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THREE DIMENSIONAL DYNAMICS

THREE DIMENSIONAL DYNAMICS

A VECTORIAL TREATMENT

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PREFACE

This book is based on a course of lectures on "Three Dimensional Dynamics" which I have given to Honours students in Mathematics in the University College of Wales, Aberystwyth, over a number of years. The scope of the book is a study of the three dimensional motion of a particle and of a rigid body when developed from first principles and when treated vectorially. The material covered is thus largely that to be found in A. S. Ramsey's *Dynamics*, Volume II, and Lamb's *Higher Mechanics* with the exclusion of the more advanced portions on analytical dynamics such as the theory of Lagrange's equations, small oscillations, and Hamilton's equations.

Although the basic treatment is vectorial, no mention has been made of tensors. This is because the properties of tensors and the laws of tensor algebra are, on the whole, more difficult to master than are those of vectors, and it has been found that students are often put off if they have to master too much that is new in the way of abstract theory in one course. One has, therefore, to strike a balance between the mastery of a technique and that of the subject itself. Since tensors are not so vital to the study of three dimensional dynamics as are vectors, they have been omitted. The theory of moments of inertia and that of Euler's equations are thus treated, in the main, by the traditional methods.

A short treatment of three dimensional statics has been included as an Appendix; this was thought to be appropriate since statics, although usually treated as a distinct branch of mechanics, is nevertheless simply a special case of dynamics, the conditions for the equilibrium of a particle or of a rigid body being directly deducible from the corresponding equations of motion.

The book is intended for students in their second and third years at a University and who are reading for an Honours degree. It is hoped that it may serve as a useful introduction to the vectorial study of the motion of a particle and of a rigid body; to this end a number of illustrative examples have been worked out in the text, while a set of selected examples has been included at the end of each chapter.

With a view to helping the student to trace the historical development of the subject, a short biographical note has been added as a footnote

PREFACE

wherever a person's name occurs in the text in connection with any theorem or result, while a complete list of names, arranged in chronological order and with dates, is given at the end of the book. In the case of Newton no biographical note has been given, as it was felt that all students of applied mathematics should be familiar with his achievements and discoveries through their school work.

I wish to thank the following for permission to use questions taken from examination papers: The Registry of the University of Cambridge, The Oxford University Press, the Senate of the University of Liverpool, and the Registrar of the University of Wales. Finally, it is my pleasure to thank Professor I. N. Sneddon for his advice and encouragement.

C. E. E.

University College of Wales,
Aberystwyth

May 1957

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CHAPTER I

VECTOR ALGEBRA*

1. Definition and representation

A *vector* may be defined as a quantity which requires for its complete specification both a magnitude and a direction and, in addition, is such that it combines with one or more such quantities according to the laws of addition, subtraction, and multiplication to be defined below. It is clear that a vector, since it possesses both a magnitude and a direction, may be represented diagrammatically by means of a segment of a straight line, the length of the segment being proportional to the magnitude of the vector, according to some convenient scale, whilst its direction is taken to be parallel to that of the vector.

A vector may be either *free* or *localized*. A *free* vector is one whose direction in space is prescribed, but whose point of application and line of application are not; an example of this type of vector is the translational displacement of a rigid body, which may be represented equally well by the displacement of any individual point of it. A *localized* vector also has its direction prescribed, but, in addition, has either its line of application prescribed (e.g. a force acting on a rigid body), or both its point of application and its line of application prescribed (e.g. a force acting on an elastic body). If only its line of application is prescribed, then a *localized* vector is also said to be a *sliding* vector; if both its point of application and its line of application are prescribed, then it is said to be a *bound* vector.

To denote a vector symbolically we shall adopt the standard notation and shall represent it by a single symbol printed in **bold-face** type, for instance, $\mathbf{R}$ denoting a vector quantity whose magnitude or *module* is R ; alternatively we shall denote it by the terminal points of its representative line, taken in the proper order, with a bar placed above, $\overline{AB}$, for example, denoting the vector represented in magnitude and direction by the line AB , the direction of the vector being from A to B . In this second notation the order in which the letters occur is important, for the vectors represented by $\overline{AB}$ and $\overline{BA}$ are not equivalent; although they have the same magnitudes, they have different directions. For two *free* vectors to be identical they must have the same magnitudes

* For an outline of the historical development of vector algebra, see Weatherburn, *Elementary Vector Analysis*, pp. xvii-xxvi.

and directions; if *localized*, then they must also have the same line of action, and, in the case of a *bound* vector, the same point of application.

If the magnitude of a vector is unity, then the vector is said to be a *unit vector*.

A *zero vector*, or *null vector*, is one whose module is zero.

In the following paragraphs we shall discuss the laws of vector algebra as applied to systems of free vectors. Such vectors are defined completely by their magnitude and their direction and may always be regarded, if necessary, as being concurrent. Although we shall consider only systems of free vectors, no difficulty will be experienced when, in later chapters, we come to apply the results so obtained to systems involving localized vectors.

2. Addition and subtraction of vectors

2.1. Addition. Vector quantities add according to the *parallelogram law*, i.e. if $\mathbf{a}$ is a vector represented by the directed line segment $\overline{OA}$

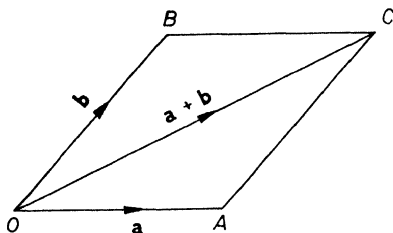


Figure 1.1

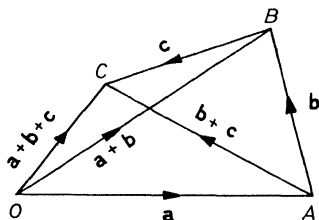


Figure 1.2

and $\mathbf{b}$ is a vector represented by $\overline{OB}$, then their sum $\mathbf{a} + \mathbf{b}$ is defined as the vector represented by $\overline{OC}$, where $OACB$ is the completed parallelogram (Fig. 1.1).*

It can be deduced immediately from the above definition of vector addition that the following laws hold:

(i) *Commutative Law*: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, i.e. the result of adding two vector quantities is independent of the order in which the operation is carried out. From this result it follows that, if a point undergoes two successive displacements, then the two displacements may also be regarded as taking place simultaneously.

* Not all quantities possessing both a magnitude and a direction are vector quantities, for not all directed quantities obey the above law of addition, e.g. finite rotations of a rigid body free to rotate about a fixed point do not (see Chap. III, § 1.3). The necessary and sufficient condition for a directed quantity to be a vector is that it should combine with another directed quantity of the same kind according to the parallelogram law.

(ii) *Associative Law*: $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$, i.e. in the addition of more than two vectors by the repeated use of the above law, the order in which the vectors are associated is irrelevant.

The results are clearly illustrated in Figs. 1.1 and 1.2.

2.2. Subtraction. To subtract a vector $\mathbf{a}$ from a vector $\mathbf{b}$, we add to $\mathbf{b}$ a vector having the same magnitude as $\mathbf{a}$, but whose direction is directly opposite to that of $\mathbf{a}$. Thus if $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ (Fig. 1.3), the vector $\mathbf{b} - \mathbf{a}$ is a vector equal and parallel to that represented by the diagonal $\overrightarrow{AB}$. This construction implies that $(-\mathbf{a})$ is the vector having the same magnitude as $\mathbf{a}$, but whose direction is the opposite

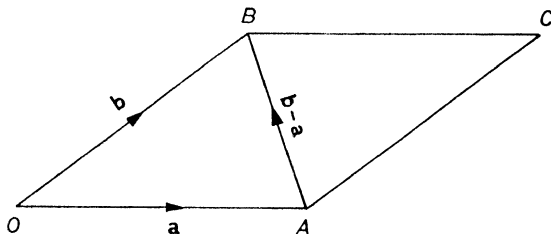


Figure 1.3

of that of $\mathbf{a}$. The law of vector subtraction is therefore such that the equation

$$\mathbf{a} - \mathbf{a} = \mathbf{0}$$

holds, just as the similar relation holds for scalars.

It should be mentioned here, perhaps, that *free* vectors, provided that they are of the same kind, can always be added and subtracted; in the case of *localized* vectors, however, it is only when their lines of application are concurrent, or, in the case of *bound* vectors, when they have a common point of application, that these operations can be carried out.

3. Multiplication of a vector by a scalar

The vector $\mathbf{a} + \mathbf{a}$ is clearly a vector having the same direction as the vector $\mathbf{a}$ but of twice its length. Thus it is denoted by $2\mathbf{a}$. Similarly $m\mathbf{a}$ is a vector having the same direction as $\mathbf{a}$ but of length ma . In particular, if $\mathbf{a}$ is a unit vector, then $m\mathbf{a}$ is a vector having the direction of $\mathbf{a}$ and of length m .

It immediately follows from this definition that

$$m(n\mathbf{a}) = n(m\mathbf{a}) = nm\mathbf{a}.$$

It can also be shown quite simply, using the properties of similar triangles,* that

$$m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}.$$

* See Rutherford, *Vector Methods*, Chap. I, p. 3.

For let ABC be the triangle (Fig. 1.4) such that $\overline{AB} = \mathbf{a}$, $\overline{BC} = \mathbf{b}$. Then we have, by the law of vector addition,

$$\mathbf{a} + \mathbf{b} = \overline{AC}.$$

Now draw the triangle $A'B'C'$ such that $\overline{A'B'} = m\mathbf{a}$, $\overline{B'C'} = m\mathbf{b}$. Then, by the same law, we have

$$m\mathbf{a} + m\mathbf{b} = \overline{A'C'}.$$

But since

$$A'B':AB = B'C':BC$$

by construction, the triangles are similar; from which it follows that

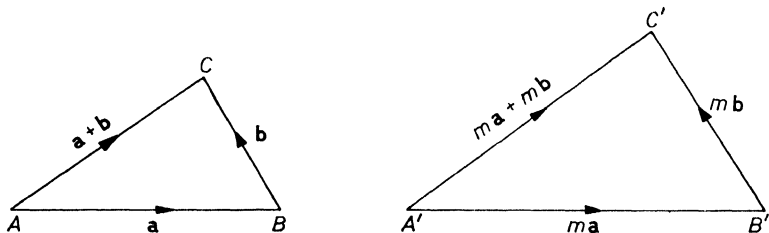


Figure 1.4

$A'C'$ and AC are parallel (and so have the same direction), whilst $A'C' = mAC$. Hence

$$\overline{A'C'} = m\overline{AC},$$

i.e.

$$m\mathbf{a} + m\mathbf{b} = m(\mathbf{a} + \mathbf{b}).$$

4. Components of a vector

If a vector $\mathbf{r}$ can be represented as the sum of a number of vectors,

$$\mathbf{r} = \mathbf{a} + \mathbf{b} + \dots + \mathbf{s},$$

then we say that $\mathbf{r}$ can be resolved into the components $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ..., $\mathbf{s}$.

It is important to realize that any vector can be resolved *uniquely* into three components parallel to any *three* given directions *which are not coplanar*. To show this, let the given vector be represented by $\overline{OP}$, and let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be three unit vectors localized at O and parallel respectively to the three given directions. Through P draw PN parallel to $\mathbf{k}$ (Fig. 1.5) to intersect the plane defined by $\mathbf{i}$ and $\mathbf{j}$ in N ; through N draw lines respectively parallel to $\mathbf{j}$ and $\mathbf{i}$ to cut the axes defined by $\mathbf{i}$ and $\mathbf{j}$ in A and B respectively. If we now complete the parallelepiped having concurrent sides OA , OB and OP as its diagonal, then we have

$$\begin{aligned} \mathbf{r} &= \overline{OP} = \overline{OA} + \overline{AN} + \overline{NP} \\ &= \overline{OA} + \overline{OB} + \overline{OC}. \end{aligned}$$

The required components $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ are thus the sides of the parallelepiped on $\overline{OP}$ as diagonal, respectively parallel to the given directions.

It is clear from the above construction that the resolution into three components, respectively parallel to the given directions, must be unique.*

The important case of resolution into three components is that in which the three directions are mutually orthogonal. In this case the magnitudes of the components $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ are equal to the orthogonal projections of $\overline{OP}$ in the same directions. If we take a set of coordinate

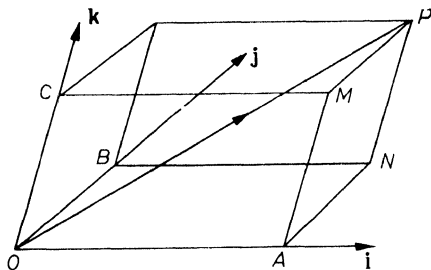


Figure 1.5

axes $OXYZ$ parallel to the three given directions and if (x, y, z) are the coordinates of P with respect to these axes, then we have

$$\overline{OA} = xi, \quad \overline{OB} = yj, \quad \overline{OC} = zk,$$

and hence

$$\mathbf{r} = xi + yj + zk.$$

This is termed the representation of $\mathbf{r}$ in terms of its Cartesian components.

More generally, if $\mathbf{a}$ is a given vector and (a_x, a_y, a_z) are its orthogonal projections in the directions of the coordinate axes respectively, then

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}.$$

(It should be noticed that it is only in the case in which the directions are mutually orthogonal that the components in the respective directions have the same lengths as the orthogonal projections of the vector in the corresponding directions.)

5. Linearly independent vectors

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three vectors and if no real values of the scalar coefficients α , β , γ can be found such that

$$\alpha \mathbf{a} + \beta \mathbf{b} + \gamma \mathbf{c} = \mathbf{0},$$

then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are said to be linearly independent.

* A vector may, of course, be resolved into three components, *all of which lie in the same plane*, in an infinite number of ways.

If three vectors are linearly independent, then they must be *non-coplanar*. For, if they were coplanar, then any one of them could be expressed as a linear function of the other two, i.e. a relation of the above type would exist between them. Conversely, any three non-coplanar vectors are linearly independent. For, if they were not so, then a relation of the above type must exist between them. Let the relation be such that γ , say, is different from zero. Then we could write

$$\mathbf{c} = -\frac{1}{\gamma}(\alpha\mathbf{a} + \beta\mathbf{b}),$$

so that $\mathbf{c}$ must lie in the plane defined by $\mathbf{a}$ and $\mathbf{b}$, i.e. $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar, which is contrary to the initial hypothesis. Thus any three non-coplanar vectors must be linearly independent.

From the fact that a vector can always be uniquely resolved into three components respectively parallel to three non-coplanar vectors, i.e. three linearly independent vectors, it follows that if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three linearly independent vectors and $\mathbf{d}$ is a fourth vector, then $\mathbf{d}$ can always be expressed in the form

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},$$

where α , β , γ have real values. Expressions for α , β , γ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are derived in § 8 below (eqn 1.10).

6. Products of two vectors

The operation of multiplication, applied to two vector quantities, may be defined in two distinct ways. When defined in one way the result

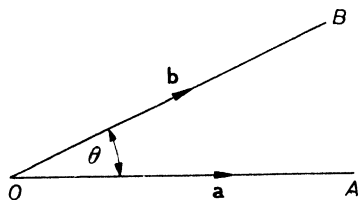


Figure 1.6

is a scalar quantity, in the other a vector quantity. The former is therefore known as the *scalar product*, the latter as the *vector product*, of two vectors.

6.1. Scalar product ($\mathbf{a} \cdot \mathbf{b}$). This is defined as the product of the absolute value of either into the orthogonal projection of the other on it. It clearly has no directional properties and so is a scalar quantity.

Let $\mathbf{a}$ and $\mathbf{b}$ be the two vectors, represented by $\overline{OA}$ and $\overline{OB}$ respectively, and let θ be the angle between them (Fig. 1.6), then the *scalar product* is given by the relation

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta.$$

It follows that

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a},$$

so that the scalar product obeys the *Commutative Law*.

Special Cases

(i) When $\theta = 0^\circ$, i.e. when the two vectors are coincident or parallel,

$$\mathbf{a} \cdot \mathbf{b} = ab;$$

(ii) when $\theta = 90^\circ$, i.e. when the two vectors are orthogonal,

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

Hence, if $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are three unit vectors defining an orthogonal system of reference, then we have

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1;$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0, \quad \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0, \quad \mathbf{k} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = 0.$$

6.1.1. Expression for the scalar product in terms of the x, y, z -components of $\mathbf{a}$ and $\mathbf{b}$. From the definition of $\mathbf{a} \cdot \mathbf{b}$ as the length of $\mathbf{a}$ multiplied by the projected length of $\mathbf{b}$ on $\mathbf{a}$, it follows that if $\mathbf{a}, \mathbf{b}, \mathbf{c}$, are three vectors, then, since the projection of $\mathbf{b} + \mathbf{c}$ on $\mathbf{a}$ is the algebraic sum of the projections of $\mathbf{b}$ and $\mathbf{c}$ on $\mathbf{a}$,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

The scalar product thus obeys the *Distributive Law*, and we may write

$$\mathbf{a} \cdot (m\mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}).$$

Consequently, writing

$$\mathbf{a} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}), \quad \mathbf{b} = (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}),$$

we have

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \cdot (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\ &= a_x b_x (\mathbf{i} \cdot \mathbf{i}) + a_x b_y (\mathbf{i} \cdot \mathbf{j}) + a_x b_z (\mathbf{i} \cdot \mathbf{k}) \\ &\quad + a_y b_x (\mathbf{j} \cdot \mathbf{i}) + a_y b_y (\mathbf{j} \cdot \mathbf{j}) + a_y b_z (\mathbf{j} \cdot \mathbf{k}) \\ &\quad + a_z b_x (\mathbf{k} \cdot \mathbf{i}) + a_z b_y (\mathbf{k} \cdot \mathbf{j}) + a_z b_z (\mathbf{k} \cdot \mathbf{k}) \\ &= a_x b_x + a_y b_y + a_z b_z. \end{aligned} \tag{1.1}$$

Alternatively, this result could have been obtained as follows. Let (l_1, m_1, n_1) , (l_2, m_2, n_2) be the direction cosines of the lines OA , OB respectively: then

$$\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

But

$$(l_1, m_1, n_1) = (a_x/a, a_y/a, a_z/a), \quad (l_2, m_2, n_2) = (b_x/b, b_y/b, b_z/b),$$

and hence

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= ab \cos \theta \\ &= ab \frac{(a_x b_x + a_y b_y + a_z b_z)}{ab} \\ &= a_x b_x + a_y b_y + a_z b_z. \end{aligned}$$

Since the quantity $ab \cos \theta$ is in no way dependent upon the orientation of the chosen system of coordinate axes, it follows that $a_x b_x + a_y b_y + a_z b_z$ is an *invariant* with respect to all such systems.

An example of a physical quantity which may be represented by a scalar product is the work done by a force. Suppose we denote the force by $\mathbf{F}$, then, if its point of application undergoes a small displacement $\delta \mathbf{r}$, the work done by the force in this displacement, being equal to the product of the displacement and the component of the force in the direction of the displacement, will be given by

$$\delta W = \mathbf{F} \cdot \delta \mathbf{r}.$$

If (X, Y, Z) are the components of $\mathbf{F}$ and $(\delta x, \delta y, \delta z)$ those of $\delta \mathbf{r}$, when referred to a set of orthogonal axes, then, by the above theory, we also have

$$\delta W = X\delta x + Y\delta y + Z\delta z.$$

6.2. Vector product ($\mathbf{a} \wedge \mathbf{b}$). This is defined as the vector which is perpendicular to both the vectors $\mathbf{a}$ and $\mathbf{b}$, whose magnitude is $ab \sin \theta$,

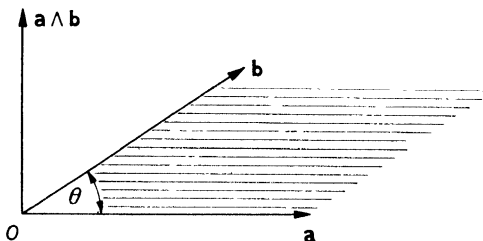


Figure 1.7

and so is numerically equal to the area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$, and whose sense is that of a right-handed screw rotating from $\mathbf{a}$ to $\mathbf{b}$ (Fig. 1.7). We shall denote it by $\mathbf{a} \wedge \mathbf{b}$, although other notations which are frequently used are $\mathbf{a} \times \mathbf{b}$ and $[\mathbf{a}\mathbf{b}]$.

From the definition it follows that

$$\mathbf{a} \wedge \mathbf{b} = -(\mathbf{b} \wedge \mathbf{a}),$$

so that the vector product does not obey the *Commutative Law*.

Special Cases

(i) When $\theta = 0^\circ$, so that $\mathbf{a}$ and $\mathbf{b}$ are coincident or parallel, then

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{0};$$

(ii) when $\theta = 90^\circ$, and the vectors are orthogonal, then

$$\mathbf{a} \wedge \mathbf{b} = ab\mathbf{n},$$

where $\mathbf{n}$ is a unit vector at right angles to both $\mathbf{a}$ and $\mathbf{b}$ and whose direction is that of the vector product $\mathbf{a} \wedge \mathbf{b}$.

⤵ If $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ form a set of unit vectors, mutually orthogonal, we therefore have

$$\mathbf{i} \wedge \mathbf{i} = \mathbf{j} \wedge \mathbf{j} = \mathbf{k} \wedge \mathbf{k} = \mathbf{0};$$

$$\mathbf{i} \wedge \mathbf{j} = -(\mathbf{j} \wedge \mathbf{i}) = \mathbf{k}, \quad \mathbf{j} \wedge \mathbf{k} = -(\mathbf{k} \wedge \mathbf{j}) = \mathbf{i}, \quad \mathbf{k} \wedge \mathbf{i} = -(\mathbf{i} \wedge \mathbf{k}) = \mathbf{j}.$$

The last three results hold, of course, only if $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ define a right-handed triad, i.e. the relative positions of the axes are such that a clockwise rotation about the axis 'i' through a right angle brings the axis 'j' into the original position of the axis 'k', and similarly for rotations about the other axes.

6.2.1. Components of the vector product. These may be obtained either by first showing that the vector product obeys the distributive law and then proceeding as in the case of the scalar product, or, alternatively, from the properties of the vector product, viz. that $\mathbf{a} \wedge \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ and has a magnitude $ab \sin \theta$. Both methods are considered.

First Method. To show that the vector product obeys the distributive law, we have to show that, if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are any three vectors,

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}.$$

Let Γ be a plane at right angles to $\mathbf{a}$; then all three vectors $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{a} \wedge \mathbf{c}$, and $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c})$ will be parallel to this plane. In Fig. 1.8 let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be represented by $\overline{OA}, \overline{OB}, \overline{OC}$ respectively, so that $\mathbf{b} + \mathbf{c}$ is represented by $\overline{OD}$, and let the plane of the paper represent Γ . Then, if $\overline{OM}_1, \overline{OM}_2$ are the orthogonal projections of $\mathbf{b} + \mathbf{c}$ on this plane, the projection of $\mathbf{b} + \mathbf{c}$ on Γ is $\overline{OM}$, where $\overline{OM}$ is the diagonal of the parallelogram on $\overline{OM}_1$ and $\overline{OM}_2$. It is clear that the lengths of OM_1, OM_2 , and OM are given by

$$OM_1 = b \sin \angle AOB, \quad OM_2 = c \sin \angle AOC,$$

$$OM = |\mathbf{b} + \mathbf{c}| \sin \angle AOD.$$

Now let this parallelogram be rotated through a right angle and at the same time let its dimensions be altered in the ratio $a:1$. Let M'_1, M'_2 , and M' be the new positions of M_1, M_2 , and M ; then clearly, from the definition of the vector product, $\overline{OM}'_1, \overline{OM}'_2$, and $\overline{OM}'$ have the

magnitudes and directions of the respective vector products $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{a} \wedge \mathbf{c}$, and $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c})$, and so represent them completely. But $\overline{OM'}$ is the diagonal of the parallelogram $OM'_1M'_2$ so that

$$\overline{OM'} = \overline{OM'_1} + \overline{OM'_2},$$

i.e. $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$.

Having obtained this result we can now determine the components of $\mathbf{a} \wedge \mathbf{b}$ respectively parallel to the coordinate axes defined by the unit

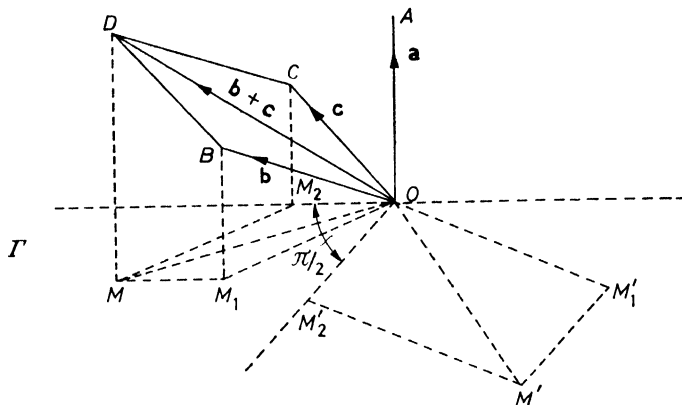


Figure 1.8

vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. For, in terms of the x, y, z -components of $\mathbf{a}$ and $\mathbf{b}$, we have

$$\mathbf{a} \wedge \mathbf{b} = (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \wedge (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}),$$

which may be expanded, in view of the above result, giving

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= a_x b_x (\mathbf{i} \wedge \mathbf{i}) + a_x b_y (\mathbf{i} \wedge \mathbf{j}) + a_x b_z (\mathbf{i} \wedge \mathbf{k}) \\ &\quad + a_y b_z (\mathbf{j} \wedge \mathbf{i}) + a_y b_y (\mathbf{j} \wedge \mathbf{j}) + a_y b_z (\mathbf{j} \wedge \mathbf{k}) \\ &\quad + a_z b_x (\mathbf{k} \wedge \mathbf{i}) + a_z b_y (\mathbf{k} \wedge \mathbf{j}) + a_z b_z (\mathbf{k} \wedge \mathbf{k}) \\ &= (a_y b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}. \end{aligned} \quad (1.2)$$

The x, y, z -components of $\mathbf{a} \wedge \mathbf{b}$ are thus:

$$(a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x).$$

The expression for $\mathbf{a} \wedge \mathbf{b}$ in terms of these components (eqn 1.2) is most easily memorized when written in the form of a determinant, viz.

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

Second Method. For convenience, let us denote the vector product $\mathbf{a} \wedge \mathbf{b}$ by $\mathbf{p}$, its components by (p_x, p_y, p_z) . Then, if (l, m, n) are its direction cosines,

$$(l, m, n) = (p_x/p, p_y/p, p_z/p),$$

where

$$p = ab \sin \theta.$$

But $\mathbf{p}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, so that, if (l_1, m_1, n_1) , (l_2, m_2, n_2) are the direction cosines of $\mathbf{a}$ and $\mathbf{b}$ respectively, then

$$ll_1 + mm_1 + nn_1 = 0,$$

$$ll_2 + mm_2 + nn_2 = 0,$$

which, in terms of the x, y, z -components of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{p}$, become

$$p_x a_x + p_y a_y + p_z a_z = 0,$$

$$p_x b_x + p_y b_y + p_z b_z = 0.$$

Consequently (p_x, p_y, p_z) must be such that

$$\frac{p_x}{a_y b_z - a_z b_y} = \frac{p_y}{a_z b_x - a_x b_z} = \frac{p_z}{a_x b_y - a_y b_x} = \kappa, \quad (1.3)$$

where κ is some constant. To determine (p_x, p_y, p_z) we have therefore to evaluate κ .

Now from (1.3) we have

$$p_x = \kappa(a_y b_z - a_z b_y), \quad p_y = \kappa(a_z b_x - a_x b_z), \quad p_z = \kappa(a_x b_y - a_y b_x). \quad (1.4)$$

Also

$$p_x^2 + p_y^2 + p_z^2 = a^2 b^2 \sin^2 \theta. \quad (1.5)$$

Substituting for (p_x, p_y, p_z) from (1.4), we therefore have

$$\kappa^2 \{ (a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 + (a_x b_y - a_y b_x)^2 \} = a^2 b^2 \sin^2 \theta.$$

But the left-hand side of this equation, by Lagrange's identity, equals

$$\begin{aligned} \kappa^2 \{ (a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) - (a_x b_x + a_y b_y + a_z b_z)^2 \} \\ = \kappa^2 \{ a^2 b^2 - a^2 b^2 \sin^2 \theta \} \\ = \kappa^2 a^2 b^2 \sin^2 \theta. \end{aligned}$$

Therefore $\kappa = \pm 1$. It is necessary, however, for κ to have either one sign or the other. To decide upon the choice of sign, we need only apply the above results to a known case. Thus, if we take

$$\mathbf{a} = \mathbf{i} = (1, 0, 0), \quad \mathbf{b} = \mathbf{j} = (0, 1, 0),$$

then, as we know, we shall have

$$\mathbf{p} = \mathbf{a} \wedge \mathbf{b} = \mathbf{k} = (0, 0, 1).$$

Hence, substituting in the relation

$$\kappa = \frac{p_z}{a_x b_y - a_y b_x}$$

(eqn 1.3), we have $\kappa = 1$, so that the positive sign must be taken.* The expressions for the components of $\mathbf{p}$ now follow from (1.4) and are identical with those obtained above.

A physical quantity which possesses the characteristics of a vector and which may be represented by means of a vector product is a couple. Let the couple consist of forces $(-\mathbf{F}, \mathbf{F})$ acting in the plane of the paper, say, and let O and O' be points on their respective lines of action (Fig. 1.9). Let $\overline{OO'} = \mathbf{r}$. Then the magnitude of the couple, as defined by its moment, is given by

$$\begin{aligned} G &= F \cdot OM \\ &= F \cdot OO' \sin \theta, \end{aligned}$$

where OM is the perpendicular from O on to the line of action of $\mathbf{F}$ and $\theta = \angle OO'M$. The axis of the couple is perpendicular to the

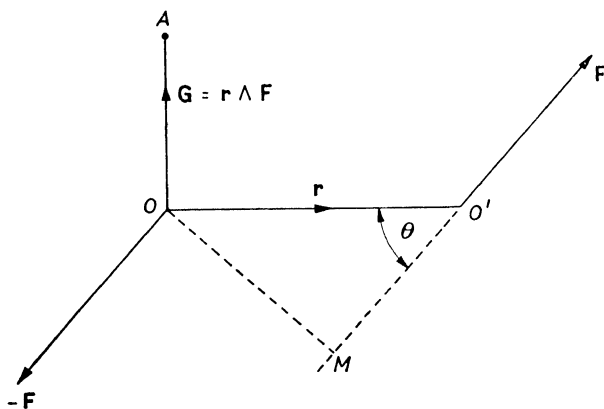


Figure 1.9

plane of the paper and, by the usual convention as to its sign, is either out of or into the paper according as to whether the couple would tend to turn a body on which it acts in a counter-clockwise or clockwise direction. (In Fig. 1.9, the axis of the couple points out of the paper.) Consequently, if we represent the couple by a line segment OA of suitable length and having the appropriate direction, then $\overline{OA}$ will clearly have the magnitude and direction of the vector product $\mathbf{r} \wedge \mathbf{F}$. Since the effect of two or more couples acting on a body, the forces comprising the couples being such that they pass through O and O' , is a couple whose magnitude and direction are given by compounding

* Actually, the choice of sign depends upon whether one adopts a right-handed or left-handed system of coordinate axes; with a left-hand system, i.e. one such that a counter-clockwise rotation about the 'i' axis brings the 'j' axis into the original position of the 'k' axis, one would have $\kappa = -1$.

the representative lines of the individual couples according to the parallelogram law of addition, a couple is a vector quantity. In addition, we know that the forces of a couple may be moved about and have their magnitudes and directions altered in any way provided only that the plane defined by the forces retains its direction and that the moment of the couple is unchanged as regards both its magnitude and its direction*. A couple may thus be equally well represented by any line segment drawn perpendicular to the plane defined by the forces and having the appropriate magnitude and direction. It follows, therefore, that a couple is a *free* vector. A couple $(-\mathbf{F}, \mathbf{F})$ is thus completely represented by the *free* vector whose magnitude and direction are those of $\mathbf{r} \wedge \mathbf{F}$, where $\mathbf{r} = \overline{OO'}$, O and O' being *any* two points on the respective lines of action of the forces.

7. Products of three vectors

7.1. Scalar product of a vector and the vector product of two other vectors $(\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c})^\dagger$. This is known as the *triple scalar product*. It is also written $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$.

To evaluate it, we have, using (1.1) in conjunction with the expressions for the components of the vector product obtained above,

$$\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c} = a_x(b_y c_z - b_z c_y) + a_y(b_z c_x - b_x c_z) + a_z(b_x c_y - b_y c_x),$$

or, more conveniently,

$$[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] \equiv \mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Since interchanging two adjacent rows of a determinant changes its sign but otherwise leaves it unaltered, it follows that

$$[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = -[\mathbf{a} \ \mathbf{c} \ \mathbf{b}] = [\mathbf{c} \ \mathbf{a} \ \mathbf{b}] = -[\mathbf{c} \ \mathbf{b} \ \mathbf{a}] = [\mathbf{b} \ \mathbf{c} \ \mathbf{a}] = -[\mathbf{b} \ \mathbf{a} \ \mathbf{c}].$$

Geometrical Interpretation. By drawing the parallelepiped having $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{c}$ for concurrent sides, as in Fig. 1.10, it is seen that $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ represents its volume. For $\mathbf{b} \wedge \mathbf{c} = bc \sin \theta \mathbf{n}$, where θ is the angle between $\mathbf{b}$ and $\mathbf{c}$ and $\mathbf{n}$ is a unit vector perpendicular to $\mathbf{b}$ and $\mathbf{c}$, so that

$$\begin{aligned} [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] &\equiv \mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c} = bc \sin \theta (\mathbf{a} \cdot \mathbf{n}) \\ &= bc \sin \theta \times AM \\ &= \text{Area of base} \times \text{perpendicular height} \\ &= \text{Volume of parallelepiped.} \end{aligned}$$

* See appendix §2.

† It should be noticed that it is not necessary to bracket $\mathbf{b} \wedge \mathbf{c}$ in any way, since no ambiguity of meaning arises if no brackets are inserted; $\mathbf{b}$ can be associated only with $\mathbf{c}$, for $(\mathbf{a} \cdot \mathbf{b}) \wedge \mathbf{c}$ would have no meaning.

If $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 0$, the volume of the parallelepiped is zero. This means that, provided $\mathbf{b}$ and $\mathbf{c}$ are not coincident, $\mathbf{a}$ must be coplanar with $\mathbf{b}$ and $\mathbf{c}$. That this is so also follows directly from the properties of the scalar product; for $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c}$ will be zero when $\mathbf{a}$ is at right angles to $\mathbf{b} \wedge \mathbf{c}$, i.e. when $\mathbf{a}$ lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.

It should be noted that since a determinant vanishes if any two of its rows are identical, the triple scalar product also vanishes if any two of the vectors occurring in it are parallel.

From the above it follows that the necessary and sufficient condition for three vectors, no two of which are parallel, to be coplanar is that their triple scalar product should be zero. Since, as we have seen, the condition for three vectors to be linearly independent is that they

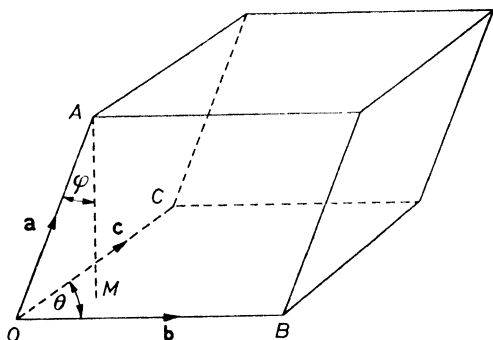


Figure 1.10

should be non-coplanar, the condition for the linear independence of three vectors, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, no two of which are parallel, is that $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] \neq 0$.

7.2. The vector product of a vector and the vector product of two other vectors $[\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})]$. This is known as the *triple vector product*. It is clearly a vector at right angles to $\mathbf{b} \wedge \mathbf{c}$ and so must be coplanar with $\mathbf{b}$ and $\mathbf{c}$. It must be possible therefore to express it in the form

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \beta \mathbf{b} - \gamma \mathbf{c}, \quad (1.6)$$

where β and γ are scalar coefficients.

To determine β and γ , and so the expression for $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$, let us write

$$\mathbf{p} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}).$$

Then, since

$$\mathbf{a} = (a_x, a_y, a_z), \quad \mathbf{b} \wedge \mathbf{c} = (b_y c_z - b_z c_y, b_z c_x - b_x c_z, b_x c_y - b_y c_x),$$

we have, applying the formulae for the components of a vector product,

$$\begin{aligned} p_x &= a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z) \\ &= b_x(a_y c_y + a_z c_z) - c_x(a_y b_y + a_z b_z) \\ &= b_x(a_x c_x + a_y c_y + a_z c_z) - c_x(a_x b_x + a_y b_y + a_z b_z) \\ &= b_x(\mathbf{a} \cdot \mathbf{c}) - c_x(\mathbf{a} \cdot \mathbf{b}). \end{aligned}$$

But, since $\mathbf{p}$ is of the form (1.6),

$$p_x = \beta b_x - \gamma c_x;$$

hence

$$\beta = \mathbf{a} \cdot \mathbf{c}, \quad \gamma = \mathbf{a} \cdot \mathbf{b}.$$

Therefore,

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})^*.$$

In the case of the triple vector product it is important that the vector product occurring in it ($\mathbf{b} \wedge \mathbf{c}$ above) should be clearly indicated by means of brackets, for it is easily verified that

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) \neq (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}.$$

For, applying the above expansion, we have

$$\begin{aligned} (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} &= -\{\mathbf{c} \wedge (\mathbf{a} \wedge \mathbf{b})\} \\ &= -\{\mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a})\} \\ &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}). \end{aligned}$$

But this clearly cannot be equivalent to the above expression for $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ unless $\mathbf{c}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$, which is impossible unless $\mathbf{a}$ and $\mathbf{c}$ are identical, in which case the two expressions are clearly equivalent. Hence it follows that the manner in which the vectors are associated within the triple vector product is important.

8. Scalar and vector products of two vector products

In applying vector algebra to problems in Mechanics one sometimes encounters the scalar and vector products of two vector products. Although they can be evaluated easily on the basis of the above theory, it will be useful to obtain expressions for them here.

(i) *Scalar Product* $[(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d})]$. If we write $\mathbf{a} \wedge \mathbf{b} = \mathbf{p}$, then we have

$$\begin{aligned} (\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) &= \mathbf{p} \cdot (\mathbf{c} \wedge \mathbf{d}) \\ &= \mathbf{c} \cdot (\mathbf{d} \wedge \mathbf{p}) \\ &= \mathbf{c} \cdot [\mathbf{d} \wedge (\mathbf{a} \wedge \mathbf{b})] \\ &= \mathbf{c} \cdot [\mathbf{a}(\mathbf{d} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{d} \cdot \mathbf{a})] \\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}). \end{aligned} \tag{1.7}$$

* The above proof is based on the expansion of the vector product in terms of its components. It is, however, possible to obtain this result using a purely vector derivation—see, for instance, Milne, *Vectorial Mechanics*, Chap. I, § 31, p. 18; also, *Math. Gaz.*, Vol. XXXV, No. 311 (1951), p. 50.

This may be conveniently written in the form of a determinant, viz.

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}.$$

If we put $\mathbf{c} = \mathbf{a}$, $\mathbf{d} = \mathbf{b}$, then (1.7) becomes

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{b}),$$

or

$$(\mathbf{a} \wedge \mathbf{b})^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2,$$

which, in its more familiar form, is Lagrange's identity, already used above.

(ii) *Vector Product* $[(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d})]$. This is a vector at right angles to $\mathbf{a} \wedge \mathbf{b}$ and so coplanar with $\mathbf{a}$ and $\mathbf{b}$. It may be expressed, therefore, as a linear function of $\mathbf{a}$ and $\mathbf{b}$. Similarly, it is coplanar with $\mathbf{c}$ and $\mathbf{d}$ and so, also, may be expressed as a linear function of $\mathbf{c}$ and $\mathbf{d}$.

To obtain these expressions, let us write $\mathbf{a} \wedge \mathbf{b} = \mathbf{p}$ as above. Then we have

$$\begin{aligned} (\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d}) &= \mathbf{p} \wedge (\mathbf{c} \wedge \mathbf{d}) \\ &= \mathbf{c}(\mathbf{b} \cdot \mathbf{d}) - \mathbf{d}(\mathbf{p} \cdot \mathbf{c}) \\ &= \mathbf{c}[\mathbf{a} \mathbf{b} \mathbf{d}] - \mathbf{d}[\mathbf{a} \mathbf{b} \mathbf{c}]. \end{aligned} \quad (1.8)$$

Alternatively, writing $\mathbf{c} \wedge \mathbf{d} = \mathbf{p}$, we have

$$\begin{aligned} (\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d}) &= (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{p} \\ &= -\{\mathbf{p} \wedge (\mathbf{a} \wedge \mathbf{b})\} \\ &= -\{\mathbf{a}(\mathbf{b} \cdot \mathbf{p}) - \mathbf{b}(\mathbf{a} \cdot \mathbf{p})\} \\ &= \mathbf{b}[\mathbf{a} \mathbf{c} \mathbf{d}] - \mathbf{a}[\mathbf{b} \mathbf{c} \mathbf{d}]. \end{aligned} \quad (1.9)$$

If we equate these two expressions, then we obtain a relation between the four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, viz.

$$[\mathbf{a} \mathbf{b} \mathbf{d}]\mathbf{c} - [\mathbf{a} \mathbf{b} \mathbf{c}]\mathbf{d} = [\mathbf{a} \mathbf{c} \mathbf{d}]\mathbf{b} - [\mathbf{b} \mathbf{c} \mathbf{d}]\mathbf{a}$$

Hence we can write

$$\mathbf{d} = \frac{[\mathbf{b} \mathbf{c} \mathbf{d}]\mathbf{a} - [\mathbf{a} \mathbf{c} \mathbf{d}]\mathbf{b} + [\mathbf{a} \mathbf{b} \mathbf{d}]\mathbf{c}}{[\mathbf{a} \mathbf{b} \mathbf{c}]}, \quad (1.10)$$

provided that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are such that $[\mathbf{a} \mathbf{b} \mathbf{c}]$ does not vanish, i.e. provided that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are not coplanar. From this it follows that any vector $\mathbf{d}$ may always be expressed as a linear function of three other vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ except in the case in which $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar, i.e. provided that they are linearly independent, a result in agreement with that obtained in § 5 above.

9. Division of vectors

Since a vector can be multiplied by a vector to give either a scalar or a vector quantity, one would expect to be able to define two possible

operations of division. This, however, is not so, the operation of division being avoided in vector theory.

To see why this is so, let us consider the following possibilities:

(i) Let m be a scalar quantity, $\mathbf{a}$ a given vector. Then one could define the quotient of m by $\mathbf{a}$ to be that vector such that, when multiplied by $\mathbf{a}$ scalarly, gives m . Let $\mathbf{b}$ be this vector; then $\mathbf{b}$ would have to satisfy the equation

$$\mathbf{a} \cdot \mathbf{b} = m. \quad (1.11)$$

But, if $\mathbf{c}$ is any vector orthogonal to $\mathbf{a}$, so that

$$\mathbf{a} \cdot \mathbf{c} = 0,$$

then the vector $\mathbf{b} + \lambda \mathbf{c}$, where λ is a scalar coefficient, would also satisfy (1.11). Consequently there is no unique vector $\mathbf{b}$ which satisfies the equation.

(ii) Let $\mathbf{a}$ and $\mathbf{c}$ be two given vectors. Then the quotient of $\mathbf{c}$ by $\mathbf{a}$ could be defined as that vector $\mathbf{b}$ which, when multiplied by $\mathbf{a}$ vectorially, gives $\mathbf{c}$, i.e. $\mathbf{b}$ would have to satisfy the equation

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{c}. \quad (1.12)$$

But, since $\mathbf{a} \wedge \mathbf{b}$ is a vector at right angles to $\mathbf{a}$, this equation could only have a meaning if $\mathbf{c}$ were also at right angles to $\mathbf{a}$. But $\mathbf{c}$ is a given vector and so need not be at right angles to $\mathbf{a}$. Consequently, in general, no vector $\mathbf{b}$ can be found to satisfy (1.12). But even if $\mathbf{c}$ were orthogonal to $\mathbf{a}$, so that a vector $\mathbf{b}$ could be found to satisfy the above equation, the vector $\mathbf{b} + \lambda \mathbf{a}$ obtained by adding to $\mathbf{b}$ an arbitrary vector parallel to $\mathbf{a}$, would also satisfy (1.12). Thus (1.12) cannot, in any case, have a unique solution.

In view of the fact that neither of these operations leads to a unique vector, the operation of division by a vector has no place in vector theory.

10. Differentiation of a vector with respect to a scalar

Supposing both the magnitude and the direction of a vector $\mathbf{a}$ depend upon the value of a scalar variable t ; both may change, for instance, with the time. If we denote this dependence upon t by writing $\mathbf{a} = \mathbf{a}(t)$, then the differential coefficient, or derivative, of $\mathbf{a}$ with respect to t is defined as a limiting process by the equation

$$\frac{d\mathbf{a}}{dt} = \lim_{t \rightarrow 0} \frac{\mathbf{a}(t+\delta t) - \mathbf{a}(t)}{\delta t} = \lim_{t \rightarrow 0} \frac{\delta \mathbf{a}}{\delta t}.$$

In Fig. 1.11, if $\mathbf{a}(t)$ and $\mathbf{a}(t+\delta t)$ are represented by $\overline{OA}$ and $\overline{OA'}$ respectively, then $\delta \mathbf{a}$ is represented by $\overline{AA'}$.

Since the division of a vector by a scalar does not destroy its vector nature, the derivative of a vector with respect to a scalar variable is itself a vector. This fact is perhaps best demonstrated by writing $\mathbf{a}$ in terms of its components, viz.

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k};$$

here (a_x, a_y, a_z) are scalar variables of t , while $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are unit vectors having fixed directions in space. If $(\delta a_x, \delta a_y, \delta a_z)$ now represent the

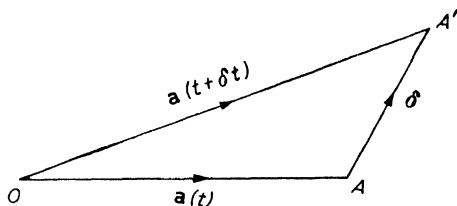


Figure 1.11

changes in (a_x, a_y, a_z) consequent upon a small change δt in t , we have

$$\delta \mathbf{a} = \delta a_x \mathbf{i} + \delta a_y \mathbf{j} + \delta a_z \mathbf{k},$$

which, on dividing by δt and proceeding to the limit, gives

$$\frac{d\mathbf{a}}{dt} = \frac{da_x}{dt} \mathbf{i} + \frac{da_y}{dt} \mathbf{j} + \frac{da_z}{dt} \mathbf{k}.$$

Thus $d\mathbf{a}/dt$ is clearly a vector.

The following results may now be deduced:

$$(i) \quad \frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}.$$

For, expressing $\mathbf{a} + \mathbf{b}$ in terms of its components,

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (a_x + b_x) \mathbf{i} + (a_y + b_y) \mathbf{j} + (a_z + b_z) \mathbf{k} \\ &= (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) + (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}). \end{aligned}$$

On differentiating, we thus have

$$\begin{aligned} \frac{d}{dt}(\mathbf{a} + \mathbf{b}) &= \left(\frac{da_x}{dt} \mathbf{i} + \frac{da_y}{dt} \mathbf{j} + \frac{da_z}{dt} \mathbf{k} \right) + \left(\frac{db_x}{dt} \mathbf{i} + \frac{db_y}{dt} \mathbf{j} + \frac{db_z}{dt} \mathbf{k} \right) \\ &= \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}. \end{aligned}$$

$$(ii) \quad \frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{b}.$$

We have

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z;$$

therefore, differentiating,

$$\begin{aligned}\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) &= \left(a_x \frac{db_x}{dt} + a_y \frac{db_y}{dt} + a_z \frac{db_z}{dt} \right) + \left(\frac{da_x}{dt} b_x + \frac{da_y}{dt} b_y + \frac{da_z}{dt} b_z \right) \\ &= \mathbf{a} \cdot \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{b}.\end{aligned}$$

Since both terms on the right-hand side are scalar products, it follows that the order in which the vectors occur in the respective products is immaterial.

$$(iii) \quad \frac{d}{dt}(\mathbf{a} \wedge \mathbf{b}) = \mathbf{a} \wedge \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \wedge \mathbf{b}.$$

Writing $\mathbf{a} \wedge \mathbf{b}$ in terms of its components,

$$\mathbf{a} \wedge \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k};$$

therefore

$$\begin{aligned}\frac{d}{dt}(\mathbf{a} \wedge \mathbf{b}) &= \left(a_y \frac{db_z}{dt} - a_z \frac{db_y}{dt} \right) \mathbf{i} + \left(a_z \frac{db_x}{dt} - a_x \frac{db_z}{dt} \right) \mathbf{j} + \left(a_x \frac{db_y}{dt} - a_y \frac{db_x}{dt} \right) \mathbf{k} \\ &\quad + \left(\frac{da_y}{dt} b_z - \frac{da_z}{dt} b_y \right) \mathbf{i} + \left(\frac{da_z}{dt} b_x - \frac{da_x}{dt} b_z \right) \mathbf{j} + \left(\frac{da_x}{dt} b_y - \frac{da_y}{dt} b_x \right) \mathbf{k} \\ &= \mathbf{a} \wedge \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \wedge \mathbf{b}.\end{aligned}$$

Since a vector product changes its sign if the order in which the vectors occur in it is inverted, it follows that the order in which $\mathbf{a}$, $\mathbf{b}$, and their derivatives occur in the above vector products must be maintained as shown.

(iv) If a vector $\mathbf{a}$ is written as the product of its module a with the direction vector $\mathbf{n}$, whose module is unity, then

$$\frac{d\mathbf{a}}{dt} = \frac{d}{dt}(a\mathbf{n}) = \frac{da}{dt}\mathbf{n} + a\frac{d\mathbf{n}}{dt}.$$

For, if (n_x, n_y, n_z) are the components of $\mathbf{n}$,

$$a\mathbf{n} = a n_x \mathbf{i} + a n_y \mathbf{j} + a n_z \mathbf{k},$$

so that

$$\begin{aligned}\frac{d}{dt}(a\mathbf{n}) &= \frac{d}{dt}(a n_x) \mathbf{i} + \frac{d}{dt}(a n_y) \mathbf{j} + \frac{d}{dt}(a n_z) \mathbf{k} \\ &= \left(\frac{da}{dt} n_x + a \frac{dn_x}{dt} \right) \mathbf{i} + \left(\frac{da}{dt} n_y + a \frac{dn_y}{dt} \right) \mathbf{j} + \left(\frac{da}{dt} n_z + a \frac{dn_z}{dt} \right) \mathbf{k} \\ &= \frac{da}{dt} (n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k}) + a \left(\frac{dn_x}{dt} \mathbf{i} + \frac{dn_y}{dt} \mathbf{j} + \frac{dn_z}{dt} \mathbf{k} \right) \\ &= \frac{da}{dt} \mathbf{n} + a \frac{d\mathbf{n}}{dt}.\end{aligned}$$

Some special results relating to the rate of change of a vector, and which may be readily deduced from the differential properties derived above, are important.

(1) If $\mathbf{a}$ is a vector of *constant* magnitude, then we have

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = \frac{d}{dt}(a^2) = 0.$$

But, from (ii),

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = 2\mathbf{a} \cdot \frac{d\mathbf{a}}{dt},$$

so that

$$\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0,$$

i.e. $d\mathbf{a}/dt$ must be perpendicular to $\mathbf{a}$.

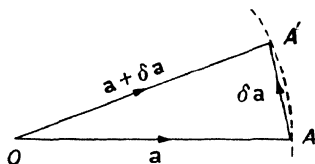


Figure 1.12

This result might also have been deduced quite simply in another way. In Fig. 1.12 let $\overline{OA}$ and $\overline{OA'}$ represent the vectors $\mathbf{a}$ and $\mathbf{a} + \delta\mathbf{a}$ corresponding to times t and $t + \delta t$ respectively; then, since $\mathbf{a}$ is a vector of constant magnitude, $\mathbf{a}$ and $\mathbf{a} + \delta\mathbf{a}$ have the same lengths, so that

$$OA = OA'.$$

Consequently, as $\delta\mathbf{a} \rightarrow 0$, A' approaches A along the arc of a spherical curve having O as centre. The limiting direction of $d\mathbf{a}/dt$ is thus along the tangent to the curve at A , and so is at right angles to $\mathbf{a}$.

(2) From (iv) above we have

$$\frac{d\mathbf{a}}{dt} = \frac{d}{dt}(a\mathbf{n}) = \frac{da}{dt}\mathbf{n} + a\frac{d\mathbf{n}}{dt}.$$

But since $\mathbf{n}$ is a unit vector, $\mathbf{n}$ and $d\mathbf{n}/dt$ are orthogonal vectors. Consequently we have

$$\left|\frac{d\mathbf{a}}{dt}\right|^2 = \left(\frac{da}{dt}\right)^2 + a^2\left|\frac{d\mathbf{n}}{dt}\right|^2,$$

from which it follows that $|d\mathbf{a}/dt|$ is not in general equal to da/dt , or, in words, *the module of the rate of change of a vector is not in general equal to the rate of change of its module.*

A simple illustration of this result is furnished by considering the motion of a point P describing a circle of radius a about a centre O . Writing $\overrightarrow{OP} = \mathbf{a}$, the velocity of P in space is given by

$$\mathbf{v} = \frac{d\mathbf{a}}{dt},$$

and hence

$$\left| \frac{d\mathbf{a}}{dt} \right|^2 = \mathbf{v}^2 = a^2 \omega^2,$$

where ω is the angular velocity of P in the circle. But, for circular motion, a is constant, so that $da/dt = 0$. Consequently

$$\left| \frac{d\mathbf{a}}{dt} \right| \neq \frac{da}{dt}.$$

11. Integration of a vector function

As in ordinary analysis, so in vector theory, the process of integration is the reverse of that of differentiation. Thus, if $\mathbf{f}(t)$ is a vector function defining a variable vector, whose magnitude and direction depend upon the scalar variable t , and if $\mathbf{F}(t)$ is a vector function of t such that

$$\mathbf{f}(t) = \frac{d}{dt} \mathbf{F}(t), \quad (1.13)$$

then $\mathbf{F}(t)$ is said to be the integral of $\mathbf{f}(t)$ with respect to t , and we write

$$\int \mathbf{f}(t) dt = \mathbf{F}(t).$$

Clearly $\mathbf{F}(t)$ is indefinite to the extent of an arbitrary constant vector $\mathbf{C}$, since, if $\mathbf{F}(t)$ satisfies (1.13), $\mathbf{F}(t) + \mathbf{C}$ also satisfies (1.13). In general, therefore, if $\mathbf{F}(t)$ is a vector function defined by (1.13), then

$$\int \mathbf{f}(t) dt = \mathbf{F}(t) + \mathbf{C},$$

where $\mathbf{C}$ is termed the constant of integration.

The above applies when the integrand is a vector function. In the application of vector theory one has, also, frequently to evaluate integrals in which the integrand has the form of a scalar product, and thus, while expressed in terms of vectors, the integrand is a scalar function. In such cases the integral is also a scalar function and the constant of integration a scalar constant.

As examples illustrating the above, we have the following :

$$(i) \quad \int \left\{ \mathbf{a}(t) \wedge \frac{d^2 \mathbf{a}(t)}{dt^2} \right\} dt = \mathbf{a}(t) \wedge \frac{d\mathbf{a}(t)}{dt} + \mathbf{C},$$

$$(ii) \quad \int \left\{ \mathbf{a}(t) \cdot \frac{d\mathbf{a}(t)}{dt} \right\} dt = \frac{1}{2} (\mathbf{a}(t) \cdot \mathbf{a}(t)) + C \\ = \frac{1}{2} a^2(t) + C,$$

$\mathbf{C}$ being a vector constant, C a scalar constant.

When integrals of the above types occur in problems of a physical nature, then $\mathbf{C}$ and C , in general, are determined from the initial conditions of the problem.

12. Moment of a vector

Such a quantity is frequently met with in the application of vector algebra to Mechanics. It may be defined with respect to either (i) a point, or (ii) a line. The definition with respect to a line, however, is dependent upon that with respect to a point.

12.1. Moment with respect to a point. Let O be a given point and let $\mathbf{F}$ denote either a *bound* vector or a *sliding* vector (see § 1). Then, if P is a point on the line of action of $\mathbf{F}$, the moment of $\mathbf{F}$ with respect to O is defined by the equation

$$\mathbf{M} = \mathbf{r} \wedge \mathbf{F},$$

where $\mathbf{r} = \overrightarrow{OP}$ (Fig. 1.13).

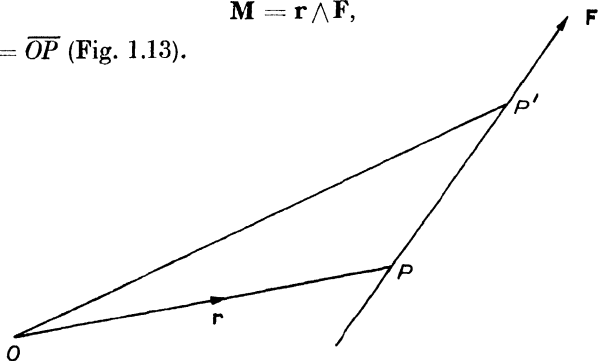


Figure 1.13

In the above definition P is some point on the line of action of $\mathbf{F}$. We shall now show that $\mathbf{M}$, as defined, is independent of the position of P on $\mathbf{F}$, i.e. for a given point O , $\mathbf{M}$ depends only upon the magnitude and the position of the line of action of $\mathbf{F}$. To do this, let P' be some other point on the line of action of $\mathbf{F}$. Then, using the above definition, we have

$$\begin{aligned} \mathbf{M} = \mathbf{r} \wedge \mathbf{F} &= \overrightarrow{OP} \wedge \mathbf{F} = (\overrightarrow{OP'} + \overrightarrow{P'P}) \wedge \mathbf{F} \\ &= \overrightarrow{OP'} \wedge \mathbf{F} - \overrightarrow{PP'} \wedge \mathbf{F} \\ &= \overrightarrow{OP'} \wedge \mathbf{F}, \end{aligned}$$

since $\overrightarrow{PP'}$ and $\mathbf{F}$ are collinear vectors, i.e. $\mathbf{M}$ is independent of the position of P .

It is seen that the moment of a vector with respect to a point, being represented by a vector product, is itself a vector. Further, whilst

its magnitude and direction are defined in terms of those of $\mathbf{r}$ and $\mathbf{F}$, yet, as a quantity, it is associated essentially with the point O ; it must therefore be regarded as a *localized* vector, its point of application being at O .

12.2. Moment with respect to a line. Let O be a point on the given line whose direction is defined by the unit vector $\boldsymbol{\lambda}$, and let P be a point on the line of action of $\mathbf{F}$. Then the moment of $\mathbf{F}$ with respect to the given line is defined to be the component of the moment of $\mathbf{F}$ with respect to O in the direction of the line. It is therefore a scalar quantity being defined by the equation

$$M_{\lambda} = \boldsymbol{\lambda} \cdot \mathbf{M} = \boldsymbol{\lambda} \cdot (\mathbf{r} \wedge \mathbf{F}),$$

where $\mathbf{r} = \overrightarrow{OP}$. Since $\mathbf{M}_{\lambda}$ has the form of a triple scalar product, the moment of $\mathbf{F}$ with respect to the line vanishes when $\boldsymbol{\lambda}$, $\mathbf{r}$, and $\mathbf{F}$ are coplanar, i.e. when the line $\boldsymbol{\lambda}$ and $\mathbf{F}$ are coplanar.

We have already shown that $\mathbf{M}$, the moment of $\mathbf{F}$ with respect to O ,

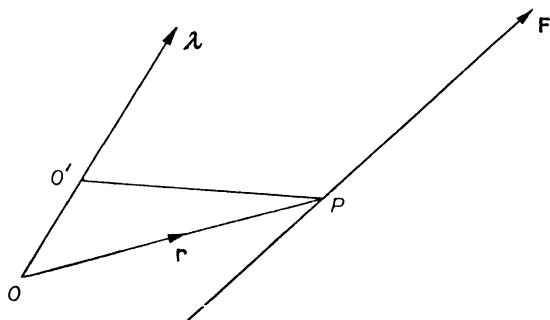


Figure 1.14

is independent of the position of the point P on the line of action of $\mathbf{F}$. We shall now show that the component of $\mathbf{M}$ with respect to the line is independent of the position of O on the line. For let O' be some other point on $\boldsymbol{\lambda}$ (Fig. 1.14). Then we have

$$\begin{aligned} M_{\lambda} &= \boldsymbol{\lambda} \cdot (\mathbf{r} \wedge \mathbf{F}) = \boldsymbol{\lambda} \cdot (\overrightarrow{OP} \wedge \mathbf{F}) = \boldsymbol{\lambda} \cdot \{(\overrightarrow{OO'} + \overrightarrow{O'P}) \wedge \mathbf{F}\} \\ &= \boldsymbol{\lambda} \cdot (\overrightarrow{OO'} \wedge \mathbf{F}) + \boldsymbol{\lambda} \cdot (\overrightarrow{O'P} \wedge \mathbf{F}) \\ &= \boldsymbol{\lambda} \cdot (\overrightarrow{O'P} \wedge \mathbf{F}), \end{aligned}$$

since $\overrightarrow{OO'}$ and $\boldsymbol{\lambda}$ are coincident. M_{λ} is thus independent of the positions of both O and P on their respective lines, i.e. it depends only upon the magnitude and the position of the line of action of $\mathbf{F}$ and the position of the line $\boldsymbol{\lambda}$.

Examples of physical quantities which involve the moment of a vector and which we shall meet with in later chapters, are:

(i) *The moment of a force with respect to a point or a line.* If $\mathbf{M}$ denotes the moment of the force with respect to a point O and if (X, Y, Z) are the components of the force $\mathbf{F}$ and (x, y, z) those of $\mathbf{r}$ ($= \overline{OP}$) with respect to a set of rectangular axes through O , then we have

$$\begin{aligned}\mathbf{M} &\equiv (M_x, M_y, M_z) = \mathbf{r} \wedge \mathbf{F} \\ &= (x, y, z) \wedge (X, Y, Z) \\ &= (yZ - zY, zX - xZ, xY - yX),\end{aligned}$$

the components of $\mathbf{M}$ being the respective moments of $\mathbf{F}$ with respect to the coordinate axes OX, OY, OZ .

(ii) *The moments of the linear momentum of a particle with respect to a point and a line.* That with respect to a point is termed the *moment of momentum* of the particle with respect to the point. It is usually denoted by the symbol $\mathbf{H}$, so that, if $\mathbf{r}$ is the vector displacement of the particle from a point O and if we adopt the same convention as to sign as in the case of the moment of a force above, then we shall have

$$\mathbf{H} = \mathbf{r} \wedge m\mathbf{v}.$$

where m is the mass of the particle and $\mathbf{v}$ its velocity.

The moment of the linear momentum with respect to any line through O is the component of $\mathbf{H}$ in the direction of the line; it is termed the moment of momentum of the particle with respect to the line.

13. Equations of a line and a plane

To conclude this chapter on the properties of vectors we shall derive the equations of a line and a plane when expressed in vector notation, since we shall need to use them in later work.

13.1. Line. Let O be a fixed point, P a variable point, and let $\mathbf{r}(x, y, z)$ be the vector displacement of P with respect to O . Let $\mathbf{r}_0(x_0, y_0, z_0)$ denote the position of some fixed point P_0 on the line whose direction is that of the unit vector $\mathbf{t}$, its direction cosines being (l, m, n) . Then the position vector of any other point on this line will be of the form

$$\mathbf{r} = \mathbf{r}_0 + \lambda \mathbf{t}, \quad (1.14)$$

where λ is a variable scalar coefficient (Fig. 1.15). As λ takes different values, $\mathbf{r}$ determines the position of different points on the line. Equation (1.14) is therefore the vector equation of the straight line. It is equivalent to the three scalar equations

$$x = x_0 + \lambda l, \quad y = y_0 + \lambda m, \quad z = z_0 + \lambda n,$$

which, on the elimination of λ , give the usual equations for a straight line in Cartesian coordinates, viz.

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}. \quad (1.15)$$

A modified form of (1.13) is also useful. If we write (1.13) in the form

$$\mathbf{r} - \mathbf{r}_0 = \lambda \mathbf{t},$$

and multiply both sides by $\mathbf{t}$, forming the vector product, then we have

$$(\mathbf{r} - \mathbf{r}_0) \wedge \mathbf{t} = \lambda(\mathbf{t} \wedge \mathbf{t}) = \mathbf{0}. \quad (1.16)$$

Since the module of the vector $(\mathbf{r} - \mathbf{r}_0) \wedge \mathbf{t}$ is zero only if its components are separately zero, it follows that (1.16) is equivalent to the three equations

$$\begin{aligned} (y-y_0)n - (z-z_0)m &= 0, & (z-z_0)l - (x-x_0)n &= 0, \\ (x-x_0)m - (y-y_0)l &= 0, \end{aligned}$$

which are, of course, simply equations (1.15) above. Thus

$$(\mathbf{r} - \mathbf{r}_0) \wedge \mathbf{t} = \mathbf{0} \quad (1.17)$$

is an alternative form of the equation of a straight line.

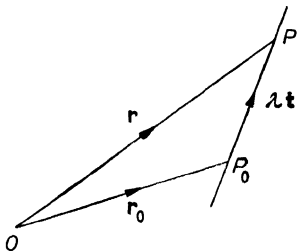


Figure 1.15

This form could also have been deduced directly from the figure; for, if P is to be a point on the line, then $\overline{P_0P}$ must be coincident with $\mathbf{t}$, and hence we must have

$$(\mathbf{r} - \mathbf{r}_0) \wedge \mathbf{t} = \mathbf{0}.$$

13.2. Plane. Let P_0 be a point on the plane whose vector position with respect to an origin O is $\mathbf{r}_0$ (Fig. 1.16). Let the plane be defined by two unit vectors $\mathbf{s}$ and $\mathbf{t}$, both localized at P_0 . Then, if $P(\mathbf{r})$ is a variable point, it lies on the plane provided that $\overline{P_0P}$ is at right angles to the normal to the plane, i.e. to the vector product $\mathbf{s} \wedge \mathbf{t}$. The condition for P to be on the plane is thus that the scalar product

$$(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{s} \wedge \mathbf{t}) = 0. \quad (1.18)$$

(This condition could, of course, have been written down at once by using the fact, already stated above [see § 7.1] that the necessary and sufficient condition for three vectors to be coplanar is that a triple scalar product formed from them should vanish.) Eqn (1.18) is there-

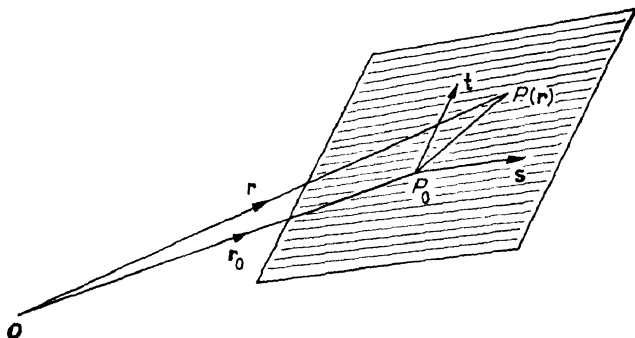


Figure 1.16

fore the equation of the plane when expressed in vector notation. The equivalent scalar equation is

$$(x-x_0)l+(y-y_0)m+(z-z_0)n=0,$$

where (l,m,n) are proportional to the components of the vector $\mathbf{s} \wedge \mathbf{t}$.

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CHAPTER II

MOTION OF A PARTICLE

1. Equations of motion

Consider a particle of mass m moving with a velocity $\mathbf{v}$. Then its equation of motion is given by

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}\left(m\frac{d\mathbf{r}}{dt}\right), \quad (2.1)$$

where $\mathbf{F}$ is the resultant of the forces acting on the particle and $\mathbf{r}$ is its vector displacement with respect to some fixed origin O .

In the case in which m is constant, (2.1) reduces to

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\frac{d^2\mathbf{r}}{dt^2}. \quad (2.2)$$

Equations (2.1) and (2.2) are simply vector expressions of Newton's Second Law of Motion, viz. that the rate of change of the linear momentum of a particle is equal in magnitude and direction to the resultant force acting on the particle. The equivalent Cartesian equations are

$$X = \frac{d}{dt}\left(m\frac{dx}{dt}\right), \quad Y = \frac{d}{dt}\left(m\frac{dy}{dt}\right), \quad Z = \frac{d}{dt}\left(m\frac{dz}{dt}\right),$$

or, in the case in which m is constant,

$$X = m\frac{d^2x}{dt^2}, \quad Y = m\frac{d^2y}{dt^2}, \quad Z = m\frac{d^2z}{dt^2};$$

here (X, Y, Z) are the components of $\mathbf{F}$ and (x, y, z) those of $\mathbf{r}$ respectively parallel to a set of orthogonal axes through O .

1.1. Integrals of the motion

1. *Energy Integral.* If the particle undergoes a small displacement $\delta\mathbf{r}$ in an interval of time δt , then we have, on multiplying (2.2) scalarly by $\delta\mathbf{r}$,

$$\begin{aligned} \mathbf{F} \cdot \delta\mathbf{r} &= m \frac{d\mathbf{v}}{dt} \cdot \delta\mathbf{r} \\ &= m \frac{d\mathbf{v}}{dt} \cdot \frac{d\mathbf{r}}{dt} \delta t \\ &= m \left(\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} \right) \delta t. \end{aligned}$$

This yields, on integrating with respect to t over a finite interval of time (t_1, t_2) in which the particle moves from the position $\mathbf{r}_1$ to the position $\mathbf{r}_2$,

$$\begin{aligned}\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} &= \int_{t_1}^{t_2} m \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \right) dt \\ &= \frac{1}{2} m (\mathbf{v}_2 \cdot \mathbf{v}_2) - \frac{1}{2} m (\mathbf{v}_1 \cdot \mathbf{v}_1).\end{aligned}$$

If we write

$$T = \frac{1}{2} m (\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2} m v^2,$$

then this may be written

$$T_2 - T_1 = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}.$$

We thus have the general result: *the change in the kinetic energy in any interval of time is equal to the total work done on the particle by the forces acting on it in the same interval of time.*

Let $\mathbf{F}_i$ ($i = 1, 2, \dots, n$) denote the forces acting on the particle, so that $\mathbf{F} = \sum \mathbf{F}_i$. Then in the case in which all the $\mathbf{F}_i$ can be derived from potential energy functions V_i ($i = 1, 2, \dots, n$), so that

$$\mathbf{F}_i = iX_i + jY_i + kZ_i = -\nabla V_i \quad (i = 1, 2, \dots, n),$$

where $\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$,

(i, j, k) being three unit vectors respectively parallel to the coordinate axes through O , or, alternatively, in the case in which those forces which cannot be derived from potential functions do no work in a small displacement, we have

$$\begin{aligned}T_2 - T_1 &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} (\sum \mathbf{F}_i) \cdot d\mathbf{r} \\ &= - \int_{\mathbf{r}_1}^{\mathbf{r}_2} (\sum \nabla V_i) \cdot d\mathbf{r} \\ &= - \int_{\mathbf{r}_1}^{\mathbf{r}_2} \left(\frac{\partial V_i}{\partial x} dx + \frac{\partial V_i}{\partial y} dy + \frac{\partial V_i}{\partial z} dz \right) \\ &= - \left[\sum V_i \right]_{\mathbf{r}_1}^{\mathbf{r}_2}.\end{aligned}$$

If we write $V = \sum V_i$, then this may be written

$$T_2 - T_1 = V_1 - V_2,$$

so that

$$T_2 + V_2 = T_1 + V_1, \quad (2.3)$$

i.e., expressed in words, *the sum of the kinetic and potential energies of the particle remains constant during the motion*. Equation (2.3) thus expresses the law of conservation of energy for the motion.

Since in any case of particle motion T will be a scalar function of the coordinates together with their first derivatives with respect to the time, while V will be a scalar function of the positional coordinates alone, it follows that, when the motion is such that the total energy remains constant, then the equation

$$T + V = \text{constant}$$

represents a first integral of the motion.

2. *Angular Momentum Integral.* An important quantity associated with the motion of a particle is the moment of its linear momentum with respect to a point. As already stated in Chapter I, § 12, it is termed the *moment of momentum* or the *angular momentum* of the particle with respect to the point. If we denote the moment of momentum of the particle with respect to the origin O by $\mathbf{H}$, then

$$\mathbf{H} = \mathbf{r} \wedge m\mathbf{v}.$$

Its rate of change is thus given by

$$\frac{d\mathbf{H}}{dt} = \frac{d\mathbf{r}}{dt} \wedge m\mathbf{v} + \mathbf{r} \wedge \frac{d}{dt}(m\mathbf{v}),$$

or, since $\frac{d\mathbf{r}}{dt} = \mathbf{v}$, while $\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}$ by (2.1),

$$\frac{d\mathbf{H}}{dt} = \mathbf{v} \wedge m\mathbf{v} + \mathbf{r} \wedge \mathbf{F},$$

and hence, since the vector product of two parallel vectors vanishes,*

$$\frac{d\mathbf{H}}{dt} = \mathbf{r} \wedge \mathbf{F}. \quad (2.4)$$

We thus have the result: *the rate of change of the moment of momentum of a particle with respect to a fixed point is equal to the moment of the resultant of the forces acting on the particle with respect to the same point.*

Since $\mathbf{F}$ is the resultant of the forces acting on the particle, it follows that the right-hand side of (2.4) cannot in general vanish: it can vanish only if $\mathbf{F}$ either vanishes or, alternatively, passes through O (i.e. $\mathbf{F}$ and $\mathbf{r}$ are coincident). Thus $\mathbf{H}$ cannot, in general, be a

* Strictly, reduces to a null vector.

constant of the motion. Suppose, however, that we multiply (2.4) scalarly by a unit vector λ localized at O ; then we have

$$\lambda \cdot \frac{d\mathbf{H}}{dt} = \lambda \cdot \mathbf{r} \wedge \mathbf{F}.$$

The right-hand side of this equation represents the component of the resultant moment of the forces acting on the particle about the line through O whose direction is that of λ . If λ is such that

$$\lambda \cdot \mathbf{r} \wedge \mathbf{F} = 0,$$

i.e., if λ , $\mathbf{r}$, and $\mathbf{F}$ are coplanar, then the component of the resultant moment with respect to this line vanishes, and we have

$$\lambda \cdot \frac{d\mathbf{H}}{dt} = 0. \quad (2.5)$$

If, in addition, λ is a constant unit vector, i.e. it has a constant direction in space, then we can write

$$\lambda \cdot \frac{d\mathbf{H}}{dt} = \frac{d}{dt}(\lambda \cdot \mathbf{H}),$$

so that (2.5) becomes

$$\frac{d}{dt}(\lambda \cdot \mathbf{H}) = 0.$$

This may be integrated immediately giving

$$\lambda \cdot \mathbf{H} = \text{constant}, \quad (2.6)$$

i.e. the component of the moment of the momentum of the particle with respect to the line through O , whose direction is that of λ , is constant throughout the motion.

The function $\lambda \cdot \mathbf{H}$ ($= \lambda \cdot \mathbf{r} \wedge m\mathbf{v}$) will be in all cases a scalar function of the positional coordinates of the particle together with their rates of change. It therefore represents another first integral of the motion.

Since a particle moving in space, and not subject to constraints, requires three independent coordinates to define its position, it has three degrees of freedom, and the above integrals, when they exist, will not be sufficient to determine the motion completely; one further first integral of the motion, clearly, will be required. This may be supplied by a second integral of the type (2.6). If, however, the particle is subject to a constraint, e.g. if it is constrained to move on a given surface, then the number of its degrees of freedom is reduced by one, and two independent coordinates suffice to define its position on the surface. In this case the above integrals of the motion, provided they exist, will be sufficient for the determination of its motion. If the particle is constrained to move along a curve, then only one independent coordinate is required to define its position, namely its distance measured along

the curve from some fixed point on the curve. In this case one first integral only is required for the determination of its motion. Consequently, if the curve is smooth, then the motion may be determined by means of the energy integral alone, provided the forces acting on the particle are such that this exists.

To illustrate the use of the above integrals we shall consider in detail the motion of a particle in contact with a smooth surface of revolution. Before proceeding to do so, however, we shall first derive expressions for the components of the velocity and of the acceleration of a particle when its position is defined in terms of either (i) *cylindrical coordinates* (r, θ, z) , or (ii) *spherical polar coordinates* (r, θ, ϕ) , these systems being the ones most readily applicable to the three-dimensional motion of a particle.

2. Components of velocity and acceleration

2.1. Cylindrical coordinates. Let $OXYZ$ be a set of right-handed rectangular axes fixed in space and let the coordinates of a general point

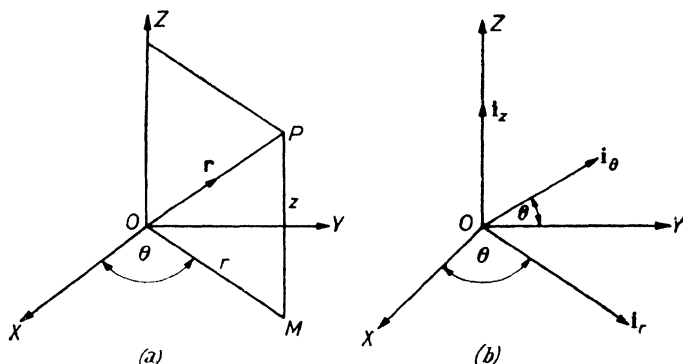


Figure 2.1

P in cylindrical coordinates be (r, θ, z) , where $OM = r$, $MP = z$, $\angle MOX = \theta$ (Fig. 2.1(a)). Let $(\mathbf{i}_r, \mathbf{i}_\theta, \mathbf{i}_z)$ be three unit vectors, $\mathbf{i}_r$ in the direction of OM , $\mathbf{i}_\theta$ in the plane OXY and at right angles to OM in the direction of θ increasing, and $\mathbf{i}_z$ in the direction of the upward vertical (Fig. 2.1(b)). Then, in terms of these vectors, we have

$$\mathbf{r} = \overline{OP} = r\mathbf{i}_r + z\mathbf{i}_z,$$

and hence

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{i}_r + \frac{dz}{dt}\mathbf{i}_z + r\frac{d\mathbf{i}_r}{dt} + z\frac{d\mathbf{i}_z}{dt},$$

or, if we use the dot notation to denote the rate of change of a *scalar* variable with respect to the time,

$$\mathbf{v} = \dot{r}\mathbf{i}_r + \dot{z}\mathbf{i}_z + r\frac{d\mathbf{i}_r}{dt} + z\frac{d\mathbf{i}_z}{dt}. \quad (2.7)$$

But, if $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are three unit vectors in the directions of OX, OY, OZ respectively, then

$$\mathbf{i}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad \mathbf{i}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}, \quad \mathbf{i}_z = \mathbf{k},$$

and hence, $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ being constant unit vectors in space,

$$\frac{d\mathbf{i}_r}{dt} = \mathbf{i}_\theta \dot{\theta}, \quad \frac{d\mathbf{i}_\theta}{dt} = -\mathbf{i}_r \dot{\theta}, \quad \frac{d\mathbf{i}_z}{dt} = \mathbf{0}. \quad (2.8)$$

On substituting these values in (2.7), we therefore have, omitting the terms involving the null vector,

$$\mathbf{v} = \dot{r}\mathbf{i}_r + r\dot{\theta}\mathbf{i}_\theta + \dot{z}\mathbf{i}_z; \quad (2.9)$$

the components of $\mathbf{v}$ in the directions of r , θ , and z increasing are thus

$$(v_r, v_\theta, v_z) = (\dot{r}, r\dot{\theta}, \dot{z}) \quad (2.10)$$

respectively.

To obtain the components of the acceleration we have, differentiating (2.9) with respect to t ,

$$\mathbf{f} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{i}_r + \frac{d}{dt}(r\dot{\theta})\mathbf{i}_\theta + \ddot{z}\mathbf{i}_z + \dot{r}\frac{d\mathbf{i}_r}{dt} + r\dot{\theta}\frac{d\mathbf{i}_\theta}{dt} + \dot{z}\frac{d\mathbf{i}_z}{dt},$$

which becomes, in virtue of (2.8),

$$\begin{aligned} \mathbf{f} &= \ddot{r}\mathbf{i}_r + \frac{d}{dt}(r\dot{\theta})\mathbf{i}_\theta + \ddot{z}\mathbf{i}_z + \dot{r}\dot{\theta}\mathbf{i}_\theta - r\dot{\theta}^2\mathbf{i}_r \\ &= (\ddot{r} - r\dot{\theta}^2)\mathbf{i}_r + \left\{ \frac{d}{dt}(r\dot{\theta}) + \dot{r}\dot{\theta} \right\} \mathbf{i}_\theta + \ddot{z}\mathbf{i}_z \\ &= (\ddot{r} - r\dot{\theta}^2)\mathbf{i}_r + \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta})\mathbf{i}_\theta + \ddot{z}\mathbf{i}_z. \end{aligned}$$

The component accelerations in the directions of r , θ , increasing are therefore given by

$$(f_r, f_\theta, f_z) = \left(\ddot{r} - r\dot{\theta}^2, \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta}), \ddot{z} \right). \quad (2.11)$$

2.2. Spherical polar coordinates. If the spherical polar coordinates of a general point P are (r, θ, ϕ) as shown (Fig. 2.2), then, introducing three orthogonal vectors $(\mathbf{i}_r, \mathbf{i}_\theta, \mathbf{i}_\phi)$ in the directions of r , θ , and ϕ increasing respectively and proceeding as for cylindrical

co-ordinates, the components of the velocity and of the acceleration in these directions, in terms of (r, θ, ϕ) and their rates of change, are found to be

$$(v_r, v_\theta, v_\phi) = (\dot{r}, r\dot{\theta}, r\dot{\phi} \sin \theta), \quad (2.12)$$

$$(f_r, f_\theta, f_\phi) = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2 \theta, \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta}) - r\dot{\phi}^2 \sin \theta \cos \theta,$$

$$\frac{1}{r} \frac{d}{dt}(r^2\dot{\phi}) + 2r\dot{\theta}\dot{\phi} \cos \theta) \quad (2.13)$$

respectively.*

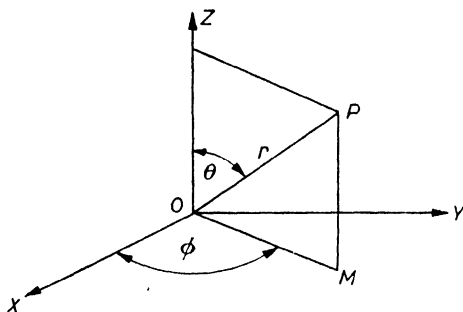


Figure 2.2

3. Motion of a heavy particle on a smooth surface of revolution

To consider the motion of a particle on a general surface of revolution we shall use the cylindrical coordinates (r, θ, z) . In the case in which the surface is a right circular cone, however, spherical polar coordinates are the more convenient ones to use. We shall consider this motion in detail in § 4.

3.1. Equations of energy and angular momentum. Consider a particle moving on a smooth surface of revolution whose axis is vertical, the equation of the surface in cylindrical coordinates being

$$z = f(r).$$

The only forces which we shall assume to be acting on the particle are its weight and the reaction of the surface. We shall further assume that the particle moves on the inner surface; the treatment for a particle moving on the outside of such a surface is, of course, similar.

Since the surface is smooth the reaction clearly does no work in any displacement of the particle. Consequently, the total energy of

* A simpler derivation of these components, based on the theory of rotating axes, is given in Chap. IV, § 1.3.

the particle remains constant during the motion. If referred to *the horizontal plane through the origin* (Fig. 2.3), the potential energy of the particle is given by

$$V = mgz,$$

while the kinetic energy of the particle is given by (see eqn 2.10)

$$T = \frac{1}{2}m(v_r^2 + v_\theta^2 + v_z^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2).$$

The energy equation for the motion thus becomes

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + mgz = \text{constant},$$

or, since m is constant,

$$\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 + 2gz = 2E, \quad (2.14)$$

where E is the total energy per unit mass of the particle.

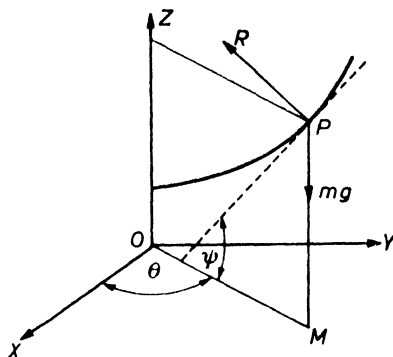


Figure 2.3

An angular momentum integral also exists. Since the reaction always lies in the axial plane through P , both the forces acting on the particle are coplanar with OZ and $\mathbf{r}$ ($= \overline{OP}$) and so have no moment about OZ ; consequently, the angular momentum of the system with respect to this axis is constant. Thus we have, as an integral of the motion,

$$mrv_\theta = mr^2\dot{\theta} = \text{constant},$$

or, omitting m ,

$$r^2\dot{\theta} = h. \quad (2.15)$$

3.2. Differential equation of the motion in terms of r or z . If we eliminate $\dot{\theta}$ between (2.14) and (2.15), then we have

$$\dot{r}^2 + \frac{h^2}{r^2} + \dot{z}^2 + 2gz = 2E. \quad (2.16)$$

Since $z = f(r)$,

$$\dot{z} = f'(r)\dot{r},$$

and hence (2.16) may be written

$$\dot{r}^2 + \frac{h^2}{r^2} + f'(r)\dot{r}^2 + 2gf(r) = 2E,$$

$$\text{or} \quad \{1 + f'(r)^2\}\dot{r}^2 + \frac{h^2}{r^2} + 2gf(r) = 2E, \quad (2.17)$$

giving a differential equation for the determination of r as a function of t .

