'N}{Q'O}; \quad \therefore \text{P, V}', \text{Q}' \text{ are collinear.}$$

Which proves the first statement.

The second statement is obvious, since before rabatment a line and its projection intersect on the axis, and the position of the point of intersection is unaffected by rotation about the axis.

§ 9. The principles (1) and (2) above enable us to draw the actual form of the projection of any plane figure given the position of the vertex of projection and the planes π, π'. For these determine the position of the vanishing lines (that in the plane of the figure is the most important for our purpose) and the axis of projection.

We have then the following simple construction for the projection of a point.

Let V′ be the rabatted vertex of projection and P the point whose projection is required. Draw any line V′N meeting the vanishing line of π at N. Let PN meet the axis of projection in O. Then OQ′ parallel to V′N will meet PV′ in Q′, the required point.

Note. In practice V′N is often drawn perpendicular to the vanishing line, though this is not *necessary*. So long as OQ′ is parallel to it and the rest of the construction is as above, we shall evidently arrive at the same position for Q′ whatever angle V′N makes with this line.

Referring to § 6, Chapter III., we see that by virtue of properties (1) and (2) **the figure and its rabatted projection are in plane perspective,** the rabatted vertex of projection and the axis of projection being respectively the centre (or pole) of perspective and the axis of perspective.

Drawing Exercises. (Axes of coordinates are rectangular.)

1. Given as rabatted vertex $(-6, 0)$, axis of projection $x=0$ and vanishing line in the plane of the figure $x+4=0$, find the projections of the points $(-2, 0)$, $(-4, 2)$, $(-3, 2)$, $(2, -4)$ and of the lines $x+2y-4=0$, $x+1=0$, $x+y+2=0$, $x-y+1=0$.

Ans. $(+2, 0)$, pt. at ∞ in the direction $y=x$, $(6, 8)$, $(-\frac{2}{3}, -\frac{8}{3})$;
$2x-y+2=0$, $3x=2$, $x-y-2=0$, $3x+2y-2=0$.

2. Given that $(1, -8)$ projects into itself, $(-5, 1)$ projects into $(13, 7)$, $(-3, -2)$ projects to infinity and the line $x+5=0$ projects into $x=13$. Find the axis of projection, rabatted vertex and vanishing lines.

Ans. $x=1$, $(1, 3)$, $x=-3$ and $x=5$.

3. Given that the points $(-2, 1)$, $(-1, 3)$ of one figure correspond respectively to the points $(8, 6)$, $(11, 3)$ of another figure in plane perspective with it and that the axis of perspective is parallel to the y-axis, find the pole of perspective and the vanishing lines.

Ans. $(2, 3)$ and $x=0$, $x=5$.

§ 10. Projection of a Quadrilateral into a Square.

As an example of the application of these methods, let it be required to carry out the details of the problem discussed in § 6.

It was there shown that the required projection would be effected if the angles FBE, GKH (Fig. 43) were projected into right angles and simultaneously FH, the third diagonal, projected to infinity.

FH must therefore be the vanishing line, and by § 4 the vertex of projection must be either of the points of intersection of circles on FE and GH as diameters. Let V be one such point.

Take any line parallel to FH as the axis of projection. Then the point E, where AB cuts the vanishing line, projects to infinity, and is therefore the point at infinity on the projection of AB. This projection must therefore be parallel to VE. Moreover (principle (2), § 8), AB and its projection must cut the axis of projection in the same point. The projection of AB is therefore obtained by drawing a parallel to VE through the point where AB cuts the axis of projection.

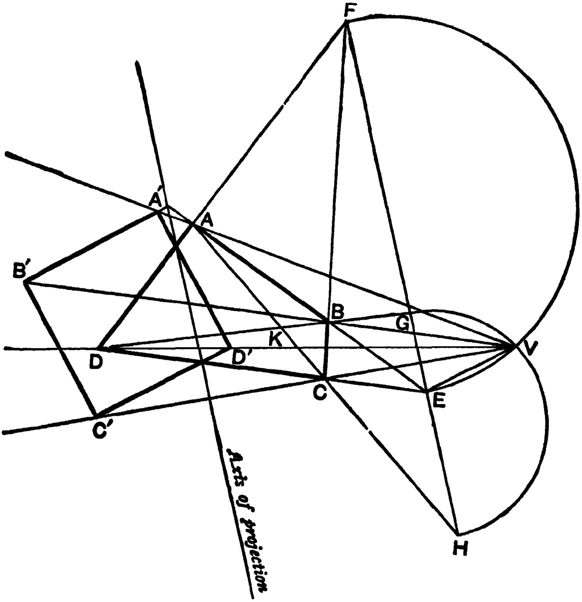


FIG. 43.

Similarly the other sides may be drawn. The accuracy of the drawing may be checked by seeing whether the corresponding points of the original and the projected figures are collinear with V as they should be by principle (1) of § 8. A′B′C′D′ is the required projection.

It should be noticed that the square can be drawn of any specified size when the directions of the sides have been found. For all that is necessary is to place a segment A″B″ of the given length of side, parallel to A′B′, so that its ends are on VA, VB.

§ 11. Parallel and Orthogonal Projection.

In parallel projection the **vertex is at infinity** and the projectors are all parallel (Chap. II. § 3).

The vanishing line in the plane of the figure is also at infinity. But since straight lines which intersect on the vanishing line project into parallel straight lines (§ 3), we have in Parallel Projection

'**Parallel straight lines project into parallel straight lines.**'

This is otherwise evident from Chap. I. § 5, Ex. 6.

In parallel projection the length of a line segment parallel to the axis of projection is unaltered by projection. This follows at once from Ex. 7, § 5, Chap. I.

The lengths of other segments are in general altered.

The fact of the vertex and vanishing line being at infinity introduces sundry modifications in the practical drawing of projections, which make it desirable that this case should be discussed a little more in detail.

First let it be noted that **it is perhaps more convenient here to rabat the plane of the figure into the plane of projection.**

We have, in constructing the projection of a given figure, necessarily the following data or their equivalent:

(1) The axis of projection and the angle α between the plane of the figure and the plane of projection.

(2) The angle β which the projectors make with the plane of projection.

(3) The angle γ which the orthogonal projections of the projectors on the plane of projection make with the axis of projection.

Consider the given diagram (Fig. 44), π and π' being as usual the planes of the figure and of projection respectively.

PP′ is a projector, P being a point of the figure and P′ its projection.

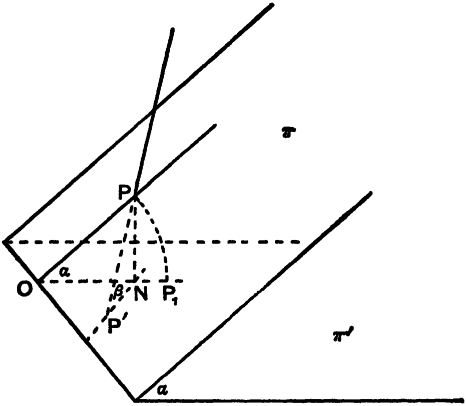


FIG. 44.

If PN is perpendicular to the plane π', NP′ is the orthogonal projection of the projector, and makes an angle γ with the axis of projection.

The angle PP′N is β.

If NO is drawn perpendicular to the axis of projection and OP joined, it is easily proved that OP is also perpendicular to the axis of projection and the angle NOP $= \alpha$.

Now by rabatment P moves in the plane NOP over the arc of a circle, centre O, to the position P_1.

P_1 and P′ are therefore corresponding points.

Remembering then that

(1) the joins of all pairs of corresponding points are parallel to P_1P',

(2) corresponding lines intersect on the axis of projection,

we are in a position to construct the projections of any other points of the figure.

If the projection is orthogonal, P′ *coincides with* N, *and* P_1 *and* N *are corresponding points.*

In any problem in which equivalent data are supplied, it is necessary to obtain the positions of the axis and a pair of corresponding points. This is usually an easy matter with the help of the above figure.

In parallel projection it is convenient for constructional purposes, in obtaining the relations between the parts, to rotate the triangle PNP′ about PN so that P′ comes into the plane OPN.

§ 12. Examples on Orthogonal Projection.

1\. The orthogonal projection of (5, 0) is (3, 0) and of (7, −2) is (4, −2). Find the axis of projection and the angle between the planes.

2\. The orthogonal projection of $4x+5y=20$ is $4x+3y=12$, and the distance between the points (2, 1) and (2, −1) is unaltered by projection. Find the axis of projection and the angle between the planes.

3\. The orthogonal projection of $x=0$ is $y=x$ and of the point (4, 3) is (3, 1). Find the axis and angle as before.

4\. A given plane makes an angle θ with the plane of projection. A straight line on it makes an angle ϕ with the plane of projection.

Draw a diagram of the rabatted figure showing the line, its projection, and the axis of projection.

5\. Solve a similar problem to Ex. 4 if the line makes an angle ϕ with the axis of projection instead of with the plane of projection.

As an example of the methods of § 11 let it be required to construct the orthogonal projection of an equilateral triangle of given side, whose plane makes an angle θ, and one side of which makes an angle a, with the horizontal plane.

Draw a line x (Fig. 45) to represent the axis of projection, and from any point O in it draw ON perpendicular to it.

Draw OP, making the angle NOP equal to θ.

With centre O and any radius OP, describe the arc PP′, cutting ON in P′. If N is the foot of the perpendicular from P on OP′, N and P′ are corresponding points when the figure is rabatted into the plane of projection.

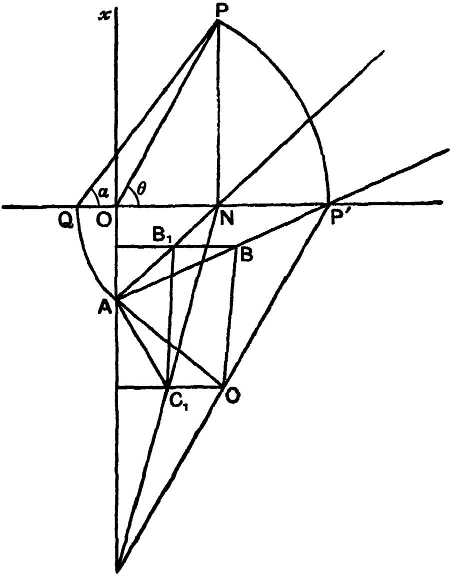


FIG. 45.

From P draw PQ, making the angle PQN equal to α.

With centre N and radius NQ, describe a circle cutting x in A. NA is the orthogonal projection of one side of the triangle.

[This may be easily seen: for supposing x and ON in the horizontal plane, and the triangle PON turned about ON into the vertical plane; also the triangle PNQ rotated about NP till Q coincides with A. We should get a solid figure in which AP could be taken as the direction of the side of the triangle, satisfying the given conditions.]

After rabatment AP becomes AP′.

Along AP′ then measure AB equal to a side of the given triangle, and describe the equilateral triangle ABC.

The construction may be completed by using the principles that the joins of corresponding points are perpendicular to x and the intersections of corresponding sides are on x.

AB_1C_1 is the triangle required.

§ 13. Orthogonal Projection of an Area.

We shall conclude this chapter with an important theorem which will be needed in Chapter IX.

The orthogonal projection of the area of a closed figure is equal to the product of the given area and the cosine of the angle between the plane of the figure and the plane of projection.

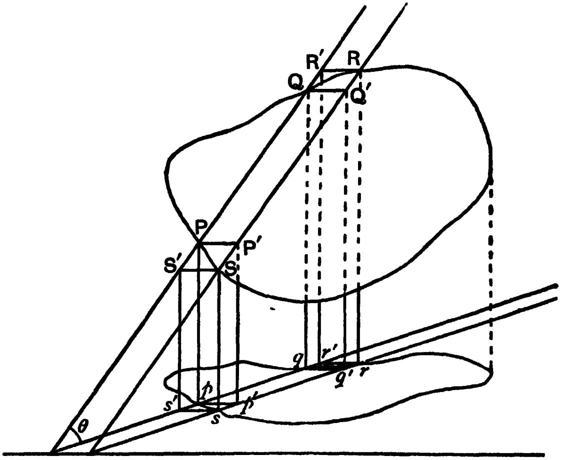


Fig. 46.

Divide the area and its projection into narrow strips by planes perpendicular to the axis of projection.

Let PQRS, *pqrs* (Fig. 46) be a strip and its projection.

From P, Q, R, S, p, q, r, s draw parallels PP', …, pp' … to the axis of projection, forming thus rectangles $PP'Q'Q$, $SS'R'R$, etc.

The area of the projection $pp'q'q$ of any such rectangle $PP'Q'Q$ is obtained evidently by multiplying the latter by the cosine of the angle θ between the planes.

But area $PP'Q'Q <$ area $PQRS <$ area $RR'S'S$,

and area $pp'q'q <$ area $pqrs <$ area $rr's's$.

The second set of extremes are each $\cos\theta$ times the first set.

This holds for all such strips.

Adding, therefore, we have

sum of areas of all 'inscribed' rectangles such as $PP'Q'Q$ < area of whole curvilinear figure < sum of areas of all 'circumscribed' rectangles such as $RR'S'S$,

and a similar result for the projected figure.

Increasing the number of such strips and diminishing their breadth indefinitely, the sums of the two sets of rectangles tend to equality in the original and also in the projected figure.

For in either figure the difference between these two sums is equal to the area of the longest of the strips—a quantity which vanishes ultimately.

But either set in the projected figure is $\cos\theta$ times the corresponding set in the original figure.

∴ ultimately the area of the projected figure is $\cos\theta \times$ area of the original figure.

Examples on Chapter VII.

DRAWING EXERCISES.

1. The lines $y=x$, $x+4y=40$ project respectively into $x=2y$ and $3x+4y=40$. Also the point (4, 4) projects into (12, 6). Find the axis, the position of the rabatted vertex of projection and the vanishing line in the plane of the figure.

2. The line $y=0$ projects into itself, the line $y=\frac{1}{2}x$ into a perpendicular line and the point (2, 1) to infinity. Find the rabatted vertex of projection.

3. The line $x=4$ projects to infinity; the straight lines $y=\frac{1}{4}x$ and $y=-x$ into two straight lines at right angles and the point (4, 1) into a point at infinity in the direction $y=-2x$. Find the rabatted vertex and axis of projection.

4. The pole of perspective of two figures in plane perspective is the origin; the axis of perspective is $x=3$; the points $A_1(1, 0)$ and $A_2(4, 0)$ are corresponding.

Construct the vanishing lines and the triangle $P_2Q_2R_2$ corresponding to $P_1Q_1R_1$, where P_1, Q_1, R_1 are the points $(-2, 1)$, $(0, -1)$, $(1, 2)$ respectively.

5. The following six points are given by their polar coordinates: $A_1(4, 0°)$, $B_1(2, 15°)$, $C_1(3{\cdot}5, 40°)$, $A_2(4{\cdot}5, 0°)$, $B_2(8{\cdot}5, 15°)$, $C_2(7, 40°)$.

Construct (*a*) the axis of perspective of the triangles $A_1B_1C_1$, $A_2B_2C_2$; (*b*) the two vanishing lines; (*c*) the quadrilateral of the second figure which corresponds to the square whose corners are (3, 0°), (3, 90°), (3, 180°), (3, 270°) in the first figure.

6. Shew how to project a set of concurrent straight lines into lines parallel to a given direction, such that three lines of the set project into parallels at specified distances apart.

For example, project $\left\{\begin{array}{c}2x-y-4=0\\ x-y-3=0\\ 2x+y=0\end{array}\right\}$ into parallels to the axis of x, so that the first shall be at unit distance from the second and the second at unit distance from the third.

7. ABC is an equilateral triangle of side 3 inches. AB is produced to D, where AD=4 inches, and AC is produced to E, where AE=4·5 inches. Draw an isosceles right-angled triangle A′B′C′, whose hypotenuse is 2·5 inches, in plane perspective with ABC, the line DE being projected to infinity.

8. A horizontal regular octagon of side 1 inch is projected from a vertex 2 inches vertically above the centre of the octagon. Draw the projection of the octagon on a vertical plane bisecting at right angles the ray from the centre to one of the vertices of the octagon.

9. The axis of x being taken as the vanishing line, construct an equilateral triangle of side 2 units which is in plane perspective with the triangle whose vertices are (1, 2), (2·5, 2·5), (3, 1). Construct the pole and axis of perspective for this case.

10. A pyramid standing on a regular hexagon of side 1 inch as base has its apex 3 inches above the centre of the hexagon. Draw a section of the pyramid by a plane through a line joining two non-consecutive (but not opposite) vertices of the base and cutting the vertical through the apex 1 inch below the latter.

11. Construct a plane perspective relation (*i.e.* pole and axis) which will transform the pairs of lines ($y=0$, $y=x$) and ($y=-x$, $y=\frac{1}{2}x$) into pairs of rectangular lines, the line $x=4$ into the line at infinity, and the segment joining the points (2, 0), (6, 0) into a segment of equal length.

DRAWING EXERCISES IN ORTHOGONAL PROJECTION.

12. Draw the orthogonal projection on a horizontal plane of an equilateral triangle of side 1·5 inches, whose plane makes an angle of 45° with the horizon and one side of which passes through a given point and is inclined at an angle of 30° to the horizontal plane.

13. Construct the orthogonal projection of a square ABCD of side $2\frac{1}{2}$ inches upon a plane through A, making an angle of 38° with the plane of the square, and such that the orthogonal projection of AB makes an angle of 25° with the line of intersection of the two planes.

14. The plane of a regular hexagon of side 2 inches is inclined at 60° to the horizontal plane, and two adjacent vertices of the hexagon are 0·6 inch and 1·6 inches respectively above the horizontal plane. Draw the orthogonal projection of the hexagon on the horizontal plane.

GENERAL EXAMPLES.

15. Prove by projection the theorem of § 5, Chap. III.

[Project the point of concurrence (or for the converse the line of collinearity) to infinity.]

16. Project a quadrilateral into a rhombus.

17. Shew how to cut a pyramid on any quadrilateral base so that the section is a parallelogram.

18. Draw the orthogonal projection on a horizontal plane of a given equilateral triangle whose vertices are at given heights above the plane.

19. Draw the orthogonal projection on a horizontal plane of a given equilateral triangle, the heights above the plane of two vertices being given, and also the angle of inclination of the plane of the triangle to the horizon.

CHAPTER VIII.

THE CONIC.

§ 1. DEF. The **order** of a curve is the number of **points**, real, coincident, or imaginary, in which any straight line cuts the curve.

DEF. The **class** of a curve is the number of tangents, real, coincident, or imaginary, which can be drawn from any point to the curve.

§ 2. PROP. **The circle is a curve of the second order and of the second class.**

Take a straight line cutting the circle in two points. Let it move directly away from the centre but keeping always parallel to itself. The two points of section will travel towards one another till they become indefinitely close together, in which position the line is a tangent. As the line continues to move in the manner indicated, visible points of intersection with the circle cease to exist. For the sake of continuity, however, we say that the line intersects the circle in a pair of imaginary points.

Again, from an external point, we can draw two tangents to a circle. Let the point move towards the centre of the circle. As it approaches the circumference the tangents from it to the circle tend more and more nearly to coincide, and become indistinguishable when the point is on the circumference. When the point moves inside the circle, no visible tangents can be drawn from it to the circle. For the sake of continuity, as before, we say that a pair of tangents exists, though imaginary.

NOTE. These conceptions of imaginary points and tangents serve to bring Pure Geometry into line with Analytical Geometry. In the latter subject the points of intersection of a straight line and a circle and the tangents from a point to a circle are determined in each case by a quadratic equation whatever straight line or point be taken. The real, coincident or imaginary elements, just considered, correspond to the real, equal or imaginary roots of the quadratic.

§ 3. Preservation of Tangency, Order and Class after Projection.

A tangent to a curve projects into a tangent to the projection of the curve.

For a tangent is the straight line joining two infinitely near points on the curve. Its projection will therefore be a straight line joining two infinitely near points on the projected curve. Clearly, also, the number of points in which any straight line cuts a curve is equal to the number of points in which the projection of the line cuts the projection of the curve.

Similarly for tangents from a point to the curve. We conclude that:

The order and class of a curve are unaltered by projection.

§ 4. The Conic.

DEF. **A conic is the projection of a circle, any vertex and plane of projection being taken.**

In other words, a conic is the section by any plane of the double cone (not necessarily right circular) generated by a line of unlimited length which moves so as always to pass through a given point and to intersect the circumference of a circle whose plane does not contain the point.

§ 5. Species of Conics.

In projecting a circle, some straight line in the plane of the circle is projected to infinity. This is the vanishing line of the plane.

We have to consider three cases:

(1) The vanishing line cuts the circle in real points.

(2) „ „ touches the circle.

(3) „ „ cuts the circle in imaginary points.

In the first case, the conic is called a Hyperbola, in the second a Parabola and in the third an Ellipse.

NOTE. The forms of these curves may be realised experimentally in the following simple manner.

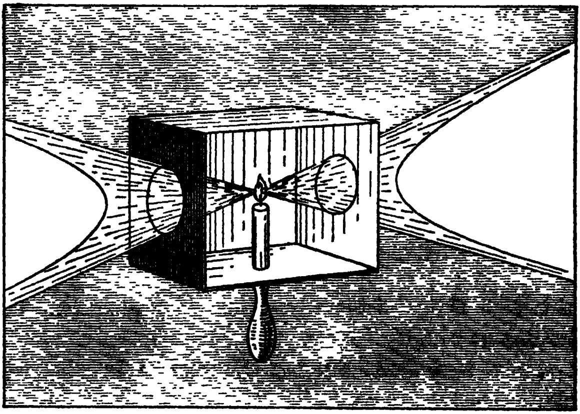


FIG. 47.

In two opposite (parallel) sides of a cardboard box cut a pair of equal circular holes. Place a source of light (the more nearly approximating to a point source the better) at the middle point of the line joining the centres of the holes.

A double oblique cone of light issues from the box, and by holding the apparatus close to a vertical board, as in Fig. 47, and slowly rotating the box, the boundaries of the portions of the board illuminated will assume successively the different species of conics.

It should be noticed in each case what the position of the vanishing line is with respect to either of the circular holes. It is of course the line of intersection of a vertical plane through the bright point with the plane of that circle. (See frontispiece.)

GENERAL FEATURES OF THE CONICS.

§ 6. Case I. **The Hyperbola.**

This curve consists of two branches which are the respective projections of the two portions into which the vanishing line divides the circle.

There are **two** points on this species of conic at infinity, viz. the projections of the points in which the line cuts the circle.

Case II. **The Parabola.**

This curve has one branch only, extending in both directions to infinity.

In this case the vanishing line touches the circle. **Therefore the line at infinity touches the parabola.**

Conversely, **if a conic touches the line at infinity it is a parabola.**

Case III. **The Ellipse.**

This is a closed oval curve and has no real points at infinity.

Fig. 48 shows the three species of conics below the circle figures, of which they are the respective projections. *l* is the vanishing line.

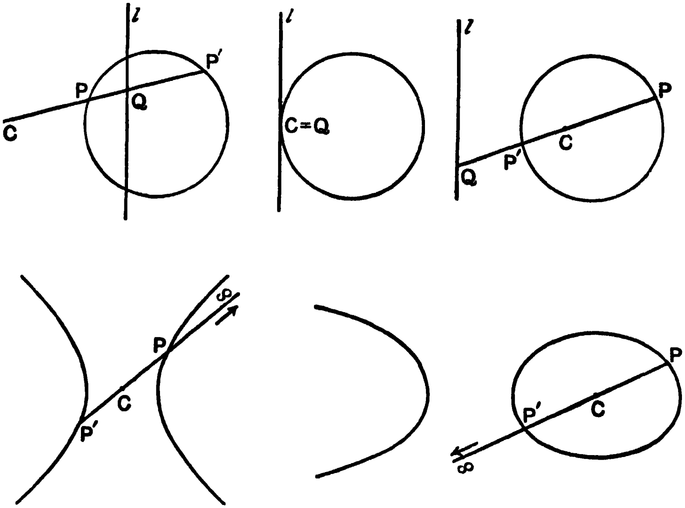


FIG. 48.

§ 7. General Properties.

I. Any straight line cuts a conic in two points, real, coincident, or imaginary.

From any point two tangents, real, coincident, or imaginary, can be drawn to any conic.

These follow at once by projection from the circle (§§ 2 and 3).

A conic is therefore a curve of the second order and of the second class.

II. The locus of the intersection of tangents at the ends of chords of a conic which pass through a fixed point is a straight line.

This theorem has been proved for the circle in Chap. VI § 1. It follows at once for the conic by projection, since tangents project into tangents (Fig. 49). (Compare Fig. 28.)

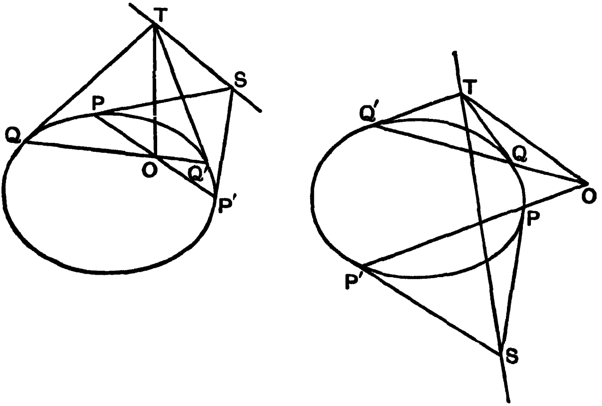


FIG. 49.

DEF. This straight line is called the Polar of the fixed point with respect to the conic. The point itself is called the Pole.

In the figures O and TS are respectively pole and polar for the ellipse, O being any point in the plane of the curve.

III. **If the polar of a point A with respect to a conic passes through B, the polar of B passes through A.**

It was proved in § 3, Chap. VI., that if the locus of intersection of tangents at the ends of chords of a circle through a fixed point A pass through another fixed point B, then the locus of the intersection of tangents at the ends of chords which pass through B is a line which passes through A.

Projecting the theorem in this form we have the above result.

But it should be noticed in this, as in Theorem II., that **the circle proof is not projective step by step.**

We assert that the geometrical **facts** which were proved to be true in the case of the circle hold good also for the conic, since lines and their intersections project into lines and their intersections, and tangents into tangents; but it does not follow that the **steps of the reasoning** by which these facts were established in the case of the circle project into steps of a corresponding argument in the case of the conic.

In Fig. 49 the polar of O passes through T, and the polar of T passes through O.

We may now define **conjugate points** and **conjugate lines** for the conic.

As in the case of the circle, **two points are said to be conjugate with respect to the conic when each lies on the polar of the other.**

Two straight lines are said to be conjugate with respect to the conic when each contains the pole of the other.

For example, in the preceding figure, T and O are conjugate points, and TO, TS are conjugate lines through T.

IV. **On any straight line we can determine an infinite number of pairs of points conjugate with respect to the conic.** For the point on the line *l* which is conjugate to P is the point where the polar of P meets *l*.

So also similarly **through any point we can draw an infinite number of pairs of conjugate lines.**

It follows from this, or by projection from the circle, that

V. **Self-conjugate or self-polar triangles—such that each side is the polar of the opposite angular point—exist for the conic.**

For example, the triangle OPQ in Fig. 50.

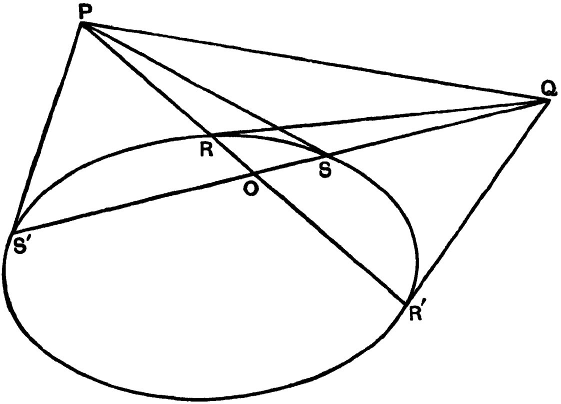


Fig. 50.

VI. **The fundamental harmonic property of pole and polar with respect to the conic,** viz.:

Any chord of a conic through a given point is cut harmonically by the point, the curve and the polar of the point.	**Any straight line, the join of a point on it to its pole, and the tangents from the point to the conic, form a harmonic pencil.**

These follow at once from the theorems of §§ 9, 10, Chap. VI., for the circle, since

(1) pole and polar for the circle project into pole and polar for the conic;

(2) harmonic ranges and pencils project into harmonic ranges and pencils.

In Fig. 50 the ranges PROR′, QSOS′ are harmonic. In fact, if any chord be drawn through a vertex of the triangle OPQ, that vertex and the point in which the chord meets the opposite side of the triangle are harmonically conjugate to the points in which the chord cuts the conic.

This property may also be expressed in terms of conjugate points and lines thus:

A pair of conjugate points together with the points where their join cuts the conic form a harmonic range.	**A pair of conjugate lines together with the tangents from their point of intersection to the conic form a harmonic pencil.**

This property is most important, and will be used constantly in the next few chapters.

VII. **The polars of a harmonic range with respect to a conic form a harmonic pencil.**

Project § 8, Chap. VI., using the fact that harmonic ranges and pencils project into harmonic ranges and pencils.

VIII. Since pole and polar properties are thus transferable to the conic and properties of incidence and tangency are unchanged, **we can transfer the theorems of** §§ 11, 12, 13, Chap. VI., **at once to the conic.**

In the case of § 10, we thereby get a simple **construction for the polar of any point P with respect to a conic by the use of the ruler only.**

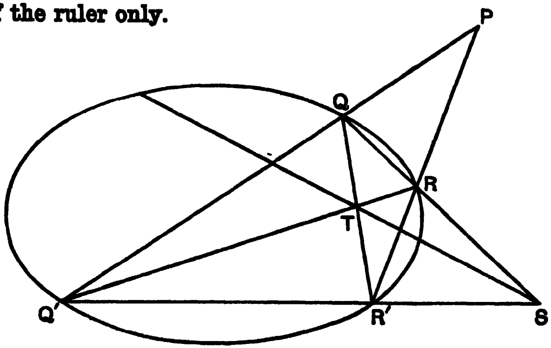


FIG. 51.