Alternatively, if the equation of the surface is given by

$$r = F(z),$$

then we can eliminate r from (2.16), the resulting differential equation in z being

$$\{1 + F'(z)^2\}\dot{z}^2 + \frac{h^2}{F(z)^2} + 2gz = 2E. \quad (2.18)$$

In general, the equation of the surface is given most simply in the form $z = f(r)$, but sometimes it is convenient to use (2.18) in preference to (2.17) (see § 3.7 below).

3.3. Nature of the motion. From (2.17) we have

$$\{1 + f'(r)^2\}\dot{r}^2 = 2g\left\{\frac{1}{2g}\left(2E - \frac{h^2}{r^2}\right) - f(r)\right\},$$

$$\text{or} \quad \{1 + f'(r)^2\}\dot{r}^2 = 2g\{G(r) - f(r)\}, \quad (2.19)$$

$$\text{on writing} \quad G(r) = \frac{1}{2g}\left(2E - \frac{h^2}{r^2}\right). \quad (2.20)$$

Now $\{1 + f'(r)^2\}$ cannot be zero at any point on the surface; consequently $\dot{r}$ vanishes only when r is such that

$$G(r) = f(r). \quad (2.21)$$

Since the motion must also be such that

$$\{1 + f'(r)^2\}\dot{r}^2 \geq 0,$$

it follows that the motion must be confined to that region, or regions, of the surface for which

$$G(r) \geq f(r). \quad (2.22)$$

Suppose we plot the curves of intersection of the surfaces of revolution defined by $z = f(r)$, $z = G(r)$ (eqn 2.20), with an axial plane through OZ . If the surface $z = f(r)$ is concave upwards, which is appropriate to the particle's motion on the inner surface, then the general forms of the curves will be as represented by the continuous and dotted lines respectively in Fig. 2.4. In order that the inequality (2.22) may hold, it follows that the motion must be confined to the zone of the surface

represented by the arc AB . But no matter what form the surface $z = f(r)$ may have, the surface $z = G(r)$ will always intersect it in one or more zones bounded by horizontal circles. Thus, for given initial conditions, i.e. subject to given values of E and h , the condition for motion on the surface to be possible is that a zone of the surface should be more remote from OZ than the corresponding zone of $z = G(r)$. If a particle does move in contact with a smooth surface of revolution it is clear that the motion will be confined to a zone bounded by the two horizontal circles $r = r_1$, $r = r_2$, where r_1 and r_2 are two adjacent solutions of (2.21). One should realize, however, that the above treatment is

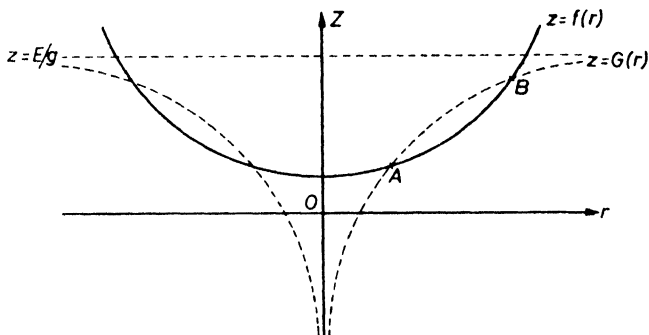


Figure 2.4

based upon the equations of energy and angular momentum alone. No account has been taken of the sign of R , the reaction of the surface, which, of course, must be positive for motion on the surface to be possible. The above result is true, therefore, only if subject to the proviso that at no point within the region (r_1, r_2) does R vanish and change its sign; if this happens, then the particle leaves the surface.

3.4. Condition that the particle should rise or fall initially when projected horizontally. At the points A and B the particle is moving horizontally, since $\dot{r} = 0$ at these points. Consequently, if the particle is projected horizontally, then the initial position of the particle must lie on one of the bounding circles. Since, for motion on the surface to be possible, the zone on which the particle moves must be more remote from the z -axis than the corresponding zone of $z = G(r)$, it follows that whether the particle rises or falls initially depends upon whether the initial position corresponds to the circle A or B respectively, i.e. whether the initial conditions are such that

$$G'(r) \geq f'(r)$$

initially.

This condition could also have been deduced directly from (2.19). For, differentiating with respect to t , we have as a property of the motion

$$2\{1+f'(r)^2\}\dot{r}\ddot{r}+2f'(r)f''(r)\dot{r}^3=2g\{G'(r)-f'(r)\}\dot{r},$$

or, since $\dot{r}$ is not in general zero,

$$\{1+f'(r)^2\}\ddot{r}+f'(r)f''(r)\dot{r}^2=g\{G'(r)-f'(r)\}.$$

But, when the particle is moving horizontally, $\dot{r}=0$. Therefore, at such a point, we have

$$\{1+f'(r)^2\}\ddot{r}=g\{G'(r)-f'(r)\}. \quad (2.23)$$

Consequently, since the particle either rises or falls according as $\dot{r} \geq 0$, the condition for the particle to rise or fall initially is that

$$G'(r) \geq f'(r)$$

initially.

From (2.20), which defines the function $G(r)$, the above condition thus becomes, if $r=r_0$ initially,

$$\frac{h^2}{gr_0^3} \geq f'(r_0),$$

or, since $h^2=r_0^2v_0^2$,

$$\frac{v_0^2}{f'(r_0)} \geq gr_0.$$

But $f'(r_0) = \tan \psi_0$, where ψ_0 is the angle which the tangent to the meridian curve through the initial point makes with the horizontal (Fig. 2.3). In terms of ψ_0 , the condition thus becomes

$$v_0^2 \cot \psi_0 \geq gr_0.$$

3.5. Condition for circular motion. If the particle moves in a horizontal circle of radius a , say, then the initial conditions must be such that the points A and B in Fig. 2.4 coincide. This means that $G(r)$ and $f(r)$ are tangential at $r=a$. The conditions for circular motion are thus that

$$G(r)=f(r), \quad G'(r)=f'(r)$$

at $r=a$, these two conditions ensuring that both $\dot{r}$ and $\ddot{r}$ should be zero (see (2.19) and (2.23)). From (2.20) the conditions, in terms of E and h , thus become

$$2E-h^2/a^2=2gf(a), \quad h^2=ga^3f'(a). \quad (2.24)$$

From the second of these conditions, it is seen that for circular motion to be possible the form of the surface must be such that $f'(a)$ is positive.

3.6. Stability of circular motion. In discussing the stability of circular motion we make use of the equation, obtained by

differentiating (2.17), the equation of energy. This gives ($r \neq 0$).

$$\{1 + f'(r)^2\}\dot{r} + f'(r)f''(r)\dot{r}^2 - h^2/r^2 + gf'(r) = 0. \quad (2.25)$$

Now let the particle, which is assumed to be moving uniformly in a circle of radius a , be subjected to a small impulse applied in the meridian plane. This will cause the particle to move away from its circular orbit; it will not, however, produce any change in h , the angular momentum with respect to the vertical OZ . Consequently (2.25) also holds for the disturbed motion.

To study the nature of this motion, suppose we write

$$r = a + x,$$

so that in the disturbed motion

$$\dot{r} = \dot{x}, \quad \ddot{r} = \ddot{x}.$$

Then, on substituting for r , $\dot{r}$, and $\ddot{r}$ in (2.25), the differential equation in x becomes

$$\{1 + f'(a+x)^2\}\ddot{x} + f'(a+x)f''(a+x)\dot{x}^2 - \frac{h^2}{(a+x)^3} + gf'(a+x) = 0.$$

If the disturbance is small, then, when x is small, $\dot{x}$ and $\ddot{x}$ are also small. Thus, neglecting terms of the second order of smallness, the equation governing the behaviour of x in the immediate neighbourhood of $x = 0$ becomes

$$\{1 + f'(a)^2\}\ddot{x} - (h^2/a^3)(1 - 3x/a) + gf'(a) + gxf''(a) = 0,$$

or, since $h^2 = ga^3f'(a)$ (eqn 2.24),

$$\{1 + f'(a)^2\}\ddot{x} + (g/a)(3f'(a) + af''(a))x = 0,$$

i.e.

$$\ddot{x} + \frac{g(3f'(a) + af''(a))}{a(1 + f'(a)^2)}x = 0.$$

The disturbed motion will be simple harmonic, therefore, provided that

$$3f'(a) + af''(a) > 0.$$

This, therefore, is the condition for the stability of circular motion.

It is seen that if this condition is satisfied, then the period of oscillation of the particle about the circle $r = a$ in the disturbed motion will be given by

$$T = 2\pi \sqrt{\left\{ \frac{a(1 + f'(a)^2)}{g(3f'(a) + af''(a))} \right\}} \quad (2.26)$$

That a circular motion is stable, as regards disturbances of this type, could have been established also, in a general way, by studying the change in $G(r)$ consequent upon the application of the impulse. For, if

$[G(r)]_d$ denotes the function $G(r)$ corresponding to the disturbed motion, then we should have

$$\begin{aligned} [G(r)]_d &= G(r) + \delta G(r) = \frac{1}{2g} \left\{ 2(E + \delta E) - \frac{h^2}{r^2} \right\} \\ &= \frac{1}{2g} \left(2E - \frac{h^2}{r^2} \right) + \frac{\delta E}{g}, \end{aligned}$$

so that
$$\delta G(r) = \frac{\delta E}{g}.$$

But δE is the instantaneous change in the kinetic energy per unit mass of the particle due to the impulse, there being, of course, no instantaneous change in the potential energy, so that

$$\delta E = \frac{1}{2} \delta v^2,$$

where δv is the instantaneous velocity given to the particle in the meridian plane. Thus $\delta G(r)$ is positive and independent of r . The effect of the impulse, therefore, is to displace the $G(r)$ curve from its original position through a distance

$$\delta G(r) = \frac{\delta E}{g} = \frac{\delta v^2}{2g}$$

parallel to OZ . The $G(r)$ curves for the two motions will thus be as shown in Fig. 2.5, the continuous curve representing the $G(r)$ function

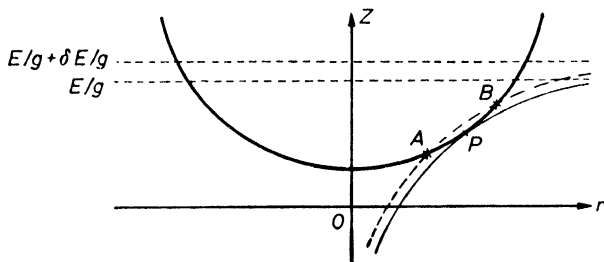


Figure 2.5

corresponding to the circular motion, the broken line the $G(r)$ for the disturbed motion. Thus, in the disturbed motion, the particle will oscillate about the circle $r = a$, the motion being confined to the small zone represented by the arc AB in Fig. 2.5. The original motion, therefore, is stable.

The above result has been obtained assuming the impulse to be applied in the meridian plane. If the impulse is not applied in this plane, but is applied horizontally along the tangent to the path, so that h is

increased to $(h + \delta h)$, then the disturbed motion will no longer be oscillatory about $r = a$. To investigate this case, let us proceed as above and consider the change in $G(r)$ arising from the application of the impulse. Writing $h = h_0$ for the undisturbed motion, then we have

$$\begin{aligned} [G(r)]_a = G(r) + \delta G(r) &= \frac{1}{2g} \left\{ 2(E + \delta E) - \left(\frac{h_0 + \delta h}{r^2} \right)^2 \right\} \\ &= \frac{1}{2g} \left(2E - \frac{h_0^2}{r^2} \right) + \frac{1}{g} \left(\delta E - \frac{h_0 \delta h}{r^2} - \frac{\delta h^2}{2r^2} \right), \end{aligned}$$

so that

$$\delta G(r) = \frac{1}{g} \left(\delta E - \frac{h_0 \delta h}{r^2} - \frac{\delta h^2}{2r^2} \right).$$

But δE , being the instantaneous increase in the kinetic energy per unit mass of the particle, is given by

$$\delta E = \delta v \left(v + \frac{\delta v}{2} \right).$$

Also

$$h_0 = av, \quad \delta h = a\delta v.$$

Hence, inserting these values for δE and δh in the above expression, we have

$$\begin{aligned} \delta G(r) &= \frac{1}{g} \left\{ v\delta v + \frac{\delta v^2}{2} - \frac{a^2 v \delta v}{r^2} - \frac{a^2 \delta v^2}{2r^2} \right\} \\ &= \frac{1}{g} \left\{ \left(v\delta v + \frac{\delta v^2}{2} \right) \left(1 - \frac{a^2}{r^2} \right) \right\}, \end{aligned}$$

or, putting $r = a + x$ and treating x as small,

$$\delta G(r) = \frac{2v\delta vx}{ga},$$

when small quantities of the third order are neglected. We thus see that $\delta G(r)$ vanishes when $x = 0$. In addition, if we assume the impulse to be such that δv is positive, then $\delta G(r)$ is positive when x is positive and negative when x is negative. Thus the $G(r)$ curve for the disturbed motion will be, in this case, as indicated by the broken line in Fig. 2.6. The disturbed motion will be confined, therefore, to the zone PP' , the particle rising initially.

The form of the $G(r)$ curve for the disturbed motion could also have been deduced from the fact that the new motion is the same as that which would result from projecting the particle horizontally with a velocity $(v + \delta v)$. For it follows from the above theory (see § 3.4) that the $G(r)$ curve for the disturbed motion must pass through P , the point of projection, which must be a point on one of the bounding circles. But, if δv is positive, $\delta h = a\delta v$ is also positive, and hence in the disturbed motion we have

$$h^2 = (h_0 + \delta h)^2 > ga^3 f'(a).$$

The particle therefore rises as a result of the impulse, and the disturbed motion will be confined to the zone PP' as indicated in Fig. 2.6.

If one investigates the problem in detail, putting

$$r = a + x, \quad h = h_0 + \delta h$$

in (2.25), then it is found that the disturbed motion is one of simple harmonic motion about the circle

$$r = a + \delta a,$$

$$\text{where} \quad \delta a = \frac{2\delta h a f'(a)}{h\{3f'(a) + af''(a)\}},$$

the period being the same as in the case in which $\delta h = 0$ (eqn 2.26).

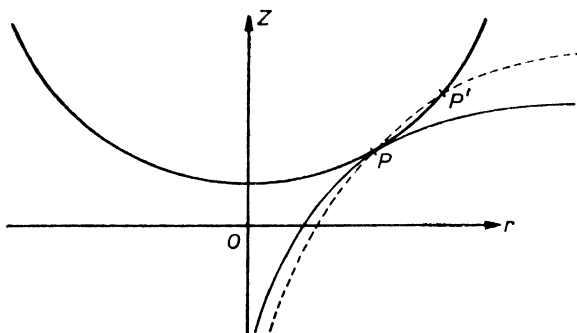


Figure 2.6

If the particle is subjected to a small impulse applied in the tangential plane but inclined to the horizontal, then the nature of the ensuing motion may be deduced by combining the disturbed motions arising from the application of an impulse in the meridian plane and a tangential impulse respectively. Since the disturbed motions are stable in both cases, it follows that a circular motion of the particle is stable with respect to any small impulse applied in the tangential plane.

3.7. Time between two points. From (2.17)

$$\dot{r} = \frac{dr}{dt} = \pm \frac{1}{\{1 + f'(r)^2\}^{1/2}} \left\{ 2E - \frac{h^2}{r^2} - 2gf(r) \right\}^{1/2},$$

and hence the time between any two points P_1 and P_2 is given by

$$t = \pm \int_{P_1}^{P_2} \frac{\{1 + f'(r)^2\}^{1/2}}{\left\{ 2E - \frac{h^2}{r^2} - 2gf(r) \right\}} dr = \pm \int_{P_1}^{P_2} \frac{\{1 + f'(r)^2\}^{1/2}}{[2g\{G(r) - f(r)\}]^{1/2}} dr,$$

the + or - sign being taken according as to whether r is increasing or decreasing with t .

Alternatively, if we use (2.18) instead of (2.17), then, since

$$\{1+F'(z)^2\}z^2 = 2E - 2gz - \frac{h^2}{F'(z)^2},$$

we have

$$t = \pm \int_{P_1}^{P_2} \frac{\{1+F'(z)^2\}^{1/2}}{\left\{2E - \frac{h^2}{F'(z)^2} - 2gz\right\}^{1/2}} dz.$$

Application. For purposes of actual calculation the second expression involving $F(z)$ is in general the simpler one to use. As an application we shall consider the case of a particle moving on the smooth inner surface of a sphere. The general theory is of interest since it is identical with that of the motion of the spherical pendulum,* and is also similar to that of the motion of a point on the axis of a symmetrical top spinning on a rough horizontal plane (see Chap. VIII, § 2.3).

Let the radius of the sphere be a . Then we have, the centre of the sphere being taken as origin,

$$a^2 = r^2 + z^2,$$

so that

$$r = F(z) = (a^2 - z^2)^{1/2},$$

and hence

$$1 + F'(z)^2 = 1 + \frac{z^2}{a^2 - z^2} = \frac{a^2}{a^2 - z^2}.$$

The expression for t thus becomes

$$\begin{aligned} t &= \pm a \int_{P_1}^{P_2} \frac{dz}{\{2E(a^2 - z^2) - 2gz(a^2 - z^2) - h^2\}^{1/2}} \\ &= \pm \frac{a}{\sqrt{(2g)}} \int_{P_1}^{P_2} \frac{dz}{\{g(z)\}^{1/2}}, \end{aligned}$$

where

$$g(z) = \frac{E}{g}(a^2 - z^2) - z(a^2 - z^2) - \frac{h^2}{2g}.$$

Now $g(z)$ is clearly a cubic in z . To obtain the nature of its roots we have:

$$\begin{aligned} \text{when } z &= -\infty, g(z) \text{ is } -ve, \\ ,, \quad z &= -a, g(z) \text{ is } -ve, \\ ,, \quad z &= +a, g(z) \text{ is } -ve, \\ ,, \quad z &= +\infty, g(z) \text{ is } +ve; \end{aligned}$$

* See Synge and Griffith, *Principles of Mechanics* (2nd ed.), p. 373, § 13.3.

furthermore, $g(z)$ must be positive for those values of z within the range $(-a \leq z \leq a)$ corresponding to the motion. Therefore the form of

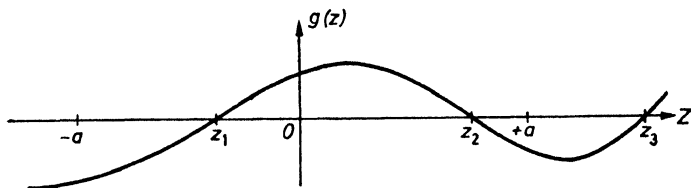


Figure 2.7

$g(z)$ must be as shown in Fig. 2.7. Thus, if z_1, z_2, z_3 are the three roots of $g(z) = 0$, such that

$$z_1 \leq z_2 \leq z_3,$$

then both z_1 and z_2 lie between $\pm a$, while $z_3 > a$. Consequently the expression for t may be written

$$t = \pm \frac{a}{\sqrt{(2g)}} \int_{P_1}^{P_2} \frac{dz}{\{(z-z_1)(z_2-z)(z_3-z)\}^{1/2}}.$$

The right-hand side of this expression is an elliptic integral. To reduce it to standard form, we introduce the new variable u defined by

$$u = \sqrt{(z-z_1)},$$

so that

$$dz = 2u du.$$

Then we have

$$t = \pm \frac{a}{\sqrt{(2g)}} \int_{P_1}^{P_2} \frac{dz}{\{g(z)\}^{1/2}} = \pm \sqrt{\left(\frac{2a^2}{g}\right)} \int_{P_1}^{P_2} \frac{du}{\{(z_2-z_1-u^2)(z_3-z_1-u^2)\}^{1/2}}. \quad (2.27)$$

Now write

$$v = \frac{u}{\sqrt{(z_2-z_1)}} = \sqrt{\left(\frac{z-z_1}{z_2-z_1}\right)},$$

$$k = \sqrt{\left(\frac{z_2-z_1}{z_3-z_1}\right)}, \quad p = \sqrt{\{(g/2a^2)(z_3-z_1)\}}.$$

Then, in terms of v , (2.27) becomes

$$pt = \pm \int_{P_1}^{P_2} \frac{dv}{\{(1-v^2)(1-k^2v^2)\}^{1/2}}. \quad (2.28)$$

Thus t is given by an elliptic integral of the first kind.

Since v is positive and < 1 throughout the motion, we may put $v = \sin \phi$, when (2.28) transforms to

$$pt = \pm \int_{P_1}^{P_2} \frac{d\phi}{\sqrt{(1-k^2 \sin^2 \phi)}} = \pm \{F(k, \phi_2) - F(k, \phi_1)\}.$$

From (2.28) we can also determine z as a function of t . In terms of Jacobi's Elliptic Functions* we have, in general, introducing t_0 as a constant of integration,

$$z - z_1 = (z_2 - z_1) \operatorname{sn}^2[p(t - t_0)],$$

$$z_2 - z = (z_2 - z_1) \operatorname{cn}^2[p(t - t_0)],$$

$$z_3 - z = (z_3 - z_1) \operatorname{dn}^2[p(t - t_0)],$$

any of which may be used to determine z .

From the equation expressing the constancy of the angular momentum, viz.

$$r^2 \dot{\theta} = h,$$

it follows that $\dot{\theta}$ ($= h/r^2$) either increases continuously as r decreases or decreases continuously as r increases, but never vanishes. Also, we

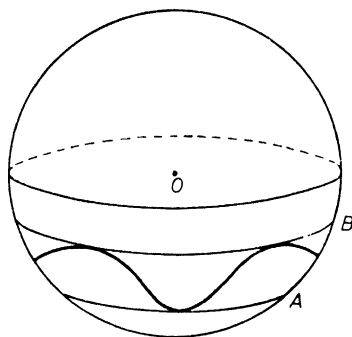


Figure 2.8

know from the above theory that the motion of the particle will be confined to a zone bounded by two horizontal circles on which $\dot{r} = 0$. Consequently, for points on the bounding circles, we shall have $\dot{r} = 0$, $\dot{\theta} \neq 0$, so that the path will be tangential to the bounding circles as shown in Fig. 2.8.

* See, for instance, Jeffreys and Jeffreys, *Methods of Mathematical Physics* (2nd ed.), Chap. 25, § 25.03; also, Synge and Griffith, *Principles of Mechanics* (2nd ed.), p. 364, § 13.1.

The time between the two bounding circles will be given by (eqn 2.28)

$$\begin{aligned}
 pt_{AB} &= \pm \int_{v(z_1)}^{v(z_2)} \frac{dv}{\{(1-v^2)(1-k^2v^2)\}^{1/2}} \\
 &= \pm \int_0^1 \frac{dv}{\{(1-v^2)(1-k^2v^2)\}^{1/2}} \\
 &= \pm \int_0^{\pi/2} \frac{d\phi}{(1-k^2 \sin^2 \phi)^{1/2}} \\
 &= \pm F(k, \pi/2).
 \end{aligned}$$

The period of the motion is thus given by

$$T = \frac{2F(k, \pi/2)}{p}.$$

3.8. Expression for the reaction R . From Fig. 2.9 we clearly

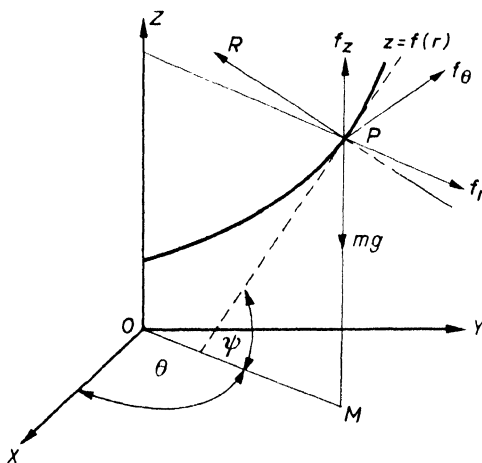


Figure 2.9

have, on resolving in the direction of the inward normal,

$$R - mg \cos \psi = m(f_z \cos \psi - f_r \sin \psi),$$

so that

$$\begin{aligned}
 R &= m(g \cos \psi + f_z \cos \psi - f_r \sin \psi) \\
 &= m(g + f_z - f_r \tan \psi) \cos \psi \\
 &= \frac{m\{g + f_z - f_r f'(r)\}}{\{1 + f'(r)^2\}^{1/2}} \quad (z = f(r)),
 \end{aligned}$$

which becomes, on substituting for f_r and f_z (eqn 2.11),

$$R = \frac{m\{g + \ddot{z} - (\ddot{r} - r\dot{\theta}^2)f'(r)\}}{\{1 + f'(r)^2\}^{1/2}}.$$

But

$$\dot{z} = f'(r)\dot{r}, \quad \ddot{z} = f'(r)\ddot{r} + f''(r)\dot{r}^2.$$

Hence, substituting for $\dot{z}$ and $\ddot{z}$, we have

$$\begin{aligned} R &= \frac{m\{g + f''(r)\dot{r}^2 + r\dot{\theta}^2 f'(r)\}}{\{1 + f'(r)^2\}^{1/2}} \\ &= \frac{m\{g + f''(r)\dot{r}^2 + h^2 f'(r)(1/r^3)\}}{\{1 + f'(r)^2\}^{1/2}}, \end{aligned}$$

where $\dot{r}^2$ is given by (2.17).

If one wishes to retain z as the independent variable instead of r , then, resolving vertically, we have

$$R \cos \psi - mg = mf_z$$

so that ($f_z = \ddot{z}$)

$$\begin{aligned} R &= \frac{m(g + \ddot{z})}{\cos \psi} \\ &= \frac{m(g + \ddot{z})\{1 + F'(z)^2\}^{1/2}}{F'(z)}, \quad (r = F(z)). \end{aligned}$$

On substituting for $\ddot{z}$ in terms of z , using (2.18), the expression for R in terms of z thus becomes

$$R = \frac{m\{gF'(z) - \dot{z}^2 F''(z) + h^2(1/F(z)^3)\}}{\{1 + F'(z)^2\}^{1/2}}.$$

4. Motion of a heavy particle on a smooth right circular cone with axis vertical and vertex downwards

The motion of a particle on a right circular cone may be treated either by expressing the motion in terms of cylindrical coordinates, as above,* or, alternatively, by the use of spherical polar coordinates. With the latter system of coordinates, since θ , the polar angle, is equal to α , the semi-vertical angle of the cone, throughout the motion, the equations of energy and angular momentum become (see eqns 2.12. and 2.13)

$$\frac{1}{2}m(\dot{r}^2 + r^2 \sin^2 \alpha \dot{\phi}^2) + mgr \sin \alpha = \text{constant},$$

$$mr^2 \sin^2 \alpha \dot{\phi} = \text{constant},$$

or

$$\dot{r}^2 + r^2 \sin^2 \alpha \dot{\phi}^2 + 2gr \cos \alpha = 2E,$$

$$r^2 \sin^2 \alpha \dot{\phi} = h.$$

* See Ramsey, *Dynamics*, Pt. II, p. 86, § 4.26.

Thus, eliminating $\dot{\phi}$ from these equations, we have

$$\dot{r}^2 + \frac{h^2}{r^2 \sin^2 \alpha} + 2gr \cos \alpha = 2E. \quad (2.29)$$

The general properties of the motion may now be obtained by reasoning similar to that employed above, but it must be remembered that r is now measured along the generator of the cone through P and not in the OXY plane.

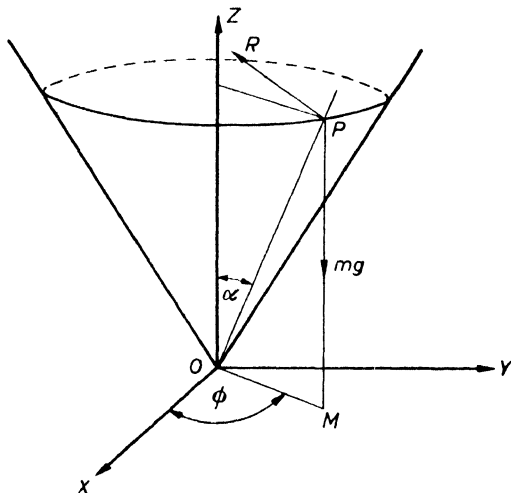


Figure 2.10

To obtain the equation of the path, we have, on differentiating (2.29)

$$2\dot{r}\ddot{r} - \frac{2h^2}{r^3 \sin^2 \alpha} \dot{r} + 2g\dot{r} \cos \alpha = 0,$$

and hence, since $\dot{r} = 0$ cannot be a general property of the motion,

$$\ddot{r} - \frac{h^2}{r^3 \sin^2 \alpha} + g \cos \alpha = 0. \quad (2.30)$$

If we now change the independent variable to ϕ , then, since

$$\begin{aligned} \dot{r} &= \frac{dr}{dt} = \frac{dr}{d\phi} \cdot \frac{d\phi}{dt} = \frac{h}{r^2 \sin^2 \alpha} \frac{dr}{d\phi}, \\ \ddot{r} &= \frac{d^2 r}{dt^2} = \left\{ \frac{h}{r^2 \sin^2 \alpha} \frac{d^2 r}{d\phi^2} - \frac{2h}{r^3 \sin^2 \alpha} \left(\frac{dr}{d\phi} \right)^2 \right\} \frac{h}{r^2 \sin^2 \alpha} \\ &= \frac{h^2}{r^4 \sin^4 \alpha} \left\{ \frac{d^2 r}{d\phi^2} - \frac{2}{r} \left(\frac{dr}{d\phi} \right)^2 \right\}, \end{aligned}$$

the (r, ϕ) equation of the path of the particle becomes, on substituting for $\ddot{r}$ in (2.30),

$$\frac{h^2}{r^4 \sin^4 \alpha} \left\{ \frac{d^2 r}{d\phi^2} - \frac{2}{r} \left(\frac{dr}{d\phi} \right)^2 \right\} - \frac{h^2}{r^3 \sin^2 \alpha} + g \cos \alpha = 0. \quad (2.31)$$

The angle ϕ , occurring in this equation, is measured on the horizontal plane, and not on the surface of the cone. Let P' be a neighbouring point to P on the particle's path, and let $\angle P'OP = \delta\phi'$. Also let OM and OM' be the orthogonal projections of OP and OP' on the plane $z = 0$. Then, since

$$\sin \alpha (\text{area } \triangle POP') = \text{area } \triangle MOM',$$

we have

$$\sin \alpha (\tfrac{1}{2} r^2 \delta\phi') = \tfrac{1}{2} r^2 \sin^2 \alpha \delta\phi,$$

and hence

$$\delta\phi' = \sin \alpha \delta\phi.$$

Thus, replacing ϕ by ϕ' as the independent variable in (2.31), we have

$$\frac{d^2 r}{d\phi'^2} - \frac{2}{r} \left(\frac{dr}{d\phi'} \right)^2 - r + \frac{r^4 g \cos \alpha \sin^2 \alpha}{h^2} = 0. \quad (2.32)$$

This, therefore, is the polar equation of the path of the particle relative to the surface.

Suppose we now put $r = 1/u$, so that

$$\frac{dr}{d\phi'} = -\frac{1}{u^2} \frac{du}{d\phi'}, \quad \frac{d^2 r}{d\phi'^2} = -\frac{1}{u^2} \frac{d^2 u}{d\phi'^2} + \frac{2}{u^3} \left(\frac{du}{d\phi'} \right)^2.$$

Then, after substitution, (2.32) reduces to

$$\frac{d^2 u}{d\phi'^2} + u = \frac{g \cos \alpha \sin^2 \alpha}{h^2 u^2}.$$

It follows, therefore, that, if the cone be developed into a plane, the trace of the path on the surface will be the same as if the particle were in motion in a plane under the action of a constant central force.*

5. Motion of a charged particle

An important example of particle motion is that of a charged particle in an electromagnetic field. If the electric and magnetic intensities of the field are denoted by $\mathbf{E}$ and $\mathbf{H}$ respectively, then the electromagnetic force per unit charge acting on the particle is given by

$$\mathbf{E} + \frac{1}{c} (\mathbf{v} \wedge \mathbf{H}),$$

where $\mathbf{v}$ is the velocity of the particle and $\mathbf{E}$ and $\mathbf{H}$ are measured in Gaussian units, c being the velocity of light. Consequently, if m is

* See, for instance, Lamb, *Dynamics*, p. 263, § 90.

the mass of the particle, e is its charge, and $\mathbf{r}$ is its vector displacement at time t with respect to a fixed origin O , then its equation of motion is given by*

$$m \frac{d^2 \mathbf{r}}{dt^2} = e \left\{ \mathbf{E} + \frac{1}{c} \left(\frac{d\mathbf{r}}{dt} \wedge \mathbf{H} \right) \right\}. \quad (2.33)$$

Suppose we consider the case in which $\mathbf{E}$ and $\mathbf{H}$ are at right angles. Then, multiplying (2.33) scalarly by $\mathbf{H}$, we have

$$\frac{d^2 \mathbf{r}}{dt^2} \cdot \mathbf{H} = 0,$$

i.e. the acceleration is always at right angles to the direction of $\mathbf{H}$. If $\mathbf{H}$ has a constant direction, then it follows that the component velocity of the particle parallel to $\mathbf{H}$ is constant throughout the motion and equal to its initial value in this direction. Let this velocity be u . Then we can regard (2.33) as defining the motion of the particle in a plane at right angles to $\mathbf{H}$, the plane itself travelling forwards parallel to the direction of $\mathbf{H}$ with velocity u .

Now let us suppose that $\mathbf{E}$ and $\mathbf{H}$ are both constants and that u is zero. Then, integrating (2.33), we have

$$m \frac{d\mathbf{r}}{dt} = e\mathbf{E}t + \frac{e}{c}(\mathbf{r} \wedge \mathbf{H}) + \mathbf{C},$$

where $\mathbf{C}$ is a constant of integration. ($\mathbf{C}$, $\mathbf{E}$, and $d\mathbf{r}/dt$ are, of course, coplanar.) If the motion is such that $d\mathbf{r}/dt = \mathbf{0}$ when $\mathbf{r} = \mathbf{0}$, $t = 0$, then $\mathbf{C}$ vanishes and hence

$$\frac{d\mathbf{r}}{dt} = \frac{e}{m} \left\{ \mathbf{E}t + \frac{1}{c}(\mathbf{r} \wedge \mathbf{H}) \right\}.$$

Substituting this expression for $d\mathbf{r}/dt$ in (2.33), the differential equation, which determines the motion, thus becomes

$$\begin{aligned} m \frac{d^2 \mathbf{r}}{dt^2} &= e\mathbf{E} + \frac{e^2}{mc}(\mathbf{E} \wedge \mathbf{H})t + \frac{e^2}{mc^2}\{(\mathbf{r} \wedge \mathbf{H}) \wedge \mathbf{H}\} \\ &= e\mathbf{E} + \frac{e^2}{mc}(\mathbf{E} \wedge \mathbf{H})t + \frac{e^2}{mc^2}\{(\mathbf{r} \cdot \mathbf{H})\mathbf{H} - H^2 \mathbf{r}\}. \end{aligned}$$

$$\text{i.e.} \quad \frac{d^2 \mathbf{r}}{dt^2} + \frac{e^2 H^2}{m^2 c^2} \mathbf{r} = \frac{e}{m} \mathbf{E} + \frac{e^2}{m^2 c} (\mathbf{E} \wedge \mathbf{H})t,$$

* This equation is that appropriate to non-relativistic motion, when the speed of the particle is small compared with that of light; for speeds comparable with that of light, the relativistic equation of motion must be used, the left-hand side of (2.33) being replaced by

$$\frac{d}{dt} \left(\frac{m}{\sqrt{1-v^2/c^2}} \frac{d\mathbf{r}}{dt} \right).$$

since $\mathbf{r} \cdot \mathbf{H} = 0$. If we now write

$$\mathbf{R} = \mathbf{r} - \frac{mc^2}{eH^2} \mathbf{E}, \quad (2.34)$$

then this becomes

$$\frac{d^2 \mathbf{R}}{dt^2} + \frac{e^2 H^2}{m^2 c^2} \mathbf{R} = \frac{e^2}{m^2 c} (\mathbf{E} \wedge \mathbf{H}) t. \quad (2.35)$$

The complete solution of this equation consists of the general solution of the equation

$$\frac{d^2 \mathbf{R}}{dt^2} + \frac{e^2 H^2}{m^2 c^2} \mathbf{R} = \mathbf{0} \quad (2.36)$$

together with a particular integral. Now the general solution of (2.36) is given by

$$\mathbf{R} = \mathbf{A} \cos \omega t + \mathbf{B} \sin \omega t \quad (\omega^2 = e^2 H^2 / m^2 c^2),$$

where $\mathbf{A}$ and $\mathbf{B}$ are two arbitrary vector constants, while a particular integral of (2.35) is clearly

$$\mathbf{R} = \frac{c}{H^2} (\mathbf{E} \wedge \mathbf{H}) t.$$

The complete solution of (2.35) is thus given by

$$\mathbf{R} = \frac{c}{H^2} (\mathbf{E} \wedge \mathbf{H}) t + \mathbf{A} \cos \omega t + \mathbf{B} \sin \omega t,$$

so that the general expression for $\mathbf{r}$ is (eqn 2.34)

$$\mathbf{r} = \frac{mc^2}{eH^2} \mathbf{E} + \frac{c}{H^2} (\mathbf{E} \wedge \mathbf{H}) t + \mathbf{A} \cos \omega t + \mathbf{B} \sin \omega t.$$

Putting in the above initial conditions, $\mathbf{r} = \mathbf{0}$, $d\mathbf{r}/dt = \mathbf{0}$ when $t = 0$, the values of $\mathbf{A}$ and $\mathbf{B}$ are thus

$$\mathbf{A} = -\frac{mc^2}{eH^2} \mathbf{E}, \quad \mathbf{B} = -\frac{c}{\omega H^2} (\mathbf{E} \wedge \mathbf{H}),$$

and the expression for $\mathbf{r}$ becomes

$$\begin{aligned} \mathbf{r} &= \frac{mc^2}{eH^2} \mathbf{E} (1 - \cos \omega t) + \frac{c}{\omega H^2} (\mathbf{E} \wedge \mathbf{H}) (\omega t - \sin \omega t) \\ &= \frac{e}{m\omega^2} \left\{ \mathbf{E} (1 - \cos \omega t) + \frac{(\mathbf{E} \wedge \mathbf{H})}{H} (\omega t - \sin \omega t) \right\}. \end{aligned}$$

If we now choose a set of orthogonal axes $OX Y$ such that OX is in the

direction of $\mathbf{E}$, OY in the negative direction of $\mathbf{E} \wedge \mathbf{H}$ (Fig. 2.11(a)), then we have

$$x = \frac{eE}{m\omega^2}(1 - \cos \omega t),$$

$$y = -\frac{eE}{m\omega^2}(\omega t - \sin \omega t).$$

The path of the particle in the OXY plane is thus a cycloid with its cusps along the axis of OY as shown in Fig. 2.11(b).

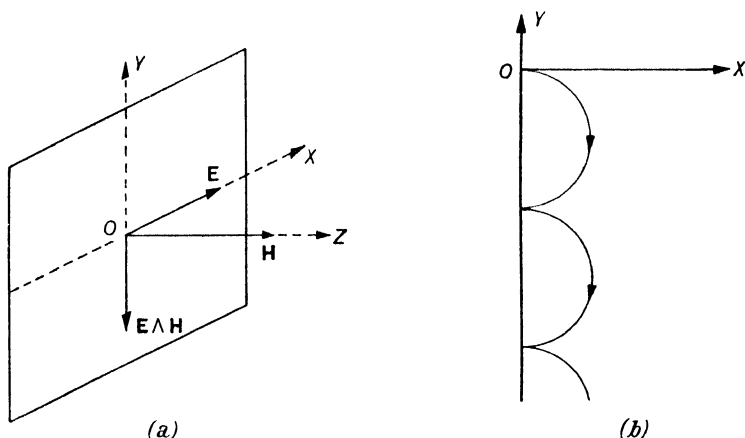


Figure 2.11

A more complete discussion of the motion of a charged particle in an electromagnetic field is to be found in Synge and Griffith, *Principles of Mechanics*, 2nd ed., p. 381, § 13.4.

In many problems involving the motion of a particle it is advantageous to refer the motion to a moving frame of reference. Before considering the general theory of such motion we shall first consider the kinematical motion of a rigid body. This necessitates the introduction of the idea of angular velocity and the establishment of its vector nature. We shall return to the consideration of the equations of motion of a particle, when referred to a set of rotating axes, in Chap. IV.

Examples

1. A particle of mass m moves on the smooth surface of revolution having as its equation $z^2 = r^2 - a^2$. The particle is projected horizontally with a velocity $2\sqrt{ag}$ along the surface from a point whose

distance from the axis of symmetry is $\sqrt{2a}$. Show that the particle moves between the two horizontal circles whose radii are given by $r^2 = 2a^2$ and $r^2 = 2(2 + \sqrt{2})a^2$.

2. A heavy particle of mass m moves on the smooth inner surface of a sphere of radius a , its greatest and least depths below the centre being $a/2$ and $a/4$. Show that when the depth below the centre is z , the normal reaction of the sphere is

$$3mg(z + a/2)/a.$$

3. A heavy particle is projected with a velocity V from a point on the smooth outer surface of a paraboloid of revolution, of latus rectum $4a$, whose axis is vertical and vertex upwards; the point of projection is at a depth a below the vertex, the direction of projection being in the tangent plane to the surface and inclined at an angle α to the line of greatest slope. Show that the particle remains on the surface indefinitely provided

$$4ag > V^2(1 + \sin^2 \alpha).$$

4. One end of a light elastic string, of natural length c and modulus λ , is attached to a fixed point A and to its other end is attached a small heavy particle of mass m . The string passes through a small hole at the highest point of a hollow sphere of radius a , the centre of the sphere being at a distance $a + c$ vertically below A , and the particle is projected along the smooth inner surface of the sphere. Show that, as long as the particle remains in contact with the sphere, the energy equation is given by

$$\dot{r}^2 + \dot{z}^2 + \frac{h^2}{r^2} - 2gz \left(1 - \frac{a\lambda}{gmc}\right) = \text{constant},$$

where r is the radial distance of the particle from the vertical diameter of the sphere, z is the distance of the particle below O , the centre of the sphere, and h has its usual meaning.

If the particle is projected horizontally from a point whose angular displacement from the downward vertical through O is α , show that the particle rises initially provided that

$$h^2 > g \left(1 - \frac{a\lambda}{gmc}\right) a^3 \sin^3 \alpha \tan \alpha.$$

5. Two particles A and B , the mass of A being twice that of B , are connected by a light inextensible string. The particle A is free to move on the smooth inner surface of a circular cone of semi-vertical angle $\frac{1}{3}\pi$, which has its axis vertical and vertex downwards. The string passes through a small hole at the vertex of the cone, the string having sufficient length that the particle B is never in contact with the cone.

Initially the particle B is hanging freely, while the particle A is held at a distance a from the vertex of the cone and projected horizontally. Show that B begins to rise initially provided the velocity of projection is such that

$$4h^2 > 3ga^3.$$

6. A particle of mass m and charge e is released from rest and proceeds to move under the influence of a uniform electric field of intensity $\mathbf{E}$ and a uniform magnetic field of intensity $\mathbf{H}$ not parallel to $\mathbf{E}$. Prove that the particle describes a cycloid in a plane, perpendicular to $\mathbf{H}$, which moves parallel to itself with constant acceleration.

Show, also, that if a set of rectangular axes $OXYZ$ is taken, OX being in the direction of $\mathbf{E}$, OY in the negative direction of $\mathbf{E} \wedge \mathbf{H}$, and OZ a mutual perpendicular such that $OXYZ$ is right-handed, then the initial motion is given approximately by

$$\begin{aligned} x &= \frac{cEt^2}{2m} - \frac{e^3EH^2 \cos^2 \alpha t^4}{24m^3c^2}, \\ y &= -\frac{c^2EH \cos \alpha t^3}{6m^2c}, \\ z &= \frac{e^3EH^2 \sin 2\alpha t^4}{48m^3c^2}, \end{aligned}$$

where α is the angle which $\mathbf{H}$ makes with OX and t is the time from rest.

CHAPTER III

KINEMATICAL MOTION OF A RIGID BODY

1. Finite displacements

In discussing the kinematical motion of a rigid body, we consider the general character of its motion without being concerned with the nature of the forces which produce it. We shall start by studying the finite displacement of a rigid body which has one point fixed, and then proceed to a study of the most general displacement of such a body.

1.1. Displacements about a fixed point. Euler's theorem. The important theorem relating to finite displacements of a rigid body having one point fixed is one due to Euler* and which is named after him. It may be expressed as follows: *If a body has one point O fixed, then any displacement of the body from one given position to another is equivalent to a rotation about some fixed axis through O .*

To prove the theorem, let us imagine O to be the centre of two coincident spherical surfaces one of which is fixed in the body and the other in space. Let the radius vector to a point Q of the body cut the spherical surfaces in the point P ; then the motion of every point Q of the body may be represented by that of the corresponding point P .

Let the displacements on the spherical surface fixed in space of two such points A and B , resulting from a given displacement of the body, be AA' and BB' respectively. Then the displacement on this surface of any other point P may be found quite simply by constructing on $A'B'$ as base a spherical triangle $A'P'B'$ similar and equal to APB , for clearly† it will be represented by the arc PP' . Let D and E be the middle points of the arcs AA' and BB' , and let the arcs of great circles be drawn through D and E perpendicular to AA' and BB' respectively. Their point of intersection C will be clearly a unique point. Then, considering the triangles ACB and $A'CB'$, we have

$$CA = CA', \quad CB = CB',$$

* *Leonard Euler* (1707–1783) has been described as the most prolific mathematician of all time. He was born at Basle in Switzerland, but did most of his mathematical work in the Research Academies of St. Petersburg and Berlin. Although essentially a pure mathematician, he made important contributions in the applied fields of astronomy, mechanics, and hydrodynamics.

† For a proof of the theorem that on the same base, and on the same side of it, there cannot be two triangles on the same sphere which are congruent, see any textbook on spherical geometry.

and, since the bases are of necessity equal,

$$AB = A'B'.$$

The two triangles, therefore, are congruent, and the displacement of C is zero. But O is a fixed point, so that its displacement is in any case zero, and hence it follows, the displacements of both O and C being zero, that the displacement of every point on OC is zero. Consequently it follows that, by a rotation about the axis OC , A can be

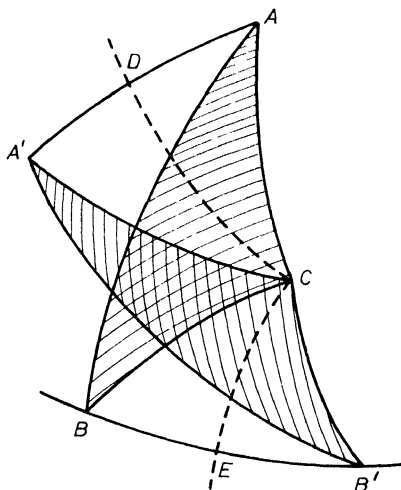


Figure 3.1

brought to A' and B to B' , and hence, since the position of a rigid body is clearly fixed if the positions of three non-collinear points of it are known, every point of the body from its initial to its final position. Similarly for any other displacement. It follows, therefore, that every displacement of a rigid body having one point fixed is equivalent to a rotation about a unique axis through the fixed point, and the theorem is proved.*

1.2. Positive and negative rotations. It is clear that a rotation has both a magnitude and a direction, its magnitude being the angle through which the body is rotated, its direction being determined by the direction of the axis together with the sense of the rotation about it. If the rotation about an axis OJ is related to the direction *from* O to J in the same way as the sense of the rotation is related to that of advance in a right-handed screw motion, then it is counted positive; in the

* For a proof based on vector analysis, see Milne, *Vectorial Mechanics*, Chap. VIII, p. 145, § 170

reverse direction, negative. A finite rotation may therefore be represented by a segment OA marked off on the axis of rotation, the length of the segment being chosen to represent the angle of rotation according to some convenient scale, and the direction from O to A being so chosen that the rotation is right-handed with respect to it. Thus in Fig. 3.2,

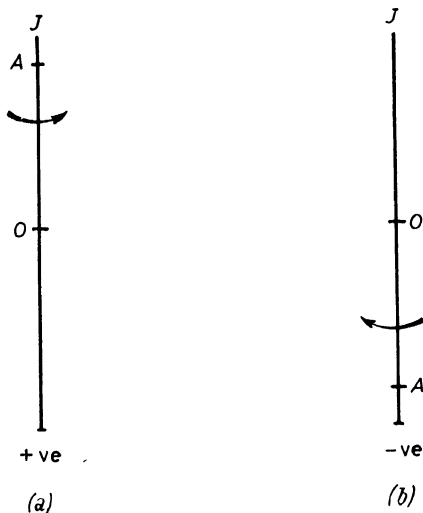


Figure 3.2

if OJ represents the axis of rotation, then a positive rotation is represented by a line segment OA as in (a), a negative rotation by a line segment OA as in (b). It follows, therefore, that a finite rotation may be represented both in magnitude and direction by the segment of a straight line.

1.3. Composition of finite rotations. Suppose we consider the result of applying to a rigid body two successive rotations, a rotation α about an axis OA , followed by a rotation β about an axis OB , A and B being the points in which the respective axes meet the spherical surface fixed in space and having O as its centre. We wish to find the angle of rotation and the axis of rotation corresponding to the equivalent single rotation.

Suppose we construct on AB as base the spherical triangles ACB , $AC'B$ having the angles at A each equal to $\alpha/2$, and the angles at B each equal to $\beta/2$ (Fig. 3.3). Then, it is clear from the figure that the rotation α about OA brings a point of the body from C to C' , and that the subsequent rotation β about OB takes it back to C . (Triangles ACB and $AC'B$ are, of course, congruent.) Hence OC is the axis of

rotation corresponding to the rotation equivalent to the two given ones. It should be noted that had the order of the rotations about OA and OB been inverted, however, then the resultant displacement would have been equivalent to a rotation about OC' . We have here, therefore, an instance of the operation of addition which does not follow the *commutative law*. Consequently, finite rotations about intersecting axes, though directed quantities, do not obey the parallelogram law of addition and, therefore, do not add as vectors. Finite rotations,

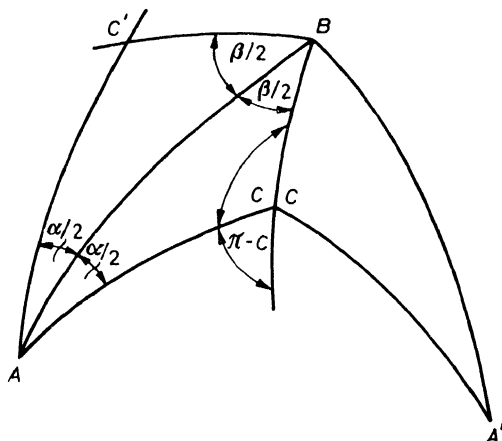


Figure 3.3

therefore, cannot be regarded as vector quantities, and the laws of vector algebra cannot be applied to them.

We have found the axis of the resultant rotation; it remains to find the angle. To do this, construct on the opposite side of BC a triangle $A'BC$ symmetrically equal to ABC . Since the rotation about OA does not affect a point initially at A , whilst the rotation about OB brings it to A' , the required angle is ACA' or $2(\pi - C)$. Since a rotation through four right angles does not affect the position of a body, this is equivalent to a rotation $-2C$ about the same axis.

1.4. The most general displacement of a rigid body. Consider any point O of the body and let it be moved into its new position O' by means of a translational displacement of the body as a whole. Then, since the position of a rigid body is known if the positions of any three non-collinear points of it are known, it follows that if we bring any two points of the body not collinear with O' into their final positions, the body, clearly, will have been brought from its initial to its final position. But, by Euler's theorem, any displacement of the body about O' as a fixed point must be equivalent to a rotation about some unique axis

through O' . Consequently we have the result: *Every displacement of a rigid body may be represented by a combination of the following motions: (i) a motion of translation whereby every particle undergoes a displacement equal and parallel to some assumed point O rigidly connected to the body and called the base point; (ii) a motion of rotation of the whole body about some unique axis through this assumed point.*

Since these motions are completely independent, it is evident that their order may be inverted, i.e. we may first rotate the body and then translate it. The two displacements are thus *commutative*. We may also suppose them to take place simultaneously.

Initially, any point of a body may be chosen as base point. Consequently any given displacement may be brought about in an infinite variety of ways.

It is at once obvious that the translational displacement of a body will vary with the base point chosen. But, in the case of the rotational displacement, this is not so, for it is possible to show that no matter what the choice of base point, the axes of rotation corresponding to all base points are parallel and the angles of rotation equal. (For proof see, for instance, Routh, *Elementary Rigid Dynamics*, p. 186, §§ 220–224.) Our main interest, however, is not in finite displacements but in the infinitesimally small displacements arising from the motion of the body in a small interval of time. We shall not prove, therefore, the above result for finite displacements, but only for infinitesimal displacements. Such displacements we consider in the next paragraph.

2. Infinitesimal displacements

Infinitesimal displacements are much simpler to treat analytically than are finite displacements since, as we shall see, the effect of successive small rotational displacements may be found by means of the parallelogram law of vector addition, whereas finite rotations, as we have seen, cannot be added in this way. We shall consider first the theory of infinitesimal rotations, then the theory of infinitesimal displacements in general, and finally, by considering the rate of change of such displacements in the limit when the displacements tend to zero, the instantaneous motion of a rigid body.

2.1. Infinitesimal rotations. It is clear that an infinitesimal rotation is defined when we know the axis about which the rotation takes place and the angle, having regard, of course, to the sense. An infinitesimal rotation, like a finite rotation, may thus be indicated completely by means of a segment AB , say, measured along the axis, the length of the segment being related to the angle of rotation $\delta\theta$ by a relation of the form

$$\delta\theta = \epsilon AB,$$

where ϵ is an infinitesimally small quantity, and the direction from A to B is so chosen that the rotation is right-handed with respect to it. The exact position of AB along the axis is, of course, not important.

2.2. Displacement of a rigid body due to an infinitesimal rotation. To find the displacement of a point P of a rigid body due to an infinitesimal rotation of the body through an angle $\delta\theta$ about a given axis, let us first consider the displacement of P when the rotation is a finite one. Let θ be the angle of rotation, the axis of rotation passing through a given point O , its direction being specified by means of the unit vector $\mathbf{a}$. Let P be a general point of the body, and let $\overline{OP} = \mathbf{r}$. Then, as a result of the rotation, P will go to the point P' in the plane through P perpendicular to the axis of the rotation such that

$$PP' = 2PM \sin \theta/2,$$

where PM is the perpendicular from P on to the axis of rotation, PP' making an angle $\frac{1}{2}(\pi - \theta)$ with the plane OPM (Fig. 3.4). Consequently,

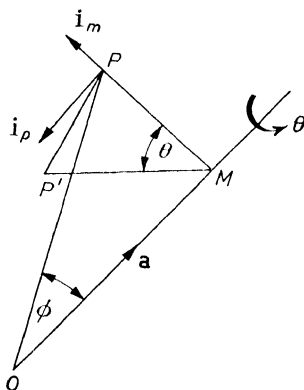


Figure 3.4

if $\mathbf{i}_m$ is a unit vector in the direction of $\overline{MP}$ and $\mathbf{i}_p$ a unit vector in a perpendicular direction as shown, then we have

$$\overline{PP'} = 2PM \sin (\theta/2) \{ \cos (\theta/2) \mathbf{i}_p - \sin (\theta/2) \mathbf{i}_m \},$$

But
$$\mathbf{i}_p = \frac{\mathbf{a} \wedge \mathbf{r}}{r \sin \phi} = \frac{\mathbf{a} \wedge \mathbf{r}}{PM} \quad (\phi = \angle POM),$$

$$\mathbf{i}_m = \mathbf{i}_p \wedge \mathbf{a} = \frac{(\mathbf{a} \wedge \mathbf{r}) \wedge \mathbf{a}}{PM}.$$

Hence

$$\begin{aligned} \overline{PP'} &= 2 \sin (\theta/2) \{ (\mathbf{a} \wedge \mathbf{r}) \cos \theta/2 - [(\mathbf{a} \wedge \mathbf{r}) \wedge \mathbf{a}] \sin \theta/2 \} \\ &= (\mathbf{a} \wedge \mathbf{r}) \sin \theta + \{ \mathbf{a} \wedge (\mathbf{a} \wedge \mathbf{r}) \} (1 - \cos \theta). \end{aligned} \quad (3.1)$$

In the case in which θ is small ($\approx \delta\theta$), then, neglecting small quantities of the second order in $\delta\theta$, we can write

$$\sin \theta \approx \delta\theta, \quad \cos \theta \approx 1.$$

Consequently, substituting into (3.1), we have, in the case of small rotations,

$$\overline{PP'} = \delta\theta(\mathbf{a} \wedge \mathbf{r}),$$

i.e. provided second powers of small quantities are neglected, the displacement of the point P of the body, where $\overline{OP} = \mathbf{r}$, is perpendicular to the plane POM , its magnitude and direction being those of the vector product $\delta\theta(\mathbf{a} \wedge \mathbf{r})$.

2.3. Vector nature of infinitesimal rotations. If $\delta\mathbf{r}$ is the displacement of P due to a small rotation $\delta\theta$ about the axis of rotation defined by $\mathbf{a}$, then

$$\delta\mathbf{r} = \delta\theta(\mathbf{a} \wedge \mathbf{r}).$$

But, if the components of $\mathbf{a}$ when referred to a set of orthogonal axes are (l, m, n) , then we can write

$$\mathbf{a} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k},$$

where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ have their usual meanings, so that

$$\begin{aligned} \delta\mathbf{r} &= \delta\theta (l\mathbf{i} + m\mathbf{j} + n\mathbf{k}) \wedge \mathbf{r} \\ &= l \delta\theta(\mathbf{i} \wedge \mathbf{r}) + m \delta\theta(\mathbf{j} \wedge \mathbf{r}) + n \delta\theta(\mathbf{k} \wedge \mathbf{r}), \end{aligned}$$

i.e. the displacement of the body is the same as that which would arise if the body were given small rotations $l\delta\theta$, $m\delta\theta$, $n\delta\theta$ about the coordinate axes respectively in any chosen order. Thus a small rotation may be resolved into components in the same way as a vector quantity, which implies that small rotations combine according to the parallelogram law. It follows, therefore, that the combined effect of applying two successive small rotations $(\delta\theta_1, \delta\theta_2)$ about two intersecting axes, defined by the unit vectors $(\mathbf{a}_1, \mathbf{a}_2)$, to a rigid body is equivalent to that of a small rotation of magnitude $\delta\theta$ about an axis $\mathbf{a}$ where

$$\delta\theta_1 \mathbf{a}_1 + \delta\theta_2 \mathbf{a}_2 = \delta\theta \mathbf{a},$$

the vector $\delta\theta \mathbf{a}$ being defined in magnitude and direction by the diagonal of the parallelogram having adjacent sides proportional to $\delta\theta_1 \mathbf{a}_1$ and $\delta\theta_2 \mathbf{a}_2$ respectively (see § 2.1). We thus have the result that *small rotations are vectors*. Two or more small rotations about intersecting axes applied to a rigid body may be regarded, therefore, as taking place either successively or simultaneously.

In the ensuing theory, we shall denote the vector nature of a small rotation by writing

$$\delta\theta \mathbf{a} = \delta\boldsymbol{\theta}.$$

2.4. General infinitesimal displacement of a rigid body. It follows from § 1.4 that, just as in the case of a finite displacement, so any small displacement of a rigid body may be regarded as consisting of the infinitesimal translational displacement of some chosen base point together with a small rotation $\delta\theta$ about some unique axis through the base point. As stated above, the translational displacement, being that of the base point, varies in general with the choice of base point; we shall now prove the result that the angles of rotation are equal, and the axes of rotation are parallel, no matter what point is chosen as base point.

Let the small displacement of the body be equivalent to the translational displacement $\delta\mathbf{s}_O$ of a point O of the body plus a rotation $\delta\theta$ about an axis through O . Then, if P is a general point of the body such that $\overline{OP} = \mathbf{r}$, the displacement of P will be given by

$$\delta\mathbf{r} = \delta\mathbf{s}_O + \delta\theta \wedge \mathbf{r}. \quad (3.2)$$

When referred to a new base point O' , let the displacement of the body be equivalent to a translational displacement $\delta\mathbf{s}_{O'}$ plus a rotation $\delta\theta'$ about an axis through O' . Then, in the same way, the displacement of P will be given by

$$\delta\mathbf{r} = \delta\mathbf{s}_{O'} + \delta\theta' \wedge \mathbf{r}', \quad (3.3)$$

where $\mathbf{r}' = \overline{O'P}$. But, in terms of the first representation, $\delta\mathbf{s}_{O'}$, being the displacement of O' , must also be given by

$$\begin{aligned} \delta\mathbf{s}_{O'} &= \delta\mathbf{s}_O + \delta\theta \wedge \overline{OO'} \\ &= \delta\mathbf{s}_O + \delta\theta \wedge (\mathbf{r} - \mathbf{r}'). \end{aligned}$$

Consequently, substituting this expression for $\delta\mathbf{s}_{O'}$ in (3.3), we have

$$\delta\mathbf{r} = \delta\mathbf{s}_O + \delta\theta \wedge (\mathbf{r} - \mathbf{r}') + \delta\theta' \wedge \mathbf{r}'.$$

This is equivalent to the expression for $\delta\mathbf{r}$, as given by (3.2), only if

$$\delta\theta \wedge (\mathbf{r} - \mathbf{r}') + \delta\theta' \wedge \mathbf{r}' = \delta\theta \wedge \mathbf{r},$$

i.e. only if

$$\delta\theta' \wedge \mathbf{r}' = \delta\theta \wedge \mathbf{r}'.$$

This requires that the relation between $\delta\theta'$ and $\delta\theta$ be of the form

$$\delta\theta' = \delta\theta + k\mathbf{r}',$$

where k is an arbitrary scalar coefficient. But $\delta\theta'$, whilst possibly depending on the position of O' , can in no way be dependent upon the position of P . Hence k must be zero. We have the result, therefore, that

$$\delta\theta' = \delta\theta,$$

i.e. the rotational displacements are the same, both as regards magnitude and direction, whatever the choice of base point.

3. Instantaneous motion of a rigid body

Suppose that we consider the displacement of the body during a small interval of time $(t, t + \delta t)$. Let this displacement be equivalent to the translational displacement $\delta \mathbf{s}_O$ of a point O of the body plus a rotation $\delta \boldsymbol{\theta}$ about an axis through O , the latter being a unique vector as regards both magnitude and direction. Then, if P is a general point of the body such that $\overline{OP} = \mathbf{r}$, the displacement of P in the interval δt will be given by

$$\delta \mathbf{r} = \delta \mathbf{s}_O + \delta \boldsymbol{\theta} \wedge \mathbf{r},$$

and hence the velocity of the point P at time t , being defined by the relation

$$\mathbf{v}_P = \lim_{\delta t \rightarrow 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt},$$

will be given by

$$\begin{aligned} \mathbf{v}_P &= \lim_{\delta t \rightarrow 0} \left\{ \frac{\delta \mathbf{s}_O}{\delta t} + \frac{\delta \boldsymbol{\theta}}{\delta t} \wedge \mathbf{r} \right\}, \\ &= \mathbf{v}_O + \boldsymbol{\omega} \wedge \mathbf{r}, \end{aligned}$$

where $\mathbf{v}_O$ is the velocity of the point O at time t , and $\boldsymbol{\omega}$ is the vector defined by the relation

$$\boldsymbol{\omega} = \lim_{\delta t \rightarrow 0} \frac{\delta \boldsymbol{\theta}}{\delta t}. \quad (3.4)$$

In the same way, if we choose some other point O' of the body as base point, then the velocity of P will be given by

$$\mathbf{v}_P = \mathbf{v}_{O'} + \boldsymbol{\omega} \wedge \mathbf{r}',$$

where $\mathbf{v}_{O'}$ is the velocity of O' and $\mathbf{r}' = \overline{O'P}$.

The vector $\boldsymbol{\omega}$, defined by (3.4) above, being independent of the choice of base point, depends upon the motion of the body alone and so is completely defined when the motion of the body at time t is known. It is termed the *angular velocity of the body* at time t . Although it can be defined without reference to a base point, yet it cannot be regarded as a free vector. For, in determining the motion of a point of the body at any instant, the axis of the angular velocity vector must be specified, since the effect of an angular velocity about a given axis is not equivalent to that of an equal angular velocity about a parallel axis.* In this connection, therefore, the angular velocity vector is a *sliding* vector in just the same way as a small rotation is a *sliding* vector.

* It is equivalent to an equal angular velocity about a parallel axis together with a velocity of translation.

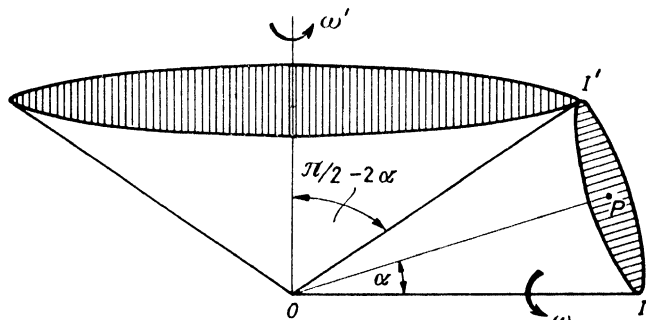
Summing up, we thus have the result: *the instantaneous motion of a rigid body is equivalent, in general, to a velocity of translation plus an angular velocity, the velocity of translation being that of some arbitrary point O of the body, whilst the axis of the angular velocity passes through O ; the velocity of translation varies with the choice of O , but the angular velocity is the same, both as regards magnitude and direction, whatever the choice of O .*

In the case of a rigid body having one point O fixed, then taking O as base point, it follows, from the above, that the instantaneous motion of such a body is equivalent to an angular velocity about a unique axis through O^* ; the axis is termed *the instantaneous axis of rotation*.

3.1. Examples

(1) *A right-circular cone of semi-vertical angle α rolls on a horizontal plane with its vertex at O . A second cone of semi-vertical angle $(\frac{1}{2}\pi - 2\alpha)$ rotates, with its axis vertical and vertex at O , in rolling contact with the first cone. If the line of contact of the first cone with the plane makes a complete circuit around the vertical in time T , determine the angular velocity of the second cone.*

Solution. Since the cone of semi-vertical angle α is in rolling contact with the ground, it follows that any point on OI , the line of contact, must be instantaneously at rest. OI is thus, at any instant, the axis of rotation for the cone.



Let ω be the angular velocity of the cone. Then the velocity of a point P on its axis will be given by

$$\mathbf{v} = \omega \wedge \overline{OP} = \omega OP \sin \alpha \mathbf{p},$$

where $\mathbf{p}$ is a unit vector at right angles to the plane IOP in the

* A vector proof of this result, which is independent of Euler's Theorem, is given in Milne, *Vectorial Mechanics*, Chap. X, p. 163, § 198.

appropriate direction. But the cone makes one complete circuit around the vertical in time T . Hence we have

$$T = \frac{2\pi OP \cos \alpha}{v} = \frac{2\pi \cos \alpha}{\omega \sin \alpha},$$

so that
$$\omega = \frac{2\pi}{T} \cot \alpha. \quad (1)$$

Now consider the motion of the second cone. Since the two cones are in rolling contact, it follows that there must be no relative motion between the cones along their line of contact. Now the velocity of any point I' on the line of contact, regarded as a point on the first cone, is given by

$$\mathbf{v}' = \boldsymbol{\omega} \wedge \overline{OI'} = \omega OI' \sin 2\alpha \mathbf{p}. \quad (2)$$

Regarded as a point on the second cone, the velocity of I' will be given by

$$\mathbf{v}' = \boldsymbol{\omega}' \wedge \overline{OI'} = -\omega' OI' \sin (\tfrac{1}{2}\pi - 2\alpha) \mathbf{p}, \quad (3)$$

if $\boldsymbol{\omega}'$ is the angular velocity of the second cone about its axis which is vertical.

Equating the expressions for $\mathbf{v}'$, given by (2) and (3), we must therefore have

$$\omega' \sin (\tfrac{1}{2}\pi - 2\alpha) = -\omega \sin 2\alpha,$$

so that
$$\omega' = -\frac{\omega \sin 2\alpha}{\cos 2\alpha} = -\omega \tan 2\alpha.$$

The negative sign means that if the angular velocity of the first cone about OI is positive, i.e. it corresponds to a right-handed rotational motion about OI , then the rotational motion of the second cone about its axis must be in a negative direction, i.e. it must be left-handed.

On substituting for ω from (1), the magnitude of the angular velocity of the second cone, the question of sign being ignored, is thus given by

$$\omega' = \frac{2\pi}{T} \cot \alpha \tan 2\alpha.$$

(2) *A sphere of radius a is pressed between two perfectly rough parallel planes and made to revolve with uniform angular speeds Ω_1 , Ω_2 about fixed axes perpendicular to their planes. Determine the motion of the sphere, and show that the path of the centre of the sphere is a circle described with uniform speed.*

Solution. Let the upper plane rotate with an angular speed Ω_1 about an axis Γ_1 , and let the lower plane rotate about the axis Γ_2 with an angular speed Ω_2 . Let a plane parallel to the given planes and

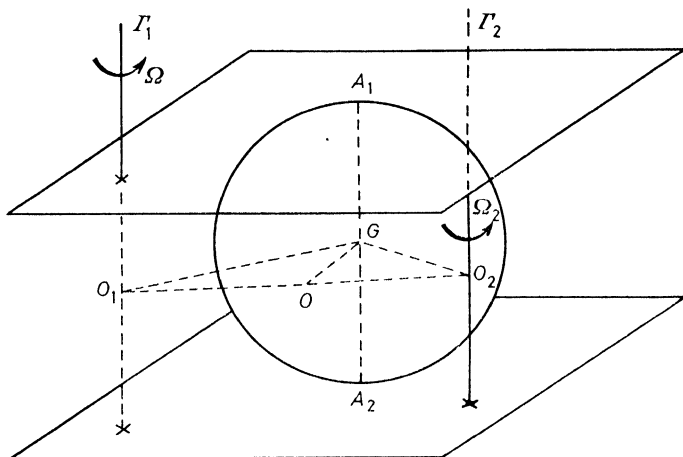
passing through G , the centre of the sphere, cut the axes of rotation in O_1 and O_2 respectively, and let

$$\overline{O_1G} = \mathbf{r}_1, \quad \overline{O_2G} = \mathbf{r}_2.$$

Let A_1 and A_2 be the points of contact of the sphere with the planes. Then, if $\mathbf{n}$ is a unit vector perpendicular to the planes, we have

$$\overline{GA_1} = a\mathbf{n}, \quad \overline{GA_2} = -a\mathbf{n},$$

where a is the radius of the sphere.



Now consider the motion of the points A_1 and A_2 . Since the sphere is in rolling contact with the planes, it follows that the point A_1 , regarded as a point of the sphere, must have the same velocity as the corresponding point of the plane. Similarly for the point A_2 . To express these results mathematically, we must be able to write down the velocities of A_1 and A_2 , therefore, (i) as points of their respective planes, (ii) as points of the sphere. The former can be written down at once, for clearly we have:

Velocity of A_1 regarded as a point of the upper plane

$$\begin{aligned} &= \boldsymbol{\Omega}_1 \wedge \overline{O_1A_1} \\ &= \boldsymbol{\Omega}_1 \wedge (\overline{O_1G} + \overline{GA_1}) \\ &= \boldsymbol{\Omega}_1 \wedge \overline{O_1G}, \quad (\boldsymbol{\Omega}_1 \wedge \overline{GA_1} = \mathbf{0}), \\ &= \boldsymbol{\Omega}_1(\mathbf{n} \wedge \mathbf{r}_1). \end{aligned} \tag{1}$$

Similarly :

Velocity of A_2 regarded as a point of the lower plane

$$= \boldsymbol{\Omega}_2(\mathbf{n} \wedge \mathbf{r}_2). \tag{2}$$

To write down the velocities of A_1 and A_2 when regarded as points of the sphere, the motion of the sphere must be known. This can be defined in terms of the velocity of a chosen base point together with the angular velocity of the sphere. The most convenient point to choose as base point is the centre of the sphere. Let $\mathbf{v}$ be the velocity of G , therefore, and $\boldsymbol{\omega}$ the angular velocity of the sphere. Then we shall have:

$$\text{Velocity of } A_1 \text{ regarded as a point of the sphere} = \mathbf{v} + (\boldsymbol{\omega} \wedge \mathbf{a}\mathbf{n}), \quad (3)$$

$$\text{Velocity of } A_2 \quad ,, \quad ,, \quad ,, \quad ,, \quad ,, \quad ,, \quad = \mathbf{v} - (\boldsymbol{\omega} \wedge \mathbf{a}\mathbf{n}). \quad (4)$$

Therefore, equating (1) and (3), and (2) and (4), we have

$$\mathbf{v} + \boldsymbol{\omega} \wedge \mathbf{a}\mathbf{n} = \Omega_1(\mathbf{n} \wedge \mathbf{r}_1), \quad (5)$$

$$\mathbf{v} - \boldsymbol{\omega} \wedge \mathbf{a}\mathbf{n} = \Omega_2(\mathbf{n} \wedge \mathbf{r}_2). \quad (6)$$

We have now two vector equations to determine the two vector unknowns, $\mathbf{v}$ and $\boldsymbol{\omega}$.

On adding the above equations, we obtain $\mathbf{v}$; for we have

$$2\mathbf{v} = \Omega_1(\mathbf{n} \wedge \mathbf{r}_1) + \Omega_2(\mathbf{n} \wedge \mathbf{r}_2),$$

so that

$$\mathbf{v} = \frac{1}{2}\{\mathbf{n} \wedge (\Omega_1\mathbf{r}_1 + \Omega_2\mathbf{r}_2)\}. \quad (7)$$

Since all the quantities on the right-hand side are known, $\mathbf{v}$ can be determined.

To determine $\boldsymbol{\omega}$ we have, on subtracting (6) from (5),

$$2(\boldsymbol{\omega} \wedge \mathbf{a}\mathbf{n}) = \Omega_1(\mathbf{n} \wedge \mathbf{r}_1) - \Omega_2(\mathbf{n} \wedge \mathbf{r}_2),$$

i.e.

$$\mathbf{a}(\boldsymbol{\omega} \wedge \mathbf{n}) = \frac{1}{2}\{\mathbf{n} \wedge (\Omega_1\mathbf{r}_1 - \Omega_2\mathbf{r}_2)\},$$

or

$$\{\mathbf{a}\boldsymbol{\omega} + \frac{1}{2}(\Omega_1\mathbf{r}_1 - \Omega_2\mathbf{r}_2)\} \wedge \mathbf{n} = \mathbf{0}.$$

The general solution of this equation is

$$\mathbf{a}\boldsymbol{\omega} + \frac{1}{2}(\Omega_1\mathbf{r}_1 - \Omega_2\mathbf{r}_2) = \lambda\mathbf{n},$$

or

$$\mathbf{a}\boldsymbol{\omega} = \lambda\mathbf{n} - \frac{1}{2}(\Omega_1\mathbf{r}_1 - \Omega_2\mathbf{r}_2),$$

where λ is an arbitrary constant. Thus, unless the initial conditions of the motion are known, $\boldsymbol{\omega}$ is indeterminate to the extent of an arbitrary vertical component.

To determine the path of G , we make use of (7). Let O be the point in O_1O_2 such that

$$\Omega_1\overline{O_1O} + \Omega_2\overline{O_2O} = 0. \quad (8)$$

Then, by a well-known result, we have

$$\begin{aligned} \Omega_1\mathbf{r}_1 + \Omega_2\mathbf{r}_2 &= (\Omega_1 + \Omega_2)\overline{OG} \\ &= (\Omega_1 + \Omega_2)\mathbf{r}, \end{aligned}$$

say, where $\overline{OG} = \mathbf{r}$. But O is clearly a uniquely defined point in O_1O_2 , and hence a fixed point in space. We can therefore write

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}.$$

Consequently (7) may be written in the form

$$\frac{d\mathbf{r}}{dt} = \frac{1}{2}(\Omega_1 + \Omega_2)(\mathbf{n} \wedge \mathbf{r}). \quad (9)$$

But, if a point P describes a circle of radius r and centre O with an angular velocity Ω , then its velocity at any instant is given by

$$\mathbf{v} = \Omega \wedge \mathbf{r} = \Omega(\mathbf{n} \wedge \mathbf{r}),$$

where $\overline{OP} = \mathbf{r}$ and $\mathbf{n}$ is a unit vector perpendicular to its plane. Thus its motion is defined completely by the vector equation

$$\frac{d\mathbf{r}}{dt} = \Omega(\mathbf{n} \wedge \mathbf{r}). \quad (10)$$

Comparing (9) and (10), we thus see that the path of G is a circle having the point O in O_1O_2 , defined by (8), as centre, the angular speed with which G describes the circle being constant and equal to $\frac{1}{2}(\Omega_1 + \Omega_2)$.

Note. Eqn (10), which is the description of circular motion in vector notation, is important in vectorial mechanics and should be remembered; we shall have occasion to use it in later chapters.

3.2. Screw motion. Equation of central axis. Since the velocity of translation associated with the representation of the instantaneous motion of a rigid body varies, as regards both magnitude and direction, with the choice of base point, while the axis of the angular velocity vector, which with the velocity of translation defines the instantaneous motion of the body, has the same direction whatever the choice of base point, the question naturally arises: can a base point be found such that the direction of the corresponding velocity of translation is in the direction of the angular velocity vector? We shall see that, no matter what the motion (provided that it is not a pure translational one), such a base point can always be found. When referred to this point as base point, the motion is defined as being a *screw motion*, the axis of the angular velocity vector being termed the *central axis* of the motion.

To show that the above representation of the motion is always possible, let O be some point of the body and let the instantaneous motion of the body at time t be defined in terms of the translational velocity $\mathbf{v}_O$ of O together with an angular velocity ω . Let O' be some other point of the body and let $\mathbf{v}'$ be its velocity at time t . If the motion is to be equivalent to a screw motion with O' as base, then O'

must be such that $\mathbf{v}'$ and $\boldsymbol{\omega}$ are parallel. But, if O' is such that this is so, then the velocity of any point on the line through O' whose direction is that of $\boldsymbol{\omega}$ (i.e. of any point on the *central axis* corresponding to the motion) will also be parallel to $\boldsymbol{\omega}$. Thus the *central axis* corresponds to the locus of points such that $\mathbf{v}'$ and $\boldsymbol{\omega}$ are parallel. Using this property, we can now obtain the vector equation for the instantaneous position of the central axis.

With O as base point, the velocity of O' ($\overline{OO'} = \mathbf{r}$) will be given by

$$\mathbf{v}' = \mathbf{v}_0 + \boldsymbol{\omega} \wedge \mathbf{r}. \quad (3.5)$$

Applying the above condition, O' will thus be a point on the central axis provided that $\mathbf{r}$ is such that

$$\mathbf{v}' = p\boldsymbol{\omega},$$

where p is some scalar coefficient, i.e. provided that

$$\boldsymbol{\omega} \wedge \mathbf{v}' = \mathbf{0}.$$

Substituting for $\mathbf{v}'$ from (3.5), the condition for O' to be a point on the central axis is thus that $\mathbf{r}$ should satisfy the equation

$$\boldsymbol{\omega} \wedge \mathbf{v}_0 + \{\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})\} = \mathbf{0},$$

which becomes, on expanding the second term

$$\boldsymbol{\omega} \wedge \mathbf{v}_0 + \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \mathbf{r}) - \mathbf{r}\omega^2 = \mathbf{0};$$

on rearranging, this may be written

$$\mathbf{r} - \frac{\boldsymbol{\omega} \wedge \mathbf{v}_0}{\omega^2} = \boldsymbol{\omega} \frac{(\boldsymbol{\omega} \cdot \mathbf{r})}{\omega^2}.$$

Multiplying both sides of this equation by $\boldsymbol{\omega}$ and forming the vector product, $\mathbf{r}$ must therefore be such that

$$\left(\mathbf{r} - \frac{\boldsymbol{\omega} \wedge \mathbf{v}_0}{\omega^2} \right) \wedge \boldsymbol{\omega} = \frac{(\boldsymbol{\omega} \cdot \mathbf{r})}{\omega^2} (\boldsymbol{\omega} \wedge \boldsymbol{\omega}),$$

i.e. it must satisfy the equation

$$\left(\mathbf{r} - \frac{\boldsymbol{\omega} \wedge \mathbf{v}_0}{\omega^2} \right) \wedge \boldsymbol{\omega} = \mathbf{0}. \quad (3.6)$$

But we know that the vector equation of a straight line which passes through a point $\mathbf{r}_0$, and whose direction is defined by the unit vector $\mathbf{t}$, is given by (Chap. I, eqn 1.17)

$$(\mathbf{r} - \mathbf{r}_0) \wedge \mathbf{t} = \mathbf{0}.$$

Hence (3.6) defines a straight line passing through the point

$$\mathbf{r} = \mathbf{r}_0 = \frac{\boldsymbol{\omega} \wedge \mathbf{v}_0}{\omega^2} \quad (3.7)$$

whose direction is parallel to that of $\boldsymbol{\omega}$. It is, therefore, the equation of the *central axis*. It is clear that the point $\mathbf{r}_O$, as defined above, always exists, no matter what values $\boldsymbol{\omega}$ and $\mathbf{v}_O$ may have. Thus, in general, any given motion possesses a *central axis*. The one exception is that for which $\boldsymbol{\omega} = \mathbf{0}$, $\mathbf{v}_O \neq \mathbf{0}$ corresponding to a pure translational motion of the body.

Equations (3.6) and (3.7) enable the position of the *central axis* to be easily determined. It is clearly the line parallel to $\boldsymbol{\omega}$ which passes through the point O' on the perpendicular through O to the plane defined by $\boldsymbol{\omega}$ and $\mathbf{v}_O$ and drawn in the direction of the vector product $\boldsymbol{\omega} \wedge \mathbf{v}_O$, its distance from O being given by

$$OO' = \left| \frac{\boldsymbol{\omega} \wedge \mathbf{v}_O}{\omega^2} \right| = \frac{v_O \sin \theta}{\omega},$$

where θ is the angle between $\boldsymbol{\omega}$ and $\mathbf{v}_O$ (Fig. 3.5).

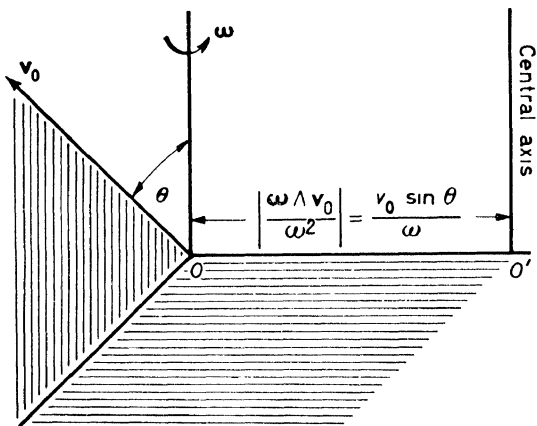


Figure 3.5

To determine the screw motion equivalent to the given motion, we have yet to determine the velocity of a point on the central axis. To do this, we shall make use of an invariance property of the motion, though other methods may be used (e.g. $\mathbf{v}' = \mathbf{v}_O + \boldsymbol{\omega} \wedge \mathbf{r}_O$). If $\mathbf{v}$ is the velocity of a general point P of the body, where $\overline{OP} = \mathbf{r}$, then

$$\mathbf{v} = \mathbf{v}_O + \boldsymbol{\omega} \wedge \mathbf{r}.$$

Hence, multiplying by $\boldsymbol{\omega}$ scalarly, we have

$$\boldsymbol{\omega} \cdot \mathbf{v} = \boldsymbol{\omega} \cdot [\mathbf{v}_O + \boldsymbol{\omega} \wedge \mathbf{r}] = \boldsymbol{\omega} \cdot \mathbf{v}_O.$$

Consequently $\boldsymbol{\omega} \cdot \mathbf{v}$ is a scalar invariant of the motion, i.e. it has the

same value for all points of the body. But, if P is a point on the central axis, then $\mathbf{v}_P$ and $\boldsymbol{\omega}$ are parallel. Applying the above property, we therefore have,

$$\omega v_P = \boldsymbol{\omega} \cdot \mathbf{v}_O,$$

$$\text{so that} \quad v_P = \frac{\boldsymbol{\omega} \cdot \mathbf{v}_O}{\omega} = v_O \cos \theta, \quad (3.8)$$

i.e. the velocity of a point on the central axis is equal to the component of $\mathbf{v}_O$ in the direction of $\boldsymbol{\omega}$.

We have established the result, therefore, that the instantaneous motion of a rigid body is equivalent to that of a screw motion consisting of a velocity of translation $(\boldsymbol{\omega} \cdot \mathbf{v}_O)/\omega$ along the line defined by (3.6) together with an angular velocity about it. The ratio of the linear velocity of a point on the central axis to the angular velocity is termed the *pitch* of the screw; it is given by

$$p = \frac{\boldsymbol{\omega} \cdot \mathbf{v}_O}{\omega^2}. \quad (3.9)$$

3.2.1. Condition for motion to reduce to a pure rotation. If the motion is equivalent to a pure rotation, then, for any point on the axis of rotation, we must have

$$\mathbf{v}_O + \boldsymbol{\omega} \wedge \mathbf{r} = \mathbf{0}.$$

This, therefore, must represent the equation of the axis. On multiplying by $\boldsymbol{\omega}$ vectorially, and proceeding as above, we again find that the locus of $\mathbf{r}$ is given by

$$\left(\mathbf{r} - \frac{\boldsymbol{\omega} \wedge \mathbf{v}_O}{\omega^2} \right) \wedge \boldsymbol{\omega} = \mathbf{0},$$

i.e. if the motion reduces to a pure rotation, then the axis of the rotation must correspond to the central axis of the motion. The central axis thus becomes the *instantaneous axis of rotation* corresponding to the motion. The condition for the motion defined by $\mathbf{v}_O$ and $\boldsymbol{\omega}$ to reduce to a pure rotation is thus that v_P (eqn 3.8) should be zero. It follows that the condition for this to happen is that

$$\boldsymbol{\omega} \cdot \mathbf{v}_O = 0,$$

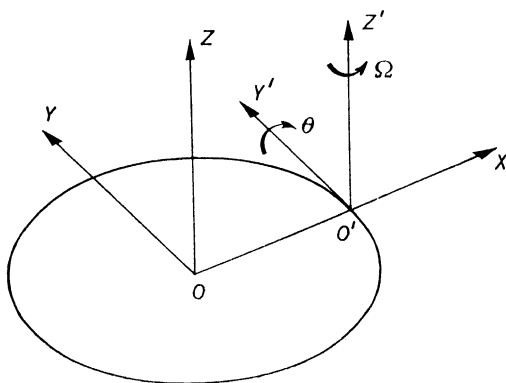
i.e. that $\mathbf{v}_O$ and $\boldsymbol{\omega}$ should be at right angles.

Examples

(1) *An aeroplane rotates about its axis and at the same time describes a circular path of radius a in such a way that its axis remains horizontal. If O is the centre of its circular path and $OXYZ$ a set of orthogonal axes such that at any instant OX is at right angles to the axis of the aeroplane,*

OY is parallel to the axis, and OZ is along the vertical through O , show that the instantaneous motion of the aeroplane is equivalent to a screw motion and find the equation of the surface generated by its central axis when referred to these axes.

Solution. The axis of the aeroplane will be along the tangent to its circular path and will point in the direction of its motion. If the aeroplane describes its circular path in a counter-clockwise direction, then the instantaneous positions of the axes $OXYZ$ will be as shown. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be a set of unit vectors respectively parallel to the instantaneous positions of OX , OY , OZ and let O' be the point of intersection of OX with the axis of the aeroplane. We shall refer the motion, in the first place, to the instantaneous position of O' as base point.



If the aeroplane describes the circle with an angular speed Ω , whilst its angular speed about its axis is θ , then its angular velocity at any instant will be given by

$$\boldsymbol{\omega} = \theta \mathbf{j} + \Omega \mathbf{k},$$

so that

$$\omega^2 = \theta^2 + \Omega^2.$$

Also, if $\mathbf{v}'_O$ is the velocity of O' , then

$$\mathbf{v}'_O = a\Omega \mathbf{j}.$$

The motion will be equivalent, therefore, at any instant to a screw motion, the vector equation of its axis being (eqn 3.6)

$$\left\{ \mathbf{r} - \frac{(\theta \mathbf{j} + \Omega \mathbf{k}) \wedge a\Omega \mathbf{j}}{\theta^2 + \Omega^2} \right\} \wedge (\theta \mathbf{j} + \Omega \mathbf{k}) = \mathbf{0},$$

or

$$\left(\mathbf{r} + \frac{a\Omega^2}{\theta^2 + \Omega^2} \mathbf{i} \right) \wedge (\theta \mathbf{j} + \Omega \mathbf{k}) = \mathbf{0}.$$

If the components of $\mathbf{r}$, referred to a parallel set of axes through O' , are (x', y', z') , then the above equation may be written

$$\left(x' + \frac{a\Omega^2}{\theta^2 + \Omega^2}, y', z'\right) \wedge (0, \theta, \Omega) = (0, 0, 0),$$

so that its equivalent scalar equations are

$$\left. \begin{aligned} y'\Omega - z'\theta &= 0 & (a) \\ -\Omega\left(x' + \frac{a\Omega^2}{\theta^2 + \Omega^2}\right) &= 0 & (b) \\ \theta\left(x' + \frac{a\Omega^2}{\theta^2 + \Omega^2}\right) &= 0 & (c) \end{aligned} \right\}. \quad (1)$$

Since both Ω and θ are different from zero, (1(b)) and (1(c)) are equivalent to the single equation

$$x' + \frac{a(\Omega/\theta)^2}{1 + (\Omega/\theta)^2} = 0. \quad (2)$$

But, from (1(a)),

$$\Omega/\theta = z'/y',$$

so that, on eliminating Ω/θ from (2), we have

$$x' + \frac{a(z'/y')^2}{1 + (z'/y')^2} = 0,$$

or

$$x'(y'^2 + z'^2) + az'^2 = 0. \quad (3)$$

Thus, no matter what the values of Ω and θ , the central axis must lie on the surface defined by this equation. When referred to axes $OXYZ$, its equation will be given by (3) with $(x-a, y, z)$ replacing (x', y', z') respectively; the equation of the surface generated by the central axis, when referred to these axes, is thus

$$ay^2 = x(y^2 + z^2).$$

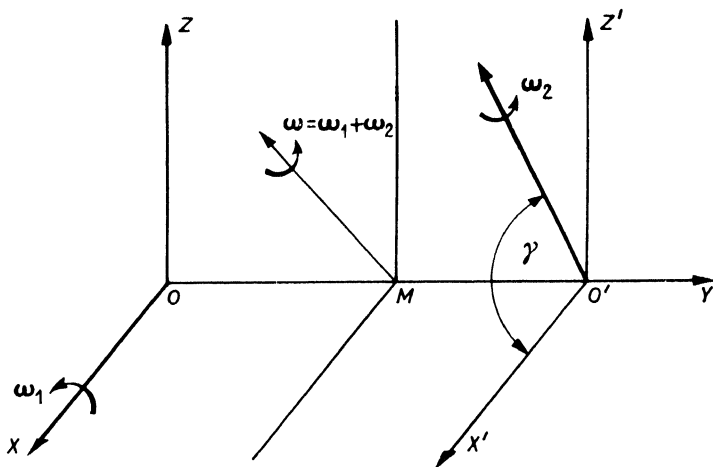
(2) *A body has angular velocities ω_1, ω_2 about two non-intersecting axes. If OO' is the shortest distance between them, show that the axis of the equivalent screw intersects OO' in the point which divides it in the ratio*

$$\omega_2(\omega_2 + \omega_1 \cos \gamma) : \omega_1(\omega_1 + \omega_2 \cos \gamma),$$

where γ is the angle between ω_1 and ω_2 .

Solution. Suppose we take a set of coordinate axes as shown, viz. OX in the direction of ω_1 , OY along OO' , and OZ a mutual perpendicular such that $OXYZ$ forms a right-handed system. Let $O'X', O'Z'$ be

axes respectively parallel to OX, OZ so that ω_2 makes an angle γ with $O'X'$ as shown.



Now the resultant motion of the body is equivalent to that arising from the superimposing of the motions defined by ω_1 and ω_2 . But the motion due to the angular velocity ω_2 about an axis through O' is equivalent to the motion due to an equal angular velocity about a parallel axis through O together with the motion due to a velocity of translation given in magnitude and direction by the vector product $\omega_2 \wedge \overline{O'O}$ (see footnote, p. 62). Hence, taking O as base point, the resultant motion of the body will be equivalent to the motion in which the body has an angular velocity

$$\omega = \omega_1 + \omega_2 \quad (1)$$

about an axis through O , whilst O has a velocity of translation given by

$$\begin{aligned} \mathbf{v}_O &= \omega_2 \wedge \overline{O'O} = \omega_2 \wedge -h\mathbf{j} \\ &= h\mathbf{j} \wedge \omega_2, \end{aligned} \quad (2)$$

where $\mathbf{j}$ is a unit vector in the direction of $\overline{OO'}$ and $h = OO'$. The position of the axis of the equivalent screw will thus be given by the vector equation

$$\left(\mathbf{r} - \frac{\omega \wedge \mathbf{v}_O}{\omega^2} \right) \wedge \omega = 0,$$

where ω and $\mathbf{v}_O$ are defined by (1) and (2). Since both ω and $\mathbf{v}_O$ lie

in the OXZ plane, it therefore passes through the point M in OO' where

$$\begin{aligned}\overline{OM} &= \frac{\boldsymbol{\omega} \wedge \mathbf{v}_O}{\omega^2} = \frac{(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \wedge (h\mathbf{j} \wedge \boldsymbol{\omega}_2)}{\omega^2} \\ &= \frac{h\mathbf{j}\{\boldsymbol{\omega}_2 \cdot (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\} - \boldsymbol{\omega}_2\{h\mathbf{j} \cdot (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\}}{\omega^2} \\ &= \frac{h(\omega_1\omega_2 \cos \gamma + \omega_2^2)\mathbf{j}}{\omega^2},\end{aligned}$$

since $\mathbf{j} \cdot (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) = 0$, $\mathbf{j}$ being at right angles to both $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. Thus

$$OM = \frac{h(\omega_1\omega_2 \cos \gamma + \omega_2^2)}{\omega^2},$$

where

$$\omega^2 = \omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos \gamma.$$

The point M therefore divides OO' in the ratio

$$h\left(\frac{\omega_1\omega_2 \cos \gamma + \omega_2^2}{\omega^2}\right) : h\left(1 - \frac{\omega_1\omega_2 \cos \gamma + \omega_2^2}{\omega^2}\right),$$

i.e. in the ratio

$$\omega_2(\omega_2 + \omega_1 \cos \gamma) : \omega_1(\omega_1 + \omega_2 \cos \gamma).$$

The pitch of the equivalent screw is given, as we know from the general theory (eqn 3.9), by the expression

$$p = \frac{\boldsymbol{\omega} \cdot \mathbf{v}_O}{\omega^2},$$

so that, on substituting for $\boldsymbol{\omega}$ and $\mathbf{v}_O$, we have

$$\begin{aligned}p &= \frac{(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \cdot (h\mathbf{j} \wedge \boldsymbol{\omega}_2)}{\omega^2} \\ &= \frac{h\mathbf{j} \cdot [\boldsymbol{\omega}_2 \wedge (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)]}{\omega^2} \\ &= \frac{h\mathbf{j} \cdot (\boldsymbol{\omega}_2 \wedge \boldsymbol{\omega}_1)}{\omega^2} \\ &= \frac{h\mathbf{j} \cdot \omega_1\omega_2 \sin \gamma \mathbf{j}}{\omega^2} \\ &= \frac{h\omega_1\omega_2 \sin \gamma}{\omega^2}.\end{aligned}$$

Since p and $\boldsymbol{\omega}$ are now known, the velocity of a point on the central axis may be obtained from the relation

$$\mathbf{v}_P = p\boldsymbol{\omega}.$$

The equivalent screw has thus been completely determined.

Examples

1. A circular cone, of semi-vertical angle β , rotates about its axis OZ with uniform angular speed Ω . An inner circular cone, of semi-vertical angle α , rolls without slipping inside the former cone, so that its axis OC revolves around OZ with the angular speed $k\Omega$. Show that ω , the angular speed of the inner cone, is given by

$$\omega = \Omega \sin \beta / \sin (\beta - \theta),$$

where θ is the angle that the instantaneous axis of rotation of the inner cone makes with OZ .

Show, also, that θ is given by the equation

$$\cot (\beta - \theta) = (1 - k) \cot \alpha + k \cot \beta.$$

2. Two concentric spherical surfaces, of radii a and b , rotate with angular speeds ω_1 , ω_2 respectively about fixed perpendicular diameters. A sphere, placed between them, rolls in contact with both. Determine the vector expression for the velocity of its centre, and hence show that the path of its centre is a circle which it describes with an angular speed

$$(a^2\omega_1^2 + b^2\omega_2^2)^{1/2} / (a + b).$$

Obtain the corresponding result when the diameters about which the spheres rotate are not perpendicular.

3. A circular cone, of semi-vertical angle α and height h , is rolling without slipping on the floor of a lift which is descending with speed V . The centre C of the base of the cone describes a circle, relative to the lift, with speed v . Show that the axis of the equivalent screw motion at any instant is parallel to the generator of the cone in contact with the lift and at a distance $Vh \sin \alpha / v$ from it.

4. A rigid body is given angular velocities ω_1 and ω_2 respectively about a pair of non-intersecting perpendicular lines, together with linear velocities $p\omega_1$ and $p\omega_2$ respectively along these lines. The shortest distance between the lines is h . Determine the position of the central axis of the resultant screw, and show that the magnitude of the velocity along the axis is

$$\frac{p\omega_1^2 \pm h\omega_1\omega_2 + p\omega_2^2}{(\omega_1^2 + \omega_2^2)^{1/2}}.$$

5. A rigid body has simultaneous angular velocities about the lines $y = a$, $z = -a$; $z = a$, $x = -a$; $x = a$, $y = -a$. If the equivalent screw motion has a pitch a , show that its axis lies on the plane

$$x + y + z = 0.$$

6. The motion of a rigid body is compounded of angular velocities about the lines

$$x = y - z - a = 0, \quad y = z - x - a = 0, \quad z = x - y - a = 0.$$

Prove that (i) it cannot reduce to a translation, (ii) if it reduces to a rotation, the instantaneous axis of rotation lies on the surface

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = a^2.$$

CHAPTER IV

THEORY OF ROTATING AXES

In the previous chapter, we saw how the motion of a rigid body may be defined in terms of the linear velocity of a chosen point of the body together with an angular velocity. In future, our main problem will be the determination of the motion of a particle, or of a rigid body, when the forces producing the motion are given; in other words, we shall be principally concerned from now on with the kinetics of a system moving under the action of prescribed forces.

In determining the motion of a particle, and also in the solution of problems involving the motion of a rigid body, it is frequently convenient to refer the motion to a set of axes rotating in some prescribed manner. In this chapter we shall derive the general expression for the acceleration of a moving point when referred to such a system, and then utilize the expression so obtained in the solution of some problems of particle motion, including that of a particle whose motion is referred to a set of axes fixed relative to the earth, the earth's rotation being taken into account.

1. Rate of change of displacement referred to rotating axes

Let P be a point moving in space and let its vector position, referred to a fixed origin O , at time t be $\mathbf{r}$. Let $O123$ be a system of rectangular axes which rotates about O with an angular velocity $\boldsymbol{\theta}$, so that at time t the components of $\mathbf{r}$ with respect to these axes are (r_1, r_2, r_3) . In a time δt let P move to P' , and let the position at time $(t + \delta t)$ of the point, assumed rigidly attached to the rotating axes, which was coincident with P at time t , be P'' (Fig. 4.1).

Then we have

$$\overline{PP'} = \overline{PP''} + \overline{P''P'},$$

where $\overline{P''P'}$ represents the displacement of P relative to the rotating axes in time δt . But, if $\delta\boldsymbol{\theta}$ is the angular displacement of the coordinate axes, regarded as comprising a rigid frame, in the interval of time δt , then

$$\overline{PP''} = \delta\boldsymbol{\theta} \wedge \mathbf{r} = \boldsymbol{\theta} \delta t \wedge \mathbf{r},$$

and hence

$$\overline{PP'} = (\boldsymbol{\theta} \wedge \mathbf{r}) \delta t + \overline{P''P'}.$$

The velocity of P in space will thus be given by

$$\begin{aligned}\mathbf{v} &= \lim_{\delta t \rightarrow 0} \frac{\overline{PP'}}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{(\boldsymbol{\theta} \wedge \mathbf{r})\delta t}{\delta t} + \lim_{\delta t \rightarrow 0} \frac{\overline{P''P'}}{\delta t} \\ &= \boldsymbol{\theta} \wedge \mathbf{r} + \lim_{\delta t \rightarrow 0} \frac{\overline{P''P'}}{\delta t}.\end{aligned}$$

In this relation $\lim_{\delta t \rightarrow 0} (\overline{PP'}/\delta t)$ represents the rate of change of $\mathbf{r}$ in space at time t , and so may be defined in terms of its components with respect to any set of coordinate axes fixed in space, whilst $\lim_{\delta t \rightarrow 0} (\overline{P''P'}/\delta t)$ represents the rate of change of $\mathbf{r}$ with respect to the moving axes. Consequently, if we choose to refer the rate of change of $\mathbf{r}$ in space to

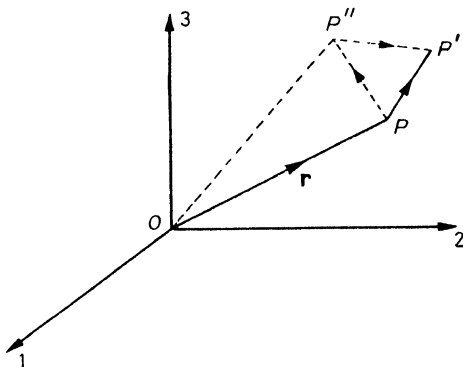


Figure 4.1

the set of axes fixed in space which are coincident with $O123$ at time t , and if we let d/dt denote the rate of change with respect to these axes, $\partial/\partial t$ the rate of change with respect to the moving axes, then at time t we shall have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial t} + (\boldsymbol{\theta} \wedge \mathbf{r}). \quad (4.1)$$

Since a vector can always be represented by a displacement, it follows that the above result, viz.

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial t} + \boldsymbol{\theta} \wedge \mathbf{r}, \quad (4.2)$$

may be interpreted as a general result expressing the relation between

the rates of change of any vector $\mathbf{r}$ referred to fixed and moving axes respectively.*

If, in problems which involve the use of rotating axes, we restrict the use of the symbols d/dt and $\partial/\partial t$ to the above meanings, then in operator notation we have the result

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{\theta} \wedge. \quad (4.3)$$

1.1 Acceleration of a moving point. If we apply the above operator (eqn 4.3) to the vector $\mathbf{v}$, then we have

$$\begin{aligned} \mathbf{f} = \text{acceleration of } P &= \frac{d\mathbf{v}}{dt} \\ &= \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\theta} \wedge \mathbf{v} \\ &= \frac{\partial}{\partial t} \left\{ \frac{\partial \mathbf{r}}{\partial t} + \boldsymbol{\theta} \wedge \mathbf{r} \right\} + \boldsymbol{\theta} \wedge \left\{ \frac{\partial \mathbf{r}}{\partial t} + \boldsymbol{\theta} \wedge \mathbf{r} \right\} \\ &= \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2 \left(\boldsymbol{\theta} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \frac{\partial \boldsymbol{\theta}}{\partial t} \wedge \mathbf{r} + \{ \boldsymbol{\theta} \wedge (\boldsymbol{\theta} \wedge \mathbf{r}) \}. \quad (4.4) \end{aligned}$$

1.2. Expressions for $\mathbf{v}$ and $\mathbf{f}$ referred to a moving origin. In deriving the above expressions for $\mathbf{v}$ and $\mathbf{f}$, we have taken the origin O to be fixed. If O is also in motion, then, in order to obtain expressions for the velocity and acceleration of P in space, we must add to the

* Eqn (4.2) may also be derived as follows. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be unit vectors in the directions of the axes $O1, O2, O3$ respectively. Then, if (r_1, r_2, r_3) are the components of $\mathbf{r}$ with respect to these axes,

$$\mathbf{r} = r_1 \mathbf{i} + r_2 \mathbf{j} + r_3 \mathbf{k}.$$

Differentiating with respect to t , we therefore have, since $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are not fixed in space,

$$\frac{d\mathbf{r}}{dt} = \frac{dr_1}{dt} \mathbf{i} + \frac{dr_2}{dt} \mathbf{j} + \frac{dr_3}{dt} \mathbf{k} + r_1 \frac{d\mathbf{i}}{dt} + r_2 \frac{d\mathbf{j}}{dt} + r_3 \frac{d\mathbf{k}}{dt}.$$

But $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ may be regarded as fixed to the rotating frame, so that

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\theta} \wedge \mathbf{i}, \quad \frac{d\mathbf{j}}{dt} = \boldsymbol{\theta} \wedge \mathbf{j}, \quad \frac{d\mathbf{k}}{dt} = \boldsymbol{\theta} \wedge \mathbf{k}.$$

Hence, substituting in the above equation, we have

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \frac{dr_1}{dt} \mathbf{i} + \frac{dr_2}{dt} \mathbf{j} + \frac{dr_3}{dt} \mathbf{k} + r_1 (\boldsymbol{\theta} \wedge \mathbf{i}) + r_2 (\boldsymbol{\theta} \wedge \mathbf{j}) + r_3 (\boldsymbol{\theta} \wedge \mathbf{k}) \\ &= \frac{dr_1}{dt} \mathbf{i} + \frac{dr_2}{dt} \mathbf{j} + \frac{dr_3}{dt} \mathbf{k} + \boldsymbol{\theta} \wedge (r_1 \mathbf{i} + r_2 \mathbf{j} + r_3 \mathbf{k}) \\ &= \frac{\partial \mathbf{r}}{\partial t} + \boldsymbol{\theta} \wedge \mathbf{r}, \end{aligned}$$

$\partial \mathbf{r} / \partial t$ representing the rate of change of $\mathbf{r}$ with respect to the rotating frame.

expressions (4.1) and (4.4) respectively the velocity $\mathbf{v}_O$ and the acceleration $\mathbf{f}_O$ of O , so that we have

$$\mathbf{v} = \mathbf{v}_O + \frac{d\mathbf{r}}{dt}, \quad (4.5)$$

$$\mathbf{f} = \mathbf{f}_O + \frac{d\mathbf{v}}{dt}. \quad (4.6)$$

1.3. Expressions for the components of acceleration in spherical polar coordinates. A method for the derivation of these expressions has already been given in Chap. II, § 2.2. We shall now derive them using the above theory.*

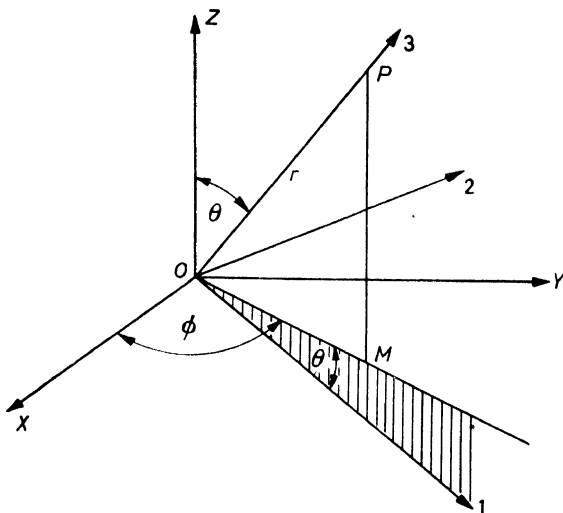


Figure 4.2

Let $OXYZ$ be a set of orthogonal axes fixed in space, and let the spherical polar coordinates of a general point P be (r, θ, ϕ) , where $r = OP$, $\theta = \angle POZ$, and $\phi = \angle MOX$, M being the foot of the perpendicular from P on to the OXY plane (Fig. 4.2). Now choose a set of rotating axes $O123$ such that $O1$ lies in the plane ZOP , is at right angles to OP and makes an angle θ with OM as shown; $O3$ is along OP ; and $O2$ is a mutual perpendicular axis such that $O123$ forms a right-

* This method of derivation is given in Jeffreys and Jeffreys, *Methods of Mathematical Physics*, 2nd ed, § 3.131, p. 109.

handed system. ($O2$, being at right angles to the plane ZOP , will clearly lie in the OXY plane.)

Now consider the velocity of the point P . To apply the above theory (eqn 4.1), we have to determine θ , the angular velocity of the rotating axes. If $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are unit vectors in the directions of $O1$, $O2$, $O3$ respectively, then this is defined by

$$\theta = \dot{\theta}\mathbf{j} + \dot{\phi}\mathbf{z},$$

where $\mathbf{z}$ is a unit vector in the direction of the upward vertical OZ , and $\dot{\theta}$ and $\dot{\phi}$ are the rates of change of the angles θ and ϕ respectively.*

But

$$\mathbf{z} = -\sin \theta \mathbf{i} + \cos \theta \mathbf{k}.$$

Hence

$$\begin{aligned}\theta &= \dot{\theta}\mathbf{j} + \dot{\phi}(-\sin \theta \mathbf{i} + \cos \theta \mathbf{k}) \\ &= -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\phi} \cos \theta \mathbf{k},\end{aligned}$$

so that

$$(\theta_1, \theta_2, \theta_3) = (-\dot{\phi} \sin \theta, \dot{\theta}, \dot{\phi} \cos \theta)$$

when referred to the axes $O123$. Also the components of $\mathbf{r}$ ($= \overline{OP}$), referred to the same axes, are $(0, 0, r)$. Hence, expressing θ and $\mathbf{r}$ in (4.1) in terms of their components, we have†

$$\begin{aligned}\mathbf{v}(v_1, v_2, v_3) &= \frac{d}{dt}(0, 0, r) + (-\dot{\phi} \sin \theta, \dot{\theta}, \dot{\phi} \cos \theta) \wedge (0, 0, r) \\ &= (0, 0, \dot{r}) + (r\dot{\theta}, r\dot{\phi} \sin \theta, 0) \\ &= (r\dot{\theta}, r\dot{\phi} \sin \theta, \dot{r}).\end{aligned}\tag{4.7}$$

These are the component velocities of P parallel respectively to $O1$, $O2$, $O3$.

* In this and the following chapters, we shall restrict the use of the dot notation to denote the rate of change of a scalar quantity; the rate of change of a vector quantity $\mathbf{a}$ with respect to a set of *fixed* axes will always be denoted by $d\mathbf{a}/dt$.

† It should be understood that d/dt and $\partial/\partial t$ are different operational symbols only when the quantity operated upon is a vector, in which case the two symbols have their own distinct meanings. When, however, the $\mathbf{r}$ in the expression

$$\frac{\partial \mathbf{r}}{\partial t} + \theta \wedge \mathbf{r}$$

is replaced by its scalar components and written in the form

$$\frac{\partial}{\partial t}(r_1, r_2, r_3) + (\theta_1, \theta_2, \theta_3) \wedge (r_1, r_2, r_3),$$

then the $\partial/\partial t$ is no longer operating on a vector quantity as such, but on the equivalent triad of scalar quantities (r_1, r_2, r_3) . The $\partial/\partial t$ in the above expression may therefore be replaced by d/dt , the standard symbol for denoting the rate of change of a scalar quantity, so that we have

$$\frac{\partial}{\partial t}(r_1, r_2, r_3) \equiv \frac{d}{dt}(r_1, r_2, r_3) = (\dot{r}_1, \dot{r}_2, \dot{r}_3).$$

To obtain the corresponding components of acceleration, since

$$\mathbf{f} = \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\theta} \wedge \mathbf{v},$$

we have, on substituting for $\boldsymbol{\theta}$ and $\mathbf{v}$,

$$(f_1, f_2, f_3) = \frac{d}{dt}(r\dot{\theta}, r\dot{\phi} \sin \theta, \dot{r}) + (-\dot{\phi} \sin \theta, \dot{\theta}, \dot{\phi} \cos \theta) \wedge (r\dot{\theta}, r\dot{\phi} \sin \theta, \dot{r}),$$

so that

$$\left. \begin{aligned} f_1 &= \frac{d}{dt}(r\dot{\theta}) + (\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin \theta \cos \theta) = r(\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2) + 2\dot{r}\dot{\theta} \\ f_2 &= \frac{d}{dt}(r\dot{\phi} \sin \theta) + (r\dot{\theta}\dot{\phi} \cos \theta + \dot{r}\dot{\phi} \sin \theta) = \frac{1}{r \sin \theta} \frac{d}{dt}(r^2 \sin^2 \theta \dot{\phi}) \\ f_3 &= \ddot{r} - r\dot{\phi}^2 \sin^2 \theta - r\dot{\theta}^2 = \ddot{r} - r(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \end{aligned} \right\} \quad (4.8)$$

2. Application to particle kinetics

Let the moving point P in § 1 denote the position of a particle of mass m moving under the action of a resultant force $\mathbf{F}$. Then, referred to a set of moving axes whose origin is at rest, the vector equation of motion of the particle is

$$m \left\{ \frac{\partial^2 \mathbf{r}}{\partial t^2} + \frac{\partial \boldsymbol{\theta}}{\partial t} \wedge \mathbf{r} + 2 \left(\boldsymbol{\theta} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \boldsymbol{\theta} \wedge (\boldsymbol{\theta} \wedge \mathbf{r}) \right\} = \mathbf{F}. \quad (4.9)$$

This can be written in the form

$$m \frac{\partial^2 \mathbf{r}}{\partial t^2} = \mathbf{F} - m \left\{ \frac{\partial \boldsymbol{\theta}}{\partial t} \wedge \mathbf{r} + 2 \left(\boldsymbol{\theta} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \boldsymbol{\theta} \wedge (\boldsymbol{\theta} \wedge \mathbf{r}) \right\}.$$

If $\boldsymbol{\theta}$ is constant, equal to $\boldsymbol{\Omega}$ say, then this reduces to

$$m \frac{\partial^2 \mathbf{r}}{\partial t^2} = \mathbf{F} - m \left\{ 2 \left(\boldsymbol{\Omega} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}) \right\}.$$

The motion relative to the rotating frame of reference is therefore seen to be the same as if the frame were at rest and two extra forces were applied to the particle, viz.

$$(i) \text{ a force: } -2m \left(\boldsymbol{\Omega} \wedge \frac{\partial \mathbf{r}}{\partial t} \right),$$

$$(ii) \text{ a force: } -m \{ \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}) \}.$$

The former is clearly perpendicular to the angular velocity of the rotating frame and to the velocity of P relative to the frame; it is known as the *Coriolis force*.* The latter is perpendicular to the axis of rotation, lies

* *Gustave Gaspard de Coriolis* (1792–1843) was a French mathematician who made many contributions in the field of mechanics. In 1829 he published his *Traité de la mécanique des corps solides*, a second edition appearing in 1844.

in the plane containing OP and the axis of rotation, is directed outward (see Fig. 4.3), and has a magnitude given by

$$\Omega^2 r \sin \theta = \Omega^2 PN;$$

it is therefore equivalent to the *centrifugal force*.

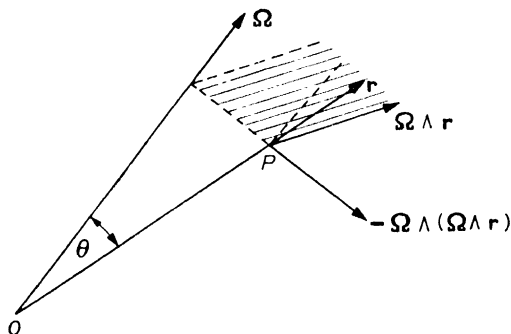
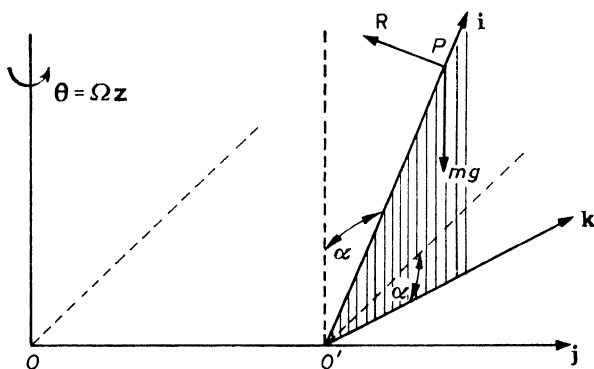


Figure 4.3

If O is neither at rest nor in uniform motion, but has an acceleration $\mathbf{f}_O$, then a term $m\mathbf{f}_O$ must be added to the left-hand side of (1.9).

2.1. Example. A particle slides in a smooth straight tube which rotates with constant angular speed Ω about a vertical axis which does not intersect the tube. If OO' is the shortest distance between the vertical



axis and the tube, and if initially the particle is projected from O' along the tube with a speed v_0 , determine the position of the particle after a time t .

Solution. Let the axis of the tube make an angle α with the upward vertical through O' , and let $\mathbf{i}$ be a unit vector in the direction of the

tube, $\mathbf{j}$ a unit vector in the direction of $\overline{OO'}$, and $\mathbf{k}$ a unit vector in a mutually perpendicular direction such that $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ form a right-handed triad. Let $\mathbf{z}$ be a unit vector in the direction of the upward vertical. Then the angular velocity of the rotating axes, having their origin at O' and defined by the unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, is given by

$$\boldsymbol{\theta} = \Omega \mathbf{z}.$$

Also, if $\mathbf{f}_{O'}$ is the acceleration of O' in space at any instant and $OO' = a$, then

$$\mathbf{f}_{O'} = -a\Omega^2 \mathbf{j},$$

Hence the equation of motion of the particle, when referred to the above axes, becomes

$$m \left\{ \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\Omega \left(\mathbf{z} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \Omega^2 [\mathbf{z} \wedge (\mathbf{z} \wedge \mathbf{r})] - a\Omega^2 \mathbf{j} \right\} = \mathbf{R} - mg\mathbf{z}, \quad (1)$$

where $\mathbf{r} = \overline{O'P}$ and $\mathbf{R}$ is the reaction between the tube and the particle.

But

$$\frac{\partial \mathbf{r}}{\partial t} = \dot{r}\mathbf{i}, \quad \frac{\partial^2 \mathbf{r}}{\partial t^2} = \ddot{r}\mathbf{i}.$$

Substituting into (1), the equation of motion thus becomes

$$m\{\ddot{r}\mathbf{i} + 2\Omega\dot{r}(\mathbf{z} \wedge \mathbf{i}) + \Omega^2[(\mathbf{z} \cdot \mathbf{r})\mathbf{z} - \mathbf{r}] - a\Omega^2 \mathbf{j}\} = \mathbf{R} - mg\mathbf{z}. \quad (2)$$

But the position of the particle at time t is known if we know the position of the particle along the tube as a function of t . To determine the position of the particle, we therefore require a scalar differential equation in r which does not involve the reaction. This may be obtained by multiplying (2) scalarly by $\mathbf{i}$, for, $\mathbf{R}$ being at right angles to the axis of the tube, $\mathbf{R} \cdot \mathbf{i} = 0$, and we therefore have

$$m\{\ddot{r} - \Omega^2(\mathbf{z} \cdot \mathbf{r})(\mathbf{z} \cdot \mathbf{i}) - \Omega^2(\mathbf{r} \cdot \mathbf{i})\} = -mg(\mathbf{z} \cdot \mathbf{i}).$$

Since

$$\mathbf{z} \cdot \mathbf{r} = r \cos \alpha, \quad \mathbf{z} \cdot \mathbf{i} = \cos \alpha, \quad \mathbf{r} \cdot \mathbf{i} = r,$$

this reduces to

$$\ddot{r} + r\Omega^2 \sin^2 \alpha = -g \cos \alpha, \quad (3)$$

which is the desired equation.

If we write (3) in the form

$$\ddot{r} - \Omega^2 \sin^2 \alpha \left(r - \frac{g \cos \alpha}{\Omega^2 \sin^2 \alpha} \right) = 0,$$

then it is readily seen that the most general solution of the equation is given by

$$r - \frac{g \cos \alpha}{\Omega^2 \sin^2 \alpha} = A \cosh (\Omega \sin \alpha t) + B \sinh (\Omega \sin \alpha t),$$

where A and B are two constants to be evaluated. But initially $\mathbf{r} = 0$, $\dot{\mathbf{r}} = v_0$. Using these conditions, we therefore have

$$A = -\frac{g \cos \alpha}{\Omega^2 \sin^2 \alpha}, \quad B = \frac{v_0}{\Omega^2 \sin^2 \alpha},$$

so that the solution for $\mathbf{r}$ becomes

$$\mathbf{r} = \frac{g \cos \alpha}{\Omega^2 \sin^2 \alpha} \{1 - \cosh (\Omega \sin \alpha t)\} + \frac{v_0}{\Omega^2 \sin^2 \alpha} \sinh (\Omega \sin \alpha t).$$

In deriving the expression for the acceleration of a moving point when referred to a set of rotating axes (eqn 4.4), it is assumed that $\boldsymbol{\theta}$ is the angular velocity of the rotating axes relative to a set of axes *which are fixed in space*. In the above example, however, the motion is referred to a set of axes which is rotating with an angular velocity $\boldsymbol{\theta} = \Omega \mathbf{z}$ with respect to a set of axes fixed relative to the earth. But the earth is itself rotating in space. Consequently these latter axes do not fulfil the above condition. The expression (4.4) above for $\mathbf{f}$, in which $\boldsymbol{\theta}$ is taken to be the angular velocity of a set of axes rotating with respect to a set of axes fixed relative to the earth, can be regarded, therefore, as being reasonably accurate only provided the effects of the earth's rotation are negligible. This point is discussed in greater detail in § 3 below.

2.2. Motion of a particle on a rotating plane curve. Suppose we apply the above theory to the case of a particle whose motion is restricted to a plane curve which rotates about an axis perpendicular to its plane. We shall assume the speed of rotation to be constant, its magnitude being equal to Ω .

Let the plane of the curve intersect the axis of rotation in O , and let us choose a set of axes $O123$ such that $O1$ and $O2$ are in the plane of the curve and are fixed relative to it, while $O3$ coincides with the axis of rotation (Fig. 4.4). Then, referred to such a set of rotating axes, the acceleration of P will be given by

$$\begin{aligned} \mathbf{f} &= \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2 \left(\boldsymbol{\theta} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \boldsymbol{\theta} \wedge (\boldsymbol{\theta} \wedge \mathbf{r}) \\ &= \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2 \left(\boldsymbol{\theta} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) - \mathbf{r}(\boldsymbol{\theta} \cdot \boldsymbol{\theta}), \end{aligned} \quad (4.10)$$

since $\mathbf{r} \cdot \boldsymbol{\theta} = 0$, $\boldsymbol{\theta}$ being at right angles to $\mathbf{r}$, where $\boldsymbol{\theta}$ is, as above, the angular velocity of the rotating axes.

Here $\partial \mathbf{r} / \partial t$ and $\partial^2 \mathbf{r} / \partial t^2$ are, respectively, the velocity and acceleration of P relative to the curve. The direction of the relative velocity at any instant will clearly be along the tangent to the curve at that instant. Let its magnitude be v . Then it follows, from the known properties of

the motion of a particle on a plane curve, that $\partial^2 \mathbf{r} / \partial t^2$ is equivalent to a component $v dv/ds (= dv/dt)$ in the direction of the forward tangent to the curve (s being the displacement of P along the curve from some fixed point on the curve) and a component v^2/ρ along its inward normal, ρ being the radius of curvature. Consequently, if $\mathbf{i}_t$, $\mathbf{i}_n$, $\mathbf{i}_r$ are three unit vectors in the directions of the forward tangent to the curve, its *outward* drawn normal, and the radius vector respectively, then we

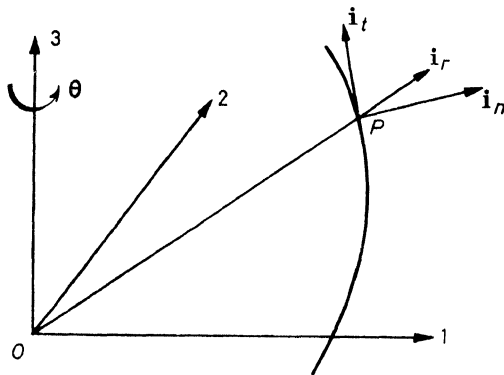


Figure 4.4

shall have from (4.10), since $\boldsymbol{\theta} = \Omega \mathbf{k}$ where $\mathbf{k}$ is a unit vector in the direction of the axis of rotation so that $\mathbf{k} \wedge \mathbf{i}_t = -\mathbf{i}_n$,

$$\mathbf{f} = \left(v \frac{dv}{ds} \mathbf{i}_t - \frac{v^2}{\rho} \mathbf{i}_n \right) - 2\Omega v \mathbf{i}_n - \Omega^2 r \mathbf{i}_r. \quad (4.11)$$

But (see Fig. 4.5)

$$\mathbf{i}_r = \mathbf{i}_n \cos \phi + \mathbf{i}_t \sin \phi = \mathbf{i}_n r \frac{d\theta}{ds} + \mathbf{i}_t \frac{dr}{ds},$$

where θ is the angular displacement of P from an axis in the plane of the curve (say $O1$) and fixed relatively to it. Therefore, substituting into (4.11), we have

$$\mathbf{f} = \left(v \frac{dv}{ds} - \Omega^2 r \frac{dr}{ds} \right) \mathbf{i}_t - \left(\frac{v^2}{\rho} + 2\Omega v + \Omega^2 r^2 \frac{d\theta}{ds} \right) \mathbf{i}_n.$$

The acceleration of the particle at any instant may thus be resolved into two components:

(i) a component

$$f_t = v \frac{dv}{ds} - \Omega^2 r \frac{dr}{ds}$$

along the forward tangent to the curve,

(ii) a component

$$f_n = \frac{v^2}{\rho} + 2\Omega v + \Omega^2 r^2 \frac{d\theta}{ds}$$

along the *inward*-drawn normal to the curve.

If M is the foot of the perpendicular from O on to the tangent to the

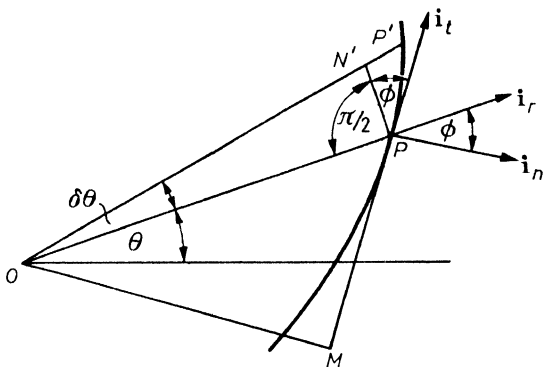


Figure 4.5

curve at P and if p is the length of this perpendicular, then we have

$$OM = p = r \cos \phi = r^2 \frac{d\theta}{ds}.$$

Consequently, the above expression for f_n may also be written

$$f_n = \frac{v^2}{\rho} + 2\Omega v + \Omega^2 p.$$

2.2.1. Application to the motion of a bead on a smooth wire.

Suppose we apply the above results to the motion of a bead free to move on a smooth plane horizontal wire which rotates about a vertical axis with a constant angular speed, the only forces acting on the particle being its weight and the reaction of the curve; then the scalar equations of the motion will be given by

$$\begin{aligned} m \left\{ v \frac{dv}{ds} - \Omega^2 r \frac{dr}{ds} \right\} &= 0, \\ m \left\{ \frac{v^2}{\rho} + 2\Omega v + \Omega^2 p \right\} &= R_n, \end{aligned}$$

where R_n is the component of the reaction in the direction of the *inward* normal, R_t being zero since the curve is smooth. If the polar equation of the curve is $r = f(\theta)$ (θ being measured from a polar axis

fixed relative to the curve), then the integration of these equations leads to the determination of the motion.

In a position of relative equilibrium v must be zero. Such positions are therefore given by

$$dr/ds = \sin \phi = 0.$$

The positions of relative equilibrium thus correspond to those points for which $\phi = 0$, i.e. for those points at which the tangent to the curve is at right angles to the radius vector through the point. They are therefore those points for which r has an extreme value, and are thus given by the equation

$$dr/d\theta = f'(\theta) = 0.$$

To study the stability of a position of relative rest, let $\theta = \theta_0$ correspond to such a position, and suppose the particle to be slightly displaced from this position. Then, since the curve is smooth, the disturbed motion will be governed by the equation

$$v \frac{dv}{ds} - \Omega^2 r \frac{dr}{ds} = 0. \quad (4.12)$$

Let us now determine the form which this equation assumes when θ is such that the motion is in the immediate neighbourhood of $\theta = \theta_0$. Since

$$ds^2 = dr^2 + r^2 d\theta^2 = \{r^2 + f'(\theta)^2\} d\theta^2,$$

we have

$$r \frac{dr}{ds} = \frac{r}{\{r^2 + f'(\theta)^2\}^{1/2}} \frac{dr}{d\theta}.$$

If we write

$$\theta = \theta_0 + \alpha,$$

and treat α as being a small quantity, then, in the immediate neighbourhood of $\theta = \theta_0$, we have, since $f'(\theta_0) = 0$,

$$\begin{aligned} r &= f(\theta_0 + \alpha) = f(\theta_0) + \alpha f'(\theta_0) + \dots = f(\theta_0), \\ \frac{dr}{d\theta} &= f'(\theta_0 + \alpha) = f'(\theta_0) + \alpha f''(\theta_0) + \dots = \alpha f''(\theta_0), \end{aligned}$$

provided second-order terms in α are neglected, and hence

$$\{r^2 + f'(\theta)^2\}^{1/2} = \{r^2 + f'(\theta_0 + \alpha)^2\}^{1/2} = f(\theta_0).$$

Consequently, to this degree of approximation,

$$r \frac{dr}{ds} = \alpha f''(\theta_0). \quad (4.13)$$

Also

$$v \frac{dv}{ds} = \frac{dv}{dt}.$$

But

$$v = \frac{ds}{dt} = \{r^2 + f'(\theta)^2\}^{1/2} \frac{d\theta}{dt} = \{r^2 + f'(\theta_0 + \alpha)^2\}^{1/2} \frac{d\alpha}{dt},$$

so that

$$\frac{dv}{dt} = \frac{d}{d\alpha} \{r^2 + f'(\theta_0 + \alpha)^2\}^{1/2} \left(\frac{d\alpha}{dt} \right)^2 + \{r^2 + f'(\theta_0 + \alpha)^2\}^{1/2} \frac{d^2\alpha}{dt^2}.$$

But for a *small disturbance* both $d\alpha/dt$ and $d^2\alpha/dt^2$ will be small quantities and hence, to the above degree of approximation, the term involving $(d\alpha/dt)^2$ can be neglected, while we may replace the second term by

$$r_0 \frac{d^2\alpha}{dt^2},$$

where $r_0 = f(\theta_0)$. To the degree of approximation assumed, we may therefore write

$$v \frac{dv}{ds} = r_0 \frac{d^2\alpha}{dt^2}. \quad (4.14)$$

Thus, substituting for $r dr/ds$ and $v dv/ds$ from (4.13) and (4.14) into (4.12), the equation which governs the motion of the particle in the immediate neighbourhood of the position of relative rest $\theta = \theta_0$, resulting from a small disturbance from this position, is given by

$$\frac{d^2\alpha}{dt^2} - \frac{\Omega^2 f''(\theta_0)}{r_0} \alpha = 0.$$

The motion will therefore be simple harmonic about $\theta = \theta_0$, and the position of relative equilibrium a stable one, provided that $f''(\theta_0)$ is negative (r_0 must, of course, be positive), i.e. provided that $\theta = \theta_0$ corresponds to a maximum value of r .

3. Motion relative to the earth

It was mentioned at the end of § 2.1 that, when we refer the motion of a particle to a set of axes fixed relative to the earth, we are, in fact, referring the motion to a set of axes rotating in space, since the earth itself is rotating. We shall now investigate the magnitude of the error introduced if the earth's rotation is neglected.

Consider a particle moving near the earth's surface. One of the forces acting on it must be its weight. Now this can be defined in either of two ways. In the first place we may define the weight of the particle or body as being the force with which the earth attracts it; with this definition the line of action of the weight is along the earth's radius through the particle or, in the case of a body, through its centre of gravity. On the other hand, we may define the weight of a

body as being either the pull exerted on a spring balance, or the tension in a supporting string, when the body is at rest relative to the earth. This is the definition used in practice. But, when a body is at rest relative to the earth, it is in reality describing a circle about the earth's axis, and so must have an acceleration toward this axis, this acceleration being produced by the joint effect of the earth's attraction and the tension of the supporting string. It follows, therefore, that the weight of the body as measured in practice is not equal, in general, to the earth's attraction; except at the poles, or at a point on the equator, its line of action must be inclined to that of the earth's attraction.

To find the amount of this deviation, let P be the bob of a pendulum of weight $\mathbf{W}$ (as measured by the tension in its supporting string), and

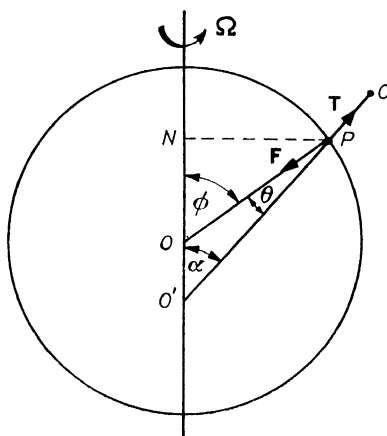


Figure 4.6

C its point of support. CP is, of course, small when compared with the earth's radius OP . The direction of the 'plumb line' PC is called the vertical at P . Let $\mathbf{F}$ denote the earth's attraction on the bob, m the mass of the bob, a the earth's radius, and let CP, PO make angles α, ϕ with the earth's axis ($\phi = \frac{1}{2}\pi - \lambda$, where λ is the latitude of P) (Fig. 4.6). Then, considering the relative equilibrium of the bob, we have

$$\mathbf{F} + \mathbf{T} = m\Omega^2 \overline{PN}, \quad (4.15)$$

where Ω is the angular speed of the earth about its axis of rotation. From this equation we have, on multiplying by $\mathbf{F}$ and forming the vector product,

$$\mathbf{F} \wedge \mathbf{T} = m\Omega^2 (\mathbf{F} \wedge \overline{PN}),$$

while, on multiplying through by $\mathbf{F}$ and forming the scalar product, we get

$$\mathbf{F} \cdot \mathbf{F} + \mathbf{F} \cdot \mathbf{T} = m\Omega^2(\mathbf{F} \cdot \overline{PN}).$$

If θ is the deviation of CP from the earth's radius, we therefore have, in virtue of the above equations,

$$\begin{aligned} \tan \theta &= \tan(\pi - \angle CPO) \\ &= -\tan \angle CPO \\ &= -\frac{|\mathbf{F} \wedge \mathbf{T}|}{\mathbf{F} \cdot \mathbf{T}} \\ &= \frac{m\Omega^2|\mathbf{F} \wedge \overline{PN}|}{\mathbf{F} \cdot \mathbf{F} - m\Omega^2(\mathbf{F} \cdot \overline{PN})}. \end{aligned}$$

Thus

$$\begin{aligned} \tan \theta &= \frac{m\Omega^2 F \cdot PN \sin \angle OPN}{F^2 - m\Omega^2 F \cdot PN \cos \angle OPN} \\ &= \frac{m\Omega^2 a \sin \phi \cos \phi}{F - m\Omega^2 a \sin^2 \phi}, \end{aligned}$$

which becomes, if we write $F = mg'$ for convenience,

$$\tan \theta = \frac{a\Omega^2 \sin \phi \cos \phi}{g' - a\Omega^2 \sin^2 \phi} = \frac{a\Omega^2 \sin \phi \cos \phi}{g' \left(1 - \frac{a\Omega^2}{g'} \sin^2 \phi\right)}.$$

But, taking the earth's radius to be 4000 miles, $a\Omega^2$ in ft/sec units is approximately $1/9$, so that, assuming g' to be equal to 32.2 ft/sec (g' is, of course, the acceleration of a freely falling body, the only force acting on such a body, if air resistance is neglected, being the earth's attraction), $a\Omega^2/g'$ is approximately $1/290$. Thus the term $(a\Omega^2 \sin^2 \phi)/g'$ is small compared with unity for all values of ϕ , and we have, therefore, neglecting squares and higher powers of $a\Omega^2/g'$,

$$\tan \theta = \frac{a\Omega^2}{g'} \sin \phi \cos \phi;$$

hence

$$\theta \approx \frac{1}{290} \sin \phi \cos \phi = \frac{1}{580} \sin^2 \phi.$$

The deviation vanishes, therefore, when $\phi = 0$ or $\pi/2$ and is a maximum when $\phi = 45^\circ$, its value in this position being

$$\theta_{max} = \frac{1}{580} \text{ radians.}$$

It is thus seen that the deviation is quite small whatever the latitude of P .

3.1. Equation of motion for a particle moving near the earth's surface. Consider a particle moving near the earth's surface in latitude λ . We shall suppose that in addition to $\mathbf{F}$, the earth's gravitational pull, it is acted upon by other forces having a resultant $\mathbf{R}$. Let us refer the motion to a set of axes fixed relative to the earth, their origin being a point O on the earth's surface in latitude λ . Then, since

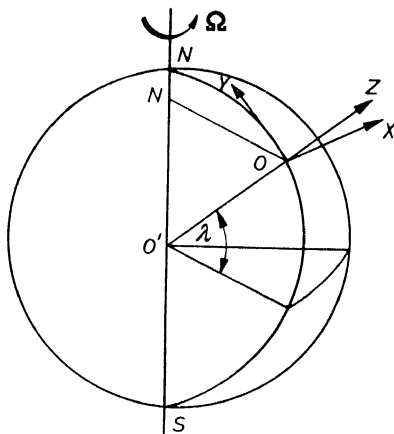


Figure 4.7

the acceleration of O in space is $\Omega^2 \overline{ON}$ (Fig. 4.7) while the axes themselves are rotating in space with an angular velocity Ω , the equation of motion of the particle becomes, on applying (4.9) after the inclusion of a term to account for the motion of O ,

$$\mathbf{R} + \mathbf{F} = m \left\{ \Omega^2 \overline{ON} + \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2 \left(\Omega \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \Omega \wedge (\Omega \wedge \mathbf{r}) \right\}. \quad (4.16)$$

But, if we assume that the particle never moves far from the earth's surface, then we can assume $\mathbf{F}$ to be constant and to have the same magnitude and direction as it would have if the particle were at O ; if, in addition, we define the weight of the particle as being the force equal and opposite to the tension in a supporting string when the particle is suspended in equilibrium from a fixed point on the vertical through O , then, replacing $\mathbf{T}$ by $-\mathbf{W}$ in (4.15), we have

$$\mathbf{F} - \mathbf{W} = m \Omega^2 \overline{ON}.$$

Hence, substituting for $m \Omega^2 \overline{ON}$ in (4.16), the equation of motion becomes

$$\mathbf{R} + \mathbf{F} = \mathbf{F} - \mathbf{W} + m \left\{ \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2 \left(\Omega \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \Omega \wedge (\Omega \wedge \mathbf{r}) \right\},$$

or
$$\mathbf{R} + \mathbf{W} = m \left\{ \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2 \left(\boldsymbol{\Omega} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}) \right\}.$$

If we write $\mathbf{W} = m\mathbf{g}$, then this becomes

$$\mathbf{R} + m\mathbf{g} = m \left\{ \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2 \left(\boldsymbol{\Omega} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}) \right\}. \quad (4.17)$$

(It should be noted that the $\mathbf{g}$ occurring in this equation is not, of course, the true acceleration due to the earth's attraction, since this is defined by the equation

$$\mathbf{F} = m\mathbf{g}';$$

instead, it represents the *apparent* acceleration due to gravity, its direction being that of the plumb line at O .)

In the above equation, $\boldsymbol{\Omega}$ is a small quantity whose magnitude is of the order of 0.73×10^{-4} radians per second. Consequently, if we neglect the term involving $\boldsymbol{\Omega}^2$ in comparison with the other terms in the equation, then (4.17) reduces to

$$\mathbf{R} + m\mathbf{g} = m \left\{ \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2 \left(\boldsymbol{\Omega} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) \right\}, \quad (4.18)$$

where $\mathbf{g}$ may now be regarded as the true acceleration due to gravity, since to the above degree of approximation g and g' are equal,* while $\theta = 0$.

3.1.1. Motion of a projectile. With the high muzzle velocities now possible in modern gunnery, it is essential that the effect of the earth's rotation be taken into account, in the case of long-range fire, if reasonable accuracy is to be achieved. We shall work in detail, therefore, an example of this nature and obtain an estimate of the target error which would be likely to arise if the earth's rotation were ignored.

Example. A shot is fired with velocity $\mathbf{V}$ due East from a point on the earth's surface whose latitude is $\lambda^\circ N$, the angle of projection being α .

* The difference in the magnitudes of g and g' may be obtained by resolving in the direction of PO (Fig. 4.6); this gives

$$mg' - mg \cos \theta = m\Omega^2 ON \sin \phi = m\alpha\Omega^2 \sin^2 \phi,$$

or, since θ is a small quantity of the order $\alpha\Omega^2/g'$,

$$g' - g = \alpha\Omega^2 \sin^2 \phi,$$

neglecting $(\alpha\Omega^2/g')^2$, so that

$$g = g' \left(1 - \frac{\alpha\Omega^2}{g'} \sin^2 \phi \right).$$

Show that the point where the shot again meets the horizontal plane through the point of projection is at a distance approximately

$$4v^3\Omega \sin^2 \alpha \cos \alpha \sin \lambda/g^2$$

South of a line drawn Eastwards from the point of projection.

Solution. If we ignore the resistance of the air, then we shall have $\mathbf{R} = \mathbf{0}$, and hence, neglecting terms of $O(\Omega^2)$, the equation of motion becomes

$$m\left\{\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\left(\boldsymbol{\Omega} \wedge \frac{\partial \mathbf{r}}{\partial t}\right)\right\} = m\mathbf{g},$$

or, omitting m ,

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\left(\boldsymbol{\Omega} \wedge \frac{\partial \mathbf{r}}{\partial t}\right) = \mathbf{g}. \quad (1)$$

Since both $\boldsymbol{\Omega}$ and $\mathbf{g}$ are constant vectors with respect to axes fixed relative to the earth, this equation can be immediately integrated giving

$$\frac{\partial \mathbf{r}}{\partial t} + 2(\boldsymbol{\Omega} \wedge \mathbf{r}) = \mathbf{g}t + \mathbf{A},$$

where $\mathbf{A}$ is an arbitrary vector, so that

$$\frac{\partial \mathbf{r}}{\partial t} = -2(\boldsymbol{\Omega} \wedge \mathbf{r}) + \mathbf{g}t + \mathbf{A}. \quad (2)$$

On substituting this expression for $\partial \mathbf{r} / \partial t$ in (1), we now have

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\{\boldsymbol{\Omega} \wedge [-2(\boldsymbol{\Omega} \wedge \mathbf{r}) + \mathbf{g}t + \mathbf{A}]\} = \mathbf{g},$$

which, on neglecting the term involving Ω^2 , becomes

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2(\boldsymbol{\Omega} \wedge \mathbf{g})t + 2(\boldsymbol{\Omega} \wedge \mathbf{A}) = \mathbf{g}.$$

This again can be integrated directly giving

$$\frac{\partial \mathbf{r}}{\partial t} + (\boldsymbol{\Omega} \wedge \mathbf{g})t^2 + 2(\boldsymbol{\Omega} \wedge \mathbf{A})t = \mathbf{g}t + \mathbf{B}, \quad (3)$$

which, after a further integration, gives

$$\mathbf{r} + \frac{1}{3}(\boldsymbol{\Omega} \wedge \mathbf{g})t^3 + \{(\boldsymbol{\Omega} \wedge \mathbf{A}) - \frac{1}{2}\mathbf{g}\}t^2 = \mathbf{B}t + \mathbf{C}, \quad (4)$$

$\mathbf{B}$ and $\mathbf{C}$ being two additional arbitrary vectors.

The three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ have to be determined in terms of the initial conditions. Since we know only the values of $\mathbf{r}$ and $\partial \mathbf{r} / \partial t$ initially, it follows that the three cannot be independent. Let us take

the initial point of projection as origin, so that initially $\mathbf{r} = \mathbf{0}$, $\partial\mathbf{r}/\partial t = \mathbf{V}$. Then, from (2), (3) and (4), we have

$$\mathbf{A} = \mathbf{B} = \mathbf{V}, \quad \mathbf{C} = \mathbf{0},$$

so that

$$\mathbf{r} + \frac{1}{3}(\boldsymbol{\Omega} \wedge \mathbf{g})t^3 + \{(\boldsymbol{\Omega} \wedge \mathbf{V}) - \frac{1}{2}\mathbf{g}\}t^2 = \mathbf{V}t; \quad (5)$$

hence we have found $\mathbf{r}$ as a function of t .

To determine the southward deviation of the trajectory during the time of flight, we have to express the above equation in terms of its equivalent scalar equations. To do so, let us choose a system of orthogonal axes such that OZ is vertical (i.e. it is in the direction of the plumb line at O^*), OX is horizontal and points due East, while OY is a mutual perpendicular such that $OXYZ$ forms a right-handed system (it will lie, therefore, in the meridian plane and point due North) (Fig. 4.7). Then we have

$$\begin{aligned} \mathbf{r} &= (x, y, z), \\ \boldsymbol{\Omega} &= (0, \Omega \cos \lambda, \Omega \sin \lambda), \\ \mathbf{g} &= (0, 0, -g), \\ \mathbf{V} &= (V \cos \alpha, 0, V \sin \alpha), \end{aligned}$$

so that

$$\begin{aligned} \boldsymbol{\Omega} \wedge \mathbf{g} &= (-\Omega g \cos \lambda, 0, 0), \\ \boldsymbol{\Omega} \wedge \mathbf{V} &= (\Omega V \sin \alpha \cos \lambda, \Omega V \cos \alpha \sin \lambda, -\Omega V \cos \alpha \cos \lambda). \end{aligned}$$

Hence, substituting these values into (5), we have

$$\begin{aligned} (x, y, z) + \frac{1}{3}(-\Omega g \cos \lambda, 0, 0)t^3 \\ + \{(\Omega V \sin \alpha \cos \lambda, \Omega V \cos \alpha \sin \lambda, -\Omega V \cos \alpha \cos \lambda) - \frac{1}{2}(0, 0, -g)\}t^2 \\ = (V \cos \alpha, 0, V \sin \alpha)t, \end{aligned}$$

so that the equivalent scalar equations become

$$\left. \begin{aligned} x - \frac{1}{3}\Omega g \cos \lambda t^3 + \Omega V \sin \alpha \cos \lambda t^2 &= V \cos \alpha t \\ y + \Omega V \cos \alpha \sin \lambda t^2 &= 0 \\ z - (\Omega V \cos \alpha \cos \lambda + \frac{1}{2}g)t^2 &= V \sin \alpha t \end{aligned} \right\}. \quad (6)$$

From the last of these it follows that $z = 0$ when either $t = 0$ or

$$t = \frac{2V \sin \alpha}{2V\Omega \cos \alpha \cos \lambda + g}.$$

* Since we are neglecting terms of the order Ω^2 , the vertical at O may be considered as coinciding with the earth's radius through O .

The latter is the time of flight. At this instant we have

$$\begin{aligned} y &= -\Omega V \cos \alpha \sin \lambda t^2 \\ &= -\Omega V \cos \alpha \sin \lambda \left\{ \frac{4V^2 \sin^2 \alpha}{g^2} \left(1 - \frac{2\Omega V \cos \alpha \cos \lambda}{g} + \dots \right) \right\} \\ &= -\frac{4\Omega V^3 \sin^2 \alpha \cos \alpha \sin \lambda}{g^2}, \end{aligned}$$

when terms of $O(\Omega^2)$ are neglected. The shot thus meets the horizontal plane at a point distant $4\Omega V^3 \sin^2 \alpha \cos \alpha \sin \lambda / g^2$ South of OX , i.e. South of East.

To get an estimate of the above effect, suppose we take

$$V = 3000 \text{ ft/sec}, \quad \alpha = 30^\circ, \quad \lambda = 45^\circ;$$

then the range of the projectile will be approximately 46 miles, while the above deviation will be approximately 326 yds, which is certainly not negligible and well worth taking into account when considering gunnery performance.

3.2. Foucault's pendulum.* This is a simple pendulum. It is termed Foucault's pendulum when used to demonstrate the earth's rotation, the underlying theory being that of determining the nature of its motion for small displacements from the vertical when suspended from a fixed point in latitude λ , the earth's rotation being taken into account.

To study its nature, suppose we refer the motion to O , the position of the bob of the pendulum when hanging at rest. Let C be the point of suspension, and let $WXYZ$ be a set of rectangular axes fixed relative to the earth such that OZ is in the direction of the upward vertical through the point of suspension, while OX, OY both lie in the horizontal plane through O and point due East and due North respectively (Fig. 4.8(a)). We shall neglect terms of $O(\Omega^2)$, so that OZ may be regarded as coincident with the earth's radius through O .

Since the only force acting on the bob, apart from the earth's attraction, is the tension in the supporting string, we have $\mathbf{R} = \mathbf{T}$ (eqn 4.18), so that the equation of motion of the bob becomes, if $\overline{OP} = \mathbf{r}$, P being the bob of the pendulum,

$$\mathbf{T} + m\mathbf{g} = m \left\{ \frac{\partial^2 \mathbf{r}}{\partial t^2} + 2 \left(\boldsymbol{\Omega} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) \right\}. \quad (4.19)$$

* *Jean Bernard Léon Foucault* (1819–1868) was a French scientist who carried out researches in many branches of both pure and applied physics. One of his important experiments involved the determination of the velocity of light, which he showed to be greater in air than in water, a result which was taken as establishing the wave theory of light. Today he is chiefly remembered for his pendulum experiment, first carried out in Paris in 1851, by means of which he demonstrated the earth's rotation.

Let us now put

$$\mathbf{r} = \mathbf{z} + \boldsymbol{\rho},$$

where $\mathbf{z}$ represents the vertical displacement $\overline{MP}$ of the bob and $\boldsymbol{\rho}$ its horizontal displacement $\overline{NP}$ (Fig. 4.8(b)). Then, on substituting for $\mathbf{r}$ and dividing through by m , (4.19) becomes

$$\frac{\partial^2 \mathbf{z}}{\partial t^2} + \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} + 2 \left\{ \boldsymbol{\Omega} \wedge \left(\frac{\partial \mathbf{z}}{\partial t} + \frac{\partial \boldsymbol{\rho}}{\partial t} \right) \right\} = \frac{T}{m} \frac{(l-z)}{l} \mathbf{k} - \frac{T}{m} \frac{\boldsymbol{\rho}}{l} - g \mathbf{k}, \quad (4.20)$$

where $\mathbf{k}$ is a unit vector in the direction of OZ and l is the length of the pendulum. But, for small displacements of the pendulum from the vertical, z is a small quantity of the second order, whereas ρ is a

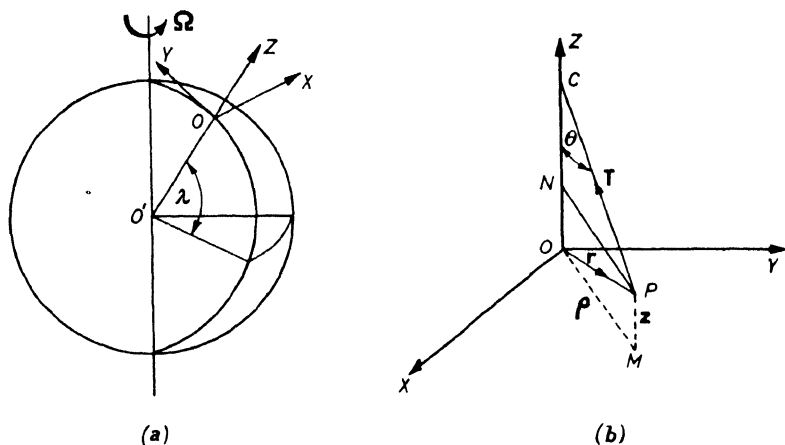


Figure 4.8

small quantity of the first order. Therefore, neglecting terms involving $\mathbf{z}$ ($= z\mathbf{k}$) as being small in comparison with those involving $\boldsymbol{\rho}$, (4.20) reduces to

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} + 2 \left(\boldsymbol{\Omega} \wedge \frac{\partial \boldsymbol{\rho}}{\partial t} \right) = \left(\frac{T}{m} - g \right) \mathbf{k} - \frac{T}{m} \frac{\boldsymbol{\rho}}{l}. \quad (4.21)$$

We can also write

$$\boldsymbol{\Omega} = \Omega \sin \lambda \mathbf{k} + \Omega \cos \lambda \mathbf{j},$$

where $\mathbf{j}$ is a unit vector in the direction of OY . On substituting into (4.21), we therefore have

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} + 2\Omega \sin \lambda \left(\mathbf{k} \wedge \frac{\partial \boldsymbol{\rho}}{\partial t} \right) + 2\Omega \cos \lambda \left(\mathbf{j} \wedge \frac{\partial \boldsymbol{\rho}}{\partial t} \right) = \left(\frac{T}{m} - g \right) \mathbf{k} - \frac{T}{m} \frac{\boldsymbol{\rho}}{l}. \quad (4.22)$$

But, since $\mathbf{j}$ and $\partial\mathbf{p}/\partial t$ are both vectors in the horizontal plane, $\mathbf{j} \wedge \partial\mathbf{p}/\partial t$ must be a vertical vector parallel to $\mathbf{k}$, while $\mathbf{k} \wedge \partial\mathbf{p}/\partial t$ must be a horizontal vector. Consequently (4.22) must be equivalent to the two equations

$$\left. \begin{aligned} \frac{\partial^2 \mathbf{p}}{\partial t^2} + 2\Omega \sin \lambda \left(\mathbf{k} \wedge \frac{\partial \mathbf{p}}{\partial t} \right) &= \frac{T}{m} \frac{\mathbf{p}}{l} & (a) \\ \left\{ \left(\frac{T}{m} - g \right) - 2\Omega \cos \lambda \left| \mathbf{j} \wedge \frac{\partial \mathbf{p}}{\partial t} \right| \right\} \mathbf{k} &= \mathbf{0} & (b) \end{aligned} \right\} \quad (4.23)$$

From the second of these we have

$$\frac{T}{m} = g + 2\Omega \cos \lambda \left| \mathbf{j} \wedge \frac{\partial \mathbf{p}}{\partial t} \right|.$$

On substituting this value for T in (4.23(a)), the differential equation in $\mathbf{p}$ for the determination of the motion thus becomes

$$\frac{\partial^2 \mathbf{p}}{\partial t^2} + 2\Omega \sin \lambda \left(\mathbf{k} \wedge \frac{\partial \mathbf{p}}{\partial t} \right) = -\frac{\mathbf{p}}{l} \left(g + 2\Omega \cos \lambda \left| \mathbf{j} \wedge \frac{\partial \mathbf{p}}{\partial t} \right| \right),$$

or, since the second term on the right-hand side, being of the order of ρ^2 , is of the second order of smallness and so may be neglected,

$$\frac{\partial^2 \mathbf{p}}{\partial t^2} + 2\Omega \sin \lambda \left(\mathbf{k} \wedge \frac{\partial \mathbf{p}}{\partial t} \right) + n^2 \mathbf{p} = \mathbf{0}, \quad (4.24)$$

where $n^2 = g/l$.

To interpret the motion represented by this differential equation, consider a particle moving in a plane under the action of a central force of attraction $n^2 \mathbf{r}$. Then, referred to a set of axes fixed in space and having their origin at the centre of attraction O , the equation of motion of the particle is

$$\frac{d^2 \mathbf{r}}{dt^2} + n^2 \mathbf{r} = \mathbf{0},$$

and the path of the particle is an ellipse having its centre at O and described with a periodic time $2\pi/n$.

If the same motion be referred to a set of axes rotating about the perpendicular to the plane through O with a uniform angular speed Ω , then, since the acceleration of the particle when referred to these axes is given by

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\Omega \left(\mathbf{k} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) - \Omega^2 \mathbf{r},$$

the corresponding equation of motion would be

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\Omega \left(\mathbf{k} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + (n^2 - \Omega^2) \mathbf{r} = \mathbf{0},$$

where $\mathbf{k}$ is a unit vector perpendicular to the plane of the motion. But the actual path of the particle is, of course, the same in both cases. Consequently we have the general result that the solution of the differential equation of the type

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + 2\Omega \left(\mathbf{k} \wedge \frac{\partial \mathbf{r}}{\partial t} \right) + \mu^2 \mathbf{r} = \mathbf{0},$$

the motion being referred to a set of axes rotating with an angular speed Ω about an axis perpendicular to the plane of the motion, is the central ellipse which, when referred to a set of fixed axes, corresponds to the solution of the equation

$$\frac{d^2 \mathbf{r}}{dt^2} + (\mu^2 + \Omega^2) \mathbf{r} = \mathbf{0}.$$

If we now apply the above interpretation to (4.24), we see that the path of the bob is the central ellipse fixed in space, the differential equation of which, when referred to fixed axes, is

$$\frac{d^2 \mathbf{p}}{dt^2} + (n^2 + \Omega^2 \sin^2 \lambda) \mathbf{p} = \mathbf{0}.$$

To an observer moving with the earth the path of the bob thus appears to be that of an ellipse, described with a periodic time

$$T = 2\pi / (n^2 + \Omega^2 \sin^2 \lambda)^{1/2} \approx 2\pi / n,$$

which rotates about the vertical with an angular speed $-\Omega \sin \lambda$.

A visual demonstration of this result may be seen in the entrance hall of the Science Museum, South Kensington. Here a simple pendulum is suspended from a point vertically above the centre of a circular design traced on the floor. The pendulum oscillates to and fro in a vertical plane through the point of suspension, the direction of the plane being, of course, fixed in space. Owing to the rotation of the earth, however, this plane appears to rotate relative to the circle about the vertical through O , whereas, in fact, it is the circle which is rotating relative to the plane of oscillation. As mentioned in the footnote on p. 96, this proof of the rotation of the earth was first demonstrated by Foucault.

Examples

1. A smooth tube of narrow bore is bent into a circle of radius a . The circle is fixed in a vertical plane which is constrained to rotate with constant angular speed Ω about a fixed vertical axis; the normal to the plane through the centre C of the circle intersects this axis in a fixed point O , and the distance OC is constant and equal to h . A particle of mass m , which is free to move within the tube, is slightly

displaced from a position of rest relative to the tube at its highest point. Show that, when the angle between the radius to the particle and the upward vertical is θ , the speed of the particle relative to the tube is

$$\{2ga(1 - \cos \theta) - a^2\Omega^2 \sin^2 \theta\}^{1/2}.$$

2. A particle moves under gravity on a smooth plane, inclined at an angle α to the horizontal, which rotates with small angular speed Ω about a fixed vertical axis. The axis intersects the plane in the point O . Taking rectangular axes OXY fixed in the plane such that OX points down the plane in the direction of the line of greatest slope, prove that, if at time $t = 0$ the particle starts from rest at O , its deviation from OX at time t is given approximately by

$$\frac{1}{6}\Omega g t^3 \sin 2\alpha,$$

terms of $O(\Omega^2)$ being neglected.

3. A shot is fired with velocity V towards the East from a point on the earth's surface in latitude $\lambda^\circ\text{N}$, the angle of projection being α . Find the position of the point of impact of the shot with the horizontal plane through the point of projection, and show that the increase in range due to the rotation of the earth is

$$\frac{4\Omega V^3}{3g^2} \cos \lambda \sin \alpha (1 - \frac{4}{3} \sin^2 \alpha),$$

Ω being the angular speed of the earth's rotation.

4. A particle P , of unit mass, moves with simple harmonic motion $x = a \cos nt$ in a smooth E-W groove on the earth's surface in latitude $\lambda^\circ\text{N}$. If Ω is the earth's angular velocity, show that, provided that terms in Ω^2 are neglected, the reaction of the groove has a horizontal component at right angles to the motion of magnitude $2an\Omega \sin \lambda \sin nt$, and a vertical component whose magnitude fluctuates harmonically with an amplitude $2an\Omega \cos \lambda$.

5. A particle is displaced slightly from the position of stable equilibrium inside a smooth spherical surface, fixed relative to the earth, and is then released. Prove that after a time t the particle oscillates approximately in a line making an angle $\Omega t \sin \lambda$ with the original line of oscillation.

6. A simple pendulum of length l is suspended from a fixed point near the earth's surface in latitude $\lambda^\circ\text{N}$. If the pendulum describes a circular cone of small angle, show that the period of rotation relative to the earth depends upon the direction of rotation about the vertical, the difference between the two periods being approximately $4\pi\Omega \sin \lambda/n$, when terms of $O(\Omega^2)$ are neglected.

CHAPTER V

MOMENTS OF INERTIA

The dynamical behaviour of a rigid body, when acted upon by a given system of forces, is determined by its shape and constitution. The dependence of the motion on these properties is expressed through a number of constants which occur in the equations of motion and are characteristic of the body. They are termed *the moments and the products of inertia* of the body*. Before proceeding to consider the kinetics of a rigid body, it will be convenient to consider their general properties.

1. Definition of moments and of products of inertia

1.1. Moments of inertia. The moment of inertia of a system of particles with respect to a plane, line, or point is defined as the sum of the products formed by multiplying the mass of each particle of the system into the square of the perpendicular distance of the particle from the plane, line, or point (or, merely, the square of the distance in the case of a point). If m_i ($i = 1, 2, \dots, n$) are the respective masses of the particles and p_i ($i = 1, 2, \dots, n$) the appropriate perpendicular distances, then the moment of inertia has the form

$$I = m_1 p_1^2 + m_2 p_2^2 + \dots + m_n p_n^2 = \sum m_i p_i^2.$$

In the case of a rigid body, this sum passes over into the definite integral

$$I = \int_{\Omega} p^2 dm,$$

where $dm = \rho d\Omega$, ρ being the mass per unit volume of the body, and the suffix Ω denotes that the integration is taken over the whole of the volume occupied by the body.

If I_x , I_y , I_z denote the moments of inertia with respect to the planes $x = 0$, $y = 0$, $z = 0$ respectively, then by the above definition

* Alternatively, they are termed the *quadratic moments* of the body.

$$\left. \begin{aligned} I_x &= \sum m_i x_i^2 \rightarrow \int_{\Omega} x^2 dm \\ I_y &= \sum m_i y_i^2 \rightarrow \int_{\Omega} y^2 dm \\ I_z &= \sum m_i z_i^2 \rightarrow \int_{\Omega} z^2 dm \end{aligned} \right\}; \quad (5.1)$$

similarly the moments of inertia with respect to the x, y, z -axes respectively are given by

$$\left. \begin{aligned} A &= \sum m_i (y_i^2 + z_i^2) \rightarrow \int_{\Omega} (y^2 + z^2) dm \\ B &= \sum m_i (z_i^2 + x_i^2) \rightarrow \int_{\Omega} (z^2 + x^2) dm \\ C &= \sum m_i (x_i^2 + y_i^2) \rightarrow \int_{\Omega} (x^2 + y^2) dm \end{aligned} \right\}. \quad (5.2)$$

From these results it is evident that

$$A = I_y + I_z, \quad B = I_z + I_x, \quad C = I_x + I_y.$$

The moment of inertia with respect to the origin is

$$\begin{aligned} I_0 &= \sum m_i (x_i^2 + y_i^2 + z_i^2) \rightarrow \int_{\Omega} (x^2 + y^2 + z^2) dm, \\ &= I_x + I_y + I_z \\ &= \frac{1}{2}(A + B + C). \end{aligned} \quad (5.3)$$

Since the choice of coordinate system is entirely arbitrary, it follows that:

(i) *The moment of inertia with respect to any line is the sum of the moments of inertia with respect to any two mutually perpendicular planes which pass through that line.*

(ii) *The moment of inertia with respect to any point is equal to the sum of the moments of inertia with respect to any three mutually perpendicular planes which pass through that point, or to one half the sum of the moments of inertia with respect to any three mutually perpendicular lines which pass through that point.*

1.2. Products of inertia. Expressions similar to (5.1) above, in which the squares of the coordinates are replaced by the product of the rectangular coordinates with respect to two perpendicular planes, are termed the products of inertia with respect to these planes. The

products of inertia with respect to the coordinate planes taken in pairs are thus given by

$$\sum m_i y_i z_i, \quad \sum m_i z_i x_i, \quad \sum m_i x_i y_i,$$

or, in the case of a continuous distribution, by

$$\int_{\Omega} yz \, dm, \quad \int_{\Omega} zx \, dm, \quad \int_{\Omega} xy \, dm;$$

we shall denote them by the letters F , G , H respectively.*

Since

$$(y-z)^2 \geq 0,$$

it follows that

$$y^2 + z^2 \geq 2yz,$$

and therefore

$$\int_{\Omega} (y^2 + z^2) dm \geq 2 \int_{\Omega} yz \, dm,$$

i.e.

$$A \geq 2F;$$

similarly

$$B \geq 2G, \quad C \geq 2H.$$

Thus

$$I_0 = \frac{1}{2}(A+B+C) \geq (F+G+H).$$

2. Radius of gyration

Suppose the entire mass M of the body, or of the system of particles, to be concentrated into a single particle of mass M at a distance k from the plane, line, or point with respect to which the moment of inertia is I . Then, if k is so chosen that

$$Mk^2 = I,$$

k is called the *radius of gyration* with respect to that plane, line, or point.

The *principal radius of gyration* with respect to a given direction (the direction being that of either a given plane or a given line, as the case may be) is the radius of gyration with respect to the parallel plane, or line, which passes through the centre of mass.

2.1. Minimal property of a principal radius of gyration. In order to establish this property, we shall derive first the following general theorem relating to a set of particles:

Consider the scalar function

$$Q(x, y, z) = Q(\mathbf{r}) - a(\mathbf{r} \cdot \mathbf{r}) + b(\boldsymbol{\lambda} \cdot \mathbf{r})(\boldsymbol{\mu} \cdot \mathbf{r}), \quad (5.4)$$

where a, b are two scalar constants and $\boldsymbol{\lambda}, \boldsymbol{\mu}$ two arbitrary unit vectors,

* D , E , and F are also commonly used.

and let m be the mass of a particle of the system situated at the point $P(\mathbf{r})$; then

$$\sum mQ(\mathbf{r}) = MQ(\mathbf{r}_G) + \sum mQ(\mathbf{r}'),$$

where $M = \sum m$, $\mathbf{r}_G$ is the vector displacement of G , the centre of mass, with respect to the origin and $\mathbf{r}'$ is the vector displacement of P with respect to G . (Fig. 5.1).*

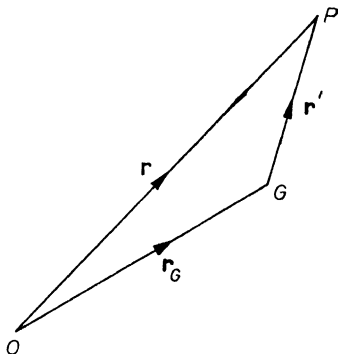


Figure 5.1

To prove this result, since

$$\mathbf{r} = \mathbf{r}_G + \mathbf{r}',$$

we have, on substituting for $\mathbf{r}$ in expression (5.4),

$$\begin{aligned} Q(x,y,z) = Q(\mathbf{r}) &= a(\mathbf{r}_G + \mathbf{r}')(\mathbf{r}_G + \mathbf{r}') + b(\boldsymbol{\lambda} \cdot \overline{\mathbf{r}_G + \mathbf{r}'})(\boldsymbol{\mu} \cdot \overline{\mathbf{r}_G + \mathbf{r}'}) \\ &= a\{\mathbf{r}_G \cdot \mathbf{r}_G + 2(\mathbf{r}_G \cdot \mathbf{r}') + \mathbf{r}' \cdot \mathbf{r}'\} + b\{(\boldsymbol{\lambda} \cdot \mathbf{r}_G)(\boldsymbol{\mu} \cdot \mathbf{r}_G) \\ &\quad + (\boldsymbol{\lambda} \cdot \mathbf{r}_G)(\boldsymbol{\mu} \cdot \mathbf{r}') + (\boldsymbol{\lambda} \cdot \mathbf{r}')(\boldsymbol{\mu} \cdot \mathbf{r}_G) + (\boldsymbol{\lambda} \cdot \mathbf{r}')(\boldsymbol{\mu} \cdot \mathbf{r}')\}. \end{aligned}$$

Consequently, multiplying by m and summing over all the particles of the system, we have

$$\begin{aligned} \sum mQ(\mathbf{r}) &= (\sum m)\{a(\mathbf{r}_G \cdot \mathbf{r}_G) + b(\boldsymbol{\lambda} \cdot \mathbf{r}_G)(\boldsymbol{\mu} \cdot \mathbf{r}_G)\} + \sum m\{a(\mathbf{r}' \cdot \mathbf{r}') + b(\boldsymbol{\lambda} \cdot \mathbf{r}')(\boldsymbol{\mu} \cdot \mathbf{r}')\} \\ &\quad + [2a\mathbf{r}_G + b\{(\boldsymbol{\lambda} \cdot \mathbf{r}_G)\boldsymbol{\mu} + (\boldsymbol{\mu} \cdot \mathbf{r}_G)\boldsymbol{\lambda}\}] \cdot (\sum m\mathbf{r}') \\ &= MQ(\mathbf{r}_G) + \sum mQ(\mathbf{r}'), \end{aligned}$$

since $\sum m\mathbf{r}' = \mathbf{0}$, thus proving the theorem.

* On putting $\mathbf{r} = (x,y,z)$, $\boldsymbol{\lambda} = (l,m,n)$, $\boldsymbol{\mu} = (l',m',n')$, the expression for $Q(\mathbf{r})$ becomes identical with the most general homogeneous function of the second degree in (x,y,z) . The above theorem is thus equivalent to that proved in Lamb, *Statics* (Cambridge University Press, 1933), Chap. VIII, § 73; also Routh, *Elementary Rigid Dynamics* (MacMillan and Co., 1913), Chap. I, § 14.