Through P draw any two chords PQQ′, PRR′ cutting the conic in Q, Q′, R, R′ (Fig. 51).

The intersections of QR, Q′R′ and of QR′, Q′R give the other two diagonal points S, T of the quadrilateral QQ′R′R, and the join of these is the required polar.

If P is an external point, we have thus a *construction for drawing two tangents to a conic*, for ST cuts the conic at the points of contact.

IX. For the same reasons as in VIII. we can extend to the conic the method of reciprocation explained in Chap. VI.

A fuller discussion of this will be given in Chap. XVIII.

For the present it will suffice to state that we can replace the circle of § 14, Chap. VI., by any conic. This is usually called the **base conic.** This conic, as in the case of the circle, is its own reciprocal.

We can then obtain the reciprocals of theorems

(1) about rectilineal figures only, taking any conic as **base conic**;

(2) about rectilineal figures connected with the conic, taking the conic itself as base,

it being understood that descriptive properties only are considered.

Corresponding theorems will be ranged in dual form, but as before the **proofs will be independent.** It is again suggested that **these should be studied independently first and correlated by reciprocation or otherwise afterwards.**

§ 8. We now proceed to find **what amount of symmetry is possessed by the conic sections.**

In Chap. V. §§ 12, 13, symmetry about a point and symmetry about a line were discussed. A curve possessing the first kind of symmetry when turned in its own plane about the point through two right angles coincides with itself. If it possesses the second kind of symmetry it can be brought into coincidence with itself after rotation through two right angles about the line as an axis.

§ 9. We have seen that the vanishing line and its pole with respect to the circle project respectively into the line at infinity and its pole with respect to the conic.

Def. **The pole of the line at infinity with respect to the conic is called the centre of the conic.**

When the vanishing line does not touch the circle, its pole projects into a finite point in the plane of the conic. When it is a tangent, however, its pole is the point of contact, and therefore, like other points on the vanishing line, projects into a point at infinity.

We have, therefore, that **the Hyperbola and Ellipse have each a single finite centre, but that of the Parabola is at infinity.** The two former curves are for this reason commonly called '**central conics.**'

The appropriateness of the term 'centre' appears from the following property:

All chords through the centre of a conic are bisected there.

Referring to Fig. 48, if l be the vanishing line and C be its pole with respect to the circle, PP′ **any** chord through C cutting the vanishing line at Q is divided harmonically at the points C and Q (Chap. VI. § 9).

In the projected figure the range is still harmonic, but as Q is at infinity, $PC = CP'$.

Central conics therefore possess the first kind of symmetry indicated in § 8.

§ 10. Def. **A straight line passing through the centre of a conic is called a diameter.**

All diameters are therefore bisected at the centre.

From the conjugate property of pole and polar (§ 7, section

III.) it follows that the *diameters of a conic are the polars with respect to it of points on the line at infinity.*

PROP. **The locus of the middle points of any system of parallel chords of a conic is a diameter.**

Considering as before the conic as the projection of a circle, let T′ be any point on the vanishing line l, and let the polar of T′ with respect to the circle cut any chord T′QQ′ in V. Let C be the pole of the vanishing line. Then C projects into the centre of the conic. Fig. 52 shows the circle figure and the projected figure.

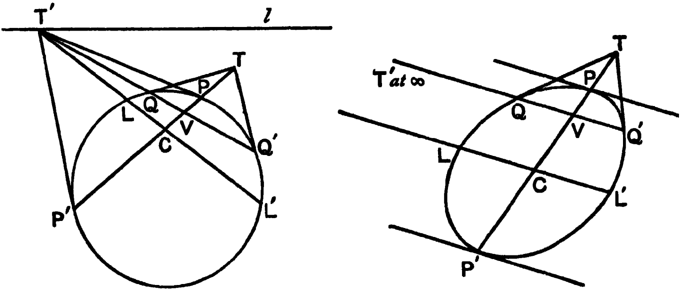


FIG. 52.

From the former we have T′QVQ′ a harmonic range.

The projected range is therefore also harmonic.

But T′ is projected to infinity; therefore in the projected figure QV = VQ′.

Moreover, the system of chords of the circle through T′ projects into a system of chords of the conic passing through the same point at infinity, *i.e.* a system of parallel chords of the conic.

The locus of the middle points of a system of parallel chords is therefore the polar of a point on the line at infinity, *i.e.* **a diameter.**

COR. 1 **The tangents at the ends of any diameter are parallel,** for they are the projections of the tangents from T′.

COR. 2. **The tangents at the ends of any one of the system of parallel chords intersect on the diameter.**

This follows at once from the reciprocal property of pole and polar. For in the circle figure the tangents at Q and Q′ intersect at T on the polar of T′; and the latter becomes by projection the diameter PCP′.

COR. 3. **If two tangents be drawn to a conic, the join of the centre to their point of intersection bisects the chord of contact.**

COR. 4. **If the tangents at the ends of any such chord QQ′ meet in T, and V be the middle point of QQ′, $CV.CT = CP^2$, where P is the point where the diameter CT cuts the conic.** For if P′ be the other end of the diameter CP (Fig. 52), the range P′VPT is harmonic (Chap. VIII. § 7, VI.); and as PC = CP′, we have $CV.CT = CP^2$ (Chap. IV. § 4 (2)).

DEF. **Any one of the bisected chords of the parallel system is called a double ordinate of the diameter which bisects them.**

§ 11. Diameters of a Parabola.

Since the centre of a parabola is at infinity, **all diameters of a parabola are parallel.** Also **each diameter is the locus of the middle points of a system of chords parallel to the tangent at the finite extremity of that diameter.**

It is well to consider the case of the parabola a little more in detail, as some useful properties of the curve follow readily from the same figure.

Let T, T′ (Fig. 53) be the poles with respect to the circle of QQ′, PP′ respectively, and let T′P′ be the vanishing line. The circle projects into a parabola. The chords through T′ become the system of parallel chords bisected by the diameter through

P. Also since TPVP′ is a harmonic range and P′ goes to infinity in the projected figure, TP = PV.

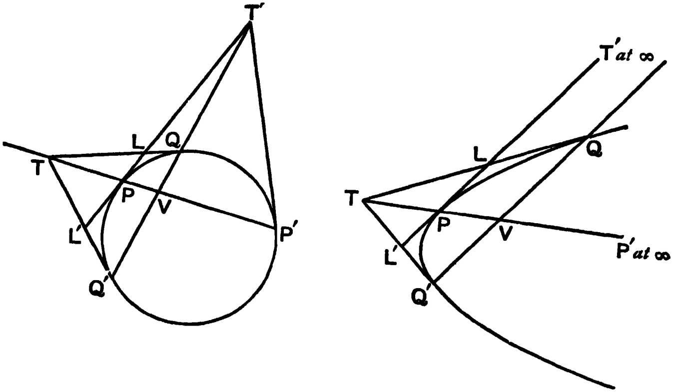


Fig. 53.

Cor. 1. If LL′ be the portion of the tangent at P which is intercepted between TQ, TQ′, then LP = PL′.

Cor. 2. The straight line midway between a point and its polar with respect to a parabola touches the curve.

§ 12. Def. **Conjugate lines through the centre of a conic are called conjugate diameters.**

Prop. **Conjugate diameters of a conic are such that each bisects all chords parallel to the other.**

Take any point X on the vanishing line, and let the polar of X cut this line in Y (Fig. 54), (circle figure).

The polar of Y will pass through X and form with the polar of X and the vanishing line a self-conjugate triangle CXY.

Draw any two chords XQQ′ and YRR′ through X and Y, cutting the polars of X and Y respectively in V and U.

Then XQVQ′ and YRUR′ are harmonic ranges.

Now project the whole figure.

The chords through X and Y project into systems of parallel chords. Since in the projected figures ∞ QVQ′ and ∞ RUR′ are harmonic ranges, QV = VQ′ and RU = UR′. CX and CY being conjugate lines at the centre are conjugate diameters.

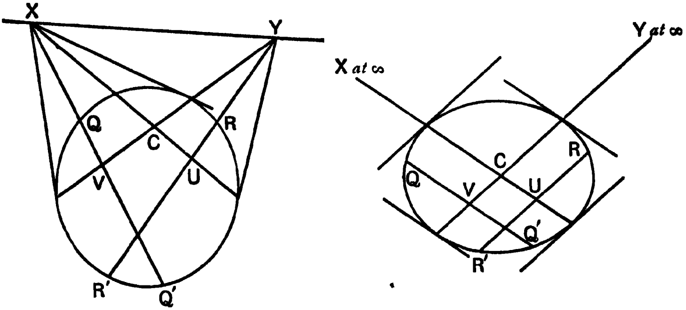


FIG. 54.

COR. **Any chord of a conic and the join of its middle point to the centre determine the directions of a pair of conjugate diameters.**

Ex. Draw the figures for the cases in which the vanishing line cuts, and touches, the circle.

§ **13.** Since an infinite number of pairs of conjugate lines can be drawn through a point, a central conic has an infinite number of pairs of conjugate diameters.

It will now be proved that **there exists one and only one pair of conjugate diameters at right angles.** This pair is called the **axes of the conic.**

Consider, as before, the conic as the projection of a circle. Let V be the vertex of projection, XY the vanishing line (Fig. 55). Describe a circle to have its centre on XY and to pass through V and cut the given circle orthogonally.

[This may be done as follows: Any circle cutting the circle whose centre is O orthogonally and passing through V will also pass through K, the inverse of V with respect to the given circle (Chap. V. § 3). The centre of the required circle is therefore the point of intersection of XY with the straight line bisecting VK at right angles.]

This circle will cut XY in a pair of conjugate points with respect to the circle. For if OX cuts the circle in L, L is the inverse of X with respect to the circle whose centre is O, and as XLY is a right angle, LY is the polar of X. X and Y are therefore conjugate points with respect to this circle, and the polars of X and Y will be a pair of conjugate lines. These will project into conjugate lines with respect to the conic, the angle between which is equal to the angle which XY subtends at V (Chap. VII. § 3), *i.e.* a right angle. Also C, the point of intersection of the polars of X and Y, being the pole of XY, projects into the centre of the conic. One pair of conjugate lines for the circle, therefore, projects into a pair of rectangular conjugate diameters for the conic

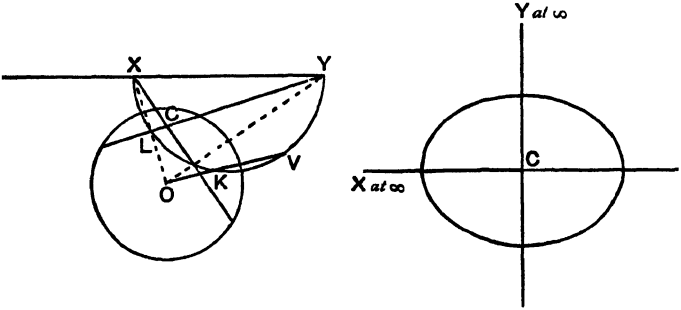


FIG. 55.

Moreover, this construction is unique, for only one circle can be drawn through V having its centre on XY and cutting the given circle orthogonally. Hence **only one pair of rectangular conjugate diameters exists.**

Fig. 55 shows the case in which the circle projects into an ellipse.

The other cases, in which the vanishing line respectively cuts, and touches, the circle, are represented in Figs. 56*a* and *b*.

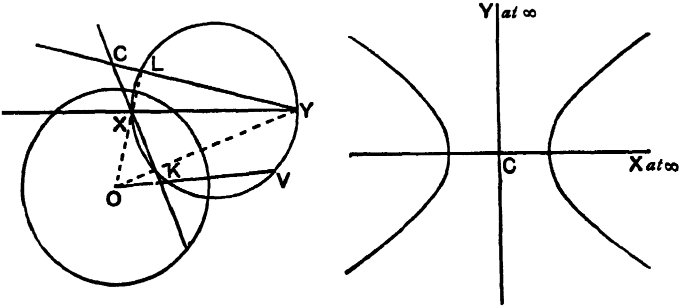


FIG. 56*a*.

In the former (hyperbola), since of the two conjugate lines which project into the axes, one cuts the circle in points which are on opposite sides of the vanishing line, and the other does not

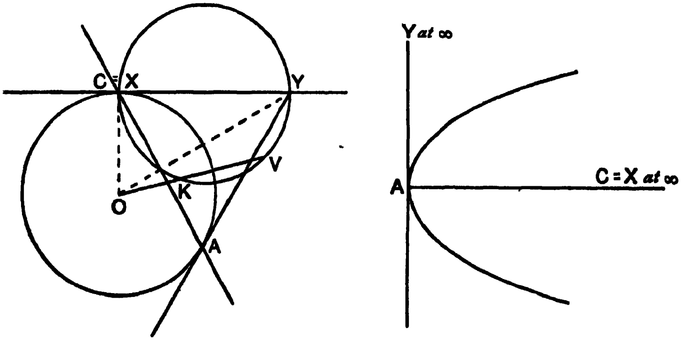


FIG. 56*b*.

cut the circle in real points, one axis in the projected figure cuts both branches of the curve and the other cuts neither.

In the latter case (parabola) one conjugate line is the vanishing line itself, and the other passes through its point of contact with the circle. The parabola, therefore, has but one real finite axis, the other axis being at infinity. If the second conjugate line cut the circle in A, AY is a tangent to the circle and projects into the tangent to the parabola at the finite end of its axis.

§ 14. The Symmetry of the Conic. Lengths of Axes.

From § 12 of this chapter, it follows that each axis is an axis of symmetry, as shown in the figures.

Central conics, therefore, possess double symmetry.

Moreover, the tangents at the points where the axes cut the curve are perpendicular to those axes.

These points are called the **Vertices.**

The **lengths of the axes** are the intercepts made by the curve on them. They are *both* real only in the case of the ellipse.

On this account it is customary to speak of the **major and minor axes** of an ellipse.

But in the case of the hyperbola, since one axis only meets the curve in real points, these terms would be inappropriate. Hence the use of the terms '**transverse**' and '**conjugate**' to denote respectively the axes of the hyperbola which cut the curve in real and imaginary points.

The tangents at the vertices are at right angles to the axes.

Also it is easily seen that the *polar of any point on an axis is at right angles to that axis* (see § 10, Cor. 2). **So that at every point on an axis one pair of conjugate lines are orthogonal**—viz., the axis itself, and a line at right angles to it through the point.

The particular cases of § 10, Cors. 2 and 3, where the diameter is an axis of the curve, should be carefully noted.

§ 15. Asymptotes.

DEF. **An asymptote of a curve is a tangent whose point of contact is at infinity, but which is not itself wholly at infinity.**

Since the points at infinity on a conic are the projections of those in which the vanishing line cuts the circle, it follows that an ellipse can have no real asymptotes.

Since the pole of the line at infinity is the centre of the conic, **the asymptotes are the tangents from the centre to the curve.**

Such tangents are real and finite in the case of the hyperbola, but coincide at infinity in the case of the parabola.

The hyperbola is therefore the only conic possessing real asymptotes. Moreover, the curve lies wholly within one pair of opposite angles between these, since the circle of which it is the projection lies entirely between the tangents at the points where the vanishing line cuts it and these tangents project into the asymptotes.

§ 16. Some Properties of the Asymptotes.

I. **They form with any pair of conjugate diameters a harmonic pencil.**

This follows at once from § 7, section VI., of the present chapter, since the asymptotes are tangents from the centre.

COR. **The axes of a conic bisect the angles between the asymptotes.** For they are a particular orthogonal pair of conjugate diameters. Hence the result follows by § 7, Chap. IV.

II. **If any straight line cut the hyperbola and its asymptotes, the intercepts between the curve and the asymptotes are equal.**

Take any chord QQ′ (Fig. 57) and join its middle point V to the centre C. Then QQ′ is parallel to the diameter conjugate to CV (§ 12, Cor.).

Let DCD′ be this diameter. Since it forms with CV and the asymptotes a harmonic pencil, it will not cut the curve in real points.

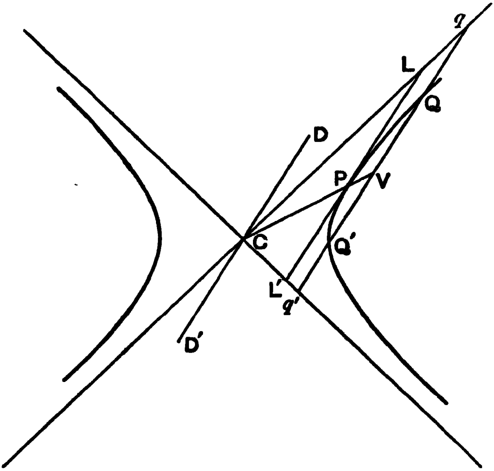


FIG. 57.

But if QQ′ cut the asymptotes in q and q', by the previous prop. $\infty qVq'$ is a harmonic range, and therefore $qV = Vq'$. But $QV = VQ'$.

$$\therefore \quad qQ = q'Q'. \qquad \text{Q.E.D.}$$

COR. **The portion of a tangent intercepted between the asymptotes is bisected at the point of contact.**

For let the chord move parallel to itself till it becomes a tangent in the position LPL′.

We have in the limit when Q and Q′ coincide at P,

$$LP = PL'.$$

Other properties of the asymptotes will be developed later.

§ **17.** It should be noticed that the last article gives an **easy method of obtaining any number of points on a hyper-**

bola, given the asymptotes and one point on the curve. For if Q be the given point, we can draw through it any chord qq' terminated by the asymptotes, and mark off Q′ on it, so that $q\mathrm{Q}=q'\mathrm{Q}'$. Q′ will then be another point on the curve.

§ 18. Particular and Degenerate Cases of the Conic.

A circle is, of course, a particular case of an ellipse whose major and minor axes are of equal length.

A rectangular hyperbola is one whose asymptotes are at right angles.

A pair of straight lines is a degenerate case of a hyperbola when the curve ultimately coincides with its asymptotes. It may be obtained by taking the plane of projection to pass through the vertex of projection.

Two parallel straight lines is a degenerate case of a parabola when the plane of projection is indefinitely close to the plane through the vertex of projection and the vanishing line.

A pair of points will also be seen to be a degenerate form of the conic (see Chap. XIV. § 5).

§ **19.** The student is advised at this stage to make himself familiar with the forms of the different species of conics, and their relations to asymptotes, axes and the like, by drawing to a sufficiently large scale, according to the method of § 8 of the last chapter, the projections of the circle indicated in the examples at the end of this chapter.

The following will serve as an illustration:

Ex. The base of a cone is the horizontal circle $x^2+y^2=25$. *The vertex is* 3 *units vertically above the point* (3, 1). *It is required to draw the section of the cone by a plane passing through the line* $x+y=0$, *and inclined at an angle of* 30° *to the horizon.* [The plane cuts the join of the vertex to (3, 1) *between* these points.]

NOTE. The proviso in brackets is necessary to distinguish the plane in question from the other plane, which can be drawn through the same line at the same elevation to the horizon.

To construct points on the section we must find the position in the plane of the circle of the axis of projection, vanishing line and vertex of projection when the latter is rabatted into this plane.

This is most effectively done by considering the plan and elevation of the cone.

The **plan** is the orthogonal projection on the horizontal plane, which is the plane of the figure to be projected.

The **elevation** is the orthogonal projection on a vertical plane.

The most convenient vertical plane to select is that which passes through the vertex of projection and is perpendicular to the axis of projection, for this is the plane in which the vertex of projection moves during rabatment.

Points in the plan and elevation will be distinguished by the suffixes 1 and 2.

Draw then the axes of x and y, the circle $x^2+y^2=25$ and the line $x+y=0$, which is the axis of projection; mark in the point V_1 (3, 1), which is the plan of the vertex of projection.

Draw V_1N_1 perpendicular to $x+y=0$.

At a convenient distance draw a straight line L_2N_2 at right angles to $x+y=0$, intersecting it at N_2. Set off along it from N_2 a length equal to N_1V_1, and draw a line at right angles 3 units in length. The extremity of this is V_2, the elevation of the vertex of projection.

Draw from N_2 a line inclined at 30° to L_2N_2.

This and N_2N_1 represent the lines in which the plane of projection cuts the vertical and horizontal planes, and are called the **traces** of this plane on these.

Draw V_2L_2 parallel to the former trace and L_2L_1 parallel to N_2N_1.

Then L_2L_1 projects to infinity on the plane of section, and is the vanishing line.

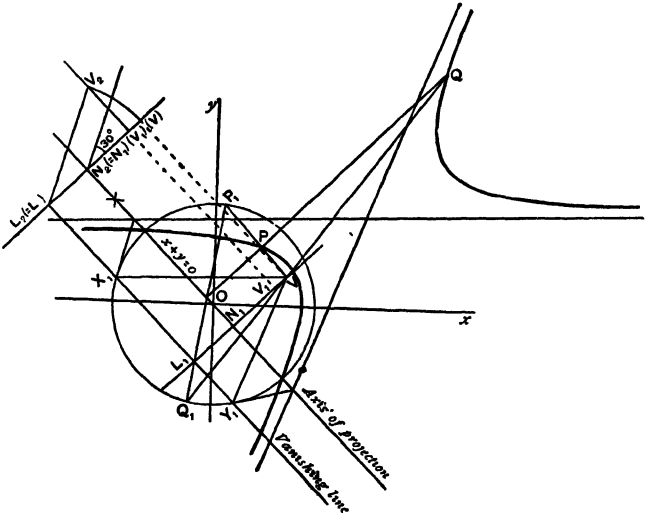


FIG. 58.

V_2 moves in the elevation figure over an arc of a circle, centre L_2, to V the rabatted vertex.

We have therefore all that is required to construct the projected figure.

Take any chord $P_1L_1Q_1$ through L_1.

The projections of P_1 and Q_1 are, by the method of § 8, Chap. VII., the points where VP_1, VQ_1 are cut by a parallel to VL_1 through the intersection of P_1Q_1 with the axis of projection. Any number of other pairs of points can be similarly obtained.

The curve is a hyperbola, since the vanishing line cuts the circle in real points X_1, Y_1.

The asymptotes can be drawn in at once, for they are the parallels to VX_1, VY_1 through the points where the tangents at X_1 and Y_1 meet the axis of projection.

Ex. 1. Construct the angle of inclination of the plane to the horizon, so that the section may be a parabola, and draw the section in this case.

Ex. 2. Construct the section when the angle the plane makes with the horizon is 15°.

§ 20. Having given the positions of the axis of projection, vanishing line and vertex of projection, when rabatted into the plane of the figure, we might employ the construction of § 13 to obtain the axes. This furnishes an independent check upon the accuracy of the previous work, as of course they must bisect the angles between the asymptotes (when such exist).

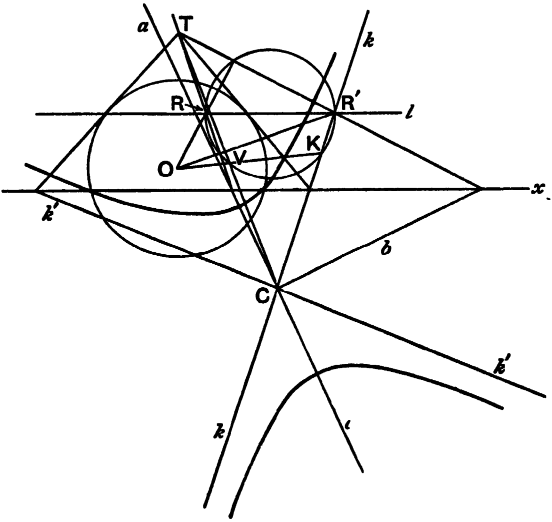


FIG. 59.

In the accompanying figures (59, 60, 61) both axes and asymptotes are shown. In each of these

O is the centre of the circle whose projection is drawn.

V is the rabatted vertex. K its inverse with respect to this circle.

x the axis of projection.

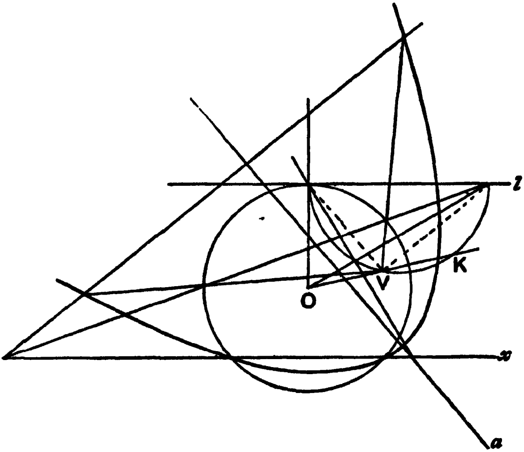


FIG. 60.

l the vanishing line in the plane of the circle.

a, *b* the axes of the conic.

k, *k′* the asymptotes of the conic.

C the centre of the conic.

For convenience we repeat the construction of § 13 for the axes.

Take the inverse of V with respect to the circle, centre O. Through V and this point describe a circle having its centre on the vanishing line. (This cuts the circle, centre O, orthogonally.) Join O to the points R, R′, where this circle cuts the vanishing line. Join R′ and R to the points where these lines cut the circle again, and let the joining lines meet in T. Then TR, TR′ are a pair of conjugate lines with respect to the circle, centre O, such that the intercept RR′ on the vanishing line between them subtends a right angle at V. They therefore project into conjugate lines for the conic at right angles, which, as proved in § 13, are the axes. We get the axes then by joining to C the points where TR, TR′ meet the axis of projection.

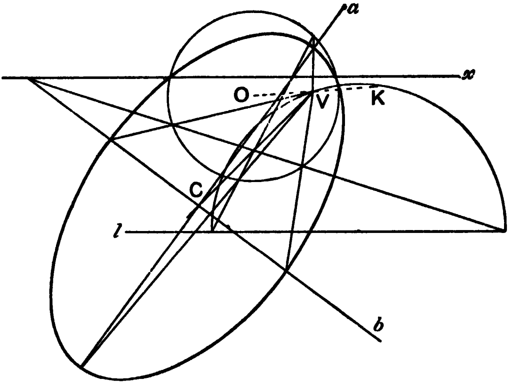


FIG. 61.

Examples on Chapter VIII.

DRAWING EXERCISES.

1. Draw the actual form of the section of a right circular cone of height 10 inches and radius of base 3 inches, made by a plane inclined at 30° to the base, the line of intersection of the planes being a tangent to the base.

2. The circle $x^2+y^2=9$ is projected from a point 5 units vertically above the point (4, 0), the plane of xy being horizontal. Draw the projection of the circle upon the vertical plane through the line $y=2$, this projection being rabatted about the latter line into the horizontal plane.

3. Draw a figure in plane perspective with the circle $x^2+(y-2)^2=4$, being given that the origin and $(-1, 1)$ are corresponding points, that

the lines $y=0$, $y=2x$ are to be projected into perpendicular lines, and the line $y=4$ is to be projected to infinity.

4. The horizontal circle $x^2+y^2=(7\cdot5)^2$ is to be projected from a point 9 units vertically above the point (0, 4) on to a vertical plane passing through $2x+y-6=0$. Construct the asymptotes of the projection and one point on the curve.

5. A circle of radius 2 inches is projected upon a plane through its centre C making an angle of 45° with the plane of the circle from a vertex V, which is such that CV is $2\frac{1}{2}$ inches, and which lies in a plane perpendicular both to the plane of the circle and the plane of projection and makes with these planes angles of 20° and 25° respectively. Draw the projection.

6. Draw a figure in plane perspective with the circle $(x-4)^2+y^2=4$, given that the axis of y is to be projected to infinity, that the pairs of lines ($x+y=3$, $4x-3y=12$) and ($x-3y=3$, $4x+y=12$) are to become pairs of lines at right angles, and that the distance between the points (2, 0), (6, 0) is to remain unaltered by projection.

7. The angles AOB, BOC, COD all measured in the same sense are 30°, 15°, 15° respectively, and (OB, OC), (OA, OD) are pairs of conjugate diameters of a hyperbola. Construct a plane perspective relation which will transform this hyperbola into a circle, and at the same time the line perpendicular to OB at a distance 2 inches from O into the line at infinity, the distance between the two points on OA each one inch from O transforming into a length 1·5 inches.

8. The base of a right circular cone is $1\frac{1}{2}$ inches in radius and the height of the cone is 3 inches. A generator of the cone is bisected at P: draw a section of the cone by a plane which contains P and a diameter of the base which is inclined at 45° to the radius to the foot of the bisected generator.

9. A square ABCD is described about a circle of diameter 3 inches whose plane is horizontal. The figure is projected from a point O vertically above D, and distant 2 inches from D, upon a plane through AB at right angles to OA. Draw the projection of the figure.

10. Draw circles, centres C and C′, with radii 2 in. and 1·5 in.; the distance CC′ being 2·5 in. Consider the circles as bases of two cones with a common vertex V, 3 in. vertically above C. Draw any plane section yielding two parabolas.

RIDERS.

11. PQ, PR are two tangents to a conic, PO bisecting the angle QPR meets QR in O. Through O any chord Q′OR′ of the conic is drawn: prove that the angle Q′PR′ is also bisected by PO.

12. If a parallelogram circumscribes an ellipse, its diagonals are the directions of conjugate diameters.

13. A conic is drawn touching a given ellipse at the extremities P, D of conjugate diameters, and passing through the centre C. Prove that the tangent at C is parallel to PD.

14. The polar of a point T is parallel to the diameter conjugate to CT, C being the centre of the conic.

15. If two tangents to a hyperbola from an external point touch the same branch of the curve, the point must be situated in the space between the curve and its asymptotes: if outside, they will touch opposite branches. Investigate this from the point of view of projection.

16. If P, P′ be the extremities of a diameter, and the tangent at Q meets the tangents at P, P′ in L, L′ respectively, prove that

$$\frac{PL}{P'L'}=\frac{LQ}{L'Q'}.$$

17. The chords which join any point on a central conic to the ends of a diameter are parallel to a pair of conjugate diameters. (Such are called '*supplemental chords.*')

18. Through two fixed points lines are drawn parallel to conjugate diameters of a conic. Show that the locus of their intersection is a similar and similarly situated conic.

19. The lines drawn from the ends of any chord of a hyperbola, parallel to the line joining the centre to the middle point of the chord, to meet the asymptotes, are equal.

20. TP, TQ are two tangents to a conic and CP′, CQ′ are radii from the centre respectively parallel to them. Prove that PQ is parallel to P′Q′.

21. Through C, the middle point of a chord AB of a conic, two other chords PCP′, QCQ′ are drawn. PQ′ and P′Q cut AB in X and Y respectively. Prove that AX = BY.

22. Prove that the locus of the centres of ellipses inscribed in the same quadrilateral is a straight line (Newton).

23. If a triangle self-conjugate with respect to a conic be given and one point on (or tangent to) the conic, then three other points (or tangents) are known.

CHAPTER IX.

CARNOT'S THEOREM.

§ 1. Carnot's Theorem.

If the sides of a triangle ABC cut a conic in X_1, X_2, Y_1, Y_2, Z_1, Z_2, then

$$AZ_1 . AZ_2 . BX_1 . BX_2 . CY_1 . CY_2 = AY_2 . AY_1 . CX_2 . CX_1 . BZ_2 . BZ_1$$

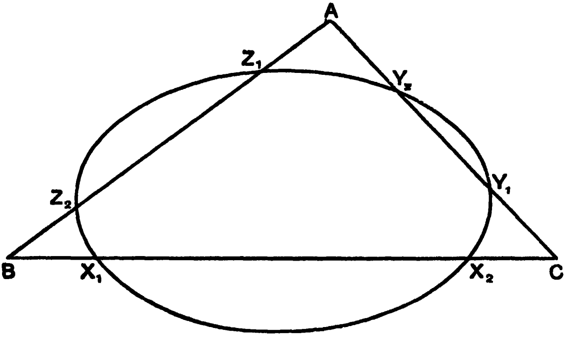


FIG. 62*a*.

This (Fig. 62*a*) is evidently true for the circle, for

$$AZ_1 . AZ_2 = AY_2 . AY_1, \text{ etc.}$$

It remains to prove that the relation is unaltered by projection.

Let small letters refer to the circle diagram and capitals, as in the figure, to its projection.

Let V be the vertex of projection, and consider the projection of bcx_1x_2 (Fig. 62*b*).

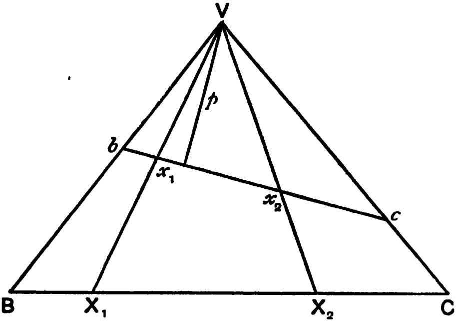


FIG. 62*b*.

Let p be the length of the perpendicular from V on bc.

Then $bx_1 . p =$ twice $\triangle b\mathrm{V}x_1 = b\mathrm{V} . \mathrm{V}x_1 \sin b\mathrm{V}x_1$, and similar relations hold for bx_2, cx_1 and cx_2.

From these, $$\frac{bx_1 . bx_2}{cx_2 . cx_1} = \frac{b\mathrm{V}^2 . \mathrm{V}x_1 . \mathrm{V}x_2 \sin b\mathrm{V}x_1 \sin b\mathrm{V}x_2}{c\mathrm{V}^2 . \mathrm{V}x_2 . \mathrm{V}x_1 \sin c\mathrm{V}x_1 \sin c\mathrm{V}x_2}.$$

$$\therefore \frac{az_1 . az_2}{bz_2 . bz_1} . \frac{bx_1 . bx_2}{cx_2 . cx_1} . \frac{cy_1 . cy_2}{ay_2 . ay_1}$$
$$= \frac{b\mathrm{V}^2}{c\mathrm{V}^2} . \frac{c\mathrm{V}^2}{a\mathrm{V}^2} . \frac{a\mathrm{V}^2}{b\mathrm{V}^2} . \frac{\sin a\mathrm{V}z_1 \sin a\mathrm{V}z_2}{\sin b\mathrm{V}z_2 \sin b\mathrm{V}z_1}$$
$$\times \frac{\sin b\mathrm{V}x_1 . \sin b\mathrm{V}x_2}{\sin c\mathrm{V}x_2 . \sin c\mathrm{V}x_1} . \frac{\sin c\mathrm{V}y_1 . \sin c\mathrm{V}y_2}{\sin a\mathrm{V}y_2 . \sin a\mathrm{V}y_1},$$

an expression involving the mutual inclinations of the rays only.

Similarly $\frac{\mathrm{AZ}_1 . \mathrm{AZ}_2}{\mathrm{BZ}_2 . \mathrm{BZ}_1} . \frac{\mathrm{BX}_1 . \mathrm{BX}_2}{\mathrm{CX}_2 . \mathrm{CX}_1} . \frac{\mathrm{CY}_1 . \mathrm{CY}_2}{\mathrm{AY}_2 . \mathrm{AY}_1} =$ the same expression, since BVX_1 and $b\mathrm{V}x_1$ are the same angle, etc.

∴ the two products are equal. But the small-letter product = unity for the circle.

∴ the required result follows for the conic.

§ 2. The theorem of the last article is very important and admits of a number of applications. Perhaps the most useful of these is (the so-called) Newton's theorem, which may be enunciated as follows:

If chords be drawn through any point in the plane of a conic to meet the curve, the ratio of the rectangles of their segments depends only upon the directions in which the chords are drawn, not upon the position of the particular point through which they are drawn.

We have to show, in fact, that if TPP', tpp' and TQQ', tqq' (Fig. 63) be any two pairs of parallel chords, then

$$\frac{\mathsf{TP}.\mathsf{TP}'}{\mathsf{TQ}.\mathsf{TQ}'}=\frac{tp.tp'}{tq.tq'}.$$

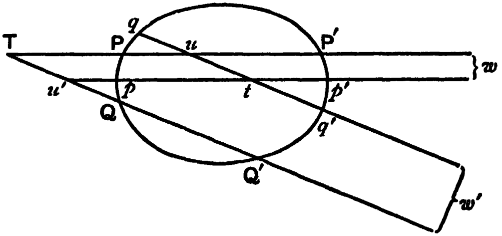


Fig. 63.

Let w, w' be the infinitely distant points of intersection of the parallels, and let qq' meet PP' at u and pp' meet QQ' at u'.

Then, applying Carnot's theorem to the triangle tuw,

$$\frac{uq.uq'}{tq.tq'}\cdot\frac{tp.tp'}{wp.wp'}\cdot\frac{w\mathsf{P}.w\mathsf{P}'}{u\mathsf{P}.u\mathsf{P}'}=1.$$

But $w\mathsf{P}=wp$ and $w\mathsf{P}'=wp'$.

$$\therefore\ \frac{tp.tp'}{tq.tq'}=\frac{u\mathsf{P}.u\mathsf{P}'}{uq.uq'}.$$

Similarly, applying the same theorem to the triangle Tuw', we have

$$\frac{\mathrm{TP}.\mathrm{TP}'}{u\mathrm{P}.u\mathrm{P}'}\cdot\frac{uq.uq'}{w'q.w'q'}\cdot\frac{w'\mathrm{Q}.w'\mathrm{Q}'}{\mathrm{TQ}.\mathrm{TQ}'}=1.$$

Also $w'q=w'\mathrm{Q}$ and $w'\mathrm{Q}'=w'q'$.

$$\therefore\ \frac{\mathrm{TP}.\mathrm{TP}'}{\mathrm{TQ}.\mathrm{TQ}'}=\frac{u\mathrm{P}.u\mathrm{P}'}{uq.uq'}.$$

$$\therefore\ \frac{\mathrm{TP}.\mathrm{TP}'}{\mathrm{TQ}.\mathrm{TQ}'}=\frac{tp.tp'}{tq.tq'}.$$

Cor. If the chords move parallel to themselves, until they become tangents, the rectangles of the segments of the chords become the **squares on the parallel tangents.**

Hence the ratio of the rectangles of the segments of the chords = the ratio of the squares on the parallel tangents.

§ **3.** We proceed to apply this to some very important particular cases.

It has been shown that in a central conic a diameter bisects all chords parallel to its conjugate.

Such chords are sometimes called '*double ordinates*' of the bisecting diameter.

I. Prop. **In any central conic, if QVQ′ be a double ordinate of the diameter PCP′, $\frac{\mathrm{QV}^2}{\mathrm{PV}.\mathrm{VP}'}$ is a ratio which is constant for all such ordinates.**

Consider a diameter and its family of double ordinates. The above theorem then follows immediately from § 2, taking T at V (Figs. 64*a* and *b*), but *there is a distinction between the cases of the ellipse and hyperbola.*

In the ellipse it appears by taking T at C that the constant ratio is equal to $\frac{\mathrm{CD}^2}{\mathrm{CP}^2}$, where CD is the length of the semiiameter conjugate to CP (Fig. 64*a*) the **lengths of both dia-ieters are here definite.** We have, therefore, **for the ellipse,**

$$\frac{QV^2}{PV \, . \, VP'} = \frac{CD^2}{CP^2}.$$

In the **hyperbola,** however, the ratio $\frac{QV^2}{PV \, . \, VP'}$ is negative, nd CD does not meet the curve in real points.

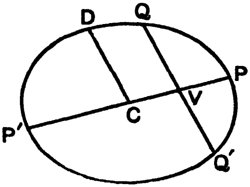


FIG. 64*a*.

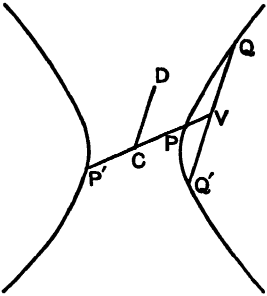


FIG. 64*b*.

The truth of the latter statement is manifest on examining 'ig. 56*a*. For **any** pair of conjugate lines through C with espect to the circle will project into conjugate diameters of he hyperbola. Drawing any line through C, the conjugate ne is the line joining C to its pole; and this pole is inside he circle if the line is outside, and vice versa. It follows hat one of the pair of conjugate lines through C cuts the ircle, while the other does not. Accordingly, if a diameter f the hyperbola be drawn cutting the curve, the conjugate iameter will not cut the curve, and therefore **cannot be said o have any definite length.**

It is convenient, nevertheless, to **measure off a real length** D along this diameter, such that

$$\frac{QV^2}{PV \, . \, VP'} = -\frac{CD^2}{CP^2} \text{ (Fig. 64}b\text{).}$$

CD thus defined will then, **in the case of the hyperbola,** in future be called the length of the semi-diameter conjugate to CP.

We can now add a further corollary to § 2, viz. that **if two tangents be drawn to a central conic the ratio of their lengths is equal to that of the lengths of the parallel semi-diameters.** For the ratio of the squares in each case is equal to the ratio of the rectangles of the segments of parallel chords.

II. Prop. **In the parabola, if QVQ′ be a double ordinate of a diameter PV, QV^2/PV is a constant for all ordinates of that diameter.**

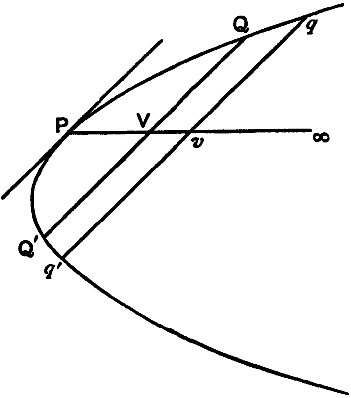


Fig. 65.

Let QVQ′, *qvq′* (Fig. 65) be any two ordinates of the diameter PV. Then, since PV meets the conic again at infinity, Newton's theorem gives

$$\frac{QV^2}{PV \, . \, V\infty} = \frac{qv^2}{Pv \, . \, v\infty}. \quad \text{'Also } \frac{V\infty}{v\infty} = \text{unity}.$$

$$\therefore \ \frac{QV^2}{PV} = \frac{qv^2}{Pv}. \quad \therefore \ \frac{QV^2}{PV} = \text{constant}.$$

This constant is of course a length, and it will be seen shortly how to determine it.

§ 4. The Cartesian Equations of the different species of Conic referred to their Principal Axes.

Take the particular cases of Theorems I. and II. of § 3, when the diameters are the axes of the curves.

Let PNP′ (Fig. 66) be the double ordinate of ACA′, the transverse or major axis of a central conic. Then PNP′ is perpendicular to ACA′. A and A′ are the vertices of the curve.

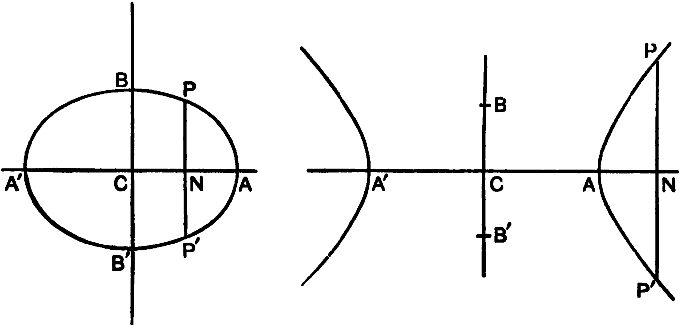


FIG. 66.

We have $\frac{\mathrm{PN}^2}{\mathrm{AN}\,.\,\mathrm{NA}'}=\frac{\mathrm{BC}^2}{\mathrm{AC}^2}$ for the ellipse or $-\frac{\mathrm{BC}^2}{\mathrm{AC}^2}$ for the hyperbola, BC being the actual length of the semi-minor axis in the former case, and the length **measured off** along the conjugate axis in the latter, as explained in § 3.

Taking ACA′, BCB′ as the axes of x and y respectively, if P be (x, y), AN . NA′ $= a^2 - x^2$ in both cases, and the above theorems give as the required Cartesian equations referred to the principal axes,

$$\frac{y^2}{a^2-x^2}=\frac{b^2}{a^2} \quad \text{or} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{(ellipse)}$$

$$\text{and} \quad \frac{y^2}{a^2-x^2}=-\frac{b^2}{a^2} \quad \text{or} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{(hyperbola).}$$

Ex. **Plot (*a*) the ellipse, (*b*) the hyperbola for which $a=4$, $b=3$.**

For the parabola, take the axis of the curve as the axis of x, and the tangent at its extremity A (the vertex) as the axis of y. Then PNP′ (Fig. 67), being the double ordinate and P(x, y) as before, we have $\frac{PN^2}{AN}$ = constant, or in symbols y^2 = constant × x, the required Cartesian equation.

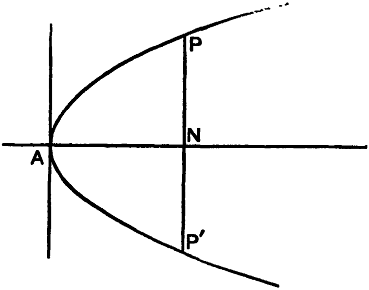


FIG. 67.

Ex. Plot the parabolas for which this constant (*a*) equals 2, (*b*) equals $\frac{1}{2}$.

The properties

$$\frac{PN^2}{AN . NA'} = \frac{BC^2}{AC^2} \text{ (for the ellipse)},$$

$$\frac{PN^2}{AN . NA'} = -\frac{BC^2}{AC^2} \text{ (for the hyperbola)},$$

$$PN^2 = \text{constant} \times AN \text{ (for the parabola)},$$

are fundamental.

It has just been shown that they lead at once to Cartesian equations, from which the curves can be at once plotted without ambiguity, and **accordingly they may be taken as defining, or characteristic, properties of the several species of conics.**

Cor. If PM be an ordinate of the minor axis of an ellipse, it may be shown in like manner (or deduced from the above) that

$$\frac{PM^2}{BM . MB'}=\frac{AC^2}{BC^2}.$$

A similar relation holds for the hyperbola.

It should be noted that equations of the same form are obtained when any pair of conjugate diameters CP, CD (in the case of a central conic), or a diameter and the tangent at its extremity (in the case of a parabola), are taken as *oblique* axes of x and y. We have then, putting $CV=x$, $QV=y$, $CP=a'$, $CD=b'$, as the symbolic forms of Prop. I. § 3,

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1 \text{ (ellipse)}, \quad \frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1 \text{ (hyperbola)}.$$

Analytically, we have from the last equation when $x=0$, $y=\pm b'i$, where i is $\sqrt{-1}$, so that in the case of the hyperbola we may consider that the diameter CD meets the curve in **imaginary** points at a distance $\pm b'i$ from the centre, where $b'=CD$.

The form for the parabola is the same as above, where PV is x and QV is y, but, of course, the constant is different.

§ 5. We are now in a position to resume the discussion of **asymptotic properties of the hyperbola.**

III. **The length of the portion of the tangent at P intercepted between the asymptotes is equal to 2CD,** where CD is the length **measured along** the conjugate diameter to CP (§ 3, I.). We may conveniently call CD the "pseudo-conjugate" diameter to CP.

Let LPL′ (Fig. 68) be the intercept in question, and QVQ′ a double ordinate of the diameter cutting the asymptotes at q, q'.

By Newton's theorem, since the asymptote may be regarded as a tangent from either L or q,

$$\frac{LP^2}{qQ \cdot qQ'} = \frac{L\infty^2}{q\infty^2} = 1. \quad \therefore \ LP^2 = qQ \cdot qQ' = qV^2 - QV^2.$$

But, by similar triangles, $\dfrac{qV}{LP} = \dfrac{CV}{CP}$.

Hence $\dfrac{qV^2 - LP^2}{LP^2} = \dfrac{CV^2 - CP^2}{CP^2} = \dfrac{PV \cdot P'V}{CP^2} = \dfrac{QV^2}{CD^2}$ (by § 3).

But, as above, $qV^2 - LP^2 = QV^2$.

$$\therefore \ LP = CD \quad \text{and} \quad LL' = 2CD.$$

This property may also be proved as in Ex. 2 at the end of this chapter.

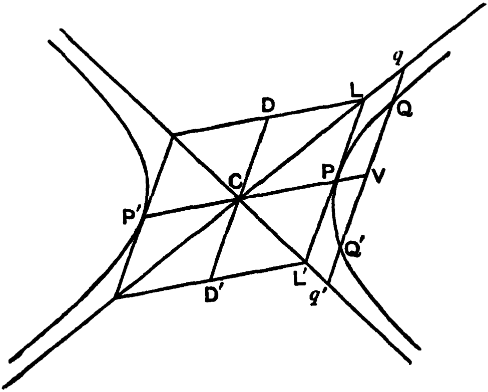


FIG. 68.

IV. It follows that if D′ be taken on the other side of C on the conjugate diameter to CP, so that CD′ = CD, **the tangents at P and P′ together with parallels through D and D′ to CP, form a parallelogram of which the asymptotes are the diagonals.** (Fig. 68.)

The particular case of this, in which the diameters are the principal axes ACA′, BCB′, is very important inasmuch as it gives us the easiest **construction for drawing the asymptotes** as follows:

Mark the points A, A′, B, B′ as in § 4 on the axes (Fig. 69), and draw parallels to the axes through them, thus forming a rectangle.

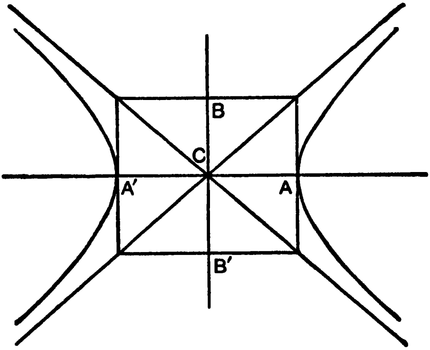


FIG. 69.

The diagonals of this rectangle (which of course intersect at the centre) are the asymptotes.

We shall speak of BC as the length of the (pseudo-) conjugate axis of the hyperbola, though B is not a point on the curve.

DEF. **The hyperbola which has the same asymptotes, but BCB′ as transverse and ACA′ as conjugate axis, is called the conjugate hyperbola.** It is confined to the other pair of angles between the asymptotes and, as is easily seen, touches at D and D′, the above-mentioned parallelogram.

V. **The area of the triangle cut off from the asymptotes by any tangent is constant.**

First Proof. Let a be half the angle between the asymptotes

(Fig. 70) and LR, L′R′ be perpendicular to the transverse **axis.** Then $CL \cos\alpha = CR$, $CL' \cos\alpha = CR'$, and since $LP = PL'$,

$$CR + CR' = 2CN.$$

Hence $$CL + CL' = 2CN \sec\alpha.$$

Similarly $$CL - CL' = 2PN \operatorname{cosec}\alpha.$$

$$\therefore\ CL \,.\, CL' = CN^2 \sec^2\alpha - PN^2 \operatorname{cosec}^2\alpha.$$

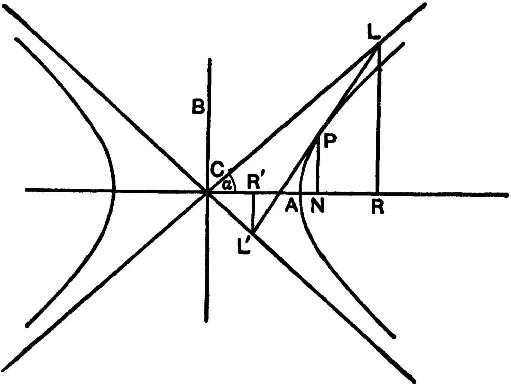


FIG. 70.

But $\dfrac{PN^2}{AN \,.\, NA'} = -\dfrac{BC^2}{AC^2}$ and $AN \,.\, NA' = AC^2 - CN^2$.

$$\therefore\ \frac{PN^2}{CN^2 - AC^2} = \frac{BC^2}{AC^2} = \tan^2\alpha = \frac{\sec^2\alpha}{\operatorname{cosec}^2\alpha}.$$

$$\therefore\ CN^2 \sec^2\alpha - PN^2 \operatorname{cosec}^2\alpha = AC^2 \sec^2\alpha$$

and $$CL \,.\, CL' = AC^2 \sec^2\alpha = \text{constant}.$$

But the angle LCL′ is constant.

∴ the triangle LCL′ is of constant area.

Second Proof. Let LPL′, MQM′ be tangents at two neai points P, Q on the hyperbola, cutting the asymptotes at L, L′ M, M′ (Fig 71), and one another at T.

Then $LP = PL'$ and $MQ = QM'$.

Denoting the opposite angles at T by θ,

$$\triangle \mathrm{LTM} - \triangle \mathrm{L'TM'} = \tfrac{1}{2}(\mathrm{TL}\,.\,\mathrm{TM} - \mathrm{TL'}\,.\,\mathrm{TM'})\sin\theta$$
$$= \tfrac{1}{2}\{(\mathrm{PL}+\mathrm{PT})(\mathrm{QM}-\mathrm{QT}) - (\mathrm{PL}-\mathrm{PT})(\mathrm{QM}+\mathrm{QT})\}\sin\theta$$
$$= (\mathrm{PT}\,.\,\mathrm{QM} - \mathrm{QT}\,.\,\mathrm{PL})\sin\theta.$$

But applying Carnot's theorem to the triangle LTM,

$$\frac{\mathrm{M}\infty\,.\,\mathrm{LP}\,.\,\mathrm{TQ}}{\mathrm{MQ}\,.\,\mathrm{TP}\,.\,\mathrm{L}\infty} = 1, \qquad \therefore\ \mathrm{LP}\,.\,\mathrm{TQ} = \mathrm{MQ}\,.\,\mathrm{TP}.$$

Therefore the triangles LTM, L'TM' are equal.

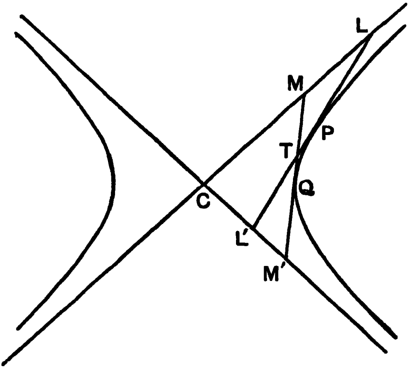


FIG. 71.

Hence $\triangle$ CMM' $=\triangle$ CLL', and any tangent cuts off from the asymptotes a triangle of constant area. To find this constant, consider the triangle cut off by the tangent at A; the area of this is (by IV.) AC . BC.

Ex. Verify that the two expressions found for this constant area agree.

§ 6. Properties of a Diameter of a Hyperbola and its Pseudo-conjugate.

(*a*) It follows at once from the above, that the area of the parallelogram in IV. is constant.

For it is obviously four times that of the triangle LCL'.

(*b*) The difference of the squares of these diameters is constant.

For (Fig. 72) PD bisects CL in K. If LZ be perpendicular to PD,

$$CP^2 - CD^2 = DL^2 - LP^2 = 2DP \, . \, KZ.$$

But $DP = CL'$ and $KZ = \frac{1}{2} CL \cos 2\alpha$, where α has the same meaning as in the last article.

$$\therefore \; CP^2 - CD^2 = CL \, . \, CL' \cos 2\alpha = \text{constant.}$$

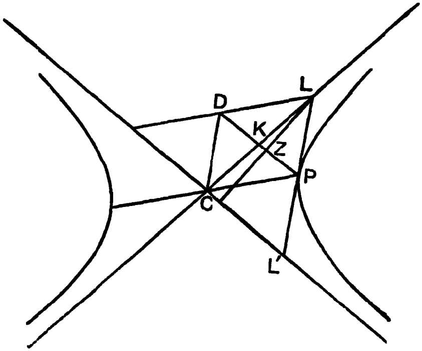


FIG. 72.

This constant is evidently $a^2 - b^2$, as is seen by taking P and D at A and B respectively.

Corresponding properties hold for the ellipse, but are best established by separate treatment, as in the following articles.

§ 7. The Ellipse can always be obtained as the Orthogonal Projection of a Circle.

In orthogonal projection the vertex is at infinity and the projectors are all perpendicular to the plane of projection.

Moreover, the projected length of any segment parallel to the axis of projection is unaltered, whilst that of a segment perpendicular to it is diminished in the ratio $\cos\theta : 1$.

Now let a circle be projected orthogonally on to another plane.

The centre of the circle projects into the centre C of the projection; for the line at infinity of the one plane projects into the line at infinity of the other, and the centre is the pole of the line at infinity.

Let small letters refer to the circle, and capitals to the projected figure.

Then the diameter aca' (Fig. 73), parallel to the axis of projection, projects into ACA′ of equal length (§ 11, Chap. VII.).

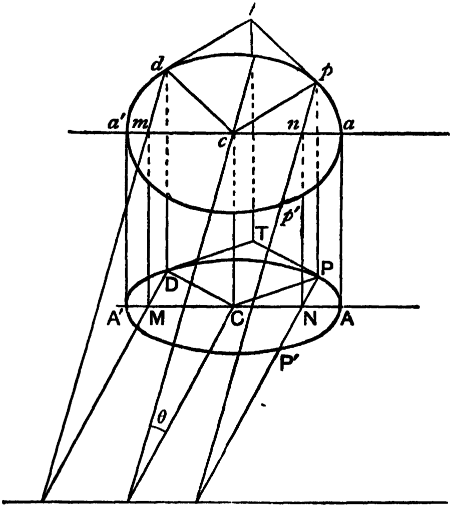


FIG. 73.

Further, a bisected ordinate pnp' of this diameter projects into a corresponding bisected ordinate PNP′, but $\mathrm{PN} = pn \cos \theta$ Also $\mathrm{AN} = an$ and $\mathrm{NA'} = na'$.

Now, in the circle $pn^2 = an \, . \, na'$.

$\therefore$ in the projection $\cos^2\theta = \frac{PN^2}{AN \cdot NA'} = \frac{BC^2}{AC^2}$.

$\therefore$ $\cos\theta$ is the ratio of the minor to the major axis of the projection.

Conversely, to project a circle into an ellipse, the lengths of whose axes are a and b, we have merely to project orthogonally on to a plane, making an angle θ with that of the circle, where $\cos\theta = b/a$.

Ex. Project orthogonally a circle into an ellipse of eccentricity $\frac{3}{4}$.

§ 8. In any Ellipse, if CP and CD are Conjugate Semi-diameters, $CP^2 + CD^2$ is constant.

Since the tangent at the end of any diameter is parallel to the conjugate diameter, and since parallels project into parallels in orthogonal projection (Chap VII. § 11), the pair of diameters of the circle which project into conjugate diameters in the ellipse must possess the same property. But in the circle a diameter is at right angles to the tangent at its extremity. Therefore the diameters of the circle which project into conjugate diameters of the ellipse must be at right angles.

Referring to Fig. 73, if dm, DM be corresponding ordinates of the circle and ellipse respectively, since the angle pcd is a right angle, the triangles cpn, cdm are congruent;

$$\therefore\ pn = cm \quad \text{and} \quad dm = cn.$$

$$\begin{aligned}\therefore\ CP^2 + CD^2 &= CN^2 + PN^2 + CM^2 + DM^2\\ &= cn^2 + pn^2\cos^2\theta + cm^2 + dm^2\cos^2\theta\\ &= cn^2 + pn^2\cos^2\theta + pn^2 + cn^2\cos^2\theta\\ &= (pn^2 + cn^2)(1 + \cos^2\theta) = cp^2(1 + \cos^2\theta)\\ &\qquad = \text{constant}.\end{aligned}$$

$$\therefore\ CP^2 + CD^2 = \text{constant}$$

Taking the conjugate diameters as the axes, this constant is seen to be $a^2 + b^2$.

§ 9. The Area of the Parallelogram formed by Tangents to an Ellipse at the ends of Conjugate Diameters is constant.

The area of the projection of a closed figure is equal to the area of the original figure multiplied by the cosine of the angle between the plane of the figure and the plane of projection (§ 13, Chap. VII.).

Apply this to the circle.

Since the tangents at p and d project into those at P and D, the parallelogram CDTP is the projection of the square *cdtp*.

We have therefore

$$\text{area of parallelogram CDTP} = cp'^2 \times \cos\theta$$

$$= \text{constant for any pair of conjugate diameters CP, CD.}$$

If p be the length of the perpendicular from C on the tangent at P, we have therefore

$$\text{area of CDTP} = p\,.\,\text{CD} = \text{constant},$$

and taking CP and CD as the semi-axes, this constant is seen to be ab.