Having obtained the above result, we shall now deduce the following properties regarding principal radii of gyration and products of inertia. Suppose we put:

(1) $\lambda = \mu$, $a = 1$, $b = -1$. Then, if we regard λ as defining a line through the origin O , we have

$$\begin{aligned} Q(\mathbf{r}) &= r^2 - (\lambda \cdot \mathbf{r})(\lambda \cdot \mathbf{r}) = (\lambda \wedge \mathbf{r}) \cdot (\lambda \wedge \mathbf{r})^* \\ &= p^2, \end{aligned}$$

where p is the perpendicular distance from the point $P(\mathbf{r})$ to the line ' λ '. Thus we have

$$\sum mQ(\mathbf{r}) = \sum mp^2 = I,$$

where I is the moment of inertia of the system with respect to the same line. But, from the above theorem, it follows that

$$\begin{aligned} I &= M(\lambda \wedge \mathbf{r}_G) \cdot (\lambda \wedge \mathbf{r}_G) + \sum m(\lambda \wedge \mathbf{r}') \cdot (\lambda \wedge \mathbf{r}') \\ &= Md^2 + I_G, \end{aligned} \tag{5.5}$$

where d is the perpendicular distance from G on to the line ' λ ' and I_G is the moment of inertia of the system with respect to a line through G parallel to λ . We thus have the well-known result, termed the *Theorem of Parallel Axes*, which states that *the moment of inertia of a system about any axis is equal to the moment of inertia of the system with respect to a parallel axis through the centre of mass together with the product of the total mass of the system and the square of the perpendicular distance between the axes.*

Since $I = Mk^2$, $I_G = Mk_G^2$, where k and k_G are the radii of gyration with respect to the parallel axes through O and G respectively defined by λ , (5.5) may also be written

$$Mk^2 = Md^2 + Mk_G^2,$$

so that

$$k^2 = d^2 + k_G^2.$$

But d^2 must always be positive or zero; hence it follows that *a principal radius of gyration is smaller than the radius of gyration with respect to any parallel axis.*

(2) $\lambda = \mu$, $a = 0$, $b = 1$. Then, if we regard λ as being a unit vector perpendicular to a given plane through O , we have

$$Q(\mathbf{r}) = (\lambda \cdot \mathbf{r})(\lambda \cdot \mathbf{r}) = p^2,$$

where p is the perpendicular distance from the point $P(\mathbf{r})$ to the plane, so that

$$\sum mQ(\mathbf{r}) = \sum mp^2 = I_p,$$

* See Chap. I, § 8 (i).

where I_p is the moment of inertia of the system with respect to the given plane. Using the above theorem, we thus have

$$\begin{aligned} I_p &= M(\lambda \cdot \mathbf{r}_G)(\lambda \cdot \mathbf{r}_G) + \sum m(\lambda \cdot \mathbf{r}')(\lambda \cdot \mathbf{r}') \\ &= Md^2 + I_G, \end{aligned}$$

so that $k^2 = d^2 + k_G^2$,

where d is the perpendicular distance from the centre of mass to the given plane and k and k_G are the radii of gyration with respect to the given plane and the parallel plane through the centre of mass respectively. It is seen that a principal radius of gyration is again a minimum.

(3) $a = 1, b = 0$. Then we have

$$Q(\mathbf{r}) = r^2,$$

so that

$$\sum mQ(\mathbf{r}) = \sum mr^2 = I_0,$$

where I_0 is the moment of inertia of the system with respect to the origin. Proceeding as above, we obtain

$$I_0 = Mr_G^2 + \sum mr'^2,$$

so that

$$k^2 = r_G^2 + k_G^2,$$

where k and k_G are the radii of gyration with respect to the origin and the centre of mass respectively; from this it follows that *the radius of gyration with respect to the centre of mass is smaller than the radius of gyration with respect to any other point.*

(4) $a = 0$, while λ and μ are taken to be unit vectors perpendicular to two orthogonal planes through O , so that $\lambda \cdot \mu = 0$. Then we have

$$Q(\mathbf{r}) = (\lambda \cdot \mathbf{r})(\mu \cdot \mathbf{r}) = p_\lambda p_\mu,$$

where p_λ and p_μ are the perpendicular distances from P to the planes through O at right angles to λ and μ respectively, and hence

$$\sum mQ(\mathbf{r}) = \sum m(\lambda \cdot \mathbf{r})(\mu \cdot \mathbf{r}) = \sum mp_\lambda p_\mu = I_{\lambda\mu},$$

where $I_{\lambda\mu}$ is the product of inertia of the system with respect to the two planes. Applying the above result, we thus have

$$\begin{aligned} I_{\lambda\mu} &= M(\lambda \cdot \mathbf{r}_G)(\mu \cdot \mathbf{r}_G) + \sum m(\lambda \cdot \mathbf{r}')(\mu \cdot \mathbf{r}') \\ &= Mp_\lambda p_\mu + \sum mp'_\lambda p'_\mu, \end{aligned} \tag{5.6}$$

where p'_λ, p'_μ are the perpendicular distances from P to parallel planes through G . We thus have the result: *the product of inertia with respect to any two perpendicular planes is equal to the product of inertia with respect to a parallel set of planes through the centre of mass plus the product of inertia with respect to the original planes of a particle of mass M situated at the centre of mass.*

Since the product $p'_\lambda p'_\mu = (\boldsymbol{\lambda} \cdot \mathbf{r}_G)(\boldsymbol{\mu} \cdot \mathbf{r}_G)$ may be either positive or negative, depending upon the position of the centre of mass, there is, in this case, no minimal principle involved.

3. Moment of inertia with respect to a variable line through a fixed point O

Let the point O be taken as the origin of a system of rectangular coordinates, OQ any line through O having (l, m, n) as its direction cosines,

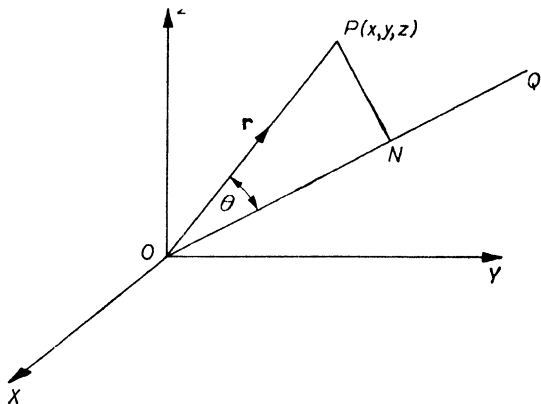


Figure 5.2

$P(x, y, z)$ any point of the body, and PN the perpendicular from P to the line OQ (Fig. 5.2). Then, if $r = OP$ and $\theta = \angle PON$, we have

$$PN = r \sin \theta = |(\boldsymbol{\lambda} \wedge \mathbf{r})|,$$

where $\boldsymbol{\lambda}$ is a unit vector in the direction of OQ , so that

$$\begin{aligned} p^2 &= PN^2 = (\boldsymbol{\lambda} \wedge \mathbf{r}) \cdot (\boldsymbol{\lambda} \wedge \mathbf{r}) \\ &= \{(l, m, n) \wedge (x, y, z)\}^2 \\ &= \{mz - ny, nx - lz, ly - mz\}^2 \\ &= l^2(y^2 + z^2) + m^2(z^2 + x^2) + n^2(x^2 + y^2) \\ &\quad - 2mnyz - 2nlzx - 2lmxy. \end{aligned}$$

Multiplying by m_i and summing over all the particles of the system, we thus have

$$\begin{aligned} I &= \sum m_i p^2 = l^2 \sum m_i (y^2 + z^2) + m^2 \sum m_i (z^2 + x^2) + n^2 \sum m_i (x^2 + y^2) \\ &\quad - 2mn \sum m_i yz - 2nl \sum m_i zx - 2lm \sum m_i xy \\ &= Al^2 + Bm^2 + Cn^2 - 2Fmn - 2Gnl - 2Hlm; \end{aligned}$$

the moment of inertia of the system with respect to the line OQ may

thus be expressed in terms of the direction cosines of OQ and the moments and products of inertia of the system with respect to the coordinate axes and planes.

4. The momental ellipsoid or ellipsoid of inertia

A geometrical interpretation of the above result can be given. Consider the quadric whose equation is

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy = M\epsilon^4, \quad (5.7)$$

where ϵ is any arbitrary length (the reason for taking the fourth power of ϵ is to keep the dimensions of the constant term consistent with those of the other terms occurring in the equation), and $M = \sum m$, the total mass of the system. If r is the radius of this quadric in the direction OQ , then, on replacing (x, y, z) in (5.7) by (lr, mr, nr) respectively, we have

$$I = M\epsilon^4/r^2,$$

i.e. *the moment of inertia about any radius varies inversely as the square of its length*. Since I is necessarily positive, the above equation for the determination of r always has real roots; thus every radius meets the surface in real points, and the quadric is therefore an ellipsoid. It is called the *momental ellipsoid* with respect to the point O .

If the surface be referred to its principal axes, then the product terms in (5.7) vanish and the equation takes the form

$$Ax^2 + By^2 + Cz^2 = M\epsilon^4.$$

These special axes are termed the *principal axes of inertia* at O , and A, B, C the corresponding moments of inertia with respect to them, the *principal moments of inertia* at O . Since every ellipsoid has at least three principal axes, so every material system has at least three principal axes of inertia and hence three principal moments of inertia.

The mutually orthogonal planes which pass through the principal axes are termed the *principal planes of inertia* at O . It is seen that the products of inertia ($\sum myz$, $\sum mzx$, $\sum mxy$) with respect to these planes, taken in pairs, all vanish.

It should be noticed that the possible forms which can be assumed by the momental ellipsoid are restricted in number. This arises from the fact that A, B, C must always be such that

$$B+C > A, \quad C+A > B, \quad A+B > C$$

(see eqns 5.2). The greatest principal moment must therefore be less than the sum of the other two.*

* An exception occurs when the distribution is a two dimensional one. For, taking the plane of the distribution to be the plane $z = 0$, then in this case $A+B = C$.

4.1. Determination of the principal moments of inertia and the principal axes. The principal axes of the ellipsoid clearly coincide with those radii which cut the ellipsoid orthogonally. If we write the equation of the ellipsoid in the form

$$\phi(x, y, z) \equiv Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy = M\epsilon^4,$$

then the direction cosines of the normal at a point $P(x, y, z)$ will be proportional to

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right),$$

while the direction cosines of the radius vector to P are proportional to (x, y, z) . Consequently the point P corresponds to the extremity of a principal axis provided that its coordinates are such that

$$\frac{\partial \phi / \partial x}{x} = \frac{\partial \phi / \partial y}{y} = \frac{\partial \phi / \partial z}{z}. \quad (5.8)$$

But, if the numerator and denominator of the first of these ratios are both multiplied by x , those of the second by y , and those of the third by z , and the three numerators added and the three denominators added, then the common value of the above ratios is equal to

$$\frac{x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} + z \frac{\partial \phi}{\partial z}}{x^2 + y^2 + z^2} = \frac{2\phi}{r^2} = \frac{2M\epsilon^4}{r^2} = 2I,$$

since $\phi(x, y, z)$ is a homogeneous quadratic expression. Evaluating the partial differential coefficients in (5.8), the point (x, y, z) therefore corresponds to the extremity of a principal axis provided that

$$\frac{Ax - Hy - Gz}{x} = \frac{-Hx + By - Fz}{y} = \frac{-Gx - Fy + Cz}{z} = I,$$

i.e. provided (x, y, z) satisfy the three equations

$$\left. \begin{aligned} (A-I)x - Hy - Gz &= 0 \\ -Hx + (B-I)y - Fz &= 0 \\ -Gx - Fy + (C-I)z &= 0 \end{aligned} \right\}. \quad (5.9)$$

But, if these equations are to yield consistent solutions, then it is necessary, since (x, y, z) cannot all be zero, that

$$\begin{vmatrix} A-I & -H & -G \\ -H & B-I & -F \\ -G & -F & C-I \end{vmatrix} = 0.$$

This is a cubic in I . The three values of I corresponding to its three roots determine the moments of inertia of the system with respect to the three axes coincident with the three principal axes of the momental

ellipsoid at O , i.e. they are the three principal moments of inertia at O . If these three values of I are now substituted in turn into (5.9), then the principal axis of inertia corresponding to any one of these values is defined by the intersection of any two of the planes, corresponding to this value of I , defined by (5.9).

4.2. Special Cases. (In this paragraph A, B, C refer to principal moments of inertia.)

(1) If two of the principal moments at O are equal, say $A = B$, then the momental ellipsoid is an ellipsoid of revolution about OZ as axis, and the system is said to have *kinetic symmetry* with respect to this axis. All diameters perpendicular to this axis are principal axes. Such a system is said to be *uniaxial*.*

(2) If $A = B = C$, the momental ellipsoid is a sphere. All axes through O are in this case principal axes and the moment of inertia with respect to each is the same. The system is accordingly said to be *kinetically symmetrical* with respect to O .

5. Central ellipsoid

If the point O coincides with the centre of mass, then the momental ellipsoid is termed the *central ellipsoid*.

5.1. Principal property of the central ellipsoid. Let O be an arbitrary point and suppose we take it to be the origin of a system of coordinate axes. If these coincide with the principal axes at O , then the equation of the momental ellipsoid, when referred to them, will be of the form

$$Ax^2 + By^2 + Cz^2 = Me^4.$$

Now suppose that we take a new set of orthogonal axes $OX'Y'Z'$ such that OZ' coincides with OZ . Then we shall have $z = z'$, while x and y will be linear functions of x' and y' . Consequently, if we refer the above ellipsoid to these axes, then its equation will contain a term involving the product $x'y'$ but no terms involving $z'x'$ and $z'y'$. It follows, therefore, that, if a given axis is to be a principal axis at a point O , then the product of inertia of the distribution with respect to the plane through O , at right angles to the axis, and *any* other plane which contains the axis must vanish. But, if this condition is satisfied for any two given planes, say the planes $x = 0$, $y = 0$, the given axis being taken as OZ , so that

$$\sum mxz = 0, \quad \sum myz = 0,$$

then it clearly follows that

$$\sum m(ax + by)z = 0$$

* This term was first used by Routh.

for all values of the coefficients a and b , i.e. the condition is satisfied for all planes which pass through the axis. Consequently we have the result: *if $OXYZ$ is a system of orthogonal axes passing through a point O , then the necessary and sufficient conditions for OZ to be a principal axis at O is that the products of inertia of the distribution with respect to the plane $z = 0$ and any other two planes which pass through OZ should vanish.*

Now suppose that OZ is a principal axis at some point O , so that

$$\sum mxz = 0, \quad \sum myz = 0, \quad (5.10)$$

and let O' be some other point on OZ at a distance h from O . Then, so that the z -axis should also be a principal axis at O' , it is necessary that

$$\sum mx(z-h) = 0, \quad \sum my(z-h) = 0,$$

and therefore, in virtue of (5.10), that

$$h \sum mx = 0, \quad h \sum my = 0.$$

But these conditions are satisfied if, and only if, the z -axis passes through the centre of mass of the system. We thus have the result: *A principal axis of inertia of the central ellipsoid is a principal axis of inertia for all its points. Conversely, if any axis is a principal axis of inertia for any two of its points, then it passes through the centre of mass and so is a principal axis of inertia at all its points.*

If the moment of inertia about any axis through the centre of mass is known, then that about any parallel axis can be determined; for, by the theorem of parallel axes, it is simply obtained by the addition of the quantity Md^2 , where d is the distance between the two axes. For this reason the central ellipsoid is one of special importance.

6. Ellipsoid of gyration

Let a, b, c be the radii of gyration with respect to the principal axes at some point O , so that

$$A = Ma^2, \quad B = Mb^2, \quad C = Mc^2.$$

Then the moment of inertia about an axis through O in the direction (l, m, n) will be given by

$$\begin{aligned} I &= Al^2 + Bm^2 + Cn^2 \\ &= M(a^2l^2 + b^2m^2 + c^2n^2) \\ &= Mp^2, \end{aligned}$$

where p is the length of the perpendicular drawn from O in the direction (l, m, n) to a tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

This ellipsoid is therefore called the *ellipsoid of gyration*.

6.1. Determination of the principal axes at a given point P when the principal moments of inertia and principal axes of inertia at the centre of mass are known. Let a system of orthogonal axes $GXYZ$ be chosen to coincide with the principal axes at G , the centre of mass, and let the coordinates of P with respect to these axes be (x_1, y_1, z_1) . Let $A = Ma^2$, $B = Mb^2$, $C = Mc^2$ be the principal moments of inertia at G . Then the moments and products of inertia with respect to a set of parallel axes and coordinate planes through the point P are (eqns 5.5 and 5.6)

$$\left. \begin{aligned} A' &= A + M(y_1^2 + z_1^2), & F' &= My_1z_1 \\ B' &= B + M(z_1^2 + x_1^2), & G' &= Mz_1x_1 \\ C' &= C + M(x_1^2 + y_1^2), & H' &= Mx_1y_1 \end{aligned} \right\}. \quad (5.11)$$

The equation of the momental ellipsoid with respect to a set of axes $PX'Y'Z'$ respectively parallel to $GXYZ$ will thus be given by

$$A'x'^2 + B'y'^2 + C'z'^2 - 2F'y'z' - 2G'z'x' - 2H'x'y' = M\epsilon^4.$$

Its principal axes are thus determined by the equations

$$\left. \begin{aligned} (A' - I)x' - H'y' - G'z' &= 0 \\ -H'x' + (B' - I)y' - F'z' &= 0 \\ -G'x' - F'y' + (C' - I)z' &= 0 \end{aligned} \right\}, \quad (5.12)$$

where I assumes the values given by the roots of the cubic

$$\begin{vmatrix} A' - I & -H' & -G' \\ -H' & B' - I & -F' \\ -G' & -F' & C' - I \end{vmatrix} = 0,$$

the three values being the moments of inertia with respect to the principal axes at P . On substituting the values for A' , B' , C' , F' , G' , H' as given by (5.11), the determinantal equation in I thus becomes

$$\begin{vmatrix} M(a^2 + y_1^2 + z_1^2) - I & -Mx_1y_1 & -Mx_1z_1 \\ -Mx_1y_1 & M(b^2 + z_1^2 + x_1^2) - I & -My_1z_1 \\ -Mx_1z_1 & -My_1z_1 & M(c^2 + x_1^2 + y_1^2) - I \end{vmatrix} = 0.$$

If we now make the substitution

$$M(GP^2) - I \equiv M(x_1^2 + y_1^2 + z_1^2) - I = M\lambda,$$

say, this reduces to

$$\begin{vmatrix} -x_1^2 + (a^2 + \lambda) & -x_1y_1 & -x_1z_1 \\ -x_1y_1 & -y_1^2 + (b^2 + \lambda) & -y_1z_1 \\ x_1z_1 & -y_1z_1 & -z_1^2 + (c^2 + \lambda) \end{vmatrix} = 0. \quad (5.13)$$

But the general expansion of a determinant of the form

$$\Delta = \begin{vmatrix} x_{11} + a_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} + a_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} + a_{33} \end{vmatrix}$$

is given by*

$$\begin{aligned} \Delta = & a_{11}a_{22}a_{33} + x_{11}a_{22}a_{33} + x_{22}a_{33}a_{11} + x_{33}a_{11}a_{22} \\ & + |x_{11}x_{22}|a_{33} + |x_{11}x_{33}|a_{22} + |x_{22}x_{33}|a_{11} + |x_{11}x_{22}x_{33}|. \end{aligned}$$

Consequently, expanding (5.13) using this result, it follows, since

$$|x_{11}x_{22}x_{33}| = \begin{vmatrix} (-x_1 & -y_1 & -z_1) \times x_1 \\ (-x_1 & -y_1 & -z_1) \times y_1 \\ (-x_1 & -y_1 & -z_1) \times z_1 \end{vmatrix}$$

so that $|x_{11}x_{22}|$, $|x_{22}x_{33}|$, $|x_{33}x_{11}|$, and $|x_{11}x_{22}x_{33}|$ are all zero, that the cubic equation in λ becomes

$$\begin{aligned} (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) - x_1^2(b^2 + \lambda)(c^2 + \lambda) \\ - y_1^2(c^2 + \lambda)(a^2 + \lambda) - z_1^2(a^2 + \lambda)(b^2 + \lambda) = 0, \end{aligned}$$

which, on dividing through by $(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)$, takes the form

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1. \quad (5.14)$$

We therefore see that the principal moments of inertia at P are given by

$$I = M(GP^2 - \lambda), \quad (5.15)$$

where λ takes the values λ_1 , λ_2 , λ_3 which define the three surfaces through P confocal with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the ellipsoid of gyration at G .

To find the principal axes at P , we note that their direction cosines are proportional to the values of (x', y', z') as given by (5.12) when I assumes the above values (eqn 5.15). Consequently it follows that they will be proportional to the minors of the first row of (5.13) when λ takes the values $\lambda_1, \lambda_2, \lambda_3$ in turn. They are thus proportional to

$$(b^2 + \lambda)(c^2 + \lambda) - z_1^2(b^2 + \lambda) - y_1^2(c^2 + \lambda), \quad x_1y_1(c^2 + \lambda), \quad x_1z_1(b^2 + \lambda),$$

or, on dividing through by $(b^2 + \lambda)(c^2 + \lambda)$, to

$$1 - \frac{z_1^2}{c^2 + \lambda} - \frac{y_1^2}{b^2 + \lambda}, \quad \frac{x_1y_1}{b^2 + \lambda}, \quad \frac{x_1z_1}{c^2 + \lambda} \quad (\lambda = \lambda_1, \lambda_2, \lambda_3),$$

* See, for instance, Aitken, *Determinants and Matrices* (University Mathematical Texts, Oliver and Boyd, Edinburgh, 1948), p. 87, eqn (2).

i.e. in virtue of (5.14), to

$$\frac{x_1^2}{a^2+\lambda}, \quad \frac{x_1 y_1}{b^2+\lambda}, \quad \frac{x_1 z_1}{c^2+\lambda},$$

i.e. to

$$\frac{x_1}{c^2+\lambda}, \quad \frac{y_1}{b^2+\lambda}, \quad \frac{z_1}{c^2+\lambda}. \quad (5.16)$$

The principal axes at P thus correspond to the normals to the above three confocal surfaces passing through P . The condition that a line may be a principal axis at some point of its length thus becomes the condition that it may be a normal at some point to one of a family of confocal quadrics.

It follows from the above that the principal moments and principal axes at any point P may be determined if the ellipsoid of gyration at G is known.

6.2. Special cases

(1) *Two of the principal moments at G are equal.* This is the case of kinetic symmetry with respect to an axis through G . Suppose we take $B = C$, so that $b^2 = c^2$. Then, taking the coordinate axes to coincide with a set of principal axes at G , the equation of the ellipsoid of gyration will be of the form

$$\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1. \quad (5.17)$$

In this case the equation for the determination of the confocal quadrics passing through the point $P(x, y, z)$ becomes

$$\frac{x^2}{a^2+\lambda} + \frac{y^2+z^2}{b^2+\lambda} = 1,$$

which for any particular values of (x, y, z) is a quadratic in λ . Let its two roots be denoted by λ_1 and λ_2 respectively. Then to the one root will correspond an ellipsoid of revolution, to the other an hyperboloid of revolution. Further, these surfaces will cut orthogonally, the curve of intersection being a circle. The surfaces confocal to (5.17) thus consist of (i) a family of confocal ellipsoids, (ii) a family of confocal hyperboloids, the two families being mutually orthogonal. Two of the principal axes at P are the normals to the respective members of these families which pass through P . The third principal axis must therefore be the axis mutually orthogonal to them; it will thus be parallel to the yz -plane. The principal moments at P are therefore

$$M(GP^2 - \lambda_1), \quad M(GP^2 - \lambda_2), \quad M(GP^2 + b^2),$$

the last value being obtained by the theorem of parallel axes.

(2) *The three principal moments at G are equal ($A = B = C = Ma^2$).* This is the case of kinetic symmetry with respect to G . In this case the ellipsoid of gyration is a sphere, and the confocal surfaces are concentric spheres. If P is any point in space, then, if GP is a principal axis at G , we know that it will also be a principal axis at P . Therefore one principal axis is the line GP itself, in other words, the normal to the surface through P concentric with the sphere of gyration. The other principal axes must therefore be a pair of orthogonal axes passing through P and lying in the tangent plane to this sphere at P . But since the moments of inertia about all lines through G are the same, and since all lines through P in the tangent plane at P are at the same perpendicular distance from G , it follows, by the theorem of parallel axes, that the moments of inertia about all lines through P perpendicular to GP are the same, so that the system must be kinetically symmetrical with respect to GP at P . The principal axes at P thus consist of GP and any two mutually orthogonal lines through P .

The principal moments of inertia will be Ma^2 with respect to GP , and $M(a^2 + GP^2)$ with respect to the other two axes. That with respect to GP is easily seen to be equivalent to $M(GP^2 - \lambda)$, where λ is the root of the equation

$$\frac{x^2 + y^2 + z^2}{a^2 + \lambda} = 1,$$

(x, y, z) being the coordinates of P , as we should expect from the general theory described above.

7. Examples

(1) *The coordinate axes $OXYZ$ can be brought into coincidence with the principal axes at a point O of a body by a small rotation θ . Show that the equation of the momental ellipsoid referred to $OXYZ$ is*

$$Ax^2 + By^2 + Cz^2 + 2(B - C)\theta_1 yz + 2(C - A)\theta_2 zx + 2(A - B)\theta_3 xy = M\epsilon^4,$$

where A, B, C are the principal moments at O and $(\theta_1, \theta_2, \theta_3)$ are the components of θ with respect to the coordinate axes.

Solution. Let OX', OY', OZ' be the principal axes at O . Then, referred to these axes, the equation of the momental ellipsoid at O is

$$Ax'^2 + By'^2 + Cz'^2 = M\epsilon^4.$$

Consequently, if the transformation relations between the (x', y', z') and the (x, y, z) -coordinates can be obtained, then, on substituting for (x', y', z') in terms of (x, y, z) in the above equation, the resulting equation will be that of the momental ellipsoid when referred to the axes $OXYZ$. To determine these relations, let $\mathbf{R}(x', y', z')$ be the vector position of a general point P of the body with respect to the axes $OX'Y'Z'$ and

let $\mathbf{r}(x,y,z)$ be the vector position of the same point when referred to $OXYZ$. Imagine the body as undergoing a rotation $(-\boldsymbol{\theta})$. Then, after undergoing this rotation, its principal axes will coincide with the axes $OXYZ$. Thus, if P' is the new position of P , then the coordinates of P' referred to $OXYZ$ will be (x',y',z') . Also we shall have

$$\overline{OP'} = \overline{OP} + \overline{PP'},$$

i.e. $\mathbf{R}(X,Y,Z) = \mathbf{r}(x,y,z) - \boldsymbol{\theta}(\theta_1, \theta_2, \theta_3) \wedge \mathbf{r}(x,y,z)$.

The transformation of coordinates from (x',y',z') to (x,y,z) will thus be described by the scalar equations

$$(x',y',z') = (x - \theta_2 z + \theta_3 y, y - \theta_3 x + \theta_1 z, z - \theta_1 y + \theta_2 x).$$

Referred to the $OXYZ$ axes, the equation of the momental ellipsoid thus becomes

$$A(x + \theta_2 z + \theta_3 y)^2 + B(y - \theta_3 x + \theta_1 z)^2 + C(z - \theta_1 y + \theta_2 x)^2 = M\epsilon^4,$$

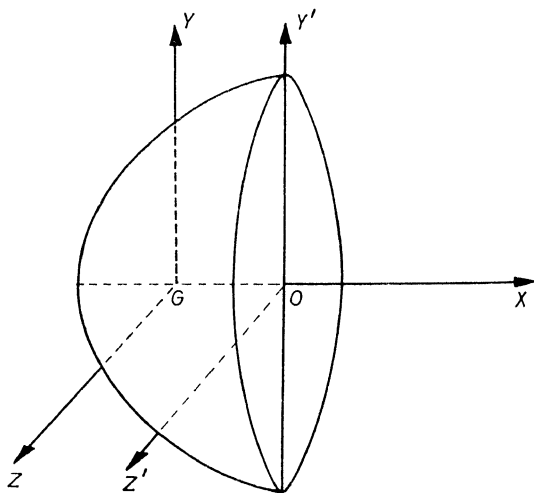
or, neglecting second-order terms,

$$Ax^2 + By^2 + Cz^2 + 2(B-C)\theta_1 yz + 2(C-A)\theta_2 zx + 2(A-B)\theta_3 xy = M\epsilon^4.$$

(2) *To find the principal axes and the principal moments of inertia at a point on the rim of a uniform thin hemispherical shell.*

Solution. Let M be the mass of the hemisphere, a its radius, G its centre of mass, and O the centre of its rim. We shall use the results of § 6.1 above.

Let P be the point on the rim, and let $OX'Y'Z'$ be a set of orthogonal axes such that OX' is perpendicular to the rim, OY' is in the direction of OP , and OZ' is a mutual perpendicular. Also let $GXYZ$ be a set of



parallel axes through G . Then, since the hemisphere is symmetrical with respect to GO , (GX, GY, GZ) are the principal axes at G . Once the moments of inertia with respect to these axes have been found, the principal moments of inertia and the principal axes at P can be determined.

The moments of inertia of the hemisphere with respect to $GXYZ$ can easily be calculated. To evaluate them, consider a spherical shell of mass M , centre O , and radius a . If I_O is its moment of inertia with respect to its centre, then we have, from the general properties of moments of inertia (§ 1.1, eqn 5.3),

$$A+B+C = 2I_O,$$

or, since $A = B = C$ for a spherical shell:

$$\begin{aligned} \text{Moment of inertia of spherical shell of mass } M \text{ about a diameter} \\ = \frac{2}{3}I_O = \frac{2}{3}Ma^2. \end{aligned}$$

The moments of inertia of the hemisphere with respect to OX', OY', OZ' may now be deduced; they will be equal to $\frac{2}{3}Ma^2$, where M is now the mass of the hemisphere. Consequently we have:

$$\text{Moment of inertia of hemisphere about } GX = \frac{2}{3}Ma^2,$$

$$\begin{aligned} \text{Moments of inertia of hemisphere about } GX, GY &= \frac{2}{3}Ma^2 - M(GO)^2 \\ &= \frac{2}{3}Ma^2 - \frac{1}{4}Ma^2 \\ &= \frac{5}{12}Ma^2. \end{aligned}$$

The equation of the ellipsoid of gyration for the hemisphere will thus be given by

$$\frac{x^2}{\frac{2}{3}a^2} + \frac{y^2+z^2}{\frac{5}{12}a^2} = 1.$$

Also the coordinates of P with respect to the axes through G are $(\frac{1}{2}a, a, 0)$. Thus the quadrics through P confocal with the above ellipsoid are determined by the values of λ given by the equation

$$\frac{(\frac{1}{2}a)^2}{\frac{2}{3}a^2 + \lambda} + \frac{a^2}{\frac{5}{12}a^2 + \lambda} = 1,$$

$$\text{i.e.} \quad \frac{1}{4}a^2(\frac{5}{12}a^2 + \lambda) + a^2(\frac{2}{3}a^2 + \lambda) = (\frac{2}{3}a^2 + \lambda)(\frac{5}{12}a^2 + \lambda),$$

$$\text{or} \quad \lambda^2 - (\frac{1}{6}a)^2\lambda - \frac{7}{144}a^2 = 0.$$

On solving this equation for λ , we find that the roots of the quadratic are given by

$$(\lambda_1, \lambda_2) = \frac{1 \pm 6\sqrt{2}}{12}.$$

The principal moments of inertia at P are thus given by:

(I_1, I_2) (about principal axes which lie in the GXY plane)

$$= M(GP^2 - \lambda) \quad (\lambda = \lambda_1, \lambda_2)$$

$$= Ma^2 \left(\frac{5}{4} - \frac{1 \pm 6\sqrt{2}}{12} \right)$$

$$= Ma^2 \left(\frac{7 \pm 3\sqrt{2}}{6} \right),$$

I_3 (about the principal axis parallel to GZ)

$$= \frac{5}{12}Ma^2 + M(GP^2)$$

$$= \frac{5}{12}Ma^2 + \frac{5}{4}Ma^2$$

$$= \frac{5}{3}Ma^2.$$

The directions of the principal axes corresponding to I_1, I_2 may now be obtained by determining the direction cosines of the normals at P to the two confocal quadrics defined by the equation

$$\frac{x^2}{\frac{2}{3}a^2 + \lambda} + \frac{y^2 + z^2}{\frac{5}{12}a^2 + \lambda} = 1$$

in which λ takes the values λ_1, λ_2 in turn; they will be proportional to (eqn 5.16)

$$\left(\frac{a/2}{\frac{2}{3}a^2 + \lambda}, \frac{a}{\frac{5}{12}a^2 + \lambda}, 0 \right) \quad (\lambda = \lambda_1, \lambda_2).$$

In this example it is simpler to use the above method rather than the direct method of § 4.1, since in this case the principal axes at G , together with the moments of inertia with respect to them, can be determined quite easily; to have used the method of § 4.1 would have involved the determination of the moments of inertia with respect to a set of orthogonal axes having their origin at P together with the products of inertia with respect to planes defined by these axes. By using the above method, the determination of the products of inertia is avoided. In other cases, however, it may be simpler to use the direct method.

Examples

1. The principal moments of a uniaxial body at its centre of mass are A, A, C , and the direction cosines of the axis of symmetry with respect to rectangular axes through the centre of mass are (l, m, n) . Prove that its moments and products of inertia with respect to these axes are

$$A + (C - A)l^2, \quad A + (C - A)m^2, \quad A + (C - A)n^2,$$

and

$$(A - C)mn, \quad (A - C)nl, \quad (A - C)lm,$$

respectively.

2. Find the principal moments of inertia at the vertex of a cone of semi-vertical angle α , height h , and mass M , and show that the moment of inertia about a generator is

$$\frac{3}{4}Mh^2 \sin^2 \alpha (1 + \frac{1}{5} \sec^2 \alpha).$$

3. Three uniform rods OA, OB, OC are each of unit length and mass; OB is normal to the plane COA , and the angle COA is 30° . Show that the principal moments of inertia at O are $(\frac{2}{3}, (4 \pm \sqrt{3})/6)$, and determine the directions of the principal axes.

4. The edges of the base of a uniform rectangular block of mass M are each of length $2a$, and the height of the block is $2a\sqrt{5}$. Find the principal moments of inertia, and also the directions of the principal axes, at a corner of the base.

5. The principal moments of inertia of a solid of revolution at its centre of mass are A, A, C , where $C > A$. Show that there are two points on the axis of the body such that the system is kinetically symmetrical with respect to either point.

6. Show that a uniform right circular cone, whose altitude is equal to the diameter of its base, is kinetically symmetrical with respect to its centre of mass, and determine the principal moments of inertia and the directions of the principal axes at a point on the rim of its base.

CHAPTER VI

EQUATIONS OF MOTION OF A RIGID BODY

Having discussed the properties of the moments of inertia of a rigid body, we shall now consider the derivation of the equations of motion of such a body when moving three-dimensionally. We shall consider first the motion of a system of mutually interacting particles, and then, having obtained the equations of motion appropriate to such a system, deduce the form which these equations take when applied to a rigid body.

1. Equations of motion of a system of particles. Principles of linear and angular momentum

The equations which govern the motion of a single particle are

$$m \frac{d\mathbf{v}}{dt} \equiv m \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}, \quad (6.1)$$

and
$$\frac{d\mathbf{H}}{dt} = \mathbf{r} \wedge \mathbf{F}, \quad (6.2)$$

where $\mathbf{r}$ is the vector displacement of the particle with respect to a fixed origin O , $\mathbf{F}$ is the resultant force acting on the particle, and $\mathbf{H} = \mathbf{r} \wedge m\mathbf{v}$ is the moment of momentum of the particle with respect to O (see Chap. II, eqns (2.2) and (2.4)). The second of these equations expresses the fact that the rate of change of the moment of momentum of the particle with respect to O is equal to the moment of the resultant force acting on the particle with respect to the same point.

Suppose we now apply these results for a single particle to the motion of a system of mutually interacting particles. Let there be n particles in the system, and let their masses be denoted by $m_1, m_2, \dots, m_n$, the mass of the i th particle being m_i , its vector displacement with respect to O being $\mathbf{r}_i$, and its velocity $\mathbf{v}_i$. Further, suppose that we denote the resultant force acting on m_i from outside the system by $\mathbf{F}_i$, and the force which the particle m_k exerts on m_i by $\mathbf{F}_{ik}$. Then the equations of motion of the particles are given by eqn (6.1)

$$\frac{d}{dt}(m_i \mathbf{v}_i) \equiv \frac{d}{dt} \left(m_i \frac{d\mathbf{r}_i}{dt} \right) = \mathbf{F}_i + \sum_k \mathbf{F}_{ik} \quad (i = 1, 2, \dots, n). \quad (6.3)$$

We shall call $\mathbf{F}_i$ ($i = 1, 2, \dots, n$) the *external* forces, the $\mathbf{F}_{ik}$'s the *internal* forces of the system. With the aid of Newton's third law, the following results may now be obtained:

(1) If we add equations (6.3), so that we have

$$\frac{d}{dt} \sum_i (m_i \mathbf{v}_i) = \sum_i \mathbf{F}_i + \sum_i \sum_k' \mathbf{F}_{ik}, \quad (6.4)$$

then, since according to Newton's third law $\mathbf{F}_{ik} = -\mathbf{F}_{ki}$, the internal forces cancel out in pairs and we obtain

$$\frac{d}{dt} \sum_i (m_i \mathbf{v}_i) = \sum_i \mathbf{F}_i. \quad (6.5)$$

The vector sum of the individual linear momentum vectors is called the *resultant or total linear momentum* of the system. The above result thus states that: *The rate of change of the total linear momentum of the system is equal to the resultant of the external forces acting on the system and is independent of the internal forces.*

(2) If we consider the rate of change of $\mathbf{H}_i (= \mathbf{r}_i \wedge m_i \mathbf{v}_i)$, the moment of momentum of the i th particle with respect to O , then we have from (6.2)

$$\frac{d\mathbf{H}_i}{dt} = \mathbf{r}_i \wedge (\mathbf{F}_i + \sum_k' \mathbf{F}_{ik}).$$

Therefore, summing over all the particles, we have

$$\begin{aligned} \sum_i \frac{d\mathbf{H}_i}{dt} &= \sum_i \{ \mathbf{r}_i \wedge (\mathbf{F}_i + \sum_k' \mathbf{F}_{ik}) \} \\ &= \sum_i (\mathbf{r}_i \wedge \mathbf{F}_i) + \sum_i (\mathbf{r}_i \wedge \sum_k' \mathbf{F}_{ik}). \end{aligned} \quad (6.6)$$

But

$$\mathbf{r}_i \wedge \mathbf{F}_{ik} = -(\mathbf{r}_k \wedge \mathbf{F}_{ki}), \quad (6.7)$$

and hence the second term on the right-hand side of (6.6), which involves the internal forces, drops out. We are therefore left with the equation

$$\frac{d\mathbf{H}}{dt} = \sum_i (\mathbf{r}_i \wedge \mathbf{F}_i), \quad (6.8)$$

where

$$\mathbf{H} = \sum_i (\mathbf{r}_i \wedge m_i \mathbf{v}_i) \quad (6.9)$$

is the resultant moment of momentum of the system, a result which states that: *The rate of change of the resultant moment of momentum of the system of particles with respect to an arbitrary fixed point O is equal to the resultant moment of the external forces with respect to the same point and is independent of the internal forces.*

1.1. Independence of the translational motion of the centre of mass and the motion relative to the centre of mass

(1) *Translational Motion of the Centre of Mass.* The motion of any system can be defined by the translational motion of some chosen base point together with the motion relative to this base point. Suppose we

choose the base point to be the centre of mass, or centre of gravity, of the system.* Its position with respect to the fixed origin O is defined vectorially by the equation

$$M\mathbf{r}_G = \sum_i m_i \mathbf{r}_i,$$

where $M = \sum_i m_i$ is the total mass of the system. If we now assume that the individual masses $m_1, m_2, \dots, m_n$ are all constant, then we shall have

$$M \frac{d\mathbf{r}_G}{dt} = \sum_i m_i \frac{d\mathbf{r}_i}{dt},$$

or, writing $d\mathbf{r}_G/dt = \mathbf{v}_G$,

$$M\mathbf{v}_G = \sum_i m_i \mathbf{v}_i,$$

i.e. *the total linear momentum of the system is the same as that of a particle of mass M whose motion is that of the centre of mass of the system.* From the above equation we have, on differentiating,

$$M \frac{d\mathbf{v}_G}{dt} = \frac{d}{dt} \sum_i m_i \mathbf{v}_i,$$

and consequently (6.5) becomes

$$M \frac{d\mathbf{v}_G}{dt} \equiv M \frac{d^2 \mathbf{r}_G}{dt^2} = \sum \mathbf{F}_i, \quad (6.10)$$

i.e. *the motion of the centre of mass of the system obeys the same law of motion as that for a single particle having a mass equal to the total mass of the system and acted upon by a set of forces equal and parallel to the external forces applied to the system.*

(2) *Motion relative to the Centre of Mass.* Let us now consider the motion of the system relative to its centre of mass. Suppose we write

$$\mathbf{r}_i = \mathbf{r}_G + \mathbf{r}'_i, \quad (6.11)$$

where $\mathbf{r}_G$ represents the vector displacement of the centre of mass with respect to O , and $\mathbf{r}'_i$ is the vector displacement of m_i with respect to the centre of mass. Now the resultant moment of momentum of the system is given by eqn (6.9)

$$\mathbf{H} = \sum_i (\mathbf{r}_i \wedge m_i \mathbf{v}_i).$$

* The centre of mass and the centre of gravity of a system are coincident when the gravitational field acting on the system is uniform throughout the region occupied by the system. For systems ordinarily met with, this may always be regarded as being true in the case of the earth's gravitational field. Throughout the book, we shall therefore use the term centre of mass, although both terms are in all cases appropriate.

This may be expressed as the sum of two terms, one representing the moment of momentum of the system relative to the centre of mass, the other the moment of momentum with respect to the origin of a hypothetical particle of mass M moving with the centre of mass. For, on substituting for $\mathbf{r}_i$ from (6.11), we have

$$\begin{aligned}\mathbf{H} &= \sum_i \{(\mathbf{r}_G + \mathbf{r}'_i) \wedge m_i \mathbf{v}_i\} \\ &= \sum_i (\mathbf{r}_G \wedge \sum_i m_i \mathbf{v}_i) + \sum_i (\mathbf{r}'_i \wedge m_i \mathbf{v}_i).\end{aligned}$$

But

$$\sum_i m_i \mathbf{v}_i = M \mathbf{v}_G,$$

so that

$$\mathbf{H} = \mathbf{H}_G + (\mathbf{r}_G \wedge M \mathbf{v}_G), \quad (6.12)$$

where

$$\begin{aligned}\mathbf{H}_G &= \sum_i (\mathbf{r}'_i \wedge m_i \mathbf{v}_i) \\ &= \sum_i \{ \mathbf{r}'_i \wedge m_i (\mathbf{v}_G + \mathbf{v}'_i) \} \\ &= (\sum_i m_i \mathbf{r}'_i) \wedge \mathbf{v}_G + \sum_i (\mathbf{r}'_i \wedge m_i \mathbf{v}'_i) \\ &= \sum_i (\mathbf{r}'_i \wedge m_i \mathbf{v}'_i) \quad (\sum_i m_i \mathbf{r}'_i = \mathbf{0})\end{aligned} \quad (6.13)$$

represents the moment of momentum of the system with respect to the centre of mass. (It should be noticed that the velocity occurring in the expression (6.13) for $\mathbf{H}_G$ is the velocity of m_i relative to the centre of mass, not its actual velocity; thus $\mathbf{H}_G$ may be expressed either by $\sum_i (\mathbf{r}'_i \wedge m_i \mathbf{v}_i)$ or $\sum_i' (\mathbf{r}'_i \wedge m_i \mathbf{v}'_i)$.) On differentiating (6.12), we thus have

$$\begin{aligned}\frac{d\mathbf{H}}{dt} &= \frac{d\mathbf{H}_G}{dt} + M \frac{d}{dt} (\mathbf{r}_G \wedge \mathbf{v}_G) \\ &= \frac{d\mathbf{H}_G}{dt} + M \left(\frac{d\mathbf{r}_G}{dt} \wedge \mathbf{v}_G \right) + M \left(\mathbf{r}_G \wedge \frac{d\mathbf{v}_G}{dt} \right).\end{aligned}$$

But $d\mathbf{r}_G/dt = \mathbf{v}_G$, so that the second term vanishes; we are therefore left with

$$\frac{d\mathbf{H}}{dt} = \frac{d\mathbf{H}_G}{dt} + \left(\mathbf{r}_G \wedge M \frac{d\mathbf{v}_G}{dt} \right),$$

or, by virtue of (6.10),

$$\frac{d\mathbf{H}}{dt} = \frac{d\mathbf{H}_G}{dt} + \mathbf{r}_G \wedge \sum_i \mathbf{F}_i. \quad (6.14)$$

On the other hand, we have from (6.8)

$$\begin{aligned}\frac{d\mathbf{H}}{dt} &= \sum_i (\mathbf{r}_i \wedge \mathbf{F}_i) \\ &= \sum_i \{ (\mathbf{r}_G + \mathbf{r}'_i) \wedge \mathbf{F}_i \} \\ &= \mathbf{r}_G \wedge \sum_i \mathbf{F}_i + \sum_i (\mathbf{r}'_i \wedge \mathbf{F}_i).\end{aligned} \quad (6.15)$$

Therefore, equating (6.14) and (6.15), we have

$$\frac{d\mathbf{H}_G}{dt} + \mathbf{r}_G \wedge \sum_i \mathbf{F}_i = \mathbf{r}_G \wedge \sum_i \mathbf{F}_i + \sum_i (\mathbf{r}'_i \wedge \mathbf{F}_i),$$

$$\text{i.e.} \quad \frac{d\mathbf{H}_G}{dt} = \sum_i (\mathbf{r}'_i \wedge \mathbf{F}_i) = \mathbf{M}_G, \text{ say} \quad (6.16)$$

where $\mathbf{M}_G$ represents the resultant moment of the external forces with respect to the centre of mass. We therefore have the result: *The rate of change of the resultant moment of momentum of the system with respect to the centre of mass is equal to the resultant moment of the external forces acting on the system with respect to the same point.*

It is seen that this result is identical in form with that obtained when moments are taken with respect to a fixed origin, the mathematical expressions for the two results being:

$$\frac{d\mathbf{H}}{dt} = \sum_i (\mathbf{r}_i \wedge \mathbf{F}_i), \quad \mathbf{H} = \sum_i (\mathbf{r}_i \wedge m_i \mathbf{v}_i) \quad \begin{array}{l} \text{(with respect to a fixed} \\ \text{origin),} \end{array}$$

$$\frac{d\mathbf{H}_G}{dt} = \sum_i (\mathbf{r}'_i \wedge \mathbf{F}_i), \quad \mathbf{H}_G = \sum_i (\mathbf{r}'_i \wedge m_i \mathbf{v}'_i) \quad \begin{array}{l} \text{(with respect to the centre} \\ \text{of mass).} \end{array}$$

The above theory shows that the translational motion of the centre of mass and the motion relative to the centre of mass are independent of one another. This fact is important, since its use greatly facilitates the determination of the motion of a system consisting of two or more particles; in particular, it is important in considering the motion of a rigid body, as we shall see in § 2. The simplification introduced is in part due to the fact that, in applying the second result, we are referring the moments to a point moving with the system instead of to a point fixed in space.

We shall now investigate whether the above result, that the rate of change of the moment of momentum of a system with respect to a point is equal to the moment of the external forces with respect to the same point, holds when moments are taken about a moving point other than the centre of mass.

1.2. Rate of change of the moment of momentum with respect to a moving origin. Let O be a moving origin and let its velocity be $\mathbf{v}_0$. Then the velocity in space of the particle, whose vector displacement with respect to O is $\mathbf{r}$, is

$$\mathbf{v} = \mathbf{v}_0 + \frac{d\mathbf{r}}{dt}. \quad (6.17)$$

If $\mathbf{H}$ is the resultant moment of momentum of the system with respect to O , then

$$\mathbf{H} = \sum_i (\mathbf{r}_i \wedge m_i \mathbf{v}_i).$$

$$\begin{aligned} \text{Thus } \frac{d\mathbf{H}}{dt} &= \sum_i \left(\frac{d\mathbf{r}_i}{dt} \wedge m_i \mathbf{v}_i \right) + \sum_i \left(\mathbf{r}_i \wedge m_i \frac{d\mathbf{v}_i}{dt} \right) \\ &= \sum_i \{ (\mathbf{v}_i - \mathbf{v}_0) \wedge m_i \mathbf{v}_i \} + \sum_i \left(\mathbf{r}_i \wedge m_i \frac{d\mathbf{v}_i}{dt} \right) \end{aligned}$$

on substituting for $d\mathbf{r}_i/dt$ from (6.17),

$$\begin{aligned} &= -\sum_i (\mathbf{v}_0 \wedge m_i \mathbf{v}_i) + \sum_i \left(\mathbf{r}_i \wedge m_i \frac{d\mathbf{v}_i}{dt} \right) \\ &= -(\mathbf{v}_0 \wedge \sum_i m_i \mathbf{v}_i) + \sum_i \left(\mathbf{r}_i \wedge m_i \frac{d\mathbf{v}_i}{dt} \right). \end{aligned} \quad (6.18)$$

$$\begin{aligned} \text{But } \sum_i \left(\mathbf{r}_i \wedge m_i \frac{d\mathbf{v}_i}{dt} \right) &= \sum_i \{ \mathbf{r}_i \wedge (\mathbf{F}_i + \sum_k' \mathbf{F}_{ik}) \} \\ &= \sum_i (\mathbf{r}_i \wedge \mathbf{F}_i) \end{aligned}$$

in virtue of (6.7). Hence, writing

$$\mathbf{p} = \sum_i m_i \mathbf{v}_i = M \mathbf{v}_G \quad (6.19)$$

for the total linear momentum of the system, (6.18) reduces to

$$\frac{d\mathbf{H}}{dt} + \mathbf{v}_0 \wedge \mathbf{p} = \sum_i (\mathbf{r}_i \wedge \mathbf{F}_i). \quad (6.20)$$

This represents the general result for the rate of change of the moment of momentum with respect to a moving origin. If the moving origin corresponds to the centre of mass, then it follows immediately from (6.19) that

$$\mathbf{v}_0 \wedge \mathbf{p} = \mathbf{v}_G \wedge M \mathbf{v}_G = \mathbf{0},$$

$$\text{so that we have } \frac{d\mathbf{H}}{dt} = \frac{d\mathbf{H}_G}{dt} = \sum_i (\mathbf{r}_i \wedge \mathbf{F}_i),$$

in agreement with the result obtained above (eqn 6.16). In addition, $\mathbf{v}_0 \wedge \mathbf{p}$ will also vanish if the velocity of O is such that $\mathbf{v}_0 = k \mathbf{v}_G$ where k is a scalar coefficient, i.e. if the moving origin, while not being coincident with the centre of mass, yet has a velocity which is parallel to that of the centre of mass.

Summing up, we thus see that $\mathbf{v}_0 \wedge \mathbf{p}$ vanishes, and that the rate of change of the moment of momentum of the system with respect to a

point may be equated to the moment of the external forces with respect to the same point, if:

- (i) $\mathbf{v}_0 = \mathbf{0}$, so that the point is fixed in space;
- (ii) $\mathbf{v}_0 = k\mathbf{v}_G$, so that either
 - (a) the point coincides with the centre of mass ($k = 1$), or
 - (b) the point is not coincident with the centre of mass, but has a velocity parallel to that of the centre of mass ($k \neq 1$).

1.3. Equation of energy. The equation of motion of the i th particle of the system is given by

$$m_i \frac{d^2 \mathbf{r}_i}{dt^2} = \mathbf{F}_i + \sum_k' \mathbf{F}_{ik}.$$

Multiplying this equation scalarly by $d\mathbf{r}_i/dt$, then we have

$$m_i \frac{d^2 \mathbf{r}_i}{dt^2} \cdot \frac{d\mathbf{r}_i}{dt} = (\mathbf{F}_i + \sum_k' \mathbf{F}_{ik}) \cdot \frac{d\mathbf{r}_i}{dt},$$

and hence, on summing over all the particles of the system,

$$\sum_i m_i \left(\frac{d^2 \mathbf{r}_i}{dt^2} \cdot \frac{d\mathbf{r}_i}{dt} \right) = \sum_i \left\{ (\mathbf{F}_i + \sum_k' \mathbf{F}_{ik}) \cdot \frac{d\mathbf{r}_i}{dt} \right\}.$$

Integrating this equation with respect to t , we thus have

$$\begin{aligned} \frac{1}{2} \sum_i m_i \left(\frac{d\mathbf{r}_i}{dt} \cdot \frac{d\mathbf{r}_i}{dt} \right) &= \int \left\{ \sum_i (\mathbf{F}_i + \sum_k' \mathbf{F}_{ik}) \cdot \frac{d\mathbf{r}_i}{dt} \right\} dt + \text{constant} \\ &= \sum_i \int \{ (\mathbf{F}_i + \sum_k' \mathbf{F}_{ik}) \cdot d\mathbf{r}_i \} + \text{constant}. \end{aligned}$$

The left-hand side of this equation is simply the total kinetic energy of the system. Hence, writing

$$T = \frac{1}{2} \sum_i m_i \left(\frac{d\mathbf{r}_i}{dt} \cdot \frac{d\mathbf{r}_i}{dt} \right),$$

then, on integrating over an interval of time (t_0, t_1) , we have the result

$$\begin{aligned} T_1 - T_0 &= \sum_i \int_{t_0}^{t_1} \{ (\mathbf{F}_i + \sum_k' \mathbf{F}_{ik}) \cdot d\mathbf{r}_i \} \\ &= \sum_i \int_{t_0}^{t_1} (\mathbf{F}_i \cdot d\mathbf{r}_i) + \sum_i \int_{t_0}^{t_1} (\sum_k' \mathbf{F}_{ik}) \cdot d\mathbf{r}_i, \end{aligned} \quad (6.21)$$

or, expressed in words: *the change in the kinetic energy of the system in any interval of time is equal to the total work done on the system by both the internal and external forces acting on the system during the same interval of time.*

In the event of the displacement of the system during the interval being such that the mutual distances between the respective particles remain unchanged, then the work done by the internal forces will be zero, and in this case (6.21) reduces to

$$T_1 - T_0 = \sum_i \int_0^1 (\mathbf{F}_i \cdot d\mathbf{r}_i),$$

so that the increase in the kinetic energy in the interval is now equal to the work done on the system by the external forces alone. If, in addition, the forces $\mathbf{F}_i$ can be derived from potential energy functions V_i such that

$$\mathbf{F}_i = -\nabla V_i,$$

where

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z},$$

($\mathbf{i}, \mathbf{j}, \mathbf{k}$) having their usual meanings, then we shall have

$$\begin{aligned} T_1 - T_0 &= - \int_0^1 \sum_i (\nabla V_i \cdot d\mathbf{r}_i) \\ &= - \int_0^1 \sum_i \left(\frac{\partial V_i}{\partial x} dx + \frac{\partial V_i}{\partial y} dy + \frac{\partial V_i}{\partial z} dz \right) \\ &= \sum_i (V_{i0} - V_{i1}). \end{aligned}$$

Writing $V = \sum_i V_i$, we thus have

$$T_1 - T_0 = V_0 - V_1,$$

i.e.

$$T_1 + V_1 = T_0 + V_0 = \text{constant}, \quad (6.22)$$

a result which expresses the law of conservation of energy for the system.

1.4. Expression for the kinetic energy. The kinetic energy of a system of particles may be conveniently expressed as the sum of two terms, one involving the motion of the centre of mass of the system, the other the motion relative to the centre of mass.

To obtain the expression, we have,

$$\begin{aligned} 2T &= \sum_i m_i (\mathbf{v}_i \cdot \mathbf{v}_i) \\ &= \sum_i m_i \{ (\mathbf{v}_G + \mathbf{v}'_i) \cdot (\mathbf{v}_G + \mathbf{v}'_i) \}, \end{aligned}$$

where $\mathbf{v}'_i (= d\mathbf{r}'_i/dt)$ is the velocity of the i th particle relative to the centre of mass,

$$\begin{aligned} &= \sum_i m_i \{(\mathbf{v}_G \cdot \mathbf{v}_G) + 2(\mathbf{v}_G \cdot \mathbf{v}'_i) + (\mathbf{v}'_i \cdot \mathbf{v}'_i)\} \\ &= (\mathbf{v}_G \cdot \mathbf{v}_G) \sum_i m_i + 2\mathbf{v}_G \cdot (\sum_i m_i \mathbf{v}'_i) + \sum_i m_i (\mathbf{v}'_i \cdot \mathbf{v}'_i) \\ &= Mv_G^2 + 2\mathbf{v}_G \cdot (\sum_i m_i \mathbf{v}'_i) + \sum_i m_i v_i'^2. \end{aligned}$$

But
$$\sum_i m_i \mathbf{v}'_i = \sum_i m_i \frac{d\mathbf{r}'_i}{dt} = \frac{d}{dt} (\sum_i m_i \mathbf{r}'_i) = \mathbf{0}.$$

Therefore we have

$$T = \frac{1}{2} M v_G^2 + \frac{1}{2} \sum_i m_i v_i'^2, \quad (6.23)$$

or, expressed in words: *the kinetic energy of a system of particles is equal to the kinetic energy of a hypothetical particle, supposed moving with the centre of mass of the system and having a mass equal to the total mass of the system, together with the kinetic energy of the system in its motion relative to the centre of mass.*

2. Motion of a rigid body

Since a rigid body in reality consists of a system of mutually interacting particles, it should be possible to use the above results to determine its motion.

In order to apply them, suppose we look at the position in a little more detail. Any substance is, of course, made up of a large number of molecules, each molecule consisting of one or more atoms; the atoms themselves consist of electrons in motion around hard, compact nuclei composed of neutrons and protons, each nucleus having its own particular number of satellite electrons. A rigid body is characterized by the fact that, no matter what the forces acting upon it, the distances between the centres of mass of the molecules which compose the body, together with their relative positions, remain unchanged.

Having described, in brief, the physical structure of a rigid body, let us now consider the application of the above results to the motion of such a body. The simplest case involving the motion of a rigid body is that in which the body has one point fixed. Its motion at any instant is defined, as we know (Chap. III, § 3), by an angular velocity $\boldsymbol{\omega}$ about an instantaneous axis passing through the fixed point, and to determine the motion we have thus to determine $\boldsymbol{\omega}$ as a function of t . Since only one vector unknown is involved, one vector equation in $\boldsymbol{\omega}$ should suffice for the determination of the motion. To obtain

this equation, it is natural, in view of the nature of the motion, to start with the moment of momentum equation of § 1 above, viz.

$$\frac{d\mathbf{H}}{dt} = \sum_i (\mathbf{r}_i \wedge \mathbf{F}_i), \quad [(6.8)]$$

where $\mathbf{H}$ is the moment of momentum of a system of interacting particles with respect to the fixed point O (eqn 6.8), and investigate what form this equation assumes when the system is a rigid body.

To derive the form of this equation, let the body consist of n molecules and let m_i be the mass of the i th molecule, $\mathbf{r}_{Gi}$ the vector displacement of its centre of mass from O , m_{si} the mass of one of the constituent particles of the molecule, $\mathbf{r}_{si}$ its vector displacement with respect to O , and $\mathbf{r}'_{si}$ its vector displacement with respect to the centre of mass of the molecule. Also let $\mathbf{F}_{si}$ represent an *external* force (i.e. a force from outside the system) acting on m_{si} . Then, provided that all the internal forces within the body are assumed to be in the nature of central forces, (6.8) above, when applied to such a system, becomes

$$\frac{d\mathbf{H}}{dt} = \sum_i \sum_s (\mathbf{r}_{si} \wedge \mathbf{F}_{si}), \quad (6.24)$$

$$\text{where} \quad \mathbf{H} = \sum_i \sum_s (\mathbf{r}_{si} \wedge m_{si} \mathbf{v}_{si}), \quad \mathbf{v}_{si} = \frac{d\mathbf{r}_{si}}{dt}. \quad (6.25)$$

But

$$\mathbf{r}_{si} = \mathbf{r}_{Gi} + \mathbf{r}'_{si}.$$

Hence

$$\begin{aligned} \sum_i \sum_s (\mathbf{r}_{si} \wedge \mathbf{F}_{si}) &= \sum_i \sum_s \{(\mathbf{r}_{Gi} + \mathbf{r}'_{si}) \wedge \mathbf{F}_{si}\} \\ &= \sum_i \sum_s (\mathbf{r}_{Gi} \wedge \mathbf{F}_{si}) + \sum_i \sum_s (\mathbf{r}'_{si} \wedge \mathbf{F}_{si}) \\ &= \sum_i (\mathbf{r}_{Gi} \wedge \sum_s \mathbf{F}_{si}) + \sum_i \sum_s (\mathbf{r}'_{si} \wedge \mathbf{F}_{si}) \\ &= \sum_i (\mathbf{r}_{Gi} \wedge \mathbf{F}_i) + \sum_i \sum_s (\mathbf{r}'_{si} \wedge \mathbf{F}_{si}), \end{aligned} \quad (6.26)$$

where $\mathbf{F}_i = \sum_s \mathbf{F}_{si}$ is the total external force acting on the i th molecule.*

Also

$$\begin{aligned} \mathbf{H} &= \sum_i \sum_s (\mathbf{r}_{si} \wedge m_{si} \mathbf{v}_{si}) \\ &= \sum_i \sum_s \{(\mathbf{r}_{Gi} + \mathbf{r}'_{si}) \wedge m_{si} (\mathbf{v}'_{si} + \mathbf{v}_{Gi})\} \quad \left(\mathbf{v}'_{si} = \frac{d\mathbf{r}'_{si}}{dt}, \mathbf{v}_{Gi} = \frac{d\mathbf{r}_{Gi}}{dt} \right) \end{aligned}$$

* It should be realized that $\mathbf{F}_i$ does not include the interactions between the i th molecule and the other molecules of the system, since these forces do not enter into (6.24).

$$\begin{aligned}
& = \sum_i \sum_s (\mathbf{r}_{Gi} \wedge m_{st} \mathbf{v}'_{st}) + \sum_i \sum_s (\mathbf{r}_{Gi} \wedge m_{st} \mathbf{v}_{Gi}) \\
& \quad + \sum_i \sum_s (\mathbf{r}'_{si} \wedge m_{st} \mathbf{v}'_{st}) + \sum_i \sum_s (\mathbf{r}'_{si} \wedge m_{st} \mathbf{v}_{Gi}) \\
& = \sum_i \{ \mathbf{r}_{Gi} \wedge (\sum_s m_{st} \mathbf{v}'_{st}) \} + \sum_i \{ \mathbf{r}_{Gi} \wedge (\sum_s m_{st}) \mathbf{v}_{Gi} \} \\
& \quad + \sum_i \sum_s (\mathbf{r}'_{si} \wedge m_{st} \mathbf{v}'_{st}) + \sum_i \{ (\sum_s m_{st} \mathbf{r}'_{si}) \wedge \mathbf{v}_{Gi} \} \\
& = \sum_i (\mathbf{r}_{Gi} \wedge m_i \mathbf{v}_{Gi}) + \sum_i \sum_s (\mathbf{r}'_{si} \wedge m_{st} \mathbf{v}'_{st})
\end{aligned}$$

since $\sum_s m_{st} \mathbf{r}'_{si} = \mathbf{0}, \quad \sum_s m_{st} \mathbf{v}'_{st} = \frac{d}{dt} (\sum_s m_{st} \mathbf{r}'_{si}) = \mathbf{0},$

i.e. $\mathbf{H} = \sum_i \mathbf{H}_{Gi} + \sum_i \mathbf{H}'_{Gi},$ (6.27)

where $\mathbf{H}_{Gi} = \mathbf{r}_{Gi} \wedge m_i \mathbf{v}_{Gi}, \quad \mathbf{H}'_{Gi} = \sum_s (\mathbf{r}'_{si} \wedge m_{st} \mathbf{v}'_{st}).$ (6.28)

Substituting for $\sum_i \sum_s (\mathbf{r}_{st} \wedge \mathbf{F}_{st})$ and $\mathbf{H}$ from (6.26) and (6.27), into (6.24), the moment of momentum equation thus becomes

$$\frac{d}{dt} (\sum_i \mathbf{H}_{Gi}) + \frac{d}{dt} (\sum_i \mathbf{H}'_{Gi}) = \sum_i (\mathbf{r}_{Gi} \wedge \mathbf{F}_i) + \sum_i \sum_s (\mathbf{r}'_{si} \wedge \mathbf{F}_{st}). \quad (6.29)$$

This equation, however, can be simplified if we assume that the resultant forces between the molecules to be such that they act through the centres of mass of the respective molecules. For, $\mathbf{H}'_{Gi}$ being the moment of momentum of the i th molecule with respect to its centre of mass, we have, on applying the result for a system of particles given by (6.16) to this molecule,

$$\frac{d\mathbf{H}'_{Gi}}{dt} = \sum_s (\mathbf{r}'_{si} \wedge \mathbf{F}_{st}),$$

the only forces, apart from $\mathbf{F}_{st}$, acting on the molecule being either the internal forces within the molecule, which will make no contribution to this equation, or the molecular interaction forces which also do not enter in view of the above assumption. Thus (6.29) reduces to

$$\frac{d}{dt} (\sum_i \mathbf{H}_{Gi}) = \sum_i (\mathbf{r}_{Gi} \wedge \mathbf{F}_i),$$

or $\frac{d\mathbf{H}}{dt} = \sum_i (\mathbf{r}_{Gi} \wedge \mathbf{F}_i)$ (6.30)

if we write $\mathbf{H} = \sum_i \mathbf{H}_{Gi} = \sum_i \mathbf{r}_{Gi} \wedge m_i \mathbf{v}_{Gi}.$ (6.31)

Equation (6.30) has exactly the same form as (6.8). Consequently we have the result that, provided the intermolecular forces are such that they act through the centres of mass of the respective molecules,

then the result given by (6.8) for a system of mutually interacting particles can be applied to a system of mutually interacting molecules, although the molecules themselves consist of systems of interacting particles; in applying the equation, each molecule is to be regarded as being replaced by a hypothetical particle having a mass equal to that of the molecule and supposed situated at its centre of mass. We shall call such a particle a *molecular particle*. For convenience, we shall drop the suffixes and replace $\mathbf{r}_{Gi}$, $\mathbf{v}_{Gi}$, and $\mathbf{F}_i$ by $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{F}$ respectively. Then (6.30) and (6.31) become

$$\frac{d\mathbf{H}}{dt} = \sum (\mathbf{r} \wedge \mathbf{F}), \quad \mathbf{H} = \sum (\mathbf{r} \wedge m\mathbf{v}), \quad (6.32)$$

m denoting the mass of a molecular particle whose vector displacement from O is $\mathbf{r}$.*

Having obtained this result, we can now apply (6.32) to the case in which the system is a rigid body rotating about a fixed point. For in this case we can write, from our knowledge of the motion of such a body,

$$\mathbf{v} = \boldsymbol{\omega} \wedge \mathbf{r},$$

and hence

$$\begin{aligned} \mathbf{H} &= \sum \{\mathbf{r} \wedge m(\boldsymbol{\omega} \wedge \mathbf{r})\} \\ &= \sum m\{\mathbf{r} \wedge (\boldsymbol{\omega} \wedge \mathbf{r})\} \\ &= \sum m\{\boldsymbol{\omega}(\mathbf{r} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega})\} \\ &= \boldsymbol{\omega} \sum m r^2 - \sum m \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}). \end{aligned}$$

The scalar components (H_x, H_y, H_z) of $\mathbf{H}$ respectively parallel to a set of coordinate axes $OXYZ$ through O , and which are termed *the moments of momentum of the body with respect to the coordinate axes*, are thus given by

$$\begin{aligned} H_x &= \mathbf{i} \cdot \mathbf{H} = (\mathbf{i} \cdot \boldsymbol{\omega}) \sum m r^2 - \sum m (\mathbf{i} \cdot \mathbf{r})(\mathbf{r} \cdot \boldsymbol{\omega}) \\ &= \omega_x \sum m (x^2 + y^2 + z^2) - \sum m x (x\omega_x + y\omega_y + z\omega_z) \\ &= \omega_x \sum m (y^2 + z^2) - \omega_y \sum m xy - \omega_z \sum m xz \\ &= A\omega_x - H\omega_y - G\omega_z, \end{aligned}$$

with similarly

$$\begin{aligned} H_y &= \mathbf{j} \cdot \mathbf{H} = -H\omega_x + B\omega_y - F\omega_z, \\ H_z &= \mathbf{k} \cdot \mathbf{H} = -G\omega_x - F\omega_y + C\omega_z. \end{aligned}$$

* The above assumption that the intermolecular forces are in the nature of central forces is, of course, an over-simplified picture, since not all intermolecular forces are of this type. To consider the question more generally, however, would be a much more difficult problem. Consequently, it is necessary to assume that, although the intermolecular forces do not in all cases act along the lines joining their centres of mass, nevertheless (6.32) is generally true. The same assumption must also be made, of course, when considering the motion of a rigid body having no point fixed (eqns (6.34) and (6.35) below).

We thus have

$$\begin{aligned}\mathbf{H} &\equiv (H_x, H_y, H_z) \\ &= (A\omega_x - H\omega_y - G\omega_z, -H\omega_x + B\omega_y - F\omega_z, \\ &\quad -G\omega_x - F\omega_y + C\omega_z). \quad (6.33)\end{aligned}$$

On expressing (6.32) in terms of its equivalent scalar equations, we shall thus have three equations to determine $(\omega_x, \omega_y, \omega_z)$ and hence $\boldsymbol{\omega}$.

In the case of a rigid body which has no point fixed, then its motion will be known if we know its angular velocity $\boldsymbol{\omega}$ together with the velocity $\mathbf{v}_O$ of some point O of the body which is taken as base point (Chap. III, § 3). In this case, therefore, the determination of the motion involves the determination of two vectors unknown, viz. $\boldsymbol{\omega}$ and $\mathbf{v}_O$, as functions of t , so that two differential equations of motion involving these quantities are required. These may be obtained by the application of the equation of linear momentum (eqn 6.10),

$$M \frac{d\mathbf{v}_G}{dt} = \sum \mathbf{F}, \quad [(6.10)]$$

and that of moment of momentum (eqn 6.20),

$$\frac{d\mathbf{H}}{dt} + \mathbf{v}_O \wedge \mathbf{P} = \sum (\mathbf{r} \wedge \mathbf{F}), \quad [(6.20)]$$

where $\mathbf{v}_O$ is the velocity of the base point. In practice, it is convenient to take G , the centre of mass of the system, as base point, in which case (6.20) becomes, on putting $\mathbf{v}_O = \mathbf{v}_G$,

$$\frac{d\mathbf{H}_G}{dt} = \sum (\mathbf{r}' \wedge \mathbf{F}), \quad \mathbf{H}_G = \sum (\mathbf{r}' \wedge m\mathbf{v}'), \quad [(6.16)]$$

where $\mathbf{r}'$ is the vector displacement of the particle of mass m with respect to G and $\mathbf{v}'$ is its velocity relative to G . If we apply these equations to a system of mutually interacting molecules, then, by reasoning similar to the above, it can be shown that the above equations can be applied to such systems, the individual particles being replaced by molecular particles. The above equations are thus immediately applicable to the motion of a rigid body.

Since for a rigid body

$$\mathbf{v}' = \boldsymbol{\omega} \wedge \mathbf{r}',$$

we have, in this case,

$$\mathbf{H}_G = \sum m \{ \mathbf{r}' \wedge (\boldsymbol{\omega} \wedge \mathbf{r}') \}.$$

Comparing this expression with that for $\mathbf{H}$ (p. 131), it follows that the expressions for the scalar components of $\mathbf{H}_G$ will have the same form as those for (H_x, H_y, H_z) above (eqn 6.33) except that the A, B, C, F, G, H occurring in them are now the moments of inertia and products of

inertia with respect to a set of rectangular axes and planes through the centre of mass, the directions of the axes in space being fixed. When (6.10) and (6.16) are expressed in terms of their scalar equivalents, we thus have six equations to determine the six components of $\mathbf{v}_G$ and $\boldsymbol{\omega}$.

To summarize the results obtained above, we see that the equations which determine the motion of a rigid body are thus :

(1) *Rigid body rotating about a fixed point:*

$$\frac{d\mathbf{H}}{dt} = \sum (\mathbf{r} \wedge \mathbf{F}), \quad [(6.32)]$$

where $\mathbf{H}$ is the moment of momentum of the system with respect to O ; the components of $\mathbf{H}$ with respect to a set of fixed rectangular axes, having their origin at O , are given by

$$(H_x, H_y, H_z) = (A\omega_x - H\omega_y - G\omega_z, -H\omega_x + B\omega_y - F\omega_z, -G\omega_x - F\omega_y + C\omega_z), \quad [(6.33)]$$

A, B, C, F, G, H being the moments and products of inertia with respect to the coordinate axes and planes.

(2) *General motion of a rigid body:*

$$M \frac{d\mathbf{v}_G}{dt} = \sum \mathbf{F}, \quad (6.34)$$

$$\frac{d\mathbf{H}_G}{dt} = \sum (\mathbf{r}' \wedge \mathbf{F}), \quad (6.35)$$

where $\mathbf{H}_G$ is the moment of momentum of the system with respect to G , its centre of mass; the expressions for the components of $\mathbf{H}_G$ referred to rectangular axes through G , respectively parallel to a set of fixed axes, are similar to those of (6.33) except that A, B, C, F, G, H are now the moments and products of inertia with respect to a set of parallel axes and planes through G .

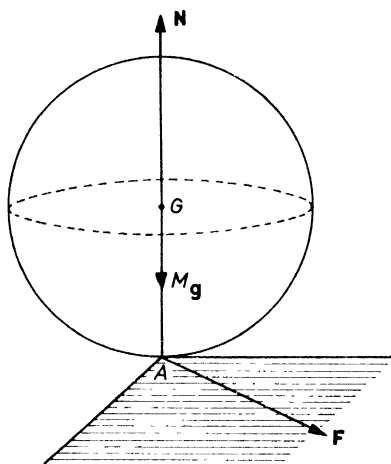
In the above equations, the rates of change of the vector quantities occurring in them always refer to rates of change with respect to a set of axes whose directions in space are fixed. This means, since the body is moving relative to these axes, that the inertia coefficients A, B, C, F, G, H will not in general be constants. This fact severely restricts the use of the above equations. They may, however, be used to determine the motion of a spherical body which either has its centre fixed, or is moving freely, and whose mass distribution is symmetrical with respect to its centre, since in this case the moments of inertia with respect to all axes through its centre are the same and the products of inertia vanish.

2.1. Examples involving motion of a sphere

(1) *Motion of a Billiard Ball.* Consider a billiard ball in motion on a rough table. We shall assume that the conditions at the commencement of the motion are such that slipping occurs initially. Then, as long as this condition lasts, the equations of motion are, applying (6.34) and (6.35) above,

$$M \frac{d\mathbf{v}}{dt} = \mathbf{N} + M\mathbf{g} + \mathbf{F}, \quad (1)$$

$$\frac{d\mathbf{H}_G}{dt} = A \frac{d\boldsymbol{\omega}}{dt} = -a(\mathbf{n} \wedge \mathbf{F}), \quad (2)$$



where A is the moment of inertia of the ball with respect to a diameter, $\mathbf{v}$ is the velocity of its centre G , $\boldsymbol{\omega}$ is its angular velocity, $\mathbf{N}$ is the normal reaction at the point of contact A , $\mathbf{F}$ is the force of friction, and $\mathbf{n}$ is a unit vector in the direction AG , the normal to the plane at the point of contact.

Since there is no motion perpendicular to the plane

$$\mathbf{N} + M\mathbf{g} = \mathbf{0},$$

and hence (1) reduces to

$$M \frac{d\mathbf{v}}{dt} = \mathbf{F}, \quad (3)$$

or, since we can write

$$\mathbf{F} = -\mu Mgs$$

where $\mathbf{s}$ is a unit vector in the direction in which slipping occurs,

$$\frac{d\mathbf{v}}{dt} = -\mu g \mathbf{s}. \quad (4)$$

We have now written down all the equations that are open to us: they must therefore suffice for a discussion of the motion.

To integrate the above equations, we first eliminate $\mathbf{F}$ from (2) and (3); this gives

$$A \frac{d\boldsymbol{\omega}}{dt} + Ma \left(\mathbf{n} \wedge \frac{d\mathbf{v}}{dt} \right) = \mathbf{0}^*,$$

or, writing $A = Mk^2$,

$$k^2 \frac{d\boldsymbol{\omega}}{dt} + a \left(\mathbf{n} \wedge \frac{d\mathbf{v}}{dt} \right) = \mathbf{0}. \quad (5)$$

We now introduce a new variable, namely, the velocity $\mathbf{v}_A$ of the point A of the ball. This is given by

$$\mathbf{v}_A = \mathbf{v} + \boldsymbol{\omega} \wedge -a\mathbf{n} = \mathbf{v} + a(\mathbf{n} \wedge \boldsymbol{\omega}).$$

If we differentiate this equation with respect to t , then we have

$$\frac{d\mathbf{v}_A}{dt} = \frac{d\mathbf{v}}{dt} + a \left(\mathbf{n} \wedge \frac{d\boldsymbol{\omega}}{dt} \right),$$

while from (5), on multiplying through vectorially by $\mathbf{n}$,

$$\begin{aligned} k^2 \left(\mathbf{n} \wedge \frac{d\boldsymbol{\omega}}{dt} \right) &= -a \left\{ \mathbf{n} \wedge \left(\mathbf{n} \wedge \frac{d\mathbf{v}}{dt} \right) \right\} \\ &= -a \mathbf{n} \left(\mathbf{n} \cdot \frac{d\mathbf{v}}{dt} \right) + a \frac{d\mathbf{v}}{dt} \\ &= a \frac{d\mathbf{v}}{dt}, \end{aligned}$$

$\mathbf{n} \cdot d\mathbf{v}/dt$ being zero since the motion of the centre of the ball is parallel to the table and so at right angles to $\mathbf{n}$. Therefore we have

$$\begin{aligned} \frac{d\mathbf{v}_A}{dt} &= \frac{d\mathbf{v}}{dt} + \frac{a^2}{k^2} \frac{d\mathbf{v}}{dt} = \frac{(k^2 + a^2)}{k^2} \frac{d\mathbf{v}}{dt} \\ &= -\frac{(k^2 + a^2)}{k^2} \mu g \mathbf{s} \end{aligned} \quad (6)$$

* This result could also have been derived directly by taking moments about A , the point of contact, using (6.20), § 1.2. Since the motion of A , regarded as a point of the plane, is equal and parallel to that of G , the term $\mathbf{v}_0 \wedge \mathbf{p} = \mathbf{v}_A \wedge M\mathbf{v}_G$ vanishes, so that

$$\frac{d\mathbf{H}_A}{dt} = \sum (\mathbf{r}_i \wedge \mathbf{F}_i) = a\mathbf{n} \wedge M\dot{\mathbf{g}};$$

also, applying the result given by (6.12) for a system of interacting particles,

$$\mathbf{H}_A = \mathbf{H}_G + \mathbf{r}_G \wedge M\mathbf{v}_G = A\boldsymbol{\omega} + a\mathbf{n} \wedge M\dot{\mathbf{g}}.$$

Substituting this expression for $\mathbf{H}_A$ into the above expression, the result follows.

on substituting for $d\mathbf{v}/dt$ from (4). But, in view of the definition of $\mathbf{s}$, we can write

$$\mathbf{v}_A = v_A \mathbf{s},$$

so that

$$\frac{d\mathbf{v}_A}{dt} = v_A \frac{d\mathbf{s}}{dt} + \frac{dv_A}{dt} \mathbf{s}.$$

Hence, substituting for $d\mathbf{v}_A/dt$ into (6), we have

$$v_A \frac{d\mathbf{s}}{dt} = - \left\{ \frac{dv_A}{dt} + \frac{(k^2 + a^2)}{k^2} \mu g \right\} \mathbf{s}, \quad (7)$$

which, on multiplying both sides scalarly by $\mathbf{s}$, gives

$$v_A \mathbf{s} \cdot \frac{d\mathbf{s}}{dt} = - \left\{ \frac{dv_A}{dt} + \frac{(k^2 + a^2)}{k^2} \mu g \right\}, \quad \mathbf{s} \cdot \mathbf{s} = 1.$$

But $\mathbf{s} \cdot d\mathbf{s}/dt = 0$ since $\mathbf{s}$ is a unit vector; so that

$$\frac{dv_A}{dt} + \frac{(k^2 + a^2)}{k^2} \mu g = 0. \quad (8)$$

Hence, from (7), we have

$$\frac{d\mathbf{s}}{dt} = \mathbf{0},$$

i.e. $\mathbf{s}$ must be a constant vector and so must have a constant direction in space. We thus have the result that the direction of slipping is constant throughout the motion. Its direction will therefore be defined completely by its initial direction, that is, by the initial direction of $\mathbf{v}_A$. In addition, it follows from (8) that dv_A/dt is negative, so that v_A is a decreasing quantity. Slipping will therefore continue until $v_A = 0$. But, integrating (8), we have

$$\begin{aligned} v_A &= -\mu g \left(\frac{k^2 + a^2}{k^2} \right) t + \text{constant} \\ &= v_{A0} - \mu g \left(\frac{k^2 + a^2}{k^2} \right) t, \end{aligned}$$

where v_{A0} is the initial value of v_A . Thus slipping continues for a time

$$t_s = \frac{k^2 v_{A0}}{(k^2 + a^2) \mu g}.$$

Having deduced the result that $\mathbf{s}$ is a constant vector, the motion and the path of the centre of the sphere may now be determined.

From (4), we have on integrating*

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\mu g \mathbf{s} t + \mathbf{C}_1, \quad (9)$$

* $\mathbf{r}$ is the horizontal vector displacement of G with respect to a fixed axis perpendicular to the plane of the table.

where $\mathbf{C}_1$ is an arbitrary constant vector. This gives $\mathbf{v}$ as a function of t , once $\mathbf{C}_1$ is known. The angular velocity of the ball may now be obtained by substituting the above expression for $\mathbf{v}$ in that for $\boldsymbol{\omega}$ obtained by integrating (5), viz.

$$k^2\boldsymbol{\omega} + a(\mathbf{n} \wedge \mathbf{v}) = \text{a constant vector} = \mathbf{C}.$$

To obtain the path of the ball, we have from (9), after integration,

$$\mathbf{r} = -\frac{1}{2}\mu g s t^2 + \mathbf{C}_1 t + \mathbf{C}_2, \quad (10)$$

say, where $\mathbf{C}_2$ is another arbitrary constant vector. If initially $\mathbf{r} = \mathbf{0}$, $\mathbf{v} = d\mathbf{r}/dt = \mathbf{v}_0$, then we shall have $\mathbf{C}_1 = \mathbf{v}_0$, $\mathbf{C}_2 = \mathbf{0}$, so that (10) becomes

$$\mathbf{r} = -\frac{1}{2}\mu g s t^2 + \mathbf{v}_0 t. \quad (11)$$

This is the equation of a parabola whose axis is in the direction opposite to that of $\mathbf{s}$. (This follows immediately if one compares this equation with that for the path of a particle moving freely under gravity.) Thus, in general, the path of the centre of the ball, as long as slipping takes place, is a parabola. It reduces to a straight line, however, if

(i) $\mathbf{v}_0$ is zero,

(ii) $\mathbf{v}_0$ and $\mathbf{s}$ are parallel; since the direction of $\mathbf{s}$ is determined by the initial conditions according to the equation

$$v_{A0}\mathbf{s} = \mathbf{v}_0 + a(\mathbf{n} \wedge \boldsymbol{\omega}_0),$$

this will be the case if the ball is struck in such a way that $\mathbf{n} \wedge \boldsymbol{\omega}_0$, is parallel to $\mathbf{v}_0$, i.e. if the axis of $\boldsymbol{\omega}_0$ is horizontal and perpendicular to $\mathbf{v}_0$.

Rolling Motion. When slipping ceases, then the motion becomes one of pure rolling. The equations of motion corresponding to this motion are

$$M \frac{d\mathbf{v}}{dt} = \mathbf{F} \quad (12)$$

and

$$Mk^2 \frac{d\boldsymbol{\omega}}{dt} = -a(\mathbf{n} \wedge \mathbf{F}), \quad (13)$$

where $\mathbf{F}$ is the force of friction at the point of contact, though no longer, of course, having its limiting value; in addition we have the kinematical condition for rolling contact, viz.

$$\mathbf{v}_A = \mathbf{v} + a(\mathbf{n} \wedge \boldsymbol{\omega}) = \mathbf{0}. \quad (14)$$

If we eliminate $\mathbf{F}$ from (12) and (13), we have

$$Mk^2 \frac{d\boldsymbol{\omega}}{dt} = -a \left(\mathbf{n} \wedge M \frac{d\mathbf{v}}{dt} \right),$$

or

$$k^2 \frac{d\boldsymbol{\omega}}{dt} = -a \left(\mathbf{n} \wedge \frac{d\mathbf{v}}{dt} \right), \quad (15)$$

which gives, as above, on multiplying both sides vectorially by $\mathbf{n}$

$$k^2 \left(\mathbf{n} \wedge \frac{d\boldsymbol{\omega}}{dt} \right) = a \frac{d\mathbf{v}}{dt}.$$

Also, we have from (14), on differentiating with respect to t ,

$$\frac{d\mathbf{v}}{dt} + a \left(\mathbf{n} \wedge \frac{d\boldsymbol{\omega}}{dt} \right) = \mathbf{0}.$$

Eliminating $d\boldsymbol{\omega}/dt$ from these equations, we thus have

$$\frac{d\mathbf{v}}{dt} = \mathbf{0}.$$

On integrating, this gives the result

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{D}_1, \quad (16)$$

say, so that

$$\mathbf{r} = \mathbf{D}_1 t + \mathbf{D}_2, \quad (17)$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are arbitrary constant vectors.