§ 10. Def. **The auxiliary circle** is the circle described on the major (or transverse) axis of a central conic as diameter.

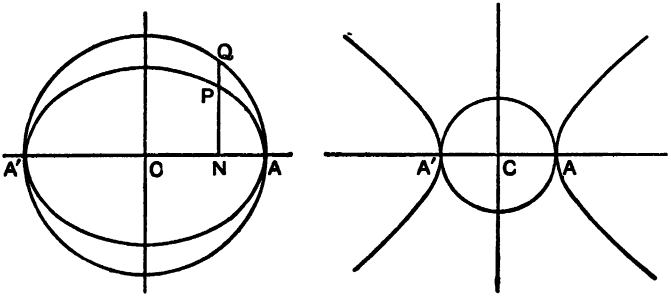


Fig. 74.

Since in the ellipse $\frac{\mathrm{PN}^2}{\mathrm{AN}\,.\,\mathrm{NA}'}=\frac{\mathrm{BC}^2}{\mathrm{AC}^2}$, if NP meet the auxiliary circle in Q (Fig. 74),

$$\mathrm{QN}^2=\mathrm{AN}\,.\,\mathrm{NA}',$$

and it follows that $$\frac{\mathrm{PN}}{\mathrm{QN}}=\frac{\mathrm{BC}}{\mathrm{AC}}.$$

Accordingly the ellipse may be obtained by reducing all ordinates of the auxiliary circle in the ratio $\frac{b}{a}$.

Ex. 1. The ends of a straight rod move on two fixed straight rods at right angles. Prove that any point on the moving rod describes an ellipse.

Ex. 2. A circular disc rolls on the inside of a circular hoop of double its radius. Prove that any point on the disc describes an ellipse.

No similar property obtains in the case of the hyperbola, as NP does not meet the auxiliary circle.

In the case of the parabola, since one vertex is at infinity, the auxiliary circle has infinite radius, and is therefore a straight line—**the tangent at the vertex.**

Historical Note. The discovery of the conic sections is due to Menaechmus, a pupil of Eudoxus and contemporary of Plato (circ. B.C. 370–330). Menaechmus' fame as a teacher led to his being appointed tutor to Alexander the Great. An anecdote of his life in this capacity is worth recording, as illustrating the continuity of human nature. The pupil complained of the length of his master's proofs; to which Menaechmus replied that there might be royal roads in various places, but in Geometry there was only 'one road for all.'

It is probable that the conic sections were first investigated as a means of furnishing a solution of the famous Delian problem of antiquity. 'To find the side of a cube whose volume is double that of a given cube,' or more generally stated, 'To insert two mean proportionals between two given straight lines.'

Putting it algebraically, the first form of the problem required x, such that $a : x = x : y = y : 2a$.

Menaechmus showed that this can be solved in two ways by the intersection of conic sections, (1) either by taking the parabolas $y^2 = 2ax$ and $x^2 = ay$, which have a common vertex and axes at right angles, (2) or by the intersection of the parabola $y^2 = 2ax$ with the rectangular hyperbola $xy = 2a^2$.

The earliest writers, Menaechmus and Aristaeus, considered the conics as obtained by sections of different cones by the same plane, not of the same cone by different planes. For example, the ellipse, parabola and hyperbola were obtained as the sections by a plane perpendicular to a generator of acute-angled, right-angled and obtuse-angled cones.

Archimedes (B.C. 287–212) effected the quadrature of the parabola; but his methods in geometry were rather quantitative than purely geometrical, and found their full expression nearly two thousand years later in the infinitesimal calculus of Newton and Leibnitz.

Euclid wrote a treatise on the conics in four books, now lost. This fact is mainly noteworthy, as these books formed the basis of a remarkable work by Apollonius, which completely overshadowed the achievements of his predecessors, and earned for itself the proud title of the 'crown of Greek geometry.' To this reference is made in a subsequent note.

The theorem with which the foregoing chapter opens, and from which most of the important results are deduced, is taken from the *Géométrie de Position* of Lazare Carnot (1753–1823), an officer in the French army during the stormy times of the Revolution. He must not be confused with his son, Sadi Carnot, the inventor of the heat-engine which bears his name.

In the elder Carnot's work, the theory of transversals is expounded (to quote Poncelet) 'for the first time in all its generality.' The above-mentioned theorem is certainly a remarkable one, and may justly be ranked as second only to that of Pascal (Chap. XV.) in the number and variety of its applications to the geometry of the conic.

Examples on Chapter IX.

1. Prove that the ellipse which touches two sides of a triangle at their middle points and also passes through the middle point of the third side, touches the third side.

2. By considering the limiting case when QQ′ moves, parallel to itself, to an infinite distance from C, verify by § 3, I. the property of § 5, III.

3. Prove by orthogonal projection that the vertices of parallelograms drawn to circumscribe an ellipse at the ends of conjugate diameters, all lie on another ellipse, and find the lengths of its axes.

4. Prove the harmonic property of pole and polar for the conic by Newton's theorem, assuming the diameter property.

5. The tangent at P to an ellipse meets the axes at T, *t*. Construct the position of P when the area of the triangle TC*t* is a minimum.

6. Triangles are circumscribed to an ellipse in such a way that their centres of gravity are always at the centre of the ellipse. Find the locus of their vertices. (Use orthogonal projection.)

7. PQ is a chord of a parabola, PT the tangent at P, and a parallel to the axis cuts the tangent in T, the curve in E and the chord in F. Prove that $\frac{TE}{EF}=\frac{FP}{FQ}$.

8. The tangents at the ends of a chord of a hyperbola intersect the asymptotes at T, T′. Show that TT′ is parallel to the chord.

9. A circle cuts a conic in four points. Prove that the six chords of intersection taken in pairs are equally inclined to the axes.

10. If two chords of a conic be equally inclined to the axes, the other chords through their intersection with the conic are equally inclined to the axes.

11. Squares inscribed in a circle have the same area, being equal to half the square on the diameter. What form does this proposition take when the circle is orthogonally projected into an ellipse?

12. Find the least parallelogram which can be circumscribed to a given ellipse and have one side along a given tangent.

13. If PP′, a tangent to a hyperbola, centre C, meet the asymptotes at P, P′, the locus of the circumcentre of the triangle PCP′ is another hyperbola whose asymptotes are at right angles to those of the given hyperbola.

14. The locus of the poles with respect to the auxiliary circle of tangents to an ellipse is a similar ellipse.

15. Project (conically) the theorem : 'The locus of the middle points of chords of a circle which pass through a fixed point is a circle.'

Deduce the locus of the middle points of chords of a conic which pass through a given point.

16. A conic cuts the sides BC, CA, AB of a triangle in X_1, X_2; Y_1, Y_2; Z_1, Z_2, respectively. Prove that if AX_1, BY_1, CZ_1 are concurrent, so are AX_2, BY_2, CZ_2.

17. State the correlative of Carnot's theorem.

CHAPTER X.

THE FOCI.

§ 1. Conjugate Lines at Right Angles.

It has already been shown (Chap. VIII. § 14) that through any point on either axis of a conic a pair of conjugate lines at right angles can be drawn, viz. the axis in question and a straight line at right angles to it through the point.

It will now be proved that points on the axis exist through which more than one pair of perpendicular conjugate lines pass.

Lemma. If any point S be taken on either axis of a conic *inside the curve*, and P be any point on the curve; if the joins of P to the vertices A and A′, produced when necessary, meet the polar of S at K and K′, SK, SK′ are a pair of conjugate lines with respect to the conic.

[A point is said to be inside or outside a conic according as it is the projection of a point inside or outside the circle of which the conic is the projection.]

For (Fig. 75) since KK′ is the polar of S, XASA′ is a harmonic range.

∴ the pencil K(XASA′) and ∴ the range K′PVA′ are harmonic.

∴ the polar of K′ passes through V.

But it also passes through S. ∴ it is KSV.

Similarly SK′ is the polar of K—which proves the lemma.

NOTE. In the case of the hyperbola, S can be taken on the transverse axis only; for it is easily seen that, according to the above definition, the limitation 'inside the curve' confines S to that side of either branch which is remote from the centre.

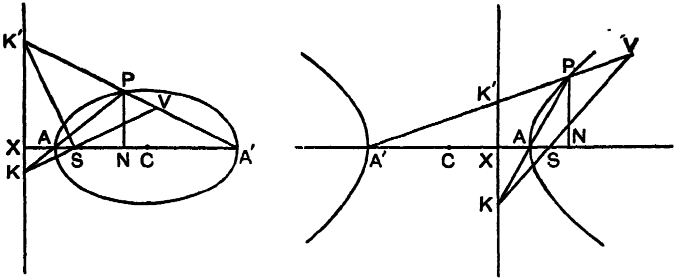


FIG. 75.

THEOREM. **If through any point S on an axis of a conic two pairs of perpendicular conjugate lines can be drawn, S must be a fixed point, and every pair of conjugate lines through it will be perpendicular.**

For, taking S on the major or transverse axis (Fig. 75),

$$\frac{PN}{NA'} = \frac{K'X}{XA'} \quad \text{and} \quad \frac{PN}{AN} = \frac{KX}{XA}.$$

$$\therefore \frac{PN^2}{AN \cdot NA'} = \frac{KX \cdot K'X}{XA \cdot XA'}.$$

But the former ratio is constant $\left(\frac{BC^2}{AC^2} \text{ (ellipse)} \right.$

$\left. \text{or } -\frac{BC^2}{AC^2} \text{ (hyperbola)}\right)$.

If SK, SK' are at right angles, $SX^2 = KX \cdot K'X$.

$$\therefore \frac{SX^2}{XA \cdot XA'} \text{ is constant.}$$

But, since $XASA'$ is a harmonic range,

$$\frac{2}{XS}=\frac{1}{XA}+\frac{1}{XA'},$$

whence $SX.CX=XA.XA'$, where C is the centre of the conic, so that $XA+XA'=2CX$.

$$\therefore \quad \frac{SX^2}{SX.CX'}, \quad i.e.\ \frac{SX}{CX} \text{ is constant} = k \text{ say.}$$

Then $$\frac{CS}{CX}=1-k.$$

But since $XASA'$ is a harmonic range, $CS.CX=CA^2$.

$$\therefore \quad \frac{CS^2}{CA^2}=\frac{CS}{CX}=1-k. \quad \therefore \quad CS^2=CA^2(1-k).$$

It follows that S must be a fixed point, and as its position is thus independent of the particular pair SK, SK' of perpendicular conjugate lines, we have the result that:

Every pair of conjugate lines through the point S on the transverse axis defined by the relation $CS^2=CA^2(1-k)$ is orthogonal.

This result is of such importance that it ought to be proved *directly*. Taking S as thus defined, and KK' its polar with respect to the conic, all that is necessary is to reverse the steps of the above argument. In this way SK and SK', which, by the lemma, are conjugate lines, are proved to be at right angles *for every position of the point* P *on the curve.* Hence *every pair of conjugate lines through* S *is orthogonal.*

The case in which S is taken on the minor or conjugate axis is dealt with in the next article.

§ 2. DEF. **A point through which every pair of lines conjugate with respect to a conic is orthogonal is called a Focus.**

DEF. **The polar of a focus with respect to the conic is called a Directrix.**

THEOREM. **In any central conic there are two real foci on the major (or transverse) axis, and no real foci on the minor (or conjugate) axis.**

In the theorem of § 1, S was taken on the former axis, and the existence of one focus established. For $1-k$ is positive, and CS is therefore a real length. By the symmetry of the curve there must be another on the same axis at the same distance from the centre, and on the other side of it.

If S had been taken on the minor axis of an ellipse, the rest of the proof would follow as before, but we should have obtained $\frac{CS^2}{CB^2}=1-k'$, where $k'=\frac{AC^2}{BC^2}$. $1-k'$ is therefore negative and CS imaginary in this case. A similar argument obtains in the case of the hyperbola.

But the non-existence of real foci on the minor (or conjugate) axis is best established by the help of the following lemma:

The line joining any two foci of a conic must be an axis. To prove this, consider the line joining two foci P, Q. The conjugate line through P is, by definition, at right angles to PQ, and it contains the pole of PQ. Similarly the conjugate line through Q is also at right angles to PQ, and contains the pole of PQ. The pole of PQ is therefore at infinity in a direction perpendicular to it, and therefore PQ is an axis.

Accordingly, if a real focus on the minor or conjugate axis existed, the line joining it to one of the real foci already found would have to be an axis.

But this is impossible.

The same argument may be applied to shew that no foci exist at points not on an axis.

To sum up.

(*a*) **A central conic has two and only two real foci situated on its transverse or major axis, and equidistant from the centre.**

(*b*) Their positions being S, S′, we have

$$CS^2 = CS'^2 = CA^2(1-k),$$

where $$k = \frac{BC^2}{AC^2} \text{ for the ellipse,}$$

but $$-\frac{BC^2}{AC^2} \text{ for the hyperbola.}$$

$$\therefore\ CS^2 = AC^2 - BC^2 \text{ (ellipse)}$$

and $$AC^2 + BC^2 \text{ (hyperbola).}$$

§ 3. DEF. **The double ordinate through a focus is called a Latus rectum.**

There are thus two latera recta for a central conic.

By the fundamental proposition (§ 4) of the preceding chapter we have, if SR is the ordinate,

$$\frac{SR^2}{AS \, . \, SA'} = \frac{BC^2}{AC^2} \text{ (ellipse)} \quad \text{or} \quad -\frac{BC^2}{AC^2} \text{ (hyperbola).}$$

But $AS \, . \, SA' = AC^2 - SC^2$ in each case.

$$\therefore\ AS \, . \, SA' = BC^2 \text{ (ellipse)} \quad \text{or} \quad -BC^2 \text{ (hyperbola).}$$

We have, therefore, in each case

$$\frac{SR^2}{BC^2} = \frac{BC^2}{AC^2} \quad \text{or} \quad SR \, . \, AC = BC^2.$$

The length of the semi-latus rectum is therefore $\frac{b^2}{a}$ in both cases of the central conic, *a* and *b* being, as usual, the lengths of the semi-axes.

§ 4. Case of the Parabola.

Taking a point S inside the curve on the axis and its polar KXK′, since XASA′ is a harmonic range and A′ is in this case at infinity, XA = AS (Fig. 76).

If P be any point on the curve, and PA and a parallel to the axis through P meet the polar of S in K′ and K respectively, and if $K'S$ meet KP in V, $KP = PV$. $\therefore$ $KPV\infty$ is a harmonic range. Therefore the polar of K passes through V, and is $K'SV$, so that, as before, SK, SK' are conjugate lines.

But $\frac{PN}{K'X} = \frac{AN}{AX}$ and $PN = KX$. $\therefore$ $\frac{PN^2}{KX \cdot K'X} = \frac{AN}{AX}$.

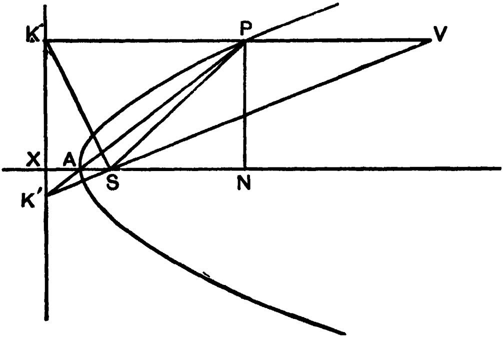


FIG. 76.

But if it be possible for SK, SK' to be orthogonal,

$$KX \cdot K'X = SX^2 = 4AX^2 = 4AS^2.$$

Hence $PN^2 = 4AS \cdot AN$. But $\frac{PN^2}{AN}$ is constant.

$\therefore$ AS is constant and **S is a fixed point.**

It follows that **every pair of conjugate rays through S is orthogonal, and therefore a parabola has one real focus and only one.**

This is what might have been expected: for regarding the parabola as the limiting case of an ellipse whose centre moves to infinity, the other focus goes to infinity likewise.

It should be noted that the **fundamental theorem for ordinate and abscissa now becomes** $PN^2 = 4AS \cdot AN$. If $AS = a$ the Cartesian equation of the parabola, referred to its axis and the tangent at its vertex as axes of coordinates, takes the form $y^2 = 4ax$.

Taking SR as the semi-latus rectum as before, and taking P at R, we have $SR^2 = 4AS^2$. $\therefore$ $SR = 2AS$.

The **latus rectum of a parabola** is therefore 4AS.

NOTE 1. The complete and most elegant treatment of foci depends upon the theory of involution, and is due to Poncelet and Plücker, who showed that a conic has four foci, the real pair given above and an imaginary pair on the minor or conjugate axis. The foregoing discussion in the special cases of the conic is intended to provide an easy introduction to the treatment of the focal properties of the curve, which are too important to allow of their postponement till the theory of involution has been mastered.

NOTE 2. If l is the length of the latus rectum, we have proved that in the parabola $PN^2 = l \, . \, AN$.

In the central conic $l = \frac{2b^2}{a}$ (§ 3).

But
$$\frac{PN^2}{AN \, . \, NA'} = \pm \frac{b^2}{a^2} = \pm \frac{l}{2a}.$$
$$\therefore \frac{PN^2}{l \, . \, AN} = \pm \frac{NA'}{2a}.$$

This is in both cases a positive ratio, less than unity in the ellipse, and greater than unity in the hyperbola. We have then $PN^2 \lesseqqgtr l \, . \, AN$, according as the conic is an ellipse, parabola, or hyperbola.

§ 5. The fundamental Focus-Directrix Property.

PROP. **If any point P be taken on a conic and PK be perpendicular to that directrix which corresponds to a focus S, SP/PK is a constant ratio (called the Eccentricity).**

Let P be any point on the curve and let PA, PA′ meet the directrix in E, E′ (Fig. 77).

Then SE, SE′ are conjugate lines at right angles.

It has been proved that if ES meets A′P in V, E′PVA′ is a harmonic range. The pencil obtained by connecting this range to S has two conjugate rays at right angles ; therefore these rays bisect the angles between the other pair of conjugate rays.

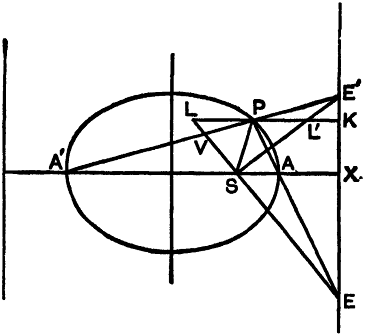


FIG. 77.

∴ angle PSV = angle A′SV = angle PLS if SV meets KP in L.

∴ SP = PL. Hence SP/PK = LP/PK = SA/AX.

Similarly if SE′ meets PK in L′, PL′ = PS, and either of the above ratios is equal to SA′/A′X.

The student should draw the figure for the case of the hyperbola.

§ 6. The Eccentricity in terms of the Semi-Axes.

Denote the eccentricity by *e*.

Then $$e = \frac{SA}{AX} = \frac{SA'}{A'X} = \frac{SA' + SA}{A'X + AX} = \frac{SA' - SA}{A'X - AX}.$$

But $$\left\{\begin{matrix} SA' \pm SA = 2CA, & A'X \pm AX = 2CX \\ SA' \mp SA = 2CS, & A'X \mp AX = 2CA \end{matrix}\right\},$$

the upper signs in each case referring to the ellipse and the lower to the hyperbola, and the absolute magnitudes of the lengths concerned alone being taken into account.

$$\therefore \ e=\frac{\mathsf{CS}}{\mathsf{CA}}=\frac{\mathsf{CA}}{\mathsf{CX}} \text{ in both cases.}$$

$$\therefore \text{ by § 2, } e^2=1-\frac{\mathsf{BC}^2}{\mathsf{AC}^2} \text{ (ellipse) or } 1+\frac{\mathsf{BC}^2}{\mathsf{AC}^2} \text{ (hyperbola).}$$

e is therefore less than 1 in the former case and greater than 1 in the latter.

For the parabola, since SA = AX, $e = 1$.

From Fig. 76 we have then SP = PK, where P is any point on the curve; for the angle KSV is a right angle and KP = PV = PS.

§ 7. The Bifocal Property.

Prop. $\left.\begin{array}{l}\textbf{In the ellipse the sum}\\ \textbf{In the hyperbola the difference}\end{array}\right\}$ **of the focal distances of any point on the curve is constant.**

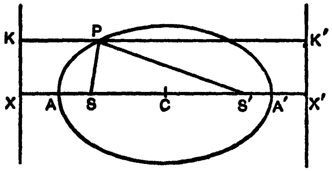


Fig. 78*a*.

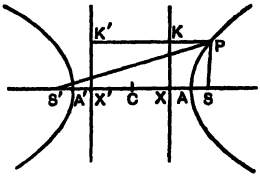


Fig. 78*b*.

For Figs. 78*a* and *b*,

$$\frac{\mathsf{S'P}}{\mathsf{PK'}}=e=\frac{\mathsf{SP}}{\mathsf{PK}}; \quad \therefore \ e=\frac{\mathsf{S'P}\pm\mathsf{SP}}{\mathsf{PK'}\pm\mathsf{PK}}=\frac{\mathsf{S'P}\pm\mathsf{SP}}{\mathsf{KK'}},$$

since PK′ + PK = KK′ in the ellipse,

but PK′ − PK = KK′ in the hyperbola.

But $KK' = XX' = 2CX$, and $CA = eCX$ for both curves.

$$\therefore\ S'P + SP = 2CA \text{ for the ellipse,}$$

and $$S'P - SP = 2CA \text{ for the hyperbola.}$$

Ex. 1. Given a focus of a conic, the corresponding directrix, and the eccentricity, shew how from the property of § 5 to construct any number of points on the curve.

Ex. 2. Construct by points in this way with a common focus 3″ from the corresponding directrix, the ellipses whose eccentricities are $\frac{1}{2}$ and $\frac{1}{3}$.

Ex. 3. Construct with a common focus and axis and concavities in the same direction the parabolas whose latera recta are respectively 2″ and 3″.

Ex. 4. Draw on the same diagram the hyperbolas whose foci are 3″ apart and whose eccentricities are $\frac{3}{2}$ and 2. Construct the asymptotes in each case.

Ex. 5. Prove that the distances of the foci from the ends of the minor axis of an ellipse are each equal to the semi-major axis.

Ex. 6. An ellipse has as its foci the centre of a given circle and a point on its circumference. Prove that the distance of the latter focus from either point of intersection of the curves is double of its distance from the nearer end of the major axis. Prove also that if this distance be half the radius of the circle, the length of the common chord of the two curves is $\sqrt{3}$ times the radius.

Ex. 7. Prove that the locus of a point which moves so that its shortest distance from the circumference of a fixed circle is always equal to its distance from a fixed straight line is a parabola.

Ex. 8. Trace the locus of a point which moves so that the sum of its distances from two fixed points 3″ apart is 5″. Shew that suitable distances for the above tracing may be obtained as the coordinates of points on that part of the line $x+y=5$ which is intercepted between the lines $x-y=\pm 3$.

Ex. 9. Trace the locus of a point which moves so that the difference of its distances from two fixed points 2″ apart is 1′. If the point is such that the difference of the squares of its distances from the same points is also 6 square inches, construct its possible positions as the intersections of two loci.

Ex. 10. From the bifocal property of central conics deduce the focus-directrix property.

§ 8. Mechanical Construction of the Conic.

We have now a simple means of constructing the curves mechanically.

Ellipse. Fasten the ends of a thread SPS′ to two fixed points S, S′ on a plane. A pencil point pressed in the loop so as to keep the two portions SP, S′P always taut will trace out on the plane an ellipse having S, S′ as foci, and major axis equal to the length of the thread.

Hyperbola. One end of a long straight rod S′L is fixed at S′, so that the rod is free to turn about it in a plane.

A thread SPL of length less than SL has one end fastened to L and the other to a fixed point S.

A pencil point pressed in the loop of the thread against the rod so as to keep the two portions of the thread taut will trace out on the plane a portion of an hyperbola. To get the other branch, fix the end of the rod at S and fasten the string at S′ (Fig. 79).

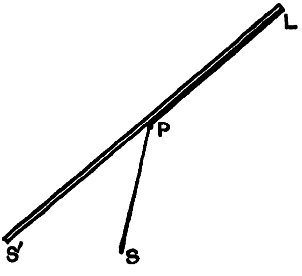


FIG. 79.

Parabola. Take a straight bar RK, bent rigidly at right angles for a short length KL. A string of length RK has one end fastened to R and the other to a fixed point S.

If then KL be made to slide along a fixed straight edge, a pencil point at P pressed against the bar and keeping the

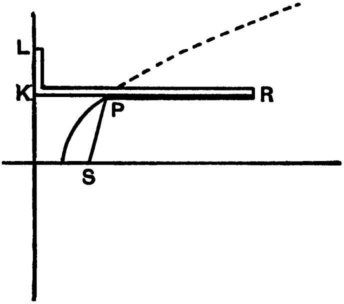


FIG. 80.

string taut will trace out a parabola, having S for focus and the straight edge as directrix. For SP is always equal to PK (Fig. 80).

§ 9. As an exercise on the application of the focus-directrix property, consider the following theorem:

The semi-latus rectum is a harmonic mean between the segments of any focal chord.

If PSP′ be any focal chord meeting the directrix in F, and PK, P′K′ perpendicular to the directrix,

$$e=\frac{SP}{PK}=\frac{SP'}{P'K'}=\frac{SR}{SX},$$

where SR is the semi-latus rectum.

But since FP′SP is a harmonic range, S being the pole of the directrix, FS is the harmonic mean between FP and FP′.

But FS : FP : FP′ = SX : PK : P′K′ = SR : SP : SP′.

∴ SR is a harmonic mean between SP and SP′.

§ 10. The Foci, Directrices, and Eccentricity in the particular and degenerate cases of the Conic.

The following statements are left to be verified by the student ·

(*a*) For the **rectangular hyperbola** $e = \sqrt{2}$.

(*b*) For the **circle** the foci coincide with the centre, the directrices are at infinity, and $e = 0$.

(*c*) For a **pair of straight lines** the foci coincide at the point of intersection of the lines, the directrices coincide in one of the bisectors of the angles between the lines, and e is the secant of half that angle between the lines of which the directrices coincide in the external bisector.

§ 11. Foci of Sections of a Right Circular Cone.

The sections of a *right circular* cone are of course the particular cases of the projection of a circle already considered, when the vertex of projection is on the line through the centre of the circle perpendicular to its plane.

But in this case a very elegant construction* has been given for the foci of such a section, which will now be explained.

Let a sphere be inscribed in the cone so as to touch the plane of section (Fig. 81).

We shall prove that the point of contact of such a sphere with the cutting plane is a focus.

Let the plane of the paper be the plane through the axis of the cone perpendicular to the plane of the section; and let it cut the section in the line AA′.

Let S be the point of contact of the inscribed sphere.

Then by symmetry S is in the plane of the paper and on AA′.

[* Due, in successive stages of completeness, to the following writers: H. Hamilton of Dublin (1798); Dandelin (Mem. Roy. Acad. Brussels, 1822); and Pierce Morton (Trans. Camb. Phil. Soc., 1829).]

The sphere touches the cone along a circle MLM′, whose plane is perpendicular to the axis of the cone.

Let the plane MLM′ cut the plane of the section in the straight line XK.

Take any point P on the section, and draw through it a plane perpendicular to the axis of the cone.

Let PN be the common section of these two planes.

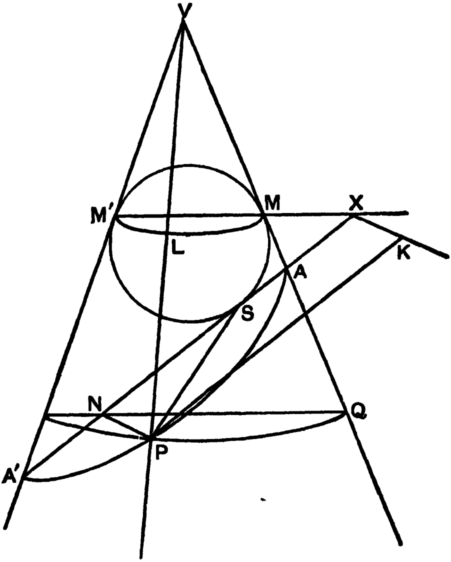


FIG. 81.

Then MLM′ and NPQ being parallel planes, NP is parallel to XK. Also each of these lines is perpendicular to AA′, for either is the common section of two planes, each of which is perpendicular to the plane of the paper.

Join VP. This line is a generator of the cone, and therefore passes through the circumference of the circle MLM′ at some point L, and touches the sphere at that point.

Now $SP = PL$, since they are tangents to the sphere,

$= MQ$, since MLM' and NPQ are parallel sections.

Also
$$\frac{MQ}{NX} = \frac{MA}{AX}.$$

But $MA = AS$ and $NX = PK$, if PK is parallel to NX.

$$\therefore \frac{SP}{PK} = \frac{SA}{AX};$$

this holds for **any point** P on the section.

∴ S satisfies the condition of § 5, and is a focus of the section, XK being the corresponding directrix.

The figure shows an elliptic section. The other focus may be obtained as the point of contact of an inscribed sphere which touches the section on the side remote from V.

In the case of the hyperbola, the other sphere is inscribed in the upper half of the double cone, having V as vertex.

For the parabola, the cutting plane is parallel to a generating line of the cone, and one focus of the section is at infinity.

Ex. 1. Draw the figures for the two cases last mentioned.

Ex. 2. Prove from the above figure that

$$SP + S'P = \text{const.} = AA',$$

S and S′ being the foci.

Historical Note. Apollonius of Perga, the 'great geometer,' was born in B.C. 247, and studied at Alexandria.

His great treatise, the Κωνικά, was divided into eight books, whereof the first four are said to have been an extension of Euclid's work.

He deals with the sections of any cone, defined as the surface swept out by a line of unlimited length, which moves so as always to pass through a fixed point and the circumference of a fixed circle, and proves the proposition stated in § 4. He proceeds to deduce that these sections fall into three species according as the square on the ordinate is less than, equal to, or greater than, the rectangle contained by the latus rectum and the abscissa (see note 2, § 4). These curves he calls respectively ellipse, parabola and hyperbola, names which signify defect, equality and excess, and by which the conic sections have ever since been known.

He was unaware of the existence of the focus in the parabola, or of the directrix in the central conic. The first mention of these appears in the works of Pappus (see note on Chap. IV.). Nevertheless, he worked out very completely a large number of the best known properties of the conic, including the problem of the number of normals that can be drawn from any point to the curve, and the position of the centre of curvature (see Chap. XI.). To give other instances, the results given in Chap. VIII. §§ 7 (VI.), 10, 12, Chap. IX. §§ 2, 5 (V.), 6, 8, and certain of the properties proved in Chap. XI.

The introduction of the term 'focus' is due to Kepler (1571-1630). (See note at the end of the next chapter.)

Examples on Chapter X.

1. Prove that a directrix of a conic may be regarded as the polar of the corresponding focus with respect to the auxiliary circle.

2. Prove that two parabolas can be drawn to pass through two given points and to have a given focus.

3. Prove that the circle on a focal distance of a point on a central conic as diameter, touches the auxiliary circle.

4. If GL be the perpendicular from G, the point where the normal at P meets the axis, on SP, then PL is equal to the semi-latus rectum.

5. A line is drawn from a focus of a hyperbola parallel to an asymptote to meet the directrix. Shew that it is equal in length to half the latus rectum and is bisected by the curve.

6. Two fixed circles which touch one another are each touched by a third variable circle. Prove that the centre of the latter may lie on a certain conic. Find the eccentricity of this conic in terms of the ratio of the radii of the fixed circles, distinguishing between the cases where the fixed circles touch externally and internally.

7. An endless string passes round the circumferences of three equal circular discs in one plane, two of which are fixed. Prove that if the third is moved so as to keep the string always tight, its centre describes an ellipse.

8. Find the locus of the foci of all parabolas which pass through two given points and have their axes in a given direction.

9. Given a focus and two points of an ellipse, find the locus of the other focus.

10. The semi-vertical angle of a right circular cone is α, and the angle between the axis and the plane of section is β. Find the eccentricity of the section.

11. The semi-vertical angle of a right cone is 30°. Find the angle made by the plane of section with the axis of the cone, if the eccentricity of the conic section is $\sqrt{\frac{2}{3}}$.

12. Prove that the square on the semi-minor axis of an elliptic section of a right circular cone is equal to the rectangle under the distances of the vertices from the axis of the cone.

13. Prove that all parallel plane sections of a cone are similar conics.

14. Through a parabola of given latus rectum passes a right circular cone of given vertical angle. Find in terms of the given quantities the distance of the vertex of the cone from the vertex of the parabola.

15. Show that the locus of vertices of right circular cones through a given parabola is an equal parabola.

16. The vertices of an elliptic section of a right-angled cone are at distances 6 and 8 inches from the vertex of the cone. Prove that the radius of the smallest sphere which can be inscribed in the cone so as to touch the plane of section is 2 inches.

CHAPTER XI.

FOCAL, TANGENT, AND NORMAL PROPERTIES.

§ 1. Prop. **Tangents to a conic subtend equal or supplementary angles at a focus.**

Let S be the focus, LL′ the corresponding directrix, and let QQ′ cut LL′ at L′, TQ, TQ′ being the tangents from T (Fig. 82).

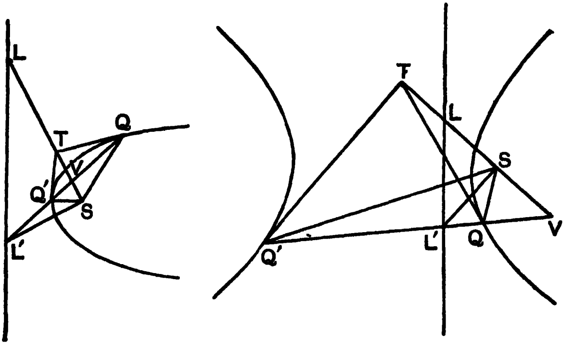


Fig. 82.

Then the polars of T and S intersect at L′. ∴ ST is the polar of L′. Let ST cut the directrix at L. Then since the polars of L′ and S intersect at L, L′S is the polar of L.

∴ SLL′ is a self-polar triangle, and SL, SL′ are a pair of conjugate lines at S.

But these are orthogonal by the definition of a focus, and if ST cuts QQ′ at V, L′Q′VQ is a harmonic range (Ch. VIII. § 7, VI.).

∴ SL, SL′ must be the internal and external bisectors of the angle QSQ′.

When the conic is a hyperbola and TQ and TQ′ are drawn to touch different branches, SL will be the external bisector, and accordingly the angles TSQ, TSQ′ will be supplementary.

Cor. ST cuts the conic at the point of contact of the tangent from L′. We have therefore

The portion of a tangent intercepted between the curve and the directrix subtends a right angle at the focus.

This also follows from the fact that the polar of a point on the directrix passes through the focus.

The equal angles subtended at the focus by tangents from this point are, accordingly, each right angles.

§ 2. The Tangent at any Point is equally inclined to the Focal Distances of its Point of Contact.

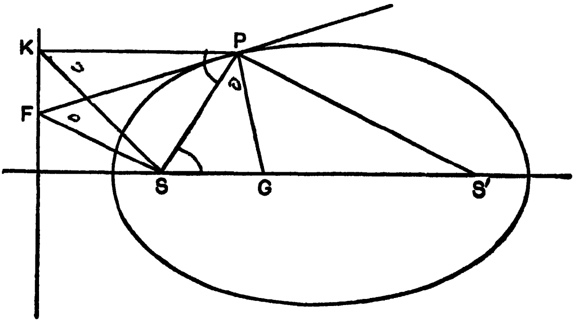


Fig. 83.

Taking the figure (83) for the ellipse, we have, by the last corollary, PSF is a right angle.

Let PG be the normal at P, and KF the directrix corresponding to S.

Then angle SPG = angle SFP, each being complementary to SPF,

= angle SKP, since SFKP is cyclic.

Also angle PSG = angle SPK.

∴ the triangles SPG, SPK are similar, and

$$\frac{\text{SG}}{\text{SP}}=\frac{\text{SP}}{\text{PK}}=e.$$

Similarly dealing with S′ and its corresponding directrix, we shall find that $\frac{\text{S}'\text{G}}{\text{S}'\text{P}}=e.$

$$\therefore\ \frac{\text{SG}}{\text{SP}}=\frac{\text{S}'\text{G}}{\text{S}'\text{P}}.$$

∴ PG bisects the angle SPS′, and therefore the tangent also is equally inclined to the focal distances.

The figure for the hyperbola is easily drawn.

Cor. 1. **In the case of the parabola** SG = SP, S′ is at infinity, and the corresponding focal distance is parallel to the axis.

We have then **in the parabola**:

The tangent is equally inclined to the axis and to the focal distance of its point of contact.

Cor. 2. If an ellipse and hyperbola are **confocal,** *i.e.* have the same foci, they intersect at right angles at every common point. For the tangents at the point of intersection to the respective curves are the external and internal bisectors of the same angle.

Fig. 84 shows such a system of confocal ellipses and hyperbolas.

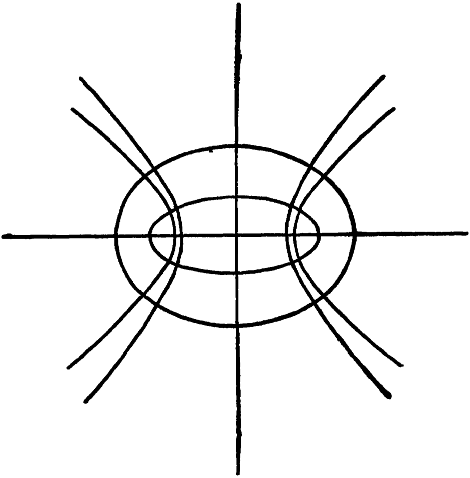


FIG. 84.

§ 3. PROP. (*a*) **The locus of the foot of the focal perpendicular on any tangent to a conic is the auxiliary circle.**

(*b*) **The product of the focal perpendiculars on a tangent is constant.**

(*c*) **The diameter parallel to a tangent cuts off from the focal distances of the point of contact intercepts each equal to the semi-major axis.**

(*a*) Let the feet of the perpendiculars from the foci S, S′ on the tangent be Y, Y′ (Figs. 85*a* and *b*). Let SY meet S′P in K, and the circle again in L.

Then, since angle SPY = angle YPK by § 2, and the angles at Y are right angles, and PY common, SY = YK and SP = PK.

But SC = CS′. $\therefore$ CY $\parallel$ S′K and $= \frac{1}{2}$S′K.

But $$S'K = S'P \pm PK = S'P \pm SP$$

the upper sign referring to Fig. (*a*) and the lower to Fig. (*b*)]

$$= AA' \text{ [for both figures].}$$

$\therefore$ $CY = CA$, and the locus of Y is the auxiliary circle.

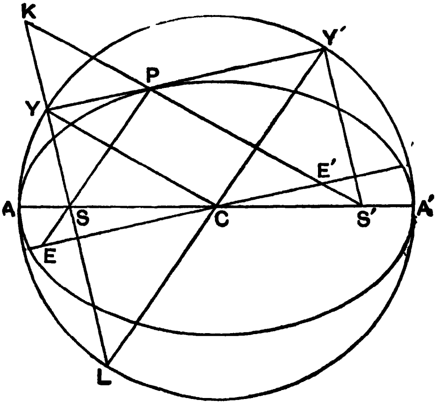


Fig. 85*a*.

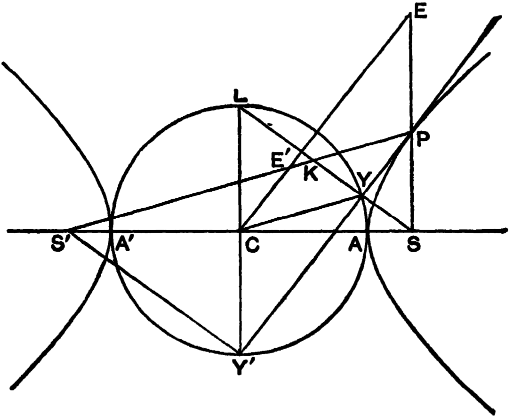


Fig. 85*b*.

(*b*) Similarly Y' is on it. Moreover $Y'YL$ is a right angle.

$\therefore$ $Y'L$ is a diameter of the circle.

$\therefore$ since the triangles $S'CY'$, SCL are congruent, $SL = S'Y'$.

$\therefore$ $SY \,.\, S'Y' = SY \,.\, SL = AS \,.\, SA' = BC^2$ (disregarding sign for the sake of brevity).

(*c*) If the diameter through $C \parallel$ to YPY' cut SP, $S'P$ in E, E', then, since CY, CY' are parallel to these lines, $PE'CY$, $PECY'$ are parallelograms, and $PE = CY' = AC$.

Similarly $PE' = AC$.

COR. 1. **In the parabola the locus of the foot of the focal perpendicular on the tangent is the tangent at the vertex.** (See § 10, Chap. IX.)

This may be proved independently, and the proof is left as an exercise.

COR. 2. Only one conic can be drawn with given foci (in the case of a parabola with given focus and axis) to touch a given straight line.

COR. 3. If radial lines be drawn from a fixed point to meet a fixed circle, and from their points of intersection with it straight lines be drawn at right angles to them, these lines will all touch a central conic, which is an ellipse or hyperbola according as the point is inside or outside the given circle; the conic has this circle for auxiliary circle.

Ex. 1. When is the envelope of such a family of lines a parabola? (Cf. Cor. 1.)

Ex. 2. What is the envelope when the fixed point is on the circle?

§ 4. The Relative Positions of Asymptotes, Directrices, Foci, and Auxiliary Circle in the Hyperbola.

Since an asymptote is a tangent, the feet of the perpendiculars from the foci on the asymptotes lie on the auxiliary circle.

It is easy to show that the directrices pass through the same points. For it has been proved that if X be the foot of the directrix, CS . CX = CA^2 (§ 1, Chap. X.).

∴ the directrix is the polar of the focus with respect to the auxiliary circle, and therefore passes through the points of contact of tangents from S. Fig. 86 shows the required relation.

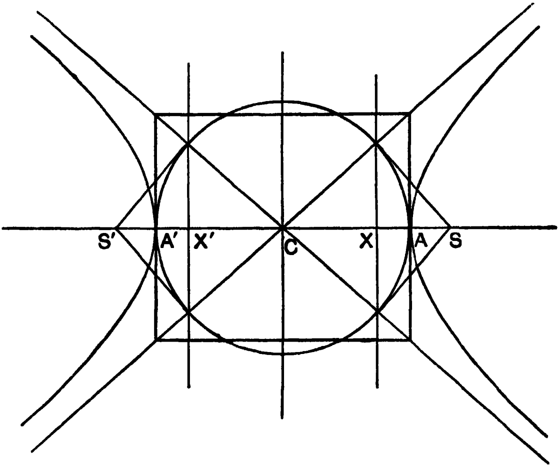


FIG. 86.

§ 5. PROP. **Tangents from an external point to a central conic are equally inclined to the focal distances of the point.**

Let SY, S′Y′ (Figs. 87*a* and *b*), SZ, S′Z′ be the perpendiculars from the foci on the tangents from T.

Then SY . S′Y′ = BC^2 = SZ . S′Z′ (§ 3 (*b*)).

$$\therefore \frac{SY}{SZ} = \frac{S'Z'}{S'Y'},$$

and angle YSZ = angle Y′S′Z′, each being equal to (in Fig. 87*b*), or supplementary to (in Fig. 87*a*), the angle YTZ.

∴ the triangles YSZ, Y′S′Z′ are similar. ∴ angle YZS = angle S′Y′Z′.

But since SYTZ is cyclic, angle YZS = angle YTS in Fig. 87*b*, but supplement of angle YTS in Fig. 87*a*; and since S′Y′TZ′ is cyclic, angle S′Y′Z′ = angle S′TZ′ in Fig. 87*b*, but supplement of angle S′TZ′ in Fig. 87*a*.

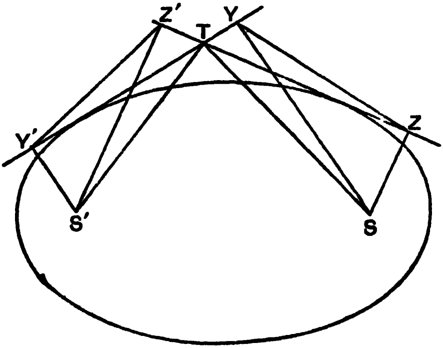


FIG. 87*a*.

∴ angle STY = angle S′TZ′ *in both cases.*

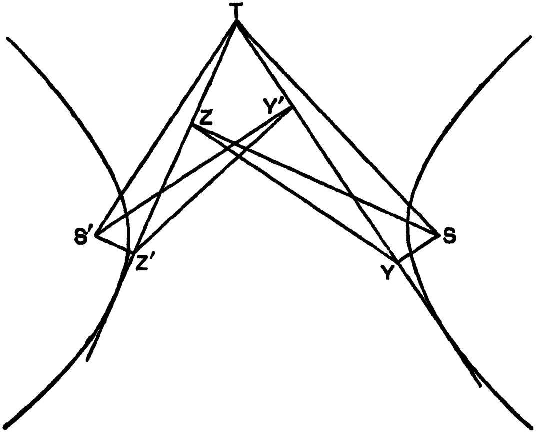


FIG. 87*b*.

NOTE 1. The proof applies equally when the tangents both touch the same branch of the hyperbola.

NOTE 2. § 2 is the particular case of this theorem when T is on the curve.

Ex. What does this property become for the parabola?

§ 6. PROP. **In a central conic, if N, G be the feet of the ordinate and normal at P respectively, and C be the centre,**

(*a*) $CG = e^2 CN$.

(*b*) **PG is inversely proportional to the central perpendicular on the tangent.**

Let the tangent at P meet the axis in T (Figs. 88*a* and *b*).

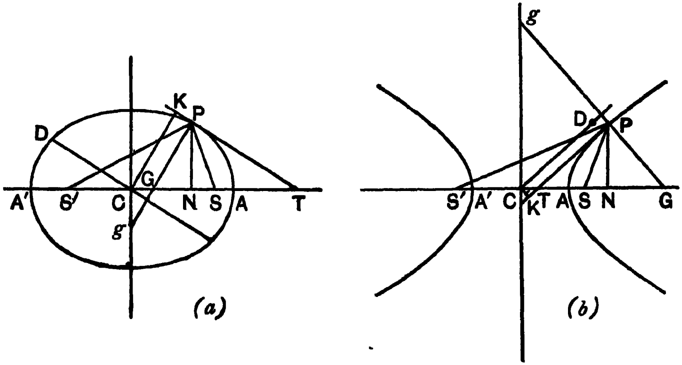


FIG. 88.

Then, since T is the pole of PN, TANA′ is a harmonic range and $CN.CT = CA^2$.

But PG and PT being the internal and external bisectors of the angle SPS′, TSGS′ is a harmonic range (§ 7, Chap. IV.).

$$\therefore CG\ CT = CS^2.$$

$$\therefore \frac{CG.CT}{CN.CT} = \frac{CS^2}{CA^2} = e^2 \text{ (§ 6, Chap. X.)}.$$

$$\therefore CG = e^2 CN.$$

(*c*) From the preceding $\frac{\mathrm{NG}}{\mathrm{CN}} = 1 - e^2$ (ellipse)

or $e^2 - 1$ (hyperbola).

∴ *in both cases* $\frac{\mathrm{NG}}{\mathrm{CN}} = \frac{b^2}{a^2}$.

But the triangles PNG, CKT are similar, CK being the central perpendicular on the tangent.

$$\therefore \frac{\mathrm{CK}}{\mathrm{CT}} = \frac{\mathrm{NG}}{\mathrm{PG}}. \quad \therefore \mathrm{PG} . \mathrm{CK} = \mathrm{NG} . \mathrm{CT}.$$

But $\frac{\mathrm{NG} . \mathrm{CT}}{\mathrm{CN} . \mathrm{CT}} = \frac{\mathrm{NG}}{\mathrm{CN}} = \frac{b^2}{a^2}$, and $\mathrm{CN} . \mathrm{CT} = a^2$. (Chap. VIII. § 10, Cor. 3 and § 14.)

$$\therefore \mathrm{PG} . \mathrm{CK} = \mathrm{NG} . \mathrm{CT} = b^2.$$

Cor. 1. It may be proved similarly that $\mathrm{P}g . \mathrm{CK} = a^2$, g being the foot of the normal on the other axis.

Cor. 2. Since $\mathrm{CK} . \mathrm{CD} = ab$ (Chap. IX. §§ 9 and 6)

and $\mathrm{CK} . \mathrm{PG} = b^2$, it follows that $\frac{\mathrm{PG}}{\mathrm{CD}} = \frac{b}{a}$.

§ **7.** As an exercise, consider the problem:

'Given a pair of conjugate diameters of an ellipse in magnitude and position, to construct the axes.'

Let KPK′, the normal at P, cut DCD′, the diameter conjugate to CP in F (Fig. 89).

Mark off $\mathrm{PK} = \mathrm{PK}' = \mathrm{CD}$ on this normal.

Then
$$\begin{aligned} \mathrm{CK}^2 &= \mathrm{CP}^2 + \mathrm{PK}^2 + 2\mathrm{FP} . \mathrm{PK} \\ &= \mathrm{CP}^2 + \mathrm{CD}^2 + 2\mathrm{FP} . \mathrm{CD} \\ &= a^2 + b^2 + 2ab. \quad \text{(Chap. IX., §§ 8 and 9.)} \end{aligned}$$

Thence $\mathrm{CK} = a + b$, and similarly $\mathrm{CK}' = a - b$.

Hence $a = \tfrac{1}{2}(\mathrm{CK} + \mathrm{CK}')$, $b = \tfrac{1}{2}(\mathrm{CK} - \mathrm{CK}')$.

As the normal at P is the perpendicular from P on CD, CK and CK′ are known, and hence the magnitudes of the axes also.

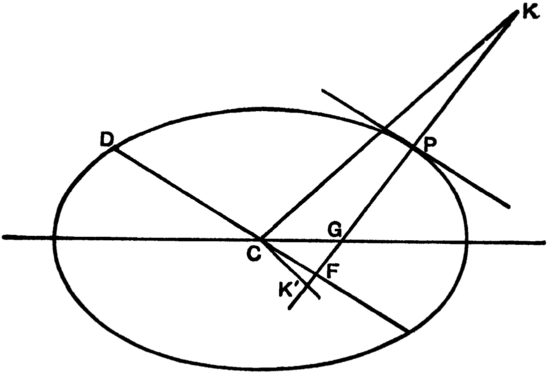


FIG. 89.

Also, by Cor. 2 above, $\frac{\mathrm{PG}}{\mathrm{CD}}=\frac{b}{a}$. $\therefore$ $\frac{\mathrm{PG}}{\mathrm{PK}}=\frac{b}{a}$.

$$\therefore\ \frac{\mathrm{PK}+\mathrm{PG}}{\mathrm{PK}-\mathrm{PG}}=\frac{a+b}{a-b}, \quad i.e.\ \frac{\mathrm{KG}}{\mathrm{GK'}}=\frac{\mathrm{CK}}{\mathrm{CK'}}, \text{ from the above.}$$

$\therefore$ CG bisects the angle KCK′.

Accordingly, when CK and CK′ are constructed as above, the internal and external bisectors of the angle between them are the axes.

Hence the latter can be constructed in magnitude and position.

§ 8. Parabola Properties.

In the **parabola**:

(1) **the subtangent is twice the abscissa**;

(2) **the subnormal is constant**;

(3) **UP, UP′ being a pair of tangents, SUP, SUP′ are similar triangles**;

(4) **the circumcircle of the triangle formed by three tangents passes through the focus.**

Let S be the focus, KX the directrix of the parabola, and A the vertex. Let the tangent, normal, and ordinate at P meet the axis at T, G, and N respectively.

Then NT is called the subtangent, and NG the subnormal, at P.

(1) TAN∞ is a harmonic range, since PN is the polar of T (Fig. 90).

∴ AT = AN, so the subtangent NT is twice the abscissa AN

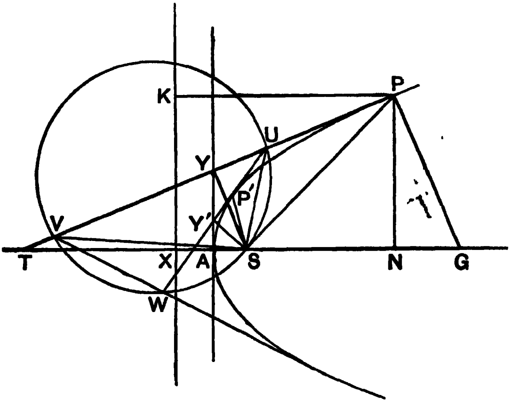


FIG. 90.

(2) SG = SP (§ 2, Cor. 1) = PK = NX. ∴ NG = SX = 2AS.

(3) Y and Y′, the feet of the perpendiculars from S on UP, UP′, lie on the tangent at the vertex (§ 3, Cor. 1).

∴ SY′YU is a cyclic quadrilateral, and angle SYY′ = angle SUY′

But angle SYY′ = angle STY = angle TPK = angle SPT.

∴ angle SPU = angle SUP′.

Also, by § 1, angle USP = angle USP′.

∴ the △s SUP, SUP′ are equiangular and similar.

(4) If UVW be a triangle formed by tangents, its circumcircle will pass through S. For, considering the tangents from V, we have by the preceding, angle SVW = angle SPV.

But SUW was proved equal to this angle.

∴ SUVW is a cyclic quadrilateral.

§ 9. The Director Circle.

The locus of a point from which orthogonal tangents can be drawn to a conic is a concentric circle.

Let TYY′, TZZ′ (Figs. 91*a* and *b*) be orthogonal tangents from T to a central conic. The feet of the perpendiculars from the foci, viz. Y, Y′, Z, Z′, lie on the auxiliary circle. Moreover, SYTZ, S′Y′TZ′ are rectangles, and $TY \cdot TY' = SZ \cdot S'Z' = BC^2$.

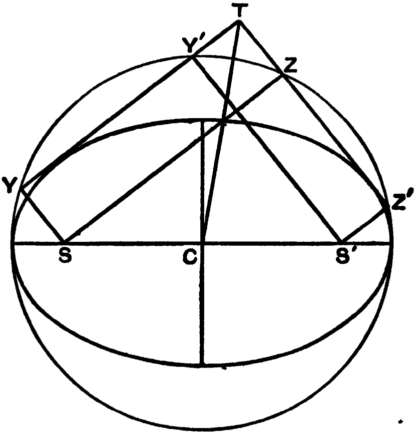


FIG. 91*a*.

Also $TY \cdot TY'$ = square of tangent from T to the auxiliary circle

$= CT^2$ – sq. of radius of aux. ⊙ (for the ellipse),

but $=$ sq. of radius of aux. ⊙ – CT^2 (for the hyperbola)

(for in the latter case the foci lie on opposite sides of any tangent, and it follows that T necessarily lies between Y and Y′, Z and Z′).

$$\therefore \quad CT^2 = AC^2 + BC^2 \text{ (for the ellipse)}$$

and

$$AC^2 - BC^2 \text{ (for the hyperbola).}$$

∴ CT is constant in both cases, and the locus of **T is a** circle.

This is called the **Director Circle.**

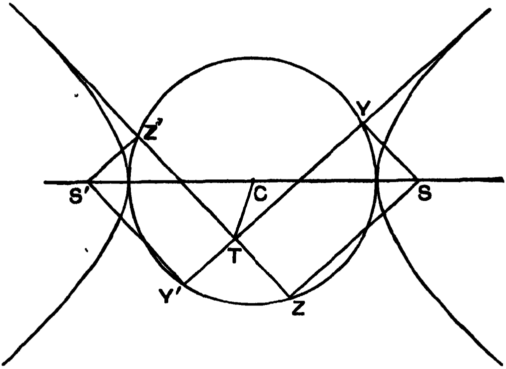


Fig. 91*b*.

It appears that an **ellipse** always has a real director **circle**; a **hyperbola** only when $AC > BC$.

In the **rectangular hyperbola** $AC = BC$, and the director circle becomes a point-circle at the centre of the curve.

In the **parabola** the circle has an infinite radius, and thus becomes a straight line.

It is easily shown that this line is the directrix. For if TP, TP′ are the tangents, SY, SY′ the focal perpendiculars on them, Y and Y′ lie on the tangent at the vertex (Fig. 92).

But SYTY′ is a rectangle and ST is bisected by YY′. T must therefore lie on a line parallel to YY′ and twice as far from the focus as YY′, *i.e.* it is on the directrix. *The director circle for a parabola is therefore the directrix.*

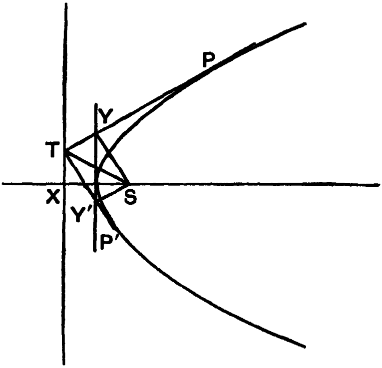


FIG. 92.

§ 10. Gaskin's Theorem.

The circumcircle of a triangle self-conjugate with respect to an ellipse cuts the director circle orthogonally.

In Fig. 93, let TRR′ be a self-conjugate triangle, and let RR′ cut the conic in Q, Q′. Let V be the middle point of QQ′. Then CVT is a straight line and $CV . CT = CP^2$, if CT cuts the conic at P.

Now $\frac{QV^2}{PV . VP'} = \frac{CD^2}{CP^2}$ (Chap. IX. § 3), CD being the semi-diameter conjugate to CP.

Also, since TR is the polar of R′, R′QRQ′ is a harmonic range.

$\therefore$ $QV^2 = VR . VR' = VK . VT$, if CT cuts the circle TRR′ in K.

Also $$PV . VP' = CP^2 - CV^2 = CV . CT - CV^2$$
(Chap. VIII., § 10, Cor. 3)
$$= CV . VT.$$

$$\therefore \frac{CD^2}{CP^2} = \frac{VK . VT}{CV . VT} = \frac{VK}{CV}.$$

$$\therefore \quad \frac{CD^2+CP^2}{CP^2}=\frac{CK}{CV} \text{ (componendo)} = \frac{CK \, . \, CT}{CV \, . \, CT}$$

But $$CP^2 = CV \, . \, CT.$$

$$\therefore \quad CK \, . \, CT = CD^2 + CP^2$$
$$= a^2 + b^2.$$

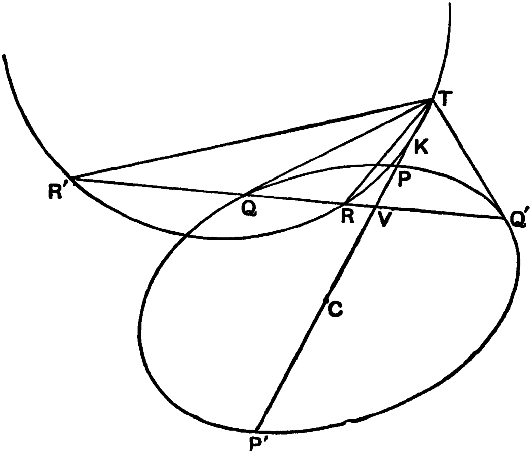


FIG. 93.

But $CK \, . \, CT$ is the square of the tangent from C to the circumcircle of TRR′.

∴ this tangent is equal to the radius of the director circle, and the two circles cut orthogonally.

∴ the circle TRR′ cuts the director circle orthogonally.

§ 11. Steiner's Theorem.

The orthocentre of the triangle formed by three tangents to a parabola lies on the directrix.

Considering the parabola as the limiting form of an ellipse whose centre is at infinity, Gaskin's Theorem assumes the form:

'The centre of the circumcircle of a triangle self-conjugate for a parabola lies on the directrix.'