If in (16) and (17) we choose $t = 0$ to correspond to the instant when pure rolling begins, and if $\mathbf{v} = \mathbf{v}_s$, $\mathbf{r} = \mathbf{r}_s$ at this instant, then we shall have $\mathbf{D}_1 = \mathbf{v}_s$, $\mathbf{D}_2 = \mathbf{r}_s$, so that (16) and (17) become

$$\left. \begin{aligned} \mathbf{v} &= \mathbf{v}_s \\ \mathbf{r} &= \mathbf{r}_s + \mathbf{v}_s t \end{aligned} \right\}. \quad (18)$$

We thus have the result that the velocity of the centre of the ball after rolling commences is constant, both in magnitude and direction, and the path of the ball is a straight line, whose direction in space is that of $\mathbf{v}_s$.

The values of $\mathbf{v}_s$ and $\mathbf{r}_s$ may be obtained from (11); they are

$$\begin{aligned} \mathbf{v}_s &= \left(\frac{d\mathbf{r}}{dt} \right)_{t=t_s} = -\mu g s t_s + \mathbf{v}_0, \\ \mathbf{r}_s &= -\frac{1}{2} \mu g s t_s^2 + \mathbf{v}_0 t_s. \end{aligned}$$

The equation of the path during rolling motion will therefore be given by (eqn 18)

$$(\mathbf{r} - \mathbf{r}_s) \wedge \mathbf{v}_s = \mathbf{0},$$

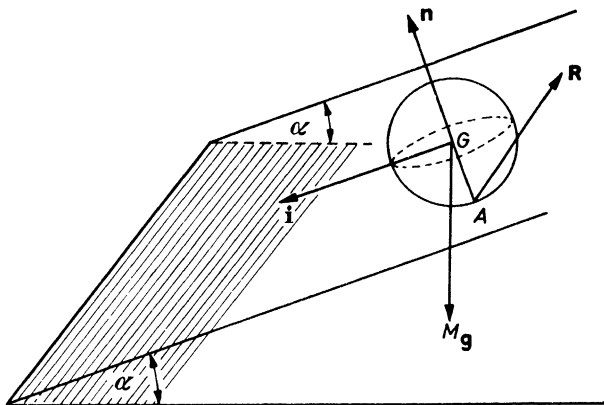
where $\mathbf{v}_s$ and $\mathbf{r}_s$ have the above values.

It should be noticed that, since $d\mathbf{v}/dt = \mathbf{0}$,

$$\mathbf{F} = \mathbf{0}$$

(eqn 12), i.e. no frictional force is necessary for rolling motion to be possible.

(2) *Rolling Motion of a Uniform Sphere on a Fixed Rough Inclined Plane.* Consider a sphere of radius a in rolling motion under gravity on a fixed inclined plane whose inclination to the horizontal is α . Let $\mathbf{v}$ be the velocity of its centre, $\boldsymbol{\omega}$ its angular velocity, and $\mathbf{R}$ the reaction between it and the plane. (In this example, it is not essential to resolve



$\mathbf{R}$ into components at right angles and perpendicular to the plane, as in Example (1), since the motion, being one of pure rolling, the equation of limiting friction, $\mathbf{F} = \mu\mathbf{N}$ is not applicable.) Then the equations of motion are

$$M\frac{d\mathbf{v}}{dt} = \mathbf{R} + M\mathbf{g}, \quad (1)$$

$$A\frac{d\boldsymbol{\omega}}{dt} = -a(\mathbf{n} \wedge \mathbf{R}), \quad (2)$$

where $\mathbf{n}$ is a unit vector perpendicular to the plane, together with the kinematical condition for rolling contact, viz.

$$\mathbf{v} + a(\mathbf{n} \wedge \boldsymbol{\omega}) = \mathbf{0}. \quad (3)$$

We have thus three equations between the three unknowns occurring in the problem, namely $\mathbf{v}$, $\boldsymbol{\omega}$, and $\mathbf{R}$, i.e. we have just sufficient equations for the solution of the problem.

To determine the motion of the sphere, and hence the equation of its path, we have, on eliminating $\mathbf{R}$ from (1) and (2),

$$A\frac{d\boldsymbol{\omega}}{dt} + Ma\left(\mathbf{n} \wedge \frac{d\mathbf{v}}{dt}\right) - Ma(\mathbf{n} \wedge \mathbf{g}) = \mathbf{0}. \quad (4)$$

We have now to eliminate $\boldsymbol{\omega}$ from (3) and (4). Multiplying (4) vectorially by $\mathbf{n}$ gives

$$A\left(\mathbf{n} \wedge \frac{d\boldsymbol{\omega}}{dt}\right) - Ma \frac{d\mathbf{v}}{dt} - Ma\{\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{g})\} = \mathbf{0};$$

also, from (3), we have

$$\frac{d\mathbf{v}}{dt} + a\left(\mathbf{n} \wedge \frac{d\boldsymbol{\omega}}{dt}\right) = \mathbf{0}.$$

Eliminating $d\boldsymbol{\omega}/dt$, we thus have, putting $A = Mk^2$,

$$(k^2 + a^2) \frac{d\mathbf{v}}{dt} + a^2\{\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{g})\} = \mathbf{0}. \quad (5)$$

This can be immediately integrated, the result being

$$(k^2 + a^2)\mathbf{v} + a^2\{\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{g})\}t = \mathbf{c}_1,$$

where $\mathbf{c}_1$ is an arbitrary constant vector. Since we can write $\mathbf{v} = d\mathbf{r}/dt$, where $\mathbf{r}$ is a vector parallel to the plane as in Ex. (1), we have on integrating again

$$(k^2 + a^2)\mathbf{r} + \frac{1}{2}a^2\{\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{g})\}t^2 = \mathbf{c}_1t + \mathbf{c}_2, \quad (6)$$

$\mathbf{c}_2$ being a second arbitrary constant vector. Also, if $\mathbf{i}$ is a unit vector parallel to a line of greatest slope and pointing up the plane, then

$$\mathbf{g} = g(\sin \alpha \mathbf{i} - \cos \alpha \mathbf{n}),$$

and hence

$$\begin{aligned} \mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{g}) &= \mathbf{n}(\mathbf{n} \cdot \mathbf{g}) - \mathbf{g} \\ &= -g \cos \alpha \mathbf{n} - g(\sin \alpha \mathbf{i} - \cos \alpha \mathbf{n}) \\ &= -g \sin \alpha \mathbf{i}. \end{aligned}$$

Consequently (6), which expresses $\mathbf{r}$ as a function of t , becomes

$$(k^2 + a^2)\mathbf{r} = \frac{1}{2}a^2g \sin \alpha t^2 + \mathbf{c}_1t + \mathbf{c}_2.$$

If $\mathbf{r} = \mathbf{0}$ when $t = 0$, then $\mathbf{c}_2 = \mathbf{0}$ and we have

$$\mathbf{r} = \frac{a^2}{k^2 + a^2}(\mathbf{c}_1t + \frac{1}{2}g \sin \alpha t^2).$$

The path is thus a parabola with its axis parallel to a line of greatest slope and pointing down the plane.

(3) *Rolling Motion of a Uniform Sphere on a Rotating Plane.* Both of the above examples involved the rolling motion of a sphere on a fixed plane. Another example involving the rolling motion of a sphere is that in which the sphere rolls in contact with a horizontal plane which is made to rotate with a constant angular velocity about an axis perpendicular to itself.

To consider this problem, let O be the point of intersection of the axis of rotation with the plane, and let $\mathbf{r}$ be the vector displacement of A , the point of contact of the sphere with the plane, from O . (It is also the vector displacement, parallel to the plane, of G , the centre of the sphere, from the axis of rotation.) Then, with the same notation as above, the equations to determine the motion of the sphere are

$$M \frac{d\mathbf{v}}{dt} = \mathbf{R} + M\mathbf{g}, \quad (1)$$

$$A \frac{d\boldsymbol{\omega}}{dt} = -a(\mathbf{n} \wedge \mathbf{R}), \quad (2)$$

together with the kinematical condition for rolling motion, which in this case is

$$\mathbf{v} + a(\mathbf{n} \wedge \boldsymbol{\omega}) = \Omega(\mathbf{n} \wedge \mathbf{r}), \quad (3)$$

if Ω is the angular velocity of rotation of the plane. We have here again three equations involving three vector unknowns, viz. $\mathbf{r}$, $\boldsymbol{\omega}$, and $\mathbf{R}$. The problem should thus be soluble.

If we eliminate $\mathbf{R}$ from (1) and (2), and put $A = Mk^2$, we have

$$k^2 \frac{d\boldsymbol{\omega}}{dt} = -a \left\{ \mathbf{n} \wedge \left(\frac{d\mathbf{v}}{dt} - \mathbf{g} \right) \right\} = -a \left(\mathbf{n} \wedge \frac{d\mathbf{v}}{dt} \right), \quad (\mathbf{n} \wedge \mathbf{g} = \mathbf{0})$$

which on multiplying by $\mathbf{n}$ vectorially gives

$$k^2 \left(\mathbf{n} \wedge \frac{d\boldsymbol{\omega}}{dt} \right) = a \frac{d\mathbf{v}}{dt}.$$

Also, from (3), we have

$$\frac{d\mathbf{v}}{dt} + a \left(\mathbf{n} \wedge \frac{d\boldsymbol{\omega}}{dt} \right) = \Omega(\mathbf{n} \wedge \mathbf{v}).$$

Eliminating $d\boldsymbol{\omega}/dt$ from these equations, we thus have

$$(k^2 + a^2) \frac{d\mathbf{v}}{dt} = k^2 \Omega(\mathbf{n} \wedge \mathbf{v}).$$

This can be immediately integrated giving

$$(k^2 + a^2) \mathbf{v} = k^2 \Omega \{ \mathbf{n} \wedge (\mathbf{r} - \mathbf{c}) \}, \quad (4)$$

where $\mathbf{c}$ is an arbitrary constant vector; or, putting $\mathbf{v} = d\mathbf{r}/dt$,

$$\frac{d\mathbf{r}}{dt} = \frac{k^2 \Omega}{k^2 + a^2} \{ \mathbf{n} \wedge (\mathbf{r} - \mathbf{c}) \}.$$

Comparing the form of this equation with that occurring in Chap. III, § 3.1 (Ex. 2, eqn 10), it is seen that the path of the centre of the sphere

is a circle of radius $|\mathbf{r}-\mathbf{c}|$ having its centre at the point $\mathbf{r} = \mathbf{c}$, and described with a uniform angular speed $k^2\Omega/(k^2+a^2)$.*

Additional problems involving the motion of a sphere, which do not lend themselves to a completely vector solution, are discussed at the end of the chapter.

2.2. Derivation of the principle of energy for a rigid body. In § 1.3 we derived the principle of the conservation of energy for a system of mutually interacting particles. If the internal forces are such that they do no work in any small displacement of the system and if the external forces acting on the system are conservative, and so can be derived from potential energy functions, then we found that throughout the motion of the system

$$T+V = \text{constant},$$

where $V = \sum V_i$ is the total potential energy of the system in the field of the external forces (eqn 6.22).

Let us now consider the same question for a system of mutually interacting molecules. We shall first obtain an expression for the

* To determine the radius of the circle and the position of its centre, one has to evaluate $\mathbf{c}$ in terms of the initial values of $\mathbf{v}$ and $\boldsymbol{\omega}$. If the sphere is at rest and is placed gently on the rotating plane such that the point of contact A is at a vector distance $\mathbf{r}_0$ from O , for instance, then an impulsive force comes into play initially at the point of contact of the sphere with the plane. The determination of the initial values of $\mathbf{v}$ and $\boldsymbol{\omega}$ thus involves the application of the equations of impulsive motion. These are discussed in full in Chap. IX. If $\mathbf{J}$ is the impulse which occurs at A , then, if $\mathbf{v}_0$ and $\boldsymbol{\omega}_0$ are the initial values of $\mathbf{v}$ and $\boldsymbol{\omega}$, the appropriate equations in the present case are

$$M\mathbf{v}_0 = \mathbf{J}, \quad (a) \quad A\boldsymbol{\omega}_0 = -a(\mathbf{n} \wedge \mathbf{J}), \quad (b)$$

together with the equation

$$\mathbf{v}_0 + a(\mathbf{n} \wedge \boldsymbol{\omega}_0) = \Omega(\mathbf{n} \wedge \mathbf{r}_0). \quad (c)$$

Eliminating $\mathbf{J}$ from (a) and (b), we have

$$A\boldsymbol{\omega}_0 = Ma(\mathbf{v}_0 \wedge \mathbf{n}),$$

which, on multiplying by $\mathbf{n}$ vectorially, gives

$$A(\mathbf{n} \wedge \boldsymbol{\omega}_0) = Ma\{\mathbf{n} \wedge (\mathbf{v}_0 \wedge \mathbf{n})\} = Mav_0.$$

Using (c), we therefore have

$$(A + Ma^2)\mathbf{v}_0 = A\Omega(\mathbf{n} \wedge \mathbf{r}_0),$$

i.e. writing $A = Mk^2$,

$$(k^2 + a^2)\mathbf{v}_0 = k^2\Omega(\mathbf{n} \wedge \mathbf{r}_0).$$

Substituting into (4), the value of $\mathbf{c}$ is thus given by the equation

$$\mathbf{n} \wedge \mathbf{r}_0 = \mathbf{n} \wedge (\mathbf{r}_0 - \mathbf{c}),$$

from which it follows that $\mathbf{c}$ must be zero. The centre of the circle thus coincides with the point O ; its radius is therefore equal to r_0 .

kinetic energy. Using the result given by (6.23), we have, with the same notation as above (§ 2),

$$\text{Total kinetic energy of the } i\text{th molecule} = T_i + T'_i,$$

where

$$T_i = \frac{1}{2} m_i (\mathbf{v}_{Gi} \cdot \mathbf{v}_{Gi}), \quad T'_i = \frac{1}{2} \sum_s m_{si} (\mathbf{v}'_{si} \cdot \mathbf{v}'_{si}).$$

Summing over all the molecules, we thus have:

$$\text{Total kinetic energy of system} = T + T',$$

where

$$T = \sum_i T_i = \frac{1}{2} \sum m_i (\mathbf{v}_{Gi} \cdot \mathbf{v}_{Gi}), \quad T' = \sum_i T'_i = \frac{1}{2} \sum_i \sum_s m_{si} (\mathbf{v}'_{si} \cdot \mathbf{v}'_{si}).$$

The former is the kinetic energy of the molecular particles of the system (see p. 131), whilst T' represents the total kinetic energy of the individual particles which together make up the molecules, in their motion relative to the centres of mass of the molecules. We shall call T' the internal kinetic energy of the system. It will, of course, be unaffected by the external forces acting on the system, its value at any instant depending entirely upon the internal forces within the molecules.

Now let us apply the results of § 1.3 to the system. We shall clearly have (eqn 6.21)

$$(T_1 + T'_1) - (T_0 + T'_0) = \text{Work done by all the forces acting on the system, both internal and external.}$$

But the internal forces of the system consist of the intermolecular forces between the molecules together with the internal forces associated with the respective molecules. Also, if the system is a rigid body, then we know that the relative positions of the molecules remain unchanged. Consequently the work done by the intermolecular forces in any displacement of the body will be zero. Further, if V'_i denotes the potential energy associated with the internal forces of the i th molecule, then the work done by such forces for the system as a whole in any interval of time will be given by

$$\sum_i V'_{i0} - \sum_i V'_{i1} = V'_0 - V'_1,$$

say. Hence, for a rigid body, we have the result

$$(T_1 + T'_1) - (T_0 + T'_0) = V'_0 - V'_1 + \sum_i \int \mathbf{F}_i \cdot d\mathbf{r}_{Gi}.$$

where $\mathbf{F}_i$ is the total external force acting on the i th molecule. But, since the motion of the particles within any molecule, relative to its centre of mass, is independent of the external forces acting on the body, then, for the relative motion alone,

$$T'_1 - T'_0 = V'_0 - V'_1.$$

Consequently, the above equation reduces to

$$T_1 - T_0 = \sum_i \int \mathbf{F}_i \cdot d\mathbf{r}_i, \quad (6.36)$$

where, for simplification of the notation, $d\mathbf{r}_i$ has been written in place of $d\mathbf{r}_{Gi}$. If the external forces are conservative, and so can be derived from potential-energy functions V_i , say, then

$$\sum_i \int \mathbf{F}_i \cdot d\mathbf{r}_i = \sum_i V_{i0} - \sum_i V_{i1} = V_0 - V_1,$$

and (6.36) becomes

$$T_1 + V_1 = T_0 + V_0 = \text{constant}, \quad (6.37)$$

which is the energy equation for a rigid body. The kinetic energy defined by the expression

$$T = \frac{1}{2} \sum m_i (\mathbf{v}_{Gi} \cdot \mathbf{v}_{Gi}) = \frac{1}{2} \sum m(\mathbf{v} \cdot \mathbf{v}),$$

$\mathbf{v}$ being written in place of $\mathbf{v}_{Gi}$, will be termed the kinetic energy of the body.

2.3. Expression for the kinetic energy of a rigid body. In order to use the energy equation (6.37) as an integral of the motion, it is essential to be able to write down an expression for T . We have already obtained such an expression when the system consists of mutually interacting particles (eqn 6.23). What we require is the corresponding expression when the system is a rigid body.

(1) *Rigid body having one point fixed.* If $\mathbf{r}$ denotes the vector displacement of a molecular particle of mass m with respect to the fixed point, then we have, since $\mathbf{v} = \boldsymbol{\omega} \wedge \mathbf{r}$,

$$2T = \sum m(\mathbf{v} \cdot \mathbf{v}) = \sum m(\boldsymbol{\omega} \wedge \mathbf{r}) \cdot (\boldsymbol{\omega} \wedge \mathbf{r}). \quad (6.38)$$

We notice that this expression is identical in form with the expression for the moment of inertia I with respect to a line, viz. $\sum m(\mathbf{r} \wedge \boldsymbol{\lambda}) \cdot (\mathbf{r} \wedge \boldsymbol{\lambda})$ (see Chap. V, § 3) with $\boldsymbol{\omega}(\omega_x, \omega_y, \omega_z)$ replacing $\boldsymbol{\lambda}(l, m, n)$. We therefore have

$$2T = A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2F\omega_y\omega_z - 2G\omega_z\omega_x - 2H\omega_x\omega_y, \quad (6.39)$$

where A, B, C, F, G, H are the moments and products of inertia with respect to a set of orthogonal coordinate axes and planes through O .

An alternative expression for T , which is sometimes useful, may be obtained as follows:

$$\begin{aligned} 2T &= \sum m(\mathbf{v} \cdot \mathbf{v}) = \sum m\{\mathbf{v} \cdot (\boldsymbol{\omega} \wedge \mathbf{r})\} \\ &= \sum m\{\boldsymbol{\omega} \cdot (\mathbf{r} \wedge \mathbf{v})\} \\ &= \boldsymbol{\omega} \cdot \sum (\mathbf{r} \wedge m\mathbf{v}) \\ &= \boldsymbol{\omega} \cdot \mathbf{H}. \end{aligned} \quad (6.40)$$

On expanding (for components of $\mathbf{H}$ see (6.33)), this reduces to (6.39) above.

(2) *Rigid Body having no point fixed.* The kinetic energy of a system of mutually interacting particles is given by eqn (6.23)

$$2T = Mv_G^2 + \sum m(\mathbf{v}' \cdot \mathbf{v}'),$$

where $\mathbf{v}'$ is the velocity of the particle of mass m relative to G , the centre of mass of the system, $\mathbf{v}_G$ being the velocity of G . But, in the case of a rigid body, $\mathbf{v}'$ for a molecular particle is given by

$$\mathbf{v}' = \boldsymbol{\omega} \wedge \mathbf{r}',$$

and hence

$$2T = Mv_G^2 + \sum m(\boldsymbol{\omega} \wedge \mathbf{r}') \cdot (\boldsymbol{\omega} \wedge \mathbf{r}').$$

The second term of this expression is identical in form with (6.38) above. We therefore have

$$2T = Mv_G^2 + (A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2F\omega_y\omega_z - 2G\omega_z\omega_x - 2H\omega_x\omega_y),$$

where A, B, C, F, G, H are now the moments and products of inertia with respect to a set of orthogonal axes and planes whose origin is at the centre of mass.

We also have

$$2T = Mv_G^2 + \boldsymbol{\omega} \cdot \mathbf{H}_G$$

analogous to (6.40) above.

2.4. Application of the principles of conservation of angular momentum and energy. Just as the moment of momentum of a particle with respect to a given axis may be conserved throughout its motion, so the moment of momentum of a rigid body with respect to a fixed direction may, under certain circumstances, be conserved (see below), a result which is known as the *principle of conservation of angular momentum* for a rigid body. This principle, when it holds, and the principle of conservation of energy, as derived above, are often extremely useful in the solution of problems involving the motion of a rigid body since, when expressed mathematically, they lead to first integrals of the equations of motion, i.e. the resulting equations involve the rates of change of the positional coordinates of the system but not their second differentials.

(1) *Motion of a Rigid Body about a Fixed Point.* Let $\mathbf{H}$ be the moment of momentum of the body with respect to a fixed point O . If $\boldsymbol{\lambda}$ is a unit vector defining a line through O , then the moment of momentum of the body with respect to the line $\boldsymbol{\lambda}$ is defined by the scalar quantity

$$H_\lambda = \boldsymbol{\lambda} \cdot \mathbf{H}.$$

Suppose we multiply the equation

$$\frac{d\mathbf{H}}{dt} = \sum (\mathbf{r} \wedge \mathbf{F}),$$

which expresses that the rate of change of the moment of momentum of the body with respect to O is equal to the algebraic sum of the moments of the external forces about O , scalarly by $\boldsymbol{\lambda}$. Then we have

$$\boldsymbol{\lambda} \cdot \frac{d\mathbf{H}}{dt} = \boldsymbol{\lambda} \cdot \sum (\mathbf{r} \wedge \mathbf{F}) = \sum \{\boldsymbol{\lambda} \cdot (\mathbf{r} \wedge \mathbf{F})\}.$$

In the event of the right-hand side of this equation vanishing, we shall have

$$\boldsymbol{\lambda} \cdot \frac{d\mathbf{H}}{dt} = 0,$$

and hence, since $\boldsymbol{\lambda}$ is a constant unit vector in space,

$$\boldsymbol{\lambda} \cdot \mathbf{H} = \text{constant}, \quad (6.41)$$

i.e. the moment of momentum of the body about the line $\boldsymbol{\lambda}$ is constant during the motion. The condition for this to happen is clearly that the algebraic sum of the moments of the external forces about the line should vanish.*

In the solution of problems involving the motion of a rigid body having one point fixed, the system being such that it possesses only two degrees of freedom (i.e. it requires only two independent coordinates to define its position) and is moving under the action of a system of conservative forces, then an equation of the type (6.41), if it exists, together with the equation expressing the constancy of the total energy will suffice to determine the motion completely.

If the body possesses three degrees of freedom, as in the case in which a rigid body moves freely under gravity about a fixed point, then two equations of the above type together with the energy equation will be sufficient for the determination of the motion.

(2) *General Motion of a Rigid Body.* If the body has no point fixed, then the motion will be defined at any instant by the motion of its centre of mass together with the rotational motion of the body relative

* In some cases it may happen that

$$\frac{d(\boldsymbol{\lambda} \cdot \mathbf{H})}{dt} = 0,$$

so that

$$\boldsymbol{\lambda} \cdot \mathbf{H} = \text{constant},$$

even if the direction of $\boldsymbol{\lambda}$ is not constant in space, e.g. the moment of momentum of a top, spinning on a rough plane, about its axis is constant. This point is discussed in detail in Chap. VIII.

to the centre of mass. The motion of the centre of mass is determined, in the usual way, by the equation

$$M \frac{d\mathbf{v}_G}{dt} = \sum \mathbf{F}, \quad (6.42)$$

while the rotational motion is governed by the equation

$$\frac{d\mathbf{H}_G}{dt} = \sum (\mathbf{r} \wedge \mathbf{F}). \quad (6.43)$$

Now a rigid body, not subject to constraints, requires six independent coordinates to specify its position; these, in general, are chosen for convenience to be the three positional coordinates of its centre of mass together with the three coordinates defining the orientation of the body relative to the centre of mass. Thus six scalar integral equations involving these six coordinates, together with their rates of change, will in general be required to determine the motion. These may be obtained by the integration of (6.42) and (6.43), each equation being equivalent to three independent scalar equations. Some of these integrals may, however, be written down at once by applying the principles of conservation of linear momentum, angular momentum, and energy, when they exist. Thus, if $\boldsymbol{\lambda}$ is a unit vector defining a constant direction in space such that

$$\boldsymbol{\lambda} \cdot M \frac{d\mathbf{v}_G}{dt} = \boldsymbol{\lambda} \cdot \sum \mathbf{F} = \sum (\boldsymbol{\lambda} \cdot \mathbf{F}) = 0,$$

then we have

$$\boldsymbol{\lambda} \cdot M \mathbf{v}_G = \text{constant}$$

as one such integral of the motion. Similarly, if a unit vector $\boldsymbol{\lambda}$ exists such that

$$\boldsymbol{\lambda} \cdot \frac{d\mathbf{H}_G}{dt} = \boldsymbol{\lambda} \cdot \sum (\mathbf{r}' \wedge \mathbf{F}) = \sum \{\boldsymbol{\lambda} \cdot (\mathbf{r}' \wedge \mathbf{F})\} = 0,$$

then we have, as in (6.41) above,

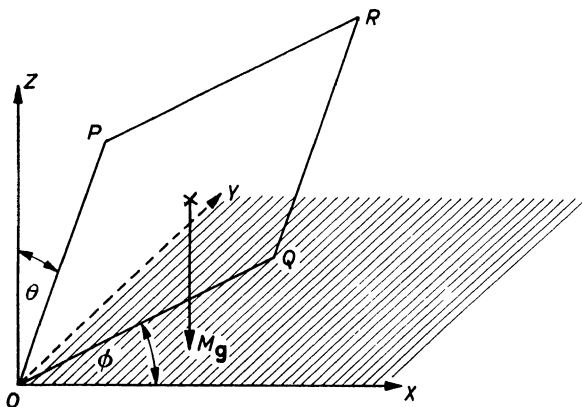
$$\boldsymbol{\lambda} \cdot \mathbf{H}_G = \text{constant}$$

as another integral of the motion. If, in addition, the body is in motion under the action of conservative forces, then the equation of energy leads to a further first integral.

If the body is subject to constraints, so that the number of independent coordinates required to define its position is reduced, then it often happens that the equations of constraint together with integrals of the above types suffice for the determination of the motion; in this case the direct integration of the equations of motion, (6.42) and (6.43), can be avoided.

Example. A square plate $OPQR$ is free to move about the corner O with the lower edge OQ sliding on a smooth horizontal plane through O . The plate is held with its plane vertical and, after being slightly displaced, is allowed to fall. Prove that, when the plate becomes horizontal, OQ will have moved through an angle $3\pi/16$ and determine the angular velocity of OP about OQ in this position.

Solution. Let ϕ be the angle that OQ makes with OX , a fixed line in the plane, θ the angle that OP makes with the upward vertical OZ . Then these two angles clearly define the position of the lamina at any instant, while their rates of change define its motion. Only two co-ordinates are thus involved, so that two first integrals of the equations of motion will suffice for the determination of the motion.



Since the total moment of the external forces about OZ is zero, all the forces acting on the lamina being vertical, one integral of the motion will be given by the constancy of the component of the moment of momentum with respect to OZ , while a second integral will be given by the constancy of the total energy, the plane on which OQ slides being smooth.

To obtain the integrals we have:

(i) *Angular Momentum Integral.* To obtain an expression for H_z , the component of the angular momentum, or the moment of momentum, about OZ , it is convenient to take a set of axes $O123$, fixed relative to the plate, such that $O1$ is along OP , $O2$ is along OQ , and $O3$ is a mutual perpendicular axis such that $O123$ is a right-handed system. Then we shall have

$$H_z = \mathbf{z} \cdot \mathbf{H},$$

where $\mathbf{z}$ is a unit vector in the direction of the upward vertical, and

$$\mathbf{H} \equiv (H_1, H_2, H_3)$$

$$= (A\omega_1 - H\omega_2 - G\omega_3, -H\omega_1 + B\omega_2 - F\omega_3, -G\omega_1 - F\omega_2 + C\omega_3),$$

A, B, C, F, G, H being the moments and products of inertia with respect to the axes and planes defined by $O123$, and $(\omega_1, \omega_2, \omega_3)$ the components of the angular velocity with respect to the same axes. But, if a is the length of a side of the plate, then

$$A = \text{Moment of inertia about } O1 = \frac{1}{3}Ma^2,$$

$$B = \text{Moment of inertia about } O2 = \frac{1}{3}Ma^2,$$

$$C = \text{Moment of inertia about } O3 = \frac{2}{3}Ma^2;$$

also the products of inertia are given by

$$F = I_{23} = 0, \quad G = I_{31} = 0, \quad H = I_{12} = \frac{1}{4}Ma^2$$

(The value of H is obtained by using the general result for the product of inertia given in Chap. V (eqn 5.6), viz.,

$$\begin{aligned} H = I_{12} &= M\bar{x}_1\bar{x}_2 + I_{G12} \\ &= M\bar{x}_1\bar{x}_2, \end{aligned}$$

since I_{G12} , the product of inertia with respect to parallel planes through the centre of mass of the plate, is by symmetry zero.) Hence we have

$$(H_1, H_2, H_3) = Ma^2 \left(\frac{\omega_1}{3} - \frac{\omega_2}{4}, -\frac{\omega_1}{4} + \frac{\omega_2}{3}, \frac{2\omega_3}{3} \right).$$

Also

$$\mathbf{z} = (\cos \theta, 0, \sin \theta),$$

when referred to the same axes, and hence

$$\begin{aligned} H_z = \mathbf{z} \cdot \mathbf{H} &= (\cos \theta, 0, \sin \theta) \cdot Ma^2 \left(\frac{\omega_1}{3} - \frac{\omega_2}{4}, -\frac{\omega_1}{4} + \frac{\omega_2}{3}, \frac{2\omega_3}{3} \right) \\ &= Ma^2 \left\{ \left(\frac{\omega_1}{3} - \frac{\omega_2}{4} \right) \cos \theta + \frac{2\omega_3 \sin \theta}{3} \right\}. \end{aligned} \quad (1)$$

To obtain the components of $\boldsymbol{\omega}$, and so evaluate H_z , we have

$$\boldsymbol{\omega} = \phi \mathbf{z} + \dot{\theta} \mathbf{j},$$

if $\mathbf{j}$ is a unit vector in the direction of OQ ; consequently

$$\begin{aligned} \boldsymbol{\omega} \equiv (\omega_1, \omega_2, \omega_3) &= \phi(\cos \theta, 0, \sin \theta) + \dot{\theta}(0, 1, 0) \\ &= (\phi \cos \theta, \dot{\theta}, \phi \sin \theta). \end{aligned} \quad (2)$$

Hence, from (1),

$$\begin{aligned} H_z &= Ma^2 \left\{ \left(\frac{\phi \cos \theta}{3} - \frac{\dot{\theta}}{4} \right) \cos \theta + \frac{2\dot{\theta} \sin^2 \theta}{3} \right\} \\ &= Ma^2 \left\{ \frac{1}{3}(\cos^2 \theta + 2 \sin^2 \theta) \phi - \frac{1}{4} \dot{\theta} \cos \theta \right\} \\ &= Ma^2 \left\{ \frac{1}{3}(1 + \sin^2 \theta) \phi - \frac{1}{4} \dot{\theta} \cos \theta \right\}. \end{aligned}$$

The angular momentum integral,

$$H_z = \text{constant},$$

thus becomes

$$Ma^2\left\{\frac{1}{3}(1 + \sin^2 \theta)\dot{\phi} - \frac{1}{4}\dot{\theta} \cos \theta\right\} = \text{constant}.$$

But initially $\dot{\theta} = \dot{\phi} = 0$, so that the constant vanishes, and we therefore have

$$\frac{1}{3}(1 + \sin^2 \theta)\dot{\phi} - \frac{1}{4}\dot{\theta} \cos \theta = 0,$$

$$\text{i.e.} \quad \dot{\phi} = \frac{3\dot{\theta} \cos \theta}{4(1 + \sin^2 \theta)}. \quad (3)$$

Thus

$$\phi = \int \frac{3 \cos \theta}{4(1 + \sin^2 \theta)} d\theta + \text{constant}.$$

To carry out the integration, put

$$\sin \theta = u,$$

so that

$$\cos \theta d\theta = du.$$

Then we have

$$\int \frac{3 \cos \theta}{4(1 + \sin^2 \theta)} d\theta = \int \frac{3du}{4(1 + u^2)} = \frac{3}{4} \tan^{-1} u = \frac{3}{4} \tan^{-1} (\sin \theta).$$

$$\text{Hence} \quad \phi = \frac{3}{4} \tan^{-1} (\sin \theta) + \text{constant}.$$

But the constant vanishes, since $\theta = \phi = 0$ initially; thus

$$\phi = \frac{3}{4} \tan^{-1} (\sin \theta).$$

When the plate becomes horizontal, then $\theta = \pi/2$, so that $\sin \theta = 1$, and hence

$$\phi(\theta = \pi/2) = \frac{3}{4} \tan^{-1} 1 = \frac{3}{4} \cdot \frac{\pi}{4} = \frac{3\pi}{16},$$

i.e. OQ has moved through an angle $3\pi/16$.

(ii) *Energy Integral.* To obtain $\dot{\theta}$, and thus to determine the motion, we make use of the energy integral,

$$T + V = \text{constant}. \quad (4)$$

Now T is given by

$$\begin{aligned} 2T = \boldsymbol{\omega} \cdot \mathbf{H} &= Ma^2 \left\{ (\omega_1, \omega_2, \omega_3) \cdot \left(\frac{\omega_1}{3} - \frac{\omega_2}{4}, -\frac{\omega_1}{4} + \frac{\omega_2}{3}, \frac{2\omega_2}{3} \right) \right\} \\ &= Ma^2 \left(\frac{\omega_1^2}{3} - \frac{\omega_1 \omega_2}{4} - \frac{\omega_1 \omega_2}{4} + \frac{\omega_2^2}{3} + \frac{2\omega_3^2}{3} \right) \end{aligned}$$

$$\begin{aligned}
&= Ma^2 \left\{ \frac{(\omega_1^2 + \omega_2^2 + 2\omega_3^2)}{3} - \frac{\omega_1 \omega_2}{2} \right\} \\
&= \frac{Ma^2}{6} \{ 2(1 + \sin^2 \theta) \dot{\phi}^2 + 2\dot{\theta}^2 - 3\dot{\theta} \dot{\phi} \cos \theta \}
\end{aligned}$$

after substituting from (2); also

$$V = \frac{1}{2} Mga \cos \theta.$$

Therefore (4) becomes

$$\frac{Ma^2}{6} \{ 2(1 + \sin^2 \theta) \dot{\phi}^2 + 2\dot{\theta}^2 - 3\dot{\theta} \dot{\phi} \cos \theta \} + Mga \cos \theta = \text{constant},$$

$$\text{i.e.} \quad 2(1 + \sin^2 \theta) \dot{\phi}^2 + 2\dot{\theta}^2 - 3\dot{\theta} \dot{\phi} \cos \theta + (6g/a) \cos \theta = C, \text{ say.} \quad (5)$$

But initially $\dot{\theta} = \dot{\phi} = 0$, $\theta = 0$; therefore

$$C = 6g/a,$$

so that (5) becomes

$$2(1 + \sin^2 \theta) \dot{\phi}^2 + 2\dot{\theta}^2 - 3\dot{\theta} \dot{\phi} \cos \theta = (6g/a)(1 - \cos \theta).$$

On substituting for $\dot{\phi}$ from (3), this leads to the equation

$$\left(2 - \frac{9 \cos^2 \theta}{8(1 + \sin^2 \theta)} \right) \dot{\theta}^2 = (6g/a)(1 - \cos \theta),$$

i.e.

$$\{ 16(1 + \sin^2 \theta) - 9 \cos^2 \theta \} \dot{\theta}^2 = (48g/a)(1 - \cos \theta)(1 + \sin^2 \theta),$$

so that

$$\dot{\theta}^2 = \frac{48g(1 - \cos \theta)(1 + \sin^2 \theta)}{a(7 + 25 \sin^2 \theta)}.$$

This equation together with (3) allow the motion of the lamina to be determined.

Putting $\theta = \pi/2$, the angular velocity of OP about OQ , when the plate becomes horizontal, is thus $\sqrt{(3g/a)}$.

The above general theory has been derived for a single body; it is clear, however, that the equations so obtained may be readily applied to systems consisting of two or more mutually interacting bodies.

2.5. Additional problems involving the rolling motion of a sphere. The examples involving the rolling motion of a sphere, discussed in § 2.1, all involved motion in contact with a plane, the system being such that $\mathbf{n}$, the unit vector through the point of contact in the direction of the normal to the plane, had a constant direction in space. If the surface on which the sphere rolls is such that the direction of $\mathbf{n}$ is not constant, however, then a complete vectorial solution of the motion is not possible.

To see why this is so, suppose we consider the case of a sphere in rolling contact with a fixed surface, the surface being such that the direction of $\mathbf{n}$ is variable. We shall assume that the only forces acting on the sphere are its weight and the reaction at the point of contact. Then, using the same notation as in § 2.1, the equations of motion are given by

$$M \frac{d\mathbf{v}}{dt} = \mathbf{R} + M\mathbf{g},$$

$$A \frac{d\boldsymbol{\omega}}{dt} = -a(\mathbf{n} \wedge \mathbf{R}),$$

together with the kinematical condition for rolling contact

$$\mathbf{v} + a(\mathbf{n} \wedge \boldsymbol{\omega}) = \mathbf{0}.$$

If we eliminate $\mathbf{R}$ from the first two of these equations, then we have

$$A \frac{d\boldsymbol{\omega}}{dt} = Ma \left\{ \mathbf{n} \wedge \left(\mathbf{g} - \frac{d\mathbf{v}}{dt} \right) \right\},$$

which, after multiplying vectorially by $\mathbf{n}$, gives

$$A \left(\mathbf{n} \wedge \frac{d\boldsymbol{\omega}}{dt} \right) = Ma \left[\mathbf{n} \wedge \left\{ \mathbf{n} \wedge \left(\mathbf{g} - \frac{d\mathbf{v}}{dt} \right) \right\} \right].$$

When $\mathbf{n}$ has a constant direction in space, then $\boldsymbol{\omega}$ can be eliminated from this equation using the kinematical relation above, after differentiating with respect to the time. If the direction of $\mathbf{n}$ is not constant, however, then on differentiating we have

$$\frac{d\mathbf{v}}{dt} + a \left(\mathbf{n} \wedge \frac{d\boldsymbol{\omega}}{dt} \right) + a \left(\frac{d\mathbf{n}}{dt} \wedge \boldsymbol{\omega} \right) = \mathbf{0},$$

and it is seen that the elimination of $\boldsymbol{\omega}$ is no longer possible. In this case, therefore, we proceed rather differently. We first multiply the kinematical relation for rolling contact by $\mathbf{n}$ vectorially; this gives

$$\mathbf{n} \wedge \mathbf{v} + a \{ \mathbf{n}(\mathbf{n} \cdot \boldsymbol{\omega}) - \boldsymbol{\omega} \} = \mathbf{0},$$

so that we have

$$\begin{aligned} a\boldsymbol{\omega} &= \mathbf{n} \wedge \mathbf{v} + a\mathbf{n}(\mathbf{n} \cdot \boldsymbol{\omega}) \\ &= \mathbf{n} \wedge \mathbf{v} + a\omega_3 \mathbf{n}, \end{aligned}$$

where $\omega_3 (= \mathbf{n} \cdot \boldsymbol{\omega})$ is the component of $\boldsymbol{\omega}$ in the direction of the normal to the surface.* On differentiating with respect to t and substituting the resulting expression for $d\boldsymbol{\omega}/dt$ into the equation obtained above, after the elimination of $\mathbf{R}$, we thus have

$$A \left\{ \left(\mathbf{n} \wedge \frac{d\mathbf{v}}{dt} \right) + \left(\frac{d\mathbf{n}}{dt} \wedge \mathbf{v} \right) + a\omega_3 \frac{d\mathbf{n}}{dt} + a \frac{d\omega_3}{dt} \mathbf{n} \right\} = Ma^2 \left\{ \mathbf{n} \wedge \left(\mathbf{g} - \frac{d\mathbf{v}}{dt} \right) \right\},$$

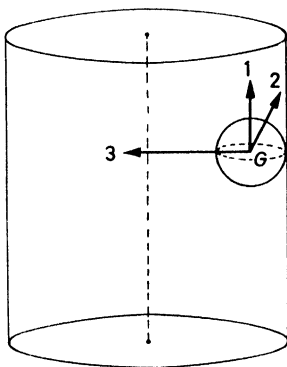
* The reason for this notation becomes clear later when we apply the theory to a particular example.

or, writing $A = Mk^2$,

$$\frac{k^2+a^2}{a^2}\left(\mathbf{n} \wedge \frac{d\mathbf{v}}{dt}\right) + \frac{k^2}{a^2}\left(\frac{d\mathbf{n}}{dt} \wedge \mathbf{v}\right) + \frac{k^2\omega_3}{a} \frac{d\mathbf{n}}{dt} + \frac{k^2}{a} \frac{d\omega_3}{dt} \mathbf{n} - (\mathbf{n} \wedge \mathbf{g}) = \mathbf{0}.$$

To effect a solution, we now replace this equation by its equivalent scalar equations. This necessitates the introduction of a set of rotating axes. If these are suitably chosen, then the scalar equations invariably involve just three scalar variables and their rates of change. These variables are ω_3 together with the two positional coordinates which define the position of the centre of the sphere relative to the surface on which it moves. The equations are therefore sufficient for the determination of the motion.* To illustrate the theory, we shall consider just one case of such motion.

Example. Motion of a uniform sphere on the interior rough surface of a fixed vertical circular cylinder.



Solution. Let the radii of the sphere and cylinder be a and b respectively. We shall refer the motion to a set of axes $G123$ whose

* The above procedure is not actually essential in the case of a sphere in rolling contact with a fixed sphere. In this case, if O is the centre of the fixed sphere, G that of the moving sphere, then the radius OG is always coincident with the normal at the point of contact. Consequently, if a and b are the radii of the respective spheres, then the velocity of G may be expressed in the form

$$\mathbf{v} = (a+b) \frac{d\mathbf{n}}{dt}.$$

As a result, the motion may be determined without first writing down the equivalent scalar equations, the scalar first integrals, necessary for the determination of the motion, being obtained as in the case of the top (see Chap. VIII); see, also, Newbould, *Analytical Method in Dynamics* (Oxford, Clarendon Press, 1946), Chap. III, § 3.3.

origin is at the centre of the sphere, the directions of the axes being such that $G1$ is vertically upwards, $G3$ is along the inward normal, and $G2$ is a mutually orthogonal axis such that $G123$ is a right-handed system, while to define the position of the sphere we introduce the two positional coordinates x and θ , where x denotes the distance of G above a fixed horizontal level and θ is the angle which a meridian plane through G makes with a meridian plane fixed in space.

To determine the scalar equations equivalent to the above equation, when referred to axes $G123$, we must first replace $d\mathbf{v}/dt$ and $d\mathbf{n}/dt$ by

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\theta} \wedge \mathbf{v}, \quad \frac{\partial \mathbf{n}}{\partial t} + \boldsymbol{\theta} \wedge \mathbf{n} = \boldsymbol{\theta} \wedge \mathbf{n} \quad \left(\frac{\partial \mathbf{n}}{\partial t} = \mathbf{0} \right)$$

respectively, $\boldsymbol{\theta}$ being the angular velocity of the rotating axes. Remembering that $\mathbf{n} \cdot \mathbf{v} = 0$, this leads to the equation

$$\left(\frac{k^2 + a^2}{a^2} \right) \left\{ \left(\mathbf{n} \wedge \frac{\partial \mathbf{v}}{\partial t} \right) - \mathbf{v}(\mathbf{n} \cdot \boldsymbol{\theta}) \right\} + \frac{k^2}{a^2} \mathbf{n}(\boldsymbol{\theta} \cdot \mathbf{v}) + \frac{k^2 \omega_3}{a} (\boldsymbol{\theta} \wedge \mathbf{n}) + \frac{k^2}{a} \frac{d\omega_3}{dt} \mathbf{n} - (\mathbf{n} \wedge \mathbf{g}) = \mathbf{0}. \quad (1)$$

The scalar equations may now be derived. We have

$$\begin{aligned} \boldsymbol{\theta} (= \dot{\theta} \mathbf{z}) &= \dot{\theta}(1, 0, 0), & \mathbf{n} \wedge \frac{\partial \mathbf{v}}{\partial t} &= \{-(b-a)\dot{\theta}, \ddot{x}, 0\} \\ \mathbf{n} &= (0, 0, 1), & \boldsymbol{\theta} \wedge \mathbf{n} &= \dot{\theta}(0, -1, 0), \\ \mathbf{g} (= -g\mathbf{z}) &= -g(1, 0, 0), & \mathbf{n} \wedge \mathbf{g} &= -g(0, 1, 0), \\ \mathbf{v} &= \{\dot{x}, (b-a)\dot{\theta}, 0\}, & \mathbf{n} \cdot \boldsymbol{\theta} &= 0, \\ \frac{\partial \mathbf{v}}{\partial t} &= \{\ddot{x}, (b-a)\ddot{\theta}, 0\}, & \boldsymbol{\theta} \cdot \mathbf{v} &= \dot{x}\dot{\theta}. \end{aligned}$$

Hence, substituting into (1), we have, after multiplying through by $a^2/(k^2 + a^2)$,

$$\begin{aligned} \{-(b-a)\ddot{\theta}, \ddot{x}, 0\} + \frac{k^2}{k^2 + a^2} (0, 0, 1) \dot{x}\dot{\theta} + \frac{ak^2\omega_3}{k^2 + a^2} \dot{\theta}(0, -1, 0) \\ + \frac{ak^2\dot{\omega}_3}{k^2 + a^2} (0, 0, 1) + \frac{a^2}{k^2 + a^2} g(0, 1, 0) = (0, 0, 0). \end{aligned}$$

The equivalent scalar equations are thus:

$$\left. \begin{aligned} \ddot{\theta} &= 0, & (a) \\ \ddot{x} - \frac{ak^2\omega_3}{k^2 + a^2} \dot{\theta} + \frac{a^2g}{k^2 + a^2} &= 0, & (b) \\ \dot{x}\dot{\theta} + a\dot{\omega}_3 &= 0. & (c) \end{aligned} \right\} \quad (2)$$

The first of these gives on integrating

$$\dot{\theta} = \text{constant} = \beta;$$

thus θ is constant during the motion. On substituting for θ in (2c), we get

$$\beta \dot{x} + a \dot{\omega}_3 = 0,$$

which can be immediately integrated giving

$$\beta x + a \omega_3 = \text{constant} = a\mu.$$

On substituting for $a\omega_3$ and θ in (2b), we finally have the equation

$$\ddot{x} - \frac{k^2\beta}{k^2+a^2}(a\mu - \beta x) + \frac{a^2g}{k^2+a^2} = 0,$$

which on rearranging takes the form

$$\ddot{x} + \frac{k^2\beta^2}{k^2+a^2} \left\{ x - \frac{a}{\beta} \left(\mu - \frac{ag}{k^2\beta} \right) \right\} = 0.$$

It is thus seen that the centre of the sphere oscillates in the cylinder about the horizontal level defined by

$$x = \frac{a}{\beta} \left(\mu - \frac{ag}{k^2\beta} \right)$$

with simple harmonic motion of period

$$T = 2\pi \sqrt{\left(\frac{k^2+a^2}{k^2\beta^2} \right)}.$$

The fact that the centre of the sphere oscillates vertically with simple harmonic motion has been given as the explanation of why a golf ball, after apparently having been 'sunk', sometimes leaves the hole to the chagrin of the golfer.

Examples

1. A particle of mass m , moving with velocity $\mathbf{v}$, strikes a stationary body, also of mass m , at a point whose position vector relative to G , the centre of mass of the body, is $\mathbf{c}$. The particle penetrates the body and at the instant when it is reduced to rest, relative to the body, its position vector with respect to G is $\mathbf{c}$. If the body possesses kinetic symmetry with respect to G , show that $\boldsymbol{\omega}$, the angular velocity of the body, at this instant is given by

$$2I\boldsymbol{\omega} + m\{\mathbf{c} \wedge (\boldsymbol{\omega} \wedge \mathbf{c})\} = m(\mathbf{a} \wedge \mathbf{v}),$$

where I is the moment of inertia of the body about any axis through G .

2. A uniform sphere of mass M and radius a rolls without slipping on a rough horizontal board, which is free to turn about a vertical axis through a point O of the board. The moment of inertia of the board

about the axis is I , and its angular velocity at any instant is Ω . Show that the equations of motion of the system are given by

$$(i) \ k^2 \frac{d\boldsymbol{\omega}}{dt} + a \left(\mathbf{n} \wedge \frac{d\mathbf{v}}{dt} \right) = \mathbf{0}, \quad (ii) \ I \frac{d\Omega}{dt} \mathbf{n} + M \left(\mathbf{r} \wedge \frac{d\mathbf{v}}{dt} \right) = \mathbf{0},$$

$$(iii) \ \mathbf{v} + a(\mathbf{n} \wedge \boldsymbol{\omega}) = \Omega(\mathbf{n} \wedge \mathbf{r}),$$

where $\boldsymbol{\omega}$ is the angular velocity of the sphere, $\mathbf{v}$ is the velocity of its centre, k is its radius of gyration about a diameter, $\mathbf{n}$ is a unit vector in the direction of the upward vertical, and $\mathbf{r}$ is the vector displacement of the point of contact of the sphere from O .

Hence show, by integrating these equations, that Ω satisfies an equation of the form

$$\{Mk^2r^2 + I(k^2 + a^2)\}\Omega - Ma(\mathbf{r} \cdot \mathbf{C}) = \text{constant},$$

where $\mathbf{C}$ is an arbitrary constant vector.

3. A uniform sphere is in rolling motion on a rough inclined plane, of angle α , which is compelled to rotate with a constant angular velocity Ω about an axis normal to itself. Show that the vector equation of the path of the centre of the sphere, when referred to a set of axes fixed in space, is given by

$$\frac{d\mathbf{r}}{dt} = \frac{k^2\Omega}{k^2 + a^2} \mathbf{n} \wedge \left\{ (\mathbf{r} - \mathbf{r}_0) + \frac{a^2g \sin \alpha t}{k^2\Omega} \mathbf{j} \right\},$$

where k is the radius of gyration of the sphere about a diameter, $\mathbf{r}$ is the vector displacement of the point of contact of the sphere from the point of intersection of the axis of rotation with the plane, $\mathbf{n}$ is a unit vector in the direction of the axis of rotation, and $\mathbf{j}$ is a unit vector in the direction of a fixed horizontal line in the plane.

Interpret this result.

4. A uniform rod of length $2a$ can turn freely about one end which is fixed. Initially it is inclined at an acute angle α to the downward drawn vertical. If it is set rotating about a vertical axis through the fixed end with an angular velocity Ω , show that the rod rotates about the axis at constant inclination α provided

$$\Omega^2 = \frac{3g}{4a \cos \alpha}.$$

If this relation is not satisfied, show that the inclination of the rod varies between α and

$$\cos^{-1}\{-n + \sqrt{(1 - 2n \cos \alpha + n^2)}\},$$

where

$$n = \frac{a\Omega^2 \sin^2 \alpha}{3g}.$$

5. A uniform rectangular lamina, whose principal moments of inertia at the centre are kA , A , $(1+k)A$, is mounted so that it can swing freely about a smooth horizontal axis which coincides with the axis about which the moment of inertia is A . The axle is carried on a frame which can rotate about a smooth vertical axis passing through the centre of the lamina. At time $t = 0$, $\theta = \pi/4$, $\dot{\theta} = 0$, $\dot{\phi} = \Omega$, where θ is the inclination of the lamina to the horizontal and ϕ is the angle through which the frame turns. Show that, in the subsequent motion,

$$\dot{\theta}^2 = \frac{(1+2k)\Omega^2 \cos 2\theta}{4(k+\cos^2 \theta)}$$

6. A uniform sphere, whose centre is G and whose radius is a , rolls on the rough inner surface of a fixed hollow sphere of radius $(a+c)$ and centre O . Obtain the equations of motion, and hence show that the condition for a steady motion in which G describes a horizontal circle of radius r with a constant angular speed Ω is

$$a^2 n^2 \geq 35rg\Omega \cot \alpha,$$

where n is the spin of the sphere about OG and α is the angle that the line OG makes with the downward vertical through O .

7. A sphere is in rolling motion on the rough inner surface of a right circular cone at rest with its axis vertical. Show that the path of its centre is similar to that of a heavy particle moving under gravity on the smooth inner surface of a vertical cone of the same angle. Show also that the vertical component of its angular velocity is constant throughout the motion.

8. A sphere of radius a is in rolling motion on the rough inner surface of a vertical right-circular cylinder of radius b , which rotates about its axis with a uniform angular velocity Ω . Show that the vertical motion of its centre is simple harmonic with period

$$T = \frac{2\pi(b-a)(a^2+k^2)}{bk^2(\Omega+n)},$$

where n is a constant.

CHAPTER VII

EULER'S EQUATIONS AND APPLICATIONS

1. Equations of motion referred to rotating axes

It has already been pointed out in Chap. VI (see § 2, p. 133) that the use of the equation

$$\frac{d\mathbf{H}}{dt} = \sum (\mathbf{r} \wedge \mathbf{F}) \quad (7.1)$$

in the solution of problems involving the motion of rigid bodies is limited by the fact that, since the motion is referred to a set of coordinate axes, the directions of which are fixed in space, the inertia coefficients occurring in the scalar components of $\mathbf{H}$ (and hence in the scalar equations equivalent to (7.1)) are variable quantities. In practice it is, in general, more convenient therefore to refer the motion to a set of axes moving in some prescribed manner such that the inertia coefficients are constant with respect to them.

Let the origin of the axes have a velocity $\mathbf{v}_O$ and suppose that they rotate with an angular velocity $\boldsymbol{\theta}$. Let $\mathbf{H}$ be the moment of momentum of the system with respect to the origin. If we denote the rate of change of a vector with respect to the rotating axes by $\partial/\partial t$, so that we have

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{\theta} \wedge$$

(see Chap. IV, eqn 4.3), then the equation of moment of momentum of the previous chapter (eqn 6.20) becomes, when referred to the rotating axes,

$$\frac{d\mathbf{H}}{dt} \equiv \frac{\partial \mathbf{H}}{\partial t} + \boldsymbol{\theta} \wedge \mathbf{H} + \mathbf{v}_O \wedge \mathbf{p} = \sum (\mathbf{r} \wedge \mathbf{F}), \quad (7.2)$$

where $\mathbf{p}$ is the linear momentum of the system.

If the origin is at rest, or coincides with the centre of mass of the system, then (7.2) reduces to

$$\frac{\partial \mathbf{H}}{\partial t} + \boldsymbol{\theta} \wedge \mathbf{H} = \sum (\mathbf{r} \wedge \mathbf{F}). \quad (7.3)$$

1.1. Application to a body having one point fixed. Euler's equations. If the body has one point O fixed, then, taking O to be

the origin, the motion is determined by (7.3). The equivalent scalar equations of motion of the body are thus:

$$\left. \begin{aligned} \frac{dH_1}{dt} + (\theta_2 H_3 - \theta_3 H_2) &= M_1 \\ \frac{dH_2}{dt} + (\theta_3 H_1 - \theta_1 H_3) &= M_2 \\ \frac{dH_3}{dt} + (\theta_1 H_2 - \theta_2 H_1) &= M_3 \end{aligned} \right\}, \quad (7.4)$$

where the suffixes 1,2,3 are used to denote the instantaneous values of the components of a vector with respect to the moving axes and $\mathbf{M}(M_1, M_2, M_3) = \sum (\mathbf{r} \wedge \mathbf{F})$ is the resultant moment about O of the external forces acting on the body. The expressions for (H_1, H_2, H_3) are given by

$$(H_1, H_2, H_3) = (A\omega_1 - H\omega_2 - G\omega_3, -H\omega_1 + B\omega_2 - F\omega_3, -G\omega_1 - F\omega_2 + C\omega_3). \quad (7.5)$$

Now suppose the rotating axes to be so chosen that they are fixed in the body, so that

$$(\theta_1, \theta_2, \theta_3) = (\omega_1, \omega_2, \omega_3).$$

Then the inertia coefficients occurring in (7.5) will be constants, and (eqns 7.4) become

$$\left. \begin{aligned} \frac{dH_1}{dt} + (\omega_2 H_3 - \omega_3 H_2) &= M_1 \\ \frac{dH_2}{dt} + (\omega_3 H_1 - \omega_1 H_3) &= M_2 \\ \frac{dH_3}{dt} + (\omega_1 H_2 - \omega_2 H_1) &= M_3 \end{aligned} \right\}. \quad (7.6)$$

If, in addition, the rotating axes are taken to coincide with the principal axes of inertia at O , then we shall have $F = G = H = 0$, so that

$$(H_1, H_2, H_3) = (A\omega_1, B\omega_2, C\omega_3).$$

On substituting these values into (7.6), the scalar equations of motion, when referred to such a set of axes, thus reduce to

$$\left. \begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= M_1 \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= M_2 \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= M_3 \end{aligned} \right\}, \quad (7.7)$$

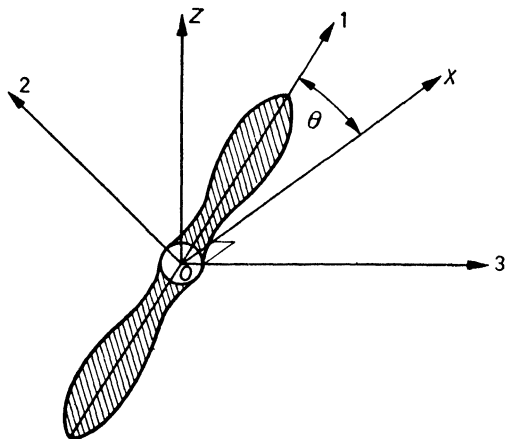
the dot being used, as in previous chapters, to denote the differentiation of a scalar quantity with respect to the time. In this form they are known as *Euler's equations of motion* of a body having one point fixed.

A similar set of equations will govern the rotational motion of a rigid body about its centre of mass if the motion is referred to a set of axes fixed in the body whose origin is at the centre of mass, the directions of the axes being such that they coincide with the principal axes at the centre of mass.

Examples

(1) *An aeroplane is moving in a horizontal circle of radius a with speed v , the axis of the propeller being horizontal. Prove that the couple acting on the propeller has a component Cnv/a in an axial vertical plane and a component $(A-B)nv/a$ in an axial plane rotating with an angular velocity $2n$, where n is the spin of the propeller, assuming the axis of the propeller to be a principal axis.*

Solution. Suppose we refer the motion of the propeller to a set of rotating axes, the origin O of the axes being taken at the centre of mass of the propeller. Let $O3$ be chosen to coincide with the axis of the propeller, $O1$ to be along the axis of the blade, and $O2$ a mutually perpendicular axis such that $O123$ is a right-handed system.



At time t , let $O1$ make an angle θ with OX , the horizontal through O in the plane of the blade, i.e. in the plane $O12$. Then, if Ω is the angular speed with which the aeroplane describes its circular motion, the angular velocity of the propeller in space, and hence the angular velocity of the rotating axes, will be given by

$$\boldsymbol{\omega} = \Omega \mathbf{z} + \theta \mathbf{k},$$

where $\mathbf{z}$ is a unit vector in the direction of the upward vertical and $\mathbf{k}$ is a unit vector in the direction of $O3$. But, referred to the rotating axes,

$$\mathbf{z} = \sin \theta \mathbf{i} + \cos \theta \mathbf{k},$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions of $O1$ and $O2$ respectively, so that

$$\boldsymbol{\omega} = \Omega \sin \theta \mathbf{i} + \Omega \cos \theta \mathbf{j} + \dot{\theta} \mathbf{k},$$

and hence

$$(\omega_1, \omega_2, \omega_3) = (\Omega \sin \theta, \Omega \cos \theta, \dot{\theta}).$$

On substituting these values into Euler's equations, the equations of motion thus become

$$\begin{aligned} A\Omega \cos \theta \dot{\theta} - (B-C)\Omega \cos \theta \dot{\theta} &= M_1, \\ -B\Omega \sin \theta \dot{\theta} - (C-A)\Omega \sin \theta \dot{\theta} &= M_2, \\ -(A-B)\Omega^2 \sin \theta \cos \theta &= M_3, \end{aligned}$$

or, since $\Omega = v/a$ while $\dot{\theta} = \text{constant} = n$,

$$(A-B)n(v/a) \cos \theta + Cn(v/a) \cos \theta = M_1, \quad (1)$$

$$(A-B)n(v/a) \sin \theta - Cn(v/a) \sin \theta = M_2, \quad (2)$$

$$-(A-B)(v/a)^2 \sin \theta \cos \theta = M_3. \quad (3)$$

These equations give the values of the component couples acting on the blade with respect to the rotating axes.

Suppose we consider the couples M_1 and M_2 . The resultant of the couple $(A-B)n(v/a) \cos \theta$ about $O1$ and the couple $(A-B)n(v/a) \sin \theta$ about $O2$ is clearly a couple of magnitude $(A-B)nv/a$ about an axis in the plane $O12$ making an angle ϕ , say, with $O1$, where

$$\tan \phi = \frac{(A-B)n(v/a) \sin \theta}{(A-B)n(v/a) \cos \theta} = \tan \theta,$$

i.e. it is equivalent to a couple of magnitude $(A-B)nv/a$ whose component forces lie in an axial plane making an angle 2θ with the horizontal plane through O . The plane of this couple thus rotates with an angular velocity $2n$.

In the same way the couple $Cn(v/a) \cos \theta$ about $O1$ together with the couple $-Cn(v/a) \sin \theta$ about $O2$ is equivalent to a resultant couple of magnitude Cnv/a about an axis making an angle $(\frac{1}{2}\pi - \psi)$ with the negative direction of $O2$, where

$$\tan \psi = \frac{Cn(v/a) \sin \theta}{Cn(v/a) \cos \theta} = \tan \theta,$$

i.e. $\psi = \theta$, so that the axis must be coincident with OX ; thus the forces which comprise the couple must always act in the vertical plane through $O3$, the axis of the propeller.

The couple represented by (3) is about the axis of the propeller. Since v/a will be small compared to n , the magnitude of this couple is small compared to the other two.

(2) *If a constant couple be applied about the axis of symmetry of a body supported at its centre of mass, and initially rotating about an axis perpendicular to that of symmetry, determine the motion completely and show that the cone described in the body by the instantaneous axis has the equation*

$$2AN(x^2+y^2) \tan^{-1}(y/x) = C(A-C)\Omega^2 z^2,$$

where N is the magnitude of the couple and Ω the initial angular speed of rotation.

Solution. Taking G , the centre of mass as origin, let the axes $G123$, fixed relative to the body, be taken such that $G3$ coincides with the axis of symmetry, while $G1$ and $G2$ are two mutually perpendicular axes forming with $G3$ a right-handed system. Then, since $A = B$ in this example, the body having an axis of symmetry, Euler's equations become

$$A\dot{\omega}_1 - (A-C)\omega_2\omega_3 = 0, \quad (1)$$

$$A\dot{\omega}_2 - (C-A)\omega_3\omega_1 = 0, \quad (2)$$

$$C\dot{\omega}_3 = N. \quad (3)$$

From the third of these it follows that

$$\frac{d\omega_3}{dt} = \frac{N}{C},$$

so that

$$\omega_3 = Nt/C + \text{constant}.$$

But initially $\omega_3 = 0$; therefore the constant vanishes, and we have

$$\omega_3 = Nt/C. \quad (4)$$

On substituting this value for ω_3 into (1) and (2), these become

$$A\dot{\omega}_1 - (N/C)(A-C)t\omega_2 = 0, \quad (5)$$

$$A\dot{\omega}_2 + (N/C)(A-C)t\omega_1 = 0. \quad (6)$$

To integrate these equations, and so determine the motion, we multiply (6) by i and add to (5); this gives

$$A(\dot{\omega}_1 + i\dot{\omega}_2) + i(N/C)(A-C)t(\omega_1 + i\omega_2) = 0,$$

or

$$\frac{dp_c}{dt} + i\lambda t p_c = 0, \quad (7)$$

if we write

$$p_c = \omega_1 + i\omega_2 \quad (8)$$

and

$$\lambda = \frac{N(A-C)}{AC}. \quad (9)$$

Equation (5) may be integrated immediately giving

$$p_c = D_c i \lambda^2,$$

where D_c , the constant of integration, is a complex number. On writing

$$D_c = D_1 + iD_2,$$

the expressions for ω_1 and ω_2 thus become

$$\omega_1 = D_1 \cos \frac{1}{2} \lambda t^2 - D_2 \sin \frac{1}{2} \lambda t^2, \quad \omega_2 = D_1 \sin \frac{1}{2} \lambda t^2 + D_2 \cos \frac{1}{2} \lambda t^2.$$

To evaluate D_1 and D_2 , suppose we choose $G1$, $G2$ to be such that $G1$ coincides with the initial position of the axis of rotation. Then initially $\omega_1 = \Omega$, $\omega_2 = 0$, so that

$$D_1 = \Omega, \quad D_2 = 0,$$

and hence

$$\omega_1 = \Omega \cos \frac{1}{2} \lambda t^2, \quad \omega_2 = \Omega \sin \frac{1}{2} \lambda t^2. \quad (10)$$

The component angular velocities ($\omega_1, \omega_2, \omega_3$) are now known as functions of t ; the motion has thus been completely determined.

To find the path of the instantaneous axis in the body, let P be a general point on the instantaneous axis, the coordinates of P being (x, y, z) when referred to the axes $G123$, which are, of course, fixed in the body. Then we shall have

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3},$$

so that

$$\frac{z^2}{\omega_3^2} = \frac{x^2 + y^2}{\omega_1^2 + \omega_2^2} = \frac{x^2 + y^2}{\Omega^2},$$

using (10), and hence

$$\frac{z^2}{x^2 + y^2} = \frac{\omega_3^2}{\Omega^2} = \frac{N^2 t^2}{C^2 \Omega^2} \quad (11)$$

by (4). But, from (10), we have

$$\tan \left(\frac{1}{2} \lambda t^2 \right) = \frac{\omega_2}{\omega_1} = \frac{y}{x},$$

so that

$$\frac{1}{2} \lambda t^2 = \tan^{-1}(y/x).$$

On substituting for t^2 in (11), we thus have

$$\frac{z^2}{x^2 + y^2} = \frac{N^2}{C^2 \Omega^2} \frac{2}{\lambda} \tan^{-1} \frac{y}{x},$$

which, on using (9), gives

$$2AN(x^2 + y^2) \tan^{-1} \frac{y}{x} = C(A - C)\Omega^2 z^2.$$

This, therefore, represents the path of the instantaneous axis in the body.

2. Euler's geometrical equations

The equations of Euler obtained above (eqns 7.7) determine $(\omega_1, \omega_2, \omega_3)$ as functions of t . They do not, however, enable one to determine the position of the body at time t unless we can also express $(\omega_1, \omega_2, \omega_3)$ in terms of a set of coordinates, which define the orientation of the body in space, together with their rates of change. In the case of a body rotating about a fixed point, or in the rotational motion of a body relative to its centre of mass, the most convenient set of coordinates to take for this purpose are *Euler's angular coordinates* (θ, ϕ, ψ) . These are illustrated in Fig. 7.1 and are defined as follows.

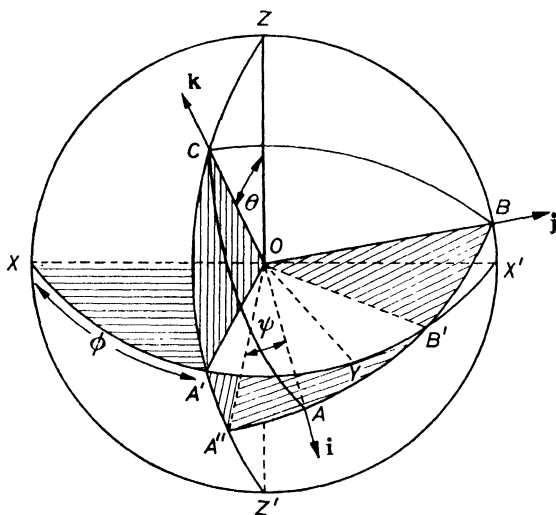


Figure 7.1

Let $OABC$ be a set of rectangular axes fixed in the body. Then θ is the angle which OC makes with an axis OZ , fixed in space, ϕ is the angle which the plane ZOC makes with a fixed plane ZOX , and ψ the angle which the plane COA , fixed in the body, makes with the plane ZOC . (Fig. 7.1 represents the intersection of the planes through O with unit sphere, centre O .) It is readily seen that the position of the body at any instant may be specified by means of these three angles; θ and ϕ define the position of the axis OC in space, while ψ defines the orientation of the body with respect to OC .

Having defined these coordinates, we shall now derive expressions for $(\omega_1, \omega_2, \omega_3)$ in terms of them and their time derivatives $(\dot{\theta}, \dot{\phi}, \dot{\psi})$. To do so, let $OXYZ$ form a right-handed system of mutually orthogonal

axes fixed in space and let $(\mathbf{I}, \mathbf{J}, \mathbf{K})$ denote unit vectors in their respective directions; let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be three unit vectors in the directions of OA, OB, OC respectively. Then, if $\boldsymbol{\omega}$ is the angular velocity of the body and $(\omega_1, \omega_2, \omega_3)$ are its components with respect to the axes $OABC$, we have

$$\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}.$$

Also, from the diagram, it is seen that

$$\boldsymbol{\omega} = \dot{\theta} \mathbf{J}' + \dot{\phi} \mathbf{K} + \dot{\psi} \mathbf{k}, \quad (7.8)$$

where $\mathbf{J}'$ is a unit vector in the direction of OB' (B' is the point of intersection of the great circles XYX' and $A''AB$, so that OB' is at right angles to the plane ZOC). But, since $\angle A''OB' = \angle A'OB' = \frac{1}{2}\pi$,

$$\begin{aligned} \mathbf{J}' &= \cos(\tfrac{1}{2}\pi - \psi) \mathbf{i} + \sin(\tfrac{1}{2}\pi + \psi) \mathbf{j} \\ &= \sin \psi \mathbf{i} + \cos \psi \mathbf{j}, \end{aligned}$$

while, if $\mathbf{I}''$ is a unit vector in the direction of OA'' ,

$$\begin{aligned} \mathbf{K} &= \cos \theta \mathbf{k} - \sin \theta \mathbf{I}'', \\ &= \cos \theta \mathbf{k} - \sin \theta (\cos \psi \mathbf{i} - \sin \psi \mathbf{j}) \\ &= -\sin \theta \cos \psi \mathbf{i} + \sin \theta \sin \psi \mathbf{j} + \cos \theta \mathbf{k}. \end{aligned}$$

Hence, substituting for $\mathbf{J}'$ and $\mathbf{K}$ into (7.8), we have

$$\begin{aligned} \boldsymbol{\omega} &= \dot{\theta}(\sin \psi \mathbf{i} + \cos \psi \mathbf{j}) + \dot{\phi}(-\sin \theta \cos \psi \mathbf{i} + \sin \theta \sin \psi \mathbf{j} + \cos \theta \mathbf{k}) + \dot{\psi} \mathbf{k} \\ &= (\dot{\theta} \sin \psi - \dot{\phi} \sin \theta \cos \psi) \mathbf{i} + (\dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi) \mathbf{j} + (\dot{\psi} + \dot{\phi} \cos \theta) \mathbf{k}. \end{aligned}$$

The desired expressions are thus

$$(\omega_1, \omega_2, \omega_3) = (\dot{\theta} \sin \psi - \dot{\phi} \sin \theta \cos \psi, \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi, \dot{\psi} + \dot{\phi} \cos \theta). \quad (7.9)$$

They are known as *Euler's geometrical equations*.

Example. A body, possessing an axis of symmetry, rotates about its centre of mass under no forces. Show that θ , the angle that the axis of symmetry makes with the axis of the resultant angular momentum, is constant, while ϕ and ψ , the other two Eulerian angles specifying the position of the body, are linear functions of the time.

Solution. Since the body has an axis of symmetry, if $G3$ is taken to coincide with this axis, then $A = B$; also $\mathbf{M} = \sum (\mathbf{r} \wedge \mathbf{F}) = \mathbf{0}$, since the body is in motion under no forces, its centre of mass being fixed. Euler's equations of motion thus become (eqns 7.7)

$$A\dot{\omega}_1 - (A - C)\omega_2\omega_3 = 0, \quad (1)$$

$$A\dot{\omega}_2 - (C - A)\omega_3\omega_1 = 0, \quad (2)$$

$$C\dot{\omega}_3 = 0. \quad (3)$$

From the third of these equations, it follows that

$$\omega_3 = \text{constant} = n,$$

say. But, since the body is in motion under no forces,

$$\frac{d\mathbf{H}}{dt} = \mathbf{0},$$

so that $\mathbf{H}$ is a constant vector, i.e. both its magnitude and its direction are constant. In addition, it is a localized vector, its axis passing through G , the centre of rotation; hence its direction defines a fixed line in space. Consequently if θ is the angle between the axis of symmetry and this line, then we shall have

$$\cos \theta = \frac{C\omega_3}{H} = \frac{Cn}{H}.$$

Since both n and H are constant, θ is also constant.

If ϕ and ψ are the other two Eulerian angles, then from the above (eqns 7.9)

$$\begin{aligned}\omega_1 &= \dot{\theta} \sin \psi - \dot{\phi} \sin \theta \cos \psi, \\ \omega_2 &= \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi.\end{aligned}$$

But, in this case, θ is constant, so that $\dot{\theta} = 0$. We therefore have

$$\omega_1 = -\dot{\phi} \sin \theta \cos \psi, \quad \omega_3 = \dot{\phi} \sin \theta \sin \psi.$$

On substituting these expressions into (1) and (2), we thus have

$$A \sin \theta (-\dot{\phi} \cos \psi + \dot{\psi} \sin \psi) - (A-C)n\dot{\phi} \sin \theta \sin \psi = 0,$$

$$A \sin \theta (\dot{\phi} \sin \psi + \dot{\psi} \cos \psi) + (C-A)n\dot{\phi} \sin \theta \cos \psi = 0,$$

or, since $\sin \theta \neq 0$,

$$\left. \begin{aligned}A\dot{\phi} \cos \psi - A\dot{\phi} \psi \sin \psi + (A-C)n\dot{\phi} \sin \psi &= 0 \\ A\dot{\phi} \sin \psi + A\dot{\phi} \psi \cos \psi - (A-C)n\dot{\phi} \cos \psi &= 0\end{aligned} \right\}. \quad (4)$$

Suppose we multiply the first of these by $\cos \psi$ and the second by $\sin \psi$. Then, adding the resulting equations, we obtain

$$A\dot{\phi} = 0.$$

Hence

$$\dot{\phi} = \text{constant} = C_1,$$

so that

$$\phi = C_1 t + C_2,$$

where C_1 and C_2 are two constants of integration. Also, since $\dot{\phi} = 0$ and $\dot{\phi} = \text{constant}$, we have, from either of (4),

$$A\dot{\psi} - (A-C)n = 0,$$

so that

$$\dot{\psi} = \frac{(A-C)}{A}n,$$

and hence

$$\psi = \frac{(A-C)}{A}nt + C_3,$$

C_3 being a further constant of integration. Thus both ϕ and ψ are linear functions of the time.*

3. Motion referred to rotating axes only one of which is fixed in the body

In the above examples the motion has been referred to a set of axes fixed in the body and rotating with it, the motion being determined by the application of Euler's equations. In some cases in which the body *possesses an axis of symmetry*, it is more convenient to refer the motion to a set of axes which rotates, but which is not fixed relative to the body. In this case, the rotational motion of the body will be given by (7.4). In order, however, that the inertia coefficients which occur in the components of $\mathbf{H}$ with respect to the rotating axes should be constant, it is necessary that the axes should be principal axes. This means that one axis coincides with the axis of symmetry and is fixed in the body.

Before applying the theory to a specific problem, we shall first determine certain relations between the components of $\boldsymbol{\theta}$, the angular velocity of the rotating axes, and those of $\boldsymbol{\omega}$, the angular velocity of the body, which hold when such a system of axes is used.

Let us choose $O3$ to be the axis fixed in the body, and let P be the point on $O3$ such that OP is unity. Then, regarding P , in the first place, as a point of the body, its velocity will be given by

$$\mathbf{v}_P = \boldsymbol{\omega} \wedge \overline{OP} = \boldsymbol{\omega} \wedge \mathbf{k},$$

where $\mathbf{k}$ is a unit vector in the direction of $O3$. But we may also regard P as being a point rigidly attached to the rotating frame $O123$. Regarded in this way, we shall have

$$\mathbf{v}_P = \boldsymbol{\theta} \wedge \mathbf{k}.$$

Equating these two expressions for $\mathbf{v}_P$, we must thus have

$$\boldsymbol{\theta} \wedge \mathbf{k} = \boldsymbol{\omega} \wedge \mathbf{k},$$

which can only be satisfied provided that

$$\boldsymbol{\theta} = \boldsymbol{\omega} + \lambda \mathbf{k},$$

where λ is an arbitrary scalar coefficient, i.e. provided that the components of $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ are such that

$$(\theta_1, \theta_2, \theta_3) = (\omega_1, \omega_2, \omega_3) + \lambda(0, 0, 1).$$

Thus it follows that, if $O3$ is fixed in the body, then we must have

$$\theta_1 = \omega_1, \quad \theta_2 = \omega_2, \quad (7.10)$$

but

$$\theta_3 \neq \omega_3. \quad (7.11)$$

* A vector treatment of this problem is given under § 5.2.

These relations are important in the application of the theory and should be remembered.

3.1. Application to the gyroscope. A *gyroscope* (or *gyrostat*), in its simplest form, consists of a rigid body possessing an axis of symmetry AA' about which it spins, the extremities of its axis being at the same time constrained to rotate in a circular frame, which itself may be free to rotate about a diameter OO' (Fig. 7.2).

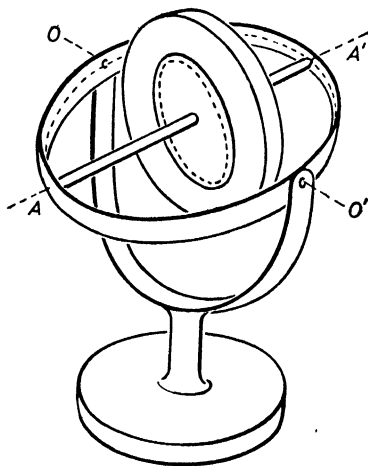


Figure 7.2

We shall now derive an important property of the gyroscope. If the frame is *rigidly* mounted in a horizontal position at some point on the earth's surface and the gyroscope given a spin about its axis, then, owing to the rotation of the earth, the axis of the gyroscope is able to remain in relative rest only when its axis is orientated along the N-S line.*

To show that this is so, let the frame of the gyroscope be fixed with its plane horizontal at a point on the earth's surface whose latitude,

* The word '*gyroscope*' was originally used to denote an instrument in which an effect, arising from the rotation of the earth, could be measured. It was coined by Foucault, who used such an instrument to verify the fact of the earth's rotation, first demonstrated by means of his pendulum experiment. The word is derived from the two Greek words, *gyros* meaning 'revolution', and *skopien* meaning 'to view'. The term gyroscope is now used, however, to denote any system in which a rapidly spinning body is mounted so that it may change the direction of its angular velocity vector. Thus a symmetrical top spinning on a rough horizontal plane may be regarded as a form of gyroscope; its motion is discussed in detail in Chap. VIII.

say, is $\lambda^\circ N$. Let $GXYZ$ be a set of orthogonal axes fixed relative to the earth such that GX , GY point due East and North respectively while GZ is in the direction of the upward vertical, and let the motion itself be referred to a set of rotating axes $G123$ having their origin at the centre of mass of the gyroscope, the directions of the axes being such that $G3$ is along the axis of the gyroscope, $G1$ coincides with GZ , and $G2$ is a mutual perpendicular such that $G123$ is a right-handed system (Fig. 7.3).

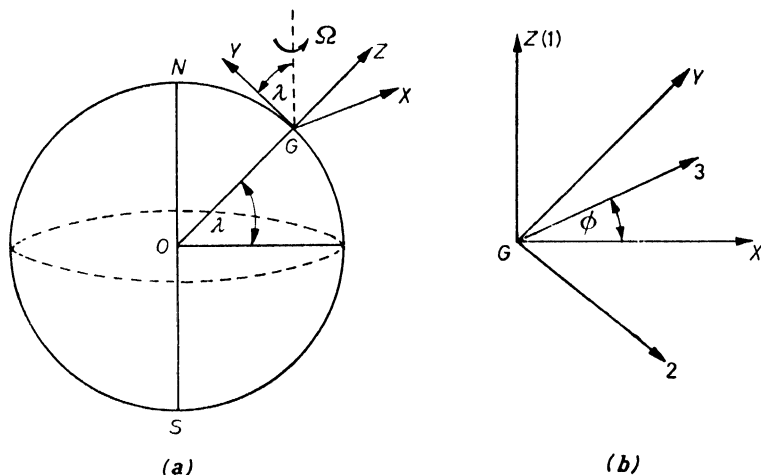


Figure 7.3

Let the position of the rotating axes be defined by the angle ϕ which $G3$ makes with GX . Then, if θ represents the angular velocity of $G123$ in space, we have

$$\dot{\theta} = \Omega + \phi \mathbf{i},$$

where Ω is the angular velocity of the earth and $\mathbf{i}$ is a unit vector in the direction of $G1$, and hence of GZ . But if $\mathbf{y}$ is a unit vector in the direction of GY ,

$$\begin{aligned}\Omega &= \Omega \sin \lambda \mathbf{i} + \Omega \cos \lambda \mathbf{y}, \\ &= \Omega \sin \lambda \mathbf{i} + \Omega \cos \lambda (\sin \phi \mathbf{k} - \cos \phi \mathbf{j}), \\ &= \Omega \sin \lambda \mathbf{i} - \Omega \cos \lambda \cos \phi \mathbf{j} + \Omega \cos \lambda \sin \phi \mathbf{k},\end{aligned}$$

$\mathbf{j}$ and $\mathbf{k}$ being unit vectors in the directions of $G2$ and $G3$ respectively. Hence

$$\theta = (\Omega \sin \lambda + \phi) \mathbf{i} - \Omega \cos \lambda \cos \phi \mathbf{j} + \Omega \cos \lambda \sin \phi \mathbf{k},$$

so that

$$(\theta_1, \theta_2, \theta_3) = (\Omega \sin \lambda + \phi, -\Omega \cos \lambda \cos \phi, \Omega \cos \lambda \sin \phi). \quad (7.12)$$

Now the scalar equations of motion of the body are given by (7.4), viz.,

$$\left. \begin{aligned} \frac{dH_1}{dt} + (\theta_2 H_3 - \theta_3 H_2) &= M_1 \\ \frac{dH_2}{dt} + (\theta_3 H_1 - \theta_1 H_3) &= M_2 \\ \frac{dH_3}{dt} + (\theta_1 H_2 - \theta_2 H_1) &= M_3 \end{aligned} \right\}. \quad [(7.4)]$$

But the only forces acting on the system are the reactions of the frame on the extremities of the axis (the centre of mass is fixed); hence, the gyroscope being constrained to rotate in a horizontal plane,

$$M_1 = 0, \quad M_2 \neq 0, \quad M_3 = 0.$$

Also, since $G3$ is an axis of symmetry, $(G1, G2, G3)$ are the principal axes at O . Consequently we have

$$(H_1, H_2, H_3) = (A\omega_1, A\omega_2, C\omega_3).$$

The scalar equations of motion for the system thus become, on substituting for (M_1, M_2, M_3) and (H_1, H_2, H_3) into (7.4),

$$A\dot{\omega}_1 + (C\theta_2\omega_3 - A\theta_3\omega_2) = 0,$$

$$A\dot{\omega}_2 + (A\theta_3\omega_1 - C\theta_1\omega_3) = M_2,$$

$$C\dot{\omega}_3 + (A\theta_1\omega_2 - A\theta_2\omega_1) = 0,$$

or, since $G3$ is fixed in the body, so that relations (7.10) and (7.11) hold,

$$\left. \begin{aligned} A\dot{\theta}_1 + (C\theta_2\omega_3 - A\theta_3\omega_2) &= 0 & (a) \\ A\dot{\theta}_2 + (A\theta_3\omega_1 - C\theta_1\omega_3) &= M_2 & (b) \\ C\dot{\omega}_3 &= 0 & (c) \end{aligned} \right\}. \quad (7.13)$$

But, from (7.13c), we have

$$\omega_3 = \text{constant} = n,$$

say. Hence (a) and (b) become, on substituting for $(\theta_1, \theta_2, \theta_3)$ from (7.12),

$$A \frac{d}{dt} (\Omega \sin \lambda + \phi) - Cn\Omega \cos \lambda \cos \phi + A\Omega^2 \cos^2 \lambda \sin \phi \cos \phi = 0,$$

$$A \frac{d}{dt} (-\Omega \cos \lambda \cos \phi) + A\Omega \cos \lambda \sin \phi (\Omega \sin \lambda + \dot{\phi}) - Cn(\Omega \sin \lambda + \dot{\phi}) = M_2,$$

i.e.

$$A\ddot{\phi} - Cn\Omega \cos \lambda \cos \phi + A\Omega^2 \cos^2 \lambda \sin \phi \cos \phi = 0,$$

$$A\Omega\dot{\phi} \cos \lambda \sin \phi + A\Omega \cos \lambda \sin \phi (\Omega \sin \lambda + \dot{\phi}) - Cn(\Omega \sin \lambda + \dot{\phi}) = M_2.$$

Neglecting terms involving Ω^2 as being small, these reduce to

$$A\dot{\phi} - Cn\Omega \cos \lambda \cos \phi = 0, \quad (7.14)$$

$$2A\Omega\dot{\phi} \cos \lambda \sin \phi - Cn(\Omega \sin \lambda + \dot{\phi}) = M_2; \quad (7.15)$$

they are the equations which define completely the behaviour of the gyroscope. From (7.14) it follows that the axis of the gyroscope will remain at rest, given $\dot{\phi} = 0$ initially, only if $\cos \phi = 0$, i.e. when $\phi = \pm \frac{1}{2}\pi$. Thus, in a position of relative rest, the axis of the gyroscope lies either along GZ or in the reverse direction. If the axis is released from rest in any other position, then it is clear that it will not remain at rest in this position, but instead will seek the N-S position. The couple acting in the gyroscope in a position of relative rest is given by (7.15).

To discuss the stability of the above positions, let ϕ_0 define a position of relative equilibrium. Then, putting

$$\phi = \phi_0 + \epsilon$$

in (7.14), the differential equation governing small disturbances from an equilibrium position becomes

$$A\ddot{\epsilon} - Cn\Omega \cos \lambda \cos(\phi_0 + \epsilon) = 0,$$

where ϵ is to be regarded as small. Neglecting second-order quantities, this reduces to

$$A\ddot{\epsilon} - Cn\Omega \cos \lambda (\cos \phi_0 - \epsilon \sin \phi_0) = 0,$$

or, since $\cos \phi_0 = 0$,

$$A\ddot{\epsilon} + Cn\Omega \epsilon \cos \lambda \sin \phi_0 = 0. \quad (7.16)$$

Hence,

(i) if $\phi_0 = \pi/2$, then (7.16) becomes

$$A\ddot{\epsilon} + Cn\Omega \epsilon \cos \lambda = 0;$$

thus the disturbed motion will be oscillatory provided n is *positive*, i.e. for the position of relative equilibrium in which the axis points due North to be stable the gyroscope must be spinning about its axis GZ in a clockwise direction.

(ii) if $\phi_0 = -\pi/2$, so that GZ points due South, then (7.16) becomes

$$A\ddot{\epsilon} - Cn\Omega \epsilon \cos \lambda = 0,$$

which represents a simple harmonic motion about the position of relative equilibrium only if n is *negative*; the condition for stability is therefore, in this case, that the gyroscope should be spinning about its axis in a counter-clockwise direction.

In both of the above cases, the period of small vibrations about a stable position of relative equilibrium is given by

$$T = 2\pi \sqrt{\frac{A}{Cn\Omega \cos \lambda}}.$$

If the frame of the gyroscope is mounted in latitude λ , so that its axis of symmetry is free to move in the meridian plane, then it can be shown, in the same way, that, in a position of relative equilibrium, the axis of the gyroscope lies along a line making an angle λ with the horizontal, the period of small vibrations about a stable position in this case being given by

$$T = 2\pi \sqrt{\frac{A}{Cn\Omega}}.$$

It should be mentioned that the above theory has important consequences in navigation; it is the basis of the development and use of the gyrocompass and the automatic pilot as navigational aids.*

4. Motion under no forces

In the example discussed in § 2 above (p. 165), we considered the motion of a rigid body rotating about its centre of mass when there were no forces acting on it. We shall now consider the general theory of such motion in detail. The theory will apply not only to the above case, but also to the following motions: (1) that of a rigid body rotating about a fixed point O , (2) the rotational motion of a rigid body with respect to its centre of mass, the forces acting on the body being such that the algebraic sum of their moments with respect to O , or the centre of mass as the case may be, vanishes. In case (2), this means that the resultant of the forces acting on the body must pass through the centre of mass; if the resultant vanishes, then the centre of mass will be either at rest or moving with uniform velocity.

In developing the theory, the motion considered will be that of a rigid body rotating about a fixed point; the results obtained for this motion may then be applied, where appropriate, to the corresponding case of the rotational motion of a rigid body with respect to its centre of mass by simply making the inertia coefficients which occur in them refer to axes having their origin at the centre of mass instead of at the fixed point.

If a rigid body rotates about a fixed point O under the action of a

* For a detailed technical treatment of gyroscopes and their uses, together with a complete bibliography on the subject, see Richardson, *The Gyroscope Applied* (Hutchinson's Scientific and Technical Publications, 1954).

system of forces such that their total moment about O is zero, then we have

$$\frac{d\mathbf{H}}{dt} = \sum (\mathbf{r} \wedge \mathbf{F}) = \mathbf{0}, \quad (7.17)$$

and hence $\mathbf{H}$, the moment of momentum of the body with respect to O , is a constant vector, i.e. it has both a constant magnitude and a constant direction in space.

The line through O whose direction is that of $\mathbf{H}$ is called *the invariable line*, while a plane perpendicular to it is called *an invariable plane*.

If $\boldsymbol{\omega}$ is the angular velocity of the body and $(\omega_1, \omega_2, \omega_3)$ are its components respectively parallel to the principal axes at O , then

$$\mathbf{H} \equiv (H_1, H_2, H_3) = (A\omega_1, B\omega_2, C\omega_3),$$

$$\text{so that} \quad A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = H^2 = \text{constant} \quad (7.18)$$

is an integral of the motion.

To obtain another integral, suppose we express (7.17) in terms of its equivalent scalar equations; when referred to the principal axes at O , these are (Euler's equations):

$$\left. \begin{aligned} A\dot{\omega}_1 - (B-C)\omega_2\omega_3 &= 0 \\ B\dot{\omega}_2 - (C-A)\omega_3\omega_1 &= 0 \\ C\dot{\omega}_3 - (A-B)\omega_1\omega_2 &= 0 \end{aligned} \right\}. \quad (7.19)$$

On multiplying these by $\omega_1, \omega_2, \omega_3$ respectively and adding, we have

$$A\omega_1\dot{\omega}_1 + B\omega_2\dot{\omega}_2 + C\omega_3\dot{\omega}_3 = 0,$$

which can be integrated immediately giving

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2T = \text{constant}. \quad (7.20)$$

In the present case T is, of course, the kinetic energy of the motion; otherwise it is the kinetic energy of the motion relative to the centre of mass.

It should be noted that both (7.18) and (7.20), involving as they do (θ, ϕ, ψ) and their rates of change, represent first integrals of the equations of motion, and so do not themselves define the position of the body at any instant. As we shall see, however, they do enable the path in space of any particular point of the body to be traced.

In consequence of the result that T is constant, it follows, since $2T = \boldsymbol{\omega} \cdot \mathbf{H}$ (Chap. VI, eqn 6.40), that

$$\boldsymbol{\omega} \cdot \mathbf{H} = \text{constant},$$

i.e. since $\mathbf{H}$ is itself constant, that the component of $\boldsymbol{\omega}$ in the direction of the invariable line is constant.

4.1. Poinso't's representation of the motion.* Let the equation of a momental ellipsoid referred to the principal axes at O be

$$Ax^2 + By^2 + Cz^2 = M\epsilon^4, \quad (7.21)$$

and let $P(\xi, \eta, \zeta)$ be the point in which the ellipsoid is cut by the

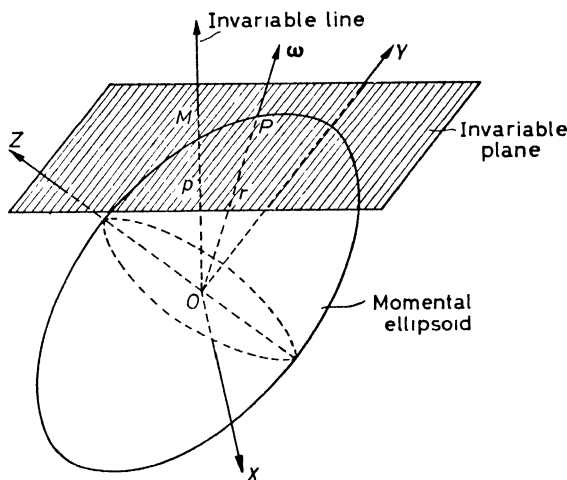


Figure 7.4

instantaneous axis of rotation (Fig. 7.4). Then the direction cosines of the normal to the tangent plane at P are proportional to

$$(A\xi, B\eta, C\zeta).$$

But, if $OP = r$, then

$$(\xi/r, \eta/r, \zeta/r) = (\omega_1/\omega, \omega_2/\omega, \omega_3/\omega), \quad (7.22)$$

so that

$$(A\xi, B\eta, C\zeta) = \frac{r}{\omega}(A\omega_1, B\omega_2, C\omega_3),$$

i.e. the normal is parallel to the invariant line. Consequently the motion is such that the tangent plane to the ellipsoid at P has a constant direction in space.

Now let p be the perpendicular distance from O to the tangent plane at P . Then we have

$$p = r \frac{(\mathbf{H} \cdot \boldsymbol{\omega})}{H\omega} = \frac{2Tr}{H\omega}. \quad (7.23)$$

* *Louis Poinso't* (1777–1859) was a French mathematician, who made many contributions in the field of Mechanics. Apart from his work on the motion of a rigid body, he originated the theory of couples, which he set forth in his '*Les Éléments de la Statique*', published in 1803.

But, since (ξ, η, ζ) is a point on the ellipsoid, we have, on substituting from (7.22) into (7.21),

$$r^2 = \frac{M\epsilon^4\omega^2}{A\omega_1^2 + B\omega_2^2 + C\omega_3^2} = \frac{M\epsilon^4\omega^2}{2T},$$

so that

$$r = \epsilon^2\omega \sqrt{\frac{M}{2T}}. \quad (7.24)$$

Hence, substituting into (7.23),

$$p = \frac{2T}{H\omega} \cdot \epsilon^2\omega \sqrt{\frac{M}{2T}} = \epsilon^2 \sqrt{\frac{2MT}{H^2}}, \quad (7.25)$$

and so is constant throughout the motion. Thus the tangent plane to the ellipsoid at P not only has a fixed direction but is also at a fixed distance from O . It is therefore a fixed plane in space. In addition, P being a point on the instantaneous axis, its velocity at the instant under consideration is zero.

We therefore have the following result: *If a rigid body is in motion about a fixed point O under the action of a system of forces such that their total moment about O is zero, then the motion may be represented by the rolling of an arbitrary momental ellipsoid for the body on an invariable plane, the angular velocity of the body at any instant being proportional to the radius of the ellipsoid in the direction of the instantaneous axis. It is known as Poincot's representation of the motion.*

4.2. The polhode and herpolhode. As the ellipsoid rolls on the invariable plane, the point of contact traces out a curve both on the ellipsoid and on the plane. The former is the curve of intersection of the instantaneous axis with the momental ellipsoid and thus defines the path of the instantaneous axis in the body; it is termed the *polhode*, the cone associated with it being the *polhode cone*. In the same way the curve traced out on the plane defines the path of the instantaneous axis in space and is termed the *herpolhode*, the associated cone being the *herpolhode cone*.

It is clear that it is possible to have an infinity of polhode and herpolhode curves, their positions in space being determined by the choice of ϵ ; the positions of the corresponding cones, however, are independent of the choice of ϵ .

From Poincot's representation, derived above, it follows that during the motion the polhode cone rolls on the herpolhode cone.

Polhode Cone. If $P(\xi, \eta, \zeta)$ is the point of intersection of the instantaneous axis with the momental ellipsoid defined by (7.21), then we have

$$A\xi^2 + B\eta^2 + C\zeta^2 = M\epsilon^4. \quad (7.26)$$

But, as we have seen, the motion is such that the length of the perpendicular from O to the tangent plane at P is constant. Hence, since

$$p^2 = M^2\epsilon^4/(A^2\xi^2 + B^2\eta^2 + C^2\zeta^2),$$

(ξ, η, ζ) must also satisfy the relation (eqn 7.25)

$$A^2\xi^2 + B^2\eta^2 + C^2\zeta^2 = M\epsilon^4 H^2/2T. \quad (7.27)$$

The polhode will thus be the curve of intersection of the two surfaces represented by (7.26) and (7.27) with (x, y, z) replacing (ξ, η, ζ) . The equation of the polhode cone will be obtained from (7.26) and (7.27) by eliminating ϵ ; its equation is therefore

$$A(A - H^2/2T)x^2 + B(B - H^2/2T)y^2 + C(C - H^2/2T)z^2 = 0. \quad (7.28)$$

Now the most general equation of a surface which passes through the polhode will clearly be of the form

$$(A^2x^2 + B^2y^2 + C^2z^2 - M\epsilon^4 H^2/2T) - \lambda(Ax^2 + By^2 + Cz^2 - M\epsilon^4) = 0, \quad (7.29)$$

where λ is an arbitrary constant. By giving λ special values, the following results may thus be obtained:

(1) If we put $\lambda = A$, then the terms in x cancel one another and the resulting equation is that of the projection of the polhode on the plane $x = 0$. Similarly the equations of the projections on the planes $y = 0$, $z = 0$ may be obtained. The projections of the polhode on the coordinate planes are thus given by:

$$\left. \begin{aligned} B(A - B)y^2 + C(A - C)z^2 &= M\epsilon^4(A - H^2/2T) & (a) \\ A(B - A)x^2 + C(B - C)z^2 &= M\epsilon^4(B - H^2/2T) & (b) \\ A(C - A)x^2 + B(C - B)y^2 &= M\epsilon^4(C - H^2/2T) & (c) \end{aligned} \right\}. \quad (7.30)$$

If we suppose $A > B > C$, then we see that (7.30a) and (7.30c) are ellipses and (7.30b) is a hyperbola; thus, in general, for any chosen values of ϵ and $H^2/2T$, the corresponding polhode consists of a closed curve, which surrounds either the axis of greatest moment or that of least moment. It is seen that the above projections are real only if $A > H^2/2T > C$, i.e. if $H^2/2T$ lies between the greatest and the least values of the principal moments of inertia.

(2) If we put $\lambda = H^2/2T$, then the constant term in (7.29) disappears and we get the equation of the polhode cone, viz. eqn (7.28),

$$A(A - H^2/2T)x^2 + B(B - H^2/2T)y^2 + C(C - H^2/2T)z^2 = 0. \quad [(7.28)]$$

The condition for the reality of this cone will be the same as that for the reality of the curves of projection in (1) above. The condition may, however, be deduced quite simply in another way; for, clearly, it can be real if, and only if, the length of the perpendicular from O to the tangent plane to the ellipsoid (eqn 7.21) lies between the greatest and

least values of the lengths of its semi-axes, i.e. it is real only if $A > H^2/2T > C$, in agreement with the above condition.

If $H^2/2T = B$, then it is seen that the cone consists of the two planes

$$z = \pm \sqrt{\left(\frac{A(A-B)}{C(B-C)}\right)}x.$$

These two planes, which pass through the y -axis, intersect the ellipsoid of inertia in two ellipses which intersect on the y -axis of the ellipsoid. These ellipses, being the curves of intersection of polhode cones with the ellipsoid, are themselves polhodes, and separate the polhodes on the surface of the ellipsoid into two classes, the one class consisting of closed curves surrounding the x -axis, the other class the closed curves surrounding the z -axis.* To pass from one class to the other it would be necessary for the path to cross these ellipses, so that $H^2/2T$ would have to assume the value B . Consequently, for any *particular* motion, for which $H^2/2T$ has, of course, a prescribed value, the polhode must consist of either a closed curve surrounding the x -axis, or of a closed curve surrounding the z -axis; the class of curve to which the polhode belongs is determined by the initial motion of the body.

Special Cases. We have seen above that, when $H^2/2T = B$, the polhode cone consists of two planes. It is of interest to consider what form the polhode cone takes for other values of $H^2/2T$. We shall assume $A > B > C$. Then we have:

(i) If $H^2/2T = A$, the cone is the line $y = 0$, $z = 0$, i.e. the instantaneous axis coincides with the axis of greatest moment throughout the motion; the polhode thus reduces to a point.

(ii) If $A > H^2/2T > B > C$, then the cone cuts the plane $x = 0$ in imaginary lines and the planes $y = 0$, $z = 0$ in real lines, so that in this case the cone surrounds the axis of greatest moment, and therefore so does the polhode.

(iii) If $H^2/2T = B$, then, as we have seen, the cone consists of two planes which pass through the axis of mean moment and cut the ellipsoid in two ellipses which separate the polhodes surrounding the axis of moment A from the polhodes surrounding the axis of moment C .

(iv) If $B > H^2/2T > C$, the cone cuts the planes $x = 0$, $y = 0$ in real lines and $z = 0$ in imaginary lines, so that the cone surrounds the axis of least moment, and so does the polhode.

(v) If $H^2/2T = C$, the cone is the line $x = 0$, $y = 0$, i.e. the instantaneous axis coincides with the axis of least moment and the polhode again reduces to a point.

* See, for instance, MacMillan, *Theoretical Mechanics—Dynamics of Rigid Bodies* (1st ed., McGraw-Hill, 1936), p. 209, Fig. 54 (continuous curves).

Herpolhode Cone. The herpolhode, as we have seen, is the path traced out by the point of contact of an ellipsoid of inertia turning about its centre and in rolling contact with an invariable plane. If M is the foot of the perpendicular from the centre on to the fixed plane and P a point on the herpolhode (Fig. 7.5), then we have (eqns 7.24 and 7.25)

$$MP^2 = r^2 - p^2 = \frac{M\epsilon^4}{2T} \left(\omega^2 - \frac{4T^2}{H^2} \right).$$

Since $4T^2/H^2$ is constant throughout the motion, and since, in addition ω has at some time during the motion both a greatest and a least value, it follows that MP will also have a greatest and a least value. The herpolhode, therefore, lies between two concentric circles. Poinso

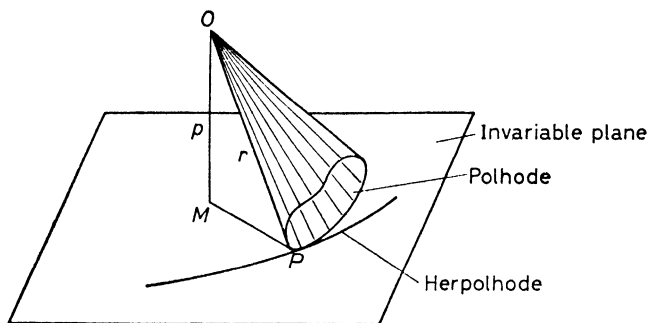


Figure 7.5

who gave the herpolhode its name, thought that the path would touch these two circles alternately and so have a snakelike appearance, which is the meaning of the word 'herpolhode'. In general, however, the curve is transcendental and is always concave towards the origin.* The name herpolhode is therefore a misnomer.

The cone traced out in space by the instantaneous axis as it traverses the herpolhode is the herpolhode cone. We shall not obtain its equation since, for this to be of any purpose, it would be necessary to refer it to a set of axes fixed in space, whereas the equations of motion (viz. Euler's equations), from which the above properties have been deduced, are all referred to axes fixed in the body and rotating with it. The nature of the herpolhode cone in the case of a solid of revolution rotating about a point on its axis is, however, discussed in § 4.4 below.

Invariable Cone. Although the invariable line is fixed in space, as the body rotates, the line clearly traces out a cone in the body. This is called the *invariable cone*.

* See, for instance, MacMillan, *Dynamics of Rigid Bodies* (1st ed.), p. 210, § 98.

To obtain its equation, we have, from (7.18) and (7.20),

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = (A\omega_1^2 + B\omega_2^2 + C\omega_3^2)H^2/2T.$$

But the direction cosines of the invariable line, when referred to the principal axes at O , are proportional to $(A\omega_1, B\omega_2, C\omega_3)$, so that if (x, y, z) are the coordinates of a point on the invariable line, when referred to the same axes, then

$$x:y:z = A\omega_1:B\omega_2:C\omega_3.$$

Consequently the equation of the invariable cone is given by

$$x^2(1-H^2/2AT) + y^2(1-H^2/2BT) + z^2(1-H^2/2CT) = 0.$$

It follows that, in its general form, it resembles that of the polhode cone discussed above (see eqn 7.28).

4.3. Stability of rotational motion. From the above discussion of the properties of the polhode and the polhode cone it follows that, if the body is set in motion so that $A > H^2/2T > C$ ($H^2/2T \neq B$), then the motion must be one in which the instantaneous axis rotates about the axis of greatest or of least moment. Further, no matter which rotation it is, this motion will continue. These axes are therefore called *the axes of permanent rotation*.

If, initially, the motion is such that either $H^2/2T = A$ or $H^2/2T = C$, then the motion is one of steady rotation with the axis of greatest or of least moment as instantaneous axis. If the motion is slightly disturbed from such a steady state, then in the new motion the path of the instantaneous axis will be that of a neighbouring polhode cone. But, for a rotation about the axis of greatest or of least moment, a neighbouring polhode will be a small closed curve surrounding the original axis of rotation. Consequently, in the disturbed motion, the instantaneous axis will never depart far from its position in the undisturbed motion. A motion in which the body rotates steadily either about the axis of greatest or of least moment is thus stable.

More generally, if the original motion is one in which the instantaneous axis rotates either about the axis of greatest or of least moment, then the motion is stable provided that the dimensions of the corresponding polhode are small; for, if this is so, the disturbed motion will be that corresponding to a neighbouring polhode and its deviation from the original will always be small. If the polhode corresponding to the original motion is not small, however, then the disturbed motion may be quite different from that of the original. This will happen if $(B-H^2/2T)$ changes its sign as a result of the disturbance. For this to occur $(B-H^2/2T)$ in the original motion must be small, and the polhode for the motion must be an elongated ellipse.

Since, when $H^2/2T = B$, the polhode cone consists of two planes through the axis of mean moment, it follows that the motion of the axis must in this case be confined to one of these planes. It can, however, be shown that, provided this relation holds, then, no matter how started, the body ultimately rotates about its axis of mean moment (see § 4.4 below). A motion in which the axis of rotation coincides with the axis of mean moment is thus a possible motion of steady rotation. If this motion undergoes a slight disturbance so that the relation $H^2/2T = B$ is destroyed, then, in the disturbed motion, the instantaneous axis will rotate either about the axis of greatest or of least moment according as, in the disturbed motion, $H^2/2T \gtrless B$. A motion of rotation about the axis of mean moment is thus unstable.

4.4. Solution of the equations of motion. When a rigid body is in motion about a fixed point O under the action of a system of forces whose total moment about O is zero, then the scalar equations governing the motion are (eqns 7.19)

$$\left. \begin{aligned} A\dot{\omega}_1 - (B-C)\omega_2\omega_3 &= 0 \\ B\dot{\omega}_2 - (C-A)\omega_3\omega_1 &= 0 \\ C\dot{\omega}_3 - (A-B)\omega_1\omega_2 &= 0 \end{aligned} \right\}. \quad [(7.19)]$$

We shall now consider the solutions of these equations when (i) $A = B$, (ii) $H^2 = 2BT$. For the solution corresponding to the general case the reader is referred to one of the standard text-books.*

(i) $A = B$. In this case (7.19) become

$$\left. \begin{aligned} A\dot{\omega}_1 - (A-C)\omega_2\omega_3 &= 0 & (a) \\ A\dot{\omega}_2 - (C-A)\omega_3\omega_1 &= 0 & (b) \\ C\dot{\omega}_3 &= 0 & (c) \end{aligned} \right\}. \quad (7.31)$$

From (7.31a) we immediately have, on integration,

$$\omega_3 = \text{constant} = n.$$

From (7.31), we also have

$$\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 + \omega_3\dot{\omega}_3 = 0,$$

so that

$$\omega = \text{constant}.$$

Thus, if γ is the angle which the instantaneous axis makes with the C -axis, then we have

$$\cos \gamma = \omega_3/\omega = n/\omega,$$

and hence, since ω_3 and ω are both constant, γ is constant. Also, if

* See, for instance, Ramsey, *Dynamics*, Vol. II, p. 280, § 11.6.

β is the angle which the instantaneous axis makes with the invariable line, then

$$\cos \beta = \frac{\boldsymbol{\omega} \cdot \mathbf{H}}{\omega H} = \frac{2T}{\omega H}.$$

Consequently, β is constant. Thus, both of these angles being constant, we have the result that, when $A = B$, both the polhode and herpolhode cones are right circular cones. Clearly this case corresponds to that in which the body is that of a solid of revolution.

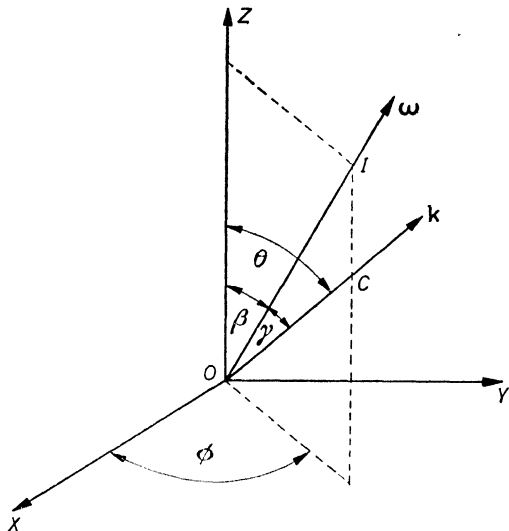


Figure 7.6

If we put $\omega_3 = n$, then (7.31a) and (7.31b) become

$$A\dot{\omega}_1 - (A-C)n\omega_2 = 0, \quad A\dot{\omega}_2 - (C-A)n\omega_1 = 0,$$

which, on eliminating ω_2 , give

$$A^2\ddot{\omega}_1 + (C-A)^2n^2\omega_1 = 0.$$

On integrating, we thus have

$$\omega_1 = p \cos(\lambda nt + \epsilon), \quad \omega_2 = -p \sin(\lambda nt + \epsilon), \quad (7.32)$$

where $\lambda = (A-C)/A$, ϵ is an arbitrary constant, and

$$p^2 = \omega_1^2 + \omega_2^2 = (H^2 - C^2)/A^2 \quad (7.33)$$

(eqn 7.18). Thus ω_1 and ω_2 are both periodic functions of t with the same period $T = 2\pi/\lambda n$.

Suppose we introduce the Eulerian angles (θ, ϕ, ψ) , the invariable line being taken as the fixed axis OZ from which θ is measured (Fig. 7.6).

Then resolving $\mathbf{H}$ along OC , the axis of symmetry, we have

$$C\omega_3 = H \cos \theta,$$

so that
$$\cos \theta = \frac{C\omega_3}{H} = \frac{Cn}{H}. \quad (7.34)$$

Thus θ is constant throughout the motion. (This also follows from the above, since $\theta = \beta \pm \gamma$, see Fig. 7.7.) Also, from Euler's geometrical equations (eqns 7.9), we have

$$\omega_1 = \dot{\theta} \sin \psi - \dot{\phi} \sin \theta \cos \psi, \quad \omega_2 = \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi,$$

where ϕ is the angle which the plane ZOC makes with a fixed plane through OZ and ψ is the angle which an axial plane fixed in the body makes with the plane ZOC ; hence, since $\dot{\theta} = 0$,

$$\omega_1 = -\dot{\phi} \sin \theta \cos \psi, \quad \omega_2 = \dot{\phi} \sin \theta \sin \psi. \quad (7.35)$$

Thus

$$\omega_1^2 + \omega_2^2 = \dot{\phi}^2 \sin^2 \theta.$$

On comparing this expression with (7.33), we therefore have

$$\dot{\phi}^2 \sin^2 \theta = p^2 = (H^2 - C^2 n^2)/A^2, \quad (7.36)$$

so that

$$\dot{\phi}^2 = \frac{H^2 - C^2 n^2}{A^2} \cdot \frac{H^2}{H^2 - C^2 n^2} = \frac{H^2}{A^2}$$

in virtue of (7.34), i.e.

$$\dot{\phi} = \pm H/A,$$

and hence

$$\phi = \pm \frac{H}{A} t + \text{constant},$$

the choice of sign being determined when the initial conditions are known.* We also see, on comparing (7.32) and (7.35) and noting that $\dot{\phi} \sin \theta (= p)$ is constant (eqn 7.36), that

$$\dot{\psi} = \lambda n t + \pi + \epsilon,$$

so that

$$\dot{\psi} = \lambda n = \frac{(A - C)}{A} n.$$

These results for ϕ and ψ are, of course, consistent with those obtained in the example considered in § 2 above.

If I is a point on the instantaneous axis, then since both the polhode and the herpolhode cones are right circular cones, and since the motion of a rigid body 'under no forces' is, by the definition of these cones,

* When we consider the same problem vectorially in § 5.2, we shall find that the positive sign has to be taken in all cases.

equivalent to a rolling of the one cone on the other, it follows that OZ , OC and OI must be coplanar. Thus ϕ will be the angle which the plane ZOI makes with a fixed plane in space and so ϕ ($= \pm H/A$) must be the angular velocity with which the instantaneous axis describes the herpolhode cone.

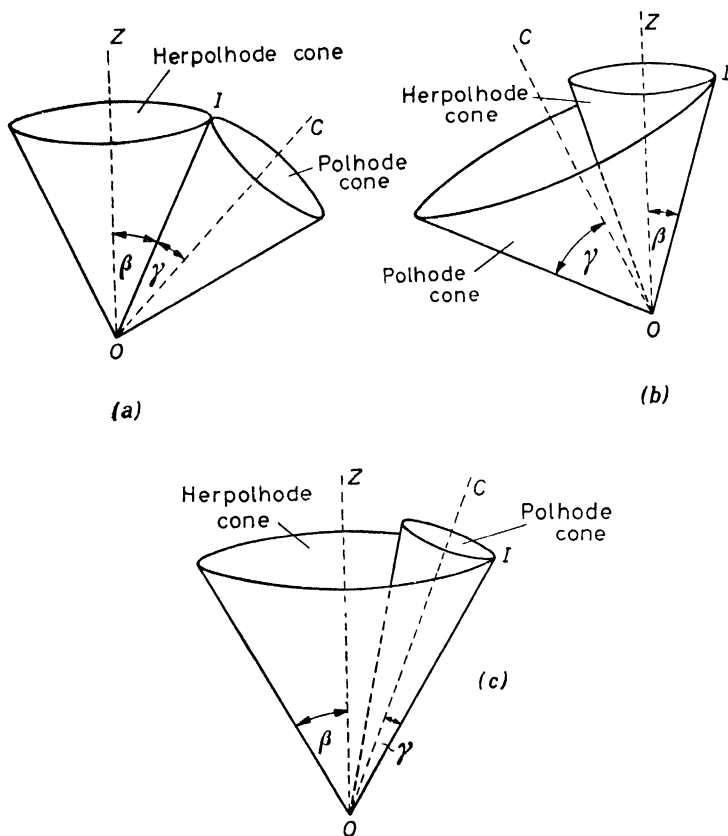


Figure 7.7

Similarly ψ will be the angle which the plane ZOI makes with a fixed plane in the body. Thus, relative to the body, the angular velocity with which the instantaneous axis describes the polhode cone is given by $-\psi = n(C-A)/A$.

The nature of the polhode and herpolhode cones may be as represented in Fig. 7.7, (a), (b), or (c).

Since in terms of Euler's angular coordinates

$$n = \dot{\psi} + \dot{\phi} \cos \theta$$

from (eqn 7.9), we have

$$\dot{\phi} = \frac{n - \dot{\psi}}{\cos \theta}.$$

But

$$\dot{\psi} = n(A-C)/A.$$

Hence, substituting for $\dot{\psi}$, it follows that, for any given motion, $\dot{\phi}$ and $\dot{\psi}$ are related by the equation

$$\dot{\phi} = \frac{Cn}{A \cos \theta} = \frac{C\dot{\psi}}{(A-C) \cos \theta}.$$

That $\dot{\phi}$ and $\dot{\psi}$ are related in this way could also have been deduced from first principles. Since the angular velocity of the body is made

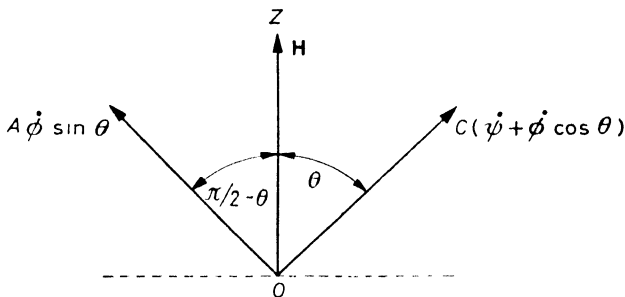


Figure 7.8

up of the angular velocity $\dot{\psi}$ about the axis of the body together with the angular velocity $\dot{\phi}$ about the invariable line OZ , it follows that at any instant $\mathbf{H}$ is the resultant of two components, a component $C(\dot{\psi} + \dot{\phi} \cos \theta)$ about the axis of symmetry and a component $A\dot{\phi} \sin \theta$ about a perpendicular axis in the plane defined by the axis of symmetry and the invariable line, these two axes being principal axes at O (see Fig. 7.8). Thus, resolving in a direction in this plane at right angles to $\mathbf{H}$, it is clear that $\dot{\phi}$ and $\dot{\psi}$ must be such that

$$A\dot{\phi} \sin \theta \cos \theta = C(\dot{\psi} + \dot{\phi} \cos \theta) \sin \theta,$$

from which the above relation follows.

The above type of motion in which the axis of symmetry of the body makes a constant angle with the invariable line and rotates around it with a uniform angular velocity is termed one of *steady precession*. Such motion is discussed in full in the next chapter.

(ii) $H^2 = 2BT$. In this case the polhode cone splits up into two planes, while the first integrals of the equations of motion become

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2T,$$

and

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = H^2 = 2BT.$$

On rearranging, these may be written

$$A\omega_1^2 + (B\omega_2^2 - 2T) + C\omega_3^2 = 0,$$

$$A^2\omega_1^2 + B(B\omega_2^2 - 2T) + C^2\omega_3^2 = 0,$$

which, on solving for $\omega_1, \omega_2, \omega_3$, give

$$\frac{\omega_1^2}{C(C-B)} = \frac{2T - B\omega_2^2}{AC(C-A)} = \frac{\omega_3^2}{A(B-A)}. \quad (7.37)$$

Thus, throughout the motion,

$$\omega_1 = \mu\omega_3, \quad (7.38)$$

where

$$\mu^2 = \frac{C(B-C)}{A(A-B)}. \quad (7.39)$$

Also, from the second of the equations of motion (eqn 7.19), we have

$$B\dot{\omega}_2 + (A-C)\omega_1\omega_3 = 0,$$

which, on substituting for ω_1 from (7.38), gives

$$B\dot{\omega}_2 + (A-C)\mu\omega_3^2 = 0.$$

But, from (7.37),

$$\omega_3^2 = \frac{(B-A)}{C(C-A)}(2T - B\omega_2^2). \quad (7.40)$$

Hence we have

$$B\dot{\omega}_2 + \mu \frac{(A-B)}{C}(2T - B\omega_2^2) = 0,$$

which may be written

$$\frac{d\omega_2}{\alpha_2^2 - \omega_2^2} + \mu \frac{(A-B)}{C} dt = 0,$$

where $\alpha_2^2 = 2T/B$. On integration, this gives

$$\frac{\omega_2}{\alpha_2} = \frac{Dc^{2\mathcal{M}} - 1}{Dc^{2\mathcal{M}} + 1},$$

where D is an arbitrary constant and

$$\lambda^2 = \mu^2 \alpha_2^2 \frac{(A-B)^2}{C^2} = \frac{(A-B)(B-C)T}{ABC}$$

If we measure t from the instant when $\omega_2 = 0$, then $D = 1$ and the expression for ω_2 becomes

$$\omega_2 = \alpha_2 \tanh \lambda t.$$

ω_1 and ω_3 can now be obtained. From (7.40) we have

$$\begin{aligned} C(A-C)\omega_3^2 &= (A-B)(2T-B\omega_2^2) \\ &= (A-B)\left(2T-B\frac{2T}{B}\tanh^2 \lambda t\right) \\ &= 2T(A-B) \operatorname{sech}^2 \lambda t, \end{aligned}$$

so that

$$\omega_3 = \alpha_3 \operatorname{sech} \lambda t,$$

and

$$\alpha_3 = \sqrt{\frac{2T(A-B)}{C(A-C)}},$$

while

$$\omega_1 = \mu\omega_3 \text{ (eqns 7.38 and 7.39).}$$

We thus have

$$(\omega_1, \omega_2, \omega_3) = (\alpha_1 \operatorname{sech} \lambda t, \alpha_2 \tanh \lambda t, \alpha_3 \operatorname{sech} \lambda t),$$

where $\alpha_1 = \mu\alpha_3$, and α_2, α_3 are defined as above. Since the three components of the angular velocity are now known as functions of t , the angular velocity of the body at any instant is thus completely defined.

It should be noticed that since, when $H^2 = 2BT$, $\omega_1 = \mu\omega_3$, where μ is a constant depending only on the inertia coefficients A, B , and C , only two of the three unknowns ($\omega_1, \omega_2, \omega_3$) may be regarded as independent. Thus the complete solution involves only two arbitrary constants of integration and not the usual three. In the above expressions these are D , which, by a suitable choice of the origin for t , has been put equal to unity, and T ; the third constant H is now related to T by the relation $H^2 = 2BT$.

Since $\operatorname{sech} \lambda t \rightarrow 0$ as $t \rightarrow \infty$, it follows that ultimately the body rotates about the axis of mean moment whatever its initial motion.

5. Vector treatment of the motion of a uniaxial body

In the case in which a body possesses symmetry with respect to an axis, so that $A = B$, say, and rotates about a point O on its axis, then in many cases the motion may be treated quite simply vectorially. This arises from the fact that when $A = B$ both $\boldsymbol{\omega}$, the angular velocity of the body, and $\mathbf{H}$, the moment of momentum of the body with respect to O , may be conveniently represented by vector expressions.

5.1. Expressions for the angular velocity and the angular momentum of a uniaxial body. Let $\mathbf{k}$ be a unit vector in the

direction of the axis of symmetry. Then, if P is the point on the axis of symmetry such that OP is unity, the velocity of P will be given by

$$\mathbf{v}_P = \boldsymbol{\omega} \wedge \mathbf{k}.$$

But we also have

$$\mathbf{v}_P = \frac{d\mathbf{k}}{dt},$$

and hence, equating the two expressions, we have

$$\boldsymbol{\omega} \wedge \mathbf{k} = \frac{d\mathbf{k}}{dt}.$$

On multiplying both sides of this equation vectorially by $\mathbf{k}$, we thus have

$$\mathbf{k} \wedge (\boldsymbol{\omega} \wedge \mathbf{k}) = \mathbf{k} \wedge \frac{d\mathbf{k}}{dt},$$

or

$$\boldsymbol{\omega}(\mathbf{k} \cdot \mathbf{k}) - \mathbf{k}(\mathbf{k} \cdot \boldsymbol{\omega}) = \mathbf{k} \wedge \frac{d\mathbf{k}}{dt},$$

i.e.

$$\boldsymbol{\omega} = \mathbf{k} \wedge \frac{d\mathbf{k}}{dt} + n\mathbf{k}, \quad (7.41)$$

where $n (= \mathbf{k} \cdot \boldsymbol{\omega})$ is the component of the angular velocity about the axis of symmetry.

To obtain the expression for the angular momentum, or moment of momentum, of the body with respect to O , we notice that $\mathbf{k} \wedge d\mathbf{k}/dt$ is the component of $\boldsymbol{\omega}$ at right angles to $\mathbf{k}$. Suppose we choose a set of orthogonal axes $O123$ such that at any instant $O3$ is in the direction of $\mathbf{k}$, $O1$ in the direction of $d\mathbf{k}/dt$ (since $\mathbf{k}$ is a unit vector, $d\mathbf{k}/dt$ is at right angles to $\mathbf{k}$), and $O2$ has the direction of $\mathbf{k} \wedge d\mathbf{k}/dt$, $O123$ thus forming a right-handed system. Then, from (7.41), we have

$$\boldsymbol{\omega} \equiv (\omega_1, \omega_2, \omega_3) = \left(0, \left|\mathbf{k} \wedge \frac{d\mathbf{k}}{dt}\right|, n\right),$$

where $\left|\mathbf{k} \wedge \frac{d\mathbf{k}}{dt}\right|$ denotes the magnitude of the component of $\boldsymbol{\omega}$ in the direction of $\mathbf{k} \wedge \frac{d\mathbf{k}}{dt}$. But the above axes are principal axes at O , the corresponding principal moments of inertia being A, A, C ; hence

$$\begin{aligned} \mathbf{H} &\equiv (H_1, H_2, H_3) = (A\omega_1, A\omega_2, C\omega_3) \\ &= \left(0, A\left|\mathbf{k} \wedge \frac{d\mathbf{k}}{dt}\right|, Cn\right). \end{aligned}$$

The angular momentum of the body will therefore be given by

$$\mathbf{H} = A\left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt}\right) + Cn\mathbf{k}. \quad (7.42)$$

From the expressions (7.41) and (7.42) for $\boldsymbol{\omega}$ and $\mathbf{H}$ respectively, it follows that $\mathbf{k}$, $\boldsymbol{\omega}$, and $\mathbf{H}$ are coplanar, both $\boldsymbol{\omega}$ and $\mathbf{H}$ lying in the plane defined by $\mathbf{k}$ and $\mathbf{k} \wedge d\mathbf{k}/dt$.

Assuming n to be positive, the above results are illustrated diagrammatically in Fig. 7.9.

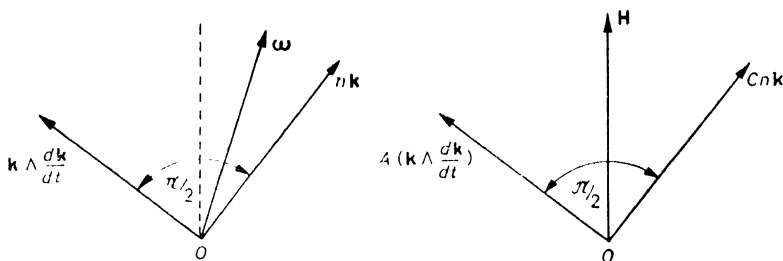


Figure 7.9

The above expressions for $\boldsymbol{\omega}$ and $\mathbf{H}$ are fundamental to the vector treatment of the motion of any uniaxial body and should be remembered.

5.2. Application: Determination of the motion of a uniaxial body rotating about a fixed point O under no forces. Let $\mathbf{h}$ be a unit vector in the direction of $\mathbf{H}$, the angular momentum with respect to O . Then, since for this type of motion $\mathbf{H}$ is constant, $\mathbf{h}$ will be a bound vector with its point of application at O and having a constant direction in space. We shall now derive the following properties of the motion.

(i) $\boldsymbol{\omega}$ is constant. Let $\mathbf{j}$ be a unit vector in the direction of $\mathbf{k} \wedge d\mathbf{k}/dt$, and suppose we write

$$\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} = p\mathbf{j},$$

so that p is the component of $\boldsymbol{\omega}$ perpendicular to $\mathbf{k}$ in the plane defined by $\mathbf{h}$ and $\mathbf{k}$ ($\mathbf{h}$, $\mathbf{k}$, and $\boldsymbol{\omega}$ are coplanar as shown above). With this notation the expressions for $\boldsymbol{\omega}$ and $\mathbf{H}$ become

$$\boldsymbol{\omega} = p\mathbf{j} + n\mathbf{k}, \quad (7.43)$$

$$\mathbf{H} = Ap\mathbf{j} + Cn\mathbf{k}. \quad (7.44)$$

If we differentiate the second of these with respect to t , we have, since $\mathbf{H}$ is constant throughout the motion,

$$Ap \frac{d\mathbf{j}}{dt} + A \frac{dp}{dt} \mathbf{j} + Cn \frac{d\mathbf{k}}{dt} + C \frac{dn}{dt} \mathbf{k} = \mathbf{0}. \quad (7.45)$$

This gives, on multiplying scalarly by $\mathbf{k}$,

$$Ap \left(\mathbf{k} \cdot \frac{d\mathbf{j}}{dt} \right) + C \frac{dn}{dt} = 0.$$

But $\mathbf{j}$ is always at right angles to $\mathbf{k}$, so that $\mathbf{k} \cdot \mathbf{j} = 0$. Differentiating with respect to t , we thus have

$$\mathbf{k} \cdot \frac{d\mathbf{j}}{dt} + \frac{d\mathbf{k}}{dt} \cdot \mathbf{j} = 0,$$

so that

$$\mathbf{k} \cdot \frac{d\mathbf{j}}{dt} = - \frac{d\mathbf{k}}{dt} \cdot \mathbf{j}.$$

But

$$p\mathbf{j} = \mathbf{k} \wedge \frac{d\mathbf{k}}{dt},$$

and hence

$$\frac{d\mathbf{k}}{dt} \cdot \mathbf{j} = \frac{1}{p} \left\{ \frac{d\mathbf{k}}{dt} \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) \right\} = 0.$$

Thus

$$\mathbf{k} \cdot \frac{d\mathbf{j}}{dt} = 0,$$

and hence

$$C \frac{dn}{dt} = 0.$$

We therefore have

$$\omega_3 = n = \text{constant}.$$

Similarly, the scalar multiplication of (7.45) by $\mathbf{j}$ leads to

$$A \frac{dp}{dt} = 0,$$

so that

$$p = \text{constant}.$$

Since

$$\omega^2 = p^2 + n^2,$$

it follows that ω is constant throughout the motion.

(ii) *The polhode and herpolhode cones are circular.* If γ is the angle that the instantaneous axis makes with the axis of symmetry, then, since

$$n = \mathbf{k} \cdot \boldsymbol{\omega} = \omega \cos \gamma$$

so that

$$\cos \gamma = n/\omega,$$

it follows that γ is constant. The polhode cone is thus right circular.

Also, from (7.42), we have, on multiplying scalarly by $\mathbf{k}$,

$$Cn = \mathbf{k} \cdot \mathbf{H} = H \cos \theta,$$

where θ is the angle that the axis of symmetry makes with the invariable line (Fig. 7.6), so that

$$\cos \theta = Cn/H;$$

hence, since C , n , and H are constant, θ is constant. But $\mathbf{h}$, $\mathbf{k}$, and $\boldsymbol{\omega}$ are coplanar. Consequently the angle $\beta = (\theta \pm \gamma)$ (Fig. 7.10), which the instantaneous axis makes with $\mathbf{h}$, will also be constant. Therefore the herpolhode cone is also right circular.

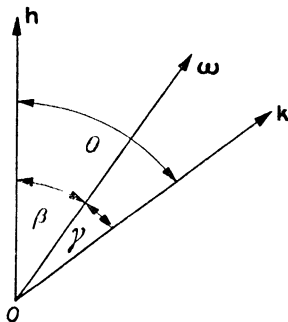


Figure 7.10

(iii) *Angular velocities of the instantaneous axis in the body and in space.* On multiplying (7.42) vectorially by $\mathbf{k}$, we have

$$\begin{aligned} \mathbf{k} \wedge \mathbf{H} &= A \left\{ \mathbf{k} \wedge \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) \right\} \\ &= -A \frac{d\mathbf{k}}{dt}, \end{aligned}$$

so that

$$\frac{d\mathbf{k}}{dt} = \frac{H}{A} (\mathbf{h} \wedge \mathbf{k}).$$

Thus $\mathbf{k}$ rotates around the invariable line $\mathbf{h}$ with a uniform angular velocity H/A . This is, therefore, the angular velocity with which the

plane containing $\mathbf{h}$, $\boldsymbol{\omega}$, and $\mathbf{k}$ rotates around $\mathbf{h}$: it is, therefore, the angular velocity with which the instantaneous axis describes the herpolhode cone. Since

$$H^2 = A^2 p^2 + C^2 n^2 = \omega^2 (A^2 \sin^2 \gamma + C^2 \cos^2 \gamma),$$

the magnitude of this angular velocity is given by

$$\Omega_h = \omega (A^2 \sin^2 \gamma + C^2 \cos^2 \gamma)^{1/2} / A.$$

To find the angular velocity with which the instantaneous axis describes the polhode cone, we have, on referring the motion to a set of axes rotating with the body,

$$\frac{\partial \mathbf{H}}{\partial t} + \boldsymbol{\omega} \wedge \mathbf{H} = \mathbf{0}, \quad \left(\frac{d\mathbf{H}}{dt} = \mathbf{0} \right),$$

i.e.

$$Ap \frac{\partial \mathbf{j}}{\partial t} + Cn \frac{\partial \mathbf{k}}{\partial t} + (p\mathbf{j} + n\mathbf{k}) \wedge (Ap\mathbf{j} + Cn\mathbf{k}) = \mathbf{0}.$$

But $\mathbf{k}$ is a constant vector with respect to these axes, so that $\partial \mathbf{k} / \partial t = \mathbf{0}$; consequently, the above equation reduces to

$$Ap \frac{\partial \mathbf{j}}{\partial t} + Apn(\mathbf{k} \wedge \mathbf{j}) + Cpn(\mathbf{k} \wedge \mathbf{j}) = \mathbf{0},$$

so that

$$\frac{\partial \mathbf{j}}{\partial t} = n \frac{(C-A)}{A} (\mathbf{k} \wedge \mathbf{j}).$$

Thus, relative to the body, $\mathbf{j}$ rotates about $\mathbf{k}$ with a uniform angular velocity

$$\Omega_p = \frac{(C-A)}{A} n = \frac{(C-A)}{A} \omega \cos \gamma.$$

It follows that this is the angular velocity of the instantaneous axis in the body.

It is seen that the above values for Ω_h and Ω_p are identical with those obtained in § 4.4 above.*

* The earth, while not strictly a sphere, is symmetrical about the polar axis, its form being that of a prolate spheroid ($A < C$). Since the moments of the forces due to the other heavenly bodies with respect to its centre are small, its rotational motion may be regarded as being similar to that of a uniaxial body under no forces. If its axis of rotation is not coincident with the polar axis, then, by the above theory, it should describe a cone about the polar axis with period $2\pi\sqrt{\{A/n(C-A)\}}$, assuming the earth to be absolutely rigid. If we put n equal to the earth's angular velocity, then since it has been established that the value of $(C-A)/A$ is 0.0033, the above result would give a period of approximately 306 days, a result known as the 'Eulerian nutation' of the earth. Some observational evidence exists that a motion of this nature takes place, although the period is longer, being of the order of 427 days.

5.3. Examples

(1) *A rigid body, having an axis of symmetry, is free to turn about its centre of mass. It is acted upon by a frictional couple $-\lambda\omega$, where ω is the angular velocity of the body and λ is a positive constant. Deduce an expression for the angular velocity as a function of the time, and show that the axis of rotation ultimately coincides with the axis of symmetry provided C , the moment of inertia about this axis, is greater than A .*

Solution. The equation governing the motion of the body, viz.

$$\frac{d\mathbf{H}}{dt} = \sum (\mathbf{r} \wedge \mathbf{F}),$$

in this case becomes, in virtue of the expressions (7.43) and (7.44) for ω and $\mathbf{H}$,

$$\frac{d}{dt}(Ap\mathbf{j} + Cn\mathbf{k}) = -\lambda(p\mathbf{j} + n\mathbf{k}),$$

or, on differentiating with respect to t ,

$$Ap\frac{d\mathbf{j}}{dt} + A\frac{dp}{dt}\mathbf{j} + Cn\frac{d\mathbf{k}}{dt} + C\frac{dn}{dt}\mathbf{k} = -\lambda(p\mathbf{j} + n\mathbf{k}).$$

If we now multiply both sides of this equation scalarly by $\mathbf{j}$, then we have

$$A\frac{dp}{dt} = -\lambda p,$$

which, on integrating, gives

$$p = D_1 e^{-\lambda t/A},$$

where D_1 is an arbitrary constant.

In the same way, multiplying both sides of the equation scalarly by $\mathbf{k}$, we have

$$C\frac{dn}{dt} = -\lambda n,$$

so that

$$n = D_2 e^{-\lambda t/C},$$

where D_2 is a second arbitrary constant.

The magnitude of the angular velocity as a function of t is thus given by

$$\omega^2 = p^2 + n^2 = D_1^2 e^{-2\lambda t/A} + D_2^2 e^{-2\lambda t/C}.$$

If γ is the angle that the instantaneous axis makes with the axis of symmetry, then

$$\begin{aligned} \tan \gamma &= p/n \\ &= (D_1/D_2) e^{-\lambda(1/A - 1/C)t}, \end{aligned}$$

using the above expressions for p and n ,

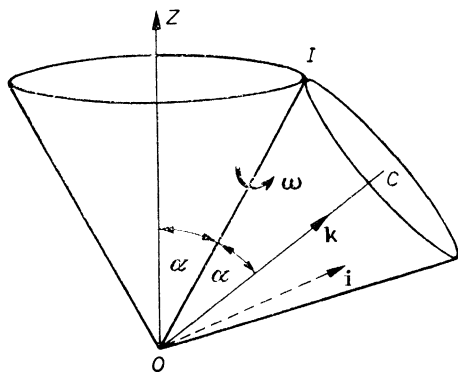
$$= (D_1/D_2) e^{-\lambda((C-A)/A)t}.$$

But, if $C > A$, then clearly $\tan \gamma \rightarrow 0$ as $t \rightarrow \infty$, i.e. $\gamma \rightarrow 0$ as $t \rightarrow \infty$, and the axis of rotation ultimately coincides with the axis of symmetry.

(2) A uniform solid of revolution moves about a point O , so that its motion may be represented by the uniform rolling of a cone of semi-vertical angle α fixed in the body upon an equal cone fixed in space, the axis of the former being the axis of revolution of the solid. Show that the forces necessary to maintain the motion are equivalent to a couple of moment

$$\frac{1}{2}\omega^2 \tan \alpha \{C + (C-A) \cos 2\alpha\},$$

where ω is the magnitude of the resultant angular velocity and A, A, C are the principal moments of inertia at the point. Show also that the axis of the equivalent couple is at right angles to the plane containing the axes of the cones.



Solution. Since the body has an axis of symmetry, ω and $\mathbf{H}$, its moment of momentum with respect to O , are given respectively by (7.41) and (7.42) above. The forces necessary to maintain the motion must thus have a total moment given by

$$\mathbf{M} = \frac{d\mathbf{H}}{dt} = A \left(\mathbf{k} \wedge \frac{d^2\mathbf{k}}{dt^2} \right) + Cn \frac{d\mathbf{k}}{dt}, \quad (1)$$

n being constant since $n = \mathbf{k} \cdot \omega = \omega \cos \alpha$. Also, we have

$$\frac{d\omega}{dt} = \mathbf{k} \wedge \frac{d^2\mathbf{k}}{dt^2} + n \frac{d\mathbf{k}}{dt},$$

so that, if we eliminate $\mathbf{k} \wedge d^2\mathbf{k}/dt^2$ from (1), then

$$\mathbf{M} = A \frac{d\omega}{dt} + n(C-A) \frac{d\mathbf{k}}{dt}. \quad (2)$$

But

$$\frac{d\mathbf{k}}{dt} = \omega \wedge \mathbf{k} = \omega \sin \alpha \mathbf{i}, \quad (3)$$

where $\mathbf{i}$ is a unit vector at right angles to the plane ZOC as shown (it is, of course, in the direction of $d\mathbf{k}/dt$). Also, if $\mathbf{a}$ is a unit vector in the direction of the instantaneous axis of rotation, then

$$\frac{d\boldsymbol{\omega}}{dt} = \omega \frac{d\mathbf{a}}{dt}. \quad (4)$$

If ω' is the angular velocity with which $\mathbf{a}$ rotates around the vertical, however, then

$$\frac{d\mathbf{a}}{dt} = \omega'(\mathbf{z} \wedge \mathbf{a}) = \omega' \sin \alpha \mathbf{i},$$

$\mathbf{z}$ being a unit vector in the direction of the upward vertical. Also, OZ , OI , and OC are always coplanar from the nature of the motion. Consequently, considering the motion of a point on OC , unit distance from O , then we have

$$\omega'(\mathbf{z} \wedge \mathbf{k}) = \omega(\mathbf{a} \wedge \mathbf{k}),$$

i.e.

$$\omega' \sin 2\alpha \mathbf{i} = \omega \sin \alpha \mathbf{i};$$

hence

$$\omega' = \omega \sin \alpha \operatorname{cosec} 2\alpha = \frac{1}{2}\omega \sec \alpha,$$

so that

$$\frac{d\mathbf{a}}{dt} = \frac{1}{2}\omega \tan \alpha \mathbf{i}. \quad (5)$$

From (4) and (5), we thus have

$$\frac{d\boldsymbol{\omega}}{dt} = \omega \frac{d\mathbf{a}}{dt} = \frac{1}{2}\omega^2 \tan \alpha \mathbf{i}. \quad (6)$$

On substituting for $d\mathbf{k}/dt$ and $d\boldsymbol{\omega}/dt$ from (3) and (6) respectively into (2), the total moment of the forces required to maintain the motion is thus given by

$$\begin{aligned} \mathbf{M} &= \frac{1}{2}A\omega^2 \tan \alpha \mathbf{i} + n(C-A)\omega \sin \alpha \mathbf{i} \\ &= \omega^2 \left\{ \frac{1}{2} \tan \alpha + (C-A) \sin \alpha \cos \alpha \right\} \mathbf{i} \\ &= \frac{1}{2}\omega^2 \tan \alpha \{A + 2(C-A) \cos^2 \alpha\} \mathbf{i} \\ &= \frac{1}{2}\omega^2 \tan \alpha \{C + (C-A) \cos 2\alpha\} \mathbf{i}. \end{aligned}$$

The forces must thus be equivalent to a couple of moment

$$\frac{1}{2}\omega^2 \tan \alpha \{C + (C-A) \cos 2\alpha\},$$

the axis of the couple being parallel to $\mathbf{i}$, i.e. at right angles to the plane ZOC .

It should be noticed, from the study of these examples, that, in order to treat the motion of a uniaxial body vectorially, it is necessary that one should be able to *express* the resultant moment with respect to

O of the forces acting on the body as the sum of three vectors respectively parallel to the three vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as defined above.

Examples

1. A uniform circular lamina rotates with constant angular velocity ω about a fixed axis which passes through its centre and makes an angle α with the normal to its plane. Show that, if a is the radius of the lamina, the reactions of the axis on the lamina are equivalent to a couple of magnitude $\frac{1}{2}Ma^2\omega^2 \sin \alpha \cos \alpha$.

2. A thin plane lamina is free to rotate about O , its centre of mass, and is acted upon by no forces except the reaction at O . OX, OY are the principal axes of inertia at O in its plane and the moments of inertia with respect to these axes are A, B . The lamina is set in motion with an angular velocity Ω about an axis OJ in its plane which makes an angle α with OX . Show that, throughout the motion, the component of the angular velocity in the plane of the lamina has a constant magnitude whilst its direction makes a constant angle θ with OX such that

$$(A+B)\dot{\theta}^2 = (A-B)(\sin^2 \theta - \sin^2 \alpha)\Omega^2.$$

3. The end of a uniform rod of length l is mounted on an axle so that the rod can turn freely in the plane perpendicular to the axle. If the axle is made to rotate in a horizontal plane with a constant angular velocity Ω , the point where the rod joins the axle remaining fixed, prove that θ , the angle that the rod makes with the downward vertical at time t , satisfies the equation

$$\ddot{\theta} = \Omega^2 \sin \theta \cos \theta + \frac{3g}{2l} \sin \theta.$$

Examine the stability of the motions in which θ is constant.

4. A solid of revolution is free to move about its centre of mass which is fixed. It is set in motion from the rest by an impulsive couple whose components along and perpendicular to the axis of symmetry are J and λJ respectively. If λ is a small quantity, show that the equation of the polhode cone is

$$A^2(x^2+y^2) - \lambda^2 C^2 z^2 = 0,$$

where A and C have their usual meanings.

5. A uniform right circular cylinder, whose length is three times the radius of its circular section and which is set spinning with an angular velocity Ω about an axis through its centre and through a point of the rim of one of its plane ends, is thrown into the air. If the resistance

of the air is neglected, prove that the times taken by the instantaneous axis to describe the herpolhode and polhode cones are in the ratio 3:5.

6. An elliptic disc, whose semi-major and minor axes are of lengths a and $a/\sqrt{2}$ respectively, is free to move about its centre which is fixed. It is set rotating with an angular velocity Ω about an axis in a plane normal to the disc and which passes through the major axis. If the initial axis of rotation makes an angle 30° with the major axis, show that at time t the component angular velocities of the disc about the principal axes at its centre are respectively

$$\Omega\sqrt{\frac{3}{2}} \operatorname{sech}(\Omega t/2), \quad \Omega\sqrt{\frac{3}{2}} \tanh(\Omega t/2), \quad \frac{1}{2}\Omega \operatorname{sech}(\Omega t/2).$$

7. A uniform rectangular lamina, whose sides are of lengths a and $2a$, is set rotating under the action of no external forces about an axis through its centre of mass G . If initially the instantaneous axis of rotation is equally inclined to the three principal axes at G , prove that the angle between the instantaneous axis and the plane of the lamina varies between the limits α and β , where $2 \cot \alpha = \cot \beta = \sqrt{5}$.

8. A gyroscopic compass is mounted in a ship which is speeding North with speed V in latitude λ . If the axis of the compass is constrained to move in a horizontal plane show that, when in a position of relative stable equilibrium, the axis points in a direction.

$$\alpha = \tan^{-1} \frac{V}{\alpha \Omega \cos \alpha}$$

to the West of North, where Ω is the angular velocity of the earth and a is its radius.

What is the corresponding result if the ship is speeding due East.

(*Hint*: In the first case, owing to the motion of the ship, λ is no longer constant, while the angular velocity of the axes of reference about the East-West line is increased by an amount V/a).

CHAPTER VIII

PRECESSIONAL MOTION

1. General Properties

We have already met with an example of precessional motion in the previous chapter (see § 4.4, Case $A = B$, p. 180). We shall now discuss the nature and general properties of such motion in detail. The theory of precessional motion is important in the field of Theoretical Physics and also in that of Applied Mechanics in which it has many applications; in particular, it is the theory underlying the use of both gyroscopic stabilizers in ships and automatic pilots.

1.1. Definition. We have already seen that if a rigid body has one point fixed and is acted upon by a system of forces, then its motion is determined by the equation

$$d\mathbf{H}/dt = \sum (\mathbf{r} \wedge \mathbf{F}) = \mathbf{M}, \quad (8.1)$$

where $\mathbf{H}$ is the moment of momentum of the body with respect to the fixed point and $\mathbf{M}$ is the resultant moment of the forces acting on the body with respect to the same point. A similar equation holds for the motion of the body with respect to its centre of mass. In any small interval of time δt , the change in the moment of momentum of the body will therefore be given by

$$\delta\mathbf{H} = \mathbf{M}\delta t, \quad (8.2)$$

i.e. the change in the moment of momentum of the body during any small interval of time $(t, t + \delta t)$ must be parallel to the instantaneous direction of $\mathbf{M}$ at time t .

Now suppose the body to be uniaxial and free to rotate about a fixed point O on its axis of symmetry OC under a given system of forces, the direction of the resultant moment of the forces with respect to O being always at right angles to the plane defined by OC and some other fixed line OZ . If the body has no spin about its axis, then (8.2) may clearly be satisfied by a motion of the body which is one of rotation about the axis of $\mathbf{M}$. If, however, the body possesses a spin n about its axis, then a contribution to the change $\delta\mathbf{H}$ required by (8.2) can arise from a motion of the body such that in the interval δt the axis OC undergoes a rotation about O in the plane defined by the instantaneous directions of $\mathbf{M}$ and OC ; for, if ϵ is the angle through which OC rotates in time δt (Fig. 8.1) and C is the moment of inertia of the body with

respect to OC , then the consequent change in $\mathbf{H}$ arising from the spin would have the desired direction and be of magnitude

$$Cn \tan \epsilon \approx Cn\epsilon.$$

A similar contribution also arises in this case from the simultaneous rotation about O of the component of $\mathbf{H}$ with respect to the principal axis in the plane ZOC at right angles to OC ; the rotation of the component with respect to the third principal axis leads, of course, to no contribution parallel to $\mathbf{M}$.

In general, the motion of a body having a spin about its axis will be such that $\delta\mathbf{H}$ is made up of the contribution arising from the rotation of the body about the axis of $\mathbf{M}$, together with those due to the rotation of the angular momentum vector. As we shall see below, however,

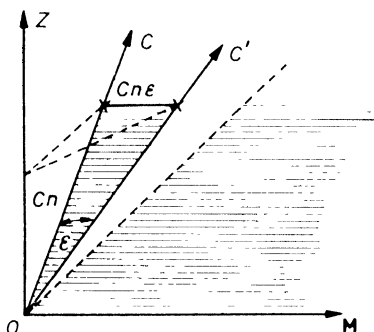


Figure 8.1

if the forces acting on the body have the appropriate moment with respect to O , then (8.2) may be satisfied by a motion in which there is no rotation about the axis of $\mathbf{M}$. In this case, if the forces acting on the body are such that the magnitude of $\mathbf{M}$ remains constant, then the motion will be one in which OC rotates about OZ with a uniform angular velocity, the angle ZOC remaining constant. Such a motion is termed one of *uniform precession* about OZ .*

1.2. Couple required to produce uniform precessional motion.

Let $\mathbf{k}$ be a unit vector in the direction of the axis of symmetry. Then the moment of momentum of the body may, as we have already seen (p. 188, eqn 7.42), be expressed in the form

$$\mathbf{H} = A(\mathbf{k} \wedge d\mathbf{k}/dt) + Cn\mathbf{k}, \quad (8.3)$$

* The uniform precessional motion met with in the previous chapter is known as that of *free precession* since it is that which arises when a uniaxial body is in motion about a fixed point under no forces. In this case, $\mathbf{M}$ being zero, the motion is such that $\delta\mathbf{H}$ is also zero.

where C is the moment of inertia of the body about OC , A the moment of inertia about a perpendicular axis through O , and n is the component of the angular velocity of the body about OC . If the motion of the

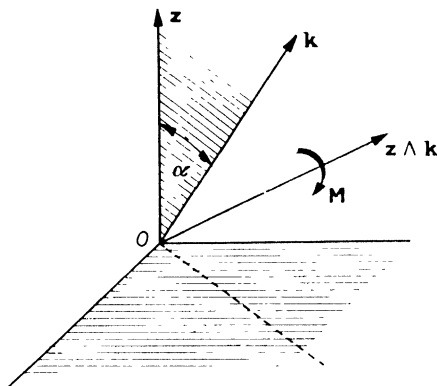


Figure 8.2

body is one of steady precession, then both n and Ω , the angular velocity of precession, are constant. Consequently we have, on differentiating (8.3) with respect to t ,

$$\frac{d\mathbf{H}}{dt} = A\left(\mathbf{k} \wedge \frac{d^2\mathbf{k}}{dt^2}\right) + Cn\frac{d\mathbf{k}}{dt}, \quad (8.4)$$

whilst, from the above definition of precessional motion, it follows that, if $\mathbf{z}$ is a unit vector in the direction of OZ ,

$$\frac{d\mathbf{k}}{dt} = \Omega(\mathbf{z} \wedge \mathbf{k}), \quad (8.5)$$

so that

$$\begin{aligned} \frac{d^2\mathbf{k}}{dt^2} &= \Omega\left(\mathbf{z} \wedge \frac{d\mathbf{k}}{dt}\right) \\ &= \Omega^2\{\mathbf{z} \wedge (\mathbf{z} \wedge \mathbf{k})\}, \quad \text{from (8.5),} \\ &= \Omega^2\{\mathbf{z}(\mathbf{z} \cdot \mathbf{k}) - \mathbf{k}\}. \end{aligned} \quad (8.6)$$

On substituting for $\frac{d\mathbf{k}}{dt}$ and $\frac{d^2\mathbf{k}}{dt^2}$ from (8.5) and (8.6) into (8.4), we thus have

$$\begin{aligned} \frac{d\mathbf{H}}{dt} &= Cn\Omega(\mathbf{z} \wedge \mathbf{k}) + A\Omega^2(\mathbf{k} \wedge \mathbf{z})(\mathbf{k} \cdot \mathbf{z}) \\ &= (Cn\Omega - A\Omega^2 \cos \alpha)(\mathbf{z} \wedge \mathbf{k}) \quad (\mathbf{k} \cdot \mathbf{z} = \cos \alpha). \end{aligned} \quad (8.7)$$

Consequently we see that a state of steady precessional motion about OZ with an angular velocity of precession Ω , the axis OC making a

constant angle α with OZ , is possible provided the forces acting on the system are such that their resultant moment about O is equivalent to that of a couple of moment

$$(Cn\Omega - A\Omega^2 \cos \alpha) \sin \alpha \quad (8.8)$$

whose axis is in the direction of the vector $\mathbf{z} \wedge \mathbf{k}$, i.e. it is perpendicular to the plane ZOC and acts in the direction shown in Fig. 8.2.*

For a uniaxial body in motion under no forces, then the angular velocity of precession, corresponding to an angle α , will be given by equating the above couple to zero, i.e. it is given by

$$\Omega = Cn/A \cos \alpha,$$

a value in agreement with that obtained for ϕ in Chap. IV, § 4.4, Case (i).

1.2.1. Motion under gravity. If the body is in motion under the action of its own weight, the only other force being the reaction at O , then the moment of these forces about O is equal to

$$h\mathbf{k} \wedge -Mg\mathbf{z} = Mgh(\mathbf{z} \wedge \mathbf{k}),$$

where $OG = h$ (Fig. 8.3). It immediately follows, therefore, that such a body can precess about the vertical, with its axis making a constant angle α with the vertical, provided the angular velocity of precession Ω is such that

$$(Cn\Omega - A\Omega^2 \cos \alpha)(\mathbf{z} \wedge \mathbf{k}) = Mgh(\mathbf{z} \wedge \mathbf{k}),$$

i.e. provided Ω satisfies the equation

$$A\Omega^2 \cos \alpha - Cn\Omega + Mgh = 0. \quad (8.9)$$

* The above result can be used to discuss the stability of the motion of a person on a bicycle. In this case, assuming the person to be riding along a straight road so that the wheels of the bicycle are spinning about their spindles, which are horizontal, then, if, through any cause, the plane of the bicycle tilts to either side, each spindle will be acted upon by a resultant couple, the directions of the axes of the couples being horizontal and at right angles to the respective spindles; the forces producing the couples arise from the reactions between the wheels and the ground and the reactions of the frame on the spindles. As a result of these couples, each wheel will tend to undergo a motion of precession about the vertical. If Ω is the angular velocity of precession of the front wheel, say, and n is the angular velocity of the wheel about its axis, then the magnitude of the couple acting on the wheel must be $Cn\Omega$, assuming the angle of tilt to be small (eqn 8.8, with $\alpha = \pi/2$). Consequently it follows that for the same couple, i.e. for the same amount of tilting, the rotational motion of the wheel about the vertical will be greater the smaller the value of n , i.e. the slower the motion the greater the 'wobble', or, in other words, the faster one goes the greater the stability.

By the above reasoning one can also see why one leans inwards when rounding a bend: it is clearly in order to produce the couple necessary for the precessional motion involved in the change of direction.

This is a quadratic in Ω , its two solutions being given by

$$\Omega = \frac{Cn \pm \sqrt{(C^2n^2 - 4AMgh \cos \alpha)}}{2A \cos \alpha}.$$

These values are real if and only if

$$C^2n^2 \geq 4AMgh \cos \alpha.$$

This, therefore, is the condition that a body, possessing an axis of symmetry and rotating about a fixed point on its axis under the action of its own weight, may be able to precess uniformly about the vertical with its axis making a constant angle α with the vertical. Further, if this condition is satisfied, then, unless $C^2n^2 = 4AMgh \cos \alpha$, there are two possible angular velocities of precession, the directions of

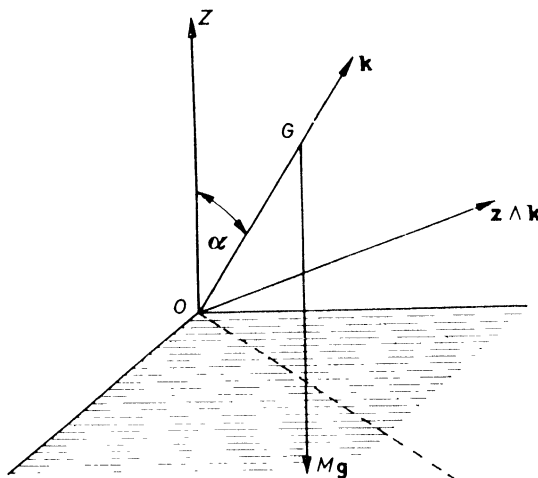


Figure 8.3

rotation in both cases being the same. They are termed the *fast* and *slow* precessions.

Having obtained the conditions necessary for a uniaxial body to precess uniformly about an axis, we could next proceed to discuss the general question of the stability of such motion. This, however, is better left until later (see § 2.7). Instead, we shall illustrate the general theory by considering in detail the motion of a top spinning on a horizontal plane. If the plane is sufficiently rough to prevent slipping, then the point of contact becomes a fixed point and the theory is similar to that above; if the plane is smooth, then the point of contact is no longer fixed. Since, however, in this case both the reaction at the

point of contact and the weight act vertically, the motion of G will be confined to a vertical line, and a state of steady precessional motion about the vertical is still possible. We shall consider both of these motions.

2. Top spinning on a rough plane

2.1. Vector treatment. If O is the point of contact, then taking moments about O , we shall have, as a general result,

$$\frac{d}{dt} \left\{ A \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) + Cn\mathbf{k} \right\} = Mgh(\mathbf{z} \wedge \mathbf{k}),$$

$$\text{or} \quad A \left(\mathbf{k} \wedge \frac{d^2\mathbf{k}}{dt^2} \right) + Cn \frac{d\mathbf{k}}{dt} + C \frac{dn}{dt} \mathbf{k} = Mgh(\mathbf{z} \wedge \mathbf{k}). \quad (8.10)$$

This is the vector differential equation which governs the motion of the top.

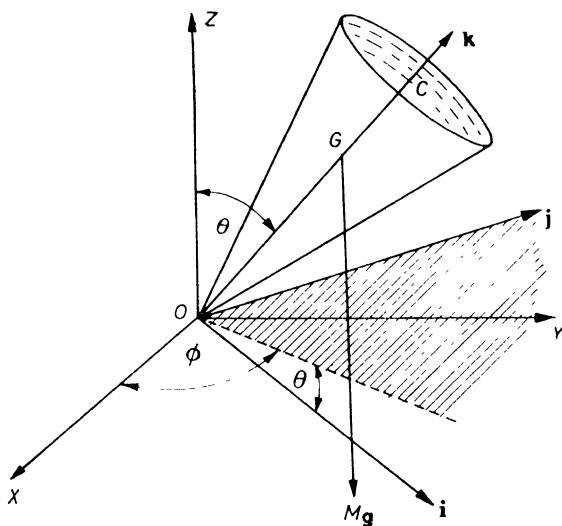


Figure 8.4

Now a body having one point fixed has three degrees of freedom; three positional coordinates are therefore required to define its position, and hence its motion, at any instant of time. The most natural system of coordinates to use in this case is that consisting of Euler's angular coordinates (θ, ϕ, ψ) . To determine the motion completely, we thus require three scalar integrals of the above equation involving these

coordinates and their time rates of change. The three integrals may be obtained as follows:

(1) *Integral of Axial Spin.* If we multiply the above equation scalarly by $\mathbf{k}$, then we have, since $\mathbf{k} \cdot d\mathbf{k}/dt = 0$ and $\mathbf{k} \cdot \mathbf{k} = 1$,

$$C \frac{dn}{dt} = 0,$$

so that

$$n = \text{constant},$$

i.e. the spin of the top about its axis is constant. The equivalent result expressed in terms of Euler's angles is (see Chap. VII, eqn 7.9)

$$\dot{\psi} + \dot{\phi} \cos \theta = \text{constant}.$$

(2) *Angular Momentum Integral.* Putting $dn/dt = 0$ in (8.10), then the differential equation of motion becomes

$$A \left(\mathbf{k} \wedge \frac{d^2 \mathbf{k}}{dt^2} \right) + Cn \frac{d\mathbf{k}}{dt} = Mgh(\mathbf{z} \wedge \mathbf{k}). \quad (8.11)$$

On multiplying this equation scalarly by $\mathbf{z}$, we have

$$A \mathbf{z} \cdot \left(\mathbf{k} \wedge \frac{d^2 \mathbf{k}}{dt^2} \right) + Cn \left(\mathbf{z} \cdot \frac{d\mathbf{k}}{dt} \right) = 0,$$

which can be integrated immediately giving

$$A \mathbf{z} \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) + Cn(\mathbf{z} \cdot \mathbf{k}) = \text{constant}.$$

If we write this in the form

$$\mathbf{z} \cdot \left\{ A \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) + Cn\mathbf{k} \right\} = \text{constant} = D, \text{ say}, \quad (8.12)$$

then we see that this integral is simply the mathematical expression of the fact that the component of the moment of momentum of the top about the vertical through O is constant.* It is termed the *integral of angular momentum* of the motion.

* The above equation can be deduced more simply, perhaps, by writing the equation of motion in the form

$$\frac{d\mathbf{H}}{dt} = Mgh(\mathbf{z} \wedge \mathbf{k}).$$

For, multiplying this scalarly by $\mathbf{z}$, we have

$$\begin{aligned} \mathbf{z} \cdot \frac{d\mathbf{H}}{dt} &= Mgh\{\mathbf{z} \cdot (\mathbf{z} \wedge \mathbf{k})\} \\ &= 0, \end{aligned}$$

and hence on integrating, $\mathbf{z}$ being a unit vector whose direction is constant,

$$\mathbf{z} \cdot \mathbf{H} = \mathbf{z} \cdot \left\{ A \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) + Cn\mathbf{k} \right\} = \text{constant}.$$

To express the above integral in terms of Euler's angles, suppose we introduce two additional unit vectors $\mathbf{i}$ and $\mathbf{j}$ localized at O such that $\mathbf{i}$ is in the plane ZOC and at right angles to $\mathbf{k}$, whilst $\mathbf{j}$ is a mutually orthogonal vector such that $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ form a right-handed triad, the respective directions of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ being as shown in Fig. 8.4. Then, referred to these directions, we have

$$\mathbf{z} = (-\sin \theta, 0, \cos \theta), \quad \mathbf{k} = (1, 0, 0), \quad d\mathbf{k}/dt = (\dot{\theta}, \dot{\phi} \sin \theta, 0)^*, \quad (8.13)$$

and hence

$$\mathbf{z} \cdot \mathbf{k} \wedge \frac{d\mathbf{k}}{dt} = \begin{vmatrix} -\sin \theta & 0 & \cos \theta \\ 0 & 0 & 1 \\ \dot{\theta} & \dot{\phi} \sin \theta & 0 \end{vmatrix} = \dot{\phi} \sin^2 \theta.$$

Consequently (8.12) becomes

$$A\dot{\phi} \sin^2 \theta + Cn \cos \theta = \text{constant} = D. \quad (8.14)$$

(3) *Energy Integral.* If we multiply (8.11) scalarly by $\mathbf{k} \wedge d\mathbf{k}/dt$, then we have

$$A \left(\mathbf{k} \wedge \frac{d^2 \mathbf{k}}{dt^2} \right) \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) + Cn \frac{d\mathbf{k}}{dt} \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) = Mgh(\mathbf{z} \wedge \mathbf{k}) \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right),$$

or

$$A \left(\mathbf{k} \wedge \frac{d^2 \mathbf{k}}{dt^2} \right) \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) - Mgh \left\{ (\mathbf{z} \cdot \mathbf{k}) \left(\mathbf{k} \cdot \frac{d\mathbf{k}}{dt} \right) - (\mathbf{k} \cdot \mathbf{k}) \left(\mathbf{z} \cdot \frac{d\mathbf{k}}{dt} \right) \right\} = 0,$$

i.e. since $\mathbf{k} \cdot d\mathbf{k}/dt = 0$,

$$A \left(\mathbf{k} \wedge \frac{d^2 \mathbf{k}}{dt^2} \right) \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) + Mgh \left(\mathbf{z} \cdot \frac{d\mathbf{k}}{dt} \right) = 0.$$

This again can be integrated immediately, the result being

$$A \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right)^2 + 2Mgh(\mathbf{z} \cdot \mathbf{k}) = \text{constant} = E, \text{ say.}$$

This is simply the energy equation which expresses the fact that the total energy remains constant throughout the motion, the first term being equal to the kinetic energy (minus $\frac{1}{2}Cn^2$)† and the second term

* The components of $d\mathbf{k}/dt$ are simply the velocity components of a point P on the axis unit distance from O .

† Using the result $2T = \boldsymbol{\omega} \cdot \mathbf{H}$ (Chap. VI, eqn 6.40), the kinetic energy is given by

$$\begin{aligned} 2T &= \left\{ A \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) + Cn\mathbf{k} \right\} \cdot \left\{ \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) + n\mathbf{k} \right\} \\ &= A \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right)^2 + Cn^2. \end{aligned}$$

representing the potential energy. Expressed in terms of Euler's coordinates, the result becomes, in virtue of (8.13),

$$A(\dot{\theta}^2 + \phi^2 \sin^2 \theta) + 2Mgh \cos \theta = E. \quad (8.15)$$

We have thus obtained three equations between the three angular coordinates (θ, ϕ, ψ) and their first differentials with respect to the time. These three equations are sufficient for the complete determination of the motion.

2.2. Integrals of the motion derived non-vectorially. For a body rotating about a fixed point, its vector equation of motion, when referred to a set of axes rotating with an angular velocity $\boldsymbol{\theta}$, is given by

$$\frac{\partial \mathbf{H}}{\partial t} + (\boldsymbol{\theta} \wedge \mathbf{H}) = \mathbf{M}, \quad (8.16)$$

where $\mathbf{M} = \sum (\mathbf{r} \wedge \mathbf{F})$. If the body is uniaxial and $O3$ is chosen to coincide with the axis of symmetry, then, since

$$\mathbf{H} \equiv (H_1, H_2, H_3) = (A\omega_1, A\omega_2, C\omega_3),$$

the scalar equations equivalent to (8.16) are

$$\left. \begin{aligned} A\dot{\omega}_1 + (C\theta_2\omega_3 - A\theta_3\omega_2) &= M_1 \\ A\dot{\omega}_2 + (A\theta_3\omega_1 - C\theta_1\omega_3) &= M_2 \\ C\dot{\omega}_3 + (A\theta_1\omega_2 - A\theta_2\omega_1) &= M_3 \end{aligned} \right\}. \quad (8.17)$$

In this case, we also have, $O3$ being fixed in the body (see Chap. VII, p. 167, § 3, eqns 7.10 and 7.11),

$$\theta_1 = \omega_1, \quad \theta_2 = \omega_2, \quad \theta_3 \neq \omega_3.$$

Substituting these values into (8.17), the equations of motion thus become

$$\left. \begin{aligned} A\dot{\omega}_1 + (C\omega_2\omega_3 - A\theta_3\omega_2) &= M_1 \\ A\dot{\omega}_2 + (A\theta_3\omega_1 - C\omega_1\omega_3) &= M_2 \\ C\dot{\omega}_3 &= M_3 \end{aligned} \right\}. \quad (8.18)$$

Now let us apply these equations to the motion of the top. Choosing the axes $O123$ to coincide with the directions of the unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ as defined above (Fig. 8.4), then we have

$$\boldsymbol{\theta} = \phi \mathbf{z} + \theta \mathbf{j},$$

so that

$$(\theta_1, \theta_2, \theta_3) = (-\phi \sin \theta, \theta, \phi \cos \theta),$$

and hence

$$\omega_1 = -\dot{\phi} \sin \theta, \quad \omega_2 = \dot{\theta}.$$

Also, we have

$$M_1 = 0, \quad M_2 = Mgh \sin \theta, \quad M_3 = 0.$$

Substituting into (8.18), the equations of motion thus become

$$\left. \begin{aligned} A \frac{d}{dt}(-\dot{\phi} \sin \theta) + C\dot{\theta}\omega_3 - A\dot{\theta}\dot{\phi} \cos \theta &= 0 & (a) \\ A \frac{d\dot{\theta}}{dt} - A\dot{\phi}^2 \sin \theta \cos \theta + C\omega_3\dot{\phi} \sin \theta &= Mgh \sin \theta & (b) \\ C\dot{\omega}_3 &= 0 & (c) \end{aligned} \right\}. \quad (8.19)$$

These equations, on integration, yield the desired first integrals of the motion.