For the directrix is the limiting form of the director circle, and can only cut the above circle orthogonally when it is a diameter.

But in the parabola the straight line midway between a pole and its polar and parallel to the latter is a tangent (Chapter VIII. § 11, Cor. 2). Hence the lines joining the middle points of the sides of a self-conjugate triangle are tangents to the parabola.

Also the centre of the circumcircle of the self-conjugate triangle is the orthocentre of the triangle formed by joining the middle points of the sides.

∴ the orthocentre of the triangle formed by three tangents is on the directrix.

Ex. From properties of the parabola prove (assuming that only one parabola can be drawn to touch four given straight lines) that

(1) the circumcircles of the four triangles formed by four given straight lines are concurrent;

(2) the orthocentres of these triangles are collinear.

§ 12. Curvature.

Consider two curves touching one another at a point A. Draw the common tangent at A, and from a near point P on it draw PQq at right angles to PA to cut the curves at Q and q respectively.

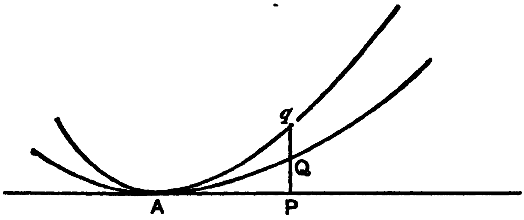


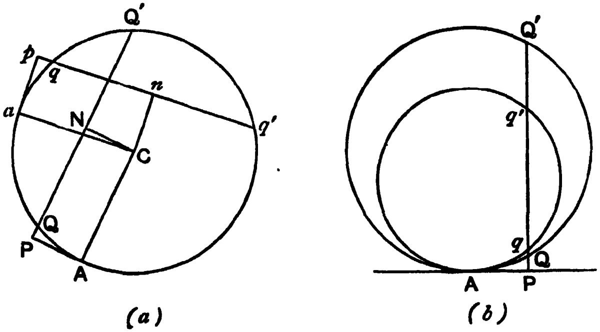
FIG. 94.

Then PQ, Pq measure the deflections of the curves from the tangent line (Fig. 94).

Consider the ratio $\frac{PQ}{Pq}$. If it were unity for all positions of P near A, the curves would coincide in that neighbourhood, and their curvatures would be the same.

We shall therefore not be inconsistent if we **define the ratio of the curvatures of the curves at A in the general case as the limiting value of the ratio $\frac{PQ}{Pq}$ when P moves up to A.**

Consider the circle. If any two points A, a (Fig. 95a) be taken on it and AP, ap be the tangents, PQ, pq at right angles to these to meet the circle in Q, q; then if $AP = ap$, it is easily proved that $PQ = pq$.



(*a*) (*b*)

FIG. 95.

For if C be the centre and CN, Cn perpendicular to PQ, pq, $AP = CN$ and $ap = Cn$.

∴ $CN = Cn$ and the chords QQ′, qq' are equal.

∴ PQ and pq, being the differences between the halves of these chords and the radius, are equal to one another.

Hence the application of the above definition gives a result in accordance with the obvious fact that **the curvature of the circle is the same at every point on it.**

Moreover, **in different circles the curvatures are inversely as the radii.**

For let the circles touch at A; also let AP, PQ as above refer to the one circle and AP, Pq to the other. Let PQ, Pq produced cut their respective circles in Q$'$, q' (Fig. 95b).

Then $$\mathrm{PQ}\,.\,\mathrm{PQ}' = \mathrm{AP}^2 = \mathrm{P}q\,.\,\mathrm{P}q'.$$

Hence $$\frac{\mathrm{PQ}}{\mathrm{P}q} = \frac{\mathrm{P}q'}{\mathrm{PQ}'}.$$

Ultimately, when P moves up to A, PQ$'$, Pq' become the diameters of the circles.

Hence the ultimate value of the ratio $\frac{\mathrm{PQ}}{\mathrm{P}q}$ is in the inverse ratio of the radii.

This also agrees with common experience, for of circles touching at a common point, that with the larger radius lies flatter to the tangent.

Since the curvature of a circle is the same at every point, this curve is eminently suitable as a standard of comparison; also, since the curvature of a circle is inversely proportional to the radius, we can conveniently take the **reciprocal of the radius** as the **measure of that curvature.**

§ **13.** Now consider the **curvature of any curve.**

The curve may be regarded as the limit of a closely inscribed polygon when the number of sides is indefinitely increased and the magnitude of each indefinitely diminished. The limiting positions of consecutive angular points of this polygon will be called '**consecutive points**' on the curve.

The line joining two such consecutive points is the limiting position of a side of a polygon, and is ultimately a tangent to the curve.

Through the same two points an infinite number of circles can be drawn, each having this tangent in common with the curve.

But if we take **three** consecutive points, only **one** circle can be drawn to pass through these.

Moreover, **this circle will have the same curvature as the curve**; for evidently we have ultimately $\frac{PQ}{Pq}=1$, as Q and q then coincide.

Hence this circle is called the **circle of curvature of the curve** at the point, and its radius the **radius of curvature. The reciprocal of this radius is the measure of the curvature of the curve at that point.**

DEF. Any chord of the circle of curvature passing through its point of contact with the curve is called a **chord of curvature** of the curve at that point.

§ 14. To find an Expression for the Chord of Curvature of a Curve in any Given Direction.

Let PQQ′ be a circle touching the curve at P and passing through a neighbouring point Q (Fig. 96). When Q moves up to P, this is the circle of curvature of the curve at P. Let PK be a chord of the circle in the given direction. Through Q draw Q′QT parallel to PK to cut the tangent at P in T.

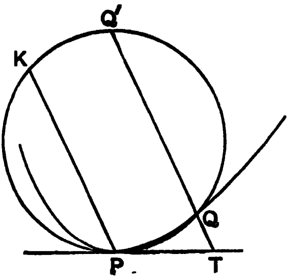


FIG. 96.

Then $TP^2 = TQ \, . \, TQ'$, and $\text{Limit}\left(\frac{TP^2}{TQ}\right) = \text{Limit}\,(TQ') = PK$, when Q moves up to P.

Hence $\text{Lt}\,\frac{TP^2}{TQ}$ is the expression required, T being a near point on the tangent at P and TQ drawn in the given direction to meet the curve.

§ 15. Chord of Curvature through the Centre, and Radius of Curvature at any Point of a Central Conic.

Let T be a near point to P on the tangent at P and Q on the conic.

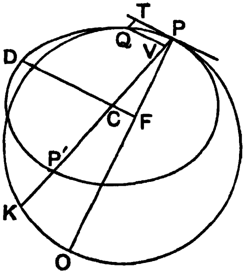


FIG. 97*a*.

Then, in Figs. 97*a* and *b*, PK the required chord of curvature through P

$$= \text{Lt}\,\frac{TP^2}{TQ} = \text{Lt}\,\frac{QV^2}{PV}, \text{ where } QV \text{ is parallel to } TP.$$

But $$\frac{QV^2}{PV \, . \, VP'} = \pm\frac{CD^2}{CP^2} \quad \left(\begin{array}{l}\text{upper sign—ellipse} \\ \text{lower } \;,, \;\text{—hyperbola}\end{array}\right),$$

where CD is the semi-diameter conjugate to CP.

$$\therefore \; PK = \text{Lt}\,\frac{QV^2}{PV} = \pm\frac{CD^2}{CP^2} \cdot \text{Lt}\,VP'$$

$$= \frac{CD^2}{CP^2} \cdot 2CP \text{ (in both cases) } = \frac{2CD^2}{CP}.$$

But if PFO be the normal at P and PO the diameter of curvature, since CKOF is cyclic,

$$PF \,.\, PO = PC \,.\, PK = 2CD^2 \text{ in absolute magnitude.}$$

$$\therefore \; PO = 2CD^2/PF.$$

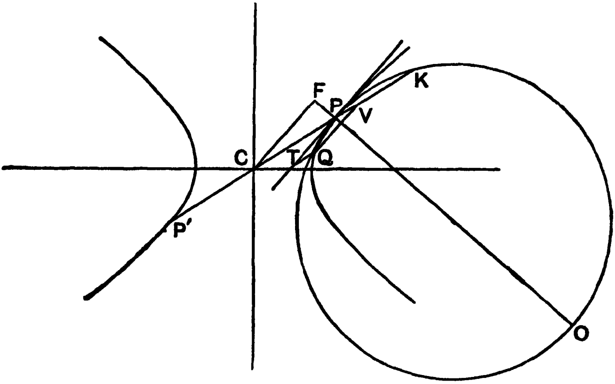


FIG. 97b.

Hence the radius of curvature at any point P of a central

$$\text{conic} = \frac{\text{sq. of semi-diameter parallel to the tangent at P}}{\text{central perpendicular on this tangent}}.$$

§ 16. The Chord of Curvature through the Focus S at any Point P of a Parabola is 4SP.

Let TQQ′ be parallel to PP′, the focal chord (Fig. 98). Let *tpt′* be the tangent parallel to PP′, and let it be cut at *t* and *t′* by the tangents at P and P′ which intersect at right angles at *k* on the directrix (§ 9).

Now the triangles S*pt*, S*t*P being similar (§ 8 (3)),

$$\frac{tp}{tP} = \frac{Sp}{St} = \frac{St}{SP}. \quad \therefore \; \frac{tp^2}{tP^2} = \frac{Sp}{SP}.$$

Also, by Newton's Theorem, § 2, Chap. IX.,

$$\frac{\mathrm{TP}^2}{\mathrm{TQ}\,.\,\mathrm{TQ}'} = \frac{t\mathrm{P}^2}{tp^2} = \text{(from the above) } \frac{\mathrm{SP}}{\mathrm{S}p}.$$

$$\therefore \text{ chord of curvature through } \mathrm{S} = \mathrm{Lt}\,\frac{\mathrm{TP}^2}{\mathrm{TQ}} = \frac{\mathrm{SP}}{\mathrm{S}p}\,.\,\mathrm{Lt}\,\mathrm{TQ}' = \frac{\mathrm{SP}}{\mathrm{S}p}\,.\,\mathrm{PP}'.$$

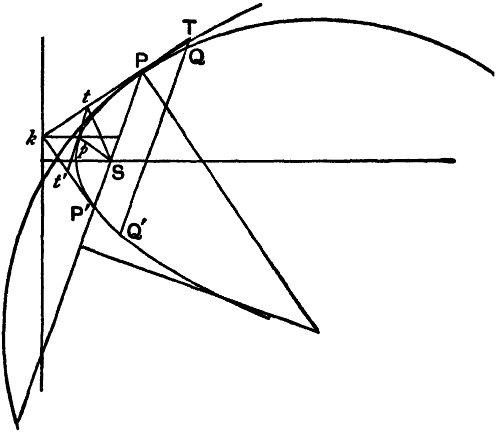


FIG. 98.

But $\mathrm{PP}' = 2tt' = 4tp$ (§ 11, Chap. VIII.)

$= 4pk$, since tkt' is a right angle,

$= 4\mathrm{S}p$.

$\therefore$ chord of curvature through $\mathrm{S} = 4\mathrm{SP}$.

From this expressions for the diameter of curvature may be obtained.

NOTE. It will be subsequently proved that a circle cannot cut a conic in more than four points. Three of these being coincident at P in the above figures, it will cut the conic in one other point only. The line joining P to this point is called the chord of curvature at P.

Historical Note. A flood of light was thrown upon the study of Geometry, and in particular on the geometry of the conic, by Kepler's enunciation of his doctrine of continuity. By drawing attention to the fact that the ellipse, parabola and hyperbola might be obtained in succession by continuous change, he was enabled to coordinate and unify results already known, particularly in relation to the foci and asymptotes of conics, and to give the whole theory a compactness which, up to his time, it had never possessed.

Some idea of a focus in the parabola had been possessed by Pappus (circa A.D. 300). Kepler showed that this was to be expected by considering the curve as the limiting case of an ellipse, and further that the second focus was at infinity. The name 'focus' was introduced by him, and owes its appropriateness to the fact (a consequence of the geometrical property of § 2, Chap. XI.) that rays of light proceeding from one focus of an ellipse and reflected at the curve converge to the other.

The systematic development of Kepler's theories was, however, the work of Desargues, who, as has been stated, inaugurated a new era in Geometry.

The focus and directrix property (§ 5, Chap. X.) appears first in a fragmentary way in the 'Collections' of Pappus, but remained unnoticed as a defining property of the curves till the time of Newton.

Examples on Chapter XI.

DRAWING EXERCISES.

1. A hyperbola has the origin for centre and the axis of x for one asymptote. Its eccentricity is 2 and the length of its transverse axis 4 units. Construct the vertex of the hyperbola and draw its latera recta. How many solutions of the problem are there?

2. A hyperbola has the origin for one focus, the line $y=x+2$ for one asymptote and the line $x+1=0$ for a tangent. Construct its centre, remaining asymptote, axes, and one point on the curve.

3. A hyperbola has two conjugate diameters inclined at 70° to one another. The real semi-diameter is 3 inches in length, the imaginary semi-diameter 2 inches in length. Construct the axes, foci, etc.

4. Shew how to draw a conic given a focus and three tangents. How many solutions are there?

5. The angle between the asymptotes of a hyperbola is 60° and the conjugate axis is 2 inches long. Construct the positions of foci and directrices, and find any number of points on the curve.

6. An ellipse is drawn on paper. Shew how to construct the centre, axes, foci and directrices.

7. A hyperbola is drawn on paper. Shew how to find its centre, and construct its axes and asymptotes.

RIDERS.

8. The tangent at P to a hyperbola is at right angles to an asymptote. The focal distances SP, S′P cut this asymptote in Q, Q′. Shew that SQ, S′Q′ are each equal to the transverse semi-axis.

9. SY, SZ are the perpendiculars from one focus S of an ellipse on the tangents from an external point T. Prove that the join of T to the other focus is perpendicular to YZ.

10. P, N, G having their usual meanings in the parabola, prove that the length of the tangent drawn from the vertex A to a circle whose centre is G and radius GP is equal to AN.

11. The envelope of a chord of a circle which subtends a right angle at a fixed point in the plane of the circle, and inside it, is an ellipse having the centre and the fixed point as foci.

12. Straight lines from a point equally inclined (in opposite senses) to the tangents from the point to a conic touch a confocal conic.

13. PQQ′P′ is any chord of an ellipse cutting the curve in Q, Q′ and the directrices in P, P′. Prove that PQ, P′Q′ subtend equal angles at the pole of the chord.

14. The director circle of an ellipse and one point on the curve are given in position and magnitude. Prove that the ellipse touches a fixed ellipse.

15. In the last example prove that the envelopes corresponding to different magnitudes of the director circle are confocal.

16. The tangents to an ellipse from P are PQ, PR, and O is the orthocentre of the triangle PQR. Prove that O, P are conjugate points with respect to the director circle of the ellipse.

17. Prove that the chord of curvature at any point of a conic and the tangent at that point are equally inclined to the axes.

18. An ellipse has a given focus and touches two given straight lines. Prove that the loci of the other focus and the centre are straight lines, and that the envelope of the minor axis is a parabola.

19. A series of ellipses is drawn to pass through a given point and to have a common focus and equal transverse axes. Show that they all touch another ellipse having the given point and the given focus as foci.

20. The circles of curvature at the ends P, D of a pair of conjugate diameters of an ellipse meet the curve in Q, R respectively. Prove that PR is parallel to DQ.

21. Prove the following construction for the centre of curvature **X** at a point P on a central conic. From G, the foot of the normal, draw GY perpendicular to PG to meet S′P in Y; YX perpendicular to S′P will meet PG in X.

8. Prove that the locus of corresponding rays of two *equal* pencils, whose corresponding angles are measured in opposite senses, is a rectangular hyperbola having the vertices of the pencils at the ends of a diameter.

9. Shew that the locus of the vertex of a triangle of given base, whose base angles differ by a constant angle, is a rectangular hyperbola. Find the asymptotes.

10. ACA′, BCB′ are two conjugate diameters of an ellipse. The tangents at A and B meet at D. Prove that if EF, any parallel to CD, cut BD, BC in E, F, then AE, A′F intersect at a point on the curve. Deduce a method of constructing an ellipse, given a pair of conjugate diameters in magnitude and position.

11. A and B are fixed points and P a variable point on a hyperbola. Prove that the length intercepted on an asymptote between AP and BP is constant.

12. Two angles of constant magnitude turn about fixed vertices in such a manner that the intersection of two of their sides describes a fixed line. Shew that the intersection of the other two describes a conic passing through the fixed vertices. (Newton.)

13. The locus of the middle point of the portion intercepted on a variable tangent to a parabola between two fixed tangents is a straight line (the tangent parallel to the join of the points of contact of the fixed tangents).

14. Prove property V. of § 5, Chap. IX., by considering the vanishing points of a pair of homographic ranges determined by a variable tangent to a hyperbola on the asymptotes.

15. If the tangent at P meet the tangents at the ends of A, A′ of a diameter of a central conic in L, L′, AL.A′L′ is constant. Hence prove that if any other tangent cut the tangents at A, A′ in M, M′, the point of intersection of LM′, L′M lies on AA′.

16. Prove the theorems of § 2 direct from the conic by making use of § 1, Chap. XI. (B. W. Horne.)

17. Two pairs of tangents from a pair of conjugate points are cut by any other tangent in a harmonic range.

18. If two conics are such that a triangle inscribed in one is circumscribed to the other, an infinite number of such triangles exist.

CHAPTER XV.

PASCAL'S AND BRIANCHON'S THEOREMS.

§ 1. (*a*) Pascal's Theorem.

If a hexagon be inscribed in a conic, the points of intersection of pairs of opposite sides are collinear.

(*b*) Brianchon's Theorem.

If a hexagon circumscribe a conic, the joins of opposite angular points are concurrent.

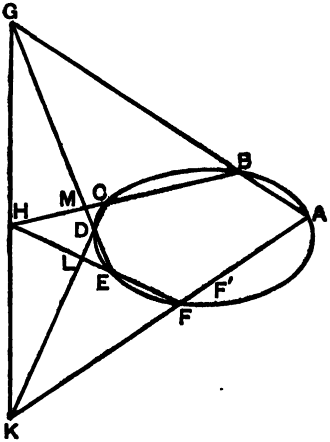


FIG. 115*a*.

Let A, B, C, D, E, F be the angular points of the hexagon (Fig. 115*a*).

Let *a*, *b*, *c*, *d*, *e*, *f* be the sides of the hexagon (Fig. 115*b*).

Then, by the theorem of § 2, Chap. XIV.,

$$B\{CDEA\}=F\{CDEA\}.$$

Then, by the theorem of § 2, Chap. XIV.,

$$b\{cdea\}=f\{cdea\}$$

(using this notation as before to denote the range determined on the line *b* or *f* by the lines *c*, *d*, *e*, *a*).

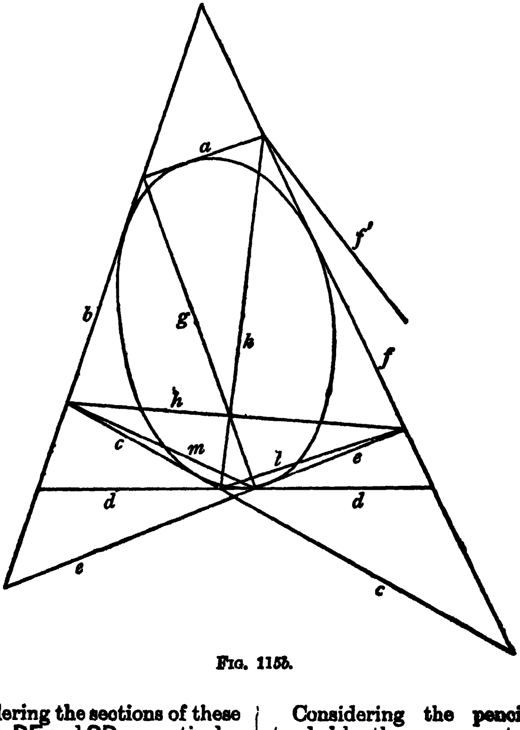


FIG. 115*b*.

Considering the sections of these pencils by DE and CD respectively, we have

$$\{MDEG\}=\{CDLK\}.$$

But these ranges have a common point D.

∴ by § 2, Chap. XIII., MC, EL, GK are concurrent.

Considering the pencils subtended by these ranges at (*de*) and (*cd*) respectively, we have

$$\{mdeg\}=\{cdlk\}.$$

But these pencils have a common ray *d*.

∴ by § 2, Chap. XIII., the points (*mc*), (*el*), (*gk*) are collinear.

Def. The line GHK is called the **Pascal line** of the hexagon ABCDEF.

§ 2. (*a*) **Second Proof.**

Consider the hexagon as determined by the vertices and the intersections of corresponding rays of the two pencils $\{abcd\}$, $\{a'b'c'd'\}$ (Fig. 116*a*).

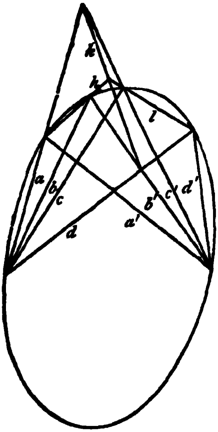


Fig. 116*a*.

Then $\{a'b'c'd'\}$

$=\{abcd\}$ (§ 2 (*a*), Chap. XIV.)

$=\{badc\}$ (§ 2, Chap. XII.).

∴ the joins of cross-intersections of corresponding rays pass through the cross-centre (§ 5, Chap. XIII.).

But these joins are

$(a'a)$ to $(b'b)$, viz. h,

$(b'd)$ to (ac'), viz. k,

$(c'c)$ to $(d'd)$, viz. l.

∴ h, k, l are concurrent.

Def. The point of concurrence of g, h, k is called the **Brianchon point** of the hexagon *abcdef.*

(*b*) **Second Proof.**

Consider the hexagon as determined by the bases and joins of corresponding points of the two ranges {ABCD}, {A′B′C′D′} (Fig. 116*b*).

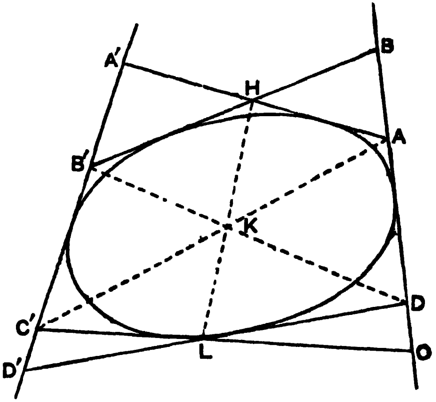


Fig. 116*b*.

Then {A′B′C′D′}

={ABCD} (§ 2 (*b*), Chap. XIV.)

={BADC} (§ 2, Chap. XII.).

∴ the intersections of cross-joins of corresponding rays are collinear in the cross-axis (§ 5, Chap. XIII.).

But these intersections are

A′A with B′B, viz. H,

B′D with AC′, viz. K,

C′C with D′D, viz. L.

∴ H, K, L are collinear.

§ 3. (*a*) *Conversely*. **If the intersections of opposite sides of a hexagon are collinear, its six vertices lie on a conic.**

For, let a conic be drawn through A, B, C, D, E. If it does not pass through F, let it cut AF in F′ (Fig. 115*a*).

Then, since A, B, C, D, E, F′ lie on a conic, the intersections of AB and DE, of BC and EF′ and of CD and F′A are collinear.

But this Pascal line is GK, for the two points G and K on it are the intersections of opposite sides of the hexagon ABCDEF.

∴ F′E and FE must both meet BC on GK. This can only be the case when F and F′ coincide.

(*b*) *Conversely*. **If the diagonals of a hexagon are concurrent, its six sides touch a conic.**

For, let a conic be drawn to touch five of the sides *a*, *b*, *c*, *d*, *e*. If it does not touch *f*, from the point (*af*) draw a tangent *f′* to the conic (Fig. 115*b*).

Then, since *a*, *b*, *c*, *d*, *e*, *f′* touch a conic, the joins of (*ab*) and (*de*), of (*bc*) and (*ef′*) and of (*cd*) and (*f′a*) are concurrent.

But this Brianchon point is the same as that of the hexagon *abcdef*, for two diagonals *g* and *k* of each are identical.

∴ (*fe*) and (*f′e*) must both lie on the join of (*bc*) to this Brianchon point. This can only happen when *f* and *f′* coincide.

§ 4. Deductions from Pascal's and Brianchon's Theorems.

The above theorems are capable of very wide application in the geometry of the conic, owing to the unrestricted nature of the 'hexagon' which figures in each theorem.

To begin with, any order of taking the six points (or lines) will give us a Pascal (or Brianchon) hexagon, provided no distinction is made between any given order and its reverse. The number of possible orders is, under these circumstances, $\frac{1}{2}\underline{|5}$ or 60.

We thus get 60 Pascal or Brianchon hexagons from six given points or lines. Each of these has its Pascal line (or Brianchon point as the case may be).

Important particular cases arise when consecutive vertices or sides of the hexagons coincide.

In the former case the side of the hexagon joining such a pair of ultimately coincident vertices becomes a tangent to the conic at the point of coincidence.

In the latter the vertex of the hexagon, which is the point of intersection of such a pair of ultimately coincident tangent sides, becomes the point of contact of the tangent to the conic in which they coincide.

§ 5. Hence we may apply the theorems in such cases as the following:

(*a*) **Given five points on a conic to find the tangent at any one of them.**

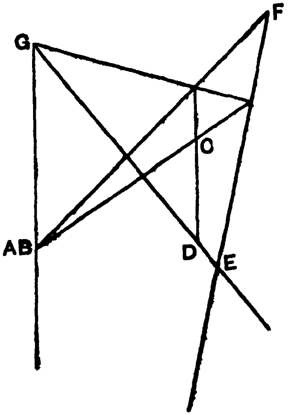


Fig. 117*a*.

Let A, C, D, E, F (Fig. 117*a*) be the five points on the conic. Then, considering A as two ultimately coincident points A, B, we have the Pascal line of the hexagon ABCDEF given by the intersections of BC, EF and CD, FA.

If therefore DE meets this line in G, AB passes through G, and thus AG is the tangent at A.

(*b*) **Given five tangents to a conic to find the point of contact of any one of them.**

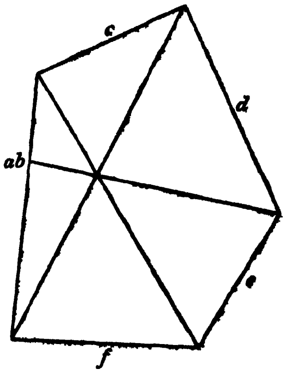


Fig. 117*b*.

Let *a*, *c*, *d*, *e*, *f* (Fig. 117*b*) be the five tangents to the conic. Then, considering a sixth tangent *b* ultimately coinciding with *a*, we have the Brianchon point of the hexagon *abcdef* given by the joins of (*bc*), (*ef*) and (*cd*), (*fa*).

If (*de*) be joined to this point, the joining line will pass through (*ab*), which is ultimately the point of contact of *a*.

§ 6. As another example we may take the following:

(*a*) **If a quadrangle (4-point) be inscribed in a conic, the tangents at opposite angular points meet on the third diagonal of the quadrangle.**

For consider the Pascal hexagon ABCDEF (Fig. 118*a*), wherein A and B coincide ultimately, as also D and E. The third diagonal is the Pascal line, and the theorem follows at once.

(*b*) **If a quadrilateral (4-side) be circumscribed to a conic, the line joining the points of contact of opposite sides passes through the intersections of the diagonals.**

For consider the Brianchon hexagon *abcdef* (Fig. 118*b*), wherein *a*, *b* are a pair of ultimately coincident tangents, as also *d* and *e*. The intersection of diagonals is the Brianchon point, and the theorem follows at once.

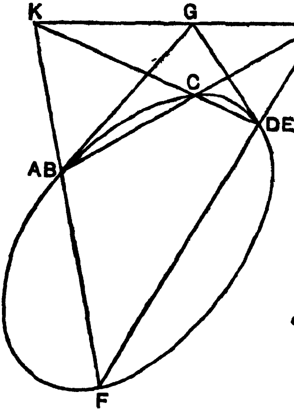


FIG. 118*a*.

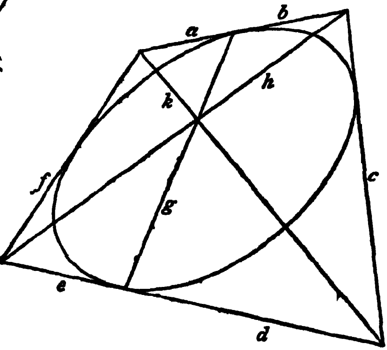


FIG. 118*b*.

The above results establish the theorem found in § 13, Chap. VI., as extended to the conic in Chap. VIII.

§ 7. NOTE. In obtaining the intersections which give the Pascal line or Brianchon point, it is sometimes a little puzzling to a beginner in dealing with re-entrant hexagons to determine 'opposite sides' or 'opposite angular points.' In such cases it is often a saving of time to write down the elements of the hexagon (vertices or sides as the case may be) in the order in which they are to be considered, repeating the first again at the end, *e.g.* AFDCEBA for the Pascal hexagon AFDCEB Opposite elements are consecutive pairs in order, missing one letter between each: *i.e.* the Pascal line is given by the intersections of AF and CE, FD and EB, and DC and BA.

§ 8. We conclude this chapter by proving a couple of useful theorems, which will supply additional instances of the application of the results of the last chapter.

I. **If the six vertices of two triangles lie on a conic, the six sides touch a conic, and conversely.**

If ABC, A′B′C′ be the two triangles (Fig. 119), we have, by § 2 (*a*) of the last chapter, $A\{B'BCC'\} = A'\{B'BCC'\}$.

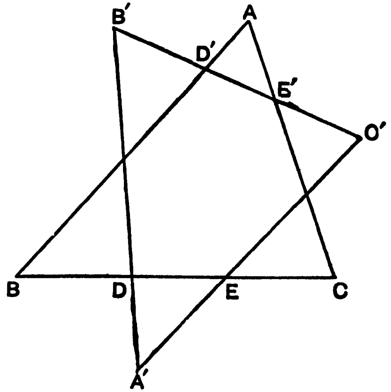


Fig. 119.