From the third, we have

$$\omega_3 = \text{constant} = n.$$

Replacing ω_3 by n , (8.19a) and (8.19b) become

$$A \frac{d}{dt}(\dot{\phi} \sin \theta) + A\dot{\theta}\dot{\phi} \cos \theta - Cn\dot{\theta} = 0, \quad (8.20)$$

$$A\ddot{\theta} - A\dot{\phi}^2 \sin \theta \cos \theta + Cn\dot{\phi} \sin \theta = Mgh \sin \theta. \quad (8.21)$$

On multiplying (8.20) by $\sin \theta$, then we have

$$A \left\{ \frac{d}{dt}(\dot{\phi} \sin \theta) \sin \theta + \dot{\theta}\dot{\phi} \sin \theta \cos \theta \right\} - Cn\dot{\theta} \sin \theta = 0,$$

which can be integrated immediately giving

$$A\dot{\phi} \sin^2 \theta + Cn \cos \theta = \text{constant} = D \quad [(8.14)]$$

in agreement with (8.14).

To obtain the energy integral, we have, on multiplying (8.21) by $\dot{\theta}$,

$$A\dot{\theta}\ddot{\theta} - A\dot{\phi}^2 \sin \theta \cos \theta \dot{\theta} + Cn\dot{\phi} \sin \theta \dot{\theta} = Mgh \sin \theta \dot{\theta}.$$

On substituting for $Cn\dot{\theta}$ from (8.20), this reduces to

$$A\dot{\theta}\ddot{\theta} + A \frac{d}{dt}(\dot{\phi} \sin \theta)\dot{\phi} \sin \theta = Mgh \sin \theta \dot{\theta},$$

which gives on integration (8.15), viz.

$$A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + 2Mgh \cos \theta = \text{constant} = E. \quad [(8.15)]$$

2.3. Properties of the motion. If we eliminate $\dot{\phi}$ from (8.14) and (8.15), then we have

$$A\dot{\theta}^2 + A \sin^2 \theta \frac{(D - Cn \cos \theta)^2}{A^2 \sin^4 \theta} + 2Mgh \cos \theta = E,$$

or

$$A\dot{\theta}^2 = E - 2Mgh \cos \theta + \frac{(D - Cn \cos \theta)^2}{A \sin^2 \theta}. \quad (8.22)$$

Writing

$$z = \cos \theta,$$

so that

$$\dot{z} = -\sin \theta \dot{\theta}, \quad \dot{z}^2 = (1-z^2)\dot{\theta}^2,$$

this may be written

$$A\dot{z}^2 = E(1-z^2) - 2Mghz(1-z^2) - (D - Cnz)^2/A.$$

The right-hand side of this equation is a cubic in z , so that the equation

$$f(z) = E(1-z^2) - 2Mghz(1-z^2) - (D - Cnz)^2/A = 0 \quad (8.23)$$

has three roots. To obtain the general form of $f(z)$, we know in the first place that for a value of z , say $z = z_0$ ($-1 < z_0 < 1$), corresponding to a position of the top during its motion for which $\dot{\theta} \neq 0$, $f(z)$ must of necessity be positive. We thus have the following table:

$z = -\infty$,	$f(z)$ is -ve and infinite,
$z = -1$,	$f(z)$ is -ve,
$z = z_0$,	$f(z)$ is +ve,
$z = 1$,	$f(z)$ is -ve (or zero, if $D = Cn$),
$z = +\infty$,	$f(z)$ is +ve and infinite.

Consequently, if z_1, z_2, z_3 represent the three roots of (8.23) such that $z_1 < z_2 < z_3$, then, in the general case ($D \neq Cn$), it follows that z_1

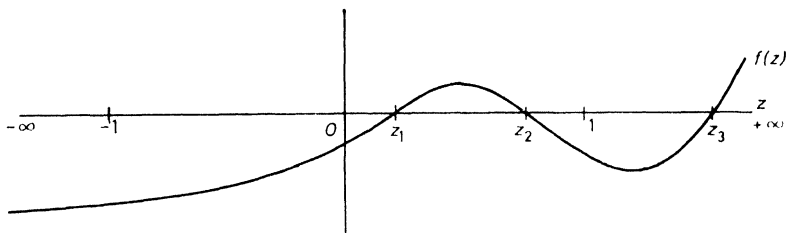


Figure 8.5

and z_2 both lie between $z = -1$ and $z = 1$, while z_3 is greater than unity. The general form of $f(z)$ is thus as shown in Fig. 8.5.

Since $\dot{\theta}^2$ must always be positive during the motion, it follows that the axis of the top will be confined to the region such that $\theta_1 \geq \theta \geq \theta_2$, where θ_1 and θ_2 are the values of θ corresponding to $z = z_1$ and $z = z_2$ respectively.

If $D = Cn$, then one root of the above cubic is $z = 1$. In this case, therefore, the axis of the top either passes through the upward vertical or becomes vertical at some stage of its motion.

It should be mentioned here, perhaps, that, in discussing the motion of the top spinning on a rough plane, z is restricted to the values $0 \leq z \leq 1$. If, however, the top is pivoted at its vertex, then z may assume values such that $-1 \leq z \leq 1$. When this is so D may be such that $D \neq Cn$, $D = Cn$, or $D = -Cn$. In the first case the axis of the top never assumes the vertical position, in the second it assumes the upward vertical position, whilst, in the last, it assumes the downward vertical position, but not the upward. In what follows we shall consider the nature of the motion when $D \neq Cn$ and when $D = Cn$; the nature of the motion for the case in which $D = -Cn$ can, of course, be readily deduced from that in which $D = Cn$.

In order to study the motion of the top in detail, it is possibly better to consider the behaviour of the function $g(z)$, defined by the equation

$$g(z) = A\dot{\theta}^2 = \frac{f(z)}{1-z^2}, \quad (8.24)$$

rather than $f(z)$ itself. The exact expression for $g(z)$ is given by

$$g(z) = E - 2Mghz - \frac{(D - Cnz)^2}{A(1-z^2)}. \quad (8.25)$$

It is clear that we are concerned only with the behaviour of $g(z)$ in the

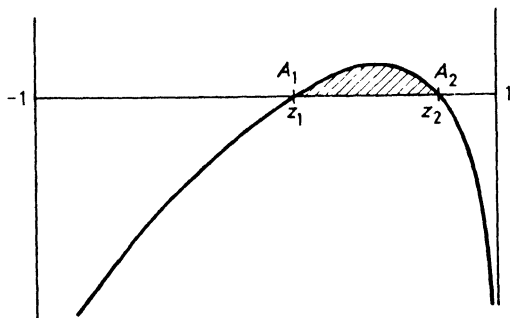


Figure 8.6

region $-1 \leq z \leq 1$. There are a number of different cases to consider depending upon the initial conditions.

Case I. Initial conditions are such that $D \neq Cn$ while $\dot{\theta}$ is not, in general, zero during the motion.* Since in this case the equation $f(z) = 0$ has two roots within the region $-1 < z < 1$ (Fig. 8.5), $g(z) = 0$ (see eqn 8.24) will also have two roots within the same region; further $g(z)$ is clearly asymptotic at $-\infty$ to both $z = -1$ and $z = 1$. Within the region $-1 < z < 1$, it thus has the general

* Initial conditions are the initial values of θ , $\dot{\theta}$, ϕ , and n .

form as shown in Fig. 8.6, the portion of the curve corresponding to the motion being represented by the shaded region.

If θ_1 and θ_2 are the two roots of $g(z) = 0$ within this region, and if P is the point of intersection of the axis of the top with unit sphere having its centre at O , then the path traced out by P will touch two horizontal circles on the sphere, defined by $\theta = \theta_1$ and $\theta = \theta_2$, alternately.

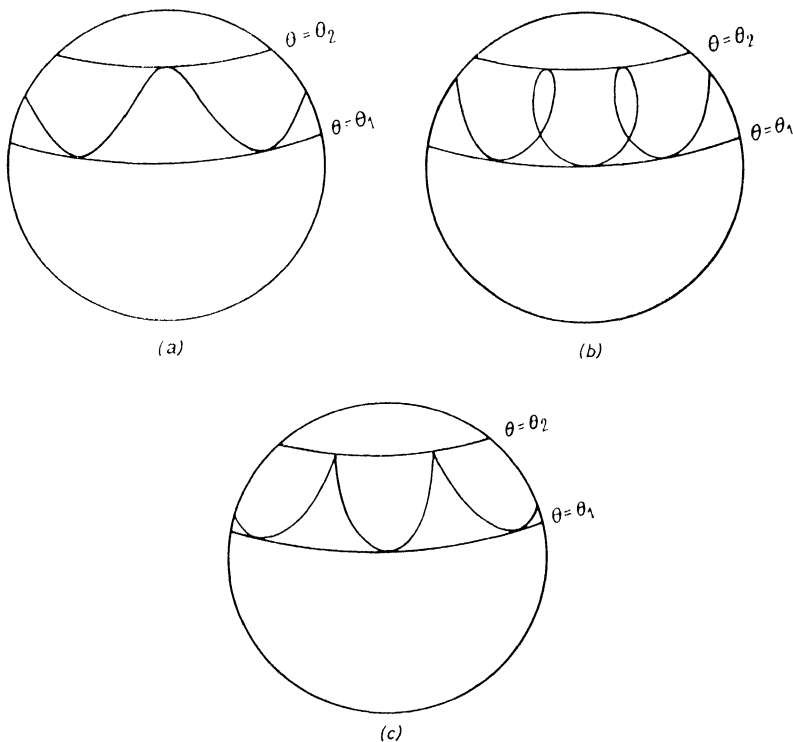


Figure 8.7

If $\dot{\phi}$ does not vanish when either $\theta = \theta_1$ or $\theta = \theta_2$, then the path of P will be tangential to the bounding circles as shown in Figs. 8.7 (a) or (b), case (b) corresponding to that in which the sign of $D - Cnz$ (and hence that of $\dot{\phi}$ (eqn 8.14)) changes between z_1 and z_2 .

If $\dot{\theta}$ and $\dot{\phi}$ vanish simultaneously at any point, then $\ddot{\phi}$ also vanishes at the same point (eqn 8.14). Consequently, at such a point, the path of P will be orthogonal to a bounding circle and so will have a cusp. Since, when $\dot{\theta}$ and $\dot{\phi}$ are both zero, the axis is instantaneously at rest,

it follows that the ensuing motion of the axis, immediately afterwards, must be in a downward direction and not in an upward one, i.e. a cusp can occur only at a point of contact with the upper of the two bounding circles.

That this is so can also be shown analytically. For, from (8.14), we have

$$\phi = \frac{D - Cnz}{1 - z^2},$$

so that $\phi = 0$ when

$$D - Cnz = 0. \quad (8.26)$$

But, if this equation is satisfied by a particular value of z , it will not be satisfied by any other value of z . Consequently a cusp can occur on only one of the bounding circles. To determine on which it occurs, suppose we differentiate $g(z)$ (eqn 8.25); then we have

$$g'(z) = -2Mgh + 2Cn \frac{(D - Cnz)}{A(1 - z^2)} - 2z \frac{(D - Cnz)^2}{A(1 - z^2)^2}.$$

But, at a cusp, (8.26) holds, and hence at such a point

$$g'(z) = -2Mgh,$$

i.e. the slope of $g(z)$ is negative. The path of P can therefore have a cusp, if at all, along the circle corresponding to $z = z_2$, i.e. along the upper bounding circle. The nature of the path of P , in this case, is thus as shown in Fig. 8.7(c).

If initial conditions are such that $z = z_0$ while $\dot{\theta} = 0$, then $z = z_0$ must coincide with one of the points A_1 or A_2 in Fig. 8.5. The question as to whether the axis of the top rises or falls initially (the case of circular motion, corresponding to a state of steady precession, is discussed in § 2.4 below) depends on the sign of $g'(z_0)$. If this is positive, then z_0 must coincide with A_1 and the axis of the top will rise initially; if negative, then z_0 coincides with A_2 and the axis of the top descends.

The angular velocity $\dot{\theta}$ is termed the *angular velocity of nutation*, and the complete period between two successive instants at which the axis of the top is passing through the same value of θ with the same velocity, the *period of nutation*.

Case II. Initial conditions are such that $D = Cn$ and $\dot{\theta} \neq 0$ when $z = 1$. In this case, since $D = Cn$, $g(z)$ has the form

$$\begin{aligned} g(z) &= E - 2Mghz - \frac{C^2 n^2 (1 - z)^2}{A(1 - z^2)} \\ &= E - 2Mghz - \frac{C^2 n^2}{A} \frac{(1 - z)}{(1 + z)}. \end{aligned}$$

Further, the equation $g(z) = 0$ is now a quadratic, one of its roots

being greater than unity. The general nature of the curve for $g(z)$ in the range $-1 \leq z \leq 1$ is thus as shown in Fig. 8.8. It intersects

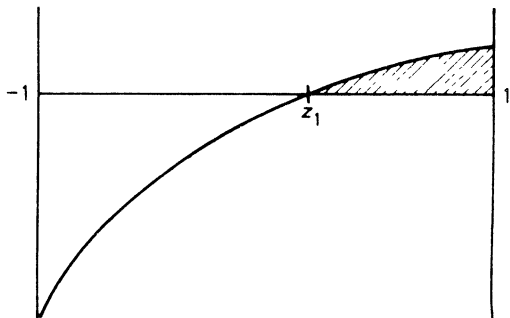


Figure 8.8

the z -axis at one point between $z = -1$ and $z = 1$, and the motion of the axis will thus be confined to the region corresponding to

$$z_1 \leq z \leq 1,$$

the axis of the top passing periodically through the vertical and touching the circle $\theta = \theta_1$. The projection of the path of P on a horizontal plane will thus be as in Fig. 8.9.

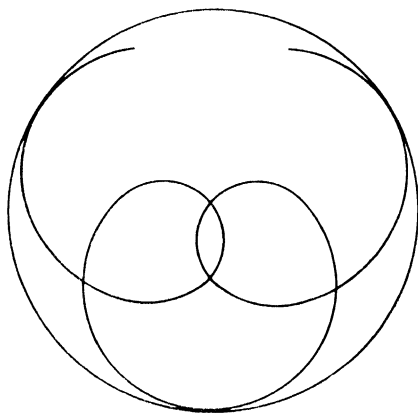


Figure 8.9

Case III. Initial conditions are such that $D = Cn$ and $\dot{\theta} = 0$ when $z = 1$. In this case, since $\dot{\theta} = 0$ when $z = 1$, E can be evaluated. From (8.22), on putting $D = Cn$, $\dot{\theta} = 0$, we have $E = 2Mgh$, and hence

$$g(z) = (1-z) \left\{ 2Mgh - \frac{C^2 n^2}{A(1+z)} \right\}. \quad (8.27)$$

It is asymptotic at $-\infty$ to $z = -1$, and also passes through $z = 1$. Its general behaviour, therefore, may be as indicated in Fig. 8.10, curves (a), (b), (c). The question as to which of the curves is applicable to the motion depends upon the sign of the slope of $g(z)$ at $z = 1$. Differentiating (8.27), we have

$$g'(z) = -\left\{2Mgh - \frac{C^2 n^2}{A(1+z)}\right\} + (1-z) \frac{C^2 n^2}{A(1+z)^2},$$

so that

$$g'(1) = -2Mgh + \frac{C^2 n^2}{2A} = \frac{C^2 n^2 - 4AMgh}{2A}. \quad (8.28)$$

The curves (a), (b), (c) thus correspond to the cases in which

$$C^2 n^2 \gtrless 4AMgh$$

respectively, (b) being tangential to the z -axis at $z = 1$.

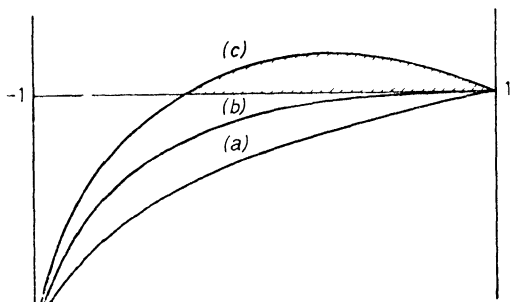


Figure 8.10

Now a top can clearly spin with its axis vertical no matter what the value of n . If $C^2 n^2 \geq 4AMgh$, then it is seen that, subject to the conditions $D = Cn$, $\dot{\theta} = 0$, the only possible motion is one where the axis stays vertical; i.e. initial values different from zero cannot be found for θ , $\dot{\theta}$, and $\dot{\phi}$ such that $E = Mgh$, $D = Cn$, and, at the same time, $\dot{\theta} = 0$ when $\theta = 0$. If, however, $C^2 n^2 < 4AMgh$, then an alternative motion is possible. In this case the $g(z)$ curve cuts the z -axis not only when $z = 1$ but also for a value of z given by

$$z_1 = \frac{C^2 n^2 - 2AMgh}{2AMgh}.$$

(Since we are assuming $C^2 n^2 < 4AMgh$, this value of z is such that $-1 \leq z_1 \leq 1$ and so corresponds to a real value of θ .) Thus, if initially the top is held with its axis in this position, subject to the condition $\dot{\theta} = 0$, and, at the same time, is given a velocity of precession such that

the condition $D = Cn$ is satisfied, then, when released, the axis of the top begins to rise and during the ensuing motion will steadily approach the vertical.

Suppose we investigate the nature of this motion when nearing the vertical. We have, from (8.24) and (8.27),

$$A\dot{\theta}^2 = (1-z)\left\{2Mgh - \frac{C^2n^2}{A(1+z)}\right\}.$$

Putting

$$z = \cos \theta = 1 - \theta^2/2,$$

then we have

$$\begin{aligned} A\dot{\theta}^2 &= \frac{\theta^2}{2}\left\{2Mgh - \frac{C^2n^2}{A(2-\theta^2/2)}\right\} \\ &= \frac{\theta^2}{2}\left\{2Mgh - \frac{C^2n^2}{2A}\left(1 + \frac{\theta^2}{4} + \dots\right)\right\}, \end{aligned}$$

so that, neglecting terms of $O(\theta^4)$,

$$\dot{\theta}^2 = \theta^2\left\{\frac{4AMgh - C^2n^2}{4A^2}\right\}.$$

Since, in the present case, $C^2n^2 < 4AMgh$, this may be written

$$\dot{\theta}^2 = p^2\theta^2,$$

where

$$p^2 = \frac{4AMgh - C^2n^2}{4A^2} (> 0),$$

so that

$$\dot{\theta} = \pm p\theta.$$

Consequently we have

$$\frac{d\theta}{\theta} = \pm p dt,$$

and hence

$$\theta = Be^{\pm pt}, \quad (8.29)$$

where B is a constant of integration. Now during the motion, when the axis of the top is approaching the vertical, $\dot{\theta}$ must clearly be negative. Therefore, choosing the negative sign in (8.29), the behaviour of θ near the vertical will be given by

$$\theta = Be^{-pt},$$

i.e. it decreases exponentially with the time. Thus θ can vanish only when t becomes infinite, i.e. the axis of the top approaches the vertical asymptotically with the time.

When a top spins with its axis vertical, then it is referred to as the upright or *sleeping* top.

Although a top can spin theoretically with its axis vertical whatever the value of n , if slightly disturbed from this position, then the nature of the disturbed motion, and hence the stability of the original motion, varies according to the value of n . To study the nature of the disturbed motion, suppose the disturbing force to be a small impulsive force applied in an axial plane. The values of D and n will remain unchanged in the new motion, the effect of the disturbing force being to increase E by an amount δE . If $g_d(z)$ denotes the function $g(z)$ corresponding to the disturbed motion, then we shall have (eqn 8.25)

$$g_d(z) = g(z) + \delta E;$$

the $g_d(z)$ curves will thus be as represented by the broken lines in

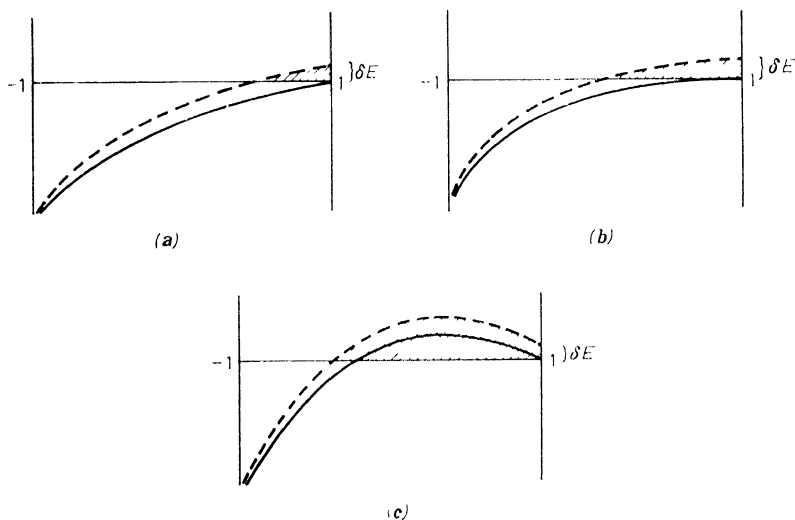


Figure 8.11

Fig. 8.11 (a), (b), and (c), obtained by displacing the corresponding $g(z)$ curve for the original motion through a distance δE in a direction perpendicular to the z -axis. It is clear that in each case the motion will be of the type described in Case II.

In *Case (a)* ($C^2 n^2 > 4AMgh$), it can be seen that, in a general way, the original motion is stable, the motion of the axis in the disturbed motion being restricted to the shaded region. Since, when $D = Cn$ and $\theta = 0$ when $z = 1$,

$$g'(1) = (C^2 n^2 - 4AMgh)/2A$$

(eqn 8.28), it follows that the slope of $g(z)$ at $z = 1$ increases as

($C^2n^2 - 4AMgh$) increases. Consequently, for the same impulsive energy increase δE , the greater Cn , the greater the value of $g'(1)$, and hence the smaller the range of the disturbed motion, i.e. the greater Cn , the greater the stability of the original motion.

In *Case (b)* ($C^2n^2 = 4AMgh$) the disturbed motion would appear from the diagram (Fig. 8.11 (b)) to be of the same general type as that in *Case (a)*. As we shall see in § 2.6, where we discuss the same problem vectorially, this is not so, the disturbed motion in *Case (a)* being oscillatory about the vertical, while in this case the deviation increases linearly with the time. The motion must therefore be regarded as unstable.

In *Case (c)* ($C^2n^2 < 4AMgh$) the motion is unstable, and no matter how small the disturbing impulse, the deviation of the axis from the vertical in the disturbed motion will never be less than that for the non-vertical motion represented by the broken line in Fig. 8.11(c).

2.4. Steady precessional motion. This is characterized by the fact that θ remains constant, so that $\dot{\theta} = 0$, $\ddot{\theta} = 0$ throughout the motion.

From the general equation

$$A\dot{\theta}^2 = g(z), \quad [(8.24)]$$

we have

$$2A\dot{\theta} \ddot{\theta} = g'(z)\dot{z},$$

or, since $\dot{z} = -\sin \theta \dot{\theta} = \sqrt{1-z^2}\dot{\theta}$,

$$2A\dot{\theta} \ddot{\theta} = -g'(z)\sqrt{1-z^2}\dot{\theta}. \quad (8.30)$$

But this is a result which holds no matter what the nature of the motion, whether $\dot{\theta} = 0$ or not, and hence we must have, as a general property of the motion,

$$2A\ddot{\theta} = -g'(z)\sqrt{1-z^2}. \quad (8.31)$$

For $z = z_0$ to correspond to a motion of steady precession, therefore, we must have, putting $\dot{\theta} = 0$, $\ddot{\theta} = 0$,

$$g(z_0) = 0, \quad g'(z_0) = 0.$$

In a state of steady precession the two solutions of the equation $g(z) = 0$, which one normally gets in the general case (Fig. 8.6, p. 208), thus merge into a single value $z = z_0$, and $g(z)$ becomes tangential to the z -axis at this point (Fig. 8.12). Differentiating the general expression for $g(z)$ (eqn 8.25), the equation $g'(z_0) = 0$ thus becomes

$$Mgh - \frac{Cn(D - Cnz_0)}{A(1-z_0^2)} + \frac{z_0(D - Cnz_0)^2}{A(1-z_0^2)^2} = 0. \quad (8.32)$$

But, if $\dot{\phi} = \Omega$ is the angular velocity of the axis about the vertical in the state of steady precession, then

$$D - Cnz_0 = A\Omega \sin^2 \theta_0 = A\Omega(1 - z_0^2),$$

and condition (8.32) becomes*

$$Az_0\Omega^2 - Cn\Omega + Mgh = 0. \quad [(8.9)]$$

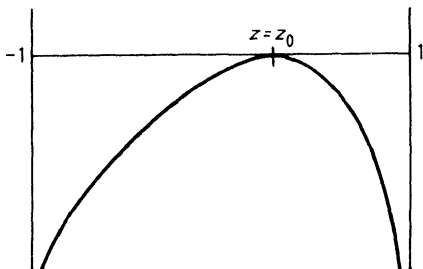


Figure 8.12

This is identical with (8.9), showing that for any set of values of n and z_0 , there are two possible values of Ω , viz.

$$\Omega = \frac{Cn \pm \sqrt{(C^2n^2 - 4AMghz_0)}}{2Az_0}, \quad (8.33)$$

which are real provided

$$C^2n^2 \geq Mghz_0.$$

The above is the general result for $z_0 < 1$, $D \neq Cn$. It should be noted that it does not apply to the case $z_0 = 1$, since, when $z_0 = 1$, then $D = Cn$ and the form of the $g(z)$ curve is quite different from that given in Fig. 8.12 (see Fig. 8.10).

2.5. Stability of steady precessional motion. Let $z = z_0$ correspond to a state of steady precession, so that $g(z_0) = 0$, $g'(z_0) = 0$, and suppose the top to be slightly disturbed from this position by an impulse applied to the top in an axial plane through the vertical. Such an impulse will not produce any instantaneous changes in either D or n , only in E . Consequently D and n retain their values in the disturbed motion; in addition, $g'(z)$ also retains its form, since it depends upon D and n but is independent of E .

To discuss the nature of the disturbed motion, let us write

$$\theta = \theta_0 + \epsilon, \quad z = z_0 + \delta,$$

δ being the small change in z ($= \cos \theta$) consequent upon a small change ϵ in θ when $\theta = \theta_0$. Then, substituting these expressions for θ and

* This condition could also have been deduced from (8.21), the second of the scalar equations of motion, by putting $\dot{\theta} = \ddot{\theta} = 0$, $\dot{\phi} = \Omega$.

z in equation (8.31) for θ , the equation governing the behaviour of ϵ in the neighbourhood of $z = z_0$ is given by

$$\begin{aligned} 2A\ddot{\epsilon} &= -\{1-(z_0+\delta)^2\}^{1/2}g'(z_0+\delta) \\ &= -(1-z_0^2)^{1/2}\left\{1-\frac{z_0\delta}{(1-z_0^2)}\right\}\{g'(z_0)+g''(z_0)\delta\} \\ &= -(1-z_0^2)^{1/2}g''(z_0)\delta \end{aligned} \quad (8.34)$$

on neglecting small quantities of the second order, $g'(z_0)$ being zero. But

$$z_0+\delta = \cos(\theta_0+\epsilon) = \cos\theta_0 \cos\epsilon - \sin\theta_0 \sin\epsilon,$$

so that

$$\delta = -\sin\theta_0 \epsilon = (1-z_0^2)^{1/2}\epsilon$$

to the same degree of approximation. Hence, substituting into (8.34), the differential equation in ϵ becomes

$$\ddot{\epsilon} - \frac{(1-z_0^2)}{2A}g''(z_0)\epsilon = 0. \quad (8.35)$$

Since $(1-z_0^2) > 0$ ($z_0 \neq 1$), this represents a motion of oscillation

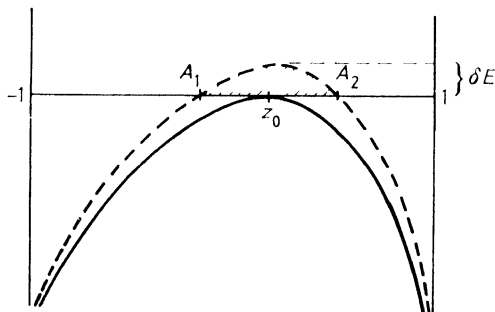


Figure 8.13

about $z = z_0$, provided $g''(z_0) < 0$. But, from Fig. 8.12, it is clear that this condition is always satisfied, $z = z_0$ corresponding to a maximum value of $g(z)$. Thus a motion of steady precession is stable.

The same general result regarding the stability could also have been deduced from the nature of the $g(z)$ curve. When a small impulse is applied as above, then its effect is to increase E by an amount δE , D and n remaining unchanged. The $g(z)$ function corresponding to the disturbed motion will thus be represented, as in § 2.3 when discussing the stability of the vertical motion, by the broken line in Fig. 8.13. In the disturbed motion the axis of the top thus oscillates through the region bounded by the horizontal circles corresponding to the points A_1

and A_2 as in Case I, § 2.3, above. The original motion of precession must therefore be stable.

The period of oscillation, or the *nutation*, for small disturbances will be given by

$$T = 2\pi/p,$$

where

$$p^2 = -\frac{(1-z_0^2)g''(z_0)}{2A} \\ = \frac{(Cn-2A\Omega z_0)^2 + A^2\Omega^2(1-z_0^2)}{A^2}, \quad (8.36)$$

after D has been eliminated from the expression for $g''(z_0)$ by means of the relation

$$D - Cnz_0 = A\Omega(1-z_0^2),$$

the condition for the existence of the state of steady precession.

2.6. Vertical top. Stability. Vector treatment. The stability of a top spinning about its axis which is vertical was discussed in general terms in § 2.3. We shall now consider the same problem vectorially.

For a top moving under gravity on a rough horizontal plane, the vector equation of motion is ((8.10) with $dn/dt = 0$)

$$A\left(\mathbf{k} \wedge \frac{d^2\mathbf{k}}{dt^2}\right) + Cn \frac{d\mathbf{k}}{dt} - Mgh(\mathbf{z} \wedge \mathbf{k}) = \mathbf{0}. \quad (8.37)$$

A particular solution of this equation is clearly

$$\mathbf{k} = \text{constant} = \mathbf{z},$$

which describes a top spinning with its axis vertical, D being equal to Cn (eqn 8.12).

Now let us consider the stability of this motion. We shall suppose the motion disturbed by a small impulse applied in an axial plane so that n (and also D) remains unchanged. To consider the nature of the disturbed motion, let us write

$$\mathbf{k} = \mathbf{z} + \mathbf{p},$$

where $\mathbf{p}$ is to be regarded as a vector whose magnitude is of the first order in θ and is such that, to this degree of approximation, $\mathbf{z} \cdot \mathbf{p} = 0$. Then, replacing $\mathbf{k}$ by this expression, (8.37) reduces, when terms of the second order are neglected, to

$$A\left(\mathbf{z} \wedge \frac{d^2\mathbf{p}}{dt^2}\right) + Cn \frac{d\mathbf{p}}{dt} - Mgh(\mathbf{z} \wedge \mathbf{p}) = \mathbf{0}. \quad (8.38)$$

Suppose we now refer the motion to a set of axes rotating about the vertical with an angular velocity Ω . Then we shall have

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= \frac{\partial\mathbf{p}}{\partial t} + \Omega(\mathbf{z} \wedge \mathbf{p}), \\ \frac{d^2\mathbf{p}}{dt^2} &= \frac{\partial^2\mathbf{p}}{\partial t^2} + 2\Omega\left(\mathbf{z} \wedge \frac{\partial\mathbf{p}}{\partial t}\right) + \Omega^2\{\mathbf{z}(\mathbf{p} \cdot \mathbf{z}) - \mathbf{p}\} \\ &\quad - \frac{\partial^2\mathbf{p}}{\partial t^2} + 2\Omega\left(\mathbf{z} \wedge \frac{\partial\mathbf{p}}{\partial t}\right) - \Omega^2\mathbf{p},\end{aligned}$$

since $\mathbf{z} \cdot \mathbf{p} = 0$. Consequently (8.38) becomes, on substituting these expressions for $d\mathbf{p}/dt$ and $d^2\mathbf{p}/dt^2$,

$$A\left(\mathbf{z} \wedge \frac{\partial^2\mathbf{p}}{\partial t^2}\right) + (Cn - 2A\Omega)\frac{\partial\mathbf{p}}{\partial t} + (Cn\Omega - Mgh - A\Omega^2)(\mathbf{z} \wedge \mathbf{p}) = \mathbf{0},$$

or, after multiplying vectorially by $\mathbf{z}$,

$$A\frac{\partial^2\mathbf{p}}{\partial t^2} - (Cn - 2A\Omega)\left(\mathbf{z} \wedge \frac{\partial\mathbf{p}}{\partial t}\right) + (Cn\Omega - Mgh - A\Omega^2)\mathbf{p} = \mathbf{0}; \quad (8.39)$$

this, by choosing $\Omega = Cn/2A$, so that $Cn - 2A\Omega = 0$, reduces to

$$\frac{\partial^2\mathbf{p}}{\partial t^2} + \frac{(C^2n^2 - 4AMgh)}{4A^2}\mathbf{p} = \mathbf{0}. \quad (8.40)$$

The nature of the disturbed motion in the neighbourhood of the vertical, and hence the stability of the original motion, may now be obtained by considering the solutions of this equation for different values of n . We have the following cases:

(i) $C^2n^2 > 4AMgh$. For large values of n , such that this inequality is obeyed, (8.40) takes the form

$$\frac{\partial^2\mathbf{p}}{\partial t^2} + p^2\mathbf{p} = \mathbf{0},$$

where

$$p^2 = (C^2n^2 - 4AMgh)/4A^2.$$

Relative to the rotating frame, $\mathbf{p}$ is thus periodic with respect to t , its general form being given by

$$\mathbf{p} = \mathbf{A} \cos pt + \mathbf{B} \sin pt,$$

where $\mathbf{A}$ and $\mathbf{B}$ are two arbitrary constant vectors. Taking the initial conditions to be

$$\mathbf{p} = \mathbf{0}, \quad \frac{\partial\mathbf{p}}{\partial t} = \mathbf{v}_0,$$

then we have

$$\mathbf{A} = \mathbf{0}, \quad \mathbf{B} = \mathbf{v}_0/p,$$

so that

$$= \frac{\mathbf{v}_0}{p} \sin pt.$$

Relative to the rotating axes, the point P on the axis, such that OP is unity, thus performs simple harmonic vibrations along the line defined by the vector $\mathbf{v}_0$, the period of vibration being $T = 2\pi/p$, while the line itself rotates about the vertical with an angular velocity $\Omega = Cn/2A$. The maximum deviation of P from the vertical is thus v_0/p , and the vertical motion is clearly stable.

(ii) $C^2n^2 = 4AMgh$. If the value of n satisfies this relation, then (8.40) reduces to

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} = \mathbf{0},$$

and the general solution of $\boldsymbol{\rho}$ is given by

$$\boldsymbol{\rho} = \mathbf{A}t + \mathbf{B}.$$

Subject to the same initial conditions as above, we thus have

$$\boldsymbol{\rho} = \mathbf{v}_0 t.$$

Relative to the rotating axes, the path of P is again the straight line defined by $\mathbf{v}_0$; the deviation of P from the vertical, however, now increases linearly with the time. Although the deviation from the vertical will still be small for a small disturbing force, yet, if oscillatory motion is taken as the test of stability, then, on this basis, the original motion must be regarded as unstable.

(iii) $C^2n^2 < 4AMgh$. For values of n satisfying this inequality, (8.40) becomes

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} - p^2 \boldsymbol{\rho} = \mathbf{0} \quad \left(p^2 = \frac{4AMgh - C^2n^2}{4A^2} \right).$$

The general solution of this equation is

$$\boldsymbol{\rho} = \mathbf{A}e^{pt} + \mathbf{B}e^{-pt}.$$

Putting in the initial conditions as above, we have

$$\mathbf{A} = \mathbf{B} = \mathbf{v}_0/2p,$$

so that $\boldsymbol{\rho}$, as a function of t , is given by

$$\boldsymbol{\rho} = \frac{\mathbf{v}_0}{2p}(e^{pt} - e^{-pt}) = \frac{\mathbf{v}_0}{p} \sinh pt.$$

The path of P , relative to the rotating axes, is again the straight line defined by $\mathbf{v}_0$, while the distance of P from the axis increases with the time hyperbolically. The vertical motion is thus again unstable.

Summing up, we can say, on the basis of the above, that it is only when $C^2n^2 > 4AMgh$ that the motion of a symmetrical top, spinning with its axis vertical, is stable.

In the above theory Ω has been put equal to $Cn/2A$ as a matter of convenience, in order that the term in $\mathbf{z} \wedge \partial \mathbf{p} / \partial t$ in (8.39) should vanish. That such a value of Ω should occur in the theory, however, can be explained on physical grounds. If the impulse is applied in an axial plane, then $D = Cn$ in the disturbed motion. But we know that

$$D = A\dot{\phi} \sin^2 \theta + Cn \cos \theta.$$

For small θ , putting $\sin^2 \theta = \theta^2$, $\cos \theta = 1 - \theta^2/2$, D will thus be given by

$$D = A\dot{\phi}\theta^2 + Cn(1 - \theta^2/2),$$

which is equal to Cn only if

$$\dot{\phi} = Cn/2A.$$

Thus, for small θ , the axis of the top precesses about the vertical with a constant angular velocity $\Omega = Cn/2A$; the reason for this angular velocity occurring in the theory in the above manner thus becomes clear.

2.7. Stability of precessional motion. Vector treatment.

The equation which determines the angular velocity of precession corresponding to a given value of n and a given inclination of the axis (eqn 8.9) was derived vectorially in § 1.2.1. To discuss the general question of the stability of this type of motion, however, it is better to derive this equation, which is the condition for the existence of steady precessional motion, in another way, using the theory of rotating axes, as we did in the previous paragraph.

If the motion is referred to a set of axes rotating around the vertical with a uniform angular velocity Ω , then, when referred to these axes, we shall have, as in § 2.6,

$$\frac{d\mathbf{k}}{dt} = \frac{\partial \mathbf{k}}{\partial t} + \Omega(\mathbf{z} \wedge \mathbf{k}), \quad (8.41)$$

$$\begin{aligned} \frac{d^2 \mathbf{k}}{dt^2} &= \frac{\partial}{\partial t} \left\{ \frac{\partial \mathbf{k}}{\partial t} + \Omega(\mathbf{z} \wedge \mathbf{k}) \right\} + \Omega \left\{ \mathbf{z} \wedge \left(\frac{\partial \mathbf{k}}{\partial t} + \Omega(\mathbf{z} \wedge \mathbf{k}) \right) \right\} \\ &= \frac{\partial^2 \mathbf{k}}{\partial t^2} + 2\Omega \left(\mathbf{z} \wedge \frac{\partial \mathbf{k}}{\partial t} \right) + \Omega^2 \{ \mathbf{z}(\mathbf{k} \cdot \mathbf{z}) - \mathbf{k} \}, \end{aligned}$$

and hence the equation of motion of the top, viz.

$$A \left(\mathbf{k} \wedge \frac{d^2 \mathbf{k}}{dt^2} \right) + Cn \frac{d\mathbf{k}}{dt} + Mgh(\mathbf{k} \wedge \mathbf{z}) = \mathbf{0},$$

becomes, on substituting these expressions for $d\mathbf{k}/dt$ and $d^2\mathbf{k}/dt^2$,

$$A \left(\mathbf{k} \wedge \frac{\partial^2 \mathbf{k}}{\partial t^2} \right) + \{ Cn - 2A\Omega(\mathbf{k} \cdot \mathbf{z}) \} \frac{\partial \mathbf{k}}{\partial t} + \{ A\Omega^2(\mathbf{k} \cdot \mathbf{z}) - Cn\Omega + Mgh \} (\mathbf{k} \wedge \mathbf{z}) = \mathbf{0} \quad (8.42)$$

($\mathbf{k} \cdot \partial \mathbf{k} / \partial t = 0$, see (8.41)). But, if the top is precessing around the vertical with an angular velocity Ω , then, with respect to this frame, $\mathbf{k}$ must be a constant vector. Putting $\partial \mathbf{k} / \partial t = \mathbf{0}$, $\partial^2 \mathbf{k} / \partial t^2 = \mathbf{0}$ in (8.42), $\mathbf{k}$ can therefore be constant with respect to the rotating frame only if

$$\{A\Omega^2(\mathbf{k} \cdot \mathbf{z}) - Cn\Omega + Mgh\}(\mathbf{k} \wedge \mathbf{z}) = \mathbf{0},$$

i.e. since $\mathbf{k} \wedge \mathbf{z}$ is not in general zero, provided Ω is such that

$$A\Omega^2 \cos \alpha - Cn\Omega + Mgh = 0 \quad (\mathbf{k} \cdot \mathbf{z} = \cos \alpha).$$

This equation, which determines the possible angular velocities of precession for given values of n and α , is, of course, identical with that already obtained above (eqn 8.9).

To determine the stability of a state of steady precessional motion, let such a state be defined by $\mathbf{k} = \mathbf{k}_0$, and let it be disturbed by the application of a small impulse.* Then we have to study the nature of the disturbed motion when $\mathbf{k}$ differs but slightly from $\mathbf{k}_0$. To do this, we write, as in § 2.6 above,

$$\mathbf{k} = \mathbf{k}_0 + \boldsymbol{\rho},$$

where $\boldsymbol{\rho}$ is to be regarded as a vector whose magnitude is of the first order of small quantities and is such that $\mathbf{k}_0 \cdot \boldsymbol{\rho} = 0$ to this degree of approximation. Then, referred to a frame rotating around the vertical with the angular velocity of precession Ω , corresponding to the state of steady motion, we shall have, since $\mathbf{k}_0$ is constant with respect to this frame,

$$\frac{\partial \mathbf{k}}{\partial t} = \frac{\partial \boldsymbol{\rho}}{\partial t}, \quad \frac{\partial^2 \mathbf{k}}{\partial t^2} = \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2}.$$

Hence, substituting for $\mathbf{k}$, $\partial \mathbf{k} / \partial t$, and $\partial^2 \mathbf{k} / \partial t^2$ in (8.42), the equation governing the behaviour of $\boldsymbol{\rho}$ becomes, on neglecting small quantities of the second order and assuming the line of application of the impulse to be such that n remains unchanged,

$$A\left(\mathbf{k}_0 \wedge \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2}\right) + \{Cn - 2A\Omega(\mathbf{k}_0 \cdot \mathbf{z})\} \frac{\partial \boldsymbol{\rho}}{\partial t} + A\Omega^2(\boldsymbol{\rho} \cdot \mathbf{z})(\mathbf{k}_0 \wedge \mathbf{z}) = \mathbf{0}. \quad (8.43)$$

But

$$\mathbf{z} = \mathbf{k}_0 \cos \alpha - \mathbf{i}_0 \sin \alpha,$$

where at any instant $\mathbf{i}_0$ is the position of $\mathbf{i}$ as defined in § 2, p. 204 (see, also, Fig. 8.4) corresponding to $\mathbf{k} = \mathbf{k}_0$. Consequently, substituting for $\mathbf{z}$, (8.43) becomes

$$A\left(\mathbf{k}_0 \wedge \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2}\right) + (Cn - 2A\Omega \cos \alpha) \frac{\partial \boldsymbol{\rho}}{\partial t} + A\Omega^2 \sin^2 \alpha (\boldsymbol{\rho} \cdot \mathbf{i}_0)(\mathbf{k}_0 \wedge \mathbf{i}_0) = \mathbf{0},$$

* $\mathbf{k}_0$ makes a constant angle with the vertical but is not, of course, a constant vector in space.

or, multiplying vectorially by $\mathbf{k}_0$ and remembering that $\mathbf{k}_0 \cdot \boldsymbol{\rho} = 0$,

$$A \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} - (Cn - 2A\Omega \cos \alpha) \left(\mathbf{k}_0 \wedge \frac{\partial \boldsymbol{\rho}}{\partial t} \right) + A\Omega^2 \sin^2 \alpha (\boldsymbol{\rho} \cdot \mathbf{i}_0) \mathbf{i}_0 = \mathbf{0}. \quad (8.44)$$

In this equation $\boldsymbol{\rho}$, since it is to be regarded as a vector perpendicular to $\mathbf{k}_0$, is a two-dimensional vector with respect to the rotating frame. Owing to the presence of the third term, which involves the component of $\boldsymbol{\rho}$ in the direction defined by $\mathbf{i}_0$, the equation is not directionally independent. In consequence, the equation cannot be solved vectorially. To integrate it, we shall therefore replace it by its equivalent scalar equations. If P_0 is the point on the axis of the top such that $\overline{OP_0} = \mathbf{k}_0$, then, taking a set of orthogonal axes having P_0 as origin and respectively parallel to the directions defined by $(\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$ (see Fig. 8.4),

$$\mathbf{k}_0 = (0, 0, 1), \quad \boldsymbol{\rho} = (x, y, 0), \quad \frac{\partial \boldsymbol{\rho}}{\partial t} = (\dot{x}, \dot{y}, 0).$$

Writing

$$\frac{Cn - 2A\Omega \cos \alpha}{A} = \lambda,$$

the equivalent scalar equations are thus

$$\left. \begin{aligned} \ddot{x} - \lambda \dot{y} + \Omega^2 \sin^2 \alpha x &= 0 & (a) \\ \ddot{y} + \lambda \dot{x} &= 0 & (b) \end{aligned} \right\}. \quad (8.45)$$

These are readily integrated. After first integrating (8.45*b*), which yields

$$\dot{y} = -\lambda x + A_1, \quad (8.46)$$

the final expressions for x and y are given by

$$\left. \begin{aligned} x &= B_1 \cos pt + B_2 \sin pt + \frac{A_1 \lambda}{p^2} & (a) \\ y &= -\frac{\lambda}{p} (B_1 \sin pt - B_2 \cos pt) - \frac{A_1 \lambda^2}{p^2} t + A_2 & (b) \end{aligned} \right\}, \quad (8.47)$$

where

$$p^2 = \lambda^2 + \Omega^2 \sin^2 \alpha = \frac{(Cn - 2A\Omega \cos \alpha)^2 + A^2 \Omega^2 \sin^2 \alpha}{A^2}, \quad (8.48)$$

and A_1, A_2, B_1, B_2 are four constants of integration. Only two of them, however, are independent owing to the coupling which occurs between (8.45*a*) and (8.45*b*) arising from the term involving $\mathbf{k}_0 \wedge \partial \boldsymbol{\rho} / \partial t$ in (8.44).

If, as in § 2.5, the disturbing impulse is chosen so that D , the angular momentum about the vertical, remains constant, and so must act in an

axial plane through OZ , then the initial conditions for the disturbed motion are

$$x = y = 0, \quad \dot{x} \neq 0, \quad \dot{y} = 0.$$

Substituting these values into (8.46) and (8.47), then the expressions for x and y for the disturbed motion are found to be

$$x = B_2 \sin pt, \quad y = \frac{B_2 \lambda}{p} (\cos pt - 1).$$

It is seen that the disturbed motion is periodic, the locus of the point P on the axis of the top such that OP is unity being the ellipse whose equation, when referred to the $\mathbf{i}_0$ - and $\mathbf{j}_0$ -axes, is

$$\frac{x^2}{B_2^2} + \frac{(y + B_2 \lambda/p)^2}{(B_2 \lambda/p)^2} = 1;$$

the ellipse itself rotates about the axis $\mathbf{k}_0$ with an angular velocity Ω and is described with a periodic time $2\pi/p$. It follows that a motion of steady precession is thus stable.

The above result for the periodic time (eqn 8.48) is identical with that obtained previously in § 2.5 (eqn 8.36) with $z_0 = \cos \alpha$. Using the vector treatment, however, it is seen that we are able to show, not only that the motion of steady precession is stable, but also how the axis of the top moves in space in the disturbed motion. To this extent the vector treatment is superior to the non-vector one.*

3. Top spinning on a smooth horizontal plane

Let $\mathbf{R}$ be the reaction between the plane and the top at the point of contact. Then, applying the principle of linear momentum, we have, if $\mathbf{v}$ is the velocity of G , the centre of mass,

$$M \frac{d\mathbf{v}}{dt} = \mathbf{R} - Mgz;$$

* It should be noted that the treatment of the motion of the top, as given in this chapter, may also be applied to the motion of the spherical pendulum. If l is the length of the pendulum and m is its mass, then, on putting $C = 0$, $A = ml^2$, $h = l$ in the vector equation of motion of the top, we obtain the equation

$$ml^2 \left(\mathbf{k} \wedge \frac{d^2 \mathbf{k}}{dt^2} \right) + mgl(\mathbf{k} \wedge \mathbf{z}) = \mathbf{0},$$

which is the vector equation of motion for the spherical pendulum (see Milne, *Vectorial Mechanics*, § 259). On integrating this equation, as for the top, we get the integrals of angular momentum and of energy, viz.

$$ml^2 \sin^2 \theta \dot{\phi} = ml^2(1 - z^2)\dot{\phi} = h(\text{or } D),$$

$$ml^2 \dot{\theta}^2 = ml^2 \frac{\dot{z}^2}{1 - z^2} = E - 2mglz - \frac{h^2}{ml^2(1 - z^2)},$$

where $z = \cos \theta$. These may now be dealt with as in §§ 2.3–2.5, while the condition for uniform circular motion, together with the question of its stability, may also be discussed vectorially as in § 2.7.

also, taking moments about G , we have

$$\frac{d\mathbf{H}_G}{dt} = (-h\mathbf{k} \wedge \mathbf{R}),$$

where $\mathbf{k}$ is a unit vector in the direction of $\overline{OG}$. Eliminating $\mathbf{R}$ from these equations, we thus have

$$\frac{d\mathbf{H}_G}{dt} = -h\left\{\mathbf{k} \wedge \left(M\frac{d\mathbf{v}}{dt} + Mg\mathbf{z}\right)\right\}. \quad (8.49)$$

But

$$\mathbf{H}_G = A\left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt}\right) + Cn\mathbf{k},$$

where C is the moment of inertia of the top about its axis and A is its moment of inertia about a line through G perpendicular to its axis. (8.49) therefore becomes

$$A\left(\mathbf{k} \wedge \frac{d^2\mathbf{k}}{dt^2}\right) + Cn\frac{d\mathbf{k}}{dt} + C\frac{dn}{dt}\mathbf{k} = -Mh\left\{\mathbf{k} \wedge \left(\frac{d\mathbf{v}}{dt} + g\mathbf{z}\right)\right\}, \quad (8.50)$$

where, in consequence of the plane being smooth, so that G moves along a fixed vertical line during the motion,

$$\mathbf{v} = \frac{d}{dt}(h \cos \theta)\mathbf{z} = h\frac{d}{dt}(\mathbf{z} \cdot \mathbf{k})\mathbf{z}. \quad (8.51)$$

(8.50) may be integrated as in the case of a top spinning on a rough plane. Proceeding as in § 2, we obtain the following integrals:

(1) *Integral of Axial Spin.* Multiplying scalarly by $\mathbf{k}$, we have

$$C\frac{dn}{dt} = 0,$$

so that

$$n = \text{constant}.$$

(2) *Angular Momentum Integral.* Putting $dn/dt = 0$, (8.50) becomes

$$A\left(\mathbf{k} \wedge \frac{d^2\mathbf{k}}{dt^2}\right) + Cn\frac{d\mathbf{k}}{dt} = -Mh\left\{\mathbf{k} \wedge \left(\frac{d\mathbf{v}}{dt} + g\mathbf{z}\right)\right\}. \quad (8.52)$$

On multiplying scalarly by $\mathbf{z}$, $d\mathbf{v}/dt$ being parallel to $\mathbf{z}$, this gives,

$$\mathbf{z} \cdot \left\{A\left(\mathbf{k} \wedge \frac{d^2\mathbf{k}}{dt^2}\right) + Cn\frac{d\mathbf{k}}{dt}\right\} = 0,$$

which can be immediately integrated giving

$$\mathbf{z} \cdot \left\{A\left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt}\right) + Cn\mathbf{k}\right\} = \text{constant},$$

a result which expresses the fact that the angular momentum of the top about the vertical through G is constant during the motion.

(3) *Energy Integral.* On multiplying (8.52) scalarly by $\mathbf{k} \wedge d\mathbf{k}/dt$, after substituting for $d\mathbf{v}/dt$ using (8.51), we have, since $\frac{d\mathbf{k}}{dt} \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) = 0$,

$$A \left(\mathbf{k} \wedge \frac{d^2\mathbf{k}}{dt^2} \right) \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) = -Mh \left\{ h \left(\mathbf{z} \cdot \frac{d^2\mathbf{k}}{dt^2} \right) + g \right\} \left\{ \left(\mathbf{k} \wedge \mathbf{z} \right) \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) \right\},$$

i.e. after expanding the right-hand side,

$$A \left(\mathbf{k} \wedge \frac{d^2\mathbf{k}}{dt^2} \right) \cdot \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right) + Mh \left\{ h \left(\mathbf{z} \cdot \frac{d^2\mathbf{k}}{dt^2} \right) + g \right\} \left(\mathbf{z} \cdot \frac{d\mathbf{k}}{dt} \right) = 0.$$

This can be integrated as it stands giving

$$A \left(\mathbf{k} \wedge \frac{d\mathbf{k}}{dt} \right)^2 + Mh^2 \left(\mathbf{z} \cdot \frac{d\mathbf{k}}{dt} \right)^2 + 2Mgh(\mathbf{z} \cdot \mathbf{k}) = \text{constant},$$

which expresses the constancy of the total energy.

Having obtained these three integrals, if we now introduce the Eulerian angles (θ, ϕ, ψ) to define the position of the top relative to G , then we shall have three scalar equations connecting these variables and their rates of change. The remainder of the theory is therefore as in § 2.3.

3.1. Precessional motion. If the top has a motion of steady precession, then G remains stationary. To obtain Ω , the angular velocity of precession corresponding to such a motion, the axis making an angle α with the vertical, we can proceed either as in § 2.7 above, or, alternatively, make use of the fact that in a state of steady precession

$$\begin{aligned} \frac{d\mathbf{k}}{dt} &= \Omega(\mathbf{z} \wedge \mathbf{k}), \\ \frac{d^2\mathbf{k}}{dt^2} &= \Omega \left(\mathbf{z} \wedge \frac{d\mathbf{k}}{dt} \right) = \Omega^2 \{ \mathbf{z} \wedge (\mathbf{z} \wedge \mathbf{k}) \}. \end{aligned}$$

On substituting these values into (8.52), we have, after putting $d\mathbf{v}/dt = 0$,

$$A\Omega^2 \mathbf{k} \wedge \{ \mathbf{z} \wedge (\mathbf{z} \wedge \mathbf{k}) \} - Cn\Omega(\mathbf{k} \wedge \mathbf{z}) + Mgh(\mathbf{k} \wedge \mathbf{z}) = \mathbf{0},$$

or

$$\{ A\Omega^2(\mathbf{k} \cdot \mathbf{z}) - Cn\Omega + Mgh \} (\mathbf{k} \wedge \mathbf{z}) = \mathbf{0}.$$

The possible angular velocities of precession are therefore given by

$$A\Omega^2 \cos \alpha - Cn\Omega + Mgh = 0.$$

It is seen that this equation has exactly the same form as (8.9) for a top spinning on a rough plane; the only difference is that A now represents the moment of inertia about a transverse axis through G instead

of through the point of contact O . The condition for steady precessional motion to be possible is again

$$C^2 n^2 \geq 4AMgh \cos \alpha.$$

Further problems, which involve precessional motion and which are treated by vector methods, are to be found in Milne, *Vectorial Mechanics*, Chapter XVII.*

Examples

1. A right circular cone of mass M , whose axis makes an angle α with the vertical, has its vertex smoothly pivoted to a fixed point O and moves with the rim of its base on a smooth table. If the points of contact lie on a circle of radius a , prove that the pressure on the table is

$$\sin \alpha (Mgh + Cn\Omega - A\Omega^2 \cos \alpha)/a.$$

where Ω is the angular velocity of rotation of the vertical plane containing the axis of symmetry.

2. A symmetrical top is spinning on a rough horizontal plane. If Ω_1 , Ω_2 are the two possible values of the angular velocity of precession with the axis inclined at a constant angle α to the vertical, show that, in order to maintain a constant precessional angular velocity Ω with the same constant value of α , a couple of magnitude

$$A \sin \alpha \cos \alpha (\Omega - \Omega_1)(\Omega - \Omega_2)$$

must be applied in the vertical plane through the axis of the top in a direction tending to decrease α .

3. A symmetrical top is spinning with its vertex in contact with a rough horizontal plane. Initially the axis of the top is at rest and makes an angle $\pi/3$ with the upward vertical, the spin about the axis being

$$2\sqrt{(AMgh/C^2)}.$$

Prove that, after the axis is released, its inclination to the vertical varies between $\pi/3$ and $\pi/2$, and that in the latter position the velocity of precession is $\sqrt{(Mgh/A)}$.

4. A symmetrical top is free to move about a fixed point O of its axis. The axis is projected without vertical motion so that the

* An interesting example of a uniaxial body spinning in contact with a rough plane is that of the 'tippel-top'. This is a body whose shape is that of a segment of a sphere (greater than a hemisphere) with a slender cylindrical peg protruding from the centre of the plane portion of its surface. When spun with the spherical portion of its surface in contact with a rough plane, the top rolls and eventually spins with the extremity of its peg in contact with the plane, its spherical portion being uppermost. For the theory of its motion, see D. G. Parkyn, "The Inverting Top", *Math. Gaz.*, Vol. XL, No. 334 (1956), p. 260.

precession Ω and the spin n of the top are connected by the relation $Cn\Omega = 2Mgh$. Show that, if $2A\Omega < Cn$, the axis of the top comes to rest from time to time.

If the axis of the top is horizontal when it is projected in this way, show that, when it comes to rest, it makes an angle α with the vertical, where $\cos \alpha = A\Omega/Cn$.

5. A symmetrical top is spinning with the point of its peg fixed. When the inclination of its axis to the upward vertical is α ($< \pi/2$), the rate of precession is Ω and the inclination is instantaneously stationary. Prove that the condition that the axis reaches the horizontal position is

$$2AMgh \geq C^2n^2 \cos \alpha + 2ACn\Omega \sin^2 \alpha - A^2\Omega^2 \sin^2 \alpha \cos \alpha.$$

6. A top of mass M , which can move freely about its vertex O which is fixed, is precessing uniformly with its axis of symmetry in the horizontal plane through O . If the precession is advanced by the action of an impulsive couple of moment $AMgh/Cn$ about a vertical axis, prove that, during the subsequent motion, the least inclination of the axis of symmetry to the vertical is $\cos^{-1}(2AMgh/C^2n^2)$ provided $C^2n^2 > 2AMgh$.

7. The *apex* of a symmetrical top is constrained to move in a smooth horizontal groove of radius a . Prove that a steady motion is possible in which the axis makes an angle α with the vertical and the velocity of precession Ω is given by

$$(A \sin \alpha + Mah)\Omega^2 \cos \alpha - Cn\Omega \sin \alpha + Mgh \sin \alpha = 0,$$

A being the moment of inertia of the top about a line through its apex perpendicular to its axis.

8. A sphere, of radius a and mass M , whose centre of mass coincides with its centre of figure, but whose principal moments at that point are A, A, C , is in rolling contact with a horizontal plane. If n is the angular velocity of the sphere about the axis of unequal moment, show that the equations of motion are satisfied by a steady motion in which this axis is inclined at an angle α to the vertical whilst the centre of the sphere describes a circle of radius c with a constant angular velocity Ω , where

$$c = \frac{(C-A)}{(C+Ma^2)} a \sin \alpha \cos \alpha, \quad \Omega = \frac{(C+Ma^2)}{(A+Ma^2)} n \sec \alpha.$$

CHAPTER IX

IMPULSIVE MOTION

1. Equations of impulsive motion of a particle

Consider a particle of mass m moving under the action of a force $\mathbf{F}$. Its equation of motion is given by

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}.$$

If we integrate this equation with respect to t from $t = t_1$ to $t = t_2$, say, then we have

$$m\mathbf{v}_2 - m\mathbf{v}_1 = \int_{t_1}^{t_2} \mathbf{F} dt. \quad (9.1)$$

The integral on the right-hand side is defined to be the impulse of the force $\mathbf{F}$ in the interval of time $(t_2 - t_1)$; equation (9.1) thus expresses the fact that *the impulse of the force in any interval of time is equal to the change of the linear momentum of the particle in the same interval of time.*

The result stated above is true no matter what the length of the interval $(t_2 - t_1)$ may be, whether it is large or small; in particular, it will hold for infinitesimally small intervals of time. Suppose, therefore, that we consider a very large force acting for an infinitesimally short interval of time, the magnitude of the force being such that it produces a finite change of momentum in the given interval. Such a force is termed an *impulsive force*, and is defined by the change of momentum which it produces. Mathematically, an impulsive force may therefore be defined by the limit of the integral occurring on the right-hand side of (9.1) as the interval $\tau = (t_2 - t_1)$ is indefinitely diminished. If we write

$$\mathbf{J} = \lim_{\tau \rightarrow 0} \int_0^{\tau} \mathbf{F} dt, \quad (9.2)$$

then we have

$$\mathbf{J} = m\mathbf{v}_2 - m\mathbf{v}_1. \quad (9.3)$$

$\mathbf{J}$ is termed the *instantaneous impulse* exerted on the particle at time $t = t_1 (= t_2)$.

In order for the limiting value of the above integral to be different from zero, it is clear that, as $\tau = (t_2 - t_1)$ is indefinitely diminished, $\mathbf{F}$

must become very large. Consequently, in calculating the effect of an impulsive force, the effect of any ordinary *finite* forces, which may be acting on the particle at the same time, may be neglected; in addition, since the velocity of the particle, although it changes, remains finite, it follows that the integral

$$\lim_{\tau \rightarrow 0} \int_0^{\tau} \mathbf{v} \, dt$$

is zero, i.e. the change in the position of the particle in the short interval of time $\tau = (t_2 - t_1)$ may also be ignored. The vector $\mathbf{J}$ defined by (9.2) and (9.3) may thus be regarded as localized at the position of the particle at the instant at which the sudden change of motion takes place.

The above result has been obtained by considering the equation expressing the rate of change of the linear momentum of the particle. Now let us consider the equation governing the rate of change of its moment of momentum. If O is a fixed point and $\mathbf{H}$ is the moment of momentum of the particle with respect to it, then we have

$$\frac{d\mathbf{H}}{dt} = \mathbf{r} \wedge \mathbf{F}, \quad (9.4)$$

where $\mathbf{r}$ is the vector displacement of the particle with respect to O . Integrating over an interval of time $(t_2 - t_1)$, we thus have

$$\mathbf{H}_2 - \mathbf{H}_1 = \int_{t_1}^{t_2} (\mathbf{r} \wedge \mathbf{F}) dt. \quad (9.5)$$

Now suppose the interval $\tau = (t_2 - t_1)$ to be made indefinitely small, whilst, at the same time, $\mathbf{F}$ is so increased that the integral

$$\int_{t_1}^{t_2} \mathbf{F} \, dt$$

remains finite and different from zero. Then, as we have seen above, we may regard $\mathbf{r}$ as remaining constant during this interval. Consequently (9.5) may be written

$$\mathbf{H}_2 - \mathbf{H}_1 = \mathbf{r} \wedge \left(\lim_{\tau \rightarrow 0} \int_0^{\tau} \mathbf{F} \, dt \right) = \mathbf{r} \wedge \mathbf{J}.$$

In deducing this result we started with the equation expressing the rate of change of the moment of momentum of a particle with respect to a fixed point. If, instead of taking a fixed point, we consider the

rate of change of the moment of momentum with respect to a moving point O whose velocity is $\mathbf{v}_0$, then (9.4) has to be replaced by (see Chap. VI, § 1.2, p. 125)

$$\frac{d\mathbf{H}}{dt} + \mathbf{v}_0 \wedge m\mathbf{v} = \mathbf{r} \wedge \mathbf{F}.$$

On integration this gives

$$\mathbf{H}_2 - \mathbf{H}_1 + \int_{t_1}^{t_2} (\mathbf{v}_0 \wedge m\mathbf{v}) dt = \int_{t_1}^{t_2} (\mathbf{r} \wedge \mathbf{F}) dt. \quad (9.6)$$

But the velocity of O is clearly independent of the forces acting on the particle. Consequently, in the limit, when the interval $\tau = (t_2 - t_1)$ is taken to be infinitesimally small, we can assume $\mathbf{v}_0$ to remain constant during the interval, so that (9.6) becomes, on proceeding to the limit $\tau \rightarrow 0$,

$$\mathbf{H}_2 - \mathbf{H}_1 + m\mathbf{v}_0 \wedge \left(\lim_{\tau \rightarrow 0} \int_0^\tau \mathbf{v} dt \right) = \mathbf{r} \wedge \left(\lim_{\tau \rightarrow 0} \int_0^\tau \mathbf{F} dt \right),$$

or

$$\mathbf{H}_2 - \mathbf{H}_1 = \mathbf{r} \wedge \mathbf{J},$$

since $\lim_{\tau \rightarrow 0} \int_0^\tau \mathbf{v} dt = \mathbf{0}$ (see above).

We thus have the result: *if an instantaneous impulse is applied to a particle, then the instantaneous increase in its moment of momentum with respect to a point O in space, whether O is at rest or in motion, is equal to the moment of the impulse with respect to the same point.*

2. Extension to a system of interacting particles

For a system of mutually interacting particles, the equations governing the rates of change of the total linear momentum and the total moment of momentum of the system are (see Chap. VI, eqns 6.4 and 6.6)

$$\frac{d}{dt} \left(\sum_i m_i \mathbf{v}_i \right) = \sum_i \mathbf{F}_i + \sum_i \sum_k' \mathbf{F}_{ik},$$

$$\frac{d}{dt} \left\{ \sum_i (\mathbf{r}_i \wedge m_i \mathbf{v}_i) \right\} = \sum_i (\mathbf{r}_i \wedge \mathbf{F}_i) + \sum_i (\mathbf{r}_i \wedge \sum_k' \mathbf{F}_{ik}),$$

respectively.

If the forces occurring in the above equations are replaced by impulsive forces, then, since there are no instantaneous changes in the positions of the particles, the corresponding equations for the instan-

taneous changes in the total linear momentum and the total moment of momentum for the system are

$$\sum_i (m_i \mathbf{v}_i)_2 - \sum_i (m_i \mathbf{v}_i)_1 = \sum_i \mathbf{J}_i + \sum_i \sum_k' \mathbf{J}_{ik},$$

$$\sum_i \mathbf{r}_i \wedge \{(m_i \mathbf{v}_i)_2 - (m_i \mathbf{v}_i)_1\} = \sum_i (\mathbf{r}_i \wedge \mathbf{J}_i) + \sum_i (\mathbf{r}_i \wedge \sum_k' \mathbf{J}_{ik}),$$

where
$$\mathbf{J}_i = \lim_{\tau \rightarrow 0} \int_0^\tau \mathbf{F}_i dt \quad (i = 1, 2, \dots, n)$$

represent the impulses applied to the system and

$$\mathbf{J}_{ik} = \lim_{\tau \rightarrow 0} \int_0^\tau \mathbf{F}_{ik} dt \quad \begin{pmatrix} i = 1, 2, \dots, n \\ k = 1, 2, \dots, n \end{pmatrix} \quad (k \neq i)$$

are the internal impulses brought into play as a result of the application of the external impulses. The suffixes 1 and 2 refer, as above, to the motion immediately before and immediately after the application of the impulses respectively. But the internal impulses will consist of pairs of equal and opposite impulses, so that, for any system of mutually interacting particles,

$$\sum_i \sum_k' \mathbf{J}_{ik} = \mathbf{0}, \quad \sum_i (\mathbf{r}_i \wedge \sum_k' \mathbf{J}_{ik}) = \mathbf{0}.$$

Consequently, since

$$\sum_i m_i \mathbf{v}_i = M\mathbf{v},$$

where $M = \sum_i m_i$ and $\mathbf{v}$ represents the velocity of the centre of mass of the system, the equations governing the instantaneous change in the motion of the system due to the application of a set of impulsive forces become

$$M\mathbf{v}_2 - M\mathbf{v}_1 = \sum_i \mathbf{J}_i, \quad (9.7a)$$

$$\mathbf{H}_2 - \mathbf{H}_1 \equiv \sum_i (\mathbf{r}_i \wedge m_i \mathbf{v}_i)_2 - \sum_i (\mathbf{r}_i \wedge m_i \mathbf{v}_i)_1 = \sum_i (\mathbf{r}_i \wedge \mathbf{J}_i), \quad (9.7b)$$

the last equation being true whether the origin with respect to which moments are taken is either at rest or in motion. We thus have the following results: *when a system of mutually interacting particles is acted upon by a set of instantaneous impulses then (i) the instantaneous change in the total linear momentum of the system is equal to the vector sum of the applied impulses, (ii) the instantaneous increase in the total moment of momentum of the system with respect to any point, which may be either at rest or in motion, is equal to the vector sum of the moments of the impulses with respect to the point.*

It follows from (9.7b) that, if the total moment of the applied

impulses with respect to any point vanishes, then the total moment of momentum of the system with respect to this point remains constant.

Another important result also follows from (9.7*b*). If λ is a unit vector defining a line through O , the point about which moments are taken, then, on multiplying this equation scalarly by λ , we have

$$\lambda \cdot \mathbf{H}_2 - \lambda \cdot \mathbf{H}_1 = \lambda \cdot \sum_i (\mathbf{r}_i \wedge \mathbf{J}_i).$$

If the impulses and their points of application are such that

$$\lambda \cdot \sum_i (\mathbf{r}_i \wedge \mathbf{J}_i) = 0,$$

then we shall have

$$\lambda \cdot \mathbf{H}_2 = \lambda \cdot \mathbf{H}_1,$$

i.e. the component of the moment of momentum with respect to the line λ remains constant.

3. Application to a rigid body

The above results for a system of mutually interacting particles will hold, of course, if the system on which the impulses act is a rigid body. In this case the motion of the body is known if we know the velocity of some point of it (usually the centre of mass) together with the angular velocity of the body. If the impulses are known, then the velocity of the centre of mass immediately after the application of the impulses will be determined by means of (9.7*a*), while the angular velocity may be obtained by applying (9.7*b*), moments being taken with respect to the centre of mass.* Thus the above equations allow the instantaneous change in the motion of a rigid body, resulting from the application of given impulses, to be determined.

We shall now consider their application to some particular types of motion.

3.1. General equations for a single body. Let $\mathbf{v}$ be the velocity of G , the centre of mass of the body, ω the angular velocity, and suppose the body to be acted upon by a set of impulses $\mathbf{J}_i$ whose points of application are at distances $\mathbf{r}_i$ from G . Then, if the suffixes 1 and 2 denote the values of $\mathbf{v}$ and $\mathbf{H}$, the moment of momentum of the system with respect to the centre of mass, immediately before and immediately after the application of the impulses, the equations which determine the change of motion will be given by

$$M(\mathbf{v}_2 - \mathbf{v}_1) = \sum_i \mathbf{J}_i, \quad (9.8)$$

$$\mathbf{H}_2 - \mathbf{H}_1 = \sum_i (\mathbf{r}_i \wedge \mathbf{J}_i). \quad (9.9)$$

* Any other convenient point may also, of course, be chosen, the angular velocity of the body being a property of the motion without reference to any particular point.

The scalar equations equivalent to these are:

$$M(v_{2x}-v_{1x}) = \sum_i \mathbf{J}_{ix},$$

$$M(v_{2y}-v_{1y}) = \sum_i \mathbf{J}_{iy},$$

$$M(v_{2z}-v_{1z}) = \sum_i \mathbf{J}_{iz},$$

$$A(\omega_{2x}-\omega_{1x})-H(\omega_{2y}-\omega_{1y})-G(\omega_{2z}-\omega_{1z}) = \sum_i (y_i \mathbf{J}_{iz}-z_i \mathbf{J}_{iy}),$$

$$-H(\omega_{2x}-\omega_{1x})+B(\omega_{2y}-\omega_{1y})-F(\omega_{2z}-\omega_{1z}) = \sum_i (z_i \mathbf{J}_{iz}-x_i \mathbf{J}_{iz}),$$

$$-G(\omega_{2x}-\omega_{1x})-F(\omega_{2y}-\omega_{1y})+C(\omega_{2z}-\omega_{1z}) = \sum_i (x_i \mathbf{J}_{iy}-y_i \mathbf{J}_{iz});$$

the axes of reference are any convenient, instantaneous, set of orthogonal axes through G .

Provided the impulses and their points of application are known, then the above equations are just sufficient to determine the change of motion.

If the impulses are not known, but $\mathbf{v}_i$, the velocities of their points of application in the new motion, are prescribed, then we have a further set of equations of the type

$$\mathbf{v}_2 + (\boldsymbol{\omega}_2 \wedge \mathbf{r}_i) = \mathbf{v}_i. \quad (9.10)$$

These, in conjunction with (9.8) and (9.9), or their scalar equivalents, allow the impulses, as well as the change in motion, to be determined.

As we shall see below, in certain cases, the solution of (9.8), (9.9), and (9.10) may be effected vectorially; in general, however, the equivalent scalar equations have to be used.

3.2. General result for a body possessing kinetic symmetry with respect to its centre of mass. Consider a rigid body in motion, and let it be acted upon by an instantaneous impulse $\mathbf{J}$ whose point of application P is at a vector distance $\mathbf{r}$ from G . Then the instantaneous change in the linear momentum of the body will be given by

$$\mathbf{J} = M(\mathbf{v}_2 - \mathbf{v}_1), \quad (9.11)$$

while, if $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are the corresponding angular velocities of the body, we shall have, considering the instantaneous change in the moment of momentum of the body with respect to G ,

$$\mathbf{r} \wedge \mathbf{J} = I_2 \boldsymbol{\omega}_2 - I_1 \boldsymbol{\omega}_1, \quad (9.12)$$

where I_1 and I_2 are the moments of inertia of the body about the respective instantaneous axes of rotation, regarded as localized at G , immediately before and immediately after the application of the impulse. We also have, if $\mathbf{v}_P$ is the velocity of P immediately after the application of the impulse,

$$\mathbf{v}_2 + (\boldsymbol{\omega}_2 \wedge \mathbf{r}) = \mathbf{v}_P. \quad (9.13)$$

We thus have three equations which may be used to determine any three of the seven vector quantities $\mathbf{v}_1$, $\mathbf{v}_2$, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\mathbf{J}$, $\mathbf{r}$, and $\mathbf{v}_P$; the other four quantities being regarded as known.

If we eliminate $\mathbf{J}$ from (9.11) and (9.12), then we have

$$M\{\mathbf{r} \wedge (\mathbf{v}_2 - \mathbf{v}_1)\} = I_2 \boldsymbol{\omega}_2 - I_1 \boldsymbol{\omega}_1$$

which, after substituting for $\mathbf{v}_2$ from (9.13), gives

$$M\{\mathbf{r} \wedge [\mathbf{v}_P - (\boldsymbol{\omega}_2 \wedge \mathbf{r})] - (\mathbf{r} \wedge \mathbf{v}_1)\} = I_2 \boldsymbol{\omega}_2 - I_1 \boldsymbol{\omega}_1,$$

or

$$M\{(\mathbf{r} \wedge \mathbf{v}_P) - \boldsymbol{\omega}_2 r^2 + \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}_2) - (\mathbf{r} \wedge \mathbf{v}_1)\} = I_2 \boldsymbol{\omega}_2 - I_1 \boldsymbol{\omega}_1,$$

i.e.

$$\boldsymbol{\omega}_2(I_2 + Mr^2) = I_1 \boldsymbol{\omega}_1 + M\{\mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}_2) + (\mathbf{r} \wedge \mathbf{v}_P) - (\mathbf{r} \wedge \mathbf{v}_1)\}. \quad (9.14)$$

But, from (9.12), we have, on multiplying scalarly by $\mathbf{r}$,

$$I_2(\mathbf{r} \cdot \boldsymbol{\omega}_2) = I_1(\mathbf{r} \cdot \boldsymbol{\omega}_1),$$

so that, substituting for $(\mathbf{r} \cdot \boldsymbol{\omega}_2)$ in (9.14),

$$\boldsymbol{\omega}_2 = \frac{I_1 I_2 \boldsymbol{\omega}_1 + M\{I_1(\mathbf{r} \cdot \boldsymbol{\omega}_1)\mathbf{r} + I_2(\mathbf{r} \wedge \mathbf{v}_P) - I_2(\mathbf{r} \wedge \mathbf{v}_1)\}}{I_2(I_2 + Mr^2)}. \quad (9.15)$$

In the general case, the right-hand side of this equation does not determine $\boldsymbol{\omega}_2$, however, since I_2 cannot be determined until the direction of $\boldsymbol{\omega}_2$ itself is known. In the case in which the body possesses kinetic symmetry with respect to its centre of mass, then the moments of inertia about all lines through G are the same, so that $I_2 = I_1 = I$, say. Consequently, for such a body, the difficulty does not exist, and (9.15) reduces to

$$\boldsymbol{\omega}_2 = \frac{I \boldsymbol{\omega}_1 + M\{\mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}_1) + (\mathbf{r} \wedge \mathbf{v}_P) - (\mathbf{r} \wedge \mathbf{v}_1)\}}{I + Mr^2}.$$

Thus, in this case, $\boldsymbol{\omega}_2$ may be determined if $\mathbf{v}_1$, $\boldsymbol{\omega}_1$, $\mathbf{r}$, and $\mathbf{v}_P$ are known.

In the special case in which $\mathbf{v}_1 = \mathbf{0}$ and the impulse is such that it brings the point P to rest, so that $\mathbf{v}_P = \mathbf{0}$, the motion of the body initially being such that it is rotating about an axis through its centre of mass which is at rest, then the new angular velocity will be given by

$$\boldsymbol{\omega}_2 = \frac{I \boldsymbol{\omega}_1 + M \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}_1)}{I + Mr^2}. \quad (9.16)$$

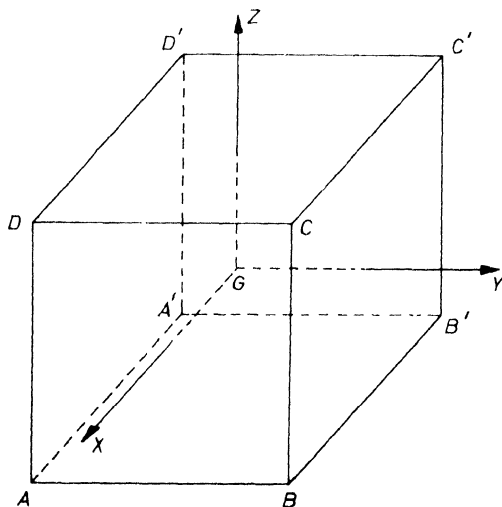
The velocity of the centre of mass in the new motion, and hence the new motion itself ($\boldsymbol{\omega}_2$ being known), may now be determined by means of (9.13). The impulse required to bring about this instantaneous change in the motion is given by (9.11).

Since $\mathbf{v}_P = \mathbf{0}$, it follows that the motion immediately after the application of the impulse must be equivalent to a rotational motion

about an axis through P , the magnitude and direction of the angular velocity being those of $\boldsymbol{\omega}_2$. But, from (9.16), it is seen that, in this case, $\boldsymbol{\omega}_2$ is represented by the sum of two vectors, one of which has the direction of $\boldsymbol{\omega}_1$ and the other that of $\mathbf{r}$. Thus the new axis of rotation lies in the plane defined by $\boldsymbol{\omega}_1$ and $\mathbf{r}$, i.e. it lies in a plane which passes through G . If, in addition, $\mathbf{r} \cdot \boldsymbol{\omega}_1 \neq 0$ (if $\mathbf{r} \cdot \boldsymbol{\omega}_1 = 0$, $\boldsymbol{\omega}_2$ will be parallel to $\boldsymbol{\omega}_1$), then it will intersect the axis of rotation corresponding to the motion immediately before the application of the impulse, the angle θ between the two axes being given by

$$\theta = \tan^{-1} \frac{\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2}{\boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_2}.$$

Example. A uniform cube, of edge $2a$ and mass M , is rotating in space about a diagonal with angular velocity Ω . One of the corners, not on the axis of rotation, is suddenly brought to rest by a frictionless constraint. Show that the angular velocity of the body is suddenly changed to $(\Omega\sqrt{17})/11$ and determine the magnitude of the impulse required.



Solution. From symmetry considerations, it follows that the three lines through G , the centre of mass, mutually orthogonal to one another and perpendicular to the faces of the cube are principal axes at G . Since the moments of inertia about these three lines are equal, it also follows that the momental ellipsoid corresponding to the point G is a sphere. The moments of inertia about all lines through G are thus

equal, so that a cube is kinetically symmetrical with respect to its centre of mass. We may therefore determine the new angular velocity by applying the above theory.

Let AA' be the initial axis of rotation, and let B be the point which is suddenly fixed. Then, since G is at rest initially, while B is at rest after the application of the impulse, we have, putting $\boldsymbol{\omega}_1 = \boldsymbol{\Omega}$ in (9.16),

$$\boldsymbol{\omega}_2 = \frac{I\boldsymbol{\Omega} + M\mathbf{r}(\mathbf{r} \cdot \boldsymbol{\Omega})}{I + Mr^2}, \quad (1)$$

where $\mathbf{r} = \overline{GB}$.

Suppose we choose a set of axes $GXYZ$ as shown. Then we shall have

$$\mathbf{r} = a(1, -1, 1), \quad \boldsymbol{\Omega} = \Omega \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \quad (2)$$

so that

$$\mathbf{r} \cdot \boldsymbol{\Omega} = -a\Omega/\sqrt{3}. \quad (3)$$

Also

$$I = \frac{2}{3}Ma^2, \quad (4)$$

so that

$$I + Mr^2 = \frac{2}{3}Ma^2 + 3Ma^2 = \frac{11}{3}Ma^2. \quad (5)$$

Hence, substituting from (2), (3), (4), and (5) into (1), we have

$$\begin{aligned} \boldsymbol{\omega}_2 &= \frac{3}{11Ma^2} \left\{ \frac{2}{3}Ma^2\Omega \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) - \frac{Ma^2\Omega}{\sqrt{3}}(1, -1, 1) \right\} \\ &= \frac{\Omega\sqrt{3}}{11} \left\{ \frac{2}{3}(-1, 1, 1) - (1, -1, 1) \right\} \\ &= \frac{\Omega\sqrt{3}}{11} \left(-\frac{5}{3}, \frac{5}{3}, -\frac{1}{3} \right). \end{aligned} \quad (6)$$

Therefore

$$\begin{aligned} \omega_2 &= \frac{\Omega\sqrt{3}}{11} \left(\frac{25}{9} + \frac{25}{9} + \frac{1}{9} \right)^{1/2} \\ &= \frac{\Omega\sqrt{17}}{11}. \end{aligned}$$

The impulse applied at B , necessary to bring B to rest in the new motion, is given by

$$\mathbf{J} = M\mathbf{v}_2 \quad (\mathbf{v}_1 = \mathbf{0}).$$

But, since $\mathbf{v}_B = \mathbf{0}$,

$$\mathbf{v}_2 + (\boldsymbol{\omega}_2 \wedge \mathbf{r}) = \mathbf{0},$$

where $\mathbf{r}$ and $\boldsymbol{\omega}_2$ are given by (2) and (6), so that we have

$$\begin{aligned}\mathbf{v}_2 &= \frac{\Omega\sqrt{3}}{11}(-\frac{5}{3}, \frac{5}{3}, -\frac{1}{3}) \wedge a(1, -1, 1) \\ &= \frac{a\Omega\sqrt{3}}{33}\{(-5, 5, -1) \wedge (1, -1, 1)\} \\ &= \frac{a\Omega\sqrt{3}}{33}(-4, 4, 0).\end{aligned}$$

Hence

$$\mathbf{J} = \frac{4\sqrt{3}}{33}Ma\Omega(-1, 1, 0).$$

4. A body is in motion when a line in the body is suddenly fixed. To find the new angular velocity of rotation.

In order to fix a line in the body it is necessary to apply impulses to two points of the line, the impulses being such that they bring their respective points of application to rest. But these impulses, since they intersect the line, will clearly have no moment about it, and hence, by the result obtained in § 2 above, the moment of momentum of the system with respect to the line will be the same in both motions, i.e. we shall have, if I is the moment of inertia of the body with respect to the given line and ω_2 the new angular velocity about it,

$$I\omega_2 = \text{a known quantity.} \quad (9.17)$$

But, since the new motion is completely defined if ω_2 is known, it follows that the new motion may be completely determined by means of the above scalar equation.

In order to make use of this result, it is essential to be able to determine the moment of momentum of a moving system with respect to a given line. This is most conveniently done in the case of a rigid body if the motion of the body is expressed in terms of the linear motion of its centre of mass together with the rotational motion relative to the centre of mass. The required expression is obtained below.

4.1. Expression for the moment of momentum of a body with respect to a line

(1) *General case.* In Chapter VI, § 1.1, when considering the motion of a system of mutually interacting particles, we derived the result that $\mathbf{H}_O$, the moment of momentum of the system with respect to a point O , may be expressed in the form

$$\mathbf{H}_O = \mathbf{H}_G + (\mathbf{r}_G \wedge M\mathbf{v}_G), \quad (9.18)$$

where $\mathbf{v}_G$ is the velocity of G , the centre of mass of the system, $\mathbf{r}_G = \overline{OG}$, $M = \sum m$, and

$$\mathbf{H}_G = \sum (\mathbf{r}' \wedge m\mathbf{v}) \equiv \sum (\mathbf{r}' \wedge m\mathbf{v}')$$

is the moment of momentum of the system with respect to G , $\mathbf{r}'$ and $\mathbf{v}'$ respectively denoting the position and velocity of a particle of mass m of the system relative to G (see equations 6.12 and 6.13).