Taking the ranges determined by these pencils on B′C′, BC respectively, we have $\{B'D'E'C'\} = \{DBCE\}$.

Hence, if a conic touch AB, AC, A′B′, A′C′ and one of the two BC, B′C′ (see § 6 (*b*), Chap. XIV.), it must, by § 7 (*b*) of that chapter, touch the other.

To prove the converse, we have simply to reverse the steps of the argument, using §§ 2 (*b*) and 7 (*a*).

II. **If two triangles be self-conjugate with respect to a conic, their six vertices lie on a conic and their six sides touch a conic.**

Taking the figure as before, the triangles being ABC, A′B′C′, the cross-ratio of the pencil $A\{B'BCC'\}$ is equal to that of the range formed by the four poles (§ 1 of last chapter).

The pole of AB′ is E, for it is the intersection of the polars of A and B′, *i.e.* of BC and A′C′. Similarly the pole of AC′ is D.

Hence $A\{B'BCC'\} = \{ECBD\} = \{BDEC\} = \{DBCE\}$

(Chap. XII. § 2).

But $\{DBCE\} = A'\{B'BCC'\}$.

$\therefore\ A\{B'BCC'\} = A'\{B'BCC'\}$.

Hence the six vertices lie on a conic, and by I. the six sides touch a conic.

Historical Note. Pascal was born at Clermont and died at Paris in 1662, at the early age of 39. His remarkable theorem, to which he gave the name of the 'mystic hexagram,' was published in an essay on Conics, written when he was only 16!

It has been well said that his fame rests rather upon the promise of his genius than upon its actual accomplishments, as for a considerable portion of his short life he abandoned mathematical studies and lived as a recluse.

Brianchon was a captain in the French army and a contemporary of Poncelet, with whom he occasionally collaborated. His theorem, which was deduced from Pascal's by a theory of polars worked out by himself, first drew attention to the Principle of Duality, which subsequently received many extensions at the hands of Poncelet, Gergonne, Chasles and others.

Examples on Chapter XV.

1. Employ Pascal's theorem to prove Ex. 9, Chapter XIII.

2. A conic touches the sides BC, CA, AB of a triangle at A′, B′, C′. Shew that AA′, BB′, CC′ are concurrent.

3. In the last example, if AA′, BB′, CC′ meet the conic in X, Y, Z, prove that (A′B′XC′), (B′C′YA′), (C′A′ZB′) subtend harmonic pencils at any point on the conic.

4. Apply Pascal's theorem to prove that if a conic touches two sides of a triangle at their middle points and passes through the middle point of the third side, it touches the third side. [The Pascal line is at infinity.]

5. If ABC be any triangle inscribed in a conic, S any point on the curve, AA′, BB′ chords intersecting in O, prove that SA′, BC and SB′, CA intersect in points collinear with O.

Taking the conic as a circle and O as the orthocentre of ABC, prove Steiner's theorem, § 11, Chap. XI.; also that SO is bisected by the pedal line of S.

6. Apply Pascal's theorem to prove that when a rectangular hyperbola circumscribes a triangle, it passes through the orthocentre; and conversely, that all conics through the vertices of a triangle and its orthocentre must be rectangular hyperbolas.

7. Prove that the two triangles obtained by taking alternate sides of a Pascal hexagon are in perspective.

8. Apply Brianchon's theorem to prove Steiner's theorem, that if a triangle circumscribes a parabola its orthocentre is on the directrix. [Consider the hexagon formed by the six tangents a, b, c, a', b' (respectively perpendicular to a, b) and the line at infinity.]

9. B and C are the points of contact of tangents from a point A to a conic; B′, C′ those of tangents from A′; A′C′, A′B′ meet BC in b, c.

Prove that A{BC′B′C} = {BbcC} = A′{BC′B′C} and A, B, C, A′, B′, C′ lie on a conic.

10. Prove that the Pascal lines of the 60 hexagons obtainable by taking six points on a conic in all possible orders, intersect three by three.

11. Shew likewise that the 60 Brianchon points obtainable from six tangents to a conic are collinear three by three.

12. OQ and OR are tangents to a conic at Q and R, and any tangent meets them at K and L. If any point X be taken on QR, XK and XL are conjugate lines (Russell).

CHAPTER XVI.

SELF-CORRESPONDING ELEMENTS.

§ 1. Homographic Ranges on a Conic.

Two ranges of corresponding points A, B, C, ..., A′, B′, C′, ... on a conic are said to be homographic when they subtend homographic pencils at any point of the conic.

It has already been seen that in order to determine completely two homographic pencils, two triads of arbitrarily chosen corresponding rays must be given.

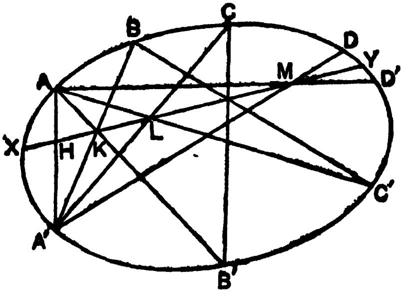


Fig. 120.

Take then any six points A, B, C, A′, B′, C′ on a conic (Fig. 120), and let the first three correspond respectively to the second three.

It is required to find points D and D′ on the conic so that A, B, C, D, A′, B′, C′, D′ subtend pencils of equal cross-ratio at any point of the conic.

Consider the Pascal hexagon AB′CA′BC′: the intersections of AB′, A′B; BC′, B′C; CA′, C′A are collinear.

Let this line of collinearity cut the conic in X, Y.

Take any point D on the conic and let XY cut the rays of the pencil $A'\{ABCD\}$ in H, K, L M.

Then D′ corresponding to D is the point where AM cuts the conic.

For $A'\{ABCD\} = \{HKLM\} = A\{A'B'C'D'\}$.

∴ by Chap. XIV. § 2, A, B, C, D and A′, B′, C′, D′ subtend equi-cross pencils at **any** point of the conic.

But it is further necessary to show that, when any number of pairs of corresponding points have thus been found, XY is the common homographic axis whatever set of four is chosen.

Taking B′ and B, C′ and C, D′ and D successively as the vertices of the pencils above considered, instead of A′ and A, it follows that every pair of cross-joins of corresponding points intersects on XY, since each such pair of equi-cross pencils has a common ray. And as at least three of the known points in each set must be used to determine fresh pairs of corresponding points, every pair of cross-joins of corresponding points must intersect on XY.

§ 2. Self-corresponding Points.

From the previous construction it is clear that X and Y are points which correspond to themselves, so that

$$\{ABCXY\ldots\} = \{A'B'C'XY\ldots\},$$

employing the usual notation for homographic ranges.

§ 3. Applying the Principle of Duality, we can formulate a corresponding theory for homographic sets of tangents to a conic.

Two sets of tangents are said to be homographic when they determine homographic ranges on any tangent to the conic.

The joins of cross-intersections of such tangents are concurrent in a Brianchon point, and the tangents from this point of concurrence to the conic are the **self-corresponding lines** of the two sets. The verification of these facts is left as an exercise.

Homographic ranges of points and sets of tangents of this kind are called **ranges and pencils of the second order.**

§ 4. Determination of the Self-corresponding Points of two Homographic Ranges on the same Straight Line.

Let A, B, C, ..., A′, B′, C′, ... be corresponding points of the two ranges on the line *l*.

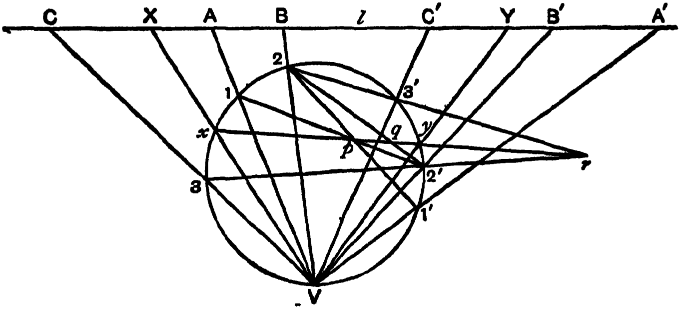


FIG. 121.

Let V be a point on any conic, and let VA, VA′, etc., cut the conic in A_1, A_1', etc.

Then A_1, B_1, C_1, ..., A_1', B_1', C_1', ... are homographic ranges on the conic. Construct their self-corresponding points on the conic, and let the lines joining V to these cut the straight line a in X, Y.

X, Y will be the self-corresponding points of the two ranges.

This method is so important that a detailed proof of the construction is appended.

In practice a circle is the most convenient conic to take; and for brevity we shall denote A_1, A_1'; B_1, B_1'; by 1, 1′; 2, 2′; etc.

The cross-axis is constructed as usual by joining the intersections of (12′, 1′2) and (23′, 2′3).

Let (12′, 1′2), (23′, 2′3) be denoted by p, r, and let 22′ cut the axis in q.

Let the axis cut the circle in x, y and Vx, Vy cut l in X, Y.

$$\begin{aligned}\text{Then }\ V\{ABCX\} = V\{123x\} &= 2'\{123x\} \quad \text{(Chap. XIV. § 2)}\\ &= \{pqrx\}\\ &= 2\{pqrx\}\\ &= 2\{1'2'3'x\}\\ &= V\{1'2'3'x\} \text{ as above}\\ &= V\{A'B'C'X\}.\end{aligned}$$

∴ X is a common point of the two ranges.

So similarly is Y.

These points are real, coincident or imaginary, according as the homographic axis X_1Y_1 cuts, touches or does not meet the circle.

The foregoing problem involves the intersection of a straight line and a circle; in other words, the use of both ruler and compass. It is thus one of the second order, a problem of the first order being one which can be solved by the use of the ruler only. Algebraically, a first order problem involves the solution of a simple equation, a second order problem that of a quadratic.

§ 5. Homographic Pencils at the same or different Vertices.

The common rays of two homographic pencils at the **same vertex** may be found, as in the last article, by cutting the pencils by any conic passing through the vertex and finding the common points of the resulting homographic ranges determined on this conic. In the case of homographic pencils at different vertices instead of common rays, we shall have two pairs of corresponding **parallel** rays. For transfer one pencil to the vertex of the other by drawing through that vertex rays parallel to those of the first pencil. We thus revert to the last case, and can determine the common rays. These and parallels to them through the first vertex are the pairs of rays required.

§ 6. Application of these Results to the Conic.

The theory of the common elements of two homographic forms enables us to complete the discussion of the **conic as the locus of the intersection of corresponding rays of two homographic pencils at different vertices.** We can, for instance, **determine the asymptotes** of a conic thus given, and so obtain a criterion of its species.

For each point on the curve is the intersection of a pair of corresponding rays. Points at infinity will therefore occur whenever a pair of corresponding rays are parallel.

Now two such pairs, or one, or none at all can be drawn according as the cross-axis of the pencils transferred to a common vertex cuts the circle of construction through that vertex in real and distinct, coincident or imaginary points. The conic is correspondingly a hyperbola, parabola or ellipse.

In the first case, the lines joining the vertex to the points where the cross-axis cuts the circle give the **directions of the asymptotes of the hyperbola.**

In the second case, the cross-axis touches the circle, and the line joining the vertex to the point of contact gives the **direction of the axis of the parabola.**

In the third case, the asymptotes are imaginary, and the curve is an ellipse.

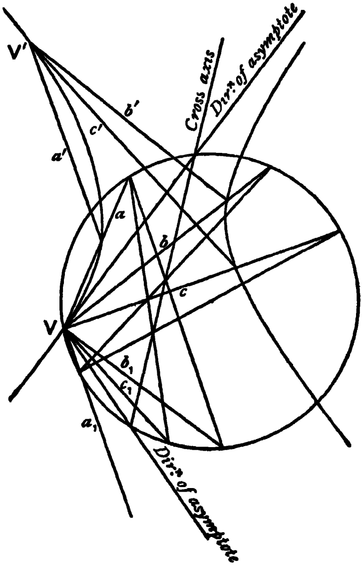


FIG. 122.

It should be noticed that if the conic is a rectangular hyperbola, the asymptotes are at right angles, and the **cross-axis is a diameter of the circle of construction of § 4.**

Figs. 122 and 123 illustrate the first two of these cases.

In these figures V and V′ are the vertices of two homographic pencils $(abc\ldots)$, $(a'b'c'\ldots)$. The latter pencil is transferred to V as vertex by drawing the parallels a_1, b_1, c_1, ... and the cross-axis found in the usual manner.

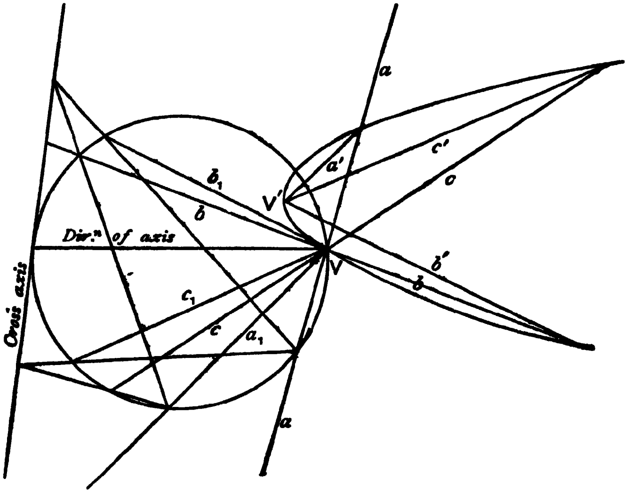


FIG. 128.

§ 7. We conclude this chapter with a theorem of which § 5 (*b*), Chap. XIV., is a particular case.

The envelope of the joins of corresponding points of two homographic ranges on a conic is a conic having double contact with the given conic.

Let A, B, ... A′, B′, ... be the homographic ranges and XY the cross-axis. Then AB′, A′B meet at P on XY.

The polar of P with respect to the conic passes through O the point of intersection of AA′ and BB′ (Chap. VIII. § 7 (VIII.)), also through K the point of intersection of the tangents at X and Y (def. of polar). ∴ it is KO.

∴ the pencils K(PXOY), O(PAKB′) are harmonic, and consequently homographic. But they have a common ray OK.

∴ P, Q, R are collinear, Q and R being the points of intersection of OA and OB′, with the tangents at X and Y respectively (Chap. XIII., § 2 (*b*)).

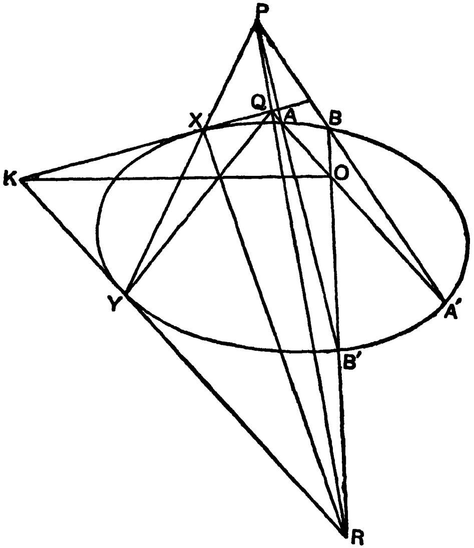


FIG. 124.

But since K{PXOY} is harmonic, by the property of the complete quadrilateral, KO passes through the intersection of XR and YQ.

It follows, by the converse of Brianchon's Theorem, that a conic can be drawn touching KX and KY at X and Y, and also AA′ and BB′ (ROQXKY is the Brianchon hexagon). Considering the conic as determined by the first five of these tangents, viz. two coincident in KX, two in KY and AA′, it may similarly be shewn that it touches CC′, and so on. Which proves the theorem.

A corresponding theorem holds for the homographic sets of tangents referred to in § 3.

Examples on Chapter XVI.

DRAWING EXERCISES.

1. A, A′; B, B′; C, C′ are three pairs of corresponding points of two homographic ranges on the same straight line.

Measuring from A always in the same sense, $AA'=1$, $AB=1\cdot5$, $AB'=2\cdot5$, $AC'=4$, $AC=6$.

Construct the self-corresponding points of the two ranges and the point of each range which corresponds to the point at infinity on the other range.

2. A, B, C, D are four points such that $AB=3$ cm., $BC=1$ cm., $CD=2$ cm. ABC, BAD are corresponding triads of two projective ranges. Find their self-corresponding points.

3. a, b, c are three rays through O whose equations are $4y+3x=0$, $5y+x=0$, $5y-x=0$ respectively. a', b', c' are three other rays through O whose equations are $2y-x=0$, $y+x=0$, $3y-4x=0$.

Find rays d such that $\{abcd\}=\{a'b'c'd\}$.

4. Given O(0, 0), O′(3, 0), A(−1, 4), B(2, 2), C(6, 5), the axes being rectangular, O{ABC}, O′{ABC} define two projective pencils. Construct the rays of the first pencil that are parallel to the corresponding rays of the second pencil.

5. Draw a triangle ABC whose vertices A, B, C lie respectively on $x=2$, $x=5$ and $x=9$, and whose sides AB, BC, CA pass through the points (1, 2), (11, 3), (6, 4).

6. If O be the origin of coordinates and A, B the points (5, 0), (0, 4) respectively, inscribe in the triangle OAB a triangle PQR such that P lies on OA, Q on OB, R on BA and PQ is parallel to $y+3x=0$, QR to $3y-x=0$ and RP to $y=x$.

RIDERS.

7. Two *equal* pencils in the same plane have a pair of imaginary double rays when the corresponding equal angles are measured in the same sense, but have two real and perpendicular double rays when the equal angles are measured in opposite senses.

8. A projective relation on a straight line is given by a pair of corresponding points A, A′ and the double points X and Y. Construct other pairs of corresponding points.

9. A projective relation between two pencils at a common vertex is given by two corresponding rays a, a' and the double rays x, y. Construct other corresponding rays.

10. In a projective relation between two pencils at a common vertex, two pairs of corresponding rays are given, and one double ray. Find the other double ray.

Solve the corresponding problem for projective point-rows.

11. A system of conics touches two given lines at given points. Prove that the axes of the system determine on the normals at the given points similar homographic ranges.

Deduce the envelope of these axes.

12. A variable diameter of a conic meets a given line in P, and through P a line is drawn parallel to the conjugate diameter. Prove that the envelope of this line is a parabola.

13. If O, O′ be two fixed points on a conic and P, P′ two others such that {OPO′P′} is constant, all such pairs as P, P′ belong to a homographic system of points on the conic, of which O, O′ are the double points.

14. Investigate a corresponding theorem for a homographic set of tangents.

CHAPTER XVII.

THE CONSTRUCTION OF A CONIC FROM GIVEN CONDITIONS.

§ 1. It has been proved that a conic can, in general, be drawn to pass through five given points or to touch five given straight lines. We proceed to discuss practical methods (of projective character) for determining any number of points on the curve when five such conditions, or their equivalent, are given.

The dual method of presentation will be followed where appropriate.

§ 2. **Given five points on a conic to construct the conic by points (Maclaurin's method).**

Let A, B, C, D, E be the given points (Fig. 125*a*). Through A draw any two straight lines x, y.

Join BE, CE cutting x, y respectively in E_1, E_2; and let BD, CD cut the same lines in D_1, D_2 respectively. Let E_1E_2, D_1D_2 meet in O. Then, if any line through O cut x, y in F_1, F_2 respectively, BF_1, CF_2 will intersect in a point F on the curve.

Given five tangents to a conic to construct the conic by tangents.

Let a, b, c, d, e be the given lines (Fig. 125*b*). On a take any two points X, Y.

Let the joins of (be) and (ce) to X, Y be denoted by e_1, e_2 respectively, and those of (bd) and (cd) to the same points be denoted by d_1, d_2 respectively. Let the join of (e_1e_2), (d_1d_2) be o. Then, if the joins of any point on o to X, Y be f_1, f_2, the line joining (bf_1) to (cf_2) will be a tangent to the curve.

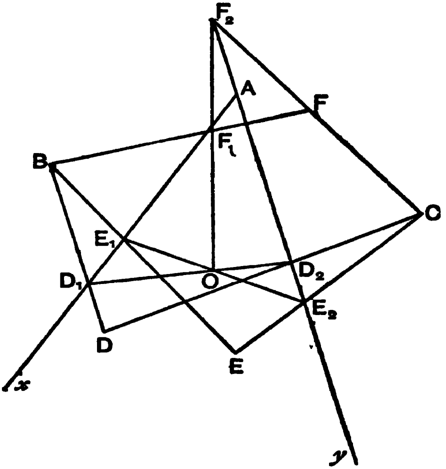


FIG. 125*a*.

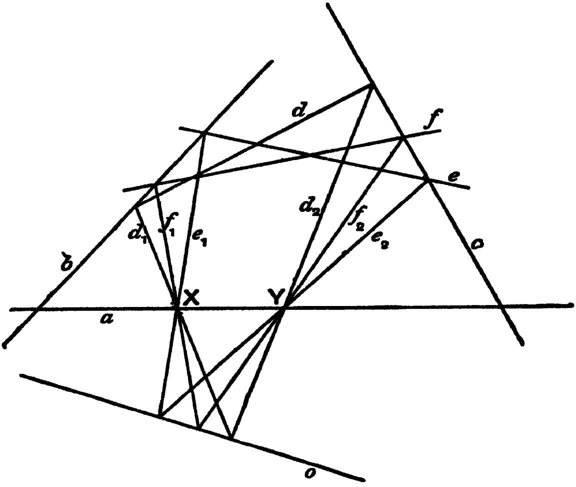


FIG. 125*b*.

Thus, for every straight line we draw through O, we obtain a fresh point on the curve.

The principle of the construction is that homographic pencils at vertices B and C are completely determinable when two triads of corresponding rays BA, BD, BE and CA, CD, CE are known.

The ranges determined by these on x, y, having a common point, are in perspective. So we have

$$\begin{aligned} \mathrm{B}\{\mathrm{ADEF}\} &= \{\mathrm{AD_1E_1F_1}\} \\ &= \mathrm{O}\{\mathrm{AD_1E_1F_1}\} = \{\mathrm{AD_2E_2F_2}\} \\ &= \mathrm{C}\{\mathrm{ADEF}\}. \end{aligned}$$

∴ by § 5, Chap. XIV., F is a point on the curve.

Thus, for every point we take on o, we obtain a fresh tangent to the curve.

The principle of the construction is that homographic ranges on straight lines b, c are completely determinable when two triads of corresponding points (ba), (bd), (be) and (ca), (cd), (ce) are known.

The pencils subtended by these at X, Y, having a common ray, are in perspective. So we have

$$\begin{aligned} b\{adef\} &= \{ad_1e_1f_1\} = o\{ad_1e_1f_1\} \\ &= \{ad_2e_2f_2\} = c\{adef\}. \end{aligned}$$

∴ by § 5, Chap. XIV., f is a tangent to the curve.

§ 3. Pascal's and Brianchon's theorems may also be used for these problems as follows:

Given five points A, B, C, D, E on a conic to determine the remaining point in which the conic meets any line through A.

Given five tangents a, b, c, d, e to a conic to determine the remaining tangent that can be drawn to the conic from any point on a.

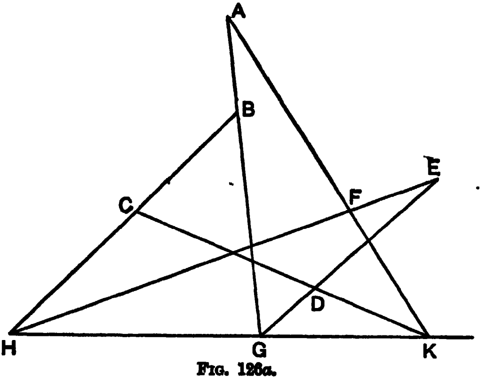


FIG. 126*a*.

Let AF (Fig. 126*a*) be any line through A, and let F be the point to be found where the conic cuts AF.

Take any point on a (Fig. 126*b*), and let f be the tangent to be drawn from it to the conic.

The Pascal line for ABCDEF is the join of the intersections of AB and DE, and CD and FA.

If these be G and K, and GK cut BC in H, EH will cut AF in the required point.

Taking other radial lines through A, further points on the curve may be obtained and the conic constructed by points.

The Brianchon point for *abcdef* is the intersection of the joins of (*ab*) and (*de*), and (*cd*) and (*fa*).

If these be *g* and *k*, and if the join of (*gk*) and (*bc*) be *h*, the join of (*eh*) to the point taken on *a* will be the tangent required.

Taking other points on *a*, further tangents to the curve may be obtained and the conic constructed by tangents.

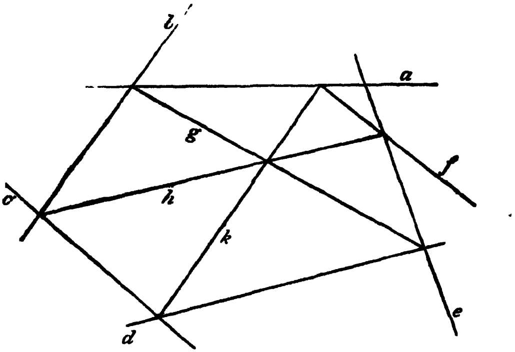


FIG. 126*b*.

§ 4. The particular cases of Pascal's and Brianchon's theorems are of service in solving such problems as the following:

Ex. 1. (*a*) To construct a conic given four points and the tangent at one of them.

(*b*) To construct a conic given four tangents and the point of contact of one of them.

It should be noted that to be given a tangent and its point of contact is to be given two ultimately coincident points or tangents. So that the conditions given are equivalent to five points or five tangents.

The details of the construction are exactly as in § 5, Chap. XV., and are left to the student.

Ex. 2. (*a*) Construct a conic given 3 points and the tangents at two of them.

(*b*) Construct a conic given 3 tangents and the points of contact of two of them.

§ 5. **Any line parallel to an asymptote meets the conic in one point at infinity.** So to be given the **direction of an asymptote** is equivalent to being given a point on the curve at infinity, viz. the point at infinity on any line drawn in the given direction (see Chap. I. § 6).

To be given an asymptote in position is, of course, to be given a tangent whose point of contact is at infinity.

Every parabola touches the line at infinity. So that **a parabola may be drawn to satisfy four conditions**, *i.e.* to pass through four points or touch four lines.

The **direction of the axis** being given is equivalent to two such conditions; for a tangent and its point of contact—both at infinity—are known. We proceed to apply these principles to the following examples.

§ 6. **To construct a conic given the directions of both asymptotes and three points on the curve.**

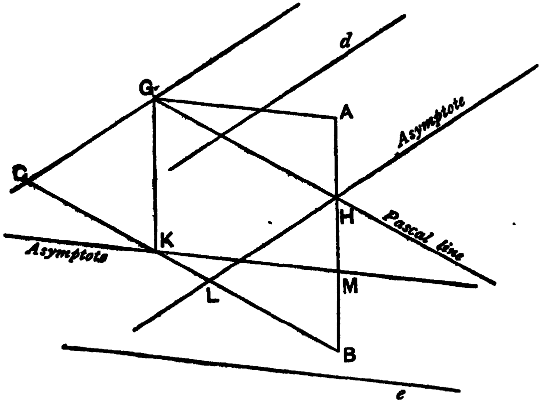


FIG. 127.

Let A, B, C be the three given points and *d*, *e* the straight lines giving the directions of the asymptotes. Then two more points on the curve, D_∞ at infinity on d and E_∞ at infinity on e, are known. If F_∞ be another point on the curve at infinity, either $D_\infty F_\infty$ or $E_\infty F_\infty$ are the actual asymptotes.

Taking the former case, consider the Pascal hexagon $ABCD_\infty F_\infty E_\infty$.

The Pascal line is the join of the intersections of AB and $D_\infty F_\infty$, BC and $F_\infty E_\infty$, CD_∞ and $E_\infty A$.

The first will be the point where the asymptote required cuts AB. The second is the point at infinity on BC. The Pascal line is therefore parallel to BC.

The third is the point G of intersection of a parallel through C to d with a parallel through A to e.

It follows that a parallel to BC through G will cut AB in a point on the asymptote parallel to d; this asymptote may accordingly be drawn.

Again, taking F_∞ as on the other asymptote, consider the Pascal hexagon $ABCD_\infty E_\infty F_\infty$.

AB and $D_\infty E_\infty$ meet at the point at infinity on AB. BC and $E_\infty F_\infty$ meet at the point where the required asymptote cuts BC.

CD_∞ and $F_\infty A$ cut at G as above. A parallel through G to AB therefore cuts BC at a point on the asymptote required.

Both asymptotes are therefore determined.

This problem has been followed out in detail to illustrate the modifications necessary when the data include points or tangents at infinity, and also because the construction affords a verification of a property of the asymptotes already established.

For if the asymptotes cut AB, BC in H, L and M, K, we have, since GL, GB, GM are parallelograms, $CL = GH = KB$; and $HB = GK = AM$, whence $AH = BM$.

In other words, the portions of a chord intercepted between a conic and its asymptotes are equal (§ 16 (II.), Chapter VIII.).

By the use of this property any number of other points on a curve may be found now that the asymptotes are determined, and the curve may accordingly be drawn.

Exs. Construct a conic

(1) Given one asymptote, the direction of the other, and two more points on the curve.

(2) Given one asymptote, the direction of the other, and a tangent and its point of contact.

(3) Given one asymptote, the direction of the other, and two more tangents.

(4) Given the directions of both asymptotes, two points on the curve and the tangent at one of them.

(5) Given the directions of both asymptotes, two tangents and the point of contact of one of them.

§ 7. To construct a parabola given three tangents and the direction of the axis.

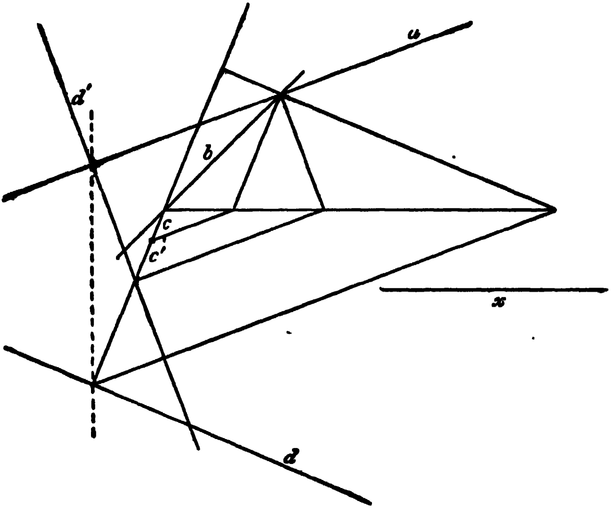


FIG. 128.

Let *a*, *b*, *c* be three tangents, *x* the straight line giving the direction of the axis.

We may consider the curve as possessing two consecutive tangents at infinity, the intersection of which is the point at infinity on x.

Let e_∞, f_∞ be these tangents.

Let it be required to find the point of contact of one of the given tangents, say c.

Consider the Brianchon hexagon $abcc'e_\infty f_\infty$, where c' is the consecutive tangent to c. (The required point of contact is the intersection of c and c'.)

The Brianchon point is given by the intersection of a line through (ab) parallel to c or c', with a line through (bc) parallel to x (since x passes through $e_\infty f_\infty$).

A line through this point parallel to a will cut c at the required point of contact.

Similarly, the point of contact of b may be found by considering the Brianchon hexagon $abb'ce_\infty f_\infty$, b' being the consecutive tangent to b.

Again, we may construct the directrix in position if we can find tangents d and d' perpendicular respectively to c and a. The directrix is then the join of (cd) and (ad').

Consider the Brianchon hexagon $abcde_\infty f_\infty$. The diagonals are respectively (1) a line through (ab) perpendicular to c—for (de_∞) is a point at infinity in this direction, (2) a line through (bc) parallel to x, (3) a line through (cd) parallel to a.

The first two determine the Brianchon point.

A parallel through it to a meets c in a point on d.

The latter tangent is therefore determined.

Similarly, d' may be constructed by considering the Brianchon hexagon $abcd'e_\infty f_\infty$, and the directrix is the dotted line in the figure.

Knowing the directrix and two points on the curve, the focus is readily obtained and any number of points can be constructed.

§ 8. Applications involving the Determination of Common Elements of two Homographic Forms.

(*a*) To find the points in which a given line meets a five-point conic.

Let A, B, C, D, E be the given points.

Then the points C, D, E joined to A and B give two triads of rays which determine on the given line corresponding triads of points C_1, D_1, E_1 and C_2, D_2, E_2.

Let X, Y be the common points of the homographic ranges which can be constructed with these as corresponding triads.

Then $A\{CDEX\}=A\{C_1D_1E_1X\}$
$=A\{C_2D_2E_2X\}=B\{CDEX\}$.

$\therefore$ X is a point on the conic.

So similarly is Y.

(*b*) To find the tangents from a given point to a five-tangent conic.

Let *a*, *b*, *c*, *d*, *e* be the given tangents.

Then the tangents *c*, *d*, *e* determine on *a* and *b* two triads of points which, joined to the given point, give corresponding triads of rays c_1, d_1, e_1 and c_2, d_2, e_2.

Let x, y be the common rays of the homographic pencils which can be constructed having these as corresponding triads.

Then $a\{cdex\}=a\{c_1d_1e_1x\}$
$=a\{c_2d_2e_2x\}=b\{cdex\}$.

$\therefore$ x is a tangent to the conic.

So similarly is y.

§ 9. To construct the centre of a five-point or five-tangent conic.

Suppose the conic determined by the five points A, B, C, D, E.

By Pascal's theorem, as in § 3, determine the point F in which a line through A parallel to CD meets the conic.

The join of the middle points of AF and CD is then a diameter (Chap. VIII. § 10).

Similarly, another diameter may be found and the intersection of the two gives the centre.

[If the diameters are parallel, the conic is, of course, a parabola.]

If we are given five tangents *a*, *b*, *c*, *d*, *e*, by Brianchon's theorem we can construct the tangent *f* parallel to *a*. This may be done by the method of § 3, the point (*af*) being at infinity on *a*.

Similarly a tangent *g* parallel to *b* may be found. The centre of the parallelogram formed by *a*, *f*, *b*, *g* is the centre of the conic.

We can now construct the asymptotes of a five-tangent conic. For by the last article we can construct the tangents from the centre just found, and these are the asymptotes required.

§ 10. To construct a parabola through four points.

This problem has practically been solved in § 6 of the last chapter, but for clearness the construction is given in detail here.

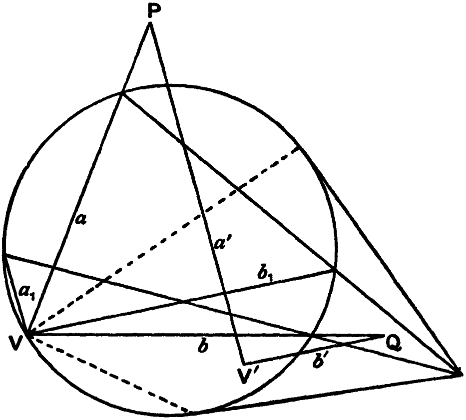


FIG. 129.

Let V, V′, P, Q be the given points; denote the rays VP, VQ, V′P, V′Q by a, b, a', b' respectively, and draw a_1, b_1 through V parallel to a', b'.

Now the cross-axis for two homographic pencils at vertex V touches the circle (of § 4, Chap. XVI.) when the conic is a parabola.

Describing then this circle, which may be any through V cutting the rays a, b, a_1, b_1, we know one point on the cross-axis of the ranges determined on this circle, viz. the point of intersection of the joins of the points where a and b_1, a_1 and b meet the circle. The cross-axis may therefore be either of the tangents from this point to the circle.

Hence two parabolas may be described to pass through the given points, and the directions of their axes are those of the dotted lines in the figure, joining V to the points of contact.

We have now four points and the direction of the axis. By § 5 we have therefore sufficient data to determine by Pascal's theorem any number of points on the curve.

Historical Note. No series of sketches of this kind would be complete without some brief notice of the life of Newton.

The greatest British mathematician and one of the greatest mathematicians of all time was born near Grantham in 1642, and died at the age of 85. He studied at Trinity College, Cambridge, under Barrow, whom he succeeded as Lucasian Professor of Mathematics in the University. Both in extent and variety his achievements in pure and applied mathematics were amazing, considering that he accomplished classical researches in algebra, theory of equations, geometry, dynamics, attractions, and optics, founded the infinitesimal calculus and, greatest of all, established the law of gravitation.

To confine our attention to his contributions to geometry; his connection with the 'locus ad quatuor lineas' has been noticed; he gave an organic description of curves, of which Ex. 12, Chapter XIV., is a particular case; discussed the projection of cubics; the theory of curvature; and what is most commonly remembered in connection with his geometrical work, established the connection between the conic sections and the orbits of the heavenly bodies.

His researches were the means of inaugurating a British school of mathematics, to which Maclaurin may be considered to belong.

Colin Maclaurin, Professor of Mathematics at Aberdeen in 1717, at the age of 19, and subsequently at Edinburgh, continued the geometrical work of Newton on the conic sections and higher plane curves, wrote a treatise on fluxions, and discovered some notable theorems on attractions and figure of the earth. He earned for himself a high reputation as a geometer. Lagrange, indeed, compared his work to all that was best in that of Archimedes—an encomium as high as it was well deserved.

Examples on Chapter XVII.

DRAWING EXERCISES.

(*Take axes of coordinates rectangular and scale in inches.*)

1. By Maclaurin's method draw a conic through the five points
$(4, 0), (3, 4), (0, 3), (-2, 0), (0, -3)$.

2. Also draw a conic through
$(-5, 5), (-4\cdot5, -2), (2\cdot5, 5), (0, 4), (1, 0)$.

3. Draw a conic touching the lines $x=0$, $y=0$, $x=4$, $y=2$, $5x+3y=5$. Find the centre of this conic.

4. ABC is an equilateral triangle of 4″ side. P is a point on AB between A and B and distant 1″ from A. Q is on AC between A and C and distant 2·5″ from A. Construct by tangents the conic which touches AB at P, AC at Q, and also touches BC.

5. Construct by tangents the conic which has the origin for one focus and touches the three lines $x=2$, $\frac{x}{9}+\frac{y}{5}+\frac{1}{2}=0$, $x+y=3$.

6. OX, OY are two axes inclined at an angle of 50°. A hyperbola touches OX at a point 9 cm. from O and OY at a point 8 cm. from O. The hyperbola has an asymptote in the direction OV, where the angle VOX$=35°$ and VOY$=85°$. Construct the centre of the hyperbola and sketch the curve.

7. A conic passes through $(5, -2)$, $(0, 3)$ and $(1, -2)$. Its asymptotes are parallel to $x+2y=0$ and $3x-y=0$. Draw the asymptotes in position and find two more points on the curve.

8. A parabola passes through the points $(0, 0)$, $(-1, 4)$ and $(1, 2)$, and its axis is parallel to $y=x$. Find the position of the axis and the tangent at the vertex.

9. A parabola touches the x-axis at (3, 0) and also touches the lines $y=4x$, $x+y=2$.

Construct the points of contact of these two tangents and the directrix of the parabola.

10. Construct the asymptotes of the conic in Qn. 1.

11. Construct the asymptotes of the conic in Qn. 3.

12. Draw a parabola through the four points

$$(-2, 0), \ (0, 2), \ (2, 2), \ (4, -4).$$

GENERAL CONSTRUCTIONS AND RIDERS.

13. Only one parabola can be drawn to touch four given lines.

14. Given two points on a conic, one asymptote and the direction of the other, find the centre.

15. Given three tangents to a parabola and the point of contact of one of them, find the direction of the axis.

16. Given the asymptotes of a conic and one tangent, construct the point of contact of that tangent by Brianchon's theorem.

17. A conic is given by five tangents. Construct the pole of a given line with respect to it.

18. Given two tangents to a parabola and their points of contact, construct the directrix by Brianchon's theorem.

19. With the data of Ex. 17 construct, by metrical methods, the focus and axis, and check the preceding result.

20. Given three points on a conic and a pole and its polar, construct the conic.

21. Shew that each of a system of conics through four points has a pair of conjugate diameters parallel to the axes of the two parabolas which can be drawn through the four points.

22. Shew that all conics given by four points or four tangents have a common self-conjugate triangle.

CHAPTER XVIII.

RECIPROCATION.

§ 1. In this chapter we shall develop somewhat the method of reciprocation explained in Chapters VI. and VIII.

It has been shewn that from a given property of a figure composed of points and lines we can derive another property of a corresponding figure composed of lines and points, which are respectively the polars and poles of the points and lines of the first figure with respect to a conic.

This conic was called the **base conic,** and when the given theorem is one concerning points and lines *alone* may be any conic chosen arbitrarily. When the theorem is about a conic we take the conic itself as base conic.

From any known theorem we are thus able to write down a reciprocal theorem **whose truth we can at once assert**; and we have thus a useful and valuable method of extending our knowledge of geometrical properties.

An abundance of examples of this kind of reciprocation has been given in the theorems enunciated and proved in dual form in the preceding chapters. To take only three illustrations:

(1) The harmonic property of the complete quadrangle and the harmonic property of the complete quadrilateral are reciprocal.

Vertices of the one reciprocate into sides of the other, diagonal points into diagonal lines, and harmonic ranges in the one figure become harmonic pencils in the other.

(2) Theorems (*a*) and (*b*) of § 5, Chap. XIV., are reciprocal. For by § 1 of that chapter homographic ranges reciprocate into homographic pencils.

(3) If we reciprocate Pascal's theorem with respect to the conic, we have Brianchon's theorem, and conversely.

§ 2. We have now to consider the reciprocal of a circle or conic theorem taking **some other** circle or conic as base.

Let it be required to find the reciprocal of a conic C with respect to a conic X as base.

If we take the polars with respect to X of points on C, we get a family of straight lines which generate **as an envelope** a curve C′, which is called the **polar reciprocal of C with respect to X.**

It is easy to prove that the polar reciprocal of C is also the locus of the poles with respect to X of tangents to C.

For take two points P, Q on C and let p and q be their polars with respect to X.

By definition the polar reciprocal C′ touches p and q. Moreover, the line PQ is the polar of the point (pq) with respect to X.

Let Q move along the conic C till it coincides with P. PQ becomes ultimately the tangent at P. But in this case p and q also move up to coincidence and the point (pq) coincides ultimately with either of the points in which p and q touch their envelope C′.

Hence points on C′ are the poles with respect to X of tangents to C.

C and C′ therefore justify their title of reciprocal curves.

§ 3. We now prove that

The reciprocal C′ of a conic C with respect to a conic X is a conic.

For take four fixed points P, Q, R, S on the conic C.

If K be any fifth point on C, we know that K{PQRS} is constant (§ 2 (*a*), Chap. XIV.).

If *k*, *p*, *q*, *r*, *s* be the polars of K, P, Q, R, S with respect to X, we have (using § 1, Chap. XIV.), that if *p*, *q*, *r*, *s* be four fixed tangents to C′ and *k* any fifth tangent, the range *k*{*pqrs*} has a constant cross-ratio.

∴ by § 7 (*b*), Chap. XIV., the envelope of *k* is a conic, *i.e.* C′, the polar reciprocal of C, is a conic.

§ 4. Prop. **Pole and polar with respect to C reciprocate into polar and pole with respect to C′, the reciprocal conic.**

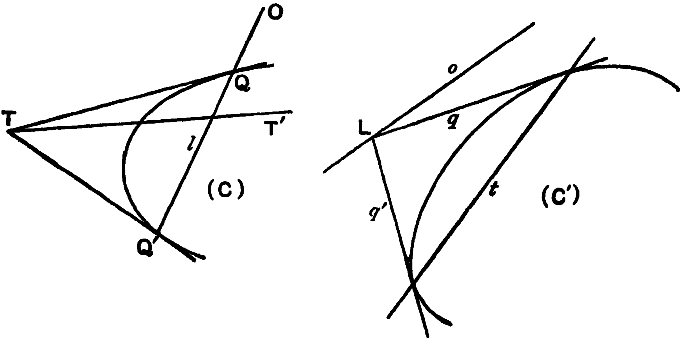


Fig. 180.

(For the sake of clearness the base conic X is not shewn in the figure.)

Let O be any point in the plane of the conic C.

Then to O corresponds as its reciprocal with respect to the base conic a line *o* in the plane of the conic C′.

To a line l through O corresponds a point L on o.

If Q, Q′ are the points where the conic C cuts this line, we have correspondingly q, q', the tangents to C′, which intersect at this point.

In the figure C tangents at Q and Q′ intersect on a fixed line TT′, **the polar of O with respect to C.**

In the figure C′ the join of the points (tq), (tq') passes through a fixed point, **the pole of o with respect to C′.**

∴ pole and polar for the given conic reciprocate into polar and pole for the reciprocal conic.

COR. I. Conjugate lines for C reciprocate into conjugate points for C′, and conversely.

COR. II. A self-conjugate 'three-side' with respect to C reciprocates into a self-conjugate 'three-point' with respect to C′, and conversely.

§ **5.** A number of interesting developments arise when we take the base conic X to be a circle. We have first the theorem:

The reciprocal of a circle with respect to a circle whose centre is S as base conic is a conic having S as focus, and is an ellipse, parabola or hyperbola, according as this centre is inside, upon or outside the given circle.

[As in the case of inversion, the radius of the 'base circle' is a matter of small consequence; the position of the centre is the all-important consideration, and on this account reciprocation with respect to a circle is sometimes called 'point reciprocation.']

Let the dotted circle be the base. Let O be the centre of the circle to be reciprocated and P the pole of any tangent QL with respect to the dotted circle.

Let KX be the polar of O with respect to the same circle.

Then KX is a fixed line, and we require the locus of P.

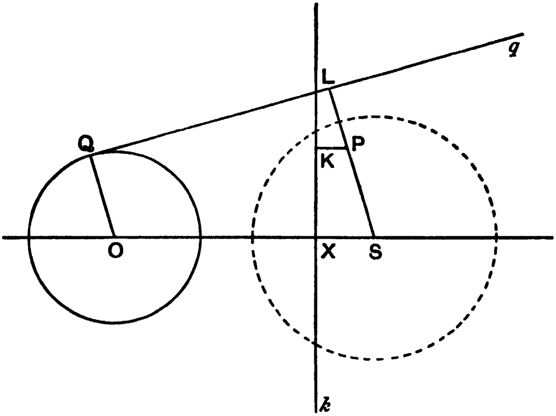


FIG. 181.

By Salmon's Theorem (§ 4, Chap. VI.),

$$\frac{SP}{SO}=\frac{PK}{OQ}.$$

$$\therefore\ \frac{SP}{PK}=\frac{SO}{OQ}=\text{constant}.$$

The locus of P is thus a conic whose **focus is S** and corresponding **directrix the polar of the centre of the given circle with respect to the base circle.**

The eccentricity is $\lesseqgtr 1$, according as $SO \lesseqgtr OQ$, *i e.* according as S is within, on or outside the circle whose centre is O.

Also conversely, the reciprocal of a conic with respect to any circle whose centre is at its focus is a circle.

For if O be the reciprocal of its directrix and QL that of any point P on the conic as before,

$$\frac{SO}{OQ}=\frac{SP}{PK}=\text{eccentricity}=\text{const.}$$

But SO is constant.

∴ OQ is constant. ∴ QL touches a circle with centre O.

§ 6. If we draw tangents from S to the given circle, the reciprocals of these tangents will be points at infinity on the reciprocal curve.

Also the points of contact with the given circle reciprocate into the tangents to the reciprocal curve at these infinitely distant points.

Hence it follows that

'*The asymptotes of the conic are the reciprocals of the points of contact of the tangents from* S *to the given circle.*'

Since the asymptotes intersect at the centre, we have also

'*The centre is the reciprocal of the polar of* S *with respect to the given circle.*'

Ex. 1. Obtain the same result by considering the centre as the pole of the line at infinity.

Ex. 2. Verify the condition of § 5, determining the species of the reciprocal conic by considering the reality or otherwise of the tangents from S.

Ex. 3. Prove that the reciprocal of a parabola with respect to a point on its directrix is a rectangular hyperbola passing through the point. (Prove that it is a conic whose asymptotes are at right angles.)

Ex. 4. Prove that the reciprocal of a conic with respect to a point on its director circle is a rectangular hyperbola; and conversely, that of a rectangular hyperbola with respect to any point is a conic whose director circle passes through that point.

§ 7. The Reciprocal of the Angle between two Lines.

We can with advantage apply the method of reciprocation to many metrical theorems in the case in which the base conic is a circle.

For the **reciprocal of the angle between two lines** is easily proved to be **the angle subtended by the join of the reciprocal points at the centre of the base circle.**

For (Fig. 29), if a, b be the reciprocals of points A, B with respect to the circle whose centre is C, CA and CB are respectively perpendicular to a and b. Also, it is obvious from the same figure that the angle between CA and b is equal to the angle between CB and a. Therefore the angle between any line and the join of C to some point A is equal to the angle between the join of C to the pole of that line and the polar of A—a result which is sometimes useful.

Ex. 1. The reciprocal of a triangle with respect to its orthocentre is another triangle having the same orthocentre.

Ex. 2. Reciprocate for any point the theorem: 'The perpendiculars from the vertices of a triangle on the opposite sides are concurrent.'

Ex. 3. Reciprocate the theorem: 'The lines which bisect the sides of a triangle and are at right angles to them are concurrent.'

§ 8. Let us now apply these results to one or two examples of reciprocation.

Let it be required to reciprocate the theorem:

'Two tangents to a circle are equally inclined to their chord of contact.'

Reciprocate with regard to any point S.

The circle becomes a conic with S as focus.

Tangents to the circle become points on the conic.

The chord of contact becomes the point of intersection of the tangents.

Hence, by the last article, the reciprocal theorem is seen to be

'Two tangents to a conic subtend equal angles at the focus'—a known property (§ 1, Chap. XI.).

Again reciprocate the theorem:

'Angles in the same segment of a circle are equal.'

For purposes of reciprocation, perhaps this is best stated as follows:

'If A and B be two fixed points on a circle and P a variable point on it, the angle APB is constant.'

In the reciprocal figure with respect to any point S, *a*, *b* are fixed tangents, *p* a variable tangent.

Corresponding to AP, BP in the given figure, we have the points (*ap*), (*bp*) in the reciprocal figure.

Hence, by § 7, the reciprocal theorem is:

'The portion of a variable tangent intercepted between two fixed tangents to a conic subtends a constant angle at the focus.'

This result is easily verified from the last.

§ 9. PROP. **A system of coaxal circles can be reciprocated into a system of confocal conics.**

Whatever point is taken, the circles will reciprocate into conics having one focus in common.

To ensure their having a second focus in common, it is necessary that they should have a common centre.

This requires that the centre of reciprocation should be such that its polar with respect to all circles of the system should be the same (§ 6).

But the limiting points of a coaxal system are inverse points for every circle of the system (Chap. V. § 19). The polar of either therefore passes through the other (Chap. VI. § 2), and these are common pole and polar for all circles of the system.

Hence, if L be a limiting point, reciprocation with respect to L gives a system of conics having both a common centre and a common focus, *i.e.* a confocal system.

Ex. 1. Prove that the radical axis reciprocates into the other focus of the confocals.

Ex. 2. Shew how conversely to reciprocate a set of confocal conics into coaxal circles.

Ex. 3. 'If two sets of circles are such that each member of the one set cuts each member of the other set orthogonally, the two sets are each coaxal.' Reciprocate this theorem with respect to a real limiting point.

Ex. 4. Prove one of the two following theorems and deduce the truth of the other by reciprocation:

A common tangent to two coaxal circles subtends a right angle at either limiting point.	Confocal conics cut one another at right angles.

Ex. 5. Consider similarly the theorems:

A common tangent to two circles is cut harmonically by a coaxal circle.	If two confocal conics intersect at P, their tangents at P form a harmonic pencil with the tangents from P to any other confocal.

§ 10. Reciprocation with respect to a Conic.

We can determine the species of the reciprocal conic by the aid of the following theorem:

The reciprocal conic is an ellipse, parabola or hyperbola, according as the centre of the base conic lies within, on or outside the given conic.

For if we draw tangents from this point to the given conic, the reciprocals of these tangents are points at infinity on the reciprocal curve, and the points of contact of these tangents are the reciprocals of the tangents at these points at infinity on the reciprocal curve, *i.e.* the asymptotes.

Now these are only real points of contact when the tangents are real, *i.e.* when their point of intersection lies outside the given conic. The reciprocal curve then has real asymptotes, and is therefore a hyperbola. When the point from which the tangents are drawn lies on the given conic, the asymptotes coincide at infinity and the reciprocal curve is a parabola. When it is inside the given conic, the asymptotes are imaginary, since no real tangents can be drawn, and the reciprocal conic is an ellipse.

§ 11. Frégier's Theorem.

Chords of a conic which subtend a right angle at a fixed point O on the curve pass through another fixed point.

To prove this, let us build up the reciprocal theorem in detail, placing corresponding constituents facing one another in parallel columns.

Thus, reciprocating with respect to O, we have:

Conic passing through O.	Conic touching the line at infinity (*i.e.* a parabola).
Chord	Point of intersection of two tangents
which subtends a right angle at O	which cut at right angles
a fixed point.	a fixed line.

Collecting the results in the right-hand column, we have the theorem:

'Tangents to a parabola which cut at right angles intersect on a fixed line,' viz. the directrix. This has been proved in § 9, Chap. XI.

Ex. Prove by reciprocation that the second fixed point is on the normal at O to the conic.

§ **12.** We shall conclude this chapter with another example.

[The student will find it a helpful exercise to arrange the argument, as in the preceding article, in parallel columns.]

Prove by reciprocation the theorem:

'*If a rectangular hyperbola circumscribes a triangle, it passes through its orthocentre.*'

Take the orthocentre O as the centre of reciprocation, a circle being the base conic.

The reciprocal of a rectangular hyperbola passing through O is a conic touching the line at infinity, *i.e.* a parabola (§ 6).

Also, since the points at infinity on a rectangular hyperbola subtend a right angle at any point on the curve, the tangents from O to the reciprocal curve are at right angles.

O is therefore on the directrix of the parabola.

Also it is the orthocentre of the reciprocal triangle (§ 7, Ex. 1).

Hence the reciprocal theorem is:

'*The orthocentre of a triangle circumscribed to a parabola lies on the directrix,*' which is Steiner's Theorem (§ 11, Chap. XI.).

We may extend this theorem a little to prove another property of the rectangular hyperbola.

Consider the triangle as fixed and a variable parabola touching its sides.

We know that the circumcircle of a triangle formed by three tangents to a parabola passes through the focus (§ 8, Chap. XI.).

Also the polar of a point O on the directrix of a parabola passes through the focus S and is perpendicular to the join of O to the focus (Chap. XI. § 1).

Hence the envelope of this polar is a conic whose focus is O (§ 3, Cor. 3, Chap. XI.), and has the above circumcircle as auxiliary circle.

Reciprocating this with respect to O, since the polar of the origin reciprocates into the centre of the rectangular hyperbola, we have:

'*The locus of the centre of a rectangular hyperbola circumscribing a triangle is a circle.*'

It is easy to show that *this circle is the nine-points' circle of the triangle.*

For, referring back, let the triangle of tangents to the parabola be PQR (Fig. 132).

Since the conic whose focus is O has the circle PQR as auxiliary circle, it touches straight lines through P, Q, R, the vertices of the triangle, respectively perpendicular to OP, OQ, OR (§ 3, Chap. XI.).

But each of these lines may be looked upon as the join of a vertex to the point at infinity on the opposite side, and this join subtends a right angle at O.

By § 7, the reciprocal of such a line is the point of intersection of a side of the reciprocal triangle with the perpendicular on it from the opposite angular point.

Hence the reciprocal theorem finally becomes

'The centre-locus of a rectangular hyperbola circumscribing a triangle is a circle which passes through the feet of the perpendiculars from the vertices on the opposite sides of the triangle,' *i.e.* the nine-points' circle of the triangle.

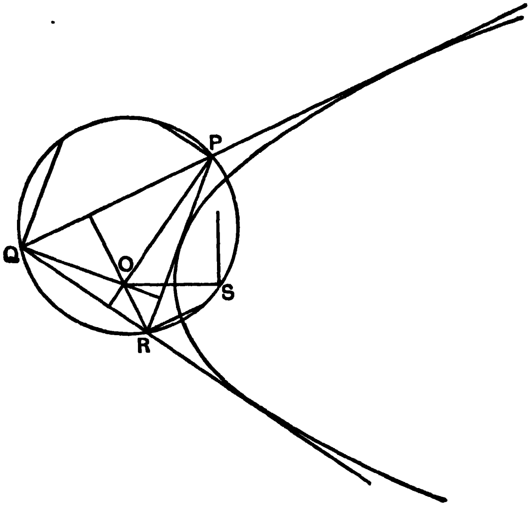


Fig. 182.

Examples on Chapter XVIII.

1. Reciprocate the theorems:

(α) The angle between a tangent to a circle and any chord through its point of contact is equal to the angle in the alternate segment of the circle.

(β) The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

(γ) The angle in a semi-circle is a right angle.

(δ) The tangent to a circle is perpendicular to the radius through its point of contact.

(ϵ) The feet of the perpendiculars from any point on the circumcircle of a triangle on the sides are collinear. (Reciprocate with respect to the point.)

(ζ) The bisectors of the angles of a triangle meet three by three in four points.

2. Reciprocate (1) the normal at any point of a conic, (2) an axis of a conic.

3. Reciprocate with respect to O the following theorem:
'The feet of the four normals from O to a central conic lie on a rectangular hyperbola which passes through O.'

4. Shew also by reciprocating, as in the last example, that the tangents to a conic at the feet of the four normals touch a parabola, which touches the axes.

5. Through any given point on a given tangent to a parabola two lines are drawn equally inclined to the fixed tangent, and so that one of them touches the parabola. Prove that the envelope of the other is a parabola whose directrix passes through the focus of the original parabola.

6. Reciprocate with respect to a focus the following theorem:
'Tangents to a central conic at right angles to one another intersect on a fixed circle.'
Prove the reciprocal theorem.

7. A circle is drawn through the centre of a rectangular hyperbola. Prove that the polar reciprocal of this circle with respect to the rectangular hyperbola is a parabola whose focus is at the centre of the rectangular hyperbola.

8. Reciprocate the theorem of Ex. 13, Chap. VI., with respect to a limiting point.

9. A system of circles passes through two points A, B. A given straight line through A is cut by one of the circles at P. Prove that the tangent at P for different circles of the system envelopes a parabola whose focus is B. Prove this and reciprocate.

10. Reciprocate with respect to the fixed point the theorem of Cor. 3, § 3, Chap. XI.

11. Construct the axis and directrix of the parabola in Ex. 7.

12. Two conics are such that each is its own polar reciprocal with regard to the other. Prove that they have double contact.

13. Prove that all conics passing through the vertices of a triangle and its orthocentre are rectangular hyperbolas.
Hence obtain another proof of the result established in the latter half of § 12.

14. Prove, by the use of properties of conics, that the nine-points' circles of the four triangles obtained by taking as their vertices three out of four given points, are concurrent.

15. Prove the following theorem and write down its reciprocal:

The locus of the pole of a given straight line with respect to a system of coaxal circles is a hyperbola passing through the limiting points of the system and whose asymptotes are respectively perpendicular to the given line and the line of centres, intersecting these lines in points which are the images of one another in the radical axis.

APPENDIX ON MATHEMATICAL INSTRUMENTS.

In addition to the mechanical contrivances described in the text for the construction of the conic sections, reference may be made to the following:

(1) Rottsieper's appliance for drawing a hyperbola (*Zeitschrift für Math.*, vol. 61 (1912), p. 74), depending on § 17, p. 114.

(2) The common ellipsograph. The principle of this apparatus, which was known to Pappus, is contained in Ex. 1, p. 140.

(3) Jürges' instrument for drawing a conic (*Zeitschrift für Math.*, vol. 38 (1893), p. 350), which depends on the fundamental projective property of § 5, p. 219.

I am indebted to Prof. E. T. Whittaker, F.R.S., for drawing my attention to Nos. (1) and (3).

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