The above result will clearly apply if the system is a rigid body. We thus have the result: *the moment of momentum of a rigid body with respect to any point O is equal to the moment of momentum of the body with respect to the centre of mass plus the moment of momentum with respect to O of a fictitious particle supposedly situated at the centre of mass and moving with it and having a mass equal to that of the body.*

If the motion is such that G is at rest, then it follows that, no matter what the point O ,

$$\mathbf{H}_O = \mathbf{H}_G.$$

Now let O be a point on the given line. If the unit vector $\boldsymbol{\lambda}$ defines the direction of the line in space, then, on multiplying (9.18) scalarly by $\boldsymbol{\lambda}$, we have

$$\boldsymbol{\lambda} \cdot \mathbf{H}_O = \boldsymbol{\lambda} \cdot \mathbf{H}_G + \boldsymbol{\lambda} \cdot (\mathbf{r}_G \wedge M\mathbf{v}_G). \quad (9.19)$$

$\boldsymbol{\lambda} \cdot \mathbf{H}_O$ is termed the *moment of momentum of the system with respect to the given line*.

Since O is *any* point on the given line, this is justified only provided the above expression is independent of the choice of O . To show that this is so, let O' be some other point on the line, so that we can write

$$\overline{OO'} = k\boldsymbol{\lambda},$$

where k is a scalar coefficient. Then we shall have (eqn 9.18)

$$\mathbf{H}_{O'} = \mathbf{H}_G + (\mathbf{r}'_G \wedge M\mathbf{v}_G),$$

where $\mathbf{r}'_G = O'G$. But

$$\mathbf{r}'_G = \mathbf{r}_G - \overline{OO'} = \mathbf{r}_G - k\boldsymbol{\lambda},$$

and hence

$$\begin{aligned} \mathbf{H}_{O'} &= \mathbf{H}_G + (\mathbf{r}_G - k\boldsymbol{\lambda}) \wedge M\mathbf{v}_G \\ &= \{\mathbf{H}_G + (\mathbf{r}_G \wedge M\mathbf{v}_G)\} - k(\boldsymbol{\lambda} \wedge M\mathbf{v}_G) \\ &= \mathbf{H}_O - k(\boldsymbol{\lambda} \wedge M\mathbf{v}_G). \end{aligned}$$

If we now multiply scalarly by $\boldsymbol{\lambda}$, we have

$$\boldsymbol{\lambda} \cdot \mathbf{H}_{O'} = \boldsymbol{\lambda} \cdot \mathbf{H}_O - k\boldsymbol{\lambda} \cdot (\boldsymbol{\lambda} \wedge M\mathbf{v}_G),$$

i.e.

$$\boldsymbol{\lambda} \cdot \mathbf{H}_{O'} = \boldsymbol{\lambda} \cdot \mathbf{H}_O,$$

as desired.

Expressed in words, (9.19) states that: *the moment of momentum of a system about a given line is equal to the moment of momentum of the system about a parallel line through the centre of mass of the system together with the moment of momentum about the given line of a fictitious particle supposedly situated at the centre of the mass of the system and moving with it and having a mass equal to the total mass of the system.*

If the centre of mass is at rest, then the above result reduces to

$$\lambda.H_O = \lambda.H_G,$$

i.e. the moment of momentum of the system about any line is equal to the moment of momentum of the system about a parallel line through the centre of mass.

(2) *Expression when the motion is one of rotation about an axis fixed in the body.* Suppose the new motion to arise from the sudden fixing of some axis in the body, the original motion being one of rotation about an axis, this axis being released at the same instant as the new axis is

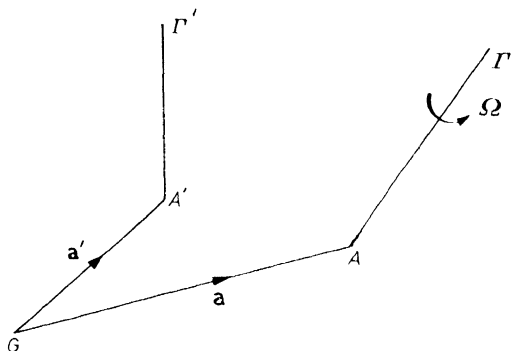


Figure 9.1

fixed. In such cases, as already stated, the new motion will be completely defined by the angular velocity about the new axis of rotation, so that the scalar equation obtained by expressing the constancy of the moment of momentum of the system about the new axis of rotation will suffice to determine the motion (eqn 9.17). To enable this result to be applied, we shall now obtain an expression for the moment of momentum of a rigid body about a given line when the motion of the body consists of a rotation about some other axis.

Let the body be rotating with an angular velocity Ω about the line Γ whose direction is defined by the unit vector λ , and let Γ' be some other line whose direction is defined by the unit vector λ' , say (Fig. 9.1). We wish to find the moment of momentum of the body about Γ' .

Let A be some point on Γ and A' some point on Γ' , and let their vector displacements with respect to G , the centre of mass of the body, be $\mathbf{a}$ and $\mathbf{a}'$ respectively. Then the velocity of G is given by

$$\mathbf{v}_G = -(\Omega \wedge \mathbf{a}) = -\Omega(\lambda \wedge \mathbf{a}).$$

Hence, applying the expression for $\mathbf{H}$ above (eqn 9.18), we have

$$\begin{aligned}\mathbf{H}_{A'} &= \mathbf{H}_G + \{-\mathbf{a}' \wedge -M\Omega(\lambda \wedge \mathbf{a})\} \\ &= \mathbf{H}_G + M\Omega\{\mathbf{a}' \wedge (\lambda \wedge \mathbf{a})\}.\end{aligned}$$

The moment of momentum of the system about Γ' is thus given by

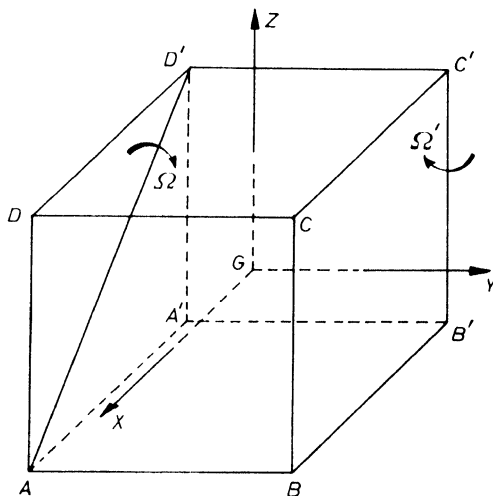
$$\begin{aligned}\lambda' \cdot \mathbf{H}_{A'} &= \lambda' \cdot \mathbf{H}_G + M\Omega \lambda' \cdot \{\mathbf{a}' \wedge (\lambda \wedge \mathbf{a})\} \\ &= \lambda' \cdot \mathbf{H}_G + M\Omega \{(\lambda' \cdot \lambda)(\mathbf{a}' \cdot \mathbf{a}) - (\lambda' \cdot \mathbf{a}')(\lambda \cdot \mathbf{a})\} \\ &= \lambda' \cdot \mathbf{H}_G + M\Omega \{(\lambda' \wedge \mathbf{a}') \cdot (\lambda \wedge \mathbf{a})\},\end{aligned}$$

which is the desired expression.

If either $\mathbf{a}$ or $\mathbf{a}'$ is zero, so that either Γ or Γ' passes through G , then the second term on the right-hand side vanishes and the above result becomes

$$\lambda' \cdot \mathbf{H}_{A'} = \lambda' \cdot \mathbf{H}_G.$$

Example. $ABCD$ and $A'B'C'D'$ are two parallel faces of a cube of side $2a$. The cube is rotating with an angular velocity Ω about the diagonal



AD' of the side $AA'DD'$. Show that the moment of momentum about the edge $B'C'$ is $Ma^2\Omega/3\sqrt{2}$, where M is the mass of the cube.

If the axis of rotation AD' is suddenly released and at the same instant the edge $B'C'$ is fixed, determine the new angular velocity of the cube.

Solution. Let G be the centre of mass of the cube, and take a set of axes $GXYZ$ perpendicular to the sides as shown. Then if A is the moment of inertia of the cube about any axis through G ,

$$\mathbf{H} \equiv (A\omega_1, A\omega_2, A\omega_3) = A\left(-\frac{\Omega}{\sqrt{2}}, 0, \frac{\Omega}{\sqrt{2}}\right),$$

where $A = 2Ma^2/3$. Also

$$\boldsymbol{\lambda} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \boldsymbol{\lambda}' = (0, 0, 1);$$

$$\mathbf{a} = \overline{GA} = (a, -a, -a), \quad \mathbf{a}' = \overline{GB'} = (-a, a, -a).^*$$

Hence, applying the general expression obtained above, we have

Moment of momentum about $B'C'$

$$\begin{aligned} &= \boldsymbol{\lambda}' \cdot \mathbf{H} + M\Omega(\boldsymbol{\lambda}' \wedge \mathbf{a}') \cdot (\boldsymbol{\lambda} \wedge \mathbf{a}) \\ &= (0, 0, 1) \cdot \frac{2}{3}Ma^2\left(-\frac{\Omega}{\sqrt{2}}, 0, \frac{\Omega}{\sqrt{2}}\right) \\ &\quad + M\Omega\left\{[(0, 0, 1) \wedge (-a, a, -a)] \cdot \left[\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \wedge (a, -a, -a)\right]\right\} \\ &= \frac{2Ma^2\Omega}{3\sqrt{2}} + M\Omega\left\{(-a, -a, 0) \cdot \left(\frac{a}{\sqrt{2}}, 0, \frac{a}{\sqrt{2}}\right)\right\} \\ &= \frac{2Ma^2\Omega}{3\sqrt{2}} - \frac{Ma^2\Omega}{\sqrt{2}} \\ &= -\frac{Ma^2\Omega}{3\sqrt{2}}, \end{aligned}$$

i.e.

Moment of momentum about $B'C' = \frac{Ma^2\Omega}{3\sqrt{2}}$ in a clockwise direction.

To find the new angular velocity about $B'C'$ when AD' is released and $B'C'$ is fixed, we make use of the fact that the moment of momentum about $B'C'$ remains unchanged in the new motion. Consequently, if Ω' is the new angular velocity and I is the moment of inertia of the cube about $B'C'$, then we must have

$$I\Omega' = \frac{Ma^2\Omega}{3\sqrt{2}},$$

* We have here chosen A and B' as the points on AD' and $B'C'$ defined by $\mathbf{a}$ and $\mathbf{a}'$ respectively. But other points, equally convenient, would have been D' and C' , or even the mid-points of AD' and $B'C'$.

the direction of Ω' being opposite to that of Ω . But, by the theorem of parallel axes,

$$I = A + M.2a^2 = \frac{8}{3}Ma^2 + 2Ma^2 = \frac{14}{3}Ma^2.$$

Hence

$$\Omega' = \Omega/8\sqrt{2},$$

its direction about $B'C'$ being clockwise.

5. General theorems

Consider a system of n particles, the masses of which are $m_i (i = 1, 2, \dots, n)$, and suppose the system to be acted upon by a set of simultaneous impulses. Let the resultant of the external impulses acting on m_i be $\mathbf{J}_i$, and the resultant of the internal impulses acting on the same particle be $\mathbf{J}'_i$. (The latter are brought into play, of course, as a result of the action of the $\mathbf{J}_i$.) Then, considering the instantaneous changes in the linear momenta of the respective particles, we have

$$m_i(\mathbf{v}_{i2} - \mathbf{v}_{i1}) = \mathbf{J}_i + \mathbf{J}'_i \quad (i = 1, 2, \dots, n), \quad (9.20)$$

where $\mathbf{v}_{i1}$ and $\mathbf{v}_{i2}$ denote the velocities of m_i immediately before and immediately after the application of the impulses.

Suppose we now multiply each of the above equations scalarly by a small displacement $\delta \mathbf{r}_i$ and add their respective sides. Then we have

$$\sum_i m_i(\mathbf{v}_{i2} - \mathbf{v}_{i1}) \cdot \delta \mathbf{r}_i = \sum_i (\mathbf{J}_i + \mathbf{J}'_i) \cdot \delta \mathbf{r}_i. \quad (9.21)$$

Now this equation, since it has been obtained by simply multiplying both sides of the constituent equations in (9.20) scalarly by equal vector quantities and adding, must hold no matter what values the $\delta \mathbf{r}_i$ may have. Thus they may be chosen to correspond to any possible small *actual* displacements of the system, or, on the other hand, they may be entirely arbitrary.

By a suitable choice of the $\delta \mathbf{r}_i$, the right-hand side of (9.21) can be considerably simplified. If we use the notation of § 1, then we may write

$$\begin{aligned} \sum_i (\mathbf{J}_i + \mathbf{J}'_i) \cdot \delta \mathbf{r}_i &= \sum_i \left\{ \lim_{\tau \rightarrow 0} \int_0^\tau (\mathbf{F}_i + \mathbf{F}'_i) dt \right\} \cdot \delta \mathbf{r}_i \\ &= \lim_{\tau \rightarrow 0} \int_0^\tau \left\{ \sum_i (\mathbf{F}_i + \mathbf{F}'_i) \cdot \delta \mathbf{r}_i \right\} dt. \end{aligned}$$

But the $\mathbf{F}'_i$'s occurring in this expression will consist of pairs of equal and opposite forces; also, if the $\delta \mathbf{r}_i$'s have been so chosen that the

mutual distance between any two particles of the system remains unchanged, then we know that for such displacements

$$\sum_i (\mathbf{F}'_i \cdot \delta \mathbf{r}_i) = 0.$$

Consequently, with this choice of the $\delta \mathbf{r}_i$'s, (9.21) reduces to

$$\begin{aligned} \sum_i m_i (\mathbf{v}_{i2} - \mathbf{v}_{i1}) \cdot \delta \mathbf{r}_i &= \lim_{\tau \rightarrow 0} \int_0^\tau \sum_i (\mathbf{F}_i \cdot \delta \mathbf{r}_i) dt \\ &= \sum_i \left(\lim_{\tau \rightarrow 0} \int_0^\tau \mathbf{F}_i dt \right) \cdot \delta \mathbf{r}_i, \end{aligned}$$

i.e.

$$\sum_i m_i (\mathbf{v}_{i2} - \mathbf{v}_{i1}) \cdot \delta \mathbf{r}_i = \sum_i (\mathbf{J}_i \cdot \delta \mathbf{r}_i). \quad (9.22)$$

Although the terms involving the internal impulses no longer occur, the term $\sum (\mathbf{J}_i \cdot \delta \mathbf{r}_i)$ can still involve the impulses due to any constraints which may be acting on the system. If, however, the constraints are frictionless and the displacements are so chosen that they are along the surface of the constraint, then the terms arising from the constraints will also disappear, and the term on the right-hand side of (9.22) will involve the external impulses alone.

When the displacements have been so chosen that this is the case, then they are termed *virtual displacements*, the term virtual meaning that the displacements are completely arbitrary except in so far that they must comply with the geometrical constraints which act on the system and are such that the relative positions of the particles of the system remain unchanged. With this choice of displacements, (9.22), when written in the form

$$\sum_i \{m_i (\mathbf{v}_{i2} - \mathbf{v}_{i1}) - \mathbf{J}_i\} \cdot \delta \mathbf{r}_i = 0, \quad (9.23)$$

is known as the *Equation of Virtual Work* for the system when acted upon by a set of instantaneous impulses. In particular, it will clearly apply if the system consists either of a rigid body or of a number of rigid bodies mutually connected by inextensible constraints, e.g. a framework of jointed rods.

We shall now use the above result to derive some general theorems governing the instantaneous change in the motion of a system of mutually interacting particles when acted upon by a set of impulses. In deriving them, it should be borne in mind that the condition for (9.22) to hold is that the $\delta \mathbf{r}_i$ occurring in the equation should correspond to any *possible* small displacement of the system in which the particles retain their relative positions. Clearly, provided that this condition is

satisfied, then the displacements may be taken to refer to the small displacement of the system corresponding to the motion of the system in a small element of time δt either immediately before or immediately after the application of the impulses. Such displacements will clearly be permissible ones if the system consists of either a rigid body or a set of rigid bodies mutually connected. In what follows, we shall assume that the system is such that this is always so. Consequently we may write

$$\delta \mathbf{r}_i = \mathbf{v}_i \delta t,$$

and (9.22) becomes

$$\sum_i m_i (\mathbf{v}_{i2} - \mathbf{v}_{i1}) \cdot \mathbf{v}_i = \sum_i (\mathbf{J}_i \cdot \mathbf{v}_i), \quad (9.24)$$

where the $\mathbf{v}_i$, without an additional suffix, may be chosen to correspond to the motion of the system immediately before the application of the impulses, immediately after the application of the impulses, or to some other motion consistent with the above conditions regarding the $\delta \mathbf{r}_i$ being satisfied.

To simplify the notation, we shall drop, in the remainder of the chapter, the suffix i , and let the suffixes 1 and 2 above denote velocities immediately before and immediately after the application of the impulses respectively, while $\mathbf{v}$ replaces $\mathbf{v}_i$. With this modification in the notation, (9.24) thus becomes

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v} = \sum (\mathbf{J} \cdot \mathbf{v}). \quad (9.25)$$

5.1. Changes in the kinetic energy produced by a set of impulses

Theorem. *If any set of instantaneous impulses is applied to a moving system, the change in the kinetic energy is equal to the sum of the scalar products of the impulses and the vector mean of the velocities of their points of application immediately before and after the action of the impulses.*

Taking $\mathbf{v}$ in (9.25) to correspond to the actual motion of the system immediately before the application of the impulses, then, on replacing $\mathbf{v}$ by $\mathbf{v}_1$, (9.25) becomes

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v}_1 = \sum (\mathbf{J} \cdot \mathbf{v}_1). \quad (9.26)$$

Similarly, by taking $\mathbf{v}$ to correspond to the motion immediately after the application of the impulses, we have

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v}_2 = \sum (\mathbf{J} \cdot \mathbf{v}_2). \quad (9.27)$$

Hence, adding (9.26) and (9.27), we have

$$\sum m(\mathbf{v}_2 \cdot \mathbf{v}_2 - \mathbf{v}_1 \cdot \mathbf{v}_1) = \sum (\mathbf{J} \cdot \mathbf{v}_1 + \mathbf{J} \cdot \mathbf{v}_2)$$

or

$$\frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) - \frac{1}{2} \sum m(\mathbf{v}_1 \cdot \mathbf{v}_1) = \sum \mathbf{J} \cdot \frac{(\mathbf{v}_1 + \mathbf{v}_2)}{2},$$

thus proving the theorem.

5.2. Kelvin's theorem.* *If any points of a connected system are suddenly set in motion with prescribed velocities, the kinetic energy of the resulting motion is less than that of any other kinematically possible motion in which the given points have the same prescribed velocities.†*

To prove the theorem, let $\mathbf{v}_2$ denote the velocity of a particle of the system in the actual motion, including the prescribed velocities of specified points of the system, and let $\sum'$ denote a summation extending only to those particles whose motion is prescribed. Let $\mathbf{J}$ denote the impulses applied to the particles to give them their prescribed velocities. Then, choosing $\mathbf{v}$ to correspond to the actual motion of the system immediately after it has been set in motion, we have, on replacing $\mathbf{v}$ by $\mathbf{v}_2$ in (9.25),

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v}_2 = \sum' (\mathbf{J} \cdot \mathbf{v}_2),$$

and hence, since the system is initially at rest so that $\mathbf{v}_1 = \mathbf{0}$,

$$\sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) = \sum' (\mathbf{J} \cdot \mathbf{v}_2). \quad (9.28)$$

But we are also free to choose $\mathbf{v}$ to correspond to a kinematically possible motion. If we denote the velocities corresponding to such a motion by $\mathbf{v}'_2$, then we have, in like manner,

$$\sum m(\mathbf{v}_2 \cdot \mathbf{v}'_2) = \sum' (\mathbf{J} \cdot \mathbf{v}_2), \quad (9.29)$$

the velocities of the given points being the same in both motions. Hence, equating the left-hand sides of (9.28) and (9.29), we have

$$\begin{aligned} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) &= \sum m(\mathbf{v}_2 \cdot \mathbf{v}'_2) \\ &= \frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) + \frac{1}{2} \sum m(\mathbf{v}'_2 \cdot \mathbf{v}_2) - \frac{1}{2} \sum m(\mathbf{v}'_2 - \mathbf{v}_2) \cdot (\mathbf{v}'_2 - \mathbf{v}_2), \end{aligned}$$

i.e.

$$\frac{1}{2} \sum m(\mathbf{v}'_2 \cdot \mathbf{v}'_2) - \frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) = \frac{1}{2} \sum m(\mathbf{v}'_2 - \mathbf{v}_2) \cdot (\mathbf{v}'_2 - \mathbf{v}_2). \quad (9.30)$$

Since the right-hand side of this equation is essentially positive, the theorem follows.

Since any kinematically possible motion may vary by an infinitesimal amount from another kinematically possible motion, we may replace

* *Lord Kelvin* (William Thomson) (1824–1907) was born in Belfast and died at Largs, Scotland. He was both a mathematician and a physicist and was appointed to the Chair of Natural Philosophy at Glasgow when only twenty-two years of age. His most important researches were in the fields of thermodynamics and applied electricity. Altogether he published more than three hundred original papers and was regarded as the foremost scientist of his time.

† By 'any kinematically possible motion' is to be understood a motion in which the points whose velocities are prescribed in the actual motion have these same velocities, while the motion as a whole is consistent with the mutual distances between the particles remaining unchanged.

the $\mathbf{v}'_2$ in the above equation by $\mathbf{v}_2 + \delta\mathbf{v}_2$. If we do this, then (9.30) reduces to

$$\sum m(\mathbf{v}_2 \cdot \delta\mathbf{v}_2) = 0,$$

or

$$\delta \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) = 0,$$

i.e. the actual motion corresponds to a stationary value of the kinetic energy. The kinetic energy of the actual motion is thus a minimum.

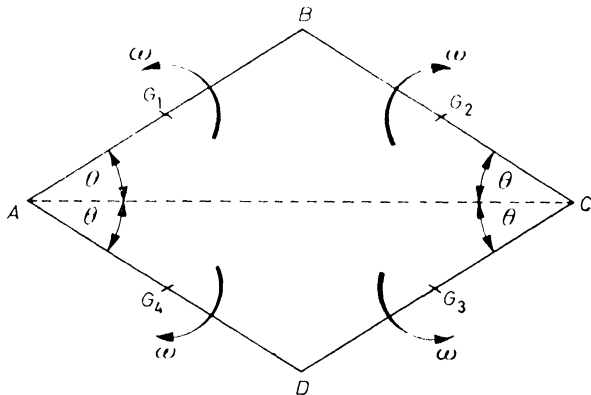
Example. A framework consisting of four uniform rods, smoothly jointed at their ends to form a rhombus $ABCD$, is at rest on a smooth horizontal plane, the angle at A being 2θ . If it is jerked into motion by an impulsive force applied at A in the direction AC , such that the velocity imparted to A is v , show that the angular velocity ω imparted to the rods is given by

$$(1 + 3 \sin^2 \theta) \omega = (3v/2a) \sin \theta,$$

where $2a$ is the length of each rod.

Solution. Since the impulse at A acts in the direction AC , the direction of v will also be that of AC . Further the angular velocity imparted to each rod will be as indicated in the diagram.

Let G_1, G_2, G_3, G_4 be the centres of mass of the four rods. Then choosing AC and a perpendicular axis through A in the plane of the



framework as axes of reference, we have:

$$\text{Coordinates of } G_1 = (a \cos \theta, a \sin \theta),$$

$$\text{Coordinates of } G_2 = (3a \cos \theta, a \sin \theta).$$

Therefore:

$$\begin{aligned} \text{Velocity components of } G_1 &= (v - a \sin \theta \dot{\theta}, a \cos \theta \dot{\theta}) \\ &= (v - a\omega \sin \theta, a\omega \cos \theta), \end{aligned}$$

$$\begin{aligned}\text{Velocity components of } G_2 &= (v-3a \sin \theta \dot{\theta}, a \cos \theta \dot{\theta}) \\ &= (v-3a\omega \sin \theta, a\omega \cos \theta).\end{aligned}$$

T , the kinetic energy of the whole framework, is thus given by

$$2T = 2m\{(v-a\omega \sin \theta)^2 + a^2\omega^2 \cos^2 \theta + (v-3a\omega \sin \theta)^2 + a^2\omega^2 \cos^2 \theta\} + 4I\omega^2, \quad (1)$$

where I is the moment of inertia of each rod with respect to an axis perpendicular to the plane of the motion and passing through its centre of mass. But, in the actual motion, ω must be such that T is a minimum. Therefore, differentiating (1) with respect to ω , the equation to determine ω becomes

$$\begin{aligned}\frac{\partial T}{\partial \omega} &= 2m\{-a \sin \theta (v-a\omega \sin \theta) - 3a \sin \theta (v-3a\omega \sin \theta) \\ &\quad + 2a^2\omega \cos^2 \theta\} + 4I\omega = 0,\end{aligned}$$

i.e.

$$\omega(I + ma^2 \cos^2 \theta + 5ma^2 \sin^2 \theta) = 2ma v \sin \theta. \quad (2)$$

Also

$$I = ma^2/3.$$

Substituting this value of I into (2), ω is thus given by

$$\omega(1 + 3 \sin^2 \theta) = \frac{3v}{2a} \sin \theta.$$

5.3. Bertrand's theorem.* *If a system in motion is acted upon by any impulses, the kinetic energy of the subsequent motion is greater than if this system were subjected to any additional constraints (of a kind which do no work in a small displacement) and acted upon by the same impulses.*

If $\mathbf{J}$ denotes the impulse applied to the system when the constraints are absent, then we have (eqn 9.25)

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v} = \sum (\mathbf{J} \cdot \mathbf{v}). \quad (9.31)$$

Also, since the additional constraints are of the type which do no work, the impulses arising from the constraints will not figure in the equation similar to (9.31) for the constrained motion. Consequently, if $\mathbf{v}'_2$ denotes the velocity of the mass m in this motion, then we shall have

$$\sum m(\mathbf{v}'_2 - \mathbf{v}_1) \cdot \mathbf{v} = \sum (\mathbf{J} \cdot \mathbf{v}).$$

* *Joseph Louis François Bertrand* (1822–1902) was born in Paris. He was an infant prodigy, being awarded the Doctorate of Science at the age of seventeen. He published numerous papers in both pure mathematics and mathematical physics and held professorships in these subjects. His final appointment was that of Professor of Mathematical Physics in the Collège de France to which he was appointed in 1862 and which he held until his death.

But we may clearly take $\mathbf{v}$ in both equations to correspond to the actual motion immediately after the imposition of the constraints. Consequently, putting $\mathbf{v} = \mathbf{v}'_2$, we have

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v}'_2 = \sum (\mathbf{J} \cdot \mathbf{v}'_2),$$

$$\sum m(\mathbf{v}'_2 - \mathbf{v}_1) \cdot \mathbf{v}'_2 = \sum (\mathbf{J} \cdot \mathbf{v}'_2).$$

On equating the left-hand sides, we thus have

$$\sum m(\mathbf{v}'_2 - \mathbf{v}_1) \cdot \mathbf{v}'_2 = \sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v}'_2,$$

i.e.

$$\begin{aligned} \sum m(\mathbf{v}'_2 \cdot \mathbf{v}'_2) &= \sum m(\mathbf{v}_2 \cdot \mathbf{v}'_2) \\ &= \frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) + \frac{1}{2} \sum m(\mathbf{v}'_2 \cdot \mathbf{v}'_2) - \frac{1}{2} \sum m(\mathbf{v}_2 - \mathbf{v}'_2) \cdot (\mathbf{v}_2 - \mathbf{v}'_2). \end{aligned}$$

Thus

$$\frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) - \frac{1}{2} \sum m(\mathbf{v}'_2 \cdot \mathbf{v}'_2) = \frac{1}{2} \sum m(\mathbf{v}_2 - \mathbf{v}'_2) \cdot (\mathbf{v}_2 - \mathbf{v}'_2), \quad (9.32)$$

and the theorem is proved.

The expression occurring on the right-hand side of (9.32) is termed the *kinetic energy of the relative motion* of the two motions defined by the velocities $\mathbf{v}_2$ and $\mathbf{v}'_2$ respectively. It is thus seen that the difference between the kinetic energies in the two states of motion is equal to the kinetic energy of their relative motion.

It should be noticed that, in the above treatment, the $\mathbf{v}'_2$'s refer to a motion of the system when subject to frictionless constraints.* They therefore refer to the motion of a system which differs kinematically from the given one. Thus we cannot, in this case, replace $\mathbf{v}'_2$ in the above expression by $\mathbf{v}_2 + \delta\mathbf{v}_2$ as we did in the case of Kelvin's theorem, since it does not follow that frictionless constraints can be imposed in such a way that the $\mathbf{v}_2$'s and the $\mathbf{v}'_2$'s corresponding to the different systems differ from one another by infinitesimal amounts. Thus Bertrand's theorem involves no stationary property.

The motion resulting from the sudden imposition of a set of impulses on a moving system can, however, be associated with the stationary value of a certain function as is proved in the following theorem.

5.4. Robin's theorem.† *If frictionless constraints are suddenly imposed on a moving system simultaneously with the application of a set of impulses $\mathbf{J}$, the values of the velocities immediately afterwards are such as to give a stationary value to the expression.*

$$\frac{1}{2} \sum m\{(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1)\} - \sum \mathbf{J} \cdot (\mathbf{v}_2 - \mathbf{v}_1). \quad (9.33)$$

* The term *frictionless constraint* is here used to denote a constraint which does no work in a small displacement.

† *Victor Gustave Robin* (1855–1897) was born in Paris. He was a mathematical physicist of note who died tragically early. His collected works were published posthumously in three volumes between the years 1899 and 1903.

Since the constraints imposed are frictionless, then we have (eqn 9.25)

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v} = \sum (\mathbf{J} \cdot \mathbf{v}).$$

Now $\mathbf{v}$ may clearly be taken to correspond to the motion immediately after the imposition of the constraints, so that we have, on putting $\mathbf{v} = \mathbf{v}_2$,

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v}_2 = \sum (\mathbf{J} \cdot \mathbf{v}_2). \quad (9.34)$$

We may also take $\mathbf{v}$ to correspond to any other kinematically possible motion compatible with the constraints. If $\mathbf{v}'_2$ represents such a motion, then we also have

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v}'_2 = \sum (\mathbf{J} \cdot \mathbf{v}'_2). \quad (9.35)$$

Subtracting (9.34) from (9.35), we therefore have

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}'_2 - \mathbf{v}_2) = \sum \mathbf{J} \cdot (\mathbf{v}'_2 - \mathbf{v}_2). \quad (9.36)$$

But, since any kinematically possible motion may vary by an infinitesimal amount from another kinematically possible motion, we may, as in the case of Kelvin's theorem, replace $\mathbf{v}'_2$ by $\mathbf{v}_2 + \delta\mathbf{v}_2$. Then, neglecting terms of second order, (9.36) becomes

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \delta\mathbf{v}_2 - \sum \mathbf{J} \cdot \delta\mathbf{v}_2 = 0,$$

$$\text{or} \quad \delta\left\{\frac{1}{2}\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1) - \sum \mathbf{J} \cdot (\mathbf{v}_2 - \mathbf{v}_1)\right\} = 0.$$

The actual motion thus corresponds to a stationary value of the expression

$$\frac{1}{2}\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1) - \sum \mathbf{J} \cdot (\mathbf{v}_2 - \mathbf{v}_1),$$

and the theorem is proved.

That the stationary value corresponds to a minimum value can easily be shown. For we have

$$\begin{aligned} & \sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}'_2 - \mathbf{v}_2) \\ &= \sum m(\mathbf{v}_2 \cdot \mathbf{v}'_2) - \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) - \sum m(\mathbf{v}_1 \cdot \mathbf{v}'_2) + \sum m(\mathbf{v}_1 \cdot \mathbf{v}_2) \\ &= \sum m(\mathbf{v}_2 \cdot \mathbf{v}'_2) - \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) + \frac{1}{2}\sum m(\mathbf{v}'_2 - \mathbf{v}_1)^2 - \frac{1}{2}\sum m(\mathbf{v}_2 - \mathbf{v}_1)^2 \\ & \quad - \frac{1}{2}\sum m(\mathbf{v}'_2 \cdot \mathbf{v}'_2) + \frac{1}{2}\sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) \\ &= \frac{1}{2}\sum m(\mathbf{v}'_2 - \mathbf{v}_1)^2 - \frac{1}{2}\sum m(\mathbf{v}_2 - \mathbf{v}_1)^2 - \frac{1}{2}\sum m(\mathbf{v}'_2 \cdot \mathbf{v}'_2) \\ & \quad - \frac{1}{2}\sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) + \sum m(\mathbf{v}_2 \cdot \mathbf{v}'_2) \\ &= \frac{1}{2}\sum m(\mathbf{v}'_2 - \mathbf{v}'_1)^2 - \frac{1}{2}\sum m(\mathbf{v}_2 - \mathbf{v}_1)^2 \\ & \quad - \frac{1}{2}\sum m\{(\mathbf{v}'_2 \cdot \mathbf{v}'_2) + (\mathbf{v}_2 \cdot \mathbf{v}_2) - 2(\mathbf{v}_2 \cdot \mathbf{v}'_2)\} \\ &= \frac{1}{2}\sum m(\mathbf{v}'_2 - \mathbf{v}_1)^2 - \frac{1}{2}\sum m(\mathbf{v}_2 - \mathbf{v}_1)^2 - \frac{1}{2}\sum m(\mathbf{v}'_2 - \mathbf{v}_2)^2. \end{aligned}$$

Using this result, (9.36) may be written

$$\begin{aligned} & \left\{\frac{1}{2}\sum m(\mathbf{v}'_2 - \mathbf{v}_1)^2 - \sum \mathbf{J} \cdot (\mathbf{v}'_2 - \mathbf{v}_1)\right\} - \left\{\frac{1}{2}\sum m(\mathbf{v}_2 - \mathbf{v}_1)^2 - \sum \mathbf{J} \cdot (\mathbf{v}_2 - \mathbf{v}_1)\right\} \\ & \quad = \frac{1}{2}\sum m(\mathbf{v}'_2 - \mathbf{v}_2)^2. \end{aligned}$$

The right-hand side of this equation is essentially positive, and the value of the expression

$$I' = \frac{1}{2} \sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1) - \sum \mathbf{J} \cdot (\mathbf{v}_2 - \mathbf{v}_1) \quad (9.37)$$

for the actual motion is thus less than that for any other kinematically possible motion compatible with the constraints.

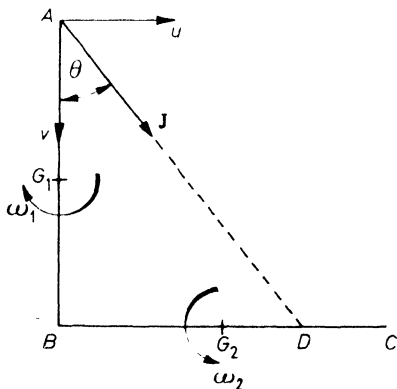
Special Cases

(1) *Only impulses of the type $\mathbf{J}$ are imposed on the system, the frictionless constraints being absent.* It follows, by reasoning similar to that above, that the new motion will also be such as to give a stationary value to the expression (9.33).

(2) *The impulses of the type $\mathbf{J}$ are absent, the only impulses acting on the system being those arising from the sudden application of the frictionless constraints.* The above result then shows that the new velocities resulting from the application of these constraints must be such as to make the kinetic energy of the relative motion a minimum.

Examples

(1) *Two equal uniform rods AB, BC of length $2a$ are freely jointed at B and lie on a smooth horizontal plane, the angle at B being $\pi/2$, while BC is free to rotate about its end C which is fixed. An impulsive blow is given to A towards D , a point in BC such that $DC = \frac{1}{4}BC$. Find the initial motions of the rods.*



Solution. Let $\mathbf{J}$ be the impulse applied at A . Since the end C of the rod BC is fixed, it follows that simultaneously with the application of $\mathbf{J}$ at A an impulse comes into play at C . Furthermore, since C is at rest in the new motion, this impulse is of the type which does no work when the small displacements $\delta \mathbf{r}$ are chosen to correspond to the

motion immediately after the application of the impulse at A . The initial motion of the system may thus be found by applying Robin's theorem, the angular velocities of the rods being such that they give a stationary value to the expression

$$\Gamma = \frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{J} \cdot \mathbf{v}_A), \quad (1)$$

$\mathbf{v}_A$ being the velocity of the point A .

To evaluate this expression, let u and v be the component velocities of A at right angles to AB and along AB respectively, ω_1 and ω_2 , the angular velocities of the rods, being as shown. Then, if G_1, G_2 are the mid-points of AB and BC respectively, we have:

Component velocities of G_1 parallel to u and v respectively

$$= (u - a\omega_1, v);$$

Component velocities of G_2 parallel to u and v respectively

$$= (u - 2a\omega_1, v - a\omega_2).$$

The first term in (1), being equivalent to the kinetic energy of the system, is thus given by

$$\frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) = \frac{1}{2} m \{ (u - a\omega_1)^2 + v^2 + (u - 2a\omega_1)^2 + (v - a\omega_2)^2 \} + \frac{1}{2} I (\omega_1^2 + \omega_2^2), \quad (2)$$

where I is the moment of inertia of either rod about a perpendicular axis through its centre of mass. But, since A is a fixed point, we must clearly have

$$u - 2a\omega_1 = 0, \quad v - a\omega_2 = 0,$$

so that

$$u = 2a\omega_1, \quad v = a\omega_2. \quad (3)$$

Hence, substituting for u and v , (2) becomes

$$\begin{aligned} \frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) &= \frac{1}{2} m (a^2 \omega_1^2 + 4a^2 \omega_2^2 + a^2 \omega_2^2) + \frac{1}{6} m a^2 (\omega_1^2 + \omega_2^2) \\ &= \frac{2}{3} m a^2 (\omega_1^2 + 4\omega_2^2). \end{aligned} \quad (4)$$

We also have

$$\begin{aligned} \mathbf{J} \cdot \mathbf{v}_A &= J \sin \theta \cdot u + J \cos \theta \cdot v \quad (\theta = \angle BCD) \\ &= \frac{3}{5} J \cdot u + \frac{4}{5} J \cdot v \\ &= \frac{2}{5} a J (3\omega_1 + 4\omega_2). \end{aligned} \quad (5)$$

Hence, substituting from (4) and (5), expression (1) for Γ becomes

$$\Gamma = \frac{2}{3} m a^2 (\omega_1^2 + 4\omega_2^2) - \frac{2}{5} a J (3\omega_1 + 4\omega_2).$$

Γ is thus a function of the two independent variables ω_1 and ω_2 .

Differentiating with respect to these, the equations to determine the angular velocities are thus:

$$\frac{\partial \Gamma}{\partial \omega_1} = \frac{4}{3}ma^2\omega_1 - \frac{6}{5}aJ = 0,$$

$$\frac{\partial \Gamma}{\partial \omega_2} = \frac{1}{3}ma^2\omega_2 - \frac{8}{5}aJ = 0.$$

Hence

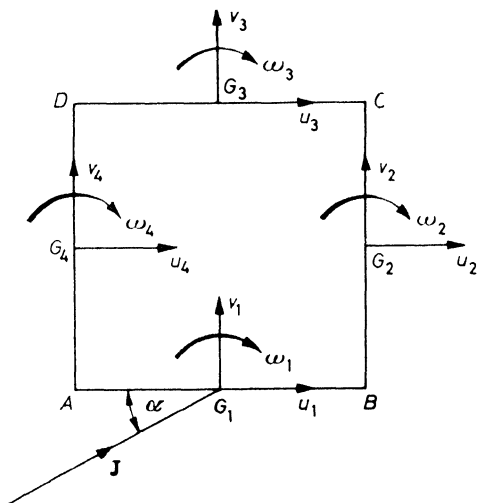
$$\omega_1 = \frac{9J}{10ma}, \quad \omega_2 = \frac{3J}{10ma},$$

while, from (3), u and v , the component velocities of C , are given by

$$u = 2a\omega_1 = \frac{9J}{5m}, \quad v = 2a\omega_2 = \frac{3J}{5m}.$$

The motion of each rod has thus been determined.

(2) *Four equal uniform rods of mass m are freely jointed together to form a square. An impulse $\mathbf{J}$ is applied at the mid-point of one of the rods*



along a line which makes an angle α with the rod and lies in the plane of the square. Show that the kinetic energy created is

$$\frac{J^2}{16m}(5 - 3 \sin^2 \alpha).$$

Solution. In this example no impulsive forces of constraint come into play, the only impulse acting being the single applied impulse $\mathbf{J}$. The problem may therefore be solved by applying Robin's theorem, *Special Case (1)*.

Let G_1, G_2, G_3, G_4 be the mid-points of AB, BC, CD, DA respectively, and let their component velocities be $(u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4)$ as shown. Also, let $\omega_1, \omega_2, \omega_3, \omega_4$ be the angular velocities of the four rods, their directions being in all cases clockwise. Then, since the four rods are of equal length, it follows that

$$\omega_1 = \omega_3, \quad \omega_2 = \omega_4.$$

Also, only two of the eight component velocities $(u_1, v_1), \dots, (u_4, v_4)$ can be regarded as independent. Choosing these to be (u_1, v_1) , then we have:

Velocity components of $G_1 = (u_1, v_1)$,

$$\begin{aligned} \text{,,} \quad \text{,,} \quad \text{,,} \quad B &= (u_1, v_1 - a\omega_1), \\ \text{,,} \quad \text{,,} \quad \text{,,} \quad G_2 &= (u_1 + a\omega_2, v_1 - a\omega_1), \\ \text{,,} \quad \text{,,} \quad \text{,,} \quad C &= (u_1 + 2a\omega_2, v_1 - a\omega_1), \\ \text{,,} \quad \text{,,} \quad \text{,,} \quad G_3 &= (u_1 + 2a\omega_2, v_1 - a\omega_1 + a\omega_3) \\ &= (u_1 + 2a\omega_2, v_1), \\ \text{,,} \quad \text{,,} \quad \text{,,} \quad D &= (u_1 + 2a\omega_2, v_1 - a\omega_1 + 2a\omega_3) \\ &= (u_1 + 2a\omega_2, v_1 + a\omega_1), \\ \text{,,} \quad \text{,,} \quad \text{,,} \quad G_4 &= (u_1 + 2a\omega_2 - a\omega_4, v_1 + a\omega_1) \\ &= (u_1 + a\omega_2, v_1 + a\omega_1). \end{aligned}$$

To evaluate T eqn 9.37, since $\mathbf{v}_1 = 0$, we have

$$\begin{aligned} \frac{1}{2} \sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1) &= \frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) \\ &= \text{kinetic energy of system } (= T); \end{aligned}$$

$$\text{also } \sum \mathbf{J} \cdot (\mathbf{v}_2 - \mathbf{v}_1) = \mathbf{J} \cdot \mathbf{v}_2. \quad \text{Hence, } T = \mathbf{J} \cdot \mathbf{v}_2.$$

But it follows, using the above expressions for the velocity components, that

$$\begin{aligned} T &= \frac{1}{2} m \{ (u_1^2 + v_1^2) + (u_1 + a\omega_2)^2 + (v_1 - a\omega_1)^2 + (u_1 + 2a\omega_1)^2 + v_1^2 \\ &\quad + (u_1 + a\omega_2)^2 + (v_1 + a\omega_1)^2 \} + \frac{1}{2} I (2\omega_1^2 + 2\omega_2^2) \\ &= \frac{1}{2} m \{ u_1^2 + 2v_1^2 + 2(u_1 + a\omega_2)^2 + (v_1 - a\omega_1)^2 + (u_1 + 2a\omega_2)^2 \\ &\quad + (v_1 + a\omega_1)^2 \} + \frac{1}{3} m a^2 (\omega_1^2 + \omega_2^2), \quad (1) \end{aligned}$$

since $I = ma^2/3$, while

$$\mathbf{J} \cdot \mathbf{v}_2 = J \cos \alpha u_1 + J \sin \alpha v_1.$$

Hence,

$$\begin{aligned} T &= \frac{1}{2} m \{ u_1^2 + 2v_1^2 + 2(u_1 + a\omega_2)^2 + (v_1 - a\omega_1)^2 + (u_1 + 2a\omega_2)^2 + (v_1 + a\omega_1)^2 \} \\ &\quad + \frac{1}{3} m a^2 (\omega_1^2 + \omega_2^2) - (J \cos \alpha u_1 + J \sin \alpha v_1). \end{aligned}$$

Differentiating this expression partially with respect to $u_1, v_1, \omega_1, \omega_2$, the equations to determine the motion thus become

$$\frac{\partial \Gamma}{\partial u_1} = m\{u_1 + 2(u_1 + a\omega_2) + (u_1 + 2a\omega_2)\} - J \cos \alpha = 0, \quad (2)$$

$$\frac{\partial \Gamma}{\partial v_1} = m\{2v_1 + (v_1 - a\omega_1) + (v_1 + a\omega_1)\} - J \sin \alpha = 0, \quad (3)$$

$$\frac{\partial \Gamma}{\partial \omega_1} = m\{-a(v_1 - a\omega_1) + a(v_1 + a\omega_1)\} + \frac{2}{3}ma^2\omega_1 = 0, \quad (4)$$

$$\frac{\partial \Gamma}{\partial \omega_2} = m\{2a(u_1 + a\omega_2) + 2a(u_1 + 2a\omega_2)\} + \frac{2}{3}ma^2\omega_2 = 0. \quad (5)$$

On solving these, we have, from (2), (3), (4), and (5) respectively:

$$u_1 = J \cos \alpha / 4m - a\omega_2, \quad (6)$$

$$v_1 = J \sin \alpha / 4m, \quad (7)$$

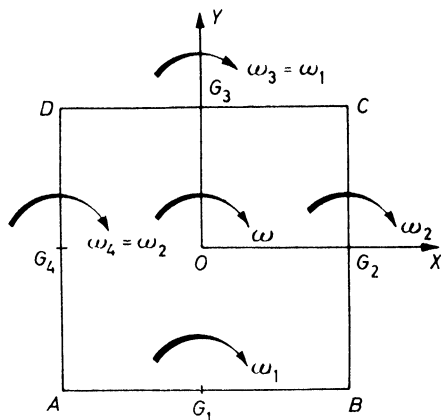
$$\omega_1 = 0, \quad (8)$$

$$5a\omega_2 = -3u_1. \quad (9)$$

From (6) and (9) we now have

$$u_1 = 5J \cos \alpha / 8m, \quad a\omega_2 = -3J \cos \alpha / 8m.$$

The four quantities $u_1, v_1, \omega_1, \omega_2$ have thus been evaluated in terms of J , and the motion of the system thus determined. The desired value of T is obtained on substituting the above values of $u_1, v_1, \omega_1, \omega_2$ into (1).



(3) Four equal rods are smoothly jointed in the form of a square $ABCD$, which is rotating with angular velocity ω about the perpendicular to the plane of the square through its centre. The mid-point of one of the rods is suddenly fixed. Show that immediately afterwards the angular velocity of the rod is unaltered, while that of an adjacent rod is $2\omega/5$.

Solution. In this case the only impulse acting on the system is that arising from the imposition of a frictionless constraint, when the mid-point of one of the rods is fixed. The problem may therefore be solved by means of Robin's theorem, *Special Case* (2).

Let G_1, G_2, G_3, G_4 be the mid-points of the rods, G_1 being the point which is suddenly fixed. Let ω_1 be the new angular velocities of AB and CD , ω_2 the new angular velocities of BC and DA . Then, if $2a$ is the length of each rod, we have, on taking orthogonal axes OX, OY as shown,

	Component velocities before fixing	Component velocities after fixing
G_1	$(-a\omega, 0)$	$(0, 0)$
G_2	$(0, -a\omega)$	$(a\omega_2, -a\omega_1)$
G_3	$(a\omega, 0)$	$(2a\omega_2, 0)$
G_4	$(0, a\omega)$	$(a\omega_2, a\omega_1)$

The kinetic energy of the relative motion is thus given by*

$$T_{\text{rel}} = \frac{1}{2}ma^2\{\omega^2 + \omega_2^2 + (\omega_1 - \omega)^2 + (2\omega_2 + \omega)^2 + \omega_2^2 + (\omega_1 - \omega)^2\} \\ + I\{(\omega_1 - \omega)^2 + (\omega_2 - \omega)^2\}.$$

But $I = ma^2/3$. Therefore

$$T_{\text{rel}} = \frac{1}{2}ma^2\{\omega^2 + 2\omega_2^2 + 2(\omega_1 - \omega)^2 + (2\omega_2 - \omega)^2\} \\ + \frac{1}{3}ma^2\{(\omega_1 - \omega)^2 + (\omega_2 - \omega)^2\}.$$

* In the two dimensional motion of a uniform rod, if, immediately before the application of an impulse, (u, v) are the orthogonal components of the velocity of its centre of mass, along and perpendicular to its length, and ω is its angular velocity, while the corresponding quantities immediately after are (u', v') and ω' respectively, then the relative kinetic energy of the two motions is given by

$$T_{\text{rel}} = \frac{1}{2}m\{(u' - u)^2 + (v' + v)^2\} + \frac{1}{2}I(\omega' - \omega)^2.$$

That this is so readily follows from the fact that, by the general definition, if x is the distance of an element of the rod of mass δm from its centre, then,

$$T_{\text{rel}} = \frac{1}{2}\sum \delta m[(u' - u)^2 + \{(v' + x\omega') - (v + x\omega)^2\}], \\ = \frac{1}{2}\sum \delta m\{(u' - u)^2 + (v' - v)^2 - 2x(v' - v)(\omega' - \omega) + x^2(\omega' - \omega)^2\} \\ = \frac{1}{2}[(\sum \delta m)\{(u' - u)^2 + (v' - v)^2\} - 2(v' - v)(\omega' - \omega)\sum \delta m x + (\omega' - \omega)^2\sum \delta m x^2] \\ = \frac{1}{2}m\{(u' - u)^2 + (v' - v)^2\} + \frac{1}{2}I(\omega' - \omega)^2,$$

since $\sum \delta m x = 0$ and $I = \sum \delta m x^2$.

Differentiating with respect to ω_1 and ω_2 , the equations to determine the angular velocities are thus:

$$\begin{aligned}\frac{1}{ma^2} \frac{\partial \Gamma}{\partial \omega_1} &= 2(\omega_1 - \omega) + \frac{2}{3}(\omega_1 - \omega) = 0, \\ \frac{1}{ma^2} \frac{\partial \Gamma}{\partial \omega_2} &= 2\omega_2 + 2(2\omega_2 - \omega) + \frac{2}{3}(\omega_2 - \omega) = 0,\end{aligned}$$

or, on collecting terms,

$$\omega_1 - \omega = 0, \quad 5\omega_2 - 2\omega = 0.$$

Thus

$$\omega_1 = \omega, \quad \omega_2 = 2\omega/5.$$

5.5. Carnot's theorem.* *When constraints of the type which do no work are suddenly imposed on a system, kinetic energy is always lost.*

Since the constraints are frictionless, (9.25) reduces to

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v} = 0. \quad (9.38)$$

But, clearly, we may take $\mathbf{v}$ to correspond to the motion of the system immediately *after* the imposition of the constraints. Replacing $\mathbf{v}$ by $\mathbf{v}_2$, therefore, we have

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v}_2 = 0,$$

or

$$\begin{aligned}\sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) &= \sum m(\mathbf{v}_1 \cdot \mathbf{v}_2) \\ &= \frac{1}{2} \sum m(\mathbf{v}_1 \cdot \mathbf{v}_1) + \frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) - \frac{1}{2} \sum m(\mathbf{v}_1 - \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2),\end{aligned}$$

i.e.

$$\frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) - \frac{1}{2} \sum m(\mathbf{v}_1 \cdot \mathbf{v}_1) = -\frac{1}{2} \sum m(\mathbf{v}_1 - \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2).$$

Since the right-hand side of this equation is always negative, the result follows.

It is seen that the amount of kinetic energy lost is equal to the kinetic energy of the relative motion.

5.6. Converse of Carnot's theorem. *If constraints of the type which do no work are suddenly removed, kinetic energy is always gained.*

Equation (9.38) again holds. In this case, however, in order for the small displacements, occurring in the equation of virtual work (9.22) or (9.23), to be consistent with the condition that the constraints should do no work, they must clearly be chosen to correspond to the motion

* Nicolas Léonard Sadi Carnot (1796–1832) was an officer in the French army and also a scientist. His main research work was in thermodynamics, principally on heat engines, but he also made important contributions in the field of mechanics.

immediately *before* the removal of the constraints. Hence, writing $\mathbf{v} = \mathbf{v}_1$ in (9.38), we have

$$\sum m(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{v} = 0,$$

leading to the result

$$\frac{1}{2} \sum m(\mathbf{v}_2 \cdot \mathbf{v}_2) - \frac{1}{2} \sum m(\mathbf{v}_1 \cdot \mathbf{v}_1) = \frac{1}{2} \sum m(\mathbf{v}_1 - \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2),$$

thus proving the result.

The frictionless constraints, introduced in Carnot's theorem, may take the form of uniting material systems, previously free, by such means as strings, hinges, etc., the impulsive reactions from which do no work in the immediate subsequent motion. The removal of the constraints considered in the converse of Carnot's theorem is illustrated by the explosion of a shell, when kinetic energy is clearly generated consequent on the fragmentation. Another example would be the sudden breaking of the rope in a tug-of-war contest between two teams.

Examples

1. A uniform right circular cone, of vertical angle 2α , is rotating freely about its axis when a point on the rim of its base is suddenly fixed. Prove that the axis about which it proceeds to rotate makes with the axis of the cone an angle

$$\tan^{-1} \frac{5 \tan \alpha}{2 + 3 \tan^2 \alpha}.$$

2. A uniform solid cube, of mass M and edge $2a$, is spinning freely about a diagonal through one corner O , when suddenly an edge OA becomes fixed. Show that the kinetic energy is reduced to one twelfth of its initial value.

3. A uniform solid cube, of mass M and edge $2a$, is rotating freely about the centre O of one of its faces, the component angular velocities parallel to three adjacent edges being each equal to $\Omega/\sqrt{3}$. The point O is suddenly set free, and a corner A of the opposite face is fixed. Determine the impulsive reaction at A and the new angular velocity of the cube.

4. A uniform square plate $ABCD$ of side length $2a$, free to move in any manner, is spinning about the diagonal AC with angular velocity Ω . Show that the corner B may be reduced instantaneously to rest by a suitable blow applied at the middle point of AB normal to the plate. Prove that the angular velocity of the plate immediately after the application of the blow is $\Omega\sqrt{5/8}$ about the line joining B to the middle point of CD .

5. Three equal uniform rods AB, BC, CD freely jointed at B and C are placed so as to form three sides of a square $ABCD$. If A is suddenly jerked parallel to the direction CB , show that the angular velocity of OB is initially seven times that of CD .

6. Two uniform rods AB, BC of masses m, m' are smoothly hinged at B and rest on a smooth horizontal table in the same straight line. The end A is fixed and the mid-point of BC is in contact with a smooth peg. If an impulse J be applied to the mid-point of AB , perpendicular to its length and in such a way that BC is pressed against the constraining peg at its mid-point, show that the kinetic energy of the system immediately after the application of the impulse is

$$3J^2/8(m+m').$$

7. Equal uniform rods AB, BC, CD , each of length $2a$ and mass m , are smoothly hinged at B and C . They rest on a smooth horizontal table with angles ABC, BCD each $\pi/2$ and AB and CD are on opposite sides of BC . An impulse J is applied at A in the direction AC . Prove that the initial angular velocities of the rods are in the ratios $-49:12:7$, and that the kinetic energy generated is

$$55J^2/56m.$$

8. Four uniform rods, each of mass m and length $2a$, are freely jointed together to form a rhombus $ABCD$. At the instant when the framework is square and rotating in its own plane about A with an angular velocity ω as if rigid, A is set free and the mid-point of CD is fixed. Show that the loss of kinetic energy is

$$26ma^2\omega^2/5.$$

APPENDIX

THREE DIMENSIONAL STATICS

1. Conditions of equilibrium of a rigid body.

Although statics is treated, in general, as a distinct branch of mechanics, it is, nevertheless, simply a special case of dynamics. In statics one is required to find either the system of forces which will keep a body at rest in a given position or the position in which a body will remain at rest when acted upon by a given system of forces. If a body is to be at rest when acted upon by a system of forces the conditions to be satisfied may readily be deduced from the equations of motion of such a body.

As we have seen in Chapter VI, equations (6.33) and (6.34), the equations of motion of a body moving under a given system of forces $\mathbf{F}$, are

$$M \frac{d\mathbf{v}}{dt} = \sum \mathbf{F} \quad (\text{A.1})$$

$$\frac{d\mathbf{H}}{dt} = \sum (\mathbf{r} \wedge \mathbf{F}), \quad (\text{A.2})$$

where $\mathbf{v}$ is the velocity of G , the centre of mass of the body, and $\mathbf{H}$ is the moment of momentum of the body with respect to G . The former determines the translational motion of the centre of mass, while the latter determines the rotational motion of the body. In the event of the forces being such that

$$\left. \begin{aligned} \sum \mathbf{F} &= \mathbf{O}, & (a) \\ \sum (\mathbf{r} \wedge \mathbf{F}) &= \mathbf{O}, & (b) \end{aligned} \right\} \quad (\text{A.3})$$

then

$$\frac{d\mathbf{v}}{dt} = \mathbf{O}, \quad \frac{d\mathbf{H}}{dt} = \mathbf{O},$$

so that both $\mathbf{v}$ and $\mathbf{H}$ have either constant magnitudes and directions or, alternatively, are zero. The conditions which must be satisfied by the forces acting on a body, if the body is either at rest or has a uniform motion (i.e. one in which $\mathbf{v}$ and $\boldsymbol{\omega}$, the angular velocity, are both constant vectors), are therefore given by (A.3).

Now consider the case in which the body is at rest, and suppose we take moments about some point O of the body. Let the vector position

of P , the point of application of an external force acting on the body, with respect to O be $\mathbf{r}'$. Then taking moments about O , we have, since $\mathbf{r} = \overline{GO} + \mathbf{r}'$,

$$\begin{aligned}\sum(\mathbf{r}' \wedge \mathbf{F}) &= \sum\{(\mathbf{r} - \overline{GO}) \wedge \mathbf{F}\} \\ &= \sum(\mathbf{r} \wedge \mathbf{F}) - \overline{GO} \wedge \sum \mathbf{F}.\end{aligned}\quad (\text{A.4})$$

Hence, if equations (A.3) are satisfied, then it follows that

$$\sum(\mathbf{r}' \wedge \mathbf{F}) = \mathbf{O},$$

so that the vector sum of the moments of the forces with respect to O vanishes, while, conversely, if

$$\sum \mathbf{F} = \mathbf{O}, \quad \sum(\mathbf{r}' \wedge \mathbf{F}) = \mathbf{O},$$

then

$$\sum(\mathbf{r} \wedge \mathbf{F}) = \mathbf{O},$$

so that equations (A.3) are satisfied. Consequently it follows that the conditions, necessary and sufficient, for a body to remain at rest when acted upon by a system of forces $\mathbf{F}$ are

$$\sum \mathbf{F} = \mathbf{O}, \quad (a) \quad \sum(\mathbf{r} \wedge \mathbf{F}) = \mathbf{O}, \quad (b) \quad (\text{A.5})$$

where $\mathbf{r}$ now denotes the vector position of the point of application of an external force with respect to any arbitrarily chosen point of the body.

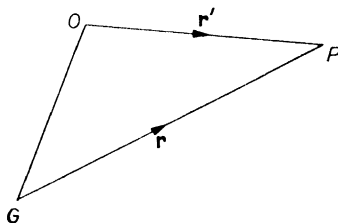


Figure A.1

Expressed in words, we thus have the result: *If a rigid body is at rest, then the necessary and sufficient conditions for it to remain at rest when acted upon by a system of forces are (i) the vector sum of the forces should vanish, and (ii) the vector sum of the moments of the forces with respect to any arbitrarily chosen point O of the body should also vanish.*

If (x, y, z) , (X, Y, Z) are the components of $\mathbf{r}$ and $\mathbf{F}$ respectively with respect to a set of orthogonal axes having their origin at O , then the scalar equations equivalent to (A.5) are

$$\left. \begin{aligned} \sum X &= \sum Y = \sum Z = 0, \\ \sum(yZ - zY) &= \sum(zX - xZ) = \sum(xY - yX) = 0. \end{aligned} \right\} \quad (\text{A.6})$$

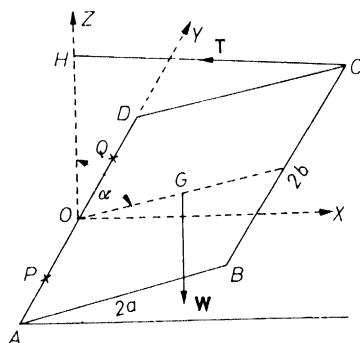
Since, for a body to be in equilibrium, the vector sum of the moments of the forces acting on the body with respect to any point of the body

must be zero, it follows that the algebraic sum of the moments of the forces about any axis through an arbitrarily chosen point of the body must also be zero.

1.1. Examples

(1) A rectangular lamina $ABCD$ of edges $AB = 2a$, $AD = 2b$, has two smooth hinges at points P and Q on the horizontal edge AD equidistant from O , the mid-point of AD . The lamina is kept at an angle α with the vertical plane through AD by a light string joining a fixed point H , vertically above O , to the corner C of the lamina. To find the tension in the string.

Solution. Let T represent the tension in the string, and let $OXYZ$ be a set of mutually orthogonal axes such that OX is horizontal and at right angles to AD , OY is in the direction of AD , and OZ is vertically upwards, the system being right-handed. Since we are not interested in the reactions at the hinges but only require the tension in the string,



one scalar equation involving T will suffice for its determination. This equation may be obtained by expressing the fact that, in the position of equilibrium, the algebraic sum of the moments of the forces acting on the lamina about OY must be zero, i.e. that

$$M_y = \mathbf{j} \cdot \mathbf{M}_0 = 0,$$

where $\mathbf{j}$ is a unit vector in the direction of OY and $\mathbf{M}_0$ is the vector sum of the moments of the forces with respect to OY .

Now $\mathbf{M}_0$ is given by

$$\mathbf{M}_0 = \overline{OG} \wedge \mathbf{W} + \overline{OC} \wedge \mathbf{T},$$

and hence

$$M_y = \mathbf{j} \cdot \mathbf{M}_0 = \mathbf{j} \cdot (\overline{OG} \wedge \mathbf{W}) + \mathbf{j} \cdot (\overline{OC} \wedge \mathbf{T}).$$

But

$$\mathbf{j} = (0, 1, 0), \quad \mathbf{W} = (0, 0, -W), \quad \mathbf{T} = (T_1, T_2, T_3),$$

$$\overline{OG} = (a \sin \alpha, 0, a \cos \alpha), \quad \overline{OC} = (2a \sin \alpha, b, 2a \cos \alpha),$$

and the expression for M_y thus becomes

$$M_y = \begin{vmatrix} 0 & 1 & 0 \\ a \sin \alpha & 0 & a \cos \alpha \\ 0 & 0 & -W \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ 2a \sin \alpha & b & 2a \cos \alpha \\ T_1 & T_2 & T_3 \end{vmatrix}$$

$$= aW \sin \alpha + 2aT_1 \cos \alpha - 2aT_3 \sin \alpha.$$

Hence, putting $M_y = 0$, we have

$$T_3 \sin \alpha - T_1 \cos \alpha = W \sin \alpha. \quad (1)$$

But, since $OH = h$, the coordinates of H are $(0, 0, h)$, while those of C are $(2a \sin \alpha, b, 2a \cos \alpha)$. Consequently the direction cosines of CH are given by

$$(l, m, n) = \left(\frac{-2a \sin \alpha}{c}, -\frac{b}{c}, \frac{h - 2a \cos \alpha}{c} \right) \quad (C = CH),$$

and hence

$$T_1 = Tl = -2a \frac{\sin \alpha}{c} T,$$

$$T_3 = Tn = \frac{(h - 2a \cos \alpha)}{c} T.$$

Substituting for T_1 and T_3 into (1), we thus have

$$\frac{2T}{c} \{ (h - 2a \cos \alpha) \sin \alpha + 2a \sin \alpha \cos \alpha \} = W \sin \alpha,$$

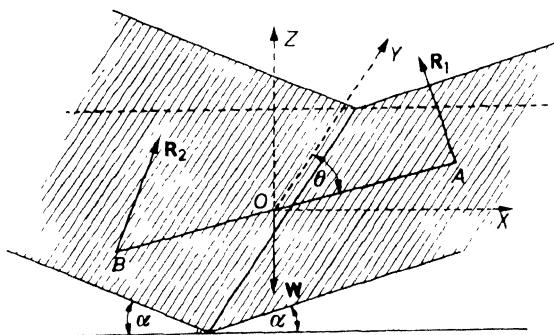
i.e.

$$T = cW/2h.$$

(2) *Two smooth planes intersect in a horizontal line and are inclined at the same angle α to the horizontal. A uniform rod of weight W and length $2a$ is placed between them in a horizontal position making an angle θ with their line of intersection. Show that a horizontal couple of magnitude $Wa \cos \theta \tan \alpha$ is required to maintain equilibrium.*

Solution. Let AB be the rod, O its mid-point. Choose a set of axes $OXYZ$ such that OX is horizontal and at right-angles to the line of intersection of the planes, OY is parallel to the line of intersection, and OZ is vertically upwards, the direction of the axes being such that the system is right-handed.

Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be the reactions of the planes on the rod at A and B respectively, and let $\mathbf{G}$ be the couple which must be applied to the rod



to maintain equilibrium. Then, applying the general conditions for equilibrium, moments being taken with respect to O , we must have

$$\mathbf{R}_1 + \mathbf{R}_2 + \mathbf{W} = \mathbf{0}, \quad (1)$$

$$\overline{OA} \wedge \mathbf{R}_1 + \overline{OB} \wedge \mathbf{R}_2 + \mathbf{G} = \mathbf{0}. \quad (2)$$

Now

$$\mathbf{R}_1 = (-R_1 \sin \alpha, 0, R_1 \cos \alpha),$$

$$\mathbf{R}_2 = (R_2 \sin \alpha, 0, R_2 \cos \alpha),$$

$$\mathbf{W} = (0, 0, -W),$$

$$\mathbf{G} = (G_1, G_2, G_3);$$

also

$$\overline{OA} = -\overline{OB} = (a \sin \theta, a \cos \theta, 0).$$

Applying (1), the equivalent scalar equations are thus:

$$(R_2 - R_1) \sin \alpha = 0,$$

$$(R_1 + R_2) \cos \alpha - W = 0.$$

These give

$$R_1 = R_2 = R, \quad (3)$$

say,

$$R = W/2 \cos \alpha. \quad (4)$$

Applying (2), we have, using (3) and (4),

$$(a \sin \theta, a \cos \theta, 0) \wedge \{(-R \sin \alpha, 0, R \cos \alpha) - (R \sin \alpha, 0, R \cos \alpha)\} \\ + (G_1, G_2, G_3) = (0, 0, 0),$$

or

$$(a \sin \theta, a \cos \theta, 0) \wedge (-2R \sin \alpha, 0, 0) + (G_1, G_2, G_3) = (0, 0, 0),$$

so that the equivalent scalar equations are:

$$G_1 = 0, \quad G_2 = 0, \quad 2Ra \cos \theta \sin \alpha + G_3 = 0.$$

Thus

$$G_1 = G_2 = 0,$$

while

$$\begin{aligned} G_3 &= -2Ra \cos \theta \sin \alpha \\ &= -Wa \cos \theta \tan \alpha, \end{aligned}$$

on substituting for R from (4). Thus, for equilibrium to be maintained, a horizontal couple of magnitude $Wa \cos \theta \tan \alpha$, acting in a $\searrow$ direction, must be applied to the rod.

Note. It should be realised that, in any problem, once the vector equations (A.5) for the equilibrium of the body have been written down and a suitable set of coordinate axes chosen, so that the components of the various vectors occurring in them can be evaluated, then the writing down of the scalar equations required for the solution of the problem becomes a purely routine process and constitutes a standard technique for the solution of statical problems.

1.2. Equilibrium of a body under the action of three forces.

Suppose a body to be in equilibrium under the action of three forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$. Let their points of application be P_1, P_2, P_3 , and let the vector displacements of P_2 and P_3 with respect to P_1 be $\mathbf{r}_2$ and $\mathbf{r}_3$ respectively. Then, for equilibrium, we must have

$$\Sigma \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}, \quad (\text{A.7})$$

while, taking moments with respect to P_1 ,

$$\Sigma (\mathbf{r} \wedge \mathbf{F}) = \mathbf{r}_2 \wedge \mathbf{F}_2 + \mathbf{r}_3 \wedge \mathbf{F}_3 = \mathbf{0}. \quad (\text{A.8})$$

Multiplying (8) scalarly by $\mathbf{r}_2$, we must thus have

$$\mathbf{r}_2 \cdot (\mathbf{r}_3 \wedge \mathbf{F}_3) = 0, \quad (\text{A.9})$$

i.e. since in general no two of the vectors occurring in the scalar product need be parallel, $\mathbf{F}_3$ must lie in the plane $P_1P_2P_3$ defined by $\mathbf{r}_2$ and $\mathbf{r}_3$. Alternatively (A.9) expresses the fact that the moment of $\mathbf{F}_3$ with respect to the line P_1P_2 must be zero. In the same way, it can be shown that $\mathbf{F}_1$ and $\mathbf{F}_2$ must lie in the plane $P_1P_2P_3$. Thus, in order for a body to be in equilibrium under the action of three forces, the forces must be co-planar.

Also, if we multiply (A.7) vectorially by $\mathbf{F}_1$ say, then we must have

$$\mathbf{F}_1 \wedge (\mathbf{F}_2 + \mathbf{F}_3) = \mathbf{0}.$$

This can be satisfied if (i) $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ are all parallel, (ii) if they are not all parallel, then provided the resultant of $\mathbf{F}_2$ and $\mathbf{F}_3$ is parallel to $\mathbf{F}_1$. But, if this condition holds, then, for the body to be in equilibrium so

that the moment of the resultant with respect to P_1 vanishes, as required by the general conditions of equilibrium, the resultant must pass through P_1 , i.e. if the forces are not all parallel then for equilibrium they must be concurrent.

Thus we have the general result: *if a rigid body is in equilibrium under the action of three forces, then the forces must be coplanar and either concurrent or parallel.*

2. Equivalence of two sets of forces

Two sets of forces are defined to be equivalent if they produce the same effect when acting upon the same body. If $\mathbf{F}$ and $\mathbf{F}'$ denote the two sets of forces, and $\mathbf{r}$ and $\mathbf{r}'$ the vector displacements of their respective points of application with respect to G , the centre of mass of the body, then it follows from (A.1) and (A.2) that the two systems will be equivalent provided

$$(i) \sum \mathbf{F} = \sum \mathbf{F}' \quad (ii) \sum (\mathbf{r} \wedge \mathbf{F}) = \sum (\mathbf{r}' \wedge \mathbf{F}'). \quad (\text{A.10})$$

The two sets will also be equivalent if $\sum \mathbf{F} = \sum \mathbf{F}'$, while the vector sums of their moments with respect to any given fixed point are the same, since, if these conditions hold, equations (A.10) are also satisfied.

From the above, it follows that, in the case of a couple, since

$$\sum \mathbf{F} = \mathbf{0}$$

while

$$\sum (\mathbf{r} \wedge \mathbf{F}) = \mathbf{G} = \text{moment of the couple}$$

no matter what the point about which moments are taken, any two couples having the same vector moment are under all circumstances equivalent. Similarly, it follows that the forces and the arm of a couple may be varied in any way one pleases, or the couple moved about in any manner one wishes, without its effect being altered, provided only that its vector moment remains unchanged.

3. Reduction of a system of forces to a wrench

Consider a force $\mathbf{F}$ acting at a point P and let O be some other point. Suppose we introduce two equal and opposite forces, $\mathbf{F}$ and $-\mathbf{F}$, to act at O (Fig. A.2). It follows (§ 1.2) that the system consisting of the single force $\mathbf{F}$ at P and the system consisting of the force $\mathbf{F}$ at P together with the forces $(\mathbf{F}, -\mathbf{F})$ at O will be equivalent, the conditions for equivalence as set out above being satisfied. We thus have the result: *a force $\mathbf{F}$ acting at a point P is equivalent to a force $\mathbf{F}$ acting at any other point O together with a couple of moment $(\overline{OP} \wedge \mathbf{F})$.* Since any force may be replaced in this way by an equal and parallel force acting at an arbitrary point O together with a couple, we have the general result: *a*

system of forces $\mathbf{F}_1, \mathbf{F}_2, \dots$, acting at the points $P_1, P_2, \dots$, will be equivalent to a single force $\mathbf{F}$ acting at an arbitrary point O together with a couple $\mathbf{G}$, where $\mathbf{F} = \sum \mathbf{F}_i$ is the resultant of the forces acting at O and $\mathbf{G} = \sum (\overline{OP}_i \wedge \mathbf{F}_i)$ is the resultant couple. It is clear that while the magnitude and direction of $\mathbf{F}$ are independent of the choice of O , those of $\mathbf{G}$ are not. It should also be realised that $\mathbf{F}$ is a vector localized at O , but $\mathbf{G}$, although its magnitude depends upon the position of O , is a free vector.

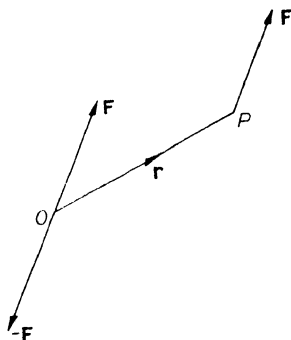


Figure A.2

Now suppose we compare the above theory with that for the motion of a rigid body resulting from the superposing of two or more angular velocities. In Chapter III, § 3, we found that an angular velocity $\boldsymbol{\omega}$ about an axis through a point P is equivalent to an equal and parallel angular velocity $\boldsymbol{\omega}$ about an axis through an arbitrary point O together with a velocity of translation $\mathbf{v}_0$ given by

$$\mathbf{v}_0 = \boldsymbol{\omega} \wedge \overline{PO} = \overline{OP} \wedge \boldsymbol{\omega},$$

while, more generally, the motion resulting from simultaneous angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \dots$, about axes passing through the points $P_1, P_2, \dots$, respectively is equivalent to the motion resulting from an angular velocity

$$\boldsymbol{\omega} = \sum \boldsymbol{\omega}_i$$

about an axis through an arbitrary point O together with a velocity of translation given by

$$\mathbf{v}_0 = \sum (\overline{OP} \wedge \boldsymbol{\omega}_i);$$

the resultant angular velocity $\boldsymbol{\omega}$ is independent of the choice of O , but $\mathbf{v}_0$ varies, both as regards its magnitude and direction, with the choice of O .

If we compare the above expressions for $\mathbf{F}$ and $\mathbf{G}$ with those for $\boldsymbol{\omega}$ and $\mathbf{v}_0$, then it is seen that the two representations are completely analogous with $\boldsymbol{\omega}$ and $\mathbf{v}_0$ replacing $\mathbf{F}$ and $\mathbf{G}$. In view of this correspondence between the equivalent representations in the two cases, one can readily deduce certain results for a system of forces. Thus, just as it was found possible to represent the motion of a rigid body by an equivalent screw motion in which the velocity of translation has a direction parallel to that of the resultant angular velocity, so it is possible to replace a system of forces by a single force together with a couple, the axis of the couple being parallel to the direction of the resultant force. This representation is termed a *wrench*, the line of action of the force $\mathbf{F}$ in this representation being again termed the *central axis*. If a system of forces is equivalent to a force $\mathbf{F}$ acting at a point O together with a couple $\mathbf{G}$, then, proceeding as in Chapter III, the vector equation of the central axis of the equivalent wrench is given by

$$\left(\mathbf{r} - \frac{\mathbf{F} \wedge \mathbf{G}}{F^2}\right) \wedge \mathbf{F} = \mathbf{0}$$

(see eqn. 3.6), while the pitch of the wrench, being defined as the ratio of the magnitude of the resultant couple to that of the resultant force, is given by

$$p = \mathbf{F} \cdot \mathbf{G} / F^2$$

(equation 3.9). The analogous quantities entering into the respective representations are shown in the accompanying table.

	<i>Instantaneous Motion of a Rigid Body</i>	<i>Force System</i>
<i>Basic Vectors associated with a base point O</i>	<i>Resultant Angular Velocity</i> $\boldsymbol{\omega}$. <i>Resultant Transla- tional Velocity</i> $\mathbf{v}_0$	<i>Resultant Force</i> $\mathbf{F}$ <i>Resultant Couple</i> $\mathbf{G}$
<i>Invariants of the system</i>	$\boldsymbol{\omega} \cdot \boldsymbol{\omega} = \omega^2$, $\boldsymbol{\omega} \cdot \mathbf{v}_0$	$\mathbf{F} \cdot \mathbf{F} = F^2$, $\mathbf{F} \cdot \mathbf{G}$
<i>Simplest Representation</i>	<i>Screw Motion</i> (Pitch = $\boldsymbol{\omega} \cdot \mathbf{v}_0 / \omega^2$)	<i>Wrench</i> (Pitch = $\mathbf{F} \cdot \mathbf{G} / F^2$)

In the motion of a rigid body, the condition for the instantaneous motion of the body to be equivalent to a pure rotation was (§ 3.2.1)

$$\boldsymbol{\omega} \cdot \mathbf{v}_0 = 0;$$

a system of forces may similarly be replaced by a single force if

$$\mathbf{F} \cdot \mathbf{G} = 0.$$

4. Work done by a wrench

The above representations may be used to derive an expression for the work done by a system of forces acting on a rigid body when the body undergoes an infinitesimal displacement.

Let the forces acting on the body reduce to a wrench of intensity $\mathbf{F}$ and pitch p , and let the small displacement which the body undergoes be equivalent to that resulting from a screw motion of intensity $\boldsymbol{\omega}$ and pitch p' supposed acting for an interval of time δt . Let OO' be the shortest distance between the two axes.

Suppose we take a set of axes $OXYZ$ having O as origin, OX along the axis of the wrench, OY in the direction of OO' , and OZ a mutual orthogonal axis such that $OXYZ$ forms a right-handed system. If O' lies to the right of O , then the system will be as shown (Fig. A.3). Let the angle which the axis of the screw makes with the OXY plane be γ .

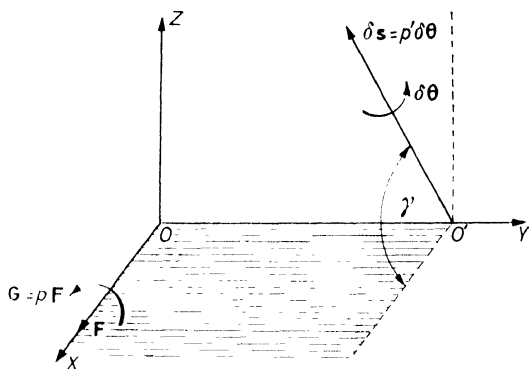


Figure A.3

Now the force $\mathbf{F}$ acting at O together with the couple $\mathbf{G}$ ($= p\mathbf{F}$) will be equivalent to a force $\mathbf{F}$ acting at O' together with a couple $\mathbf{G}'$ whose magnitude and direction are those of the resultant of $\mathbf{G}$ and a couple whose moment is equal to that of $\mathbf{F}$ acting at O with respect to O' . If $OO' = h$, and if $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are unit vectors in the directions of the co-ordinate axes respectively, we shall thus have

$$\begin{aligned}
 \mathbf{G}' &= \mathbf{G} + (\overline{OO'} \wedge \mathbf{F}) \\
 &= \mathbf{G} - (hj \wedge F\mathbf{i}) \\
 &= \mathbf{G} + Fh(\mathbf{i} \wedge \mathbf{j}) \\
 &= \mathbf{G} + Fh\mathbf{k}.
 \end{aligned}$$

Also the displacement of the body in the interval of time δt will be equivalent to a translational displacement of amount $\delta \mathbf{s}$, where

$$\delta \mathbf{s} = \mathbf{v} \delta t = \mathbf{p}' \omega \delta t$$

together with a rotational displacement of amount $\delta \boldsymbol{\theta} = \boldsymbol{\omega} \delta t$. Consequently, considering the forces acting on the body as being represented by the force $\mathbf{F}$ at O' together with the couple $\mathbf{G}'$, then in the small displacement of the body defined by the infinitesimal vectors $\delta \mathbf{s}$ and $\delta \boldsymbol{\theta}$, the work done by the forces will be given by

$$\begin{aligned} \delta W &= \mathbf{F} \cdot \delta \mathbf{s} + \mathbf{G}' \cdot \delta \boldsymbol{\theta} \\ &= \mathbf{F} \cdot \delta \mathbf{s} + (\mathbf{G} + Fh\mathbf{k}) \cdot \delta \boldsymbol{\theta} \\ &= (\mathbf{F} \cdot \mathbf{p}' \boldsymbol{\omega}) \delta t + (p\mathbf{F} + Fh\mathbf{k}) \cdot \boldsymbol{\omega} \delta t \\ &= \{(\mathbf{F} \cdot \boldsymbol{\omega})p' + (\mathbf{F} \cdot \boldsymbol{\omega})p + Fh(\mathbf{k} \cdot \boldsymbol{\omega})\} \delta t \\ &= \{F\omega(p+p') \cos \gamma + Fh\omega \sin \gamma\} \delta t \\ &= F\omega \delta t \{(p+p') \cos \gamma + h \sin \gamma\}. \end{aligned}$$

If the point O were to lie to the right of O' instead of to the left, as in the diagram, then it is easily seen that the corresponding result would be

$$\delta W = F\omega \delta t \{(p+p') \cos \gamma - h \sin \gamma\}.$$

The work done by the wrench, therefore, is given by

$$\delta W = F\omega \delta t \{(p+p') \cos \gamma \pm h \sin \gamma\},$$

according as to whether O' lies to the right or left of O .

Examples

1. A heavy rectangular trap-door $ABCD$ of weight W hangs with its plane vertical and its line of hinges AD inclined at an angle α above the horizontal. It is opened through an angle θ and kept in this position by a horizontal force F applied at the corner B in a direction perpendicular to the plane of the door when hanging freely. Show that in the equilibrium position

$$F = \frac{1}{2} W \cos \alpha \tan \theta.$$

2. A tripod consists of three equal uniform bars, each of length a and weight W , which are freely jointed at one extremity, their middle points being joined by strings, each of length $\frac{1}{2}a$. The tripod is placed with its free ends in contact with a smooth horizontal plane and a weight $3nW$ is attached to the common joint; show that the tension in each string is

$$(2n+1)W/\sqrt{6}.$$

3. A uniform rod, of length $2l$ and weight W , rests in a horizontal position with its ends on two smooth planes; each plane is inclined at an angle α to the horizontal and their line of intersection is horizontal and makes an angle θ with the rod. The rod is held in equilibrium by a force P acting through its mid-point, and inclined to the downward vertical at an acute angle ϕ , together with a couple. Show that the angle which the axis of the couple makes with the vertical is

$$\cot^{-1}\{\cos \theta \tan^2 \alpha (\cot \phi + \frac{W}{P} \operatorname{cosec} \phi)\},$$

and prove that, if $P > W \tan \alpha$, then equilibrium is possible only if

$$\phi \leq \alpha + \sin^{-1}(W \sin \alpha / P).$$

4. A uniform rod of weight W can turn freely about a hinge at one end and rests with the other end against a rough vertical wall, making an angle β with the normal to the wall. Show that this end may rest anywhere on the arc of a circle of angle $2 \tan^{-1}(\mu \cot \beta)$, where μ is the coefficient of friction, and that, at the extremities of the arc, the normal reaction of the wall is $\frac{1}{2}W/\sqrt{(\mu^2 + \tan^2 \beta)}$.

5. A uniform circular cylinder of weight W and length $2l$ is pivoted freely at the centre O of its base. It rests obliquely against a smooth horizontal rail, being supported by a force F acting through the centre of the face not containing O , F being parallel to the rail. Show that

$$F = W \cos \alpha \cos \gamma / 2 \sin^2 \alpha,$$

where γ is the inclination of the axis of the cylinder to the vertical and α is the angle between the axis of the cylinder and the rail.

Show, also, that the magnitude of the reaction at the rail is

$$\frac{Wl(1 - \cos^2 \alpha - \cos^2 \gamma)^{1/2}}{\lambda \sin \alpha},$$

where λ is the perpendicular distance of the point of contact from the base of the cylinder.

6. A, B, C are three points on the axes OX, OY, OZ respectively at distances a from the origin. Forces $P, 2P, 3P$ act at A, B, C parallel to BC, CA, AB respectively. Show that they are equivalent to a single force and determine the position of its line of action.

HISTORICAL NOTE

The names of certain mathematicians and mathematical physicists have occurred in the text in connection with important results; as an aid to a better appreciation of the development of the subject, they are here listed in chronological order.

Newton, Sir Isaac	(1642-1727)
Euler, Leonard	(1707-1783)
Poinsot, Louis	(1777-1859)
Coriolis, Gustave Gaspard de	(1792-1843)
Carnot, Nicolas Leonard Sadi	(1796-1832)
Foucault, Jean Bernard Léon	(1819-1868)
Bertrand, Joseph Louis François	(1822-1902)
Kelvin, Lord (William Thomson)	(1824-1907)
Robin, Victor Gustave	(1853-1897)

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